A study of local symmetric properties and geometric constants in Banach spaces

Souvik Ghosh
(Index No.: 83/21/Maths./27)

THIS THESIS IS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS
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JADAVPUR UNIVERSITY
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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**A study of local symmetric properties and geometric constants in Banach spaces**" submitted by **Souvik Ghosh** who got his name registered on 19/07/2021 (**Index No.: 83/21/Maths./27**) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own research work under the supervision of **Prof. Kallol Paul**, Department of Mathematics, Jadavpur University, Kolkata 700032, India and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

KPaul 16.06

(Prof. Kallol Paul)
(Signature of the Supervisor
and date with official seal)

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Dedicated to my parents

Mrs. Hiranmoyee Ghosh

and

Mr. Samir Kumar Ghosh

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Souvik Ghosh

Abstract

This dissertation explores the symmetricity of elements in Banach spaces and in the space of all bounded linear operators with respect to various notions of orthogonality. We analyze left and right symmetric elements in Banach spaces with respect to T-orthogonality, and extend our investigation to the space of bounded linear operators through numerical radius orthogonality (nr-orthogonality). It is shown that the only nr-left symmetric bounded operator on a Hilbert space is the zero operator, while no nonzero compact normal operator on an infinite-dimensional Hilbert space can be nr-right symmetric. Some necessary and sufficient conditions are established separately for nr-left and nr-right symmetric operators defined on Banach spaces. We further examine symmetricity with respect to norm derivative orthogonality, i.e., ρ-orthogonality. In this context, we prove that a two-dimensional Banach space is a strictly convex Radon plane if every element is ρ-symmetric. Additionally, we characterize Hilbert spaces via ρ-symmetricity for the spaces with dimensions greater than or equal to three. A complete classifications of ρ-symmetric elements in ℓ_1^n and ℓ_∞^n are also provided. Apart from symmetricity we also study several important geometric constants in Banach spaces. We show that elements attaining the James constant must be isosceles orthogonal to each other. We explore the relationship between the modulus of convexity and approximate isosceles orthogonality. To measure the quantitative difference between Birkhoff-James orthogonality and ρ-orthogonality, we introduce a new constant revealing that a Banach space is uniformly non-square if this constant is less than 0.5. We compute the exact value of this constant for spaces whose unit ball is a regular $2n$-gon. In the end, we explore another local geometric constant both in general Banach spaces and in the space of all bounded linear operators. By investigating the properties of this local constant, we obtain sufficient conditions under which the collection of smooth points forms an open set in a Banach space and characterize approximate smoothness in the space of all bounded linear operators.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Motivation and historical backgrounds

In the 3rd century BC, the Greek mathematician Euclid laid the foundation for the systematic study of geometry in his renowned work *'Elements'*. The geometry developed in this work came to be known as Euclidean geometry, and the corresponding mathematical setting is referred to as Euclidean space. The journey of Euclidean space started from here. One of the fundamental notion of an Euclidean space is the concept of angle between two vectors, based on which one can define the perpendicularity between them. The natural generalization of an Euclidean space is the inner product space, where the concept of perpendicularity or orthogonality of two elements can be defined in terms of the inner product of the space. This idea of orthogonality is immensely helpful in the study of the geometry of inner product spaces. However, a general normed linear space lacks such a straightforward definition of orthogonality. To overcome this, several mathematicians have introduced various notions of orthogonality, such as Birkhoff-James orthogonality, isosceles orthogonality, Roberts orthogonality, Pythagorean orthogonality and so on. These definitions actually generalize the classical inner product orthogonality but they are not equivalent to one another in a normed linear space. Each brings a unique perspective that contributes to a deeper understanding of the geometric structure and local analytic properties (e.g., smoothness, convexity, symmetry, best approximation) of a normed linear space.

Among the various types of orthogonality notions, Birkhoff-James orthogonality, arguably the most important one, has been extensively explored, both in normed linear spaces and in the context of bounded linear operators, for investigating local and global geometric and analytic characteristics. In this dissertation we focus on the study of geometric constants introduced by various orthogonality notions along with the study of local symmetricity and smoothness in the setting of Banach spaces as well as in the space of bounded linear operators.

1.2 Basic geometric properties

Understanding the geometric structure of the unit ball of a normed linear space is one of the most significant study in functional analysis and its applications. Various notions such as smoothness, strict convexity, and exposed points are used to characterize the behavior of normed linear spaces in both the local and global sense. These properties influence important aspects such as the uniqueness of best approximations, differentiability of the norm, duality mappings, and optimization theory. We begin by recalling some foundational definitions that will be used throughout the discussion.

Let $\mathbb{X}$ denote a normed linear space over the field of real numbers unless specified otherwise and let $\mathbb{X}^*$ be the dual of $\mathbb{X}$. The notation $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ stand for the unit ball and the unit sphere of $\mathbb{X}$, respectively. We use the term 'dim($\mathbb{X}$)' for dimension of the space $\mathbb{X}$. We take the symbol $\mathbb{H}$ to denote inner product spaces. $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ denotes the space of all bounded linear operators defined from $\mathbb{X}$ to $\mathbb{Y}$. Whenever $\mathbb{X} = \mathbb{Y}$, we write $\mathbb{L}(\mathbb{X})$ instead of $\mathbb{L}(\mathbb{X}, \mathbb{X})$ for simplicity.

Definition 1.1. *(i)* ***Extreme point:*** *Suppose that C is a convex subset of a normed linear space $\mathbb{X}$. A point $x \in C$ is said to be an extreme point of C, i.e., $x \in \operatorname{Ext}(C)$ if $x = (1-t)y + tz$, for some $0 < t < 1$ and $y, z \in C$ then it implies $x = y = z$.*

(ii) ***Exposed point:*** *Let C be a convex subset of a normed linear space $\mathbb{X}$. Then $x \in C$ is said to be an exposed point of C, i.e., $x \in \operatorname{Exp}(C)$ if there exists $x^* \in S_{\mathbb{X}^*}$ such that $x^*(x) = 1 > x^*(y)$ for all $y \neq x$.*

(iii) ***Strictly convex:*** *A normed linear space $\mathbb{X}$ is said to be strictly convex if each element of the unit sphere is an extreme point of the unit ball of $\mathbb{X}$.*

(iv) ***Smoothness:*** *Let $\mathbb{X}$ be a normed linear space and let $x \in S_{\mathbb{X}}$. x is said to be smooth if there exists a unique supporting functional of the unit ball at x, i.e., there exists unique $x^* \in S_{\mathbb{X}^*}$ such that $x^*(x) = 1$.*

If each element of the unit sphere of $\mathbb{X}$ is smooth, then $\mathbb{X}$ is called a smooth.

Regarding the notion of extreme points we would like to note one of the most fundamental theory in Banach space geometry.

Theorem 1.1. ***Krein-Milmann Theorem:*** *Let C be a nonempty compact convex subset of a normed linear space. Then C is the closed convex hull of its extreme points, i.e.,* $C = \overline{\operatorname{conv}\{\operatorname{Ext}(C)\}}$.

Next, we mention two notions which are stronger than strict convexity and smoothness in an infinite-dimensional normed linear space.

Definition 1.2. *Let $\mathbb{X}$ be a normed linear space. Then*

(i) Given $\epsilon \in [0,2]$,

$$\delta_{\mathbb{X}}(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\right\}$$

is called the modulus of convexity of $\mathbb{X}$. If $\delta_{\mathbb{X}}(\epsilon) > 0$ for each $\epsilon > 0$ then we say $\mathbb{X}$ is uniformly convex.

(ii) Given any $\epsilon \geq 0$, the function

$$\rho_{\mathbb{X}}(\epsilon) = \sup\left\{\frac{\|x+\epsilon y\| + \|x-\epsilon y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}$$

is called the modulus of smoothness. $\mathbb{X}$ is said to be uniformly smooth if $\lim_{\epsilon \to 0^+} \frac{\rho_{\mathbb{X}}(\epsilon)}{\epsilon} = 0$.

In finite-dimensional Banach space uniform convexity and uniform smoothness coincide with strict convexity and smoothness, respectively. For infinite-dimensional case the following results hold true.

Theorem 1.2. *(i) (Milman-Pettis theorem [61, Th. 5.2.12]) Every uniformly convex Banach space is reflexive.*
(ii) (Šmulian [61, Th. 5.5.13]) Every uniformly smooth Banach space is reflexive.

Definition 1.3. ***(Uniform non-squareness)****[41] Let $\mathbb{X}$ be a normed linear space. Then the unit ball of $\mathbb{X}$ is said to be uniformly non-square if there is a positive number δ such that there do not exist any members x, y of the unit ball satisfying $\|\frac{1}{2}(x+y)\| > 1-\delta$ and $\|\frac{1}{2}(x-y)\| > 1-\delta$.*

In his seminal paper ([41]), James showed that a Banach space is reflexive if its unit ball is uniformly non-square.
It is important to note that the properties discussed thus far pertain to general Banach spaces. We now introduce a local property relevant to elements of the space of all bounded linear

operators namely the numerical range. The concept of the numerical range (also known as the field of values) has its roots in early 20th-century functional analysis and operator theory. It was first introduced by Otto Toeplitz in 1918 in the context of linear transformations on Hilbert spaces. For a complex Hilbert space $\mathbb{H}$ and $T \in \mathbb{L}(\mathbb{H})$, the numerical range is defined as

$$W(T) = \big\{\langle Tx, x\rangle : x \in \mathbb{H}, \|x\| = 1\big\}.$$

This set captures some geometric information about the operator such as the range of the spectrum of T. Toeplitz observed that $W(T)$ is always a convex subset of the complex plane a result now known as the Toeplitz-Hausdorff Theorem, later rigorously proven by Hausdorff in 1919. For a rigorous proof see [36, Th. 1.1-2]. In order to extend the idea of the numerical range to Banach spaces, mathematicians introduced the spatial numerical range, which, for simplicity, we will refer to as the numerical range.

Definition 1.4. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X})$ the space of all bounded linear operators.*

- Numerical range *of T is denoted by $W(T)$ and is defined by*

$$W(T) = \{x^*(Tx) : \|x\| = 1 = \|x^*\|, x^*(x) = 1\}.$$

- Numerical radius *of T is defined by*

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

Numerical radius of an operator in a complex Banach space induces a norm on $\mathbb{L}(\mathbb{X})$. However, the numerical radius need not be a norm in case of real scalar field. For more information on this topic readers can visit [15, 17, 31, 36] and the references therein.

1.3 Various orthogonality notions in normed linear spaces

There are several notions of orthogonality in the setting of normed linear spaces. Though each of them is a generalization of usual inner product orthogonality, they are non-equivalent in normed linear spaces. We mention some of them here which are well-known. The first one was introduced in [74] by B. D. Roberts in 1934.

Definition 1.5. *(**Roberts' orthogonality**, 1934) Let $\mathbb{X}$ be a normed linear space and let $x, y \in \mathbb{X}$. Then x is said to be Roberts orthogonal to y ($x \perp_R y$) if $\|x + \alpha y\| = \|x - \alpha y\|$, for any scalar α.*

It is easy to observe that given any $x, y \in \mathbb{X}$, $x \perp_R y \implies \alpha x \perp_R \beta y$, for all $\alpha, \beta \in \mathbb{R}$. Also, $x \perp_R y \implies y \perp_R x$. In other words, Roberts' orthogonality is homogeneous and symmetric in nature.

One year later(1935), G. Birkhoff introduced another notion of orthogonality in [16].

Definition 1.6. *(**Birkhoff**, 1935) Let x, y are two elements of a normed linear space $\mathbb{X}$. Then x is said to be Birkhoff orthogonal to y ($x \perp_B y$) if $\|x + \lambda y\| \geq \|x\|$, for any scalar λ.*

In [42, 44], R. C. James studied this orthogonality notion in details and characterized it to make this orthogonality free from the norm structure. From then on it is called *Birkhoff-James orthogonality*. This dual space connection makes Birkhoff-James orthogonality more appreciated in general settings.

Lemma 1.1. *[42, Th. 2.1] Let $x, y \in \mathbb{X}$ then $x \perp_B y$ if and only if there exists $x^* \in S_{\mathbb{X}^*}$ such that $x^*(x) = 1$ and $x^*(y) = 0$.*

From the above characterization one can observe that for any given nonzero $x \in \mathbb{X}$ there exists a hyperplane H in $\mathbb{X}$ such that $x \perp_B H$. Although this orthogonality is homogeneous but lacks symmetricity. More details on symmetric properties of Birkhoff-James orthogonality are discussed in Section 1.4.

James [44] introduced two more orthogonalities in normed linear spaces namely, isosceles orthogonality and Pythagorean orthogonality.

Definition 1.7. *(**Isosceles orthogonality**, 1947) Let $x, y \in \mathbb{X}$. x is said to be isosceles orthogonal to y ($x \perp_I y$) if $\|x + y\| = \|x - y\|$.*

Clearly, isosceles orthogonality is symmetric, whereas it lacks homogeneity in general. In fact, James [44] proved that isosceles orthogonality is homogeneous in $\mathbb{X}$ if and only if the norm of $\mathbb{X}$ is induced by an inner product.

Definition 1.8. *(**Pythagorean orthogonality**, 1947) In a normed linear space $\mathbb{X}$, a vector x is said to be Pythagorean orthogonal to a vector y ($x \perp_P y$) if $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.*

Note that Pythagorean orthogonality is symmetric. In [44], it is shown that Pythagorean orthogonality is homogeneous in $\mathbb{X}$ if and only if $\mathbb{X}$ is an inner product space.

Next we would like to mention a comparative result regarding the above two orthogonalities which eventually characterizes the inner product space.

Theorem 1.3. *[6] Let $\mathbb{X}$ be a normed linear space. Then the following are equivalent:*

(i) $x \perp_I y \implies x \perp_B y$, *for all* $x, y \in \mathbb{X}$.

(ii) $x \perp_P y \implies x \perp_B y$, *for all* $x, y \in \mathbb{X}$.

(iii) $x \perp_I y \implies x \perp_P y$, *for all* $x, y \in \mathbb{X}$.

(iv) $\mathbb{X}$ *is a Hilbert space.*

1.4 Symmetricity with respect to Birkhoff-James orthogonality

Birkhoff-James orthogonality is not symmetric in general. In other words, $x \perp_B y$ does not always necessarily imply that $y \perp_B x$ in a Banach space. Let us observe this by the following figure.

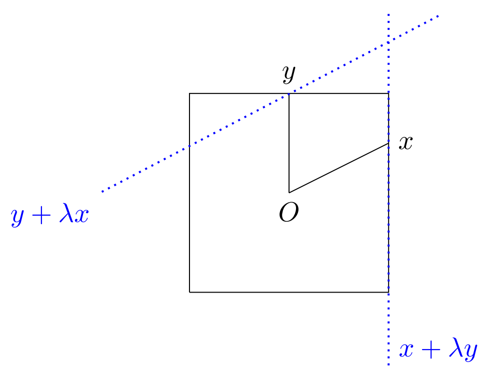

Figure 1.1: Unit ball of $(\mathbb{R}^2, \|\cdot\|_\infty)$

The above square (Fig. 1.1) is taken as the unit ball of $\mathbb{R}^2$ with respect to the maximum norm. Let $x = (1, \frac{1}{2})$ and $y = (0, 1)$. One can clearly see from the picture that $\|x + \lambda y\| \geq \|x\|$, for any $\lambda \in \mathbb{R}$. Hence $x \perp_B y$. On the other hand, $\|y + \lambda x\| < \|y\|$, for some $\lambda \in \mathbb{R}$ this implies $y \not\perp_B x$. In [25, 43] Day and James separately proved that for a normed linear space with dimension greater than or equal to 3, symmetric property of Birkhoff-James orthogonality induces an inner product.

Theorem 1.4. *Suppose that $\mathbb{X}$ is a Banach space with dimension greater than or equal to three. Birkhoff-James orthogonality is symmetric if and only if $\mathbb{X}$ is a Hilbert space.*

In case of two-dimensional Banach spaces, there are spaces, not necessarily Hilbert, in which Birkhoff-James orthogonality is symmetric. These types of spaces are called Radon plane due to mathematician Johann Radon. Up to this point we look for the symmetricity in normed linear spaces from global point of view. In [91], Sain introduced the notion of local symmetricity in a Banach space.

Definition 1.9. *Let $\mathbb{X}$ be a normed linear space. An element $x \in \mathbb{X}$ is said to be left symmetric if $x \perp_B y \implies y \perp_B x$, for all $y \in \mathbb{X}$. On the other hand, x is said to be right symmetric if $y \perp_B x \implies x \perp_B y$, for all $y \in \mathbb{X}$. The element x is said to be symmetric if it is both left and right symmetric.*

This local symmetric property of an element has been studied extensively over the years in Banach spaces as well as in the space of bounded linear operators. We note some of the important results:

Theorem 1.5. *[71, Prop. 2.1] Let $\mathbb{X}$ be a Banach space and let $x \in \mathbb{X}$ be nonzero. Then the following results hold true:*

(i) If x is right symmetric and smooth then x is left symmetric.

(ii) If $\mathbb{X}$ is strictly convex and x is left symmetric then x is right symmetric.

(iii) If $\mathbb{X}$ is strictly convex and x is left symmetric then x is smooth.

In [19], Chattopadhyay et al. completely characterized the left and right symmetric points in ℓ_p^n spaces.

Theorem 1.6. *[19, Th. 2.7] Let $\mathbb{X} = \ell_p^n$, where $p \in (1, \infty) \setminus \{2\}$ and $n \geq 2$. Then the following are equivalent:*

(i) $x \in \mathbb{X}$ is left symmetric

(ii) $x \in \mathbb{X}$ is right symmetric

(iii) Either x has only nonzero coordinate, which is unimodular or x has only two nonzero coordinates having the values $\pm \frac{1}{2^{\frac{1}{p}}}$.

This characterization of symmetric points of ℓ_p^n helps to classify the isometric group of ℓ_p^n spaces in a more elementary way.

Theorem 1.7. *[19, Th. 2.11] Let $\mathbb{X} = \ell_p^n$, where $p \in (1, \infty) \setminus \{2\}$ and $n \geq 2$. Then $T \in \mathbb{L}(\mathbb{X})$ is isometry if and only if T is a signed permutation.*

The characterization of local symmetric points in $\mathbb{L}(\mathbb{H})$ studied in [32, 34] and then completely solved by Tŭrnsek in [93].

Theorem 1.8. *[93, Cor. 3.4, Th, 4.4] Let $\mathbb{H}$ be a Hilbert space and let $T \in \mathbb{L}(\mathbb{H})$. Then the following results hold true:*

(i) T is left symmetric if and only if $T = 0$

(ii) T is right symmetric if and only if T is a scalar multiple of a isometry or a coisometry.

In general Banach space settings the complete characterization for local symmetric nature of an operator still remains unsolved, though with some additional condition on the Banach space there are some classifications of left and right symmetric operators. For this one can see [33, 34, 71, 64, 85]. In the next section we discuss some geometric constants and their relevance in the study of geometry of Banach spaces.

1.5 Some geometric constants and their applications

To understand the structure of the unit ball of a normed linear space, geometric constants play a significant role. In particular, study of these constants indicates how far is a Banach space away from the Hilbert space structure from a quantitative perspective. There are many geometric constants introduced over the years. For some survey on this topic readers can go through [5, 56] and the references therein. Here we would like to mention a few of them.

Definition 1.10. *[24] Let $\mathbb{X}$ be a Banach space. The Jordan-von Neumann constant is defined by*

$$C_{NJ}(\mathbb{X}) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} : x, y \in \mathbb{X} \textit{ not both zero}\right\}.$$

We note some fundamental properties of Jordan-von Neumann constant.

Theorem 1.9. *Given a Banach space $\mathbb{X}$,*

(i) $1 \leq C_{NJ}(\mathbb{X}) \leq 2$.

(ii) $C_{NJ}(\mathbb{X}) = 1$ *if and only if $\mathbb{X}$ is a Hilbert space ([48]).*

(iii) $\mathbb{X}$ is uniformly non-square if and only if $C_{NJ}(\mathbb{X}) < 2$.

(iv) $C_{NJ}(\mathbb{X}) = C_{NJ}(\mathbb{X}^*)$.

Other than these many properties of this constant has been explored, see [4]. On the other hand, in 1990 Gao and Lau introduced the James constant to study the fact that how much a unit ball of a Banach space is "uniformly non-square".

Definition 1.11. *[30] Given a Banach space $\mathbb{X}$, the James constant is defined by*

$$J(\mathbb{X}) = \sup\{\min\{\|x+y\|, \|x-y\|\} : \|x\| = \|y\| = 1\}.$$

We mention some interesting properties of the James constant.

Theorem 1.10. *[30] For a Banach space $\mathbb{X}$,*

(i) $\sqrt{2} \leq J(\mathbb{X}) \leq 2$.

(ii) For $\dim(\mathbb{X}) \geq 3$, $J(\mathbb{X}) = \sqrt{2}$ *if and only if* $\mathbb{X}$ *is Hilbert space.*

(iii) $J(\mathbb{X}) < 2$ *if and only if* $\mathbb{X}$ *is uniformly non-square.*

The problem of characterizing the space $\mathbb{X}$ with $\dim(\mathbb{X}) = 2$ and $J(\mathbb{X}) = \sqrt{2}$ is not completely solved. In this direction few works have been done. (see e.g., [56]). One can look into [49] for more information regarding the above two constants and their applicability.
Up to this point, we have observed that notions of orthogonality have not played a direct role in defining any geometric constants. However, in [47], Joly introduced the concept of the rectangular constant, which explores the geometry of a Banach space by focusing on elements that are connected through Birkhoff-James orthogonality. We now recall its definition.

Definition 1.12. *[47] For a Banach space $\mathbb{X}$ with $\dim(\mathbb{X}) \geq 2$, the rectangular constant is defined as follows:*

$$\mu(\mathbb{X}) = \sup\left\{\frac{\|x\| + \|y\|}{\|x+y\|} : x, y \text{ not both zero and } x \perp_B y, \right\}.$$

We note from [47] that $\sqrt{2} \leq \mu(\mathbb{X}) \leq 3$ and $\mu(\mathbb{X}) = \sqrt{2}$ if and only if $\mathbb{X}$ is a Hilbert space. Later this constant and its generalized notions have been studied thoroughly. For more information on this topic one may see [11, 26, 76].
Thereafter many geometric constants have been developed over the years not only to understand the structure of the unit ball but also to find the quantitative differences between two distinct orthogonality notions in a Banach space. Among them we recall the following two well-known constants.

Definition 1.13. *Let $\mathbb{X}$ be a Banach space. Then*

- *[46]* $D(\mathbb{X}) = \inf\left\{\inf \|x + \lambda y\| : \|x\| = \|y\| = 1, x \perp_I y\right\}$.
- *[70]* $BR(\mathbb{X}) = \sup\left\{\frac{\|x+\alpha y\| - \|x-\alpha y\|}{\alpha} : \|x\| = \|y\| = 1, \alpha > 0, x \perp_B y\right\}$.

The constant $D(\mathbb{X})$ studies the quantitative difference between Birkhoff-James orthogonality and isosceles orthogonality whereas, $BR(\mathbb{X})$ estimates the difference between Birkhoff-James orthogonality and Roberts orthogonality. There are so many constants developed for the same purpose. Readers can go through [9, 40] and the references therein. In the next section we give an outline of the content of thesis.

1.6 Outline of the thesis

This dissertation is organized into six chapters. The first chapter serves as an introduction, presenting fundamental notations and preliminary results concerning various geometric aspects of Banach spaces.

Chapter 2 examines the concept of T-orthogonality within Banach spaces and analyzes how it connects to several geometric features, such as strict convexity, smoothness, and reflexivity. We explore the notions of left and right symmetric elements with respect to T-orthogonality. Additionally, the chapter provides a characterization of Hilbert spaces among Banach spaces based on this orthogonality framework.

In Chapter 3, the discussion centers around the symmetricity with respect to numerical radius orthogonality. We observe that in $\mathbb{L}(\mathbb{H})$, there is no nonzero left symmetric elements. We also prove that a nonzero compact normal operator on an infinite-dimensional complex Hilbert space cannot be right symmetric. We extend our study in the setting of Banach space and obtain some necessary and sufficient condition separately for left and right symmetric elements of $\mathbb{L}(\mathbb{X})$. In particular, we characterize left and right symmetric elements in $\mathbb{L}(\ell_1^n)$ and $\mathbb{L}(\ell_\infty^n)$.

Chapter 4 studies the James constant $J(\mathbb{X})$, an important geometric quantity associated with a normed space $\mathbb{X}$, and explore its connection with isosceles orthogonality. We prove that if $J(\mathbb{X})$ is attained for unit vectors $x, y \in \mathbb{X}$, then $x \perp_I y$. We also show that if $\mathbb{X}$ is a two-dimensional polyhedral Banach space, then $J(\mathbb{X})$ is always attained at an extreme point z of the unit ball of $\mathbb{X}$, so that $J(\mathbb{X}) = \|z+y\| = \|z-y\|$, where $\|y\| = 1$ and $z \perp_I y$. This gives a clear and practical approach for calculating the James constant in two-dimensional polyhedral Banach spaces. In addition, we explore the connection between modulus of convexity and approximate isosceles orthogonality within normed linear spaces.

Chapter 5 investigates a specific geometric constant that quantifies the difference between orthogonality defined by the norm derivative (ρ-orthogonality) and Birkhoff-James orthogonality. We explore the relation between various geometric properties and this constant. This chapter also includes an analysis of left and right symmetric elements in Banach spaces with respect to ρ-orthogonality. We characterize the inner product space among Banach spaces via symmetricity of ρ-orthogonality. Moreover, we provide a complete description of left and right symmetric points with respect to ρ-orthogonality in ℓ_1^n and ℓ_∞^n.

The final chapter, i.e., Chapter 6 delves into the concepts of smooth and approximately smooth points in Banach spaces and in the space of bounded linear operators. Under some additional properties, we obtain a sufficient condition for the openness of the collection of all smooth points in $\mathbb{X}$. Next we study the approximate smoothness of the elements of the space of bounded linear operators.

To ensure clarity and coherence, each chapter begins with a concise motivation, relevant definitions and notational conventions to make it self-contained and accessible.

CHAPTER 2

T-ORTHOGONALITY AND THE STUDY OF SYMMETRY IN COMPLEX BANACH SPACES

2.1 Introduction

Our main aim in this chapter is to study the notion of T-orthogonality in the setting of Banach space and explore its relation with the geometric properties of the Banach space. In particular, we investigate the local symmetricity in Banach spaces with respect to T-orthogonality. Before proceeding further we introduce the requisite notations and terminologies.

Let $\mathbb{X}$ be a Banach space over the real or complex field $\mathbb{C}$, stated accordingly and let $\mathbb{X}^*$ denotes the dual space of $\mathbb{X}$. For a complex number λ, the real and imaginary part of λ is denoted by $\Re\lambda$ and $\Im\lambda$ respectively. Suppose $\mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ denotes the collection of all bounded linear operators from $\mathbb{X}$ to $\mathbb{X}^*$ and $\mathbb{L}(\mathbb{X})$ denotes the space of all bounded linear operators on $\mathbb{X}$. Let $S_{\mathbb{X}}$ and $B_{\mathbb{X}}$ denote the unit sphere and unit ball of the space $\mathbb{X}$ i.e., $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$

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- D. Sain, **S. Ghosh**, and K. Paul, *On T-orthogonality in Banach spaces*, Colloq. Math., **172** (2023), no. 2, 231-242. DOI: 10.4064/cm8962-11-2022.

and $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$. The space $\mathbb{X}$ is said to be strictly convex if the unit sphere $S_{\mathbb{X}}$ does not contain a non-trivial line segment, i.e., if $(1-t)x + ty \in S_{\mathbb{X}}$ for some $t \in (0,1)$ and $x, y \in S_{\mathbb{X}}$, then $x = y$. An element $x \neq 0$ is said to be a smooth point of the space $\mathbb{X}$ if there exists a unique functional $f \in \mathbb{X}^*$ such that $f(x) = \|f\|\|x\|$ and $\|f\| = \|x\|$. The space $\mathbb{X}$ is said to be smooth if every non-zero element of the space $\mathbb{X}$ is a smooth point. Following [92], T-orthogonality in a Banach space $\mathbb{X}$ can be defined as follows.

Definition 2.1. *Let $\mathbb{X}$ be a Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$. For $x, y \in \mathbb{X}$, x* is T-orthogonal to y, *written as $x \perp_T y$, if $Tx(y) = 0$. We write $Tx(y) = (Tx, y)$.*

T-orthogonality is said to be *symmetric* if $(Tx, y) = 0$ implies $(Ty, x) = 0$, for all $x, y \in \mathbb{X}$. The operator T is said to be *symmetric* if $(Tx, y) = (Ty, x)$, for all $x, y \in \mathbb{X}$. An element $x \in \mathbb{X}$ is said to be *T-isotropic* if $(Tx, x) = 0$. When the operator T is clear from the context, we use the term 'isotropic' instead of 'T-isotropic'. We here introduce the following definitions.

Definition 2.2. *Let $\mathbb{X}$ be a Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$. Let $\theta \in [0, 2\pi)$. For $x, y \in \mathbb{X}$,* x is said to be T-orthogonal to y in the direction of θ, *written as $x \perp_{T_\theta} y$, if $\cos\theta\, \Re(Tx, y) + \sin\theta\, \Im(Tx, y) = 0$. For brevity, we write $(T_\theta x, y) = \cos\theta\, \Re(Tx, y) + \sin\theta\, \Im(Tx, y)$. We say $y \in x^+_{T_\theta}$ if $(T_\theta x, y) \geq 0$ and $y \in x^-_{T_\theta}$ if $(T_\theta x, y) \leq 0$. In case $\mathbb{X}$ is a real Banach space we write $y \in x^+_T$ if $(Tx, y) \geq 0$ and $y \in x^-_T$ if $(Tx, y) \leq 0$.*

Next we introduce the notion of left symmetric and right symmetric points with respect to T-orthogonality.

Definition 2.3. *Let $\mathbb{X}$ be a Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$. We say that T*-orthogonality is left symmetric at $x \in \mathbb{X}$, *if $x \perp_T y$ implies $y \perp_T x$ for each $y \in \mathbb{X}$. In short, we write '$\perp_T$ is left symmetric at x'.*

Further for each $\theta \in [0, 2\pi)$, we say that T-orthogonality is left symmetric at x in the direction of θ *if $x \perp_{T_\theta} y$ implies $y \perp_{T_\theta} x$ for each $y \in \mathbb{X}$. In short, we write '$\perp_{T_\theta}$ is left symmetric at x'.*

Definition 2.4. *Let $\mathbb{X}$ be a Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$. We say that T*-orthogonality is right symmetric at $x \in \mathbb{X}$, *if $y \perp_T x$ implies $x \perp_T y$ for each $y \in \mathbb{X}$. In short, we write '$\perp_T$ is right symmetric at x'.*

Further for each $\theta \in [0, 2\pi)$, we say that T-orthogonality is right symmetric at x in the direction of θ *if $y \perp_{T_\theta} x$ implies $x \perp_{T_\theta} y$ for each $y \in \mathbb{X}$. In short, we write '$\perp_{T_\theta}$ is right symmetric at x'.*

We need the notion of Birkhoff-James orthogonality [16, 44] to study the geometric properties of the Banach space using T-orthogonality. For $x, y \in \mathbb{X}$, x is said to be Birkhoff-James orthogonal to y, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$. The marvelous paper by James [42] nicely characterizes the properties like smoothness, strict convexity, reflexivity using Birkhoff-James orthogonality.

In this chapter we study the symmetric properties of $\perp_T$ and $\perp_{T_\theta}$ and explore the relations between them. We also characterize the geometric properties of the space like smoothness and strict convexity using the notion of T-orthogonality. We also obtain necessary and sufficient condition for an operator $A \in \mathbb{L}(\mathbb{X})$ preserving T-orthogonality in terms of newly introduced notion of T-isometry. An operator $A \in \mathbb{L}(\mathbb{X})$ is said to be T-isometry if it satisfies $T = A^*TA$, where A^* is the adjoint of A. Finally we characterize Hilbert spaces among Banach spaces using T-orthogonality.

2.2 Local symmetric properties in Banach spaces

We begin this section by noting the following basic properties in the form of an easy proposition. The proofs are omitted as they can be obtained by applying elementary geometric and algebraic reasoning.

Proposition 2.1. *Let $\mathbb{X}$ be a Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$.*

(i) Given any $x, y \in \mathbb{X}$, there exists $\theta \in [0, 2\pi)$ such that $x \perp_{T_\theta} y$.

(ii) For $x, y \in \mathbb{X}$, $x \perp_T y$ if and only if $x \perp_{T_\theta} y$ and $x \perp_{T_\phi} y$ for some θ and ϕ, where $\theta - \phi \neq 0, \pi$.

(iii) For $x, y \in \mathbb{X}$, $x \perp_T y$ implies that $x \perp_{T_\theta} y$, for each θ. However, there are operators T for which $x \perp_{T_\theta} y$, for some particular θ, whereas $x \not\perp_T y$.

(iv) For each $x, y \in \mathbb{X}$ and $\theta, \phi \in [0, 2\pi)$, we have, $(T_\theta x, e^{i\phi} y) = (T_{\theta-\phi} x, y) = (T_\theta(e^{i\phi} x), y)$ and so $x \perp_{T_\theta} e^{i\phi} y \Leftrightarrow x \perp_{T_{\theta-\phi}} y \Leftrightarrow e^{i\phi} x \perp_{T_\theta} y$.

(v) $\perp_T$ is right additive i.e., if $x \perp_T y, x \perp_T z$ then $x \perp_T (y + z)$.

(vi) $\perp_T$ is left additive i.e., if $x \perp_T z, y \perp_T z$ then $(x + y) \perp_T z$.

An elementary example of $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ such that $x \perp_{T_\theta} y$, for some particular θ, whereas $x \perp_T y$ does not hold is as follows:

Remark 2.1. *Consider the two-dimensional complex Hilbert space $\mathbb{C}^2$. Let $T : \mathbb{C}^2 \longrightarrow (\mathbb{C}^2)^*$ be defined as $T(1,0) = 7i(1,0)^* + (0,1)^*$ and $T(0,1) = 2(1,0)^* + 3i(0,1)^*$, where $(1,0)^*(z_1, z_2) = z_1$*

and $(0,1)^*(z_1,z_2) = z_2$, *for all* $(z_1,z_2) \in \mathbb{C}^2$. *Then* $(0,1) \perp_{T_{\pi/4}} (1/2,-1/3)$ *but* $(0,1) \not\perp_T (1/2,-1/3)$.

Proposition 2.2. *Let* $\mathbb{X}$ *be a complex Banach space and let* $T \in L(\mathbb{X},\mathbb{X}^*)$ *be bijective. Suppose that* $x \in \mathbb{X}$ *and* $Tx \neq 0$. *If* $\perp_T$ *is left symmetric at* x *then there exists a scalar* λ *such that* $(Tx,y) = \lambda(Ty,x)$, *for all* $y \in \mathbb{X}$.

Proof. Clearly there exists $0 \neq z \in \mathbb{X}$ such that $(Tx,z) \neq 0$. Then $(Tx,z) = \lambda(Tz,x)$ for some scalar λ. We claim that $(Tx,y) = \lambda(Ty,x)$, for all $y \in \mathbb{X}$. As before we can write $\mathbb{X} = Ker\,Tx \oplus \langle\{z\}\rangle$. Let $y \in \mathbb{X}$. Then $y = \alpha z + v$, for some scalar α and $v \in Ker\,Tx$. Now $(Tx,v) = 0$ implies $(Tv,x) = 0$ by left symmetricity of $\perp_T$ at x. Then $(Tx,y) = \alpha(Tx,z) + (Tx,v) = \alpha\lambda(Tz,x) + \lambda(Tv,x) = \lambda(Ty,x)$. This completes the proof. □

Theorem 2.2. *Let* $\mathbb{X}$ *be a complex Banach space and let* $T \in \mathbb{L}(\mathbb{X},\mathbb{X}^*)$. *Suppose that* $x \in \mathbb{X}$ *and* $\theta \in [0,2\pi)$. *If* $\perp_{T_\theta}$ *is left symmetric at* x *then* $\perp_T$ *is left symmetric at* x.

Proof. Let $\gamma \in [0,2\pi)$. We first claim that $\perp_{T_\gamma}$ is left symmetric at x. Let $x \perp_{T_\gamma} y$. Then by Proposition 2.1 (iv), we get $0 = (T_\gamma x, y) = (T_\theta x, e^{i(\theta-\gamma)}y)$. So we get $(T_\theta(e^{i(\theta-\gamma)}y, x) = 0$, by left symmetricity of T_θ at x. Again by Proposition 2.1 (iv), we get, $(T_\gamma y, x) = (T_\theta(e^{i(\theta-\gamma)}y, x) = 0$, and hence $y \perp_{T_\gamma} x$. Thus $\perp_{T_\gamma}$ is left symmetric at x. Now we show that $\perp_T$ is left symmetric at x. Let $x \perp_T y$. Then $x \perp_{T_\theta} y$ for any $\theta \in [0,2\pi)$ and by left symmetricity of $\perp T_\theta$ we get, $y \perp_{T_\theta} x$. Then by Proposition 2.1 (ii), we get, $y \perp_T x$. □

In this following example we show that the converse is not true in general.

Example 2.3. *Consider the two-dimensional complex Hilbert space* $\mathbb{C}^2$. *Every element of* $\mathbb{C}^2$ *can be written as* (z_1,z_2), *where* $z_1,z_2 \in \mathbb{C}$. *Let* $T : \mathbb{C}^2 \longrightarrow (\mathbb{C}^2)^*$ *be defined as* $T(1,0) = i(0,1)^*$ *and* $T(0,1) = (1,0)^* + 3i(0,1)^*$, *where* $(1,0)^*(z_1,z_2) = z_1$ *and* $(0,1)^*(z_1,z_2) = z_2$, *for all* $(z_1,z_2) \in \mathbb{C}^2$. *Clearly,* $(1,0)$ *is an isotropic vector and* $\perp_T$ *is both left and right symmetric at* $(1,0)$. *Now for* $(1,i) \in \mathbb{C}^2$ *it is straightforward to see that* $(1,0) \perp_{T_{\pi/2}} (1,i)$ *whereas* $(1,i) \not\perp_{T_{\pi/2}} (1,0)$. *Therefore,* $\perp_{T_{\pi/2}}$ *is not left symmetric at* $(1,0)$.

Remark 2.4. *The above example shows that for isotropic vector* x *the converse of Theorem 2.2 may not hold true. However, if* x *is a nonisotropic vector then following Proposition 2.2, we get a scalar* λ *such that* $(Tx,y) = \lambda(Ty,x)$ *for all* $y \in \mathbb{X}$. *In case* λ *is real, then* $\perp_T$ *is left symmetric at* x *will imply that* $\perp_{T_\theta}$ *is left symmetric at* x *for all* $\theta \in [0,2\pi)$.

Theorem 2.5. *Let* $\mathbb{X}$ *be a complex Banach space and let* $T \in \mathbb{L}(\mathbb{X},\mathbb{X}^*)$. *Then the followings hold:*

(i) For any nonisotropic vector $x \in \mathbb{X}$, $\perp_T$ is left symmetric at x if and only if $\perp_T$ is right symmetric at x.

(ii) If T is bijective then for any nonzero isotropic vector $x \in \mathbb{X}$, $\perp_T$ is left symmetric at x if and only if $\perp_T$ is right symmetric at x.

Proof. (i) We have $Tx(x) \neq 0$. Assume $\perp_T$ is left symmetric at x. We show that $\perp_T$ is right symmetric at x. If possible let $y \perp_T x$ but $x \not\perp_T y$. Then $Tx(y) \neq 0$ and so $Tx(y) = \alpha Tx(x)$ for some scalar α. This implies $Tx(y - \alpha x) = 0$ so that $x \perp_T (y - \alpha x)$. Since $\perp_T$ is left symmetric at x so we get $(y - \alpha x) \perp_T x$. Thus $T(y - \alpha x)(x) = 0 \Rightarrow Ty(x) = \alpha Tx(x) = Tx(y)$. This leads to a contradiction. Thus $\perp_T$ is right symmetric at x. Analogously one can show that if $\perp_T$ is right symmetric at x then $\perp_T$ is left symmetric at x.

(ii) Assume $\perp_T$ is left symmetric at x where $x \neq 0$ and $(Tx, x) = 0$. If possible let there exist $z \in \mathbb{X}$ such that $z \perp_T x$ but $x \not\perp z$. Then $z \notin Ker\,Tx$ and Tx being a linear functional on $\mathbb{X}$, we get $\mathbb{X} = Ker\,Tx \oplus \langle\{z\}\rangle$. Let $w \in \mathbb{X}$. Then $w = \alpha z + v$ where $v \in Ker\,Tx$ and α is a scalar. So $Tx(v) = 0$ implies $x \perp_T v$ and hence $v \perp_T x$, by left symmetricity of $\perp_T$ at x. Thus $Tv(x) = 0$. This shows that $Im\,T \subset \{x\}^0$ where $\{x\}^0$ is the annihilator of x. Then by bijectivity of T we get, $\mathbb{X}^* = \{x\}^0$ which contradicts the fact that $x \neq 0$. Thus $\perp_T$ is right symmetric at x. Next we assume that $\perp_T$ is right symmetric at x where $x \neq 0$ and $(Tx, x) = 0$. If possible let there exist $z \in \mathbb{X}$ such that $x \perp_T z$ but $z \not\perp x$. Then $Tz(x) \neq 0$ and so $Tz \notin \{x\}^0$. Now $\mathbb{X}^* = \{x\}^0 \oplus \langle\{Tz\}\rangle$. Let $u \in \mathbb{X}$. Then $Tu \in \mathbb{X}^*$ and so $Tu = g + \alpha Tz$ for some $g \in \{x\}^0$ and for some scalar α. Since T is bijective so there exists unique $v \in \mathbb{X}$ such that $Tv = g$ and $Tv(x) = 0$. This implies $v \perp_T x$ and by right symmetricity of $\perp_T$ we get, $x \perp_T v$. Now $Tu = Tv + \alpha Tz$ implies $u = v + \alpha z$. Then $Tx(u) = Tx(v) + Tx(z) = 0$ so that $x \perp_T u$. Thus for each $u \in \mathbb{X}$, $Tx(u) = 0$ and so $Tx = 0$. This implies $x = 0$, by injectivity of T. This is a contradiction. Thus $\perp_T$ is left symmetric at x. $\square$

In the following example we show that in Theorem 2.5 (ii), the 'bijectivity' condition on T can not be omitted.

Example 2.6. *Let ℓ_2^2 be the 2-dimensional real Hilbert space. Consider $T : \ell_2^2 \longrightarrow (\ell_2^2)^*$ is defined as $T(1,0) = (1,-1)^*$, where $(1,-1)^*(x,y) = x - y$. and $T(0,1) = (2,-2)^*$, where $(2,-2)^*(x,y) = 2x - 2y$. Clearly, T is not bijective and $(1,1)$ is an isotropic vector. It is easy to check that $\perp_T$ is left symmetric at $(1,1)$. On the other hand, observe that $(1,0) \perp_T (1,1)$ whereas $(1,1) \not\perp_T (1,0)$. This implies that $\perp_T$ is not right symmetric at $(1,1)$.*

Our next lemma deals with an interesting observation about nonisotropic vectors.

Lemma 2.1. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$. Suppose that $x \in \mathbb{X}$ is a nonisotropic vector at which $\perp_T$ is left (right) symmetric. Then $(Tx, y) = (Ty, x)$, for all $y \in \mathbb{X}$.*

Proof. Since $\perp_T$ is left symmetric at x, $(Tx, y) = 0$ implies $(Ty, x) = 0$. Suppose that $(Tx, y) \neq 0$. It is easy to observe that for $\alpha \in \mathbb{C}$, $(Tx, \alpha x + y) = \alpha(Tx, x) + (Tx, y)$. Choosing $\alpha = \frac{-(Tx,y)}{(Tx,x)}$ we obtain that $(Tx, \alpha x + y) = 0$. As $\perp_T$ is left symmetric at x,

$$(T(\alpha x + y), x) = 0 \implies \alpha(Tx, x) + (Ty, x) = 0 \implies (Tx, y) = (Ty, x).$$

This proves our lemma. □

Before we proceed to our next theorem we would like to note that the above lemma is not necessarily true for isotropic vector. The following example illustrates our claim.

