STUDY OF SOME CONNECTIONS ON ALMOST CONTACT, PARA CONTACT AND CONTACT MANIFOLDS

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by

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under the guidance of

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CERTIFICATE

Date: / / 2025

This is to certify that the following thesis entitled **Study of Some Connections on Almost Contact, Para Contact and Contact manifolds** is being submitted by **Mr. Arup Kumar Mallick** for the award of the degree of **Doctor of Philosophy** to the **Department of Mathematics**, **Jadavpur University,** is a record of bonafide research work carried out by him under our guidance and supervision. **Mr. Arup Kumar Mallick** has followed the rules and regulations proposed by **Jadavpur University** for the fulfillment of the requirements for the degree of **Doctor of Philosophy in Science**.

The results of the thesis have not been submitted to any University or Institution for the award of any Degree or Diploma.

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DECLARATION

I declare that

(a) the work done as part of the thesis is original and has been done by me under the guidance of my supervisor Prof. Arindam Bhattacharyya and co-supervisor Dr. Barnali Laha;

(b) I have not submitted the work for any degree or diploma to any other University or Institute;

(c) I have conformed to ethical norms and guidelines while writing the thesis;

(d) I have followed the guidelines provided by the University in preparing the thesis;

(e) Whenever I have used example materials from other sources for reference, I have given them due credit by citing them in the text of the thesis, giving their details in the references.

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Dedicated to my parents

Shri Kali Das Mallick and Smt. Pratima Mallick

&

my wife Rupa Das Mallick and son Archit Mallick

List of Publications

Published Papers

(i) Arup Kumar Mallick and Arindam Bhattacharyya. A (k,μ)-contact metric manifold as an η-Einstein soliton. *Korean Journal of Mathematics*, 32(2):315–328, 2024.

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(iv) Barnali Laha and Arup Kumar Mallick. A study of Kenmotsu manifolds with semi-symmetric metric connection. *International Journal of Mathematical Combinatorics*, 2:8–20, 2022.

(v) Barnali Laha and Arup Kumar Mallick. Some results on LP-Sasakian manifolds with semi-symmetric metric connection. *Bulletin of Calcutta Mathematical Society*, 114(6):841–852, 2022.

Accepted Papers

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(ii) Arup Kumar Mallick and Arindam Bhattacharyya. $*$-Conformal η-Einstein soliton on Trans-Sasakian manifold, 2025.

(iii) Arup Kumar Mallick, Nandan Bhunia and Arindam Bhattacharyya. Conformal η-Einstein soliton on (k,μ)-contact metric manifold, 2024.

(iv) Arup Kumar Mallick, Brnali Laha and Arindam Bhattacharyya. Conformal η-Einstein soliton on 3-dimensional LP-Sasakian manifold, 2023.

(v) Arup Kumar Mallick, Brnali Laha and Arindam Bhattacharyya. Conformal η-Einstein soliton on ε-Kenmotsu manifold, 2023.

Contents

List of Symbols and Abbreviations

Symbols:

$\mathbb{R}$	Set of real numbers
R	Riemannian curvature tensor
R^*	Scalar curvature tensor
λ	Smooth function
λ^*, μ^*	Constants
H_i^*	Vetor fields
T^*	Torsion tensor
T	Energy momentum tensor
E	Einstein tensor
ω	Conformal pressure
σ	Energy density
σ^*	Scalar function
ρ	Isotropic pressure
κ	Gravitational constant

Abbreviations:

RS	Ricci soliton
ηRS	η-Ricci soliton
CηRS	conformal η-Ricci soliton
CRS	conformal Ricci soliton
$*$-RS	$*$-Ricci soliton
$*$-CηRS	$*$-conformal η-Ricci soliton
ES	Einstein soliton
ηE	η-Einstein
ηES	η-Einstein soliton
CηES	conformal η-Einstein soliton
CES	conformal Einstein soliton
$*$-CηES	$*$-conformal η-Einstein soliton
TFVF	Torse forming Vecor field
LP-S	Lorentzian para-Sasakian

Chapter 1

Introduction

In mathematics, particularly in differential geometry and topology, a manifold is a topological space that locally resembles Euclidean space of a certain dimension, known as the manifold's dimension. For example, a line and a circle are one-dimensional manifolds, a plane and a sphere (the surface of a ball) are two-dimensional manifolds, and this concept extends to higher-dimensional spaces.
A manifold M is a topological space where

(i) M is a Hausdorff space,

(ii) each point $h_1^* \in M$ has a neighborhood which is homeomorphic to an open subset V of $\mathbb{R}^n$, i.e., M is locally Euclidean,

(iii) M is second countable, i.e., M has a countable basis of open sets.

Riemannian geometry is an extension of the differential geometry of surfaces in E^3. It is impossible to discuss Riemannian geometry without acknowledging the foundational works in the field, like the writings of C. F. Gauss (1825, 1827) and B. Riemann (1854), G. Darboux's comprehensive summary (1894) of nineteenth and early twentieth-century work, and E. Cartan's lectures (1946), in which the method of moving frames became a powerful and exciting tool in differential geometry. Two key concepts in Riemannian geometry are geodesics and curvature. In this context, a Riemannian manifold or Riemannian space (M, g) is a real differentiable manifold M in which each tangent space is endowed with an inner product g, called the Riemannian metric, which varies smoothly from point to point. The metric g is a symmetric tensor field of type (0,2) on M. Essentially, a Riemannian manifold is a differentiable manifold where the tangent space at each point is a finite-dimensional Euclidean space.
An affine connection on a Riemannian manifold is a mathematical tool that enables the differentiation of vector fields along other vector fields, thereby allowing us to define key geometric concepts such as covariant derivatives, geodesics, and curvature on the

manifold.
A Riemannian manifold (M^n, g) is a smooth manifold equipped with a Riemannian metric g. An affine connection ∇ is a compatible with the Riemannian structure if it satisfies:

(i) $H_1^* \left[g(H_2^*, H_3^*)\right] = g(\nabla_{H_1^*} H_2^*, H_3^*) + g(H_2^*, \nabla_{H_1^*} H_3^*)$,

(i) $\nabla_{H_1^*} H_2^* - \nabla_{H_2^*} H_1^* = [H_1^*, H_2^*]$.

An affine connection ∇ on a Riemannian manifold (M^n, g) of dimension n is called the Levi-Civita connection or Riemannian connection if it satisfies the following properties:

$$\nabla g = 0 \ \ and \ \ T^* = 0,$$

where T^* is a torsion tensor.
In 1924, Friedmann and Schouten proposed the notion of a semi-symmetric linear connection on a differential manifold [32]. Hayden in 1932 gave the idea of metric connection with torsion on Riemannian manifold in [44]. In 1970, K. Yano [92] introduced a semi-symmetric metric connection on Riemannian manifold. Later many researchers [3, 20, 35, 75, 84, 92] also worked on semi-symmetric metric connection on Riemannian manifold.
Let (M^n, g) be a Riemannian manifold with the Levi-Civita connection ∇.
A linear connection ∇ on (M^n, g) is semi-symmetric if its torsion tensor T^* can be written as

$$T^*(H_1^*, H_2^*) = \eta(H_2^*)H_1^* - \eta(H_1^*)H_2^*, \tag{1.1}$$

for all vector fields H_1^*, H_2^*, where η is a 1-form on the manifold.
A connection $\bar{\nabla}$ is called semi-symmetric metric connection if $\bar{\nabla} g = 0$.
A contact manifold is a type of manifold that is closely related to symplectic geometry, but it has a different structure. More specifically, a contact manifold is a differentiable manifold M equipped with a contact form η, which is a 1-form that satisfies a certain non-degeneracy condition.
A contact manifold is a smooth manifold M of odd dimension $(2n+1)$ together with a 1-form η such that the contact condition holds. This means that the 1-form η is "non-degenerate" in a particular way. The condition ensures that the distribution of hyperplanes defined by the kernel of η is maximally non-integrable, giving the contact structure its geometric significance.
An almost contact manifold is a generalization of a contact manifold where the non-degeneracy condition on the contact form is relaxed. While a contact manifold is equipped with a 1-form that satisfies a strict non-degeneracy condition (the contact condition), an almost contact manifold only requires a weaker condition, making it a more

flexible structure.

An almost contact manifold is a smooth manifold M of odd dimension $(2n+1)$ that is equipped with a triplet (η,ϕ,ξ), where η is a 1-form on M, often called the almost contact form, ξ is a vector field, called the Reeb vector field, which satisfies $\eta(\xi)=1$. These conditions ensure that ξ is not arbitrary, but specifically aligned with ϕ is a $(1,1)$-tensor field, often called the structure tensor.

A $(2n+1)$-dimensional smooth manifold $\left(M^{2n+1},g\right)$ is called an almost contact metric manifold with structure (ϕ,ξ,η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and a Riemannian metric g if

$$\phi^2(H_1^*)=-H_1^*+\eta(H_1^*)\xi,\ \eta(\xi)=1,\ \eta(\phi H_1^*)=0,\ \phi\xi=0, \tag{1.2}$$

$$g(\phi H_1^*,\phi H_2^*)=g(H_1^*,H_2^*)-\eta(H_1^*)\eta(H_2^*),\quad g(H_1^*,\xi)=\eta(H_1^*),\quad g(\xi,\xi)=1. \tag{1.3}$$

Then M^{2n+1} becomes an almost contact metric manifold furnished with an almost contact metric structure (ϕ,ξ,η,g), i.e.,

$$g(H_1^*,\phi H_2^*)=-g(\phi H_1^*,H_2^*). \tag{1.4}$$

An almost contact metric structure enhance a contact metric structure if

$$d\eta(H_1^*,H_2^*)=g(H_1^*,\phi H_2^*). \tag{1.5}$$

In a contact metric manifold M^{2n+1}, we define the (1,1)-tensor field h by $2hH_1^*=\left(\mathcal{L}_\xi\phi\right)(H_1^*)$, where $\mathcal{L}_\xi$ denotes Lie differentiation in the direction of the vector field ξ. The tensor h is symmetric, such that

$$h\xi=0,\ h\phi=-\phi h,\ tr(h)=0,\ tr(\phi h)=0, \tag{1.6}$$

$$\left(\nabla_{H_1^*}\eta\right)H_2^*=g(H_1^*,\phi H_2^*)-g(H_1^*,\phi hH_2^*), \tag{1.7}$$

$$\nabla_{H_1^*}\xi=-\phi H_1^*-\phi hH_1^*, \tag{1.8}$$

$$h^2=(k-1)\phi^2, \tag{1.9}$$

and

$$\text{rank}\phi=2n, \tag{1.10}$$

for every $H_1^*,H_2^*\in\chi(M)$.

A paracontact manifold is a geometric structure that generalizes both almost contact manifolds and contact manifolds. It is a type of almost contact structure, but with a specific twist on the properties of the structure tensor ϕ. Like almost contact manifolds, paracontact manifolds are also defined by a triplet (η, ϕ, ξ), but the key difference lies in how the structure tensor behaves with respect to the metric and the other components. A n-dimensional manifold is known as a Lorentzian almost paracontact manifold with structure (ϕ, ξ, η, g), where a 1-form η, a (1,1) tensor field ϕ, a contravariant vector field ξ and a Lorentzian metric g satisfy the relations

$$\phi^2(H_1^*) = H_1^* + \eta(H_1^*)\xi,\ \eta(\xi) = -1, \tag{1.11}$$

$$g(\phi H_1^*, \phi H_2^*) = g(H_1^*, H_2^*) + \eta(H_1^*)\eta(H_2^*),\ g(H_1^*, \xi) = \eta(H_1^*). \tag{1.12}$$

In the Lorentzian almost paracontact manifold, the following relations hold:

$$g(H_1^*, \phi H_2^*) = g(H_2^*, \phi H_1^*),\ \phi\xi = 0,\ \eta(\phi H_1^*) = 0. \tag{1.13}$$

In 1982, R.S. Hamilton [37, 36] introduced the Ricci flow as a method to obtain a canonical metric on a smooth manifold. The Ricci flow is an evolution equation that applies to a Riemannian metric $g(t)$ on a smooth manifold M. It is defined by the following equation:

$$\frac{\partial g}{\partial t} = -2R_t, \tag{1.14}$$

where R_t is the Ricci tensor with respect to a Riemannian metric $g(t)$.

A soliton is a unique type of solution to certain partial differential equations that preserves its shape and structure over time. Solitons are often associated with integrable systems and exhibit stability and they do not change as they evolve. Solitons have important applications in various areas of mathematics and physics, fluid dynamics, nonlinear waves, quantum field theory, and general relativity.

In differential geometry, solitons refer to specific geometric structures or solutions to geometric flows that preserve their shape or evolve in a stable manner over time. These solitons arise in the context of geometric partial differential equations, which describe the evolution of geometric objects like metrics, curvatures, or submanifolds.

Solitons are typically associated with geometric flows such as the Ricci flow, mean curvature flow, and harmonic map flow. These solitons represent steady or self-similar solutions to these flows, and they play a crucial role in understanding the behavior of geometric structures under evolution.

A smooth manifold M, equipped with a Riemannian metric g, is known as a Ricci soliton (RS) if it moves only by a one parameter family of diffeomorphism and scaling. For the RS there exists a constant λ^* and a smooth vector field V on M that satisfies the following equation:

$$\mathcal{L}_V g + 2R_t = 2\lambda^* g, \tag{1.15}$$

where $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V and λ^* is a constant. The RS exhibits expanding, steady and shrinking behaviour depending on $\lambda^* < 0$, $\lambda^* = 0$, $\lambda^* > 0$ respectively.

A RS is a generalization of an Einstein metric which moves only by an one-parameter group diffeomorphisms and scaling [37].

A.E. Fischer [31] in 2005, developed the concept of conformal Ricci flow equation which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M is defined by the equation [31]

$$\frac{\partial g}{\partial t} + 2\left(R_t + \frac{g}{n}\right) = -\omega g, \qquad R^*(g) = -1, \tag{1.16}$$

where M is considered as a smooth closed connected oriented manifold, $R^*(g)$ is the scalar curvature of the manifold.

In 2015, N. Basu and A. Bhattacharyya [7] introduced the notion of conformal Ricci soliton (CRS) as a generalization of the RS and the equation is given by

$$\mathcal{L}_V g + 2R_t = \left[2\lambda^* - \left(\omega + \frac{2}{n}\right)\right] g, \tag{1.17}$$

where ω is the conformal pressure, which is a nondynamical scalar field.

The concept of η-Ricci soliton (ηRS) introduced by J.T. Cho and M. Kimura [25], and later C. Calin and M. Crasmareanu [18] studied it on Hopf hyper-surfaces in complex space forms. A Riemannian manifold is said to admit an ηRS if for a smooth vector field V, the metric g satisfies the following equation

$$\mathcal{L}_V g + 2R_t + 2\lambda^* g + 2\mu^* \eta \otimes \eta = 0, \tag{1.18}$$

where λ^*, μ^* are constants.

In 2018, A.M. Blaga [9] proposed that a Riemannian manifold admits an η-Einstein soliton (ηES) if the equation satisfies

$$\mathcal{L}_V g + 2R_t + (2\lambda^* - R^*) g + 2\mu^* \eta \otimes \eta = 0, \tag{1.19}$$

where R^* is the scalar curvature of the metric g. For $\mu^* = 0$, (1.19) reduces to Einstein soliton (ES) [19].

In 2018, M.D. Siddiqui [76] introduced the concept of a conformal η-Ricci soliton

(CηRS) and the equation is given by

$$\mathcal{L}_V g + 2R_t + \left[2\lambda^* - \left(\omega + \frac{2}{n}\right)\right] g + 2\mu^* \eta \otimes \eta = 0. \tag{1.20}$$

In 2021, Roy et al. [69] introduced a conformal Einstein soliton (CES) on n-dimensional manifold M and it is defined by

$$\mathcal{L}_V g + 2R_t + \left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right)\right] g = 0. \tag{1.21}$$

A n-dimensional Riemannian manifold (M, g) is called a conformal η-Einstein soliton (CηES) [67] if

$$\mathcal{L}_V g + 2R_t + \left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right)\right] g + 2\mu^* \eta \otimes \eta = 0. \tag{1.22}$$

If $\mu^* = 0$, then it becomes conformal Einstein soliton (CES).
A Riemannian (or semi-Riemannian) metric g on M is called a $*$-Ricci soliton ($*$-RS) [45], if

$$\mathcal{L}_V g + 2R_t^* + 2\lambda g = 0. \tag{1.23}$$

A Riemannian (or semi-Riemannian) metric g on M is called a $*$-conformal η-Ricci soliton ($*$-CηRS) [71] , if

$$\mathcal{L}_V g + 2R_t^* + \left(2\lambda - \left(\omega + \frac{2}{n}\right)\right) g + 2\mu \eta \otimes \eta = 0. \tag{1.24}$$

Now we define the notion of $*$-conformal η-Einstein soliton ($*$-CηES) as:

$$\mathcal{L}_V g + 2R_t^* + \left[2\lambda - R^* + \left(\omega + \frac{2}{n}\right)\right] g + 2\mu \eta \otimes \eta = 0. \tag{1.25}$$

The energy momentum tensor, accordance with Einstein's field equation is fundamental as it sheds light on the curvature of spacetime, playing a very important role in the theory of relativity. In general relativity, spacetime is conceptualized as a connected 4-dimension semi-Riemannian manifold with Lorentzian metric g characterized by $(-,+,+,+)$.

For a perfect fluid, the energy-momentum tensor T is [60]

$$\rho[g(H_1^*, H_2^*) + \eta(H_1^*)\eta(H_2^*)] + \sigma\eta(H_1^*)\eta(H_2^*) = T(H_1^*, H_2^*), \tag{1.26}$$

where σ is energy density and ρ is isotropic pressure in fluids,

$g(H_1^*,\xi) = -\eta(H_1^*)$ and $g(\xi,\xi) = -1$.

If $\rho = \rho(\sigma)$, then perfect fluid spacetime is said to be isentropic [43] and perfect fluid spacetime represents a dark energy era [23] for $\sigma + \rho = 0$. We have the Einstein's field equations [60]

$$R_t(H_1^*,H_2^*) = \frac{R^*}{2} g(H_1^*,H_2^*) + \kappa T(H_1^*,H_2^*), \tag{1.27}$$

where κ is the gravitational constant and cosmological constant is zero.
Put together (1.26) and (1.27) we get

$$R_t(H_1^*,H_2^*) = \left(\kappa\rho + \frac{R^*}{2}\right) g(H_1^*,H_2^*) + (\rho+\sigma)\kappa\eta(H_1^*)\eta(H_2^*). \tag{1.28}$$

1.1 Prerequisite and Related Work

Here we have discussed some definitions and lemma required for further work:

Definition 1.1.1. *A Riemannian manifold is said to be an η-Einstein (ηE) if its Ricci tensor R_t satisfies the form*

$$R_t(H_1^*,H_2^*) = ag(H_1^*,H_2^*) + b\eta(H_1^*)\eta(H_2^*), \tag{1.29}$$

where a,b are smooth functions.

Definition 1.1.2. *A vector field H_1^* on a Riemannian manifold $M^n(\phi,\xi,\eta,g)$ is said to be contact vector field if*

$$(\mathcal{L}_{H_1^*}\eta)(H_2^*) = \sigma^*\eta(H_2^*), \tag{1.30}$$

where σ^* is a scalar function on M and $\mathcal{L}_{H_1^*}$ denote the Lie derivative along H_1^*. H_1^* is called strict contact vector field if $\sigma^* = 0$. A relation between the curvature tensor R and $\bar{R}$ of type $(1,3)$ of the connections ∇ and $\bar{\nabla}$ respectively is given by [93]

$$\bar{R}(H_1^*,H_2^*)H_3^* = R(H_1^*,H_2^*)H_3^* + g(H_2^*,H_3^*)H_1^* - g(H_3^*,H_1^*)H_2^*. \tag{1.31}$$

Also Ricci tensor satisfies

$$\bar{R}_t(H_2^*,H_3^*) = R_t(H_2^*,H_3^*) - 2g(H_2^*,H_3^*) + 2\eta(H_3^*)\eta(H_2^*) + g(\phi H_2^*,H_3^*), \tag{1.32}$$

where $\bar{R}_t$ and R_t are Ricci tensor of M with respect to semi-symmetric metric connections

$\bar{\nabla}$ *and the Levi-Civita connection* ∇*, respectively. Also we have*

$$\bar{R}_t(H_2^*,H_3^*) = R_t(H_2^*,H_3^*) + 2ng(H_2^*,H_3^*). \tag{1.33}$$

Definition 1.1.3. *A Riemannian manifold with respect to semi-symmetric connection is said to be a* ϕ*-recurrent manifold if there exists a non-zero* 1*-form* B *such that*

$$\phi^2((\bar{\nabla}_{H_4^*}R)(H_1^*,H_2^*)H_3^*) = B(H_4^*)R(H_1^*,H_2^*)H_3^*. \tag{1.34}$$

Definition 1.1.4. *A Riemannian manifold* (M^n,g) *is called* ϕ*-generalized recurrent [26], if its curvature tensor* R *satisfies the condition*

$$\begin{aligned}\phi^2((\nabla_{H_4^*}R)(H_1^*,H_2^*)H_3^*) &= A(H_4^*)R(H_1^*,H_2^*)H_3^* \\ &+ B(H_4^*)[g(H_2^*,H_3^*)H_1^* - g(H_1^*,H_3^*)H_2^*],\end{aligned} \tag{1.35}$$

where A *and* B *are two* 1*-forms,* B *is non zero and these are defined by*

$$g(H_4^*,\rho_1) = A(H_4^*), g(H_4^*,\rho_2) = B(H_4^*).$$

Here ρ_1 *and* ρ_2 *being the vector fields associated to the* 1*-forms* A *and* B *respectively.*

Definition 1.1.5. *A Riemannian manifold is said to be an extended generalized* ϕ*-recurrent manifold if its curvature tensor* R *satisfies the relation*

$$\begin{aligned}\phi^2(\nabla_{H_4^*}R)(H_1^*,H_2^*)H_3^*) &= A(H_4^*)\phi^2(R(H_1^*,H_2^*)H_3^*) \\ &+ B(H_4^*)\phi^2[g(H_2^*,H_3^*)H_1^* - g(H_1^*,H_3^*)H_2^*],\end{aligned} \tag{1.36}$$

where A,B *are two non-vanishing 1-forms such that* $g(H_4^*,\rho_1) = A(H_4^*)$ *and* $g(H_4^*,\rho_2) = B(H_4^*)$*, for all* $H_4^* \in \chi(M)$*, with* ρ_1 *and* ρ_2 *being the vector fields associated to be* 1*-forms* A *and* B*, respectively [62].*

Definition 1.1.6. *[33] A Riemannian manifold* (M^n,g) *is said to have Codazzi type Ricci tensor* R_t*, if* R_t *is non-zero and satisfies the following relation*

$$(\nabla_{H_1^*}R_t)(H_2^*,H_3^*) = (\nabla_{H_2^*}R_t)(H_1^*,H_3^*). \tag{1.37}$$

Definition 1.1.7. *[33] A Riemannian manifold* (M^n,g) *is said to have cyclic parallel Ricci tensor* R_t*, if* R_t *is non-zero and satisfies the following relation*

$$(\nabla_{H_1^*}R_t)(H_2^*,H_3^*) + (\nabla_{H_2^*}R_t)(H_1^*,H_3^*) + (\nabla_{H_3^*}R_t)(H_1^*,H_2^*) = 0. \tag{1.38}$$

Definition 1.1.8. *A Riemannian manifold (M^n, g) is said to be an Einstein semi-symmetric [79] if $R.E = 0$, then satisfies the following condition*

$$E(R(H_1^*, H_2^*)H_3^*, H_4^*) + E(H_3^*, R(H_1^*, H_2^*)H_4^*) = 0, \tag{1.39}$$

where E is an Einstein tensor given by

$$E(H_1^*, H_2^*) = R_t(H_1^*, H_2^*) - \frac{R^*}{n} g(H_1^*, H_2^*). \tag{1.40}$$

Definition 1.1.9. *[64] A Riemannian manifold is said to be Ricci-recurrent if it satisfies the following relation*

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = \eta(H_1^*) R_t(H_2^*, H_3^*), \tag{1.41}$$

where η is a 1-form on M. If the 1-form η is identically zero on M, then the Ricci-recurrent manifold is said to be a Ricci-symmetric manifold, that is, the Ricci tensor is covariant constant.

Definition 1.1.10. *[54] A Riemannian manifold (M^n, g) is said to be cyclic η-recurrent Ricci tensor R_t if R_t is non-zero and satisfies the following relation*

$$\begin{aligned}
&(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) + (\nabla_{H_2^*} R_t)(H_1^*, H_3^*) + (\nabla_{H_3^*} R_t)(H_1^*, H_2^*) \\
&= \eta(H_1^*) R_t(H_2^*, H_3^*) + \eta(H_2^*) R_t(H_1^*, H_3^*) + \eta(H_3^*) R_t(H_1^*, H_2^*).
\end{aligned} \tag{1.42}$$

Definition 1.1.11. *The concircular curvature tensor in a n-dimensional Riemannian manifold is defined by [90]*

$$C(H_1^*, H_2^*)H_3^* = R(H_1^*, H_2^*)H_3^* - \frac{R^*}{n(n-1)} \left[g(H_2^*, H_3^*)H_1^* - g(H_1^*, H_3^*)H_2^* \right]. \tag{1.43}$$

The manifold (M^n, g) is called ξ-concircularly flat if $C(H_1^*, H_2^*)\xi = 0$.

Definition 1.1.12. *A vector field V on a n-dimensional Riemannian manifold is said to be Torse forming vector field (TFVF) [91] if*

$$\nabla_{H_2^*} V = f H_2^* + \gamma(H_2^*) V, \tag{1.44}$$

where f is a smooth function and γ is a 1-form.

Definition 1.1.13. *A Riemannian manifold is said to be Ricci semi symmetric if $R(H_1^*, H_2^*).R_t = 0$, i.e., it satisfies the following relation*

$$R_t(R(H_1^*, H_2^*)H_3^*, H_4^*) + R_t(H_3^*, R(H_1^*, H_2^*)H_4^*) = 0. \tag{1.45}$$

Again the manifold satisfies the ξ-Ricci symmetric condition, i.e., $R(\xi, H_1^*).R_t = 0$, then

$$R_t(R(\xi, H_1^*)H_2^*, H_3^*) + R_t(H_2^*, R(\xi, H_1^*)H_3^*) = 0. \tag{1.46}$$

Definition 1.1.14. *In an almost contact manifold M of dimension $n \geq 3$, the conharmonic curvature tensor $\bar{L}$ with respect to semi-symmetric metric connection $\bar{\nabla}$ is given by [62, 77, 83]*

$$\begin{aligned}\bar{L}(H_1^*, (H_2^*)H_3^* = \bar{R}(H_1^*, (H_2^*)H_3^* - \frac{1}{n-2}[\bar{R}_t(H_2^*, H_3^*)H_1^* \\ - \bar{R}_t(H_1^*, H_3^*)H_2^* + g(H_2^*, H_3^*)\bar{Q}H_1^* - g(H_1^*, H_3^*)\bar{Q}H_2^*],\end{aligned} \tag{1.47}$$

where $\bar{R}, \bar{Q}$ are the Riemannian curvature tensor and the Ricci operator with respect to semi-symmetric connection $\bar{\nabla}$, respectively.
A conharmonic curvature tensor $\bar{L}$ with respect to semi-symmetric metric connection $\bar{\nabla}$ is said to be flat if it vanishes identically with respect to the connection $\bar{\nabla}$.

Definition 1.1.15. *In a Riemannian manifold (M^n, g), the Koszul's formula is given by*

$$\begin{aligned}2g(\nabla_{H_1^*}H_2^*, H_3^*) = H_1^* g(H_2^*, H_3^*) + H_2^* g(H_3^*, H_1^*) - H_3^* g(H_1^*, H_2^*) \\ - g(H_1^*, [H_2^*, H_3^*]) + g(H_1^*, [H_3^*, H_1^*]) + g(H_3^*, [H_1^*, H_2^*]).\end{aligned} \tag{1.48}$$

Also the relation of Riemmanian curvature tensor is

$$R(H_1^*, H_2^*)H_3^* = \nabla_{H_1^*}\nabla_{H_2^*}H_3^* - \nabla_{H_2^*}\nabla_{H_1^*}H_3^* - \nabla_{[H_1^*, H_2^*]}H_3^*. \tag{1.49}$$

Lemma 1.1.1. *The $*$-Ricci tensor on a 3-dimensional trans-Sasakian manifold is given by [41],*

$$R_t^*(H_1^*, H_2^*) = R_t(H_1^*, H_1^*) - (\alpha^2 - \beta^2)[g(H_1^*, H_2^*) + \eta(H_1^*)\eta(H_2^*)]. \tag{1.50}$$

1.2 Organization of the Thesis

This thesis is organized into six Chapters, each exploring various types of manifolds admitting different soliton structures with respect to semi-symmetric metric connections. Chapter 2, investigates (k,μ)-contact metric manifolds.

- Section 2.3 discusses $(2n+1)$-dimensional (k,μ)-contact metric manifolds that admit an ηES and it is shown that for these solitons, the scalar curvature of (k,μ)-contact metric manifolds is constant and the solitons represent shrinking, steady and expanding under same curvature conditions.

- Sections 2.4–2.6 focus on such manifolds satisfying various curvature conditions, including $R(H_1^*,H_2^*)R_t = 0$. It is established that a $(2n+1)$-dimensional (k,μ)-contact metric manifold satisfying an η-Einstein soliton is locally isometric to the Riemannian product $E^{n+1} \times S^{n+1}(4)$ for $n > 1$ and flat for $n = 1$.

- Section 2.7 provides a concrete example of a manifold admitting an ηES.

- Sections 2.8–2.11 study CηRS and their geometric properties under additional conditions. Also we obtained that the Ricci tensor of a (k,μ)-contact metric manifold with torse forming vector field takes the form of an η-Einstein under ηES.

- Sections 2.12–2.16 offer further examples and examine Einstein semi-symmetric cases and CηES under various Ricci-type conditions.

Chapter 3, is devoted to Kenmotsu and ε-Kenmotsu manifolds.

- Sections 3.3–3.6 analyze n-dimensional ε-Kenmotsu manifolds admitting CηES under structural constraints like Codazzi type, cyclic parallel, and cyclic η-recurrent Ricci tensors.

- Section 3.7 presents an illustrative example of a manifold admitting CηES.

- Sections 3.8 and 3.9 examine the behavior of geometric vector fields and extended generalized ϕ-recurrent structures under semi-symmetric connections.We shall proof every contact vector field on a Kenmotsu manifold leaving the Ricci tensor with respect to semi-symmetric connection invariant is a strict contact vector field.

- Sections 3.10–3.11 explore conharmonic curvature tensors and include examples.

Chapter 4, focuses on trans-Sasakian manifolds.

- Section 4.3 characterizes $*$-CηRS on 3-dimensional trans-Sasakian manifolds and it becomes an ηE manifold.

- Sections 4.4 explore the pointwise collinearity of vector fields. It is demonstrated that a 3-dimensional trans-Sasakian manifold admitting $*$-CηRS is an ηE manifold and the potential vector field is pointwise collinear with the characteristic vector field.

- Sections 4.5-4.9 explore the relationships between soliton structures and geometric conditions like cyclic parallel Ricci tensor, ξ-Ricci conformally semi-symmetric. We obtain that a 3-dimensional trans-Sasakian manifold admitting $*$-CηES is an Einstein manifold.

- Section 4.10 provides a detailed example of a trans-Sasakian manifold admitting $*$-CηES.

Chapter 5, examines LP-S manifolds.

- Sections 5.3–5.6 study 4-dimensional LP-S manifolds admitting CηRS under ξ-Ricci semi-symmetric and conformally semi-symmetric conditions, as well as their interactions with TFVF and perfect fluid spacetimes. Also we shall prove that a LP-S manifold represent a dark-energy era.

- Sections 5.7–5.11 give illustrative examples and explore geometric vector fields and extended generalized ϕ-recurrent conditions. We show that an extended generalized ϕ-recurrent LP-S manifold with respect to semi-symmetric metric connection is an Einstein manifold.

- Sections 5.12–5.15 analyze LP-S manifolds admitting CηES, including Codazzi type Ricci tensors and Einstein semi-symmetric cases. It is revealed that a manifold represent a dark-energy era. In addition we show that a 4-dimensional Ricci-recurrent LP-S manifold reduces to a Minkowski spacetime.

In Chapter 6, presents the conclusion and outlines future research directions.
It summarizes the main findings, contributions, and suggests extensions of the work that could lead to further theoretical developments and applications in differential geometry.

