SOME PROBLEMS ASSOCIATED WITH GRAPHS RELATED TO ALGEBRAIC STRUCTURES

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled **“Some problems associated with graphs related to algebraic structure”** submitted by **Sri Soumen Pradhan** who got his name registered on **22nd March, 2021 (Index No.: 40/21/Maths./27)** for the award of **PhD** (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Sukhendu Kar** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

02/6/2025
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Dedicated to
my beloved parents
Sri Santosh Pradhan
and
Smt. Jyotsna Pradhan

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Abstract

Algebraic graph theory is an intriguing area of mathematics that investigates the connections between graphs and matrices. Algebraic graph theory affords efficient techniques for analyzing and understanding many different structures, encompassing social and computer networks. In this thesis, we constructed new graphs over groups, rings, and other algebraic structures and explored their features, particularly different topological indices of graph-theoretic and algebraic properties. The thesis consists of 7 chapters.

In **Chapter 1**, we discuss the historical backdrop and objective of this effort, as well as the theoretical preliminaries and some important results from the literature reviewed.

In **Chapter 2**, we define a new algebraic graph by adding new condition on square element graph, $\mathbb{S}_q(G)$ over a group G, which is called the order two element graph over a group G. We denote this graph by $\mathbb{S}_2(G)$. In this chapter, we characterize those commutative groups G for which the graph $\mathbb{S}_2(G)$ is connected. We also characterize the structures of the graph $\mathbb{S}_2(\mathbb{Z}_n)$ for any positive integer n. Moreover we look into the structure and other graph-theoretic properties of the symmetric group S_n, the dihedral group D_n for any positive integer n. Finally, we give a list of finite groups of order ≤ 16 with their corresponding order two element graph structures.

In **Chapter 3**, we explore the Merrifield-Simmons index, independence number, and domination number of the order two element graph over a group. We start by establishing some lovely results on the Merrifield-Simmons index of the order two element graph over a group. The product of two order two element graphs over two

groups is then described. Additionally, we examine some significant findings on the domination number of order two element graph over a group.

In **Chapter 4**, we discuss a number of distance-based topological indices over the order two element graph of a group, including the Wiener index, Hyper-Wiener index, Harary index, and Gutman index. We ascertain the precise formula for the Wiener index, Hyper-Wiener index, Harary index, and Gutman index of the order two element graph over the finite groups $\mathbb{Z}_n$ and D_n.

In **Chapter 5**, we explore the co-maximal graph over a commutative ring R with identity which is denoted by $\Gamma'(R)$. In this chapter, we characterize Merrifield-Simmons Index of a co-maximal graph over a commutative ring. Furthermore, we characterised the Merrifield-Simmons index of co-maximal graph over the ring $\mathbb{Z}_n$.

In **Chapter 6**, we explore the zero-divisor graph over a commutative ring R. This graph is denoted by $\Gamma(R)$. In this chapter, we compute the Merrifield-Simmons index of the zero-divisor graph over the ring $\mathbb{Z}_n$ (the ring of integer modulo n), for $n = p^m, pq, p^2q, p^2q^2, pqr$, where p, q, r are distinct prime numbers. Furthermore, we derive the Merrifield-Simmons index for the zero-divisor graph $\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >)$. Additionally, we compute the Merrifield-Simmons index of zero divisor graphs of some small finite commutative rings.

Finally in **Chapter-7**, we focus on two distance based topological indices, the sum-connectivity Gourava inex and the product-connectivity Gourava index of a zero-divisor graph over a commutative ring. We compute the sum-connectivity Gourava index and the product-connectivity Gourava index of the zero-divisor graph over the ring $\mathbb{Z}_n$, for $n = p^m, pq, p^2q, p^2q^2, pqr$, where p, q are distinct prime numbers. Furthermore, we compute the sum and product connectivity Gourava index for the zero-divisor graph $\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >)$. In the final section, we compute the sum and product connectivity Gourava index of zero divisor graphs from several small finite commutative rings.

Contents

List of Symbols

$\mathbb{N}$	Set of all natural numbers
$\mathbb{N}_0$	Set of all non-negative integers
$\mathbb{Z}$	Set of all integers
$\mathbb{Q}$	Set of all rationals
$\mathbb{R}$	Set of all real numbers
$\mathbb{Z}_n$	Congruent classes of integers modulo n
D_n	Dihedral group of order $2n$
S_n	Symmetric group of order $n!$
$SU(2)$	Group of the Pauli Matrices
M_{16}	Modular or Isanowa group of order 16
Dic_4	Dicyclic group of degree 4
SD_2	Semidihedral group of degree 2
G/H	The quotient group of G by H
$U(R)$	The set of all unit elements of the ring R
$\mathbb{F}_n$	Finite field having exactly n elements
$\|A\|$	Cardinality of a set A
$A \cup B$	Union of the sets A and B
$A \cap B$	Intersection of the sets A and B
$A \setminus B$	All elements which are in set A but not in set B
$A \triangle B$	Symmetric difference between the sets A and B
$\phi(n)$	Euler's phi function
$P(S)$	The power set of the set S

$Z(R)$	Set of zero-divisors of a ring R
$J(R)$	Jacobson radical of a ring R
$U(R)$	Set of units of a ring R
$r(R)$	Set of regular elements of a ring R
$Char(R)$	Characteristics of a ring R
S_G	Set of square elements of a group G
G_2	Set of order two elements of a group G
$gcd(a,b)$	Greatest common divisor of integers a, b
$a \leftrightarrow b$	Vertices a, b are adjacent
$a \leftarrow (P) \rightarrow b$	P is a path between vertices a and b
$d(v)$	Degree of a vertex v
$diam(\mathcal{G})$	Diameter of a graph $\mathcal{G}$
$girth(\mathcal{G})$	Girth of a graph $\mathcal{G}$
$d(a,b)$	Length of the shortest path between vertices a, b
$\overline{\mathcal{G}}$	Complement of a graph $\mathcal{G}$
$k\mathcal{G}$	k copies of a graph $\mathcal{G}$
K_n	Complete graph of n vertices
$K_{m,n}$	Complete bipartite graph where partite sets have sizes m and n
$\mathbb{S}_q(G)$	The Square element graph over a group G
$\mathbb{S}_2(G)$	The order two element graph over a group G
$\Gamma'(R)$	Co-maximal graph over a ring R
$\Gamma'_1(R)$	The subgraph of $\Gamma'(R)$ whose vertices are the units of R
$\Gamma'_2(R)$	The subgraph of $\Gamma'(R)$ whose vertices are the non-units of R
$\Gamma(R)$	The zero divisor graph over a ring R
$Z(R)$	The set of zero-divisors of the ring R
$Z^*(R)$	The set of non-zero zero-divisors of the ring R
P_n	Path with exactly n vertices
C_n	Cycle with exactly n vertices

$dom(\mathcal{G})$	Domination number of a graph $\mathcal{G}$
$\alpha(\mathcal{G})$ *or* $u(\mathcal{G})$	The independence number of a graph $\mathcal{G}$
$i(\mathcal{G})$	Merrifield-Simmons index of a graph $\mathcal{G}$
$\mathcal{G} + H$	Disjoint union of two graphs $\mathcal{G}, H$
$\omega(\mathcal{G})$	Clique number of a graph $\mathcal{G}$
$N(v)$	The set of neighbours of a vertex v in a graph $\mathcal{G}$
$\mathcal{G}_1 \vee \mathcal{G}_2$	The join of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$
$\mathcal{G}_1 \square \mathcal{G}_2$	The Cartesian product of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$
$\mathcal{G}_1 \times \mathcal{G}_2$	The direct product of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$
$\mathcal{G}_1 \boxtimes \mathcal{G}_2$	The strong product of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$
$W(\mathcal{G})$	The Wiener index of the graph $\mathcal{G}$
$WW(\mathcal{G})$	The Hyper-Wiener index of the graph $\mathcal{G}$
$H(\mathcal{G})$	The Harary index of the graph $\mathcal{G}$
$Gut(\mathcal{G})$	The Gutman index of the graph $\mathcal{G}$
$M_1(\mathcal{G})$	The first Zgreb index of the graph $\mathcal{G}$
$M_2(\mathcal{G})$	The second Zagreb index of the graph $\mathcal{G}$
$GO_1(\mathcal{G})$	The first Gourava index of the graph $\mathcal{G}$
$GO_2(\mathcal{G})$	The first Gourava index of the graph $\mathcal{G}$
$S(\mathcal{G})$	The sum connectivity index of the graph $\mathcal{G}$
$SGO(\mathcal{G})$	The sum connectivity Gourava index of the graph $\mathcal{G}$
$\chi(\mathcal{G})$	The Randic connectivity index of the graph $\mathcal{G}$
$PGO(\mathcal{G})$	The product connectivity Gourava index of the graph $\mathcal{G}$

Chapter 1

Introduction and Preliminaries

Chapter 1

Introduction, preliminaries and prerequisites

1.1 Introduction

A city originally known as Konigsberg, situated beside the Pregolya River in Prussia. Seven bridges were built across the river in 1700. The river divided the city into four land masses, including the island of Kneiphopf. These four regions were connected by seven bridges. These seven bridges are known as Blacksmith's Bridge, Connecting Bridge, Green Bridge, Merchant's Bridge, Wooden Bridge, High Bridge, and Honey Bridge [1]. The city's residents enjoyed strolling across these bridges, but no matter how hard they tried, no one could walk a path that crossed each bridge exactly once. Several mathematical studies were done on this problem. Then a Swiss mathematician Leonhard Euler thought about this problem, and on 26th August 1735, Euler presented a mathematical demonstration of his argument to a member of the Petersburg Academy. Also, he published the result in the form of a paper in 1736 under the title 'Solution Problematis ad Geometriam Situs Pertinentis' (which translates to 'The solution to a problem relating to the geometry of position') [1]. The method he used to solve this problem is considered to be the birth of graph theory.

Initially, Euler's concepts and 'graph theory' were primarily useful for solving puzzles and evaluating games and other forms of entertainment. However, by the mid-1800s, people realized that graphs could be used to model a wide range of societal phenomena. For example, the ‘Four Color Map Conjecture’, proposed by DeMorgan in 1852, was a well-known problem that appeared unrelated to graph theory. According to the conjecture, four is the maximum number of colors required to color any map with different-colored bordering sections. This conjecture is simply expressed in terms of graph theory, and many scholars employed this approach over the dozen decades that the problem remained unanswered. It is worth noting that 200 years after Euler’s paper on Konigsber’s bridge problem was published, a mathematician named Konig became the first to write a book on graph theory, titled ’Theorie der endlichen und unendlichen Graphen’. The term ‘graph’ was invented by J.J. Sylvester in 1878.

Graph theory has been the most helpful topic since 1878 and remains so today. Graph theory has applications in a wide range of disciplines, including mathematics, computer science, engineering, biology, physics, social sciences, operations research, linguistics, chemistry, geography, neuroscience, and cybersecurity. Mathematics has several subfields related to graph theory, including geometric graph theory, extremal graph theory, probabilistic graph theory, combinatorial graph theory, topological graph theory, and algebraic graph theory. Subfields of computer science include algorithms and data structures, network design and analysis, cryptography, machine learning and artificial intelligence, database management, and so on. Engineering subfields include electrical and electronics engineering (circuit analysis, network theory), transportation engineering (route optimization), and software engineering (dependency graph). Biology has several subfields: genetics (gene networks), systems biology (protein interaction networks), and ecology (food webs and ecosystem modeling). The subfields of physics include statistical mechanics, quantum computing, and network physics. Sociology (social network analysis), economics (market graphs), and political science (voting and influence networks) are three subfields of social sciences. Operations research subfields include optimization problems (such as the traveling salesman problem and the minimum spanning trees) and supply chain management.

Linguistic subfields include syntax trees and semantic networks. Chemistry's subfields include molecular graphs (structure representation) and computational chemistry. Geography has three subfields: geographic information system (GIS), urban planning, and logistics. Neural networks are a branch of neuroscience that studies brain connection graphs. The subfields of cybersecurity involve network security, malware analysis, and intrusion detection.

It is important to understand that graph theory has a natural link to algebra. Using matrices to represent graphs, such as adjacency matrices or incidence matrices, connects linear algebra to graph theory. However, investigations on a graph's automorphism group require abstract algebra (specifically group theory). The study of algebraic graph invariants, including the chromatic polynomial, is a crucial aspect of algebraic graph theory. This field applies algebraic approaches to solve many graph theoretical problems.

Abstract algebra has been most precisely connected to graph theory since it defines multiple graphs over different algebraic structures. In this situation, select an appropriate algebraic structure and define the adjacency condition for the graph by choosing a subset of the algebraic structure as a vertex set and applying an algebraic operation. Following that, we investigated this graph's graph-theoretic properties. On the other hand, an appropriate graph operation can be used to create an algebraic structure of a family of graphs. Thus, graph-theoretic methods are used to study algebraic structures. The potential to examine this interaction has prompted mathematicians to design various graphs over different algebraic structures, and it has now become a very active topic of study.

Graphs have been defined throughout various fields of algebraic graph theory. There are three major branches of algebraic graph theory. Using linear algebra is the first branch of algebraic graph theory, which studies graphs in relation to linear algebra. This branch focuses on the spectrum of the adjacency matrix, or Laplacian matrix of a graph. This part is also known as spectral graph theory. Using group theory is the second branch of algebraic graph theory, which studies graphs in relation to semigroups, posets, groups, automorphism groups, and geometric group theory. Beyond from these, this branch studies graphs in connection to semirings, rings, and other structures. The third branch of algebraic graph theory is the study of graph invariants, which focuses on the algebraic features of graph invariants such as knot invariants and chromatic polynomials.

In 1874, Alexander Crum Brown, a pioneer in chemical structure theory, predicted that chemistry will become a part of applied mathematics while remaining an experimental science. Mathematics may assist to rationalize experimental results, identify study areas, and predict new discoveries. A topological graph index, also known as a molecular descriptor, is a mathematical formula that may be applied to any graph representing a molecular structure. This index enables you to analyse mathematical values and study some physicochemical properties of a molecule. As a result, it provides a cost-effective and time-saving alternative to laboratory experiments. Topological indices can be classified in many ways, the most useful of which represent the particular elements of graph structure to their total value. There are various types of indices, including degree-based, distance-based, additive, multiplicative, spectral, and many others. Harold Wiener introduced the first topological index in 1947.

Cayley introduced the concept of Cayley graphs defined over groups in 1878, when he presented a graphic representation of groups. In terms of ring-defined graphs, the beginning point appears to be Beck's introduction of zero-divisor graphs in 1988. Beck's definition of zero-divisor graphs was later changed, and several researchers have been studying them extensively. Zero-divisor graphs have not only been generalized to non-commutative rings, semirings, posets, and other structures, but they have also inspired the development of new graph types such as total graphs, unit graphs, and co-maximal graphs.

In this thesis, we define some new graphs over groups and investigate their many characteristics, particularly the interplay of graph-theoretic and algebraic properties. Furthermore, we explored the graph-theoretic and algebraic properties of particular predefined graphs in several algebraic structures using various topological indices.

1.2 Some preliminaries on algebraic structure

This section contains preliminary definitions and notions of some algebraic structures that are relevant to this thesis.

Let G be a non-empty set. A *binary operation* on G is a function from $G \times G$ into G. A *group* is an algebraic structure on the order pair $(G, *)$, where $*$ is a binary operation from $G \times G$ into G such that the following axioms hold:
i) $*$ is associative on G.
ii) There exists an element e in G such that for all $a \in G$, $a * e = a = e * a$ (existence of identity).
iii) For all $a \in G$, there exists $b \in G$ such that $a * b = b * a$ (existence of an inverse). The elements a and b are called inverses of each other. The inverse of an element is denoted by a^{-1}. In this thesis the group $(G, *)$ treated as G.

A group $(G, *)$ is called a *finite group* if G has only finite number of elements, otherwise it is called an *infinite group.* The group $(G, *)$ is called *commutative or abelian group* if $*$ is commutative on G, i.e. $a * b = b * a$ for all $a, b \in G$. The group $(G, *)$ is called *non-commutative group* if $*$ is not commutative on G. For any element $a \in G$ and $n \in \mathbb{N}$, we define a^n as $\underbrace{a * a * \cdots * a}_{\text{n-times}}$. For a negative integer n, $a^n = (a^{-n})^{-1}$. For any element a in a group G, define the *order* of a to be the smallest positive integer n such that $a^n = e$, and denote this integer by $\circ(a)$. In this case a is said to be of order n. If no such positive power of a exist such that $a^n = e$, the order of a is defined to be infinity and a is said to be of *infinite order.*

Let G be a group. We define S_G as a *square element set* of G by $S_G = \{x^2 : x \in G\}$. Then $S_G \subseteq G$. Let $G_2 \subseteq G$ and we define G_2 as an *order two element set* of G by $G_2 = \{x \in G : x^2 = e\}$. Let $(G, *)$ be a group and H be a nonempty subset of G. Then $(H, *)$ is called a *subgroup* of $(G, *)$ if $(H, *)$ is a group under the binary operation $*$ restricted to H. A group G is *cyclic* if G can be generated by a single element, i.e., there is some element $x \in G$ such that $G = \{x^n : n \in \mathbb{Z}\}$ (where the binary operation is multiplicative). In additive notation G is cyclic if $G = \{nx : n \in \mathbb{Z}\}$. In both cases, we write $G =< x >$ and say G is generated by x (and x is a generator of G). Also, for any non-empty subset S of a group G, $< S >$ denotes the subgroup of G generated by S, which is defined to be the intersection of all subgroups of G containing S.

Let H be a subgroup of a group G and $a \in G$. The sets $aH = \{ah : h \in H\}$ and $Ha = \{ha : h \in H\}$ are called *the left coset* and right coset of H in G, respectively. Any element of a coset is called a representative for the coset. Let G be a group. A subgroup H of G is said to be a *normal subgroup* of G if $aH = Ha$ for all $a \in G$. Let H be a normal subgroup of a group G. Denote the set of all left cosets $\{aH : a \in G\}$ by G/H and define an operation $*$ on G/H by for all $aH, bH \in G/H$, $(aH)*(bH) = abH$. Then $(G/H, *)$ forms a group under the operation $*$, which is called *quotient group or factor group.*

Let $(G_1, *_1)$ and $(G_2, *_2)$ be groups and f a function from G_1 into G_2. Then f is called a *group homomorphism* of G_1 into G_2 if for all $a, b \in G_1$, $f(a *_1 b) = f(a) *_2 f(b)$. A group homomorphism f of a group G_1 into a group G_2 is called an *isomorphism* of G_1 onto G_2 if f is one-one and onto. We write $G_1 \simeq G_2$ and say that G_1 and G_2 are isomorphic. An isomorphism of a group G onto G is called a *group automorphism.* For a group G, $Aut(G)$, denotes the set of all automorphisms of G. In fact, $(Aut(G), \circ)$ forms a group, where $\circ$ is the composition of mappings. The *direct product* $G_1 \times G_2 \times \cdots \times G_n$ of groups $G_1, G_2, \cdots, G_n$ with operations $*_1, *_2, \cdots, *_n$, respectively, is the set of *n-tuples* $(g_1, g_2, \cdots, g_n)$ where $g_i \in G_i$ with operation defined component-wise: $(g_1, g_2, \cdots, g_n) * (h_1, h_2, \cdots, h_n) = (g_1 *_1 h_1, g_2 *_2 h_2, \cdots, g_n *_n h_n)$. Let H and K be groups and let φ be a group homomorphism from K into $Aut(H)$. Let $\bullet$ denote the action of K on H determined by φ. Let G be the set of ordered pairs (h, k) with $h \in H$ and $k \in K$ and define the following multiplication on G: $(h_1, k_1)(h_2, k_2) = (h_1 k_1 \bullet h_2, k_1 k_2)$. Then G forms a group under the above operation and it is called the *semidirect product* of H and K with respect to φ and it is denoted by $H \rtimes_\varphi K$ or simply $H \rtimes K$.

A *ring* is an ordered triple $(R, +, .)$ such that R is a non-empty set and $+$ and $.$ are two binary operations on R satisfying the following axioms:

i) $(R,+)$ is an abelian group.

ii) $(R,.)$ is a semigroup.

iii) the distributive laws hold in R: for all $a,b \in R$, $(a+b).c = (a.c)+(b.c)$ and $a.(b+c) = (a.b)+(a.c)$.

The zero element of the ring $(R,+,.)$ is the additive identity element of the additive group $(R,+)$, which is denoted by 0. The additive inverse of an element $a \in R$ is denoted by $-a$. Also, for any $a,b \in R$, $a+(-b)$ is written as $a-b$. A ring R is commutative if multiplication is commutative. A ring R is called a *finite ring* if the number of elements in R is finite; otherwise R is called an *infinite ring.* If the multiplicative identity element exists in $(R,.)$, then the ring $(R,+,.)$ is called *ring with unity or identity* and the multiplicative identity element is denoted by 1.

A non-zero element a of a ring R is called a *zero divisor* if there is a non-zero element b in R such that either $ab=0$ or $ba=0$. Let R be a ring with identity $1 \neq 0$. An element u of R is called a *unit* in R if there is some v in R such that $uv = vu = 1$. A ring R with unity 1 is called a *division ring* if every non-zero element of R is a unit. A *field* is a commutative division ring. A commutative ring with unity is called an *integral domain* if it has no zero-divisors. If there exists a positive integer n such that for all $a \in R$, $na = 0$, then the smallest such positive integer is called the *characteristic* of R. If no such positive integer exists, then R is said to be of characteristic zero. An element a in a ring R is called *idempotent* if $a^2 = a$ and *nilpotent* if $a^n = 0$ for some positive integer n.

Let R and R' be rings. Define $+$ and $.$ on $R \times R'$ by for all $(a,b),(c,d) \in R \times R'$, $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b).(c,d) = (a.c, b.d)$. Then $(R \times R', +, .)$ forms a ring, which is called the *direct sum* of R and R', and is denoted by $R \oplus R'$. Let R be a commutative ring with identity. The set of formal symbols $R[x] = \{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 : a_i \in R, n$ is a non-negative integer$\}$ is called the *ring of polynomials* over R in the indeterminate x.

Let $(R,+,.)$ be a ring. Let S be a non-empty subset of R. Then $(S,+,.)$ is called a *subring* of $(R,+,.)$ if $(S,+)$ is a subgroup of $(R,+)$ and for all $x,y \in S$, $x.y \in S$. A non-empty subset I of a ring R is called a *left ideal* of R if for all $a, b \in I$ and for all $r \in R$, $a - b \in I, ra \in I$. Similarly, a non-empty subset I of a ring R is called a *right ideal* of R if for all $a, b \in I$ and for all $r \in R$, $a - b \in I, ar \in I$. A non-empty subset I of a ring R is called a *(two-sided) ideal* of R if I is both a left and a right ideal of R. An ideal I of a ring R is called a *proper* ideal if $I \neq R$. For a non-empty subset S of a ring R, $< S >$ denotes the ideal generated by S, which is defined to be the intersection of ideals of R containing S. If R is a ring and I is an ideal of R, then the ring $(R/I, +, .)$, where $R/I = \{a + I : a \in R\}$ is called the *quotient ring* of R by I.

Let $(R, +, .)$ and $(R', +', .')$ be rings and f a function from R into R'. Then f is called a *ring homomorphism* of R into R' if $f(a + b) = f(a) +' f(b)$ and $f(a.b) = f(a).'f(b)$ for all $a, b \in R$. A ring homomorphism f of a ring R into a ring R' is called a *monomorphism* if f is one-one; an *epimorphism* if f is onto R', and is an *isomorphism* if f is one-one and onto. An isomorphism of a ring R onto R is called an *automorphism*. Two rings R and R' are said to be *isomorphic* if there exists an isomorphism of R onto R'. A ring R is said to be embedded in a ring R' if there exists a monomorphism of R into R'.

Let R be an integral domain. Suppose $r \in R$ is nonzero and is not a unit. Then r is called *irreducible* in R if whenever $r = ab$ with $a, b \in R$, at least one of a or b must be a unit in R. Otherwise r is said to be *reducible*. A nonzero non-unit element $p \in R$ is called *prime* in R if p divides ab for any $a, b \in R$, then either p divides a or p divides b. Two elements a and b of R differing by a unit are said to be *associate* in R (i.e., $a = ub$ for some unit u in R). Let R be a ring and M be an ideal of R. Then M is called a *maximal ideal* of R if $M \neq R$ and there does not exist any ideal I of R such that $M \subset I \subset R$. The *Jacobson radical* of a ring R, denoted by $J(R)$, is the set $J(R) = \cap\{M : M \text{ is a maximal ideal of } R\}$. A commutative ring with identity that has a unique maximal ideal is called a *local ring*.

A field containing exactly n number of elements is denoted by $\mathbb{F}_n$. Let R be an integral domain. A field F is called a *quotient field* of R if there exists a subring R_1 of F such that R is isomorphic to R_1 and for all $x \in F$, there exists $a, b \in R_1$ with

$b \neq 0$ such that $x = ab^{-1}$.

For other definitions and results on groups and rings, one may look at [18], [19], [23], [31], [36] respectively.

1.3 Graph-theoretic preliminaries

This section covers the fundamentals of graph theory, including definitions and terminologies which are required for this thesis.

A *linear graph (or simply a graph)* $\mathcal{G}$ is a triple (V, E, f), where $V = V(\mathcal{G}) = \{v_1, v_2, \cdots\}$ consists of a set of objects (non-empty) called the *set of vertices or vertex set*, and $E = E(\mathcal{G}) = \{e_1, e_2, \cdots\}$ is a set (may be empty) called the *set of edges or edge set*, and f is a function called an *incidence function* that assigns to each edge $e_k \in E$, a 1-element subset $\{v_i\}$ of Vor a 2-element subset $\{v_i, v_j\}$ of V. The vertices v_i, v_j associated with edge e_k are called the *end vertices* of e_k. The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices. Often this diagram itself is referred to as the graph.

A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges. If u and v are two end vertices of an edge e, the vertices u and v are said to be *adjacent* and we write $e = uv$ (or $e = vu$). The notation $u \leftrightarrow v$ denotes that the vertices u, v are adjacent to each other. We specify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices for an edge e with endpoints u and v. The *null graph* is the graph without any edges.

If the vertex and edge sets of a graph are both finite, then the graph is called a *finite graph.* It is possible to visually represent a finite graph graphically. Each vertex is represented by a dot in the plane in the standard pictorial representation, and a line is drawn to connect the dots that correspond to any two neighboring vertices. It is important to remember that a diagram is merely a representation of a graph, even though the diagram of a graph is identified as the graph itself. Although there are other ways to represent graphs, such as adjacency lists, incidence matrices, adjacency matrices, etc., the visual representation is particularly significant since it makes it possible to see the various characteristics of the graph.

Two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are *isomorphic* if there is a one-one correspondence between the vertices of $\mathcal{G}_1$ and those of $\mathcal{G}_2$ such that the number of edges joining any two vertices of $\mathcal{G}_1$ is equal to the number of edges joining the corresponding vertices of $\mathcal{G}_2$. A bijection $f : V \to V$ of a graph $\mathcal{G} = (V, E)$ to itself such that for any two vertices $u, v \in V$, $(u, v) \in E \iff (f(u), f(v)) \in E$, is called a *graph automorphism.*

The *degree* of a vertex v of a graph $\mathcal{G}$ is the number of edges incident with v, and is written as $deg(v)$ or simply $d(v)$; in calculating the degree of v, we usually make the convention that a loop at v contributes 2 (rather than 1) to the degree of v. A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 is called an *end-vertex* or *pendant vertex.* A graph in which all vertices are of equal degree is called a *regular graph* (or simply a *regular*). A graph is called *k-regular* if the degree of all vertices are of degree k. The *complement graph* $\overline{\mathcal{G}}$ of a simple graph $\mathcal{G}$ is the simple graph with vertex set $V(\mathcal{G})$ defined by $uv \in E(\overline{\mathcal{G}})$ if and only if $uv \notin E(\mathcal{G})$.

A graph $\mathcal{H}$ is said to be a *subgraph* of a graph $\mathcal{G}$ if all the vertices and all the edges of $\mathcal{H}$ are in $\mathcal{G}$, and each edge of $\mathcal{H}$ has the same end vertices in $\mathcal{H}$ as in $\mathcal{G}$. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is called an *induced subgraph* of $\mathcal{G}$ if and only if $E(\mathcal{H})$ consists of all edges of $\mathcal{G}$ whose endpoints belog to $V(\mathcal{H})$. The *neighbourhood* of a vertex u in a graph $\mathcal{G}$ is the subgraph of $\mathcal{G}$ induced by all vertices adjacent to u, i.e., the graph composed of the vertices adjacent to u and all edges connecting vertices adjacent to u, and is denoted by $N(u)$. For any vertex u in a graph $\mathcal{G}$, $\mathcal{G} - u$ represents the graph obtained by removing vertex u from the original graph $\mathcal{G}$. Essentially, it is the graph that results when we delete vertex u from all its connections within the graph $\mathcal{G}$. Similarly, $\mathcal{G} - uv$ represents the graph obtained by removing the edge between vertices u and v from $\mathcal{G}$. Also, $\mathcal{G} - N(u)$ represents the graph obtained by removing from graph $\mathcal{G}$ the entire neighbourhood of vertex u i.e., all vertices that are directly connected to u, along with the edges connecting them to u.

A *walk* is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. No edge appears more than once in a walk. A vertex, however, may appear more than once. A walk is called a *closed walk* if its begin and end vertices are same. A walk that is not closed is called an *open walk*. An open walk in which no vertices appears more than once is called a *path*. The number of edges in a path is called the *length of a path*. A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a *circuit*. That is, a circuit is a closed, non-intersecting walk. A *trail* is a walk if all its edges are distinct. A *cycle* is a closed path and a circuit is a closed trail. A graph having no cycle is called *acyclic*. The length of the shortest cycle in a graph $\mathcal{G}$ is called the *girth* of the graph, denoted by $girth(\mathcal{G})$.

A *Euler path* is a path that uses every edge of a graph exactly once. A graph that consists of an Euler path is called an *Euler graph*. A circuit containing all edges of a graph is an *Eulerian circuit*. A graph is called *Eulerian* if it has an Eulerian circuit or if it is edgeless. If there exists a closed walk in a connected graph that visits every vertex of the graph exactly once (except starting vertex) without repeating the edges then such a graph is called as a *Hamiltonian graph* and the closed walk is called *Hamiltonian circuit*.

A graph is a *connected graph* if, for each pair of vertices, there exists at least one path which joins them otherwise called a *disconnected graph*. A disconnected graph consists of two or more connected subgraphs. Each of these connected subgraphs is called a *connected component* or simply a *component*. A connected graph is called *2-connected* if the graph remains connected after the deletion of any arbitrary vertex (along with the edges incident on it). A *complete graph* is a undirected graph where every pair of distinct vertices is connected by a unique edge. A complete graph with n number of vertices is denoted by K_n. A *clique* is a complete subgraph. The number of vertices in a largest clique of a graph $\mathcal{G}$ is its *clique number*. A *bipartite graph* is a particular kind of graph in which all of the edges join the vertices of one set to the vertices of the other set. Its vertices can be divided into two separate and disjoint sets. In other words, no edges exist that link vertices belonging to the same collection. If a graph $\mathcal{G}$ is bipartite then its vertex set $V(\mathcal{G})$ can be divided into two disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$. In a *complete bipartite graph*, every vertex in V_1 is connected to every vertex in V_2, and $K_{m,n}$ is made up of two sets of vertices, V_1 and V_2, with $|V_1|$ and $|V_2|$ having m and n vertices, respectively.

A set of vertices of a graph is *independent or stable* if the vertices are pairwise non-adjacent. The number of vertices in the largest independent set is called the *independence number* of a given graph, which is denoted by conventionally α. If a graph on n vertices contains a clique of $n - \alpha$ vertices and the rest α vertices is a stable set, where every vertex within the clique is linked to every vertex in the stable set, then the graph is called a *complete split graph* and is denoted by $CS(n, \alpha)$, $1 \leq \alpha \leq n - 1$. A *star graph* is a complete bipartite graph $K_{1,n}$. A graph is called *planar* if it can be drawn in a plane in such a way so that no two edges cross each other except (possibly) at endpoints.

Let $\mathcal{G}$ be a graph and u, v be any two vertices of $\mathcal{G}$. Then the *distance* between two vertices u and v in the graph $\mathcal{G}$ is the number of edges in a shortest path in $\mathcal{G}$, which is denoted by $d(u, v)$. There can exist more than one shortest path between two vertices. If there does not exist any path between u and v, then $d(u, v)$ is denoted by ∞ . Also, for any vertex u in a graph $\mathcal{G}$, $d(u, u) = 0$.

In graph theory, a number that remains unchanged under graph automorphism is termed a graphical invariant. This is often accepted as a structural invariant that is important in graph theory. There are lots of graph-theoretical invariants of molecular structures (graph-based indices also known as topological indices or molecular descriptors) in both mathematics and chemistry literature. This studies look at the relations of such quantities with certain characteristics of the molecules in question. For any graph $\mathcal{G}$, the general formula for these invariants is given by:

$$TI(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})} F(d(u), d(v))$$

Here, $F(x, y)$ represents a function with the property $F(x, y) = F(y, x)$ and $d(u)$, $d(v)$ represent the degree of vertices u and v in $\mathcal{G}$ respectively. A topological index is a numerical parameter derived from the graph's structure. These indices are valuable for establishing correlations between the structure of a molecular compound and its physico-chemical properties, as discussed in [29].

The first and second Zagreb indices (as defined in [29]) of a molecular graph $\mathcal{G}$ are given by:

$$M_1(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})} (d(u) + d(v)),\ M_2(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})} d(u)d(v)$$

Motivated by the definition of Zagreb indices and their wide applications, Kulli introduced the first Gourava index, denoted as $GO_1(\mathcal{G})$, for a molecular graph $\mathcal{G}$ in [34]. This index is defined as:

$$GO_1(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})} [d(u) + d(v) + d(u)d(v)]$$

The second Gourava index $GO_2(\mathcal{G})$ in [34] of a molecular graph $\mathcal{G}$ is defined as

$$GO_2(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})} (d(u) + d(v))d(u)d(v)$$

In [59], Zhou and Trinajstic introduced the sum connectivity index $S(\mathcal{G})$ for a graph $\mathcal{G}$. This index is defined as follows:

$$S(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})} \frac{1}{\sqrt{d(u) + d(v)}}$$

Motivated by the definitions of the first Gourava index and the sum connectivity index, Kulli introduced the sum connectivity Gourava index, denoted as $SGO(\mathcal{G})$ in [33], for a graph $\mathcal{G}$. This index is defined as follows:

$$SGO(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})} \frac{1}{\sqrt{d(u) + d(v) + d(u)d(v)}}$$

The Randić connectivity index, $\chi(\mathcal{G})$, of a graph $\mathcal{G}$, established by Randić in [43] and defined as one of the most well-known and often used topological indices, is defined as

$$\chi(\mathcal{G}) = \sum_{uv \in E(\mathcal{G})} \frac{1}{\sqrt{d(u)d(v)}}$$

In [35], Kulli introduced the product connectivity Gourava index of a graph in the following manner, which is inspired by the definition of the Randić connectivity index: The product connectivity Gourava index, $PGO(\mathcal{G})$, of a molecular graph $\mathcal{G}$ is described as

$$PGO(\mathcal{G}) = \sum_{uv \in E(\mathcal{G})} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

Wiener index: As an oldest topological index, the Wiener index of a (molecular) graph $\mathcal{G}$, first introduced by chemist Harold Wiener [53] in 1947, was defined as

$$W(\mathcal{G}) = \sum_{\{u,v\} \subseteq V(\mathcal{G})} d(u,v) = \frac{1}{2} \sum_{v \in V(\mathcal{G})} d_{\mathcal{G}}(v), \text{ where } d_{\mathcal{G}}(v) = \sum_{u \in V(\mathcal{G})} d(u,v).$$

Hyper-Wiener index: The hyper-Wiener index of acyclic graphs was introduced by Milan Randic in 1993. It is defined as

$$WW(\mathcal{G}) = \frac{1}{2}W(\mathcal{G}) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(\mathcal{G})} d^2(u,v) = \frac{1}{2}W(\mathcal{G}) + \frac{1}{4} \sum_{v \in V(\mathcal{G})} d^2_{\mathcal{G}}(v),$$

where $d^2(u,v) = d(u,v)^2$.

Harary index: The Harary index has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The Harary index of a connected graph $\mathcal{G}$ is defined as follows:

$$H(\mathcal{G}) = \sum_{\{u,v\} \subseteq V(\mathcal{G})} \frac{1}{d(u,v)} = \frac{1}{2} \sum_{v \in V(\mathcal{G})} \frac{1}{d_{\mathcal{G}}(v)}, \text{ where } \frac{1}{d_{\mathcal{G}}(v)} = \sum_{u \in V(\mathcal{G})} \frac{1}{d(u,v)}$$

Gutman index: The Gutman index is a natural extension of the Wiener index. The Gutman index of a finite connected graph $\mathcal{G}$ is defined as

$$Gut(\mathcal{G}) = \sum_{\{u,v\} \subseteq V(\mathcal{G})} d(u)d(v)d(u,v)$$

The “Merrifield-Simmons index" is a topological index first introduced by American chemists Richard E. Merrifield and Howard E. Simmons in a series of publications in the early 1980s. They devised a topological approach to structural chemistry, and the Merrifield-Simmons index was defined as the number of independent vertex sets (sets of vertices that are not adjacent) in a molecular graph. The concept was expanded in their 1989 book, “Topological Methods in Chemistry" in [51]. More precisely, Merrifield-Simmons index (defined in [28]) of a graph $\mathcal{G} = (V, E)$, denoted by $i(\mathcal{G})$, is the total number of independent sets (including the empty set) of vertices of $\mathcal{G}$ and is defined by

$$i(\mathcal{G}) = \sum_{k=0}^{|V|} i_k(\mathcal{G}), \text{ where } i_0(\mathcal{G}) = 1,\ i_1(\mathcal{G}) = \text{number of vertices in } \mathcal{G} = |V|,$$

$i_2(\mathcal{G})$ = total number pair of vertices in V such that they are non-adjacent.

$i_3(\mathcal{G})$ = total number of $3 - tuple$ vertices in V such that they are pairwise non-adjacent.

$i_k(\mathcal{G})$ = total number of $k - tuple$ vertices in V such that they are pairwise non-adjacent.

For other graph-theoretic results, terminologies and definitions, one may look at [9], [30], [52] and [55].

1.4 Literature review and some important results.

In this section, we provide a brief overview of previous works that have connected to graphs with various algebraic structures. This thesis focuses on square element graphs, co-maximal graphs, and zero-divisor graphs, which inspired us to define the new graphs and analyze the pre-defined graphs under consideration.

Many algebraic graphs defined over groups, among of them, total graphs [3] and power graphs [8] deserve special mention. The square element graph $\mathbb{S}q(R)$, defined over a finite commutative ring by Sen Gupta and Sen [48], is perhaps the first instance where the set of squares of an algebraic structure is directly used for defining a graph. Latter, Sen Gupta and Sen generalized the square element graph by defining it over any ring [47]. Later, B. Biswas et al. studied on connectedness of square element graphs over arbitrary ring in [15]. They introduced the square element graph $\mathbb{S}q(S)$ over a semigroup S and studied some of its properties in [16]. Later, Biswas et al.

studied more properties on square element graph of square-subtract rings in [11].

The definition of square element graph is as follows:

Definition 1.4.1. *(Square element graph over ring)[48] Let* R *be a finite commutative ring. The square element graph* $\mathbb{S}_q(R)$ *of* R *is a simple undirected graph with all the non-zero elements of* R *as its vertices, and two distinct vertices* x *and* y *are adjacent if and only if* $xy = t^2$ *for some* $t \in R \setminus \{0\}$.

Definition 1.4.2. *(Square element graph over semigroup)[16] Let* S *be a semigroup. The square element graph* $\mathbb{S}_q(S)$ *of* S *is a simple undirected graph whose vertex set consists precisely of all the non-zero elements of* S*, and two distinct vertices* a *and* b *are adjacent if and only if either* ab *or* ba *belongs to the set* $\{t^2 : t \in S\} \setminus \{1\}$*, where* 1 *is the identity of the semigroup (if it exists).*

The set of squares in a group shows intriguing graph-theoretic structures in [16]. The following are some very significant results on square element graphs over semigroups:

Theorem 1.4.3. *(Theorem 3.4, [16]) For* $n \geq 3$, $\mathbb{S}q(S_n)$ *is a disjoint union of* $\overline{mK_1 + (\frac{n!-2m}{4})K_2}$ *and* $\overline{(p+1)K_1 + (\frac{n!-2p-2}{4})K_2}$*, where* p *is the number of those permutations in* S_n *which are the products of an even number of disjoint 2-cycles and* m *is the number of those permutations in* S_n *which are the products of an odd number of disjoint 2-cycles.*

Theorem 1.4.4. *(Theorem 4.2, [16]) If* n *is an odd integer, then*

$$\mathbb{S}q(D_n) \cong \overline{K_1 + (\frac{n-1}{2})K_2} + K_n.$$

Theorem 1.4.5. *(Theorem 4.3, [16]) If* n *is an even integer, then*

$$\mathbb{S}q(D_n) \cong \begin{cases} \overline{2(K_1 + (\frac{n-2}{4})K_2)} + 2K_{\frac{n}{2}} & \textit{if } \frac{n}{2} \textit{ is odd} \\ \\ \overline{\frac{n}{4}K_2} + \overline{2K_1 + (\frac{n-4}{4})K_2} + 2K_{\frac{n}{2}} & \textit{if } \frac{n}{2} \textit{ is even.} \end{cases}$$

Theorem 1.4.6. *[52] Independent set of a graph is a clique of its complement graph & vice-versa.*

In 1995, Sharma and Bhatwadekar [49] introduced the co-maximal graph $\Gamma'(R)$ for a commutative ring R with identity. In this graph, the vertices represent elements of the ring, and two distinct vertices, a and b, are adjacent if and only if $Ra+Rb=R$. Subsequent research by Maimani et al. [37] delved deeper into the properties of these co-maximal graphs, coining the term "co-maximal graph of R". Furthermore, B. Biswas, S. Kar, and M.K. Sen also contributed to this field by investigating the generalized co-maximal graph in [12] and [13]. Additionally, the authors of [37] examined properties of subgraphs $\Gamma'_1(R)$, $\Gamma'_2(R)$, and $\Gamma'_2(R)\setminus J(R)$, where $\Gamma'_1(R)$ represents the subgraph whose vertices are the units of R, $\Gamma'_2(R)$ includes the non-unit elements of R, and $\Gamma'_2(R)\setminus J(R)$ consists of non-unit elements of R that are not in the Jacobson radical $J(R)$.

The definition of co-maximal graph is as follows:

Definition 1.4.7. *[49] Let R be a commutative ring with identity. The co-maximal graph $\Gamma'(R)$ of R is a simple undirected graph with vertices as elements of R, where two distinct vertices x and y are adjacent if and only if $Rx+Ry=R$.*

Theorem 1.4.8. *(Theorem 6.1.2, [10]) Let R be a finite ring (not necessarily commutative) with a multiplicative identity $1\neq 0$ whose zero-divisors form an additive group J. Then*

(i) J is the Jacobson radical of R;

(ii) $|R|=p^{nr}$ and $|J|=p^{(n-1)r}$, for some prime p and some positive integer r and n;

(iii) $J^n=(0)$;

(iv) the characteristic of the ring R is p^k for some integer $1\leq k\leq n$;

(v) if the characteristic is p^n, then R will be commutative.

Now we come to the zero-divisor graph, which is most likely the beginning point for graph-ring associations. Beck[7] introduced the zero-divisor graph in 1988. He considered the coloring of a commutative ring by viewing it as a graph. He defined a graph over a commutative ring R having a vertex set that includes all elements of R. Two distinct vertices are adjacent if and only if their product is zero. This graph is considered the first form of the zero-divisor graph. Beck conjectured that the clique number and the chromatic number are always equal for the graph he developed.

Three years later, a counterexample provided by D.D. Anderson and Naseer [6] disproved Beck's conjecture. They also looked at a few more of the graph's characteristics that Beck had defined. D.F. Anderson and P. Livingston [5] revised the definition of the zero-divisor graph in 1999. They defined the zero-divisor graph as we know it today. With the use of graph theory, the set of zero-divisors was better illustrated.

The definition of zero-divisor graph is as follows:

Definition 1.4.9. *[5] Let R be a commutative ring with identity and let $Z(R)$ be its set of zero-divisors. The zero-divisor graph $\Gamma(R)$ of R is a simple undirected graph (V, E) where $V = Z(R) \setminus \{0\}$, i.e., the set of nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$.*

In accordance with Beck, who used all elements of R as vertices in the graph he defined, this definition obviously limits the vertex set to the set of nonzero zero-divisors of R. Redmond later extended the zero-divisor graphs to any ring, meaning they are not always commutative.