Example 2.7. *Let $T : \ell_2^2 \longrightarrow (\ell_2^2)^*$ be defined as $T(1, 0) = (0, 1)^*$ and $T(0, 1) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^*$, where ℓ_2^2 is 2-dimensional real Hilbert space. It is easy to verify that $(1, 0)$ is an isotropic vector and $\perp_T$ is both left and right symmetric at $(1, 0)$. For any $(\alpha, \beta), (x, y) \in \ell_2^2$, $(T(\alpha, \beta), (x, y)) = \frac{\beta}{\sqrt{2}}x + (\alpha + \frac{\beta}{\sqrt{2}})y$. Then $(T(\alpha, \beta), (1, 0)) = \frac{\beta}{\sqrt{2}}$ and $(T(1, 0), (\alpha, \beta)) = \beta$. Therefore, whenever $\beta \neq 0$, we have $(T(\alpha, \beta), (1, 0)) \neq (T(1, 0), (\alpha, \beta))$.*

In our next theorem we characterize $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ which are symmetric. Note that the following theorem is an improvement of [92, Lem. 1].

Theorem 2.8. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ be nonzero. Then T is symmetric if and only if the followings hold true:*

(i) There exists a nonisotropic vector $x \in \mathbb{X}$.

(ii) $\perp_T$ is left(right) symmetric at y, where y is a nonisotropic vector in $\mathbb{X}$.

Proof. First we prove the sufficient part of the theorem. Suppose that $z, w \in \mathbb{X}$ are two arbitrary elements in $\mathbb{X}$. If one of them is nonisotropic then from Lemma 2.1, we get $(Tz, w) = (Tw, z)$. Let z, w both be isotropic. Then from [92, Lem. 1] we have either $x + z$ or $x - z$ is nonisotropic. Without loss of generality we assume that $x + z$ is nonisotropic. Therefore, again using Lemma 2.1, we obtain

$$(T(x + z), w) = (Tw, x + z) \implies (Tx, w) + (Tz, w) = (Tw, x) + (Tw, z).$$

Since x is nonisotropic, $(Tx, w) = (Tw, x)$. Therefore, $(Tz, w) = (Tw, z)$. Hence T is symmetric. Let us now prove the necessary part. Clearly (ii) holds if T is symmetric. We just prove (i).

Suppose on the contrary that every vector in $\mathbb{X}$ is isotropic, i.e., $(Tx, x) = 0$, for all $x \in \mathbb{X}$. Take $u, v \in \mathbb{X}$. Then $(T(u+v), u+v) = 0$ which implies $(Tu, v) = -(Tv, u)$. As T is symmetric, it can be readily seen that $(Tu, v) = 0$. As u, v are chosen arbitrarily, T is zero. This contradiction proves the necessary part of the theorem.

□

Theorem 2.9. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$. Suppose that $x \in \mathbb{X}$. Then $\perp_T$ is left symmetric at x if and only if there exists $\phi_0 \in [0, 2\pi)$ such that the followings hold true:*

$$y \in x^+_{T_\theta} \implies x \in y^+_{T_{\theta-\phi_0}} \text{ and } y \in x^-_{T_\theta} \implies x \in y^-_{T_{\theta-\phi_0}}, \text{ for every } \theta \in [0, 2\pi).$$

Proof. Let us first prove the sufficient part of the theorem. Let $y \in \mathbb{X}$ be such that $(Tx, y) = 0$. This implies that $y \in x^+_{T_\theta} \cap x^-_{T_\theta}$, for all $\theta \in [0, 2\pi)$. Then there exists $\phi_0 \in [0, 2\pi)$ such that $x \in y^+_{T_{\theta-\phi_0}} \cap y^-_{T_{\theta-\phi_0}}$, i.e., $x \in y^+_{T_\theta} \cap y^-_{T_\theta}$. Therefore, $(Ty, x) = 0$. This proves the sufficient part. We next prove the necessary part. If $Tx = 0$ then the result holds trivially. Assume $Tx \neq 0$. Then there exists $z \in \mathbb{X}$ such that $(Tx, z) \neq 0$. We consider the following two cases.
Case-I: Let x is nonisotropic. Then from Lemma 2.1 we note that $(Tx, y) = (Ty, x)$, for all $y \in \mathbb{X}$. This clearly proves our result.
Case-II: Let x be isotropic. As $Tx \neq 0$, and $(Tx, z) \neq 0$ this implies that $z \neq \pm x$. If $(Tz, x) \neq 0$ then using the similar argument as given in Proposition 2.2 that there exists a scalar λ such that $(Tx, u) = \lambda (Tu, x)$, for all $u \in \mathbb{X}$. Let $y \in x^+_{T_\theta}$. Then $(T_\theta x, y) \geq 0$ and by using Proposition 2.1 (iv) we get, $(T_{\theta-\phi_0} y, x) \geq 0$, where $\lambda = \mid \lambda \mid e^{i\phi_0}$. This shows that $x \in y^+_{T_{\theta-\phi_0}}$. Similarly we can show that $y \in x^-_{T_\theta} \implies x \in y^-_{T_{\theta-\phi_0}}$. On the other hand, if $(Tz, x) = 0$ and as $z \neq \pm x$ then given any $u \in \mathbb{X}$, $u = \alpha z + v$, for some $v \in \ker Tx$. This implies $(Tu, x) = 0$ as $\perp_T$ is left symmetric at x. Therefore, given any $u \in \mathbb{X}$, $x \in u^+_{T_\theta} \cap u^-_{T_\theta}$, for every $\theta \in [0, 2\pi)$. This completes the necessary part.

□

The following corollary is an immediate consequence of the last theorem.

Corollary 2.1. *Let $\mathbb{X}$ be a real Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$. Suppose that $x \in \mathbb{X}$. Then $\perp_T$ is left symmetric at x if and only if either of the following hold true:*

1. *$y \in x^+_T \implies x \in y^+_T$ and $y \in x^-_T \implies x \in y^-_T$.*
2. *$y \in x^+_T \implies x \in y^-_T$ and $y \in x^-_T \implies x \in y^+_T$.*

Proof. To show the necessary part note from Theorem 2.9 that for real Banach space either $\phi_0 = 0$ or $\phi_0 = \pi$. In case of $\phi_0 = 0$, we have (i) whereas for $\phi_0 = \pi$, we obtain (ii).
The sufficient part follows easily from the definition of left symmetricity of $\perp_T$ at x. $\square$

We say "$\perp_T$ is left symmetric in $\mathbb{X}$" if $\perp_T$ is left symmetric at x, for each $x \in \mathbb{X}$. We note the following remark in this regard.

Remark 2.10. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$. Using Theorem 2.8 together with Theorem 2.9, it can be concluded that $\perp_T$ is left symmetric in $\mathbb{X}$ if and only if either of the following holds true:*

1. $y \in x^+_{T_\theta} \implies x \in y^+_{T_\theta}$ *and* $y \in x^-_{T_\theta} \implies x \in y^-_{T_\theta}$.
2. $y \in x^+_{T_\theta} \implies x \in y^-_{T_\theta}$ *and* $y \in x^-_{T_\theta} \implies x \in y^+_{T_\theta}$, *for every* $\theta \in [0, 2\pi)$.

On a more precise note we can observe that if there exists a nonisotropic vector in $\mathbb{X}$ then $\perp_T$ is left symmetric in $\mathbb{X}$ if and only if $y \in x^+_{T_\theta} \implies x \in y^+_{T_\theta}$ and $y \in x^-_{T_\theta} \implies x \in y^-_{T_\theta}$, whereas if all the vectors in $\mathbb{X}$ are isotropic then $\perp_T$ is left symmetric in $\mathbb{X}$ if and only if $y \in x^+_{T_\theta} \implies x \in y^-_{T_\theta}$ and $y \in x^-_{T_\theta} \implies x \in y^+_{T_\theta}$, for every $\theta \in [0, 2\pi)$, for every $\theta \in [0, 2\pi)$.

2.3 Some geometric properties

In this section we observe some local geometric properties from the perspective of T-orthogonality. Let us note that $x^\perp = \{y \in \mathbb{X} : x \perp_B y\}$ and $x^{\perp_T} = \{y \in \mathbb{X} : x \perp_T y\}$.

Proposition 2.3. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ be bijective. Then $\mathbb{X}$ is reflexive if and only if for any nonzero $x \in \mathbb{X}$, there exist $z \in \mathbb{X}$ such that $z \perp_B x^{\perp_T}$.*

Proof. It is easy to observe that $x^{\perp_T}$ is a hyperspace of $\mathbb{X}$. Since $\mathbb{X}$ is reflexive, there exist a $z \in \mathbb{X}$ such that $z \perp_B x^{\perp_T}$. Conversely, Let H be a hyperspace in $\mathbb{X}$. Consider $x^* \in \mathbb{X}^*$ as the corresponding functional of H such that ker $x^* = H$. As T is bijective, let $Tw = x^*$, for some $w \in \mathbb{X}$. Clearly, $w^{\perp_T} = H$. Then there exist $v \in \mathbb{X}$ such that $v \perp_B H$. Then from [42] we conclude that $\mathbb{X}$ is reflexive. $\square$

Theorem 2.11. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ be bijective. Then the following hold true:*

(i) A nonzero element $x \in \mathbb{X}$ is smooth if and only if there exists a unique (upto scalar multiplication) $z \in \mathbb{X}$ such that $z^{\perp_T} = x^\perp$.

(ii) The space $\mathbb{X}$ is strictly convex if and only if for every $x \in \mathbb{X}$, there exists atmost one (upto scalar multiplication) $z \in S_{\mathbb{X}}$ such that $z \perp_B x^{\perp_T}$.

Proof. (i) Let $y \in x^{\perp}$. Since x is smooth, $J(x) = \{x^*\}$ and $x^*(y) = 0$. We note that T is bijective and therefore, there exists $z \in \mathbb{X}$ such that $Tz = x^*$. This implies that $z \perp_T y$. Thus we obtain $x^{\perp} \subset z^{\perp_T}$. Observe that $x^{\perp} = \ker x^* = z^{\perp_T} = (\lambda z)^{\perp_T}$. Therefore, there exists a unique (upto scalar multiplication) $z \in \mathbb{X}$ such that $z^{\perp_T} = x^{\perp}$. Conversely, suppose that x is a nonzero element of $\mathbb{X}$. Let $x \perp_B u$ and $x \perp_B v$, for some $u, v \in \mathbb{X}$. Then there exists a unique $z \in \mathbb{X}$ such that $z \perp_T u$ and $z \perp_T v$. As $Tz \in \mathbb{X}^*$, $z \perp_T u + v$, which implies $x \perp_B u + v$. Thus from [42], we obtain that x is a smooth point.

(ii) Let $\mathbb{X}$ be strictly convex and let $x \in \mathbb{X}$ be a nonzero element. Then $M_{Tx} = \emptyset$ or $M_{Tx} = \{e^{i\theta} z\}$, where $z \in S_{\mathbb{X}}$ and $\theta \in [0, 2\pi)$. It is easy to verify that when $M_{Tx} = \emptyset$, there exists no such $z \in S_{\mathbb{X}}$ such that $z \perp_B x^{\perp_T}$. Let $M_{Tx} = \{e^{i\theta} z\}$. This implies that $|Tx(z)| = \|Tx\|$, i.e., $\mu \frac{Tx}{\|Tx\|} \in J(z)$, for some $\mu \in S_{\mathbb{C}}$. Thus for any $y \in x^{\perp_T}$, we get $z \perp_B y$. In other words, $z \perp_B x^{\perp_T}$. Now if we assume that $z' \in S_{\mathbb{X}}$ with $z' \neq \gamma z$, where $\gamma \in S_{\mathbb{C}}$ such that $z' \perp_B x^{\perp_T}$ then one can easily show that $z' \in M_{Tx}$. This leads to a contradiction that $\mathbb{X}$ is strictly convex.

Conversely, Suppose on the contrary that $\mathbb{X}$ is not strictly convex. Then there exists $u, v (u \neq \lambda v) \in S_{\mathbb{X}}$ such that $J(u) \cap J(v) \neq \emptyset$, where $\lambda \in S_{\mathbb{C}}$. Let us take $x^* \in J(u) \cap J(v)$ and let $Tw = x^*$, for some $w \in \mathbb{X}$. Clearly, $w^{\perp_T} = \ker x^*$. Therefore, it can be readily seen that $u \perp_B w^{\perp_T}$ as well as $v \perp_B w^{\perp_T}$. This contradicts our hypothesis and proves our theorem. □

Remark 2.12. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ be bijective. From Theorem 2.11(ii) and Proposition 6.8, we can conclude that if $\mathbb{X}$ is reflexive and strictly convex then for any $x \in \mathbb{X}$, there exists a unique (upto scalar multiplication) $z \in S_{\mathbb{X}}$ such that $z \perp_B x^{\perp_T}$ whereas if $\mathbb{X}$ is non-reflexive and strictly convex then there exists a $w \in \mathbb{X}$ such that for no $z \in \mathbb{X}$, $z \perp_B w^{\perp_T}$ holds true.*

For a given Banach space $\mathbb{X}$ it is a natural question to ask which operators preserves T-orthogonality. In [55] it is proved that a linear operator on $\mathbb{X}$ preserves '$\perp_B$' if and only if the operator is a positive multiple of an isometry. We next characterize the operators preserving '$\perp_T$'.

Theorem 2.13. *Let $\mathbb{X}$ be a complex Banach space and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ be bijective. Suppose that $A \in \mathbb{L}(\mathbb{X})$. Then for any $x, y \in \mathbb{X}$, $x \perp_T y \iff Ax \perp_T Ay$ if and only if A is a scalar multiple of T-isometry.*

Proof. First we prove the necessary part of the theorem. Suppose that $x \perp_T y \iff Ax \perp_T Ay$, for all $x, y \in \mathbb{X}$. This implies that for any nonzero $x \in \mathbb{X}$, $(Tx, y) = 0 \implies (TAx, Ay) = 0$. This gives us $(TAx, Ay) = 0 \implies (A^*TAx, y) = 0$. Therefore, $y \in \ker Tx$ implies $y \in \ker A^*TAx$, for every $y \in \mathbb{X}$. Since T is bijective, for all nonzero $x \in \mathbb{X}$, we obtain $\ker Tx = \ker A^*TAx$, i.e., $Tx = \beta_x A^*TAx$, for some $\beta_x \in \mathbb{C}$. Now we claim that β_x is a fixed constant. Suppose that $u, v \in \mathbb{X}$ be two arbitrary nonzero vector. It is easy to check that if $v = \alpha u$, where $\alpha \in \mathbb{C}$ then $Tv = \beta_u A^*TAv$. Let u, v be independent. Then

$$\begin{aligned} Tu + Tv &= T(u+v) \\ &= \beta_{u+v} A^*TA(u+v) \\ &= \beta_{u+v} A^*TAu + \beta_{u+v} A^*TAv \\ &= \frac{\beta_{u+v}}{\beta_u} Tu + \frac{\beta_{u+v}}{\beta_v} Tv. \end{aligned}$$

As T is bijective and u, v are linearly independent, $\beta_u = \beta_v = \beta_{u+v}$. So our claim is established. Therefore, we obtain that $T = \beta A^*TA$, as desired.
To prove the sufficient part of the theorem let us assume that $T = \lambda A^*TA$, for some $\lambda \in \mathbb{C}$. Then for any $x, y \in \mathbb{X}$, with $x \perp_T y$ we get

$$(Tx, y) = 0 \iff (\lambda A^*TAx, y) = 0 \iff \lambda(TAx, Ay) = 0 \iff Ax \perp_T Ay.$$

Hence our theorem is proved. □

Now in case of real Banach spaces with $\dim(\mathbb{X}) \geq 3$, we are going to show that T-orthogonality and Birkhoff-James orthogonality coincides only in Hilbert spaces.

Theorem 2.14. *Let $\mathbb{X}$ be a real Banach space and let $\dim(\mathbb{X}) \geq 3$. Then $\mathbb{X}$ is a Hilbert space if and only if there exists a $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ such that $\perp_T = \perp_B$.*

Proof. To prove the necessary part we assume $\mathbb{X}$ is a real Hilbert space. Let us consider the map $T : \mathbb{X} \longrightarrow \mathbb{X}^*$ defined by $Tx(y) = \langle y, x \rangle$, for each $y \in \mathbb{X}$. Then clearly T is a bounded linear operator. Clearly, $x \perp_T y \iff x \perp_B y$, for all $x, y \in \mathbb{X}$. Therefore, $\perp_T = \perp_B$.
Let us now prove the sufficient part of the theorem. Suppose that $u, v, w \in \mathbb{X}$ be such that $u \perp_B w$ and $v \perp_B w$. Since $\perp_T = \perp_B$, we have $u \perp_T w$ and $v \perp_T w$. Then by left additivity of $\perp_T$, we get $u + v \perp_T w$ and so $u + v \perp_B T$. This implies that $\perp_B$ is left additive. Therefore, from [43, Th. 2] we conclude that $\mathbb{X}$ is a Hilbert space. □

Remark 2.15. *Analogous characterization for Hilbert spaces in complex case follows easily: Let $\mathbb{X}$ be a complex Banach space and let $\dim(\mathbb{X}) \geq 3$. Then $\mathbb{X}$ is a Hilbert space if and only if there exists a conjugate linear operator $T : \mathbb{X} \to \mathbb{X}^*$ such that $\perp_T = \perp_B$.*

Our next result is the characterization of two-dimensional real Euclidean spaces out of all $\ell_p^2(\mathbb{R})$ spaces. Before we note the following lemma which will be useful for our next theorem.

Lemma 2.2. *Let* $\mathbb{X} = \ell_p^2(\mathbb{R})$, *where* $1 \leq p \leq \infty$. *Then* $p = 2$ *If and only if* $(\alpha, \beta) \perp_B (\beta, -\alpha)$, *for all* $(\alpha, \beta) \in \mathbb{X}$.

Theorem 2.16. *Let* $\mathbb{X} = \ell_p^2(\mathbb{R})$. *Then* $p = 2$ *if and only if there exists an operator* $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ *such that* $\perp_T = \perp_B$.

Proof. Since the necessary part is immediate, we only prove the sufficient part. Let $e_1, e_2 \in S_{\ell_p^2}$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We note that $e_1 \perp_B e_2$, $e_2 \perp_B e_1$ and $(e_1 + e_2) \perp_B (e_1 - e_2)$. Let $e_1^*, e_2^* \in \ell_q^2$ be such that $e_i^*(e_j) = \delta_{ij}$, where $i \in \{1, 2\}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we write $Te_1 = ae_1^* + be_2^*$ and $Te_2 = ce_1^* + de_2^*$, for some $a, b, c, d \in \mathbb{R}$. Since $\perp_T = \perp_B$, it is easy to observe that $Te_1 = ae_1^*$ and $Te_2 = de_2^*$. Now using $(e_1 + e_2) \perp_T (e_1 - e_2)$, we obtain $a = d$. So $Te_1 = ae_1^*$ and $Te_2 = ae_2^*$. Now for any $(\alpha, \beta) \in \mathbb{X}$, $(T(\alpha e_1 + \beta e_2), (\beta e_1 - \alpha e_2)) = 0$. This implies that $(\alpha e_1 + \beta e_2) \perp_B (\beta e_1 - \alpha e_2)$, i.e., $(\alpha, \beta) \perp_B (\beta, -\alpha)$, for all $(\alpha, \beta) \in \ell_p^2$. Then from Lemma 2.2, we have $p = 2$. This completes the proof of the theorem. □

Theorem 2.17. *Let* $\mathbb{X}$ *be a two-dimensional real Banach space. Then* $\mathbb{X}$ *is a Hilbert space if and only if the following conditions hold true:*

(i) There exists a $T \in \mathbb{L}(\mathbb{X}, \mathbb{X}^*)$ *such that* $\perp_T = \perp_B$.

(ii) There exists $u, v \in \mathbb{S}_{\mathbb{X}}$ *such that* $u \perp_B v$, $v \perp_B u$, $(u + v) \perp_B (u - v)$ *and* $\|(\gamma + \kappa\delta)u + (\delta - \kappa\gamma)v\| = \|(\gamma - \kappa\delta)u + (\delta + \kappa\gamma)v\|$, *for all* $\gamma, \delta, \kappa \in \mathbb{R}$.

Proof. Clearly, if $\mathbb{X}$ is Hilbert space then (i) and (ii) holds. We only prove the sufficient part of the theorem. Since $\perp_T = \perp_B$, it is easy to note that $Tu = \alpha u^*$ and $Tv = \beta v^*$, where $\alpha, \beta \in \mathbb{R}$ and $u^*(u) = 1, v^*(v) = 1$. As $(u + v) \perp_B (u - v)$, we have $(T(u + v), (u - v)) = 0$, which gives us $\alpha = \beta$. Now for any $\gamma, \delta \in \mathbb{R}$, it can be seen that $(T(\gamma u + \delta v), (\delta u - \gamma v)) = 0$. From Proposition 2.1(v) we observe that $\mathbb{X}$ is smooth and therefore, $(\gamma u + \delta v)^{\perp} = \kappa(\delta u - \gamma v)$, i.e., $(\gamma u + \delta v) \perp_B \kappa(\delta u - \gamma v)$. Also we observe that $\|(\gamma u + \delta v) + \kappa(\delta u - \gamma v)\| = \|(\gamma + \kappa\delta)u + (\delta - \kappa\gamma)v\| = \|(\gamma - \kappa\delta)u + (\delta + \kappa\gamma)v\| = \|(\gamma u + \delta v) - \kappa(\delta u - \gamma v)\|$. This implies that $(\gamma u + \delta v) \perp_I \kappa(\delta u - \gamma v)$. Thus we get $\perp_B \implies \perp_I$. Therefore, from [6, Chapter 4], it can be concluded that $\mathbb{X}$ is Hilbert space. □

We complete this chapter with the following observation.

Theorem 2.18. *Let* $\mathbb{H}$ *be a complex Hilbert space. Then there exists no* $T \in \mathbb{L}(\mathbb{H}, \mathbb{H}^*)$ *such that* $\perp_T = \perp_B$.

Proof. Suppose on the contrary that there exists $T \in \mathbb{L}(\mathbb{H}, \mathbb{H}^*)$ such that $\perp_T = \perp_B$. For any nonzero $x \in \mathbb{H}$, we have $x \perp_T y \iff x \perp_B y$, for all $y \in \mathbb{H}$. This implies that $(Tx, y) = 0 \iff x^*(y) = 0$, where x^* is the support functional of x. Therefore, one can see that $\ker Tx = \ker\ x^*$. This gives us $Tx = \lambda_x x^*$, for some $\lambda_x \in \mathbb{C}$. One can easily observe that T is bijective and therefore, $\lambda_x \neq 0$. Now choosing an $\alpha \in \mathbb{C}$ with nonzero imaginary part, it can be clearly seen that $T(\alpha x) = \bar{\alpha} Tx$. Thus we get T is not linear. This contradicts $T \in \mathbb{L}(\mathbb{H}, \mathbb{H}^*)$. Hence the theorem. □

CHAPTER 3

NUMERICAL RADIUS ORTHOGONALITY: SYMMETRY OF BOUNDED LINEAR OPERATORS

3.1 Introduction

The asymmetric nature of Birkhoff-James orthogonality in a general Banach space plays a key role in determining the geometry of the space. Moreover, a Banach space of dimension at least 3 is a Hilbert space if and only if Birkhoff-James orthogonality is symmetric [43]. The study of symmetric points with respect to Birkhoff-James orthogonality was initiated recently in [91]. Thereafter, many authors have studied left symmetric and right symmetric operators in the setting of Hilbert spaces as well as Banach spaces. In this chapter, we plan to study such operators with respect to numerical radius orthogonality. We first mention the relevant notations and terminologies.

Content of this chapter is based on the following article:

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Letters $\mathbb{X}$ and $\mathbb{H}$ stand for a Banach space and a Hilbert space, respectively, over the real or the complex field. Let $\mathbb{L}(\mathbb{X})$ (resp. $\mathbb{L}(\mathbb{H})$) denote the Banach algebra of all bounded linear operators on $\mathbb{X}$ (resp. $\mathbb{H}$). Let $B_{\mathbb{X}}$ (resp. $B_{\mathbb{H}}$) and $S_{\mathbb{X}}$ (resp. $S_{\mathbb{H}}$) denote the closed unit ball and the unit sphere of the space $\mathbb{X}$ (resp. $\mathbb{H}$) respectively. The real part and imaginary part of a complex number z is denoted by $\Re z$ and $\Im z$ respectively. $\mathbb{X}^*$ denotes the dual of $\mathbb{X}$. The collection of all supporting functionals at x is denoted by $J(x)$, i.e., $J(x) = \{x^* : x^* \in S_{\mathbb{X}^*}, x^*(x) = \|x\|\}$. Observe that $J(x)$ is a weak*-compact convex subset of $S_{\mathbb{X}^*}$. For $T \in \mathbb{L}(\mathbb{H})$, the numerical range of of T, denoted by $W(T)$, is defined as $W(T) = \{\langle Tx, x\rangle : x \in S_{\mathbb{H}}\}$, which is a convex subset of the scalar field by the famous Toeplitz-Hausdorff Theorem. The numerical radius of T, denoted by $w(T)$, is defined as the radius of the smallest circle with center at origin containing the numerical range, i.e., $w(T) = \sup\{|\langle Tx, x\rangle| : x \in S_{\mathbb{H}}\}$. It is easy to verify that $w(\cdot)$ defines a norm on $\mathbb{L}(\mathbb{H})$ when the field is complex and this does not hold true if the field is real. For $T \in \mathbb{L}(\mathbb{X})$, the numerical range and the numerical radius of T are respectively defined as $W(T) = \{x^*Tx : x \in S_{\mathbb{X}}, x^* \in S_{\mathbb{X}^*}, x^*(x) = 1\}$ and $w(T) = \sup\{|x^*Tx| : x \in S_{\mathbb{X}}, x^* \in S_{\mathbb{X}^*}, x^*(x) = 1\}$. Unlike Hilbert space, the numerical range is not necessarily convex, in general, for bounded linear operators on a Banach space, see [67]. For more information on numerical range and numerical radius, the readers may see the recent books [15, 31]. Motivated from Birkhoff-James orthogonality, the notion of numerical radius orthogonality in the space of operators is introduced in [65, 79] as follows: For $T, A \in \mathbb{L}(\mathbb{H})$ or $\mathbb{L}(\mathbb{X})$, T is said to be numerical radius orthogonal to A if $w(T + \lambda A) \geq w(T)$ for all scalars λ and we write it as $T \perp_w A$. For more information on the numerical radius orthogonality and related results one can see [62, 79]. The study of left and right symmetric operators with respect to the Birkhoff-James orthogonality was initiated in [91] and followed by many authors [33, 34, 54, 71, 85, 87, 88]. We are interested in studying the left and right symmetric operators with respect to the numerical radius orthogonality in the space of bounded linear operators. For this purpose, let us first introduce the following definitions.

Definition 3.1. *Let $T \in \mathbb{L}(\mathbb{X})$. Then T is said to be nr-left symmetric if for $A \in \mathbb{L}(\mathbb{X})$, we have $T \perp_w A \Rightarrow A \perp_w T$. Similarly, T is said to be nr-right symmetric if for $A \in \mathbb{L}(\mathbb{X})$, we have $A \perp_w T \Rightarrow T \perp_w A$. An operator T is said to be nr-symmetric if it is both nr-left and nr-right symmetric.*

In the study of left and right symmetric operators with respect to Birkhoff-James orthogonality, the norm attainment set of an operator plays an important role. Recall that for $T \in \mathbb{L}(\mathbb{X})$, the norm attainment set of T, denoted by M_T, is defined as $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. For more on norm attainment set one can see [89, 90]. In due course of the current study, we will see that the numerical radius attainment set also plays a central role in the study of left and right nr-symmetric operators. We make a note of the following definitions.

Definition 3.2. *Let $T \in \mathbb{L}(\mathbb{X})$. Then the numerical radius attainment set of T, denoted by $M_{w(T)}$, is defined as*

$$M_{w(T)} = \{x \in S_{\mathbb{X}} : \exists\ x^* \in J(x) \text{ such that } |x^*Tx| = w(T)\}.$$

For brevity we take the help of following notations: $M_{W(T)} = \{(x, x^*) \in S_{\mathbb{X}} \times S_{\mathbb{X}^*} : x^*(x) = 1, |x^*Tx| = w(T)\}$ and $\mathcal{M}_{W(T)} = \{(x, x^*) \in \text{Ext}\,(B_{\mathbb{X}}) \times \text{Ext}\,(B_{\mathbb{X}^*}) : |x^*(x)| = 1, x^*Tx = w(T)\}$. We also write $\mathcal{J}(x) = \{x^* \in S_{\mathbb{X}^*} : |x^*(x)| = 1\}$.

This chapter is divided into five sections including the introductory one. In sections two and three, we study nr-left and nr-right symmetric operators on Hilbert spaces. In section four, we study nr-left and nr-right symmetric operators on Banach spaces. In section five, we study the above mentioned problem on some particular Banach spaces. Before we end this section, we mention two important theorems on numerical radius orthogonality of operators which will be used later on.

Lemma 3.1. *[65, Th. 2.3] Let $\mathbb{H}$ be a complex Hilbert space and $T, A \in \mathbb{L}(\mathbb{H})$. Then $T \perp_w A$ if and only if for each $\theta \in [0, 2\pi)$, there exists a sequence $\{x_n^\theta\} \subset S_{\mathbb{H}}$ such that following two conditions hold true:*

(i) $\lim_{n\to\infty} |\langle Tx_n^\theta, x_n^\theta\rangle| = w(T)$.

(ii) $\lim_{n\to\infty} \Re\{e^{-i\theta}\langle Tx_n^\theta, x_n^\theta\rangle\langle Ax_n^\theta, x_n^\theta\rangle\} = 0$.

Theorem 3.1. *[79, Th. 2.3] Let $\mathbb{X}$ be a finite-dimensional Banach space and let $T, A \in \mathbb{L}(\mathbb{X})$ be nonzero. Then the following conditions are equivalent:*
(1) $T \perp_w A$
(2) $0 \in \text{conv}\{\overline{(x^*Tx)}(x^*Ax) : (x, x^*) \in M_{W(T)}\}$.

Theorem 3.2. *[62, Th. 3.1] Let $\mathbb{X}$ be a finite-dimensional Banach space. Let $T, A \in \mathbb{L}(\mathbb{X})$. Then the following are equivalent.*
(i) $T \perp_w A$
(ii) $0 \in \text{conv}\{x^*Ax : (x, x^*) \in \mathcal{M}_{W(T)}\}$.

3.2 Nr-left symmetric operators on Hilbert spaces

In this section, our first goal is to prove that a positive definite operator on a Hilbert space is not nr-left symmetric. In this direction, we first prove the following lemma, which is of independent interest.

Lemma 3.2. *Let $\mathbb{H}$ be a Hilbert space. Let $T \in \mathbb{L}(\mathbb{H})$ be a positive operator. Suppose that $M_T = \emptyset$. Then there exists an orthonormal sequence $\{e_n\}$ in $\mathbb{H}$ such that $\|Te_n\| \to \|T\|$ and $\langle Te_n, e_n\rangle \to \|T\|$.*

Proof. Since T is a positive operator, there exists an operator B on $\mathbb{H}$ such that $T = B^*B$. Note that $\|T\| = \|B\|^2$. Indeed, $\|T\| = \|B^*B\| \leq \|B\|^2$. On the other hand, since for each $x \in S_{\mathbb{H}}$,

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle B^*Bx, x\rangle = \langle Tx, x\rangle \leq \|Tx\|,$$

we get $\|B\|^2 \leq \|T\|$. Now, since T does not attain its norm so, $\|T\|$ can't be an eigenvalue of T. This implies form [75, Th. VII.10] that $\|T\| \in \sigma_{ess}(T)$, where $\sigma_{ess}(T)$ is the essential spectrum of T. Following [75, Th. VII.12] (Weyl's criterion) we observe that there exists an orthonormal sequence $\{e_n\}$ in $\mathbb{H}$ such that $\|Te_n\| \to \|T\|$. Therefore,

$$\begin{aligned}
\|Te_n\|^2 &= \langle Te_n, Te_n\rangle \\
&= \langle Te_n, B^*Be_n\rangle \\
&= \langle BTe_n, Be_n\rangle \\
&\leq \|B\|\|Te_n\|\|Be_n\| \\
\Rightarrow \|Te_n\| &\leq \|B\|\|Be_n\| \leq \|B\|^2 = \|T\| \\
\Rightarrow \|T\| &= \lim \|Te_n\| \leq \|B\| \lim \|Be_n\| \leq \|T\| \\
\Rightarrow \|B\| \lim \|Be_n\| &= \|T\| = \|B\|^2 \\
\Rightarrow \lim \|Be_n\| &= \|B\| \\
\Rightarrow \lim\langle Be_n, Be_n\rangle &= \|B\|^2 \\
\Rightarrow \lim\langle B^*Be_n, e_n\rangle &= \|B\|^2 = \|T\| \\
\Rightarrow \lim\langle Te_n, e_n\rangle &= \|T\|,
\end{aligned}$$

hence the proof. □

We are now ready to prove the desired result. We would like to mention that in [93, Prop. 3.2], Trunšek first proved that a positive operator on a Hilbert space is not $r-$left symmetric (with respect to operator norm), which consequently proves that the operator is not left symmetric (with respect to operator norm). In the following theorem, we prove that a positive definite operator on a Hilbert space is not nr-left symmetric. Moreover, we provide an alternative proof of [93, Prop. 3.2] for positive definite case.

Theorem 3.3. *Let $\mathbb{H}$ be a Hilbert space and let $T \in \mathbb{L}(\mathbb{H})$ be positive definite. Then T is neither nr-left symmetric nor left symmetric.*

Proof. We prove this theorem by considering the following two cases:
Case-I: Let $M_T \neq \emptyset$. If $M_T = S_{\mathbb{H}}$ then due to positive definiteness of T one can observe that $T = \lambda I$, for some $\lambda > 0$. Take any $z_0 \in S_{\mathbb{H}}$ arbitrary but fixed. We consider $A \in \mathbb{L}(\mathbb{H})$ such that $Ax = \langle x, z_0\rangle z_0$, for all $x \in \mathbb{H}$. Then for any $y_0 \in z_0^{\perp}$, we have $Ay_0 = 0$. As $M_T = M_{w(T)} = S_{\mathbb{H}}$, $y_0 \in M_{w(T)}$. This implies from Theorem 3.1 that $T \perp_w A$. Note that $M_A = M_{w(A)} = \{\mu z_0 : |\mu| = 1\}$. Therefore, $A \not\perp_w T$. Now assume that $M_T \neq S_{\mathbb{H}}$. As T is positive definite, $M_{w(T)} \subsetneq S_{\mathbb{H}}$. Then there exists $y' \in S_{\mathbb{H}} \setminus M_{w(T)}$ such that $y' \in span\, M_{w(T)}{}^{\perp}$. Now considering $A \in \mathbb{L}(\mathbb{H})$ as $Ax = \langle x, y'\rangle y'$ we obtain that $T \perp_w A$ whereas, from the positive definiteness of T we get $A \not\perp_w T$. This shows that T is not nr-left symmetric. As T is self-adjoint, we have $T \perp_w A \implies T \perp_B A$ (see [65, Prop. 2.2(i)]). From this we conclude that T is not left symmetric.
Case-II: Suppose that $M_T = \emptyset$. Then from Lemma 3.2, choose an orthonormal sequence $\{e_n\}$ in $\mathbb{H}$ such that $\|Te_n\| \to \|T\|$ and $\langle Te_n, e_n\rangle \to \|T\|$. Since $\|T\| > 0$, there exists $n_0 \in \mathbb{N}$ such that $\langle Te_{n_0}, e_{n_0}\rangle > 0$. Define $A \in \mathbb{L}(\mathbb{H})$ as follows

$$Az = \langle z, e_{n_0}\rangle e_{n_0}, \text{ for all } z \in \mathbb{H}.$$

Notice that $\|Az\|^2 = |\langle z, e_{n_0}\rangle|^2 = |\langle Az, z\rangle|$ and $|\langle z, e_{n_0}\rangle|^2 \leq \|z\|^2$, where the equality holds if and only if $z = \lambda e_{n_0}$ for some scalar λ. Thus,

$$M_A = M_{w(A)} = \{\lambda e_{n_0} : |\lambda| = 1\}.$$

Clearly, for all $n \neq n_0$, $Ae_n = \langle e_n, e_{n_0}\rangle e_{n_0} = 0$. Thus $\langle Te_n, Ae_n\rangle = 0$ for all $n \neq n_0$. Now, from Lemma 3.1 it follows that $T \perp_w A$ and from [14, Rem 3.1], we get $T \perp_B A$. However, A is a compact operator and $\langle Te_{n_0}, Ae_{n_0}\rangle = \langle Te_{n_0}, e_{n_0}\rangle \neq 0$. Therefore, from [71, Th. 2.2(ii)], it follows that $A \not\perp_B T$. Now, observe that $A = A^*$. Therefore, if $A \perp_w T$, then from [65, Prop. 2.2(i)], we get $A \perp_B T$, which is a contradiction. Thus, $A \not\perp_w T$. Hence, T is neither nr-left symmetric nor left symmetric. □

Our next goal is to prove that the zero operator is the only nr-left symmetric operator on a complex Hilbert space, provided that the operator attains its numerical radius. We need the following lemma to prove this.

Lemma 3.3. *Let $\mathbb{H}$ be a Hilbert space. Suppose that $T \in \mathbb{L}(\mathbb{H})$ is nr-left symmetric and $M_{w(T)} \neq \emptyset$. Let $x \in M_{w(T)}$ and $x \perp y$. Then $Ty \perp y$ and $T^*y \perp y$.*

Proof. We prove the theorem for complex Hilbert space. Analogously, the result can be proved for real Hilbert space. Without loss of generality, assume that $\|y\| = 1$. Define $A \in \mathbb{L}(\mathbb{H})$ as follows

$$Az = \langle z, y \rangle y, \text{ for all } z \in \mathbb{H}.$$

Then clearly, A is a compact operator, $M_{w(A)} = \{\lambda y : |\lambda| = 1\}$ and $w(A) = 1$. Since $Ax = 0$, from Lemma 3.1, we have $T \perp_w A$. Therefore, $A \perp_w T$, since T is nr-left symmetric. Choose $\theta \in [0, 2\pi)$. Then using Lemma 3.1, we get a sequence $\{x_{n_\theta}\}$ from $S_{\mathbb{H}}$ such that

$$\lim |\langle Ax_{n_\theta}, x_{n_\theta} \rangle| = w(A) = 1 \text{ and}$$

$$\lim \Re\{e^{-i\theta} \langle Ax_{n_\theta}, x_{n_\theta} \rangle \overline{\langle Tx_{n_\theta}, x_{n_\theta} \rangle}\} \geq 0.$$

Since $B_{\mathbb{H}}$ is weakly compact, there exists $x_\theta \in B_{\mathbb{H}}$ such that $\{x_{n_\theta}\}$ weakly converges to x_θ. Thus, $Ax_{n_\theta} \to Ax_\theta$ and $\langle Ax_{n_\theta}, x_{n_\theta} \rangle \to \langle Ax_\theta, x_\theta \rangle$. Therefore, $|\langle Ax_\theta, x_\theta \rangle| = w(A)$, which implies that $\|x_\theta\| = 1$ and $x_\theta \in M_{w(A)}$. Now, since $\mathbb{H}$ is Kadets-Klee, $\{x_{n_\theta}\}$ weakly converges to x_θ and $\|x_{n_\theta}\| \to \|x_\theta\|$, we get $x_{n_\theta} \to x_\theta$. Hence, $Tx_{n_\theta} \to Tx_\theta$. Now, from $x_\theta \in M_{w(A)}$ we get $x_\theta = \lambda y$ for some scalar λ with $|\lambda| = 1$. Therefore,

$$\begin{aligned} \lim \Re\{e^{-i\theta} \langle Ax_{n_\theta}, x_{n_\theta} \rangle \overline{\langle Tx_{n_\theta}, x_{n_\theta} \rangle}\} &\geq 0 \\ \Rightarrow \Re\{e^{-i\theta} \langle Ax_\theta, x_\theta \rangle \overline{\langle Tx_\theta, x_\theta \rangle}\} &\geq 0 \\ \Rightarrow \Re\{e^{-i\theta} \langle Ay, y \rangle \overline{\langle Ty, y \rangle}\} &\geq 0 \\ \Rightarrow \Re\{e^{-i\theta} \overline{\langle Ty, y \rangle}\} &\geq 0. \end{aligned}$$

This is true for each $\theta \in [0, 2\pi)$. In particular, considering $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, we get $\Re\{\overline{\langle Ty, y \rangle}\} = 0$ and $\Re\{i\overline{\langle Ty, y \rangle}\} = 0$. Thus, $\langle Ty, y \rangle = 0$. Therefore, $\langle T^*y, y \rangle = \langle y, Ty \rangle = 0$. This completes the proof of the theorem. □

Now, we prove our promised result.

Theorem 3.4. *Let $\mathbb{H}$ be a complex Hilbert space. Suppose $T \in \mathbb{L}(\mathbb{H})$ be such that $M_{w(T)} \neq \emptyset$. Then T is nr-left symmetric if and only if T is the zero operator.*

Proof. If T is the zero operator, then clearly T is nr-left symmetric. We only prove the converse part. Suppose that T is nr-left symmetric and $x \in M_{w(T)}$. Then from Lemma 3.3, it follows that $\langle Ty, y\rangle = 0 = \langle T^*y, y\rangle$ for all $y \in x^\perp$. Since $x^\perp$ itself is a complex Hilbert space, we have $Ty = T^*y = 0$ for all $y \in x^\perp$. Choose $y \in x^\perp \cap S_{\mathbb{H}}$. Let $y \perp H_y$, where H_y is a hyperspace in $x^\perp$. Now, each $z \in \mathbb{H}$ can be uniquely written in the form $z = ax + by + h$, where a, b are scalars and $h \in H_y$. Define $A \in \mathbb{L}(\mathbb{H})$ as follows

$$Az = A(ax + by + h) = (a + b)y + bx.$$

Observe that A is a compact operator, and $A = A^*$. It is obvious that $M_A \subseteq S_{\mathbb{H}} \cap span\{x, y\}$. Moreover, A is not a scalar multiple of an isometry in $span\{x, y\}$. Now, it is easy to check that $M_A = \{\lambda z : |\lambda| = 1\}$, for some $z = ax + by$, where a, b are nonzero scalars. Indeed,

$$\|Au\| > \|Ay\| > \|Ax\|, \text{ where } u = \frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}}x + \frac{\sqrt{3+\sqrt{5}}}{\sqrt{5+\sqrt{5}}}y,$$

proves that $x, y \notin M_A$. Now, $\langle Ax, x\rangle = \langle y, x\rangle = 0$ implies that $T \perp_w A$. Therefore, $A \perp_w T$. Since $A = A^*$, from [65, Prop. 2.2(i)], we get $A \perp_B T$. Thus,

$$\begin{aligned}
\langle Az, Tz\rangle &= 0\\
\Rightarrow \langle z, Tz\rangle &= 0\\
\Rightarrow \langle ax + by, aTx\rangle &= 0 \text{ (since } Ty = 0)\\
\Rightarrow |a|^2\langle x, Tx\rangle + b\overline{a}\langle y, Tx\rangle &= 0\\
\Rightarrow |a|^2\langle x, Tx\rangle + b\overline{a}\langle T^*y, x\rangle &= 0\\
\Rightarrow |a|^2\langle x, Tx\rangle &= 0 \text{ (since } T^*y = 0)\\
\Rightarrow \langle Tx, x\rangle &= 0\\
\Rightarrow w(T) &= 0.
\end{aligned}$$

Thus, T is the zero operator. Hence the proof. □

3.3 Nr-right symmetric operators on Hilbert space

Let us begin this section with a necessary condition for a normal operator to be nr-right symmetric.