Chapter 2

(k,μ)-contact metric manifold

2.1 Introduction

In 1995, Blair et al. [14] introduced the notion of contact metric manifold with characteristic vector field ξ belonging to the (k,μ) distribution and such type of manifold is called (k,μ)-contact metric manifold. They obtained several results and a full classification of this manifold has been given by Boeckx [16].
A (k,μ)-contact metric manifold (M^{2n+1},g) is known [28] to exist where the curvature tensor R, in the direction of the characteristic vector field ξ, satisfies the equation $R(H_1^*,H_2^*)\xi = 0$ for any tangent vector fields H_1^*, H_2^* on M^{2n+1}. For instance, the tangent sphere bundle of a flat Riemannian manifold possesses such a structure [10]. By applying a D-homothetic deformation [82] on M^{2n+1} with the equation $R(H_1^*,H_2^*)\xi = 0$, a novel class of contact metric manifolds that fulfills the condition

$$R(H_1^*,H_2^*)\xi = k\{\eta(H_2^*)H_1^* - \eta(H_1^*)H_2^*\} + \mu\{\eta(H_2^*)hH_1^* - \eta(H_1^*)hH_2^*\},\ k,\ \mu \in \mathbb{R} \tag{2.1}$$

where h represents the Lie differentiation of ϕ in the direction of ξ and R is the curvature tensor. A notable characteristic of this class is that the equation's type remains unchanged under a D-homothetic deformation.
A contact metric manifold that satisfies the aforementioned relation (2.1) is known as a (k,μ)-contact metric manifold. This class of manifolds encompasses both Sasakian and non-Sasakian manifolds. In the case of Sasakian manifolds, $k = 1$, resulting in $h = 0$. However, for non-Sasakian manifolds, $k < 1$. Examples of such manifolds can be found in all dimensions. Notably, the tangent sphere bundles of Riemannian manifolds with constant sectional curvature c, excluding $c = 1$, serve as characteristic examples of non-Sasakian (k,μ)-contact metric manifolds. Particularly in the 3-dimensional case, this class includes the Lie group $SO(3)$, $SL(2,R)$, $SU(2)$, $O(1,2)$, $E(2)$, $E(1,1)$ with a

left invariant metric [14]. For additional examples and a comprehensive classification of such manifolds, we refer to the mentioned paper [14]. It is worth noting that the papers also discuss contact metric manifolds with ξ belonging to the (k,μ)-nullity distribution [17, 48, 49, 86, 87] along with numerous other studies on this topic. For the real constants $k,\ \mu$, the (k,μ)-nullity distribution of a contact metric manifold forms a distribution [17]

$$\begin{aligned} N(k,\mu): p \to N_p(k,\mu) =& [H_3^* \in TpM : R(H_1^*,H_2^*)H_3^* \\ &= k\{g(H_2^*,H_3^*)H_1^* - g(H_1^*,H_3^*)H_2^*\} \\ &\quad + \mu\{g(H_2^*,H_3^*)hH_1^* - g(H_1^*,H_3^*)hH_2^*\}], \end{aligned} \tag{2.2}$$

for each $H_1^*,\ H_2^*,\ H_3^* \in T_pM$.
Consequently, if the characteristic vector field ξ belongs to the (k,μ)- nullity distribution, the above relation holds true. If $\xi \in N(k)$, we classify the manifold as an $N(k)$ contact metric manifold [13]. For $k=1$, then the manifold is Sasakian, and if $k=0$, the manifold is locally isometric to the product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$ [11], where n is the dimenssion of the manifold. In a (k,μ)-contact metric manifold, the manifold becomes an $N(k)$-contact manifold for $\mu=0$.

2.2 Preliminaries

A $(2n+1)$-dimensional (k,μ)-contact metric manifold, we have the following relations hold from [14, 16]

$$\begin{aligned} \eta\,(R(H_1^*,H_2^*)H_3^*) =& k\,[g(H_2^*,H_3^*)\eta(H_1^*) - g(H_1^*,H_3^*)\eta(H_2^*)] \\ &+ \mu\,[g(hH_2^*,H_3^*)\eta(H_1^*) - g(hH_1^*,H_3^*)\eta(H_2^*)]\,, \end{aligned} \tag{2.3}$$

$$R(\xi,H_1^*)H_2^* = k\,[g(H_1^*,H_2^*)\xi - \eta(H_2^*)H_1^*] + \mu\,[g(hH_1^*,H_2^*)\xi - \eta(H_2^*)hH_1^*]\,, \tag{2.4}$$

$$R(\xi,H_1^*)\,\xi = k\,[\eta(H_1^*)\xi - H_1^*] - \mu hH_1^*, \tag{2.5}$$

$$R_t(\phi H_1^*,\phi H_2^*) = R_t(H_1^*,H_2^*) - 2nk\eta(H_1^*)\eta(H_2^*) - 2(2n-2+\mu)g(hH_1^*,H_2^*), \tag{2.6}$$

$$\begin{aligned} R_t(H_1^*,H_2^*) =& (2n-2-n\mu)\,g(H_1^*,H_2^*) + (2-2n+2nk+n\mu)\,\eta(H_1^*)\eta(H_2^*) \\ &+ (2n-2+\mu)\,g(hH_1^*,H_2^*), \end{aligned} \tag{2.7}$$

$$R^* = 2n(2n-2+k-n\mu), \tag{2.8}$$

$$R_t(H_1^*, \xi) = 2nk\eta(H_1^*), \tag{2.9}$$

$$R_t(\xi, \xi) = 2nk, \tag{2.10}$$

$$Q\xi = 2nk\xi. \tag{2.11}$$

2.3 $(2n+1)$-dimensional (k,μ)-contact metric manifold admitting an ηES

Here we consider (k,μ)-contact metric manifold (M^{2n+1}, g) admitting an ηES. In the first part, we try to characterize the nature of the soliton by calculating the condition under which an ηES is shrinking, steady or expanding on a $(2n+1)$-dimensional (k,μ)-contact metric manifold.
Now, we state the following theorems:

Theorem 2.3.1. *If a $(2n+1)$-dimensional (k,μ)-contact metric manifold (M^{2n+1}, g) is Ricci symmetric and admits an ηES $(g, \xi, \lambda^*, \mu^*)$, then $\mu^* = 0$ and the constant scalar curvature $R^* = 2\lambda^* + 4kn$. Furthermore, the soliton is shrinking, steady and expanding for $R^* < 4kn$, $R^* = 4kn$ and $R^* > 4kn$, respectively.*

Proof. Let us consider a (k,μ)-contact metric manifold (M^{2n+1}, g) admitting an ηES $(g, \xi, \lambda^*, \mu^*)$. Then from the equation (1.19), we have

$$(\mathcal{L}_\xi g)(H_1^*, H_2^*) + 2R_t(H_1^*, H_2^*) + (2\lambda^* - R^*)g(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) = 0. \tag{2.12}$$

From (2.12), we get

$$2R_t(H_1^*, H^*) = -(\mathcal{L}_\xi g)(H_1^*, H_2^*) - (2\lambda^* - R^*)g(H_1^*, H_2^*) - 2\mu^*\eta(H_1^*)\eta(H_2^*). \tag{2.13}$$

Now, with the help of (1.8), we have

$$(\mathcal{L}_\xi g)(H_1^*, H_2^*) = -2g(\phi h H_1^*, H_2^*). \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$R_t(H_1^*, H_2^*) = \left(\frac{R^*}{2} - \lambda^*\right) g(H_1^*, H_2^*) - \mu^*\eta(H_1^*)\eta(H_2^*) + g(\phi h H_1^*, H_2^*). \tag{2.15}$$

Putting $H_2^* = \xi$ in (2.15), we get

$$R_t(H_1^*, \xi) = \left(\frac{R^*}{2} - \lambda^* - \mu^*\right) \eta(H_1^*). \tag{2.16}$$

Comparing the equations (2.9) and (2.16), we have

$$2kn\eta(H_1^*) = \left(\frac{R^*}{2} - \lambda^* - \mu^*\right)\eta(H_1^*).$$

Since η is a non-zero 1-form, it becomes

$$R^* = 2\lambda^* + 2\mu^* + 4kn. \tag{2.17}$$

It is well known that,

$$(\nabla_{H_1^*}R_t)(H_2^*, H_3^*) = H_1^*(R_t(H_2^*, H_3^*)) - R_t(\nabla_{H_1^*}H_2^*, H_3^*) - R_t(H_2^*, \nabla_{H_1^*}H_3^*). \tag{2.18}$$

Using the equation (2.15) and (2.18), we achieve

$$(\nabla_{H_1^*}R_t)(H_2^*, H_3^*) = -\mu^*[\eta(H_3^*)(\nabla_{H_1^*}\eta)H_2^* + \eta(H_2^*)(\nabla_{H_1^*}\eta)H_3^*], \tag{2.19}$$

Using equation (1.7), the above equation becomes

$$\begin{aligned}(\nabla_{H_1^*}R_t)(H_2^*, H_3^*) = &-\mu^*\left[\eta(H_3^*)\left(g(H_1^*, \phi H_2^*) - g(H_1^*, \phi h H_2^*)\right)\right] \\ &-\mu^*\left[\eta(H_2^*)\left(g(H_1^*, \phi H_3^*) - g(H_1^*, \phi h H_3^*)\right)\right].\end{aligned} \tag{2.20}$$

If the manifold M^{2n+1} is Ricci symmetric, then $\nabla R_t = 0$.
Therefore the equation (2.20) reduces to

$$\begin{aligned}&\mu^*\left[\eta(H_3^*)\left(g(H_1^*, \phi H_2^*) - g(H_1^*, \phi h H_2^*)\right)\right] \\ &+\mu^*\left[\eta(H_2^*)\left(g(H_1^*, \phi H_3^*) - g(H_1^*, \phi h H_3^*)\right)\right] = 0,\end{aligned} \tag{2.21}$$

for all vector fields $H_1^*, H_2^*, H_3^* \in \chi(M)$.
Putting $H_3^* = \xi$ in the equation (2.21), we have

$$\mu^*\left[g(H_1^*, \phi H_2^*) - g(H_1^*, \phi h H_2^*)\right] = 0. \tag{2.22}$$

Then μ^*=0 as $g(\phi H_1^*, H_2^*) \neq g(H_1^*, \phi h H_2^*)$.
Equation (2.17) reduces to

$$R^* = 2\lambda^* + 4kn. \tag{2.23}$$

From (2.23), we can conclude the following:

(i) if $\lambda^* < 0$, then $R^* < 4kn$ implies the soliton is shrinking.

(ii) if $\lambda^* = 0$, then $R^* = 4kn$ implies the soliton is steady.

(iii) if $\lambda^* > 0$, then $R^* > 4kn$ implies the soliton is expanding.

This completes the proof. □

Theorem 2.3.2. *If the metric of a (2n+1)-dimensional (k,μ)-contact metric manifold is an ηES and the Ricci tensor is Ricci-recurrent , then the constant scalar curvature $R^* = 2(\lambda^* + \mu^*)$.*

Proof. From equations (1.41) and (2.20), we obtain

$$\begin{aligned}
&-\mu^* [\eta(H_3^*)(g(H_1^*, \phi H_2^*) - g(H_1^*, \phi h H_2^*))] \\
&-\mu^* [\eta(H_2^*)(g(H_1^*, \phi H_3^* - g(H_1^*, \phi h H_3^*))] \\
&= \eta(H_1^*) R_t(H_2^*, H_3^*).
\end{aligned} \tag{2.24}$$

Putting $H_2^* = H_3^* = \xi$ in the equation (2.24) and using the equation (2.16), we obtain

$$\left(\frac{R^*}{2} - \lambda^* - \mu^*\right)\eta(H_1^*) = 0. \tag{2.25}$$

Since η is 1-form, the above equation becomes

$$R^* = 2(\lambda^* + \mu^*).$$

Thus, the proof is complete. □

Theorem 2.3.3. *If a $(2n+1)$-dimensional (k,μ)-contact metric manifold (M^{2n+1}, g) admits an ηES (g, v, λ^*, μ^*) such that the vector field v is pointwise collinear with ξ (i.e., v is a constant multiple of ξ), then the manifold (M^{2n+1}, g) becomes an ηE manifold of constant scalar curvature $R^* = 2(\lambda^* + \mu^* + 2kn)$.*

Proof. Considering a (k,μ)-contact metric manifold (M^{2n+1}, g) that admits an ηES (g, v, λ^*, μ^*) such that v is parallel to ξ, that is, $v = c\xi$ for some function c, and using this in equation (1.19), it follows that

$$(\mathcal{L}_{c\xi} g)(H_1^*, H_2^*) + 2R_t(H_1^*, H_2^*) + (2\lambda^* - R^*)g(H_1^*, H_2^*) + 2\mu^* \eta(H_1^*)\eta(H_2^*) = 0,$$

which gives

$$\begin{aligned}
&cg(\nabla_{H_1^*}\xi, H_2^*) + (H_1^* c)\eta(H_2^*) + cg(\nabla_{H_2^*}\xi, H_1^*) + (H_2^* c)\eta(H_1^*) \\
&+ 2R_t(H_1^*, H_2^*) + (2\lambda^* - R^*)g(H_1^*, H_2^*) + 2\mu^* \eta(H_1^*)\eta(H_2^*) = 0.
\end{aligned} \tag{2.26}$$

Using (1.8) in the equation (2.26), we get

$$-cg(\phi H_1^*, H_2^*) - cg(\phi h H_1^*, H_2^*) + (H_1^* c)\eta(H_2^*) - cg(\phi H_2^*, H_1^*) - cg(\phi h H_2^*, H_1^*)$$
$$+ (H_2^* c)\eta(H_1^*) + 2R_t(H_1^*, H_2^*) + (2\lambda^* - R^*)g(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) = 0. \tag{2.27}$$

Substituting $H_2^* = \xi$ in (2.27), we have

$$(H_1^* c) + (2\lambda^* - R^* + \xi c + 4kn + 2\mu^*)\eta(H_1^*) = 0. \tag{2.28}$$

If

$$(2\lambda^* - R^* + \xi c + 4kn + 2\mu^*) = 0,$$

then $H_1^* c = 0$, that is, c is constant. This implies $\xi c = 0$. From equation (2.28), we obtain

$$R^* = 2\lambda^* + 2\mu^* + 4kn. \tag{2.29}$$

Since c is constant, equation (2.27) becomes

$$R_t(H_1^*, H_2^*) = \left(\frac{R^*}{2} - \lambda^*\right) g(H_1^*, H_2^*) - \mu^*\eta(H_1^*)\eta(H_2^*). \tag{2.30}$$

Hence the result. □

2.4 ηES on $(2n+1)$-dimensional (k,μ)-contact metric manifold satisfying $R(H_1^*, H_2^*).R_t = 0$

In this section, first we consider a (k,μ)-contact metric manifold (M^{2n+1}, g) that admits an ηES $(g, \xi, \lambda^*, \mu^*)$ and the manifold satisfies the curvature condition
$R(H_1^*, H_2^*).R_t = 0$.
On the basis of the above condition we can state the following theorems:

Theorem 2.4.1. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits an ηES $(g, \xi, \lambda^*, \mu^*)$. If the manifold satisfies the curvature condition $R(H_1^*, H_2^*).R_t = 0$, then the manifold admit a constant scalar curvature $R^* = 2\lambda^* + 4kn + \frac{2\mu^*}{2n+1}$ and the soliton is shrinking, steady and expanding as*

(i) $R^* < 4kn + \frac{2\mu^*}{2n+1}$,

(ii) $R^* = 4kn + \frac{2\mu^*}{2n+1}$,

(iii) $R^* > 4kn + \frac{2\mu^*}{2n+1}$.

Proof. Setting $H_4^* = \xi$ in (1.45), we obtain

$$R_t(R(H_1^*, H_2^*)H_3^*, \xi) + R_t(H_3^*, R(H_1^*, H_2^*)\xi) = 0. \tag{2.31}$$

Using equations (2.1), (2.3) and (2.9) in (2.31), we get

$$\begin{aligned}
&2nk(k\,[g(H_2^*, H_3^*)\eta(H_1^*) - g(H_1^*, H_3^*)\eta(H_2^*)]) \\
&+ 2nk(\mu\,[g(hH_2^*, H_3^*)\eta(H_1^*) - g(hH_1^*, H_3^*)\eta(H_2^*)]) \\
&+ R_t(H_3^*, k\,\{\eta(H_2^*)H_1^* - \eta(H_1^*)H_2^*\} + \mu\,\{\eta(H_2^*)hH_1^* - \eta(H_1^*)hH_2^*\}) = 0,
\end{aligned} \tag{2.32}$$

which implies,

$$\begin{aligned}
&[2nk^2 g(H_2^*, H_3^*) - kR_t(H_2^*, H_3^*) + 2nk\mu g(hH_2^*, H_3^*) - \mu R_t(hH_2^*, H_3^*)]\eta(H_1^*) \\
&+ [kR_t(H_1^*, H_3^*) - 2nk^2 g(H_1^*, H_3^*) + \mu R_t(H_3^*, hH_1^*) - 2nk\mu g(hH_1^*, H_3^*)]\eta(H_2^*) = 0.
\end{aligned} \tag{2.33}$$

Taking $H_1^* = \xi$ in the above equation, then it reduces to

$$kR_t(H_2^*, H_3^*) + \mu R_t(hH_2^*, H_3^*) = 2nk^2 g(H_2^*, H_3^*) + 2nk\mu g(hH_2^*, H_3^*). \tag{2.34}$$

Now, replacing H_1^* by hH_1^* in (2.7), we get

$$\begin{aligned}
R_t(hH_1^*, H_2^*) = {} & (2n-2-n\mu)\,g(hH_1^*, H_2^*) - (k-1)\,(2n-2+\mu)\,g(H_1^*, H_2^*) \\
& + (k-1)\,(2n-2+\mu)\,\eta(H_1^*)\eta(H_2^*).
\end{aligned} \tag{2.35}$$

From (2.34) and (2.35), we obtain

$$\begin{aligned}
R_t(H_2^*, H_3^*) = {} & \left[2kn + \frac{k-1}{k}(2n-2+\mu)\mu\right] g(H_2^*, H_3^*) \\
& + \left[2n\mu - \frac{1}{k}(2n-2-n\mu)\mu\right] g(hH_2^*, H_3^*) \\
& - \left(\frac{k-1}{k}\right)(2n-2+\mu)\mu\eta(H_2^*)\eta(H_3^*).
\end{aligned} \tag{2.36}$$

If $\left[2n\mu - \frac{1}{k}(2n-2-n\mu)\mu\right] = 0$,
that is, $\mu = 0$ and $\left[2n - \frac{1}{k}(2n-2-n\mu)\right] \neq 0$, then (2.36) becomes

$$R_t(H_2^*, H_3^*) = 2kng(H_2^*, H_3^*). \tag{2.37}$$

Let us assume that the Einstein semi-symmetric $(2n+1)$-dimensional (k,μ)-contact metric manifold admits an ηES (g,ξ,λ^*,μ^*). Then equation (2.15) holds and combining (2.15) with the equation (2.37), we obtain

$$2kn(2n+1) = (2n+1)\left(\frac{R^*}{2} - \lambda^*\right) - \mu^*, \tag{2.38}$$

that is,

$$R^* = 2\lambda^* + 4kn + \frac{2\mu^*}{2n+1}. \tag{2.39}$$

From (2.39), we can conclude the following:

(i) if $\lambda^* < 0$, then $R^* < 4kn + \dfrac{2\mu^*}{2n+1}$ implies the soliton is shrinking.

(ii) if $\lambda^* = 0$, then $R^* = 4kn + \dfrac{2\mu^*}{2n+1}$ implies the soliton is steady.

(iii) if $\lambda^* > 0$, then $R^* > 4kn + \dfrac{2\mu^*}{2n+1}$ implies the soliton is expanding.

□

Theorem 2.4.2. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits an ηES (g,ξ,λ^*,μ^*). If the manifold is Ricci semi-symmetric, then the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$.*

Proof. Again from (2.34) and (2.35), we obtain

$$\begin{aligned} &k(2n-2-n\mu)g(H_2^*,H_3^*) + k(2-2n+2nk+n\mu)\eta(H_2^*)\eta(H_3^*) \\ &+k(2n-2+\mu)g(hH_2^*,H_3^*) = \left[2k^2n+(k-1)(2n-2+\mu)\mu\right] g(H_2^*,H_3^*) \\ &+[2kn\mu+(2n-2-n\mu)\mu]\, g(hH_2^*,H_3^*) - (k-1)(2n-2+\mu)\mu\eta(H_2^*)\eta(H_3^*). \end{aligned} \tag{2.40}$$

Comparing the both sides, we get
$\mu=0,\ k=0$. Hence the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$. □

2.5 ηES on $(2n+1)$-dimensional (k,μ)-contact metric manifold satisfying ξ-Ricci confomally semi-symmetric condition

In this part, we study a (k,μ)-contact metric manifold (M^{2n+1},g) that admits an ηES (g,ξ,λ^*,μ^*) and the manifold satisfies the ξ-Ricci conformally semi-symmetric condition i.e., $C(\xi,H_1^*).R_t = 0$, then

$$R_t(C(\xi,H_1^*)H_2^*,H_3^*) + R_t(H_2^*,C(\xi,H_1^*)H_3^*) = 0. \tag{2.41}$$

Now we can state the following theorem:

Theorem 2.5.1. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits an ηES (g,ξ,λ^*,μ^*). If the manifold satisfies the curvature condition $C(\xi,H_1^*).R_t = 0$, then the manifold admits a constant scalar curvature $R^* = 2\lambda^* + 4kn + \dfrac{2\mu^*}{2n+1}$.*

Proof. From equation (1.43), we find

$$C(\xi,H_1^*)H_2^* = R(\xi,H_1^*)H_2^* - \frac{R^*}{2n(2n+1)}\left[g(H_1^*,H_2^*)\xi - \eta(H_2^*)H_1^*\right]. \tag{2.42}$$

Using (2.4) in (2.42), we have

$$\begin{aligned} C(\xi,H_1^*)H_2^* = &\left[k - \frac{R^*}{2n(2n+1)}\right]\left[g(H_1^*,H_2^*)\xi - \eta(H_2^*)H_1^*\right] \\ &+ \mu\left[g(hH_1^*,H_2^*)\xi - \eta(H_2^*)hH_1^*\right]. \end{aligned} \tag{2.43}$$

Similarly,

$$\begin{aligned} C(\xi,H_1^*)H_3^* = &\left[k - \frac{R^*}{2n(2n+1)}\right]\left[g(H_1^*,H_3^*)\xi - \eta(H_3^*)H_1^*\right] \\ &+ \mu\left[g(hH_1^*,H_3^*)\xi - \eta(H_3^*)hH_1^*\right]. \end{aligned} \tag{2.44}$$

Using equations (2.43), (2.44) in (2.41), we obtain

$$\begin{aligned} &\left[k - \frac{R^*}{2n(2n+1)}\right] R_t(\left[g(H_1^*,H_2^*)\xi - \eta(H_2^*)H_1^*\right],H_3^*) \\ &+ \left[k - \frac{R^*}{2n(2n+1)}\right] R_t(\left[g(H_1^*,H_3^*)\xi - \eta(H_3^*)H_1^*\right],H_2^*) \\ &+ R_t(\mu\left[g(hH_1^*,H_2^*)\xi - \eta(H_2^*)hH_1^*\right],H_3^*) \\ &+ R_t(\mu\left[g(hH_1^*,H_3^*)\xi - \eta(H_3^*)hH_1^*\right],H_2^*) = 0, \end{aligned} \tag{2.45}$$

which implies

$$\begin{aligned}&\left[k-\frac{R^*}{2n(2n+1)}\right][2kng(H_1^*,H_2^*)\eta(H_3^*)-R_t(H_1^*,H_3^*)\eta(H_2^*)]\\&+\left[k-\frac{R^*}{2n(2n+1)}\right][2kng(H_1^*,H_3^*)\eta(H_2^*)-R_t(H_1^*,H_2^*)\eta(H_3^*)]\\&+\mu\,[2kng(hH_1^*,H_2^*)\eta(H_3^*)-R_t(hH_1^*,H_3^*)\eta(H_2^*)]\\&+\mu\,[2kng(hH_1^*,H_3^*)\eta(H_2^*)-R_t(hH_1^*,H_2^*)\eta(H_3^*)]=0.\end{aligned}\tag{2.46}$$

Setting $H_3^*=\xi$ in (2.46) and using (2.9), we get

$$\begin{aligned}&\left[k-\frac{R^*}{2n(2n+1)}\right][2kng(H_1^*,H_2^*)-R_t(H_1^*,H_2^*)]\\&+\mu\,[2kng(hH_1^*,H_2^*)-R_t(hH_1^*,H_2^*)]=0.\end{aligned}\tag{2.47}$$

Using equation (2.35) in (2.47), we have

$$\begin{aligned}&\left[k-\frac{R^*}{2n(2n+1)}\right]R_t(H_1^*,H_2^*)=(2kn-2n+2+n\mu)\mu g(hH_1^*,H_2^*)\\&+\left\{2kn\left[k-\frac{R^*}{2n(2n+1)}\right]+(k-1)(2n-2+\mu)\mu\right\}g(H_1^*,H_2^*)\\&-(k-1)(2n-2+\mu)\mu\eta(H_1^*)\eta(H_2^*).\end{aligned}\tag{2.48}$$

If $[2kn-2n+2+n\mu]\,\mu=0$,
that is, $\mu=0$ and $[2kn-2n+2+n\mu)]\neq 0$, then (2.48) becomes

$$R_t(H_1^*,H_2^*)=2kng(H_1^*,H_2^*).\tag{2.49}$$

Let us assume that the Einstein semi-symmetric $(2n+1)$-dimensional (k,μ)-contact metric manifold admits an ηES (g,ξ,λ^*,μ^*). Then equation (2.15) holds and using equations (2.15) and (2.49), we obtain

$$2kn(2n+1)=(2n+1)\left(\frac{R^*}{2}-\lambda^*\right)-\mu^*,\tag{2.50}$$

that is,

$$R^*=2\lambda^*+4kn+\frac{2\mu^*}{2n+1}.\tag{2.51}$$

This concludes the proof. □

2.6 ηES on $(2n+1)$-dimensional (k,μ)-contact metric manifold with TFVF

According to this section we state and prove the following theorem:

Theorem 2.6.1. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits an ηES (g,ξ,λ^*,μ^*) with TFVF ξ, then the manifold becomes an ηE manifold.*

Proof. Let us consider a (k,μ)-contact metric manifold (M^{2n+1},g) admitting an ηES (g,ξ,λ^*,μ^*) and assume that Reeb vector field ξ of the manifold is a TFVF. Then ξ being a TFVF, from equation (1.44), we infer that

$$\nabla_{H_2^*}\xi = fH_2^* + \gamma(H_2^*)\xi. \tag{2.52}$$

Using equation (1.8) and taking inner product with ξ, we obtain

$$g(\nabla_{H_2^*}\xi,\xi) = -(\phi+\phi h)\eta(H_2^*). \tag{2.53}$$

Taking inner product in equation (2.52), with ξ we have

$$g(\nabla_{H_2^*}\xi,\xi) = f\eta(H_2^*) + \gamma(H_2^*). \tag{2.54}$$

The equations (2.53) and (2.54), give us

$$\gamma = -(\phi+\phi h+f). \tag{2.55}$$

Thus for a TFVF ξ in (k, μ)-contact metric manifold, we obtain

$$\nabla_{H_2^*}\xi = f(H_2^* - \eta(H_2^*)\xi) - (\phi+\phi h)\eta(H_2^*)\xi. \tag{2.56}$$

Since (g,ξ,λ^*,μ^*) is an ηES, from equation (1.19), we have

$$\begin{aligned} &g(\nabla_{H_1^*}\xi,H_2^*) + g(\nabla_{H_2^*}\xi,H_1^*) + 2R_t(H_1^*,H_2^*)\\ &+(2\lambda^*-R^*)g(H_1^*,H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) = 0. \end{aligned} \tag{2.57}$$

Using (2.56) in the above equation, we obtain

$$R_t(H_1^*,H_2^*) = \left[\frac{R^*}{2} - (\lambda^*+f)\right]g(H_1^*,H_2^*) + (\phi+\phi h+f-\mu^*)\eta(H_1^*)\eta(H_2^*). \tag{2.58}$$

This means that the manifold is an ηE manifold. $\square$

2.7 Example of a (k,μ)-contact metric manifold admitting an ηES

Let us consider $M = \left\{(h_1^*,h_2^*,h_3^*) \in \mathbb{R}^3, (h_1^*,h_2^*,h_3^*) \neq (0,0,0)\right\}$ be a three-dimensional manifold [46] admitting an ηES (g,ξ,λ^*,μ^*). The vector fields w_1,w_2,w_3 are linearly independent in $\mathbb{R}^3$ so as

$[w_1,w_2] = (1+\beta)w_3$, $[w_3,w_1] = (1-\beta)w_2$, $[w_2,w_3] = 2w_1$,

where $\beta = \pm\sqrt{1-k}$ is a real number.
We define the Riemannian metric g by
$g(w_1,w_2) = g(w_2,w_3) = g(w_1,w_3) = 0$ and $g(w_1,w_1) = g(w_2,w_2) = g(w_3,w_3) = 1$.
Let 1-form η defined by
$\eta(H_1^*) = g(H_1^*,w_1)$.
The (1,1) tensor field ϕ is defined as

$$\phi(w_1) = 0, \phi(w_2) = w_3, \phi(w_3) = -w_2.$$

Using the linearity of ϕ and g, we have

$$\eta(w_1) = 1,$$

$$\phi^2(H_1^*) = -H_1^* + \eta(H_1^*)w_1,$$

and

$$g(\phi H_1^*,\phi H_2^*) = g(H_1^*,H_2^*) - \eta(H_1^*)\eta(H_2^*),$$

for each $H_1^*,H_2^* \in \chi(M)$. Furthermore
$hw_1 = 0, hw_2 = \beta w_2$, and $hw_3 = -\beta w_3$.
Using equation (1.48), we can calculate

$$\nabla_{w_1}w_1 = 0, \nabla_{w_1}w_2 = 0, \nabla_{w_1}w_3 = 0,$$

$$\nabla_{w_2}w_1 = -(1+\beta)w_3, \nabla_{w_2}w_2 = 0, \nabla_{w_2}w_3 = (1+\beta)w_1,$$

$$\nabla_{w_3}w_1 = (1-\beta)w_2, \nabla_{w_3}w_2 = -(1-\beta)w_1, \nabla_{w_3}w_3 = 0.$$

Using these we can verify $\nabla_{H_1^*}\xi = -\phi H_1^* - \phi h H_1^*$ for $w_1 = \xi$. Hence the manifold is a contact metric manifold with the contact structure (ϕ,ξ,η,g) .
Also from the relation of Riemmanian curvature tensor we can calculate the following

components

$$R(w_1,w_1)w_1 = 0, R(w_1,w_2)w_1 = -(1-\beta^2)w_2, R(w_1,w_2)w_2 = (1-\beta^2)w_1,$$

$$R(w_1,w_2)w_3 = 0, R(w_2,w_3)w_1 = 0, R(w_2,w_3)w_3 = -(1-\beta^2)w_2,$$

$$R(w_1,w_3)w_1 = (1-\beta^2)w_3, R(w_1,w_3)w_2 = 0, R(w_1,w_3)w_3 = (1-\beta^2)w_1,$$

$$R(w_2,w_1)w_1 = -(1-\beta^2)w_2, R(w_3,w_1)w_1 = (1-\beta^2)w_3, R(w_2,w_3)w_2 = (1-\beta^2)w_3.$$

From these curvature tensors, we can calculate the components of Ricci tensors as follows:

$$R_t(w_1,w_1) = 2(1-\beta^2), R_t(w_2,w_2) = 0, R_t(w_3,w_3) = 0.$$

From equation (2.37), we can obtain

$$R_t(w_3,w_3) = 2kng(w_3,w_3) = 2kn.$$

By equating both the values of $R_t(w_3,w_3)$, we get

$$k = 0.$$

Hence the manifold (M^3,g) is locally isometric to the product $E^2(0)\times S^1(4)$.
Again, we can calculate equation (2.15)

$$R_t(w_3,w_3) = \left[\frac{R^*}{2} - (\lambda^* + \mu)\right].$$

Therefore,

$$\left[\frac{R^*}{2} - (\lambda^* + \mu)\right] = 0,$$

which implies that,

$$R^* = 2(\lambda^* + \mu^*).$$

Since $k = 0$, equation (2.17) reduces to

$$R^* = 2(\lambda^* + \mu^*).$$

Hence the constants λ^* and μ^* satisfies equation (2.17) and so g defines an ηES on (k,μ)-contact manifold M.
Further, putting $k = 0$ in (2.23), we can calculate

$$\lambda^* = \frac{R^*}{2}.$$

Thus the soliton (g,ξ,λ^*) on (k,μ)-contact manifold is shrinking, steady and expanding as $R^* < 0$, $R^* = 0$ and $R^* > 0$, respectively.

2.8 $(2n+1)$-dimensional (k,μ)-contact metric manifold admitting CηRS

In this segment, we think about (k,μ)-contact metric manifold (M^{2n+1},g) admitting CηRS. At first we try to characterize the nature of the soliton by calculating the condition under which a CηRS is shrinking, steady or expanding on a $(2n+1)$-dimensional (k,μ)-contact metric manifold.
Now, we state the following theorems:

Theorem 2.8.1. *If a $(2n+1)$-dimensional (k,μ)-contact metric manifold (M^{2n+1},g) is Ricci symmetric and admits a CηRS (g,ξ,λ^*,μ^*), then $\mu^*=0$ and $\lambda^* = \frac{\omega}{2}+\frac{1}{2n+1}-2kn$. Furthermore, the soliton is shrinking, steady and expanding for $\omega < 4kn-\frac{2}{2n+1}$, $\omega = 4kn-\frac{2}{2n+1}$ and $\omega > 4kn-\frac{2}{2n+1}$, respectively.*

Proof. Let us consider a (k,μ)-contact metric manifold (M^{2n+1},g) admitting a CηRS (g,ξ,λ^*,μ^*). Then from the equation (1.20), we have

$$\begin{aligned}&(\mathcal{L}_\xi g)(H_1^*,H_2^*)+2R_t(H_1^*,H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)\\&+\left[2\lambda^*-\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*)=0.\end{aligned} \tag{2.59}$$

From (2.59), we get

$$\begin{aligned}2R_t(H_1^*,H_2^*)=&-(\mathcal{L}_\xi g)(H_1^*,H_2^*)-2\mu^*\eta(H_1^*)\eta(H_2^*)\\&-\left[2\lambda^*-\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*).\end{aligned} \tag{2.60}$$

Now, with the help of (1.8), we have

$$(\mathcal{L}_\xi g)(H_1^*,H_2^*)=-2g(\phi hH_1^*,H_2^*). \tag{2.61}$$

From (2.60) and (2.61) , we obtain

$$\begin{aligned}R_t(H_1^*,H_2^*)=&\left[\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-\lambda^*\right]g(H_1^*,H_2^*)\\&-\mu^*\eta(H_1^*)\eta(H_2^*)+g(\phi hH_1^*,H_2^*).\end{aligned} \tag{2.62}$$

Putting $H_2^* = \xi$ in (2.62), we get

$$R_t(H_1^*, \xi) = \left[\left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - \lambda^* - \mu^*\right] \eta(H_1^*). \tag{2.63}$$

Comparing the equations (2.9) and (2.63), we have

$$2kn\eta(H_1^*) = \left[\left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - \lambda^* - \mu^*\right] \eta(H_1^*).$$

Since η is a non-zero 1-form, it becomes

$$\lambda^* + \mu^* = \left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - 2kn. \tag{2.64}$$

From (2.22), μ^*=0 as $g(\phi H_1^*, H_2^*) \neq g(H_1^*, \phi h H_2^*)$.
Equation (2.64) reduces to

$$\lambda^* = \left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - 2kn. \tag{2.65}$$

From (2.65), we can conclude the following:

(i) if $\lambda^* < 0$, then $\omega < 4kn - \frac{2}{2n+1}$ implies the soliton is shrinking.