The undirected zero-divisor graph over any ring was defined by him as follows:

Definition 1.4.10. *[46] Let R be a ring. The undirected zero-divisor graph $\overline{\Gamma}(R)$ over R is a simple undirected graph with vertex set $Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if either $xy = 0$ or $yx = 0$.*

Chapter 2

Order two element graph over a group

Chapter 2

Order two element graph over a group

2.1 Introduction

The study of graphs in relation to various areas of algebra is part of algebraic graph theory. From the perspectives of algebra and graph theory, the relationship between the algebraic and graph-theoretic structures has produced a number of intriguing results. Furthermore, the square element graph over semigroups and its other previously established variations primarily depend on the semigroup's operation and the ring's operations corresponding adjacency condition. First Sen Gupta and Sen defined square element graph over a ring in [48]. Later, they introduced the square element graph $\mathbb{S}q(S)$ over a semigroup S and studied some of its properties in [16]. This motivated us to define a new graph by appropriately adding a new condition to the definition of a square element graph over a group, allowing the operation of a group to be utilized in the new graph's adjacency condition. Also, we extend the definition of square element graph over a group instead of a semigroup. The graph is defined as follows:

Definition 2.1.1. *Let G be a group. Let $\mathcal{G} = (V, E)$ be a simple undirected graph whose vertex set consists of all elements of G and two distinct elements u and v are adjacent if and only if either uv or $vu \in \{t^2 : t \in G\} \cup \{a \in G : a^2 = e\} \setminus \{e\}$, where e is the identity element of G. We called this graph as* order two element graph *over* *the group G and we denote this simple undirected graph $\mathcal{G}$ by $\mathbb{S}_2(G)$.*

We denote S_G as the set of all squares in the group G and G_2 denotes the set of elements of the group G whose order is 2.

In this chapter, we generalize the concept of square element graph $\mathbb{S}q(G)$ over a group G. Clearly, $\mathbb{S}_q(G)$ is a subgraph of $\mathbb{S}_2(G)$. Here, we first characterize commutative groups G for which the graph $\mathbb{S}_2(G)$ is connected. Then we characterize the structures of $\mathbb{S}_2(\mathbb{Z}_n)$ for any positive integer n. Afterwards, we discuss the connectedness of $\mathbb{S}_2(D_n)$ and $\mathbb{S}_2(S_n)$. Finally, we provide a list of finite groups of order ≤ 16 and their corresponding graph structures.

For the order two element graph $\mathbb{S}_2(G)$, we have the following important observations :

Result - I: Let S be a non-empty set and consider the group $G = (P(S), \triangle)$. Then $\mathbb{S}_2(G)$ is a complete graph, since order of every non-identity element of G is 2.

Result - II: The graph $\mathbb{S}_2(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2)$ is complete, since every non-identity element of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ is of order 2.

2.2 Connectedness of $\mathbb{S}_2(G)$ over finite commutative group G

In this section, we mainly characterize those finite commutative group G for which the graph $\mathbb{S}_2(G)$ is connected.

First we consider the following examples :

Example 2.2.1. *The graphs $\mathbb{S}q(\mathbb{Z}_6)$ and $\mathbb{S}_2(\mathbb{Z}_6)$ are shown in Figure 2.1 and Figure 2.2*

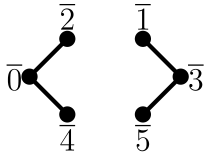

Figure 2.1: $\mathbb{S}q(\mathbb{Z}_6)$

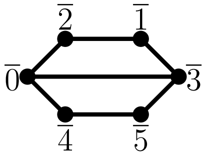

Figure 2.2: $\mathbb{S}_2(\mathbb{Z}_6)$

We observe that $\mathbb{S}q(\mathbb{Z}_6)$ *is not connected but* $\mathbb{S}_2(\mathbb{Z}_6)$ *is connected.*

Example 2.2.2. *Consider the group* $G = \mathbb{Z}_4$*. Now the graph* $\mathbb{S}_2(\mathbb{Z}_4)$ *is shown in Figure 2.3:*

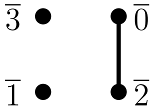

Figure 2.3: $\mathbb{S}_2(\mathbb{Z}_4)$

From above, it follows that the order two element graph $\mathbb{S}_2(\mathbb{Z}_4)$ *is not connected. So, we find that the order two element graph* $\mathbb{S}_2(G)$ *is not connected for some finite commutative group* G*.*

Proposition 2.2.3. *Let* G *be a finite commutative group. If the graph* $\mathbb{S}_2(G)$ *is connected, then the graph* $\mathbb{S}_2(\mathbb{Z}_2 \times G)$ *is connected.*

Proof. Let $(\overline{0}, a)$ and $(\overline{0}, b)$ be two arbitrary vertices in $\mathbb{S}_2(\mathbb{Z}_2 \times G)$. Since $\mathbb{S}_2(G)$ is connected, there exists a path between a and b. Let the path be $a \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow \cdots \leftrightarrow a_n \leftrightarrow b$. Then in $\mathbb{S}_2(\mathbb{Z}_2 \times G)$, $(\overline{0}, a) \leftrightarrow (\overline{0}, a_1) \leftrightarrow (\overline{0}, a_2) \leftrightarrow \cdots \leftrightarrow (\overline{0}, a_n) \leftrightarrow (\overline{0}, b)$ is a path between $(\overline{0}, a)$ and $(\overline{0}, b)$. Similarly, we find that there exists a path between $(\overline{1}, a)$ and $(\overline{1}, b)$. It can be easily verified that the subgraphs induced by the subsets $A_0 = \{(\overline{0}, x) \in \mathbb{Z}_2 \times G : x \in G\}$ and $A_1 = \{(\overline{1}, x) \in \mathbb{Z}_2 \times G : x \in G\}$ form two connected subgraphs of the graph $\mathbb{S}_2(\mathbb{Z}_2 \times G)$. Now consider $(\overline{0}, e) \in A_0$ and $(\overline{1}, e) \in A_1$. Then $(\overline{0}, e)(\overline{1}, e) = (\overline{0} + \overline{1}, e) = (\overline{1}, e)$ and $o((\overline{1}, e)) = 2$. Therefore, $(\overline{0}, e) \leftrightarrow (\overline{1}, e)$. So $\mathbb{S}_2(\mathbb{Z}_2 \times G) = \mathbb{S}_2(A_0 \cup A_1)$ is a connected graph. $\square$

Proposition 2.2.4. *Let G be a finite commutative group and n be a non-negative integer. If the graph $\mathbb{S}_2(G)$ is connected, then $\mathbb{S}_2(\mathbb{Z}_{2(2n+1)} \times G)$ is connected.*

Proof. Let $G' = \mathbb{Z}_{2(2n+1)} \times G$. Then $G'_2 =$ order two elements in G'.
Let $(\overline{0}, a)$ and $(\overline{0}, b)$ be two arbitrary vertices in $\mathbb{S}_2(\mathbb{Z}_{2(2n+1)} \times G)$. Since $\mathbb{S}_2(G)$ is connected, there exists a path between a and b. Let the path be $a \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow \cdots \leftrightarrow a_n \leftrightarrow b$. Then in $\mathbb{S}_2(\mathbb{Z}_{2(2n+1)} \times G)$, $(\overline{0}, a) \leftrightarrow (\overline{0}, a_1) \leftrightarrow (\overline{0}, a_2) \leftrightarrow \cdots \leftrightarrow (\overline{0}, a_n) \leftrightarrow (\overline{0}, b)$ is a path between $(\overline{0}, a)$ and $(\overline{0}, b)$. So the subgraph induced by the subset $O = \{(\overline{0}, x) \in \mathbb{Z}_{2(2n+1)} \times G : x \in G\}$ forms a connected subgraph. Consider any vertex $(\overline{2r}, x)$ of $A_0 = \{(\overline{2r}, x) \in \mathbb{Z}_{2(2n+1)} \times G : x \in G, r = 0, 1, 2, ..., 2n\}$. Then $(\overline{2r}, x) \leftrightarrow (\overline{0}, x^{-1}) \in O$ as $(\overline{2r}, x)(\overline{0}, x^{-1}) = (\overline{2r}, e) = (\overline{r}, e)^2 \in S_G$. Thus any vertex from $A_0 \setminus O$ is adjacent to some vertex of O. So the subgraph induced by A_0 is a connected subgraph. Consider any vertex $(\overline{2r+1}, x)$ of $A_1 = \{(\overline{2r+1}, x) \in \mathbb{Z}_{2(2n+1)} \times G : x \in G, r = 0, 1, ..., 2n\}$. Then $(\overline{2r+1}, x) \leftrightarrow (\overline{2n-2r}, x^{-1}) \in A_0$ as $(\overline{2r+1}, x)(\overline{2n-2r}, x^{-1}) = ((\overline{2r+1})(\overline{2n-2r}), e) = (\overline{2n+1}, e) \in G'_2$ as $(\overline{2n+1}, e)^2 = (\overline{0}, e)$ in G'. Thus any vertex from A_1 is adjacent to some vertex of A_0. Hence $\mathbb{S}_2(\mathbb{Z}_{2(2n+1)} \times G) = \mathbb{S}_2(A_0 \cup A_1)$ is a connected graph. □

Remark 2.2.5. *Let G be a finite commutative group and n be a positive integer. Then the graph $\mathbb{S}_2(\mathbb{Z}_{2(2n)} \times G)$ is not connected even though the graph $\mathbb{S}_2(G)$ is connected.*

Consider the subsets $A_0 = \{(\overline{2r}, x) \in \mathbb{Z}_{2(2n)} \times G : x \in G, r = 0, 1, 2, ..., 2n-1\}$ and $A_1 = \{(\overline{2r+1}, x) \in \mathbb{Z}_{2(2n)} \times G : x \in G, r = 0, 1, ..., 2n-1\}$. It can be easily verified that the subgraphs induced by A_0 and A_1 are two connected subgraphs. But no vertex of A_0 is adjacent to a vertex of A_1. So, $\mathbb{S}_2(\mathbb{Z}_{2(2n)} \times G)$ is not connected.

Proposition 2.2.6. *Let G be a finite commutative group. If the graph $\mathbb{S}_2(G)$ is connected, then $\mathbb{S}_2(\mathbb{Z}_m \times G)$ is connected, where $m = 4n+1$ or $m = 4n+3$ for some $n \in \mathbb{N}$.*

Proof. If $m = 4n+1$ or $4n+3$, then $\mathbb{Z}_m$ is a group of odd order. So each element of $\mathbb{Z}_m$ is square element. Let $(\overline{0}, a)$ and $(\overline{0}, b)$ be two arbitrary vertices in $\mathbb{S}_2(\mathbb{Z}_m \times G)$. Since $\mathbb{S}_2(G)$ is connected, there exists a path between a and b. Let the path be

$a \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow \cdots \leftrightarrow a_n \leftrightarrow b$. Then in $\mathbb{S}_2(\mathbb{Z}_m \times G)$, $(\overline{0}, a) \leftrightarrow (\overline{0}, a_1) \leftrightarrow (\overline{0}, a_2) \leftrightarrow \cdots \leftrightarrow (\overline{0}, a_n) \leftrightarrow (\overline{0}, b)$ is a path between $(\overline{0}, a)$ and $(\overline{0}, b)$. So the subgraph induced by the subset $A_0 = \{(\overline{0}, x) \in \mathbb{Z}_m \times G : x \in G, m = 4n+1 \textit{ or } 4n+3\}$ forms a connected subgraph. Then the other vertices from $\mathbb{Z}_m \times G \setminus A_0$ look like $(\overline{2r}, b)$ or $(\overline{2r+1}, b)$ for $\overline{2r}, \overline{2r+1} \in \mathbb{Z}_m$ and $b \in G$. Now $(\overline{2r}, b)(\overline{0}, b^{-1}) = (\overline{2r}, e) = (\overline{r}, e)^2$. So $(\overline{2r}, b) \leftrightarrow (\overline{0}, b^{-1}) \in A_0$. Again $(\overline{2r+1}, b)(\overline{0}, b^{-1}) = (\overline{2r+1}, e)$. Since m is odd and $\overline{2r+1} \in \mathbb{Z}_m$, so every vertex of $\mathbb{Z}_m$ is a square of some element of $\mathbb{Z}_m$. Then $\overline{2r+1} = \overline{r_1}^2$ for some $\overline{r_1} \in \mathbb{Z}_m$. Thus $(\overline{2r+1}, e) = (\overline{r_1}, e)^2$. So $(\overline{2r+1}, b) \leftrightarrow (\overline{0}, b^{-1}) \in A_0$. Hence $\mathbb{S}_2(\mathbb{Z}_m \times G)$ is a connected graph, where $m = 4n+1$ or $m = 4n+3$. □

Proposition 2.2.7. *The graph* $\mathbb{S}_2(\mathbb{Z}_{4m} \times \mathbb{Z}_{4n})$ *is not connected for any positive integer* $4m$, $4n$.

Proof. Consider the two vertices $(\overline{2}, \overline{2})$ and $(\overline{1}, \overline{1})$. It can be easily checked that there is no path between $(\overline{2}, \overline{2})$ and $(\overline{1}, \overline{1})$. So the graph $\mathbb{S}_2(\mathbb{Z}_{4m} \times \mathbb{Z}_{4n})$ is not connected for any positive integer $4m$, $4n$. □

Proposition 2.2.8. *Let* G *be a finite commutative group. Then the graph* $\mathbb{S}_2(G)$ *is connected if and only if* $G \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k}$, *where* $r_i \in \{x \in \mathbb{N} : x = 4n+1 \textit{ or } 4n+2 \textit{ or } 4n+3\}$.

Proof. If $G \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k}$, where $r_i \in \{x \in \mathbb{N} : x = 4n+1 \text{ or } 4n+2 \text{ or } 4n+3\}$, then by Result - II, Proposition 2.2.3, Proposition 2.2.4 and Proposition 2.2.6, the graph $\mathbb{S}_2(G)$ is connected.

Conversely, suppose that $\mathbb{S}_2(G)$ is connected. Since G is a finite commutative group, so by the Fundamental Theorem of finite commutative group, we have $G \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k}$, where each r_i is of prime power. If possible, let $G \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k}$, where at least one $r_i = 4n$ for some $n \in \mathbb{N}$. Without loss of generality, let $r_1 = 4n$. Then $G \cong \mathbb{Z}_{4n} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k}$. If $S_2(\mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k})$ is connected, then by Remark 2.2.5, $S_2(\mathbb{Z}_{4n} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k})$ is not connected. Again if $S_2(\mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k})$ is not connected, then obviously $S_2(\mathbb{Z}_{4n} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k})$ is not connected which contradicts our assumption. Hence $G \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k}$, where no $r_i = 4n$. So $G \cong \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_k}$, where $r_i \in \{x \in \mathbb{N} : x = 4n+1 \text{ or } 4n+2 \text{ or } 4n+3\}$. □

In this chapter, we consider the finite cyclic group $(\mathbb{Z}_n, +) = (\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{n-1}, +)$ as $\mathbb{Z}_n = \{e, x, x^2, x^3, \cdots, x^{n-1}\}$ for better understanding the group structure of the group $\mathbb{Z}_n$ as well as the graph structure of the graph $\mathbb{S}_2(\mathbb{Z}_n)$. The following examples give us clear visualization of the order two element graph over the group $\mathbb{Z}_n$ for $n = 2, 3, 4, 5, 6, 7, 8$.

Example 2.2.9. *Let* $G = \mathbb{Z}_2 = \{e, x\}$. *Then* $S_G = \{e\}$ *and* $G_2 = \{x\}$. *Then we construct the order two element graph* $\mathbb{S}_2(\mathbb{Z}_2)$ *corresponding to the group* $\mathbb{Z}_2$, *which is as follows.*

Figure 2.4: $\mathbb{S}_2(\mathbb{Z}_2) \cong K_2$

Example 2.2.10. *Let* $G = \mathbb{Z}_3 = \{e, x, x^2\}$. *Then* $S_G = \{e, x, x^2\}$ *and* $G_2 = \phi$. *Then we can draw the order two element graph* $\mathbb{S}_2(\mathbb{Z}_3)$ *corresponding to the group* $\mathbb{Z}_3$ *in the following manner.*

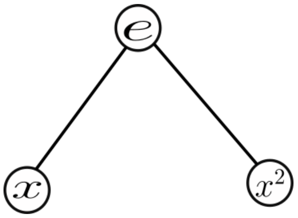

Figure 2.5: $\mathbb{S}_2(\mathbb{Z}_3) \cong \overline{K_1 + K_2}$

Example 2.2.11. *Let* $G = \mathbb{Z}_4 = \{e, x, x^2, x^3\}$. *Then* $S_G = \{e, x^2\}$ *and* $G_2 = \{x^2\}$. *Then the order two element graph* $\mathbb{S}_2(\mathbb{Z}_4)$ *of the group* $\mathbb{Z}_4$ *is shown in Figure 2.3. Also, the graph* $\mathbb{S}_2(\mathbb{Z}_4)$ *is graph-isomorphic to the graph* $2K_1 + K_2$.

Example 2.2.12. *Let* $G = \mathbb{Z}_5 = \{e, x, x^2, x^3, x^4\}$*. Then* $S_G = \{e, x, x^2, x^3, x^4\}$ *and* $G_2 = \phi$*. Then we draw the order two element graph* $\mathbb{S}_2(\mathbb{Z}_5)$ *corresponding to the group* $\mathbb{Z}_5$*, which is as follows.*

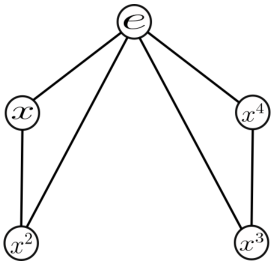

Figure 2.6: $\mathbb{S}_2(\mathbb{Z}_5) \cong \overline{K_1 + 2K_2}$

Example 2.2.13. *Let* $G = \mathbb{Z}_6 = \{e, x, x^2, x^3, x^4, x^5\}$*. Then* $S_G = \{e, x^2, x^4\}$ *and* $G_2 = \{x^3\}$*. Then the order two element graph* $\mathbb{S}_2(\mathbb{Z}_6)$ *of the group* $\mathbb{Z}_6$ *is shown in Figure 2.2.*

Example 2.2.14. *Let* $G = \mathbb{Z}_7 = \{e, x, x^2, x^3, x^4, x^5, x^6\}$*. Then* $S_G = \{e, x, x^2, x^3, x^4, x^5, x^6\}$ *and* $G_2 = \phi$*. Then we construct the order two element graph* $\mathbb{S}_2(\mathbb{Z}_7)$ *corresponding to the group* $\mathbb{Z}_7$*, which is as follows.*

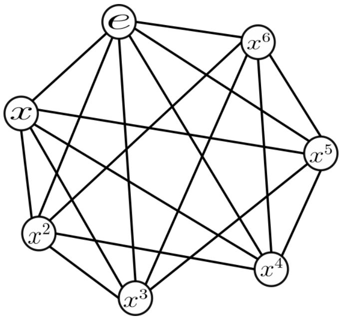

Figure 2.7: $\mathbb{S}_2(\mathbb{Z}_7) \cong \overline{K_1 + 3K_2}$

Example 2.2.15. *Let* $G = \mathbb{Z}_8 = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7\}$*. Then* $S_G = \{e, x^2, x^4, x^6\}$ *and* $G_2 = \{x^4\}$*. Then we can figure out the order two element graph* $\mathbb{S}_2(\mathbb{Z}_8)$ *corresponding to the group* $\mathbb{Z}_8$ *in the following manner.*

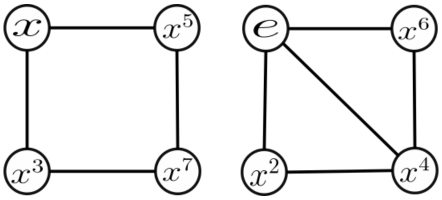

Figure 2.8: $\mathbb{S}_2(\mathbb{Z}_8) \cong C_4 + \overline{2K_1 + K_2}$

From examples 2.2.9, 2.2.10, 2.2.11, 2.2.12, 2.2.13, 2.2.14 and 2.2.15 we have the following theorem on $\mathbb{Z}_n$.

Theorem 2.2.16. *The graph $\mathbb{S}_2(\mathbb{Z}_n)$ is connected if n or $\frac{n}{2}$ is odd positive integer where as the graph $\mathbb{S}_2(\mathbb{Z}_n)$ is disconnected if $\frac{n}{2}$ is even positive integer and*

$$\mathbb{S}_2(\mathbb{Z}_n) = \begin{cases} \overline{K_1 + (\frac{n-1}{2})K_2} & if\ n\ is\ odd\ positive\ integer. \\ \overline{2K_1 + (\frac{n}{4} - 1)K_2} + \overline{\frac{n}{4}K_2} & if\ \frac{n}{2}\ is\ even\ positive\ integer. \end{cases}$$

Proof. **Case I:** Let n be an odd positive integer and $G = \mathbb{Z}_n = \{e, x, x^2, x^3, ..., x^{n-1}\}$. Then the set of square elements of the group G is given by $S_G = \{e, x, x^2, x^3, ..., x^{n-1}\}$ and the set of order two elements of G is $G_2 = \emptyset$. We notice that in $\mathbb{S}_2(G)$, 'e' is adjacent with the rest of '$n-1$' vertices. So in the complement graph, 'e' form a K_1. Also x^i and x^{n-i} or x^{i-1} are not adjacent in $\mathbb{S}_2(G)$. Thus there are $\frac{n-1}{2}$ number of pairs x^i and x^{i-1} which are not adjacent. So in the complement graph, they form $(\frac{n-1}{2})K_2$. Hence $\mathbb{S}_2(G) \cong \overline{K_1 + (\frac{n-1}{2})K_2}$ when n is odd positive integer.

Case II: Let $\frac{n}{2}$ be an odd positive integer and $G = \mathbb{Z}_n = \{e, x, x^2, x^3, ..., x^{n-1}\}$. Let us assume that $E = \{e, x^2, x^4, x^6, ..., x^{n-2}\}$ and $O = \{x, x^3, x^5, ..., x^{n-1}\}$. Then $S_G = \{e, x^2, x^4, x^6, ..., x^{2n}\}$ and $G_2 = \{x^{\frac{n}{2}}\}$. Clearly, $x^{\frac{n}{2}} \in O$ but $x^{\frac{n}{2}} \notin E$ as $\frac{n}{2}$ is odd positive integer. Then the subgraph induced by E and the subgraph induced by O are connected subgraphs of $\mathbb{S}_2(\mathbb{Z})$. Also since there exists an edge between e and $x^{\frac{n}{2}}$, where $e \in E$ and $x^{\frac{n}{2}} \in O$, so the graph $\mathbb{S}_2(\mathbb{Z})$ is connected. Hence the graph $\mathbb{S}_2(\mathbb{Z})$ is connected if $\frac{n}{2}$ is an odd positive integer.

Case III: Let $\frac{n}{2}$ be an even positive integer. Suppose $G = \mathbb{Z}_n = \{e, x, x^2, x^3, ..., x^{n-1}\}$ for $\frac{n}{2}$ is an even positive integer. Then $S_G = \{x^2, x^4, x^6, ..., x^{2n}\}$ and $G_2 = \{x^{\frac{n}{2}}\} \subset S_G$. Let $A = \{e, x^2, x^4, x^6, ..., x^{n-2}\}$ and $B = \{x, x^3, x^5, ..., x^{n-1}\}$. Then it can be easily verified that the subgraph induced by A and the subgraph induced by B are two disjoint connected subgraphs of $\mathbb{S}_2(\mathbb{Z}_n)$. Now in the subgraph induced by A, 'e' and '$x^{\frac{n}{2}}$' are adjacent with all elements of A except each other. So in the complement graph of the subgraph induced by A, 'e' and '$x^{\frac{n}{2}}$' form K_1 separately i.e. form $2K_1$. Now the number of vertices of the subgraph induced by A is $\frac{n}{2}$. Here x^i and x^{n-i} are not adjacent for all x^i and x^{n-i} are in A. There are $(\frac{n}{4} - 1)$ pair of vertices which form $(\frac{n}{4} - 1)K_2$ in the complement graph of of the subgraph induced by A. Now in the subgraph induced by B, there are $\frac{n}{4}$ number of pair of vertices x^i and x^{n-i} which are not adjacent each other. So in the complement graph of the subgraph induced by B, they form $\frac{n}{4}K_2$. Hence $\mathbb{S}_2(\mathbb{Z}_n) \cong \overline{2K_1 + (\frac{n}{4} - 1)K_2} + \overline{\frac{n}{4}K_2}$ when $\frac{n}{2}$ is an even positive integer. □

2.3 Connectedness of $\mathbb{S}_2(G)$ over finite non-commutative group G

In this section, we try to establish some important results on order two element graph over some non-commutative groups like S_n and D_n. First we understand connectedness and graph structure of the order two element graph over the group S_n through some following examples.

Example 2.3.1. *Let $G = S_2 = \{e, \rho\}$. Then $S_G = \{e\}$ and $G_2 = \{\rho\}$. Now, we draw the order two element graph $\mathbb{S}_2(S_2)$ which is shown in Figure 2.9.*

Example 2.3.2. *Let $G = S_3 = \{e, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$, where $e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\rho_1 = (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\rho_2 = (13) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $\rho_3 = (23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\rho_4 = (123) =$*

Figure 2.9: $\mathbb{S}_2(S_2) \cong K_2$

$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, $\rho_5 = (132) = \begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$. *Then* $S_G = \{e, \rho_4, \rho_5\}$ *and* $G_2 = \{\rho_1, \rho_2, \rho_3\}$. *Now, we construct the order two element graph* $\mathbb{S}_2(S_3)$ *which is shown in Figure 2.10.*

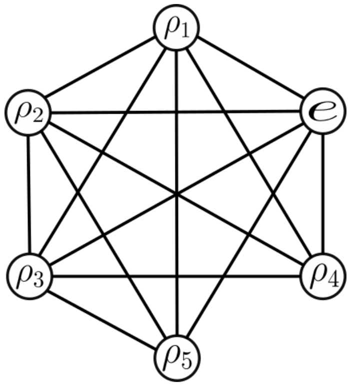

Figure 2.10: $\mathbb{S}_2(S_3) \cong \overline{4K_1 + K_2}$

Theorem 2.3.3. *[16] For* $n \geq 3$, $\mathbb{S}q(S_n)$ *is a disjoint union of* $\overline{mK_1 + (\frac{n!-2m}{4})K_2}$ *and* $\overline{(p+1)K_1 + (\frac{n!-2p-2}{4})K_2}$, *where* p *is the number of those permutations in* S_n *which are the products of an even number of disjoint 2-cycles and* m *is the number of those permutations in* S_n *which are the products of an odd number of disjoint 2-cycles.*

From examples 2.3.1, 2.3.2 and the above theorem, we have the following proposition.

Proposition 2.3.4. *The graph* $\mathbb{S}_2(S_n)$ *is connected.*

Proof. From Theorem 2.3.3, it follows that the square element graph $\mathbb{S}q(S_n)$ contains two connected components which are induced by A_n and $S_n \setminus A_n$. We have already noticed that $\mathbb{S}q(S_n)$ is subgraph of $\mathbb{S}_2(S_n)$. Now for $(abc) \in A_n$ and $(bc) \in S_n \setminus A_n$, we find that $(bc)(abc) = (ac)$. Since $o(ac) = 2$, it follows that $(bc) \leftrightarrow (abc)$. Thus the graph $\mathbb{S}_2(S_n)$ is connected. $\square$

We prove the general statement that $\mathbb{S}_2(S_n)$ is connected for $n \geq 1$ but we can not give any graphical structure for $\mathbb{S}_2(S_n)$ for $n \geq 4$. For readers, they can establish (if possible) the general graph structure of $\mathbb{S}_2(S_n)$ for $n \geq 4$.

Now, we provide more examples of order two element graph over D_n for $n = 2, 3, 4$ so that we can understand and visualize the graph-theoretic properties of $\mathbb{S}_2(D_n)$.

Example 2.3.5. *Let $G = D_2 = \{e, x, y, xy\}$. Then $S_G = \{e\}$ and $G_2 = \{x, y, xy\}$. Using these information, we draw the order two element graph $\mathbb{S}_2(D_2)$, which is shown in Figure 2.11.*

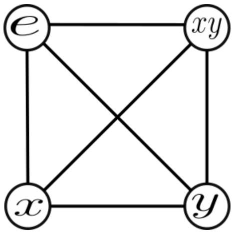

Figure 2.11: $\mathbb{S}_2(D_2) \cong K_4$

Example 2.3.6. *Let $G = D_3 = \{e, x, x^2, y, xy, x^2y\}$. Then $S_G = \{e, x, x^2\}$ and $G_2 = \{y, xy, x^2y\}$. Then we construct the order two element graph $\mathbb{S}_2(D_3)$ corresponding to the group D_3, which is shown in Figure 2.12.*

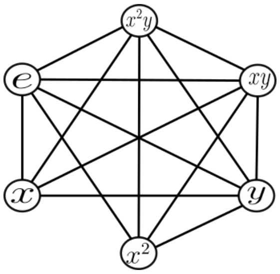

Figure 2.12: $\mathbb{S}_2(D_3) \cong \overline{4K_1 + K_2}$

Example 2.3.7. *Let* $G = D_4 = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$. *Then* $S_G = \{e, x^2\}$ *and* $G_2 = \{x^2, y, xy, x^2y, x^3y\}$. *Then one can see the Figure 2.14 to understand the graphical structure of the order two element graph* $\mathbb{S}_2(D_4)$ *corresponding to the group* D_4.

Theorem 2.3.8. *[16] If* n *is an odd integer, then* $\mathbb{S}q(D_n) \cong \overline{K_1 + (\frac{n-1}{2})K_2 + K_n}$.

Theorem 2.3.9. *[16] If* n *is an even integer, then*

$$\mathbb{S}q(D_n) \cong \begin{cases} 2\overline{(K_1 + (\dfrac{n-2}{4})K_2)} + 2K_{\frac{n}{2}} & \text{if } \dfrac{n}{2} \text{ is odd} \\ \\ \overline{\dfrac{n}{4}K_2} + \overline{2K_1 + (\dfrac{n-4}{4})K_2} + 2K_{\frac{n}{2}} & \text{if } \dfrac{n}{2} \text{ is even.} \end{cases}$$

From examples 2.3.5, 2.3.5, 2.3.7 and above two theorems, we have the following theorems regarding the connectedness and graphical structure of the graph $\mathbb{S}_2(D_n)$.

Theorem 2.3.10. *For any integer* $n \in \mathbb{N}$, *the graph* $\mathbb{S}_2(D_n)$ *is connected.*

Proof. Let $D_n = \{< a, b > : \ a^n = b^2 = e \text{ and } ab = ba^{n-1}\}$. Then we can write $D_n = \{a, a^2, a^3, \ldots, a^{n-1}, a^n (= e), b, ba, ba^2, \ldots, ba^{n-1}\}$.

Case I: Let n be odd. Clearly, the set of all squares of D_n is given by $H = \{e, a, a^2, \ldots, a^{n-1}\}$, which is a cyclic subgroup of odd order. From Theorem 2.3.8, we have the subgraphs induced by these 2 cosets H, bH form 2 connected components. Now take $e \in H$ and $b \in bH$. Since $o(eb) = o(b) = 2$, so we have $e \leftrightarrow b$ in $\mathbb{S}_2(D_n)$. Thus the graph $\mathbb{S}_2(D_n)$ is connected.

Case II : Let n be even. Now the set of all squares of D_n is given by $H = \{a^2, a^4, \ldots, a^{n-2}, e\}$. It is easy to see that there are 4 distinct cosets H, aH, bH, baH of H which partition the group D_n. Clearly, $aH = \{a, a^3, a^5, a^7, \ldots, a^{n-1}\}$, $bH = \{ba^2, ba^4, ba^6, \ldots, ba^{n-2}\}$ and $baH = \{ba, ba^3, ba^5, ba^7, \ldots, ba^{n-1}\}$. From Theorem 2.3.9, we have the subgraphs induced by these 4 cosets H, aH, bH, baH form 4 connected components. Consider 4 vertices $e \in H, a \in aH, b \in bH$ and $ab \in abH$.

Since $o(b) = o(ab) = 2$, it follows that $e \leftrightarrow b \leftrightarrow a$ and $e \leftrightarrow ab$. So the graph $\mathbb{S}_2(D_n)$ is connected. $\square$

Theorem 2.3.11. *If n is an odd positive integer, then $\mathbb{S}_2(D_n) \cong \overline{(n+1)K_1 + (\frac{n-1}{2})K_2}$.*

Proof. Let $G = D_n = \{< x, y > : \ x^n = e = y^2 \text{ and } yx = x^{-1}y\}$. Then we can represent the group G by $G = \{e, x, x^2, x^3, ..., x^{n-1}, y, xy, x^2y, x^3y, ..., x^{n-1}y\}$. Let n be an odd integer. Then $S_G = \{e, x, x^2, x^3, ..., x^{n-1}\}$ and $G_2 = \{y, xy, x^2y, x^3y, ..., x^{n-1}y\}$. So in $\mathbb{S}_2(D_n)$, all elements $e, y, xy, x^2y, x^3y, ..., x^{n-1}y$ are adjacent to each others. Thus in the complement graph of $\mathbb{S}_2(D_n)$, they form $(n+1)K_1$. Now rest of the vertices i.e. $x, x^2, x^3, ..., x^{n-1}$ are adjacent to each others except their inverse elements as $x^i(x^i)^{-1} = e$ but $e \notin S_G$ or $e \notin G_2$. So there are $(\frac{n-1}{2})$ pair of vertices which are not adjacent to each others. Hence in the complement graph of $\mathbb{S}_2(D_n)$, they form $(\frac{n-1}{2})K_2$. Thus for an odd positive integer n, we have $\mathbb{S}_2(D_n) \cong \overline{(n+1)K_1 + (\frac{n-1}{2})K_2}$. $\square$

Proposition 2.3.12. *Let G be a group, S_G be the set of squares of G and G_T be the set of those elements of G whose order is 2 but not in S_G. Then in the graph $\mathbb{S}_2(G)$, the degree of any vertex is given by*

$$deg(x) = \begin{cases} |S_G \cup G_T| - 1, & \text{if } x^2 = e \\ |S_G \cup G_T| - 2, & \text{otherwise.} \end{cases}$$

Proof. Suppose that $G = \{e, a_1, \ldots, a_{n-1}, c_1, c_2, ..., c_r, x_1, x_2, \ldots, x_k\}$ and we consider $S_G \cup G_T = \{e, a_1, \ldots, a_{n-1}, c_1, c_2, ..., c_r\}$, where $o(c_i) = 2$ and $a_j = t_j^2$ for some $t_j \in G$. Clearly, e is adjacent to precisely the other elements of $S_G \cup G_T$. So $deg(e) = n+r-1$. For an arbitrary element $g \in G$, $G = g^{-1}G = \{g^{-1}e, g^{-1}a_1, \ldots, g^{-1}a_{n-1}, g^{-1}c_1, g^{-1}c_2, ..., g^{-1}c_r, g^{-1}x_1, g^{-1}x_2, \ldots, g^{-1}x_k\}$. Now we show that g is not adjacent to $g^{-1}x_i$ for any $x_i \in G \setminus (S_G \cup G_T)$. If possible, let $g \leftrightarrow g^{-1}x_k$ for some $x_k \in G \setminus (S_G \cup G_T)$. Now $gg^{-1}x_k$ is not a square or $o(x_k) \neq 2$ and $g^{-1}x_kg \notin S_G$ as if $g^{-1}x_kg = a^2 \in S_G$ that is $x_k = ga^2g^{-1} = (gag^{-1})^2 \in S_G$ which is a contradiction as $x_k \in G \setminus (S_G \cup G_T)$ or $o(g^{-1}x_kg) = o(x_k) \neq 2$.

Case 1: If $g^2 = b_j$ for some $b_j \in S_G \cup G_T$ i.e. $g = g^{-1}b_j$, then $g \leftrightarrow g^{-1}x_i$ for $x_i(\neq b_j) \in S_G \cup G_T$. So $deg(g) \geq n+r-2$. Also g is not adjacent to $g^{-1}x$ for

$x = e, b_j, x_1, x_2, \ldots, x_k$. Thus $deg(g) \leq (n+k+r) - (k+2) = n+r-2$. Therefore, $deg(g) = n+r-2 = |S_G \cup G_T| - 2$.

Case 2: If $g^2 = e$, then $g \leftrightarrow g^{-1}b_i$ for $b_i \in S_G \cup G_T \setminus \{e\}$. So $deg(g) \geq n-1$. Also g is not adjacent to $g^{-1}x$ for $x = e, x_1, x_2, \ldots, x_k$. Hence $deg(g) \leq (n+r+k)-(k+1) = n+r-1$. Therefore, $deg(g) = n+r-1 = |S_G \cup G_T| - 1$. □

2.4 Order two element graph over the groups up to order 16

In this section, we characterize the connectedness of $\mathbb{S}_2(G)$, where G is a group of small order up to 16. From [23], we can get the group structures up to order 15. From [54, 20], we see that there are 14 groups of order 16 up to isomorphism. The details of these 14 group structures are as follows :

1. Integer mod 16 : $\mathbb{Z}_{16}$

$$\begin{aligned} \mathbb{Z}_{16} &= \{x^\alpha : x^{16} = e\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12}, x^{13}, x^{14}, x^{15}\} \end{aligned}$$

2. Direct product of $\mathbb{Z}_8$ and $\mathbb{Z}_2$: $\mathbb{Z}_8 \times \mathbb{Z}_2$

$$\begin{aligned} \mathbb{Z}_8 \times \mathbb{Z}_2 &= \{x^\alpha y^\beta : x^8 = y^2 = e, xy = yx\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

3. Direct product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$: $\mathbb{Z}_4 \times \mathbb{Z}_4$

$$\begin{aligned} \mathbb{Z}_4 \times \mathbb{Z}_4 &= \{x^\alpha y^\beta : x^4 = y^4 = e, xy = yx\} \\ &= \{e, x, x^2, x^3, y, y^2, y^3, xy, xy^2, xy^3, x^2y, x^2y^2, x^2y^3, x^3y, x^3y^2, x^3y^3\} \end{aligned}$$

4. Direct product of $\mathbb{Z}_4$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$: $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\begin{aligned} \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 &= \{x^\alpha y^\beta z^\gamma : x^4 = y^2 = z^2 = e, xy = yx, yz = zx, yz = zy\} \\ &= \{e, x, x^2, x^3, y, z, xy, x^2y, x^3y, xz, x^2z, x^3z, yz, xyz, x^2yz, x^3yz\} \end{aligned}$$

5. Direct product of $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_2$ and $\mathbb{Z}_2$: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\begin{aligned} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 &= \{a^\alpha b^\beta c^\gamma d^\delta : a^2 = b^2 = c^2 = d^2 = e, ab = ba, ca = ac, \\ &\qquad da = ad, cb = bc, db = bd, dc = cd\} \\ &= \{e, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, bcd, acd, abcd\} \end{aligned}$$

6. Direct product of D_4 and $\mathbb{Z}_2$: $D_4 \times \mathbb{Z}_2$

$$\begin{aligned} D_4 \times \mathbb{Z}_2 &= \{a^\alpha b^\beta c^\gamma : a^4 = b^2 = c^2 = e, ba = a^{-1}b, ca = ac, cb = bc\} \\ &= \{e, a, a^2, a^3, b, c, ab, a^2b, a^3b, ac, a^2c, a^3c, bc, abc, a^2bc, a^3bc\} \end{aligned}$$

7. Semi-direct product of Klein 4-Group and $\mathbb{Z}_4$: $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$

$$\begin{aligned} (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4 &= \{a^\alpha b^\beta : a^4 = b^4 = e, ba = a^{-1}b^{-1}\} \\ &= \{e, a, a^2, a^3, b, b^2, b^3, ab, ab^2, ab^3, a^2b, a^2b^2, a^2b^3, a^3b, a^3b^2, a^3b^3\} \end{aligned}$$

8. Direct product of $\mathbb{Q}_8$ and $\mathbb{Z}_2$: $\mathbb{Q}_8 \times \mathbb{Z}_2$

$$\begin{aligned} \mathbb{Q}_8 \times \mathbb{Z}_2 &= \{x^\alpha y^\beta z^\gamma : x^2 = y^2, x^4 = y^4 = z^2 = e, yx = x^{-1}y, zx = xz, zy = yz\} \\ &= \{e, x, x^2, x^3, y, y^3, z, xz, x^2z, x^3z, yz, y^3z, xy, x^3y, xyz, x^3yz\} \end{aligned}$$

9. Semi-direct product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$: $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$

$$\begin{aligned} \mathbb{Z}_4 \rtimes \mathbb{Z}_4 &= \{a^\alpha b^\beta : a^4 = b^4 = e, ba = a^{-1}b\} \\ &= \{e, a, a^2, a^3, b, b^2, b^3, ab, a^2b, a^3b, ab^2, a^2b^2, a^3b^2, ab^3, a^2b^3, a^3b^3\} \end{aligned}$$

10. Group of the Pauli Matrices : $SU(2)$

$$\begin{aligned} SU(2) &= \{x^\alpha y^\beta z^\gamma : x^4 = y^2 = z^2 = e, yx = xy, zx = xz, zy = x^2yz\} \\ &= \{e, x, x^2, x^3, y, z, xy, xz, x^2y, x^2z, x^3y, x^3z, yz, xyz, x^2yz, x^3yz\} \end{aligned}$$

11. Modular or Isanowa group of order 16 : M_{16}

$$\begin{aligned} M_{16} &= \{x^\alpha y^\beta : x^8 = y^2 = e, yx = x^5y\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

12. Dicyclic Group of Degree 4 : Dic_4

$$\begin{aligned} Dic_4 &= \{x^\alpha y^\beta : x^4 = y^2; x^8 = y^4 = e, yx = x^{-1}y\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

13. Semidihedral group of degree 2 : SD_2

$$\begin{aligned} SD_2 &= \{x^\alpha y^\beta : x^8 = y^2 = e, yx = x^3y\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

14. Dihedral group of degree 8 : D_8

$$\begin{aligned} D_8 &= \{x^\alpha y^\beta : x^8 = y^2 = e, yx = x^{-1}y\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

Diagrams of order two element graph of small groups up to order 16:

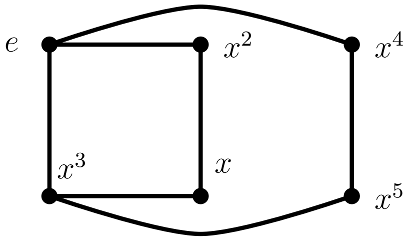

Figure 2.13: $\mathbb{S}_2(\mathbb{Z}_6)$

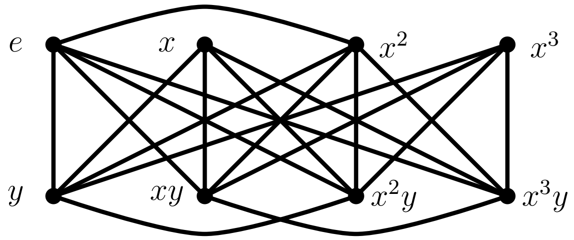

Figure 2.14: $\mathbb{S}_2(D_4)$

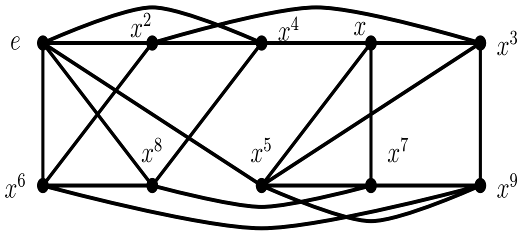

Figure 2.15: $\mathbb{S}_2(\mathbb{Z}_{10})$

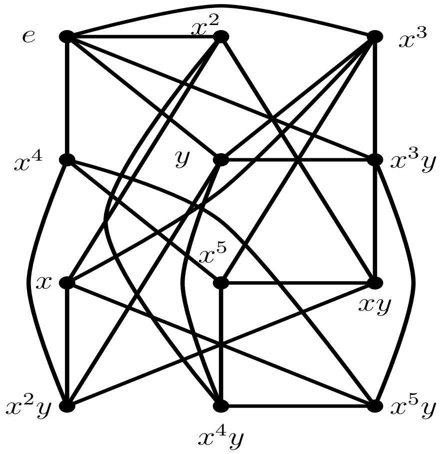

Figure 2.16: $\mathbb{S}_2(\mathbb{Z}_6 \times \mathbb{Z}_2)$

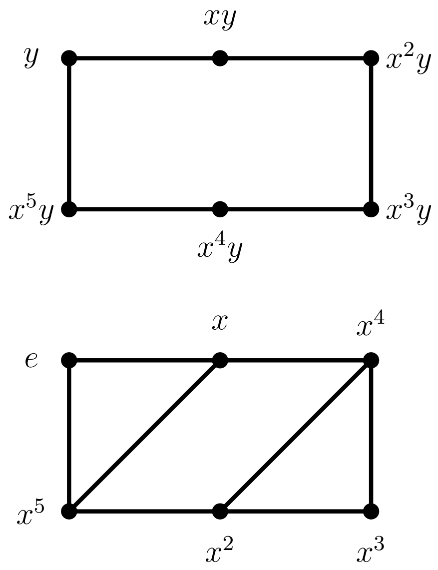

Figure 2.17: Complement graph of $\mathbb{S}_2(D_6)$

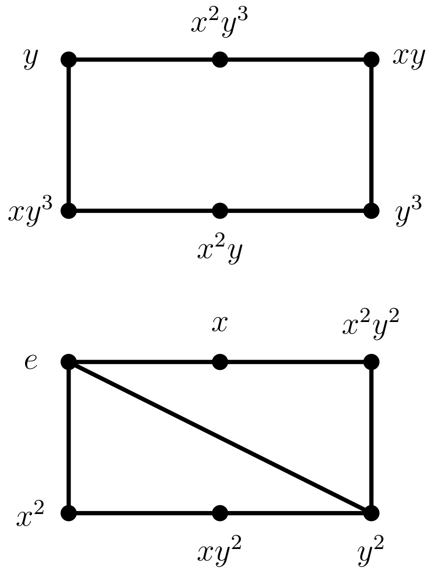

Figure 2.18: $\mathbb{S}_2(\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$