Theorem 3.5. *Let $\mathbb{H}$ be a finite-dimensional complex Hilbert space and $T \in \mathbb{L}(\mathbb{H})$ be a normal operator with $w(T) = 1$. Suppose T is nr-right symmetric. Then T must be a unitary.*

Proof. Since T is a normal operator, there exist a unitary operator U and a diagonal operator D such that $T = U^*DU$. Then $w(D) = 1$. Assume that

$$D = diag(d_1, \dots, d_n).$$

If possible, suppose that T is not a unitary. Then D is not a unitary. Therefore, there exists $j \in \{1, 2, \dots n\}$ such that $|d_j| < 1$. For our convenience, we assume that $|d_1| = 1$ and $|d_2| < 1$. Now, consider the operators

$$B = diag(d_1, d_2) \text{ and } C = diag(d_1, d),$$

where $d = -\frac{d_2}{|d_2|}$ if $d_2 \neq 0$ and $d = -1$ if $d_2 = 0$. Observe that $M_{w(B)} = \{\lambda e_1 : |\lambda| = 1\}$ and $e_1, e_2 \in M_{w(C)}$. Moreover,

$$\overline{\langle Ce_1, e_1\rangle}\langle Be_1, e_1\rangle = |d_1|^2 \text{ and } \overline{\langle Ce_2, e_2\rangle}\langle Be_2, e_2\rangle \in \{-|d_2|^2, -1\}.$$

Therefore, there exists $t \in [0, 1]$ such that

$$t\overline{\langle Ce_1, e_1\rangle}\langle Be_1, e_1\rangle + (1-t)\overline{\langle Ce_2, e_2\rangle}\langle Be_2, e_2\rangle = 0.$$

Now, by [62, Th. 3.3], we get $C \perp_w B$. However, since $\overline{\langle Be_1, e_1\rangle}\langle Ce_1, e_1\rangle = |d_1|^2 \neq 0$, again from [62, Th. 3.3], we conclude that $B \not\perp_w C$. Now, consider the operator

$$A = diag(d_1, d, d_3, \dots, d_n).$$

We show that $A \perp_w D$ but $D \not\perp_w A$. Note that for each scalar λ,

$$w(A + \lambda D) \geq w(C + \lambda B) \geq w(C) = 1 = w(A).$$

Thus, $A \perp_w D$. On the other hand, since $|d_2| < 1$, there exists $0 < \lambda < 1$ such that $|d_2| + \lambda < 1$. Now, for this λ,

$$\begin{aligned} w(D - \lambda A) &= \max\{|d_1 - \lambda d_1|, |d_2 - \lambda d|, |d_3 - \lambda d_3|, \dots, |d_n - \lambda d_n|\} \\ &< 1 = w(D). \end{aligned}$$

Hence, $D \not\perp_w A$. Now, $A \perp_w D$ gives that $U^*AU \perp_w U^*DU$, i.e., $U^*AU \perp_w T$, whereas $D \not\perp_w A$

gives that $U^*DU \not\perp_w U^*AU$, i.e., $T \not\perp_w U^*AU$. Thus, T is not nr-right symmetric, which is a contradiction. This completes the proof. □

The following corollary easily follows from Theorem 3.5.

Corollary 3.1. *Let $\mathbb{H}$ be a finite-dimensional complex Hilbert space. Suppose $T \in \mathbb{L}(\mathbb{H})$ is a positive operator such that T is not a scalar multiple of the identity operator. Then T is not nr-right symmetric.*

Proof. Clearly, T is a normal operator. Since $T \neq \lambda I$ for any scalar λ, T has at least two positive eigenvalues d_1, d_2 such that $d_1 \neq d_2$. Thus, $\frac{1}{w(T)}T$ is not a unitary. Now, from Theorem 3.5, it follows that $\frac{1}{w(T)}T$ is not nr-right symmetric. Therefore, T is not nr-right symmetric, completing the proof. □

Next, we present a class of operators which are not nr-right symmetric.

Theorem 3.6. *Let $\mathbb{H}_1, \mathbb{H}_2$ be complex Hilbert spaces. Suppose $A \in \mathbb{L}(\mathbb{H}_1), B \in \mathbb{L}(\mathbb{H}_2)$ such that either A or B is not nr-right symmetric. Then*

$$T = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$$

is not nr-right symmetric.

Proof. First assume that A is not nr-right symmetric. Then there exists $S \in \mathbb{L}(\mathbb{H}_1)$ with $w(S) = w(T)$ such that $S \perp_w A$ but $A \not\perp_w S$. Choose a scalar λ_0 such that $|\lambda_0| < 1$ and $w(A + \lambda_0 S) < w(A)$. Now, consider the operator

$$P = \begin{bmatrix} S & O \\ O & -\overline{\lambda_0}B \end{bmatrix}.$$

Then by [13, p. 10],

$$w(P) = \max\{w(S), w(-\overline{\lambda_0}B)\} = \max\{w(T), |\lambda_0|w(B)\} = w(T).$$

Now, from

$$\begin{aligned} w(P + \lambda T) &= \max\{w(S + \lambda A), w(-\overline{\lambda_0}B + \lambda B)\} \\ &\geq w(S + \lambda A) \\ &\geq w(S) = w(T) = w(P), \end{aligned}$$

we get $P \perp_w T$. On the other hand,

$$\begin{aligned} w(T+\lambda_0 P) &= \max\{w(A+\lambda_0 S), w(B-|\lambda_0|^2 B)\} \\ &< \max\{w(A), w(B)\} = w(T). \end{aligned}$$

Therefore, $T \not\perp_w P$, i.e., T is not nr-right symmetric. Similarly, B is not nr-right symmetric implies that T is not nr-right symmetric, completing the proof of the theorem. □

Our next goal is to provide another necessary condition for an operator to be nr-right symmetric. We require the following lemma for this.

Lemma 3.4. *Let $\mathbb{H}$ be a two-dimensional complex Hilbert space. Suppose $T \in \mathbb{L}(\mathbb{H})$ is a self-adjoint operator and $T \neq \lambda I$ for any scalar λ. Then T is not nr-right symmetric.*

Proof. Without loss of generality, we may assume that $w(T) = 1$. Since T is a self-adjoint operator, there exist a unitary operator U and a diagonal operator D such that $T = U^*DU$. Then $w(D) = 1$. Let $D = diag(d_1, d_2)$. Clearly, $d_1 \neq d_2$, since $T \neq \lambda I$ for any scalar λ. Without loss of generality, assume that $d_1 = 1$. If $d_2 > 0$, then D is a positive operator. Therefore, from Corollary 3.1, we can conclude that D is not nr-right symmetric, i.e., T is not nr-right symmetric. So assume that $d_2 \leq 0$. Here, we consider the cases $-1 < d_2$ and $d_2 = -1$ separately.

First consider the case $-1 < d_2 \leq 0 \leq -d_2 < 1$. Then $M_{w(D)} = \{\lambda e_1 : |\lambda| = 1\}$. Since $\langle De_1, e_1\rangle = 1$ and $\langle De_2, e_2\rangle = d_2 \leq 0$, there exists $t \in [0,1]$ such that

$$t\overline{\langle Ie_1, e_1\rangle}\langle De_1, e_1\rangle + (1-t)\overline{\langle Ie_2, e_2\rangle}\langle De_2, e_2\rangle = 0.$$

Thus, by [62, Th. 3.3], $I \perp_w D$. However, since $\langle Ie_1, e_1\rangle \neq 0$, from [62, Th. 3.3], we get $D \not\perp_w I$.

Now, consider the case $d_2 = -1$. Consider the operators

$$A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \text{ and } V = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}.$$

Observe that V is a unitary operator and

$$VAV^* = \begin{bmatrix} 0 & 2i \\ 0 & 0 \end{bmatrix}.$$

Therefore, $w(A) = w(VAV^*) = 1$. Moreover, $x = \frac{1}{\sqrt{2}}(1,1) \in M_{w(A)}$. Since $\langle Dx, x\rangle = 0$, from

[62, Th. 3.3], we get $A \perp_w D$. Note that $M_{w(D)} = \{\lambda e_1, \lambda e_2 : |\lambda| = 1\}$. Now, for any $t \in [0,1]$,

$$t\overline{\langle De_1, e_1\rangle}\langle Ae_1, e_1\rangle + (1-t)\overline{\langle De_2, e_2\rangle}\langle Ae_2, e_2\rangle = 1 > 0.$$

Therefore, from [62, Th. 3.3], we get $D \not\perp_w A$.
Thus, in each case, D is not nr-right symmetric, which proves that T is not nr-right symmetric, completing the proof of the lemma. □

The following theorem is a generalization of Lemma 3.4.

Theorem 3.7. *Let $\mathbb{H}$ be a finite-dimensional complex Hilbert space and $T \in \mathbb{L}(\mathbb{H})$ be a self-adjoint operator. Suppose that $T \neq \lambda I$ for any scalar λ. Then T is not nr-right symmetric.*

Proof. If $\dim(\mathbb{H}) = 2$, then the result follows from Lemma 3.4. So assume that $\dim(\mathbb{H}) > 2$. Since T is a self-adjoint operator, there exist a unitary operator U and a diagonal operator D such that $T = U^*DU$. Without loss of generality, assume that

$$w(T) = w(D) = 1 \text{ and } D = diag(1, d_2, \ldots, d_n),$$

where $d_j \in \mathbb{R}$ and $|d_j| \leq 1$ for all $2 \leq j \leq n$. Since for each scalar λ, $T \neq \lambda I$, i.e., $D \neq \lambda I$, we may assume that $d_2 \neq 1$. Let

$$D_1 = diag(1, d_2) \text{ and } D_2 = (d_3, \ldots, d_n).$$

Then

$$D = \begin{bmatrix} D_1 & O \\ O & D_2 \end{bmatrix}.$$

Observe that D_1 is self-adjoint and $D_1 \neq \lambda I$ for any scalar λ. Therefore, by Lemma 3.4, D_1 is not nr-right symmetric. Now, from Theorem 3.6, it follows D is not nr-right symmetric, i.e., T is not so. This completes the proof. □

The next corollary proves that the converse of Theorem 3.5 is not true.

Corollary 3.2. *Let $\mathbb{H}$ be a finite-dimensional complex Hilbert space and $T \in \mathbb{L}(\mathbb{H})$ be a unitary operator. Suppose $T^* = \frac{\alpha}{\overline{\alpha}}T$ for some scalar α but $T \neq \lambda I$ for any scalar λ. Then T is not nr-right symmetric.*

Proof. Since $T^* = \frac{\alpha}{\overline{\alpha}}T$ for some scalar α, $(\alpha T)^* = \alpha T$. Moreover, $\alpha T \neq \lambda I$ for any scalar λ. Therefore, from Theorem 3.7, we get αT is not nr-right symmetric, i.e., T is not nr-right symmetric. □

In the following theorem, we prove that nonzero compact normal operators on infinite-dimensional complex Hilbert space are not nr-right symmetric.

Theorem 3.8. *Let $\mathbb{H}$ be an infinite-dimensional complex Hilbert space. Let $T \in \mathbb{L}(\mathbb{H})$ be a compact normal operator with $w(T) = 1$. Then T is not nr-right symmetric.*

Proof. Noting that in an infinite-dimensional Hilbert space there does not exist any unitary operator and using the spectral theorem for compact normal operators, we get eigenvalues λ_1, λ_2 of T such that $|\lambda_1| = 1$ and $|\lambda_2| < 1$. Suppose $e_i \in S_{\mathbb{H}}$ is an eigenvector of T corresponding to λ_i for $i = 1, 2$. Let $X = span\{e_1, e_2\}$, $T|_X = A,\ T|_{X^\perp} = B$. Then

$$T = \begin{bmatrix} A & O \\ O & B \end{bmatrix} \in \mathbb{L}(X \oplus X^\perp).$$

Then A is a normal operator, $w(A) = 1$ and A is not an unitary. Therefore, from Theorem 3.5, we get A is not nr-right symmetric. Thus, from Theorem 3.6, it follows that T is not nr-right symmetric. This completes the proof. □

In the next theorem, we provide another class of operators which are not nr-right symmetric. Recall that an operator $A \in \mathbb{L}(\mathbb{H})$ is said to be a coisometry if A^* is an isometry.

Theorem 3.9. *Let $\mathbb{H}$ be a complex Hilbert space. Suppose $(O \neq)\ A \in \mathbb{L}(\mathbb{H})$ is neither a scalar multiple of an isometry nor a scalar multiple of a coisometry. Then the operator matrix*

$$T = \begin{bmatrix} O & A \\ O & O \end{bmatrix} \in \mathbb{L}(\mathbb{H} \oplus \mathbb{H})$$

is not nr-right symmetric.

Proof. From [93, Cor. 4.5], it follows that A is not right symmetric (with respect to operator norm). Therefore, there exists an operator $C \in \mathbb{L}(\mathbb{H})$ such that $C \perp_B A$ but $A \not\perp_B C$. Consider

$$S = \begin{bmatrix} O & C \\ O & O \end{bmatrix} \in \mathbb{L}(\mathbb{H} \oplus \mathbb{H}).$$

It follows from [1, Lemma 2], that for each scalar λ,

$$w(S + \lambda T) = \frac{1}{2}\|C + \lambda A\| \geq \frac{1}{2}\|C\| = w(S).$$

Thus, $S \perp_w T$. However, since $A \not\perp_B C$, there exists a scalar λ_0 such that $\|A + \lambda_0 C\| < \|A\|$.

Therefore,

$$w(T+\lambda_0 S) = \frac{1}{2}\|A+\lambda_0 C\| < \frac{1}{2}\|A\| = w(T).$$

Thus, $T \not\perp_w S$ and T is not nr-right symmetric. This completes the proof. □

3.4 Properties of symmetric operators on Banach spaces

In this section, first we note the following easy sufficient condition for nr-right symmetric operators in $\mathbb{L}(\mathbb{X})$.

Theorem 3.10. *Let $\mathbb{X}$ be a finite-dimensional Banach space. If for each $x \in \operatorname{Ext}(B_{\mathbb{X}})$ and $x^* \in \operatorname{Ext}(J(x))$, there exists a unimodular constant μ such that $(x, \mu x^*) \in \mathcal{M}_{W(T)}$ then T is nr-right symmetric. In particular, the identity operator $I \in \mathbb{L}(\mathbb{X})$ is nr-right symmetric.*

Proof. Let $A \in \mathbb{L}(\mathbb{X})$ be such that $A \perp_w T$. From Theorem 3.2, we have $0 \in \operatorname{conv}\{(x^*Tx) : (x, x^*) \in \mathcal{M}_{W(A)}\}$. Then applying Carathéodory theorem there exist $t_1, t_2, t_3 \geq 0$ with $\sum_{i=1}^{3} t_i = 1$ such that

$$\sum_{i=1}^{3} t_i(x_i^* T x_i) = 0, \quad \text{where} \quad (x_i, x_i^*) \in \mathcal{M}_{W(A)}. \tag{3.1}$$

Suppose that $(x_i, \mu_i x_i^*) \in \mathcal{M}_{W(T)}$, where $|\mu_i| = 1$ and $1 \leq i \leq 3$. This implies that $x_i^* T x_i = \overline{\mu}_i w(T)$. Thus from (1) it is easy to see that $\sum_{i=1}^{3} t_i \mu_i = 0$. Now

$$\begin{aligned} \sum_{i=1}^{3} t_i \mu_i x_i^* A x_i &= \sum_{i=1}^{3} t_i \mu_i w(A) \\ &= w(A)(\sum_{i=1}^{3} t_i \mu_i) \\ &= 0. \end{aligned}$$

Therefore, $0 \in \operatorname{conv}\{(x^*Ax) : (x, x^*) \in \mathcal{M}_{W(T)}\}$. Using Theorem 3.2 again, we conclude that $T \perp_w A$, i.e., T is nr-right symmetric. This establishes the first part of the theorem. The second part of the theorem follows trivially from the first part. □

Using Theorem 3.10, we give an example of a nr-right symmetric operator other than the scalar multiples of the identity operator.

Example 3.11. *Consider $\mathbb{X} = \ell_1^n$ over the real field $\mathbb{R}$ and let $T \in \mathbb{L}(\ell_1^n)$ be defined as*

$$Te_i = e_i, \quad 1 \leq i \leq r,$$

$$= \quad -e_i, \quad r+1 \leq i \leq n,$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ *with* 1 *in* i*-th position and* 0 *elsewhere. It is straightforward to see that* $\mathcal{M}_{W(T)} = \{\pm(e_i, e_{i_k}^*) : 1 \leq i \leq r, e_{i_k}^* \in \operatorname{Ext}(J(e_i))\} \cup \{\pm(e_i, -e_{i_k}^*) : r+1 \leq i \leq n, e_{i_k}^* \in \operatorname{Ext}(J(e_i))\}$. *This implies that for every* $x \in \operatorname{Ext}(B_{\mathbb{X}})$ *and every* $x^* \in \operatorname{Ext}(J(x))$, *either* $(x, x^*) \in \mathcal{M}_{W(T)}$ *or* $(x, -x^*) \in \mathcal{M}_{W(T)}$. *Therefore, from Theorem 3.10, it is evident that* T *is nr-right symmetric.*

We next introduce the following definition of sign determining extreme support functionals which will be crucial in the rest of our study.

Definition 3.3. *Let* $\mathbb{X}$ *be a finite-dimensional real Banach space. Let* $x \in \operatorname{Ext}(B_{\mathbb{X}})$ *and let* $T \in \mathbb{L}(\mathbb{X})$. *We say that* $\{x_\alpha^*\}_{\alpha\in\Lambda} \subset \operatorname{Ext}(J(x))$ *is a set of sign determining extreme support functionals for* T *at* x *if for any* $v \in \mathbb{X}$, *the conditions (i) and (ii) together imply the condition (iii), where*
(i) there exists $x_0^* \in \operatorname{Ext}(J(x)) \setminus \{x_\alpha^*\}_{\alpha\in\Lambda}$ *such that* $|x_0^*(v)| = \sup\{|x^*(v)| : x^* \in J(x)\}$.
(ii) $sgn(x_\alpha^*(v)) = sgn(x_\alpha^*(Tx))$, *for all* $\alpha \in \Lambda$.
(iii) $sgn(x_0^*(v)) = sgn(x_0^*(Tx))$.

Using the above definition, we present a sufficient condition for nr-right symmetric operators on a finite-dimensional real Banach space.

Theorem 3.12. *Let* $\mathbb{X}$ *be a finite-dimensional real Banach space and let* $T \in \mathbb{L}(\mathbb{X})$. *If for each* $x \in \operatorname{Ext}(B_{\mathbb{X}})$, *there exists a set of sign determining extreme support functionals* $\{x_\alpha^*\}_{\alpha\in\Lambda}$ *for* T *at* x *such that either* $(x, x_\alpha^*) \in \mathcal{M}_{W(T)}$ *or* $(x, -x_\alpha^*) \in \mathcal{M}_{W(T)}$, *for all* $\alpha \in \Lambda$. *Then* T *is nr-right symmetric.*

Proof. Let $A \in \mathbb{L}(\mathbb{X})$ be such that $A \perp_w T$. Suppose on the contrary that $T \not\perp_w A$. Clearly, $w(A) > 0$. Applying Theorem 3.2, we observe that exactly one of the following holds true:
(i) $x^*(Ax) > 0$, for all $(x, x^*) \in \mathcal{M}_{W(T)}$.
(ii) $x^*(Ax) < 0$, for all $(x, x^*) \in \mathcal{M}_{W(T)}$.
Without loss of generality we assume that (i) holds. Since $A \perp_w T$, once again applying Theorem 3.2, we obtain that there exists $(x_0, x_0^*) \in \mathcal{M}_{W(A)}$ such that $x_0^*(Tx_0) \leq 0$. Clearly, $(x_0, x_0^*) \notin \mathcal{M}_{W(T)}$. Suppose that $\{x_\gamma^*\}_{\gamma\in\Gamma}$ is a set of sign determining extreme support functionals for T at x_0 such that either $(x_0, x_\gamma^*) \in \mathcal{M}_{W(T)}$ or $(x_0, -x_\gamma^*) \in \mathcal{M}_{W(T)}$, for all $\gamma \in \Gamma$. Observe that
(a) $x_0^* \in \operatorname{Ext}(J(x_0)) \setminus \{x_\gamma^*\}_{\gamma\in\Gamma}$ such that $|x_0^*(Ax_0)| = \sup\{|x^*(Ax_0)| : x^* \in J(x_0)\}$.
(b) $sgn(x_\gamma^*(Tx_0)) = sgn(x_\gamma^*(Ax_0))$, for all $\gamma \in \Gamma$.
Therefore, from the sign determining property of $\{x_\gamma^*\}_{\gamma\in\Gamma}$, we obtain that $sgn(x_0^*(Tx_0) =$

$sgn(x_0^*(Ax_0))$. We note that since $x_0^*(Ax_0) = w(A) > 0$, we get $x_0^*(Tx_0) > 0$, which contradicts our assumption that $x_0^*(Tx_0) \leq 0$. This proves our theorem. $\square$

The following corollary is immediate from Theorem 3.12, in view of the fact that for every $x \in S_{\mathbb{X}}$ in a smooth Banach space $\mathbb{X}$, $J(x)$ is singleton.

Corollary 3.3. *Let $\mathbb{X}$ be a finite-dimensional smooth real Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be such that for each $x \in \operatorname{Ext}(B_{\mathbb{X}})$, $(x, x^*) \in \mathcal{M}_{W(T)}$, where $x^* \in J(x)$ or $x^* \in J(-x)$. Then T is nr-right symmetric.*

We next obtain a necessary condition for nr-right symmetric operators on a finite-dimensional Banach space.

Theorem 3.13. *Let $\mathbb{X}$ be a finite-dimensional Banach space and let $T \in \mathbb{L}(\mathbb{X})$ be nr-right symmetric. Then $0 \notin W(T) \setminus \operatorname{conv}\{x^*Tx : (x, x^*) \in M_{W(T)}\}$.*

Proof. Suppose on the contrary that $0 \in W(T) \setminus \operatorname{conv}\{x^*Tx : (x, x^*) \in M_{W(T)}\}$. Let us consider the identity map, $I \in \mathbb{L}(\mathbb{X})$. Since $0 \in W(T)$, there exists an $x \in S_{\mathbb{X}}$ such that $x^*Tx = 0$, for some $x^* \in J(x)$. As $(x, x^*) \in M_{W(I)}$, it follows from Theorem 3.1 that $I \perp_w T$. On the other hand we obtain that

$$\begin{aligned} & 0 \notin \operatorname{conv}\{x^*Tx : (x, x^*) \in M_{W(T)}\} \\ \implies & 0 \notin \operatorname{conv}\{(\overline{x^*Tx}) : (x, x^*) \in M_{W(T)}\} \\ \implies & 0 \notin \operatorname{conv}\{(\overline{x^*Tx})(x^*Ix) : (x, x^*) \in M_{W(T)}\}. \end{aligned}$$

Applying Theorem 3.1 again, we conclude that $T \not\perp_w I$. This leads to a contradiction to the fact that T is nr-right symmetric. $\square$

Following [79, Defn. 1.2] we note that an operator $T \in \mathbb{L}(\mathbb{X})$ is said to be nr-smooth (called as nu-smooth in [79]), if $T \perp_w A$ and $T \perp_w B$ implies $T \perp_w (A + B)$ for all $A, B \in \mathbb{L}(\mathbb{X})$. Using [79, Th. 2.5] and Theorem 3.13, the following corollary is immediate.

Corollary 3.4. *Let $\mathbb{X}$ be a finite-dimensional Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be nr-smooth with $0 \in W(T)$. Then T is not nr-right symmetric.*

We next present a necessary condition for nr-left symmetric operators on a finite-dimensional Banach space.

Theorem 3.14. *Let $\mathbb{X}$ be a finite-dimensional Banach space and let $T \in \mathbb{L}(\mathbb{X})$ be nr-left symmetric. Suppose that $(x, x^*) \in M_{W(T)}$ and $x^*(y) = 0$, for some exposed point $y \in B_{\mathbb{X}}$. Then $y \perp_B Ty$.*

Proof. Since $y \in B_{\mathbb{X}}$ is an exposed point, there exists an $g \in S_{\mathbb{X}^*}$ such that $g(y) = 1$ and $M_g = \{\lambda y : |\lambda| = 1\}$. Let us define a linear map $A : \mathbb{X} \to \mathbb{X}$ by $Az = g(z)y$. For any $z \in S_{\mathbb{X}}$ and $z^* \in J(z)$,

$$|z^*Az| = |g(z)z^*(y)| = |g(z)||z^*(y)| \leq 1.$$

In the last inequality, equality holds if and only if $|g(z)| = 1$ and $|z^*(y)| = 1$. Clearly, this is equivalent to $z = \mu y$, for some $\mu \in \mathbb{C}$ with $|\mu| = 1$. Thus we obtain that

$$M_{W(A)} = \{(\mu y, \overline{\mu} y^*) : |\mu| = 1, y^* \in J(y)\}.$$

Observe that $x^*Ax = 0$. Since $(x, x^*) \in M_{W(T)}$, applying Theorem 3.1 we obtain that $T \perp_w A$. Since T is nr-left symmetric, it follows that $A \perp_w T$. Once again applying Theorem 3.1, we deduce that

$$0 \in \text{conv}\{\overline{(v^*Av)}(v^*Tv) : (v, v^*) \in M_{W(A)}\}.$$

Observe that

$$\begin{aligned}
& 0 \in \text{conv}\{\overline{(y^*Ay)}(y^*Ty) : y^* \in J(y)\} \\
\implies \quad & 0 \in \text{conv}\{\overline{(y^*(g(y)y))}(y^*Ty) : y^* \in J(y)\} \\
\implies \quad & 0 \in \text{conv}\{\overline{(y^*(y)g(y))}(y^*Ty) : y^* \in J(y)\} \\
\implies \quad & 0 \in \text{conv}\{(y^*Ty) : y^* \in J(y)\}.
\end{aligned}$$

Therefore, by the Carathéodory theorem, there exist $t_1, t_2, t_3 \geq 0$ with $\sum_{i=1}^{3} t_i = 1$ such that

$$\sum_{i=1}^{3} t_i y_i^* Ty = 0, , \quad \text{where} \quad y_i^* \in J(y).$$

Since $J(y)$ is a convex set of $\mathbb{X}^*$, it follows that $\sum_{i=1}^{3} t_i y_i^* \in J(y)$. Thus there exists an $y^* \in J(y)$ such that $y^*Ty = 0$. From [42, Th. 2.1], we conclude that $y \perp_B Ty$. This completes the proof of the theorem.

□

The following corollary is immediate from the above Theorem.

Corollary 3.5. *Let $\mathbb{X}$ be a finite-dimensional strictly convex Banach space and let $T \in \mathbb{L}(\mathbb{X})$ be nr-left symmetric. Then $0 \in W(T)$.*

It is well known that given any $T \in \mathbb{L}(\mathbb{X})$ with $rank(T) = 1$, there exist $y \in \mathbb{X} \setminus \{\theta\}$ and $x^* \in \mathbb{X}^* \setminus \{\theta\}$ such that $Tz = x^*(z)y$, for all $z \in \mathbb{X}$. In the next result we show that $T \perp_B A \Leftrightarrow$

$T \perp_w A$ for all rank one compact operators A defined on a reflexive Banach space. Note that if $w(T) = \|T\|$ and $T \perp_w A$ holds then for each scalar λ, $\|T + \lambda A\| \geq w(T + \lambda A) \geq w(T) = \|T\|$ so that $T \perp_w A \Rightarrow T \perp_B A$.

Theorem 3.15. *Let $\mathbb{X}$ be a real reflexive Banach space. Suppose that $T \in \mathbb{L}(\mathbb{X})$ is given by $Tz = x^*(z)y$, for all $z \in \mathbb{X}$, where $y \in \mathbb{X} \setminus \{\theta\}$ and $x^* \in \mathbb{X}^* \setminus \{\theta\}$. Let $x^* \in J(y)$. If either $M_{x^*} = \{\pm \frac{1}{\|y\|} y\}$ or y is smooth, then $T \perp_w A \iff T \perp_B A$, for all compact operators A on $\mathbb{X}$.*

Proof. Note that $M_T = M_{x^*}$ and $w(T) = \|T\|$. Thus, $T \perp_w A \Rightarrow T \perp_B A$. For the reverse implication, assume that $T \perp_B A$.
First suppose that $M_{x^*} = \{\pm \frac{1}{\|y\|} y\}$. Then from [86, Th. 2.2], it follows that $Ty \perp_B Ay$, i.e., $y \perp_B Ay$. Thus, using [42, Th. 2.1], we get $y^* \in J(y)$ such that $y^*(Ay) = 0$. Observe that $(\frac{1}{\|y\|} y, y^*) \in M_{W(T)}$. Thus, from Theorem 3.1, $T \perp_w A$.
Now, suppose that y is smooth, i.e., $J(y) = \{x^*\}$. Consider $D = \{x \in S_{\mathbb{X}} : x^*(x) = 1\}$. Then clearly $M_{x^*} = D \cup (-D)$, where D is connected. Therefore, from [86, Th. 2.2], we get $Tx \perp_B Ax$ for some $x \in D$. Hence, $y \perp_B Ax$, i.e., $x^*(Ax) = 0$. Observe that $(x, x^*) \in M_{W(T)}$. Thus, from Theorem 3.1, $T \perp_w A$. This completes the proof. □

In general, describing all possible set of sign determining extreme support functionals for a given operator seems to be a difficult problem. However, in case of two-dimensional polyhedral Banach spaces, we have the following complete characterization of the same.

Theorem 3.16. *Let $\mathbb{X}$ be a two-dimensional polyhedral Banach space and let $T \in \mathbb{L}(\mathbb{X})$. Suppose that $\mathcal{S}$ is a set of sign determining extreme support functionals for T at x, where $x \in \mathrm{Ext}\,(B_{\mathbb{X}})$. Then $\mathcal{S} = \mathrm{Ext}\,(J(x))$.*

Proof. It is clear that for any $x \in \mathrm{Ext}\,(B_{\mathbb{X}})$, $|\mathrm{Ext}\,(J(x))| = 2$. Let us consider $\mathrm{Ext}\,(J(x)) = \{x_1^*, x_2^*\}$. Suppose on the contrary that $\mathcal{S} = \{x_1^*\}$. Let $\ker x_1^* \cap S_{\mathbb{X}} = \{\pm x_0\}$ and let $\ker x_2^* \cap S_{\mathbb{X}} = \{\pm y_0\}$. Since x_1^* and x_2^* are linearly independent, it follows that $\min\{\|x_0 + y_0\|, \|x_0 - y_0\|\} = \delta_0 > 0$. Let us consider $\|x_0 - y_0\| = \delta_0$. It is easy to observe that either of the following holds true:
(a) $sgn(x_2^*(Tx)) \neq sgn(x_2^*(x_0))$,
(b) $sgn(x_2^*(Tx)) \neq sgn(x_2^*(-x_0))$.
Without loss of generality we assume that (a) holds true. Note that since x_2^* is continuous, there exists a $\delta > 0$ such that for each $z \in B_\delta(x_0)$ we have $sgn(x_2^*(z)) = sgn(x_2^*(x_0))$. Choosing $\delta' = \frac{\delta_0}{2}$ it can be easily observed that $y \in B_{\delta'}(x_0) \cap S_{\mathbb{X}}$ implies $|x_2^*(y)| \geq \epsilon_0$, for some positive scalar ϵ_0. Since x_1^* is continuous, it follows that there exists $\delta'' > 0$ such that for any $y \in B_{\delta''}(x_0)$, $|x_1^*(y)| < \epsilon_0$. Taking $\delta''' = \min\{\delta, \delta', \delta''\}$, we obtain that $y \in B_{\delta'''}(x_0) \cap S_{\mathbb{X}}$ implies $|x_2^*(y)| \geq$

$|x_1^*(y)|$. Moreover, there exists an $y_1 \in B_{\delta'''}(x_0) \cap S_{\mathbb{X}}$ such that $sgn(x_1^*(y_1)) = sgn(x_1^*(Tx)$. Summarizing the above arguments we obtain the following conclusions:
(i) $|x_2^*(y_1)| \geq |x_1^*(y_1)|$,
(ii) $sgn(x_1^*(y_1)) = sgn(x_1^*(Tx))$,
(iii) $sgn(x_2^*(y_1)) \neq sgn(x_2^*(Tx))$.
This implies that $\mathcal{S}$ is not a set of sign determining extreme support functional for T at x, which is a contradiction. Hence the theorem. □

We end this section with the following theorem which gives a necessary condition for an operator on a Banach space to be nr-left symmetric.

Theorem 3.17. *Let $\mathbb{X}$ be a strictly convex, finite-dimensional Banach space. Let $T \in \mathbb{L}(\mathbb{X})$ be nr-left symmetric. Suppose $x \in M_{w(T)}$ be such that $y \perp_B x$ for some smooth point $y \in S_{\mathbb{X}}$. Then $y \perp_B Ty$.*

Proof. Let $J(y) = \{y^*\}$. Then from $y \perp_B x$ and [42, Th. 2.1], we get $y^*(x) = 0$. Define $A \in \mathbb{L}(\mathbb{X})$ by $Az = y^*(z)y$ for all $z \in \mathbb{X}$. Then $x^*(Ax) = 0$. Therefore, $T \perp_w A$, which implies that $A \perp_w T$. From [62, Th. 3.1], it follows that there exist $t_i \in [0, 1]$ and $y_i^* \in S_{X^*}, y_i \in S_X$ for $i = 1, 2, 3$ such that $t_1 + t_2 + t_3 = 1, |y_i^*(y_i)| = 1, y_i^*(Ay_i) = w(A)$ and $\sum_{i=1}^3 t_i y_i^*(Ty_i) = 0$. Clearly, $y_i \in M_{w(A)}$, i.e., $y_i = \lambda_i y$ for $i = 1, 2, 3$, where $|\lambda_i| = 1$. Now, $y_i^*(Ay_i) = w(A)$ implies that $y_i^*(\lambda_i y) = 1$. Thus $\lambda_i y_i^* \in J(y) = \{y^*\}$ and so $y_i^* = \overline{\lambda_i} y^*$. Therefore $\sum_{i=1}^3 t_i y_i^*(Ty_i) = 0 \Rightarrow \sum_{i=1}^3 t_i \overline{\lambda_i} y^*(\lambda_i Ty) = 0 \Rightarrow \sum_{i=1}^3 t_i y^*(Ty) = 0 \Rightarrow y^*(Ty) = 0$. It now follows from [42, Th. 2.1] that $y \perp_B Ty$.

□

3.5 Nr-symmetry on some particular Banach spaces

In this section we completely characterize the nr-left symmetric and nr-right symmetric operators defined on ℓ_1^n and ℓ_∞^n defined over the real field. Recall that the numerical index of a Banach space $\mathbb{X}$ is defined as $n(X) = \inf\{w(T) : T \in S_{\mathbb{L}(\mathbb{X})}\}$. From [50], we know that $n(\ell_1^n) = n(\ell_\infty^n) = 1$. If $\mathbb{X} = \ell_1^n$ or ℓ_∞^n, then $w(T) = \|T\|$, for each $T \in \mathbb{L}(\mathbb{X})$ and so Birkhoff-James orthogonality in the sense of numerical radius norm coincides with usual Birkhoff-James orthogonality in $\mathbb{L}(\mathbb{X})$. Thus T is nr-left symmetric (resp. nr-right symmetric) iff T is left symmetric (resp. right symmetric). Although the left symmetric and right symmetric operators on ℓ_1^n and ℓ_∞^n have been studied in [33, 34], here we discuss the situation with nr-left symmetric and nr-right symmetric operators on ℓ_1^n with an alternative analytic approach. To do so we need the following lemmas.

Lemma 3.5. *Let $x_1^*, x_2^* \in \operatorname{Ext}(B_{(\ell_1^n)^*})$ be such that $x_1^* \neq \pm x_2^*$. Then there exists $u \in \ell_1^n$ satisfying the following three:*
(i) u is smooth point.
(ii) $x_1^*(u) = 0$.
(iii) $x_2^*(u) = \|u\| = \sup\{|y^*(u)| : y^* \in \operatorname{Ext}(B_{(\ell_1^n)^*})\}$.

Proof. Since $(\ell_1^n)^*$ is isometrically isomorphic to ℓ_∞^n, we assume that $x_1^* = (\alpha_{1k})_{1\leq k\leq n}$ and $x_2^* = (\alpha_{2k})_{1\leq k\leq n}$, where $x_1^*(e_k) = \alpha_{1k}$ and $x_2^*(e_k) = \alpha_{2k}$. It is easy to see that if we choose $u = (u_k)_{1\leq k\leq n} \in \ell_1^n$ such that $sgn(u_k) = sgn(\alpha_{2k})$, for each $k \in \{1, 2, \ldots, n\}$ then $x_2^*(u) = \|u\| = \sup\{|y^*(u)| : y^* \in \operatorname{Ext}(B_{(\ell_1^n)^*})\}$. Thus (iii) holds true. Now let us consider two sets A_1 and A_2, where

$$A_1 = \{k \in \{1, 2, \ldots, n\} : sgn(\alpha_{1k}) = sgn(\alpha_{2k})\},$$

$$A_2 = \{k \in \{1, 2, \ldots, n\} : sgn(\alpha_{1k}) = -sgn(\alpha_{2k})\}.$$

Note that since $x_1^* \neq \pm x_2^*$, $A_1, A_2 \neq \emptyset$. Also, $A_1 \cup A_2 = \{1, 2, \ldots, n\}$. Let $|A_1| = n_0$ and this implies $|A_2| = n - n_0$. Moreover, we observe that

$$x_1^*(u) = \sum_{k=1}^{n} sgn(\alpha_{1k}) sgn(u_k)|u_k| = \sum_{k=1}^{n} sgn(\alpha_{1k}) sgn(\alpha_{2k})|u_k| = \sum_{k\in A_1} |u_k| - \sum_{k\in A_2} |u_k|.$$

Now we choose $|u_k| = \frac{1}{n_0}$, when $k \in A_1$ and $|u_k| = \frac{1}{n-n_0}$, when $k \in A_2$. One can easily verify that $x_1^*(u) = 0$. Therefore, (ii) holds true. It is easy to see that u is a smooth point in ℓ_1^n and hence (i) is satisfied. This proves the lemma. □

Lemma 3.6. *Let $x_1 \in \operatorname{Ext}(B_{\ell_\infty^n})$, where $n \geq 3$. Then there exist a set $\{y_1, y_2, \ldots, y_n\} \subset \operatorname{Ext}(B_{\ell_\infty^n}) \setminus \{\pm x_1\}$, which is linearly independent.*

Proof. Suppose that $x_1 = (x_{11}, x_{12}, \ldots, x_{1n}) \in \operatorname{Ext}(B_{\ell_\infty^n})$, where $x_{1j} = \pm 1$, for all $j \in \{1, 2, \ldots, n\}$. Let us choose a set $\{y_1, y_2, \ldots, y_n\}$ such that

$$\begin{aligned} y_1 &= (-x_{11}, x_{12}, \ldots, x_{1n}), \\ y_2 &= (x_{11}, -x_{12}, \ldots, x_{1n}), \\ \cdot &= \cdots\cdots \end{aligned}$$

$$\begin{aligned} \cdot &= \cdots\cdots \\ y_n &= (x_{11}, x_{12}, \ldots, -x_{1n}). \end{aligned}$$

Clearly, for each $1 \leq i \leq n$, $y_i \in \operatorname{Ext}(B_{\ell_\infty^n}) \setminus \{\pm x_1\}$. Since $n \geq 3$, it is easy to see that the set $\{y_1, y_2, \ldots, y_n\}$ is linearly independent. This proves the lemma.

□

We observe that Lemma 3.6 essentially exhibits a stronger form of the following result, in the particular case of ℓ_∞^n.

Proposition 3.1. *[63, Th. 3.5] Let $\mathbb{X}$ be a finite-dimensional polyhedral Banach space and let $x \in \operatorname{Ext}(B_{\mathbb{X}})$. Then there exists a set $\{x_1^*, x_2^*, \ldots, x_n^*\} \in \operatorname{Ext}(\mathcal{J}(x))$, which is linearly independent.*

In view of Lemma 3.6 and Proposition 3.1, the following remark seems pertinent.

Remark 3.18. *Unlike Proposition 3.1, Lemma 3.6 is not necessarily true for any given Banach space. This is easy to see in case of ℓ_1^n. Indeed, $\operatorname{Ext}(B_{\ell_1^n}) = \{\pm e_1, \pm e_2, \ldots, \pm e_n\}$, from which it is evident that Lemma 3.6 is not valid in this case.*

In the next theorem we show that there exists no nonzero nr-left symmetric operators on $\ell_1^n(\mathbb{R})$, whenever $n \geq 3$. To prove that the following lemma is essential.

Lemma 3.7. *Let $T \in \mathbb{L}(\ell_1^n)$ be nonzero and nr-left symmetric. Then the following two hold true:*
(i) $\mathcal{M}_{W(T)} = \{\pm(e_i, e_{i_r}^*)\}$, for some fixed $e_i \in \operatorname{Ext}(B_{\ell_1^n})$ and $e_{i_r}^* \in \operatorname{Ext}(\mathcal{J}(e_i))$.
(ii) For any $(e_j, e_{j_k}^*) \in \operatorname{Ext}(B_{\ell_1^n}) \times \operatorname{Ext}(\mathcal{J}(e_j))$ satisfying $|e_{j_k}^* T e_j| < w(T)$, it follows that $e_{j_k}^* T e_j = 0$.

Proof. Suppose on the contrary that there exists $(e_j, e_{j_s}^*) \in \mathcal{M}_{W(T)}$ such that $(e_j, e_{j_s}^*) \neq \pm(e_i, e_{i_r}^*)$. If $e_{j_s}^* \neq \pm e_{i_r}^*$ then applying Lemma 3.5, we note that there exists a $u \in \ell_1^n$ such that it satisfies the following three conditions:
(a) u is a smooth point.
(b) $e_{i_r}^*(u) = 0$.
(c) $e_{j_s}^*(u) = \|u\|$.
Now we define a linear map $A : \ell_1^n \to \ell_1^n$ by $Ae_j = u$ and $Ae_k = 0$, for any $k \in \{1, 2, \ldots, n\} \setminus \{j\}$. Clearly, $w(A) = \|u\|$. Since $u \in \ell_1^n$ is a smooth point, it follows that $\mathcal{M}_{W(A)} = \pm\{(e_j, e_{j_s}^*)\}$. As $e_{j_s}^* T e_j \neq 0$, we get from Theorem 3.2 that $A \not\perp_w T$, whereas $e_{i_r}^*(Ae_i) = 0$ implies $T \perp_w A$. This contradicts the fact that T is nr-left symmetric. If $e_{j_s}^* = e_{i_r}^*$ but $e_j \neq \pm e_i$ then using similar

argument as given above we conclude the same. Thus the proof of (i) is completed.
To prove (ii), let us assume, contrary to our claim that there exists $(e_j, e^*_{j_k}) \in \text{Ext}\,(B_{\ell_1^n}) \times \text{Ext}\,(\mathcal{J}(e_j))$ satisfying $|e^*_{j_k}Te_j| < w(T)$ such that $e^*_{j_k}Te_j \neq 0$. Then following similar argument as given in the proof of (i), we can construct an $A \in \mathbb{L}(\ell_1^n)$ such that $T \perp_w A$ where $A \not\perp_w T$. This contradiction proves (ii). This completes the proof of the lemma. □

Theorem 3.19. *Let $T \in \mathbb{L}(\ell_1^n)$, where $n \geq 3$. Then T is nr-left symmetric if and only if T is the zero operator.*

Proof. The sufficient part is trivial. We only prove the necessary part of the theorem. From Lemma 3.7 we get $\mathcal{M}_{W(T)} = \{\pm(e_i, e^*_{i_r})\}$, for some fixed $e_i \in \text{Ext}\,(B_{\ell_1^n})$ and $e^*_{i_r} \in \text{Ext}\,(\mathcal{J}(e_i))$ and $e^*_{j_k}Te_j = 0$, where $j \in \{1, 2, \ldots, n\} \setminus \{i\}$ and $e^*_{j_k} \in \text{Ext}\,(\mathcal{J}(e_j))$. Therefore, applying Proposition 3.1 we get $Te_j = 0$, for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. Moreover, we see again from Claim (ii) of Lemma 3.7 that $e^*_{i_k}Te_i = 0$, for all $k \neq r$. Since $n \geq 3$, from Lemma 3.6, we obtain that there exist $e^*_{i_{k_1}}, e^*_{i_{k_2}}, \ldots, e^*_{i_{k_n}} \in \text{Ext}\,(\mathcal{J}(e_i)) \setminus \{\pm e^*_{i_r}\}$, which are linearly independent. This implies that $Te_i = 0$. Thus we obtain that T is the zero operator. This completes the proof of the theorem.

□

Note that the above theorem holds only for $n \geq 3$. In the next theorem we characterize the nr-left symmetric operators in $\mathbb{L}(\ell_1^2)$.

Theorem 3.20. *Let $T \in \mathbb{L}(\ell_1^2)$. T is nr-left symmetric if and only if the matrix representation of T with respect to the standard ordered basis is one of the following forms:*

$$(i) \begin{pmatrix} \lambda & 0 \\ \lambda & 0 \end{pmatrix}, (ii) \begin{pmatrix} \lambda & 0 \\ -\lambda & 0 \end{pmatrix}, (iii) \begin{pmatrix} 0 & \lambda \\ 0 & \lambda \end{pmatrix}, (iv) \begin{pmatrix} 0 & \lambda \\ 0 & -\lambda \end{pmatrix},$$

where $\lambda \in \mathbb{R}$.