(ii) if $\lambda^* = 0$, then $\omega = 4kn - \frac{2}{2n+1}$ implies the soliton is steady.

(iii) if $\lambda^* > 0$, then $\omega > 4kn - \frac{2}{2n+1}$ implies the soliton is expanding.

We have thus completed the proof. □

Theorem 2.8.2. *If the metric of a $(2n+1)$-dimensional (k, μ)-contact metric manifold is a $C\eta RS$ and the Ricci tensor is Ricci-recurrent, then the scalars λ^* and μ^* related by* $\lambda^* + \mu^* = \left(\frac{\omega}{2} + \frac{1}{2n+1}\right)$.

Proof. Putting $H_2^* = H_3^* = \xi$ in the equation (2.24) and using the equation (2.63), we obtain

$$\left[\left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - (\lambda^* + \mu^*)\right] \eta(H_1^*) = 0.$$

Since η is 1-form, the above equation becomes

$$\lambda^* + \mu^* = \left(\frac{\omega}{2} + \frac{1}{2n+1}\right).$$

Hence the result. □

Theorem 2.8.3. *If a $(2n+1)$-dimensional (k,μ)-contact metric manifold (M^{2n+1},g) admits a CηRS (g,ν,λ^*,μ^*) such that the vector field ν is pointwise collinear with ξ (i.e., ν is a constant multiple of ξ), then the manifold (M^{2n+1},g) becomes an η-Einstein manifold of constants λ^* and μ^* related by $\lambda^*+\mu^*=\frac{\omega}{2}+\frac{1}{2n+1}-2kn$.*

Proof. Considering a (k,μ)-contact metric manifold (M^{2n+1},g) that admits a CηRS (g,ν,λ^*,μ^*) such that ν is parallel to ξ, i.e., $\nu=c\xi$ for some scalar c, and using this in equation (1.20), it follows that

$$\begin{aligned}&(\mathcal{L}_{c\xi}g)(H_1^*,H_2^*)+2R_t(H_1^*,H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)\\&+\left[2\lambda^*-\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*)=0,\end{aligned}$$

which gives

$$\begin{aligned}&cg(\nabla_{H_1^*}\xi,H_2^*)+(H_1^*c)\eta(H_2^*)+cg(\nabla_{H_2^*}\xi,H_1^*)+(H_2^*c)\eta(H_1^*)+2R_t(H_1^*,H_2^*)\\&+\left[2\lambda^*-\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)=0.\end{aligned}\tag{2.66}$$

Using (1.8) in the equation (2.66), we get

$$\begin{aligned}&-cg(\phi H_1^*,H_2^*)-cg(\phi hH_1^*,H_2^*)+(H_1^*c)\eta(H_2^*)-cg(\phi H_2^*,H_1^*)\\&-cg(\phi hH_2^*,H_1^*)+(H_2^*c)\eta(H_1^*)+2R_t(H_1^*,H_2^*)\\&+\left[2\lambda^*-\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)=0.\end{aligned}\tag{2.67}$$

Substituting $H_2^*=\xi$ in (2.67), we have

$$(H_1^*c)+\left[2\lambda^*-\left(\omega+\frac{2}{2n+1}\right)+\xi c+4kn+2\mu^*\right]\eta(H_1^*)=0.\tag{2.68}$$

If

$$\left[2\lambda^*-\left(\omega+\frac{2}{2n+1}\right)+\xi c+4kn+2\mu^*\right]=0,$$

then $H_1^*c=0$, i.e., c is constant. This implies $\xi c=0$. From equation (2.68), we obtain

$$\lambda^*+\mu^*=\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-2kn.$$

Since c is constant, equation (2.67) becomes

$$R_t(H_1^*,H_2^*)=\left[\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-\lambda^*\right]g(H_1^*,H_2^*)-\mu^*\eta(H_1^*)\eta(H_2^*).$$

□

2.9 CηRS on $(2n+1)$-dimensional (k,μ)-contact metric manifold satisfying $R(H_1^*,H_2^*).R_t = 0$

In this section, first we consider a (k,μ)-contact metric manifold (M^{2n+1},g) that admits a conformal η-Ricci soliton (g,ξ,λ^*,μ^*) and the manifold satisfies the curvature condition $R(H_1^*,H_2^*).R_t = 0$.
So, based on the above condition we can state the following theorem:

Theorem 2.9.1. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits a CηRS (g,ξ,λ^*,μ^*). If the manifold satisfies the curvature condition $R(H_1^*,H_2^*).R_t = 0$, then the manifold admits a constant $\lambda^* = \frac{\omega}{2} + \frac{1}{2n+1} - \frac{\mu^*}{2n+1} - 2kn$ and the soliton is shrinking, steady and expanding as*

(i) $\omega < 4kn - \frac{2}{2n+1} - \frac{2\mu^*}{2n+1}$,

(ii) $\omega = 4kn - \frac{2}{2n+1} - \frac{2\mu^*}{2n+1}$,

(iii) $\omega > 4kn - \frac{2}{2n+1} - \frac{2\mu^*}{2n+1}$.

Proof. Let us posit that the Einstein semi-symmetric $(2n+1)$-dimensional (k,μ)- contact metric manifold admits a CηRS (g,ξ,λ^*,μ^*). Then equation (2.62) holds and combining (2.62) with the equation (2.37), we obtain

$$\lambda^* = \frac{\omega}{2} + \frac{1}{2n+1} - \frac{\mu^*}{2n+1} - 2kn. \tag{2.69}$$

From (2.69), we can conclude the following:

(i) if $\lambda^* < 0$, then $\omega < 4kn - \frac{2}{2n+1} - \frac{2\mu^*}{2n+1}$ implies the soliton is shrinking.

(ii) if $\lambda^* = 0$, then $\omega = 4kn - \frac{2}{2n+1} - \frac{2\mu^*}{2n+1}$ implies the soliton is steady.

(iii) if $\lambda^* > 0$, then $\omega > 4kn - \frac{2}{2n+1} - \frac{2\mu^*}{2n+1}$ implies the soliton is expanding.

Therefore, the proof is complete. □

2.10 CηRS on $(2n+1)$-dimensional (k,μ)-contact metric manifold satisfying ξ-Ricci conformally semi-symmetric condition

Now, we consider a (k,μ)-contact metric manifold admits a CηRS and it satisfies the ξ-Ricci conformally semi-symmetric condition, i.e., $C(\xi,H_1^*).R_t=0$.
So, we can state and prove of the next theorem:

Theorem 2.10.1. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits a CηRS (g,ξ,λ^*,μ^*). If the manifold satisfies the curvature condition $C(\xi,H_1^*).R_t=0$, then the manifold admits a constant $\lambda^*=\frac{\omega}{2}+\frac{1}{2n+1}-\frac{\mu^*}{2n+1}-2kn$.*

Proof. Let's suppose that the Einstein semi-symmetric $(2n+1)$-dimensional (k,μ)-contact metric manifold admits a CηRS (g,ξ,λ^*,μ^*). Then equation (2.62) holds and from equations (2.49) and (2.62), we get

$$\lambda^*=\frac{\omega}{2}+\frac{1}{2n+1}-\frac{\mu^*}{2n+1}-2kn.$$

This completes the proof. □

2.11 CηRS on $(2n+1)$-dimensional (k,μ)-contact metric manifold with TFVF

In this section, we examine a (k,μ)-contact metric manifold admits a CηRS with ξ is a TFVF.
Due to above condition we prove the following theorem:

Theorem 2.11.1. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits a CηRS (g,ξ,λ^*,μ^*) with ξ is a TFVF, then the manifold becomes an ηE manifold.*

Proof. Since (g,ξ,λ^*,μ^*) is a CηRS, from equation (1.20), we have

$$\begin{aligned}&g(\nabla_{H_1^*}\xi,H_2^*)+g(\nabla_{H_2^*}\xi,H_1^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)\\&+2R_t(H_1^*,H_2^*)+\left[2\lambda^*-\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*)=0.\end{aligned}$$

Using (2.56) in the above equation, we obtain

$$R_t(H_1^*,H_2^*)=\left[\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-(\lambda^*+f)\right]g(H_1^*,H_2^*)+(\phi+\phi h+f-\mu^*)\eta(H_1^*)\eta(H_2^*).$$

Hence the result is proved. □

2.12 Example of a (k,μ)-contact metric manifold admitting a CηRS

From Example 2.7, we can calculate equation (2.62)

$$R_t(w_3,w_3)=\left[\left(\frac{\omega}{2}+\frac{1}{3}\right)-(\lambda^*+\mu^*)\right].$$

Using $R_t(w_3,w_3)=0$, the above equation becomes

$$\left[\left(\frac{\omega}{2}+\frac{1}{3}\right)-(\lambda^*+\mu^*)\right]=0,$$

which implies that,

$$\lambda^*+\mu^*=\left(\frac{\omega}{2}+\frac{1}{3}\right).$$

Since $k=0$, equation (2.64) reduces to

$$\lambda^*+\mu^*=\left(\frac{\omega}{2}+\frac{1}{3}\right).$$

Hence the constants λ^* and μ^* satisfies equation (2.64) and so g defines a CηRS on (k,μ)-contact manifold M.
Further, putting $k=0$ in (2.65), we can calculate

$$\lambda^*=\left(\frac{\omega}{2}+\frac{1}{3}\right).$$

Thus the Ricci soliton (g,ξ,λ^*) on (k,μ)-contact manifold is shrinking, steady and expanding as $\left(\frac{\omega}{2}+\frac{1}{3}\right)<0$, $\left(\frac{\omega}{2}+\frac{1}{3}\right)=0$ and $\left(\frac{\omega}{2}+\frac{1}{3}\right)>0$, respectively.

Next we move to our next section where we have proved few results:

2.13 $(2n+1)$-dimensional (k,μ)-contact metric manifold admitting CηES

Here we consider (k,μ)-contact metric manifold (M^{2n+1},g) admitting CηES. In the first part we try to characterize the nature of the soliton by calculating the condition under which a CηES is shrinking, steady or expanding on a $(2n+1)$-dimensional (k,μ)-contact metric manifold.
Now, we state the following theorems:

Theorem 2.13.1. *If a $(2n+1)$-dimensional (k,μ)-contact metric manifold admits a CηES (g,ξ,λ^*,μ^*), then the manifold (M^{2n+1},g) admit a constant scalar curvature $R^* = \frac{2n+1}{2n-1}\omega + \frac{2(2n+1)}{2n-1}\lambda^* - \frac{2}{2n-1}\mu^* + \frac{2}{2n-1}$. Furthermore, the soliton is shrinking, steady and expanding for $\omega > 2(n-1)\mu^* + 2kn(2n-1) - \frac{2}{2n+1}$, $\omega = 2(n-1)\mu^* + 2kn(2n-1) - \frac{2}{2n+1}$ and $\omega < 2(n-1)\mu^* + 2kn(2n-1) - \frac{2}{2n+1}$, respectively.*

Proof. Let us consider a (k,μ)-contact metric manifold (M^{2n+1},g) admitting a CηES (g,ξ,λ^*,μ^*). Then from the equation (1.22), we have

$$\begin{aligned}(\mathcal{L}_\xi g)(H_1^*,H_2^*) &+ 2R_t(H_1^*,H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*)\\ &+ \left[2\lambda^* - R^* + \left(\omega + \frac{2}{2n+1}\right)\right] g(H_1^*,H_2^*) = 0. \end{aligned} \tag{2.70}$$

From (2.70), we get

$$\begin{aligned}2R_t(H_1^*,H_2^*) =& -(\mathcal{L}_\xi g)(H_1^*,H_2^*) - 2\mu^*\eta(H_1^*)\eta(H_2^*)\\ &- \left[2\lambda^* - R^* + \left(\omega + \frac{2}{2n+1}\right)\right] g(H_1^*,H_2^*). \end{aligned} \tag{2.71}$$

Now, with the help of (1.8), we have

$$\begin{aligned}(\mathcal{L}_\xi g)(H_1^*,H_2^*) =& -\left[g(\phi H_1^*,H_2^*) + g(\phi h H_1^*,H_2^*)\right]\\ &- \left[g(H_1^*,\phi H_2^*) + g(\phi h H_2^*,H_1^*)\right]. \end{aligned} \tag{2.72}$$

From (2.71) and (2.72), we obtain

$$\begin{aligned} R_t(H_1^*,H_2^*) = & -\mu^*\eta(H_1^*)\eta(H_2^*) + g(\phi h H_1^*,H_2^*) \\ & + \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - \lambda^*\right] g(H_1^*,H_2^*). \end{aligned} \tag{2.73}$$

Putting $H_2^* = \xi$ in (2.73), we get

$$R_t(H_1^*,\xi) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - \lambda^* - \mu^*\right]\eta(H_1^*). \tag{2.74}$$

Comparing the equations (2.9) and (2.74), we have

$$2kn\eta(H_1^*) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - \lambda^* - \mu^*\right]\eta(H_1^*).$$

Since η is a non-zero 1-form, which gives

$$R^* = \omega + 2\lambda^* + 2\mu^* + 4kn + \frac{2}{2n+1}. \tag{2.75}$$

Taking an orthonormal basis $\{w_i : i = 1,2,\dots,2n+1\}$ of (M^{2n+1},g) and then setting $H_1^* = H_2^* = w_i$ in the equation (2.73) and taking summation over i, we get

$$R^* = \frac{2n+1}{2n-1}\omega + \frac{2(2n+1)}{2n-1}\lambda^* + \frac{2}{2n-1}\mu^* + \frac{2}{2n-1}. \tag{2.76}$$

Finally combining equations (2.75) and (2.76), we get

$$\lambda^* = -\frac{\omega}{2} + (n-1)\mu^* + kn(2n-1) - \frac{1}{2n+1}. \tag{2.77}$$

From (2.77), we can conclude the following:

(i) If $\lambda^* < 0$, then $\omega > 2(n-1)\mu^* + 2kn(2n-1) - \frac{2}{2n+1}$ implies the soliton is shrinking.

(ii) If $\lambda^* = 0$, then $\omega = 2(n-1)\mu^* + 2kn(2n-1) - \frac{2}{2n+1}$ implies the soliton is steady.

(iii) If $\lambda^* > 0$, then $\omega < 2(n-1)\mu^* + 2kn(2n-1) - \frac{2}{2n+1}$ implies the soliton is expanding.

This completes the proof. □

Theorem 2.13.2. *If the metric of a $(2n+1)$-dimensional Ricci symmetric (k,μ)-contact metric manifold is a $C\eta ES$, then $\mu^*=0$ and the constant scalar curvature $R^*=\frac{2n+1}{2n-1}\omega+\frac{2(2n+1)}{2n-1}\lambda^*+\frac{2}{2n-1}$.*

Proof. From (2.22), $\mu^*=0$, since $g(\phi H_1^*, H_2^*)\neq g(H_1^*,\phi h H_2^*)$.
Equation (2.76) reduces to

$$R^*=\frac{2n+1}{2n-1}\omega+\frac{2(2n+1)}{2n-1}\lambda^*+\frac{2}{2n-1}.$$

This completes the proof. □

Theorem 2.13.3. *If the metric of a $(2n+1)$-dimensional (k,μ)-contact metric manifold is a $C\eta ES$ and the Ricci tensor is Ricci-recurrent, then the constant scalar curvature $R^*=\omega+2\lambda^*+2\mu^*+\frac{2}{2n+1}$.*

Proof. Setting $H_2^*=H_3^*=\xi$ in the equation (2.24) and using the equation (2.73), we obtain

$$\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-\lambda^*-\mu^*\right]\eta(H_1^*)=0, \tag{2.78}$$

which gives

$$R^*=\omega+2\lambda^*+2\mu^*+\frac{2}{2n+1}.$$

Hence the result is proved. □

Theorem 2.13.4. *If a $(2n+1)$-dimensional (k,μ)-contact metric manifold (M^{2n+1},g) admits a $C\eta ES$ (g,v,λ^*,μ^*) such that the vector field v is pointwise collinear with ξ (i.e. v is a constant multiple of ξ), then the manifold (M^{2n+1},g) becomes an ηE manifold of constant scalar curvature $R^*=2\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)+2\lambda^*+2\mu^*+4kn$.*

Proof. Considering a (k,μ)-contact metric manifold (M^{2n+1},g) that admits a $C\eta$ES (g,v,λ^*,μ^*) such that v is parallel to ξ, that is, $v=a\xi$ for some scalar a, and using this in equation (1.22), it follows that

$$\begin{aligned}(\mathcal{L}_{a\xi}g)(H_1^*,H_2^*)+2R_t(H_1^*,H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)\\+\left[2\lambda^*-R^*+\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*)=0,\end{aligned}$$

which gives

$$\begin{aligned}&ag(\nabla_{H_1^*}\xi,H_2^*)+(H_1^*a)\eta(H_2^*)+ag(\nabla_{H_2^*}\xi,H_1^*)+(H_2^*a)\eta(H_1^*)+2R_t(H_1^*,H_2^*)\\&+\left[2\lambda^*-R^*+\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)=0.\end{aligned} \tag{2.79}$$

Using (1.8) in the equation (2.79), we get

$$-ag(\phi H_1^*,H_2^*)-ag(\phi hH_1^*,H_2^*)+(H_1^*a)\eta(H_2^*)-ag(\phi H_2^*,H_1^*)$$
$$-ag(\phi hH_2^*,H_1^*)+(H_2^*a)\eta(H_1^*)+2R_t(H_1^*,H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)$$
$$+\left[2\lambda^*-R^*+\left(\omega+\frac{2}{2n+1}\right)\right]g(H_1^*,H_2^*)=0. \tag{2.80}$$

Substituting $H_2^*=\xi$ in (2.80), we have

$$(H_1^*a)+\left[2\lambda^*-R^*+\left(\omega+\frac{2}{2n+1}\right)+\xi a+4kn+2\mu^*\right]\eta(H_1^*)=0. \tag{2.81}$$

If

$$\left[2\lambda^*-R^*+\left(\omega+\frac{2}{2n+1}\right)+\xi a+4kn+2\mu^*\right]=0,$$

then $H_1^*a=0$, that is, a is constant. This implies $\xi a=0$. From equation (2.81), we obtain

$$R^*=2\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)+2\lambda^*+2\mu^*+4kn. \tag{2.82}$$

Since a is constant, equation (2.80) becomes

$$R_t(H_1^*,H_2^*)=\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-\lambda^*\right]g(H_1^*,H_2^*)-\mu^*\eta(H_1^*)\eta(H_2^*). \tag{2.83}$$

Hence the result. □

2.14 CηES on (k,μ)-contact metric $(2n+1)$-dimensional manifold with Codazzi type and cyclic parallel Ricci tensor

The focus of this segment is to investigate CηES on (k,μ)-contact metric manifolds admitting Ricci tensor is of Codazzi type and cyclic parallel.
We now state and prove the following theorems:

Theorem 2.14.1. *Let (M^{2n+1},g) be a (k,μ)-contact metric manifold admitting a CηES (g,ξ,λ^*,μ^*) with the Ricci tensor is of Codazzi type. Then the Ricci tensor of the manifold takes the form*

$$R_t(H_1^*,H_2^*)=\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-\lambda^*\right]g(H_1^*,H_2^*)-\mu^*\eta(H_1^*)\eta(H_2^*)+g(\phi hH_1^*,H_2^*)$$

and the scalar curvature $R^* = \frac{2n+1}{2n-1}\omega + \frac{2(2n+1)}{2n-1}\lambda^* + \frac{2}{2n-1}\mu^* + \frac{2}{2n-1}$.

Proof. We consider a (k,μ)-contact metric manifold (M^{2n+1},g) admitting a CηES (g,ξ,λ^*,μ^*). On taking covariant derivative of equation (2.73), we get

$$(\nabla_{H_1^*}R_t)(H_2^*,H_3^*) = -\mu^*[\eta(H_3^*)(\nabla_{H_1^*}\eta)H_2^* + \eta(H_2^*)(\nabla_{H_1^*}\eta)H_3^*]. \tag{2.84}$$

Using equation (1.7), the above equation becomes

$$\begin{aligned}(\nabla_{H_1^*}R_t)(H_2^*,H_3^*) =& -\mu^*[\eta(H_3^*)(g(H_1^*,\phi H_2^*) - g(H_1^*,\phi h H_2^*))]\\ &-\mu^*[\eta(H_2^*)(g(H_1^*,\phi H_3^*) - g(H_1^*,\phi h H_3^*)].\end{aligned} \tag{2.85}$$

Now interchanging H_1^* and H_2^* in equation (2.85), we have

$$\begin{aligned}(\nabla_{H_2^*}R_t)(H_1^*,H_3^*) =& -\mu^*[\eta(H_3^*)(g(H_2^*,\phi H_1^*) - g(H_2^*,\phi h H_1^*))]\\ &-\mu^*[\eta(H_1^*)(g(H_2^*,\phi H_3^*) - g(H_2^*,\phi h H_3^*))].\end{aligned} \tag{2.86}$$

On using (1.37), we obtain from equations (2.85) and (2.86)

$$\begin{aligned}&\mu^*[\eta(H_3^*)(g(H_1^*,\phi H_2^*) - g(H_1^*,\phi h H_2^*))]\\ &+\mu^*[\eta(H_2^*)(g(H_1^*,\phi H_3^*) - g(H_1^*,\phi h H_3^*))]\\ &=\mu^*[\eta(H_3^*)(g(H_2^*,\phi H_1^*) - g(H_2^*,\phi h H_1^*))]\\ &+\mu^*[\eta(H_1^*)(g(H_2^*,\phi H_3^*) - g(H_2^*,\phi h H_3^*))].\end{aligned} \tag{2.87}$$

Using $g(H_1^*,\phi H_2^*) = -g(\phi H_1^*,H_2^*)$, we have from (2.87)

$$\begin{aligned}&\mu^*[2\eta(H_3^*)g(H_1^*,\phi H_2^*) - 2\eta(H_3^*)g(H_1^*,\phi h H_2^*) + \eta(H_1^*)g(H_2^*,\phi H_3^*)\\ &-\eta(H_1^*)g(H_2^*,\phi h H_3^*) - \eta(H_2^*)g(H_1^*,\phi H_3^*) + \eta(H_2^*)g(H_1^*,\phi h H_3^*)] = 0.\end{aligned} \tag{2.88}$$

If $\mu^* \neq 0$, then from the equation (2.73), we obtain

$$\begin{aligned}R_t(H_1^*,H_2^*) =& -\mu^*\eta(H_1^*)\eta(H_2^*) + g(\phi h H_1^*,H_2^*)\\ &+\left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{2n+1}\right) - \lambda^*\right] g(H_1^*,H_2^*).\end{aligned} \tag{2.89}$$

Taking an orthonormal basis $\{w_i : i = 1,2,\ldots,2n+1\}$ of (M^{2n+1},g) and then setting $H_1^* = H_2^* = w_i$ in the equation (2.89) and taking summation over i, we get

$$R^* = \frac{2n+1}{2n-1}\omega + \frac{2(2n+1)}{2n-1}\lambda^* + \frac{2}{2n-1}\mu^* + \frac{2}{2n-1}.$$

This completes the proof. $\square$

Theorem 2.14.2. *Let (M^{2n+1},g) be a (k,μ)-contact metric manifold admits a CηES (g,ξ,λ^*,μ^*). If the manifold has cyclic parallel Ricci tensor, then the Ricci tensor R_t takes the form*

$$R_t(H_1^*,H_2^*)=\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-\lambda^*\right]g(H_1^*,H_2^*)-g(\phi hH_1^*,H_2^*)$$

and the constant scalar curvature $R^*=\dfrac{2n+1}{2n-1}\omega+\dfrac{2(2n+1)}{2n-1}\lambda^*+\dfrac{2}{2n-1}.$

Proof. We consider a (k,μ)-contact metric manifold (M^{2n+1},g) admitting a cyclic parallel Ricci tensor and a CηES (g,ξ,λ^*,μ^*). On putting $H_1^*=H_3^*,H_2^*=H_1^*,H_3^*=H_2^*$ we obtain from (2.85),

$$\begin{aligned}(\nabla_{H_3^*}R_t)(H_1^*,H_2^*)=&-\mu^*[\eta(H_2^*)(g(H_3^*,\phi H_1^*)-g(H_3^*,\phi hH_1^*))\\&+\eta(H_1^*)(g(H_3^*,\phi H_2^*)-g(H_3^*,\phi hH_2^*)].\end{aligned}\tag{2.90}$$

Using equations (2.85), (2.86) and (2.90) in equation (1.38), we get

$$\begin{aligned}&\mu^*[\eta(H_3^*)(g(H_1^*,\phi H_2^*)-g(H_1^*,\phi hH_2^*))+\eta(H_2^*)(g(H_1^*,\phi H_3^*)-g(H_1^*,\phi hH_3^*))]\\&+\mu^*[\eta(H_3^*)(g(H_2^*,\phi H_1^*)-g(H_2^*,\phi hH_1^*))+\eta(H_1^*)(g(H_2^*,\phi H_3^*)-g(H_2^*,\phi hH_3^*))]\\&+\mu^*[\eta(H_2^*)(g(H_3^*,\phi H_1^*)-g(H_3^*,\phi hH_1^*))+\eta(H_1^*)(g(H_3^*,\phi H_2^*)-g(H_3^*,\phi hH_2^*))]=0,\end{aligned}$$

which gives

$$\begin{aligned}&\mu^*[\eta(H_3^*)g(H_1^*,\phi hH_2^*)+\eta(H_2^*)g(H_1^*,\phi hH_3^*)+\eta(H_3^*)g(H_2^*,\phi hH_1^*)\\&+\eta(H_1^*)g(H_2^*,\phi hH_3^*)+\eta(H_2^*)g(H_3^*,\phi hH_1^*)+\eta(H_1^*)g(H_3^*,\phi hH_2^*)]=0.\end{aligned}\tag{2.91}$$

Taking $H_3^*=\xi$ in equation (2.91), we obtain

$$2\mu^*g(H_1^*,\phi hH_2^*)=0.\tag{2.92}$$

Since $g(H_1^*,\phi hH_2^*)\neq 0$, then $\mu^*=0$, we have from (2.89)

$$R_t(H_1^*,H_2^*)=\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{2n+1}\right)-\lambda^*\right]g(H_1^*,H_2^*)-g(\phi hH_1^*,H_2^*).\tag{2.93}$$

Taking an orthonormal basis $\{w_i:i=1,2,\dots,2n+1\}$ of (M^{2n+1},g) and then setting

$H_1^* = H_2^* = w_i$ in the equation (2.93) and taking summation over i, we get

$$R^* = \frac{2n+1}{2n-1}\omega + \frac{2(2n+1)}{2n-1}\lambda^* + \frac{2}{2n-1}.$$

Hence the proof is concluded. □

2.15 Einstein Semi-Symmetric (k,μ)-contact metric manifolds (M^{2n+1}, g) admitting CηES.

We can state our next theorem:

Theorem 2.15.1. *Let $(2n+1)$-dimensional (k,μ)-contact metric manifold admits a $C\eta ES$ (g,ξ,λ^*,μ^*). If the manifold is an Einstein semi-symmetric, then the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$.*

Proof. From equations (1.39) and (1.40), we get

$$\begin{aligned} &R_t(R(H_1^*,H_2^*)H_3^*,H_4^*) + R_t(H_3^*,R(H_1^*,H_2^*)H_4^*) \\ &= \frac{R^*}{2n+1}\left[g(R(H_1^*,H_2^*)H_3^*,H_4^*) + g(H_3^*,R(H_1^*,H_2^*)H_4^*)\right]. \end{aligned} \tag{2.94}$$

Putting $H_1^* = H_3^* = \xi$ in the equation (2.94) , we obtain

$$\begin{aligned} &R_t(R(\xi,H_2^*)\xi,H_4^*) + R_t(\xi,R(\xi,H_2^*)H_4^*) \\ &= \frac{R^*}{2n+1}\left[g(R(\xi,H_2^*)\xi,H_4^*) + g(\xi,R(\xi,H_2^*)H_4^*)\right]. \end{aligned} \tag{2.95}$$

Using (2.4), (2.5) in (2.95), we have

$$\begin{aligned} &R_t(k\{\eta(H_2^*)\xi - H_2^*\} + \mu\{\eta(hH_2^*) - hH_2^*\}, H_4^*) \\ &+ R_t(\xi, k\{g(H_2^*,H_4^*)\xi - \eta(H_4^*)H_2^*\} + \mu\{g(hH_2^*,H_4^*)\xi - \eta(H_4^*)hH_2^*\}) \\ &= \frac{R^*}{2n+1} g(k\{\eta(H_2^*)\xi - H_2^*\} + \mu\{\eta(hH_2^*) - hH_2^*\}, H_4^*) \\ &+ g(\xi, k\{g(H_2^*,H_4^*)\xi - \eta(H_4^*)H_2^*\} + \mu\{g(hH_2^*,H_4^*)\xi - \eta(H_4^*)hH_2^*\}], \end{aligned} \tag{2.96}$$

which implies

$$kR_t(H_2^*,H_4^*) + \mu R_t(hH_2^*,H_4^*) = 2k^2 n g(H_2^*,H_4^*) + 2k\mu g(hH_2^*,H_4^*). \tag{2.97}$$

Now, H_1^* replace by hH_1^* in (2.7), we get

$$R_t(hH_1^*, H_2^*) = (2n-2-n\mu)\,g(hH_1^*, H_2^*) - (k-1)\,(2n-2+\mu)\,g(H_1^*, H_2^*) + (k-1)\,(2n-2+\mu)\,\eta(H_1^*)\eta(H_2^*). \tag{2.98}$$

From (2.97) and (2.98), we obtain

$$\begin{aligned} R_t(H_2^*, H_4^*) = & \left[2kn + \frac{k-1}{k}(2n-2+\mu)\mu\right] g(H_2^*, H_4^*) \\ & + \left[2\mu - \frac{1}{k}(2n-2-n\mu)\mu\right] g(hH_2^*, H_4^*) \\ & - \left(\frac{k-1}{k}\right)(2n-2+\mu)\mu\eta(H_2^*)\eta(H_4^*). \end{aligned} \tag{2.99}$$

If $\left[2\mu - \frac{1}{k}(2n-2-n\mu)\mu\right] = 0$,
that is, $\mu = 0$ and $\left[2 - \frac{1}{k}(2n-2-n\mu)\right] \neq 0$ then (2.99) becomes

$$R_t(H_2^*, H_4^*) = 2kng(H_2^*, H_4^*). \tag{2.100}$$

Again from (2.97) and (2.98), we obtain

$$\begin{aligned} & k(2n-2-n\mu)g(H_2^*, H_4^*) + k(2-2n+2nk+n\mu)\eta(H_2^*)\eta(H_4^*) \\ & + k(2n-2+\mu)g(hH_2^*, H_4^*) = \left[2k^2n + (k-1)(2n-2+\mu)\mu\right] g(H_2^*, H_4^*) \\ & + [2kn\mu + (2n-2-n\mu)\mu]\, g(hH_2^*, H_4^*) - (k-1)(2n-2+\mu)\mu\eta(H_2^*)\eta(H_4^*). \end{aligned} \tag{2.101}$$

Comparing the both sides, we get
$\mu = 0,\ k = 0.$
Hence the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$. □

2.16 Example of a (k,μ)-contact metric manifold admitting a CηES

From Example 2.7, we can calculate equation (2.73)

$$R_t(w_3,w_3) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{3}\right) - (\lambda^* + \mu^*)\right].$$

Therefore,

$$\left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{3}\right) - (\lambda^* + \mu^*)\right] = 0,$$

which implies that,

$$R^* = \omega + 2\lambda^* + 2\mu^* + \frac{2}{3}.$$

Since $k = 0$, equation (2.75) gives us

$$R^* = \omega + 2\lambda^* + 2\mu^* + \frac{2}{3}.$$

Hence the constant scalar curvature R^* satisfies equation (2.75) of theorem (2.13.1) and so g defines a CηES on (k,μ)-contact manifold M.