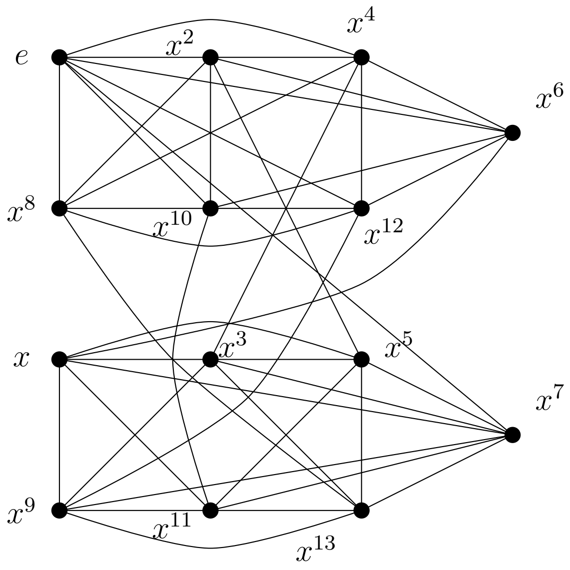

Figure 2.19: $\mathbb{S}_2(\mathbb{Z}_{14})$

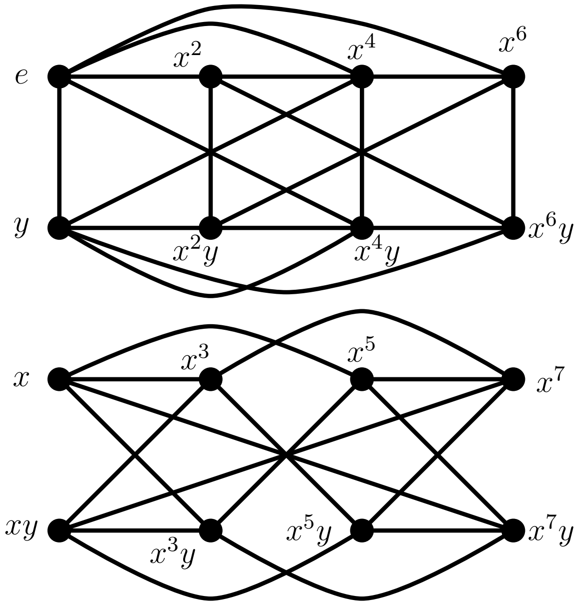

Figure 2.20: $\mathbb{S}_2(\mathbb{Z}_8 \times \mathbb{Z}_2)$

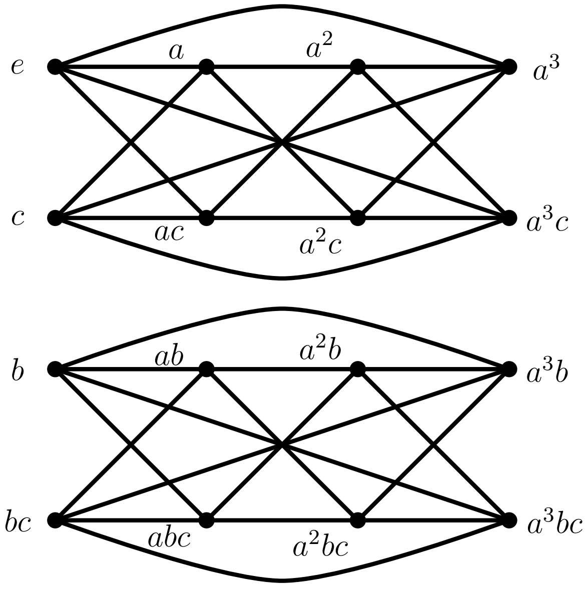

Figure 2.21: Complement graph of $\mathbb{S}_2(D_4 \times \mathbb{Z}_2)$

7. Details of the group structure of the group,

Semidirect product of Klein 4-Group and $\mathbb{Z}_4$: $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$

$$\begin{aligned}(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4 &= \{a^\alpha b^\beta : a^4 = b^4 = e, ba = a^{-1}b^{-1}\} \\ &= \{e, a, a^2, a^3, b, b^2, b^3, ab, ab^2, ab^3, a^2b, a^2b^2, a^2b^3, a^3b, a^3b^2, a^3b^3\}\end{aligned}$$

Also, $S_G = \{e, a^2, b^2, \}$ and $G_2 = \{a^2, b^2, ab, ab^3, a^2b^2, a^3b, a^3b^3\}$

Caley Table for $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$

$*$	$\mathbf{e}$	$\mathbf{a}$	$\mathbf{a^2}$	$\mathbf{a^3}$	$\mathbf{b}$	$\mathbf{b^2}$	$\mathbf{b^3}$	$\mathbf{ab}$	$\mathbf{ab^2}$	$\mathbf{ab^3}$	$\mathbf{a^2b}$	$\mathbf{a^2b^2}$	$\mathbf{a^2b^3}$	$\mathbf{a^3b}$	$\mathbf{a^3b^2}$	$\mathbf{a^3b^3}$
$\mathbf{e}$	e	a	a^2	a^3	b	b^2	b^3	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3	a^3b	a^3b^2	a^3b^3
$\mathbf{a}$	a	a^2	a^3	e	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3	a^3b	a^3b^2	a^3b^3	b	b^2	b^3
$\mathbf{a^2}$	a^2	a^3	e	a	a^2b	a^2b^2	a^2b^3	a^3b	a^3b^2	a^3b^3	b	b^2	b^3	ab	ab^2	ab^3
$\mathbf{a^3}$	a^3	e	a	a^2	a^3b	a^3b^2	a^3b^3	b	b^2	b^3	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3
$\mathbf{b}$	b	a^3b^3	a^2b	ab^3	b^2	b^3	e	a^3	a^3b	a^3b^2	a^2b^2	a^2b^3	a^2	a	ab	ab^2
$\mathbf{b^2}$	b^2	ab^2	a^2b^2	a^3b^2	b^3	e	b	ab^3	a	ab	a^2b^3	a^2	a^2b	a^3b^3	a^3	a^3b
$\mathbf{b^3}$	b^3	a^3b	a^2b^3	ab	e	b	b^2	a^3b^2	a^3b^3	a^3	a^2	a^2b	a^2b^2	ab^2	ab^3	a
$\mathbf{ab}$	ab	b^3	a^3b	a^2b^3	ab^2	ab^3	a	e	b	b^2	a^3b^2	a^3b^3	a^3	a^2	a^2b	a^2b^2
$\mathbf{ab^2}$	ab^2	a^2b^2	a^3b^2	b^2	ab^3	a	ab	a^2b^3	a^2	a^2b	a^3b^3	a^3	a^3b	b^3	e	b
$\mathbf{ab^3}$	ab^3	b	a^3b^3	a^2b	a	ab	ab^2	b^2	b^3	e	a^3	a^3b	a^3b^2	a^2b^2	a^2b^3	a^2
$\mathbf{a^2b}$	a^2b	ab^3	b	a^3b^3	a^2b^2	a^2b^3	a^2	a	ab	ab^2	b^2	b^3	e	a^3	a^3b	a^3b^2
$\mathbf{a^2b^2}$	a^2b^2	a^3b^2	b^2	ab^2	a^2b^3	a^2	a^2b	a^3b^3	a^3	a^3b	b^3	e	b	ab^3	a	ab
$\mathbf{a^2b^3}$	a^2b^3	ab	b^3	a^3b	a^2	a^2b	a^2b^2	ab^2	ab^3	a	e	b	b^2	a^3b^2	a^3b^3	a^3
$\mathbf{a^3b}$	a^3b	a^2b^3	ab	b^3	a^3b^2	a^3b^3	a^3	a^2	a^2b	a^2b^2	ab^2	ab^3	a	e	b	b^2
$\mathbf{a^3b^2}$	a^3b^2	b^2	ab^2	a^2b^2	a^3b^3	a^3	a^3b	b^3	e	b	ab^3	a	ab	a^2b^3	a^2	a^2b
$\mathbf{a^3b^3}$	a^3b^3	a^2b	ab^3	b	a^3	a^3b	a^3b^2	a^2b^2	a^2b^3	a^2	a	ab	ab^2	b^2	b^3	e

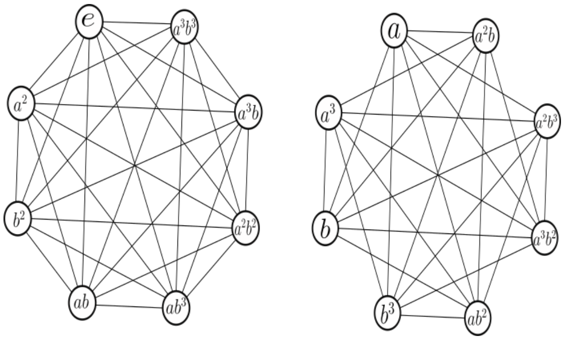

Figure 2.22: $\mathbb{S}_2((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4) \cong K_8 + \overline{4K_2}$

8. The group structure of the group, Direct product of $\mathbb{Q}_8$ and $\mathbb{Z}_2$: $\mathbb{Q}_8 \times \mathbb{Z}_2$

$$\begin{aligned} \mathbb{Q}_8 \times \mathbb{Z}_2 &= \{x^\alpha y^\beta z^\gamma : x^2 = y^2, x^4 = y^4 = z^2 = e, yx = x^{-1}y, zx = xz, zy = yz\} \\ &= \{e, x, x^2, x^3, y, y^3, z, xz, x^2z, x^3z, yz, y^3z, xy, x^3y, xyz, x^3yz\} \end{aligned}$$

Also, $S_G = \{e, x^2\}$ and $G_2 = \{x^2, z, x^2z\}$.

Caley Table for $\mathbb{Q}_8 \times \mathbb{Z}_2$																
$*$	**e**	**x**	$\mathbf{x^2}$	$\mathbf{x^3}$	**y**	$\mathbf{y^3}$	**z**	**xz**	$\mathbf{x^2z}$	$\mathbf{x^3z}$	**yz**	$\mathbf{y^3z}$	**xy**	$\mathbf{x^3y}$	**xyz**	$\mathbf{x^3yz}$
e	e	x	x^2	x^3	y	y^3	z	xz	x^2z	x^3z	yz	y^3z	xy	x^3y	xyz	x^3yz
x	x	x^2	x^3	e	xy	x^3y	xz	x^2z	x^3z	z	xyz	x^3yz	y^3	y	y^3z	yz
$\mathbf{x^2}$	x^2	x^3	e	x	y^3	y	x^2z	x^3z	z	xz	y^3z	yz	x^3y	xy	x^3yz	xyz
$\mathbf{x^3}$	x^3	e	x	x^2	x^3y	xy	x^3z	z	xz	x^2z	x^3yz	xyz	y	y^3	yz	y^3z
y	y	x^3y	y^3	xy	x^2	e	yz	x^3yz	y^3z	xyz	x^2z	z	x	x^3	xz	x^3z
$\mathbf{y^3}$	y^3	xy	y	x^3y	e	x^2	y^3z	xyz	yz	x^3yz	z	x^2z	x^3	x	x^3z	xz
z	z	xz	x^2z	x^3z	yz	y^3z	e	x	x^2	x^3	y	y^3	xyz	x^3yz	xy	x^3y
xz	xz	x^2z	x^3z	z	xyz	x^3yz	x	x^2	x^3	e	xy	x^3y	y^3z	yz	y^3	y
$\mathbf{x^2z}$	x^2z	x^3z	z	xz	y^3z	yz	x^2	x^3	e	x	y^3	y	x^3yz	xyz	x^3y	xy
$\mathbf{x^3z}$	x^3z	z	xz	x^2z	x^3yz	xyz	x^3	e	x	x^2	x^3y	xy	yz	y^3z	y	y^3
yz	yz	x^3yz	y^3z	xyz	x^2z	z	y	x^3y	y^3	xy	x^2	e	xz	x^3z	x	x^3
$\mathbf{y^3z}$	y^3z	xyz	yz	x^3yz	z	x^2z	y^3	xy	y	x^3y	e	x^2	x^3z	xz	x^3	x
xy	xy	y	x^3y	y^3	x^3	x	xyz	yz	x^3yz	y^3z	x^3z	xz	x^2	e	x^2z	z
$\mathbf{x^3y}$	x^3y	y^3	xy	y	x	x^3	x^3yz	y^3z	xyz	yz	xz	x^3z	e	x^2	z	x^2z
xyz	xyz	yz	x^3yz	y^3z	x^3z	xz	xy	y	x^3y	y^3	x^3	x	x^2z	z	x^2	e
$\mathbf{x^3yz}$	x^3yz	y^3z	xyz	yz	xz	x^3z	x^3y	y^3	xy	y	x	x^3	z	x^2z	e	x^2

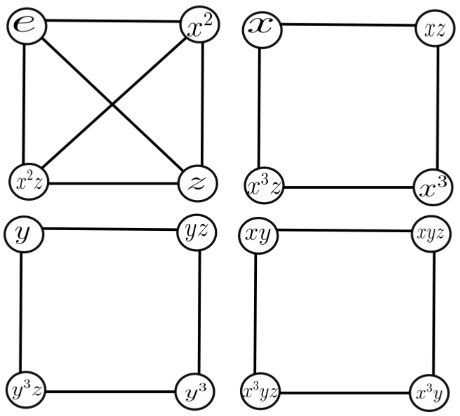

Figure 2.23: $\mathbb{S}_2(\mathbb{Q}_8 \times \mathbb{Z}_2) \cong K_4 + 3C_4$

9. Semidirect product of $\mathbb{Z}_4$ and $\mathbb{Z}_4$: $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$

$$\begin{aligned} \mathbb{Z}_4 \rtimes \mathbb{Z}_4 &= \{a^\alpha b^\beta : a^4 = b^4 = e, ba = a^{-1}b\} \\ &= \{e, a, a^2, a^3, b, b^2, b^3, ab, a^2b, a^3b, ab^2, a^2b^2, a^3b^2, ab^3, a^2b^3, a^3b^3\} \end{aligned}$$

Also, $S_G = \{e, a^2, b^2\}$ and $G_2 = \{a^2, b^2, a^2b^2\}$.

Caley Table for $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$

$*$	$\mathbf{e}$	$\mathbf{a}$	$\mathbf{a^2}$	$\mathbf{a^3}$	$\mathbf{b}$	$\mathbf{b^2}$	$\mathbf{b^3}$	$\mathbf{ab}$	$\mathbf{ab^2}$	$\mathbf{ab^3}$	$\mathbf{a^2b}$	$\mathbf{a^2b^2}$	$\mathbf{a^2b^3}$	$\mathbf{a^3b}$	$\mathbf{a^3b^2}$	$\mathbf{a^3b^3}$
$\mathbf{e}$	e	a	a^2	a^3	b	b^2	b^3	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3	a^3b	a^3b^2	a^3b^3
$\mathbf{a}$	a	a^2	a^3	e	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3	a^3b	a^3b^2	a^3b^3	b	b^2	b^3
$\mathbf{a^2}$	a^2	a^3	e	a	a^2b	a^2b^2	a^2b^3	a^3b	a^3b^2	a^3b^3	b	b^2	b^3	ab	ab^2	ab^3
$\mathbf{a^3}$	a^3	e	a	a^2	a^3b	a^3b^2	a^3b^3	b	b^2	b^3	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3
$\mathbf{b}$	b	a^3b	a^2b	ab	b^2	b^3	e	a^3b^2	a^3b^3	a^3	a^2b^2	a^2b^3	a^2	ab^2	ab^3	a
$\mathbf{b^2}$	b^2	ab^2	a^2b^2	a^3b^2	b^3	e	b	ab^3	a	ab	a^2b^3	a^2	a^2b	a^3b^3	a^3	a^3b
$\mathbf{b^3}$	b^3	a^3b^3	a^2b^3	ab^3	e	b	b^2	a^3	a^3b	a^3b^2	a^2	a^2b	a^2b^2	a	ab	ab^2
$\mathbf{ab}$	ab	b	a^3b	a^2b	ab^2	ab^3	a	b^2	b^3	e	a^3b^2	a^3b^3	a^3	a^2b^2	a^2b^3	a^2
$\mathbf{ab^2}$	ab^2	a^2b^2	a^3b^2	b^2	ab^3	a	ab	a^2b^3	a^2	a^2b	a^3b^3	a^3	a^3b	b^2	e	b
$\mathbf{ab^3}$	ab^3	b^3	a^3b^3	a^2b^3	a	ab	ab^2	e	b	b^2	a^3	a^3b	a^3b^2	a^2	a^2b	a^2b^2
$\mathbf{a^2b}$	a^2b	ab	b	a^3b	a^2b^2	a^2b^3	a^2	ab^2	ab^3	a	b^2	b^3	e	a^3b^2	a^3b^3	a^3
$\mathbf{a^2b^2}$	a^2b^2	a^3b^2	b^2	ab^2	a^2b^3	a^2	a^2b	a^3b^3	a^3	a^3b	b^3	e	b	ab^3	a	ab
$\mathbf{a^2b^3}$	a^2b^3	ab^3	b^3	a^3b^3	a^2	a^2b	a^2b^2	a	ab	ab^2	e	b	b^2	a^3	a^3b	a^3b^2
$\mathbf{a^3b}$	a^3b	a^2b	ab	b	a^3b^2	a^3b^3	a^3	a^2b^2	a^2b^3	a^2	ab^2	ab^3	a	b^2	b^3	e
$\mathbf{a^3b^2}$	a^3b^2	b^2	ab^2	a^2b^2	a^3b^3	a^3	a^3b	b^3	e	b	ab^3	a	ab	a^2b^3	a^2	a^2b
$\mathbf{a^3b^3}$	a^3b^3	a^2b^3	ab^3	b^3	a^3	a^3b	a^3b^2	a^2	a^2b	a^2b^2	a	ab	ab^2	e	b	b^2

The order two element graph $\mathbb{S}_2(\mathbb{Z}_4 \rtimes \mathbb{Z}_4)$ is graph isomorphic to $K_4 + 3C_4$.

10. Group of the Pauli Matrices : $SU(2)$

$$\begin{aligned} SU(2) &= \{x^\alpha y^\beta z^\gamma : x^4 = y^2 = z^2 = e, yx = xy, zx = xz, zy = x^2yz\} \\ &= \{e, x, x^2, x^3, y, z, xy, xz, x^2y, x^2z, x^3y, x^3z, yz, xyz, x^2yz, x^3yz\} \end{aligned}$$

Also, $S_G = \{e, x^2\}$ and $G_2 = \{x^2, y, z, x^2y, x^2z, xyz, x^3yz\}$.

Caley Table for $SU(2)$

$*$	**e**	**x**	$\mathbf{x^2}$	$\mathbf{x^3}$	**y**	**z**	**xy**	**xz**	$\mathbf{x^2y}$	$\mathbf{x^2z}$	$\mathbf{x^3y}$	$\mathbf{x^3z}$	**yz**	**xyz**	$\mathbf{x^2yz}$	$\mathbf{x^3yz}$
e	e	x	x^2	x^3	y	z	xy	xz	x^2y	x^2z	x^3y	x^3z	yz	xyz	x^2yz	x^3yz
x	x	x^2	x^3	e	xy	xz	x^2y	x^2z	x^3y	x^3z	y	z	xyz	x^2yz	x^3yz	yz
$\mathbf{x^2}$	x^2	x^3	e	x	x^2y	x^2z	x^3y	x^3z	y	z	xy	xz	x^2yz	x^3yz	yz	xyz
$\mathbf{x^3}$	x^3	e	x	x^2	x^3y	x^3z	y	z	xy	xz	x^2y	x^2z	x^3yz	yz	xyz	x^2yz
y	y	xy	x^2y	x^3y	e	yz	x	xyz	x^2	x^2yz	x^3	x^3yz	z	xz	x^2z	x^3z
z	z	xz	x^2z	x^3z	x^2yz	e	x^3yz	x	yz	x^2	xyz	x^3	x^2y	x^3y	y	xy
xy	xy	x^2y	x^3y	y	x	xyz	x^2	x^2yz	x^3	x^3yz	e	yz	xz	x^2z	x^3z	z
xz	xz	x^2z	x^3z	z	x^3yz	x	yz	x^2	xyz	x^3	x^2yz	e	x^3y	y	xy	x^2y
$\mathbf{x^2y}$	x^2y	x^3y	y	xy	x^2	x^2yz	x^3	x^3yz	e	yz	x	xyz	x^2z	x^3z	z	xz
$\mathbf{x^2z}$	x^2z	x^3z	z	xz	yz	x^2	xyz	x^3	x^2yz	e	x^3yz	x	y	xy	x^2y	x^3y
$\mathbf{x^3y}$	x^3y	y	xy	x^2y	x^3	x^3yz	e	yz	x	xyz	x^2	x^2yz	x^3z	z	xz	x^2z
$\mathbf{x^3z}$	x^3z	z	xz	x^2z	xyz	x^3	x^2yz	e	x^3yz	x	yz	x^2	xy	x^2y	x^3y	y
yz	yz	xyz	x^2yz	x^3yz	x^2z	y	x^3z	xy	z	x^2y	xz	x^3y	x^2	x^3	e	x
xyz	xyz	x^2yz	x^3yz	yz	x^3z	xy	z	x^2y	xz	x^3y	x^2z	y	x^3	e	x	x^2
$\mathbf{x^2yz}$	x^2yz	x^3yz	yz	xyz	z	x^2y	xz	x^3y	x^2z	y	x^3z	xy	e	x	x^2	x^3
$\mathbf{x^3yz}$	x^3yz	yz	xyz	x^2yz	xz	x^3y	x^2z	y	x^3z	xy	z	x^2y	x	x^2	x^3	e

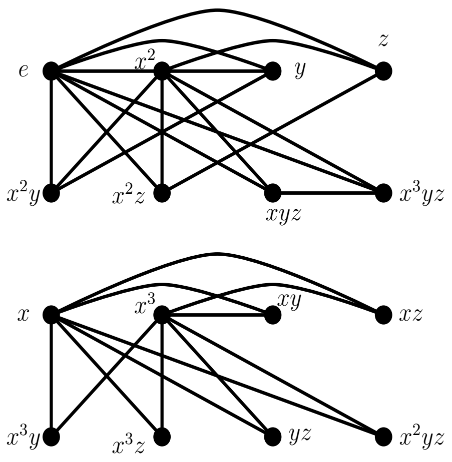

Figure 2.24: $\mathbb{S}_2(SU(2))$

11. Modular or Isanowa group of order 16 : M_{16}

$$\begin{aligned} M_{16} &= \{x^{\alpha}y^{\beta} : x^8 = y^2 = e, yx = x^5y\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

Also, $S_G = \{e, x^2, x^4, x^6\}$ and $G_2 = \{x^4, y, x^4y\}$.

Caley Table for M_{16}

$*$	$\mathbf{e}$	$\mathbf{x}$	$\mathbf{x^2}$	$\mathbf{x^3}$	$\mathbf{x^4}$	$\mathbf{x^5}$	$\mathbf{x^6}$	$\mathbf{x^7}$	$\mathbf{y}$	$\mathbf{xy}$	$\mathbf{x^2y}$	$\mathbf{x^3y}$	$\mathbf{x^4y}$	$\mathbf{x^5y}$	$\mathbf{x^6y}$	$\mathbf{x^7y}$
$\mathbf{e}$	e	x	x^2	x^3	x^4	x^5	x^6	x^7	y	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y
$\mathbf{x}$	x	x^2	x^3	x^4	x^5	x^6	x^7	e	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	y
$\mathbf{x^2}$	x^2	x^3	x^4	x^5	x^6	x^7	e	x	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	y	xy
$\mathbf{x^3}$	x^3	x^4	x^5	x^6	x^7	e	x	x^2	x^3y	x^4y	x^5y	x^6y	x^7y	y	xy	x^2y
$\mathbf{x^4}$	x^4	x^5	x^6	x^7	e	x	x^2	x^3	x^4y	x^5y	x^6y	x^7y	y	xy	x^2y	x^3y
$\mathbf{x^5}$	x^5	x^6	x^7	e	x	x^2	x^3	x^4	x^5y	x^6y	x^7y	y	xy	x^2y	x^3y	x^4y
$\mathbf{x^6}$	x^6	x^7	e	x	x^2	x^3	x^4	x^5	x^6y	x^7y	y	xy	x^2y	x^3y	x^4y	x^5y
$\mathbf{x^7}$	x^7	e	x	x^2	x^3	x^4	x^5	x^6	x^7y	y	xy	x^2y	x^3y	x^4y	x^5y	x^6y
$\mathbf{y}$	y	x^5y	x^2y	x^7y	x^4y	xy	x^6y	x^3y	e	x^5	x^2	x^7	x^4	x	x^6	x^3
$\mathbf{xy}$	xy	x^6y	x^3y	y	x^5y	x^2y	x^7y	x^4y	x	x^6	x^3	e	x^5	x^2	x^7	x^4
$\mathbf{x^2y}$	x^2y	x^7y	x^4y	xy	x^6y	x^3y	y	x^5y	x^2	x^7	x^4	x	x^6	x^3	e	x^5
$\mathbf{x^3y}$	x^3y	y	x^5y	x^2y	x^7y	x^4y	xy	x^6y	x^3	e	x^5	x^2	x^7	x^4	x	x^6
$\mathbf{x^4y}$	x^4y	xy	x^6y	x^3y	y	x^5y	x^2y	x^7y	x^4	x	x^6	x^3	e	x^5	x^2	x^7
$\mathbf{x^5y}$	x^5y	x^2y	x^7y	x^4y	xy	x^6y	x^3y	y	x^5	x^2	x^7	x^4	x	x^6	x^3	e
$\mathbf{x^6y}$	x^6y	x^3y	y	x^5y	x^2y	x^7y	x^4y	xy	x^6	x^3	e	x^5	x^2	x^7	x^4	x
$\mathbf{x^7y}$	x^7y	x^4y	xy	x^6y	x^3y	y	x^5y	x^2y	x^7	x^4	x	x^6	x^3	e	x^5	x^2

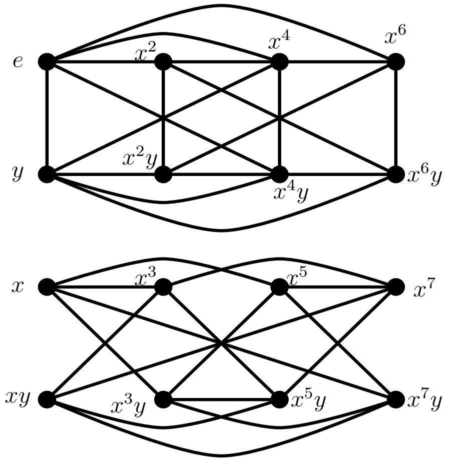

Figure 2.25: $\mathbb{S}_2(M_{16})$

12. Dicyclic Group of Degree 4 : Dic_4

$$\begin{aligned} Dic_4 &= \{x^{\alpha}y^{\beta} : x^4 = y^2; x^8 = y^4 = e, yx = x^{-1}y\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

Also, $S_G = \{e, x^2, x^4, x^6\}$ and $G_2 = \{x^4\}$.

Caley Table for Dic_4

$*$	$\mathbf{e}$	$\mathbf{x}$	$\mathbf{x^2}$	$\mathbf{x^3}$	$\mathbf{x^4}$	$\mathbf{x^5}$	$\mathbf{x^6}$	$\mathbf{x^7}$	$\mathbf{y}$	$\mathbf{xy}$	$\mathbf{x^2y}$	$\mathbf{x^3y}$	$\mathbf{x^4y}$	$\mathbf{x^5y}$	$\mathbf{x^6y}$	$\mathbf{x^7y}$
$\mathbf{e}$	e	x	x^2	x^3	x^4	x^5	x^6	x^7	y	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y
$\mathbf{x}$	x	x^2	x^3	x^4	x^5	x^6	x^7	e	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	y
$\mathbf{x^2}$	x^2	x^3	x^4	x^5	x^6	x^7	e	x	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	y	xy
$\mathbf{x^3}$	x^3	x^4	x^5	x^6	x^7	e	x	x^2	x^3y	x^4y	x^5y	x^6y	x^7y	y	xy	x^2y
$\mathbf{x^4}$	x^4	x^5	x^6	x^7	e	x	x^2	x^3	x^4y	x^5y	x^6y	x^7y	y	xy	x^2y	x^3y
$\mathbf{x^5}$	x^5	x^6	x^7	e	x	x^2	x^3	x^4	x^5y	x^6y	x^7y	y	xy	x^2y	x^3y	x^4y
$\mathbf{x^6}$	x^6	x^7	e	x	x^2	x^3	x^4	x^5	x^6y	x^7y	y	xy	x^2y	x^3y	x^4y	x^5y
$\mathbf{x^7}$	x^7	e	x	x^2	x^3	x^4	x^5	x^6	x^7y	y	xy	x^2y	x^3y	x^4y	x^5y	x^6y
$\mathbf{y}$	y	x^7y	x^6y	x^5y	x^4y	x^3y	x^2y	xy	x^4	x^3	x^2	x	e	x^7	x^6	x^5
$\mathbf{xy}$	xy	y	x^7y	x^6y	x^5y	x^4y	x^3y	x^2y	x^5	x^4	x^3	x^2	x	e	x^7	x^6
$\mathbf{x^2y}$	x^2y	xy	y	x^7y	x^6y	x^5y	x^4y	x^3y	x^6	x^5	x^4	x^3	x^2	x	e	x^7
$\mathbf{x^3y}$	x^3y	x^2y	xy	y	x^7y	x^6y	x^5y	x^4y	x^7	x^6	x^5	x^4	x^3	x^2	x	e
$\mathbf{x^4y}$	x^4y	x^3y	x^2y	xy	y	x^7y	x^6y	x^5y	e	x^7	x^6	x^5	x^4	x^3	x^2	x
$\mathbf{x^5y}$	x^5y	x^4y	x^3y	x^2y	xy	y	x^7y	x^6y	x	e	x^7	x^6	x^5	x^4	x^3	x^2
$\mathbf{x^6y}$	x^6y	x^5y	x^4y	x^3y	x^2y	xy	y	x^7y	x^2	x	e	x^7	x^6	x^5	x^4	x^3
$\mathbf{x^7y}$	x^7y	x^6y	x^5y	x^4y	x^3y	x^2y	xy	y	x^3	x^2	x	e	x^7	x^6	x^5	x^4

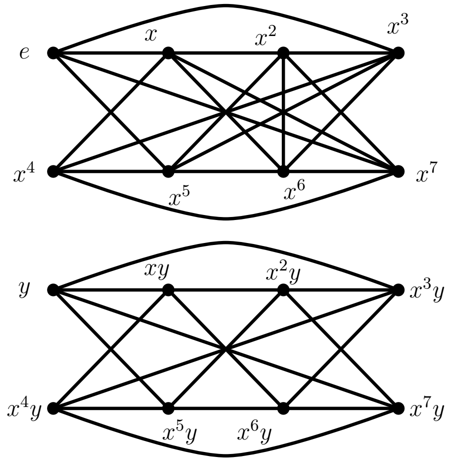

Figure 2.26: Complement graph of $\mathbb{S}_2(D_8)/\mathbb{S}_2(Dic_4)$

13. Semidihedral group of degree 2 : SD_2

$$\begin{aligned} SD_2 &= \{x^\alpha y^\beta : x^8 = y^2 = e, yx = x^3y\} \\ &= \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, y, xy, x^2y, x^3y, x^4y, x^5y, x^6y, x^7y\} \end{aligned}$$

Also, $S_G = \{e, x^2, x^4, x^6\}$ and $G_2 = \{x^4, y, x^2y, x^4y, x^6y\}$.

Caley Table for SD_2																
$*$	$\mathbf{e}$	$\mathbf{x}$	$\mathbf{x^2}$	$\mathbf{x^3}$	$\mathbf{x^4}$	$\mathbf{x^5}$	$\mathbf{x^6}$	$\mathbf{x^7}$	$\mathbf{y}$	$\mathbf{xy}$	$\mathbf{x^2y}$	$\mathbf{x^3y}$	$\mathbf{x^4y}$	$\mathbf{x^5y}$	$\mathbf{x^6y}$	$\mathbf{x^7y}$
$\mathbf{e}$	e	x	x^2	x^3	x^4	x^5	x^6	x^7	y	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y
$\mathbf{x}$	x	x^2	x^3	x^4	x^5	x^6	x^7	e	xy	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	y
$\mathbf{x^2}$	x^2	x^3	x^4	x^5	x^6	x^7	e	x	x^2y	x^3y	x^4y	x^5y	x^6y	x^7y	y	xy
$\mathbf{x^3}$	x^3	x^4	x^5	x^6	x^7	e	x	x^2	x^3y	x^4y	x^5y	x^6y	x^7y	y	xy	x^2y
$\mathbf{x^4}$	x^4	x^5	x^6	x^7	e	x	x^2	x^3	x^4y	x^5y	x^6y	x^7y	y	xy	x^2y	x^3y
$\mathbf{x^5}$	x^5	x^6	x^7	e	x	x^2	x^3	x^4	x^5y	x^6y	x^7y	y	xy	x^2y	x^3y	x^4y
$\mathbf{x^6}$	x^6	x^7	e	x	x^2	x^3	x^4	x^5	x^6y	x^7y	y	xy	x^2y	x^3y	x^4y	x^5y
$\mathbf{x^7}$	x^7	e	x	x^2	x^3	x^4	x^5	x^6	x^7y	y	xy	x^2y	x^3y	x^4y	x^5y	x^6y
$\mathbf{y}$	y	x^3y	x^6y	xy	x^4y	x^7y	x^2y	x^5y	e	x^3	x^6	x	x^4	x^7	x^2	x^5
$\mathbf{xy}$	xy	x^4y	x^7y	x^2y	x^5y	y	x^3y	x^6y	x	x^4	x^7	x^2	x^5	e	x^3	x^6
$\mathbf{x^2y}$	x^2y	x^5y	y	x^3y	x^6y	xy	x^4y	x^7y	x^2	x^5	e	x^3	x^6	x	x^4	x^7
$\mathbf{x^3y}$	x^3y	x^6y	xy	x^4y	x^7y	x^2y	x^5y	y	x^3	x^6	x	x^4	x^7	x^2	x^5	e
$\mathbf{x^4y}$	x^4y	x^7y	x^2y	x^5y	y	x^3y	x^6y	xy	x^4	x^7	x^2	x^5	e	x^3	x^6	x
$\mathbf{x^5y}$	x^5y	y	x^3y	x^6y	xy	x^4y	x^7y	x^2y	x^5	e	x^3	x^6	x	x^4	x^7	x^2
$\mathbf{x^6y}$	x^6y	xy	x^4y	x^7y	x^2y	x^5y	y	x^3y	x^6	x	x^4	x^7	x^2	x^5	e	x^3
$\mathbf{x^7y}$	x^7y	x^2y	x^5y	y	x^3y	x^6y	xy	x^4y	x^7	x^2	x^5	e	x^3	x^6	x	x^4

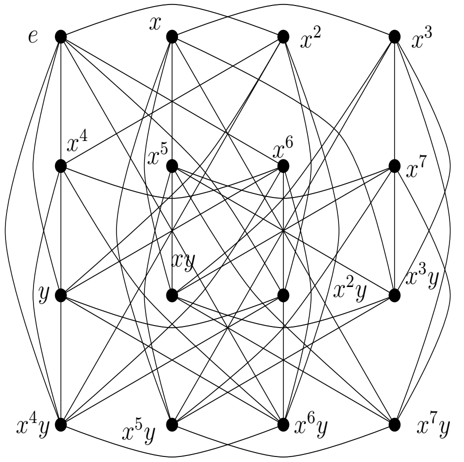

Figure 2.27: $\mathbb{S}_2(SD_2)$

No. of Vertices	**Group G**	$\mathbb{S}_2(\mathbf{G})$	**Type of** $\mathbb{S}_2(\mathbf{G})$
1	$\{e\}$	K_1	Complete, Planar
2	$\mathbb{Z}_2$	K_2	Complete, Planar
3	$\mathbb{Z}_3$	$\overline{K_1 + K_2}$	Connected, Planar
4	$\mathbb{Z}_4$	$2K_1 + K_2$	Disconnected, Planar
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	K_4	Complete, Planar
5	$\mathbb{Z}_5$	$\overline{K_1 + 2K_2}$	Connected, Planar
6	$\mathbb{Z}_6$	Figure 2.13	Connected, Planar
	D_3	$\overline{4K_1 + K_2}$	Connected
7	$\mathbb{Z}_7$	$\overline{K_1 + 3K_2}$	Connected
8	$\mathbb{Z}_8$	$C_4 + \overline{2K_1 + K_2}$	Disconnected, Planar
	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$K_4 + C_4$	Disconnected, Planar
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	K_8	Complete
	D_4	Figure 2.14	Connected
	Q_8	$6K_1 + 2K_2$	Disconnected, Planar
9	$\mathbb{Z}_9$	$\overline{K_1 + 4K_2}$	Connected
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\overline{K_1 + 4K_2}$	Connected
10	$\mathbb{Z}_{10}$	Figure 2.15	Connected
	D_5	$\overline{6K_1 + 2K_2}$	Connected
11	$\mathbb{Z}_{11}$	$\overline{K_1 + 5K_2}$	Connected
12	$\mathbb{Z}_{12}$	$\overline{2K_1 + 2K_2} + \overline{3K_2}$	Disconnected
	$\mathbb{Z}_6 \times \mathbb{Z}_2$	Figure 2.16	Connected
	A_4	$\overline{2K_1 + 5K_2}$	Connected
	D_6	Figure 2.17	Connected
	$\mathbb{Z}_4 \rtimes \mathbb{Z}_3$	Figure 2.18	Disconnected, Planar
13	$\mathbb{Z}_{13}$	$\overline{K_1 + 6K_2}$	Connected
14	$\mathbb{Z}_{14}$	Figure 2.19	Connected
	D_7	$\overline{8K_1 + 3K_2}$	Connected
15	$\mathbb{Z}_{15}$	$\overline{K_1 + 7K_2}$	Connected
16	$\mathbb{Z}_{16}$	$\overline{2K_1 + 3K_2} + \overline{4K_2}$	Disconnected
	$\mathbb{Z}_8 \times \mathbb{Z}_2$	Figure 2.20	Disconnected
	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$K_4 + 3C_4$	Disconnected, Planar
	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$K_8 + \overline{4K_2}$	Disconnected
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	K_{16}	Complete
	$D_4 \times \mathbb{Z}_2$	Figure 2.21	Connected
	$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$	$K_8 + \overline{4K_2}$	Disconnected
	$Q_8 \times \mathbb{Z}_2$	$K_4 + 3C_4$	Disconnected, Planar
	$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	$K_4 + 3C_4$	Disconnected, Planar
	$SU(2)$	Figure 2.24	Disconnected
	M_{16}	Figure 2.25	Disconnected
	Dic_4	Figure 2.26	Connected
	SD_2	Figure 2.27	Connected
	D_8	Figure 2.26	Connected

Chapter 3

Independence and Domination number of order two element graph over a group

Chapter 3

Independence and Domination number of $\mathbb{S}_2(G)$

3.1 Introduction

Algebraic graph theory opened a new horizon in the field of research in mathematics. This branch builds a beautiful relation between algebra and graph theory. Also many people analyze many topological indices like Merrified-Simmons index [21], Wiener Polarity index, Hosoya index, Hyper-Wiener index, Zagreb index etc over graphs. The square element graphs were defined over a finite commutative ring by Sen Gupta and Sen in [48]. Later Biswas et. al. [16] introduced the square element graph over a semigroup. In [41], Pradhan, Biswas and Kar introduced order two element graph over a group by adding some extra conditions with square element graph. We mainly characterize some properties of order two element graph over some finite groups.

In the first section, we establish some beautiful results of Merrified-Simmons index (studied in [21]) of order two element graph over some finite groups. In the next section, we characterize the product of two order two element graphs over two groups. In the last section, we study some results of domination number of order two element graph over some finite groups.

3.2 Merrified - Simmons index of the graph $\mathbb{S}_2(G)$ over a group G

In this section, we study about the independence number and Merrifield-Simmons index of order two element graph over some finite groups. We firstly start about the definition of independence number of a given graph, which is as follows:

Definition 3.2.1. *Independence number [52] : A set of vertices of a graph $\mathcal{G}$ is independent if the vertices are pairwise nonadjacent. The independence number of a graph $\mathcal{G}$, denoted by $u(\mathcal{G})$, is the cardinality of a largest independent set of $\mathcal{G}$.*

Now, we give an example for better understanding the definition of independence number of a given graph.

Example 3.2.2. *Let $\mathcal{G} = (V, E)$ be a graph, which is shown in the below Figure 3.1.*

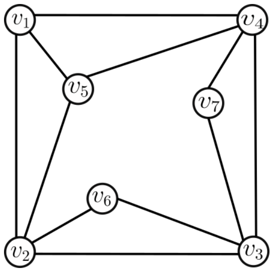

Figure 3.1

Let the vertex set of $\mathcal{G}$ be $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. One can see that there are many subsets of V like $S_1 = \{v_1, v_3\}$, $S_2 = \{v_2, v_4\}$, $S_3 = \{v_1, v_6\}$, $S_4 = \{v_1, v_7\}$, $S_5 = \{v_2, v_7\}$, $S_6 = \{v_3, v_5\}$, $S_7 = \{v_4, v_6\}$, $S_8 = \{v_5, v_6, v_7\}$, which are all independent subsets. Among of all of these independent subsets of V, S_8 is the largest independent subset of V. Therefore, the independence number of the graph $\mathcal{G}$ is the cardinality of S_4 which is equal to 3. So, $u(\mathcal{G}) = 3$.

Now, we give the definition of Merrifield-Simmons index of a graph in terms of independent sets.

Definition 3.2.3. *The Merrifield - Simmons index (defined in [28]) of a graph* $\mathcal{G} = (V, E)$*, denoted by* $i(\mathcal{G})$*, is the total number of independent sets (including the empty set) of vertices of* $\mathcal{G}$ *and is defined by* $i(\mathcal{G}) = \sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G})$*, where* $i_0(\mathcal{G}) = 1$*,* $i_1(\mathcal{G}) =$ *number of vertices in* $\mathcal{G} = |V|$*,*

$i_2(\mathcal{G}) =$ *total number of pairs of vertices in* V *such that they are non-adjacent.*

$i_3(\mathcal{G}) =$ *total number of* $3-$ *tuple vertices in* V *such that they are pairwise non-adjacent.*

$i_k(\mathcal{G}) =$ *total number of* $k-$ *tuple vertices in* V *such that they are pairwise non-adjacent.*

For better understanding the definition of Merrifield-Simmons index of a graph, we give the following example to understand the concept of finding Merrifield-Simmons index of given graph.

Example 3.2.4. *Let* $\mathcal{G} = (V, E)$ *be a graph which is shown in Figure 3.2.*

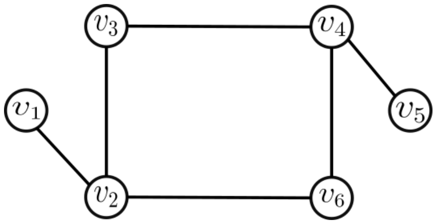

Figure 3.2

In this graph, the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$*. Then from the definition of Merrifield-Simmons index, we have*

$i_0(\mathcal{G}) = 1$, $i_1(\mathcal{G}) = |V| = 6$

$i_2(\mathcal{G}) = |\{(v_i, v_j) : v_i, v_j \in V \text{and } v_i, v_j \text{ are non-adjacent}\}|$

$= |\{(v_1, v_3), (v_1, v_4), (v_1, v_5), (v_1, v_6), (v_2, v_4), (v_2, v_5), (v_3, v_5), (v_3, v_6), (v_5, v_6)\}| = 9$

$i_3(\mathcal{G}) = |\{(v_i, v_j, v_k) : v_i, v_j, v_k \in V \text{and } v_i, v_j, v_k \text{ are pairwise non-adjacent}\}|$

$= |\{(v_1, v_3, v_5), (v_1, v_3, v_6), (v_3, v_5, v_6)\}| = 3$

$i_4(\mathcal{G}) = |\{(v_i, v_j, v_k, v_l) : v_i, v_j, v_k, v_l \in V \text{and } v_i, v_j, v_k, v_l \text{ are pairwise non-adjacent}\}|$

$= |\{(v_1, v_3, v_5, v_6)\}| = 1$

Also, $i_k(\mathcal{G}) = 0$ *for* $k \geq 5$ *i.e., we cannot find any* $5-$*tuple or* $6-$*tuple vertices so that they are pairwise adjacent.*

Therefore, $i(\mathcal{G}) = \sum_{k=0}^{|V|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + i_2(\mathcal{G}) + i_3(\mathcal{G}) + i_4(\mathcal{G}) + i_5(\mathcal{G}) + i_6(\mathcal{G})$
$= 1 + 6 + 9 + 3 + 1 + 0 + 0 = 20.$

Lemma 3.2.5. *[28] Let* $\mathcal{G} = (V, E)$ *be a graph.*
(i) If $uv \in E(\mathcal{G})$*, then* $i(\mathcal{G}) = i(\mathcal{G} - uv) - i(\mathcal{G} - \{N[u] \cup N[v]\})$
(ii) If $v \in V(\mathcal{G})$*, then* $i(\mathcal{G}) = i(\mathcal{G} - v) + i(\mathcal{G} - N[v])$
(iii) If $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_t$ *are the components of the graph* $\mathcal{G}$*, then* $i(\mathcal{G}) = \prod_{j=1}^{t} i(\mathcal{G}_j)$.