Proof. Suppose that T is nr-left symmetric. If T is the zero operator then we are done. Let T be nonzero. Then from (i) of Lemma 3.7, we get $\mathcal{M}_{W(T)} = \{\pm(e_i, e^*_{i_k})\}$ or $\mathcal{M}_{W(T)} = \{\pm(e_i, -e^*_{i_k})\}$, where $i = 1, 2$, $k = 1, 2$ and $e^*_{1_1} = (1, 1), e^*_{1_2} = (1, -1), e^*_{2_1} = (1, 1), e^*_{2_2} = (-1, 1)$. Without loss of generality assume that $\mathcal{M}_{W(T)} = \{\pm(e_1, e^*_{1_1})\}$. Then from (ii) of Lemma 3.7, we get $e^*_{1_2}Te_1 = 0, e^*_{2_1}Te_2 = 0$, and $e^*_{2_2}Te_2 = 0$. Therefore, $Te_1 = \lambda(1, 1)$ and $Te_2 = 0$, for some $\lambda \in \mathbb{R}$. Thus the matrix representation of T with respect to the standard ordered basis is

$$T = \begin{pmatrix} \lambda & 0 \\ \lambda & 0 \end{pmatrix}.$$

Therefore, the form (i) is established. Similarly, considering other possibilities of $\mathcal{M}_{W(T)}$ we get the desired forms of operator T. To show the sufficient part, let T be given in the form (i). Then clearly, $Te_1 = (\lambda, \lambda)$ and $Te_2 = 0$, where $\lambda \in \mathbb{R}$. Suppose $A \in \mathbb{L}(\ell_1^n)$ is such that $T \perp_w A$. If $\lambda = 0$, then we are done. Let $\lambda \neq 0$. Then it can be easily seen that $\mathcal{M}_{W(T)} = \{\pm(e_1, e_{1_1}^*)\}$ or $\{\pm(e_1, -e_{1_1}^*)\}$. Now following Theorem 3.2, we observe that $e_{11}^* A e_1 = 0$. Clearly, $(e_1, \pm e_{1_1}^*) \notin \mathcal{M}_{W(A)}$. Let $(e_j, e_{j_k}^*) \in \text{Ext}\,(B_{\ell_1^2}) \times \text{Ext}\,(\mathcal{J}(e_j)) \setminus (e_1, \pm e_{1_1}^*)$ be such that $(e_j, e_{j_k}^*) \in \mathcal{M}_{W(A)}$. Now since $e_{j_k}^* T e_j = 0$, it follows from Theorem 3.2 that $A \perp_w T$. Similarly, considering the other three forms (ii), (iii) and (iv) of T, we can show that T is nr-left symmetric in each case.

□

Remark 3.21. *In [33, Th. 2.3] it is proved that if $T \in \mathbb{L}(\ell_1^n)$ then T is left symmetric if and only if T attains norm at only one extreme point, image of which is a left symmetric point of ℓ_1^n and images of other extreme points are zero. However, for $n \geq 3$, there exists no nonzero left symmetric point (see [19, Th. 2.8]) and so it follows that T is left symmetric if and only if T is the zero operator.*

We now study the nr-right symmetric operators on $\ell_1^n(\mathbb{R})$. We first present a necessary condition for nr-right symmetric operators on $\ell_1^n(\mathbb{R})$.

Theorem 3.22. *Let $T \in \mathbb{L}(\ell_1^n)$ be nr-right symmetric. Then for each $e_i \in \text{Ext}\,(B_{\ell_1^n})$, there exists a set $\mathcal{S}_i \subset \text{Ext}\,(J(e_i))$ of sign determining extreme support functionals for T at e_i such that either $(e_i, e_{i_k}^*) \in \mathcal{M}_{W(T)}$ or $(e_i, -e_{i_k}^*) \in \mathcal{M}_{W(T)}$, for each $e_{i_k}^* \in \mathcal{S}_i$.*

Proof. We prove the theorem in the following two steps:
Step I: In our first step we show that for every $e_i \in \text{Ext}\,(B_{\ell_1^n})$, there exists $e_{i_k}^* \in \text{Ext}\,(B_{\ell_\infty^n})$ such that $(e_i, e_{i_k}^*) \in \mathcal{M}_{W(T)}$. Suppose on the contrary that there exists $e_p \in \text{Ext}\,(B_{\ell_1^n})$ such that $(e_p, e_{p_k}^*) \notin \mathcal{M}_{W(T)}$, for all $e_{p_k}^* \in \text{Ext}\,(B_{\ell_\infty^n})$. Now we consider the following two cases:
Case I : Let $Te_p = 0$. We define a linear map $A \in \mathbb{L}(\ell_1^n)$ by

$$\begin{aligned} Ae_j &= w(T)e_j, \; j = p \\ Ae_j &= Te_j, \; \text{otherwise.} \end{aligned}$$

Since $w(A) = \max_{1 \leq i \leq n}\{|e_{i_k}^*(Ae_i)| : (e_i, e_{i_k}^*) \in \text{Ext}\,(B_{\ell_1^n}) \times \text{Ext}\,(J(e_i))\}$, it can be easily observed that $w(A) = w(T)$ and either $(e_p, e_{p_k}^*) \in \mathcal{M}_{W(A)}$ or $(e_p, -e_{p_k}^*) \in \mathcal{M}_{W(A)}$, where $e_{p_k}^* \in \text{Ext}\,(J(e_p))$. Since $Te_p = 0$, $e_{p_k}^* Te_p = 0$, which implies $0 \in \text{conv}\{(x^*Tx) : (x, x^*) \in \mathcal{M}_{W(A)}\}$. So from Theorem 3.2 we get $A \perp_w T$. Suppose that $(e_j, e_{j_s}^*) \in \mathcal{M}_{W(T)}$, for some $j \in \{1, 2, \ldots, n\} \setminus \{p\}$ and $e_{j_s}^* \in \text{Ext}\,(\mathcal{J}(e_j))$. Then we observe that $(e_j, e_{j_s}^*) \in \mathcal{M}_{W(A)}$. Therefore, $e_{j_s}^*(Ae_j) = e_{j_s}^*(Te_j) = w(T) > 0$. Thus $0 \notin \text{conv}\{x^*Ax : (x, x^*) \in \mathcal{M}_{W(T)}\}$. Hence from

Theorem 3.2, $T \not\perp_w A$. This contradicts that T is nr-right symmetric.

Case II : Let $Te_p \neq 0$. Then from Proposition 3.1, there exists $e^*_{p_k} \in \operatorname{Ext}(\mathcal{J}(e_p))$ such that $e^*_{p_k}(Te_p) \neq 0$. We define a linear map $A \in \mathbb{L}(\ell_1^n)$ by

$$\begin{aligned} Ae_j &= sgn(e^*_{j_k}(e_j))w(T)e_j, \; j = p \\ Ae_j &= -sgn(e^*_{p_k}(Te_p))Te_j, \; \text{otherwise.} \end{aligned}$$

It is straightforward to see that $w(A) = w(T)$ and $(e_p, e^*_{p_k}) \in \mathcal{M}_{W(A)}$. Let $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(T)}$, for some $j \in \{1, 2, \ldots, n\} \setminus \{p\}$ and $e^*_{j_k} \in \operatorname{Ext}(\mathcal{J}(e_j))$. It is easy to observe that $|e^*_{j_k} Ae_j| = w(T) = w(A)$. Then either $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(A)}$ or $(e_j, -e^*_{j_k}) \in \mathcal{M}_{W(A)}$. Without loss of generality assume that $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(A)}$. This implies that

$$e^*_{j_k} Te_j = -sgn(e^*_{p_k} Te_p)(e^*_{j_k} Ae_j) = -sgn(e^*_{p_k} Te_p)w(A). \tag{3.2}$$

Thus we obtain that $sgn(e^*_{j_k} Te_j) \neq sgn(e^*_{p_k} Te_p)$. In other words, $0 \in \operatorname{conv}\{x^*Tx : (x, x^*) \in \mathcal{M}_{W(A)}\}$. Therefore, applying Theorem 3.2, we get $A \perp_w T$. On the other hand, suppose that $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(T)}$. Then applying Equation (3.2), we obtain that $sgn(e^*_{j_k} Ae_j) = -sgn(e^*_{p_k} Te_p) \neq 0$. This implies that for any $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(T)}$, the sign of $(e^*_{j_k} Ae_j)$ is same as well as nonzero. Therefore, $0 \notin \operatorname{conv}\{x^*Ax : (x, x^*) \in \mathcal{M}_{W(T)}\}$. Applying Theorem 3.2, we get $T \not\perp_w A$, which is a contradiction to the fact tat T is nr-right symmetric.

Step II: Now in the final step we show that for each $e_i \in \operatorname{Ext}(B_{\ell_1^n})$, there exists a set $\mathcal{S}_i \subset \operatorname{Ext}(J(e_i))$ of sign determining extreme support functionals for T at e_i such that either $(e_i, e^*_{i_k}) \in \mathcal{M}_{W(T)}$ or $(e_i, -e^*_{i_k}) \in \mathcal{M}_{W(T)}$, for each $e^*_{i_k} \in \mathcal{S}_i$. Suppose on the contrary that for some $p \in \{1, 2, \ldots, n\}$ there does not exist a set $\mathcal{S}_p$ of sign determining extreme support functionals for T at e_p such that either $(e_p, e^*_{p_k}) \in \mathcal{M}_{W(T)}$ or $(e_p, -e^*_{p_k}) \in \mathcal{M}_{W(T)}$, for all $e^*_{p_k} \in \mathcal{S}_p$. Then there exists an element $v \in \ell_1^n$ such that

(1) for some $e^*_{p_l} \in \operatorname{Ext}(J(e_p)) \setminus \mathcal{S}_p$, $|e^*_{p_l}(v)| = \sup\{|g(v)| : g \in J(e_p)\}$,

(2) $sgn(e^*_{p_k}(v)) = sgn(e^*_{p_k}(Te_p))$, for all $e^*_{p_k} \in \mathcal{S}_p$,

(3) $sgn(e^*_{p_t}(v)) \neq sgn(e^*_{p_t}(Te_p))$.

Now let us define a linear map $A : \ell_1^n \to \ell_1^n$ by

$$\begin{aligned} Ae_j &= v, \; j = p \\ Ae_j &= \frac{|e^*_{p_t}(v)|}{w(T)} Te_j, \; \text{otherwise.} \end{aligned}$$

It is easy to check that $w(A) = |e^*_{p_t}(v)|$. Then either $(e_p, e^*_{p_t}) \in \mathcal{M}_{W(A)}$ or $(e_p, -e^*_{p_t}) \in \mathcal{M}_{W(A)}$ and therefore from (3), we note that either $e^*_{p_t} Te_p \leq 0$ or $(-e^*_{p_t})Te_p \leq 0$, respectively. Moreover, for any $j \in \{1,2,\ldots,n\} \setminus \{p\}$, $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(A)}$ if and only if $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(T)}$. Thus $e^*_{j_k} T e_j > 0$. Now it is easy to see that $0 \in \mathrm{conv}\{x^*Tx : (x, x^*) \in \mathcal{M}_{W(A)}\}$. Applying Theorem 3.2, we obtain that $A \perp_w T$. On the other hand, suppose that $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(T)}$. If $j = p$ then from (2), we observe that $e^*_{j_k} A e_j > 0$. Also, note that for any $j \in \{1,2,\ldots,n\} \setminus \{p\}$, $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(T)}$ if and only if $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(A)}$. This gives us $e^*_{j_k} T e_j > 0$. Therefore, $0 \notin \{x^*Ax : (x, x^*) \in \mathcal{M}_{W(T)}\}$. Thus again applying Theorem 3.2, we get $T \not\perp_w A$. This contradicts that T is nr-right symmetric. This completes the proof of the theorem. □

Combining Theorem 3.12 and Theorem 3.22 together, we obtain the following characterization of nr-right symmetric operators in $\mathbb{L}(\ell_1^n)$.

Theorem 3.23. *Let $T \in \mathbb{L}(\ell_1^n)$. Then T is nr-right symmetric if and only if given any $e_j \in \mathrm{Ext}\,(B_{\ell_1^n})$, there exists a set $\mathcal{S}_j$ of sign determinig extreme support functionals for T at e_j such that either $(e_j, e^*_{j_k}) \in \mathcal{M}_{W(T)}$ or $(e_j, -e^*_{j_k}) \in \mathcal{M}_{W(T)}$, for each $e^*_{j_k} \in \mathcal{S}_j$.*

In the next theorem we explicitly characterize the nr-right symmetric operators. For this we first need the following lemma.

Lemma 3.8. *Let $\mathbb{X}$ be a finite-dimensional polyhedral Banach space. Let $\{x_1^*, x_2^*, \ldots x_k^*\} \in \mathrm{Ext}\,(B_{\mathbb{X}^*})$ be such that it contains r linearly independent elements. Suppose that $x \in \cap_{i=1}^k M_{x_i^*}$. Then the following hold true:*
(i) $x \in F$, where F is a face of $B_{\mathbb{X}}$.
(ii) $\dim(F) \leq n - r$.

Proof. Since $x \in \cap_{i=1}^k M_{x_i^*}$, without loss of generality we assume that $x \in \{y \in S_{\mathbb{X}} : 1 \leq i \leq k \text{ and } x_i^*(y) = 1\}$. We denote the above set as F_x. Clearly, F_x is convex. Let $w, z \in S_{\mathbb{X}}$ be such that $tw + (1-t)z \in F_x$, where $0 \leq t \leq 1$. Then it is easy to note that $x_i^*(w) = x_i^*(z) = 1$, for all $i \in \{1,2,\ldots,k\}$. Therefore, $w, z \in F_x$. Thus the claim (i) is satisfied. Next consider $S = span\{u - v : u, v \in F_x\}$. Clearly $u - v \in \cap_{i=1}^k \ker x_i^*$ and so $S \subset \cap_{i=1}^k \ker x_i^*$. Since the set $\{x_1^*, x_2^*, \ldots x_k^*\}$ contains r linearly independent elements so dimension of $\cap_{i=1}^k \ker x_i^*$ is $n - r$ and hence $\dim(S) = \dim(F_x) \leq n - r$.

□

Theorem 3.24. *Let $T \in \mathbb{L}(\ell_1^n)$ and let $w(T) = 1$. Then T is nr-right symmetric if and only if $Te_i = \pm e_j$, where $i, j \in \{1,2,\ldots,n\}$.*

Proof. Since the sufficient part of the proof directly follows from Theorem 3.10, we only prove the necessary part. Let $T \in \mathbb{L}(\ell_1^n)$ be nr-right symmetric. From Theorem 3.23, for each $e_i \in \mathrm{Ext}\,(B_{\ell_1^n})$ there exists a set $\mathcal{S}_i = \{e^*_{i_k}\} \subset \mathrm{Ext}\,(J(e_i))$ of sign determining extreme support functionals for T at e_i such that $(e_i, e^*_{i_k}) \in M_{W(T)}$, for all $e^*_{i_k} \in \mathcal{S}_i$. For each $1 \leq i \leq n$, let us consider the set $N_i = \{e^*_{i_p} \in \text{Ext}\,(J(e_i)) : (e_i, e^*_{i_p}) \in M_{W(T)}\}$. Clearly, $\mathcal{S}_i \subset N_i$, for each $1 \leq i \leq n$. We claim that each N_i contains n linearly independent elements. Suppose on the contrary that there exists $j \in \{1, 2, \ldots, n\}$ such that N_j contains $r(< n)$ linearly independent elements. Note that $Te_j \notin \text{Ext}\,(B_{\ell_1^n})$. Otherwise, $N_j = \text{Ext}\,(J(e_j))$ and so from Proposition 3.1 N_j has n linearly independent elements. Using Proposition 3.1, we get $e^*_{j_s} \in \text{Ext}\,(J(e_j)) \backslash span\, N_j$ such that $|e^*_{j_s} Te_j| < 1$. Since $Te_j \in \cap_{e^*_{j_p} \in N_j} M_{e^*_{j_p}}$, it follows from Lemma 3.8 that there exists a face F such that $Te_j \in F$, where $\dim(F) \leq n - r$. Moreover, $\dim(F) > 0$, as $Te_j \notin \text{Ext}\,(B_{\ell_1^n})$. Without loss of generality, assume that $F = co\{e_1, e_2, \ldots, e_m\}$. Since F is a face and $F \subset M_{e^*_{j_k}}$, it follows that for each $1 \leq t \leq m$, $sgn(e^*_{j_k} Te_j) = sgn(e^*_{j_k} e_t)$, where $e^*_{j_k} \in S_j$. Note that $M_{e^*_{j_s}} = F_s \cup -F_s$, where $F_s = \{x \in S_{\ell_1^n} : e^*_{j_s}(x) = 1\}$. Observe that $F \not\subset F_s$ and $F \not\subset -F_s$. Otherwise, we get $Te_j \in F_s$ or $Te_j \in -F_s$. This implies that $|e^*_{j_s} Te_j| = 1$, which is not possible. Again for each $1 \leq t \leq m$, either $e_t \in F_s$ or $e_t \in -F_s$. Consider $e_1 \in F_s$ and $e_m \in -F_s$. Suppose that $e^*_{j_s}(Te_j) \geq 0$. Then choosing $v = e_m$ it is straightforward to obtain the following observations:

(i) $|e^*_{j_p}(e_m)| = 1 = \sup\{|e^*_j(e_m)| : e^*_j \in J(e_j)\}$.

(ii) $sgn(e^*_{j_k}(Te_j)) = sgn(e^*_{j_k}(e_m))$, for all $e^*_{j_k} \in \mathcal{S}_j$.

(iii) $sgn(e^*_{j_s}(Te_j)) \neq sgn(e^*_{j_s}(e_m))$.

Therefore, $\mathcal{S}_j$ is not a set of sign determining extreme support functionals for T at e_j, which is a contradiction. Similarly, if $e^*_{j_s}(Te_j) \leq 0$ then choosing $v = e_1$, we get a contradiction. This proves that for each $1 \leq i \leq n$, N_i contains n linearly independent elements. Since for each $1 \leq i \leq n$, $Te_i \in \cap_{e^*_{i_k} \in N_i} M_{e^*_{i_k}}$ and N_i contains n linearly independent elements, it follows from Lemma 3.8 that there exists a face F of $B_{\ell_1^n}$ such that $Te_i \in F$ and $\dim(F) = 0$. This implies that $Te_i \in \text{Ext}\,(B_{\ell_1^n})$, for each $1 \leq i \leq n$. This completes the proof of theorem.

□

We note from Theorem 3.19 that there are no nonzero nr-left symmetric operators in $\mathbb{L}(\ell_1^n)$, whenever $n \geq 3$. Together with this if we consider Theorem 3.20 and Theorem 3.24 then we obtain the following result.

Theorem 3.25. *Let $T \in \mathbb{L}(\ell_1^n)$. Then T is nr-symmetric if and only if T is the zero operator.*

Finally we discuss characterizations of the nr-left symmetric and nr-right symmetric operators on $\ell_\infty^n(\mathbb{R})$. Recall that $w(T) = \|T\|$ for all $T \in \mathbb{L}(\ell_\infty^n(\mathbb{R}))$. Now noting that $T \perp_B A \Leftrightarrow T \perp_w A$ and using results from [34] we get the following characterizations.

Theorem 3.26. *[34, Th.2.1] Let $T \in \mathbb{L}(\ell_\infty^n)$ and $T = (t_{ij})$ be the matrix representation of T with respect to the standard ordered basis. Then T is nr-right symmetric if and only if for each*

$1 \leq i \leq n$, exactly one term of $t_{i1}, t_{i2}, ..., t_{in}$ is nonzero and of the same magnitude.

Theorem 3.27. *[34, Th.2.3] Let $T \in \mathbb{L}(\ell_\infty^2)$. Then T is nr-left symmetric if and only if T attains its norm at only one pair of extreme points, say $\pm(1,1)$, $T(1,1)$ is a left symmetric point and images of the other two extreme points are zero.*

Theorem 3.28. *[34, Th.2.5] Let $T \in \mathbb{L}(\ell_\infty^n)$, where $n \geq 3$. Then T is nr-left symmetric if and only if T is the zero operator.*

Combining the above three theorems, we immediately get the following characterization of nr-symmetric operators on ℓ_∞^n.

Theorem 3.29. *Let $T \in \mathbb{L}(\ell_\infty^n)$. Then T is nr-symmetric if and only if T is the zero operator.*

CHAPTER 4

GEOMETRIC CONSTANTS IN A NORMED LINEAR SPACE THROUGH ISOSCELES ORTHOGONALITY

4.1 Introduction

There are various geometric constants associated with a normed linear space, which are useful towards a quantitative understanding of the geometry of the space and also play an important role in the study of some other related problems of functional analysis. The James constant is one of the most prominent geometric constants associated with the space, which measures the "non-squareness" of the unit ball of a normed linear space. Our main focus is to illustrate the central role played by isosceles orthogonality, a natural generalization of the usual orthogonality in an inner product space, in studying various geometric constants, including the James constant. Before proceeding further, let us fix the notations and the terminologies.

Content of this chapter is based on the following paper:

- D. Sain, **S. Ghosh**, K. Paul, *On isosceles orthogonality and some geometric constants in a normed space*, Aequationes Math., **97**(2023), no. 1, 147-160. https://doi.org/10.1007/s00010-022-00909-y.

Let $\mathbb{X}, \mathbb{Y}$ denote real normed linear spaces. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ denote the unit ball and the unit sphere of $\mathbb{X}$, respectively. An element $x \in \mathbb{X}$ is said to be isosceles orthogonal [42] to an element $y \in \mathbb{X}$, denoted as $x \perp_I y$, if $\|x+y\| = \|x-y\|$. Geometrically it means that the length of the two diagonal vectors $\|x+y\|$ and $\|x-y\|$ of the parallelogram formed by two vectors x and y are equal. We refer the readers to [2, 5, 45] for more information related to this topic. An element $x \in \mathbb{X}$ is said to be approximate isosceles orthogonal [20] to y if for $\epsilon \in [0,1)$, $|\|x+y\|^2 - \|x-y\|^2| \leq 4\epsilon\|x\|\|y\|$, and is written as $x \perp_I^\epsilon y$. Note that approximate isosceles orthogonality is symmetric, and therefore, so is exact isosceles orthogonality.

4.2 Preliminaries

We mention the definitions of the following geometric constants, to be studied throughout this paper.

Definition 4.1. *[30] Let $\mathbb{X}$ be a normed linear space.*

(i) The James constant, denoted by $J(\mathbb{X})$, is defined as

$$J(\mathbb{X}) = \sup\Big\{ \min\{\|x+y\|, \|x-y\|\} : x, y \in S_{\mathbb{X}} \Big\}.$$

(ii) For $x \in S_{\mathbb{X}}$, the local James constant, denoted by $\beta(x)$, is defined as

$$\beta(x) = \sup\Big\{ \min\{\|x+y\|, \|x-y\|\} : y \in S_{\mathbb{X}} \Big\}.$$

(iii) The Schäffer constant, denoted by $S(\mathbb{X})$, is defined as

$$S(\mathbb{X}) = \inf\Big\{ \max\{\|x+y\|, \|x-y\|\} : x, y \in S_{\mathbb{X}} \Big\}.$$

(iv) For each $x \in S_{\mathbb{X}}$, the local Schäffer constant at x, denoted by $\alpha(\mathbb{X})$, is defined as

$$\alpha(x) = \inf\Big\{ \max\{\|x+y\|, \|x-y\|\} : y \in S_{\mathbb{X}} \Big\}.$$

We note from [30] that for a given normed linear space $\mathbb{X}$, $\sqrt{2} \leq J(\mathbb{X}) \leq 2$. Moreover, $\mathbb{X}$ is said to be uniformly non-square if and only if $J(\mathbb{X}) < 2$. It is also known that $J(\mathbb{X}) = \sqrt{2}$ whenever $\mathbb{X}$ is an inner product space but the converse is not true, in general. In [56], the authors studied the normed linear spaces with James constant $\sqrt{2}$.

Generalizations of the notions of the James constant and the local James constant, were introduced in [59] in the following way. For $\lambda \in (0,1)$, the generalized James constant, denoted by $J(\lambda, \mathbb{X})$, is defined as

$$J(\lambda, \mathbb{X}) = \sup\Big\{ \min\left\{\|\lambda x + (1-\lambda)y\|, \|\lambda x - (1-\lambda)y\|\right\} : x, y \in S_{\mathbb{X}}\Big\}$$

and for $x \in S_{\mathbb{X}}$, the generalized local James constant, denoted by $\beta(\lambda, x)$, is defined as

$$\beta(\lambda, x) = \sup\Big\{ \min\left\{\|\lambda x + (1-\lambda)y\|, \|\lambda x - (1-\lambda)y\|\right\} : y \in S_{\mathbb{X}}\Big\}.$$

We also need two other well-known geometric constants, modulus of smoothness and modulus of convexity, which are denoted by $\rho_{\mathbb{X}}(\epsilon)$ and $\delta_{\mathbb{X}}(\epsilon)$, respectively, and are defined as

$$\rho_{\mathbb{X}}(\epsilon) = \sup\Big\{1 - \frac{\|x+y\|}{2} : x, y \in S_{\mathbb{X}}, \|x-y\| \leq \epsilon\Big\},$$

$$\delta_{\mathbb{X}}(\epsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : x, y \in S_{\mathbb{X}}, \|x-y\| \geq \epsilon\Big\},$$

where $\epsilon \in [0,2]$. We note from [57, Cor. 5] that $\delta_{\mathbb{X}}$ is a continuous function on $[0,2)$ whereas from [94], $\rho_{\mathbb{X}}$ is continuous on $[0,2]$. The modulus of smoothness is also defined as:

$$\rho'_{\mathbb{X}}(\epsilon) = \sup_{x,y \in S_{\mathbb{X}}}\Big\{\frac{\|x+\epsilon y\| + \|x - \epsilon y\|}{2} - 1\Big\}.$$

or (equivalently)

$$\rho'_{\mathbb{X}}(\epsilon) = \Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| \leq \epsilon\Big\}.$$

Observe that $\rho'_{\mathbb{X}}(\epsilon)$ is not equivalent to $\rho_{\mathbb{X}}(\epsilon)$ (see [10, Th. 1]).

Given any $x, y \in \mathbb{X}$, we denote by $[x, y\rangle$ the ray passing through y and starting from x, i.e., $[x, y\rangle = \{(1-t)x + ty : t \geq 0\}$ and $[x, y]$ denotes the closed convex line segment between x and y, i.e., $[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$. Another important concept to be used in this paper is that of orientation. Following [12], we say that x precedes y in a two-dimensional Banach space $\mathbb{X}$, if $x_1 y_2 - x_2 y_1 > 0$, where $x = (x_1, y_1), y = (y_1, y_2) \in \mathbb{X}$ and in this case we write that $x \prec y$. Of course, here $\mathbb{X}$ is identified with $\mathbb{R}^2$ in the obvious way. We note from [45, Cor.2.4] that for any $x \in S_{\mathbb{X}}$ there exists a unique (except for the sign) $y \in S_{\mathbb{X}}$ such that $x \perp_I y$. In particular, whenever it is given that $x \perp_I y$, without loss of generality we can assume that $-y \prec x \prec y$. We also consider the attainment set $M_{J(\mathbb{X})}$ of the James constant:

$$M_{J(\mathbb{X})} = \{(x, y) \in S_{\mathbb{X}} \times S_{\mathbb{X}} : \min\{\|x+y\|, \|x-y\|\} = J(\mathbb{X})\}.$$

When $\mathbb{X}$ is finite-dimensional, $M_{J(\mathbb{X})} \neq \emptyset$.

We end this section by mentioning the following known results, which are essential in our works of this chapter.

Lemma 4.1. *[60, Prop. 31]* ***(monotonicity lemma):*** *Let $\mathbb{X}$ be a two-dimensional Banach space. Let $x, y, z \neq 0$, $x \neq z$, with $[0, y\rangle$ lying in between $[0, x\rangle$ and $[0, z\rangle$, and suppose that $\|y\| = \|z\|$. Then $\|x - y\| \leq \|x - z\|$. In particular, if $\mathbb{X}$ is strictly convex, then we always have strict inequality.*

Lemma 4.2. *[30, Lemma 2.2] Let $\mathbb{X}$ be a two-dimensional Banach space and let $x \in S_{\mathbb{X}}$. Then there exists a unique $y \in S_{\mathbb{X}}$ such that $\alpha(x) = \beta(x) = \|x + y\| = \|x - y\|$.*

Theorem 4.1. *[30, Th. 3.3] Let $\mathbb{X}$ be a normed linear space. Then*

$$J(\mathbb{X}) = \sup\left\{\epsilon : \epsilon < 2 - 2\delta_{\mathbb{X}}(\epsilon)\right\}.$$

Proposition 4.1. *[30, Prop. 2.8] Let $\mathbb{X}$ be two-dimensional Banach space. If $S_{\mathbb{X}}$ is affinely homeomorphic to a convex symmetric body in the two-dimensional Euclidean space $\mathbb{R}^2$ which is invariant under a rotation of $\frac{\pi}{4}$, then $J(\mathbb{X}) = \sqrt{2}$.*

Theorem 4.2. *[45, Th. 2.3] Let $\mathbb{X}$ be a two-dimensional Banach space and let $x \in \mathbb{X}$ be non-zero. Then for each number $0 \leq r \leq \|x\|$, there exists a unique $y \in\in rS_{\mathbb{X}}$ such that $x \perp_I y$. Moreover, if $\mathbb{X}$ is strictly convex then for each $r \in [0, +\infty)$, there exists a unique $y \in rS_{\mathbb{X}}$ such that $x \perp_I y$.*

4.3 James constant and Isosceles orthogonality

In [30], Gao and Lau proved that in a two-dimensional Banach space $\mathbb{X}$ if $x, y \in S_{\mathbb{X}}$ are such that $x \perp_I y$, then $\beta(x) = \beta(y) = \|x - y\| = \|x + y\|$. We begin with a proposition by establishing a similar result in the case of the generalized local James constant $\beta(\lambda, x)$, from which the above result follows directly as a particular case ($\lambda = \frac{1}{2}$).

Proposition 4.2. *Let $\mathbb{X}$ be a two-dimensional Banach space and $x, y \in S_{\mathbb{X}}$. If $x \perp_I (\frac{1-\lambda}{\lambda})y$, where $\lambda \in (0, 1)$, then $\beta(\lambda, x) = \|\lambda x + (1 - \lambda)y\| = \|\lambda x - (1 - \lambda)y\|$.*

Proof. Let $x \perp_I (\frac{1-\lambda}{\lambda})y$. Then we get, $\|\lambda x + (1-\lambda)y\| = \|\lambda x - (1-\lambda)y\|$. Clearly, for any $z \neq \pm y$ we have $(1-\lambda)z \neq \pm(1-\lambda)y$. Consider the following four sets :

$$C_1 = \{(1-\lambda)\frac{(1-t)x + ty}{\|(1-t)x + ty\|} : 0 \leq t \leq 1\},$$
$$C_2 = \{(1-\lambda)\frac{(1-t)y - tx}{\|(1-t)y - tx\|} : 0 \leq t \leq 1\},$$
$$C_3 = \{(1-\lambda)\frac{-(1-t)x - ty}{\|-(1-t)x - ty\|} : 0 \leq t \leq 1\},$$
$$C_4 = \{(1-\lambda)\frac{-(1-t)y + tx}{\|-(1-t)y + tx\|} : 0 \leq t \leq 1\},$$

whose union is the circle of radius $|1-\lambda|$ and the sets C_i intersect only at $\pm(1-\lambda)x, \pm(1-\lambda)y$. Observe that for any $z \in S_{\mathbb{X}}$, we have $(1-\lambda)z \in C_i$, for some i, $1 \leq i \leq 4$. Let us assume that $(1-\lambda)z \in C_1$. Then applying Lemma 4.1 it is straightforward to observe that $\|\lambda x + (1-\lambda)z\| \geq \|\lambda x + (1-\lambda)y\|$ whereas $\|\lambda x - (1-\lambda)z\| \leq \|\lambda x - (1-\lambda)y\|$. Therefore, we obtain, $\min\{\|\lambda x - (1-\lambda)y\|, \|\lambda x + (1-\lambda)y\|\} = \|\lambda x - (1-\lambda)y\| \geq \|\lambda x - (1-\lambda)z\| \geq \min\{\|\lambda x - (1-\lambda)z\|, \|\lambda x + (1-\lambda)z\|\}$. If $(1-\lambda)z \in C_i$, for some $i \in \{2,3,4\}$ then we can proceed similarly to conclude that $\min\{\|\lambda x - (1-\lambda)y\|, \|\lambda x + (1-\lambda)y\|\} \geq \min\{\|\lambda x - (1-\lambda)z\|, \|\lambda x + (1-\lambda)z\|\}$. As $z \in S_{\mathbb{X}}$ is arbitrary, we get $\beta(\lambda, x) = \min\{\|\lambda x - (1-\lambda)y\|, \|\lambda x + (1-\lambda)y\|\} = \|\lambda x + (1-\lambda)y\| = \|\lambda x - (1-\lambda)y\|$. ☐

To determine the value of $J(\lambda, \mathbb{X})$ of a normed linear space $\mathbb{X}$, we observe the following:

Remark 4.3. *Following Proposition 4.2, it is easy to observe that for a given $\lambda \in (0,1)$,*

$$\begin{aligned} J(\lambda, \mathbb{X}) &= \sup\Big\{\|\lambda x + (1-\lambda)y\| : x, y \in S_{\mathbb{X}}, x \perp_I (\frac{1-\lambda}{\lambda})y\Big\} \\ &= \sup\Big\{\|\lambda x - (1-\lambda)y\| : x, y \in S_{\mathbb{X}}, x \perp_I (\frac{1-\lambda}{\lambda})y\Big\}. \end{aligned}$$

Therefore, to find the generalized James constant $J(\lambda, \mathbb{X})$, for a given $\lambda \in (0,1)$, we only need to consider the subset $\{(x,y) \in S_{\mathbb{X}} \times S_{\mathbb{X}} : x \perp_I (\frac{1-\lambda}{\lambda})y\} \subseteq S_{\mathbb{X}} \times S_{\mathbb{X}}$.

In the following theorem, we study the converse of Proposition 4.2.

Theorem 4.4. *Let $\mathbb{X}$ be a strictly convex normed linear space and $x \in S_{\mathbb{X}}, \lambda \in (0,1)$. If $\beta(\lambda, x) = \min\{\|\lambda x + (1-\lambda)y\|, \|\lambda x - (1-\lambda)y\|\}$, for some $y \in S_{\mathbb{X}}$, then $x \perp_I (\frac{1-\lambda}{\lambda})y$.*

Proof. Clearly $x \neq \pm y$. Since x, y are linearly independent consider the two-dimensional subspace $\mathbb{Y} = span\ \{x, y\}$. If possible let us assume that $x \not\perp_I (\frac{1-\lambda}{\lambda})y$. Then either $\|x + (\frac{1-\lambda}{\lambda})y\| > \|x - (\frac{1-\lambda}{\lambda})y\|$ or $\|x - (\frac{1-\lambda}{\lambda})y\| > \|x + (\frac{1-\lambda}{\lambda})y\|$. Without loss of generality we assume that $\|x + (\frac{1-\lambda}{\lambda})y\| > \|x - (\frac{1-\lambda}{\lambda})y\|$ so that $\beta(\lambda, x) = \|\lambda x - (1-\lambda)y\|$. Applying Theorem 4.2, there

exists a unique $z \in S_{\mathbb{Y}}$ (except for the sign) such that $x \perp_I \frac{1-\lambda}{\lambda} z$. Observe that either
(i) the ray $[0,(1-\lambda)y\rangle$ lies in between the rays $[0,\lambda x\rangle$ and $[0,(1-\lambda)z\rangle$ or
(ii) the ray $[0,(1-\lambda)y\rangle$ lies in between the rays $[0,\lambda x\rangle$ and $[0,-(1-\lambda)z\rangle$.
Assume that (i) holds. Since $\lambda x,(1-\lambda y),(1-\lambda z) \neq \theta$ and $\|(1-\lambda)y\| = \|(1-\lambda)z\|$ applying Lemma 4.1, together with the assumption that $\mathbb{X}$ is strictly convex, we conclude that $\|\lambda x-(1-\lambda)y\| < \|\lambda x-(1-\lambda)z\| = \|\lambda x+(1-\lambda)z\|$. This implies that $\beta(\lambda,x) < \min\{\|\lambda x-(1-\lambda)z\|, \lambda x+(1-\lambda)z\|$, a contradiction to the definition of $\beta(\lambda,x)$. If (ii) holds then also we can proceed similarly. Thus we must have $x \perp_I (\frac{1-\lambda}{\lambda})y$. □

It is easy to see that $\beta(\frac{1}{2},x) = \frac{1}{2}\beta(x)$, for any $x \in S_{\mathbb{X}}$. Therefore, taking $\lambda = \frac{1}{2}$, we state the following result as a particular case of Theorem 4.4 that studies the converse of [30, Lemma 2.2(i)].

Theorem 4.5. *Let $\mathbb{X}$ be a strictly convex normed linear space and let $x_0 \in S_{\mathbb{X}}$. If $y_0 \in S_{\mathbb{X}}$ is such that $\beta(x_0) = \min\{\|x_0-y_0\|, \|x_0+y_0\|\}$, then $x_0 \perp_I y_0$.*

The following example illustrates that the condition of strict convexity in the above theorem cannot be relaxed in general.

Example 4.6. *Let $\mathbb{X} = \ell_\infty^2$ and let $x_0 = (1,0) \in S_{\mathbb{X}}$. To compute $\beta((1,0))$, we observe that any $y \in S_{\ell_\infty^2}$ can be written as either $y = (\alpha,\pm1)$ or $y = (\pm1,\alpha)$, where $-1 \leq \alpha \leq 1$. It is straightforward to observe that whenever $y = (\alpha,\pm1)$, $\min\{\|x_0-y\|_\infty, \|x_0+y\|_\infty\} = 1$. On the other hand, $\min\{\|x_0-y\|_\infty, \|x_0+y\|_\infty\} = |\alpha| \leq 1$, when $y = (\pm1,\alpha)$. Therefore, $\beta((1,0)) = 1$. Clearly, for any $y = (\alpha,\pm1)$ with $0 < \alpha < 1$, we have that*

$$\min\{\|x_0-y\|_\infty, \|x_0+y\|_\infty\} = \min\{1,|1+\alpha|\} = 1 = \beta((1,0)).$$

In particular, we observe that isosceles orthogonality is not a necessary condition for the attainment of $\beta(x)$, where $x \in S_{\mathbb{X}}$.

Remark 4.7. *For another local constant $\alpha(x)$, introduced in [30], using similar arguments as in Theorem 4.5, we conclude that if $x_0,y_0 \in S_{\mathbb{X}}$ with $\max\{\|x_0-y_0\|, \|x_0+y_0\|\} = \alpha(x_0)$ then $x_0 \perp_I y_0$, provided $\mathbb{X}$ is strictly convex.*

Regarding the attainment of the local James constant $\beta(x)$ in an arbitrary Banach space, we have already noticed that if there exist $x_0,y_0 \in S_{\mathbb{X}}$ such that $\min\{\|x_0-y_0\|, \|x_0+y_0\|\} = \beta(x_0)$ then $x_0 \not\perp_I y_0$, in general. However, as illustrated in the following theorem, we are going to observe a stronger behavior with respect to isosceles orthogonality, in the case of attainment of the corresponding global constant $J(\mathbb{X})$.

Theorem 4.8. *Let $\mathbb{X}$ be a normed linear space. Let $u, v \in S_{\mathbb{X}}$ be such that* $\min\{\|u-v\|, \|u+v\|\} = J(\mathbb{X})$. *Then $u \perp_I v$.*

Proof. We prove the theorem by considering the following two possible cases.

Case (i): Let us assume that $J(\mathbb{X}) = 2$. Then $\min\{\|u-v\|, \|u+v\|\} = 2$. It is trivial to see that $\max\{\|x+y\|, \|x-y\| : x, y \in S_{\mathbb{X}}\} \leq 2$. Therefore, it necessarily follows that $\|u+v\| = \|u-v\|$, i.e., $u \perp_I v$.

Case (ii): Suppose that $J(\mathbb{X}) < 2$. Consider the set $S = \{\epsilon \in [0,2) : \epsilon < 2 - 2\delta_{\mathbb{X}}(\epsilon)\}$, where $\delta_{\mathbb{X}}(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in S_{\mathbb{X}} \text{ and } \|x-y\| \geq \epsilon\}$. From Theorem 4.1, we observe that $\sup S = J(\mathbb{X}) < 2$. Suppose on the contrary that $u \not\perp_I v$. Without loss of generality, let us assume that $\|u+v\| > \|u-v\|$. Also, let $\|u-v\| = \epsilon_0 = J(\mathbb{X}) < 2$. Then, $1 - \frac{\|u+v\|}{2} < 1 - \frac{\epsilon_0}{2}$ implies that $\delta_{\mathbb{X}}(\epsilon_0) < 1 - \frac{\epsilon_0}{2}$, i.e., $\epsilon_0 < 2 - 2\delta_{\mathbb{X}}(\epsilon_0)$. Therefore, $\epsilon_0 \in S$. Now, from [57, Cor. 5], we note that $\delta_{\mathbb{X}}(\epsilon)$ is a continuous function on $[0,2)$. Therefore, it is easy to verify that S is an open set in $\mathbb{R}$, with its usual topology. Since $\epsilon_0 \in S$ and S is open, it follows that there exists $\mu_0 > 0$ such that $\epsilon_0 + \mu_0 \in S$, which contradicts our assumption that $\sup S = J(\mathbb{X}) = \epsilon_0$. Hence $\|u-v\| = \|u+v\|$, i.e., $u \perp_I v$, as claimed. □

Remark 4.9. *For $x \in S_{\mathbb{X}}$ and $\epsilon \in [0,1)$, let us consider the set $A(x,\epsilon) = \{y \in S_{\mathbb{X}} : x \perp_I^{\epsilon} y\}$. Now it is easy to see that for any $\epsilon > 0$, if $z \in \{\mathbb{X} \setminus A(x,\epsilon)\} \cap S_{\mathbb{X}}$ then* $\min\{\|x+z\|, \|x-z\|\} < J(\mathbb{X})$. *For, otherwise, from Theorem 4.8 we obtain $x \perp_I z$, which contradicts $z \in \{\mathbb{X} \setminus A(x,\epsilon)\} \cap S_{\mathbb{X}}$.*

In view of the Example 4.6, it is natural to speculate whether strict convexity is essential for Theorem 4.5. We negate this by means of the following explicit example, constructed with the help of Theorem 4.8.

Let us recall from [56] that for each $\theta \in \mathbb{R}$, the θ-rotation matrix $R(\theta)$ is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

A norm $\|.\|$ on $\mathbb{R}^2$ is said to be θ-invariant if $R(\theta)$ is an isometry on $(\mathbb{R}^2, \|.\|)$.

Example 4.10. *Let $\mathbb{X}$ be the two-dimensional Banach space, identified as $\mathbb{R}^2$, endowed with the norm* $\|(x,y)\| = \max\{|x|, |y|, 2^{-1/2}(|x|+|y|)\}$ *for any $(x,y) \in \mathbb{R}^2$. It is easy to verify that $S_{\mathbb{X}}$ is a regular octagon, with vertices* $\pm v_1 = \pm(1, \sqrt{2}-1), \pm v_2 = \pm(\sqrt{2}-1, 1), \pm v_3 = \pm(1-\sqrt{2}, 1), \pm v_4 = \pm(-1, \sqrt{2}-1)$. *The unit sphere is shown in the following figure:*

It is easy to see that the given norm on $\mathbb{R}^2$ is $\frac{\pi}{4}$-invariant. Let E_1 be the edge joining the vertices $-v_4, v_1$. Therefore, the following two conditions are equivalent.

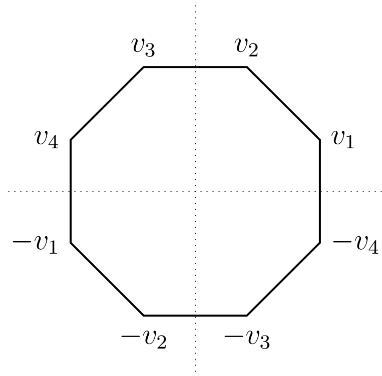

Figure 4.1: Unit ball of $\mathbb{X}$

(i) If for any $\widetilde{x} \in E_1$ there exists an $\widetilde{y} \in S_{\mathbb{X}}$ such that $\min\{\|\widetilde{x} - \widetilde{y}\|, \|\widetilde{x} + \widetilde{y}\|\} = \beta(\widetilde{x})$ *then* $\widetilde{x} \perp_I \widetilde{y}$.