Chapter 3

Kenmotsu and ε-Kenmotsu manifolds

3.1 Introduction

In [80] S. Tanno classified the connected almost contact metric manifold whose automorphism group has maximum dimension. There are three classes:

(i) Homogeneous normal contact Riemannian manifolds with constant ϕ- holomorphic sectional curvature if the sectional curvature of the plain section containing ξ, say $C(H_1^*,\xi)$>0.

(ii) Global Riemanian product of a line or a circle and a Käehlerian manifold with constant holomorphic sectional curvature, $C(H_1^*,\xi)=0$.

(iii) A warped product space $R\times_\lambda C^n$, if $C(H_1^*,\xi)<0$, where λ is a smooth function.

Manifold of class (a) are characterized by some tensor equations, it has a Sasakian structure and manifolds of class (b) are characterized by a tensorial relation admitting a cosymplectic structure. In 1972 Kenmotsu has introduced a new class of almost contact Riemannian manifolds which are nowadays called Kenmotsu manifolds[47]. He obtained some tensorial equations to characterize manifolds of class (c).

Let (M,ϕ,ξ,η,g) be a $n=2m+1$ dimensional almost contact metric manifold. Then the product $\bar{M}=\mathbf{M}\times\mathbf{R}$ has a natural almost complex structure J with the product metric G being Hermitian manifold $(\bar{M},J,G)$. The notion of trans-Sasakian manifolds was introduced by Oubina [61] in 1985. In general, a trans-Sasakian manifold $(M,\phi,\xi,\eta,g,\alpha,\beta)$ is called a trans-Sasakian manifold of type (α,β). Trans-Sasakian manifold of type $(0,\beta)$ is called β-Kenmotsu manifold. In 1932 [44] Hayden has given the notion of metric connection with torsion on Riemannian manifold. Semi-symmetric connection on Riemannian manifold was studied by K.Yano

[92] in 1970. Semi–symmetric connections on Riemannian manifold was also studied by K.S. Amur [3], S.S. Pujar, C.S. Bagewadi [4] et al. in 1976.
The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [81] introduced the notion of locally ϕ-symmetry on a Sasakian manifold. Generalizing the notion of ϕ-symmetry one of the authors in [6] introduced the notion of ϕ-recurrent Kenmotsu manifolds.
The notion of generalized recurrent manifold has been introduced by Dubey [30] . Again, the notion of generalized Ricci-recurrent manifold has been introduced and studied by De et al. [27]. A Riemannian manifold $(M^n, g), n > 2$, is called generalized recurrent [26, 30] if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{3.1}$$

where A and B are non-vanishing 1-forms defined by $A(\delta) = g(\delta, \rho_1), B(\delta) = g(\delta, \rho_2)$ and the tensor G is defined by

$$G(H_1^*, H_2^*)H_3^* = g(H_2^*, H_3^*)H_1^* - g(H_1^*, H_3^*)H_2^*, \tag{3.2}$$

for all $H_1^*, H_2^*, H_3^* \in \chi(M)$; $\chi(M)$ being the Lie algebra of smooth vector fields on M and ∇ denotes the operator of covariant differentiation with respect to the metric g. The 1-forms A and B are called the associated 1-forms of the manifold. A Riemannian manifold $(M^n, g), n > 2$, is called generalized Ricci-recurrent [21] if its Ricci tensor R_t of type $(0,2)$ satisfies the condition $\nabla R_t = A \otimes R_t + B \otimes g$, where A and B are non-vanishing 1-forms. In 2007, Özgür [62] studied generalized recurrent Kenmotsu manifolds. Recently Basari et al. [6] introduced the notion of generalized ϕ-recurrent Kenmotsu manifolds. Extending the notion of Basari et al. [6], Shaikh et al. [2] introduce the notion of extended generalized ϕ-recurrent β-Kenmotsu manifolds. In this Chapter, we have further studied and established few results on generalized ϕ-recurrent Kenmotsu manifolds.
The examination of manifolds with indefinite metrics carries significant importance in terms of the geometrization of physics and relativity. Scientists and mathematicians from various fields have consistently held an interest in studying indefinite structures on manifolds. A manifold is equipped with a geometric structure, it opens up avenues for a deeper exploration of its geometric characteristics. Within this context, various classes of submanifolds, including warped product submanifolds, biharmonic submanifolds, and singular submanifolds, serve as motivating factors for further investigation, drawing the attention of numerous researchers [51, 52, 53, 65, 66, 68, 70]. In a Riemannian

manifold (M, g), the signature of metric tensor is positive definite. Conversely, in a semi-Riemannian manifold, the signature of the metric becomes indefinite. Utilizing this indefinite metric, Bejancu et al. [8] introduced ε-Sasakian manifolds. Subsequently, Xufeng et al. [89] demonstrated that every ε-Sasakian manifold corresponds to a real hyperface within certain indefinite Kähler manifolds. Concurrently, K. Kenmotsu [47] introduced a distinct category of contact Riemannian manifolds that adhere to specific conditions, later recognized as Kenmotsu manifolds. Building on this, De et al. [28] introduced indefinite metrics on Kenmotsu manifolds, coining them as ε-Kenmotsu manifolds. More recently, Haseeb and De [40] delved into the study of ηRS in ε-Kenmotsu manifolds. Many researchers [38, 42, 78, 88] have also contributed to the exploration of ε-Kenmotsu manifolds, leading to the acquisition of numerous captivating results regarding this intriguing indefinite structure. The characteristic vector field ξ is spacelike or timelike according as the value of ε is either 1 or -1. Further, noted that for spacelike vector field ξ and ε =1, an ε-Kenmotsu manifold becomes usual Kenmotsu manifold.

3.2 Preliminaries

A n-dimensional smooth manifold (M, g) is said to be an ε-almost contact metric manifold [8] if it admits a (1,1) tensor field ϕ, ξ is a characteristic vector field, η is a 1-form and g is an indefinite metric such that

$$\phi^2(H_1^*) = -H_1^* + \eta(H_1^*)\xi, \quad \eta(\xi) = 1, \tag{3.3}$$

$$\eta(H_1^*) = \varepsilon g(H_1^*, \xi), \quad g(\xi, \xi) = \varepsilon, \tag{3.4}$$

$$g(H_1^*, \phi H_2^*) = -g(\phi H_1^*, H_2^*), \tag{3.5}$$

$$g(\phi H_1^*, \phi H_2^*) = g(H_1^*, H_2^*) - \varepsilon\eta(H_1^*)\eta(H_2^*). \tag{3.6}$$

Also, we have

$$rank\phi = n - 1, \tag{3.7}$$

$$\phi\xi = 0, \quad \eta(\phi H_1^*) = 0. \tag{3.8}$$

If

$$d\eta(H_1^*, H_2^*) = g(H_1^*, \phi H_2^*). \tag{3.9}$$

Then the manifold (M,g) is said to be an ε-contact metric manifold.
An ε-contact metric manifold is called an ε-Kenmotsu manifold [28] if

$$\left(\nabla_{H_1^*}\phi\right)H_2^* = -g(\phi H_1^*, H_2^*) - \varepsilon\eta(H_2^*)\phi H_1^*, \tag{3.10}$$

where ∇ denotes the Levi-Civita connection of M.
Further an ε-almost contact metric manifold is an ε-Kenmotsu manifold if and only if

$$\nabla_{H_1^*}\xi = \varepsilon\left(H_1^* - \eta\left(H_1^*\right)\xi\right). \tag{3.11}$$

Again an ε-Kenmotsu manifold M satisfies

$$\left(\nabla_{H_1^*}\eta\right)H_2^* = g(H_1^*, H_2^*) - \varepsilon\eta\left(H_1^*\right)\eta\left(H_2^*\right), \tag{3.12}$$

$$\eta\left(R(H_1^*, H_2^*)H_3^*\right) = \varepsilon\left[g(H_1^*, H_3^*)\eta(H_2^*) - g(H_2^*, H_3^*)\eta(H_1^*)\right], \tag{3.13}$$

$$R\left(H_1^*, H_2^*\right)\xi = \eta\left(H_1^*\right)H_2^* - \eta\left(H_2^*\right)H_1^*, \tag{3.14}$$

$$R\left(\xi, H_1^*\right)H_2^* = \eta\left(H_2^*\right)H_1^* - \varepsilon g\left(H_1^*, H_2^*\right)\xi, \tag{3.15}$$

$$R\left(\xi, H_1^*\right)\xi = -R\left(H_1^*, \xi\right)\xi = H_1^* - \eta\left(H_1^*\right)\xi, \tag{3.16}$$

$$R_t(\phi H_1^*, \phi H_2^*) = R_t(H_1^*, H_2^*) + \varepsilon(n-1)\eta(H_1^*)\eta(H_2^*), \tag{3.17}$$

$$R_t\left(H_1^*, \xi\right) = -(n-1)\eta\left(H_1^*\right), \tag{3.18}$$

$$R_t\left(\xi, \xi\right) = -(n-1), \tag{3.19}$$

$$Q\xi = -\varepsilon(n-1)\xi. \tag{3.20}$$

In Kenmotsu manifold, ε=1 and the above equations become

$$\phi^2\left(H_1^*\right) = -H_1^* + \eta\left(H_1^*\right)\xi, \;\; \eta\left(\xi\right) = 1, \tag{3.21}$$

$$\eta\left(H_1^*\right) = g\left(H_1^*, \xi\right), \;\; g\left(\xi, \xi\right) = 1, \tag{3.22}$$

$$g\left(H_1^*, \phi H_2^*\right) = -g\left(\phi H_1^*, H_2^*\right), \tag{3.23}$$

$$g\left(\phi H_1^*, \phi H_2^*\right) = g\left(H_1^*, H_2^*\right) - \eta\left(H_1^*\right)\eta\left(H_2^*\right). \tag{3.24}$$

Also, we have

$$rank\phi = n-1, \tag{3.25}$$

$$\phi\xi = 0, \;\; \eta\left(\phi H_1^*\right) = 0, \tag{3.26}$$

$$\left(\nabla_{H_1^*}\phi\right)H_2^* = -g(\phi H_1^*, H_2^*) - \eta(H_2^*)\phi H_1^*, \tag{3.27}$$

where ∇ denotes the Levi-Civita connection of M.

$$\nabla_{H_1^*}\xi = (H_1^* - \eta\,(H_1^*)\,\xi)\,. \tag{3.28}$$

Again Kenmotsu manifold M satisfies

$$\left(\nabla_{H_1^*}\eta\right)H_2^* = g(H_1^*,H_2^*) - \eta\,(H_1^*)\,\eta\,(H_2^*)\,, \tag{3.29}$$

$$\eta\,(R(H_1^*,H_2^*)H_3^*) = [g(H_1^*,H_3^*)\eta(H_2^*) - g(H_2^*,H_3^*)\eta(H_1^*)]\,, \tag{3.30}$$

$$R\,(H_1^*,H_2^*)\,\xi = \eta\,(H_1^*)\,H_2^* - \eta\,(H_2^*)\,H_1^*, \tag{3.31}$$

$$R\,(\xi,H_1^*)\,H_2^* = \eta\,(H_2^*)\,H_1^* - g\,(H_1^*,H_2^*)\,\xi, \tag{3.32}$$

$$R\,(\xi,H_1^*)\,\xi = -R\,(H_1^*,\xi)\,\xi = H_1^* - \eta\,(H_1^*)\,\xi, \tag{3.33}$$

$$R_t(\phi H_1^*,\phi H_2^*) = R_t(H_1^*,H_2^*) + (n-1)\eta(H_1^*)\eta(H_2^*), \tag{3.34}$$

$$R_t\,(H_1^*,\xi) = -(n-1)\eta\,(H_1^*)\,, \tag{3.35}$$

$$R_t\,(\xi,\xi) = -(n-1), \tag{3.36}$$

$$Q\xi = -(n-1)\xi, \tag{3.37}$$

for all $H_1^*,H_2^*,H_3^* \in \chi\,(M)$, where R is the curvature tensor, R_t is the Ricci tensor of type (0,2) and $g(QH_1^*,H_2^*)$=$R_t(H_1^*,H_2^*)$, Q is the Ricci operator of the manifold.

3.3 n-dimensional ε-Kenmotsu manifold admitting CηES

Here we consider ε-Kenmotsu manifold (M^n, g) admitting CηES. In the first part we try to characterize the nature of the soliton by calculating the condition under which a CηES is shrinking, steady or expanding on n-dimensional ε-Kenmotsu manifold.
Now, we state the following theorems:

Theorem 3.3.1. *If a n-dimensional ε-Kenmotsu manifold admits a CηES $(g, \xi, \lambda^*, \mu^*)$, then the manifold (M^n, g) becomes an ηE manifold of constant scalar curvature $R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}\mu^*$. Further*
1. *if ξ is spacelike, then the soliton is shrinking, steady and expanding as*

   (i) $\omega > (n-3)\mu^* + \frac{1}{n}(n-2)(n+1-n^2)$,

   (ii) $\omega = (n-3)\mu^* + \frac{1}{n}(n-2)(n+1-n^2)$,

   (iii) $\omega < (n-3)\mu^* + \frac{1}{n}(n-2)(n+1-n^2)$.

2. *if ξ is timelike, then the soliton is shrinking, steady and expanding as*

   (i) $\omega > (1-n)\mu^* + \frac{1}{n}(n-2)(n^2-n+1)$,

   (ii) $\omega = (1-n)\mu^* + \frac{1}{n}(n-2)(n^2-n+1)$,

   (iii) $\omega < (1-n)\mu^* + \frac{1}{n}(n-2)(n^2-n+1)$.

Proof. Let us consider an ε-Kenmotsu manifold (M^n, g) admitting a CηES $(g, \xi, \lambda^*, \mu^*)$. Then from the equation (1.22), we have

$$(\mathcal{L}_\xi g)(H_1^*, H_2^*) + 2R_t(H_1^*, H_2^*) + 2\mu^* \eta(H_1^*)\eta(H_2^*) + \left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right)\right] g(H_1^*, H_2^*) = 0. \tag{3.38}$$

From (3.38), we get

$$2R_t(H_1^*, H_2^*) = -(\mathcal{L}_\xi g)(H_1^*, H_2^*) - 2\mu^* \eta(H_1^*)\eta(H_2^*) - \left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right)\right] g(H_1^*, H_2^*). \tag{3.39}$$

We have,

$$(\mathcal{L}_\xi g)(H_1^*, H_2^*) = g\left(\nabla_{H_1^*}\xi, H_2^*\right) + g\left(\nabla_{H_2^*}\xi, H_1^*\right). \tag{3.40}$$

Now, with the help of (3.11), we infer that

$$(\mathcal{L}_{\xi} g)(H_1^*, H_2^*) = 2\varepsilon g(H_1^*, H_2^*) - 2\eta(H_1^*)\eta(H_2^*). \tag{3.41}$$

From equatios (3.39) and (3.41) , we obtain

$$R_t(H_1^*, H_2^*) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{n}\right) - (\lambda^* + \varepsilon)\right] g(H_1^*, H_2^*) \\ - (\mu^* - 1)\eta(H_1^*)\eta(H_2^*). \tag{3.42}$$

Putting $H_2^* = \xi$ in (3.42), acquire

$$R_t(H_1^*, \xi) = \left[\frac{\varepsilon R^*}{2} - \varepsilon\left(\frac{\omega}{2} + \frac{1}{n}\right) - (\varepsilon\lambda^* + \mu^*)\right]\eta(H_1^*). \tag{3.43}$$

From the equations (3.18) and (3.43), we deduce that

$$-(n-1)\eta(H_1^*) = \left[\frac{\varepsilon R^*}{2} - \varepsilon\left(\frac{\omega}{2} + \frac{1}{n}\right) - \varepsilon\lambda^* - \mu^*\right]\eta(H_1^*).$$

Since η is a non-zero 1-form, then above equation becomes

$$R^* = \omega + 2\lambda^* + (2\mu^* - 2n + 2)\varepsilon + \frac{2}{n}. \tag{3.44}$$

Taking an orthonormal basis $\{w_i : i = 1, 2, \ldots, n\}$ of (M^n, g) and then setting $H_1^* = H_2^* = w_i$ in the equation (3.42) and taking summation over i, we obtain

$$R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}\mu^*. \tag{3.45}$$

Finally combining equations (3.44) and (3.45), we get

$$\lambda^* = -\frac{\omega}{2} + \frac{1}{2}(n\varepsilon - 2\varepsilon - 1)\mu^* + \frac{1}{2}(1-n)(n-2)\varepsilon + \frac{1}{2n}(n-2). \tag{3.46}$$

From (3.46), we can conclude the following:

1. if ξ is spacelike with

(i) $\omega > (n-3)\mu^* + \frac{1}{n}(n-2)(n+1-n^2)$ implies the soliton is shrinking.

(ii) $\omega = (n-3)\mu^* + \frac{1}{n}(n-2)(n+1-n^2)$ implies the soliton is steady.

(iii) $\omega < (n-3)\mu^* + \frac{1}{n}(n-2)(n+1-n^2)$ implies the soliton is expanding.

2. if ξ is timelike with

(i) $\omega > (1-n)\mu^* + \frac{1}{n}(n-2)(n^2-n+1)$ implies the soliton is shrinking.

(ii) $\omega = (1-n)\mu^* + \frac{1}{n}(n-2)(n^2-n+1)$ implies the soliton is steady.

(iii) $\omega < (1-n)\mu^* + \frac{1}{n}(n-2)(n^2-n+1)$implies the soliton is expanding.

This completes the proof. □

Theorem 3.3.2. *If a n-dimensional Ricci symmetric space ε-Kenmotsu manifold admits $C\eta ES$, then $\mu^* = 1$ and the constant scalar curvature $R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}$.*

Proof. Using the equations (2.18) and (3.42), we get

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = -(\mu^* - 1)\left[\eta(H_3^*)(\nabla_{H_1^*}\eta)H_2^* + \eta(H_2^*)(\nabla_{H_1^*}\eta)H_3^*\right]. \tag{3.47}$$

From equations (3.12) and (3.47), we obtain

$$\begin{aligned}(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = &-(\mu^* - 1)\left[\eta(H_3^*)\left(g(H_1^*, H_2^*) - \varepsilon\eta(H_1^*)\eta(H_2^*)\right)\right.\\ &\left.+\eta(H_2^*)\left(g(H_1^*, H_3^*) - \varepsilon\eta(H_1^*)\eta(H_3^*)\right)\right].\end{aligned} \tag{3.48}$$

If the manifold M^n is Ricci symmetric, then $\nabla R_t = 0$ and equation (3.48) becomes

$$\begin{aligned}&-(\mu^* - 1)\left[\eta(H_3^*)\left(g(H_1^*, H_2^*) - \varepsilon\eta(H_1^*)\eta(H_2^*)\right)\right.\\ &\left.+\eta(H_2^*)\left(g(H_1^*, H_3^*) - \varepsilon\eta(H_1^*)\eta(H_3^*)\right)\right] = 0.\end{aligned} \tag{3.49}$$

Substituting $H_3^* = \xi$ in the equation (3.49), we get

$$(\mu^* - 1)\, g(\phi H_1^*, \phi H_2^*) = 0. \tag{3.50}$$

Since $g(\phi H_1^*, \phi H_2^*) \neq 0$, μ^*=1.
Putting the value of μ^* in (3.45), we have

$$R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}.$$

This concludes the proof. □

Theorem 3.3.3. *If a n-dimensional ε-Kenmotsu manifold is a CηES and the Ricci tensor R_t is Ricci-recurrent, then the constant scalar curvature $R^* = \omega + 2\lambda^* + 2\mu^* + 2(\varepsilon - 1) + \frac{2}{n}$.*

Proof. Let us consider the Ricci tensor is Ricci-recurrent i.e., $\nabla R_t = \eta \otimes R_t$.
Then from the equations (1.41) and (3.48), we obtain

$$\begin{aligned} &-(\mu^* - 1)\,[\eta(H_3^*)\,(g(H_1^*, H_2^*) - \varepsilon\eta(H_1^*)\eta(H_2^*)) \\ &+ \eta(H_2^*)\,(g(H_1^*, H_3^*) - \varepsilon\eta(H_1^*)\eta(H_3^*))] = \eta(H_1^*)R_t(H_2^*, H_3^*). \end{aligned} \tag{3.51}$$

Putting $H_2^* = H_3^* = \xi$ in (3.51) and using the equation (3.42), we have

$$\left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{n}\right) - (\lambda^* + \mu^* + \varepsilon - 1)\right]\eta(H_1^*) = 0, \tag{3.52}$$

which gives

$$R^* = \omega + 2\lambda^* + 2\mu^* + 2(\varepsilon - 1) + \frac{2}{n}. \tag{3.53}$$

Thus, the proof is complete. □

Theorem 3.3.4. *Let M be a n-dimensional ε-Kenmotsu manifold admitting a CηES $(g, \nu, \lambda^*, \mu^*)$ such that ν is a pointwise collinear with ξ, then ν is a constant multiple of ξ and the manifold (M^n, g) becomes an ηE manifold of constant scalar curvature $R^* = \omega + 2\lambda^* + \frac{2}{n}$.*

Proof. Let us consider an ε-Kenmotsu manifold (M^n, g) that admits a CηES $(g, \nu, \lambda^*, \mu^*)$ such that ν is parallel to ξ, that is, $\nu = m\xi$ for some function m, then equation (1.22) becomes,

$$\begin{aligned} &(\mathcal{L}_{m\xi} g)(H_1^*, H_2^*) + 2R_t(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) \\ &+ \left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right)\right] g(H_1^*, H_2^*) = 0. \end{aligned}$$

The foregoing equation gives us

$$\begin{aligned} &mg(\nabla_{H_1^*}\xi, H_2^*) + \varepsilon(H_1^* m)\eta(H_2^*) + mg(\nabla_{H_2^*}\xi, H_1^*) + \varepsilon(H_2^* m)\eta(H_1^*) \\ &+ 2R_t(H_1^*, H_2^*) + \left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right)\right] g(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) = 0. \end{aligned} \tag{3.54}$$

Using (3.11) in the equation (3.54), we obtain

$$
\begin{aligned}
&mg(\varepsilon(H_1^* - \eta(H_1^*)\xi), H_2^*) + \varepsilon(H_1^* m)\eta(H_2^*) + \varepsilon(H_2^* m)\eta(H_1^*)\\
&+ mg(\varepsilon(H_2^* - \eta(H_2^*)\xi), H_1^*) + 2R_t(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*)\\
&+ \left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right)\right] g(H_1^*, H_2^*) = 0,
\end{aligned} \tag{3.55}
$$

which implies

$$
\begin{aligned}
&2\varepsilon mg(H_1^*, H_2^*) - 2m\eta(H_1^*)\eta(H_2^*) + \varepsilon(H_1^* m)\eta(H_2^*) + \varepsilon(H_2^* m)\eta(H_1^*)\\
&+ 2R_t(H_1^*, H_2^*) + \left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right)\right] g(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) = 0.
\end{aligned} \tag{3.56}
$$

Putting $H_2^* = \xi$ in (3.56) and using (3.18), we get

$$
\begin{aligned}
&\varepsilon(H_1^* m) + 2\left[\mu^* - (n-1)\right]\eta(H_1^*)\\
&+ \varepsilon\left[2\lambda^* - R^* + \left(\omega + \frac{2}{n}\right) + \xi m\right]\eta(H_1^*) = 0.
\end{aligned} \tag{3.57}
$$

Again putting $H_1^* = \xi$ in (3.57) entails that

$$
\varepsilon(\xi m) = \varepsilon\left[\frac{R^*}{2} - \lambda^* - \left(\frac{\omega}{2} + \frac{1}{n}\right)\right] + (n-1) - \mu^*. \tag{3.58}
$$

Equations (3.57) and (3.58) together imply

$$
\varepsilon(H_1^* m) + \varepsilon\left[\lambda^* - \frac{R^*}{2} + \left(\frac{\omega}{2} + \frac{1}{n}\right)\right]\eta(H_1^*) = 0. \tag{3.59}
$$

If

$$
\left[\lambda^* - \frac{R^*}{2} + \left(\frac{\omega}{2} + \frac{1}{n}\right)\right] = 0,
$$

then $H_1^* m = 0$, that is, m is constant. This implies $\xi m = 0$.
Therefore the equation (3.59) reduces to

$$
R^* = \omega + 2\lambda^* + \frac{2}{n}. \tag{3.60}
$$

Since m is constant, equation (3.55) gives

$$
R_t(H_1^*, H_2^*) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{n}\right) - \lambda^*\right] g(H_1^*, H_2^*) - \mu^*\eta(H_1^*)\eta(H_2^*). \tag{3.61}
$$

Hence the result. □

3.4 CηES on ε-Kenmotsu manifold (M^n, g) with Codazzi type, cyclic parallel and cyclic η-recurrent Ricci tensor

The focus of this section is to study CηES on ε-Kenmotsu manifolds (M^n, g) admitting Ricci tensor is of Codazzi type, cyclic parallel and cyclic η-recurrent .
We now state and prove the following theorems:

Theorem 3.4.1. *Let an ε-Kenmotsu manifold (M^n, g) admits a CηES $(g, \xi, \lambda^*, \mu^*)$. If the Ricci tensor R_t is of Codazzi type then the manifold becomes an Einstein manifold and the constant scalar curvature* $R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}$.

Proof. Let us consider the manifold (M^n, g) having Codazzi type of Ricci tensor admits a CηES $(g, \xi, \lambda^*, \mu^*)$. Taking covariant derivative of (3.42), we have

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = -(\mu^* - 1)\,[\eta(H_3^*)(\nabla_{H_1^*}\eta)H_2^* + \eta(H_2^*)(\nabla_{H_1^*}\eta)H_3^*]. \tag{3.62}$$

Using equation (3.12) in (3.62), we get

$$\begin{aligned}(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) &= -(\mu^* - 1)\,[\eta(H_3^*)g(H_1^*, H_2^*) \\ &+ \eta(H_2^*)g(H_1^*, H_3^*) - 2\varepsilon\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)].\end{aligned} \tag{3.63}$$

Interchanging H_1^* and H_2^* in equation (3.63), we obtain

$$\begin{aligned}(\nabla_{H_2^*} R_t)(H_1^*, H_3^*) &= -(\mu^* - 1)\,[\eta(H_3^*)g(H_1^*, H_2^*) \\ &+ \eta(H_1^*)g(H_2^*, H_3^*) - 2\varepsilon\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)].\end{aligned} \tag{3.64}$$

Again using (3.63) and (3.64) in (1.37), we get

$$\begin{aligned}&-(\mu^* - 1)\,[\eta(H_3^*)g(H_1^*, H_2^*) + \eta(H_2^*)g(H_1^*, H_3^*) - 2\varepsilon\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)] \\ &= -(\mu^* - 1)\,[\eta(H_3^*)g(H_1^*, H_2^*) + \eta(H_1^*)g(H_2^*, H_3^*) - 2\varepsilon\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)],\end{aligned} \tag{3.65}$$

that is,

$$(\mu^* - 1)\,[g(H_1^*, H_3^*)\eta(H_2^*) - g(H_2^*, H_3^*)\eta(H_1^*)] = 0. \tag{3.66}$$

If $\mu^* = 1$, then the equation (3.42) reduces to

$$R_t(H_1^*, H_2^*) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{n}\right) - (\lambda^* + \varepsilon)\right] g(H_1^*, H_2^*). \tag{3.67}$$

Taking an orthonormal basis $\{w_i, i = 1, 2, \ldots, n\}$ of (M^n, g) and then setting $H_1^* = H_2^* =$

w_i in the equation (3.67) and taking summation over i, we get

$$R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}.$$

This completes the proof. □

Theorem 3.4.2. *Let (M^n, g) be an ε-Kenmotsu manifold admitting a CηES $(g, \xi, \lambda^*, \mu^*)$. If the manifold has cyclic parallel Ricci tensor, then the manifold becomes an Einstein manifold and the constant scalar curvature takes the form $R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}$.*

Proof. Let us consider an ε-Kenmotsu manifold (M^n, g) admitting a cyclic parallel Ricci tensor and a CηES $(g, \xi, \lambda^*, \mu^*)$. On substituting $H_1^* = H_3^*, H_2^* = H_1^*, H_3^* = H_2^*$ in (3.63), we reach

$$\begin{aligned}(\nabla_{H_3^*} R_t)(H_1^*, H_2^*) = &-(\mu^* - 1)\,[\eta(H_2^*)g(H_1^*, H_3^*) \\ &+ \eta(H_1^*)g(H_2^*, H_3^*) - 2\varepsilon\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)].\end{aligned} \tag{3.68}$$

Using (3.63), (3.64) and (3.68) in equation (1.38), we get

$$\begin{aligned}&-2\,(\mu^* - 1)\,[\eta(H_3^*)g(H_1^*, H_2^*) + \eta(H_2^*)g(H_1^*, H_3^*) \\ &+ \eta(H_1^*)g(H_2^*, H_3^*) - 3\varepsilon\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)] = 0.\end{aligned} \tag{3.69}$$

Setting $H_3^* = \xi$ in equation (3.69), we get

$$2\,(\mu^* - 1)\,g(\phi H_1^*, \phi H_2^*) = 0. \tag{3.70}$$

The foregoing equation implies μ^*=1. Therefore (3.42) reduces to

$$R_t(H_1^*, H_2^*) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{n}\right) - (\lambda^* + \varepsilon)\right] g(H_1^*, H_2^*). \tag{3.71}$$

Taking an orthonormal basis $\{w_i, i = 1, 2, \ldots, n\}$ of (M^n, g) and then setting $H_1^* = H_2^* = w_i$ in the equation (3.71) and taking summation over i, we get

$$R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}.$$

This completes the proof. □

We can state our next theorem:

Theorem 3.4.3. *Let* (M^n, g) *be an* ε*-Kenmotsu manifold admitting a C*η*ES* $(g, \xi, \lambda^*, \mu^*)$*. If the manifold has cyclic* η*-recurrent Ricci tensor, then the constant scalar curvature* $R^* = \omega + 2\lambda^* + 2\varepsilon\mu^* + \frac{2}{n}$.