Theorem 3.2.6. *[52] Independent set of a graph is a clique of its complement graph & vice-versa.*

Using the Theorem 3.2.6, we have the following lemma.

Lemma 3.2.7. *Let* $\mathcal{G}$ *be any graph and for any positive integer* $n > 1$*, we have* $i_n(\mathcal{G}) =$ *number of* K_n *in* $\overline{\mathcal{G}}$.

Proposition 3.2.8. $i(\overline{K_1 + K_2}) = 4.$

Proof. Using the Theorem 3.2.6 and Lemma 3.2.7, we have $i(\overline{K_1 + K_2}) = i_0(\overline{K_1 + K_2}) + i_1(\overline{K_1 + K_2}) + i_2(\overline{K_1 + K_2}) + i_3(\overline{K_1 + K_2}) = 1 + 3 + 1 = 5$, as $i_0(\overline{K_1 + K_2}) = 1$, $i_1(\overline{K_1 + K_2}) =$ number of vertices in $\overline{K_1 + K_2} = 3$, $i_2(\overline{K_1 + K_2}) = 1$ and $i_k(\overline{K_1 + K_2}) = 0$ for $k > 2$. □

Lemma 3.2.9. *From Lemma 3.2.7, we have* $i_2(\overline{K_1 + \frac{n-1}{2} K_2}) = \dfrac{n-1}{2}.$

Theorem 3.2.10. *[41] The graph* $\mathbb{S}_2(\mathbb{Z}_n)$ *is connected if* n *or* $\frac{n}{2}$ *is an odd positive integer where as the graph* $\mathbb{S}_2(\mathbb{Z}_n)$ *is disconnected if* $\frac{n}{2}$ *is an even positive integer and*

$$\mathbb{S}_2(\mathbb{Z}_n) = \begin{cases} \overline{K_1 + (\frac{n-1}{2})K_2} & \text{if } n \text{ is odd positive integer.} \\ \overline{2K_1 + (\frac{n}{4} - 1)K_2} + \overline{\frac{n}{4}K_2} & \text{if } \frac{n}{2} \text{ is even positive integer.} \end{cases}$$

Theorem 3.2.11. *For an integer n,*

$$i(\mathbb{S}_2(\mathbb{Z}_n)) = \begin{cases} \dfrac{3n+1}{2} & if\ n\ is\ an\ odd\ nubmer. \\ \dfrac{3n(3n+4)}{16} & if\ \frac{n}{2}\ is\ an\ even\ number. \end{cases}$$

Proof. **Case I:** Let n be an odd number and from Theorem 3.2.10, $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n) = \overline{K_1 + (\frac{n-1}{2})K_2}$. Then $i_0(\mathcal{G}) = 1$; $i_1(\mathcal{G}) =$ number of vertices in $\mathcal{G} = n$; $i_2(\mathcal{G}) =$ number of K_2 in $K_1 + (\frac{n-1}{2})K_2 = \dfrac{n-1}{2}$; $i_3(\mathcal{G}) = i_3(\overline{K_1 + (\frac{n-1}{2})K_2}) =$ number of K_3 in $K_1 + (\frac{n-1}{2})K_2 = 0$ and $i_k(\mathcal{G}) = 0 \ \forall\ k \geq 3$.

Now $i(\mathcal{G}) = \displaystyle\sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + i_2(\mathcal{G}) + i_3(\mathcal{G}) + \cdots + i_n(\mathcal{G})$
$= 1 + n + \dfrac{n-1}{2} + 0 + \cdots + 0 = \dfrac{3n+1}{2}$.

Case II: Suppose $\frac{n}{2}$ is even number. Then from Theorem 3.2.10,
$\mathbb{S}_2(\mathbb{Z}_n) = \overline{2K_1 + (\frac{n}{4} - 1)K_2} + \overline{\frac{n}{4}K_2}$.

Let $G_1 = \overline{2K_1 + (\frac{n}{4} - 1)K_2}$ and $G_2 = \overline{\frac{n}{4}K_2}$. Here $\mid G_1 \mid = \dfrac{n}{2}$ and $\mid G_2 \mid = \dfrac{n}{2}$.

Now, $i(\mathbb{S}_2(\mathbb{Z}_n)) = i(\overline{2K_1 + (\frac{n}{4} - 1)K_2} + \overline{\frac{n}{4}K_2}) = i(G_1 + G_2) = i(G_1) \times i(G_2)$ [From Lemma 3.2.5]
$= [i_0(G_1) + i_1(G_1) + i_2(G_1) + 0] \times [i_0(G_2) + i_1(G_2) + i_2(G_2) + 0]$
$= (1 + \dfrac{n}{2} + \dfrac{n}{4} - 1) \times (1 + \dfrac{n}{2} + \dfrac{n}{4}) = \dfrac{3n(3n+4)}{16}$. □

Theorem 3.2.12. *Let $\mathcal{G}$ be any simple graph with n vertices. Then*

$$i_2(\mathcal{G}) = \frac{1}{2} \sum_{v \in V(\mathcal{G})} \{(n-1) - d(v)\}$$

.

Proof. For any simple graph $\mathcal{G}$, let $\mid V(\mathcal{G}) \mid = n$ and $\mid E(\mathcal{G}) \mid = m$.
Then $i_2(\mathcal{G}) = \binom{n}{2} - m = \dfrac{n(n-1)}{2} - m$
$= \dfrac{1}{2}\{n(n-1) - \displaystyle\sum_{v \in V(\mathcal{G})} d(v)\}$, Since $\displaystyle\sum_{v \in V(\mathcal{G})} d(v) = 2m$.
$= \dfrac{1}{2} \displaystyle\sum_{v \in V(\mathcal{G})} \{(n-1) - d(v)\}$. □

Now we explain graphically, how independent set works on $\mathbb{S}_2(\mathbb{Z}_{4m+2})$ for any $m \in \mathbb{N}$ in the following example.

Example 3.2.13. *Let* $G = \mathbb{Z}_{10} = \{e, x, x^2, x^3, \cdots, x^9\}$ *and* $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_{10})$*. Then the order two element graph* $\mathbb{S}_2(\mathbb{Z}_{10})$ *is as follows:*

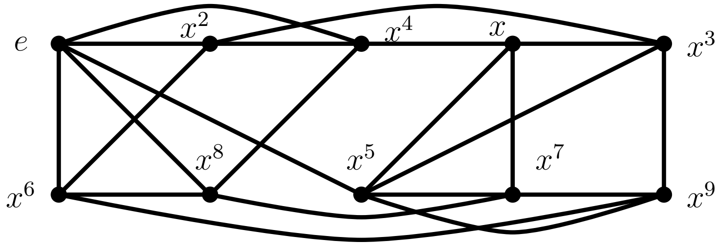

Figure 3.3: $\mathbb{S}_2(\mathbb{Z}_{10})$

From the Figure 3.3, we obtain

$i_0(\mathcal{G}) = 1$, $i_1(\mathcal{G}) = 10$,

$i_2(\mathcal{G}) = |\{(v_i, v_j) : v_i, v_j \in \mathbb{Z}_{10}$ *and* v_i, v_j *are non-adjacent*$\}|$

$= |\{(e,x),(e,x^3),(e,x^7),(e,x^9),(x,x^2),(x,x^6),(x,x^8),(x,x^9),(x^2,x^5),(x^2,x^7),$ $(x^2,x^8),(x^2,x^9),(x^3,x^4),(x^3,x^6),(x^3,x^7),(x^3,x^8),(x^4,x^5),(x^4,x^6),(x^4,x^7),(x^4,x^9),$ $(x^5,x^6),(x^5,x^8),(x^6,x^7),(x^8,x^9)\}| = 24$

$i_3(\mathcal{G}) = |\{(v_i, v_j, v_k) : v_i, v_j, v_k \in \mathbb{Z}_{10}$ *and* v_i, v_j, v_k *are pairwise non-adjacent*$\}|$

$= |\{(e,x,x^9),(e,x^3,x^7),(e,x^3,x^7),(x,x^2,x^8),(x,x^2,x^9),(x,x^6,x^9),(x^2,x^5,x^8),$ $(x^2,x^8,x^9),(x^3,x^4,x^6),(x^3,x^4,x^7),(x^3,x^6,x^7),(x^4,x^5,x^6)\}| = 12$

$i_4(\mathcal{G}) = |\{(v_i, v_j, v_k, v_l) : v_i, v_j, v_k, v_l \in \mathbb{Z}_{10}$ *and* v_i, v_j, v_k, v_l *are pairwise non-adjacent*$\}|$

$= |\{(x,x^2,x^8,x^9),(x^3,x^4,x^6,x^7)\}| = 2.$

Also, we can not find any $5-$*tuples vertices so that they are pairwise adjacent in* $\mathcal{G}$*. Therefore,* $i(\mathcal{G}) = \sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + i_2(\mathcal{G}) + i_3(\mathcal{G}) + i_4(\mathcal{G}) + \sum_{k=5}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = 1 + 10 + 24 + 12 + 2 + 0 = 49.$

From the above example, we understand the graphical structure and visualize the graph $\mathbb{S}_2(\mathbb{Z}_{4m+2})$ and we have the following lemmas.

Lemma 3.2.14. $i_2(\mathbb{S}_2(\mathbb{Z}_n)) = 4m(m+1)$*, where* $n = 4m + 2$ *for any positive integer* m*.*

Proof. Consider the graph $\mathbb{S}_2(\mathbb{Z}_n)$, where $n = 4m + 2$ for any positive integer m. Let $G = \mathbb{Z}_{4m+2} = \{e, x, x^2, x^3, ..., x^{4m+1}\}$. Then $S_G = \{e, x^2, x^4, ..., x^{4m}\}$ and $G_2 = \{x^{2m+1}\}$. Now the vertex 'e' is adjacent to $x^2, x^4, ..., x^{4m}$ and x^{2m+1}. So, $d(e) = 2m + 1$. Also the vertex 'x^{2m+1}' is adjacent to $x, x^3, x^5, ..., x^{2m-1}, x^{2m+3}, ..., x^{4m+1}$ and 'e'. So, $d(x^{2m+1}) = 2m + 1$. Since x^i for $i = 1, 3, 5, ..., 4m + 1$ is adjacent to x^{2m+1-i} and $x, x^3, ..., x^{4m+1}$ except x^{4m+2-i}. So $d(x^i) = 2m$. Similarly, x^j for $j = 2, 4, ..., 4m$ is adjacent to $x^2, x^4, ..., x^{4m}$ except x^{4m+2-j}. So $d(x^j) = 2m$. So, $d(v) = 2m \; \forall v \in \mathbb{S}_2(\mathbb{Z}_n) \setminus \{e, x^{2m+1}\}$. Then from Theorem 3.2.12, $i_2(\mathbb{S}_2(\mathbb{Z}_n)) = \frac{1}{2}\{2(4m+2-1-(2m+1))+4m(4m+2-1-2m)\} = \frac{1}{2}(4m+8m^2+4m) = 4m(m+1)$. □

Lemma 3.2.15. *Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$ for $n = 4m+2$, $m \in \mathbb{N}$. Then $i_3(\mathcal{G}) = 2m(2m-1)$.*

Proof. Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$ and $G = \mathbb{Z}_n$ for $n = 4m + 2$, $m \in \mathbb{N}$. Then $G = \mathbb{Z}_{4m+2} = \{e, x, x^2, x^3, ..., x^{4m+1}\}$. Now, $S_G = \{e, x^2, x^4, x^6, ..., x^{4m}\}$ and $G_2 = \{x^{2m+1}\}$. Let $A = V(\mathbb{S}_2(G)) \setminus S_G \cup G_2$, where $V(\mathbb{S}_2(G))$ is the vertex set of $\mathbb{S}_2(G)$. Then $A = \{x, x^3, x^5, ..., x^{2m-1}, x^{2m+3}, ..., x^{4m+1}\}$. So $|S_G| = 2m+1$ and $|A| = 2m$. Thus we have $i_3(\mathcal{G}) = |\{(x^i, x^j, x^k) : x^i, x^j, x^k \textit{ are not adjacent to each other}\}|$.

Now, $A_1 = \{(e, x^i, x^j) : i+j = 4m+2 \textit{ and } x^i, x^j \in A\} = \{(e, x, x^{4m+1}), (e, x^3, x^{4m-1}), ..., (e, x^{2m-1}, x^{2m+3})\}$. As there are m number of pairs (x^i, x^j) in A so that $i + j = 4m + 2$. So, $|A_1| = m$.

Similarly, let $A_2 = \{(x^{2m+1}, x^i, x^j) : i + j = 4m + 2 \textit{ and } x^i, x^j \in S_G \setminus \{e\}\}$. Then $|A_2| = m$.

Later, consider $A_3 = \{(x^i, x^j, x^k) : j + k = 4m + 2 \textit{ but neither } i + j = 2m + 1 \textit{ nor } i + k = 2m + 1 \textit{ and } x^i \in S_G \setminus \{e\} \;\&\; x^j, x^k \in A\}$. Now, if we fix x^i and count the pairs (x^j, x^k) so that $j + k = 4m + 2$. Then we get m number of pairs in A. But there exists one pair like $(x^i, x^{2m+1-i}, x^{2m+1+i})$ which does not belong to A_3. So, we get $m - 1$ number of pairs for fixed x^i. Now, if we run x^i over $S_G \setminus \{e\}$ we get the cardinality of A_3. So, $|A_3| = 2m(m - 1)$.

Similarly, let $A_4 = \{(x^i, x^j, x^k) : j + k = 4m + 2 \textit{ but neither } i + j = 2m + 1 \textit{ nor } i + k = 2m + 1 \textit{ and } x^i \in A \;\&\; x^j, x^k \in S_G \setminus \{e\}\}$. Now, if we fix x^i and count the pairs

(x^j, x^k) so that $j + k = 4m + 2$. Then we get m number of pairs in $S_G \setminus \{e\}$. But there exists one pair like $(x^i, x^{2m+1-i}, x^{2m+1+i})$ which does not belong to A_4. So, we get $m-1$ number of pairs for fixed x^i. Now, if we run x^i over A we get the cardinality of A_4. So, $|A_4| = 2m(m-1)$.

Therefore, $i_3(\mathbb{S}_2(\mathbb{Z}_{4m+2})) = |A_1| + |A_2| + |A_3| + |A_4| = m + m + 2m(m-1) + 2m(m-1) = 2m(2m-1)$. □

Lemma 3.2.16. *Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$ for $n = 4m + 2$, $m \in \mathbb{N}$. Then $i_4(\mathcal{G}) = m(m-1)$. Also $i_k(\mathcal{G}) = 0$ for $k > 4$.*

Proof. Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$ and $G = \mathbb{Z}_n$ for $n = 4m + 2$, $m \in \mathbb{N}$. Then $G = \mathbb{Z}_{4m+2} = \{e, x, x^2, x^3, ..., x^{4m+1}\}$. Now, $S_G = \{e, x^2, x^4, x^6, ..., x^{4m}\}$ and $G_2 = \{x^{2m+1}\}$. Let $A = V(\mathbb{S}_2(G)) \setminus S_G \cup G_2$. Then $|S_G| = 2m + 1$ and $|A| = 2m$. Since, $i_4(\mathcal{G}) = |\{(x^i, x^j, x^k, x^l) : x^i, x^j, x^k, x^l \textit{ are not adjacent to each other}\}|$.
$= |\{(x^i, x^j, x^k, x^l) : i + j = k + l = 4m + 2 \textit{ but sum of any two of } i, j, k, l \neq 2m + 1 \;\&\; x^i, x^j \in S_G \textit{ and } x^k, x^l \in A\}|$.

Now, the number of pairs (x^i, x^j) from S_G with $i+j = 4m+2$ is m. Also the number of pairs (x^k, x^l) from A with $k + l = 4m + 2$ is m. But among of these, one 4-tuple is found so that (x^i, x^j, x^k, x^l) is not independent as the sum of any two of i, j, k, l is $2m+1$. Then $|\{(x^i, x^j, x^k, x^l) : i+j = k+l = 4m+2 \textit{ but sum of any two of } i, j, k, l \neq 2m + 1 \;\&\; x^i, x^j \in S_G \textit{ and } x^k, x^l \in A\}| = m(m-1)$. So, $i_4(\mathbb{S}_2(\mathbb{Z}_{4m+2})) = m(m-1)$.

Now, $i_k(\mathcal{G})$ = number of k-tuples so that these k vertices are not adjacent to each other. In that case we can choose vertices only from S_G, G_2 or A. But we can not choose three vertices from any one of these sets. If so at least two of them are adjacent. So we can not find any k-tuples pairs for $k > 4$. Hence $i_k(\mathcal{G}) = 0$ for $k > 4$. □

Theorem 3.2.17. *For any positive integer m, $i(\mathbb{S}_2(\mathbb{Z}_{4m+2})) = 9m^2 + 5m + 3$.*

Proof. Let $\mathcal{G} = \mathbb{S}_2(G)$, where $G = \mathbb{Z}_{4m+2}$ and $i(\mathcal{G}) = \sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G})$.

Then $i_0(\mathcal{G}) = 1$ and $i_1(\mathcal{G})$ = number of vertices in $\mathcal{G} = 4m + 2$. Now From Lemma 3.2.14, Lemma 3.2.15 and Lemma 3.2.16, we have $i_2(\mathbb{S}_2(\mathbb{Z}_{4m+2})) = 4m(m+1)$, $i_3(\mathbb{S}_2(\mathbb{Z}_{4m+2})) = 2m(2m-1)$, $i_4(\mathbb{S}_2(\mathbb{Z}_{4m+2})) = m(m-1)$ and $i_k(\mathbb{S}_2(\mathbb{Z}_{4m+2})) = 0$ for $k > 4$.

Thus $i(\mathcal{G}) = \sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + i_2(\mathcal{G}) + i_3(\mathcal{G}) + i_4(\mathcal{G}) + \cdots + i_{|V(\mathcal{G})|}(\mathcal{G})$.
$= 1 + 4m + 2 + 4m(m+1) + 2m(2m-1) + m(m-1) + 0 + \cdots + 0$
$= 9m^2 + 5m + 3$. □

Theorem 3.2.18. *[41] If n is an odd integer, then* $\mathbb{S}_2(D_n) \cong \overline{(n+1)K_1 + (\frac{n-1}{2})K_2}$.

Theorem 3.2.19. *Let* $\mathcal{G} = \mathbb{S}_2(D_n)$*, where n is an odd integer. Then* $i(\mathcal{G}) = \dfrac{5n+1}{2}$.

Proof. Let $\mathcal{G} = \mathbb{S}_2(D_n) \cong \overline{(n+1)K_1 + (\frac{n-1}{2})K_2}$ for n is an odd integer.
Then $i(\mathcal{G}) = \sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G})$.

Now, $i_0(\mathcal{G}) = 1$; $i_1(\mathcal{G}) = 2n$;
$i_2(\mathcal{G}) = i_2(\overline{(n+1)K_1 + (\frac{n-1}{2})K_2})$ = Number of K_2 in $(n+1)K_1 + (\frac{n-1}{2})K_2 = \dfrac{n-1}{2}$;
$i_3(\mathcal{G}) = i_3(\overline{(n+1)K_1 + (\frac{n-1}{2})K_2})$ = *Number of* K_3 *in* $(n+1)K_1 + (\frac{n-1}{2})K_2 = 0$
and $i_k(\mathcal{G}) = 0 \ \forall\ k \geq 3$. Thus $i(\mathcal{G}) = \sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = 1 + 2n + \dfrac{n-1}{2} = \dfrac{5n+1}{2}$. □

In the next two examples, we will show graphically how independent sets work on $\mathbb{S}_2(D_n)$ for even positive integer n.

Example 3.2.20. *Let* $G = D_4 = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$ *and* $\mathcal{G} = \mathbb{S}_2(D_4)$*. Then the order two element graph* $\mathbb{S}_2(D_4)$ *is as follows:*

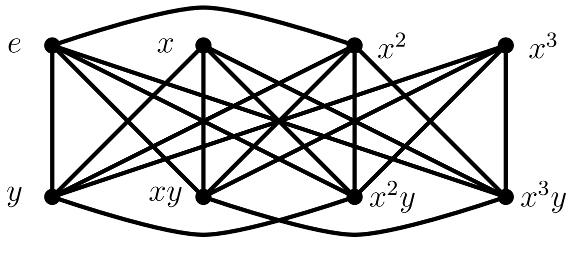

Figure 3.4: $\mathbb{S}_2(D_4)$

From the Figure 3.4, we obtain

$i_0(\mathcal{G}) = 1$, $i_1(\mathcal{G}) = 8$,

$i_2(\mathcal{G}) = |\{(v_i, v_j) : v_i, v_j \in D_4$ *and* v_i, v_j *are non-adjacent*$\}|$

$= |\{(e, x), (e, x^3), (x, x^2), (x, x^3), (x^2, x^3), (y, xy), (y, x^3y), (xy, x^2y), (x^2y, x^3y)\}|$

$= 9$

$i_3(\mathcal{G}) = |\{(v_i, v_j, v_k) : v_i, v_j, v_k \in D_4$ *and* v_i, v_j, v_k *are pairwise non-adjacent*$\}|$

$= |\{(e, x, x^3)\}| = 1$

Also, we can not find any $4-$ *tuples vertices so that they are pairwise adjacent in* $\mathcal{G}$*. Therefore,* $i(\mathcal{G}) = \sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + i_2(\mathcal{G}) + i_3(\mathcal{G}) + \sum_{k=4}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = 1 + 8 + 9 + 1 + 0 = 19.$

Example 3.2.21. *Let* $G = D_6 = \{e, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y\}$ *and* $\mathcal{G} = \mathbb{S}_2(D_6)$*. Then the order two element graph* $\mathbb{S}_2(D_6)$ *is as follows:*

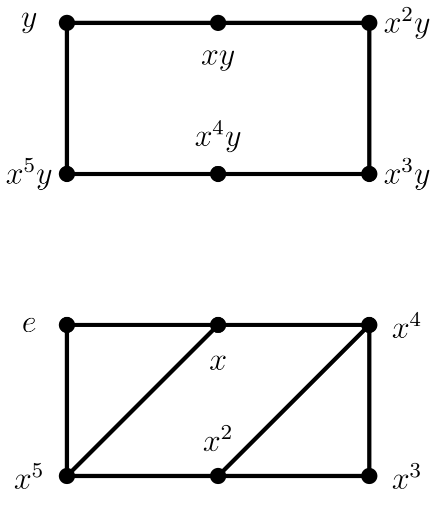

Figure 3.5: Complement graph of $\mathbb{S}_2(D_6)$

From the Figure 3.5, we obtain

$i_0(\mathcal{G}) = 1$, $i_1(\mathcal{G}) = 12$,

$i_2(\mathcal{G}) = |\{(v_i, v_j) : v_i, v_j \in D_6$ *and* v_i, v_j *are non-adjacent*$\}|$

$= |\{(e, x), (e, x^5), (x, x^4), (x, x^5), (x^2, x^3), (x^2, x^4), (x^2, x^5), (x^3, x^4), (y, xy), (y, x^5y), (xy, x^2y), (x^2y, x^3y), (x^3y, x^4y), (x^4y, x^5y)\}| = 14$

$i_3(\mathcal{G}) = |\{(v_i, v_j, v_k) : v_i, v_j, v_k \in D_6$ *and* v_i, v_j, v_k *are pairwise non-adjacent*$\}|$

$= |\{(e, x, x^5), (x^2, x^3, x^4)\}| = 2$

Also, we can not find any $4-$ *tuples vertices so that they are pairwise adjacent in* $\mathcal{G}$. *Therefore,* $i(\mathcal{G}) = \sum\limits_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + i_2(\mathcal{G}) + i_3(\mathcal{G}) + \sum\limits_{k=4}^{|V(\mathcal{G})|} i_k(\mathcal{G}) = 1 + 12 + 14 + 2 + 0 = 29.$

From the above examples 3.2.20 and 3.2.21, we understand the graphical structure and visualize the graph $\mathbb{S}_2(D_4)$. Using this concept, we arrive at the following theorems.

Lemma 3.2.22. *Let* $\mathcal{G} = \mathbb{S}_2(D_n)$, *where* n *is an even integer. Then*

$$i_2(\mathcal{G}) = \begin{cases} 8m^2 - 10m + 2 & \text{if } \frac{n}{2} \text{ is an odd number, i.e., } n = 2(2m-1) = 4m-2 \\ 8m^2 + 2m - 1 & \text{if } \frac{n}{2} \text{ is an even number, i.e., } n = 2(2m) = 4m. \end{cases}$$

Proof. Let $\mathcal{G} = \mathbb{S}_2(G)$, where $G = D_n = \{<x, y>: x^n = e = y^2 \text{ and } yx = x^{-1}y\} = \{e, x, x^2, x^3, ..., x^{n-1}, y, xy, x^2y, x^3y, ..., x^{n-1}y\}$.

Case I: Suppose $n = 4m - 2$ for $m \in \mathbb{N}$.

Then $G = \{e, x, x^2, ..., x^{4m-3}, y, xy, x^2y, ..., x^{4m-3}y\}$, $S_G = \{e, x^2, x^4, x^6, ..., x^{4m-4}\}$ and $G_2 = \{x^{2m-1}, y, xy, x^2y, ..., x^{4m-3}y\}$. Let $A = V(\mathbb{S}_2(G)) \setminus S_G \cup G_2 = \{x, x^3, x^5, ..., x^{2m-3}, x^{2m+1}, ..., x^{4m-3}\}$. Then $|S_G| = 2m - 1$; $|G_2| = 4m - 1$ and $|A| = 2m - 2$. Assume that $d(v)$ is the degree of a vertex v in the complement graph of $\mathbb{S}_2(G)$. Then 'e' is not adjacent to every elements of A. So, $d(e) = 2m - 2$. Also 'x^{2m-1}' is not adjacent to every elements of $S_G \setminus \{e\}$. So, $d(x^{2m-1}) = 2m - 2$. Now if $x^i \in S_G \setminus \{e\}$, then x^i is not adjacent to x^{4m-2-i}, x^{2m-1} and every elements of A except $x^{2m-(i+1)}$. So, $d(v) = 1 + 1 + 2m - 3 = 2m - 1 \ \forall \ v \in S_G \setminus \{e\}$. Again, each element of $G_2 \setminus \{x^{2m-1}\}$ is not adjacent to $\frac{4m-4}{2}$ number of elements of $G_2 \setminus \{x^{2m-1}\}$. So, $d(v) = \frac{4m-4}{2} = 2m - 2 \ \forall \ v \in G_2 \setminus \{x^{2m-1}\}$. Again, if $x^j \in A$ then x^j is not adjacent to every elements of $S_G \setminus \{x^{2m-(j+1)}\} \cup \{x^{4m-2-j}\}$. So, $d(v) = 2m - 2 + 1 = 2m - 1$. Hence $i_2(\mathcal{G}) = \frac{1}{2}[2m-2+2m-2+(2m-2)(2m-1)+(4m-2)(2m-2)+(2m-2)(2m-1)]$ $= 2m-2+(m-1)(2m-1)+(2m-1)(2m-2)+(m-1)(2m-1) = 8m^2-10m+2.$

Case II: Let $n = 4m$ for $m \in \mathbb{N}$.

Then $G = \{e, x, x^2, ..., x^{4m-1}, y, xy, x^2y, ..., x^{4m-1}y\}$, $S_G = \{e, x^2, x^4, x^6, ..., x^{4m-2}\}\setminus \{x^{2m}\}$ and $G_2 = \{x^{2m}, y, xy, x^2y, ..., x^{4m-1}y\}$. Suppose that $A = V(\mathbb{S}_2(G))\setminus S_G \cup G_2 = \{x, x^3, x^5, ..., x^{4m-1}\}$. Then $|S_G| = 2m - 1$; $|G_2| = 4m + 1$ and $|A| = 2m$. Let $d(v)$ be the degree of a vertex v in the complement graph of $\mathbb{S}_2(G)$. Then 'e' and 'x^{2m}' are not adjacent to every elements of A. So, $d(e) = d(x^{2m}) = 2m$. Now if $x^i \in S_G \setminus \{e\}$, then x^i is not adjacent to x^{4m-i}, and every elements of A. So, $d(v) = 1 + 2m = 2m + 1$ $\forall\ v \in S_G \setminus \{e\}$. Again, each element of $G_2 \setminus \{x^{2m}\}$ is not adjacent to $\frac{4m}{2}$ number of elements of $G_2 \setminus \{x^{2m}\}$. So, $d(v) = \frac{4m}{2} = 2m\ \forall\ v \in G_2 \setminus \{x^{2m}\}$. Again, if $x^j \in A$ then x^j is not adjacent to every elements of $S_G \cup \{x^{2m}\} \cup \{x^{4m-j}\}$. So, $d(v) = 2m-1+1+1 = 2m + 1$. Hence $i_2(\mathcal{G}) = \frac{1}{2}[2m + 2m + (2m - 2)(2m + 1) + (4m)(2m) + (2m)(2m + 1)]$ $= 2m + (m - 1)(2m + 1) + (2m)(2m) + m(2m + 1) = 8m^2 + 2m - 1$. □

Lemma 3.2.23. *Let $\mathcal{G} = \mathbb{S}_2(D_n)$, where n is an even integer. Then*

$$i_3(\mathcal{G}) = \begin{cases} 2m^2 - 2m & \text{if } \frac{n}{2} \text{ is an odd number} > 3, \text{ i.e., } n = 2(2m - 1) = 4m - 2 \\ 4m^2 - 2m & \text{if } \frac{n}{2} \text{ is an even number} > 2, \text{ i.e., } n = 2(2m) = 4m. \end{cases}$$

Also, $i_k(\mathcal{G}) = 0$ for $k \geq 4$.

Proof. Let $\mathcal{G} = \mathbb{S}_2(G)$, where $G = D_n = \{< x, y >: x^n = e = y^2 \textit{ and } yx = x^{-1}y\} = \{e, x, x^2, x^3, ..., x^{n-1}, y, xy, x^2y, x^3y, ..., x^{n-1}y\}$.

Case I: Suppose $n = 4m - 2$ for $m \geq 3$.

Then $G = \{e, x, x^2, ..., x^{4m-3}, y, xy, x^2y, ..., x^{4m-3}y\}$, $S_G = \{e, x^2, x^4, x^6, ..., x^{4m-4}\}$ and $G_2 = \{x^{2m-1}, y, xy, x^2y, ..., x^{4m-3}y\}$. Let $A = V(\mathbb{S}_2(G)) \setminus S_G \cup G_2 = \{x, x^3, x^5, ..., x^{2m-3}, x^{2m+1}, ..., x^{4m-3}\}$. Then $|S_G| = 2m-1$; $|G_2| = 4m-1$ and $|A| = 2m-2$. Now, we are going to find 3-tuples vertices in $V(\mathbb{S}_2(G))$ so that they are pairwise non-adjacent.

Let $A_1 = \{(e, x^i, x^{n-i}) : x^i, x^{n-i} \in A\}$. Then $|A_1| = m - 1$.
$A_2 = \{(x^{2m-1}, x^i, x^{n-i}) : x^i, x^{n-i} \in S_G \setminus \{e\}\}$. Then $|A_2| = m - 1$.
$A_3 = \{(x^i, x^j, x^k) : x^i \in S_G \setminus \{e\}, x^j, x^k \in A$ with $j + k = n, i + j \neq 2m - 1, i + k \neq 2m - 1\}$. Then $|A_3| = \frac{2m-2}{2} \times \frac{2m-2}{2} = (m - 1)^2$.
$A_4 = \{(x^i, x^j, x^k) : x^i \in A, x^j, x^k \in S_G \setminus \{e\}$ with $j + k = n, i + j \neq 2m - 1, i + k \neq 2m - 1\}$. Then $|A_4| = \frac{2m-2}{2} \times \frac{2m-2}{2} = (m - 1)^2$.

Also, we can not find any $3-tuples$ vertices from $\{y, xy, x^2y, ..., x^{4m-3}y\}$ so that they are pairwise adjacent. Therefore, $i_3(\mathcal{G}) = m-1+m-1+(m-1)^2+(m-1)^2+0 = 2m^2-2m$ for $m \geq 3$.

Case II: Let $n = 4m$ for $m > 1$.

Then $G = \{e, x, x^2, ..., x^{4m-1}, y, xy, x^2y, ..., x^{4m-1}y\}$, $S_G = \{e, x^2, x^4, x^6, ..., x^{4m-2}\}\setminus\{x^{2m}\}$ and $G_2 = \{x^{2m}, y, xy, x^2y, ..., x^{4m-1}y\}$. Suppose that $A = V(\mathbb{S}_2(G))\setminus S_G \cup G_2 = \{x, x^3, x^5, ..., x^{4m-1}\}$. Then $|S_G| = 2m-1$; $|G_2| = 4m+1$ and $|A| = 2m$. Now, we are going to find $3-tuples$ vertices in $V(\mathbb{S}_2(G))$ so that they are pairwise non-adjacent.

Let $A_1 = \{(e, x^i, x^{n-i}) : x^i, x^{n-i} \in A\}$. Then $|A_1| = m$.

$A_2 = \{(x^{2m}, x^i, x^{n-i}) : x^i, x^{n-i} \in A\}$. Then $|A_2| = m$.

$A_3 = \{(x^i, x^j, x^k) : x^i \in S_G \setminus \{e\}, x^j, x^k \in A \text{ with } j+k = n\}$.
Then $|A_3| = (2m-2) \times \frac{2m}{2} = 2m(m-1)$.

$A_4 = \{(x^i, x^j, x^k) : x^i \in A, x^j, x^k \in S_G \setminus \{e\} \text{ with } j+k = n\}$.
Then $|A_4| = 2m \times \frac{2m-2}{2} = 2m(m-1)$.

Also, we can not find any $3-tuples$ vertices from $\{y, xy, x^2y, ..., x^{4m-1}y\}$ so that they are pairwise adjacent. Therefore, $i_3(\mathcal{G}) = m+m+2m(m-1)+2m(m-1)+0 = 4m^2-2m$ for $m > 1$.

Moreover, we can not find any $k - tuples$ vertices from G such that they are pairwise adjacent. Therefore, $i_k(\mathcal{G}) = 0$ for $k \geq 4$. □

Theorem 3.2.24. *Let $\mathcal{G} = \mathbb{S}_2(D_n)$, where n is an even integer. Then*

$$i(\mathcal{G}) = \begin{cases} 10m^2-4m-1 & \text{if } \frac{n}{2} \text{ is an odd number} > 3, \text{ i.e., } n = 2(2m-1) = 4m-2 \\ 12m^2+8m & \text{if } \frac{n}{2} \text{ is an even number} > 2, \text{ i.e., } n = 2(2m) = 4m. \end{cases}$$

Proof. Since $i(\mathcal{G}) = \sum_{k=0}^{|V(\mathcal{G})|} i_k(\mathcal{G})$. Now, $i_0(\mathcal{G}) = 1$;

$$i_1(\mathcal{G}) = |V(\mathcal{G})| = \begin{cases} 8m-4 & \text{if } \frac{n}{2} \text{ is an odd number, that is } n = 2(2m-1) = 4m-2 \\ 8m & \text{if } \frac{n}{2} \text{ is an even number, that is } n = 2(2m) = 4m. \end{cases}$$

From Lemma 3.2.22,

$$i_2(\mathcal{G}) = \begin{cases} 8m^2 - 10m + 2 & if\ \frac{n}{2}\ is\ an\ odd\ number\ i.e.,\ n = 2(2m-1) = 4m-2 \\ 8m^2 + 2m - 1 & if\ \frac{n}{2}\ is\ an\ even\ number\ i.e.,\ n = 2(2m) = 4m. \end{cases}$$

From Lemma 3.2.23,

$$i_3(\mathcal{G}) = \begin{cases} 2m^2 - 2m & if\ \frac{n}{2}\ is\ an\ odd\ number > 3\ i.e.,\ n = 2(2m-1) = 4m-2 \\ 4m^2 - 2m & if\ \frac{n}{2}\ is\ an\ even\ number > 2\ i.e.,\ n = 2(2m) = 4m. \end{cases}$$

And $i_k(\mathcal{G}) = 0$ for $k > 3$. So, $i(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + i_2(\mathcal{G}) + i_3(\mathcal{G}) + 0$.

$$= \begin{cases} 10m^2 - 4m - 1 & if\ \frac{n}{2}\ is\ an\ odd\ number > 3,\ that\ is\ n = 2(2m-1) = 4m-2 \\ 12m^2 + 8m & if\ \frac{n}{2}\ is\ an\ even\ number > 2,\ that\ is\ n = 2(2m) = 4m. \end{cases}$$

□

3.3 Product of graphs $\mathbb{S}_2(G_1)$ and $\mathbb{S}_2(G_2)$ over two groups G_1 and G_2

Let us recall different standard notations of products of two graphs.

Definition 3.3.1. *[8] Product of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$:*

(i) The Cartesian product $\mathcal{G}_1 \Box \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as follows:

$V(\mathcal{G}_1 \Box \mathcal{G}_2) = V(\mathcal{G}_1) \times V(\mathcal{G}_2)$ and $(g_1, g_2) \sim (g_1', g_2')$ if and only if either $g_1 = g_1'$ and $g_2 \sim g_2'$ or $g_1 \sim g_1'$ and $g_2 = g_2'$.

(ii) The direct product $\mathcal{G}_1 \times \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as follows:

$V(\mathcal{G}_1 \times \mathcal{G}_2) = V(\mathcal{G}_1) \times V(\mathcal{G}_2)$ and $(g_1, g_2) \sim (g_1', g_2')$ if and only if $g_1 \sim g_1'$ and $g_2 \sim g_2'$.

(iii) The strong product $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ of $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as follows:

$V(\mathcal{G}_1 \boxtimes \mathcal{G}_2) = V(\mathcal{G}_1) \times V(\mathcal{G}_2)$ and $(g_1, g_2) \sim (g_1', g_2')$ if and only if either $g_1 = g_1'$ and $g_2 \sim g_2'$ or $g_1 \sim g_1'$ and $g_2 = g_2'$ or $g_1 \sim g_1'$ and $g_2 \sim g_2'$.

Note : Here $g_1 \sim g_1'$ means g_1 is adjacent to g_1'.

Now we have the following result which follows from the above definition.

Corollary 3.3.2. *Let G_1 and G_2 be any two groups. Then direct product of $\mathbb{S}_2(G_1) \times \mathbb{S}_2(G_2)$ is complete if and only if $\mathbb{S}_2(G_1)$ and $\mathbb{S}_2(G_2)$ are complete.*

Example 3.3.3. *Consider G is cyclic group of order 4. Let $G = \{e, x, x^2, x^3\}$ and $H = \{e, x^2\}$. Then $G/H = \{H, xH\}$. Then the graphs $\mathbb{S}_2(G)$ and $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$ are shown below:*

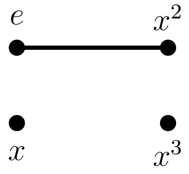

Figure 3.6: $\mathbb{S}_2(G)$

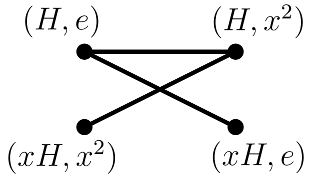

Figure 3.7: $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$

From the above Figure 3.6 and Figure 3.7, it shows that $\mathbb{S}_2(G)$ and $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$ are not graph isomorphic as $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$ has two vertices of degree 2 but $\mathbb{S}_2(G)$ has no vertices of degree 2.

Example 3.3.4. *Consider G is cyclic group of order 9. Let $G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ and $H = \{e, x^3, x^6\}$. Then $G/H = \{H, xH, x^2H\}$. Then the graphs $\mathbb{S}_2(G)$ and $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$ are shown below:*

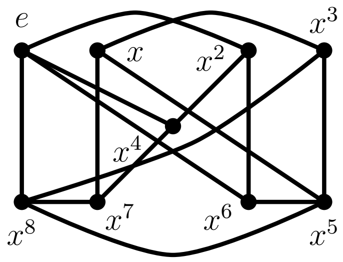

Figure 3.8: $\mathbb{S}_2(G)$

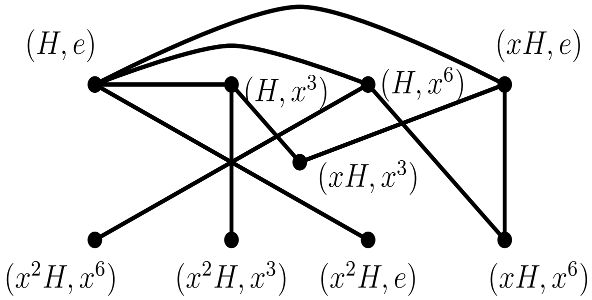

Figure 3.9: $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$

From Figure 3.8 and Figure 3.9, it shows that $\mathbb{S}_2(G)$ and $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$ are not graph isomorphic as $\mathbb{S}_2(G)$ has two vertices of degree 4 but $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$ has one vertex of degree 4.

From the above examples we have if G is any cyclic group of finite order except the order $4m+2$ for $m \in \mathbb{N}$ and H be any subgroup of G then $\mathbb{S}_2(G)$ and $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$ are not graph isomorphic. Now we have a following theorem.

Theorem 3.3.5. *Let G be any finite cyclic group of order $4m+2$, $m \in \mathbb{N}$. If H be any subgroup of G. Then $\mathbb{S}_2(G)$ is graph isomorphic to the Cartesian product $\mathbb{S}_2(G/H)\square\mathbb{S}_2(H)$ of $\mathbb{S}_2(G/H)$ and $\mathbb{S}_2(H)$.*

Proof. Let G be any cyclic group of order $4m+2$, $m \in \mathbb{N}$ and $G =< x >$. Let H be any subgroup of G with $\mid H \mid= m_1$. Then H is a normal subgroup of G and $H =< x^p >$, where p is a divisor of $4m+2$ and $0 \leq p < 4m+2$. Also G/H is a cyclic group and $G/H =< xH >$. Now, we define a mapping $f : G \to G/H \times H$ by $f(x^k) = (x^kH, x^{pk})$, where $0 \leq k < 4m+2$. Let a and b be any two element of G such that a and b are adjacent to each other in $\mathbb{S}_2(G)$. Now, let $a = x^{k_1}$ and $b = x^{k_2}$. Since a and b are adjacent to each other then either $o(ab) = 2$ or $ab = t^2$ for some $t \in G$. So $o(x^{k_1+k_2}) = 2$ or $x^{k_1+k_2} = x^{2k_3}$, $(t = x^{k_3} \in G)$. Now $f(a) = f(x^{k_1}) = (x^{k_1}H, x^{pk_1})$ and $f(b) = f(x^{k_2}) = (x^{k_2}H, x^{pk_2})$. Let $x^{k_1}H = x^{k_2}H$ then $x^{pk_1}.x^{pk_2} = x^{p(k_1+k_2)} = x^{2pk_3} = (x^{pk_3})^2 = t_1^2$ for some $t_1 \in H$. Again, let $x^{pk_1} = x^{pk_2}$ then $x^{k_1}H.x^{k_2}H = x^{k_1+k_2}H = x^{2k_3}H = (x^{k_3}H)^2$, $x^{k_3}H \in G/H$. So if a, b are adjacent to each other then $f(a)$, $f(b)$ are also adjacent to each other. Therefore the mapping f is well defined.

One-One: Let $a \neq b$ for $a, b \in G$ where $a = x^{r_1}$, $b = x^{r_2}$ with $r_1 \neq r_2$ and $0 \leq r_1, r_2 < 4m+2$. Now, let us assume that $f(a) = f(b) \Rightarrow f(x^{r_1}) = f(x^{r_2}) \Rightarrow (x^{r_1}H, x^{pr_1}) = (x^{r_2}H, x^{pr_2}) \Rightarrow x^{r_1}H = x^{r_2}H$ and $x^{pr_1} = x^{pr_2}$. Now, if $x^{r_1}H = x^{r_2}H \Rightarrow x^{r_1-r_2} \in H \Rightarrow x^{r_1-r_2} = x^{ps_1}$ where $0 \leq s_1 < m_1 =\mid H \mid< 4m+2$. If $s_1 = 0$ then $r_1 - r_2 = 0$ as $r_1, r_2 < m$. So $r_1 = r_2$ a contradiction. Let $s_1 \neq 0$, then $x^{r_1-r_2-ps_1} = e$ where e is the identity element of G. $\Rightarrow r_1 - r_2 - ps_1 = (4m+2)k_1$, for some $k_1 \in \mathbb{N}$. Now $0 \leq r_1, r_2 < 4m+2$, $0 < p < 4m+2$ and $0 < s_1 < m_1 < 4m+2$. $\Rightarrow r_1 - r_2 - ps_1 < 4m+2 \Rightarrow r_1 - r_2 - ps_1 \neq (4m+2)k_2$ for some $k_2 \in \mathbb{N}$ So if $a \neq b$ then $f(a) \neq f(b)$ for any $a, b \in G$. Hence f is one-one.