(ii) If for any $\widetilde{x} \in S_{\mathbb{X}}$ there exists an $\widetilde{y} \in S_{\mathbb{X}}$ such that $\min\{\|\widetilde{x} - \widetilde{y}\|, \|\widetilde{x} + \widetilde{y}\|\} = \beta(\widetilde{x})$ *then* $\widetilde{x} \perp_I \widetilde{y}$.

We will show that (i) holds true. Any $u \in E_1$ can be written as $u = (1, \gamma)$, where $|\gamma| \leq \sqrt{2}-1$. Also, given any $u = (1, \gamma) \in E_1$, we have $v = \pm(-\gamma, 1) \in S_{\mathbb{X}}$ such that $u \perp_I v$. From Lemma 4.2, we obtain that $\beta(v) = \|u - v\| = \sqrt{2}$. On the other hand, from Proposition 4.1 it follows that $J(\mathbb{X}) = \sqrt{2}$. This implies that $(u, v) \in M_{J(\mathbb{X})}$.
Since $\beta(\widetilde{x}) = \sqrt{2}$ for any $\widetilde{x} \in E_1$, it is easy to see that $\min\{\|\widetilde{x} + \widetilde{y}\|, \|\widetilde{x} - \widetilde{y}\|\} = \beta(\widetilde{x})$ *implies that $(\widetilde{x}, \widetilde{y}) \in M_{J(\mathbb{X})}$. Now applying Theorem 4.8, we conclude that $\widetilde{x} \perp_I \widetilde{y}$.*
In particular, Theorem 4.5 may indeed hold true for certain Banach spaces which are not strictly convex.

As a complementary notion of the James constant, we may also consider the Schäffer constant, in view of Theorem 4.8. It can be shown similarly by using the method from [30, Th. 3.3] that:

$$S(\mathbb{X}) = \inf\{\epsilon : \epsilon > 2 - 2\rho_{\mathbb{X}}(\epsilon)\}.$$

Recall that $\rho_{\mathbb{X}}(\epsilon)$ is continuous and convex (see [94]) on $[0, 2]$. Therefore, applying similar methods as used in the Theorem 4.8 we obtain the following result.

Theorem 4.11. *Let $\mathbb{X}$ be a normed linear space. Let $u, v \in S_{\mathbb{X}}$ be such that* $\min\{\|u - v\|, \|u + v\|\} = S(\mathbb{X})$. *Then $u \perp_I v$.*

Next we show that in a two-dimensional polyhedral Banach space, the James constant is always attained at one of the extreme points of the unit ball. To achieve this we need the following lemma.

Lemma 4.3. *Let $\mathbb{X}$ be a two-dimensional Banach space. Let $v_1, v_2 \in S_{\mathbb{X}}$ be such that $v_1 \neq \pm v_2$ and $v_1 \prec v_2$. Suppose that $w_1, w_2 \in S_{\mathbb{X}}$ are such that $v_i \perp_I w_i$ and $-w_i \prec v_i \prec w_i$, for $i \in \{1, 2\}$. Then $w_1 \prec w_2$.*

Proof. It follows from Theorem 4.2 that $w_1 \neq \pm w_2$. Suppose on the contrary that $w_1 \nprec w_2$. Then $w_2 \prec w_1$. Therefore, the only possibility is that $v_1 \prec v_2 \prec w_2 \prec w_1 \prec -v_1$. This implies that the ray $[0, w_2\rangle$ lies in between the rays $[0, v_1\rangle$ and $[0, w_1\rangle$ and the ray $[0, v_2\rangle$ lies in between the rays $[0, v_1\rangle$ and $[0, w_2\rangle$. Now applying Lemma 4.1 we get,

$$\|v_1 - w_2\| \leq \|v_1 - w_1\| = \|v_1 + w_1\| \leq \|v_1 + w_2\|,$$

and

$$\|v_1 - w_2\| = \|w_2 - v_1\| \geq \|w_2 - v_2\| = \|w_2 + v_2\| \geq \|w_2 + v_1\| = \|v_1 + w_1\|.$$

Thus $\|v_1 + w_2\| = \|v_1 - w_2\|$, which shows that v_1 is isosceles orthogonal to w_2. This is a contradiction as $w_1 \neq \pm w_2$. Therefore, $w_1 \prec w_2$, as desired. □

The following important remark, which is immediate from the above lemma, is also relevant for the proof of our next theorem.

Remark 4.12. *Let $\mathbb{X}$ be a two-dimensional Banach space. Let $v_1, v_2 \in S_{\mathbb{X}}$ be such that $v_1 \neq \pm v_2$ and let $w_1, w_2 \in S_{\mathbb{X}}$ be such that $v_i \perp_I w_i$ and let $-w_i \prec v_i \prec w_i$, for $i \in \{1, 2\}$. Without loss of generality we can assume that $v_1 \prec v_2$. Suppose $v \in S_{\mathbb{X}}$ is such that the ray $[0, v\rangle$ lies in between the rays $[0, v_1\rangle$ and $[0, v_2\rangle$, which implies that $v_1 \prec v \prec v_2$. From Lemma 4.3, it can be concluded that $w_1 \prec w \prec w_2$. In other words, the ray $[0, w\rangle$ lies in between the rays $[0, w_1\rangle$ and $[0, w_2\rangle$, where $v \perp_I w$.*

We are now in a position to prove the following result.

Theorem 4.13. *Let $\mathbb{X}$ be a two-dimensional polyhedral Banach space. Then there exists $z \in \operatorname{Ext}(B_{\mathbb{X}})$ such that $\beta(z) = J(\mathbb{X})$, i.e., $\|z + y\| = \|z - y\| = J(\mathbb{X})$, where $y \in S_{\mathbb{X}}$ and $z \perp_I y$.*

Proof. Let v_1, v_2 be two extreme points of $B_{\mathbb{X}}$ such that $v_1 \prec v_2$ and $tv_1 + (1-t)v_2 \in S_{\mathbb{X}}$, for all $t \in [0, 1]$. Then there exist $w_1, w_2 \in S_{\mathbb{X}}$ such that $v_1 \perp_I w_1, v_2 \perp_I w_2$ and $w_1 \prec w_2$, by using Lemma 4.3. We consider the following two cases:

Case 1 : At first we consider that $\lambda w_1 + (1-\lambda) w_2 \subset S_{\mathbb{X}}$, for all $\lambda \in [0, 1]$. For any $v \in [v_1, v_2]$, we can write $v = t_0 v_1 + (1-t_0) v_2$, for some $t_0 \in [0, 1]$. Take $w \in S_{\mathbb{X}}$ such that $v \perp_I w$. By virtue of Remark 4.12, it follows that the ray $[0, w\rangle$ lies in between the rays $[0, w_1\rangle$ and $[0, w_2\rangle$. Now, if $w = t_0w_1 + (1 - t_0)w_2$, then using Lemma 4.2, we get

$$\begin{aligned}
\beta(v) &= \|v - w\| \\
&= \|t_0v_1 + (1 - t_0)v_2 - t_0w_1 - (1 - t_0)w_2\| \\
&\leq t_0\|v_1 - w_1\| + (1 - t_0)\|v_2 - w_2\| \\
&= t_0\beta(v_1) + (1 - t_0)\beta(v_2).
\end{aligned}$$

If $w \neq t_0w_1 + (1 - t_0)w_2$ then either $\|v - w\| \leq \|v - (t_0w_1 + (1 - t_0)w_2)\|$ or $\|v - w\| > \|v - (t_0w_1 + (1 - t_0)w_2)\|$. Applying Lemma 4.1, it is straightforward to see that in the latter case we have $\|v + w\| \leq \|v + t_0w_1 + (1 - t_0)w_2\|$. Therefore, we get,

$$\begin{aligned}
\beta(v) &= \|v \pm w\| \\
&= \|t_0v_1 + (1 - t_0)v_2 \pm w\| \\
&\leq \|t_0v_1 + (1 - t_0)v_2 \pm (t_0w_1 - (1 - t_0)w_2)\| \\
&\leq t_0\|v_1 \pm w_1\| + (1 - t_0)\|v_2 \pm w_2\| \\
&= t_0\beta(v_1) + (1 - t_0)\beta(v_2).
\end{aligned}$$

Therefore, $\beta(v) \leq \max\{\beta(v_1), \beta(v_2)\}$.

Case 2 : Let $\{\lambda w_1 + (1 - \lambda)w_2 : \lambda \in [0, 1]\} \not\subset S_{\mathbb{X}}$. Assume that there exist k extreme points $x_1, x_2, \ldots, x_k$ lying in between the rays $[0, w_1\rangle$ and $[0, w_2\rangle$ such that $w_1 \prec x_1 \prec x_2 \prec \ldots \prec x_k \prec w_2$. Then following Remark 4.12, we get $z_1, z_2, \ldots, z_k \in [v_1, v_2]$ such that $v_1 \prec z_1 \prec z_2 \prec \ldots \prec z_k \prec v_2$ and $z_i \perp_I x_i$, for $1 \leq i \leq k$. Considering the segments $[v_1, z_1]$, $[z_1, z_2], \ldots, [z_k, v_2]$ in place of $[v_1, v_2]$ and applying similar arguments as in Case 1, we get, for any $v \in [v_1, v_2]$,

$$\begin{aligned}
\beta(v) &\leq \max\{\beta(v_1), \beta(z_1), \ldots, \beta(z_k), \beta(v_2)\} \\
&= \max\{\beta(v_1), \beta(x_1), \ldots, \beta(x_k), \beta(v_2)\}.
\end{aligned}$$

Therefore, we observe that for any $v \in [v_1, v_2]$, there exists $z \in E_{\mathbb{X}}$ such that $\beta(v) \leq \beta(z)$. As v_1, v_2 are chosen arbitrarily, we can conclude that for any $v \in S_{\mathbb{X}}$ there exists $z \in \text{Ext}\,(B_{\mathbb{X}})$ such that $\beta(v) \leq \beta(z)$. This completes the proof of the theorem.

□

The following remark is immediate from Theorem 4.13.

Remark 4.14. *Let $\mathbb{X}$ be a two-dimensional polyhedral Banach space. Suppose that $\pm v_1, \pm v_2, \ldots, \pm v_m$ are the extreme points of $B_{\mathbb{X}}$. From Theorem 4.13, it can be easily seen that to find the James constant $J(\mathbb{X})$, we only need to deal with the extreme points of the unit ball of $\mathbb{X}$. Indeed, we can compute the James constant $J(\mathbb{X})$ in a more efficient way by the formula:*

$$J(\mathbb{X}) = \max_{1\leq i\leq m} \beta(v_i) = \max\{\|v_i + w_i\| : 1 \leq i \leq m, w_i \in S_{\mathbb{X}} \ \text{ and } \ v_i \perp_I w_i\}.$$

In the following example, we will show the applicability of Theorem 4.13 towards explicitly computing the James constant, as described in Remark 4.14.

Example 4.15. *Consider a two-dimensional polyhedral Banach space $\mathbb{X}$ whose unit sphere is an irregular hexagon, as shown in the following figure:*

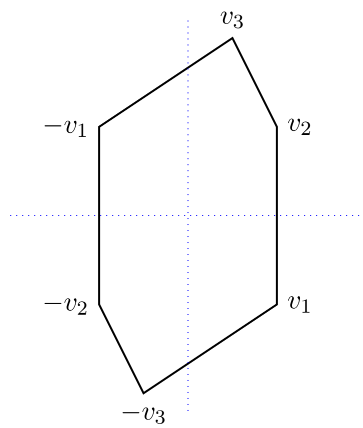

Figure 4.2: unit ball of $\mathbb{X}$

The vertices of $B_{\mathbb{X}}$ are $\pm v_1 = \pm(1,-1), \pm v_2 = \pm(1,1), \pm v_3 = \pm(\frac{1}{2}, 2)$. Clearly, $\beta(x) = \beta(-x)$, for any $x \in \mathbb{X}$, so that we only need to calculate $\beta(1,-1), \beta(1,1), \beta(\frac{1}{2},2)$. By a straightforward computation, we have $(1,-1) \perp_I \pm(\frac{9}{13}, \frac{21}{13})$, $(1,1) \perp_I \pm(-\frac{5}{17}, \frac{25}{17})$ and $(\frac{1}{2}, 2) \perp_I \pm(1, -\frac{2}{7})$. Using Lemma 4.2, we obtain that

$$\begin{aligned} \beta(1,-1) &= \|(1,-1) + (\frac{9}{13}, \frac{21}{13})\| = \|(\frac{22}{13}, \frac{8}{13})\| = \frac{22}{13}, \\ \beta(1,1) &= \|(1,1) + (-\frac{5}{17}, \frac{25}{17})\| = \|(\frac{12}{17}, \frac{42}{17})\| = \frac{22}{17}, \\ \beta(\frac{1}{2},2) &= \|(\frac{1}{2},2) + (1,-\frac{2}{7}) = \|(\frac{3}{2}, \frac{12}{7})\| = \frac{11}{7}. \end{aligned}$$

Thus $J(\mathbb{X}) = \max\{\frac{22}{13}, \frac{22}{17}, \frac{11}{7}\} = \frac{22}{13}$.

The above example illustrates that the problem of finding the James constant in a two-dimensional polyhedral Banach space $\mathbb{X}$ is equivalent to calculating the local constant $\beta(x)$ only for the finitely many extreme points of $B_{\mathbb{X}}$.

4.4 Approximate isosceles orthogonality and modulus of convexity

In this section, we study approximate isosceles orthogonality and its role in the attainment of the modulus of convexity, an important geometric constant associated with a given normed linear space. We begin with the following basic observation.

Proposition 4.3. *Let $\mathbb{X}$ be a normed linear space and let $x, y \in S_{\mathbb{X}}$ with $x \neq \pm y$. Then there exists an $\epsilon \in [0, 1)$ such that $x \perp_I^{\epsilon} y$.*

Proof. If $x \perp_I y$ then we are done by taking $\epsilon = 0$. Suppose that $x \not\perp_I y$. Since $x \neq \pm y$, it follows that $|\|x+y\|^2 - \|x-y\|^2| = 4 - \epsilon_0$, for some $0 < \epsilon_0 < 4$. Therefore, choosing $\epsilon \in [\frac{4-\epsilon_0}{4}, 1)$ we conclude that $|\|x+y\|^2 - \|x-y\|^2| \leq 4\epsilon$, i.e., $x \perp_I^{\epsilon} y$. ☐

For a given $\epsilon \in [0, 2]$, let us define the set:

$$M_{\delta_{\mathbb{X}}(\epsilon)} = \Big\{(x, y) \in S_{\mathbb{X}} \times S_{\mathbb{X}} : 1 - \frac{\|x+y\|}{2} = \delta_{\mathbb{X}}(\epsilon)\Big\}.$$

$M_{\delta_{\mathbb{X}}(\epsilon)}$ is called the attainment set of $\delta_{\mathbb{X}}(\epsilon)$, for any $\epsilon \in [0, 2]$. It is clear that whenever $\mathbb{X}$ is finite-dimensional, $M_{\delta_{\mathbb{X}}(\epsilon)} \neq \emptyset$. Our next result shows that the attainment of $\delta_{\mathbb{X}}(\epsilon)$ is closely related to approximate isosceles orthogonality.

Theorem 4.16. *Let $\mathbb{X}$ be a normed linear space. Let $M_{\delta_{\mathbb{X}}(\epsilon)} \neq \emptyset$, for some $\epsilon \in (0, 2)$. Then there exists $(u_0, v_0) \in M_{\delta_{\mathbb{X}}(\epsilon)}$ such that $u_0 \perp_I^{\epsilon_0} v_0$, where $\epsilon_0 = |1 + \delta_{\mathbb{X}}(\epsilon)^2 - 2\delta_{\mathbb{X}}(\epsilon) - \frac{\epsilon^2}{4}| \in [0, 1)$.*

Proof. Suppose that $(u, v) \in M_{\delta_{\mathbb{X}}(\epsilon)}$. Since $\epsilon \in (0, 2)$, it is clear that $u \neq \pm v$. Consider the set $P_u = \{w \in S_{\mathbb{X}} : \|u - w\| = \epsilon\}$. We claim that there exists $w' \in P_u$ such that $(u, w') \in M_{\delta_{\mathbb{X}}(\epsilon)}$. If $v \in P_u$ then our claim holds true. Let us now assume that $v \notin P_u$. Suppose on the contrary that the claim is not true. Then clearly, $\delta_{\mathbb{X}}(\epsilon) < 1 - \frac{\|u+w\|}{2}$ for all $w \in P_u$, i.e., $\|u + v\| > \|u + w\|$. Considering the two-dimensional subspace $\mathbb{Y} = span\{u, v\}$ and applying Lemma 4.1, we obtain that $\|u - v\| \leq \|u - w\|$ for all $w \in P_u$. As $v \notin P_u$, we have $\|u - v\| < \|u - w\|$ for all $w \in P_u$, which is a contradiction to the fact that $\|u - v\| \geq \epsilon$. This establishes our claim. It is now easy to observe that there exists $(u_0, v_0) \in M_{\delta_{\mathbb{X}}(\epsilon)}$ such that $\|u_0 - v_0\| = \epsilon$. This implies that

$|\|u_0+v_0\|^2-\|u_0-v_0\|^2| = 4|1+\delta_{\mathbb{X}}(\epsilon)^2-2\delta_{\mathbb{X}}(\epsilon)-\frac{\epsilon^2}{4}|$. Let $\epsilon_0 = |1+\delta_{\mathbb{X}}(\epsilon)^2-2\delta_{\mathbb{X}}(\epsilon)-\frac{\epsilon^2}{4}|$. Then $0 \leq \epsilon_0 < 1$ and $|\|u_0+v_0\|^2-\|u_0-v_0\|^2| = 4\epsilon_0$, which shows that $u_0 \perp_I^{\epsilon_0} v_0$.

□

In case $\mathbb{X}$ is strictly convex, we have the following corollary to the above theorem.

Corollary 4.1. *Let $\mathbb{X}$ be a strictly convex normed linear space and let $\epsilon \in (0,2)$. If $(u,v) \in M_{\delta_{\mathbb{X}}(\epsilon)}$ then $u \perp_I^{\epsilon_0} v$, where $\epsilon_0 = |1+\delta_{\mathbb{X}}(\epsilon)^2-2\delta_{\mathbb{X}}(\epsilon)-\frac{\epsilon^2}{4}| \in [0,1)$.*

Proof. Given $\epsilon \in (0,2)$, we only need to show that for any $(u,v) \in M_{\delta_{\mathbb{X}}(\epsilon)}$, it necessarily follows that $\|u-v\| = \epsilon$. Suppose on the contrary that $\|u-v\| > \epsilon$. Consider the set $P_u = \{w \in S_{\mathbb{X}} : \|u-w\| = \epsilon\}$. Clearly, $v \notin P_u$ and for any $w \in P_u$, we have that $\|u-v\| > \|u-w\|$. Therefore, by Lemma 4.1, together with strict convexity, we get $\|u+v\| < \|u+w\|$ and so $1-\frac{1}{2}\|u+v\| > 1-\frac{1}{2}\|u+w\|$, which contradicts the fact that $\delta_{\mathbb{X}}(\epsilon) = 1-\frac{\|u+v\|}{2}$. Now proceeding similarly as in the proof of Theorem 4.16, we obtain the desired conclusion.

□

In connection with the explicit computation of $\delta_{\mathbb{X}}(\epsilon)$, the following remark seems relevant.

Remark 4.17. *For $\epsilon \in (0,2)$, let us consider the set :*

$$G_\epsilon = \{(u,v) \in S_{\mathbb{X}} \times S_{\mathbb{X}} : u \perp_I^{\epsilon_0} v \text{ and } \|u-v\| = \epsilon\},$$

where $\epsilon_0 = |1+\delta_{\mathbb{X}}(\epsilon)^2-2\delta_{\mathbb{X}}(\epsilon)-\frac{\epsilon^2}{4}|$. Clearly, G_ϵ is a closed set with respect to the usual product topology defined on $\mathbb{X} \times \mathbb{X}$. It can be readily seen that whenever $\mathbb{X}$ is finite-dimensional, there exists $(u_1,v_1) \in G_\epsilon$ such that $\delta_{\mathbb{X}}(\epsilon) = 1-\frac{\|u_1+v_1\|}{2}$. Therefore, we conclude that to find the value of $\delta_{\mathbb{X}}(\epsilon)$, for any $\epsilon \in (0,2)$, we only need to take into account the subset G_ϵ.

In [84], the authors explored the geometric structure of the approximate Birkhoff-James orthogonality set. Motivated by this, we study the same in the case of approximate isosceles orthogonality, in our next theorem. For this purpose, we consider the ϵ-approximate isosceles orthogonality set $A(x,\epsilon)$, corresponding to the vector $x \in S_{\mathbb{X}}$ and $\epsilon \in [0,1)$, as defined in Remark 4.9.
We end this chapter with the following characterization of $A(x,\epsilon)$.

Theorem 4.18. *Let $\mathbb{X}$ be a two-dimensional Banach space. Then for any $x \in S_{\mathbb{X}}$, $A(x,\epsilon) = D \cup -D$, where D is a connected subset of $S_{\mathbb{X}}$.*

Proof. We note from Theorem 4.2 that for $x \in S_{\mathbb{X}}$, there exists a unique (except for the sign) $y \in S_{\mathbb{X}}$ such that $x \perp_I y$. For each $t \in [0,1]$, let $u_t = \frac{(1-t)x+ty}{\|(1-t)x+ty\|}$ and $v_t = \frac{-(1-t)x+ty}{\|-(1-t)x+ty\|}$. Consider the sets $R = \{t \in [0,1] : x \perp_I^{\epsilon} u_t\}$, and $L = \{t \in [0,1] : x \perp_I^{\epsilon} v_t\}$. Clearly, $R, L \neq \emptyset$, since $1 \in R \cap L$. Next we prove that R and L are closed. Suppose $\{t_n\}_{n\in\mathbb{N}} \in R$ is such that $t_n \to t$. Then $x \perp_I^{\epsilon} u_{t_n}$. This implies that for every $n \in \mathbb{N}$, we have $|\|x + u_{t_n}\|^2 - \|x - u_{t_n}\|^2| \leq 4\epsilon$. As $n \to \infty$, $|\|x+u_t\|^2 - \|x-u_t\|^2| \leq 4\epsilon$. Therefore, $x \perp_I^{\epsilon} u_t$. This proves that R is closed. Similarly, it can be shown that L is also closed.

Let $t_R = \inf R$ and let $t_L = \inf L$. Then using Lemma 4.1, for any $t \in [0,1]$ with $t \geq t_R$, we get that $\|x - u_t\| \geq \|x - u_{t_R}\|$ and $\|x + u_t\| \leq \|x + u_{t_R}\|$. This gives $|\|x + u_t\|^2 - \|x - u_t\|^2| \leq |\|x + u_{t_R}\|^2 - \|x - u_{t_R}\|^2| \leq 4\epsilon$. Therefore, $x \perp_I^{\epsilon} u_t$. Similarly, one can show that for any $t \in [0,1]$ with $t \geq t_L$, $x \perp_I^{\epsilon} v_t$. Consider $D = \left\{\frac{su_{t_R}+(1-s)u_{t_L}}{\|su_{t_R}+(1-s)u_{t_L}\|} : 0 \leq s \leq 1\right\}$. Clearly, D is connected. Moreover, it is easy to see that $D \cup (-D) \subset A(x,\epsilon)$. Also, the implication $A(x,\epsilon) \subset D \cup (-D)$ is trivial from the description of the sets R and L. This completes the proof of the theorem.

□

CHAPTER 5

GEOMETRIC CONSTANT AND SYMMETRY ANALYSIS VIA NORM DERIVATIVE ORTHOGONALITY

5.1 Introduction

In Banach space geometry the norm derivative is a well-known and active area of research. In particular, our main objective in this chapter is to study the orthogonality induced by norm derivative, in short ρ-orthogonality in a Banach space. More specifically, our aim is to study a newly introduced geometric constant and symmetricity of the elements of a Banach space from the perspective of ρ-orthogonality. Since geometric constants are well-known and active area of research one can see [3, 46, 52, 56, 70, 83, 97] and the references therein. Our motivation behind introducing new constant is to investigate the difference between ρ-orthogonality and Birkhoff-James orthogonality quantitatively. Before diving into the main results let us fix the notations and terminologies.

Content of this chapter is based on the following paper:

- **S. Ghosh**, K. Paul, D. Sain, *Orthogonality induced by norm derivative: a new geometric constant and symmetry*, Aequationes Math., **99** (2025), 883–904. https://doi.org/10.1007/s00010-025-01154-9.

Letters $\mathbb{X}, \mathbb{Y}$ denote real normed linear spaces and $\mathbb{X}^*$ stands for the dual space of $\mathbb{X}$. Let $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ denote the unit ball and unit sphere of $\mathbb{X}$, respectively. $\text{Ext}\,(B_{\mathbb{X}})$ is denotes the set of all extreme point of the unit ball of $\mathbb{X}$.
Let us mention the definition of norm derivatives and the orthogonality induced by it, i.e., ρ-orthogonality (see [69]).

Definition 5.1. *Let $\mathbb{X}$ be a normed linear space and let $x, y \in \mathbb{X}$. The norm derivatives at x in the direction of y are defined as:*

$$\begin{aligned} \rho'_+(x,y) &= \|x\| \lim_{t\to 0^+} \frac{\|x+ty\|-\|x\|}{t} \\ \rho'_-(x,y) &= \|x\| \lim_{t\to 0^-} \frac{\|x+ty\|-\|x\|}{t} \\ \rho'(x,y) &= \frac{1}{2}(\rho'_+(x,y)+\rho'_-(x,y)). \end{aligned}$$

We say that x is ρ-orthogonal to y, i.e., $x \perp_\rho y$ if $\rho'(x,y) = 0$. Note that ρ-orthogonality is homogeneous, i.e., for any $\alpha, \beta \in \mathbb{R}$, $x \perp_\rho y \iff \alpha x \perp_\rho \beta y$. For further readings on this topic one can see [8, 20, 22, 23, 81].

We now introduce the following definition of a new geometric constant:

Definition 5.2. *Let $\mathbb{X}$ be a normed linear space. We define the following constant $\Gamma(\mathbb{X})$ as:*

$$\Gamma(\mathbb{X}) = \sup\big\{|\rho'(x,y)| : x, y \in S_{\mathbb{X}} \, and \, x \perp_B y\big\}.$$

For a smooth space $\mathbb{X}$, ρ-orthogonality and Birkhoff-James orthogonality are equivalent. This shows that $\Gamma(\mathbb{X}) = 0$ iff $\mathbb{X}$ is smooth (see [8, Prop. 2.2.2]). Let us first note some preliminary results which will be essential throughout this chapter.

5.2 Preliminaries

We observe some of the important results regarding the functions ρ'_+ and ρ'_-.

Lemma 5.1. *[95, Th. 2.4] Let $\mathbb{X}$ be a normed linear space. Then for $x, y \in S_{\mathbb{X}}$,*

$$\begin{aligned} \rho'_+(x,y) &= \sup\{x^*(y) : x^* \in \text{Ext}\,(J(x))\}, \\ \rho'_-(x,y) &= \inf\{x^*(y) : x^* \in \text{Ext}\,(J(x))\}. \end{aligned}$$

Lemma 5.2. *[6] Let $\mathbb{X}$ be a normed linear space. Then $x \perp_B y$ if and only if $\rho'_-(x,y) \le 0 \le \rho'_+(x,y)$.*

Apart from the above mentioned properties of the functions ρ'_+ and ρ'_-, interested readers may see [7, 81]. It is a well known fact [20, 22] that $\perp_\rho \subset \perp_B$ in any normed linear space $\mathbb{X}$. For the converse inclusion we note the following result.

Theorem 5.1. *[8, Prop. 2.2.2] Let $\mathbb{X}$ be a normed linear space. Then $\mathbb{X}$ is smooth if and only if $x \perp_B y$ implies $x \perp_\rho y$, for all $x, y \in \mathbb{X}$.*

Given any $x, y \in \mathbb{X}$, let us denote the ray passing through y starting from x as $[x, y\rangle$, which is defined by $[x, y\rangle = \{(1-t)x + ty : t \geq 0\}$. Following [12], we mention the positive orientation of a two-dimensional Banach space $\mathbb{X}$. Suppose that $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{X}$, where $\mathbb{X}$ is identified with $\mathbb{R}^2$ in the canonical way. Then we say 'x precedes y,' i.e., $x \prec y$ if $x_1y_2 - x_2y_1 > 0$. In this connection, we would like to mention a very important lemma.

Lemma 5.3. *(Monotonicity lemma) [60] Let $\mathbb{X}$ be a two-dimensional Banach space and let $x, y, z \in \mathbb{X} \setminus \{0\}$ such that $x \neq z$. Suppose that the ray $[0, y\rangle$ lies in between the rays $[0, x\rangle$ and $[0, z\rangle$ with $\|y\| = \|z\|$. Then $\|x - y\| \leq \|x - z\|$.*

Moreover, the inequality is strict if $\mathbb{X}$ is strictly convex.

Henceforth, the results of this chapter are mainly divided into three sections including the introductory part. In the second section we explore the newly defined constant $\Gamma(\mathbb{X})$ and find some connection with some other geometric properties of a Banach space. In the last section we deal with symmetricity with respect to ρ-orthogonality. There we observe the interconnection between ρ-symmetricity and symmetricity with respect to Birkhoff-James orthogonality. Further, we obtain a characterization of both ρ-left and ρ-right symmetric points. Finally we give a complete description of both the ρ-left and ρ-right symmetric points of the spaces ℓ_1^n and ℓ_∞^n.

5.3 The new constant $\Gamma(\mathbb{X})$ and its properties.

At first, we develop a bound for the constant $\Gamma(\mathbb{X})$. To do so we use the notion of $\mathcal{E}(\mathbb{X})$, introduced in [21].

Definition 5.3. *Suppose that $d : \mathbb{X} \setminus \{0\} \to \mathbb{R}$ is defined as $d(x) = \operatorname{diam}(J(x))$, where $J(x)$ is the collection of all the supporting linear functionals at x. Then $\mathcal{E}(\mathbb{X})$ is defined as*

$$\mathcal{E}(\mathbb{X}) = \sup\{d(x) : x \in S_{\mathbb{X}}\}.$$

Proposition 5.1. *For a normed linear space $\mathbb{X}$, $0 \leq \Gamma(\mathbb{X}) \leq \min\{\mathcal{E}(\mathbb{X}), \frac{1}{2}\}$.*

Proof. It is easy to see that $\Gamma(\mathbb{X}) \geq 0$. To obtain the upper bound, we first note from [8, Th. 2.1.1] that $|\rho'_{\pm}(x,y)| \leq \|x\|\|y\|$. Thus $|\rho'(x,y)| \leq 1$, for any $x, y \in S_{\mathbb{X}}$. From Lemma 5.2 we see that when $x \perp_B y$, we have $\rho'_-(x,y) \leq 0 \leq \rho'_+(x,y)$. This implies that $|\rho'(x,y)| \leq \frac{1}{2}$. Now we show $\Gamma(\mathbb{X}) \leq \mathcal{E}(\mathbb{X})$. Let us consider any two arbitrary elements $x, y \in S_{\mathbb{X}}$ such that $x \perp_B y$. From Lemma 5.1, we note that $\rho'_+(x,y) = \sup\{x^*(y) : x^* \in \operatorname{Ext}(J(x))\}$. Since $J(x)$ is weak*-compact and a convex subset of $\mathbb{X}^*$, it follows that there exists $x_0^* \in J(x)$ such that $\rho'_+(x,y) = x_0^*(y)$. Similarly, we can obtain that $\rho'_-(x,y) = x_1^*(y)$, for some $x_1^* \in J(x)$. Also, from Lemma 5.2 we note that $x_0^*(y) \geq 0 \geq x_1^*(y)$ as $x \perp_B y$. Thus we have

$$\begin{aligned}
|\rho'(x,y)| &= \frac{1}{2}|\rho'_+(x,y) + \rho'_-(x,y)| \\
&= \frac{1}{2}|x_0^*(y) + x_1^*(y)| \\
&\leq \frac{1}{2}|x_0^*(y) - x_1^*(y)| \\
&\leq \|x_0^* - x_1^*\| \leq d(x).
\end{aligned}$$

Therefore,

$$\Gamma(\mathbb{X}) = \sup\{|\rho'(x,y)| : x, y \in S_{\mathbb{X}}, x \perp_B y\} \leq \sup\{d(x) : x \in S_{\mathbb{X}}\} = \mathcal{E}(\mathbb{X}).$$

This completes the proof. □

For any smooth normed linear space we note that $\Gamma(\mathbb{X}) = 0$. On the other hand, it is easy to see that $\Gamma(\mathbb{X}) = \frac{1}{2}$, when $\mathbb{X} = \ell_\infty^n$. In fact, taking $x = (1, 1, \ldots, 1)$ and $y = (0, 0, \ldots, 1)$, we get $\rho'(x,y) = \frac{1}{2}$. Similarly we can show that $\Gamma(\mathbb{X}) = \frac{1}{2}$, when $\mathbb{X} = \ell_1^n$. Also, we give an example of an infinite-dimensional Banach space where $\Gamma(\mathbb{X}) = \frac{1}{2}$.

Example 5.2. *Let us consider the space c_0 and let $x = (1, 1, 0, \ldots, 0, \ldots) \in c_0$. Clearly, $x_1^*, x_2^* \in J(x)$, where for each $i \in \{1, 2\}$, $x_i^* \in c_0^*$ and $x_i^*(y) = y_i$, for all $y = (y_1, y_2, \ldots) \in c_0$. Take $z = (0, 1, 0, \ldots) \in c_0$. Clearly, $x \perp_B z$. Also, we have $x_1^*(z) = 0$ and $x_2^*(z) = 1$. Since $x \perp_B z$, from Lemma 5.2 we have $\rho'_-(x,z) \leq 0 \leq \rho'_+(x,z)$. Also, applying Lemma 5.1, it is easy to observe that $\rho'(x,z) = \frac{1}{2}(\rho'_+(x,z) + \rho'_-(x,z)) = \frac{1}{2}$. Now from Proposition 5.1 one can see that $\Gamma(\mathbb{X}) = \frac{1}{2}$.*

Next we prove the following theorem which will be useful to estimate the constant $\Gamma(\mathbb{X})$ in any finite-dimensional polyhedral Banach space.

Theorem 5.3. *Let $\mathbb{X}$ be an n-dimensional Banach space. Then there exists an element $z \in \operatorname{Ext}(B_{\mathbb{X}})$ such that $\Gamma(\mathbb{X}) = \rho'(z,w)$, for some $w \in S_{\mathbb{X}}$ with $z \perp_B w$.*

Proof. Using Carathéodory's theorem [77], we note that for any $x \in S_{\mathbb{X}}$, there exist $z_1, z_2, \ldots, z_{n+1} \in \operatorname{Ext}(B_{\mathbb{X}})$ such that $x = \sum_{k=1}^{n+1} \lambda_k z_k$, where $\sum_{k=1}^{n+1} \lambda_k = 1$ and $\lambda_k \geq 0$, for each $1 \leq k \leq n+1$. Suppose that $y \in S_{\mathbb{X}}$ such that $x \perp_B y$. Then it is straightforward to see that $z_k \perp_B y$, for each $1 \leq k \leq n+1$. Now

$$\begin{aligned}
2\rho'(x,y) &= \rho'_+(x,y) + \rho'_-(x,y) \\
&= \rho'_+\Big(\sum_{k=1}^{n+1} \lambda_k z_k, y\Big) + \rho'_-\Big(\sum_{k=1}^{n+1} \lambda_k z_k, y\Big) \\
&= \lim_{t\to 0^+} \frac{\| \sum_{k=1}^{n+1} \lambda_k z_k + ty\| - 1}{t} + \lim_{t\to 0^-} \frac{\| \sum_{k=1}^{n+1} \lambda_k z_k + ty\| - 1}{t} \\
&= \lim_{t\to 0^+} \frac{\| \sum_{k=1}^{n+1} \lambda_k z_k + \sum_{k=1}^{n+1} \lambda_k ty\| - 1}{t} + \lim_{t\to 0^-} \frac{\| \sum_{k=1}^{n+1} \lambda_k z_k + \sum_{k=1}^{n+1} \lambda_k ty\| - 1}{t} \\
&\leq \lim_{t\to 0^+} \frac{\sum_{k=1}^{n+1} \lambda_k \|z_k + ty\| - 1}{t} + \lim_{t\to 0^-} \frac{\sum_{k=1}^{n+1} \lambda_k \|z_k + ty\| - 1}{t} \\
&= \lim_{t\to 0^+} \frac{\sum_{k=1}^{n+1} \lambda_k \|z_k + ty\| - \sum_{k=1}^{n+1} \lambda_k}{t} + \lim_{t\to 0^-} \frac{\sum_{k=1}^{n+1} \lambda_k \|z_k + ty\| - \sum_{k=1}^{n+1} \lambda_k}{t} \\
&= \sum_{k=1}^{n+1} \lambda_k (\rho'_+(z_k, y) + \rho_-(z_k, y)) \\
&\leq 2\max\{\rho'(z_k, y)\}.
\end{aligned}$$

This clearly shows that for any $x \in S_{\mathbb{X}}$, there exists $z \in \operatorname{Ext}(B_{\mathbb{X}})$ such that $\rho'(x,y) \leq \rho'(z,y)$. This completes the proof of the theorem.

□

A normed linear space $\mathbb{X}$ is uniformly non-square if $\sup_{x,y\in S_{\mathbb{X}}} \min\{\|x-y\|, \|x+y\|\} < 2$. Note that the spaces ℓ_1^n, ℓ_∞^n are non uniformly non-square. Then it is natural to ask whether for any non uniformly non-square space $\mathbb{X}$, $\Gamma(\mathbb{X}) = \frac{1}{2}$. To proceed in this direction first we prove the following lemma. See [30, Prop. 2.6].

Lemma 5.4. *Let $\mathbb{X}$ be a two-dimensional Banach space and let $\mathbb{X}$ be non uniformly non-square. Then $\mathbb{X}$ is isometrically isomorphic to ℓ_∞^2.*

Proof. Since $\mathbb{X}$ is non uniformly non-square, it follows that there exists $x_0, y_0 \in S_{\mathbb{X}}$ such that $\min\{\|x_0 - y_0\|, \|x_0 + y_0\|\} = 2$, i.e., $\|x_0 - y_0\| = \|x_0 + y_0\| = 2$. Clearly, $x_0 \neq \pm y_0$. Define a linear map $T : \mathbb{X} \to \ell_\infty^2$ by $Tx_0 = (1,1)$ and $Ty_0 = (-1,1)$. Since $\mathbb{X}$ is two-dimensional, for any $z \in \mathbb{X}$, we have $z = \alpha x_0 + \beta y_0$, where $\alpha, \beta \in \mathbb{R}$. Then $Tz = T(\alpha x_0 + \beta y_0) = (\alpha - \beta, \alpha + \beta)$. Note that $\|(\alpha - \beta, \alpha + \beta)\|_\infty = |\alpha| + |\beta|$. Thus we only need to show that $\|\alpha x_0 + \beta y_0\| = |\alpha| + |\beta|$, for any $\alpha, \beta \in \mathbb{R}$. Since $\frac{1}{2}\|x_0 - y_0\| = \frac{1}{2}\|x_0 + y_0\| = 1$, $L[x_0, y_0] := \{(1-t)x_0 + ty_0 : 0 \leq t \leq 1\}$ and $L[x_0, -y_0]$ both are subsets of $S_{\mathbb{X}}$. Note that if $\alpha = 0$ or $\beta = 0$, then we are done. Let

$\alpha, \beta \neq 0$. Moreover, assume that $\alpha, \beta > 0$. Let $z_0 = \frac{\alpha x_0 + \beta y_0}{\|\alpha x_0 + \beta y_0\|}$. Clearly, $z_0 \in S_{\mathbb{X}}$. Consider the element $z' = \frac{\|\alpha x_0 + \beta y_0\|}{\alpha+\beta} z_0$. It is easy to see that $z' \in L[x_0, y_0]$. Since $L[x_0, y_0] \subset S_{\mathbb{X}}$, it follows that $\frac{\|\alpha x_0 + \beta y_0\|}{\alpha+\beta} = 1$, i.e., $\|\alpha x_0 + \beta y_0\| = \alpha + \beta = |\alpha| + |\beta|$. Let us now consider $\alpha > 0$ and $\beta < 0$. Then we write $z = \alpha x_0 + \beta y_0 = \alpha x_0 + \beta'(-y_0)$, where $\beta' = -\beta$. Then we get $\alpha, \beta' > 0$. Proceeding similarly as above we obtain $\|\alpha x_0 + \beta y_0\| = \|\alpha x_0 + \beta'(-y_0)\| = \alpha + \beta' = |\alpha| + |\beta|$. Also, the other cases for α and β follow similarly as above. This completes the proof.

□

It is well known that in a normed linear space the James constant, $J(\mathbb{X})$ studies the non uniformly non-squareness of the unit sphere. In the next theorem we obtain a connection between the notion of uniform non-squareness and the constant $\Gamma(\mathbb{X})$.

Theorem 5.4. *Let $\mathbb{X}$ be a finite-dimensional Banach space. Then $\mathbb{X}$ is uniformly non-square whenever $\Gamma(\mathbb{X}) < \frac{1}{2}$.*

Proof. Suppose on the contrary that $\mathbb{X}$ is not uniformly non-square. Then from [30, Th. 3.4] we note that $J(\mathbb{X}) = 2$, i.e., $\sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_{\mathbb{X}}\} = 2$. Since $\mathbb{X}$ is finite-dimensional, it follows that there exist $x_0, y_0 \in S_{\mathbb{X}}$ such that $\min\{\|x_0+y_0\|, \|x_0-y_0\|\} = 2$. Clearly, $x_0 \neq \pm y_0$. Consider the two-dimensional subspace $\mathbb{Y} = span\{x_0, y_0\}$. Then from Lemma 5.4 it follows that $\mathbb{Y}$ is isometrically isomorphic to ℓ_∞^2. As $\Gamma(\ell_\infty^2) = \frac{1}{2}$, we get $\Gamma(\mathbb{X}) \geq \Gamma(\mathbb{Y}) = \Gamma(\ell_\infty^2) = \frac{1}{2}$. Thus following Proposition 5.1, we obtain that $\Gamma(\mathbb{X}) = \frac{1}{2}$. This completes the proof of the theorem.

□

Let $\mathbb{X}$ be a normed linear space such that $M_J \neq \emptyset$, where

$$M_J := \{(x, y) \in S_{\mathbb{X}} \times S_{\mathbb{X}} : \min\{\|x - y\|, \|x + y\|\} = J(\mathbb{X})\}.$$

Then using the same arguments as in the proof of Theorem 5.4 we can show that $\mathbb{X}$ is uniformly non-square if $\Gamma(\mathbb{X}) < \frac{1}{2}$.

Remark 5.5. *(i) We give an example to show that the result is not true for an infinite dimensional space. From [41, Th. 1.1] it follows that if the unit ball of a normed linear space is uniformly non-square then the space is reflexive. Consider the non-reflexive smooth Banach space $\mathbb{X}$ as given in [61, Ex. 5.4.13]. Then $\Gamma(\mathbb{X}) = 0$ (being smooth) but $\mathbb{X}$ is not uniformly non-square (being non-reflexive).*
(ii) Next we give an example to show that the converse of Theorem 5.4 is not true, in general. Let us consider the two-dimensional Euclidean space $\mathbb{R}^2$, endowed with the norm $\ell_1 - \ell_\infty$. Let $x = (1, 0) \in \operatorname{Ext}(B_{\mathbb{X}})$ and $y = (0, 1)$. Clearly, $x \perp_B y$. It is easy to calculate that

$\rho'(x, y) = \frac{1}{2}$. *Following Proposition 5.1 together with Theorem 5.3 we conclude that* $\Gamma(\mathbb{X}) = \frac{1}{2}$, *though* $(\mathbb{R}^2, \|\cdot\|_{\ell_1 - \ell_\infty})$ *is uniformly non-square.*

Applying Theorem 5.3 we next compute the constant $\Gamma(\mathbb{X})$ for a two-dimensional Banach space whose unit sphere is a regular $2n$-gon. Let us first observe the following proposition.

Proposition 5.2. *Let* $\mathbb{X}$ *be a two-dimensional Banach space and let* $x \in S_{\mathbb{X}}$. *Suppose that* $w_1, w_2 \in S_{\mathbb{X}}$ *satisfying* $x \prec w_1 \prec w_2 \prec -x$. *Then* $\rho'(x, w_1) \geq \rho'(x, w_2)$.