Proof. Now we consider an ε-Kenmotsu manifold (M^n, g) having cyclic η-recurrent Ricci tensor and a CηES $(g, \xi, \lambda^*, \mu^*)$, then equation (3.42) holds. Taking covariant differentiation of (3.42) and using (3.63), (3.64), (3.68) in the equation (1.42), we get

$$\begin{aligned}
&-2(\mu^*-1)[\eta(H_3^*)g(H_1^*,H_2^*)+\eta(H_2^*)g(H_1^*,H_3^*)+\eta(H_1^*)g(H_2^*,H_3^*) \\
&-3\varepsilon\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)] = \left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{n}\right)-(\lambda^*+\varepsilon)\right] \\
&[\eta(H_3^*)g(H_1^*,H_2^*)+\eta(H_2^*)g(H_1^*,H_3^*)+\eta(H_1^*)g(H_2^*,H_3^*)] \\
&-3(\mu^*-1)\eta(H_1^*)\eta(H_2^*)\eta(H_3^*),
\end{aligned} \tag{3.72}$$

that is,

$$\begin{aligned}
&\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{n}\right)-(\lambda^*+\varepsilon)+2(\mu^*-1)\right] \\
&[\eta(H_3^*)g(H_1^*,H_2^*)+\eta(H_2^*)g(H_1^*,H_3^*)+\eta(H_1^*)g(H_2^*,H_3^*)] \\
&-(6\varepsilon+3)(\mu^*-1)\eta(H_1^*)\eta(H_2^*)\eta(H_3^*) = 0.
\end{aligned} \tag{3.73}$$

Taking $H_2^* = H_3^* = \xi$ in (3.73), we have

$$\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{n}\right)-(\lambda^*+\varepsilon)-\varepsilon(\mu^*-1)\right]\eta(H_1^*) = 0. \tag{3.74}$$

Since $\eta(H_1^*) \neq 0$, then the above equation reduces to

$$R^* = \omega + 2\lambda^* + 2\varepsilon\mu^* + \frac{2}{n}. \tag{3.75}$$

This completes the proof. □

3.5 CηES on ε-Kenmotsu manifold (M^n,g) satisfying ξ-Ricci symmetric tensor

In this part, we study an ε-Kenmotsu manifold (M^n,g) that admits a conformal ηES (g,ξ,λ^*,μ^*) and the manifold satisfies the curvature condition $R(\xi,H_1^*).R_t=0$.
We can state the following theorem:

Theorem 3.5.1. *Let (M^n,g) be an ε-Kenmotsu manifold admitting a CηES (g,ξ,λ^*,μ^*). If the manifold satisfies the curvature condition $R(\xi,H_1^*).R_t=0$, then the manifold becomes an Einstein manifold and the constant scalar curvature takes the form $R^*=\frac{n}{n-2}\omega+\frac{2n}{n-2}(\lambda^*+\varepsilon)+\frac{2}{n-2}$.*

Proof. We obtain on using (3.42) in (1.46)

$$\begin{aligned}&\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{n}\right)-(\lambda^*+\varepsilon)\right]g(R(\xi,H_1^*)H_2^*,H_3^*)\\&+\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{n}\right)-(\lambda^*+\varepsilon)\right]g(H_2^*,R(\xi,H_1^*)H_3^*)\\&-(\mu^*-1)\eta(R(\xi,H_1^*)H_2^*)\eta(H_3^*)-(\mu^*-1)\eta(R(\xi,H_1^*)H_3^*)\eta(H_2^*)=0. \end{aligned} \tag{3.76}$$

Using the equation (3.15) in (3.76), we get

$$\begin{aligned}&\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{n}\right)-(\lambda^*+\varepsilon)\right]g[(\eta(H_2^*)H_1^*-\varepsilon g(H_1^*,H_2^*)\xi),H_3^*]\\&-(\mu^*-1)\eta[\eta(H_2^*)H_1^*-\varepsilon g(H_1^*,H_2^*)\xi]\eta(H_3^*)\\&+\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{n}\right)-(\lambda^*+\varepsilon)]\right]g[H_2^*,(\eta(H_3^*)H_1^*-\varepsilon g(H_1^*,H_3^*)\xi)]\\&-(\mu^*-1)\eta[\eta(H_3^*)H_1^*-\varepsilon g(H_1^*,H_3^*)\xi]\eta(H_2^*)=0,\end{aligned} \tag{3.77}$$

which implies,

$$-(\mu^*-1)\left[2\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)-\varepsilon g(H_1^*,H_2^*)\eta(H_3^*)-\varepsilon g(H_1^*,H_3^*)\eta(H_2^*)\right]=0. \tag{3.78}$$

Taking $H_3^*=\xi$ in (3.78), we get

$$(\mu^*-1)g(\phi H_1^*,\phi H_2^*)=0. \tag{3.79}$$

Since $g(\phi H_1^*,\phi H_2^*)\neq 0$, $\mu^*=1$.
Putting $\mu^*=1$ in (3.42), we obtain

$$R_t(H_1^*,H_2^*)=\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{n}\right)-(\lambda^*+\varepsilon)\right]g(H_1^*,H_2^*). \tag{3.80}$$

Taking an orthonormal basis $\{w_i, i = 1, 2, \ldots, n\}$ of (M^n, g) and then setting $H_1^* = H_2^* = w_i$ in the equation (3.80) and taking summation over i, we get

$$R^* = \frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}.$$

Hence the result. □

3.6 Einstein Semi-Symmetric ε-Kenmotsu manifolds (M^n, g) admitting CηES

Now, we state the following Lemma:

Lemma 3.6.1. *An Einstein semi-symmetric ε-Kenmotsu manifold (M^n, g) is an Einstein manifold.*

Proof. A n-dimensional ε-Kenmotsu manifold satisfies the curvature condition, i.e., $R.E = 0$, then from equations (1.39) and (1.40), we have

$$R_t(R(H_1^*, H_2^*)H_3^*, H_4^*) + R_t(H_3^*, R(H_1^*, H_2^*)H_4^*)$$
$$= \frac{R^*}{n}\left[g(R(H_1^*, H_2^*)H_3^*, H_4^*) + g(H_3^*, R(H_1^*, H_2^*)H_4^*)\right]. \tag{3.81}$$

Setting $H_1^* = H_3^* = \xi$ in equation (3.81), we get

$$R_t(R(\xi, H_2^*)\xi, H_4^*) + R_t(\xi, R(\xi, H_2^*)H_4^*)$$
$$= \frac{R^*}{n}\left[g(R(\xi, H_2^*)\xi, H_4^*) + g(\xi, R(\xi, H_2^*)H_4^*)\right]. \tag{3.82}$$

Using (3.15) and (3.16) in (3.82), we obtain

$$R_t(H_2^* - \eta(H_2^*)\xi, H_4^*) + R_t(\xi, \eta(H_4^*)H_2^* - \varepsilon g(H_2^*, H_4^*)\xi)$$
$$= \frac{R^*}{n}\left[g(H_2^* - \eta(H_2^*)\xi, H_4^*) + g(\xi, \eta(H_4^*)H_2^* - \varepsilon g(H_2^*, H_4^*)\xi)\right], \tag{3.83}$$

which gives

$$R_t(H_2^*, H_4^*) = \varepsilon(1-n)g(H_2^*, H_4^*). \tag{3.84}$$

□

On the basis of Lemma (3.6.1), we can state the following theorem:

Theorem 3.6.1. *Let (M^n, g) be an ε-Kenmotsu manifold admitting a CηES $(g, \xi, \lambda^*, \mu^*)$. If the manifold is an Einstein semi-symmetric, then the manifold becomes an Einstein manifold of constant scalar curvature $R^* = \omega + 2\lambda^* + \frac{2}{n}\mu^* + 4\varepsilon - 2\varepsilon n$. Further*

1. *if ξ is spacelike, then the soliton is shrinking, steady and expanding as*

   (i) $\omega > (3n-4-n^2) - \frac{2\mu^*}{n}$,

   (ii) $\omega = (3n-4-n^2) - \frac{2\mu^*}{n}$,

   (iii) $\omega < (3n-4-n^2) - \frac{2\mu^*}{n}$.

2. *if ξ is timelike, then the soliton is shrinking, steady and expanding as*

   (i) $\omega > (n^2-3n+4) - \frac{2\mu^*}{n}$,

   (ii) $\omega = (n^2-3n+4) - \frac{2\mu^*}{n}$,

   (iii) $\omega < (n^2-3n+4) - \frac{2\mu^*}{n}$.

Proof. Let us consider that an Einstein semi-symmetric ε-Kenmotsu manifold (M^n, g) admits a CηES $(g, \xi, \lambda^*, \mu^*)$. Then equation (3.42) holds and combining (3.42) with the equation (3.84), we get

$$R^* = \omega + 2\lambda^* + \frac{2}{n}\mu^* + 4\varepsilon - 2\varepsilon n. \tag{3.85}$$

Comparing (3.45) and (3.85), we obtain

$$\frac{n}{n-2}\omega + \frac{2n}{n-2}(\lambda^* + \varepsilon) + \frac{2}{n-2}\mu^* = \omega + 2\lambda^* + \frac{2}{n}\mu^* + 4\varepsilon - 2\varepsilon n,$$

that is,

$$\lambda^* = -\frac{\omega}{2} + \frac{3n-4-n^2}{2}\varepsilon - \frac{\mu^*}{n}. \tag{3.86}$$

From (3.86), we can conclude the following :

1. if ξ is spacelike with

   (i) $\lambda^* < 0$, then $\omega > (3n-4-n^2) - \frac{2\mu^*}{n}$ implies the soliton is shrinking.

   (ii) $\lambda^* = 0$, then $\omega = (3n-4-n^2) - \frac{2\mu^*}{n}$ implies the soliton is steady.

(iii) $\lambda^* > 0$, then $\omega < (3n-4-n^2) - \frac{2\mu^*}{n}$ implies the soliton is expanding.

2. if ξ is timelike with

(i) $\lambda^* < 0$, then $\omega > (n^2-3n+4) - \frac{2\mu^*}{n}$ implies the soliton is shrinking.

(ii) $\lambda^* = 0$, then $\omega = (n^2-3n+4) - \frac{2\mu^*}{n}$ implies the soliton is steady.

(iii) $\lambda^* > 0$, then $\omega < (n^2-3n+4) - \frac{2\mu^*}{n}$ implies the soliton is expanding.

This completes the proof. □

3.7 Example of an ε-Kenmotsu manifold admitting CηES

Let $M = \left\{(h_1^*, h_2^*, h_3^*) \in \mathbb{R}^3 : h_3^* \neq 0\right\}$ be a three-dimensional manifold [39]. The vector fields
$w_1 = e^{h_3^*}\frac{\partial}{\partial h_1^*}, w_2 = e^{h_3^*}\frac{\partial}{\partial h_2^*}, w_3 = -\varepsilon\frac{\partial}{\partial h_3^*}$
are linearly independent at each point of M. Let g be the indefinite Riemannian metric defined by
$g(w_1,w_2) = g(w_2,w_3) = g(w_1,w_3) = 0$ and $g(w_1,w_1) = g(w_2,w_2) = 1, g(w_3,w_3) = \varepsilon$,
where $\varepsilon = \pm\, 1$. If η is 1-form on M then $\eta(H_3^*) = \varepsilon g(H_3^*, w_3) = \varepsilon g(H_3^*, \xi)$.
Let ϕ be the (1,1) tensor field defined by

$$\phi(w_1) = w_2, \phi(w_2) = -w_1, \phi(w_3) = 0.$$

Now the linearity of ϕ and g, it can be easily verified that

$$\phi^2(H_3^*) = -H_3^* + \eta(H_3^*)w_3,$$

$$\eta(w_3) = 1$$

and

$$g(\phi H_3^*, \phi H_4^*) = g(H_3^*, H_4^*) - \varepsilon\eta(H_3^*)\eta(H_4^*).$$

Thus for $w_3 = \xi$, the structure (ϕ, ξ, η, g) defines an indefinite almost contact metric structure on M.
Let ∇ be the Levi-Civita connection with respect to the indefinite metric g. Then we have

$[w_1,w_2]=0$, $[w_1,w_3]=\varepsilon w_1$, $[w_2,w_3]=\varepsilon w_2$.
Using (1.48), we can easily calculate that

$$\nabla_{w_1}w_1=-w_3,\nabla_{w_1}w_2=0,\nabla_{w_1}w_3=\varepsilon w_1,$$

$$\nabla_{w_2}w_1=0,\nabla_{w_2}w_2=-w_3,\nabla_{w_2}w_3=\varepsilon w_2,$$

$$\nabla_{w_3}w_1=0,\nabla_{w_3}w_2=0,\nabla_{w_3}w_3=0.$$

From the above relations, it follows that the manifold M satisfies $\nabla_{H_1^*}\xi=\varepsilon(H_1^*-\eta(H_1^*)\xi)$, for $\xi=w_3$. Hence the manifold is an ε-Kenmotsu manifold.
Applying equation (1.49), we can calculate

$$R(w_1,w_1)w_1=0,R(w_1,w_2)w_1=\varepsilon w_2,R(w_1,w_2)w_2=-\varepsilon w_1,$$

$$R(w_1,w_2)w_3=-\varepsilon w_3,R(w_2,w_3)w_1=0,R(w_2,w_3)w_3=-w_2,$$

$$R(w_1,w_3)w_1=\varepsilon w_3,R(w_1,w_3)w_2=0,R(w_1,w_3)w_3=-w_1,$$

$$R(w_2,w_1)w_1=-\varepsilon w_2,R(w_3,w_1)w_1=-\varepsilon w_3,R(w_2,w_3)w_2=\varepsilon w_3.$$

Using the above curvature tensors, we can obtain the components of Ricci tensors as follows:

$$R_t(w_1,w_1)=-\varepsilon-1,\ R_t(w_2,w_2)=-\varepsilon-1,\ R_t(w_3,w_3)=-2.$$

Putting $H_1^*=H_2^*=w_3$ in equation (3.42), we obtain

$$\begin{aligned}R_t(w_3,w_3)&=\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{3}\right)-(\lambda^*+\varepsilon)\right]g(w_3,w_3)-(\mu^*-1)\\&=\varepsilon\left[\frac{R^*}{2}-\left(\frac{\omega}{2}+\frac{1}{3}\right)-(\lambda^*+\varepsilon)\right]-(\mu^*-1).\end{aligned}$$

Comparing both the values of $R_t(w_3,w_3)$, we get

$$R^*=\omega+2\lambda^*+2\varepsilon\mu^*-4\varepsilon+\frac{2}{3}.$$

From equation (3.44), we obtain

$$R^*=\omega+2\lambda^*+2\varepsilon\mu^*-4\varepsilon+\frac{2}{3}.$$

Hence the constant scalar curvature R^* satisfies equation (3.44) of theorem 3.3.1 and so g defines a CηES on the 3-dimensional ε-Kenmotsu manifold M.

3.8 Geometric vector fields on Kenmotsu manifolds with respect to semi-symmetric metric connection

We shall now state and prove a theorem on vector field:

Theorem 3.8.1. *Every contact vector field on a Kenmotsu manifold leaving the Ricci tensor with respect to semi-symmetric connection invariant is a strict contact vector field.*

Proof. Let a contact vector field V on a Kenmotsu manifold leaves the Ricci tensor with respect to semi-symmetric metric connection invariant, i.e.,

$$\mathcal{L}_V \bar{R}_t(H_1^*, H_2^*) = 0. \tag{3.87}$$

From (3.87) we have

$$\mathcal{L}_V \bar{R}_t(H_1^*, H_2^*) = \bar{R}_t(\mathcal{L}_V H_1^*, H_2^*) + \bar{R}_t(H_1^*, \mathcal{L}_V H_2^*). \tag{3.88}$$

Putting $H_2^* = \xi$ in (3.88) we obtain

$$\mathcal{L}_V \bar{R}_t(H_1^*, \xi) = \bar{R}_t(\mathcal{L}_V H_1^*, \xi) + \bar{R}_t(H_1^*, \mathcal{L}_V \xi). \tag{3.89}$$

Setting $H_2^* = \xi$ in (1.32) we can get

$$\bar{R}_t(H_1^*, \xi) = -(n-1)\eta(H_1^*). \tag{3.90}$$

Taking Lie derivative on both the sides of the above equation and using (1.30) we can obtain

$$-(n-1)\sigma^*\eta(H_1^*) = \bar{R}_t(H_1^*, \mathcal{L}_V \xi). \tag{3.91}$$

Taking $H_1^* = \xi$ in (3.91) we find

$$\eta(\mathcal{L}_V \xi) = \sigma^*. \tag{3.92}$$

Again from (1.30) and using the definition for Lie derivative we can infer

$$-\eta(\mathcal{L}_V \xi) = \sigma^*. \tag{3.93}$$

Hence combining (3.92) and (3.93) we can conclude that $\sigma^* = 0$. Therefore the proof.

□

3.9 An extended generalized ϕ-recurrent Kenmotsu manifold (M^n, g) with respect to semi-symmetric metric connection

Under this section, we investigate the upcoming theorem:

Theorem 3.9.1. *An extended generalized ϕ-recurrent Kenmotsu manifold (M^n, g) with respect to semi-symmetric metric connection is an Einstein manifold or 1-forms A and B are related as*

$$(n-1)A(H_4^*) - 2B(H_4^*) = 0.$$

Proof. Let us consider an extended generalized ϕ-recurrent Kenmotsu manifold $(M^n, \phi, \eta, \xi, g)$ with respect to semi-symmetric metric connection. Then we have from equation (1.36)

$$\begin{aligned}&\phi^2((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^*) = A(H_4^*)\phi^2(\bar{R}(H_1^*, H_2^*)H_3^*)\\&+B(H_4^*)\phi^2[g(H_2^*, H_3^*)H_1^* - g(H_1^*, H_3^*)H_2^*].\end{aligned} \tag{3.94}$$

Using (3.2), (3.3) and (3.94) we can obtain

$$\begin{aligned}&-(\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^* + \eta((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^*)\xi\\&= A(H_4^*)[-\bar{R}(H_1^*, H_2^*)H_3^* + \eta(\bar{R}(H_1^*, H_2^*)H_3^*)\xi]\\&+B(H_4^*)[-g(H_2^*, H_3^*)H_1^* + g(H_1^*, H_3^*)H_2^*\\&+\eta(H_1^*)g(H_2^*, H_3^*)\xi - g(H_1^*, H_3^*)\eta(H_2^*)\xi].\end{aligned} \tag{3.95}$$

Taking inner product of (3.95) with H_5^* and using (3.5) we can calculate

$$\begin{aligned}&-g((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^*, H_5^*) + \eta((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^*)g(\xi, H_5^*)\\&= A(H_4^*)[-g(\bar{R}(H_1^*, H_2^*)H_3^*, H_5^*) + \eta(\bar{R}(H_1^*, H_2^*)H_3^*)g(\xi, H_5^*)]\\&+B(H_4^*)[-g(H_2^*, H_3^*)g(H_1^*, H_5^*) + g(H_1^*, H_3^*)g(H_2^*, H_5^*)\\&+\eta(H_1^*)g(H_2^*, H_3^*)g(\xi, H_5^*) - g(H_1^*, H_3^*)\eta(H_2^*)g(\xi, H_5^*)].\end{aligned} \tag{3.96}$$

From equations (1.3) and (3.96) we have

$$-g((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*,H_2^*)H_3^*,H_5^*)+\eta((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*,H_2^*)H_3^*)\eta(H_5^*)$$
$$=A(H_4^*)[-g(\bar{R}(H_1^*,H_2^*)H_3^*,H_5^*)+\eta(\bar{R}(H_1^*,H_2^*)H_3^*)\eta(H_5^*)]$$
$$+B(H_4^*)[-g(H_2^*,H_3^*)g(H_1^*,H_5^*)+g(H_1^*,H_3^*)g(H_2^*,H_5^*)$$
$$+\eta(H_1^*)g(H_2^*,H_3^*)\eta(H_5^*)-g(H_1^*,H_3^*)\eta(H_2^*)\eta(H_5^*)]. \tag{3.97}$$

Let $\{w_1,w_2,\ldots,w_n\}$ be an orthonormal basis for the tangent space of M^n at a point $p\in M^n$. Putting $H_1^*=H_5^*=w_i$ in (3.97) and taking summation over i from 1 to n, we have

$$-(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*,H_3^*)+\textstyle\sum_{i=1}^n\eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)H_3^*)\eta(w_i)$$
$$=A(H_4^*)[-\bar{R}_t(H_2^*,H_3^*)+\eta(\bar{R}(\xi,H_2^*)H_3^*)]$$
$$+B(H_4^*)[-g(H_2^*,H_3^*)-\eta(H_2^*)\eta(H_3^*)]. \tag{3.98}$$

Putting $H_3^*=\xi$ in (3.98) we get

$$-(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*,\xi)+\textstyle\sum_{i=1}^n\eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)\xi)\eta(w_i)$$
$$=A(H_4^*)[-\bar{R}_t(H_2^*,\xi)+\eta(\bar{R}(\xi,H_2^*)\xi)]$$
$$+B(H_4^*)[-g(H_2^*,\xi)-\eta(H_2^*)\eta(\xi)]. \tag{3.99}$$

On simplifying above equation we have

$$-(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*,\xi)+\sum_{i=1}^n\eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)\xi)\eta(w_i)$$
$$=-A(H_4^*)\bar{R}_t(H_2^*,\xi)-2B(H_4^*)\eta(H_2^*). \tag{3.100}$$

Taking the second term of (3.100) we can calculate

$$\eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)\xi)\eta(w_i)=g(\bar{\nabla}_{H_4^*}\bar{R}(w_i,H_2^*)\xi,\xi)$$
$$-g(\bar{R}(\bar{\nabla}_{H_4^*}w_i,H_2^*)\xi,\xi)-g(\bar{R}(w_i,\bar{\nabla}_{H_4^*}H_2^*)\xi,\xi)-g(\bar{R}(w_i,H_2^*)\bar{\nabla}_{H_4^*}\xi,\xi). \tag{3.101}$$

Let $p\in M^n$, since w_i is an orthonormal basis, so $\bar{\nabla}_{H_4^*}w_i=0$ at p. Also

$$g(\bar{R}(w_i,H_2^*)\xi,\xi)=-g(\bar{R}(\xi,\xi)H_2^*,w_i)=0. \tag{3.102}$$

Since $\nabla_{H_4^*} g = 0$, we have

$$g(\bar{\nabla}_{H_4^*}\bar{R}(w_i, H_2^*)\xi, \xi) + g(\bar{R}(w_i, H_2^*)\xi, \bar{\nabla}_{H_4^*}\xi) = 0. \tag{3.103}$$

From (3.101) and (3.103) we can obtain

$$g((\bar{\nabla}_{H_4^*}\bar{R})(w_i, H_2^*)\xi, \xi) = -g(\bar{R}(w_i, H_2^*)\xi, \bar{\nabla}_{H_4^*}\xi)$$
$$-g(\bar{R}(\bar{\nabla}_{H_4^*}w_i, H_2^*)\xi, \xi) - g(\bar{R}(w_i, \bar{\nabla}_{H_4^*}H_2^*)\xi, \xi) - g(\bar{R}(w_i, H_2^*)\bar{\nabla}_{H_4^*}\xi, \xi). \tag{3.104}$$

We also know

$$g(\bar{R}(w_i, \bar{\nabla}_{H_4^*}H_2^*)\xi, \xi) = 0 = g(\bar{R}(\bar{\nabla}_{H_4^*}w_i, H_2^*)\xi, \xi). \tag{3.105}$$

Now using (3.105) in (3.104) and using the fact that R is skew-symmetric we get

$$g((\bar{\nabla}_{H_4^*}\bar{R})(w_i, H_2^*)\xi, \xi) = 0. \tag{3.106}$$

Therefore second term of (3.100) is zero, i.e.,

$$\sum_{i=1}^{n} \eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i, H_2^*)\xi)\eta(w_i) = 0. \tag{3.107}$$

On using (3.107) in (3.100) we have

$$-(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*, \xi) = -A(H_4^*)\bar{R}_t(H_2^*, \xi) - 2B(H_4^*)\eta(H_2^*). \tag{3.108}$$

Now we know

$$(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*, \xi) = \bar{\nabla}_{H_4^*}\bar{R}_t(H_2^*, \xi) - \bar{R}_t(\bar{\nabla}_{H_4^*}H_2^*, \xi) - \bar{R}_t(H_2^*, \bar{\nabla}_{H_4^*}\xi). \tag{3.109}$$

Using (3.8), (3.11) and (3.14) in (3.109) we can get

$$(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*, \xi) = -(n-1)g(H_2^*, H_4^*) - R_t(H_2^*, H_4^*). \tag{3.110}$$

From (3.108) and (3.110) we have

$$(n-1)g(H_2^*, H_4^*) + R_t(H_2^*, H_4^*) = -A(H_4^*)\bar{R}_t(H_2^*, \xi) - 2B(H_4^*)\eta(H_2^*). \tag{3.111}$$

Putting $H_2^* = \xi$ in (3.111) we get

$$(n-1)A(H_4^*) - 2B(H_4^*) = 0. \tag{3.112}$$

Hence from (3.111) and (3.112) we can infer

$$R_t(H_2^*, H_4^*) = -(n-1)g(H_2^*, H_4^*), \tag{3.113}$$

where $a = -(n-1)$ and $b = 0$.
Therefore M^n is an Einstein manifold. □

3.10 Conharmonic curvature tensor on a Kenmotsu manifold with respect to semi-symmetric metric connection

In this part, we examine conharmonic curvature tensor on a Kenmotsu manifold under semi-symmetric condition.
Due to above condition we state the theorem:

Theorem 3.10.1. *If a $n(\geq 3)$ dimensional Kenmotsu manifold with respect to semi-symmetric metric connection admitting a conharmonic curvature tensor and a non-zero Ricci-tensor satisfies $\bar{L}(H_1^*, H_2^*)\bar{R}_t = 0$, then the modulus of non-zero eigen values of the endomorphism $\bar{Q}$ of the tangent space corresponding to $\bar{R}_t$ is zero.*

Proof. We consider a $n(n \geq 3)$dimensional Kenmotsu manifold with respect to semi-symmetric metric connection, satisfying the condition $\bar{L}(H_1^*, Y)\bar{R}_t = 0$. Then we have

$$\bar{R}_t(\bar{L}(H_1^*, H_2^*)H_3^*, H_4^*) + \bar{R}_t(H_3^*, \bar{L}(H_1^*, H_2^*)H_4^*) = 0. \tag{3.114}$$

Substituting H_1^* by ξ in the above equation we can obtain

$$\bar{R}_t(\bar{L}(\xi, H_2^*)H_3^*, H_4^*) + \bar{R}_t(H_3^*, \bar{L}(\xi, H_2^*)H_4^*) = 0. \tag{3.115}$$

Let $\bar{\lambda}^*$ be the eigen value of the endomorphism $\bar{Q}$ corresponding to an eigenvector H_1^*, then

$$\bar{Q}H_1^* = \bar{\lambda}^* H_1^*. \tag{3.116}$$

We know $g(\bar{Q}H_1^*, H_2^*) = \bar{R}_t(H_1^*, H_2^*) = \bar{\lambda}^* g(H_1^*, H_2^*)$. On using (1.31), (1.33), (1.47) and (3.115) we can calculate

$$\eta(H_3^*)\bar{R}_t(H_2^*, H_4^*) - \eta(H_4^*)\bar{R}_t(H_3^*, H_2^*) = 0. \tag{3.117}$$

Putting $H_3^* = \xi$ in (3.117) we get $\bar{R}_t(H_2^*, H_4^*) = 0$. Hence from (1.33)

$$R_t(H_2^*,H_4^*) = -2ng(H_2^*,H_4^*). \tag{3.118}$$

On putting $H_2^* = H_1^* = \xi$ in the relation $\bar{R}_t(H_1^*,H_2^*) = \bar{\lambda}^* g(H_1^*,H_2^*)$ we get $\bar{\lambda} = 0$. Therefore the theorem. □

3.11 Example of a Kenmotsu manifold with respect to semi-symmetric metric connection

Let $M = \{(h_1^*,h_2^*,h_3^*) \in \mathbb{R}^3 | (h_1^*,h_2^*,h_3^*) \neq (0,0,0)\}$ be a 3-dimensional manifold [50]. The vector fields $w_1 = h_3^* \frac{\partial}{\partial h_1^*}, w_2 = h_3^* \frac{\partial}{\partial h_2^*}, \xi = w_3 = -h_3^* \frac{\partial}{\partial h_3^*}$ are linearly independent at each point of M. We define the Rimannian metric g by

$g(w_i,w_i) = 1, g(w_i,w_j) = 0$, where $i,j \in \{1,2,3\}$ and $i \neq j$. The (1,1) tensor field ϕ is defined as

$$\phi(w_1) = -w_2, \phi(w_2) = w_1, \phi(w_3) = 0.$$

If η is 1-form then $\eta(w_3) = g(w_3,w_3) = 1$. We can easily verify by the linearity of ϕ and g that (ϕ,ξ,η,g) is an almost contact metric structure on M.
Let ∇ be the Levi-Civita connection on $\mathbb{R}^3$. Then we have

$[w_1,w_2]= 0$, $[w_1,w_3] = w_1$, $[w_2,w_3] = w_2$.

By using equation (1.48), we can find

$$\nabla_{w_1} w_1 = -w_3, \nabla_{w_1} w_2 = 0, \nabla_{w_1} w_3 = w_1,$$

$$\nabla_{w_2} w_1 = 0, \nabla_{w_2} w_2 = -w_3, \nabla_{w_2} w_3 = w_2,$$

$$\nabla_{w_3} w_1 = 0, \nabla_{w_3} w_2 = 0, \nabla_{w_3} w_3 = 0.$$

Using these we can verify $\nabla_{H_1^*} \xi = H_1^* - \eta(H_1^*)\xi$. Hence the manifold is a Kenmotsu manifold.

We consider the linear connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}_{w_i} w_j = \nabla_{w_i} w_j + \eta(w_j) w_i - g(w_i,w_j) w_3,$$

where $i, j \in \{1,2,3\}$.
From above relation we can calculate the non-zero components

$$\tilde{\nabla}_{w_1} w_1 = -2w_3, \tilde{\nabla}_{w_1} w_3 = 2w_1, \tilde{\nabla}_{w_1} w_2 = -w_3, \tilde{\nabla}_{w_2} w_3 = 2w_2.$$

Let $\bar{T}^*$ be the torsion tensor of metric connection $\tilde{\nabla}$, then we have

$$\bar{T}^*(H_1^*, H_2^*) = \eta(H_2^*)H_1^* - \eta(H_1^*)H_2^*.$$

On calculation we can see that $\bar{T}^*(H_1^*, H_2^*) = 0$.
We know that

$$(\nabla_{H_1^*} g)(H_2^*, H_3^*) = H_1^* g(H_2^*, H_3^*) - g(\nabla_{H_1^*} H_2^*, H_3^*) - g(H_2^*, \nabla_{H_1^*} H_3^*)$$

and

$$(\tilde{\nabla}_{H_1^*} g)(H_2^*, H_3^*) = H_1^* g(H_2^*, H_3^*) - g(\tilde{\nabla}_{H_1^*} H_2^*, H_3^*) - g(H_2^*, \tilde{\nabla}_{H_1^*} H_3^*).$$

Using above formula we can calculate

$$(\tilde{\nabla}_{w_1} g)(w_2, w_3) = 0 = (\tilde{\nabla}_{w_1} g)(w_3, w_2) = (\tilde{\nabla}_{w_2} g)(w_1, w_3) = (\tilde{\nabla}_{w_3} g)(w_2, w_1).$$

Therefore we can view that $(\tilde{\nabla}_{H_1^*} g)(H_2^*, H_3^*) = 0$ for all H_1^*, H_2^* and $H_3^* \in \chi(M)$. Hence $\tilde{\nabla}$ is a semi-symmetric metric connection on M.

Chapter 4

Trans-Sasakian manifold

4.1 Introduction

In 1985, J.A. Oubina [61] introduced the concept of trans-Sasakian manifolds as a new class of almost contact metric manifolds. An almost metric structure on a manifold M is defined as a trans-Sasakian structure if the product manifold $M \times R$ belongs to the class W_4, where the classification of almost Hermitian manifolds appears as a class W_4 of Hermitian manifolds closely related to locally conformal Käheler manifolds as identified by A. Gray and L.M. Hervella [34]. The class $C_5 \oplus C_6$ [61] coincides with the class of trans-Sasakian structure of type (α, β), which is a Sasakian structure when $\alpha = 1,\ \beta = 0$ and a Kenmotsu structure when $\alpha = 0,\ \beta = 1$. Several distinguised authors like D.E. Blair and J.A. Oubina [15] , J.C. Marrero [56], C.S. Bagewadi and Venkatesha [5], U.C. De and M.M. Tripathi [29], K. Kenmotsu [47], et al. also anaylzed these manifold structures.

4.2 Preliminaries

An almost contact metric manifold is called a trans-Sasakian manifold [61] if it satisfies the following condition

$$\left(\nabla_{H_1^*}\phi\right) H_2^* = \alpha\left(g\left(H_1^*, H_2^*\right)\xi - \eta\left(H_2^*\right)H_1^*\right) + \beta\left(g\left(\phi H_1^*, H_2^*\right)\xi - \eta\left(H_2^*\right)\phi H_1^*\right), \quad (4.1)$$

hold for some smooth functions α and β on M and it is called trans-Sasakian manifold of type (α, β). From (4.1) we can also conclude the following relations:

$$\nabla_{H_1^*}\xi = -\alpha\phi H_1^* + \beta\left(H_1^* - \eta\left(H_1^*\right)\xi\right), \quad (4.2)$$

$$\left(\nabla_{H_1^*}\eta\right)H_2^* = -\alpha g\left(\phi H_1^*, H_2^*\right) + \beta g\left(\phi H_1^*, \phi H_2^*\right). \tag{4.3}$$

In this chapter we have studied on 3-dimensional trans-Sasakian manifold. Moreover the curvature tensor R, the Ricci tensor R_t and the Ricci operator Q in a trans-Sasakian manifold M with respect to the Levi-Civita connection satisfies

$$\begin{aligned} R\left(H_1^*, H_2^*\right)H_3^* = &\left(\frac{R^*}{2} - 2\left(\alpha^2 - \beta^2\right)\right)\left(g\left(H_2^*, H_3^*\right)H_1^* - g\left(H_1^*, H_3^*\right)H_2^*\right) \\ &- g\left(H_2^*, H_3^*\right)\left(\frac{R^*}{2} - 3\left(\alpha^2 - \beta^2\right)\right)\eta\left(H_1^*\right)\xi \\ &+ g\left(H_1^*, H_3^*\right)\left(\frac{R^*}{2} - 3\left(\alpha^2 - \beta^2\right)\right)\eta\left(H_2^*\right)\xi \\ &- \left(\frac{R^*}{2} - 3\left(\alpha^2 - \beta^2\right)\right)\eta\left(H_2^*\right)\eta\left(H_3^*\right)H_1^* \\ &+ \left(\frac{R^*}{2} - 3\left(\alpha^2 - \beta^2\right)\right)\eta\left(H_1^*\right)\eta\left(H_3^*\right)H_2^*, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} R_t\left(H_1^*, H_2^*\right) = &\left(\frac{R^*}{2} - \left(\alpha^2 - \beta^2\right)\right)g\left(H_1^*, H_2^*\right) \\ &- \left(\frac{R^*}{2} - 3\left(\alpha^2 - \beta^2\right)\right)\eta\left(H_1^*\right)\eta\left(H_2^*\right). \end{aligned} \tag{4.5}$$

Also Riemannian curvature tensor satisfies

$$R\left(H_1^*, H_2^*\right)\xi = \left(\alpha^2 - \beta^2\right)\left(\eta\left(H_2^*\right)H_1^* - \eta\left(H_1^*\right)H_2^*\right), \tag{4.6}$$

$$R\left(\xi, H_1^*\right)H_2^* = \left(\alpha^2 - \beta^2\right)\left(g\left(H_1^*, H_2^*\right)\xi - \eta\left(H_2^*\right)H_1^*\right), \tag{4.7}$$

$$R\left(\xi, H_1^*\right)\xi = \left(\alpha^2 - \beta^2\right)\left(\eta\left(H_1^*\right)\xi - H_1^*\right), \tag{4.8}$$

$$\eta\left(R\left(H_1^*, H_2^*\right)H_3^*\right) = \left(\alpha^2 - \beta^2\right)\left(g\left(H_2^*, H_3^*\right)\eta\left(H_1^*\right) - g\left(H_1^*, H_3^*\right)\eta\left(H_2^*\right)\right). \tag{4.9}$$

From (4.5), we can infer

$$R_t\left(H_1^*, \xi\right) = 2\left(\alpha^2 - \beta^2\right)\eta\left(H_1^*\right), \tag{4.10}$$

for all $H_1^*, H_2^*, H_3^* \in \chi(M)$.

4.3 3-dimensional trans-Sasakian manifolds admitting $*$-CηRS

Here we consider 3-dimensional trans-Sasakian manifolds admitting $*$-CηRS. In the beginning we try to characterize the nature of the soliton by calculating the condition under which a $*$-CηRS is shrinking, steady or expanding on a 3-dimensional trans-Sasakian manifold.