Since $\mid G \mid=\mid G/H \mid \times \mid H \mid$, so f is also onto mapping. Then f is a graph isomorphism. Therefore, $\mathbb{S}_2(G)$ is graph isomorphic to the Cartesian product $\mathbb{S}_2(H)\Box\mathbb{S}_2(G/H)$ of $\mathbb{S}_2(H)$ and $\mathbb{S}_2(G/H)$. $\Box$

3.4 Domination number of the graph $\mathbb{S}_2(G)$ over a group G

Definition 3.4.1. *[52] Domination Number: A dominating set in a graph $\mathcal{G} = (V, E)$ is a subset of V with the property that every vertex in $\mathcal{G}$ is either in the dominating set or adjacent to a vertex that is in the dominating set. The domination number of $\mathcal{G}$ is denoted by $Domn(\mathcal{G})$ and is defined as the cardinality of a minimum dominating set of $\mathcal{G}$.*

Theorem 3.4.2. *Let G be a finite group of even order and $S_G \cup G_T$ be a subgroup of G, where S_G be the set of squares of G and G_T be the set of those elements of G whose order is 2 but not in S_G. Then $S_G \cup G_T$ is a normal subgroup of G and $\mathbb{S}_2(G)$ is a disconnected graph with $\mid G/(S_G \cup G_T) \mid$ components.*

Proof. Since G is a finite group of even order. Let $\mid G \mid= 2n$. Let $G = \{e, a_1, a_2, ..., a_m, c_1, c_2, ..., c_r, x_1, x_2, ..., x_{2n-m-r-1}\}$, where $a_i = t_i^2$ for some $t_i \in G$ and $(c_j)^2 = e$ for all $j = 1, 2, ..., r$. Let $H = S_G \cup G_T$. Then $H = \{e, a_1, a_2, ..., a_m, c_1, c_2, ..., c_r\}$. Let $B = G \setminus H = \{x_1, x_2, ..., x_{2n-m-r-1}\}$. Since H is a subgroup of G. Then $m + r + 1$ divides $2n$. Let $\frac{2n}{m+r+1} = k$. Let x_i be any element in B and $y \in H$. Now consider the element $x_i y x_i^{-1}$. If $y = a_i$ for some $i = 1, 2, ..., m$ then $a_i = t_i^2$ for some $t_i \in G$. Now $x_i y x_i^{-1} = x_i t_i^2 x_i^{-1} = (x_i t_i x_i^{-1})^2 = z_i^2$ for some $z_i = x_i t_i x_i^{-1} \in G$. So, $x_i y x_i^{-1} \in H$ for $y = a_i$. If $y = c_i$ for $i = 1, 2, ..., r$ then $(x_i y x_i^{-1})^2 = x_i y^2 x_i^{-1} = x_i c_i^2 x_i^{-1} = e$. So, $x_i y x_i^{-1} \in H$ for $y = c_i$. Therefore H is a normal subgroup of G and $\mid G/(S_G \cup G_T) \mid= k$.

Again as H is a subgroup of G then the clouser property holds on H. Now, if y_i and y_j are any two elements of H then $y_i y_j$ is also an element of H. Therefore y_i is adjacent to y_j. So, $\mathbb{S}_2(H)$ forms a subgraph of $\mathbb{S}_2(G)$. Also no element from H is adjacent to any element of B. Then $\mathbb{S}_2(H)$ forms a component. Therefore $\mathbb{S}_2(G)$ is disconnected.

Since $\mid G/H \mid = k$. Then there are k number of distinct left cosets of H in G. Our assumption is that these k number of distinct left cosets form k number of components in $\mathbb{S}_2(G)$.

If possible, let there exist vertices p, q belonging to distinct cosets x_iH and x_jH respectively such that p be adjacent to q in $\mathbb{S}_2(G)$. Suppose $p = x_iy_i$ and $q = x_jy_j$ for $y_i,\ y_j \in H$. Then $pq = x_iy_ix_jy_j = t_l^2$ for some $t_l \in G$ or $(pq)^2 = e$. Now, if $x_iy_ix_jy_j = t_l^2 = a_k$ $take$ $a_k = t_l^2$. $\implies x_iy_ix_j = a_k^2y_j^{-1} \in H \implies y_ix_j \in x_i^{-1}H = x_iH = Hx_i$ $(if\ x_i \in B \subset G \implies x_i^2 \in H \implies x_iH = x_i^{-1}H)$. $\implies x_j \in y_i^{-1}Hx_i = Hx_i = x_iH$ a contradiction as $x_j \in x_jH$ and $x_iH \cap x_jH = \phi$.

Again, if $x_iy_ix_jy_j = c_l$ $(take\ t_l^2 = c_l)$
$\implies x_iy_jx_j = c_ly_j^{-1} \in H \implies y_ix_j \in x_i^{-1}H = x_iH = Hx_i \implies x_j = y_i^{-1}Hx_i = Hx_i = x_iH$ a contradiction. As $x_j \in x_jH$ and $x_iH \cap x_jH = \phi$.

Now, if $x_iy_ix_jy_j = x_l^2$ $(take\ t_l = x_l \in B)$
$\implies x_iy_ix_j = x_k^2y_j^{-1} \in H$ as $x_l^2 \in H,\ y_j^{-1} \in H$ *and* H *is normal in* G.
$\implies y_ix_j \in x_i^{-1}H = x_iH = Hx_i \implies x_j \in y_i^{-1}Hx_i = Hx_i = x_iH$ a contradiction. Hence, these k number of distinct left cosets form k number of components in $\mathbb{S}_2(G)$.

□

Proposition 3.4.3. *[41] Let* G *be a group,* S_G *be the set of squares of* G *and* G_T *be the set of those elements of* G *whose order is* 2 *but not in* S_G*. Then in the graph* $\mathbb{S}_2(G)$*, the degree of any vertex is given by*

$$deg(x) = \begin{cases} |S_G \cup G_T| - 1, & if\ x^2 = e \\ |S_G \cup G_T| - 2, & otherwise. \end{cases}$$

Theorem 3.4.4. *Let* G *be a finite group of even order and* $S_G \cup G_T$ *be a subgroup of* G*, where* S_G *be the set of squares of* G *and* G_T *be the set of those elements of* G *whose order is 2 but not in* S_G *and let* $\mid G/S_G \cup G_T \mid = k$*. Then* $Domn(\mathbb{S}_2(G)) \leq 2k - 1$*.*

Proof. If G is a finite group of even order and $S_G \cup G_T$ is a subgroup of G, then by the above Theorem 3.4.2 we get that $S_G \cup G_T$ is a normal subgroup of G and $\mathbb{S}_2(G)$ is a disconnected graph with $\mid G/(S_G \cup G_T) \mid$ components. Suppose that $H = S_G \cup G_T$ and $|G/H| = k$. In G, consider k distinct cosets are $H, x_1H, ..., x_{k-1}H$. By the proof of above Theorem, we have the subgraph induced by $H, x_1H, ..., x_{k-1}H$ form disconnected subgraph of $S_2(G)$. In the subgraph induced by H, each vertex adjacent to e. Let x_iH be an arbitrary coset. Then $|x_iH| = |H|$ and by Proposition 3.4.3, $deg(x_i)$ is either $|H| - 1$ or $|H| - 2$. If $deg(x_i) = |H| - 1$, then each vertex of x_iH is adjacent with x_i and if $deg(x_i) = |H| - 2$, then x_i is adjacent with each vertex of x_iH except an element y (say). So in this case, $\{x_i, y\}$ is dominating set of the subgraph induced by x_iH. Since x_iH is an arbitrary coset, So Maximum cadinality of the subgraph induced by x_jH is 2. Thus the domination number $Domn(\mathbb{S}_2(G)) \leq 1 + 2(k-1) = 2k - 1$. □

Proposition 3.4.5. $Domn(\mathbb{S}_2(\mathbb{Z}_n)) = \begin{cases} 1 \; if \; n = odd \\ 3 \; if \; n = even \end{cases}$

Proof. Let $\mathcal{G} = \mathbb{S}_2(G)$, where $G = \mathbb{Z}_n = \{e, x, x^2, x^3, ..., x^{n-1}\}$.

Case-I: Let n be an odd number. Then $S_G = \{e, x, x^2, x^3, ..., x^{n-1}\}$. Now '$e$' is adjacent to rest of all vertices of $\mathbb{S}_2(\mathbb{Z}_n)$. So, $\{e\}$ is the dominating set. Thus $Domn(\mathbb{S}_2(\mathbb{Z}_n)) = 1$, when n is odd.

Case-II: Let n be an even number. Then $S_G = \{e, x^2, x^4, ..., x^{n-2}\}$ and $G_T = \{x^{\frac{n}{2}}\} \subset S_G$. Then $\{e, x^i, x^{n-i}\}$ is the dominating set where $x^i \in G \setminus (S_G \cup G_T)$. Thus $\mathbb{S}_2(\mathbb{Z}_n) = 3$, when n is even. □

Proposition 3.4.6. $Domn(\mathbb{S}_2(D_n)) = \begin{cases} 1 \; if \; n = odd \\ 3 \; if \; n = even \end{cases}$

Proof. If $n = \; odd$ then 'e' is adjacent to rest of all vertices of $\mathbb{S}_2(D_n)$. So, $\{e\}$ is the dominating set. Thus $Domn(\mathbb{S}_2(D_n)) = 1$, when n is odd. If $n = \; even$ then $\{e, x^i, x^{n-i}\}$ is the dominating set. Thus $\mathbb{S}_2(D_n) = 3$, when n is even. □

Proposition 3.4.7. *Let G be any finite group, S_G be the set of squares of G and G_T be the set of those elements of G whose order is 2 but not in S_G. Then the graph $\mathbb{S}_2(G)$ is connected if $\mid S_G \cup G_T \mid > \frac{|G|}{2}$.*

Proof. Since G is a finite group. Let $G = \{e, a_1, a_2, \cdots, a_{n-1}, c_1, c_2, \cdots, c_r, x_1, x_2, \cdots, x_k\}$ where $a_i = t_i^2$ for some $t_i \in G$ and $(c_j)^2 = e$. Then $S_G \cup G_T = \{e, a_1, a_2, \cdots, a_{n-1}, c_1, c_2, \cdots, c_r\}$. So $\mid S_G \cup G_T \mid = n + r$. Clearly '$e$' is adjacent to every elements of $S_G \cup G_T$. So, $S_G \cup G_T$ is an induced connected subgraph of $\mathbb{S}_2(G)$. Let $B = G \backslash S_G \cup G_T$. Since $\mid S_G \cup G_T \mid > \frac{|G|}{2}$. So $n + r > k$. Now $\mid B \mid \geq 2$. As if possible $x_i \in B$ then $x_i^{-1} \in B$. Now $n + r > k \implies n + r - 2 > k - 2$. From 3.4.7, we have $d(x_i) = n + r - 2$ for $x_i \in B$. So $d(x_i) > k - 2$. As x_i is not adjacent to x_i^{-1} and $\mid B \mid = k$. So maximum degree of any vertex of B is $k - 2$. But $d(x_i) > k - 2$. Therefor x_i is adjacent at least one vertex of $S_G \cup G_T$. As $S_G \cup G_T$ and B are connected then $\mathbb{S}_2(G)$ is connected. □

Proposition 3.4.8. *Let G be any finite group, S_G be the set of squares of G and G_T be the set of those elements of G whose order is 2 but not in S_G. If the graph $\mathbb{S}_2(G)$ is disconnected then $\mid S_G \cup G_T \mid \leq \frac{|G|}{2}$.*

Proof. Let the graph $\mathbb{S}_2(G)$ be disconnected. Since $S_G \cup G_T$ is connected. Then B must forms one or more than one components. So, $d(x_i) \leq k - 2$ for $x_i \in B$. Also from 3.4.3, $d(x_i) = n + r - 2$ for $x_i \in B$. Therefor $n + r - 2 \leq k - 2 \implies n + r \leq \frac{n+r+k}{2}$ $\implies \mid S_G \cup G_T \mid \leq \frac{|G|}{2}$. □

Theorem 3.4.9. *Let G be any finite group, S_G be the set of squares of G and G_T be the set of those elements of G whose order is 2 but not in S_G. If the square element graph $\mathbb{S}_2(G)$ is connected, then $Domn(\mathbb{S}_2(G)) \leq \mid G \mid - \mid S_G \cup G_T \mid +1$.*

Proof. Let $G = \{e, a_1, a_2, ..., a_{n-1}, c_1, c_2, \cdots, c_r, x_1, x_2, \cdots, x_k\}$ where $\circ(c_i) = 2$ and $a_j = t_j^2$ for some $t_j \in G$. Then 'e' is adjacent to the rest of all elements of $S_G \cup G_T$. Let $A = S_G \cup G_T = \{e, a_1, a_2, \cdots, a_{n-1}, c_1, c_2, \cdots, c_r\}$ and $B = G \setminus S_G \cup G_T = \{x_1, x_2, \cdots, x_k\}$. So, $\mathbb{S}_2(A)$ and $\mathbb{S}_2(B)$ form two induced subgraphs of $\mathbb{S}_2(G)$. Since $\mathbb{S}_2(G)$ is connected then there exists at least one path from one vertex in A to another vertex in B. Let $D = \{e, x_1, x_2, ..., x_k\}$. Then this D is a dominating set. So, $Domn(\mathbb{S}_2(G)) \leq \mid G \mid - \mid S_G \cup G_T \mid +1$. □

Theorem 3.4.10. *Let G be any finite group. If $\mathbb{S}_2(G)$ be a disconnected graph with k components and m number of isolated vertices. Then $Domn(\mathbb{S}_2(G)) \leq 1 +$*

$\sum_{i=1}^{k-1}(\mid S_i \mid -1) + m$, *where* S_i *are the components of* $\mathbb{S}_2(G)$.

Proof. Let $G = \{e, a_1, a_2, \cdots, a_{n-1}, c_1, c_2, \cdots, c_r, x_1, x_2, \cdots, x_p, y_1, y_2, \cdots, y_m\}$. Let $A = S_G \cup G_T$ then $\mid A \mid = n + r$. Let $x_1, x_2, \cdots, x_p$ form $(k-1)$ subsets so that they form $(k-1)$ connected subgraphs of $\mathbb{S}_2(G)$. Let the $(k-1)$ subsets be $S_1, S_2, \cdots, S_{k-1}$ then $\sum_{i=1}^{k-1} \mid S_i \mid = p$. Now, if we take '$e$' from A, take $(\mid S_i \mid -1)$ vertices from each S_i and take all m number of isolated vertices then the cardinality of dominating set is $1 + (\mid S_i \mid -1) + m$. So, $Domn(\mathbb{S}_2(G)) \leq 1 + \sum_{i=1}^{k-1}(\mid S_i \mid -1) + m$. □

Theorem 3.4.11. *Let* G_1 *and* G_2 *be two groups. Then the following are holds:*

1) $Domn(\mathbb{S}_2(G_1)) \times Domn(\mathbb{S}_2(G_2)) < Domn(\mathbb{S}_2(G_1 \Box G_2))$, *where* $\mathbb{S}_2(G_1 \Box G_2))$ *is the graph cartesian product of* $\mathbb{S}_2(G_1)$ *and* $\mathbb{S}_2(G_2)$.

2) $Domn(\mathbb{S}_2(G_1)) \times Domn(\mathbb{S}_2(G_2)) < Domn(\mathbb{S}_2(G_1 \times G_2))$, *where* $\mathbb{S}_2(G_1 \times G_2))$ *is the graph direct product of* $\mathbb{S}_2(G_1)$ *and* $\mathbb{S}_2(G_2)$.

3) $Domn(\mathbb{S}_2(G_1)) \times Domn(\mathbb{S}_2(G_2)) < Domn(\mathbb{S}_2(G_1 \boxtimes G_2))$, *where* $\mathbb{S}_2(G_1 \boxtimes G_2))$ *is the graph strong direct product of* $\mathbb{S}_2(G_1)$ *and* $\mathbb{S}_2(G_2)$.

Proof. 1) Let $G_1 = \{e_1, a_1, a_2, \cdots, a_{n_1}, c_1, c_2, \cdots, c_{r_1}, x_1, x_2, \cdots, x_{k_1}\}$ with $a_i = t_i^2$ for some $t_i \in G_1$ and $(c_j)^2 = e_1$ for all $j = 1, 2, \cdots, r_1$ and $G_2 = \{e_2, b_1, b_2, \cdots, b_{n_2}, d_1, d_2, \cdots, d_{r_2}, y_1, y_2, \cdots, y_{k_2}\}$ with $b_i = t_i^2$ for some $t_i \in G_2$ and $(d_j)^2 = e_1$ for all $j = 1, 2, \cdots, r_2$. Also, let $D_1 = \{e_1, x_1, x_2, \cdots, x_p\}$, where $1 < p < k_1$ and $D_2 = \{e_2, y_1, y_2, \cdots, y_q\}$, where $1 < q < k_2$. Assume that these D_1 and D_2 are the minimum dominating sets of $\mathbb{S}_2(G_1)$ and $\mathbb{S}_2(G_2)$ respectively. Then $Domn(\mathbb{S}_2(G_1)) = \mid D_1 \mid = p + 1$ and $Domn(\mathbb{S}_2(G_2)) = \mid D_2 \mid = q + 1$.

Now if possible, let $D_1 \times D_2$ be a dominating set of $\mathbb{S}_2(G_1 \Box G_2)$. Then by the definition of dominating set, every vertex of $\mathbb{S}_2(G_1 \Box G_2)$ is either in $D_1 \times D_2$ or is adjacent to a member of $D_1 \times D_2$. Now we choose one vertex of $\mathbb{S}_2(G_2)$, say b_1 which is not adjacent to $y_1, y_2, \cdots, y_q$. Now, from the definition of graph cartesian product of two graphs, (e_1, b_1) is neither in $D_1 \times D_2$ nor adjacent to any member of $D_1 \times D_2$. So the cardinality of the domination number of $\mathbb{S}_2(G_1 \Box G_2)$ is greater than $\mid D_1 \times D_2 \mid$. Hence, $Domn(\mathbb{S}_2(G_1)) \times Domn(\mathbb{S}_2(G_2)) < Domn(\mathbb{S}_2(G_1 \Box G_2))$.

Proof of 2) and 3) are same as 1). $\square$

Chapter 4

Topological indices of order two element graph over a group

Chapter 4

Topological indices of $\mathbb{S}_2(G)$

4.1 Introduction

First order two element graphs were defined over a finite group by S. Pradhan, S. Kar and B. Biswas in [41]. Later Pradhan et al. studied independence and domination number of order two element graph over a group in [42]. A graph invariant is a real number related to a graph $\mathcal{G}$ that is invariant under graph isomorphism, that is, it does not depend on the labeling or the pictorial representation of a graph. In chemistry, graph invariants are known as topological indices. Topological indices have many applications as tools for modeling chemical and other properties of molecules. The Wiener index (see in [24]), the Hyper-Wiener index (see in [32]), the Harary index (see in [58]) and the Gutman index (see in [25]) are the most studied topological, both from a theoretical point of view and applications. All of these works used topological index techniques to analyze a graph's graph-theoretic features. However, we try to use these topological indices to explore the graph-theoretic features of a graph constructed over an algebraic structure.

As an oldest topological index, the Wiener index of a (molecular) graph $\mathcal{G}$, first introduced by chemist Harold Wiener [53] in 1947. Many works have been done in Wiener index of different algebraic graphs. In [24], M. Eliasi et. al., they determine the Wiener index of graphs which are constructed by some graph operations.

Definition 4.1.1 (Wiener index)**.** *Let $\mathcal{G} = (V, E)$ be any graph. Then the Wiener index $W(\mathcal{G})$ of the graph $\mathcal{G}$ is defined as*

$$W(\mathcal{G}) = \sum_{\{u,v\}\subseteq V(\mathcal{G})} d(u,v) = \frac{1}{2}\sum_{v\in V(\mathcal{G})} d_{\mathcal{G}}(v), \text{ where } d_{\mathcal{G}}(v) = \sum_{u\in V(\mathcal{G})} d(u,v).$$

The hyper-Wiener index of acyclic graphs was introduced by Milan Randic in 1993. M. H. Khalifeh et. al., they studied the hyper-Wiener indices of the Cartesian product, composition, join and disjunction of graphs in [32]. The definition of the hyper-Wiener index is as follows:

Definition 4.1.2 (Hyper-Wiener index)**.** *Let $\mathcal{G} = (V, E)$ be a graph. Then the hyper-Wiener index $WW(\mathcal{G})$ of the graph $\mathcal{G}$ is defined as*

$$WW(\mathcal{G}) = \frac{1}{2}W(\mathcal{G}) + \frac{1}{2}\sum_{\{u,v\}\subseteq V(\mathcal{G})} d^2(u,v) = \frac{1}{2}W(\mathcal{G}) + \frac{1}{4}\sum_{v\in V(\mathcal{G})} d^2_{\mathcal{G}}(v),$$

where $d^2(u,v) = d(u,v)^2$.

The Harary index has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. Kexiang Xu and Kinkar Ch. Das, they characterized some lower and upper bounds on the Harary index of graphs with different parameters, such as clique number and chromatic number, and charaterize the extremal graphs at which the lower or upper bounds on the Harary index are attained. The definition of Harary index is defined as follows:

Definition 4.1.3 (Harary index)**.** *Let $\mathcal{G} = (V, E)$ be a connected graph. Then the Harary index of the graph $\mathcal{G}$ is defined as*

$$H(\mathcal{G}) = \sum_{\{u,v\}\subseteq V(\mathcal{G})} \frac{1}{d(u,v)} = \frac{1}{2}\sum_{v\in V(\mathcal{G})} \frac{1}{d_{\mathcal{G}}(v)}$$

The Gutman index is also known as Schltz index of second kind. The Gutman index is a natural extension of the Wiener index. V. Andova et. al., they studied among all graphs on n vertices, the star graph S_n has minimal Gutman index in [?]. In addition, they presented upper and lower bounds on Gutman index for graphs with minimal and graphs with maximal Gutman index. The Gutman index of a finite connected graph $\mathcal{G}$ is defined as follows:

Definition 4.1.4 (Gutman index). *Let $\mathcal{G} = (V, E)$ be finite connected graph. Then the Gutman index $Gut(\mathcal{G})$ of the graph $\mathcal{G}$ is defined as*

$$Gut(\mathcal{G}) = \sum_{\{u,v\}\subseteq V(\mathcal{G})} d(u)d(v)d(u,v)$$

In this chapter we mainly characterize in detail the order two element graph over the groups $\mathbb{Z}_n$ and D_n through the Wiener index, Hyper-Wiener index, Harary index and Gutman index. Also, we determine a general formulae for finding the Wiener index, Hyper-Wiener index, Harary index, and Gutman index of the order two element graph over the finite groups $\mathbb{Z}_n$ and D_n.

From the above definitions, we have the following lemma.

Lemma 4.1.5. *If $\mathcal{G}$ is a disconnected graph with $\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_k$ connected components then*
(i) $W(\mathcal{G}) = \sum_{i=1}^{k} W(\mathcal{G}_i)$, *(ii)* $WW(\mathcal{G}) = \sum_{i=1}^{k} WW(\mathcal{G}_i)$, *(iii)* $H(\mathcal{G}) = \sum_{i=1}^{k} H(\mathcal{G}_i)$,
(iv) $Gut(\mathcal{G}) = \sum_{i=1}^{k} Gut(\mathcal{G}_i)$.

4.2 Wiener, Hyper-Wiener, Harary and Gutman indices of order two element graph over $\mathbb{Z}_n$.

This section focuses on the structure of order two element graph over the group $\mathbb{Z}_n$ and provides a general formula for distance-based topological indices (Wiener, Hyper-Wiener, Harary, and Gutman index) for the graph $\mathbb{S}_2(\mathbb{Z}_n)$. To better visualize the graph $\mathbb{S}_2(\mathbb{Z}_n)$, we refer to the examples below.

Example 4.2.1. *Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_6)$ and $G = \mathbb{Z}_6 = \{e, x, x^2, x^3, x^4, x^5\}$. Then $S_G = \{e, x^2, x^4\}$ and $G_2 = \{x^3\}$. For any $v \in V(\mathcal{G})$, we have $d_{\mathcal{G}}(v) = \sum_{u\in V(\mathcal{G})} d(u, v)$. The order two element graph $\mathbb{S}_2(\mathbb{Z}_6)$ over the group $\mathbb{Z}_6$ is shown in figure 4.1.*

From Figure 4.1, since e is adjacent with x^2, x^3 and x^4, then $d(e, x^2) = 1$, $d(e, x^3) = 1$, and $d(e, x^4) = 1$. But, e is not adjacent with x and x^5, although there is a path

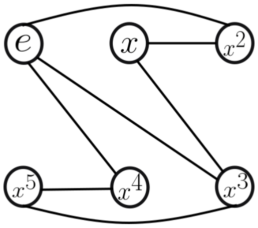

Figure 4.1: $\mathbb{S}_2(\mathbb{Z}_6)$

$e \leftrightarrow x$ of length 2 between e and x, and a path $e \leftrightarrow x^5$ of length 2 between e and x^5. So, $d(e,x) = 2$ and $d(e,x^5) = 2$. Therefore, $d_{\mathcal{G}}(e) = 2+1+1+1+2 = 7$, $d^2_{\mathcal{G}}(e) = 2^2+1^2+1^2+1^2+2^2 = 11$ and $\dfrac{1}{d_{\mathcal{G}}(e)} = \dfrac{1}{2}+\dfrac{1}{1}+\dfrac{1}{1}+\dfrac{1}{1}+\dfrac{1}{2} = 4$.

Applying the similar concepts, one can find $d_{\mathcal{G}}(x) = 8$, $d_{\mathcal{G}}(x^2) = 8$, $d_{\mathcal{G}}(x^3) = 7$, $d_{\mathcal{G}}(x^4) = 8$, $d_{\mathcal{G}}(x^5) = 8$.

Therefore, $W(\mathcal{G}) = \dfrac{1}{2}\sum\limits_{v\in V(\mathcal{G})} d_{\mathcal{G}}(v) = \dfrac{1}{2}[7+8+8+7+8+8] = 23$.

Moreover, $d^2_{\mathcal{G}}(e) = 11$, $d^2_{\mathcal{G}}(x) = 14$, $d^2_{\mathcal{G}}(x^2) = 14$, $d^2_{\mathcal{G}}(x^3) = 11$, $d^2_{\mathcal{G}}(x^4) = 14$, $d^2_{\mathcal{G}}(x^5) = 14$.

Therefore, $WW(\mathcal{G}) = \dfrac{1}{2}[\sum\limits_{\{u,v\}\in V(\mathcal{G})} d(u,v) + \sum\limits_{\{u,v\}\in V(\mathcal{G})} d^2(u,v)]$
$= \dfrac{1}{2}[\dfrac{1}{2}(7+8+8+7+8+8) + \dfrac{1}{2}(11+14+14+11+14+14)] = 31$.

Also, $\dfrac{1}{d_{\mathcal{G}}(e)} = 4$, $\dfrac{1}{d_{\mathcal{G}}(x)} = \dfrac{7}{2}$, $\dfrac{1}{d_{\mathcal{G}}(x^2)} = \dfrac{7}{2}$, $\dfrac{1}{d_{\mathcal{G}}(x^3)} = 4$, $\dfrac{1}{d_{\mathcal{G}}(x^4)} = \dfrac{7}{2}$, $\dfrac{1}{d_{\mathcal{G}}(x^5)} = \dfrac{7}{2}$.

Therefore, $H(\mathcal{G}) = \dfrac{1}{2}\sum\limits_{v\in V(\mathcal{G})} \dfrac{1}{d_{\mathcal{G}}(v)} = \dfrac{1}{2}[4+\dfrac{7}{2}+\dfrac{7}{2}+4+\dfrac{7}{2}+\dfrac{7}{2}] = 11$.

Again, $d(e) = 3 = d(x^3)$, $d(x) = d(x^2) = d(x^4) = d(x^5) = 2$.

Therefore, $Gut(\mathcal{G}) = \sum\limits_{\{u,v\}\in V(\mathcal{G})} d(u)d(v)d(u,v)$.

As $d(u)d(v)d(u,v) = d(v)d(u)d(v,u)$, then
$Gut(\mathcal{G}) = \dfrac{1}{2}[\{d(e)d(x)d(e,x)+d(e)d(x^2)d(e,x^2)+d(e)d(x^3)d(e,x^3)+d(e)d(x)d(e,x^4)+ d(e)d(x^5)d(e,x^5)\} + \{d(x)d(e)d(x,e) + d(x)d(x^2)d(x,x^2) + d(x)d(x^3)d(x,x^3) + d(x)d(x^4)d(x,x^4) + d(x)d(x^5)d(x,x^5)\} + \{d(x^2)d(e)d(x^2,e) + d(x^2)d(x)d(x^2,x) + d(x^2)d(x^3)d(x^2,x^3) + d(x^2)d(x^4)d(x^2,x^4) + d(x^2)d(x^5)d(x^2,x^5)\} + \{d(x^3)d(e)d(x^3,e)+$

$$d(x^3)d(x)d(x^3,x)+d(x^3)d(x^2)d(x^3,x^2)+d(x^3)d(x^4)d(x^3,x^4)+d(x^3)d(x^5)d(x^3,x^5)\}+ \{d(x^4)d(e)d(x^4,e)+d(x^4)d(x)d(x^4,x)+d(x^4)d(x^2)d(x^4,x^2)+d(x^4)d(x^3)d(x^4,x^3)+ d(x^4)d(x^5)d(x^4,x^5)\}+\{d(x^5)d(e)d(x^5,e)+d(x^5)d(x)d(x^5,x)+d(x^5)d(x^2)d(x^5,x^2)+ d(x^5)d(x^3)d(x^5,x^3)+d(x^5)d(x^4)d(x^5,x^4)\}]$$

$$=\frac{1}{2}[(3\times2\times2+3\times2\times1+3\times3\times1+3\times2\times1+3\times2\times2)+(3\times2\times2+2\times2\times1+ 3\times2\times1+2\times2\times2+2\times2\times2)+(3\times2\times1+2\times2\times1+2\times3\times2+2\times2\times2+2\times2\times 2)+(3\times3\times1+3\times2\times1+3\times2\times2+3\times2\times2+3\times2\times1)+(3\times2\times1+2\times2\times2+2\times 2\times2+3\times2\times2+2\times2\times1)+(3\times2\times2+2\times2\times2+2\times2\times2+3\times2\times1+2\times2\times1)]$$

$$=121.$$

Example 4.2.2. *Let $\mathcal{G}=\mathbb{S}_2(\mathbb{Z}_7)$ and $G=\mathbb{Z}_7=\{e,x,x^2,x^3,x^4,x^5,x^6\}$. Then $S_G=\{e,xx^2,x^3,x^4,x^5,x^6\}$ and $G_2=\phi$. From Figure 4.2 and similar way from 4.2.1, we have*

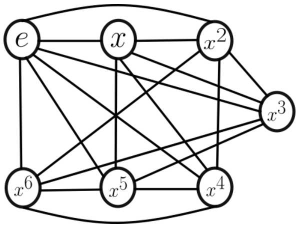

Figure 4.2: $\mathbb{S}_2(\mathbb{Z}_7)$

$W(\mathcal{G})=24$, $WW(\mathcal{G})=27$, $H(\mathcal{G})=19.5$, $Gut(\mathcal{G})=630$.

Example 4.2.3. *Let $\mathcal{G}=\mathbb{S}_2(\mathbb{Z}_8)$ and $G=\mathbb{Z}_8=\{e,x,x^2,x^3,x^4,x^5,x^6,x^7\}$. Then $S_G=\{e,x^2,x^4,x^6\}$ and $G_2=\{x^4\}$. From Figure 4.3 and similar way from 4.2.1, we have*

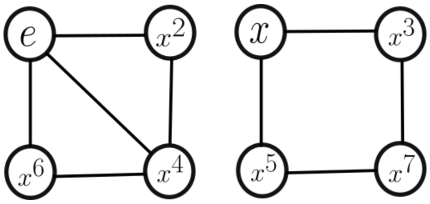

Figure 4.3: $\mathbb{S}_2(\mathbb{Z}_8)$

$W(\mathcal{G})=15$, $WW(\mathcal{G})=18$, $H(\mathcal{G})=10.5$, $Gut(\mathcal{G})=73$.

Lemma 4.2.4. *Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$, where n is any odd positive integer. Then for $x^i \in \mathbb{Z}_n$*

$$d_{\mathcal{G}}(x^i) = \begin{cases} n-1 \ if \ x^i = e \\ n \qquad if \ x^i \neq e \end{cases}$$

Proof. Let $G = \mathbb{Z}_n = \{e, x, x^2, x^3, \cdots, x^{n-1}\}$, where n is odd positive integer. Then $S_G = \{e, x, x^2, x^3, \cdots, x^{n-1}\}$ and $G_2 = \phi$. Now, e is adjacent with each x^i for $i = 1, 2, 3, \cdots, n-1$. Therefore, $d(e, x^i) = 1$. Also, x^i is adjacent with x^j with $j \neq i$ and $j \neq n-i$. Then $d(x^i, x^j) = 1$ but $d(x^i, x^{n-i}) = 2$.

Now, $d_{\mathcal{G}}(e) = \sum\limits_{x^i \in V} d(e, x^i) = (n-1) \times 1 = n-1$, where $\mathcal{G} = \mathbb{S}_2(G)$.

Also for $x^i \neq e$, $d_{\mathcal{G}}(x^i) = \sum\limits_{x^j \in V} d(x^i, x^j)$
$= d(x^i, e) + d(x^i, x) + d(x^i, x^2) + \cdots + d(x^i, x^{i-1}) + d(x^i, x^{i+1}) + \cdots + d(x^i, x^{n-i-1}) + d(x^i, x^{n-i}) + d(x^i, x^{n-i+1}) + \cdots + d(x^i, x^{n-1}) = (n-2) \times 1 + 1 \times 2 = n.$ □

Lemma 4.2.5. *Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$, where $\frac{n}{2}$ is any odd positive integer. Then for $x^i \in \mathbb{Z}_n$*

$$d_{\mathcal{G}}(x^i) = \begin{cases} \dfrac{3n-4}{2} & if \ x^i = e \ or \ x^{\frac{n}{2}} \\ \dfrac{3n-2}{2} & if \ x^i \neq e \end{cases}$$

Proof. Let $G = \mathbb{Z}_n = \{e, x, x^2, x^3, \cdots, x^{n-1}\}$, where $\frac{n}{2}$ is odd positive integer. Then $S_G = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}}\}$, where $x^{\frac{n}{2}} \notin S_G$.

Then $d(e, x^i) = \begin{cases} 1 \ \ if \ i = even \ or \ i = \frac{n}{2} \\ 2 \ \ if \ i = odd \end{cases}$.

Now, $d_{\mathcal{G}}(e) = \sum\limits_{x^i \in V} d(e, x^i) = \underbrace{(\frac{n}{2} - 1)}_{\text{even case}} \times 1 + \underbrace{1}_{\text{for } x^{\frac{n}{2}}} + \underbrace{(\frac{n}{2} - 1)}_{\text{odd case}} \times 2 = \dfrac{3n-4}{2}.$

$d_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum\limits_{x^i \in V} d(x^{\frac{n}{2}}, x^i) = \underbrace{1}_{\text{for } e} + \underbrace{(\frac{n}{2} - 1)}_{\text{odd case}} \times 1 + + \underbrace{(\frac{n}{2} - 1)}_{\text{even case}} \times 2 = \dfrac{3n-4}{2}.$

For i is odd, $d_{\mathcal{G}}(x^i) = \sum\limits_{x^j \in V} d(x^i, x^j)$
$= d(x^i, e) + [d(x^i, x) + d(x^i, x^3) + \cdots + d(x^i, x^{i-2}) + d(x^i, x^{i+2}) + \cdots + d(x^i, x^{n-i-2}) +$

$d(x^i, x^{n-i}) + d(x^i, x^{n-i+2}) + d(x^i, x^{n-1})] + [d(x^i, x^2) + d(x^i, x^4) + \cdots + d(x^i, x^{\frac{n}{2}-i-2}) + d(x^i, x^{\frac{n}{2}-i}) + d(x^i, x^{\frac{n}{2}-i+2}) + \cdots + d(x^i, x^{n-2})]$
$= 2 + \{(\frac{n}{2} - 2) \times 1 + 2\} + \{(\frac{n}{2} - 2) \times 2 + 1\} = \frac{3n-2}{2}$.

For i is even, $d_{\mathcal{G}}(x^i) = \sum\limits_{x^j \in V} d(x^i, x^j)$
$= d(x^i, e) + [d(x^i, x) + d(x^i, x^3) + \cdots + d(x^i, x^{\frac{n}{2}-i-2}) + d(x^i, x^{\frac{n}{2}-i}) + d(x^i, x^{\frac{n}{2}-i+2}) + \cdots + d(x^i, x^{n-1})] + [d(x^i, x^2) + d(x^i, x^4) + \cdots + d(x^i, x^{n-i-2}) + d(x^i, x^{n-i+2}) + \cdots + d(x^i, x^{n-2})]$
$= 1 + \{(\frac{n}{2} - 1) \times 2 + 1\} + \{(\frac{n}{2} - 3) \times 1 + 2\} = \frac{3n-4}{2}$.

Therefore, $d_{\mathcal{G}}(x^i) = \begin{cases} \frac{3n-4}{2} & if\ x^i = e\ or\ x^{\frac{n}{2}} \\ \frac{3n-2}{2} & if\ x^i \neq e \end{cases}$ □

Lemma 4.2.6. *Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$, where $\frac{n}{2}$ is any even positive integer. Then for $x^i \in \mathbb{Z}_n$*

$$d_{\mathcal{G}}(x^i) = \begin{cases} \frac{n}{2} - 1 & if\ x^i = e\ or\ x^{\frac{n}{2}} \\ \frac{n}{2} & if\ x^i \neq e \end{cases}$$

Proof. Let $G = \mathbb{Z}_n = \{e, x, x^2, x^3, \cdots, x^{n-1}\}$ and $\mathcal{G} = \mathbb{S}_2(G)$, where $\frac{n}{2}$ is even positive integer. Then $S_G = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}}\}$, where $x^{\frac{n}{2}} \in S_G$. Now, we divide the vertex set $V = G$ of $\mathcal{G}$ into two disjoint set $A = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $B = \{x, x^3, x^5, \cdots, x^{n-1}\}$. Then $\mathcal{G}_1$ with vertex set A and $\mathcal{G}_2$ with vertex set B form induced subgraphs of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$.

Now, for the graph $\mathcal{G}_1$, we have for $x^i, x^j \in A$, $d(x^i, x^j) = \begin{cases} 1 & if\ i \neq j,\ j \neq n-i \\ 2 & if\ i \neq j,\ j = n-i \end{cases}$.

Then $d_{\mathcal{G}_1}(e) = \sum\limits_{x^i \in A} d(e, x^i) = (\frac{n}{2} - 1) \times 1 = \frac{n}{2} - 1$ and $d_{\mathcal{G}_1}(x^{\frac{n}{2}}) = \sum\limits_{x^i \in A} d(x^{\frac{n}{2}}, x^i) = \frac{n}{2} - 1$.

Moreover, for $i \neq \frac{n}{2}$, $d_{\mathcal{G}_1}(x^i) = \sum\limits_{x^j \in A} d(x^i, x^j)$
$= d(x^i, e) + d(x^i, x^2) + d(x^i, x^4) + \cdots + d(x^i, x^{i-2}) + d(x^i, x^{i+2}) + \cdots + d(x^i, x^{n-i}) +$

$\cdots + d(x^i, x^{n-2})$
$= (\frac{n}{2} - 2) \times 1 + 1 \times 2 = \frac{n}{2}.$

For the graph $\mathcal{G}_2$, we have $d(x^i, x^j) = 2$ if $i \neq j$ and $\forall\ x^i, x^j \in B$. Then,

$$d_{\mathcal{G}_2}(x^i) = \sum_{x^j \in B} d(x^i, x^j)$$
$$= d(x^i, x) + d(x^i, x^3) + \cdots + d(x^i, x^{i-2}) + d(x^i, x^{i+2}) + \cdots + d(x^i, x^{n-i}) + \cdots + d(x^i, x^{n-1})$$
$$= (\frac{n}{2} - 2) \times 1 + 1 \times 2 = \frac{n}{2}. \qquad \square$$

Theorem 4.2.7. *The Wiener index of the graph $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$ is*

$$W(\mathbb{S}_2(\mathbb{Z}_n)) = \begin{cases} \dfrac{n^2-1}{2}, \ if\ n\ is\ odd \\ \dfrac{3n^2-2n-4}{4}, \ if\ \dfrac{n}{2}\ is\ odd \\ \dfrac{n^2}{4} - 1, \ if\ \dfrac{n}{2}\ is\ even \end{cases}.$$

Proof. For n is odd positive integer, we have from the Lemma 4.2.4,

$$W(\mathcal{G}) = \frac{1}{2} \sum_{u \in V(\mathcal{G})} d_{\mathcal{G}}(u) = \frac{1}{2}[d_{\mathcal{G}}(e) + \sum_{x^i \in V(\mathcal{G}) \setminus \{e\}} d_{\mathcal{G}}(x^i)]$$
$$= \frac{1}{2}[(n-1) + \sum_{x^i \in V(\mathcal{G}) \setminus \{e\}} n] = \frac{1}{2}[n - 1 + (n-1)n] = \frac{n^2-1}{2}.$$

For $\frac{n}{2}$ is odd positive integer, we have from the Lemma 4.2.5,

$$W(\mathcal{G}) = \frac{1}{2} \sum_{u \in V(\mathcal{G})} d_{\mathcal{G}}(u) = \frac{1}{2}[d_{\mathcal{G}}(e) + d_{\mathcal{G}}(x^{\frac{n}{2}}) + \sum_{\substack{x^i \in V(\mathcal{G}) \\ i = odd, i \neq \frac{n}{2}}} d_{\mathcal{G}}(x^i) + \sum_{\substack{x^i \in V(\mathcal{G}) \\ i = even, i \neq n}} d_{\mathcal{G}}(x^i)]$$
$$= \frac{1}{2}[\frac{3n-4}{2} + \frac{3n-4}{2} + (\frac{n}{2} - 1)\frac{3n-2}{2} + (\frac{n}{2} - 1)\frac{3n-2}{2}] = \frac{3n^2-2n-4}{4}.$$

For $\frac{n}{2}$ is even positive integer, we have from the Lemma 4.2.6,

$$W(\mathcal{G}) = W(\mathcal{G}_1) + W(\mathcal{G})_2 = \frac{1}{2} \sum_{u \in V(\mathcal{G}_1)} d_{\mathcal{G}_1}(u) + \frac{1}{2} \sum_{u \in V(\mathcal{G}_2)} d_{\mathcal{G}_2}(u)$$
$$= \frac{1}{2}[d_{\mathcal{G}_1}(e) + d_{\mathcal{G}_1}(x^{\frac{n}{2}}) + \sum_{\substack{x^i \in V(\mathcal{G}_1) \\ i \neq n, \frac{n}{2}}} d_{\mathcal{G}_1}(x^i)] + \frac{1}{2} \sum_{u \in V(\mathcal{G}_2)} \frac{n}{2}$$
$$= \frac{1}{2}[(\frac{n}{2} - 1) + (\frac{n}{2} - 1) + (\frac{n}{2} - 2)\frac{n}{2}] + \frac{n}{2}.\frac{n}{4} = \frac{n^2}{8} - 1 + \frac{n^2}{8} = \frac{n^2}{4} - 1. \qquad \square$$

Theorem 4.2.8. *The Hyper-Wiener index of the graph $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$ is*

$$WW(\mathbb{S}_2(\mathbb{Z}_n)) = \begin{cases} \dfrac{n^2+n-2}{2}, \ if\ n\ is\ odd \\ \dfrac{2n^2-n-4}{2}, \ if\ \dfrac{n}{2}\ is\ odd \\ \dfrac{n^2+2n-8}{4} - 1, \ if\ \dfrac{n}{2}\ is\ even \end{cases}$$

.

Proof. Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$, where $G = \mathbb{Z}_n = \{e, x, x^2, x^3, \cdots, x^{n-1}\}$.

Then $S_G = \begin{cases} \{e, x, x^2, x^3, \cdots, x^{n-1}\}\ if\ n\ is\ odd \\ \\ \{e, x^2, x^4, \cdots, x^{n-2}\}\ if\ n\ is\ even \end{cases}$ and $G_2 = \begin{cases} \phi\ if\ n\ is\ odd \\ \\ \{x^{\frac{n}{2}}\}\ if\ n\ is\ even \end{cases}$.