Proof. Note that the ray $[0, w_1\rangle$ lies in between the rays $[0, x\rangle$ and $[0, w_2\rangle$. Then applying Lemma 5.3 we obtain the following:

(1) $\|x + tw_1\| \geq \|x + tw_2\|$, when $t > 0$.

(2) $\|x + tw_1\| \leq \|x + tw_2\|$, when $t < 0$.

Therefore, it is easy to observe from (1) and (2) that

$$\lim_{t \to 0^{\pm}} \frac{\|x + tw_1\| - 1}{t} \geq \lim_{t \to 0^{\pm}} \frac{\|x + tw_2\| - 1}{t}.$$

This implies that $\rho'_{\pm}(x, w_1) \geq \rho'_{\pm}(x, w_2)$ and consequently we conclude that $\rho'(x, w_1) \geq \rho'(x, w_2)$. □

We next note that in a two-dimensional Banach space $\mathbb{X}$, $x^{\perp}$ can be described in terms of a normal cone K. To be precise, $x^{\perp} = K \cup (-K)$. Let us recall that a subset K of $\mathbb{X}$ is said to be a normal cone in $\mathbb{X}$ if it satisfies the following:

(i) $K + K \subset K$

(ii) $\alpha K \subset K$, for all $\alpha \geq 0$ and

(iii) $K \cap (-K) = \{0\}$.

We say that the cone K is determined by $x_1, x_2 \in S_{\mathbb{X}}$ if $K \cap S_{\mathbb{X}} = \left\{ \frac{(1-t)x_1 + tx_2}{\|(1-t)x_1 + tx_2\|} : 0 \leq t \leq 1 \right\}$. In particular, $K = \{\alpha x_1 + \beta x_2 : \alpha, \beta \geq 0\}$, see [84]. In the following lemma we explicitly find the points which determine the cone corresponding to the orthogonal region of a vertex for a regular $2n$-gon.

Lemma 5.5. *Let* $\mathbb{X}$ *be a two-dimensional Banach space whose unit sphere is a regular* $2n$*-gon. Let* $\{v_1, v_2, \ldots, v_{2n}\}$ *be the vertices of* $B_{\mathbb{X}}$ *such that for each* $1 \leq j \leq 2n$, $v_j = (\cos \frac{j-1}{n}\pi, \sin \frac{j-1}{n}\pi)$. *Suppose that for some* $m \in \{1, 2, \ldots, 2n\}$, $v_m^{\perp} = K \cup (-K)$, *where* K *is a normal cone determined by* w_1 *and* w_2. *Then the following holds true:*

(i) $w_1 = v_{\frac{n+2m-1}{2}}$ *and* $w_2 = v_{\frac{n+2m+1}{2}}$, *when* n *is odd.*

(ii) $w_1 = \frac{1}{2}(v_{\frac{n+2m-2}{2}} + v_{\frac{n+2m}{2}})$ *and* $w_2 = \frac{1}{2}(v_{\frac{n+2m}{2}} + v_{\frac{n+2m+2}{2}})$, *when* n *is even.*

Proof. By a straightforward computation, one can observe that given any $i \in \{1, 2, \ldots, 2n\}$ and for any $u \in [v_i, v_{i+1}]$, the supporting functional of u is given by

$$x^*(x, y) = \frac{1}{\cos\frac{\pi}{2n}}\Big(x\cos\frac{2i-1}{2n}\pi + y\sin\frac{2i-1}{2n}\pi\Big), \tag{5.1}$$

for any $(x, y) \in \mathbb{X}$. Thus $\operatorname{Ext}(J(v_m)) = \{x_1^*, x_2^*\}$, where

$$x_1^*(x, y) = \frac{1}{\cos\frac{\pi}{2n}}\Big(x\cos\frac{2m-3}{2n}\pi + y\sin\frac{2m-3}{2n}\pi\Big)$$

and

$$x_2^*(x, y) = \frac{1}{\cos\frac{\pi}{2n}}\Big(x\cos\frac{2m-1}{2n}\pi + y\sin\frac{2m-1}{2n}\pi\Big),$$

for any $(x, y) \in \mathbb{R}$. Suppose that $\ker x_i^* \cap S_{\mathbb{X}} = \{\pm w_i\}$, for each $i \in \{1, 2\}$. Observe that $v_m^{\perp} = K \cup (-K)$, where the normal cone K is determined by $\{w_1, w_2\}$. We only find w_1 as w_2 can be obtained analogously. Let $w_1 = (1-\lambda)v_j + \lambda v_{j+1} = (1-\lambda)\big(\cos\frac{j-1}{n}\pi, \sin\frac{j-1}{n}\pi\big) + \lambda\big(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}\big)$, for some $\lambda \in [0, 1]$ and for some $j \in \{1, 2, \ldots, 2n\}$. Since $x_1^*(w_1) = 0$, it follows by a simple computation that the following equation holds true:

$$(1-\lambda)\cos\frac{2j-2m+1}{2n}\pi + \lambda\cos\frac{2j-2m+3}{2n}\pi = 0. \tag{5.2}$$

From the above equation, it follows that $\frac{2j-2m+1}{2n}\pi \leq \frac{2t+1}{2}\pi \leq \frac{2j-2m+3}{2n}\pi$, for some $t \in \mathbb{N} \cup \{0\}$. This implies that $\frac{(2t+1)n-3}{2} + m \leq j \leq \frac{(2t+1)n-1}{2} + m$. Since $1 \leq j \leq 2n$ and $n \geq 2$, it is easy to see that $t \in \{0, 1\}$. Suppose that $t = 0$. Now we note the following two cases:

Case-I: Suppose that n is odd. Then clearly, either $j = \frac{n-3}{2} + m$ or $j = \frac{n-1}{2} + m$. Then putting these values in equation (2) we obtain $\lambda = 1$ or $\lambda = 0$, respectively. In both of these cases we obtain $w_1 = v_{\frac{n-1}{2}+m}$. Thus we get $w_1 = v_{\frac{n+2m-1}{2}}$. Similarly, we get $w_2 = v_{\frac{n+2m+1}{2}}$.

Case-II: Suppose that n is even. Then one can observe that $j = \frac{n-2}{2} + m$. From equation (2) it follows that $\lambda = \frac{1}{2}$. Then $w_1 = \frac{1}{2}\big(v_{\frac{n+2m-2}{2}} + v_{\frac{n+2m}{2}}\big)$. Proceeding as before we get $w_2 = \frac{1}{2}\big(v_{\frac{n+2m}{2}} + v_{\frac{n+2m+2}{2}}\big)$. This proves (ii).

If $t = 1$, then it is easy to see that $w_1 = -v_{\frac{n+2m-1}{2}}$ and $w_2 = -v_{\frac{n+2m+1}{2}}$, when n is odd. On the other hand, when n is even, we get $w_1 = -\frac{1}{2}\big(v_{\frac{n+2m-2}{2}} + v_{\frac{n+2m}{2}}\big)$ and $w_2 = -\frac{1}{2}\big(v_{\frac{n+2m}{2}} + v_{\frac{n+2m+2}{2}}\big)$. This shows that $K \cup (-K)$ is completely determined by w_1, w_2 as given in (i) and (ii). This completes the proof of the lemma. □

In the following theorem we compute the value of $\Gamma(\mathbb{X})$ whenever $\mathbb{X}$ is a two-dimensional

Banach space whose unit sphere is a regular $2n$-gon.

Theorem 5.6. *Let $\mathbb{X}$ be a two-dimensional Banach space and let $S_{\mathbb{X}}$ be a regular $2n$-gon, where $n \geq 2$. Then the following results hold true:*

(i) $\Gamma(\mathbb{X}) = \frac{\cos\frac{n-2}{2n}\pi}{2\cos\frac{\pi}{2n}}$, *when n is odd.*

(iI) $\Gamma(\mathbb{X}) = \frac{1}{4\cos\frac{\pi}{2n}}\Bigg(\cos\frac{n-3}{2n}\pi + \cos\frac{n-1}{2n}\pi\Bigg)$, *when n is even.*

Proof. Suppose $\mathbb{X}$ is such that $S_{\mathbb{X}}$ is a regular $2n$-gon with vertices $v_1, v_2, \ldots, v_{2n}$, where $v_j = \Bigg(\cos\frac{j-1}{n}\pi, \sin\frac{j-1}{n}\pi\Bigg)$, for each $j \in \{1, 2, \ldots, 2n\}$. Moreover, from Theorem 5.3 there exists an element $z \in \text{Ext}\,(B_{\mathbb{X}})$ such that $\rho'(z, y) = \Gamma(\mathbb{X})$, for some $y \in S_{\mathbb{X}}$ with $z \perp_B y$. Note that in this case for any $k \in \mathbb{N}$, $\frac{k\pi}{n}$- rotation is an isometric isomorphism on $\mathbb{X}$. Since Birkhoff-James orthogonality is preserved under isometric isomorphism [55], we only find $\rho'(z, y)$ for a fixed vertex z, where $y \in z^{\perp}$. Without loss of generality we may indeed assume that $z = v_1 = (1, 0)$. Suppose that $v_1^{\perp} = K \cup (-K)$, where K is a normal cone determined by y_1, y_2. Let us take $y \in K \cap S_{\mathbb{X}}$. Note that $v_1 \prec y_1 \preceq y \preceq y_2 \prec -v_1$. From Proposition 5.2, $\rho'(v_1, y_1) \geq \rho'(v_1, y) \geq \rho'(v_1, y_2)$. Therefore, $\Gamma(\mathbb{X}) = \max\{|\rho'(v_1, y_1)|, |\rho'(v_1, y_2)|\}$. From the definition it is easy to verify that $\rho'_+(v_1, y_2) = \rho'_-(v_1, y_1) = 0$. Thus we only find the values of $\rho'_+(v_1, y_1)$ and $\rho'_-(v_1, y_2)$. We consider the following two cases:

Case I: Suppose that n is odd. From Lemma 5.5 we see that $y_1 = v_{\frac{n+1}{2}}$ and $y_2 = v_{\frac{n+3}{2}}$. Let $\text{Ext}\,(J(v_1)) = \{x_1^*, x_2^*\}$, where $\ker x_i^* = \{\pm y_i\}$, for each $1 \leq i \leq 2$. From Equation (1) we observe that $x_1^*(x, y) = x - y\tan\frac{\pi}{2n}$ and $x_2^*(x, y) = x + y\tan\frac{\pi}{2n}$, for all $x, y \in \mathbb{R}$. Therefore, by Lemma 5.1 we have

$$\rho'_+(v_1, y_1) = x_2^*(y_1) = \cos\frac{n-1}{2n}\pi + \tan\frac{\pi}{2n}\sin\frac{n-1}{2n}\pi.$$

By simplifying, the above equation reduces to

$$\rho'_+(v_1, y_1) = x_2^*(y_1) = \frac{\cos\frac{n-2}{2n}\pi}{\cos\frac{\pi}{2n}}.$$

Also,

$$\rho'_-(v_1, y_2) = x_1^*(y_2) = -\frac{\cos\frac{n-2}{2n}\pi}{\cos\frac{\pi}{2n}}.$$

Considering these together we get:

$$|\rho'(v_1, y_1)| = |\rho'(v_1, y_2)| = \frac{\cos\frac{n-2}{2n}\pi}{2\cos\frac{\pi}{2n}}.$$

This proves (i).

Case-II: Suppose that n is even. Then from Lemma 5.5 we get $y_1 = \frac{1}{2}(v_{\frac{n}{2}} + v_{\frac{n+2}{2}})$ and $y_2 = \frac{1}{2}(v_{\frac{n+2}{2}} + v_{\frac{n+4}{2}})$. Let $\operatorname{Ext}(J(v_1)) = \{x_1^*, x_2^*\}$, where x_1^*, x_2^* are the same as in Case-I. Then

$$\rho'_+(v_1, y_1) = x_1^*(y_2) = \frac{1}{2\cos\frac{\pi}{2n}}\left(\cos\frac{n-3}{2n}\pi + \cos\frac{n-1}{2n}\pi\right).$$

Proceeding similarly we obtain that

$$\rho'_-(v_1, y_2) = x_2^*(y_1) = -\frac{1}{2\cos\frac{\pi}{2n}}\left(\cos\frac{n-3}{2n}\pi + \cos\frac{n-1}{2n}\pi\right).$$

Thus we see that

$$\Gamma(\mathbb{X}) = \max\{|\rho'(v_1, y_1)|, |\rho'(v_1, y_2)|\} = |\rho'(v_1, y_1)| = \frac{1}{2}\rho'_+(v_1, y_1).$$

This proves (ii).

Hence the proof of the theorem is completed. □

Let us now calculate the value of $\Gamma(\mathbb{X})$, for some particular two-dimensional Banach space.

Example 5.7. *(i) Let $\mathbb{X}$ be a Banach space such that $S_{\mathbb{X}}$ is a regular octagon. Then we have $n = 4$. Applying Theorem 5.6(i) we have $\Gamma(\mathbb{X}) = \frac{1}{2\sqrt{2}}$.*

(ii) Let $\mathbb{X}$ be a two-dimensional Banach space, endowed with the norm $\ell_p - \ell_1$. For any $(x, y) \in \mathbb{X}$,

$$\begin{aligned}\|(x,y)\| &= (|x|^p + |y|^p)^{\frac{1}{p}}, \text{ whenever } xy \geq 0\\ &= (|x| + |y|), \text{ whenever } xy \leq 0.\end{aligned}$$

Then $\Gamma(\mathbb{X}) = \frac{1}{2}$, where $1 \leq p \leq \infty$. It is clear that $e_1 = (1,0), e_2 = (0,1) \in \mathbb{X}$. Moreover, $\|(1,0)\| = \|(0,1)\| = 1$. Note that $\rho'_+(e_1, e_2) = \lim_{t\to 0^+} \frac{(1+t^p)^{\frac{1}{p}} - 1}{t}$. Thus we obtain that $\rho'_+(e_1, e_2) = 0$. On the other hand, $\rho'_-(e_1, e_2) = \lim_{t\to 0^-} \frac{1+|t|-1}{t}$. This implies that $\rho'_-(e_1, e_2) = -1$. Therefore, $|\rho'(e_1, e_2)| = \frac{1}{2}$. Similarly, we can show that $\Gamma(\ell_p^2 - \ell_\infty^2) = \frac{1}{2}$.

We end this section with the estimation of the constant $\Gamma(\mathbb{X})$ for uniformly convex Banach spaces.

Theorem 5.8. *Let $\mathbb{X}$ be a uniformly convex Banach space. Then $\Gamma(\mathbb{X}) < \frac{1}{2}$.*

Proof. Suppose on the contrary that $\Gamma(\mathbb{X}) = \frac{1}{2}$. Then there exist two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}} \subset S_{\mathbb{X}}$ such that $x_n \perp_B y_n$ and $|\rho'(x_n, y_n)| \to \frac{1}{2}$. Since $x_n \perp_B y_n$, it follows from Lemma 5.2 that, for each $n \in \mathbb{N}$, $-1 \leq \rho'_-(x_n, y_n) \leq 0 \leq \rho'_+(x_n, y_n) \leq 1$. This implies that either of the following two holds true:

(1) $\rho'_+(x_n, y_n) \to 1$ and $\rho'_-(x_n, y_n) \to 0$, as $n \to \infty$.

(2) $\rho'_+(x_n, y_n) \to 0$ and $\rho'_-(x_n, y_n) \to -1$, as $n \to \infty$.

Without loss of generality we assume that (1) holds true. Then from Lemma 5.1, we have $\lim_{n\to\infty} \big\{ \sup\{x_n^*(y_n) : x_n^* \in \operatorname{Ext}(J(x_n))\} \big\} = 1$. Then for each $n \in \mathbb{N}$, $\|x_n + y_n\| \geq |x_n^*(x_n + y_n)| \geq 1 + x_n^*(y_n)$. Thus

$$\begin{aligned} \|x_n + y_n\| &\geq \sup\{1 + x_n^*(y_n) : x_n^* \in \operatorname{Ext}(J(x_n))\} \\ &= 1 + \sup\{x_n^*(y_n) : x_n^* \in \operatorname{Ext}(J(x_n))\}. \end{aligned}$$

Taking the limit on both sides of the above inequality, we get that $\lim_{n\to\infty} \|x_n + y_n\| \geq 2$. Also, we have $\|x_n + y_n\| \leq 2$, for each n. This implies that $\lim_{n\to\infty} \|x_n + y_n\| = 2$. On the other hand, since for each n, $x_n \perp_B y_n$, it follows that $\|x_n - y_n\| \geq 1$. Therefore, $\|x_n - y_n\| \not\to 0$. By [61, Prop. 5.2.8], this contradicts the fact that $\mathbb{X}$ is uniformly convex. ☐

The converse of Theorem 5.8 is not true, in general. There are spaces for which $\Gamma(\mathbb{X}) < \frac{1}{2}$ but the spaces are not uniformly convex (see Theorem 5.6).

5.4 Symmetric properties of ρ-orthogonal elements.

Following the notion of left and right symmetric points with respect to Birkhoff-James orthogonality, introduced and studied in [91], we now define ρ-left and ρ-right symmetric points. Given any $x \in \mathbb{X}$, we say x is ρ-left symmetric (ρ-right symmetric) if $x \perp_\rho y$ implies $y \perp_\rho x$ ($y \perp_\rho x$ implies $x \perp_\rho y$), for all $y \in \mathbb{X}$. If x is both ρ-left and ρ-right symmetric then we say that x is ρ-symmetric. The space $\mathbb{X}$ is said to be ρ-symmetric if for any $x, y \in \mathbb{X}$, we have $x \perp_\rho y \implies y \perp_\rho x$. If $\dim(\mathbb{X}) \geq 3$ and Birkoff-James orthogonality is symmetric then the norm on $\mathbb{X}$ is induced by an inner product (see [25, 43]). However, if $\dim(\mathbb{X}) = 2$, then there exist spaces where the Birkhoff-James orthogonality is symmetric but the norm is not necessarily induced by an inner product. A two-dimensional Banach space where Birkhoff-James orthogonality is symmetric is known as the Radon plane. In this section we focus on the study of ρ-symmetric points and ρ-symmetric spaces. We begin with the following theorem.

Theorem 5.9. *Let $\mathbb{X}$ be a two-dimensional Banach space and let $\mathbb{X}$ be ρ-symmetric. Then $\mathbb{X}$ is strictly convex.*

Proof. Suppose on the contrary $\mathbb{X}$ is not strictly convex. Then there exist $u, v \in S_{\mathbb{X}}$ such that the closed line segment $L[u,v] := \{(1-t)u + tv : 0 \leq t \leq 1\}$ is a subset of the unit sphere of $\mathbb{X}$. There exists a unique $x^* \in S_{\mathbb{X}^*}$ such that $x^*(x) = 1$, for all $x \in L[u,v]$. In other words, x^* supports the line $L[u,v]$. Consider that $\ker x^* \cap S_{\mathbb{X}} = \{\pm y\}$. Then for any $x \in L[u,v]$, $x \perp_B y$. We take $x \in L(u,v)$, where $L(u,v) := \{(1-t)u + tv : 0 < t < 1\}$. Since x is a smooth point, it follows that $x \perp_\rho y$. Since $\mathbb{X}$ is ρ-symmetric, it follows that $y \perp_\rho x$. Let $\operatorname{Ext}(J(y)) = \{y_1^*, y_2^*\}$. Then one can observe using Lemma 5.1 that $y \perp_\rho w$ if and only if $w \in \ker(y_1^* + y_2^*)$. Therefore, $L(u,v) \subset \ker(y_1^* + y_2^*)$. This is a contradiction. Thus $\mathbb{X}$ is strictly convex. □

Using the above theorem we observe the following result.

Theorem 5.10. *Let $\mathbb{X}$ be a normed linear space.*

(i) Suppose that $\dim(\mathbb{X}) = 2$. *If $\mathbb{X}$ is ρ-symmetric then $\mathbb{X}$ is a Radon plane.*

(ii) Suppose $\dim(\mathbb{X}) \geq 3$. *Then $\mathbb{X}$ is ρ-symmetric if and only if $\mathbb{X}$ is an inner product space.*

Proof. (i) We prove that if $\mathbb{X}$ is ρ-symmetric then $\mathbb{X}$ is symmetric with respect to Birkhoff-James orthogonality. Suppose on the contrary that there exist $x, y \in S_{\mathbb{X}}$ such that $x \perp_B y$ but $y \not\perp_B x$. Then clearly, $y \not\perp_\rho x$. Let us consider a nonzero real number $\alpha = -\rho'(y,x)$. It is easy to see that $\rho'(y, \alpha y + x) = 0$. Take $z = \frac{\alpha y + x}{\|\alpha y + x\|} \in S_{\mathbb{X}}$. Then $y \perp_\rho z$. Since $\mathbb{X}$ is ρ-symmetric we have $z \perp_\rho y$. This implies that $z \perp_B y$. Therefore, there exists $z^* \in J(z)$ such that $y \in \ker z^*$. Also, $x \perp_B y$ implies that there exists $x^* \in J(x)$ such that $y \in \ker x^*$. Therefore, $y \in \ker z^* \cap \ker x^*$. From Theorem 5.9 we note that $\mathbb{X}$ is strictly convex. Therefore, $J(z) \cap J(x) = \emptyset$. This shows that z^* and x^* are linearly independent. Thus we obtain that $y = 0$, which is a contradiction. This implies that $\mathbb{X}$ is symmetric with respect to Birkhoff-James orthogonality and therefore it must be a Radon plane.

(ii) The sufficient part follows trivially. We prove the necessary part. Since $\mathbb{X}$ is ρ-symmetric, it follows that every two-dimensional subspace of $\mathbb{X}$ is ρ-symmetric. Then applying Theorem 5.9, every two-dimensional subspace of $\mathbb{X}$ is symmetric with respect to Birkhoff-James orthogonality. This implies that $\mathbb{X}$ is symmetric with respect to Birkhoff-James orthogonality. Hence from [25, Th. 6.4] it follows that $\mathbb{X}$ is an inner product space. □

In the next example we see that the converse of Theorem 5.10(i) is not true.

Example 5.11. *Let us consider the two-dimensional Radon plane* $(\mathbb{R}^2, \|\cdot\|_{\ell_1-\ell_\infty})$. *Observe that all the points on the unit sphere are symmetric with respect to Birkhoff-James orthogonality but there are many points which are not symmetric with respect to* ρ*-orthogonality. Note that* $(1,0) \in \mathbb{R}^2$ *is not a* ρ*-symmetric point. Indeed, take* $(-\frac{1}{3}, 1) \in \mathbb{R}^2$. *Then it is easy to see that* $(-\frac{1}{3}, 1) \perp_\rho (1,0)$ *whereas,* $(1,0) \not\perp_\rho (-\frac{1}{3}, 1)$. *So,* ρ*-orthogonality is not symmetric.*

While investigating Birkhoff-James orthogonality, the notions of x^+ and x^- were introduced in [91]. Motivated by these here we introduce the notions of $x^{\rho+}$ and $x^{\rho-}$ as follows:

Definition 5.4. *Let* $\mathbb{X}$ *be a normed linear space and let* $x, y \in \mathbb{X}$. *We say* $y \in x^{\rho+}$ *if* $\rho'(x,y) \geq 0$ *and* $y \in x^{\rho-}$ *if* $\rho'(x,y) \leq 0$.

We state the following proposition the proof of which is trivial.

Proposition 5.3. *Let* $\mathbb{X}$ *be a normed linear space and let* $x, y \in \mathbb{X}$. *Then the following relations hold true:*

(i) *Either* $y \in x^{\rho+}$ *or* $y \in x^{\rho-}$.

(ii) $x \perp_\rho y$ *if and only if* $y \in x^{\rho+}$ *and* $y \in x^{\rho-}$.

(iii) $y \in x^{\rho+}$ *implies that* $\alpha y \in (\beta x)^{\rho+}$ *for all* $\alpha, \beta > 0$.

(iv) $y \in x^{\rho+}$ *implies that* $-y \in x^{\rho-}$ *and* $y \in (-x)^{\rho-}$.

(v) $y \in x^{\rho-}$ *implies that* $\alpha y \in (\beta x)^{\rho-}$ *for all* $\alpha, \beta > 0$.

(vi) $y \in x^{\rho-}$ *implies that* $-y \in x^{\rho+}$ *and* $y \in (-x)^{\rho+}$.

With the help of the above notions, we obtain the characterization of ρ-left symmetric points.

Theorem 5.12. *Let* $\mathbb{X}$ *be any normed linear space and let* $x \in S_{\mathbb{X}}$. *Then* x *is* ρ*-left symmetric if and only if for any* $y \in S_{\mathbb{X}}$, *the following conditions hold true:*

(i) $y \in x^{\rho+}$ *implies* $x \in y^{\rho+}$

(ii) $y \in x^{\rho-}$ *implies* $x \in y^{\rho-}$.

Proof. Note that the sufficient part is easy. Indeed, let $x \perp_\rho y$. This implies that $y \in x^{\rho+} \cap x^{\rho-}$. From the hypothesis we have $x \in y^{\rho+} \cap y^{\rho-}$. Thus $y \perp_\rho x$.

To prove the necessary part we only show condition (i) as condition (ii) can be proved similarly. For this let $y \in x^{\rho+}$. This implies $\rho'(x,y) \geq 0$. If $\rho'(x,y) = 0$ then we have $\rho'(y,x) = 0$, since x is ρ-left symmetric. Thus in this case $x \in y^{\rho+}$. Next let us assume that $\rho'(x,y) > 0$. If $x = y$ then we are done. Also, note that $x \neq -y$. Thus we assume $x \neq \pm y$. Let $V = span\{x,y\}$ and let $z = y - \rho'(x,y)x \in V$. It is easy to observe that $\rho'(x,z) = 0$, i.e., $x \perp_\rho z$. Since x is ρ-left symmetric, it follows that $z \perp_\rho x$, i.e., $\rho'(z,x) = 0$. Since $y = z + \rho'(x,y)x$ and $\rho'(x,y) > 0$, it follows that the ray $[0,y\rangle$ lies in between the rays $[0,x\rangle$ and $[0,z\rangle$. Let $z' = \frac{z}{\|z\|}$. Then the ray $[0,y\rangle$ lies in between the rays $[0,x\rangle$ and $[0,z'\rangle$. Now applying Lemma 5.3 we obtain that for each $t > 0$,

$$\frac{\|z' + tx\| - 1}{t} \leq \frac{\|y + tx\| - 1}{t}.$$

Taking $t \to 0^+$ we get $\rho'_+(y,x) \geq \rho'_+(z',x)$. By using similar arguments we can show that $\rho'_-(y,x) \geq \rho'_-(z',x)$. Therefore, we have $\rho'(y,x) \geq \rho'(z',x)$. Since $\rho'(z',x) = 0$, it follows that $\rho'(y,x) \geq 0$. Therefore, (i) holds true. Hence the theorem.

□

We have already observed that the characterization of a ρ-left symmetric point holds analogously as given in [87, Th. 2.1]. It is now natural to presume that an analogous version of [87, Th. 2.2] also holds true in the case of ρ-right symmetric points. But in the following example we see that some of the ρ-right symmetric points behave otherwise.

Example 5.13. *Let us consider the space ℓ_∞^3. Suppose that $x = (1,1,\frac{1}{2}) \in S_{\ell_\infty^3}$. It is easy to observe that x is a ρ-right symmetric point. Let $y = (-\frac{1}{2},0,1) \in \ell_\infty^3$. Note that $\mathrm{Ext}\,(J(x)) = \{x_1^*, x_2^*\}$ and $\mathrm{Ext}\,(J(y)) = \{x_3^*\}$, where for each $i \in \{1,2,3\}$, $x_i^*(x) = x_i$, for all $x = (x_1,x_2,x_3) \in \ell_\infty^3$. Now applying Lemma 5.1 it is easy to obtain that $\rho'(y,x) = \frac{1}{2} > 0$ and $\rho'(x,y) = -\frac{1}{4} < 0$. This shows that $x \in y^{\rho+}$ but $y \in x^{\rho-}$.*

Remark 5.14. *Given any $x,y \in \mathbb{X}$, we say that $\perp_\rho$ has α-left (α-right) existence if there exists an $\alpha \in \mathbb{R}$ such that $\alpha x + y \perp_\rho x$ ($x \perp_\rho \alpha x + y$). Unlike Birkhoff-James orthogonality, ρ-orthogonality does not always have α-left existence. From the above example we can observe by a straightforward computation that there does not exist any $\alpha \in \mathbb{R}$ such that $\alpha x + y \perp_\rho x$. In other words, $\perp_\rho$ does not satisfy α-left existence at x. On the other hand, ρ-orthogonality always satisfies the α-right existence.*

Our next aim is to obtain a characterization of ρ-right symmetric points for which the α-left existence is guaranteed.

Theorem 5.15. *Suppose that $\mathbb{X}$ is a normed linear space and $x \in \mathbb{X}$ satisfies the α-left existence property. Then x is ρ-right symmetric if and only if for any $y \in S_{\mathbb{X}}$, the following conditions hold true:*

(i) $x \in y^{\rho+}$ *implies* $y \in x^{\rho+}$

(ii) $x \in y^{\rho-}$ *implies* $y \in x^{\rho-}$.

Proof. Since the sufficient part is easy to show, we only prove the necessary part. We prove Condition (i) as Condition (ii) can be proved similarly. Suppose on the contrary that $x \in y^{\rho+}$ but $y \notin x^{\rho+}$, for some $y \in S_{\mathbb{X}}$. This implies that $\rho'(y,x) \geq 0$ but $\rho'(x,y) < 0$. If $\rho'(y,x) = 0$ then by the ρ-right symmetricity of x we get $\rho'(x,y) = 0$. In that case we have nothing to prove. So, we consider that $\rho'(y,x) > 0$. If $x = y$ then we are done. Note that $x \neq -y$. Thus we assume $x \neq \pm y$. Let us consider the two-dimensional subspace $\mathbb{Y} = span\{x,y\}$. Since x has the α-left existence property, it follows that there exists a nonzero $\alpha \in \mathbb{R}$ such that $\rho'(\alpha x + y, x) = 0$. As x is ρ-right symmetric, we have $\rho'(x, \alpha x + y) = 0$. This implies that $\rho'(x,y) = -\frac{1}{\alpha}$. Since $\rho'(x,y) < 0$, we get $\alpha > 0$. Let us assume $\frac{\alpha x + y}{\|\alpha x + y\|} = w$. Then $y = \|\alpha x + y\| w - \alpha x$. Since $\alpha > 0$, it is easy to see that the ray $[0, w\rangle$ lies in between the rays $[0, x\rangle$ and $[0, y\rangle$. Now applying Lemma 5.3 and proceeding as in Theorem 5.12, we obtain the following:

- $\rho'_+(w,x) \geq \rho'_+(y,x)$ and
- $\rho'_-(w,x) \geq \rho'_-(y,x)$.

Combining these we get $\rho'(y,x) \leq \rho'(w,x) = 0$. This is a contradiction to the fact that $\rho'(y,x) > 0$. This completes the necessary part. □

Next, we give an example of ρ-right symmetric points satisfying α-left existence property.

Example 5.16. *Suppose that $x \in \operatorname{Ext}(B_{\ell_1^n})$. By an easy computation it can be observed that x is a ρ-right symmetric point of ℓ_1^n (also, see Theorem 5.18). Now one can check that given any $y = (y_1, y_2, \ldots, y_n) \in S_{\ell_1^n}$, there exists an $\alpha \in \mathbb{R}$ such that $\alpha x + y \perp_\rho x$. Indeed, if $x = (1, 0, \ldots, 0) \in \operatorname{Ext}(B_{\ell_1^n})$ then taking $\alpha = -y_1$ we obtain that $\alpha x + y \perp_\rho x$.*

Now we focus on the study of ρ-symmetric points (both left and right) in the classical ℓ_p^n spaces. Note that ℓ_p^n is a smooth Banach space whenever $1 < p < \infty$. Therefore, Birkhoff-James orthogonality coincides with ρ-orthogonality. Thus the characterization of ρ-left and ρ-right symmetric points in ℓ_p^n follow directly from [19]. So we only study the ρ-left and ρ-right symmetric points in ℓ_1^n and ℓ_∞^n. To do so we introduce the notations $\mathcal{Z}_x$ and $\mathcal{I}_x$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, where $\mathcal{Z}_x = \{i \in \{1, 2, \ldots, n\} : x_i = 0\}$ and $\mathcal{I}_x = \{i \in \{1, 2, \ldots, n\} :$

$|x_i| = 1\}$. Clearly, for any extreme point $x \in \ell_1^n$, $|\mathcal{Z}_x| = n-1$ and $|\mathcal{I}_x| = 1$. A point $x \in \ell_1^n$ is smooth if and only if $\mathcal{Z}_x = \emptyset$. For any extreme point $x \in \ell_\infty^n$, note that $|\mathcal{I}_x| = n$. Let us first characterize the ρ-orthogonal elements in ℓ_1^n and ℓ_∞^n.

Proposition 5.4. *Let $\mathbb{X} = \ell_p^n$, where $p = 1, \infty$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in S_{\mathbb{X}}$.*

(i) If $p = 1$, then $x \perp_\rho y$ if and only if $\sum_{i=1}^n sgn(x_i)y_i = 0$.

(ii) If $p = \infty$, then $x \perp_\rho y$ if and only if $\max_{i\in\mathcal{I}_x}\{sgn(x_i)y_i\} + \min_{i\in\mathcal{I}_x}\{sgn(x_i)y_i\} = 0$.

Proof. (i) Let $x \perp_\rho y$. Thus it can be easily observed that

$$\operatorname{Ext}(J(x)) = \Big\{u = (u_1, u_2, \ldots, u_n) \in \ell_\infty^n : u_k = sgn(x_k), \text{ when } k \notin \mathcal{Z}_x \text{ and } u_k \in \{\pm 1\}, \text{ when } k \in \mathcal{Z}_x\Big\}.$$

Since $x \perp_\rho y$, it follows that $\rho'(x,y) = 0$, i.e., $\rho'_+(x,y) = -\rho'_-(x,y)$. This implies by Lemma 5.1 that

$$\max\left\{\sum_{k=1}^n u_k y_k : u \in \operatorname{Ext}(J(x))\right\} = -\min\left\{\sum_{k=1}^n u_k y_k : u \in \operatorname{Ext}(J(x))\right\}$$
$$\implies \sum_{k\notin\mathcal{Z}_x} sgn(x_k)y_k + \sum_{k\in\mathcal{Z}_x}|y_k| = -\sum_{k\notin\mathcal{Z}_x} sgn(x_k)y_k + \sum_{k\in\mathcal{Z}_x}|y_k|.$$

Therefore, $\sum_{k\notin\mathcal{Z}_x} sgn(x_k)y_k = 0$. This means that $\sum_{k=1}^n sgn(x_k)y_k = 0$. The converse part is immediate using similar arguments as above. This completes the proof of (i).

(ii) Observe that for any $x \in S_{\ell_\infty^n}$,

$$\operatorname{Ext}(J(x)) = \big\{(0, 0, \ldots, sgn(x_i), 0, \ldots, 0) : i \in \mathcal{I}_x\big\}.$$

Since $x \perp_\rho y$, it follows from Lemma 5.1 that $\max_{i\in\mathcal{I}_x}\{sgn(x_i)y_i\} + \min_{i\in\mathcal{I}_x}\{sgn(x_i)y_i\} = 0$. This proves (ii). □

In the following theorem we give a complete description of the ρ-left symmetric points of ℓ_1^n.

Theorem 5.17. *Let $x = (x_1, x_2, \ldots, x_n) \in S_{\ell_1^n}$. Then x is ρ-left symmetric if and only if either of the following holds true:*

(i) $x \in \text{Ext}\,(B_{\ell_1^n})$.

(ii) $|x_i| = |x_j| = \frac{1}{2}$, *for some* $i, j \in \{1, 2, \ldots, n\}$ *and* $x_k = 0$, *otherwise.*

Proof. First we prove the sufficient part. Suppose that (i) holds true. Then $x_i = \pm 1$, for some $i \in \{1, 2, \ldots, n\}$ and $x_j = 0$, for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. Suppose that $x \perp_\rho y$, for some $y = (y_1, y_2, \ldots, y_n) \in S_{\ell_1^n}$. Then from Proposition 5.4 we obtain that $y_i = 0$. Therefore, $\sum_{k=1}^{n} sgn(y_k)x_k = 0$. Using Proposition 5.4 again, we obtain that $y \perp_\rho x$. Thus x is ρ-left symmetric. Now suppose that (ii) holds true. Also, assume that $x \perp_\rho y$, for some $y = (y_1, y_2, \ldots, y_n) \in S_{\ell_1^n}$. Then from Proposition 5.4 we observe that

$$\begin{aligned} sgn(x_i)y_i + sgn(x_j)y_j = 0 \quad &\implies \quad sgn(x_i)sgn(y_i) + sgn(x_j)sgn(y_j) = 0 \\ &\implies \quad sgn(y_i)sgn(x_i)|x_i| + sgn(y_j)sgn(x_j)|x_j| = 0 \\ &\implies \quad \sum_{k=1}^{n} sgn(y_k)x_k = 0. \end{aligned}$$

This proves that $y \perp_\rho x$. Thus the proof of the sufficient part is done.

Next we prove the necessary part. Suppose on the contrary that $x = (x_1, x_2, \ldots, x_n) \in S_{\ell_1^n}$ does not satisfy (i) and (ii). Then clearly, $|\mathcal{Z}_x^c| \geq 2$. Suppose that there exist $i, j \in \mathcal{Z}_x^c$ such that $|x_i| \neq |x_j|$. Let $y = (y_1, y_2, \ldots, y_n) \in \ell_1^n$ be such that $|y_i| = |y_j|$ with $sgn(y_i) = sgn(x_i)$ and $sgn(y_j) = -sgn(x_j)$ and $y_k = 0$, for all $k \in \{1, 2, \ldots, n\} \setminus \{i, j\}$. Then one can observe from Proposition 5.4 that $x \perp_\rho y$ whereas, $y \not\perp_\rho x$. Thus x is not ρ-left symmetric. Now suppose that for all $j, k \in \mathcal{Z}_x^c$, $|x_j| = |x_k|$. It is trivial to see that $|\mathcal{Z}_x^c| > 2$, otherwise (i) or (ii) will be satisfied. Without loss of generality assume that $x_j > 0$, for all $j \in \mathcal{Z}_x^c$. Suppose that $|\mathcal{Z}_x^c| = r$. Take $y = (y_1, y_2, \ldots, y_n) \in \ell_1^n$ such that $y_{k_0} = 1 - r$, for some $k_0 \in \mathcal{Z}_x^c$ and $y_j = 1$, for all $j \in \{1, 2, \ldots, n\} \setminus \{k_0\}$. Note that $\sum_{j=1}^{n} sgn(x_j)y_j = \sum_{i \in \mathcal{Z}_x^c} y_j = 0$. Thus $x \perp_\rho y$. On the other hand, we can see that $sgn(y_{k_0}) = -1$ and $sgn(y_j) = +1$, for all $j \in \{1, 2, \ldots, n\} \setminus \{k_0\}$. Since $|\mathcal{Z}_x^c| > 2$ and $|x_j|$ are equal for all $j \in \mathcal{Z}_x^c$, it follows that $\sum_{j=1}^{n} sgn(y_j)x_j \neq 0$. This gives us $y \not\perp_\rho x$, which contradicts the fact that x is ρ-left symmetric. This completes the proof of the necessary part.

□

Next we characterize the ρ-right symmetric points in ℓ_1^n.

Theorem 5.18. *Let* $x = (x_1, x_2, \ldots, x_n) \in S_{\ell_1^n}$. *Then* x *is* ρ*-right symmetric if and only if either of the following conditions holds true:*

(i) $x \in \text{Ext}\,(B_{\ell_1^n})$.

(ii) For any two nonempty disjoint sets $A, B \subset \mathcal{Z}_x^c$, $|\sum_{j\in A} x_j| \neq |\sum_{j\in B} x_j|$.

Proof. We first prove the sufficient part. Suppose that (i) holds true, i.e., $x \in \text{Ext}\,(B_{\ell_1^n})$. Then $x_i = \pm 1$, for some $1 \leq i \leq n$ and $x_j = 0$, for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. Suppose that $y \perp_\rho x$, where $y = (y_1, y_2, \ldots, y_n) \in S_{\ell_1^n}$. Then from Proposition 5.4 it is easy to see that $y_i = 0$. Since $x_j = 0$, for all $j \neq i$, it follows that $\sum_{j=1}^n sgn(x_j)y_j = 0$. Thus again from Proposition 5.4 we obtain that $x \perp_\rho y$. This proves that x is ρ-right symmetric. Now suppose that (ii) holds true. We claim that if $y \perp_\rho x$, then we have $y_i = 0$, for all $i \in \mathcal{Z}_x^c$. If possible, let $y_k \neq 0$, for some $k \in \mathcal{Z}_x^c$. Let us consider the two sets A, B as:

$$A_1 = \{j \in \mathcal{Z}_x^c : sgn(y_j) = +1\}, A_2 = \{j \in \mathcal{Z}_x^c : sgn(y_j) = -1\}.$$

Since $y_k \neq 0$, for some $k \in \mathcal{Z}_x^c$, it follows that $A_1 \cup A_2 \neq \emptyset$. Since $y \perp_\rho x$, by Proposition 5.4, $\sum_{j=1}^n sgn(y_j)x_j = 0$. Note that whenever $|A_1 \cup A_2| = 1$, we have $x_k = 0$, where $k \in \mathcal{Z}_x^c$. On the other hand, suppose that $|A_1 \cup A_2| \geq 2$. Then clearly, we obtain two sets $A, B \subset \mathcal{Z}_x^c$ such that $|\sum_{j\in A} x_j| = |\sum_{j\in B} x_j|$. Both of these cases are not possible according to our assumption. So our claim is established. Since $y_j = 0$, for all $j \in \mathcal{Z}_x^c$, it is easy to see from Proposition 5.4 that $x \perp_\rho y$, i.e., x is ρ-right symmetric.

Now we prove the necessary part. Since $x \in S_{\ell_1^n}$, we have $\mathcal{Z}_x^c \neq \emptyset$. If $|\mathcal{Z}_x^c| = 1$ then we have $x \in \text{Ext}\,(B_{\ell_1^n})$, i.e., (i) holds true. Now let $|\mathcal{Z}_x^c| \geq 2$. Suppose on the contrary that there exist two nonempty disjoint subsets A and B of $\mathcal{Z}_x^c$ such that $|\sum_{j\in A} x_j| = |\sum_{j\in B} x_j|$. Without loss of generality assume that $\sum_{j\in A} x_j = \sum_{j\in B} x_j$. Then choose $y = (y_1, y_2, \ldots, y_n) \in \ell_1^n$ such that

$$\begin{aligned} y_j &= 10^j,\ j \in A \\ &= -\frac{1}{10^j},\ j \in B \\ &= 0, \qquad j \in \{1, 2, \ldots, n\} \setminus (A \cup B). \end{aligned}$$

Note that $\sum_{j=1}^n sgn(y_j)x_j = \sum_{j\in A\cup B} sgn(y_j)x_j = 0$. Therefore, from Proposition 5.4 we obtain that $y \perp_\rho x$. But one can observe from the construction of y that $\sum_{j=1}^n sgn(x_j)y_j = \sum_{A\cup B} sgn(x_j)y_j \neq 0$. This shows that $x \not\perp_\rho y$. Thus we arrive at a contradiction to the fact that x is ρ-right symmetric. This completes the proof of the theorem. □

Combining Theorem 5.17 and Theorem 5.18 we note the following:

Theorem 5.19. *Let $x \in \ell_1^n$. Then x is a ρ-symmetric if and only if x is an extreme point of ℓ_1^n.*

In the following example we describe some ρ-left and ρ-right symmetric points of ℓ_1^n other than the extreme points.

Example 5.20. *Suppose that* $\mathbb{X} = \ell_1^4$ *and let us consider three points* $x_1, x_2, x_3 \in \mathbb{X}$ *such that* $x_1 = (\frac{1}{2}, 0, 0, -\frac{1}{2}), x_2 = (\frac{1}{2}, \frac{1}{3}, 0, -\frac{1}{4})$ *and* $x_3 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$*. From Theorem 5.17 it is easy to see that* x_1 *is* ρ*-left symmetric whereas, by Theorem 5.18,* x_2 *is* ρ*-right symmetric. On the other hand,* x_3 *is neither* ρ*-left nor* ρ*-right symmetric in* ℓ_1^4*.*

In the following two theorems we characterize the ρ-left and ρ-right symmetric points in ℓ_∞^n, respectively.