Now we state the following theorems:

Theorem 4.3.1. *A $*$-CηRS (g, V, λ^*, μ^*) on a 3-dimensional trans-Sasakian manifold (M, g, ϕ, ξ, η) is shrinking if $\mu^* > \frac{1}{2}\left(\omega + \frac{2}{3}\right)$, steady if $\mu^* = \frac{1}{2}\left(\omega + \frac{2}{3}\right)$ and expanding if $\mu^* < \frac{1}{2}\left(\omega + \frac{2}{3}\right)$.*

Proof. From equation (1.24), we have

$$(\mathcal{L}_\xi g)(H_1^*, H_2^*) + 2R_t^*(H_1^*, H_2^*) + 2\mu^* \eta(H_1^*)\eta(H_2^*) + \left[2\lambda^* - \left(\omega + \frac{2}{3}\right)\right] g(H_1^*, H_2^*) = 0. \tag{4.11}$$

Now, with the help of (4.2), we have

$$(\mathcal{L}_\xi g)(H_1^*, H_2^*) = 2\beta[g(H_1^*, H_2^*) - \eta(H_1^*)\eta(H_2^*)]. \tag{4.12}$$

From equations (4.11) and (4.12), we calculate

$$R_t^*(H_1^*, H_2^*) = -\left[\beta + \lambda^* - \frac{1}{2}\left(\omega + \frac{2}{3}\right)\right] g(H_1^*, H_2^*) + (\beta - \mu^*)\eta(H_1^*)\eta(H_2^*). \tag{4.13}$$

Using (1.50), the $*$-Ricci tensor on a 3-dimensional trans-Sasakian manifold is given by,

$$R_t^*(H_1^*, H_2^*) = R_t(H_1^*, H_2^*) - (\alpha^2 - \beta^2)[g(H_1^*, H_2^*) + \eta(H_1^*)\eta(H_2^*)]. \tag{4.14}$$

From the equations (4.13) and (4.14), it follows that

$$R_t(H_1^*, H_2^*) = \left[(\alpha^2 - \beta^2) - \beta - \lambda^* + \frac{1}{2}\left(\omega + \frac{2}{3}\right)\right] g(H_1^*, H_2^*) + (\alpha^2 - \beta^2 + \beta - \mu^*)\eta(H_1^*)\eta(H_2^*). \tag{4.15}$$

Putting $H_2^* = \xi$ in (4.15), we get

$$R_t(H_1^*,\xi) = \left[2(\alpha^2-\beta^2) - \lambda^* + \frac{1}{2}\left(\omega + \frac{2}{3}\right) - \mu^*\right]\eta(H_1^*). \tag{4.16}$$

Using (4.10) in (4.16), since $\eta(H_1^*) \neq 0$ we calculate

$$\lambda^* = \frac{1}{2}(\omega + \frac{2}{3}) - \mu^*. \tag{4.17}$$

From (4.17) we can conclude the following:

(i) If $\lambda^* < 0$, then $\mu^* > \frac{1}{2}\left(\omega + \frac{2}{3}\right)$ implies the soliton is shrinking.

(ii) If $\lambda^* = 0$, then $\mu^* = \frac{1}{2}\left(\omega + \frac{2}{3}\right)$ implies the soliton is steady.

(iii) If $\lambda^* > 0$, then $\mu^* < \frac{1}{2}\left(\omega + \frac{2}{3}\right)$ implies the solitonis expanding.

This completes the proof. □

Theorem 4.3.2. *Let* (M, g, ϕ, ξ, η) *be a* 3*-dimensional trans-Sasakian manifold. If* (g, V, λ^*, μ^*) *is a* $*$*-C*η*RS on M, then M is an* η*E manifold.*

Proof. Taking Lie derivative of (4.10) along arbitrary vector field V we have

$$R_t(\mathcal{L}_V H_1^*,\xi) + R_t(H_1^*,\mathcal{L}_V\xi) = 2(\alpha^2-\beta^2)[(\mathcal{L}_V\eta)(H_1^*) + \eta(\mathcal{L}_V H_1^*)]. \tag{4.18}$$

Using (4.17) in (1.50) we get

$$(\mathcal{L}_V g)(H_1^*,\xi) = 0. \tag{4.19}$$

Replacing $H_2^* = \xi$ in the equation (1.29) we have

$$R_t(H_1^*,\xi) = ag(H_1^*,\xi) + b\eta(H_1^*). \tag{4.20}$$

Taking Lie derivative of (4.20) along V we obtain

$$R_t(\mathcal{L}_V H_1^*,\xi) + R_t(H_1^*,\mathcal{L}_V\xi) = a(\mathcal{L}_V g)(H_1^*,\xi) + b[(\mathcal{L}_V\eta)H_1^* + \eta(\mathcal{L}_V H_1^*)]. \tag{4.21}$$

Using equations (4.18) and (4.19) in equation (4.21) we get

$$[2(\alpha^2-\beta^2) - b][(\mathcal{L}_V\eta)H_1^* + \eta(\mathcal{L}_V H_1^*)] = 0. \tag{4.22}$$

Since the second factor of above equation is non-zero therefore

$$b = 2(\alpha^2-\beta^2). \tag{4.23}$$

Again equation (4.20) reduces to

$$a+b=2(\alpha^2-\beta^2). \tag{4.24}$$

Therefore equation (1.29) becomes

$$R_t(H_1^*,H_2^*)=2(\alpha^2-\beta^2)\eta(H_1^*)\eta(H_2^*). \tag{4.25}$$

This proves that the manifold M is an ηE manifold. □

4.4 Gradient CηRS on 3-dimensional trans-Sasakian manifold

In this part we investigate trans-Sasakian manifold admitting gradient CηRS.
From [55] we have:
If (g,V,λ^*,μ^*) is a gradient CηRS on a 3-dimensional trans-Sasakian manifold (M,g,ϕ,ξ,η), then the Riemann curvature tensor R satisfies

$$R(H_1^*,H_2^*)Df=[(\nabla_{H_2^*}Q)H_1^*-(\nabla_{H_1^*}Q)H_2^*]. \tag{4.26}$$

So we give our next result:-

Theorem 4.4.1. *Let (g,V,λ^*,μ^*) be a gradient CηRS on a 3-dimensional trans-Sasakian manifold (M,g,ϕ,ξ,η). Then the potential vector field V is pointwise collinear with the characteristic vector field ξ.*

Proof. From equation (4.6) we get

$$R(H_1^*,H_2^*)\xi=(\alpha^2-\beta^2)(-\eta(H_1^*)H_2^*+\eta(H_2^*)H_1^*). \tag{4.27}$$

Taking inner product of (4.6) with Df, we have

$$g(R(H_1^*,H_2^*)\xi,Df)=(\alpha^2-\beta^2)(-(H_2^*f)\,\eta(H_1^*)+(H_1^*f)\eta(H_2^*)). \tag{4.28}$$

Again we know that

$$g(R(H_1^*,H_2^*)\xi,Df)=-g(R(H_1^*,H_2^*)Df,\xi).$$

Now equation (4.28) becomes

$$g(R(H_1^*,H_2^*)Df,\xi)=(\alpha^2-\beta^2)((H_2^*f)\,\eta(H_1^*)-(H_1^*f)\eta(H_2^*)). \tag{4.29}$$

Taking inner product of (4.26) with ξ
and using $\left(\nabla_{H_1^*}Q\right)H_2^* = \nabla_{H_1^*}QH_2^* - Q\left(\nabla_{H_1^*}H_2^*\right)$, we obtain

$$g(R(H_1^*,H_2^*)Df,\xi) = 0. \tag{4.30}$$

From (4.29) and (4.30) we calculate

$$(\alpha^2-\beta^2)((H_2^*f)\,\eta(H_1^*) - (H_1^*f)\eta(H_2^*)) = 0. \tag{4.31}$$

Since $(\alpha^2-\beta^2) \neq 0$, from equation (4.31) we have

$$H_2^*f\eta(H_1^*) = (H_1^*f)\eta(H_2^*). \tag{4.32}$$

Taking $H_2^* = \xi$ in (4.32) we get

$$\xi f\eta(H_1^*) = H_1^*f. \tag{4.33}$$

This implies

$$g(H_1^*,Df) = g(H_1^*,(\xi f)\xi), \tag{4.34}$$

for all smooth vector fields H_1^* on M. This implies

$$V = Df = (\xi f)\xi.$$

We have thus completed the proof. □

4.5 Example of a 3-dimensional trans-Sasakian manifold with respect to semi-symmetric metric connections admitting $*$-CηRS

Let $M = \{(h_1^*,h_2^*,h_3^*) \in \mathbb{R}^3 | (h_1^*,h_2^*,h_3^*) \neq (0,0,0)\}$ be a 3-dimensional manifold. The vector fields $w_1 = \frac{\partial}{\partial h_2^*}$, $w_2 = \frac{\partial}{\partial h_3^*} + 2h_2^*\frac{\partial}{\partial h_1^*}$, $\xi = w_3 = \frac{\partial}{\partial h_1^*}$ are linearly independent at each point of M. We define the Riemannian metric g by
$g(w_i,w_i) = 1$, $g(w_i,w_j) = 0$ where $i,j \in \{1,2,3\}$ and $i \neq j$. The (1,1) tensor field ϕ is defined as

$$\phi(w_1) = w_2,\ \phi(w_2) = -w_1,\ \phi(w_3) = 0.$$

If η is 1-form then $\eta(w_3) = g(w_3,w_3) = 1$. We can easily verify by the linearity of ϕ and g that (ϕ,ξ,η,g) is an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection on M. Then we have
$[w_1, w_2] = 2w_3,\ [w_1, w_3] = 0,\ [w_2, w_3] = 0.$
By using Koszul's formula (1.48) for the Riemannian metric g, we can find

$$\nabla_{w_1} w_1 = 0,\ \nabla_{w_1} w_2 = w_3,\ \nabla_{w_1} w_3 = -w_2,$$

$$\nabla_{w_2} w_1 = -w_3,\ \nabla_{w_2} w_2 = 0,\ \nabla_{w_2} w_3 = w_1,$$

$$\nabla_{w_3} w_1 = -w_2,\ \nabla_{w_3} w_2 = w_1,\ \nabla_{w_3} w_3 = 0.$$

Using these we can verify $\nabla_{H_1^*}\xi = -\alpha\phi H_1^* + \beta(H_1^* - \eta(H_1^*)\xi$. Hence the manifold is a trans-Sasakian manifold of type $(0,1)$.
Also from the relation of Riemann curvature tensor equation (1.49) we can calculate the following components

$$R(w_1, w_1)w_1 = 0,\ R(w_1, w_2)w_1 = 3w_2,\ R(w_1, w_2)w_2 = -3w_1,$$

$$R(w_1, w_2)w_3 = 0,\ R(w_2, w_3)w_1 = 0,\ R(w_2, w_3)w_3 = w_2,$$

$$R(w_1, w_3)w_1 = -w_3,\ R(w_1, w_3)w_2 = 0,\ R(w_1, w_3)w_3 = w_1,\ R(w_1, w_2)w_2 = -3w_1.$$

From these curvature tensors, we can obtain the components of Ricci tensors as follows :

$$R_t(w_1, w_1) = -2,\ R_t(w_2, w_2) = 3,\ R_t(w_3, w_3) = 2.$$

From equation (4.15), we can calculate

$$R_t(w_3, w_3) = 2(\alpha^2 - \beta^2) - \lambda^* - \mu^* + \frac{1}{2}\left(\omega + \frac{2}{3}\right).$$

By equating both the values of $R_t(w_3, w_3)$ we have

$$\lambda^* = 2(\alpha^2 - \beta^2) + \frac{\omega}{2} - \frac{5}{3} - \mu^*.$$

Thus a $*$-CηRS $(g, \xi, \lambda^*, \mu^*)$ on a 3-dimensional trans-Sasakian manifold is shrinking, steady and expanding as $\mu^* > 2(\alpha^2 - \beta^2) + \frac{\omega}{2} - \frac{5}{3}$, $\mu^* = 2(\alpha^2 - \beta^2) + \frac{\omega}{2} - \frac{5}{3}$ and $\mu^* < 2(\alpha^2 - \beta^2) + \frac{\omega}{2} - \frac{5}{3}$, respectively.

4.6 3-dimensional trans-Sasakian manifolds admitting $*$-CηES

In this section, we consider 3-dimensional trans-Sasakian manifolds admitting $*$-CηES. First we try to characterize the nature of the soliton by calculating the condition under which a $*$-CηES is shrinking, steady or expanding on a 3-dimensional trans-Sasakian manifold.
Now we state the following theorem:

Theorem 4.6.1. *A $*$-CηES (g,V,λ^*,μ^*) on a 3-dimensional trans-Sasakian manifold (M,g,ϕ,ξ,η) is shrinking if $\mu^* > \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right)$, steady if $\mu^* = \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right)$ and expanding if $\mu^* < \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right)$.*

Proof. From equation (1.25), we have

$$(\mathcal{L}_\xi g)(H_1^*,H_2^*) + 2R_t^*(H_1^*,H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) + \left[2\lambda^* - R^* + \left(\omega + \frac{2}{3}\right)\right]g(H_1^*,H_2^*) = 0. \tag{4.35}$$

Now, with the help of (4.2), we have

$$(\mathcal{L}_\xi g)(H_1^*,H_2^*) = 2\beta[g(H_1^*,H_2^*) - \eta(H_1^*)\eta(H_2^*)]. \tag{4.36}$$

From equations (4.35) and (4.36), we calculate

$$R_t^*(H_1^*,H_2^*) = (\beta - \mu^*)\eta(H_1^*)\eta(H_2^*) - \left[\beta + \lambda^* - \frac{R^*}{2} + \frac{1}{2}\left(\omega + \frac{2}{3}\right)\right]g(H_1^*,H_2^*). \tag{4.37}$$

From equation (1.50), the $*$-Ricci tensor on a 3-dimensional trans-Sasakian manifold is given by,

$$R_t^*(H_1^*,H_2^*) = R_t(H_1^*,H_2^*) - (\alpha^2 - \beta^2)[g(H_1^*,H_2^*) + \eta(H_1^*)\eta(H_2^*)]. \tag{4.38}$$

From the equations (4.37) and (4.38), it follows that

$$R_t(H_1^*,H_2^*) = \left[(\alpha^2 - \beta^2) - \beta - \lambda^* + \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right)\right]g(H_1^*,H_2^*) + (\alpha^2 - \beta^2 + \beta - \mu^*)\eta(H_1^*)\eta(H_2^*). \tag{4.39}$$

Putting $H_2^* = \xi$ in (4.39), we get

$$R_t(H_1^*, \xi) = \left[2(\alpha^2 - \beta^2) - \lambda^* + \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right) - \mu^*\right] \eta(H_1^*). \tag{4.40}$$

Using (4.10) in (4.40), since $\eta(H_1^*) \neq 0$ we calculate

$$\lambda^* + \mu^* = \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right). \tag{4.41}$$

From (4.41) we can conclude the following:

(i) if $\lambda^* < 0$, then $\mu^* > \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right)$ implies the soliton is shrinking.

(ii) if $\lambda^* = 0$, then $\mu^* = \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right)$ implies the soliton is steady.

(iii) if $\lambda^* > 0$, then $\mu^* < \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right)$ implies the solitonis expanding.

This completes the proof. □

4.7 $*$-CηES on 3-dimensional trans-Sasakian manifold admitting cyclic parallel Ricci tensor

In this segment, we are going to study $*$-CηES on trans-Sasakian manifold (M^3, g) having cyclic parallel Ricci tensor.
We now state and prove the following theorem:

Theorem 4.7.1. *Let (M^3, g) be a trans-Sasakian manifold admitting a $*$-CηES $(g, \xi, \lambda^*, \mu^*)$. Then the manifold has cyclic Ricci tensor, i.e.,*

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) + (\nabla_{H_2^*} R_t)(H_3^*, H_1^*) + (\nabla_{H_3^*} R_t)(H_1^*, H_2^*) = 0.$$

Proof. We consider a trans-Sasakian manifold (M^3, g) admitting a $*$-CηES $(g, \xi, \lambda^*, \mu^*)$. On taking covariant derivative of equation (4.39), we get

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = (\alpha^2 - \beta^2 + \beta - \mu^*)$$
$$[\eta(H_3^*)(\nabla_{H_1^*} \eta)H_2^* + \eta(H_2^*)(\nabla_{H_1^*} \eta)H_3^*]. \tag{4.42}$$

Using equation (4.3), the above equation becomes

$$\begin{aligned}(\nabla_{H_1^*}R_t)(H_2^*,H_3^*) &= (\alpha^2-\beta^2+\beta-\mu^*)[(-\alpha g(\phi H_1^*,H_2^*)\\&+\beta g(\phi H_1^*,\phi H_2^*))\eta(H_3^*)+(-\alpha g(\phi H_1^*,H_3^*)+\beta g(\phi H_1^*,\phi H_3^*))\eta(H_2^*)].\end{aligned} \tag{4.43}$$

Similarly we can obtain,

$$\begin{aligned}(\nabla_{H_2^*}R_t)(H_3^*,H_1^*) &= (\alpha^2-\beta^2+\beta-\mu^*)[(-\alpha g(\phi H_2^*,H_3^*)\\&+\beta g(\phi H_2^*,\phi H_3^*))\eta(H_1^*)+(-\alpha g(\phi H_2^*,H_1^*)+\beta g(\phi H_2^*,\phi H_1^*))\eta(H_3^*)],\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}(\nabla_{H_3^*}R_t)(H_1^*,H_2^*) &= (\alpha^2-\beta^2+\beta-\mu^*)[(-\alpha g(\phi H_3^*,H_1^*)\\&+\beta g(\phi H_3^*,\phi H_1^*))\eta(H_2^*)+(-\alpha g(\phi H_3^*,H_2^*)+\beta g(\phi H_3^*,\phi H_2^*))\eta(H_1^*)].\end{aligned} \tag{4.45}$$

Adding (4.43), (4.44), (4.45) and using (1.4) we have

$$(\nabla_{H_1^*}R_t)(H_2^*,H_3^*)+(\nabla_{H_2^*}R_t)(H_3^*,H_1^*)+(\nabla_{H_3^*}R_t)(H_1^*,H_2^*)=0. \tag{4.46}$$

This completes the proof. □

4.8 Einstein Semi-Symmetric trans-Sasakian manifolds (M^3,g) admitting $*$-CηES

Now, we think about $*$-CηES in trans-Sasakian manifold is an Einstein semi-symmetric. We investigate to characterize the nature of the soliton and prove the following theorem:

Theorem 4.8.1. *Let 3-dimensional trans-Sasakian manifold admits a $*$-CηES (g,ξ,λ^*,μ^*). If the manifold is an Einstein semi-symmetric, then the manifold represents an Einstein manifold and the soliton is shrinking, steady and expanding as*

(i) $\omega > R^* - \frac{4\beta}{3} - \frac{2\mu^*}{3} - \frac{4}{3}\left(\alpha^2-\beta^2\right) - \frac{2}{3},$

(ii) $\omega = R^* - \frac{4\beta}{3} - \frac{2\mu^*}{3} - \frac{4}{3}\left(\alpha^2-\beta^2\right) - \frac{2}{3},$

(iii) $\omega < R^* - \frac{4\beta}{3} - \frac{2\mu^*}{3} - \frac{4}{3}\left(\alpha^2-\beta^2\right) - \frac{2}{3}.$

Proof. If a 3-dimensional trans-Sasakian manifold satisfies the curvature condition $R.E = 0$, then from equations (1.39) and (1.40), we get

$$\begin{aligned} & R_t(R(H_1^*,H_2^*)H_3^*,H_4^*) + R_t(H_3^*,R(H_1^*,H_2^*)H_4^*) \\ & = \frac{R^*}{3}\left[g(R(H_1^*,H_2^*)H_3^*,H_4^*) + g(H_3^*,R(H_1^*,H_2^*)H_4^*)\right]. \end{aligned} \tag{4.47}$$

Putting $H_1^* = H_3^* = \xi$ in the equation (4.47), we obtain

$$\begin{aligned} & R_t(R(\xi,H_2^*)\xi,H_4^*) + R_t(\xi,R(\xi,H_2^*)H_4^*) \\ & = \frac{R^*}{3}\left[g(R(\xi,H_2^*)\xi,H_4^*) + g(\xi,R(\xi,H_2^*)H_4^*)\right]. \end{aligned} \tag{4.48}$$

Using (4.7), (4.8) in (4.48), we have

$$\begin{aligned} & R_t(\left(\alpha^2-\beta^2\right)\left(\eta\left(H_2^*\right)\xi - H_2^*\right),H_4^*) \\ & + R_t(\xi,\left(\alpha^2-\beta^2\right)\left(g\left(H_2^*,H_4^*\right)\xi - \eta\left(H_4^*\right)H_2^*\right)) \\ & = \frac{R^*}{3}[g(\left(\alpha^2-\beta^2\right)\left(\eta\left(H_2^*\right)\xi - H_2^*\right),H_4^*) \\ & + g(\xi,\left(\alpha^2-\beta^2\right)\left(g\left(H_2^*,H_4^*\right)\xi - \eta\left(H_4^*\right)H_2^*\right))], \end{aligned} \tag{4.49}$$

which implies

$$R_t(H_2^*,H_4^*) = 2\left(\alpha^2-\beta^2\right)g(H_2^*,H_4^*). \tag{4.50}$$

Let us assume that the Einstein semi-symmetric 3-dimensional trans-Sasakian manifold admits a $*$-CηES (g,ξ,λ^*,μ^*). Then equation (4.39) holds and combining (4.39) with the equation (4.50), we have

$$\begin{aligned} 6\left(\alpha^2-\beta^2\right) = & \; 3\left[(\alpha^2-\beta^2) - \beta - \lambda^* + \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right)\right] \\ & + (\alpha^2-\beta^2+\beta-\mu^*). \end{aligned}$$

This gives

$$\lambda^* = \frac{R^*}{2} - \frac{2\beta}{3} - \frac{\omega}{2} - \frac{\mu^*}{3} - \frac{2}{3}\left(\alpha^2-\beta^2\right) - \frac{1}{3}. \tag{4.51}$$

From (4.51), we can conclude the following:

(i) if $\lambda^* < 0$, then $\omega > R^* - \frac{4\beta}{3} - \frac{2\mu^*}{3} - \frac{4}{3}\left(\alpha^2-\beta^2\right) - \frac{2}{3}$ implies the soliton is shrinking.

(ii) if $\lambda^* = 0$, then $\omega = R^* - \frac{4\beta}{3} - \frac{2\mu^*}{3} - \frac{4}{3}\left(\alpha^2-\beta^2\right) - \frac{2}{3}$ implies the soliton is steady.

(iii) if $\lambda^* > 0$, then $\omega < R^* - \frac{4\beta}{3} - \frac{2\mu^*}{3} - \frac{4}{3}(\alpha^2 - \beta^2) - \frac{2}{3}$ implies the soliton is expanding.

$\square$

4.9 $*$-CηES on 3-dimensional trans-Sasakian manifold satisfying ξ-Ricci conformally semi-symmetric condition

Under this heading, we review a trans-Sasakisn manifold (M^3, g) that admits $*$-CηES $(g, \xi, \lambda^*, \mu^*)$ and the manifold satisfies the ξ-Ricci conformally semi-symmetric condition i.e., $C(\xi, H_1^*).R_t = 0$. On the basis of the above condition we can state and prove the following theorem:

Theorem 4.9.1. *Let 3-dimensional trans-Sasakian manifold admits $*$-CηES $(g, \xi, \lambda^*, \mu^*)$. If the manifold satisfies the curvature condition $C(\xi, H_1^*).R_t = 0$, then the manifold is an Einstein manifold.*

Proof. From equation (1.43), we find

$$C(\xi, H_1^*)H_2^* = R(\xi, H_1^*)H_2^* - \frac{R^*}{6}\left[g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*\right]. \tag{4.52}$$

Using (4.7) in (4.52), we have

$$C(\xi, H_1^*)H_2^* = \left[\alpha^2 - \beta^2 - \frac{R^*}{6}\right]\left[g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*\right]. \tag{4.53}$$

Similarly,

$$C(\xi, H_1^*)H_3^* = \left[\alpha^2 - \beta^2 - \frac{R^*}{6}\right]\left[g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*\right]. \tag{4.54}$$

Using equations (4.53), (4.54) in (2.41), we obtain

$$\begin{aligned} &\left[\alpha^2 - \beta^2 - \frac{R^*}{6}\right] R_t([g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*], H_3^*) \\ +&\left[\alpha^2 - \beta^2 - \frac{R^*}{6}\right] R_t([g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*], H_2^*) = 0, \end{aligned} \tag{4.55}$$

which implies

$$\left[\alpha^2-\beta^2-\frac{R^*}{6}\right]\left[2\left(\alpha^2-\beta^2\right)\left(g(H_1^*,H_2^*)\eta(H_3^*)+g(H_1^*,H_3^*)\eta(H_2^*)\right)\right]$$
$$-\left[\alpha^2-\beta^2-\frac{R^*}{6}\right]\left[R_t(H_1^*,H_3^*)\eta(H_2^*)-R_t(H_1^*,H_2^*)\eta(H_3^*)\right]=0. \tag{4.56}$$

Putting $H_3^*=\xi$ in (4.56) and using (4.10), we get

$$\left[\alpha^2-\beta^2-\frac{R^*}{6}\right]\left[2\left(\alpha^2-\beta^2\right)g(H_1^*,H_2^*)-R_t(H_1^*,H_2^*)\right]=0. \tag{4.57}$$

If $\left[\alpha^2-\beta^2-\frac{R^*}{6}\right]\neq 0$, then it reduces to

$$R_t(H_1^*,H_2^*)=2\left(\alpha^2-\beta^2\right)g(H_1^*,H_2^*). \tag{4.58}$$

This completes the proof. □

4.10 Example of a trans-Sasakian manifold admitting a $*$-CηES

Let us consider $M=\left\{(h_1^*,h_2^*,h_3^*)\in\mathbb{R}^3, h_2^*\neq 0\right\}$ be a three-dimensional manifold [63] admitting $*$-CηES (g,ξ,λ^*,μ^*).The vector fields $w_1=e^{2h_3^*}\frac{\partial}{\partial h_1^*}, w_2=e^{2h_3^*}\frac{\partial}{\partial h_2^*}, w_3=\frac{\partial}{\partial h_3^*}$ are linearly independent in M^3.
We define the Riemannian metric g by
$g(w_1,w_2)=g(w_2,w_3)=g(w_1,w_3)=0$ and $g(w_1,w_1)=g(w_2,w_2)=g(w_3,w_3)=1$.
Let 1-form η defined by
$\eta(H_3^*)=g(H_3^*,w_3)$,
then we get the following relations:
$\eta(w_1)=0,\eta(w_2)=0,\eta(w_3)=1$.
The (1,1) tensor field ϕ is defined as

$$\phi(w_1)=w_2,\phi(w_2)=-w_1,\phi(w_3)=0.$$

Using the linearity of ϕ and g, we have

$$\eta(w_3)=1,$$

$$\phi^2(H_3^*)=-H_3^*+\eta(H_3^*)w_3$$

and

$$g(\phi H_3^*, \phi H_4^*) = g(H_3^*, H_4^*) - \eta(H_3^*)\eta(H_4^*),$$

for each $H_3^*, H_4^* \in \chi(M)$.
Thus (ϕ, ξ, η, g) defines an almost contact metric struture on M^3. After calculation we get,
$[w_1, w_2] = 0$, $[w_2, w_1] = -2w_2$, $[w_1, w_3] = -2w_1$.
Using equation (1.48), we can calculate

$$\nabla_{w_1} w_1 = 2w_3, \nabla_{w_1} w_2 = 0, \nabla_{w_1} w_3 = -2w_1,$$

$$\nabla_{w_2} w_1 = 0, \nabla_{w_2} w_2 = 2w_3, \nabla_{w_2} w_3 = -2w_2,$$

$$\nabla_{w_3} w_1 = 0, \nabla_{w_3} w_2 = 0, \nabla_{w_3} w_3 = 0.$$

Using these we can verify that the relations (4.2) and (4.3) are satisfied. Hence the manifold is a trans-Sasakian manifold with the almost contact structure (ϕ, ξ, η, g) of type (0,2) .
From (1.49) we can calculate the following components

$$R(w_1, w_1)w_1 = 0, R(w_1, w_2)w_1 = -4w_3, R(w_1, w_2)w_2 = -4w_1,$$

$$R(w_1, w_2)w_3 = 0, R(w_2, w_3)w_1 = 0, R(w_2, w_3)w_3 = -4w_2,$$

$$R(w_1, w_3)w_1 = 4w_2, R(w_1, w_3)w_2 = 0, R(w_1, w_3)w_3 = -4w_1,$$

$$R(w_2, w_1)w_1 = 4w_3, R(w_3, w_1)w_1 = -4w_2, R(w_2, w_3)w_2 = -4w_2.$$

From these curvature tensors, we can calculate the components of Ricci tensors as follows:

$$R_t(w_1, w_1) = 0, \;\; R_t(w_2, w_2) = 0, \;\; R_t(w_3, w_3) = -8.$$

From equation (4.39), we can calculate

$$R_t(w_3, w_3) = \left[2(\alpha^2 - \beta^2) - \lambda^* - \mu^* + \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right) \right].$$

By equating both the values of $R_t(w_3, w_3)$, we obtain

$$\lambda^* + \mu^* = 2(\alpha^2 - \beta^2) + \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right) + 8.$$

From equation (4.56), we can determine

$$2(\alpha^2 - \beta^2) = -8.$$

Therefore,

$$\lambda^* + \mu^* = \frac{R^*}{2} - \frac{1}{2}\left(\omega + \frac{2}{3}\right).$$

Hence the constants λ^* and μ^* satisfy equation (4.41) of theorem (4.6.1) and so g defines a $*$-CηES on trans-Sasakian manifold M^3.

Chapter 5

LP-S manifold

5.1 Introduction

In 1976, Sato [74] introduced a structure on smooth manifold that has since gained recognition as the almost paracontact structure. This structure is analogous to the almost contact structure [12, 73] and resembles the almost contact product structure. The difference lies in the fact that almost paracontact manifolds can be both even-dimensional and odd-dimensional unlike almost contact manifolds which are always of odd-dimension.

Takahashi [80] researched about almost contact manifolds equipped with corresponding semi-Riemannian metrics. His work specifically focused on Sasakian manifolds endowed with an associated semi-Riemannian metric in 1969. The concept of LP-S manifold [1] was first introduced by Matsumoto [57] in 1989. Subsequently, Mihai and Rosca [59] independently worked on the same area and deduced various outcomes in this type of manifold. Furthermore, LP-S manifolds have been investigated by Matsumoto and Mihai [58], as well as De et al. [24, 72].

5.2 Preliminaries

A Lorentzian manifold is called an almost paracontact structure manifold if it satisfies the following condition

$$\left(\nabla_{H_1^*}\phi\right)H_2^* = g(H_1^*,H_2^*)\xi + \eta(H_2^*)H_1^* + 2\eta(H_1^*)\eta(H_2^*)\xi. \tag{5.1}$$

From (5.1) we can also conclude the following relations:

$$\nabla_{H_1^*}\xi = \phi H_1^*, \tag{5.2}$$

$$\left(\nabla_{H_1^*}\eta\right)H_2^* = \Omega(H_1^*,H_2^*) = g(\phi H_1^*,H_2^*), \tag{5.3}$$

and

$$rank\phi = n-1, \tag{5.4}$$

for all vector fields H_1^*, H_2^* on M [10]. Then the tensor field Ω is symmetric (0,2) tensor field [57], that is,

$$\Omega(H_1^*, H_2^*) = \Omega(H_2^*, H_1^*). \tag{5.5}$$

In a n-dimensional LP-S manifold, we have the following results from [85]

$$R_t(\phi H_1^*, \phi H_2^*) = R_t(H_1^*, H_2^*) + (n-1)\eta(H_1^*)\eta(H_2^*), \tag{5.6}$$

$$R(H_1^*, H_2^*)\,\xi = \eta(H_2^*)H_1^* - \eta(H_1^*)H_2^*, \tag{5.7}$$

$$R(\xi, H_1^*)\,H_2^* = g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*, \tag{5.8}$$

$$R(\xi, H_1^*)\,\xi = \eta(H_1^*)\xi - \eta(\xi)H_1^* = H_1^* + \eta(H_1^*)\xi, \tag{5.9}$$

$$\eta\,(R(H_1^*, H_2^*)H_3^*) = g(H_2^*, H_3^*)\eta(H_1^*) - g(H_1^*, H_3^*)\eta(H_2^*). \tag{5.10}$$

From (5.6), we can infer

$$R_t\,(H_1^*, \xi) = (n-1)\eta(H_1^*), \tag{5.11}$$

$$Q\xi = -(n-1). \tag{5.12}$$

Here R is the curvature tensor and R_t is the Ricci tensor.