Now, the Hyper-Wiener index of the graph $\mathcal{G}$ is defined as
$WW(\mathcal{G}) = \frac{1}{2}[\sum_{\{u,v\}\subseteq V(\mathcal{G})} d(u,v) + \sum_{\{u,v\}\subseteq V(\mathcal{G})} d^2(u,v)] = \frac{1}{2}W(\mathcal{G}) + \frac{1}{4}\sum_{v\in V(\mathcal{G})} d^2_{\mathcal{G}}(v)$.

Case-1: Let n be an odd positive integer, then by Lemma 4.2.4, we have $d^2_{\mathcal{G}}(e) = (n-1)\times 1^2 = n-1$ and $d^2_{\mathcal{G}}(x^i) = (n-2)\times 1^2 + 1\times 2^2 = n+2$.
Therefore, using Theorem 4.2.7, $WW(\mathcal{G}) = \dfrac{n^2-1}{4} + \dfrac{1}{4}[(n-1)+(n-1)(n+2)]$
$= \dfrac{n^2+n-2}{2}$.

Case-2: Let $\dfrac{n}{2}$ be an odd positive integer, then by Lemma 4.2.5, we have
$d^2_{\mathcal{G}}(e) = (\frac{n}{2}-1)\times 1^2 + 1\times 1^2 + (\frac{n}{2}-1)\times 2^2 = \dfrac{5n-8}{2}$.
$d^2_{\mathcal{G}}(x^{\frac{n}{2}}) = 1\times 1^2 + (\frac{n}{2}-1)\times 1^2 + (\frac{n}{2}-1)\times 2^2 = \dfrac{5n-8}{2}$.
For i is odd but $i \neq \dfrac{n}{2}$, $d^2_{\mathcal{G}}(x^i) = 1\times 2^2 + \{(\frac{n}{2}-2)\times 1^2 + 1\times 2^2\} + \{(\frac{n}{2}-2)\times 2^2 + 1\times 1^2\}$
$= \dfrac{5n-2}{2}$.
For i is even, $d^2_{\mathcal{G}}(x^i) = 1\times 1^2 + \{(\frac{n}{2}-1)\times 2^2 + 1\times 1^2\} + \{(\frac{n}{2}-3)\times 1^2 + 1\times 2^2\} = \dfrac{5n-2}{2}$.

Therefore, using Theorem 4.2.7, $WW(\mathcal{G}) = \dfrac{3n^2-2n-4}{8} + \dfrac{1}{4}[\dfrac{5n-8}{2} + \dfrac{5n-8}{2} + (\dfrac{n}{2}-1)\dfrac{5n-2}{2} + (\dfrac{n}{2}-1)\dfrac{5n-2}{2}] = \dfrac{2n^2-n-4}{2}$.

Case-3: Let $\dfrac{n}{2}$ is an even positive integer, then by Lemma 4.2.6, we have
For $\mathcal{G}_1$, $d^2_{\mathcal{G}_1}(e) = (\frac{n}{2}-1)\times 1^2 = \frac{n}{2}-1$, $d^2_{\mathcal{G}_1}(x^{\frac{n}{2}}) = (\frac{n}{2}-1)\times 1^2 = \frac{n}{2}-1$.
$d^2_{\mathcal{G}_1}(x^i) = (\frac{n}{2}-2)\times 1^2 + 1\times 2^2 = \dfrac{n+4}{2}$.
So, $\dfrac{1}{2}\sum_{\{u,v\}\subseteq V(\mathcal{G}_1)} d^2(u,v) = \dfrac{1}{2}[(\frac{n}{2}-1) + (\frac{n}{2}-1) + (\frac{n}{2}-2)\dfrac{n+4}{2}] = \dfrac{n^2+4n-24}{8}$.

For $\mathcal{G}_2$, $d^2_{\mathcal{G}_2}(x^i) = (\frac{n}{2} - 2) \times 1^2 + 1 \times 2^2 = \frac{n+4}{2}$.
So, $\frac{1}{2} \sum_{\{u,v\} \subseteq V(\mathcal{G}_2)} d^2(u,v) = \frac{1}{2}[\frac{n}{2} \times \frac{n+4}{2}] = \frac{n(n+4)}{8}$.

Therefore, using Theorem 4.2.7, $WW(\mathcal{G}) = \frac{n^2-4}{8} + \frac{1}{2} \times \frac{n^2+4n-24}{8} + \frac{1}{2} \times \frac{n(n+4)}{8} = \frac{n^2+2n-8}{4}$. □

Theorem 4.2.9. *The Harary index of the graph $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$ is*

$$H(\mathbb{S}_2(\mathbb{Z}_n)) = \begin{cases} \frac{(n-1)(2n-1)}{4}, & if\ n\ is\ odd \\ \frac{3n^2-4n+4}{8}, & if\ \frac{n}{2}\ is\ odd \\ \frac{n^2-3n+2}{4} - 1, & if\ \frac{n}{2}\ is\ even \end{cases}$$

.

Proof. Let $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$, where $G = \mathbb{Z}_n = \{e, x, x^2, x^3, \cdots, x^{n-1}\}$.
Then $S_G = \begin{cases} \{e, x, x^2, x^3, \cdots, x^{n-1}\}\ if\ n\ is\ odd \\ \{e, x^2, x^4, \cdots, x^{n-2}\}\ if\ n\ is\ even \end{cases}$ and $G_2 = \begin{cases} \phi\ if\ n\ is\ odd \\ \{x^{\frac{n}{2}}\}\ if\ n\ is\ even \end{cases}$.

Now, the Harary index of the graph $\mathcal{G}$ is defined as $H(\mathcal{G}) = \sum_{\{u,v\} \subseteq V(\mathcal{G})} \frac{1}{d(u,v)}$.

Case-1: Let n be an odd positive integer, then by Lemma 4.2.4, we have
$\frac{1}{d_{\mathcal{G}}(e)} = (n-1)$ and $\frac{1}{d_{\mathcal{G}}(x^i)} = \sum_{x^j \in V(\mathcal{G})} \frac{1}{d(x^i, x^j)} = (n-2) \times \frac{1}{1} + 1 \times \frac{1}{2} = \frac{2n-3}{2}$.
Therefore, using Theorem 4.2.7,
$H(\mathcal{G}) = \frac{1}{2} \sum_{x^i \in V(\mathcal{G})} \frac{1}{d_{\mathcal{G}}(x^i)} = \frac{1}{2}[n - 1 + (n-1)\frac{(2n-3)}{2}] = \frac{(n-1)(2n-1)}{4}$.

Case-2: Let $\frac{n}{2}$ be an odd positive integer, then by Lemma 4.2.5, we have
$\frac{1}{d_{\mathcal{G}}(e)} = (\frac{n}{2} - 1) \times \frac{1}{1} + 1 \times \frac{1}{1} + (\frac{n}{2} - 1) \times \frac{1}{2} = \frac{3n-2}{4}$.
$\frac{1}{d_{\mathcal{G}}(x^{\frac{n}{2}})} = 1 \times \frac{1}{1} + (\frac{n}{2} - 1) \times \frac{1}{1} + (\frac{n}{2} - 1) \times \frac{1}{2} = \frac{3n-2}{4}$.
For i is odd but $i \neq \frac{n}{2}$, $\frac{1}{d_{\mathcal{G}}(x^i)} = 1 \times \frac{1}{2} + \{(\frac{n}{2} - 2) \times \frac{1}{1} + 1 \times \frac{1}{2}\} + \{(\frac{n}{2} - 2) \times \frac{1}{2} + 1 \times \frac{1}{1}\}$
$= \frac{3n-4}{4}$.
For i is even, $\frac{1}{d_{\mathcal{G}}(x^i)} = 1 \times \frac{1}{1} + \{(\frac{n}{2} - 1) \times \frac{1}{2} + 1 \times \frac{1}{1}\} + \{(\frac{n}{2} - 3) \times \frac{1}{1} + 1 \times \frac{1}{2}\} = \frac{3n-4}{4}$.

Therefore, using Theorem 4.2.7,

$$H(\mathcal{G}) = \frac{1}{2}[\frac{1}{d_{\mathcal{G}}(e)} + \frac{1}{d_{\mathcal{G}}(x^{\frac{n}{2}})} + \sum_{\substack{x^i \in V(\mathcal{G}) \\ i=odd,\ i \neq \frac{n}{2}}} \frac{1}{d_{\mathcal{G}}(x^i)} + \sum_{\substack{x^i \in V(\mathcal{G}) \\ i=even,\ i \neq n}} \frac{1}{d_{\mathcal{G}}(x^i)}]$$

$$= \frac{1}{2}[\tfrac{3n-2}{4} + \frac{3n-2}{4} + (\frac{n}{2}-1)\frac{3n-4}{4} + (\frac{n}{2}-1)\frac{3n-4}{4}] = \frac{3n^2-4n+4}{8}.$$

Case-3: Let $\frac{n}{2}$ is an even positive integer, then by Lemma 4.2.6, we have

For $\mathcal{G}_1$, $\frac{1}{d_{\mathcal{G}_1}(e)} = (\frac{n}{2}-1) \times \frac{1}{1} = \frac{n}{2}-1$, $\frac{1}{d_{\mathcal{G}_1}(x^{\frac{n}{2}})} = (\frac{n}{2}-1) \times \frac{1}{1} = \frac{n}{2}-1$.

$\frac{1}{d_{\mathcal{G}_1}(x^i)} = (\frac{n}{2}-2) \times \frac{1}{1} + 1 \times \frac{1}{2} = \frac{n-3}{2}$.

So, $H(\mathcal{G}_1) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(\mathcal{G}_1)} \frac{1}{d(u,v)} = \frac{1}{2}[(\frac{n}{2}-1) + (\frac{n}{2}-1) + (\frac{n}{2}-2)\frac{n-3}{2}] = \frac{n^2-3n+4}{8}$.

For $\mathcal{G}_2$, $\frac{1}{d_{\mathcal{G}_2}(x^i)} = (\frac{n}{2}-2) \times \frac{1}{1} + 1 \times \frac{1}{2} = \frac{n-3}{2}$.

So, $H(\mathcal{G}_2) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(\mathcal{G}_2)} \frac{1}{d(u,v)} = \frac{1}{2}[\frac{n}{2} \times \frac{n-3}{2}] = \frac{n(n-3)}{8}$.

Therefore, using Theorem 4.2.7, $H(\mathcal{G}) = H(\mathcal{G}_1) + H(\mathcal{G}_2) = \frac{n^2-3n+4}{8} + \frac{n(n-3)}{8}$ $= \frac{n^2-3n+2}{4}$. □

Theorem 4.2.10. *The Gutman index of the graph $\mathcal{G} = \mathbb{S}_2(\mathbb{Z}_n)$ is*

$$Gut(\mathbb{S}_2(\mathbb{Z}_n)) = \begin{cases} n(n-1)^2(n-2), \ if\ n\ is\ odd \\ \dfrac{3n^4-14n^3+40n^2-72n+64}{16}, \ if\ \dfrac{n}{2}\ is\ odd\ and\ n > 6 \\ \dfrac{3n^4-22n^3+32n^2+64n-96}{32}, \ if\ \dfrac{n}{2}\ is\ even \end{cases}$$

.

Proof. Let $\mathcal{G} = \mathbb{S}_2(G)$, where $G = \mathbb{Z}_n = \{e, x, x^2, x^3, \cdots, x^{n-1}\}$.

Then $S_G = \begin{cases} \{e, x, x^2, x^3, \cdots, x^{n-1}\}\ if\ n\ is\ odd \\ \\ \{e, x^2, x^4, \cdots, x^{n-2}\}\ if\ n\ is\ even \end{cases}$ and $G_2 = \begin{cases} \phi\ if\ n\ is\ odd \\ \\ \{x^{\frac{n}{2}}\}\ if\ n\ is\ even \end{cases}$.

It is clear that if $\frac{n}{2}$ is odd then $x^{\frac{n}{2}} \notin S_G$ but if $\frac{n}{2}$ is even then $x^{\frac{n}{2}} \in S_G$.

Define for $u, v \in V(\mathcal{G})$, $d(u)d(v)d_{\mathcal{G}}(x^i) = \sum_{x^j \in V(\mathcal{G})} d(x^i)d(x^j)d(x^i, x^j)$.

Then $Gut(\mathcal{G}) = \sum_{\{u,v\} \in V(\mathcal{G})} d(u)d(v)d(u,v) = \frac{1}{2} \sum_{x^i \in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(x^i)$.

Case-1: Let n be an odd positive integer. Then $d(x^i) = \begin{cases} n-1, \ if \ x^i = e \\ n-2, \ if \ x^i \neq e \end{cases}$.

Now, using the Lemma 4.2.4,

$$d(u)d(v)d_{\mathcal{G}}(e) = \sum_{x^i \in V(\mathcal{G})} d(e)d(x^i)d(e,x^i) = d(e)\sum_{x^i \in V(\mathcal{G})} d(x^i)d(e,x^i)$$
$$= (n-1)[(n-1)(n-2)\times 1] = (n-1)^2(n-2).$$

$$d(u)d(v)d_{\mathcal{G}}(x^i) = \sum_{x^j \in V(\mathcal{G})} d(x^i)d(x^j)d(x^i,x^j)$$
$$= d(x^i)d(e)d(x^i,e)+d(x^i)d(x)d(x^i,x)+d(x^i)d(x^2)d(x^i,x^2)+\cdots+d(x^i)d(x^{i-1})d(x^i,x^{i-1})+ d(x^i)d(x^{i+1})d(x^i,x^{i+1})+\cdots+d(x^i)d(x^{n-i})d(x^i,x^{n-i})+\cdots+d(x^i)d(x^{n-1})d(x^i,x^{n-1})$$
$$= (n-2)(n-1)\times 1+(n-3)(n-2)^2\times 1+(n-2)^2\times 2 = (n-1)^2(n-2).$$

Therefore, $Gut(\mathcal{G}) = \frac{1}{2}\sum_{x^i \in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(x^i) = \frac{1}{2}d(u)d(v)d_{\mathcal{G}}(e)+\frac{1}{2}\sum_{x^i \in V(\mathcal{G})\setminus\{e\}} d(u)d(v)d_{\mathcal{G}}(x^i)$
$$= \frac{1}{2}(n-1)^2(n-2)+\frac{1}{2}(n-1)(n-1)^2(n-2) = \frac{n(n-1)^2(n-2)}{2}.$$

Case-2: Let $\frac{n}{2}$ be an odd positive integer and $n > 6$.

Then $d(x^i) = \begin{cases} \frac{n}{2}, \ if \ x^i = e \ or \ x^{\frac{n}{2}} \\ \frac{n}{2}-1, \ otherwise \end{cases}$.

Now, using the Lemma 4.2.5,

$$d(u)d(v)d_{\mathcal{G}}(e) = \sum_{x^i \in V(\mathcal{G})} d(e)d(x^i)d(e,x^i)$$
$$= d(e)\{d(x)d(e,x)+d(x^3)d(e,x^3)+\cdots+d(x^{\frac{n}{2}})d(e,x^{\frac{n}{2}})+\cdots+d(x^{n-1})d(e,x^{n-1})\}+ d(e)\{d(x^2)d(e,x^2)+d(x^4)d(e,x^4)+\cdots+d(x^{n-2})d(e,x^{n-2})\}$$
$$= \frac{n}{2}\{(\frac{n}{2}-1)(\frac{n}{2}-1)\times 2+\frac{n}{2}\times 1\}+\frac{n}{2}\{(\frac{n}{2}-1)(\frac{n}{2}-1)\times 1\} = \frac{n(3n^2-10n+12)}{8}.$$

$$d(u)d(v)d_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum_{x^i \in V(\mathcal{G})} d(x^{\frac{n}{2}})d(x^i)d(x^{\frac{n}{2}},x^i)$$
$$= d(x^{\frac{n}{2}})[d(e)d(x^{\frac{n}{2}},e)+\{d(x)d(x^{\frac{n}{2}},x)+d(x^3)d(x^{\frac{n}{2}},x^3)+\cdots+d(x^{n-1})d(x^{\frac{n}{2}},x^{n-1})\}+ d(x^{\frac{n}{2}})\{d(x^2)d(x^{\frac{n}{2}},x^2)+d(x^4)d(x^{\frac{n}{2}},x^4)+\cdots+d(x^{n-2})d(e,x^{n-2})\}]$$
$$= \frac{n}{2}[\frac{n}{2}\times 1+(\frac{n}{2}-1)(\frac{n}{2}-1)\times 1+(\frac{n}{2}-1)(\frac{n}{2}-1)\times 2] = \frac{n(3n^2-10n+12)}{8}.$$

For $i = odd$, $i \neq \frac{n}{2}$, $d(u)d(v)d_{\mathcal{G}}(x^i) = \sum_{x^j \in V(\mathcal{G})} d(x^i)d(x^j)d(x^i,x^j)$
$$= d(x^i)d(e)d(x^i,e)+\{d(x^i)d(x)d(x^i,x)+d(x^i)d(x^3)d(x^i,x^3)+\cdots+d(x^i)d(x^{i-2})d(x^i,x^{i-2})+$$

$d(x^i)d(x^{i+2})d(x^i,x^{i+2})+\cdots+d(x^i)d(x^{\frac{n}{2}})d(x^i,x^{\frac{n}{2}})+\cdots+d(x^i)d(x^{n-i})d(x^i,x^{n-i})+\cdots+$ $d(x^i)d(x^{n-1})d(x^i,x^{n-1})\}+\{d(x^i)d(x^2)d(x^i,x^2)+d(x^i)d(x^4)d(x^i,x^4)+\cdots$ $+\,d(x^i)d(x^{\frac{n}{2}-i})d(x^i,x^{\frac{n}{2}-i})+\cdots+d(x^i)d(x^{n-2})d(x^i,x^{n-2})\}$
$=\frac{n}{2}(\frac{n}{2}-1)\times 2+\{\frac{n}{2}(\frac{n}{2}-1)\times 1+(\frac{n}{2}-1)^2\times 2+(\frac{n}{2}-3)(\frac{n}{2}-1)^2\times 1\}+\{(\frac{n}{2}-1)^2\times$ $1+(\frac{n}{2}-2)(\frac{n}{2}-1)^2\times 2\}=\frac{(n-2)(3n^2-8n+16)}{8}$.

For i is even, $d(u)d(v)d_{\mathcal{G}}(x^i)=\sum\limits_{x^j\in V(\mathcal{G})} d(x^i)d(x^j)d(x^i,x^j)$
$=d(x^i)d(e)d(x^i,e)+\{d(x^i)d(x)d(x^i,x)+d(x^i)d(x^3)d(x^i,x^3)+\cdots+d(x^i)d(x^{\frac{n}{2}-i})d(x^i,x^{\frac{n}{2}-i})+$ $\cdots+d(x^i)d(x^{\frac{n}{2}})d(x^i,x^{\frac{n}{2}})\cdots+d(x^i)d(x^{n-1})d(x^i,x^{n-1})\}+\{d(x^i)d(x^2)d(x^i,x^2)+$ $d(x^i)d(x^4)d(x^i,x^4)+\cdots+d(x^i)d(x^{n-i})d(x^i,x^{n-i})+\cdots+d(x^i)d(x^{n-2})d(x^i,x^{n-2})\}$
$=\frac{n}{2}(\frac{n}{2}-1)\times 1+\{(\frac{n}{2}-1)^2\times 1+(\frac{n}{2}-1)\frac{n}{2}\times 2+(\frac{n}{2}-2)(\frac{n}{2}-1)^2\times 2\}+\{(\frac{n}{2}-1)^2\times$ $2+(\frac{n}{2}-3)(\frac{n}{2}-1)^2\times 1\}=\frac{(n-2)(3n^2-8n+16)}{8}$.

Therefore, $Gut(\mathcal{G})=\frac{1}{2}\sum\limits_{x^i\in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(x^i)$
$=\frac{1}{2}d(u)d(v)d_{\mathcal{G}}(e)+\frac{1}{2}d(u)d(v)d_{\mathcal{G}}(x^{\frac{n}{2}})+\frac{1}{2}\sum\limits_{\substack{x^i\in V(\mathcal{G})\\ i=odd,\ i\neq\frac{n}{2}}} d(u)d(v)d_{\mathcal{G}}(x^i)$
$+\frac{1}{2}\sum\limits_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq n}} d(u)d(v)d_{\mathcal{G}}(x^i)$
$=\frac{1}{2}\times\frac{n(3n^2-10n+12)}{8}+\frac{1}{2}\times\frac{n(3n^2-10n+12)}{8}+\frac{1}{2}\times(\frac{n}{2}-1)\frac{(n-2)(3n^2-8n+16)}{8}+$ $\frac{1}{2}\times(\frac{n}{2}-1)\frac{(n-2)(3n^2-8n+16)}{8}=\frac{3n^4-14n^3+40n^2-72n+64}{16}$.

Case-3: Let $\frac{n}{2}$ be an even positive integer. Then $d(x^i)=\begin{cases}\frac{n}{2}-1, & if\ x^i=e\ or\ x^{\frac{n}{2}}\\ \frac{n}{2}-2, & otherwise\end{cases}$.

Now, using the Lemma 4.2.6, for the graph $\mathcal{G}_1$,

$d(u)d(v)d_{\mathcal{G}_1}(e)=\sum\limits_{x^i\in V(\mathcal{G}_1)} d(e)d(x^i)d(e,x^i)$
$=d(e)\{d(x^2)d(e,x^2)+d(x^4)d(e,x^4)+\cdots+d(x^{\frac{n}{2}})d(e,x^{\frac{n}{2}})+\cdots+d(x^{n-2})d(e,x^{n-2})\}$
$=(\frac{n}{2}-1)\{(\frac{n}{2}-2)(\frac{n}{2}-2)\times 1+(\frac{n}{2}-1)\times 1\}=\frac{(n-2)(n^2-6n+12)}{8}$.

Similarly, $d(u)d(v)d_{\mathcal{G}_1}(x^{\frac{n}{2}}) = \sum\limits_{x^i \in V(\mathcal{G}_1)} d(x^{\frac{n}{2}})d(x^i)d(x^{\frac{n}{2}}, x^i) = \dfrac{(n-2)(n^2-6n+12)}{8}$

$$d(u)d(v)d_{\mathcal{G}_1}(x^i) = \sum_{x^j \in V(\mathcal{G}_1)} d(x^i)d(x^j)d(x^i, x^j)$$
$$= d(x^i)d(e)d(x^i, e) + d(x^i)d(x^2)d(x^i, x^2) + d(x^i)d(x^4)d(x^i, x^4) + \cdots +$$
$$d(x^i)d(x^{n-i})d(x^i, x^{n-i}) + d(x^i)d(x^{n-2})d(x^i, x^{n-2})$$
$$= (\frac{n}{2}-2)(\frac{n}{2}-1) \times 1 \times 2 + (\frac{n}{2}-3)(\frac{n}{2}-2)^2 \times 1 + (\frac{n}{2}-2)^2 \times 2 = \frac{n(n^2-6n+8)}{8}.$$

So, $Gut(\mathcal{G}_1) = \frac{1}{2}d(u)d(v)d_{\mathcal{G}_1}(e) + \frac{1}{2}d(u)d(v)d_{\mathcal{G}_1}(x^{\frac{n}{2}}) + \frac{1}{2}\sum\limits_{\substack{x^i \in V(\mathcal{G}_1)\\ i \neq n,\ i \neq \frac{n}{2}}} d(u)d(v)d_{\mathcal{G}}(x^i)$

$$= \frac{1}{2} \times \frac{(n-2)(n^2-6n+12)}{8} + \frac{1}{2} \times \frac{(n-2)(n^2-6n+12)}{8} + \frac{1}{2} \times (\frac{n}{2}-2) \times \frac{n(n^2-6n+8)}{8}$$
$$= \frac{n^4-6n^3+64n-96}{32}.$$

For the graph $\mathcal{G}_2$,

$$d(u)d(v)d_{\mathcal{G}_2}(x^i) = \sum_{x^j \in V(\mathcal{G}_2)} d(x^i)d(x^j)d(x^i, x^j)$$
$$= d(x^i)d(x)d(x^i, x) + d(x^i)d(x^3)d(x^i, x^3) + \cdots + d(x^i)d(x^{n-i})d(x^i, x^{n-i})$$
$$+ d(x^i)d(x^{n-1})d(x^i, x^{n-1})$$
$$= (\frac{n}{2}-2)^2(\frac{n}{2}-2) \times 1 + (\frac{n}{2}-2) \times 2 = \frac{n(n-4)^2}{4}$$

So, $Gut(\mathcal{G}_2) = \frac{1}{2}\sum\limits_{x^i \in V(\mathcal{G}_2)} d(u)d(v)d_{\mathcal{G}_2}(x^i) = \frac{1}{2} \times \frac{n}{2} \times \frac{n(n-4)^2}{4} = \frac{n^2(n-4)^2}{16}$

Therefore, $Gut(\mathcal{G}) = Gut(\mathcal{G}_1) + Gut(\mathcal{G}_2)$

$$= \frac{n^4-6n^3+64n-96}{32} + \frac{n^2(n-4)^2}{16} = \frac{3n^4-22n^3+32n^2+64n-96}{32}.$$ □

4.3 Wiener, Hyper-Wiener, Harary and Gutman indices of order two element graph over D_n.

In this section, we mainly discuss about the structure of order two element graph over the group D_n and established a general formula for the distance-based topological indices (Wiener, Hyper-Wiener, Harary, and Gutman index) for the graph $\mathbb{S}_2(D_n)$.

Let $G = D_n = \{(x, y) : x^n = y^2 = e \ and \ yx = x^{n-1}y\} = \{e, x, x^2, \cdots, x^{n-1}, y, xy, x^2y, \cdots, x^{n-1}y\}$ and $\mathcal{G} = \mathbb{S}_2(D_n)$. Then one can verify that $(x^iy)(x^jy) = x^{nj-j+i} =$

x^{i-j} for all $i \neq j$ and $i, j = 1, 2, \cdots, n$. For better visualization of the graph $\mathbb{S}_2(D_n)$, readers can see the following examples.

Example 4.3.1. *Let* $G = D_4 = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$ *and* $\mathcal{G} = \mathbb{S}_2(D_4)$*. Then* $S_G = \{e, x^2\}$ *and*
$G_2 = \{x^2, y, xy, x^2y, x^3y\}$*. Now, from Figure 4.4, one can verify the following.*

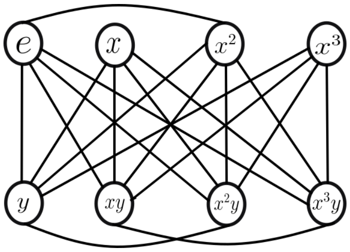

Figure 4.4: $\mathbb{S}_2(D_4)$

$W(\mathbb{S}_2(D_4)) = 37$, $WW(\mathbb{S}_2(D_4)) = 46$, $H(\mathbb{S}_2(D_4)) = \frac{97}{4}$, $Gut(\mathbb{S}_2(D_4)) = 843$.

Example 4.3.2. *Let* $G = D_5 = \{e, x, x^2, x^3, x^4, y, xy, x^2y, x^3y, x^4y\}$ *and* $\mathcal{G} = \mathbb{S}_2(D_4)$*. Then* $S_G = \{e, x, x^2, x^3, x^4\}$ *and* $G_2 = \{y, xy, x^2y, x^3y, x^4y\}$*. Now, from Figure 4.5, one can verify the following.*

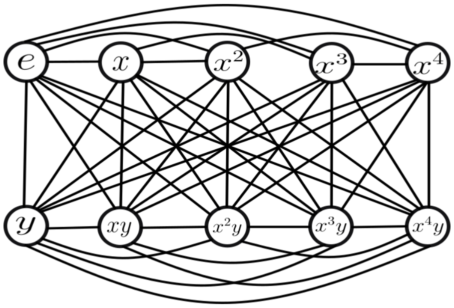

Figure 4.5: Order two element graph over D_5, $\mathbb{S}_2(D_5)$

$W(\mathbb{S}_2(D_5)) = 47$, $WW(\mathbb{S}_2(D_5)) = 49$, $H(\mathbb{S}_2(D_5)) = \frac{131}{4}$, $Gut(\mathbb{S}_2(D_5)) = 4979$.

Example 4.3.3. *Let* $G = D_6 = \{e, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y\}$ *and* $\mathcal{G} = \mathbb{S}_2(D_4)$*. Then* $S_G = \{e, x^2, x^4\}$ *and* $G_2 = \{x^3, y, xy, x^2y, x^3y, x^4y, x^5y\}$*. Now, from Figure 4.6, one can verify the following.*

$W(\mathbb{S}_2(D_6)) = 80$, $WW(\mathbb{S}_2(D_6)) = 96$, $H(\mathbb{S}_2(D_6)) = 59$, $Gut(\mathbb{S}_2(D_6)) = 5986$.

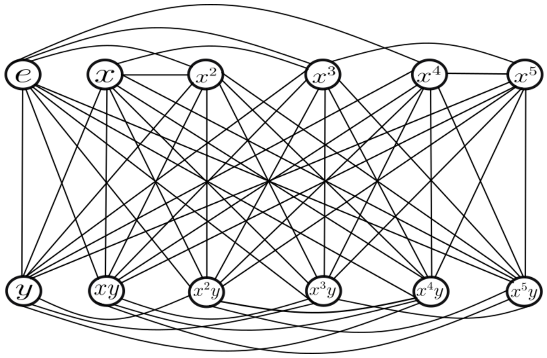

Figure 4.6: Order two element graph over D_6, $\mathbb{S}_2(D_6)$

Theorem 4.3.4. *The Wiener index of the graph $\mathbb{S}_2(D_n)$ is*

$$W(\mathbb{S}_2(D_n)) = \begin{cases} \dfrac{4n^2-n-1}{2}, \ if \ n \ is \ odd \\ \dfrac{5n^2-3n-2}{2}, \ if \ \dfrac{n}{2} \ is \ odd \\ \dfrac{5n^2-n-2}{2} \ if \ \dfrac{n}{2} \ is \ even \end{cases}$$

.

Proof. Let $G = D_n = \{e, x, x^2, \cdots, x^{n-1}, y, xy, x^2y, \cdots, x^{n-1}y\}$ and $\mathcal{G} = \mathbb{S}_2(D_n)$.

Case-1: Let n be an odd positive integer. Then $S_G = \{e, x, x^2, \cdots, x^{n-1}\}$ and $G_2 = \{y, xy, x^2y, \cdots, x^{n-1}y\}$.

Now, the degree of vertices of the graph $\mathcal{G}$ are

$$d(e) = 2n-1,$$
$$d(x^i) = 2n-2 \ \forall \ i = 1, 2, \cdots, n-1,$$
$$d(x^iy) = 2n-1 \ \forall \ i = 1, 2, \cdots, n.$$

Also, the distance between any two pair of vertices is

$$d(e, x^i) = 1 \ \forall \ i = 1, 2, \cdots, n-1,$$
$$d(e, x^iy) = 1 \ \forall \ i = 1, 2, \cdots, n,$$
$$d(x^i, x^j) = \begin{cases} 1, \ if \ i \neq j, \ j \neq n-i \\ 2, \ if \ j = n-i \end{cases}$$
$$d(x^iy, x^j) = 1 \ \forall \ i, j = 1, 2, \cdots, n,$$
$$d(x^iy, x^jy) = 1, \ if \ i \neq j \ and \ i, j = 1, 2, \cdots, n.$$

Now, $d_{\mathcal{G}}(e) = \sum\limits_{u \in V(\mathcal{G})} d(e,u) = 2n-1.$

Similarly, for each $i = 1, 2, \cdots, n$, $d_{\mathcal{G}}(x^i y) = \sum\limits_{u \in V(\mathcal{G})} d(x^i y, u) = 2n-1.$

Also, for each $i = 1, 2, \cdots, n-1$,

$$d_{\mathcal{G}}(x^i) = \sum_{u \in V(\mathcal{G})} d(x^i,u) = \sum_{\substack{x^j \in V(\mathcal{G}) \\ j \neq n-i}} d(x^i,x^j) + d(x^i,x^{n-i}) + \sum_{x^j y \in V(\mathcal{G})} d(x^i, x^j y)$$

$= (n-2) \times 1 + 2 + n \times 1 = 2n.$

Therefore, $W(\mathcal{G}) = \frac{1}{2}\sum\limits_{u \in V(\mathcal{G})} d_{\mathcal{G}}(u) = \frac{1}{2}[d_{\mathcal{G}}(e) + \sum\limits_{\substack{x^i \in V(\mathcal{G}) \\ i \neq n}} d_{\mathcal{G}}(x^i) + \sum\limits_{x^i y \in V(\mathcal{G})} d_{\mathcal{G}}(x^i y)]$

$= \frac{1}{2}[(2n-1) + (n-1) \times 2n + n(2n-1)] = \frac{4n^2-n-1}{2}.$

Case-2: Let $\frac{n}{2}$ is an odd positive integer. Then $S_G = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}}, y, xy, x^2y, \cdots, x^{n-1}y\}$. Then $x^{\frac{n}{2}} \notin S_G$.

Now, the degree of vertices of the graph $\mathcal{G}$ are

$$\begin{aligned} d(e) &= \frac{3n}{2}, \\ d(x^{\frac{n}{2}}) &= \frac{3n}{2}, \\ d(x^i) &= \frac{3n-2}{2} \ \forall\ i = 1, 2, \cdots, n-1 \text{ except } \tfrac{n}{2}, \\ d(x^i y) &= \frac{3n}{2} \ \forall\ i = 1, 2, \cdots, n. \end{aligned}$$

Also, the distance between any two pair of vertices is

$$d(e, x^i) = \begin{cases} 1, \ if\ i = even\ and\ i = \dfrac{n}{2} \\ 2, \ if\ i = odd\ and\ i \neq \dfrac{n}{2} \end{cases}$$

$$d(e, x^i y) = 1 \ \forall\ i = 1, 2, \cdots, n,$$

$$d(x^i, x^j) = \begin{cases} 1, \ if\ i+j = even\ and\ i+j = \dfrac{n}{2}\ but\ i+j \neq n \\ 2, \ if\ i+j = odd\ but\ i+j \neq \dfrac{n}{2}\ and\ i+j = n \end{cases}$$

$$d(x^i y, x^j) = 1 \ \forall\ i, j = 1, 2, \cdots, n$$

$$d(x^i y, x^j y) = \begin{cases} 1, \ if\ i-j = even\ and\ i-j = \dfrac{n}{2} \\ 2, \ if\ i-j = odd\ but\ i-j \neq \dfrac{n}{2} \end{cases}.$$

Now, $d_{\mathcal{G}}(e) = \sum\limits_{u \in V(\mathcal{G})} d(e,u) = \sum\limits_{\substack{x^i \in V(\mathcal{G}) \\ i = odd,\ i \neq \frac{n}{2}}} d(e, x^i) + d(e, x^{\frac{n}{2}}) + \sum\limits_{\substack{x^i \in V(\mathcal{G}) \\ i = even}} d(e, x^i) +$

$$\sum_{x^iy\in V(\mathcal{G})} d(e,x^iy) = (\tfrac{n}{2}-1)\times 2+1+(\tfrac{n}{2}-1)\times 1+n\times 1 = \tfrac{5n-4}{2}.$$

$$d_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum_{u\in V(\mathcal{G})} d(x^{\frac{n}{2}},u) = \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d(x^{\frac{n}{2}},x^i) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq n}} d(x^{\frac{n}{2}},x^i) + d(x^{\frac{n}{2}},e)+$$
$$\sum_{x^iy\in V(\mathcal{G})} d(x^{\frac{n}{2}},x^iy) = (\frac{n}{2}-1)\times 1+(\frac{n}{2}-1)\times 2+1+n\times 1 = \frac{5n-4}{2}.$$

For each odd i except $\dfrac{n}{2}$,

$$d_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} d(x^i,u)$$
$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq n-i}} d(x^i,x^j) + d(x^i,x^{n-i}) + \sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq \frac{n}{2}-i}} d(x^i,x^j) + d(x^i,x^{\frac{n}{2}-i})$$
$$+\sum_{x^jy\in V(\mathcal{G})} d(x^i,x^jy) = (\frac{n}{2}-2)\times 1+2+(\frac{n}{2}-1)\times 2+1+n\times 1 = \frac{5n-2}{2}.$$

For each even i except n,

$$d_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} d(x^i,u)$$
$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq \frac{n}{2}-i}} d(x^i,x^j) + d(x^i,x^{\frac{n}{2}-i}) + \sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq n-i}} d(x^i,x^j) + d(x^i,x^{n-i})$$
$$+\sum_{x^jy\in V(\mathcal{G})} d(x^i,x^jy) = (\frac{n}{2}-1)\times 2+1+(\frac{n}{2}-2)\times 1+2+n\times 1 = \frac{5n-2}{2}.$$

For each i, $d_{\mathcal{G}}(x^iy) = \displaystyle\sum_{u\in V(\mathcal{G})} d(x^iy,u) = \sum_{x^j\in V(\mathcal{G})} d(x^iy,x^j) + \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=even,\frac{n}{2}}} d(x^iy,x^jy)$

$$+\sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=odd,\ i-j\neq\frac{n}{2}}} d(x^iy,x^jy) = n\times 1+\frac{n}{2}\times 1+(\frac{n}{2}-1)\times 2 = \frac{5n-4}{2}.$$

Therefore, $W(\mathcal{G}) = \dfrac{1}{2}\displaystyle\sum_{u\in V(\mathcal{G})} d_{\mathcal{G}}(u)$

$$= \frac{1}{2}[d_{\mathcal{G}}(e)+d_{\mathcal{G}}(x^{\frac{n}{2}}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd,\ i\neq\frac{n}{2}}} d_{\mathcal{G}}(x^i) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq n}} d_{\mathcal{G}}(x^i) + \sum_{x^iy\in V(\mathcal{G})} d_{\mathcal{G}}(x^iy)]$$
$$= \frac{1}{2}[\frac{5n-4}{2}+\frac{5n-4}{2}+(\frac{n}{2}-1)\times\frac{5n-2}{2}+(\frac{n}{2}-1)\times\frac{5n-2}{2}+n\times\frac{5n-4}{2}] = \frac{5n^2-3n-2}{2}.$$

Case-3: Let $\dfrac{n}{2}$ is an even positive integer. Then $S_G = \{e,x^2,x^4,\cdots,x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}},y,xy,x^2y,\cdots,x^{n-1}y\}$. Then $x^{\frac{n}{2}}\in S_G$.

Now, the degree of vertices of the graph $\mathcal{G}$ are

$$\begin{aligned}
&d(e) = \frac{3n-2}{2},\\
&d(x^{\frac{n}{2}}) = \frac{3n-2}{2},\\
&d(x^i) = \frac{3n-4}{2}\ \forall\ i = 1, 2, \cdots, n-1 \text{ except } \tfrac{n}{2}\\
&d(x^iy) = \frac{3n-2}{2}\ \forall\ i = 1, 2, \cdots, n.
\end{aligned}$$

Also, the distance between any two pair of vertices is

$$\begin{aligned}
&d(e, x^i) = \begin{cases} 1,\ if\ i = even \\ 2,\ if\ i = odd \end{cases}\\
&d(e, x^iy) = 1\ \forall\ i = 1, 2, \cdots, n\\
&d(x^i, x^j) = \begin{cases} 1,\ if\ i + j = even\ but\ i + j \neq n \\ 2,\ if\ i + j = odd\ and\ i + j = n \end{cases}\\
&d(x^iy, x^j) = 1\ \forall\ i, j = 1, 2, \cdots, n\\
&d(x^iy, x^jy) = \begin{cases} 1,\ if\ i - j = even \\ 2,\ if\ i - j = odd \end{cases}.
\end{aligned}$$

Now, $d_{\mathcal{G}}(e) = \sum\limits_{u \in V(\mathcal{G})} d(e, u) = \sum\limits_{\substack{x^i \in V(\mathcal{G}) \\ i = odd}} d(e, x^i) + \sum\limits_{\substack{x^i \in V(\mathcal{G}) \\ i = even}} d(e, x^i) + \sum\limits_{x^iy \in V(\mathcal{G})} d(e, x^iy)$

$= \frac{n}{2} \times 2 + (\frac{n}{2} - 1) \times 1 + n \times 1 = \frac{5n-2}{2}.$

$d_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum\limits_{u \in V(\mathcal{G})} d(x^{\frac{n}{2}}, u) = \sum\limits_{\substack{x^i \in V(\mathcal{G}) \\ i = odd}} d(x^{\frac{n}{2}}, x^i) + \sum\limits_{\substack{x^i \in V(\mathcal{G}) \\ i = even}} d(x^{\frac{n}{2}}, x^i) + \sum\limits_{x^iy \in V(\mathcal{G})} d(x^{\frac{n}{2}}, x^iy)$

$= \frac{n}{2} \times 2 + (\frac{n}{2} - 1) \times 1 + n \times 1 = \frac{5n-2}{2}.$

For each odd i,

$$d_{\mathcal{G}}(x^i) = \sum_{u \in V(\mathcal{G})} d(x^i, u)$$

$$= \sum_{\substack{x^j \in V(\mathcal{G}) \\ j = odd,\ j \neq n-i}} d(x^i, x^j) + d(x^i, x^{n-i}) + \sum_{\substack{x^j \in V(\mathcal{G}) \\ j = even}} d(x^i, x^j) + \sum_{x^jy \in V(\mathcal{G})} d(x^i, x^jy)$$

$= (\frac{n}{2} - 2) \times 1 + 2 + \frac{n}{2} \times 2 + n \times 1 = \frac{5n}{2}.$

For each even i except $\frac{n}{2}, n$,

$$d_{\mathcal{G}}(x^i)=\sum_{u\in V(\mathcal{G})} d(x^i,u)$$
$$=\sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd}} d(x^i,x^j)+\sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq n-i}} d(x^i,x^j)+d(x^i,x^{n-i})+\sum_{x^jy\in V(\mathcal{G})} d(x^i,x^jy)$$
$$=\frac{n}{2}\times 2+(\frac{n}{2}-2)\times 1+2+n\times 1=\frac{5n}{2}.$$

For each i, $d_{\mathcal{G}}(x^iy)=\sum_{u\in V(\mathcal{G})} d(x^iy,u)=\sum_{x^j\in V(\mathcal{G})} d(x^iy,x^j)+\sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=even}} d(x^iy,x^jy)+$

$$\sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=odd}} d(x^iy,x^jy)=n\times 1+(\frac{n}{2}-1)\times 1+\frac{n}{2}\times 2=\frac{5n-2}{2}.$$

Therefore, $W(\mathcal{G})=\frac{1}{2}\sum_{u\in V(\mathcal{G})} d_{\mathcal{G}}(u)$

$$=\frac{1}{2}[d_{\mathcal{G}}(e)+d_{\mathcal{G}}(x^{\frac{n}{2}})+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d_{\mathcal{G}}(x^i)+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq\frac{n}{2},n}} d_{\mathcal{G}}(x^i)+\sum_{x^iy\in V(\mathcal{G})} d_{\mathcal{G}}(x^iy)]$$
$$=\frac{1}{2}[\frac{5n-2}{2}+\frac{5n-2}{2}+\frac{n}{2}\times\frac{5n}{2}+(\frac{n}{2}-2)\times\frac{5n}{2}+n\times\frac{5n-2}{2}]=\frac{5n^2-n-2}{2}. \qquad \square$$

Theorem 4.3.5. *The Hyper-Wiener index of the graph $\mathbb{S}_2(D_n)$ is*

$$WW(\mathbb{S}_2(D_n))=\begin{cases} 2n^2-1, & if\ n\ is\ odd\\ 3n^2-2n-2, & if\ \frac{n}{2}\ is\ odd\\ 3n^2-2, & if\ \frac{n}{2}\ is\ even\end{cases}$$

Proof. Let $G=D_n=\{e,x,x^2,\cdots,x^{n-1},y,xy,x^2y,\cdots,x^{n-1}y\}$ and $\mathcal{G}=\mathbb{S}_2(D_n)$.

Case-1: Let n be an odd positive integer. Then $S_G=\{e,x,x^2,\cdots,x^{n-1}\}$ and $G_2=\{y,xy,x^2y,\cdots,x^{n-1}y\}$. Now, using the degree and distance property from Theorem 4.3.4, Case-1, we have

$$d^2_{\mathcal{G}}(e)=\sum_{u\in V(\mathcal{G})} d^2(e,u)=(2n-1)\times 1^2=2n-1.$$

Similarly, for each $i=1,2,\cdots,n$, $d^2_{\mathcal{G}}(x^iy)=\sum_{u\in V(\mathcal{G})} d^2(x^iy,u)=(2n-1)\times 1^2=2n-1.$

Also, for each $i=1,2,\cdots,n-1$,

$$d^2_{\mathcal{G}}(x^i)=\sum_{u\in V(\mathcal{G})} d^2(x^i,u)=\sum_{\substack{x^j\in V(\mathcal{G})\\ j\neq n-i}} d^2_{\mathcal{G}}(x^i,x^j)+d^2(x^i,x^{n-i})+\sum_{x^jy\in V(\mathcal{G})} d^2(x^i,x^jy)$$
$$=(n-2)\times 1^2+2^2+n\times 1^2=2n+2.$$

Therefore, $\frac{1}{4}\sum\limits_{u\in V(\mathcal{G})} d^2_{\mathcal{G}}(u) = \frac{1}{4}[d^2_{\mathcal{G}}(e) + \sum\limits_{x^i\in V(\mathcal{G})} d^2_{\mathcal{G}}(x^i) + \sum\limits_{x^iy\in V(\mathcal{G})} d^2_{\mathcal{G}}(x^iy)]$
$= \frac{1}{4}[(2n-1)+(n-1)(2n+2)+n(2n-1)] = \frac{4n^2+n-3}{4}$.