Theorem 5.21. *Let* $x = (x_1, x_2, \ldots, x_n) \in S_{\ell_\infty^n}$*. Then* x *is* ρ*-left symmetric if and only if* $x_j = 0$*, for all* $j \notin \mathcal{I}_x$*.*

Proof. We first prove the sufficient part. Since $\|x\| = 1$, we have $\mathcal{I}_x \neq \emptyset$. Let $|\mathcal{I}_x| = 1$. Then $x_i = \pm 1$, for some $i \in \{1, 2, \ldots, n\}$. If $x \perp_\rho y$, for some $y \in S_{\ell_\infty^n}$, then from Proposition 5.4, we get $y_i = 0$. Therefore, $\mathcal{I}_y \subset \{1, 2, \ldots, n\} \setminus \{i\}$. Since $x_j = 0$, for all $j \in \{1, 2, \ldots, n\} \setminus \{i\}$, it follows from Proposition 5.4 that $y \perp_\rho x$. Therefore, x is ρ-left symmetric. Suppose that $|\mathcal{I}_x| \geq 2$. By Proposition 5.4 $x \perp_\rho y$ implies that

$$\max_{j \in \mathcal{I}_x} \{sgn(x_j) y_j\} + \min_{j \in \mathcal{I}_x} \{sgn(x_j) y_j\} = 0. \tag{5.3}$$

Suppose that $\max_{j \in \mathcal{I}_x} \{sgn(x_j) y_j\} = sgn(x_k) y_k$ and $\min_{j \in \mathcal{I}_x} \{sgn(x_j) y_j\} = sgn(x_l) y_l$, for some $k, l \in \mathcal{I}_x$. Then from Equation (3), it is clear that $|y_k| = |y_l|$. Now either of the following holds:

(a) $|y_k| = |y_l| = 1$, for some $k, l \in \mathcal{I}_x$.

(b) $|y_k| = |y_l| < 1$, for all $k, l \in \mathcal{I}_x$.

If (a) holds, then $k, l \in \mathcal{I}_y$ and consequently, $sgn(y_k) x_k + sgn(y_l) x_l = \max_{j \in \mathcal{I}_y} \{sgn(y_j) x_j\} + \min_{j \in \mathcal{I}_y} \{sgn(y_j) x_j\} = 0$. Thus by Proposition 5.4, we get $y \perp_\rho x$. This implies x is ρ-left symmetric. If (b) holds, then $\mathcal{I}_y \cap \mathcal{I}_x = \emptyset$. From our hypothesis observe that $sgn(y_i) x_i = 0$, for all $i \in \mathcal{I}_y$. Therefore, $y \perp_\rho x$. This also shows that x is ρ-left symmetric.

To show the necessary part suppose on the contrary that there exists $j \in \{1, 2, \ldots, n\}$ such that $0 < |x_j| < 1$. Then we take $y = (y_1, y_2, \ldots, y_n)$ such that $y_j = 1$ and $y_i = 0$, for all $i \in \{1, 2, \ldots, n\} \setminus \{j\}$. Note that $\mathcal{I}_x \cap \mathcal{I}_y = \emptyset$. Therefore, $\max_{i \in \mathcal{I}_x} \{sgn(x_i) y_i\} = 0 = \min_{i \in \mathcal{I}_x} \{sgn(x_i) y_i\}$. Using Proposition 5.4, we have $x \perp_\rho y$. On the other hand, observe that $\mathcal{I}_y = \{j\}$ and therefore, $\max_{i \in \mathcal{I}_y} \{sgn(y_i) x_i\} = \min_{i \in \mathcal{I}_y} \{sgn(y_i) x_i\} = x_j \neq 0$. Thus we get $y \not\perp_\rho x$, which contradicts that x is ρ-left symmetric. This completes the proof of the theorem. □

Theorem 5.22. *Let $x = (x_1, x_2, \ldots, x_n) \in S_{\ell_\infty^n}$. Then x is ρ-right symmetric if and only if either of the following holds true:*

(i) $x \in \text{Ext}\,(B_{\ell_\infty^n})$

(ii) for each $j \in \{1, 2, \ldots, n\} \setminus \mathcal{I}_x$, $0 < |x_j| < 1$. Moreover, $|x_j| \neq |x_k|$, for all $j, k \in \{1, 2, \ldots, n\} \setminus \mathcal{I}_x$.

Proof. To prove the sufficient part first assume that (i) holds true. Since ρ-orthogonality is preserved under the signed permutation map [96], we may without loss of generality assume that $x = (1, 1, \ldots, 1)$. Suppose that $y \perp_\rho x$, for some $y = (y_1, y_2, \ldots, y_n) \in S_{\ell_\infty^n}$. From Proposition 5.4, we observe that there exist $i, j \in \{1, 2, \ldots, n\}$ such that $y_i = 1$ and $y_j = -1$. Therefore, $\max_{i \in \mathcal{I}_x}\{sgn(x_i)y_i\} = 1$ and $\min_{i \in \mathcal{I}_x}\{sgn(x_i)y_i\} = -1$. From Proposition 5.4, we get that $x \perp_\rho y$. Thus x is ρ-right symmetric. Now suppose that (ii) holds true and $y \perp_\rho x$, for some $y \in S_{\ell_\infty^n}$. Clearly, $|\mathcal{I}_x| \leq n-1$. If $|\mathcal{I}_x| = 1$ then using Proposition 5.4 one can see that there does not exist any nonzero $y \in \ell_\infty^n$ such that $y \perp_\rho x$. Thus x is ρ-right symmetric, vacuously. Let $|\mathcal{I}_x| \geq 2$. As $y \perp_\rho x$, from Proposition 5.4, we get $\max_{i \in \mathcal{I}_y}\{sgn(y_i)x_i\} + \min_{i \in \mathcal{I}_y}\{sgn(y_i)x_i\} = 0$. This implies that $|x_j| = |x_k|$, for some $j, k \in \mathcal{I}_y$. Therefore, from the hypothesis we note that $\mathcal{I}_x \cap \mathcal{I}_y \neq \emptyset$. This implies that there exist $j, k \in \mathcal{I}_x \cap \mathcal{I}_y$ such that $sgn(x_j)y_j = 1$ and $sgn(x_k)y_k = -1$. This shows by Proposition 5.4 that $x \perp_\rho y$. Therefore, x is ρ-right symmetric.

To show the necessary part, first suppose on the contrary that $x_j = 0$, for some $i \in \{1, 2, \ldots, n\}$. Then we choose $y = (y_1, y_2, \ldots, y_n)$ such that $y_j = 1$ and $y_k = \frac{1}{10^k}$, for all $k \in \{1, 2, \ldots, n\} \setminus \{i\}$. One can clearly observe that $y \perp_\rho x$, whereas $x \not\perp_\rho y$. This contradicts that x is ρ-right symmetric. Now again we assume on the contrary that $0 < |x_j| = |x_k| < 1$, for some $j, k \in \{1, 2, \ldots, n\}$. Then we take $y \in S_{\ell_\infty^n}$ such that $y_j = sgn(x_j)$ and $y_k = -sgn(x_k)$ and $y_i = \frac{1}{10^i}$, for all $i \in \{1, 2, \ldots, n\} \setminus \{j, k\}$. Then applying Proposition 5.4, we have $y \perp_\rho x$ but $x \not\perp_\rho y$. This contradiction completes the proof of the necessary part. $\square$

Combining Theorem 5.21 and Theorem 5.22 we note that the extreme points are the only ρ-symmetric points on the unit sphere of ℓ_∞^n.

Theorem 5.23. *Let $x \in S_{\ell_\infty^n}$. Then x is a ρ-symmetric point if and only if x is an extreme point of $B_{\ell_\infty^n}$.*

We end this chapter with examples of ρ-left and ρ-right symmetric points in ℓ_∞^n, which are not extreme points.

Example 5.24. *Let $x_1 = (1, 1, 0, 0, -1)$; $x_2 = (1, \frac{1}{2}, \frac{1}{5}, -1, \frac{2}{3})$ and $x_3 = (1, -\frac{1}{3}, 1, \frac{1}{3}, \frac{1}{7})$ be three points in ℓ_∞^5. By Theorem 5.21 we observe that x_1 is a ρ-left symmetric point and from Theorem*

5.22 we get that x_2 is a ρ-right symmetric points. On the other hand, it is easy to see that x_3 is neither ρ-left symmetric nor ρ-right symmetric.

CHAPTER 6

SMOOTHNESS AND APPROXIMATE SMOOTHNESS IN BANACH SPACES AND IN THE SPACES OF BOUNDED LINEAR OPERATORS

6.1 Introduction

The present chapter deals with some new study on smoothness — an important feature considered in geometry of normed linear spaces. We develop also the concept of approximate smoothness (see [21]) in the spaces of bounded linear operators.

Letter(s) $\mathbb{X}$ (or $\mathbb{Y}$) will stand for a normed linear space over the field $\mathbb{K}$ of real or complex numbers. In some parts of the chapter (in particular in the whole Section 6.4) we confine to real spaces (i.e., $\mathbb{K} = \mathbb{R}$). We also generally assume that $\dim(\mathbb{X}) \geq 2$. By $\mathbb{X}^*$ we denote the

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- J. Chmieliński, **S. Ghosh**, K. Paul, and D. Sain, *Smoothness and approximate smoothness in normed linear spaces and operator spaces*, Ann. Funct. Anal., **16** (2025), no. 23, 1-24. https://doi.org/10.1007/s43034-025-00413-9.

dual of $\mathbb{X}$. We follow common notations: $B_{\mathbb{X}}$ and $S_{\mathbb{X}}$ stand for the closed unit ball and the unit sphere in $\mathbb{X}$, respectively. For $x \in \mathbb{X} \setminus \{\theta\}$, a functional $x^* \in S_{\mathbb{X}^*}$ is called a *supporting functional at* x if $x^*(x) = \|x\|$. The collection of all such functionals will be denoted by $J(x)$. It is always a nonempty, convex and weak*-compact subset of $S_{\mathbb{X}^*}$; $J(x)$ is also an extreme set of $B_{\mathbb{X}^*}$ and thus if $J(x) = \{x^*\}$, then $x^* \in \operatorname{Ext}(B_{\mathbb{X}^*})$ and in this case x is called a smooth point of $\mathbb{X}$. We denote the $\operatorname{Sm}\mathbb{X}$ as the collection of all smooth points in $\mathbb{X}$. For a subset C of $\mathbb{X}$, $\operatorname{diam}(C)$ denotes the 'diameter' of the set C, i.e., $\operatorname{diam}(C) = \sup\{\|x - y\| : x, y \in C\}$. We call a real finite-dimensional Banach space $\mathbb{X}$ a *polyhedral* space if the set $\operatorname{Ext}(B_{\mathbb{X}})$ is finite. $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ denotes the space of all linear and bounded operators from $\mathbb{X}$ to $\mathbb{Y}$ and $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ its subspace consisting of all compact operators (if $\mathbb{X} = \mathbb{Y}$, we write $\mathbb{L}(\mathbb{X})$ and $\mathbb{K}(\mathbb{X})$). For an operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, $M_T := \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$ denotes its (possibly empty) *norm attainment set* (in particular, M_{x^*} denotes the norm attainment set for a functional $x^* \in \mathbb{X}^*$).

6.2 Preliminaries

Let us introduce now some more advanced notions and results which will be used frequently.

First, we recall the notion of one-sided norm derivatives. For $x, y \in \mathbb{X}$ they are defined as:

$$\rho'_{\pm}(x, y) := \|x\| \lim_{\lambda \to 0^{\pm}} \frac{\|x + \lambda y\| - \|x\|}{\lambda} = \lim_{\lambda \to 0^{\pm}} \frac{\|x + \lambda y\|^2 - \|x\|^2}{2\lambda}.$$

For basic properties and applications of the above functionals we refer, e.g., to [8]. In particular, norm derivatives are used to characterize smoothness and approximate smoothness, as will be shown later.

Following the notation in [38], $\mathbb{X}$ has the *Kadets-Klee property* (KK for short) if the families of weak convergent and norm convergent sequences coincide on $S_{\mathbb{X}}$. Accordingly, $\mathbb{X}^*$ is said to have the *dual Kadets-Klee property* (KK*) if the families of w^*-convergent and norm convergent sequences coincide on $S_{\mathbb{X}^*}$. Moreover, we say $\mathbb{X}^*$ has the dual Kadets property (K*) if the weak* and norm topologies are equal on $S_{\mathbb{X}^*}$. Recently, in [37] it was shown that the properties K* and KK* are equivalent.

In an arbitrary normed linear space $\mathbb{X}$ the Birkhoff-James orthogonality relation $\perp_B$ can be defined (cf. [16]). Namely, for $x, y \in \mathbb{X}$:

$$x \perp_B y \iff \|x + \lambda y\| \geq \|x\| \quad \text{for all } \lambda \in \mathbb{K}.$$

Equivalently, $x \perp_B y$ if and only if there exists $x^* \in J(x)$ such that $x^*(y) = 0$. For a detailed study on connections of the Birkhoff-James orthogonality with the geometry of Banach spaces, readers can go through [64].

Given normed linear spaces $\mathbb{X}$ and $\mathbb{Y}$, a vector $x \in \mathbb{X}$ and a functional $y^* \in \mathbb{Y}^*$, by $y^* \otimes x$ we mean a functional on $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ given by

$$(y^* \otimes x)(T) := y^*(Tx), \qquad T \in \mathbb{L}(\mathbb{X}, \mathbb{Y}).$$

Analogously, with $x^{**} \in \mathbb{X}^{**}$ and $y^* \in \mathbb{Y}^*$ we define $x^{**} \otimes y^*$ by

$$(x^{**} \otimes y^*)(T) := x^{**}(T^* y^*), \qquad T \in \mathbb{L}(\mathbb{X}, \mathbb{Y}).$$

We will use the fact that the extremal points of the unit ball in the dual of $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ can be expressed in terms of extremal points of duals of $\mathbb{X}$ and $\mathbb{Y}$, as proved by Ruess and Stegall [80] and Lima and Olsen [58] (see also [39, Th. VI.1.3]).

Theorem 6.1 ([58, Th. 1]). *Given two Banach spaces $\mathbb{X}$ and $\mathbb{Y}$,*

$$\operatorname{Ext}(B_{\mathbb{K}(\mathbb{X},\mathbb{Y})^*}) = \{x^{**} \otimes y^* : \ x^{**} \in \operatorname{Ext}(B_{\mathbb{X}^{**}}), \ y^* \in \operatorname{Ext}(B_{\mathbb{Y}^*})\}.$$

In particular, if $\mathbb{X}$ is reflexive, then

$$\operatorname{Ext}(B_{\mathbb{K}(\mathbb{X},\mathbb{Y})^*}) = \{y^* \otimes x : \ x \in \operatorname{Ext}(B_{\mathbb{X}}), \ y^* \in \operatorname{Ext}(B_{\mathbb{Y}^*})\}.$$

For a subspace $\mathbb{V}$ of $\mathbb{X}$ we define $\mathbb{V}^{\perp} := \{x^* \in \mathbb{X}^* : \ \mathbb{V} \subset \ker x^*\}$. A closed subspace $\mathbb{V}$ is said to be an *M-ideal* in $\mathbb{X}$ if $\mathbb{X}^* = \mathbb{V}^* \oplus_1 \mathbb{V}^{\perp}$ and for any decomposition $x^* = x_1^* + x_2^*$, $x^* \in \mathbb{X}^*$, $x_1^* \in \mathbb{V}^*$, $x_2^* \in \mathbb{V}^{\perp}$ there is $\|x^*\| = \|x_1^*\| + \|x_2^*\|$. Some results in this paper will be obtained under the assumption that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is an M-ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$, applying in particular the following results from [95]. For further results on M-ideals of compact operators see, e.g., [39, 51] and the references therein.

Theorem 6.2 ([95, Lemma 3.1]). *Let $\mathbb{X}$ be a reflexive Banach space and let $\mathbb{Y}$ be any normed linear space. Assume that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is an M-ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Suppose that $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is such that* $\operatorname{dist}(T, \mathbb{K}(\mathbb{X}, \mathbb{Y})) < \|T\|$. *Then $M_T \cap \operatorname{Ext}(B_{\mathbb{X}}) \neq \emptyset$ and*

$$\operatorname{Ext}(J(T)) = \left\{y^* \otimes x \in \mathbb{K}(\mathbb{X}, \mathbb{Y})^* : \ x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}}), \ y^* \in \operatorname{Ext}(J(Tx))\right\}. \tag{6.1}$$

In the next result we restrict to real spaces.

Theorem 6.3 ([95, Th. 3.2 and Th. 3.3]). *Let $\mathbb{X}$ be a real reflexive Banach space and let $\mathbb{Y}$ be any real normed linear space. Assume that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is an M-ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$, $T, S \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$,*

$\|T\| = 1$ *and* $\operatorname{dist}(T, \mathbb{K}(\mathbb{X}, \mathbb{Y})) < 1$. *Then*

$$\rho'_+(T, S) = \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} \rho'_+(Tx, Sx); \tag{6.2}$$

$$\rho'_-(T, S) = \inf_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} \rho'_-(Tx, Sx). \tag{6.3}$$

Moreover, if $\mathbb{Y}$ *is smooth, we have*

$$\rho'_-(T, S) = \inf_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} \rho'_+(Tx, Sx). \tag{6.4}$$

6.3 Smoothness

This section is devoted to the notion of smoothness and its new characterizations.

6.3.1 Family of supporting functionals

The family $J(x)$ of all supporting functionals at x is always nonempty and may consist of a single element or may be a larger set. The size of $J(x)$ can be measured using a mapping $d\colon \mathbb{X} \to [0, 2]$:

$$d(x) := \begin{cases} \operatorname{diam}(J(x)) & \text{for } x \neq \theta; \\ 0 & \text{for } x = \theta. \end{cases}$$

We introduce also the notion

$$d(\mathbb{X}) := \sup_{x \in \mathbb{X}} d(x) = \sup_{x \in S_{\mathbb{X}}} d(x).$$

Mapping d will be used to describe smoothness and approximate smoothness. Obviously, $x \in \mathbb{X} \setminus \{\theta\}$ is a smooth point if and only if $d(x) = 0$ and $\mathbb{X}$ is smooth if and only if $d \equiv 0$, i.e., if and only if $d(\mathbb{X}) = 0$. We can express $d(x)$ using norm derivatives.

Lemma 6.1 ([21, Lemma 2.2])**.** *Let* $\mathbb{X}$ *be a normed linear space. Then*

$$d(x)\|x\| = \sup_{y \in S_X} \{\rho'_+(x, y) - \rho'_-(x, y)\}, \qquad x \in \mathbb{X}.$$

Actually, to measure the size of $J(x)$ it suffices to consider its extremal points. The result stated below will be frequently used.

Proposition 6.1. *Let $\mathbb{X}$ be a normed linear space and let $x \in \mathbb{X} \setminus \{\theta\}$. Then*

$$d(x) = \operatorname{diam}(J(x)) = \operatorname{diam}(\operatorname{Ext}(J(x))).$$

Proof. Since $J(x)$ is a weak*-compact and convex subset of $\mathbb{X}^*$, it follows from the Krein-Milman Theorem that $J(x) = \overline{\operatorname{conv}\{\operatorname{Ext}(J(x))\}}^{w^*}$. Let $u^*, v^* \in J(x)$ be arbitrary supporting functionals. We prove our result in two steps.

Step I: Suppose that $u^*, v^* \in \operatorname{conv}\{\operatorname{Ext}(J(x))\}$. Then we can write $u^* = \sum_{i=1}^k t_i x_i^*$ and $v^* = \sum_{j=1}^r s_j y_j^*$, where $x_i^*, y_j^* \in \operatorname{Ext}(J(x))$ and $t_i, s_j \geq 0$ for each $1 \leq i \leq k$, $1 \leq j \leq r$ with $\sum_{i=1}^k t_i = \sum_{j=1}^r s_i = 1$. Now

$$\begin{aligned} \|u^* - v^*\| &= \left\| \sum_{i=1}^k t_i x_i^* - \sum_{j=1}^r s_j y_j^* \right\| = \left\| \sum_{i=1}^k t_i x_i^* - \sum_{i=1}^k t_i \left(\sum_{j=1}^r s_j y_j^* \right) \right\| \\ &= \left\| \sum_{i=1}^k t_i \left(x_i^* - \sum_{j=1}^r s_j y_j^* \right) \right\| \leq \sum_{i=1}^k t_i \left\| x_i^* - \sum_{j=1}^r s_j y_j^* \right\| \\ &\leq \max_{1 \leq i \leq k} \left\| x_i^* - \sum_{j=1}^r s_j y_j^* \right\|. \end{aligned}$$

Applying similar technique as above we note that for each $i \in \{1, 2, \ldots, k\}$

$$\|x_i^* - \sum_{j=1}^r s_j y_j^*\| = \|\sum_{j=1}^r s_j x_i^* - \sum_{j=1}^r s_j y_j^*\| \leq \max_{1 \leq j \leq r} \|x_i^* - y_j^*\|.$$

This shows that

$$\|u^* - v^*\| \leq \sup\{\|x^* - y^*\| : x^*, y^* \in \operatorname{Ext}(J(x))\} = \operatorname{diam}(\operatorname{Ext}(J(x))).$$

Step II: Now suppose that $u^*, v^* \in \overline{\operatorname{conv}\{\operatorname{Ext}(J(x))\}}^{w^*}$, i.e., there exist two nets

$$\{u_\alpha^*\}_{\alpha \in J_1},\ \{v_\beta^*\}_{\beta \in J_2} \subset \operatorname{conv}\{\operatorname{Ext}(J(x))\}$$

(where J_1 and J_2 are two directed sets) such that $u_\alpha^* \xrightarrow{w^*} u^*$ and $v_\beta^* \xrightarrow{w^*} v^*$. Applying the technique given in [61, Technique 2.1.32] we observe that there exists a directed set K such that $\{u_{g(\gamma)}^*\}_{\gamma \in K}$ and $\{v_{h(\gamma)}^*\}_{\gamma \in K}$ are subnets of $\{u_\alpha^*\}_{\alpha \in J_1}$ and $\{v_\beta^*\}_{\beta \in J_2}$, respectively, where $g : K \to J_1$ and $h : K \to J_2$ are monotone mappings such that $g(K)$ and $h(K)$ are cofinal in J_1 and J_2. Note that $\{u_{g(\gamma)}^*\} \xrightarrow{w^*} u^*$ and $\{v_{h(\gamma)}^*\} \xrightarrow{w^*} v^*$. This implies $u_{g(\gamma)}^*(z) \longrightarrow u^*(z)$ and $v_{h(\gamma)}^*(z) \longrightarrow v^*(z)$, for all $z \in \mathbb{X}$. From Step I, for any $\alpha \in J_1$ and $\beta \in J_2$ we have

$\|u^*_\alpha - v^*_\beta\| \leq \operatorname{diam}(\operatorname{Ext}(J(x)))$. Therefore, we have $\|u^*_{g(\gamma)} - v^*_{h(\gamma)}\| \leq \operatorname{diam}(\operatorname{Ext}(J(x))$ for all $\gamma \in K$. This implies that for any $z \in S_{\mathbb{X}}$,

$$|u^*_{g(\gamma)}(z) - v^*_{h(\gamma)}(z)| \leq \operatorname{diam}(\operatorname{Ext}(J(x))),$$

hence

$$\lim_{\gamma \in K} |u^*_{g(\gamma)}(z) - v^*_{h(\gamma)}(z)| \leq \operatorname{diam}(\operatorname{Ext}(J(x)))$$

and

$$|u^*(z) - v^*(z)| \leq \operatorname{diam}(\operatorname{Ext}(J(x)).$$

Since $z \in S_{\mathbb{X}}$ has been chosen arbitrarily, $\|u^* - v^*\| \leq \operatorname{diam}(\operatorname{Ext}(J(x))$. Thus we conclude:

$$d(x) = \sup\{\|u^* - v^*\| : u^*, v^* \in J(x)\} \leq \operatorname{diam}(\operatorname{Ext}(J(x))) \leq \operatorname{diam}(J(x)) = d(x).$$

Therefore, $d(x) = \operatorname{diam}(J(x)) = \operatorname{diam}(\operatorname{Ext}(J(x)))$, as we wanted to demonstrate. □

Remark 6.4. *Note that compactness and Krein-Milman theorem play essential role in the above result. The equality* $\operatorname{diam}(D) = \operatorname{diam}(\operatorname{Ext}(D))$ *does not hold in general for an arbitrary closed convex set D. Indeed, for the closed unit ball B_{c_0} in c_0 (the Banach space of all sequences converging to zero) we have* $\operatorname{Ext}(B_{c_0}) = \emptyset$ *whence* $\operatorname{diam}(\operatorname{Ext}(B_{c_0}) = 0$, *whereas* $\operatorname{diam}(B_{c_0}) = 2$. *Next, consider the set* $D = \overline{\operatorname{conv}\{B_{c_0}, x_1, x_2\}} \subset \ell_\infty$ *where* $x_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots), x_2 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \ldots)$. *Then D is a closed convex subset of ℓ_∞ with* $\operatorname{Ext}(D) = \{x_1, x_2\}$ *and therefore* $\operatorname{diam}(\operatorname{Ext}(D)) = \|x_1 - x_2\| = 1 < 2 = \operatorname{diam}(D)$.

6.3.2 Smoothness vs. continuity of the mapping d

We seek for relations between continuity of d and smoothness. Let us note the following definition from [28].

Definition 6.1. *A Banach space $\mathbb{X}$ is said to be weak Asplund if every real valued continuous convex function on $\mathbb{X}$ is Gâteaux differentiable at the points of a dense G_δ subset of $\mathbb{X}$.*

Proposition 6.2. *Let $\mathbb{X}$ be a weak Asplund space. If the mapping d is continuous at $x_0 \in \mathbb{X}\backslash\{\theta\}$ then x_0 is a smooth element.*

Proof. Observe that $\operatorname{Sm}\mathbb{X}$ is precisely the set of all points of $\mathbb{X}$ at which the norm function is Gâteaux differentiable. Since $\mathbb{X}$ is a weak Asplund space, it follows that $\operatorname{Sm}\mathbb{X}$ is dense in $\mathbb{X}$. Consider then a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \operatorname{Sm}\mathbb{X}$ such that $x_n \to x_0$ as $n \to \infty$. By continuity assumption $d(x_0) = \lim_{n\to\infty} d(x_n) = 0$ which gives smoothness of x_0. □

Following Mazur's Theorem (cf. [29, Th. 8.2]) we note that every separable Banach space is also a weak Asplund space. Therefore, the corollary is immediate.

Corollary 6.1. *Let $\mathbb{X}$ be a separable Banach space. If the map d is continuous at $x \in \mathbb{X} \setminus \{0\}$, then x is smooth in $\mathbb{X}$.*

It follows that in a weak Asplund Banach space continuity of d on $\mathbb{X}$ yields smoothness of $\mathbb{X}$ (the reverse is obvious in any space). Observe that weak Asplund property is essential in Proposition 6.2, as shown below.

Example 6.5. *Let us consider the Banach space $\ell_1(\Lambda)$ of all summable real-valued sequences over an uncountable set Λ. From [73, Example 1.4(b)] we note that all the points in this space are non-smooth. This implies $\ell_1(\Lambda)$ is not weak Asplund space. Take $x = (x_\alpha)_{\alpha\in\Lambda} \in \ell_1(\Lambda)$. Thus in particular $x_\alpha = 0$, for all $\alpha \in \Lambda \setminus \mathcal{N}$, where $\mathcal{N}$ is a countable set. Moreover, consider functionals $x^* = (x^*_\alpha)_{\alpha\in\Lambda} \in \ell_\infty(\Lambda) = \ell_1(\Lambda)^*$ of the form*

$$x^*_\alpha = \begin{cases} \operatorname{sgn} x_\alpha & \text{when } \alpha \in \mathcal{N}, \\ \pm 1 & \text{when } \alpha \in \Lambda \setminus \mathcal{N}. \end{cases}$$

It is easy to see that $x^* \in J(x)$ and it follows that there exist two elements $x^*_1, x^*_2 \in J(x)$ such that $\|x^*_1 - x^*_2\| = 2$. Therefore, $d(x) = 2$ for any $x \in \ell_1(\Lambda)$, hence d is continuous on the space $\ell_1(\Lambda)$ although none of its elements is smooth.

Now, we can ask for the reverse of Proposition 6.2, i.e., whether the mapping d is continuous on $\operatorname{Sm}\mathbb{X}$. Obviously, if $\operatorname{Sm}\mathbb{X}$ is open in $\mathbb{X}$, then d as a constant function on $\operatorname{Sm}\mathbb{X}$ is continuous on this set. In general, $\operatorname{Sm}\mathbb{X}$ is only a G_δ set, not necessarily open. We are going to prove openness of $\operatorname{Sm}\mathbb{X}$ in particular spaces.

To this end we introduce the following property of a linear normed space which may be compared with a notion of roughness (cf., [29]) or strong roughness [35, Defn. 3].

Definition 6.2. *We say that a normed linear space $\mathbb{X}$ is weakly-rough if it satisfies the condition:*

$$\text{(P)} \qquad \exists\, \varepsilon_0 > 0\ \forall\, x \in \mathbb{X}: \ d(x) = 0 \text{ or } d(x) \geq \varepsilon_0.$$

The above property means that on the set of all non-smooth elements of $\mathbb{X}$ values of the mapping d are separated from zero.

Equivalently, weak-roughness means that for any sequence $\{x_n\} \subset \mathbb{X}$:

$$d(x_n) \to 0 \iff \exists\, n_0: \ d(x_n) = 0,\ n \geq n_0.$$

Obviously, each smooth space $\mathbb{X}$ is weakly-rough. We will see later that ℓ_1 as well as any finite-dimensional polyhedral Banach space have this property. On the other hand, in Example 6.9 we construct a space which is not weakly-rough.

Theorem 6.6. *Let $\mathbb{X}$ be a weakly-rough normed linear space with $\mathbb{X}^*$ satisfying the* K^* *property. Then* $\mathrm{Sm}\,\mathbb{X}$ *is open in* $\mathbb{X}$.

Proof. Suppose on the contrary that $\mathrm{Sm}\,\mathbb{X}$ is not open, whence there exists $x \in \mathrm{Sm}\,\mathbb{X}$ and a sequence $\{x_n\}$ of non-smooth points in $\mathbb{X}$ such that $x_n \longrightarrow x$ as $n \to \infty$. Since $\mathbb{X}$ is weakly-rough, for each $n \in \mathbb{N}$, we can choose two different functionals $x_n^*, y_n^* \in J(x_n)$ such that $\|x_n^* - y_n^*\| \geq \varepsilon_0 - \frac{1}{n}$. The sequences $\{x_n^*\}, \{y_n^*\}$ can be considered as nets on $B_{\mathbb{X}^*}$. From the Banach-Alaoglu theorem we know that $B_{\mathbb{X}^*}$ is weak*-compact, whence we may choose two weakly*-convergent subnets of the nets $\{x_n^*\}$ and $\{y_n^*\}$. Namely, let K be a directed set, $h\colon K \ni \alpha \mapsto h(\alpha) =: n_\alpha \in \mathbb{N}$ be a monotone mapping such that $h(K)$ is cofinal in $\mathbb{N}$, and let $\{x_{n_\alpha}^*\}_{\alpha\in K}$ be a subnet of $\{x_n^*\}$ such that $x_{n_\alpha}^* \xrightarrow{w^*} x^*$ for some $x^* \in B_{\mathbb{X}^*}$. Now we have

$$\begin{aligned} |x_{n_\alpha}^*(x_{n_\alpha}) - x^*(x)| &= |x_{n_\alpha}^*(x_{n_\alpha}) - x_{n_\alpha}^*(x) + x_{n_\alpha}^*(x) - x^*(x)| \\ &\leq \|x_{n_\alpha}^*\|\|x_{n_\alpha} - x\| + |x_{n_\alpha}^*(x) - x^*(x)|. \end{aligned}$$

For any $\epsilon > 0$ we can find an $\alpha' \in K$ such that

$$\|x_{n_\alpha} - x\| < \epsilon/2 \quad \text{and} \quad |x_{n_\alpha}^*(x) - x^*(x)| < \epsilon/2 \qquad \text{for all } n_\alpha \geq n_{\alpha'}.$$

It follows that $|x_{n_\alpha}^*(x_{n_\alpha}) - x^*(x)| < \epsilon$ for all $n_\alpha \geq n_{\alpha'}$ and this implies $x_{n_\alpha}^*(x_{n_\alpha}) \longrightarrow x^*(x)$. Since $x_{n_\alpha}^* \in J(x_{n_\alpha})$, we have $x_{n_\alpha}^*(x_{n_\alpha}) = \|x_{n_\alpha}\|$ which implies $x^*(x) = \|x\|$, hence $x^* \in J(x)$. Similarly (without loss of generality we take the same direct set K and n_α) we assume that $\{y_{n_\alpha}^*\}_{\alpha\in K}$ is a subnet of $\{y_n^*\}$ such that $y_{n_\alpha}^* \xrightarrow{w^*} y^*$ for some $y^* \in B_{\mathbb{X}^*}$, which leads to $y^*(x) = \|x\|$ and $y^* \in J(x)$. Since $x_{n_\alpha}^* \xrightarrow{w^*} x^*$ and $\|x_{n_\alpha}^*\| = \|x^*\|$ for all n_α, the K^* property of $\mathbb{X}^*$ yields $x_{n_\alpha}^* \longrightarrow x^*$. Analogously, we show that $y_{n_\alpha}^* \longrightarrow y^*$. Since $\|x_{n_\alpha}^* - y_{n_\alpha}^*\| \geq \varepsilon_0 - \frac{1}{n}$ for all n_α, we get $\|x^* - y^*\| \geq \varepsilon_0$. Thus $x^* \neq y^*$ which would mean that x is non-smooth, a contradiction. $\square$

Since the properties K^* and KK^* are equivalent we can write an equivalent version of the above theorem.

Corollary 6.2. *Let $\mathbb{X}$ be a weakly-rough normed linear space with $\mathbb{X}^*$ satisfying the* KK^* *property. Then* $\mathrm{Sm}\,\mathbb{X}$ *is open in* $\mathbb{X}$.

Since the weak topology and the weak* topology coincide in the dual of a reflexive Banach space, we note:

Corollary 6.3. *Let $\mathbb{X}$ be a reflexive weakly-rough normed linear space with $\mathbb{X}^*$ satisfying the* KK *property. Then* $\operatorname{Sm}\mathbb{X}$ *is open in* $\mathbb{X}$.

Finally, we apply Corollary 6.2 to finite-dimensional polyhedral Banach spaces.

Corollary 6.4. *Let $\mathbb{X}$ be a real finite-dimensional polyhedral Banach space. Then* $\operatorname{Sm}\mathbb{X}$ *is open in* $\mathbb{X}$.

Proof. As a finite-dimensional space $\mathbb{X}^*$ obviously satisfies the KK* property. Since the set $\operatorname{Ext}(B_{\mathbb{X}^*})$ is finite, we have

$$\varepsilon_0 := \min\{\|x^* - y^*\| : \ x^*, y^* \in \operatorname{Ext}(B_{\mathbb{X}^*}), x^* \neq y^*\} > 0.$$

Suppose that $x \neq \theta$ is a non-smooth point in $\mathbb{X}$. Let $x_1^*, x_2^* \in J(x)$ be two different supporting functionals at x; by Proposition 6.1 we may consider $x_1^*, x_2^* \in \operatorname{Ext}(J(x))$. Then (since $J(x)$ is an extremal subset of $B_{\mathbb{X}}$) it follows that $x_1^*, x_2^* \in \operatorname{Ext}(B_{\mathbb{X}^*})$ and finally $\|x_1^* - x_2^*\| \geq \varepsilon_0$. Therefore, $d(x) \geq \varepsilon_0$ whence $\mathbb{X}$ satisfies property (P). Applying Corollary 6.2 we obtain that $\operatorname{Sm}\mathbb{X}$ is open. □

It follows immediately that in a real finite-dimensional polyhedral Banach space, d is continuous on $\operatorname{Sm}\mathbb{X}$. Actually, by Proposition 6.2, continuity of d characterizes smoothness.

Theorem 6.7. *For a real finite-dimensional polyhedral Banach space $\mathbb{X}$ and $x \in \mathbb{X} \setminus \{\theta\}$, x is smooth if and only if d is continuous at x.*

The example below shows that in general the above statement need not be true, even if the space is separable.

Example 6.8. *Consider the space ℓ_1. One can easily verify that an element $x = (t_1, t_2, \ldots) \in \ell_1$ is smooth if and only if $t_n \neq 0$ for all $n = 1, 2, \ldots$. Therefore, if $x = (x_1, x_2, \ldots) \in \ell_1$ is a non-smooth point, then $x_{i_0} = 0$ for some $i_0 \in \mathbb{N}$. As ℓ_∞ is the dual of ℓ_1, we consider $x^* = (\alpha_1, \alpha_2, \ldots), y^* = (\beta_1, \beta_2, \ldots) \in J(x)$ such that $\alpha_{i_0} = 1$ and $\beta_{i_0} = -1$. Then $\|x^* - y^*\| = 2$ and thus $d(x) = 2$ for any non-smooth point in ℓ_1. Now, take any $x_0 = (t_1, t_2, t_3, \ldots) \in \operatorname{Sm}\ell_1$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \ell_1$, where $x_n = (t_1, t_2, \ldots, t_n, 0, 0, \ldots)$ for $n \in \mathbb{N}$. Then clearly $x_n \to x_0$ as $n \to \infty$ but $x_n \notin \operatorname{Sm}\ell_1$ for each $n \in \mathbb{N}$. It shows that the set $\operatorname{Sm}\ell_1$ is not open and since $d(x_0) = 0$ and $d(x_n) = 2$ for all $n \in \mathbb{N}$, it proves that d is discontinuous at any smooth point x_0.*

As showed above, $d(x) = 2$ for all non-smooth points in ℓ_1 which means that ℓ_1 is weakly-rough. Therefore, the dual of ℓ_1, i.e., ℓ_∞, cannot satisfy the K* property, whence this assumption is essential in Theorem 6.6. In turn, the following example ensures that weak-roughness property is not redundant in Theorem 6.6, even in case of finite-dimensional Banach space.

Example 6.9. *In $\mathbb{R}^2$ we consider the points*

$$x_n = \left(\frac{1}{n}, \sqrt{1 - \frac{1}{n^2}}\right) \quad \text{and} \quad y_n = \left(-\frac{1}{n}, \sqrt{1 - \frac{1}{n^2}}\right), \qquad n \in \mathbb{N}.$$

Now take the set:

$$B := \operatorname{conv}\{\pm x_n, \pm y_n : \ n \in \mathbb{N}\} \qquad \text{(see Figure 6.1).}$$

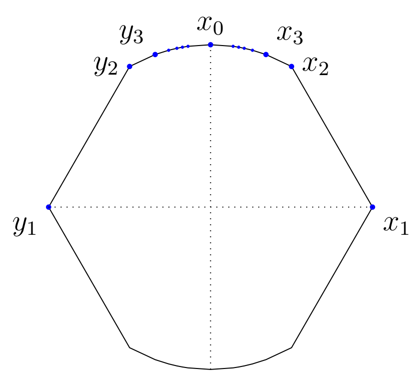

Figure 6.1: Illustration to Example 6.9, the set B

Consider the norm generated by the set B, for which B is a unit sphere, and denote by $\mathbb{X}$ the normed space $\mathbb{R}^2$ with that norm. In particular, each of x_n is a non-smooth point in $\mathbb{X}$ and $x_n \to x_0 = (0,1)$. The point x_0 is smooth; indeed, for any line passing through x_0 which is not horizontal, there exists n_0 such that for all $n \geq n_0$, all x_n or all y_n lie over this line. Hence there is only one supporting line at x_0 (the horizontal one), i.e, x_0 is smooth. Since $x_n \to x_0$, in every open neighbourhood of x_0 there are non-smooth points, hence $\operatorname{Sm}\mathbb{X}$ is not open. Obviously $\mathbb{X}$ is reflexive and $\mathbb{X}^*$ has the KK-property (as a finite-dimensional space). Thus it follows from Corollary 6.3 that $\mathbb{X}$ is not weakly-rough.

Next we give an example to show that the condition that $\mathbb{X}^*$ has K^* property in Theorem 6.6 is not necessary for $\operatorname{Sm}\mathbb{X}$ to be open in $\mathbb{X}$.

Example 6.10. *Let us consider the real Banach space ℓ_∞. Note that ℓ_∞^* does not satisfy the K^* property. We claim that $\operatorname{Sm}\ell_\infty$ is open in ℓ_∞. To this end it is enough to show that $\operatorname{Sm}\ell_\infty \cap S_{\ell_\infty}$ is open in S_{ℓ_∞}. Consider a smooth element $x = (x_n)_{n\in\mathbb{N}} \in S_{\ell_\infty}$. We want to show that x is an interior point of $\operatorname{Sm}\ell_\infty \cap S_{\ell_\infty}$. Since x is smooth, it follows from [18, Th. 2.8] that there exists* *a unique* $i_0 \in \mathbb{N}$ *such that* $|x_{i_0}| = 1$ *and* $\delta := \sup\left\{|x_j| : j \in \mathbb{N} \setminus \{i_0\}\right\} < 1$. *Consider the open set*

$$U = \left\{ y \in S_{\ell_\infty} : \ \|x - y\|_\infty < \frac{1-\delta}{2} \right\}.$$

We will show that $U \subset \operatorname{Sm} \ell_\infty \cap S_{\ell_\infty}$. *Let* $y = (y_n)_{n\in\mathbb{N}} \in U$. *Then* $\sup\{|x_j - y_j| : j \in \mathbb{N}\} < \frac{1-\delta}{2}$. *We note that* $\sup\{|y_j| : j \in \mathbb{N} \setminus \{i_0\}\} < 1$. *Indeed, if* $\sup\{|y_j| : j \in \mathbb{N} \setminus \{i_0\}\} = 1$, *then we would have*

$$\frac{1-\delta}{2} > \sup_{j\in\mathbb{N}\setminus\{i_0\}} |y_j - x_j| \geq \sup_{j\in\mathbb{N}\setminus\{i_0\}} |y_j| - \sup_{j\in\mathbb{N}\setminus\{i_0\}} |x_j| = 1 - \delta,$$

a contradiction. Thus $y \in \operatorname{Sm} \ell_\infty$ *and, consequently,* $U \subset \operatorname{Sm} \ell_\infty \cap S_{\ell_\infty}$ *which means that* x *is an interior point of* $\operatorname{Sm} \ell_\infty \cap S_{\ell_\infty}$, *as claimed.*

The openness of the set $\operatorname{Sm} \mathbb{X}$ can be alternatively shown under the assumption of yet another property of $\mathbb{X}$:

(P*) $\qquad \exists\, \theta \neq w \in \operatorname{Sm} \mathbb{X}\ \exists\, \varepsilon_0 > 0\ \forall\, x \in \mathbb{X} \setminus \operatorname{Sm} \mathbb{X}\ \exists\, x^*, y^* \in J(x) : \ |x^*(w) - y^*(w)| \geq \varepsilon_0.$

Note that this property is, in particular, satisfied by any finite-dimensional polyhedral Banach space.

Lemma 6.2. *Let* $\mathbb{X}$ *be a real finite-dimensional polyhedral Banach space. Then* (P*) *holds.*

Proof. The dual $\mathbb{X}^*$ is also a polyhedral Banach space so let

$$\operatorname{Ext}(B_{\mathbb{X}^*}) = \{\pm x_1^*, \pm x_2^*, \ldots, \pm x_k^*\}.$$

Consider the collection of all possible differences of extreme functionals:

$$R := \{y^* \in \mathbb{X}^* : y^* = x_i^* - x_j^*, \text{ for some } x_i^*, x_j^* \in \operatorname{Ext}(B_{\mathbb{X}^*}),\ i \neq j\}.$$

Clearly, R is a finite subset of $\mathbb{X}^*$. Now we consider the union of kernels of all the elements of R:

$$K := \bigcup_{y^* \in R} \ker y^*.$$

Since for any $y^* \in R$, $\ker y^*$ is nowhere dense and R is finite, it follows that K is also a nowhere dense subset of $\mathbb{X}$. Fix an arbitrary element $w_0 \in \mathbb{X} \setminus K$. Hence $|(x_i^* - x_j^*)(w_0)| \neq 0$ for all $x_i^*, x_j^* \in \operatorname{Ext}(B_{\mathbb{X}^*})$ $(i \neq j)$. Take

$$\varepsilon_0 := \min\{|(x_i^* - x_j^*)(w_0)| : x_i^*, x_j^* \in \operatorname{Ext}(B_{\mathbb{X}^*}),\ i \neq j\} > 0.$$

Therefore, for any $x \in \mathbb{X} \backslash \mathrm{Sm}\,\mathbb{X}$ there exist $x_i^*, x_j^* \in \mathrm{Ext}\,(J(x))$ $(i \neq j)$ such that $|(x_i^* - x_j^*)(w_0)| \geq \epsilon_0 > 0$. Hence $\mathbb{X}$ satisfies (P*). □

Theorem 6.11. *Let $\mathbb{X}$ be a normed linear space having property* (P*). *Then* $\mathrm{Sm}\,\mathbb{X}$ *is open in* $\mathbb{X}$.