5.3 4-dimension LP-S manifold admitting CηRS

Here we consider a LP-S manifold (M^4, g) admitting a CηRS. In the beginning we characterize the nature of the soliton by calculating the condition under which a CηRS is steady, expanding or shrinking, on a 4-dimension LP-S manifold.
We state the following theorem:

Theorem 5.3.1. *Let a LP-S manifold M^4 admits a CηRS $(g, \xi, \lambda^*, \mu^*)$, then the value of the constant $\lambda^* = \frac{\omega}{2} + \mu^* - \frac{11}{4}$. The soliton is steady, shrinking and expanding for $\omega + 2\mu^* = \frac{11}{2}$, $\omega + 2\mu^* < \frac{11}{2}$ and $\omega + 2\mu^* > \frac{11}{2}$, respectively.*

Proof. We explore a LP-S manifold (M^4, g) admitting a CηRS. From (1.20), we obtain

$$(\mathcal{L}_\xi g)(H_1^*, H_2^*) - \left[\left(\omega + \frac{1}{2}\right) - 2\lambda^*\right] g(H_1^*, H_2^*)$$
$$+ 2R_t(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) = 0. \tag{5.13}$$

From (5.13), we get

$$2R_t(H_1^*,H_2^*) = \left[\left(\omega+\frac{1}{2}\right)-2\lambda^*\right]g(H_1^*,H_2^*)$$
$$-(\mathcal{L}_\xi g)(H_1^*,H_2^*)-2\mu^*\eta(H_1^*)\eta(H_2^*). \tag{5.14}$$

Now, with the help of (5.2), we have

$$(\mathcal{L}_\xi g)(H_1^*,H_2^*) = g(\phi H_1^*,H_2^*)+g(\phi H_2^*,H_1^*). \tag{5.15}$$

From (5.14) and (5.15), we obtain

$$R_t(H_1^*,H_2^*) = \left[\left(\frac{\omega}{2}+\frac{1}{4}\right)-\lambda^*\right]g(H_1^*,H_2^*)-\mu^*\eta(H_1^*)\eta(H_2^*)-g(\phi H_1^*,H_2^*). \tag{5.16}$$

Putting $H_2^*=\xi$ in (5.16), we get

$$R_t(H_1^*,\xi) = \left[(\frac{\omega}{2}+\frac{1}{4})-\lambda^*+\mu^*\right]\eta(H_1^*). \tag{5.17}$$

Comparing the equations (5.11) and (5.17), we have

$$3\eta(H_1^*) = \left[(\frac{\omega}{2}+\frac{1}{4})-\lambda^*+\mu^*\right]\eta(H_1^*),$$

which gives

$$\lambda^* = \frac{\omega}{2}+\mu^*-\frac{11}{4}. \tag{5.18}$$

From (5.18), we conclude:

(i) If $\lambda^*=0$, then $2\mu^*+\omega=\frac{11}{2}$ implies the soliton is steady.

(ii) If $\lambda^*<0$, then $2\mu^*+\omega<\frac{11}{2}$ implies the soliton is shrinking.

(iii) If $\lambda^*>0$, then $2\mu^*+\omega>\frac{11}{2}$ implies the soliton is expanding.

This completes the proof. □

5.4 CηRS on (M^4,g) LP-S manifold satisfying ξ-Ricci semi-symmetric condition

We study a CηRS in LP-S manifold and the manifold satisfies the ξ-Ricci semi-symmetric condition, i.e., $R^*(\xi,H_1^*).R_t=0$.

So we state the next theorem:

Theorem 5.4.1. *Let us consider the case of a LP-S manifold (M^4, g) admitting a $C\eta RS$. If the manifold satisfies the ξ-Ricci semi-symmetric condition, then its becomes an Einstein manifold.*

Proof. Equations (1.46) and (5.16) entail that

$$\left[\left(\frac{\omega}{2}+\frac{1}{4}\right)-\lambda^*\right]g(R(\xi,H_1^*)H_2^*,H_3^*)-\mu^*\eta(R(\xi,H_1^*)H_2^*)\eta(H_3^*)$$
$$+\left[\left(\frac{\omega}{2}+\frac{1}{4}\right)-\lambda^*\right]g(H_2^*,R(\xi,H_1^*)H_3^*)-\mu^*\eta(R(\xi,H_1^*)H_3^*)\eta(H_2^*)$$
$$-g(\phi(R(\xi,H_1^*)H_2^*,H_3^*)-g(\phi H_2^*,R(\xi,H_1^*)H_3^*)=0. \tag{5.19}$$

We obtain on using (5.8) in (5.19)

$$\left[\left(\frac{\omega}{2}+\frac{1}{4}\right)-\lambda^*\right]g(g(H_1^*,H_2^*)\xi-\eta(H_2^*)H_1^*,H_3^*)$$
$$-\mu^*\eta(g(H_1^*,H_2^*)\xi-\eta(H_2^*)H_1^*)\eta(H_3^*)-\mu^*\eta(g(H_1^*,H_3^*)\xi$$
$$-\eta(H_3^*)H_1^*)\eta(H_2^*)-g(\phi g(H_1^*,H_2^*)\xi-\eta(H_2^*)\phi H_1^*,H_3^*)$$
$$+\left[\left(\frac{\omega}{2}+\frac{1}{4}\right)-\lambda^*\right]g(H_2^*,g(H_1^*,H_3^*)\xi-\eta(H_3^*)H_1^*)$$
$$-g(\phi H_2^*,g(H_1^*,H_3^*)\xi-\eta(H_3^*)H_1^*)=0,$$

that is,

$$\mu^*g(H_1^*,H_2^*)\eta(H_3^*)+\mu^*g(H_1^*,H_3^*)\eta(H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)\eta(H_3^*)$$
$$+\eta(H_2^*)g(\phi H_1^*,\xi)+\eta(H_3^*)g(\phi H_2^*,H_1^*)=0.$$

Putting $H_3^*=\xi$ in above equation, we find

$$\mu^*g(H_1^*,H_2^*)\eta(\xi)+\mu^*g(H_1^*,\xi)\eta(H_2^*)+2\mu^*\eta(H_1^*)\eta(H_2^*)\eta(\xi)$$
$$+\eta(H_2^*)g(\phi H_1^*,\xi)+\eta(\xi)g(\phi H_2^*,H_1^*)=0,$$

which implies

$$g(\phi H_1^*,H_2^*)+\mu^*[g(H_1^*,H_2^*)+\eta(H_1^*)\eta(H_2^*)]=0. \tag{5.20}$$

Using (5.20) in (5.16), we have

$$R_t(H_1^*,H_2^*)=\left[\left(\frac{\omega}{2}+\frac{1}{4}\right)-\lambda^*+\mu^*\right]g(H_1^*,H_2^*). \tag{5.21}$$

Hence we have proved the theorem.

□

5.5 CηRS on 4-dimension LP-S manifold satisfying ξ-Ricci conformally semi-symmetric condition

We consider a LP-S manifold (M^4, g) that admits a CηRS and it satisfies the ξ-Ricci conformally semi-symmetric condition, i.e., $C(\xi, H_1^*).R_t = 0$.
On the basis of the above condition we state and prove of the following theorem:

Theorem 5.5.1. *Let* 4*-dimension LP-S manifold admits a CηRS. If it satisfies the ξ-Ricci conformally semi-symmetric condition, then it becomes an Einstein manifold.*

Proof. From equation (1.43) reduces to

$$C(\xi, H_1^*)H_2^* = \frac{R^*}{12}\left[\eta(H_2^*)H_1^* - g(H_1^*, H_2^*)\xi\right] + R^*(\xi, H_1^*)H_2^*. \tag{5.22}$$

Using (5.8) in (5.22), we have

$$C(\xi, H_1^*)H_3^* = \left[1 - \frac{R^*}{12}\right]\left[g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*\right]. \tag{5.23}$$

Similarly,

$$C(\xi, H_1^*)H_2^* = \left[1 - \frac{R^*}{12}\right]\left[g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*\right]. \tag{5.24}$$

Using equations (5.23), (5.24) in (2.41), it is found that

$$\begin{aligned}
&\left[1 - \frac{R^*}{12}\right]\{R_t([g(H_1^*, H_3^*)\xi - \eta(H_3^*)H_1^*], H_2^*)\} \\
&+ \left[1 - \frac{R^*}{12}\right]\{R_t([g(H_1^*, H_2^*)\xi - \eta(H_2^*)H_1^*], H_3^*)\} = 0,
\end{aligned}$$

which implies

$$\begin{aligned}
&3g(H_1^*, H_3^*)\eta(H_2^*) - R_t(H_1^*, H_2^*)\eta(H_3^*) \\
&+ 3g(H_1^*, H_2^*)\eta(H_3^*) - R_t(H_1^*, H_3^*)\eta(H_2^*) = 0.
\end{aligned} \tag{5.25}$$

Setting $H_3^* = \xi$ in (5.25) and utilising (5.8), we obtain

$$R_t(H_1^*, H_2^*) - 3g(H_1^*, H_2^*) = 0. \tag{5.26}$$

Then (5.26) becomes

$$R_t(H_1^*, H_2^*) = 3g(H_1^*, H_2^*). \tag{5.27}$$

Hence the result. □

5.6 CηRS on 4-dimension LP-S manifold with TFVF

We analyze a CηRS in LP-S manifold with ξ being TFVF and we explore characterize the nature of the soliton.
Based on the above condition we state the following theorem:

Theorem 5.6.1. *Let* 4*-dimension LP-S manifold admits a CηRS with ξ being a TFVF. Then the manifold becomes an ηE manifold and the soliton is steady, expanding and shrinking for* $\mu^* = \frac{1}{4}(11-2\omega)$, $\mu^* > \frac{1}{4}(11-2\omega)$, $\mu^* < \frac{1}{4}(11-2\omega)$ *respectively.*

Proof. We examine the case of a LP-S manifold (M^4, g) admitting a Cη-RS $(g, \xi, \lambda^*, \mu^*)$ and consider that Reeb vector field ξ is a TFVF.
Then the equation (1.44) reduces to

$$\nabla_{H_2^*}\xi = fH_2^* + \gamma(H_2^*)\xi. \tag{5.28}$$

Using equation (5.2) and taking inner product with ξ, we obtain

$$g(\nabla_{H_2^*}\xi, \xi) = g(\phi H_2^*, \xi) = \eta(\phi H_2^*) = 0. \tag{5.29}$$

Taking inner product in equation (5.28), with ξ we have

$$g(\nabla_{H_2^*}\xi, \xi) = f\eta(H_2^*) - \gamma(H_2^*). \tag{5.30}$$

From equations (5.29) and (5.30), we conclude that

$$\gamma = f\eta. \tag{5.31}$$

Thus for TFVF ξ in LP-S manifold, we obtain

$$\nabla_{H_2^*}\xi = f(H_2^* + \eta(H_2^*)\xi). \tag{5.32}$$

From equation (1.20), we have

$$g(\nabla_{H_1^*}\xi, H_2^*) + g(\nabla_{H_2^*}\xi, H_1^*) + 2R_t(H_1^*, H_2^*)$$
$$+\left[2\lambda^* - \left(\omega + \frac{1}{2}\right)\right] g(H_1^*, H_2^*) + 2\mu\eta(H_1^*)\eta(H_2^*) = 0. \tag{5.33}$$

Using (5.32) in (5.33), we observe

$$R_t(H_1^*, H_2^*) = \left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - (\lambda^* + f)\right] g(H_1^*, H_2^*) - (f + \mu^*)\eta(H_1^*)\eta(H_2^*), \tag{5.34}$$

an ηE manifold. Further putting $H_2^* = \xi$ in (5.34), we obtain

$$R_t(H_1^*, \xi) = (\frac{\omega}{2} + \frac{1}{4} - \lambda^* + \mu^*)\eta(H_1^*). \tag{5.35}$$

This implies that $-(f+\mu^*)$ is an eigen value of R_t corresponding to the eigen vector ξ. Combining (5.35) with the equation (5.11), we get

$$\lambda^* = \frac{\omega}{2} + \mu^* - \frac{11}{4}. \tag{5.36}$$

From (5.36), we can conclude the following:

(i) If $\lambda^* = 0$, then $\mu^* = \frac{1}{4}(11 - 2\omega)$ implies the soliton is steady.

(ii) $\lambda^* > 0$, then $\mu^* > \frac{1}{4}(11 - 2\omega)$ implies the soliton is expanding.

(iii) $\lambda^* < 0$, then $\mu^* < \frac{1}{4}(11 - 2\omega)$ implies the soliton is shrinking.

This concludes the proof.

□

According to the above theorems we can investigate the following theorems in a perfect fluid spacetime:

Theorem 5.6.2. *Let a LP-S manifold (M^4, g) admits a CηRS. Then it represents a dark energy era for μ^*=0.*

Proof. Using equation (5.16) in (1.28), we obtain

$$\begin{aligned}&\left[(\frac{\omega}{2}+\frac{1}{4})-\lambda^*\right]g(H_1^*,H_2^*)-\mu^*\eta(H_1^*)\eta(H_2^*)-g(\phi H_1^*,H_2^*)\\&=\left(\kappa\rho+\frac{R^*}{2}\right)g(H_1^*,H_2^*)+(\rho+\sigma)\kappa\eta(H_1^*)\eta(H_2^*).\end{aligned}\tag{5.37}$$

Setting $H_1^* = H_2^* = \xi$ in (5.37), we have

$$\lambda^* = \kappa\sigma + \mu^* + \frac{\omega}{2} + \frac{1}{4} - \frac{R^*}{2}.\tag{5.38}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1,2,3,4)$ in (5.37), and adding these up,

$$\lambda^* = \frac{1}{2}(\omega + \mu^* - R^* - \kappa\rho + \kappa\sigma + \frac{1}{2}).\tag{5.39}$$

Finally combining equations (5.38) and (5.39), we get

$$\kappa(\sigma+\rho)+\mu^* = 0.\tag{5.40}$$

If μ^*=0, then equation (5.40) is reduced to

$$\sigma+\rho = 0,$$

which is the required condition for dark energy era.
This completes the proof. □

Theorem 5.6.3. *If a LP-S manifold (M^4, g) admits a CηRS satisfying ξ-Ricci semi-symmetric condition, then the manifold represents a dark energy era.*

Proof. Using equations (5.21) and (1.28), we obtain

$$\begin{aligned}&\left[\left(\frac{\omega}{2}+\frac{1}{4}\right)-\lambda^*+\mu^*\right]g(H_1^*,H_2^*)\\&=\left(\kappa\rho+\frac{R^*}{2}\right)g(H_1^*,H_2^*)+(\rho+\sigma)\kappa\eta(H_1^*)\eta(H_2^*).\end{aligned}\tag{5.41}$$

Setting $H_1^* = H_2^* = \xi$ in (5.41), we have

$$R^* = 2\mu^* + 2\kappa\sigma - 2\lambda^* + \omega + \frac{1}{2}. \tag{5.42}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1,2,3,4)$ in (5.41), and adding these up,

$$R^* = 2\mu^* - 2\lambda^* + \omega - \kappa\rho + \kappa\sigma + \frac{1}{2}. \tag{5.43}$$

Equations (5.42), (5.43) enable us to get

$$\sigma + \rho = 0.$$

As required, the statement is proven. □

Theorem 5.6.4. *Let a LP-S manifold (M^4, g) admits a CηRS with ξ being a TFVF. Then the manifold represents a dark energy era for $f + \mu^* = 0$.*

Proof. We obtain on using equation (5.34) in (1.28)

$$\begin{aligned}&\left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - (\lambda^* + f)\right] g(H_1^*, H_2^*) - (f + \mu^*)\eta(H_1^*)\eta(H_2^*) \\ &= \left(\kappa\rho + \frac{R^*}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*).\end{aligned} \tag{5.44}$$

Setting $H_1^* = H_2^* = \xi$ in (5.44), we get

$$R^* = \omega - 2\lambda^* + 2\mu^* + 2\kappa\sigma + \frac{1}{2}. \tag{5.45}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1,2,3,4)$ in (5.44), and adding these up,

$$R^* = \omega + \mu^* - 2\lambda^* - f - \kappa\rho + \kappa\sigma + \frac{1}{2}. \tag{5.46}$$

Finally combining equations (5.45) and (5.46), we get

$$\kappa(\sigma + \rho) = -(f + \mu^*). \tag{5.47}$$

If $f + \mu^* = 0$, then equation (5.47) implies

$$\sigma + \rho = 0,$$

and the criteria is fulfilled for the dark energy era.
Thus the proof is complete. □

Theorem 5.6.5. *If a LP-S manifold (M^4, g) admits a CηRS satisfying ξ-Ricci conformally semi-symmetric condition, then the manifold represents a dark energy era.*

Proof. Using equations (5.27) and (1.28), we obtain

$$3g(H_1^*, H_2^*) = \left(\kappa\rho + \frac{R^*}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*). \tag{5.48}$$

Setting $H_1^* = H_2^* = \xi$ in (5.48), we have

$$R^* = 2\kappa\sigma + 6. \tag{5.49}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1, 2, 3, 4)$ in (5.48), and adding these up,

$$R^* = 6 - \kappa\rho + \kappa\sigma. \tag{5.50}$$

Finally combining equations (5.49) and (5.50), we get

$$\sigma + \rho = 0.$$

This completes the proof. □

Now, we give an example of a LP-S manifold admitting a CηRS based on the above theorems:

5.7 Example of a 4-dimension LP-S manifold admitting a CηRS

Let $M = \left\{(h_1^*, h_2^*, h_3^*, h_4^*) \in \mathbb{R}^4\right\}$ be a 4-dimension manifold [22]. The vector fields $w_1 = \frac{h_1^*}{h_4^*}\frac{\partial}{\partial h_1^*}, w_2 = \frac{h_2^*}{h_4^*}\frac{\partial}{\partial h_2^*}, w_3 = \frac{h_3^*}{h_4^*}\frac{\partial}{\partial h_3^*}, \xi = w_4 = h_4^*\frac{\partial}{\partial h_3^*}$ are linearly independent in M^4.
We define the Lorentzian metric g by
$g(w_1, w_2) = g(w_2, w_3) = g(w_1, w_3) = g(w_1, w_4) = g(w_2, w_4) = g(w_3, w_4) = 0$
and $g(w_1, w_1) = g(w_2, w_2) = g(w_3, w_3) = 1, g(w_4, w_4) = -1.$

Then the (1,1) tensor field ϕ gives

$$\phi(w_4) = 0, \phi(w_1) = w_1, \phi(w_2) = w_2, \phi(w_3) = w_3.$$

If η is a 1-form, then $\eta(w_4) = g(w_4, w_4) = -1$. We can easily verify that g is a Lorentzian paracontact metric on M^4 and by the linearity of ϕ. Let ∇ be the Levi-Civita connection on M^4. Then we find
$[w_1, w_2] = 0$, $[w_1, w_3] = -w_1$, $[w_2, w_3] = -w_2$.

By using Koszul's formula (1.48) for the Riemannian metric g and taking $w_4 = \xi$, we can calculate

$$\nabla_{w_1} w_1 = w_4, \nabla_{w_1} w_2 = 0, \nabla_{w_1} w_3 = 0, \nabla_{w_1} w_4 = w_1,$$

$$\nabla_{w_2} w_1 = 0, \nabla_{w_2} w_2 = w_4, \nabla_{w_2} w_3 = 0, \nabla_{w_2} w_4 = w_2,$$

$$\nabla_{w_4} w_1 = 0, \nabla_{w_4} w_2 = 0, \nabla_{w_4} w_3 = 0, \nabla_{w_4} w_4 = 0.$$

Using these we can verify $\eta(\xi) = -1$ and $\nabla_{H_1^*} \xi = \phi H_1^*$ for all H_1^*. Hence M^4 is a LP-S manifold .
Also from the relation of Riemmanian curvature tensor (1.49) we can calculate the following components

$$R(w_1, w_1)w_1 = 0, R(w_1, w_2)w_2 = w_1, R(w_1, w_4)w_4 = -w_1, R(w_1, w_2)w_4 = 0,$$

$$R(w_1, w_2)w_3 = 0, R(w_2, w_3)w_3 = w_2, R(w_2, w_1)w_1 = w_2, R(w_2, w_4)w_4 = -w_2,$$

$$R(w_1, w_3)w_2 = 0, R(w_1, w_3)w_1 = -w_3, R(w_1, w_3)w_3 = w_1, R(w_3, w_4)w_4 = -w_3,$$

$$R(w_2, w_1)w_1 = w_2, R(w_3, w_1)w_1 = w_3, R(w_3, w_3)w_3 = 0, R(w_3, w_2)w_2 = w_3,$$

$$R(w_4, w_2)w_2 = w_4, R(w_4, w_3)w_3 = w_4, R(w_4, w_4)w_4 = 0, R(w_4, w_1)w_1 = w_4.$$

From above defined curvature tensors, we can obtain the components of R_t:

$$R_t(w_4, w_4) = -3, R_t(w_3, w_3) = 1, R_t(w_1, w_1) = 1, R_t(w_2, w_2) = 1.$$

From equation (5.16), we can calculate

$$R_t(w_4, w_4) = -\left[\left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] - \mu^*.$$

By equating both the values of $R_t(w_4, w_4)$, we have

$$\lambda^* = \frac{\omega}{2} + \mu^* - \frac{11}{4}.$$

Hence the constant λ^* satisfies equation (5.18) and so g defines a CηRS on the LP-S manifold M^4.

5.8 Contact vector field on LP-S manifolds with respect to semi-symmetric metric connection

We shall now state and prove a theorem on a vector field:

Theorem 5.8.1. *Every contact vector field on a LP-S manifold leaving the Ricci tensor with respect to semi-symmetric connection invariant is a strict contact vector field.*

Proof. In a LP-S manifolds, let us assume that a contact vector field V leaves the Ricci tensor with respect to semi-symmetric metric connections invariant i.e

$$\mathcal{L}_V \bar{R}_t(H_1^*, H_2^*) = 0. \tag{5.51}$$

We have from (5.51) $\mathcal{L}_V \bar{R_T}(H_1^*, H_2^*) = \bar{R}_T(\mathcal{L}_V H_1^*, H_2^*) + \bar{R}_T(H_1^*, \mathcal{L}_V H_2^*)$.
Replacing H_2^* with ξ we get

$$\mathcal{L}_V \bar{R_T}(H_1^*, \xi) = \bar{R}_T(\mathcal{L}_V H_1^*, \xi) + \bar{R_T}(H_1^*, \mathcal{L}_V \xi). \tag{5.52}$$

Putting $H_2^* = \xi$ in (1.32) we can get

$$\bar{R}_T(H_1^*, \xi) = 2(n-1)\eta(H_1^*). \tag{5.53}$$

Taking Lie derivative on both the sides of (5.53) and using definition (1.30) we have

$$2(n-1)\sigma^* \eta(H_1^*) = \bar{R}_T(H_1^*, \mathcal{L}_V \xi). \tag{5.54}$$

Putting $H_1^* = \xi$ in (5.54) we can obtain

$$-\eta(\mathcal{L}_V \xi) = \sigma^*. \tag{5.55}$$

Again from the definition (1.30) we have

$$(\mathcal{L}_V \eta)(\xi) = -\sigma^*. \tag{5.56}$$

From Lie derivative we know

$$\eta(\mathcal{L}_V \xi) = \sigma^*. \tag{5.57}$$

Therefore (5.55) and (5.57) can be true simultaneously if $\sigma^* = 0$. Hence the proof. □

5.9 On Extended Generalized ϕ-recurrent LP-S manifold with respect to semi-symmetric metric connection

In this section, we examine on extended generalized ϕ-recurrent LP-S manifolds with respect to semi-symmetric metric connections.
We can state the following theorem:

Theorem 5.9.1. *An extended generalized ϕ-recurrent LP-S manifold (M^n, g) with respect to semi-symmetric metric connection is an ηE manifold and the 1-forms A and B are related as $2(n-1)A(H_4^*) + (n+1)B(H_4^*) = 0$.*

Proof. Let us consider an extended generalized ϕ-recurrent LP-S manifold $(M^n, \phi, \eta, \xi, g)$ with respect to semi-symmetric metric connection. Then we have from equation (1.36)

$$\begin{aligned}\phi^2((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^*) = A(H_4^*)\phi^2(\bar{R}(H_1^*, H_2^*)H_3^*) \\ + B(H_4^*)\phi^2[g(H_2^*, H_3^*)H_1^* - g(H_1^*, H_3^*)H_2^*]. \qquad (5.58)\end{aligned}$$

Using (1.11) and (5.58) we can obtain

$$\begin{aligned}&(\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^* + \eta((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^*)\xi \\ &= A(H_4^*)[\bar{R}(H_1^*, H_2^*)H_3^* + \eta(\bar{R}(H_1^*, H_2^*)H_3^*)\xi] \\ &+ B(H_4^*)[g(H_2^*, H_3^*)H_1^* - g(H_1^*, H_3^*)H_2^* \\ &+ \eta(H_1^*)g(H_2^*, H_3^*)\xi - g(H_1^*, H_3^*)\eta(H_2^*)\xi]. \qquad (5.59)\end{aligned}$$

Taking inner product of (5.59) with H_5^* and using (1.12) we have

$$\begin{aligned}&g((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^*, H_5^*) + \eta((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*, H_2^*)H_3^*)g(\xi, H_5^*) \\ &= A(H_4^*)[g(\bar{R}(H_1^*, H_2^*)H_3^*, H_5^*) + \eta(\bar{R}(H_1^*, H_2^*)H_3^*)g(\xi, H_5^*)] \\ &+ B(H_4^*)[g(H_2^*, H_3^*)g(H_1^*, H_5^*) - g(H_1^*, H_3^*)g(H_2^*, H_5^*) \\ &+ \eta(H_1^*)g(H_2^*, H_3^*)g(\xi, H_5^*) - g(H_1^*, H_3^*)\eta(H_2^*)g(\xi, H_5^*)]. \qquad (5.60)\end{aligned}$$

From (1.12) and (5.61) we have

$$
\begin{aligned}
&g((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*,H_2^*)H_3^*,H_5^*)+\eta((\bar{\nabla}_{H_4^*}\bar{R})(H_1^*,H_2^*)H_3^*)\eta(H_5^*)\\
&=A(H_4^*)[g(\bar{R}(H_1^*,H_2^*)H_3^*,H_5^*)+\eta(\bar{R}(H_1^*,H_2^*)H_3^*)\eta(H_5^*)]\\
&+B(H_4^*)[g(H_2^*,H_3^*)g(H_1^*,H_5^*)-g(H_1^*,H_3^*)g(H_2^*,H_5^*)+\\
&\eta(H_1^*)g(H_2^*,H_3^*)\eta(H_5^*)-g(H_1^*,H_3^*)\eta(H_2^*)\eta(H_5^*)].
\end{aligned}
\tag{5.61}
$$

Let $\{w_1,w_2,\ldots,w_n\}$ be an orthonormal basis for the tangent space of M^n at a point $p\in M^n$. Putting $H_1^*=H_5^*=w_i$ in (5.61) and taking summation over i from 1 to n, we get

$$
\begin{aligned}
&(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*,H_3^*)+\sum_{i=1}^{n}\eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)H_3^*)\eta(w_i)\\
&=A(H_4^*)[\bar{R}_t(H_2^*,H_3^*)-\eta(\bar{R}(\xi,H_2^*)H_3^*)]+B(H_4^*)[(n+1)g(H_2^*,H_3^*)].
\end{aligned}
\tag{5.62}
$$

On putting $H_3^*=\xi$ in (5.62) we get

$$
\begin{aligned}
&(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*,\xi)+\sum_{i=1}^{n}\eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)\xi)\eta(w_i)\\
&=2(n-1)A(H_4^*)+B(H_4^*)[(n+1)g(H_2^*,H_3^*)].
\end{aligned}
$$

Second term of the above equation yields

$$
\begin{aligned}
&\eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)\xi)\eta(w_i)=-g(\bar{\nabla}_{H_4^*}\bar{R}(w_i,H_2^*)\xi,\xi)\\
&+g(\bar{R}(\bar{\nabla}_{H_4^*}w_i,H_2^*)\xi,\xi)+g(\bar{R}(w_i,\bar{\nabla}_{H_4^*}H_2^*)\xi,\xi)+g(\bar{R}(w_i,H_2^*)\bar{\nabla}_{H_4^*}\xi,\xi).
\end{aligned}
\tag{5.63}
$$

Let $p\in M^n$, since w_i is an orthonormal basis, so $\bar{\nabla}_{H_4^*}w_i=0$ at p. We have

$$
g(\bar{R}(w_i,H_2^*)\xi,\xi)=-g(\bar{R}(\xi,\xi)H_2^*,w_i)=0. \tag{5.64}
$$

Since $\nabla_{H_4^*}g=0$, we can get

$$
g(\bar{\nabla}_{H_4^*}\bar{R}(e_i,H_2^*)\xi,\xi)+g(\bar{R}(w_i,H_2^*)\xi,\bar{\nabla}_{H_4^*}\xi)=0. \tag{5.65}
$$

Also from (5.63) and (5.65) we have

$$
\begin{aligned}
g((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)\xi,\xi) &= -g(\bar{R}(w_i,H_2^*)\xi,\bar{\nabla}_{H_4^*}\xi)-g(\bar{R}(\bar{\nabla}_{H_4^*}w_i,H_2^*)\xi,\xi)\\
&\quad -g(\bar{R}(w_i,\bar{\nabla}_{H_4^*}H_2^*)\xi,\xi)-g(\bar{R}(w_i,H_2^*)\bar{\nabla}_{H_4^*}\xi,\xi).
\end{aligned}
\tag{5.66}
$$

We also have

$$g(\bar{R}(w_i,\bar{\nabla}_{H_4^*}H_2^*)\xi,\xi) = 0 = g(\bar{R}(\bar{\nabla}_{H_4^*}w_i,H_2^*)\xi,\xi). \tag{5.67}$$

Now using (5.67) in (5.66) and using skew-symmetrices of R we get

$$g((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)\xi,\xi) = 0. \tag{5.68}$$

Therefore second term of (5.63) is zero, i.e.,

$$\sum_{i=1}^{n}\eta((\bar{\nabla}_{H_4^*}\bar{R})(w_i,H_2^*)\xi)\eta(w_i) = 0. \tag{5.69}$$

We have from (5.63) in (5.69)

$$(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*,\xi) = [2(n-1)A(H_4^*)+(n+1)B(H_4^*)]\eta(H_2^*). \tag{5.70}$$

Using (5.2), (5.11) in (5.70) we obtain

$$(\bar{\nabla}_{H_4^*}\bar{R}_t)(H_2^*,\xi) = (n-1)g(H_2^*,\phi H_4^*) - \bar{R}_t(H_2^*,\phi H_4^*). \tag{5.71}$$

From (5.70) and (5.71) we have

$$(n-1)g(H_2^*,\phi H_4^*) - \bar{R}_t(H_2^*,\phi H_4^*) = [2(n-1)A(H_4^*)+(n+1)B(H_4^*)]\eta(H_2^*). \tag{5.72}$$

Using $H_2^* = \xi$ in (5.72) we have

$$2(n-1)A(H_4^*)+(n+1)B(H_4^*) = 0. \tag{5.73}$$

Now from (5.72) and (5.73) we can infer

$$R_t(H_2^*,\phi H_4^*) = ng(H_2^*,\phi H_4^*)+n\eta(H_2^*)\eta(\phi H_4^*). \tag{5.74}$$

Substituting H_4^* by ϕH_4^* in (5.74) we have

$$R_t(H_2^*,H_4^*) = ng(H_2^*,H_4^*)+n\eta(H_2^*)\eta(H_4^*), \tag{5.75}$$

where $a = n$ and $b = n$. Therefore M^n is an ηE manifold. □

Now we move to our next section:

5.10 Conharmonic Curvature tensor on a LP-S manifold with respect to semi-symmetric metric connnection

In this section we state and prove the next theorem:

Theorem 5.10.1. *If a $n(\geq 3)$dimensional LP-S manifold with respect to semi-symmetric metric connection admitting a conharmonic curvature tensor and a non-zero Ricci tensor satisfies $\bar{L}(H_1^*, H_2^*)\bar{R}_t = 0$, then the modulus of non-zero eigen values of the endomorphism $\bar{Q}$ of the tangent space corresponding to $\bar{R}_t$ is $2(n-1)$.*

Proof. We consider a $n(n \geq 3)$ dimensional LP-S manifold with respect to semi-symmetric metric connection, satisfying the condition $\bar{L}(H_1^*, H_2^*)\bar{R}_t = 0$. Then we have

$$\bar{R}_t(\bar{L}(H_1^*, H_2^*)H_3^*, H_4^*) + \bar{R}_t(H_3^*, \bar{L}(H_1^*, H_2^*)H_4^*) = 0, \tag{5.76}$$

for all $H_1^*, H_2^*, H_3^*, H_4^* \in \chi(M)$. Substituting H_1^* by ξ in the above equation we can obtain

$$\bar{R}_t(\bar{L}(\xi, H_2^*)H_3^*, H_4^*) + \bar{R}_t(H_3^*, \bar{L}(\xi, H_2^*)H_4^*) = 0. \tag{5.77}$$

Let $\bar{\lambda}^*$ be the eigen value of the endomorphism $\bar{Q}$ corresponding to an eigenvector H_1^*, then

$$\bar{Q}H_1^* = \bar{\lambda}^* H_1^*. \tag{5.78}$$

We know $g(\bar{Q}H_1^*, H_2^*) = \bar{R}_t(H_1^*, H_2^*) = \bar{\lambda}^* g(H_1^*, H_2^*)$. Therefore

$$R_t(H_1^*, H_2^*) - g(H_1^*, H_2^*) - n\eta(H_1^*)\eta(H_2^*) = \lambda^* g(H_1^*, H_2^*). \tag{5.79}$$

Putting $H_1^* = H_2^* = \xi$ in (5.79) we can calculate

$$\bar{\lambda}^* = 2(n-1). \tag{5.80}$$

Therefore the theorem. □

5.11 Example of a LP-S manifold

Let $M = \{(h_1^*, h_2^*, h_3^*) \in \mathbb{R}^3\}$ be a three-dimensional manifold [28]. The vector fields $w_1 = e^{h_3^*}\frac{\partial}{\partial h_2^*}, w_2 = e^{h_3^*}(\frac{\partial}{\partial h_1^*} + \frac{\partial}{\partial h_2^*}), \xi = w_3 = \frac{\partial}{\partial h_3^*}$ are linearly independent at each point of M. We define the Lorentzian metric g by
$g(w_1, w_1) = g(w_2, w_2) = 1, g(w_3, w_3) = -1, g(w_i, w_j) = 0$ for $i \neq j$. The (1,1) tensor field ϕ is defined as

$$\phi(w_1) = -w_1, \phi(w_2) = -w_2, \phi(w_3) = 0.$$

If η is 1-form then $\eta(w_3) = g(w_3, w_3) = -1$. We can easily verify by the linearity of ϕ and g that (ϕ, ξ, η, g) is a Lorentzian paracontact structure on M.
Let ∇ be the Levi-Civita connection on $\mathbb{R}^3$. Then we have
$[w_1, w_2]$= 0, $[w_1, w_3] = -w_1$, $[w_2, w_3] = -w_2$.
By using equation (1.48), we can find by taking $e_3 = \xi$, we can calculate

$$\nabla_{w_1} w_1 = -w_3, \nabla_{w_1} w_2 = 0, \nabla_{w_1} w_3 = -w_1,$$

$$\nabla_{w_2} w_1 = 0, \nabla_{w_2} w_2 = -w_3, \nabla_{w_2} w_3 = -w_2,$$

$$\nabla_{w_3} w_1 = 0, \nabla_{w_3} w_2 = 0, \nabla_{w_3} w_3 = 0.$$

Using these we can verify $\nabla_{H_1^*}\xi = H_1^* + \eta(H_1^*)\xi$. Hence the manifold is a LP-S manifold.
We consider the linear connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}_{w_i} w_j = \nabla_{w_i} w_j + \eta(w_j) w_i - g(w_i, w_j) w_3.$$

From above relation we can calculate the non-zero components

$$\tilde{\nabla}_{w_1} w_1 = -2w_3, \tilde{\nabla}_{w_2} w_2 = -2w_3, \tilde{\nabla}_{w_3} w_3 = 2w_3.$$

Let $\bar{T}^*$ is the torsion tensor of metric connection $\bar{\nabla}$, then using equation (1.1) we can see that $\bar{T}^*(H_1^*, H_2^*) = 0$.
We know that

$$(\nabla_{H_1^*} g)(H_2^*, H_3^*) = H_1^* g(H_2^*, H_3^*) - g(\nabla_{H_1^*} H_2^*, H_3^*) - g(H_2^*, \nabla_{H_1^*} H_3^*)$$

and

$$(\tilde{\nabla}_{H_1^*} g)(H_2^*, H_3^*) = H_1^* g(H_2^*, H_3^*) - g(\tilde{\nabla}_{H_1^*} H_2^*, H_3^*) - g(H_2^*, \tilde{\nabla}_{H_1^*} H_3^*).$$

Now using above formulae we calculate
$(\tilde{\nabla}_{w_1}g)(w_2,w_3) = 0 = (\tilde{\nabla}_{w_1}g)(w_3,w_2) = (\tilde{\nabla}_{w_2}g)(w_1,w_3) = (\bar{\nabla}_{w_3}g)(w_2,w_1)$.
Therefore we prove that $(\bar{\nabla}_{H_1^*}g)(H_2^*,H_3^*) = 0$.
Hence $\tilde{\nabla}$ is a semi-symmetric metric connection on M.