Moreover, $WW(\mathcal{G}) = \frac{1}{2}W(\mathcal{G}) + \frac{1}{4}\sum\limits_{u\in V(\mathcal{G})} d^2_{\mathcal{G}}(u) = \frac{4n^2-n-1}{4} + \frac{4n^2+n-3}{4}$
$= 2n^2-1$.

Case-2: Let $\frac{n}{2}$ is an odd positive integer. Then $S_G = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}}, y, xy, x^2y, \cdots, x^{n-1}y\}$. Then $x^{\frac{n}{2}} \notin S_G$. Now, using the degree and distance property from Theorem 4.3.4, Case-2, we have

$$d^2_{\mathcal{G}}(e) = \sum_{u\in V(\mathcal{G})} d^2(e,u) = \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd,\ i\neq\frac{n}{2}}} d^2(e,x^i) + d^2(e,x^{\frac{n}{2}}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even}} d^2(e,x^i)$$
$$+ \sum_{x^iy\in V(\mathcal{G})} d^2(e,x^iy) = (\frac{n}{2}-1)\times 2^2 + 1^2 + (\frac{n}{2}-1)\times 1^2 + n\times 1^2 = \frac{7n-8}{2}.$$

$$d^2_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum_{u\in V(\mathcal{G})} d^2(x^{\frac{n}{2}},u) = \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d^2(x^{\frac{n}{2}},x^i) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq n}} d^2(x^{\frac{n}{2}},x^i) + d^2(x^{\frac{n}{2}},e) +$$
$$\sum_{x^iy\in V(\mathcal{G})} d^2(x^{\frac{n}{2}},x^iy) = (\frac{n}{2}-1)\times 1^2 + (\frac{n}{2}-1)\times 2^2 + 1^2 + n\times 1^2 = \frac{7n-8}{2}.$$

For each odd i except $\frac{n}{2}$,

$$d^2_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} d^2(x^i,u)$$
$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq n-i}} d^2(x^i,x^j) + d^2(x^i,x^{n-i}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ j=even,\ j\neq\frac{n}{2}-i}} d^2(x^i,x^j) + d^2(x^i,x^{\frac{n}{2}-i})$$
$$+ \sum_{x^jy\in V(\mathcal{G})} d^2(x^i,x^jy) = (\frac{n}{2}-2)\times 1^2 + 2^2 + (\frac{n}{2}-1)\times 2^2 + 1^2 + n\times 1^2 = \frac{7n-2}{2}.$$

For each even i except n,

$$d^2_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} d^2(x^i,u)$$
$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq\frac{n}{2}-i}} d^2(x^i,x^j) + d^2(x^i,x^{\frac{n}{2}-i}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ j=even,\ j\neq n-i}} d^2(x^i,x^j) + d^2(x^i,x^{n-i})$$
$$+ \sum_{x^jy\in V(\mathcal{G})} d^2(x^i,x^jy) = (\frac{n}{2}-1)\times 2^2 + 1^2 + (\frac{n}{2}-2)\times 1^2 + 2^2 + n\times 1^2 = \frac{7n-2}{2}.$$

For each i,

$$d^2_{\mathcal{G}}(x^iy) = \sum_{u\in V(\mathcal{G})} d^2(x^iy,u) = \sum_{x^j\in V(\mathcal{G})} d^2(x^iy,x^j) + \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=even,\frac{n}{2}}} d^2(x^iy,x^jy)$$

$$+ \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=odd,\ i-j\neq\frac{n}{2}}} d^2(x^iy,x^jy) = n\times 1^2 + \frac{n}{2}\times 1^2 + (\frac{n}{2}-1)\times 2^2 = \frac{7n-8}{2}.$$

Therefore, $\frac{1}{4}\sum_{u\in V(\mathcal{G})} d^2_{\mathcal{G}}(u) = \frac{1}{4}[d^2_{\mathcal{G}}(e) + d^2_{\mathcal{G}}(x^{\frac{n}{2}}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd,i\neq\frac{n}{2}}} d^2_{\mathcal{G}}(x^i) \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,i\neq n}} d^2_{\mathcal{G}}(x^i) +$ $\sum_{x^iy\in V(\mathcal{G})} d^2_{\mathcal{G}}(x^iy)] = \frac{1}{4}[\frac{7n-8}{2} + \frac{7n-8}{2} + (\frac{n}{2}-1)\frac{7n-2}{2} + (\frac{n}{2}-1)\frac{7n-2}{2} + n\times\frac{7n-8}{2}] =$ $\frac{7n^2-5n-6}{4}$.

Moreover, $WW(\mathcal{G}) = \frac{1}{2}W(\mathcal{G}) + \frac{1}{4}\sum_{u\in V(\mathcal{G})} d^2_{\mathcal{G}}(u) = \frac{5n^2-3n-2}{4} + \frac{7n^2-5n-6}{4}$ $= 3n^2-2n-2$.

Case-3: Let $\frac{n}{2}$ is an even positive integer. Then $S_G = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}}, y, xy, x^2y, \cdots, x^{n-1}y\}$. Then $x^{\frac{n}{2}} \in S_G$. Now, using the degree and distance property from Theorem 4.3.4, Case-3, we have

$$d^2_{\mathcal{G}}(e) = \sum_{u\in V(\mathcal{G})} d^2(e,u) = \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d^2(e,x^i) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even}} d^2(e,x^i) + \sum_{x^iy\in V(\mathcal{G})} d^2(e,x^iy)$$

$$= \frac{n}{2}\times 2^2 + (\frac{n}{2}-1)\times 1^2 + n\times 1^2 = \frac{7n-2}{2}.$$

$$d^2_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum_{u\in V(\mathcal{G})} d^2(x^{\frac{n}{2}},u) = \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d^2(x^{\frac{n}{2}},x^i) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even}} d^2(x^{\frac{n}{2}},x^i) + \sum_{x^iy\in V(\mathcal{G})} d^2(x^{\frac{n}{2}},x^iy)$$

$$= \frac{n}{2}\times 2^2 + (\frac{n}{2}-1)\times 1^2 + n\times 1^2 = \frac{7n-2}{2}.$$

For each odd i, $d^2_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} d^2(x^i,u)$

$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq n-i}} d^2(x^i,x^j) + d^2(x^i,x^{n-i}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ j=even}} d^2(x^i,x^j) + \sum_{x^jy\in V(\mathcal{G})} d^2(x^i,x^jy)$$

$$= (\frac{n}{2}-2)\times 1^2 + 2^2 + \frac{n}{2}\times 2^2 + n\times 1^2 = \frac{7n+4}{2}.$$

For each even i except $\frac{n}{2}, n$, $d^2_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} d^2(x^i,u)$

$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd}} d^2(x^i,x^j) + \sum_{\substack{x^i\in V(\mathcal{G})\\ j=even,\ j\neq n-i}} d^2(x^i,x^j) + d^2(x^i,x^{n-i}) + \sum_{x^jy\in V(\mathcal{G})} d^2(x^i,x^jy)$$

$$= \frac{n}{2}\times 2^2 + (\frac{n}{2}-2)\times 1^2 + 2^2 + n\times 1^2 = \frac{7n+4}{2}.$$

For each i, $d^2_{\mathcal{G}}(x^iy) = \sum_{u\in V(\mathcal{G})} d^2(x^iy,u) = \sum_{x^j\in V(\mathcal{G})} d^2(x^iy,x^j) + \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=even}} d^2(x^iy,x^jy) +$

$\sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=odd}} d^2(x^iy,x^jy) = n\times 1^2 + (\frac{n}{2}-1)\times 1^2 + \frac{n}{2}\times 2^2 = \frac{7n-2}{2}.$

Therefore, $\frac{1}{4}\sum_{u\in V(\mathcal{G})} d^2_{\mathcal{G}}(u) = \frac{1}{4}[d^2_{\mathcal{G}}(e) + d^2_{\mathcal{G}}(x^{\frac{n}{2}}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d^2_{\mathcal{G}}(x^i) \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even, i\neq\frac{n}{2},n}} d^2_{\mathcal{G}}(x^i) +$

$\sum_{x^iy\in V(\mathcal{G})} d^2_{\mathcal{G}}(x^iy)] = \frac{1}{4}[\frac{7n-2}{2} + \frac{7n-2}{2} + \frac{n}{2}\times\frac{7n+4}{2} + (\frac{n}{2}-2)\frac{7n+4}{2} + n\times\frac{7n-2}{2}]$

$= \frac{7n^2+n-6}{4}.$

Moreover, $WW(\mathcal{G}) = \frac{1}{2}W(\mathcal{G}) + \frac{1}{4}\sum_{u\in V(\mathcal{G})} d^2_{\mathcal{G}}(u) = \frac{5n^2-n-2}{4} + \frac{7n^2+n-6}{4}$

$= 3n^2-2.$ □

Theorem 4.3.6. *The Harary index of the graph $\mathbb{S}_2(D_n)$ is*

$$H(\mathbb{S}_2(D_n)) = \begin{cases} \frac{6n^2-4n+1}{4}, & if\ n\ is\ odd \\ \frac{7n^2-3n+2}{4}, & if\ \frac{n}{2}\ is\ odd \\ \frac{14n^2-7n-2}{8}, & if\ \frac{n}{2}\ is\ even \end{cases}$$

Proof. Let $G = D_n = \{e, x, x^2, \cdots, x^{n-1}, y, xy, x^2y, \cdots, x^{n-1}y\}$ and $\mathcal{G} = \mathbb{S}_2(D_n)$.

Case-1: Let n be an odd positive integer. Then $S_G = \{e, x, x^2, \cdots, x^{n-1}\}$ and $G_2 = \{y, xy, x^2y, \cdots, x^{n-1}y\}$.

Now, using the degree and distance property from Theorem 4.3.4, Case-1, we have

Now, $d_{\mathcal{G}}(e) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(e,u)} = 2n-1.$

Similarly, for each $i = 1, 2, \cdots, n$, $d_{\mathcal{G}}(x^iy) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(x^iy,u)} = 2n-1.$

Also, for each $i = 1, 2, \cdots, n-1$,

$$d_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(x^i,u)} = \sum_{\substack{x^j\in V(\mathcal{G})\\ j\neq n-i}} \frac{1}{d(x^i,x^j)} + \frac{1}{d(x^i,x^{n-i})} + \sum_{x^jy\in V(\mathcal{G})} \frac{1}{d(x^i,x^jy)}$$
$$= (n-2)\times\frac{1}{1} + \frac{1}{2} + n\times\frac{1}{1} = \frac{4n-3}{2}.$$

Therefore, $W(\mathcal{G}) = \frac{1}{2}\sum_{u\in V(\mathcal{G})} d_{\mathcal{G}}(u) = \frac{1}{2}[d_{\mathcal{G}}(e) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i\neq n}} d_{\mathcal{G}}(x^i) + \sum_{x^iy\in V(\mathcal{G})} d_{\mathcal{G}}(x^iy)]$
$$= \frac{1}{2}[(2n-1) + (n-1)\times\frac{4n-3}{2} + n(2n-1)] = \frac{6n^2-4n+1}{4}.$$

Case-2: Let $\frac{n}{2}$ is an odd positive integer. Then $S_G = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}}, y, xy, x^2y, \cdots, x^{n-1}y\}$. Then $x^{\frac{n}{2}} \notin S_G$.

Now, using the degree and distance property from Theorem 4.3.4, Case-2, we have

Now, $d_{\mathcal{G}}(e) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(e,u)} = \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd,\ i\neq\frac{n}{2}}} \frac{1}{d(e,x^i)} + \frac{1}{d(e,x^{\frac{n}{2}})} + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even}} \frac{1}{d(e,x^i)} +$
$$\sum_{x^iy\in V(\mathcal{G})} \frac{1}{d(e,x^iy)} = (\frac{n}{2}-1)\times\frac{1}{2} + \frac{1}{1} + (\frac{n}{2}-1)\times\frac{1}{1} + n\times\frac{1}{1} = \frac{7n-2}{4}.$$

$d_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(x^{\frac{n}{2}},u)} = \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} \frac{1}{d(x^{\frac{n}{2}},x^i)} + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq n}} \frac{1}{d(x^{\frac{n}{2}},x^i)} + \frac{1}{d(x^{\frac{n}{2}},e)} +$
$$\sum_{x^iy\in V(\mathcal{G})} \frac{1}{d(x^{\frac{n}{2}},x^iy)} = (\frac{n}{2}-1)\times\frac{1}{1} + (\frac{n}{2}-1)\times\frac{1}{2} + \frac{1}{1} + n\times\frac{1}{1} = \frac{7n-2}{4}.$$

For each odd i except $\frac{n}{2}$,

$$d_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(x^i,u)}$$
$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq n-i}} \frac{1}{d(x^i,x^j)} + \frac{1}{d(x^i,x^{n-i})} + \sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq\frac{n}{2}-i}} \frac{1}{d(x^i,x^j)} + \frac{1}{d(x^i,x^{\frac{n}{2}-i})} +$$
$$\sum_{x^jy\in V(\mathcal{G})} \frac{1}{d(x^i,x^jy)} = (\frac{n}{2}-2)\times\frac{1}{1} + \frac{1}{2} + (\frac{n}{2}-1)\times\frac{1}{2} + \frac{1}{1} + n\times\frac{1}{1} = \frac{7n-4}{4}.$$

For each even i except n,

$$d_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(x^i,u)}$$
$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq\frac{n}{2}-i}} \frac{1}{d(x^i,x^j)} + \frac{1}{d(x^i,x^{\frac{n}{2}-i})} + \sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq n-i}} \frac{1}{d(x^i,x^j)} +$$

$$\frac{1}{d(x^i,x^{n-i})}+\sum_{x^jy\in V(\mathcal{G})}\frac{1}{d(x^i,x^jy)}=(\frac{n}{2}-1)\times\frac{1}{2}+\frac{1}{1}+(\frac{n}{2}-2)\times\frac{1}{1}+\frac{1}{2}+n\times\frac{1}{1}=\frac{7n-4}{4}.$$

For each i, $d_{\mathcal{G}}(x^iy)=\sum_{u\in V(\mathcal{G})}\frac{1}{d(x^iy,u)}=\sum_{x^j\in V(\mathcal{G})}\frac{1}{d(x^iy,x^j)}+\sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=even,\frac{n}{2}}}\frac{1}{d(x^iy,x^jy)}+$

$$\sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=odd,\ i-j\neq\frac{n}{2}}}\frac{1}{d(x^iy,x^jy)}=n\times\frac{1}{1}+\frac{n}{2}\times\frac{1}{1}+(\frac{n}{2}-1)\times\frac{1}{2}=\frac{7n-2}{4}.$$

Therefore, $W(\mathcal{G})=\frac{1}{2}\sum_{u\in V(\mathcal{G})}d_{\mathcal{G}}(u)$

$$=\frac{1}{2}[d_{\mathcal{G}}(e)+d_{\mathcal{G}}(x^{\frac{n}{2}})+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd,\ i\neq\frac{n}{2}}}d_{\mathcal{G}}(x^i)+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq n}}d_{\mathcal{G}}(x^i)+\sum_{x^iy\in V(\mathcal{G})}d_{\mathcal{G}}(x^iy)]$$

$$=\frac{1}{2}[\frac{7n-2}{4}+\frac{7n-2}{4}+(\frac{n}{2}-1)\times\frac{7n-4}{4}+(\frac{n}{2}-1)\times\frac{7n-4}{4}+n\times\frac{7n-2}{4}]=\frac{7n^2-3n+2}{4}.$$

Case-3: Let $\frac{n}{2}$ is an even positive integer. Then $S_G=\{e,x^2,x^4,\cdots,x^{n-2}\}$ and $G_2=\{x^{\frac{n}{2}},y,xy,x^2y,\cdots,x^{n-1}y\}$. Then $x^{\frac{n}{2}}\in S_G$.

Now, using the degree and distance property from Theorem 4.3.4, Case-3, we have

Now, $d_{\mathcal{G}}(e)=\sum_{u\in V(\mathcal{G})}\frac{1}{d(e,u)}=\sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}}\frac{1}{d(e,x^i)}+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=even}}\frac{1}{d(e,x^i)}+\sum_{x^iy\in V(\mathcal{G})}\frac{1}{d(e,x^iy)}$

$$=\frac{n}{2}\times\frac{1}{2}+(\frac{n}{2}-1)\times\frac{1}{1}+n\times\frac{1}{1}=\frac{7n-4}{4}.$$

$d_{\mathcal{G}}(x^{\frac{n}{2}})=\sum_{u\in V(\mathcal{G})}\frac{1}{d(x^{\frac{n}{2}},u)}=\sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}}\frac{1}{d(x^{\frac{n}{2}},x^i)}+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=even}}\frac{1}{d(x^{\frac{n}{2}},x^i)}+\sum_{x^iy\in V(\mathcal{G})}\frac{1}{d(x^{\frac{n}{2}},x^iy)}$

$$=\frac{n}{2}\times\frac{1}{2}+(\frac{n}{2}-1)\times\frac{1}{1}+n\times\frac{1}{1}=\frac{7n-4}{4}.$$

For each odd i,

$d_{\mathcal{G}}(x^i)=\sum_{u\in V(\mathcal{G})}\frac{1}{d(x^i,u)}$

$$=\sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq n-i}}\frac{1}{d(x^i,x^j)}+\frac{1}{d(x^i,x^{n-i})}+\sum_{\substack{x^j\in V(\mathcal{G})\\ j=even}}\frac{1}{d(x^i,x^j)}+\sum_{x^jy\in V(\mathcal{G})}\frac{1}{d(x^i,x^jy)}$$

$$=(\frac{n}{2}-2)\times\frac{1}{1}+\frac{1}{2}+\frac{n}{2}\times\frac{1}{2}+n\times\frac{1}{1}=\frac{7n-3}{4}.$$

For each even i except $\frac{n}{2},n$,

$$d_{\mathcal{G}}(x^i) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(x^i,u)}$$
$$= \sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd}} \frac{1}{d(x^i,x^j)} + \sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq n-i}} \frac{1}{d(x^i,x^j)} + \frac{1}{d(x^i,x^{n-i})} + \sum_{x^jy\in V(\mathcal{G})} \frac{1}{d(x^i,x^jy)}$$
$$= \frac{n}{2}\times\frac{1}{2} + (\frac{n}{2}-2)\times\frac{1}{1} + \frac{1}{2} + n\times\frac{1}{1} = \frac{7n-3}{4}.$$

For each i, $d_{\mathcal{G}}(x^iy) = \sum_{u\in V(\mathcal{G})} \frac{1}{d(x^iy,u)} = \sum_{x^j\in V(\mathcal{G})} \frac{1}{d(x^iy,x^j)} + \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=even}} \frac{1}{d(x^iy,x^jy)} +$
$\sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=odd}} \frac{1}{d(x^iy,x^jy)} = n\times\frac{1}{1} + (\frac{n}{2}-1)\times\frac{1}{1} + \frac{n}{2}\times\frac{1}{2} = \frac{7n-4}{4}.$

Therefore, $W(\mathcal{G}) = \frac{1}{2}\sum_{u\in V(\mathcal{G})} d_{\mathcal{G}}(u)$
$$= \frac{1}{2}[d_{\mathcal{G}}(e) + d_{\mathcal{G}}(x^{\frac{n}{2}}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d_{\mathcal{G}}(x^i) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq\frac{n}{2},n}} d_{\mathcal{G}}(x^i) + \sum_{x^iy\in V(\mathcal{G})} d_{\mathcal{G}}(x^iy)]$$
$$= \frac{1}{2}[\frac{7n-4}{4} + \frac{7n-4}{4} + \frac{n}{2}\times\frac{7n-3}{4} + (\frac{n}{2}-2)\times\frac{7n-3}{4} + n\times\frac{7n-4}{4}] = \frac{14n^2-7n-2}{8}.$$
□

Theorem 4.3.7. *The Gutman index of the graph $\mathbb{S}_2(D_n)$ is*

$$Gut(\mathbb{S}_2(D_n)) = \begin{cases} \frac{20n^4-42n^3+27n^2-3n-2}{2}, & if\ n\ is\ odd \\ \frac{45n^4-57n^3+60n^2-52n+32}{8}, & if\ \frac{n}{2}\ is\ odd \\ \frac{45n^4-99n^3+100n^2-12n+8}{8}, & if\ \frac{n}{2}\ is\ even \end{cases}$$

Proof. Let $G = D_n = \{e, x, x^2, \cdots, x^{n-1}, y, xy, x^2y, \cdots, x^{n-1}y\}$ and $\mathcal{G} = \mathbb{S}_2(D_n)$.

Case-1: Let n be an odd positive integer. Then $S_G = \{e, x, x^2, \cdots, x^{n-1}\}$ and $G_2 = \{y, xy, x^2y, \cdots, x^{n-1}y\}$.

Now, using the degree and distance property from Theorem 4.3.4, Case-1, we have

Now, $d(u)d(v)d_{\mathcal{G}}(e) = \sum_{x^i\in V(\mathcal{G})} d(e)d(x^i)d(e,x^i) + \sum_{x^iy\in V(\mathcal{G})} d(e)d(x^iy)d(e,x^iy)$
$$= d(e)[\sum_{x^i\in V(\mathcal{G})} d(x^i)d(e,x^i) + \sum_{x^iy\in V(\mathcal{G})} d(x^iy)d(e,x^iy)]$$
$$= (2n-1)[(n-1)(2n-2)\times 1 + n(2n-1)\times 1] = (2n-1)(4n^2-5n+2).$$

Similarly, for each $i = 1, 2, \cdots, n$,

$$d(u)d(v)d_{\mathcal{G}}(x^iy) = d(e)d(x^iy)d(x^iy, e) + \sum_{\substack{x^j \in V(\mathcal{G}) \\ j \neq n}} d(x^iy)d(x^j)d(x^iy, x^j)$$

$$+ \sum_{\substack{x^jy \in V(\mathcal{G}) \\ j \neq i}} d(x^iy)d(x^jy)d(x^iy, x^jy) = (2n-1)(2n-1) \times 1 + (n-1)(2n-1)(2n-2) \times 1 + (n-1)(2n-1)^2 \times 1 = (2n-1)(4n^2-5n+2).$$

Also, for each $i = 1, 2, \cdots, n-1$,

$$d(u)d(v)d_{\mathcal{G}}(x^i) = \sum_{\substack{x^j \in V(\mathcal{G}) \\ j \neq n-i}} d(x^i)d(x^j)d(x^i, x^j) + d(x^i)d(x^{n-i})d(x^i, x^{n-i})$$

$$+ \sum_{x^jy \in V(\mathcal{G})} d(x^i)d(x^jy)d(x^i, x^jy) = (2n-2)(2n-2)^2 \times 1 + (2n-2)^2 \times 2 + n(2n-2)(2n-1) \times 1 = (2n-2)(6n^2-5n).$$

Therefore, $Gut(\mathcal{G}) = \dfrac{1}{2} \displaystyle\sum_{w \in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(w)$

$$= \frac{1}{2}[d(u)d(v)d_{\mathcal{G}}(e) + \sum_{\substack{x^i \in V(\mathcal{G}) \\ i \neq n}} d(u)d(v)d_{\mathcal{G}}(x^i) + \sum_{x^iy \in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(x^iy)]$$

$$= \frac{1}{2}[(2n-1)(4n^2-5n+2) + (n-1)(2n-2)(6n^2-5n) + n(2n-1)(4n^2-5n+2)]$$

$$= \frac{20n^4 - 42n^3 + 27n^2 - 3n - 2}{2}.$$

Case-2: Let $\dfrac{n}{2}$ is an odd positive integer. Then $S_G = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}}, y, xy, x^2y, \cdots, x^{n-1}y\}$. Then $x^{\frac{n}{2}} \notin S_G$.

Now, using the degree and distance property from Theorem 4.3.4, Case-2, we have

Now, $d(u)d(v)d_{\mathcal{G}}(e) = \displaystyle\sum_{u \in V(\mathcal{G})} d(e)d(u)d(e, u)$

$$= d(e)[\sum_{\substack{x^i \in V(\mathcal{G}) \\ i = odd,\ i \neq \frac{n}{2}}} d(x^i)d(e, x^i) + d(x^{\frac{n}{2}})d(e, x^{\frac{n}{2}}) + \sum_{\substack{x^i \in V(\mathcal{G}) \\ i = even}} d(x^i)d(e, x^i) + \sum_{x^iy \in V(\mathcal{G})} d(x^iy)d(e, x^iy)]$$

$$= \frac{3n}{2}[(\frac{n}{2}-1)\frac{3n-2}{2} \times 2 + \frac{3n}{2} \times 1 + (\frac{n}{2}-1)\frac{3n-2}{2} \times 1 + n \times \frac{3n}{2} \times 1] = \frac{3n(15n^2-18n+12)}{8}.$$

$$d(u)d(v)d_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum_{u \in V(\mathcal{G})} d(x^{\frac{n}{2}})d(u)d(x^{\frac{n}{2}}, u)$$

$$= d(x^{\frac{n}{2}})[\sum_{\substack{x^i \in V(\mathcal{G}) \\ i = odd}} d(x^i)d(x^{\frac{n}{2}}, x^i) + \sum_{\substack{x^i \in V(\mathcal{G}) \\ i = even,\ i \neq n}} d(x^i)d(x^{\frac{n}{2}}, x^i) + d(e)d(x^{\frac{n}{2}}, e)$$

$$+ \sum_{x^iy\in V(\mathcal{G})} d(x^iy)d(x^{\frac{n}{2}},x^iy)] = \frac{3n}{2}[(\frac{n}{2}-1)\frac{3n-2}{2}\times 1+(\frac{n}{2}-1)\frac{3n-2}{2}\times 2+\frac{3n}{2}\times 1+ n\times \frac{3n}{2}\times 1] = \frac{3n(15n^2-18n+12)}{8}.$$

For each odd i except $\frac{n}{2}$,

$$d(u)d(v)d_{\mathcal{G}}(x^i) = \sum_{w\in V(\mathcal{G})} d(x^i)d(w)d(x^i,w)$$

$$= d(x^i)[\sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq n-i,\frac{n}{2}}} d(x^j)d(x^i,x^j) + d(x^{n-i})d(x^i,x^{n-i}) + d(x^{\frac{n}{2}})d(x^i,x^{\frac{n}{2}}) +$$

$$\sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq \frac{n}{2}-i,n}} d(x^j)d(x^i,x^j)+d(x^{\frac{n}{2}-i})d(x^i,x^{\frac{n}{2}-i})+d(e)d(x^i,e)+\sum_{x^jy\in V(\mathcal{G})} d(x^jy)d(x^i,x^jy)]$$

$$= \frac{3n-2}{2}[(\frac{n}{2}-3)\frac{3n-2}{2}\times 1+\frac{3n-2}{2}\times 2+\frac{3n}{2}\times 1+(\frac{n}{2}-2)\frac{3n-2}{2}\times 2+\frac{3n}{2}\times 2+ \frac{3n-2}{2}\times 1+n\times\frac{3n}{2}\times 1]$$

$$= \frac{(3n-2)(15n^2-12n+16)}{8}.$$

For each even i except n,

$$d(u)d(v)d_{\mathcal{G}}(x^i) = \sum_{w\in V(\mathcal{G})} d(x^i)d(w)d(x^i,w)$$

$$= d(x^i)[\sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq \frac{n}{2}-i,\frac{n}{2}}} d(x^j)d(x^i,x^j) + d(x^{\frac{n}{2}-i})d(x^i,x^{\frac{n}{2}-i}) + d(x^{\frac{n}{2}})d(x^i,x^{\frac{n}{2}})$$

$$+ \sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq n-i,n}} d(x^j)d(x^i,x^j)+d(x^{n-i})d(x^i,x^{n-i})+d(e)d(x^i,e)+\sum_{x^jy\in V(\mathcal{G})} d(x^jy)d(x^i,x^jy)]$$

$$= \frac{3n-2}{2}[(\frac{n}{2}-2)\frac{3n-2}{2}\times 2+\frac{3n-2}{2}\times 1+\frac{3n}{2}\times 2+(\frac{n}{2}-3)\frac{3n-2}{2}\times 1+\frac{3n}{2}\times 1+ \frac{3n-2}{2}\times 2+n\times\frac{3n}{2}]$$

$$= \frac{(3n-2)(15n^2-12n+16)}{2}.$$

For each i, $d(u)d(v)d_{\mathcal{G}}(x^iy) = \sum_{w\in V(\mathcal{G})} d(x^iy)d(w)d(x^iy,w)$

$$= d(x^iy)[\sum_{\substack{x^j\in V(\mathcal{G})\\ j\neq\frac{n}{2},n}} d(x^j)d(x^iy,x^j) + d(x^{\frac{n}{2}})d(x^iy,x^{\frac{n}{2}}) + d(e)d(x^iy,e)$$

$$+ \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=even,\frac{n}{2}}} d(x^jy)d(x^iy,x^jy) + \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=odd,\ i-j\neq\frac{n}{2}}} d(x^jy)d(x^iy,x^jy)]$$

$= \frac{3n}{2}[(\frac{n}{2}-2)\frac{3n-2}{2}\times 1+\frac{3n}{2}\times 1+\frac{3n}{2}\times 1+\frac{n}{2}\times\frac{3n}{2}\times 1+(\frac{n}{2}-1)\frac{3n}{2}\times 2]$
$= \frac{3n(15n^2-16n+8)}{8}$.

Therefore, $Gut(\mathcal{G}) = \frac{1}{2}\sum_{w\in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(u)$

$= \frac{1}{2}[d(u)d(v)d_{\mathcal{G}}(e)+d(u)d(v)d_{\mathcal{G}}(x^{\frac{n}{2}})+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd,\ i\neq\frac{n}{2}}} d(u)d(v)d_{\mathcal{G}}(x^i)+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq n}} d(u)d(v)d_{\mathcal{G}}(x^i)+$

$\sum_{x^iy\in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(x^iy)]$

$= \frac{1}{2}[\frac{3n(15n^2-18n+12)}{8}+\frac{3n(15n^2-18n+12)}{8}+(\frac{n}{2}-1)\frac{(3n-2)(15n^2-12n+16)}{8}+$
$(\frac{n}{2}-1)\frac{(3n-2)(15n^2-12n+16)}{8}+n\times\frac{3n(15n^2-16n+8)}{8}]$
$= \frac{45n^4-57n^3+60n^2-52n+32}{8}$.

Case-3: Let $\frac{n}{2}$ is an even positive integer. Then $S_G = \{e, x^2, x^4, \cdots, x^{n-2}\}$ and $G_2 = \{x^{\frac{n}{2}}, y, xy, x^2y, \cdots, x^{n-1}y\}$. Then $x^{\frac{n}{2}} \in S_G$.

Now, using the degree and distance property from Theorem 4.3.4, Case-3, we have

Now, $d(u)d(v)d_{\mathcal{G}}(e) = \sum_{w\in V(\mathcal{G})} d(e)d(w)d(e,w)$

$= d(e)[\sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d(x^i)d(e,x^i)+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=even, i\neq\frac{n}{2}}} d(x^i)d(e,x^i)+d(x^{\frac{n}{2}})d(e,x^{\frac{n}{2}})+\sum_{x^iy\in V(\mathcal{G})} d(x^iy)d(e,x^iy)]$

$= \frac{3n-2}{2}[\frac{n}{2}\times\frac{3n-4}{2}\times 2+(\frac{n}{2}-2)\frac{3n-4}{2}\times 1+\frac{3n-2}{2}\times 1+n\times\frac{3n-2}{2}\times 1]$
$= \frac{(3n-2)(15n^2-22n+12)}{8}$.

$d(u)d(v)d_{\mathcal{G}}(x^{\frac{n}{2}}) = \sum_{w\in V(\mathcal{G})} d(x^{\frac{n}{2}})d(w)d(x^{\frac{n}{2}},w)$

$= d(x^{\frac{n}{2}})[\sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d(x^i)d(x^{\frac{n}{2}},x^i)+\sum_{\substack{x^i\in V(\mathcal{G})\\ i=even, i\neq n}} d(x^i)d(x^{\frac{n}{2}},x^i)+d(e)d(x^{\frac{n}{2}},e)$

$+\sum_{x^iy\in V(\mathcal{G})} d(x^iy)d(x^{\frac{n}{2}},x^iy)] = \frac{3n-2}{2}[\frac{n}{2}\times\frac{3n-4}{2}\times 2+(\frac{n}{2}-2)\frac{3n-4}{2}\times 1+\frac{3n-2}{2}\times$
$1+n\times\frac{3n-2}{2}\times 1] = \frac{(3n-2)(15n^2-22n+12)}{8}$.

For each odd i,

$$d(u)d(v)d_{\mathcal{G}}(x^i) = \sum_{w\in V(\mathcal{G})} d(x^i)d(w)d(x^i,u)$$

$$= d(x^i)[\sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd,\ j\neq n-i}} d(x^j)d(x^i,x^j) + d(x^{n-i})d(x^i,x^{n-i}) + \sum_{\substack{x^j\in V(\mathcal{G})\\ j=even, j\neq \frac{n}{2},n}} d(x^j)d(x^i,x^j) +$$

$$d(e)d(x^i,e)$$

$$+ d(x^{\frac{n}{2}})d(x^i,x^{\frac{n}{2}}) + \sum_{x^jy\in V(\mathcal{G})} d(x^jy)d(x^i,x^jy)]$$

$$= \frac{3n-4}{2}[(\frac{n}{2}-2)\frac{3n-4}{2}\times 1+\frac{3n-4}{2}\times 2+(\frac{n}{2}-2)\frac{3n-4}{2}\times 2+\frac{3n-2}{2}\times 2+\frac{3n-2}{2}\times$$

$$2+n\times\frac{3n-2}{2}\times 1] = \frac{(3n-4)(15n^2-16n+16)}{8}.$$

For each even i except $\frac{n}{2}, n$,

$$d(u)d(v)d_{\mathcal{G}}(x^i) = \sum_{w\in V(\mathcal{G})} d(x^i)d(w)d(x^i,w)$$

$$= d(x^i)[\sum_{\substack{x^j\in V(\mathcal{G})\\ j=odd}} d(x^j)d(x^i,x^j) + \sum_{\substack{x^j\in V(\mathcal{G})\\ j=even,\ j\neq n,\frac{n}{2},n-i}} d(x^j)d(x^i,x^j) + d(e)d(x^i,e) + d(x^{\frac{n}{2}})d(x^i,x^{\frac{n}{2}}) +$$

$$d(x^{n-i})d(x^i,x^{n-i}) + \sum_{x^jy\in V(\mathcal{G})} d(x^jy)d(x^i,x^jy)]$$

$$= \frac{3n-4}{2}[\frac{n}{2}\times\frac{3n-4}{2}\times 2+(\frac{n}{2}-4)\frac{3n-4}{2}\times 1+\frac{3n-2}{2}\times 1+\frac{3n-2}{2}\times 1+\frac{3n-4}{2}\times$$

$$2+n\times\frac{3n-2}{2}\times 1] = \frac{(3n-4)(15n^2-16n+8)}{8}.$$

For each i, $d(u)d(v)d_{\mathcal{G}}(x^iy) = \sum_{w\in V(\mathcal{G})} d(x^iy)d(w)d(x^iy,w)$

$$= d(x^iy)[\sum_{\substack{x^j\in V(\mathcal{G})\\ j\neq n,\frac{n}{2}}} d(x^j)d(x^iy,x^j) + d(e)d(x^iy,e) + d(x^{\frac{n}{2}})d(x^iy,x^{\frac{n}{2}}) + \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=even}} d(x^jy)d(x^iy,x^jy)$$

$$+ \sum_{\substack{x^jy\in V(\mathcal{G})\\ i-j=odd}} d(x^jy)d(x^iy,x^jy)] = \frac{3n-2}{2}[(n-2)\frac{3n-4}{2}\times 1+\frac{3n-2}{2}\times 1+\frac{3n-2}{2}\times 1+$$

$$(\frac{n}{2}-1)\frac{3n-2}{2}\times 1+\frac{n}{2}\times\frac{3n-2}{2}\times 2] = \frac{(3n-2)(15n^2-20n+12)}{8}.$$

Therefore, $Gut(\mathcal{G}) = \frac{1}{2}\sum_{w\in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(w)$

$$= \frac{1}{2}[d(u)d(v)d_{\mathcal{G}}(e) + d(u)d(v)d_{\mathcal{G}}(x^{\frac{n}{2}}) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=odd}} d(u)d(v)d_{\mathcal{G}}(x^i) + \sum_{\substack{x^i\in V(\mathcal{G})\\ i=even,\ i\neq\frac{n}{2},n}} d(u)d(v)d_{\mathcal{G}}(x^i)$$

$$+ \sum_{x^iy\in V(\mathcal{G})} d(u)d(v)d_{\mathcal{G}}(x^iy)]$$

$$= \frac{1}{2}[\tfrac{(3n-2)(15n^2-22n+12)}{8} + \frac{(3n-2)(15n^2-22n+12)}{8} + \frac{n}{2} \times \frac{(3n-4)(15n^2-16n+16)}{8}$$

$$+ (\frac{n}{2} - 2) \times \frac{(3n-4)(15n^2-16n+8)}{8} + n \times \frac{(3n-2)(15n^2-20n+12)}{8}]$$

$$= \frac{45n^4 - 99n^3 + 100n^2 - 12n + 8}{8}. \qquad \square$$

Chapter 5

Merrifield-Simmons Index of Co-maximal graph over a commutative ring

Chapter 5

Merrifield-Simmons Index of $\Gamma'(R)$

5.1 Introduction

Algebraic graph theory finds various applications across various fields such as Mathematics, Computer Science, and Chemistry. For a commutative ring R with identity, Sharma and Bhatwadekar [49] introduced the co-maximal graph $\Gamma'(R)$ in 1995. Later studies by Maimani et al. [37] explored these co-maximal graphs' characteristics in greater detail and came up with the term "co-maximal graph of R." Furthermore, through the study of the generalized co-maximal graph in [12] and [13], B. Biswas, S. Kar, and M.K. Sen also made contributions to this field. The authors of [37] also looked at the properties of the subgraphs $\Gamma'_1(R)$, $\Gamma'_2(R)$, and $\Gamma'_2(R) \setminus J(R)$. Here, $\Gamma'_1(R)$ is the subgraph of the co-maximal graph whose vertices are the units of R, $\Gamma'_2(R)$ contains the non-unit elements of R, and $\Gamma'_2(R) \setminus J(R)$ consists of non-unit elements of R that are not in the Jacobson radical $J(R)$. The Merrifield-Simmons Index of the co-maximal graph for commutative rings with identity is the primary focus of this chapter. The definition of the co-maximal graph over a commutative ring with identity is as follows:

Definition 5.1.1. *Let R be a commutative ring with identity. Then the co-maximal graph $\Gamma'(R)$ of R with vertices representing elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$.*

In [10], it is shown that a finite commutative ring R with identity can be expressed as a direct sum of local rings. This decomposition is unique up to permutation of direct summands. If $R = \bigotimes_{j=1}^{n} R_j$, where R_j is a local ring then for any poper ideal I of R, we have $I = \bigotimes_{j=1}^{n} I_j$, where each I_j is an ideal of the ring R_j. Moreover, I is a maximal ideal in R if and only if I_k is the maximal ideal in R_k, for some $k \in \{1, ..., n\}$ and $I_j = R_j$ for $j \neq k$. From [22], a set of vertices of a graph is independent if the vertices are pairwise non-adjacent. The number of vertices in the largest independent set is called the independence number of a given graph, which is denoted conventionally by α. If a graph on n vertices contains a clique of $n - \alpha$ vertices and the rest α vertices is a stable set, where every vertex within the clique is linked to every vertex in the stable set, then the graph is called a complete split graph and is denoted by $CS(n, \alpha)$, $1 \leq \alpha \leq n - 1$.

5.2 Merrifield-Simmons index of $\Gamma'(R)$, where R is finite commutative ring with unity.

In this section, we mainly established some results on Merrifield-Simmons index over a graph and established some theorems on Merrifield-Simmons index of co-maximal graph over finite commutative ring with unity.

Join of two graphs [37]: Let $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ be two graphs with disjoint vertices set V_i and edges set E_i. The join of $\mathcal{G}_1$ and $\mathcal{G}_2$ is denoted by $\mathcal{G} = \mathcal{G}_1 \vee \mathcal{G}_2$ with vertices set $V_1 \cup V_2$ and the set of edges is $E_1 \cup E_2 \cup \{xy : x \in V_1 \text{ and } y \in V_2\}$.

Lemma 5.2.1. *If $\mathcal{G} = \mathcal{G}_1 \vee \mathcal{G}_2$ is the join of the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Then $i(\mathcal{G}) = i(\mathcal{G}_1) + i(\mathcal{G}_2)$.*

Proof. From the definition of join of the grpahs $\mathcal{G}_1$ and $\mathcal{G}_2$, x and y are adjacent to every pair of vertices $x \in V_1$ and $y \in V_2$, where V_1 and V_2 are the vertex sets of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Since there are no n-tuples of pairs of vertices $x \in V_1$ and $y \in V_2$ such that x and y are non-adjacent in $\mathcal{G}$ for $n \geq 2$. Then $i(\mathcal{G}) = i(\mathcal{G}_1) + i(\mathcal{G}_2)$. □

Lemma 5.2.2. *For a complete split graph $CS(n,\alpha)$ with α-number stable set, $i(CS(n,\alpha)) = n - \alpha + 2^{\alpha}$.*

Proof. Let $\mathcal{G}_1$ be the stable set of $CS(n,\alpha)$ whose cardinality α and $\mathcal{G}_2$ be the clique of $CS(n,\alpha)$ whose cardinality $n-\alpha$. By the definition of $CS(n,\alpha)$, $CS(n,\alpha) = \mathcal{G}_1 \vee \mathcal{G}_2$. So by Lemma 5.2.1, $i(CS(n,\alpha)) = i(\mathcal{G}_1) + i(\mathcal{G}_2) = 2^{|\mathcal{G}_1|} + |\mathcal{G}_2| = 2^{\alpha} + n - \alpha$. $\square$

Theorem 5.2.3. *[10] Let R be a finite ring (not necessarily commutative) with a multiplicative identity $1 \neq 0$ whose zero-divisors form an additive group J. Then*

(i) J is the Jacobson radical of R;

(ii) $|R| = p^{nr}$ and $|J| = p^{(n-1)r}$, for some prime p and some positive integer r and n;

(iii) $J^n = (0)$;

(iv) the characteristic of the ring R is p^k for some integer $1 \leq k \leq n$;

(v) if the characteristic is p^n, then R will be commutative.

From the above Theorem 5.2.3 follows the following theorem.

Theorem 5.2.4. *Let R be a finite commutative local ring with unity. Then the cardinality of R is equal to p^n for some prime p and some positive integer n.*

Lemma 5.2.5. *Let R be a finite commutative local ring with unity. Then $\Gamma'(R) \cong CS(p^n - m, m)$, where M is the maximal ideal of R with cardinality m and $|R| = p^n$ for any $n \in \mathbb{N}$ and prime p.*

Proof. Since R is a finite commutative local ring with unity, so $|R| = p^n$ for any $n \in \mathbb{N}$. Also, R has a unique maximal ideal, say, M with $|M| = m$. Let $U(R)$ be the set of all unit elements of R. Then $|U(R)| = p^n - m$. Since, R is a local ring, then the maximal ideal, M which is the set of non-unit elements of R. Then $R = U(R) \cup M$ is a disjoint union of $U(R)$ and M. Let $\mathcal{G} = \Gamma'(R)$. Then $\mathcal{G} = \Gamma'_1(R) \vee \Gamma'_2(R)$, where $\Gamma'_1(R)$ is a complete subgraph of $\mathcal{G}$ induced by $U(R)$ and $\Gamma'_2(R)$ is a subgraph of $\mathcal{G}$ induced by M. Clearly, $\Gamma'_1(U(R))$ forms a clique with cardinality $p^n - m$ and every vertex within the clique is adjacent to every vertex of M. Also, no two vertices are adjacent to each other within the set M. So, M is the maximaum independent set with cardinality m. Then the graph $\Gamma'(R) \cong CS(p^n - m, m)$. □

Next example shows that how the Lemma 5.2.2 and the Lemma 5.2.5 work.

Example 5.2.6. *Let* $R = \mathbb{Z}_9 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}\}$. *Then* $M =$ *maximal ideal of* $R = \{\overline{0}, \overline{3}, \overline{6}\}$ *and* $U(R) =$ *set of all unit elements of* $R = \{\overline{1}, \overline{2}, \overline{4}, \overline{5}, \overline{7}, \overline{8}\}$. *Since,* R *is a finite commutative local ring with unity, then from the Lemma 5.2.5, we have* $\Gamma'(R) \cong CS(6, 3)$.