Proof. Suppose that for some $x \in \mathrm{Sm}\,\mathbb{X}$ there exists a sequence $(x_n) \subset \mathbb{X} \setminus \mathrm{Sm}\,\mathbb{X}$ such that $x_n \to x$. Without loss of generality we may assume $\|x\| = \|x_n\| = 1$, $n = 1, 2, \ldots$. For each n, choose $x_n^*, y_n^* \in J(x_n)$ such that $|x_n^*(w) - y_n^*(w)| \geq \varepsilon_0$. By weak*-compactness of $B_{\mathbb{X}^*}$ we may find two weak*-cluster points $x^*, y^* \in B_{\mathbb{X}^*}$ of the sequences $\{x_n^*\}$ and $\{y_n^*\}$, respectively. This implies that there exists a subsequence $\{x_{n_k}^*(x)\}$ of $\{x_n^*(x)\}$ and a subsequence $\{y_{m_k}^*(x)\}$ of $\{y_n^*\}$ satisfying

$$x_{n_k}^*(x) \to x^*(x) \quad \text{and} \quad y_{m_k}^*(x) \to y^*(x).$$

As $x_{n_k} \to x$, it is easy to observe that $x^*(x) = y^*(x) = 1$ and therefore, $x^*, y^* \in J(x)$. We observe that

$$\begin{aligned} \|x^* - y^*\| &\geq \left| x^*\left(\frac{w}{\|w\|}\right) - y^*\left(\frac{w}{\|w\|}\right) \right| \\ &= \lim_{k\to\infty} \left| x_{n_k}^*\left(\frac{w}{\|w\|}\right) - y_{m_k}^*\left(\frac{w}{\|w\|}\right) \right| \\ &\geq \frac{\varepsilon_0}{\|w\|}. \end{aligned}$$

Thus $x^* \neq y^*$ whence x is non-smooth – a contradiction. The openness of $\mathrm{Sm}\,\mathbb{X}$ in $\mathbb{X}$ follows. □

6.4 Approximate smoothness

We follow the notion of *approximate smoothness* introduced in [21]. From now on, we consider only real normed linear spaces ($\mathbb{K} = \mathbb{R}$).

Definition 6.3. *In a normed linear space $\mathbb{X}$ we say that $x \in \mathbb{X} \setminus \{\theta\}$ is* approximately smooth *if $d(x) = \mathrm{diam}(J(x)) \leq \varepsilon$ with some $\varepsilon \in [0, 2)$. If the bound ε is specified, we say that x is ε*-approximately smooth (or, ε-smooth). *The space $\mathbb{X}$ is said to be approximately smooth (ε-approximately smooth) if each $x \in S_{\mathbb{X}}$ is ε_x-approximately smooth for some $\varepsilon_x \leq \varepsilon < 2$.*

In the study of geometry of Banach spaces there is a well known notion of k-smoothness (cf. [53]). Therefore, to avoid confusion, we write ε*-approximately smooth* instead of ε*-smooth.*

The following characterization was given in [21].

Lemma 6.3 ([21, Lemma 2.3]). *Let $\mathbb{X}$ be a normed linear space, $x \in \mathbb{X} \setminus \{\theta\}$ and $\varepsilon \in [0, 2)$. Then the following two conditions are equivalent:*

(i) x is ε-approximately smooth.

(ii) $\sup_{y \in S_{\mathbb{X}}} \{\rho'_+(x, y) - \rho'_-(x, y)\} \leq \varepsilon \|x\|$.

6.4.1 Approximate smoothness in the space of bounded linear operators

Definition 6.3 can equally well be applied to the space of linear bounded operators providing the notion of an *approximately smooth operator*. To show that the Definition 6.3 is meaningful for the space of bounded linear operators we provide an example of an approximately smooth operator which is not smooth.

Example 6.12. *Consider the two-dimensional vector space $\mathbb{X}$ endowed with the norm such that its unit sphere is a regular octagon. Fix $x \in \operatorname{Ext}(B_{\mathbb{X}})$ and define a linear map $T \in \mathbb{L}(\ell_2^2, \mathbb{X})$ such that $Te_1 = x$ and $Te_2 = 0$, where $\{e_1, e_2\}$ is the canonical basis of ℓ_2^2. It is clear that $M_T = \{\pm e_1\}$. Therefore, due to Theorem 6.2, $\operatorname{Ext}(J(T)) = \{y^* \otimes e_1 : \ y^* \in \operatorname{Ext}(J(x))\}$. It follows from [21, Example 4.1] that $d(x) = 2\tan\frac{\pi}{8}$. Therefore, for any $y^* \otimes e_1, z^* \otimes e_1 \in \operatorname{Ext}(J(T))$, we have (making use of Lemma 6.4) $\|y^* \otimes e_1 - z^* \otimes e_1\| = \|y^* - z^*\| \leq 2\tan\frac{\pi}{8}$. Thus T is $2\tan\frac{\pi}{8}$-approximately smooth. On the other hand, Te_1 is not a smooth point in $\mathbb{X}$, whence by Corollary 6.5 the operator T is not smooth in $\mathbb{L}(\ell_2^2, \mathbb{X})$.*

Next we note the following lemma which will be frequently used.

Lemma 6.4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be normed linear spaces.*

(1) For any $x_0 \in S_{\mathbb{X}}$ and $y_0 \in S_{\mathbb{Y}}$ there exists an operator $A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ such that $Ax_0 = y_0$.

(2) If $\mathbb{Y}$ is reflexive, then for any $x_0^* \in S_{\mathbb{X}^*}$ and $y_0^* \in S_{\mathbb{Y}^*}$ there exists an operator $A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ such that $A^* y_0^* = x_0^*$.

Proof. 1. Suppose that H is a hyperplane in $\mathbb{X}$ satisfying $x_0 \perp_B H$ (H can be taken as a kernel of any supporting functional at x_0). Define $A \colon \mathbb{X} \to \mathbb{Y}$ by $Ax_0 = y_0$ and $Ah = \theta$, for all $h \in H$. This yields $\|A\| \geq 1$. On the other hand, given any $z \in S_{\mathbb{X}}$, we have $z = \alpha x_0 + h$, for some $h \in H$. It follows $1 = \|\alpha x_0 + h\| \geq \|\alpha x_0\| = |\alpha| = \|Az\|$ and thus $\|Az\| \leq 1$ which leads to $\|A\| = 1$.

2. It follows form part 1. applied to $\mathbb{Y}^*$ and $\mathbb{X}^*$ that for fixed $x_0^* \in S_{\mathbb{X}^*}$ and $y_0^* \in S_{\mathbb{Y}^*}$ there exists $B \in \mathbb{L}(\mathbb{Y}^*, \mathbb{X}^*)$, such that $\|B\| = 1$ and $By_0^* = x_0^*$. By reflexivity of $\mathbb{Y}$, there exists

$A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ such that $A^* = B$. Hence $\|A\| = \|A^*\| = \|B\| = 1$ and $A^*y_0^* = By_0^* = x_0^*$ as claimed. □

Now, we give a necessary condition for smoothness of an operator.

Proposition 6.3. *Let $\mathbb{X}$ and $\mathbb{Y}$ be normed linear spaces and suppose that $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is ε-approximately smooth (with some $\varepsilon \in [0, 2)$). Then:*

(1) for each $x \in M_T$, Tx is ε-approximately smooth;

(2) assuming additionally that $\mathbb{Y}$ is reflexive, for any $y^* \in M_{T^*}$, $T^*(y^*)$ is ε-approximately smooth.

Note that the above statement does not prejudge about non-emptiness of M_T or M_{T^*}.

Proof. 1. Suppose on the contrary that for some $x_0 \in M_T$, Tx_0 is not ε-approximately smooth. Then there exist $y_1^*, y_2^* \in J(Tx_0)$ such that $\|y_1^* - y_2^*\| > \varepsilon$. Consider $\varphi := y_1^* \otimes x_0$ and $\psi := y_2^* \otimes x_0$ — functionals on $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. It is easy to see that $\varphi, \psi \in J(T)$. Moreover,

$$\|\varphi - \psi\| = \|y_1^* \otimes x_0 - y_2^* \otimes x_0\| = \sup_{A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} |(y_1^* - y_2^*)Ax_0|.$$

By Lemma 6.4, $\sup_{\|A\|=1} |(y_1^* - y_2^*)(Ax_0)| = \|y_1^* - y_2^*\| > \varepsilon$ and it follows $\|\varphi - \psi\| > \varepsilon$. This contradicts the fact that T is ε-approximately smooth.

2. Similarly, suppose that there exists $y_0^* \in M_{T^*}$ such that $T^*(y_0^*)$ is not ε-approximately smooth. Then there exist $x_1^{**}, x_2^{**} \in J(T^*(y_0^*))$ such that $\|x_1^{**} - x_2^{**}\| > \varepsilon$. Now for $i = 1, 2$

$$(x_i^{**} \otimes y_0^*)(T) = x_i^{**}(T^*y_0^*) = \|T^*y_0^*\| = \|T^*\| = \|T\|.$$

Therefore, $x_i^{**} \otimes y_0^* \in J(T)$ for $i = 1, 2$. Moreover, using part (2) of Lemma 6.4,

$$\|x_1^{**} \otimes y_0^* - x_2^{**} \otimes y_0^*\| = \sup_{A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} |(x_1^{**} - x_2^{**})(A^*y_0^*)| = \|x_1^{**} - x_2^{**}\| > \varepsilon.$$

This implies that $d(T) > \varepsilon$, which is a contradiction. □

Corollary 6.5. *If $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is smooth, then for each $x \in M_T$, Tx is smooth in $\mathbb{Y}$. Moreover, if $\mathbb{Y}$ is reflexive, then for each $y^* \in M_{T^*}$, $T^*(y^*)$ is smooth in X^*.*

6.4.2 Characterization of approximate smoothness by a numerical range

Let $T, S \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Following the definition of the abstract numerical range (cf. [68, 78]), we consider the following (always nonempty) set:

$$\Omega(T, S) := \{\lim y_n^*(Sx_n) : y_n^* \in S_{\mathbb{Y}^*}, x_n \in S_{\mathbb{X}} \text{ and } \lim y_n^*(Tx_n) = \|T\|\}.$$

We will use the above set for another characterization of approximate smoothness. First we note that approximate smoothness of an operator T yields some restriction on $\Omega(T, S)$.

Proposition 6.4. *Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be ε-approximately smooth with $\varepsilon \in [0, 2)$. Then* $\operatorname{diam}(\Omega(T, S)) \leq \varepsilon$ *for all $S \in B_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$.*

Proof. Suppose that there exists $S_0 \in B_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ such that $\operatorname{diam}(\Omega(T, S_0)) > \varepsilon$. Then there exist two elements $\lambda, \mu \in \Omega(T, S_0)$ such that $|\lambda - \mu| > \epsilon$. Let $\lambda = \lim y_n^*(S_0 x_n)$ and $\mu = \lim z_n^*(S_0 w_n)$, for some $y_n^*, z_n^* \in S_{\mathbb{Y}^*}$ and $x_n, w_n \in S_{\mathbb{X}}$ such that $\lim y_n^*(Tx_n) = \lim z_n^*(Tw_n) = \|T\|$. We consider sequences $\phi_n = y_n^* \otimes x_n$ and $\psi_n = z_n^* \otimes w_n$ in $B_{\mathbb{L}(\mathbb{X},\mathbb{Y})^*}$. By weak*-compactness of $B_{\mathbb{L}(\mathbb{X},\mathbb{Y})^*}$ we find two weak*-cluster points $\phi, \psi \in B_{\mathbb{L}(\mathbb{X},\mathbb{Y})^*}$ of the sequences $\{\phi_n\}$ and $\{\psi_n\}$, respectively. Thus there exist $\{n_k\}, \{m_k\} \subset \mathbb{N}$ such that $\phi_{n_k}(S_0) \to \phi(S_0)$ and $\psi_{m_k}(S_0) \to \psi(S_0)$, respectively. On the other hand we have $\lim \phi_n(T) = \lim \psi_n(T) = \|T\|$. Since ϕ, ψ are weak*-cluster points of $\{\phi_n\}, \{\psi_n\}$, respectively, it follows that $\phi(T) = \psi(T) = \|T\|$. Therefore, $\phi, \psi \in J(T)$. Now we have

$$\|\phi - \psi\| \geq |(\phi - \psi)(S_0)| = \lim_{k \to \infty} |(\phi_{n_k} - \psi_{m_k})(S_0)| = |\lambda - \mu| > \varepsilon.$$

This contradicts the fact that T is ε-approximately smooth and hence the theorem. □

A natural question arises whether the reverse implication holds true. We are able to prove it merely under some additional assumptions.

Theorem 6.13. *Let $\mathbb{X}$ be a reflexive Banach space and let $\mathbb{Y}$ be any normed linear space. Suppose that $\mathcal{K}(\mathbb{X}, \mathbb{Y})$ is an M-ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ and $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is such that* $\operatorname{dist}(T, \mathcal{K}(\mathbb{X}, \mathbb{Y})) < \|T\|$. *Then for any $\epsilon \in [0, 2)$, the following conditions are equivalent:*

(i) T is ε-approximately smooth.

(ii) $\operatorname{diam}(\Omega(T, S)) \leq \varepsilon$ *for all $S \in B_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$.*

(iii) $\operatorname{diam}(\Omega(T, S)) \leq \varepsilon$ *for all $S \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$.*

Proof. The implication (i) $\implies$ (ii) was proved in the previous result and (ii) $\implies$ (iii) is obvious. Therefore we only need to show (iii) $\implies$ (i). Suppose that $\phi, \psi \in \operatorname{Ext}(J(T))$. Then, due to Theorem 6.2, we note that $\phi = y^* \otimes x$ and $\psi = z^* \otimes w$ for some $x, w \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})$ and $y^* \in \operatorname{Ext}(J(Tx)), z^* \in \operatorname{Ext}(J(Tw))$. Observe that for any $S \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ we have $y^*(Sx), z^*(Sw) \in \Omega(T, S)$ and, since $\operatorname{diam}(\Omega(T, S)) \le \varepsilon$, $|y^*(Sx) - z^*(Sw)| \le \epsilon$. It follows then

$$\|\phi - \psi\| = \sup_{\|S\|=1} |(y^* \otimes x - z^* \otimes w)(S)| = \sup_{\|S\|=1} |y^*(Sx) - z^*(Sw)| \le \varepsilon.$$

Since $\phi, \psi \in \operatorname{Ext}(J(T))$ were arbitrarily chosen we get by Proposition 6.1, $d(x) = \operatorname{diam}(\operatorname{Ext}(J(T))) \le \varepsilon$, hence T is ε-approximately smooth. □

Immediately we get the following corollary (cf. [78, Th. 3.1] for a multilinear counterpart).

Corollary 6.6. *Under the assumption of the previous theorem, the following conditions are equivalent:*

(i) T *is smooth.*

(ii) $\forall\, S \in B_{\mathbb{L}(\mathbb{X},\mathbb{Y})}\ \exists\, \lambda_S \in \mathbb{K}: \quad \Omega(T, S) = \{\lambda_S\}$.

(ii) $\forall\, S \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}\ \exists\, \lambda_S \in \mathbb{K}: \quad \Omega(T, S) = \{\lambda_S\}$.

In the following theorem we formulate a sufficient condition for $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ to be an approximately smooth operator. Note that whenever M_T is nonempty, it can be written as a union of two disjoint sets F and $-F$. Such a decomposition is not unique, and one can try to make F as small as possible. Further, we consider the image of F and the corresponding set of supporting functionals

$$T(F) = \{Tx: \ x \in F\}, \quad J(T(F)) = \{y^* \in J(y) : y \in T(F)\}.$$

Proposition 6.5. *Let* $\mathbb{X}$ *be a reflexive Banach space and let* $\mathbb{Y}$ *be any normed linear space. Suppose that* $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ *is an* M*-ideal in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ *and* $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ *is such that* $\operatorname{dist}(T, \mathbb{K}(\mathbb{X}, \mathbb{Y})) < \|T\|$. *If the following conditions hold true:*

(i) $\operatorname{diam}(F) \le r$,

(ii) $\operatorname{diam}(J(T(F))) \le p$,

then T *is* $(r + p)$*-approximately smooth, whenever* $r + p < 2$.

Proof. We take two arbitrary elements $u^* \otimes x$ and $v^* \otimes y$ from $\operatorname{Ext}(J(T))$ represented by (6.1). Without loss of generality we assume that $x, y \in F \cap \operatorname{Ext}(B_{\mathbb{X}})$, $u^* \in \operatorname{Ext}(J(Tx))$ and $v^* \in \operatorname{Ext}(J(Ty))$. Applying (i) and (ii) we get

$$\begin{aligned} \|u^* \otimes x - v^* \otimes y\| &= \|u^* \otimes x - u^* \otimes y + u^* \otimes y - v^* \otimes y\| \\ &\leq \|u^*\|\|x - y\| + \|u^* - v^*\|\|y\| \\ &= \|x - y\| + \|u^* - v^*\| \leq r + p. \end{aligned}$$

If follows that $\operatorname{diam}(\operatorname{Ext}(J(T))) \leq r + p$ and by Proposition 6.1, $d(T) \leq r + p$. It proves that T is $(r + p)$-approximately smooth, whenever $r + p < 2$. □

Propositions 6.3 and 6.5 lead to the following corollary which is a generalization of [66, Th. 4.6] where an analogous characterization of exact smoothness of T was given.

Corollary 6.7. *Let $\mathbb{X}$ be a reflexive Banach space and let $\mathbb{Y}$ be any normed linear space such that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is an M-ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Suppose that $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ satisfies $\operatorname{dist}(T, \mathbb{K}(\mathbb{X}, \mathbb{Y})) < \|T\|$ with $M_T = \{\pm x_0\}$ and $\varepsilon \in [0, 1)$. Then T is ε-approximately smooth if and only if Tx_0 is ε-approximately smooth.*

Now, we consider the relation between $d(T)$ and $d(Tx)$ for $x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})$. Under the assumptions of Corollary 6.7 we have $d(T) = d(T(\pm x_0)) = \varepsilon$, whence

$$d(T) = \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} d(Tx).$$

In general, the above equality need not be true.

Example 6.14. *Consider the identity operator I defined on a real space ℓ_2^2. It is easy to see that $M_I = S_{\ell_2^2}$. Since ℓ_2^2 is a smooth space, it follows that $d(x) = 0$ for all $x \in M_I$. Therefore, $\sup_{x \in M_I} d(Ix) = 0$. We claim that $d(I) = 2$ whence $d(I) > \sup_{x \in M_I \cap \operatorname{Ext}(B_{\mathbb{X}})} d(Ix)$. From Lemma 6.1 we have*

$$d(I) = \sup_{S \in S_{\mathbb{L}(\ell_2^2)}} \{\rho'_+(I, S) - \rho'_-(I, S)\}.$$

Applying (6.2) and (6.3), we have for any $S \in S_{\mathbb{L}(\ell_2^2)}$:

$$\begin{aligned} \rho'_+(I, S) &= \sup_{x \in S_{\ell_2^2}} \langle x, Sx \rangle, \\ \rho'_-(I, S) &= \inf_{x \in S_{\ell_2^2}} \langle x, Sx \rangle. \end{aligned}$$

Now, consider $S_0 \in \mathbb{L}(\ell_2^2)$ *such that* $S_0x = x$ *and* $S_0y = -y$, *for some* $x, y \in S_{\ell_2^2}$ *such that* $x \perp y$. *Thus* $\|S_0\| = 1$ *and clearly* $\rho'_+(I, S_0) = 1$, $\rho'_-(I, S_0) = -1$, *hence* $d(I) = 2$.

Proposition 6.6. *Let* $\mathbb{X}$ *be a reflexive Banach space and let* $\mathbb{Y}$ *be any normed linear space such that* $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ *is an* M*-ideal in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. *Suppose that* $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ *is such that* $\operatorname{dist}(T, \mathbb{K}(\mathbb{X}, \mathbb{Y})) < \|T\|$. *Then*

$$d(T) \geq \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} d(Tx). \tag{6.5}$$

Proof. From Lemma 6.1 we have

$$d(T)\,\|T\| = \sup_{S \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} \{\rho'_+(T, S) - \rho'_-(T, S)\}.$$

Applying (6.2) and (6.3), as well as Lemma 6.4 and finally Lemma 6.1 again, we have

$$\begin{aligned}
d(T)\,\|T\| &= \sup_{S \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} \left\{ \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} \rho'_+(Tx, Sx) - \inf_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} \rho'_-(Tx, Sx) \right\} \\
&\geq \sup_{S \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} \left\{ \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} \{\rho'_+(Tx, Sx) - \rho'_-(Tx, Sx)\} \right\} \\
&= \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} \left\{ \sup_{S \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} \{\rho'_+(Tx, Sx) - \rho'_-(Tx, Sx)\} \right\} \\
&\geq \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} \left\{ \sup_{y \in S_{\mathbb{Y}}} \{\rho'_+(Tx, y) - \rho'_-(Tx, y)\} \right\} \\
&= \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} d(Tx)\|Tx\| = \sup_{x \in M_T \cap \operatorname{Ext}(B_{\mathbb{X}})} d(Tx)\,\|T\|.
\end{aligned}$$

□

Moreover, we have:

Proposition 6.7. *Let* $\mathbb{X}$ *be any normed linear space space and let* $T \in S_{\mathbb{L}(\mathbb{X}, \ell_1)}$ *be such that* $\{Tx : x \in S_{\mathbb{X}}\} \cap \operatorname{Ext}(B_{\ell_1}) \neq \emptyset$. *Then* $d(T) = \sup_{x \in M_T} d(Tx)$.

Proof. We note that $\operatorname{Ext}(B_{\ell_1}) = \{e_i\}_{i \in \mathbb{N}}$ — the canonical Schauder base of ℓ_1. We take $x \in S_{\mathbb{X}}$ such that $Tx = e_j$, for some $j \in \mathbb{N}$. Then for any $y^* \in J(e_j)$ we have

$$(y^* \otimes x)(T) = y^*(Tx) = 1 = \|T\|.$$

Thus $y^* \otimes x \in J(T)$. Following Example 6.8, $d(Tx) = d(e_j) = 2$ and we consider $y^*, z^* \in J(e_j)$ such that $\|y^* - z^*\| = 2$. Observe that $\|y^* \otimes x - z^* \otimes x\| = \|y^* - z^*\|\|x\| = 2$ which implies that $d(T) = 2$ and proves the result. □

6.4.3 Coincidence of exact and approximate smooth operators.

Whereas there exist approximately smooth but not smooth operators (cf. Example 6.12), sometimes the notions of approximate and exact smoothness coincide.

Theorem 6.15. *Let $\mathbb{H}$ be a Hilbert space,* $\dim \mathbb{H} \geq 2$ *and let $T \in \mathbb{L}(\mathbb{H})$ be a nonzero operator satisfying* $\operatorname{dist}(T, \mathbb{K}(\mathbb{H})) < \|T\|$. *Then T is approximately smooth if and only if it is smooth.*

Proof. Without loss of generality we assume that $\|T\| = 1$. Obviously, if T is smooth then it is also approximately smooth. To prove the converse, suppose that T is approximately smooth but not smooth. It is known (cf. [27, 39]) that $\mathbb{K}(\mathbb{H})$ is an M-ideal in $\mathbb{L}(\mathbb{H})$ whence it follows then from Theorem 6.2 that $M_T \neq \emptyset$ and by [66, Th. 4.6], $|M_T| > 2$, i.e., there exist two linearly independent elements $x_0, y_0 \in M_T$. By [82, Th. 2.2], $\operatorname{span}\{x_0, y_0\} \cap S_{\mathbb{X}} \subset M_T$ so we may consider $x_0, y_0 \in M_T$ such that $x_0 \perp y_0$. Then by [90, Th. 2.3] we have also $Tx_0 \perp Ty_0$.

Consider now $S_0 \in \mathbb{L}(\mathbb{H})$ such that

$$S_0 x_0 = Tx_0, \quad S_0 y_0 = -Ty_0, \quad S_0 h = \theta, \text{ for all } h \in H,$$

where $H := \operatorname{span}\{x_0, y_0\}^{\perp}$. Let $z = \alpha x_0 + \beta y_0 + h \in S_{\mathbb{H}}$ for some $\alpha, \beta \in \mathbb{R}$, $h \in H$. Then $\|S_0 z\|^2 = \|\alpha T x_0 - \beta T y_0\|^2 = |\alpha|^2 + |\beta|^2 \leq |\alpha|^2 + |\beta|^2 + \|h\|^2 = \|z\|^2 = 1$. So $\|S_0\| \leq 1$ but $\|S_0 x_0\| = \|Tx_0\| = \|T\| = 1$, hence $\|S_0\| = 1$.

In view of Theorem 6.3, for any $S \in \mathbb{L}(\mathbb{H})$ we have

$$\begin{aligned} \rho'_+(T,S) &= \sup_{x \in M_T} \langle Tx, Sx \rangle, \\ \rho'_-(T,S) &= \inf_{x \in M_T} \langle Tx, Sx \rangle, \end{aligned}$$

by which $\rho'_+(T, S_0) = 1$ and $\rho'_-(T, S_0) = -1$. Finally, from Lemma 6.3,

$$d(T) = \sup_{\|S\|=1} \{\rho'_+(T,S) - \rho'_-(T,S)\} \geq \rho'_+(T,S_0) - \rho'_-(T,S_0) = 2,$$

which means that T is not approximately smooth. This contradiction finishes the proof. □

An immediate corollary follows from the above theorem.

Corollary 6.8. *Let $T \in \mathbb{K}(\mathbb{H})$ be a nonzero operator. Then T is approximately smooth if and only if it is smooth.*

The study of coincidence of approximate smoothness and exact smoothness can be extended little further in the Banach space settings. Before getting into the results, we note the following proposition.

Proposition 6.8. *Let $\mathbb{Y}$ be any Banach space and let $T \in \mathbb{L}(\ell_1^n, \mathbb{Y})$ be a nonzero operator. If T is approximately smooth then M_T is a singleton (up to the sign).*

Proof. Without loss of generality we assume that $\|T\| = 1$. Let $\{e_k\}_{k=1}^n$ be the standard basis of ℓ_1^n. We have $\operatorname{Ext}(B_{\ell_1^n}) = \{\pm e_k : 1 \leq k \leq n\}$. As $M_T \cap \operatorname{Ext}(B_{\ell_1^n}) \neq \emptyset$ (Theorem 6.2), there exists $e_i \in M_T \cap \operatorname{Ext}(B_{\ell_1^n})$ for some $1 \leq i \leq n$. If $M_T = \{\pm e_i\}$ then we are done. Suppose on the contrary that there exists $x \neq \pm e_i$ such that $\|Tx\| = 1$. Since $x \in S_{\ell_1^n}$, we can write $x = \sum_{k=1}^n \alpha_k e_k + \sum_{k=1}^n \beta_k(-e_k)$, for some $\alpha_k, \beta_k \geq 0$ and $\sum_{k=1}^n \alpha_k + \sum_{k=1}^n \beta_k = 1$. Since $x \neq e_i$, we have for some $j \neq i$ that $\alpha_j \neq 0$ or $\beta_j \neq 0$. Suppose that $e_j \notin M_T$, i.e., $\|Te_j\| < 1$. This would lead to:

$$\begin{aligned} 1 = \|Tx\| &= \left\| T\left(\sum_{k=1}^n \alpha_k e_k + \sum_{k=1}^n \beta_k(-e_k)\right)\right\| \\ &\leq \sum_{k=1}^n \alpha_k \|Te_k\| + \sum_{k=1}^n \beta_k \|Te_k\| \\ &< \sum_{k=1}^n \alpha_k + \sum_{k=1}^n \beta_k = 1, \end{aligned}$$

a contradiction. Therefore, there exists $j \neq i$ such that $e_i, e_j \in M_T$. Now we consider an operator $S \in \mathbb{L}(\ell_1^n, \mathbb{Y})$ defined by $Se_i = \frac{Te_i}{\|Te_i\|}$, $Se_j = -\frac{Te_j}{\|Te_j\|}$ and $Se_k = \theta$ for all $k \in \{1, 2, \ldots, n\} \setminus \{i, j\}$. Clearly $\|S\| = 1$ and it follows from Theorem 6.3 that $\rho'_+(T, S) = \rho'_+(Te_i, Se_i) = 1$, $\rho'_-(T, S) = \rho'_+(Te_j, Se_j) = -1$. Applying Lemma 6.1 we obtain that $d(T) = 2$. This contradicts the fact that T is approximately smooth and proves the desired result. □

Now let us observe the following result.

Theorem 6.16. *Let $\mathbb{Y}$ be a smooth Banach space and let $T \in \mathbb{L}(\ell_1^n, \mathbb{Y})$ be a nonzero operator. Then T is approximately smooth if and only if it is smooth.*

Proof. We only need to show the necessary part. From Proposition 6.8 we note that $M_T = \{\pm e_i\}$ for some $1 \leq i \leq n$. Since $\mathbb{Y}$ is smooth, Te_i is smooth. Therefore, using [72, Th. 4.1] we get that T is smooth. □

If we consider the space $\mathbb{L}(\ell_1^n, \ell_\infty^n)$ then also we get that approximate smoothness implies smoothness.

Proposition 6.9. *Let $T \in \mathbb{L}(\ell_1^n, \ell_\infty^n)$ be a nonzero operator. Then T is approximately smooth if and only if it is smooth.*

Proof. Again we only show the necessary part. From Proposition 6.8, we get $M_T = \{\pm e_i\}$, for some $1 \leq i \leq n$. We only need to show that Te_i is smooth. Suppose on the contrary that it is not the case. Then $d(Te_i) = 2$; indeed, as Te_i is non-smooth, there exist two linearly independent functionals $y^*, z^* \in \operatorname{Ext}(J(Te_i))$. It is easy to observe that $\|y^* - z^*\| = 2 = d(Te_i)$. Therefore, using Proposition 6.6 we obtain that $d(T) \geq d(Te_i) = 2$, whence $d(T) = 2$. This contradicts the assumed approximate smoothness of T and hence the theorem. □

We end this subsection with the following theorem.

Theorem 6.17. *Let $\mathbb{Y}$ be a Banach space. Suppose that for any nonzero operator $T \in \mathbb{L}(\ell_1^n, \mathbb{Y})$, approximate smoothness of T implies that T is smooth. Then either of the following holds true:*

(i) $\mathbb{Y}$ is smooth

(ii) $\mathbb{Y}$ is not approximately smooth.

Proof. Suppose, on the contrary, that neither (i) nor (ii) hold, i.e., that $\mathbb{Y}$ is non-smooth but approximately smooth. Therefore, there exists $u \in S_{\mathbb{Y}}$ such that u is ε-approximately smooth for some $\varepsilon \in (0, 2)$. Define $S \in \mathbb{L}(\ell_1^n, \mathbb{Y})$ as $Se_1 = u$ and $Se_k = 0$ for all $k \in \{2, \ldots, n\}$. Then $M_S = \{\pm e_1\}$. Applying Corollary 6.7 we note that $d(S) = d(Se_1) = d(u) \leq \varepsilon$. Thus S is ε-approximately smooth. On the other hand it follows from from [72, Th. 4.2] that S is non-smooth. This contradicts the assumption and completes the proof. □

6.4.4 Approximate smoothness of adjoint operators

We establish a correspondence between approximate smoothness of a compact operator T and its adjoint T^*, assuming reflexivity of the target space.

Theorem 6.18. *Let $\mathbb{X}$ be a Banach space and $\mathbb{Y}$ a reflexive Banach space. Let $T \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$. Then T is ε-approximately smooth (for some $\varepsilon \in [0, 2)$) if and only if T^* is ε-approximately smooth.*

Proof. 1. *Necessity.* Suppose that T is ε-approximately smooth and T^* is not ε-approximately smooth, hence $d(T) \leq \varepsilon < d(T^*)$. Following Proposition 6.1, there exist $f, g \in \operatorname{Ext}(J(T^*))$ such that $\|f - g\| > \varepsilon$. Applying Theorem 6.1 we suppose that $f = \sigma_1 \otimes x_1^{**}$ and $g = \sigma_2 \otimes x_2^{**}$, where $\sigma_1, \sigma_2 \in \operatorname{Ext}(B_{\mathbb{Y}^{***}})$ and $x_1^{**}, x_2^{**} \in \operatorname{Ext}(B_{\mathbb{X}^{**}})$. Since $\|\sigma_1 \otimes x_1^{**} - \sigma_2 \otimes x_2^{**}\| > \varepsilon$, it follows that there exists $S \in S_{\mathbb{K}(\mathbb{Y}^*, \mathbb{X}^*)}$ such that $|(\sigma_1 \otimes x_1^{**} - \sigma_2 \otimes x_2^{**})(S)| > \varepsilon$. As $\mathbb{Y}$ is reflexive, there exists $A \in S_{\mathbb{K}(\mathbb{X}, \mathbb{Y})}$ such that $A^* = S$ and therefore,

$$|(\sigma_1 \otimes x_1^{**} - \sigma_2 \otimes x_2^{**})(A^*)| > \varepsilon. \tag{6.6}$$

Since $\mathbb{Y}$ is reflexive, so is $\mathbb{Y}^*$. Thus the canonical embedding $\psi\colon \mathbb{Y}^* \to \mathbb{Y}^{***}$ is an isometric isomorphism between $\mathbb{Y}^*$ and $\mathbb{Y}^{***}$. For $i = 1, 2$, let $y_i^* \in \mathbb{Y}^*$ be such that $\psi(y_i^*) = \sigma_i$. We note that

$$\begin{aligned}(x_i^{**} \otimes y_i^*)(T) &= (x_i^{**}T^*)(y_i^*) = \psi(y_i^*)(x_i^{**}T^*) = \sigma_i(x_i^{**}T^*) = \sigma_i(T^{**}x_i^{**}) \\ &= \sigma_i \otimes x_i^{**}(T^*) = \|T^*\| = \|T\|.\end{aligned}$$

Thus $\|x_i^{**} \otimes y_i^*\| = 1$ and $x_i^{**} \otimes y_i^* \in \mathrm{Ext}\,(J(T))$, for $i = 1, 2$. Also, from (6.6), we see

$$\begin{aligned}|(x_1^{**} \otimes y_1^* - x_2^{**} \otimes y_2^*)(A)| &= |x_1^{**}(A^*y_1^*) - x_2^{**}(A^*y_2^*)| \\ &= |\psi(y_1^*)(A^{**}x_1^{**}) - \psi(y_2^*)(A^{**}x_2^{**})| \\ &= |\sigma_1(A^{**}x_1^{**}) - \sigma_2(A^{**}x_2^{**})| \\ &= |(\sigma_1 \otimes x_1^{**} - \sigma_2 \otimes x_2^{**})(A^*)| > \varepsilon,\end{aligned}$$

which implies that $\|x_1^{**} \otimes y_1^* - x_2^{**} \otimes y_2^*\| > \varepsilon$ and hence $d(T) > \varepsilon$. This contradiction yields T^* is ε-approximately smooth.

2. *Sufficiency.* Suppose that T^* is ε-approximately smooth, but T is not ε-approximately smooth. Then, there exist $\sigma, \eta \in \mathrm{Ext}\,(J(T))$ such that $\|\sigma - \eta\| > \varepsilon$ and, by Theorem 6.1, σ, η can be represented as:

$$\sigma := x^{**} \otimes y^*, \quad \eta := u^{**} \otimes v^*, \qquad x^{**}, u^{**} \in \mathrm{Ext}\,(B_{\mathbb{X}^{**}}),\ y^*, v^* \in \mathrm{Ext}\,(B_{\mathbb{Y}^*}).$$

Thus there exists $A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}$ such that $|(x^{**} \otimes y^* - u^{**} \otimes v^*)(A)| > \varepsilon$.

Now, consider $\psi(y^*), \psi(v^*) \in \mathbb{Y}^{***}$, where ψ is the canonical isometric embedding of $\mathbb{Y}^*$ onto $\mathbb{Y}^{***}$. Then $\psi(y^*) \otimes x^{**}, \psi(v^*) \otimes u^{**} \in J(T^*)$. Indeed,

$$\begin{aligned}(\psi(y^*) \otimes x^{**})(T^*) &= \psi(y^*)(T^{**}x^{**}) = (T^{**}x^{**})(y^*) \\ &= x^{**}(T^*y^*) = (x^{**} \otimes y^*)(T) = \|T\| = \|T^*\|\end{aligned}$$

and (by reflexivity of $\mathbb{Y}$ and Lemma 6.4)

$$\begin{aligned}\|\psi(y^*) \otimes x^{**}\| &= \sup_{B \in S_{\mathbb{L}(\mathbb{Y}^*,\mathbb{X}^*)}} |(\psi(y^*) \otimes x^{**})(B)| \\ &= \sup_{A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} |(\psi(y^*) \otimes x^{**})(A^*)| \\ &= \sup_{A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} |\psi(y^*)(A^{**}x^{**})| = \sup_{A \in S_{\mathbb{L}(\mathbb{X},\mathbb{Y})}} |A^{**}x^{**}(y^*)| = 1.\end{aligned}$$

Therefore $\psi(y^*) \otimes x^{**} \in J(T^*)$ and similarly, one can show that $\psi(v^*) \otimes u^{**} \in J(T^*)$. Now, we have

$$\begin{aligned} |(\psi(y^*) \otimes x^{**} - \psi(v^*) \otimes u^{**})(A^*)| &= |\psi(y^*)(A^{**}x^{**}) - \psi(v^*)(A^{**}u^{**})| \\ &= |(A^{**}x^{**})(y^*) - (A^{**}u^{**})(v^*)| \\ &= |x^{**}(A^*y^*) - u^{**}(A^*v^*)| \\ &= |(x^{**} \otimes y^* - u^{**} \otimes v^*)(A)| > \varepsilon. \end{aligned}$$

This implies that $\|\psi(y^*) \otimes x^{**} - \psi(v^*) \otimes u^{**}\| > \varepsilon$, i.e., T^* is not ε-approximately smooth, which is a contradiction. □

Corollary 6.9. *If $\mathbb{X}$ and $\mathbb{Y}$ are finite-dimensional Banach spaces, then $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is ε-approximately smooth if and only if T^* is ε-approximately smooth.*

6.4.5 Rank-one operators

A rank-one operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ can be written in the form

$$Tx = x^*(x)w, \qquad x \in \mathbb{X}, \tag{6.7}$$

where $x^* \in \mathbb{X}^*$ and $w \in \mathbb{Y}$. We have $\|T\| = \|x^*\|\|w\|$, thus it is clear that $M_T = M_{x^*}$. The norm attainment set M_{x^*} can be written in the form $M_{x^*} = F \cup (-F)$, where F is a convex subset of $S_{\mathbb{X}}$. It follows that $M_T = F \cup (-F)$. In this context we note the following theorem as a corollary from Proposition 6.5.

Theorem 6.19. *Let $\mathbb{X}$ be a reflexive normed linear space and let $\mathbb{Y}$ be any normed linear space. Assume that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is an M-ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Suppose that $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is a rank-one operator having the form (6.7) with $M_T = F \cup (-F)$ as above. If*

(i) $\operatorname{diam}(F) = r$

(ii) w is s-approximately smooth, for some $s \in [0, 2)$,

then T is $(r + s)$-approximately smooth, whenever $r + s < 2$.

Following the above theorem we note some corollaries.

Proposition 6.10. *Let $\mathbb{X}$ be a reflexive normed linear space and let $\mathbb{Y}$ be a smooth normed space. Assume that $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ is an M-ideal in $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. Suppose that $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ is rank-one* *and has the form* (6.7) *for some* $x^* \in \mathbb{X}^*$ *and* $w \in \mathbb{Y}$. *Define* $F := \{x \in S_{\mathbb{X}} : x^*(x) = \|x\|\}$, *i.e.,* F *is a face on* $B_{\mathbb{X}}$. *Then* $M_T = F \cup (-F)$ *and* T *is* ε*-approximately smooth if and only if* $\mathrm{diam}(F) = \varepsilon$.

Proof. As $\mathbb{X}$ is reflexive, $M_T \neq \emptyset$. Then we have

$$\begin{aligned} d(T) &= \sup\{\|y^* \otimes x - y^* \otimes z\| : \ x, z \in \mathrm{Ext}\,(F),\ J(w) = \{y^*\}\} \\ &= \sup\{\|x - z\| : \ x, z \in \mathrm{Ext}\,(F)\} = \mathrm{diam}(F) \end{aligned}$$

and the assertion follows. □

Proposition 6.11. *Let* $\mathbb{X}$ *be a reflexive and strictly convex Banach space and let* $\mathbb{Y}$ *be a normed linear space. Assume that* $\mathbb{K}(\mathbb{X}, \mathbb{Y})$ *is an* M*-ideal in* $\mathbb{L}(\mathbb{X}, \mathbb{Y})$. *Suppose that* $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ *has the form* (6.7) *for some* $x^* \in \mathbb{X}^*$ *and* $w \in \mathbb{Y}$. *Then* T *is* ε*-approximately smooth if and only if* w *is* ε*-approximately smooth.*

Proof. Since $\mathbb{X}$ is reflexive and strictly convex, it follows that $\mathbb{X}^*$ is smooth. Therefore, x^* attains its norm at a unique point on $S_{\mathbb{X}}$ (up to the sign). Thus $M_T = M_{x^*} = \{\pm x_0\}$ for some $x_0 \in S_{\mathbb{X}}$. It follows that $T(x_0) = \pm w$ and we can assume that $T(x_0) = w$, hence $\mathrm{Ext}\,(J(T)) = \{y^* \otimes x_0 : y^* \in \mathrm{Ext}\,(J(w))\}$. Therefore, using Proposition 6.1 we get

$$\begin{aligned} d(T) &= \sup\{\|y^* \otimes x_0 - z^* \otimes x_0\| : \ y^*, z^* \in \mathrm{Ext}\,(J(w))\} \\ &= \sup\{\|y^* - z^*\| : \ y^*, z^* \in \mathrm{Ext}\,(J(w))\} = d(w) \end{aligned}$$

and our result is proved. □

Note that if both $\mathbb{X}^*$ and $\mathbb{Y}$ are smooth spaces, then for any $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ having the form (6.7) there is $M_T = \{\pm x_0\}$ for some $x_0 \in S_{\mathbb{X}}$ and Tx_0 is also a smooth point in $\mathbb{Y}$. Therefore, using previous arguments the following corollary can be derived.

Corollary 6.10. *Let* $\mathbb{X}$ *be reflexive and strictly convex and let* $\mathbb{Y}$ *be smooth. Suppose that* $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ *is rank-one. Assume that* $\mathcal{K}(\mathbb{X}, \mathbb{Y})$ *is an* M*-ideal in* $\mathcal{L}(\mathbb{X}, \mathbb{Y})$. *Then* T *is* ε*-approximately smooth if and only if* T *is smooth.*

The following example shows that strict convexity is an essential assumption.

Example 6.20. *Let* $\mathbb{X}$ *be the space* $\mathbb{R}^2$ *such that* $B_{\mathbb{X}}$ *is a regular hexagon with* $\mathrm{Ext}\,(B_{\mathbb{X}}) = \{\pm x_1, \pm x_2, \pm x_3\}$. *Consider* $x^* \in S_{\mathbb{X}^*}$ *that supports the segment* $[x_1, x_2]$. *Define* $T \colon \mathbb{X} \to \ell_2^2$ *by* $Tx = x^*(x)e_1$, $x \in \mathbb{X}$. *Clearly,* T *is rank-one and* $M_T = \pm[x_1, x_2]$. *Thus*

$$\mathrm{Ext}\,(J(T)) = \{y^* \otimes x : x \in M_T \cap \mathrm{Ext}\,(B_{\mathbb{X}}), y^* \in J(Tx)\} = \{e_1 \otimes x_1, e_1 \otimes x_2\}.$$

So $d(T) = \operatorname{diam}(\operatorname{Ext}(J(T))) = \|x_1 - x_2\| = 1$ *whence* T *is* 1-*approximately smooth but not smooth.*

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