5.12 4-dimensional LP-S manifold admitting CηES

Here we consider LP-S manifold (M^4,g) admitting CηES.
Now, we state and prove the following theorems:

Theorem 5.12.1. *If a LP-S manifold (M^4,g) is Ricci symmetric admits a CηES (g,ξ,λ^*,μ^*), then the manifold (M^4,g) represents a dark energy era.*

Proof. Let us consider a LP-S manifold (M^4,g) admitting a CηES (g,ξ,λ^*,μ^*). Then from the equation (1.22), we have

$$\begin{aligned}(\mathcal{L}_\xi g)(H_1^*,H_2^*) + 2R_t(H_1^*,H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*)\\ + \left[2\lambda^* - R^* + \left(\omega + \frac{1}{2}\right)\right] g(H_1^*,H_2^*) = 0.\end{aligned} \tag{5.81}$$

From (5.81), we get

$$\begin{aligned}2R_t(H_1^*,H_2^*) = -(\mathcal{L}_\xi g)(X,H_2^*) - 2\mu^*\eta(H_1^*)\eta(H_2^*)\\ - \left[2\lambda^* - R^* + \left(\omega + \frac{1}{2}\right)\right] g(H_1^*,H_2^*).\end{aligned} \tag{5.82}$$

Now, with the help of (5.2), we have

$$(\mathcal{L}_\xi g)(H_1^*,H_2^*) = g(\phi H_1^*,H_2^*) + g(\phi H_2^*,H_1^*). \tag{5.83}$$

From (5.82) and (5.83), we obtain

$$\begin{aligned}R_t(H_1^*,H_2^*) = \left[\frac{R^*}{2} - (\frac{\omega}{2} + \frac{1}{4}) - \lambda^*\right] g(H_1^*,H_2^*)\\ - \mu^*\eta(H_1^*)\eta(H_2^*) - g(\phi H_1^*,H_2^*).\end{aligned} \tag{5.84}$$

Putting $H_2^* = \xi$ in (5.84), we get

$$R_t(H_1^*,\xi) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^* + \mu^*\right] \eta(H_1^*). \tag{5.85}$$

Comparing the equations (5.11) and (5.85), we have

$$3\eta(H_1^*) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^* + \mu^*\right]\eta(H_1^*).$$

Since η is a non-zero 1-form, it becomes

$$\lambda^* - \mu^* = \frac{R^*}{2} - \frac{\omega}{2} - \frac{13}{4}. \tag{5.86}$$

Using the equations (2.18) and(5.84), we get

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = -\mu^*[\eta(H_3^*)(\nabla_{H_1^*}\eta)H_2^* + \eta(H_2^*)(\nabla_{H_1^*}\eta)H_3^*]. \tag{5.87}$$

Using equation (5.3), the above equation becomes

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = -\mu^*\left[\eta(H_3^*)g(\phi H_1^*, H_2^*) + \eta(H_2^*)g(\phi H_1^*, H_3^*)\right]. \tag{5.88}$$

If the manifold M^4 is Ricci symmetric, then $\nabla R_t = 0$.
From equation (5.88), we infer that

$$-\mu^*\left[\eta(H_3^*)g(\phi H_1^*, H_2^*) + \eta(H_2^*)g(\phi H_1^*, H_3^*)\right] = 0. \tag{5.89}$$

Putting $H_3^* = \xi$ in the equation (5.89), we have

$$\mu^* g(\phi H_1^*, H_2^*) = 0. \tag{5.90}$$

Then μ^*=0 as $g(\phi H_1^*, H_2^*) \neq 0$.
Equation (5.84) reduce to

$$R_t(H_1^*, H_2^*) = \left[\frac{R^*}{2} - (\frac{\omega}{2} + \frac{1}{4}) - \lambda^*\right] g(H_1^*, H_2^*) - g(\phi H_1^*, H_2^*). \tag{5.91}$$

Using equation (5.91) in (1.28), we obtain

$$\begin{aligned}&\left[\frac{R^*}{2} - (\frac{\omega}{2} + \frac{1}{4}) - \lambda^*\right] g(H_1^*, H_2^*) - g(\phi H_1^*, H_2^*) = \\ &\left(\kappa\rho + \frac{R^*}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*).\end{aligned} \tag{5.92}$$

Setting $H_1^* = H_2^* = \xi$ in (5.92), we have

$$\lambda^* = \kappa\sigma - \frac{\omega}{2} - \frac{1}{4}. \tag{5.93}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1,2,3,4)$ in (5.92), and adding these up,

$$\lambda^* = \frac{1}{2}\left(\kappa\sigma - \kappa\rho - \omega - \frac{1}{2}\right). \tag{5.94}$$

Finally combining equations (5.93) and (5.94), we get

$$\sigma + \rho = 0.$$

This completes the proof. □

Theorem 5.12.2. *If the metric of a 4-dimensional LP-S manifold is a CηES and the Ricci tensor is Ricci-recurrent, then the manifold represents a Minkowski spacetime.*

Proof. Using the equations (1.41) and (5.88), we obtain

$$-\mu^*\left[\eta(H_3^*)g(\phi H_1^*, H_2^*) + \eta(H_2^*)g(\phi H_1^*, H_3^*)\right] = \eta(H_1^*)S(H_2^*, H_3^*). \tag{5.95}$$

Putting $H_1^* = \xi$ in the equation (5.95), we have

$$R_t(H_2^*, H_3^*) = 0, \tag{5.96}$$

and hence R^*=0.
Consequently from equation (1.27), we obtain

$$T(H_1^*, H_2^*) = 0, \tag{5.97}$$

which represents a Minkowski spacetime. □

Theorem 5.12.3. *If a LP-S manifold (M^4, g) admits a CηES $(g, \nu, \lambda^*, \mu^*)$ such that ν is a pointwise collinear with ξ, then ν is a constant multiple of ξ and the manifold (M^4, g) becomes a dark energy era for $\mu^* = 0$.*

Proof. Considering a LP-S manifold (M^4, g) that admits a CηES $(g, \nu, \lambda^*, \mu^*)$ such that ν is parallel to ξ, that is, $\nu = b\xi$ for some function b, and using this in equation (1.27), it follows that

$$(\mathcal{L}_{b\xi} g)(H_1^*, H_2^*) + 2R_t(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*)$$
$$+\left[2\lambda^* - R^* + \left(\omega + \frac{1}{2}\right)\right] g(H_1^*, H_2^*) = 0,$$

which gives

$$bg(\nabla_{H_1^*}\xi, H_2^*) + (H_1^* b)\eta(H_2^*) + bg(\nabla_{H_2^*}\xi, H_1^*) + (H_2^* b)\eta(H_1^*)$$
$$+ 2R_t(H_1^*, H_2^*) + \left[2\lambda^* - R^* + \left(\omega + \frac{1}{2}\right)\right] g(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) = 0. \quad (5.98)$$

Using (5.2) in the equation (5.98), we get

$$bg(\phi H_1^*, H_2^*) + (H_1^* b)\eta(H_2^*) + bg(\phi H_2^*, H_1^*) + (H_2^* b)\eta(H_1^*)$$
$$+ 2R_t(H_1^*, H_2^*) + \left[2\lambda^* - R^* + \left(\omega + \frac{1}{2}\right)\right] g(H_1^*, H_2^*) + 2\mu^*\eta(H_1^*)\eta(H_2^*) = 0. \quad (5.99)$$

Substituting $H_2^* = \xi$ in (5.99) and using (5.11), we have

$$-(H_1^* b) + \left[2\lambda^* - R^* + \left(\omega + \frac{1}{2}\right) + \xi b + 4 - 2\mu^*\right] \eta(H_1^*) = 0. \quad (5.100)$$

If

$$\left[2\lambda^* - R^* + \left(\omega + \frac{1}{2}\right) + \xi b + 4 - 2\mu^*\right] = 0,$$

then $H_1^* b = 0$, i.e., b is constant. This implies $\xi b = 0$. Since b is constant, the equation (5.99) becomes

$$R_t(H_1^*, H_2^*) = \left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] g(H_1^*, H_2^*) - \mu^*\eta(H_1^*)\eta(H_2^*). \quad (5.101)$$

Using equation (5.101) in (1.28), we obtain

$$\left[\frac{R^*}{2} - (\frac{\omega}{2} + \frac{1}{4}) - \lambda^*\right] g(H_1^*, H_2^*) - \mu^*\eta(H_1^*)\eta(H_2^*)$$
$$= \left(\kappa\rho + \frac{R^*}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*). \quad (5.102)$$

Setting $H_1^* = H_2^* = \xi$ in (5.102), we have

$$\lambda^* = \mu^* + \kappa\sigma - \frac{\omega}{2} - \frac{1}{4}. \quad (5.103)$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1, 2, 3, 4)$ in (5.102), and adding these up,

$$\lambda^* = \frac{1}{2}(\mu^* + \kappa\sigma - \kappa\rho - \omega - \frac{1}{2}). \quad (5.104)$$

Finally combining equations (5.103) and (5.104), we get

$$\kappa(\sigma+\rho) = -\mu^*. \tag{5.105}$$

If μ^*= 0, then equation (5.105) is reduced to

$$\sigma+\rho = 0.$$

Hence the result. □

5.13 CηES on LP-S 4-manifold with Codazzi type Ricci tensor

In this section, we are going to study CηES on LP-S manifolds (M^4, g) having a Codazzi type of Ricci tensor.
We now state and prove the following theorem:

Theorem 5.13.1. *Let (M^4, g) be a LP-S manifold admitting a CηES $(g, \xi, \lambda^*, \mu^*)$ with the Ricci tensor is of Codazzi type. Then the Ricci tensor of the manifold becomes a dark energy era for $\mu^* = 0$.*

Proof. We consider a LP-S manifold (M^4, g) admitting a CηES $(g, \xi, \lambda^*, \mu^*)$. On taking covariant derivative of equation (5.84), we get

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = -\mu^*[\eta(H_3^*)(\nabla_{H_1^*}\eta)H_2^* + \eta(H_2^*)(\nabla_{H_1^*}\eta)H_3^*]. \tag{5.106}$$

Using equation (5.3), the above equation becomes

$$(\nabla_{H_1^*} R_t)(H_2^*, H_3^*) = -\mu^*[\eta(H_3^*)g(\phi H_1^*, H_2^*) + \eta(H_2^*)g(\phi H_1^*, H_3^*)]. \tag{5.107}$$

Now interchanging H_1^* and H_2^* in equation (5.107), we have

$$(\nabla_{H_2^*} R_t)(H_1^*, H_3^*) = -\mu^*[\eta(H_3^*)g(\phi H_2^*, H_1^*) + \eta(H_1^*)g(\phi H_2^*, H_3^*)]. \tag{5.108}$$

On using (1.37), we obtain from equations (5.107) and (5.108)

$$\begin{aligned}&\mu^*[\eta(H_3^*)g(\phi H_1^*, H_2^*) + \eta(H_2^*)g(\phi H_1^*, H_3^*)]\\ &= \mu^*[\eta(H_3^*)g(\phi H_2^*, H_1^*) + \eta(H_1^*)g(\phi H_2^*, H_3^*)].\end{aligned} \tag{5.109}$$

Using $g(\phi H_1^*, H_2^*) = g(\phi H_2^*, H_1^*)$, we have from (5.109)

$$\mu^*[\eta(H_1^*)g(\phi H_2^*, H_3^*) - \eta(H_2^*)g(\phi H_1^*, H_3^*)] = 0. \tag{5.110}$$

If $\mu^* \neq 0$, then from the equation (5.84), we obtain

$$\begin{aligned} R_t(H_1^*, H_2^*) = &\left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] g(H_1^*, H_2^*) \\ &- \mu^* \eta(H_1^*)\eta(H_2^*) - g(\phi H_1^*, H_2^*). \end{aligned} \tag{5.111}$$

Using equations (1.28) and (5.111), we get

$$\begin{aligned} &\left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right] g(H_1^*, H_2^*) - \mu^* \eta(H_1^*)\eta(H_2^*) - g(\phi H_1^*, H_2^*) \\ &= \left(\frac{R^*}{2} + k\rho\right) g(H_1^*, H_2^*) + k(\sigma + \rho)\eta(H_1^*)\eta(H_2^*). \end{aligned} \tag{5.112}$$

Setting $H_1^* = H_2^* = \xi$ in (5.112), we have

$$\lambda^* = \mu^* + \kappa\sigma - \frac{\omega}{2} - \frac{1}{4}. \tag{5.113}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* = w_i (i = 1, 2, 3, 4)$ in (5.112), and adding these up,

$$\lambda^* = \frac{1}{2}(\mu^* + \kappa\sigma - \kappa\rho - \omega - \frac{1}{2}). \tag{5.114}$$

Finally combining equations (5.113) and (5.114), we get

$$\kappa(\sigma + \rho) = -\mu^*. \tag{5.115}$$

If μ^*= 0, then equation (5.115) is reduces to

$$\sigma + \rho = 0.$$

This completes the proof. □

5.14 Einstein Semi-Symmetric LP-S manifolds (M^4, g) admitting CηES.

In this section, we investigate that the CηES on LP-S manifolds (M^4, g) having Einstein semi-symmetric Ricci tensor.
We now state and prove the following theorem:

Theorem 5.14.1. *An Einstein semi-symmetric LP-S manifold (M^4, g) is an Einstein manifold and represents a dark energy era.*

Proof. The LP-S manifold satisfies the curvature condition $R.E = 0$. From equations (1.39) and (1.40), we get

$$R_t(R(H_1^*, H_2^*)H_3^*, H_4^*) + R_t(H_3^*, R(H_1^*, H_2^*)H_4^*) \\ = \frac{R^*}{4}\left[g(R(H_1^*, H_2^*)H_3^*, H_4^*) + g(H_3^*, R(H_1^*, H_2^*)H_4^*)\right]. \tag{5.116}$$

Putting $H_1^* = H_4^* = \xi$ in the equation (5.116), we obtain

$$R_t(R(\xi, H_2^*)\xi, H_4^*) + R_t(\xi, R(\xi, H_2^*)H_4^*) \\ = \frac{R^*}{4}\left[g(R(\xi, H_2^*)\xi, H_4^*) + g(\xi, R(\xi, H_2^*)H_4^*)\right]. \tag{5.117}$$

Using (5.8), (5.9) in (5.117), we have

$$R_t(H_2^* + \eta(H_2^*)\xi, H_4^*) + R_t(\xi, g(H_2^*, H_4^*)\xi - \eta(H_4^*)H_2^*) \\ = \frac{R^*}{4}\left[g(H_2^* + \eta(H_2^*)\xi, H_4^*) + g(\xi, g(H_2^*, H_4^*)\xi - \eta(H_4^*)H_2^*)\right], \tag{5.118}$$

which implies

$$R_t(H_2^*, H_4^*) = 3g(H_2^*, H_4^*). \tag{5.119}$$

This implies that the manifold is an Einstein manifold.
Using equations (1.28) and (5.119), we obtain

$$3g(H_1^*, H_2^*) = \left(\kappa\rho + \frac{R^*}{2}\right) g(H_1^*, H_2^*) + (\rho + \sigma)\kappa\eta(H_1^*)\eta(H_2^*). \tag{5.120}$$

Setting $H_1^* = H_2^* = \xi$ in (5.120), we have

$$R^* = 2\kappa\sigma + 6. \tag{5.121}$$

Choosing a local orthonormal basis $\{w_i\}_{i=1}^4$ with respect to g, and setting $H_1^* = H_2^* =$

$w_i (i = 1,2,3,4)$ in (5.120), and adding these up,

$$R^* = 6 - \kappa\rho + \kappa\sigma. \tag{5.122}$$

Finally combining equations (5.121) and (5.122), we get

$$\sigma + \rho = 0.$$

This completes the proof. □

5.15 Example of a 4-dimensional LP-S manifold admitting a CηES

From example (5.7), we can calculate equation (5.91)

$$R_t(w_4, w_4) = -\left[\frac{R^*}{2} - \left(\frac{\omega}{2} + \frac{1}{4}\right) - \lambda^*\right].$$

By equating both the values of $R_t(w_4, w_4)$, we have

$$\lambda^* - \mu^* = \frac{R^*}{2} - \frac{\omega}{2} - \frac{13}{4}.$$

Hence the constants λ^* and μ^* satisfies equation (5.86) of theorem 5.12.1 and g defines a CηRS on the 4-dimensional LP-S manifold M^4.

Chapter 6

Conclusions and Future Directions

6.1 Conclusions

This thesis presents a thorough investigation into the geometric structures of various classes of almost contact, para-contact, and contact metric manifolds, with particular emphasis on those endowed with semi-symmetric metric connections. Through systematic and rigorous theoretical analysis, several notable geometric characteristics have been identified across different manifolds, including (k,μ)-contact metric manifolds, Kenmotsu and ε-Kenmotsu manifolds, trans-Sasakian manifolds, and LP-S manifolds. The key contributions of this work can be summarized as follows:

(i) In $(2n+1)$-dimensional (k,μ)-contact metric manifolds admitting an ηE solitons, the scalar curvature is shown to be constant.

(ii) Depending on the curvature conditions, such manifolds may exhibit shrinking, steady, or expanding soliton behaviors.

(iii) (k,μ)-contact metric manifolds admitting an ηE solitons are locally isometric to E^{n+1} for $n > 1$, and are flat when $n = 1$.

(iv) The Ricci tensor in these manifolds takes the ηE form under the influence of TFVF.

(v) Establishing the geometric conditions for ε-Kenmotsu manifolds under CηES. It was shown that such manifolds can represent Einstein spaces under conditions like Codazzi type Ricci tensor, cyclic parallel and Ricci semi-symmetry.

(vi) Trans-Sasakian manifolds admitting $*$-CηRS, $*$-CηES reduce ηE manifold and express specific collinearity between vector fields.

(vii) Investigation into the LP-S manifolds discovered their suitability with dark energy models in the framework of general relativity. These manifolds were found to support CηES and Ricci-recurrent, suggesting interesting physical properties like Minkowski spacetime.

(viii) Construction of concrete examples that illustrate the theoretical results and help to visualize the geometric intuition behind the abstract formulations.

Collectively, these findings contribute to a deeper understanding of the intricate relationships between curvature conditions and soliton structures in different classes of geometric manifolds. They also offer potential avenues for further exploration in both mathematical theory and physical applications, particularly in the context of geometric flows and general relativity.

6.2 Contributions of the Thesis

This thesis offers several original contributions to the field of differential geometry, particularly in the study of contact and para-contact structures in relation to geometric solitons and curvature conditions. The main contributions are as follows:

- A detailed investigation of conformal ηRS on (k,μ) contact metric manifolds has been carried out, including an analysis of their scalar curvature behavior under various geometric constraints.

- New results have been established for ε-Kenmotsu manifolds, especially under conformal transformations and in the presence of specific soliton structures, enriching the understanding of their geometric dynamics.

- The study presents original insights into LP-S manifolds by exploring the interaction between geometric vector fields, curvature tensors, and semi-symmetric metric connections.

- Several structural theorems have been derived, and illustrative examples have been provided to support and validate the theoretical developments.

6.3 Future Directions

- Broaden the scope of soliton analysis to encompass additional classes of manifolds, such as cosymplectic, para-Sasakian, and quasi-Sasakian geometries.

- Investigate the generalization of soliton structures to higher-dimensional settings and assess their theoretical implications.

- Examine potential applications of these solitons in the context of general relativity, with particular emphasis on anisotropic cosmological models and phenomena associated with dark energy.

- Explore Lorentzian counterparts of the studied structures and their significance in the geometric formulation of spacetime.

- Extend the analysis to include alternative geometric flows and soliton types, including but not limited to Yamabe flows and Bach-flat structures.

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A (k,μ)-CONTACT METRIC MANIFOLD AS AN η–EINSTEIN SOLITON

Arup Kumar Mallick* and Arindam Bhattacharyya

Abstract. The aim of the paper is to study an η-Einstein soliton on $(2n+1)$-dimensional (k,μ)-contact metric manifold. At first, we establish various results related to $(2n+1)$-dimensional (k,μ)-contact metric manifold that exhibit an η-Einstein soliton. Next we study some curvature conditions admitting an η-Einstein soliton on $(2n+1)$-dimensional (k,μ)-contact metric manifold. Furthermore, we consider specific conditions associated with an η-Einstein soliton on $(2n+1)$-dimensional (k,μ)-contact metric manifold. Finally, we show the existance of an η-Einstein soliton on (k,μ)-contact metric manifold.

1. Introduction

In 1995, Blair et al. [4] introduced the notion of contact metric manifold with characteristic vector field ξ belonging to the (k,μ) distribution and such type of manifold is called (k,μ)- contact metric manifold. They obtained several results and a full classification of this manifold has been given by Boeckx [8].
A contact metric manifold is known [13] to exist where the curvature tensor R, in the direction of the characteristic vector field ξ, satisfies the equation $R(X,Y)\xi = 0$ for any tangent vector field X, Y. For instance, the tangent sphere bundle of a flat Riemannian manifold possesses such a structure [5]. By applying a D-homothetic deformation [21] on M^{2n+1} with the equation $R(X,Y)\xi = 0$, A novel class of contact metric manifolds that fulfills the condition

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, k, \mu \in R \tag{1}$$

where h represents the Lie differentiation of ϕ in the direction of ξ and R is the curvature tensor. A notable characteristic of this class is that the equation's type remains unchanged under a D-homothetic deformation.
A contact metric manifold that satisfies the aforementioned relation (1) is known as a (k,μ)- contact metric manifold. This class of manifolds encompasses both Sasakian and non-Sasakian manifolds. In the case of Sasakian manifolds, $k = 1$, resulting in $h = 0$. However, for non-Sasakian manifolds, $k < 1$. Examples of such manifolds can be found in all dimensions. Notably, the tangent sphere bundles of Riemannian

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A (k, μ)-CONTACT METRIC MANIFOLD AS A CONFORMAL η-RICCI SOLITON

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Abstract. The aim of the paper is to study conformal η-Ricci soliton on $(2n+1)$-dimensional (k, μ)-contact metric manifold. At first, we establish various results related to $(2n+1)$-dimensional (k, μ)-contact metric manifold that exhibit conformal η-Ricci soliton. Next we study some curvature conditions admitting conformal η-Ricci soliton on $(2n+1)$-dimensional (k, μ)-contact metric manifold. Furthermore, we consider specific conditions associated with conformal η-Ricci soliton on $(2n+1)$-dimensional (k, μ)-contact metric manifold. Besides these geometrical point of view we consider this soliton in a perfect fluid spacetime and obtain some interesting physical properties. Finally, we show the existence of a conformal η-Ricci soliton on (k, μ)-contact metric manifold.

1. INTRODUCTION

In 1995, Blair *et al.* [4] introduced the notion of contact metric manifold with characteristic vector field ξ belonging to the (k, μ) distribution and such type of manifold is called (k, μ)-contact metric manifold. They obtained several results and a full classification of this manifold has been given by Boeckx [8].

A contact metric manifold is known [12] to exist where the curvature tensor R, in the direction of the characteristic vector field ξ, satisfies the equation $R(X, Y)\xi = 0$, for any tangent vector field X, Y. For instance, the tangent sphere bundle of a flat Riemannian manifold possesses such a structure [5]. By applying a D-homothetic deformation [25] on M^{2n+1} with the equation $R(X, Y)\xi = 0$, a novel class of contact metric manifolds that fulfills the condition

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \quad k, \mu \in R, \tag{1.1}$$

where h represents the Lie differentiation of ϕ in the direction of ξ and R is the curvature tensor. A notable characteristic of this class is that the equation's type remains unchanged under a D-homothetic deformation.

A contact metric manifold that satisfies the aforementioned relation (1.1) is known as a (k, μ)-contact metric manifold. This class of manifolds encompasses both Sasakian and non-Sasakian manifolds. In the case of Sasakian manifolds, $k = 1$, resulting in $h = 0$. However, for non-Sasakian manifolds, $k < 1$. Examples of such manifolds can be found in all dimensions. Notably, the tangent sphere bundles of Riemannian manifolds with constant sectional curvature c, excluding $c = 1$, serve as characteristic examples of non-Sasakian (k, μ)-contact metric manifolds. Particularly in the 3-dimensional case, this class includes

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Article

A Conformal η-Ricci Soliton on a Four-Dimensional Lorentzian Para-Sasakian Manifold

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Abstract: This paper focuses on some geometrical and physical properties of a conformal η-Ricci soliton (Cη-RS) on a four-dimension Lorentzian Para-Sasakian (LP-S) manifold. The first section presents an introduction to Cη-RS on LP-S manifolds, followed by a discussion of preliminary ideas about the LP-Sasakian manifold. In the subsequent sections, we establish several results pertaining to four-dimension LP-S manifolds that exhibit Cη-RS. Additionally, we consider certain conditions associated with Cη-RS on four-dimension LP-S manifolds. Besides these geometrical points of view, we consider this soliton in a perfect fluid spacetime and obtain some interesting physical properties. Finally, we present a case study of a Cη-RS on a four-dimension LP-S manifold.

Keywords: LP-S manifold; conformal η-Ricci soliton; Ricci flow; perfect fluid spacetime

MSC: 53C15; 53C25

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1. Introduction

In 1976, Sato [1] introduced a structure of smooth manifolds that has since gained recognition as an almost paracontact structure. This structure is analogous to the almost contact structure [2,3] and resembles the almost contact product structure. The difference lies in the fact that almost paracontact manifolds can be both even-dimensional and odd-dimensional, unlike almost contact manifolds, which are always odd-dimensional. Takahashi [4] researched almost contact manifolds equipped with corresponding semi-Riemannian metrics. His work specifically focused on Sasakian manifolds endowed with an associated semi-Riemannian metric in 1969. The concept of an LP-S manifold [5] was first introduced by Matsumoto [6] in 1989. Subsequently, Mihai and Rosca [7] independently worked on the same area and deduced various outcomes in this type of manifold. Furthermore, LP-S manifolds have been investigated by Matsumoto and Mihai [8], as well as De et al. [9–11]. Hamilton [12,13] introduced the concept of the Ricci flow as a means of determining a canonical metric on a smooth manifold in 1982.

The Ricci flow [12] is an evolution equation that pertains to the Riemannian metric $g(t)$ on M^d, and it is defined by

$$\frac{\partial g}{\partial t} = -2R_t,$$

where R_t is the Ricci tensor. We refer to a said manifold M^d endowed with a Riemannian metric g as a Ricci soliton [13,14] if there exists a constant λ^* and a smooth vector field W on M^d fulfilling the equation

$$(L_W g) + 2R_t = 2\lambda^* g,$$

where L_W, noted as the Lie derivative along the direction of the vector field W. The Ricci flow exhibits steady, shrinking, and expanding behaviour, depending on $\lambda^* = 0$, $\lambda^* > 0$,

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A Study of Kenmotsu Manifolds with Semi-Symmetric Metric Connection

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Abstract: The present paper aims to study semi-symmetric metric connection on Kenmotsu Manifolds. First section introduces us with the development of Kenmotsu manifolds. Next section gives us some preliminary ideas about the manifold. Here we have studied the necessary condition under which a vector field will be a strict-contact vector field. In the next section we have extended our study to generalized ϕ-recurrent $n = 2m + 1$-dimensional Kenmotsu manifold with respect to semi-symmetric metric connection. Further we have studied this manifold satisfying the condition $L.S = 0$ w.r.t semi-symmetric connection. Lastly we have cited an example of Kenmotsu manifold with semi-symmetric metric connection.

Key Words: Kenmotsu manifolds, semi-symmetric metric connection, conharmonically curvature tensor, extended generalized $\phi-$recurrent Kenmotsu manifolds.

AMS(2010): 53C15, 53C25, 53C40.

§1. Introduction

In [24], S.Tanno classified the connected almost contact metric manifold whose automorphism group has maximum dimension, which are three classes following:

a) the homogeneous normal contact Riemannian manifolds with constant $\phi-$ holomorphic sectional curvature if the sectional curvature of the plain section containing ξ, say $C(X, \xi) > 0$.

b) the global Riemanian product of a line or a circle and a Käehlerian manifold with constant holomorphic sectional curvature, $C(X, \xi) = 0$.

c) a warped product space $RX_\lambda C^n$, if $C(X, \xi) < 0$.

The manifold of class (a) are characterized by some tensor equations, it has a Sasakian structure and manifolds of class (b) are characterized by a tensorial relation admitting a cosymplectic structure. In 1972 Kenmotsu has introduced a new class of almost contact Riemannian manifolds which are nowadays called Kenmotsu manifolds [11]. He obtained some tensorial equations to characterize manifolds of class (c).

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SOME RESULTS ON LP-SASAKIAN MANIFOLDS WITH SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The object of the present paper is to study semi-symmetric metric connection on Lorentzian Para-Sasakian Manifolds. First section deals with the extensive history about introduction of LP-Sasakian manifolds. Preliminary ideas about the manifold is given in the next section which are indispensable for our derivations. In the succeeding section we present a brief survey of the necessary condition under which the Contact vector field on a Lorentzian para-Sasakian manifolds leaving the Ricci tensor with respect to semi-symmetric connection is a strict Contact vector field. In the following section we have extended our study for establishing a condition under which generalized ϕ-recurrent $n = 2m + 1$-dimensional LP-Sasakian manifold with respect to semi-symmetric metric connection will be an Einstein manifold. Further we have considered a Lorentzian para-Sasakian manifolds with respect to semi-symmetric metric connection admitting a Conharmonic Curvature tensor and a non-zero Ricci tensor satisfying $\overline{L}(X,Y)\overline{S} = 0$, and we have derived that the modulus of non-zero eigen values of the endomorphism $\overline{Q}$ of the tangent space corresponding to $\overline{S}$ is $2(n-1)$. Lastly we have cited an example of LP-Sasakian manifold with semi-symmetric metric connection.

Keywords: LP-Sasakian manifolds, semi-symmetric metric connection, Conharmonically curvature tensor, Extended generalized $\phi-$recurrent Kenmotsu manifolds.

Subject classification [2010]: 53B50, 53C15, 53C25.

1. Preliminaries. The notion of Lorentzian almost para-contact manifolds was introduced by K. Matsumoto (1989). Later on, a large number of geometers studied Lorentzian almost para-contact manifold and their different classes, viz., Lorentzian para-Sasakian manifolds and Lorentzian special para-Sasakian manifolds (Matsumoto and I. Mihai, 1988, Tarafdar and Bhattacharyya, 2003, Mihai and Rosca, 1992 and Pokhariyal, 1996). In brief, Lorentzian para-Sasakian manifolds are called LP-Sasakian manifolds. The study of LP-Sasakian manifolds has vast applications in the theory of relativity.