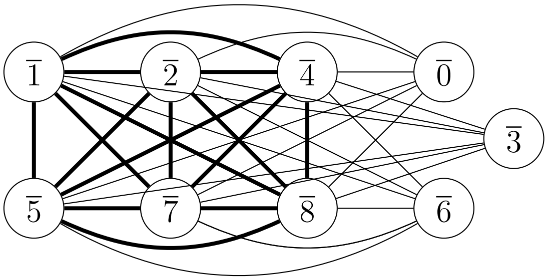

Figure 5.1: $\Gamma'(\mathbb{Z}_9)$

From the Figure 5.1, $i(\Gamma'(R)) = \sum_{k=0}^{9} i_k(\Gamma'(R))$, *where* $i_0(\Gamma'(R)) = 1$, $i_1(\Gamma'(R)) = 9$ *and* $i_k(\Gamma'(R)) =$ *Number of* $k-$ *tupple pairs of vertices from* M *in which they are pairwise non-adjacent for* $k \geq 2$. *Now,* $i_k(\Gamma'(R)) = \binom{3}{k}$ *for* $k \geq 2$. *Hence,* $i(\Gamma'(R)) = i_0(\Gamma'(R)) + i_1(\Gamma'(R)) + i_2(\Gamma'(R)) + i_3(\Gamma'(R)) = 1 + 9 + 3 + 1 = 14$.

Theorem 5.2.7. *Let* R *be a finite commutative local ring with unity. Then* $i(\Gamma'(R)) = p^n + 2^m - m$, *where* M *is the maximal ideal of* R *with* $|M| = m$ *and* $|R| = p^n$ *for any* $n \in \mathbb{N}$.

Proof. Let $\mathcal{G} = \Gamma'(R)$. From the Lemma 5.2.5, $\Gamma'(R) \cong CS(p^n - m, m)$. Then by Lemma 5.2.2, $i(\mathcal{G}) = 2^m + p^n - m$. □

If we take the ring $R \cong F_1 \times F_2$ for two finite fields F_1 and F_2, then next example shows the idea that how to find the Merrifield-Simmons index of the ring R.

Example 5.2.8. *Consider two fields* $F_1 = \mathbb{Z}_3$ *and* $F_2 = \mathbb{Z}_5$. *Let* $R = F_1 \times F_2$, $U_1 =$ *set of unit elements in* $\mathbb{Z}_3 = \{\overline{1}, \overline{2}\}$, $M_1 =$ *the maximal ideal of* $\mathbb{Z}_3 = \{\overline{0}\}$, $U_2 =$ *set of unit elements in* $\mathbb{Z}_5 = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$ *and* $M_2 =$ *the maximal ideal of* $\mathbb{Z}_5 = \{\overline{0}\}$. *Now we consider a partition:* $R = (U_1 \times U_2) \cup (U_1 \times M_2) \cup (M_1 \times U_2) \cup (M_1 \times M_2)$, *where* $U_1 \times U_2 = \{(\overline{1}, \overline{1}), (\overline{1}, \overline{2}), (\overline{1}, \overline{3}), (\overline{1}, \overline{4}), (\overline{2}, \overline{1}), (\overline{2}, \overline{2}), (\overline{2}, \overline{3}), (\overline{2}, \overline{4})\}$, $U_1 \times M_2 = \{(\overline{1}, \overline{0}), (\overline{2}, \overline{0})\}$, $M_1 \times U_2 = \{(\overline{0}, \overline{1}), (\overline{0}, \overline{2}), (\overline{0}, \overline{3}), (\overline{0}, \overline{4})\}$ *and* $M_1 \times M_2 = \{(\overline{0}, \overline{0})\}$. *From this, we observed that* $x \leftrightarrow y$ *for* $x, y \in U_1 \times U_2$ *and* $x \leftrightarrow y$ *for* $x \in U_1 \times M_2$, $y \in M_1 \times U_2$. *Also,* $x \not\leftrightarrow y$ *for either* $x, y \in U_1 \times M_2$ *or* $x, y \in M_1 \times U_2$ *or* $x, y \in M_1 \times M_2$ *and* $x \not\leftrightarrow y$ *for either* $x \in U_1 \times M_2$, $y \in M_1 \times M_2$ *or* $x \in M_1 \times U_2$, $y \in M_1 \times M_2$.

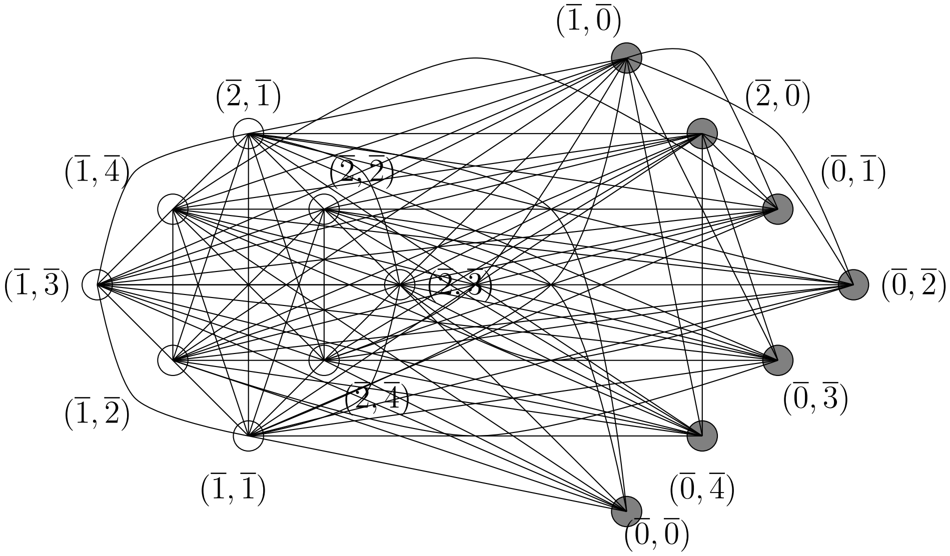

Figure 5.2: $\Gamma'(\mathbb{Z}_3 \times \mathbb{Z}_5)$

Now, from Figure 5.2, we have $i(\Gamma'(R)) = \sum_{k=0}^{|R|} i_k(\Gamma'(R))$

$$= i_0(\Gamma'(R)) + i_1(\Gamma'(R)) + \sum_{k=2}^{|R|} i_k(\Gamma'(R))$$

$$= i_0(\Gamma'(R)) + i_1(\Gamma'(R)) + \sum_{k=2}^{|U_1 \times M_2|} i_k(\Gamma'(R)) + \sum_{k=2}^{|M_1 \times U_2|} i_k(\Gamma'(R)) + \sum_{k=2}^{|M_1 \times M_2|} i_k(\Gamma'(R)) +$$

$$\sum_{k=1}^{|U_1 \times M_2|} \binom{|U_1 \times M_2|}{k} + \sum_{k=1}^{|M_1 \times U_2|} \binom{|M_1 \times U_2|}{k}$$

$$= 1 + |R| + \sum_{k=2}^{2}\binom{2}{k} + \sum_{k=2}^{4}\binom{4}{k} + \sum_{k=2}^{1}\binom{1}{k} + \sum_{k=1}^{2}\binom{2}{k} + \sum_{k=1}^{4}\binom{4}{k} = 46.$$

From the above example, we have the following theorem for arbitrary two finite fields F_1 and F_2.

Theorem 5.2.9. *Let $R \cong F_1 \times F_2$, where F_1 and F_2 are two finite fields with $|F_1| = p_1^{k_1}$ and $|F_2| = p_2^{k_2}$ for any prime numbers p_1, p_2 and $k_1, k_2 \in \mathbb{N}$. Then $i(\Gamma'(R)) = 2^{p_1^{k_1}} + 2^{p_2^{k_2}} + p_1^{k_1}p_2^{k_2} - p_1^{k_1} - p_2^{k_2} - 1$.*

Proof. Let $R = F_1 \times F_2$. Now, let U_1 = set of unit elements in R_1 with $|U_1| = p_1^{k_1} - 1$, U_2 = set of unit elements in F_2 with $|U_2| = p_2^{k_2} - 1$, M_1 = the maximal ideal of F_1 = $\{0\}$ and M_2 = the maximal ideal of $F_2 = \{0\}$. So, $R = F_1 \times F_2 = (U_1 \times U_2) \cup (U_1 \times M_2) \cup (M_1 \times U_2) \cup (M_1 \times M_2)$ with $|U_1 \times U_2| = (p_1^{k_1} - 1)(p_2^{k_2} - 1)$, $|U_1 \times M_2| = (p_1^{k_1} - 1)$, $|M_1 \times U_2| = (p_2^{k_2} - 1)$ and $|M_1 \times M_2| = 1$. Also, X and Y are adjacent to each other for every pair of vertices from $U_1 \times U_2$. Also, X and Y are not adjacent for every pair of vertices from either $U_1 \times M_2$ or $M_1 \times U_2$ or $M_1 \times M_2$. But X and Y are adjacent for every pair of vertices from $X \in U_1 \times M_2$ and $Y \in M_1 \times U_2$ and X , Y are not adjacent for every pair of vertices from either $X \in U_1 \times M_2$ and $Y \in M_1 \times M_2$ or $X \in M_1 \times U_2$ and $Y \in M_1 \times M_2$.

Now, $i(\Gamma'(R)) = \sum_{k=0}^{|R|} i_k(\Gamma'(R))$

$$= i_0(\Gamma'(R)) + i_1(\Gamma'(R)) + \sum_{k=2}^{|R|} i_k(\Gamma'(R))$$

$$= i_0(\Gamma'(R)) + i_1(\Gamma'(R)) + \sum_{k=2}^{|U_1 \times M_2|} i_k(\Gamma'(R)) + \sum_{k=2}^{|M_1 \times U_2|} i_k(\Gamma'(R)) + \sum_{k=2}^{|M_1 \times M_2|} i_k(\Gamma'(R))$$

$$+ \Big(\sum_{m=1}^{|U_1 \times M_2|} \sum_{n=1}^{|M_1 \times M_2|} \binom{|U_1 \times M_2|}{m}\binom{|M_1 \times M_2|}{n} \Big) + \Big(\sum_{m=1}^{|M_1 \times U_2|} \sum_{n=1}^{|M_1 \times M_2|} \binom{|M_1 \times U_2|}{m}\binom{|M_1 \times M_2|}{n} \Big)$$

$$= 1 + p_1^{k_1}p_2^{k_2} + \sum_{k=2}^{|U_1 \times M_2|} \binom{|U_1 \times M_2|}{k} + \sum_{k=2}^{|M_1 \times U_2|} \binom{|M_1 \times U_2|}{k} + \sum_{k=2}^{|M_1 \times M_2|} \binom{|M_1 \times M_2|}{k}$$

$$+ \Big(\sum_{m=1}^{|U_1 \times M_2|} \sum_{n=1}^{|M_1 \times M_2|} \binom{|U_1 \times M_2|}{m}\binom{|M_1 \times M_2|}{n} \Big) + \Big(\sum_{m=1}^{|M_1 \times U_2|} \sum_{n=1}^{|M_1 \times M_2|} \binom{|M_1 \times U_2|}{m}\binom{|M_1 \times M_2|}{n} \Big)$$

$$= 1 + p_1^{k_1}p_2^{k_2} + \sum_{k=2}^{p_1^{k_1}-1}\binom{p_1^{k_1}-1}{k} + \sum_{k=2}^{p_2^{k_2}-1}\binom{p_2^{k_2}-1}{k} + 0 + \sum_{k=1}^{p_1^{k_1}-1}\binom{p_1^{k_1}-1}{k} + \sum_{k=1}^{p_2^{k_2}-1}\binom{p_2^{k_2}-1}{k}$$
$$= 1 + p_1^{k_1}p_2^{k_2} + (2^{p_1^{k_1}-1} - p_1^{k_1}) + (2^{p_2^{k_2}-1} - p_2^{k_2}) + (2^{p_1^{k_1}-1} - 1) + (2^{p_2^{k_2}-1} - 1)$$
$$= 2^{p_1^{k_1}} + 2^{p_2^{k_2}} + p_1^{k_1}p_2^{k_2} - p_1^{k_1} - p_2^{k_2} - 1 \qquad \square$$

The next example explains how to find the Merreld-Simmons index of $\Gamma'(R)$, where the ring R has exactly two maximal ideals.

Example 5.2.10. *Let $R = \mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$. Let $U(R)$ = set of all unit elements in $\mathbb{Z}_6 = \{\overline{1}, \overline{5}\}$. Since, R has exactly two maximal ideals, say, M_1 and M_2, then $M_1 = \{\overline{0}, \overline{2}, \overline{4}\}$, $M_2 = \{\overline{0}, \overline{3}\}$ and $M_1 \cap M_2 = \{\overline{0}\}$. Now, each vertex within $U(R)$ is adjacent to each vertex of R. Also, no two vertices within M_1 or M_2 are adjacent to each other but each vertex within $M_1 \setminus M_1 \cap M_2$ is adjacent to each vertex within $M_2 \setminus M_1 \cap M_2$.*

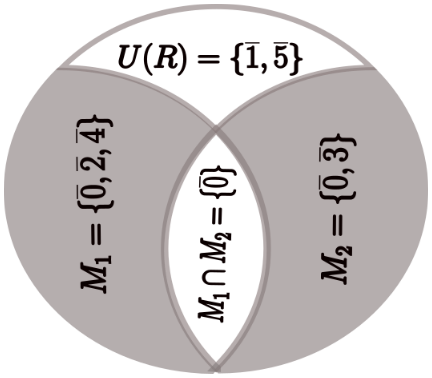

Figure 5.3: $\mathbb{Z}_6$

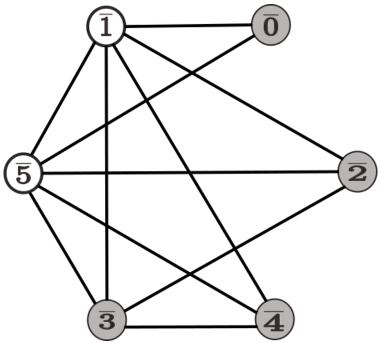

Figure 5.4: $\Gamma'(\mathbb{Z}_6)$

Now, from the Figure 5.4, we have
$i(\Gamma'(R)) = \sum_{k=0}^{|R|} i_k(\Gamma'(R)) = i_0(\Gamma'(R)) + i_1(\Gamma'(R)) + i_2(\Gamma'(R)) + i_3(\Gamma'(R)) = 1+6+4+1 = 12$.

From the above example, we have the following theorem.

Theorem 5.2.11. *Let R be a finite commutative ring with unity. If R has only two maximal ideals M_1 and M_2, then $i(\Gamma'(R)) = |R| + 2^{|M_1|} + 2^{|M_2|} - 2^{|M_1 \cap M_2|} - |M_1 \cup M_2|$.*

Proof. Let $V = R, V_1 = U(R), V_2 = M_1$ and $V_3 = M_2$. Any vertex $a \in U(R)$ is adjacent to any other vertex $b \in R \setminus \{a\}$. Since R has only two maximal ideals, M_1 and M_2, so $\Gamma_2(R) \setminus M_1 \cap M_2$ forms a complete bipartite subgraph of $\Gamma'(R)$. For two elements a and b in $J(R) = M_1 \cap M_2$, $aR + bR \neq R$, implying that a and b are not adjacent to each other. For better understanding see Figure 5.5.

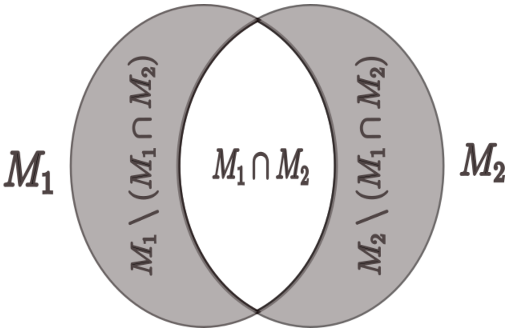

Figure 5.5

Furthermore, if either a and b are in $M_1 \setminus M_1 \cap M_2$ or both are in $M_2 \setminus M_1 \cap M_2$, then $aR + bR \neq R$, indicating that a and b are not adjacent to each other. Additionally, if $a \in J(R) = M_1 \cap M_2$ and b is in either $M_1 \setminus M_1 \cap M_2$ or $M_2 \setminus M_1 \cap M_2$, then $aR + bR \neq R$, so in this case as well, a and b are not adjacent to each other.

However, if $a \in M_1 \setminus M_1 \cap M_2$ and $b \in M_2 \setminus M_1 \cap M_2$, then $aR + bR = R$, establishing that a and b are adjacent to each other. Therefore, by Lemma 5.2.12, we have

$$i(\Gamma'(R)) = |V| + 2^{|V_2|} + 2^{|V_3|} - 2^{|V_2 \cap V_3|} - |V_2 \cup V_3|$$
$$= |R| + 2^{|M_1|} + 2^{|M_2|} - 2^{|M_1 \cap M_2|} - |M_1 \cup M_2| \qquad \square$$

Lemma 5.2.12. *Let $\mathcal{G} = (V, E)$ be any graph with $V = V_1 \cup V_2 \cup V_3$. Furthermore, $x \leftrightarrow y$ for $x \in V_1$ and $y \in V \setminus \{x\}$, $\mathcal{G}((V_2 \setminus V_2 \cap V_3) \cup (V_3 \setminus V_2 \cap V_3))$ forms a complete bipartite induced subgraph of $\mathcal{G}$ and $x \not\leftrightarrow y$ for $x, y \in V_2 \cap V_3$. Then $i(\mathcal{G}) = |V| + 2^{|V_2|} + 2^{|V_3|} - 2^{|V_2 \cap V_3|} - |V_2 \cup V_3|$.*

Proof. Let $\mathcal{G} = (V, E)$ be a graph with vertex set $V = V_1 \cup V_2 \cup V_3$, where $\mathcal{G}(V_1)$ forms a complete induced subgraph of $\mathcal{G}$. Additionally, for every $x \in V_1$ and $y \in V \setminus V_1$,

x is adjacent to y. Let V_4 be the intersection of V_2 and V_3. Then, the subgraph $\mathcal{G}((V_2\backslash V_4)\cup(V_3\backslash V_4))$ forms a complete bipartite induced subgraph of $\mathcal{G}$. Furthermore, x and y are nonadjacent for any pair of vertices in V_4.

Now, let's evaluate the Merrifield-Simmons index of $\mathcal{G}$, denoted as $i(\mathcal{G})$, as the sum of independent sets of different sizes: $i(\mathcal{G}) = \sum\limits_{k=0}^{|V|} i_k(\mathcal{G}) = i_0(\mathcal{G})+i_1(\mathcal{G})+\sum\limits_{k=2}^{|V|} i_k(\mathcal{G})$,where $i_0(\mathcal{G}) = 1$, $i_1(\mathcal{G}) = |V|$, and $i_k(\mathcal{G})$ is the number of k-tuple pairs of independent vertices from V for $k = 2, 3, \ldots, |V|$. To compute $i_k(\mathcal{G})$, we need to count all k-tuple pairs of independent vertices from V. It is sufficient to count all k-tuple pairs of vertices from V_2, V_3, and V_4 individually. In other words, we can say that it will be enough to count all k-tuple pairs of vertices from $V_2 \cup V_3$ only.

Now, $i(\mathcal{G}) = 1 + |V| + \sum\limits_{k=2}^{|V|} i_k(\mathcal{G})$

$$
\begin{aligned}
&= 1 + |V| + \sum_{k=2}^{|V_2\cup V_3|} i_k(\mathcal{G})\\
&= 1 + |V| + \sum_{k=2}^{|V_2|} i_k(\mathcal{G}) + \sum_{k=2}^{|V_3|} i_k(\mathcal{G}) - \sum_{k=2}^{|V_4|} i_k(\mathcal{G})\\
&= 1 + |V| + \sum_{k=2}^{|V_2|} \binom{|V_2|}{k} + \sum_{k=2}^{|V_3|} \binom{|V_3|}{k} - \sum_{k=2}^{|V_4|} \binom{|V_4|}{k}\\
&= 1 + |V| + (2^{|V_2|} - |V_2| - 1) + (2^{|V_3|} - |V_3| - 1) - (2^{|V_4|} - |V_4| - 1)\\
&= |V| + 2^{|V_2|} + 2^{|V_3|} - 2^{|V_4|} - (|V_2| + |V_3| - |V_4|)\\
&= |V| + 2^{|V_2|} + 2^{|V_3|} - 2^{|V_2\cap V_3|} - (|V_2| + |V_3| - |V_2 \cap V_3|)\\
&= |V| + 2^{|V_2|} + 2^{|V_3|} - 2^{|V_2\cap V_3|} - |V_2 \cup V_3|.
\end{aligned}
$$ □

Theorem 5.2.13. *Let R be a finite commutative ring with unity. If R has only three maximal ideals M_1, M_2 and M_3, then $i(\Gamma'(R)) = |R|+2^{|M_1|}+2^{|M_2|}+2^{|M_3|}-2^{|M_1\cap M_2|}-2^{|M_2\cap M_3|} - 2^{M_1\cap M_3} + 2^{|M_1\cap M_2\cap M_3|} - |M_1 \cup M_2 \cup M_3|$.*

Proof. Let R be a finite commutative ring with three maximal ideals, denoted as M_1, M_2, and M_3, where $|R| = n$, $|M_1| = m_1$, $|M_2| = m_2$, $|M_3| = m_3$, $|M_1 \cap M_2| = m_4$, $|M_2 \cap M_3| = m_5$, $|M_1 \cap M_3| = m_6$, and $|J(R)| = |M_1 \cap M_2 \cap M_3| = m_7$. If a and b are elements of $J(R)$, then $aR + bR \neq R$, implying that a and b are not adjacent to each other. Similarly, if a and b belong to either M_1, M_2 or M_3, then $aR + bR \neq R$, and hence, a and b are not adjacent in all cases. Also, if a and b belong to either $M_1 \cap M_2$, $M_2 \cap M_3$ or $M_1 \cap M_3$, then $aR + bR \neq R$, and consequently, a and b are not adjacent in all cases. However, if $a \in M_1 \setminus \{(M_1 \cap M_2) \cup (M_1 \cap M_3)\}$ and b is in either $M_2 \setminus \{(M_2 \cap M_3) \cup (M_2 \cap M_1)\}$ or $M_3 \setminus \{(M_3 \cap M_1) \cup (M_3 \cap M_2)\}$, then $aR + bR = R$, establishing that a and b are adjacent to each other in this scenario. For better understanding see Figure 5.6. Now we utilizing the set theoretical concept of $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$, we have

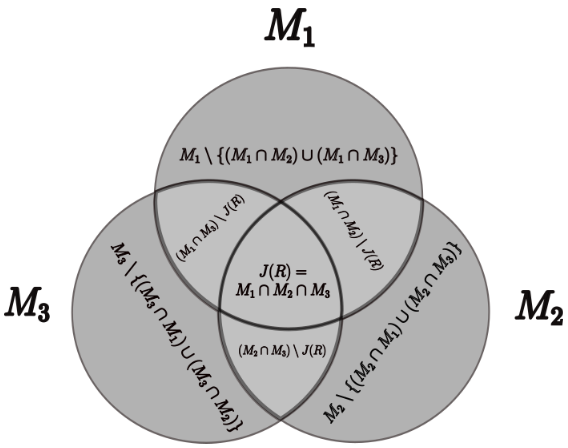

Figure 5.6

$$i(\Gamma'(R)) = \sum_{k=0}^{|R|} i_k(\Gamma'(R))$$

$$= i_0(\Gamma'(R)) + i_1(\Gamma'(R)) + \sum_{k=2}^{|R|} i_k(\Gamma'(R))$$

$$= 1 + n + \sum_{k=2}^{|M_1|} \binom{|M_1|}{k} + \sum_{k=2}^{|M_2|} \binom{|M_2|}{k} + \sum_{k=2}^{|M_3|} \binom{|M_3|}{k} - \sum_{k=2}^{|M_1 \cap M_2|} \binom{|M_1 \cap M_2|}{k} -$$
$$\sum_{k=2}^{|M_2 \cap M_3|} \binom{|M_2 \cap M_3|}{k} - \sum_{k=2}^{|M_1 \cap M_3|} \binom{|M_1 \cap M_3|}{k} + \sum_{k=2}^{|M_1 \cap M_2 \cap M_3|} \binom{|M_1 \cap M_2 \cap M_3|}{k}$$

$$= 1 + n + \sum_{k=2}^{m_1} \binom{m_1}{k} + \sum_{k=2}^{m_2} \binom{m_2}{k} + \sum_{k=2}^{m_3} \binom{m_3}{k} - \sum_{k=2}^{m_4} \binom{m_4}{k} - \sum_{k=2}^{m_5} \binom{m_5}{k} - \sum_{k=2}^{m_6} \binom{m_6}{k} +$$
$$\sum_{k=2}^{m_7} \binom{m_7}{k}$$

$$= 1 + n + \{\sum_{k=0}^{m_1} \binom{m_1}{k} - \binom{m_1}{0} - \binom{m_1}{1}\} + \{\sum_{k=0}^{m_2} \binom{m_2}{k} - \binom{m_2}{0} - \binom{m_2}{1}\} + \{\sum_{k=0}^{m_3} \binom{m_3}{k} -$$

$\binom{m_3}{0} - \binom{m_3}{1}\} - \{\sum_{k=0}^{m_4}\binom{m_4}{k} - \binom{m_4}{0} - \binom{m_4}{1}\} - \{\sum_{k=0}^{m_5}\binom{m_5}{k} - \binom{m_5}{0} - \binom{m_5}{1}\} -$
$\{\sum_{k=0}^{m_6}\binom{m_6}{k} - \binom{m_6}{0} - \binom{m_6}{1}\} + \{\sum_{k=0}^{m_7}\binom{m_7}{k} - \binom{m_7}{0} - \binom{m_7}{1}\}$
$= 1 + n + (2^{m_1} - 1 - m_1) + (2^{m_2} - 1 - m_2) + (2^{m_3} - 1 - m_3) - (2^{m_4} - 1 - m_4) - (2^{m_5} - 1 - m_5) - (2^{m_6} - 1 - m_6) + (2^{m_7} - 1 - m_7)$
$= n + 2^{m_1} + 2^{m_2} + 2^{m_3} - 2^{m_4} - 2^{m_5} - 2^{m_6} + 2^{m_7} - m_1 - m_2 - m_3 + m_4 + m_5 + m_6 - m_7$
$= |R| + 2^{|M_1|} + 2^{|M_2|} + 2^{|M_3|} - 2^{|M_1 \cap M_2|} - 2^{|M_2 \cap M_3|} - 2^{M_1 \cap M_3} + 2^{|M_1 \cap M_2 \cap M_3|} - |M_1 \cup M_2 \cup M_3|$. □

Theorem 5.2.14. *Let R be a finite commutative ring with unity. If R has n number of maximal ideals $M_1, M_2, \cdots, M_n$, then*

$$i(\Gamma'(R)) = |R| + \sum_{i=1}^{n} 2^{|M_i|} - \sum_{1 \le i < j \le n} 2^{|M_i \cap M_j|} + \sum_{1 \le i < j < l \le n} 2^{|M_i \cap M_j \cap M_l|} - \cdots + (-1)^{n-1} 2^{\bigcap_{i=1}^{n} |M_i|} - |\bigcup_{i=1}^{n} M_i|$$

Proof. Let R be a finite commutative ring with n number of maximal ideals, denoted as $M_1, M_2, \cdots, M_n$, where $J(R) = \bigcap_{i=1}^{n} M_i$. If a and b are elements of M_i for each $i = 1, 2, \cdots, n$, then $aR + bR \neq R$, implying that a and b are not adjacent to each other.

Similarly, a, b are not adjacent for all $a, b \in M_i \cap M_j$ for $i \neq j$
a, b are not adjacent for all $a, b \in M_i \cap M_j \cap M_l$ for $i \neq j \neq l$
$\vdots$
a, b are not adjacent for all $a, b \in \bigcap_{i=1}^{n-1} M_i$
a, b are not adjacent for all $a, b \in \bigcap_{i=1}^{n} M_i$

However, if $a \in M_i \setminus \{\bigcup_{l \neq i, l=1}^{k} (M_i \cap M_l)\}$ and b is in any one of $M_j \setminus \{\bigcup_{j \neq l, l=2}^{k} (M_j \cap M_l)\}$ for $i \neq j$ and $1 \le k \le n$, then $aR + bR = R$, establishing that a and b are adjacent to each other in this scenario. Now we utilizing the set theoretical concept of $|\bigcup_{i=1}^{n} A_i| =$

$\sum\limits_{i=1}^{n}|A_i| - \sum\limits_{1\le i<j\le n}|A_i\cap A_j| + \sum\limits_{1\le i<j<k\le n}|A_i\cap A_j\cap A_k| - \cdots + (-1)^{n-1}|\bigcap\limits_{i=1}^{n}A_i|$, we have

$$i(\Gamma'(R)) = \sum_{k=0}^{|R|} i_k(\Gamma'(R))$$

$$= i_0(\Gamma'(R)) + i_1(\Gamma'(R)) + \sum_{k=2}^{|R|} i_k(\Gamma'(R))$$

$$= 1 + |R| + \sum_{i=1}^{n}\sum_{k=2}^{|M_i|}\binom{|M_i|}{k} - \sum_{1\le i<j\le n}\sum_{k=2}^{|M_i\cap M_j|}\binom{|M_i\cap M_j|}{k} +$$

$$\sum_{1\le i<j<l\le n}\sum_{k=2}^{|M_i\cap M_j\cap M_l|}\binom{|M_i\cap M_j\cap M_l|}{2} - \cdots + (-1)^{n-1}\sum_{k=2}^{|\bigcap_{i=1}^{n}M_i|}\binom{|\bigcap_{i=1}^{n}M_i|}{k}$$

$$= 1+|R|+\sum_{i=1}^{n}(\sum_{k=0}^{|M_i|}\binom{|M_i|}{k}-\binom{|M_i|}{1}-1)-\sum_{1\le i<j\le n}(\sum_{k=0}^{|M_i\cap M_j|}\binom{|M_i\cap M_j|}{k}-\binom{|M_i\cap M_j|}{1}-$$

$$1) + \sum_{1\le i<j<l\le n}(\sum_{k=0}^{|M_i\cap M_j\cap M_l|}\binom{|M_i\cap M_j\cap M_l|}{2} - \binom{|M_i\cap M_j\cap M_l|}{1} - 1) - \cdots +$$

$$(-1)^{n-1}(\sum_{k=0}^{|\bigcap_{i=1}^{n}M_i|}\binom{|\bigcap_{i=1}^{n}M_i|}{k} - \binom{|\bigcap_{i=1}^{n}M_i|}{1} - 1)$$

$$= 1 + |R| + \sum_{i=1}^{n}(2^{|M_i|} - |M_i| - 1) - \sum_{1\le i<j\le n}(2^{|M_i\cap M_j|} - |M_i\cap M_j| - 1)$$

$$+ \sum_{1\le i<j<l\le n}(2^{|M_i\cap M_j\cap M_l|} - |M_i\cap M_j\cap M_l| - 1) - \cdots + (-1)^{n-1}(2^{\bigcap_{i=1}^{n}M_i} - \bigcap_{i=1}^{n}M_i - 1)$$

$$= 1+|R|+\sum_{i=1}^{n}2^{|M_i|} - \sum_{1\le i<j\le n}2^{|M_i\cap M_j|} + \sum_{1\le i<j<l\le n}2^{|M_i\cap M_j\cap M_l|} - \cdots + (-1)^{n-1}2^{|\bigcap_{i=1}^{n}M_i|}$$

$$-\{\sum_{i=1}^{n}|M_i| - \sum_{1\le i<j\le n}|M_i\cap M_j| + \sum_{1\le i<j<l\le n}|M_i\cap M_j\cap M_l| - \cdots + (-1)^{n-1}|\bigcap_{i=1}^{n}M_i|\}$$

$$-\{\sum_{i=1}^{n}1 - \sum_{1\le i<j\le n}1 + \sum_{1\le i<j<l\le n}1 - \cdots + (-1)^{n-1}\}$$

$$= |R| + \sum_{i=1}^{n}2^{|M_i|} - \sum_{1\le i<j\le n}2^{|M_i\cap M_j|} + \sum_{1\le i<j<l\le n}2^{|M_i\cap M_j\cap M_l|} - \cdots + (-1)^{n-1}2^{\bigcap_{i=1}^{n}|M_i|} -$$

$$|\bigcup_{i=1}^{n} M_i| - \{-\binom{n}{0} + \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \cdots + (-1)^{n-1}\binom{n}{n}\}$$

$$= |R| + \sum_{i=1}^{n} 2^{|M_i|} - \sum_{1\leq i<j\leq n} 2^{|M_i\cap M_j|} + \sum_{1\leq i<j<l\leq n} 2^{|M_i\cap M_j\cap M_l|} - \cdots + (-1)^{n-1} 2^{\bigcap_{i=1}^{n}|M_i|} - |\bigcup_{i=1}^{n} M_i| \qquad \square$$

5.3 Merrifield-Simmons index of $\Gamma'(\mathbb{Z}_n)$.

In this section, we mainly characterised the Merrifield-Simmons index of co-maximal graph over the ring $\mathbb{Z}_n$.

Theorem 5.3.1. *For any positive integer n,* $i(\Gamma'(\mathbb{Z}_n))$

$$= \begin{cases} 2^{p^{m-1}} + p^{m-1}(p-1) \ if \ n = p^m \ for \ prime \ p \ and \ m \in \ \mathbb{N} \\ 2^p + p + 1 \ if \ n = 2p \ for \ prime \ p \\ 2^{p^{r-1}q^{s-1}}(2^{p^r q^{s-1}} + 2^{p^{r-1}q^s} - 1) + p^r q^s \ if \ n = p^r q^s \ for \ prime \ p, q \ and \ r, s \in \mathbb{N}. \\ p_1^r p_2^s p_3^t + 2^{p_1^{r-1}p_2^{s-1}p_3^{t-1}}(2^{p_1p_2} + 2^{p_1p_3} + 2^{p_2p_3} - 2^{p_1} - 2^{p_2} - 2^{p_3} + 1) - p_1^{r-1}p_2^{s-1}p_3^{t-1}(p_1p_2 \\ +p_1p_3 + p_2p_3 - p_3 - p_2 - p_1 + 1) \ if \ n = p_1^r p_2^s p_3^t \ for \ prime \ p_1, p_2, p_3 \ and \ r, s, t \in \mathbb{N}. \end{cases}$$

Proof. **Case I:** Let $R = \mathbb{Z}_n$, where $n = p^m$ for $m \in \mathbb{N}$. Then $|U(R)| = \phi(p^m) = p^m - p^{m-1}$ and $|R \setminus U(R)| = p^m - (p^m - p^{m-1}) = p^{m-1}$. From definition of Comaximal graph, it can be observed that every pair of vertices from $R \setminus U(R)$ are nonadjacent, each pair of vertices from $U(R)$ are adjacent and if $a \in U(R)$ and $b \in R \setminus U(R)$ then $a \leftrightarrow b$. Since $R = \mathbb{Z}_{p^m}$ is finite local ring, so by Theorem 5.2.7, we have $i(\Gamma(\mathbb{Z}_{p^m})) = p^m + 2^{p^{m-1}} - p^{m-1} = 2^{p^{m-1}} + p^{m-1}(p-1)$.

Case II: Let $R = \mathbb{Z}_{2p} for \ prime \ p$. Then $\mathbb{Z}_{2p} = \mathbb{Z}_2 \times \mathbb{Z}_p$, where $\mathbb{Z}_2$ and $\mathbb{Z}_p$ are two fields. So by Theorem 5.2.9, we have $i(\Gamma(\mathbb{Z}_{2p})) = 2^2 + 2^p + 2p - 2 - p - 1 = p + 2^p + 1$.

Case III: Suppose that $R = \mathbb{Z}_{p^r q^s}$ for prime p, q and positive integers r, s. Then $\mathbb{Z}_{p^r q^s} = \mathbb{Z}_{p^r} \times \mathbb{Z}_{q^s}$, where $\mathbb{Z}_{p^r}$ and $\mathbb{Z}_{q^s}$ are two local rings. Thus $< p >$ and $< q >$ are the unique maximal ideal of $\mathbb{Z}_{p^r}$ and $\mathbb{Z}_{q^s}$ respectively. So $M_1 =< p > \times \mathbb{Z}_{q^s}$ and

$M_2 = \mathbb{Z}_{p^r} \times < q >$ are only two maximal ideals of $\mathbb{Z}_{p^r} \times \mathbb{Z}_{q^s}$. Then $M_1 \cap M_2 =< p > \times < q >$ with $|M_1| = p^{r-1}q^s$, $|M_2| = p^r q^{s-1}$ and $|M_1 \cap M_2| = p^{r-1}q^{s-1}$. Now by Theorem 5.2.11, we have

$i(\mathbb{Z}_{p^r q^s}) = i(\mathbb{Z}_{p^r} \times \mathbb{Z}_{q^s})$

$= |R| + 2^{|M_1|} + 2^{|M_2|} - 2^{|M_1 \cap M_2|} - |M_1| - |M_2| + |M_1 \cap M_2|$

$= p^r q^s + 2^{p^{r-1}q^s} + 2^{p^r q^{s-1}} - 2^{p^{r-1}q^{s-1}} - p^{r-1}q^s - p^r q^{s-1} + p^{r-1}q^{s-1}$

$= 2^{p^{r-1}q^{s-1}}(2^{(p^r - p^{r-1})q^{s-1}} + 2^{p^{r-1}(q^r - q^{s-1})} - 1) + (p^r - p^{r-1})(q^s - q^{s-1})$

Case IV: Suppose that $R = \mathbb{Z}_{p_1^r p_2^s p_3^t}$ *for prime* p_1, p_2, p_3 *and* $r, s, t \in \mathbb{N}$. Then $\mathbb{Z}_{p_1^r p_2^s p_3^t} = \mathbb{Z}_{p_1^r} \times \mathbb{Z}_{p_2^s} \times \mathbb{Z}_{p_3^t}$, where $\mathbb{Z}_{p_1^r}, \mathbb{Z}_{p_2^s}$ and $\mathbb{Z}_{p_3^t}$ are three local rings. Thus $< p_1 >$, $< p_2 >$ and $< p_3 >$ are the unique maximal ideals of $\mathbb{Z}_{p_1^r}, \mathbb{Z}_{p_2^s}$ and $\mathbb{Z}_{p_3^t}$ respectively. So $M_1 =< p_1 > \times \mathbb{Z}_{p_2^s} \times \mathbb{Z}_{p_3^t}$, $M_2 = \mathbb{Z}_{p_1^r} \times < p_2 > \times \mathbb{Z}_{p_3^t}$ and $M_3 = \mathbb{Z}_{p_1^r} \times \mathbb{Z}_{p_2^s} \times < p_3 >$ are the only three maximal ideals of $\mathbb{Z}_{p_1^r} \times \mathbb{Z}_{p_2^s} \times \mathbb{Z}_{p_3^t}$.

Now $M_1 \cap M_2 =< p_1 > \times < p_2 > \times \mathbb{Z}_{p_3^t}$, $M_1 \cap M_3 =< p_1 > \times \mathbb{Z}_{p_2^s} \times < p_3 >$, $M_2 \cap M_3 = \mathbb{Z}_{p_2^r} \times < p_2 > \times < p_3 >$ and $M_1 \cap M_2 \cap M_3 =< p_1 > \times < p_2 > \times < p_3 >$ with $|M_1| = p_1^{r-1}p_2^s p_3^t$, $|M_2| = p_1^r p_2^{s-1} p_3^t$, $|M_3| = p_1^r p_2^s p_3^{t-1}$, $|M_1 \cap M_2| = p_1^{r-1}p_2^{s-1}p_3^t$, $|M_1 \cap M_3| = p_1^{r-1}p_2^s p_3^{t-1}$, $|M_2 \cap M_3| = p_1^r p_2^{s-1}p_3^{t-1}$ and $|M_1 \cap M_2 \cap M_3| = p_1^{r-1}p_2^{s-1}p_3^{t-1}$. Now by the Theorem 5.2.13, we have

$i(\mathbb{Z}_{p_1^r p_2^s p_3^t}) = i(\mathbb{Z}_{p_1^r} \times \mathbb{Z}_{p_2^s} \times \mathbb{Z}_{p_3^t})$

$= |R| + 2^{|M_1|} + 2^{|M_2|} + 2^{|M_3|} - 2^{|M_4|} - 2^{|M_5|} - 2^{|M_6|} + 2^{|M_7|} - m_1 - m_2 - m_3 + m_4 + m_5 + m_6 - m_7$

$= p_1^r p_2^s p_3^t + 2^{p_1^{r-1} p_2^s p_3^t} + 2^{p_1^r p_2^{s-1} p_3^t} + 2^{p_1^r p_2^s p_3^{t-1}} - 2^{p_1^{r-1} p_2^{s-1} p_3^t} - 2^{p_1^{r-1} p_2^s p_3^{t-1}} - 2^{p_1^r p_2^{s-1} p_3^{t-1}} +$
$2^{p_1^{r-1} p_2^{s-1} p_3^{t-1}} - p_1^{r-1} p_2^s p_3^t - p_1^r p_2^{s-1} p_3^t - p_1^r p_2^s p_3^{t-1} + p_1^{r-1} p_2^{s-1} p_3^t + p_1^{r-1} p_2^s p_3^{t-1} + p_1^r p_2^{s-1} p_3^{t-1} -$
$p_1^{r-1} p_2^{s-1} p_3^{t-1}$

$= p_1^r p_2^s p_3^t + 2^{p_1^{r-1} p_2^{s-1} p_3^{t-1}}(2^{p_1 p_2} + 2^{p_1 p_3} + 2^{p_2 p_3} - 2^{p_1} - 2^{p_2} - 2^{p_3} + 1) - p_1^{r-1} p_2^{s-1} p_3^{t-1}(p_1 p_2 +$
$p_1 p_3 + p_2 p_3 - p_3 - p_2 - p_1 + 1)$. $\square$

Chapter 6

Merrifield-Simmons Index of zero divisor graph over finite commutative ring

Chapter 6

Merrifield-Simmons index of $\Gamma(R)$

6.1 Introduction

Algebraic graph theory is a fascinating branch of mathematics that explores the relationships between graphs and matrices. Graphs are used to represent a wide range of systems, from social networks to computer networks, and algebraic graph theory provides powerful tools for analyzing and understanding these systems. In this chapter, we will explore the fundamental concepts of algebraic graph theory, techniques for analyzing graphs using algebraic methods, and recent developments in the field. The concept of zero-divisor graph of a commutative ring was first introduced by Beck [7]. Beck was mainly interested in coloring of zero-divisor graph of commutative rings. Throughout this chapter, we will denote R as a commutative ring with unity. The set $Z(R)$ will denote as the set of zero-divisors of R and $Z^*(R) = Z(R) \setminus \{0\}$ is the set of non-zero zero-divisors of R. The zero-divisor graph of R is defined as follows:

Definition 6.1.1. *Let R be a commutative ring. Then the zero divisor graph $\Gamma(R)$ of R with vertex set $Z^*(R)$ where two distinct vertices x and y are adjacent if and only if $xy = 0$.*

In this chapter, we first compute the Merrifield-Simmons index of the zero-divisor graph over the ring $\mathbb{Z}_n$ (the ring of integer modulo n), for $n = p^m, pq, p^2q, p^2q^2, pqr$, where p, q, r are distinct prime numbers. Further, we provide a partition of the vertex set of $\Gamma(\mathbb{Z}_{pq}[x]/<x^2>)$ and connections among the vertices so that the Merrifield-Simmons index can be computed. In the last section, we compute the Merrifield-Simmons index of zero divisor graphs of some small finite commutative rings. This paper primarily focuses on studying the Merrifield-Simmons index of the zero-divisor graph $\Gamma(R)$ for commutative ring R with identity.

6.2 Merrifield-Simmons index of zero divisor graph over ring $\mathbb{Z}_n$

In this section, we provide a comprehensive overview of the Merrifield-Simmons Index and its applications in the study of zero divisor graphs over the ring $\mathbb{Z}_n$. We explore how this index can be calculated, its significance in characterizing vertices in zero divisor graphs, and the insights it offers into the algebraic properties of $\mathbb{Z}_n$. Additionally, we discuss the potential implications of the Merrifield-Simmons Index on various algebraic questions and its role in identifying specific structural patterns within zero divisor graphs.

By delving into the Merrifield-Simmons Index in the context of $\mathbb{Z}_n$, this section aims to provide researchers and mathematicians with a deeper understanding of the interplay between algebraic structures and graph theory, further advancing our knowledge of zero divisor graphs and their relevance in the study of rings modulo n.

Lemma 6.2.1. *The Merrifield-Simmons index of the zero divisor graph $\Gamma(\mathbb{Z}_{p^3})$ is $2^{(p^2-p)}+p-1$.*

Proof. Let $R=\mathbb{Z}_{p^3}$ and $\mathcal{G}=\Gamma(\mathbb{Z}_{p^3})$. The set of zero divisors of R is $V=Z^*(R)=\{p,2p,3p,...,(p^2-1)p\}$, with cardinality p^2-1. Divide the vertex set V of $\mathcal{G}$ into two disjoint subsets: V_1 and V_2, where

$$V_1=\{k_1p : k_1=1,2,3,...,p^2-1 \ and \ p\nmid k_1\}$$
$$V_2=\{k_2p^2 : k_2=1,2,3,...,p-1\}$$

with $|V_1|=p^2-p$, $|V_2|=p-1$ such that $V=V_1\cup V_2$.

It's evident that no two elements in V_1 are adjacent, and each element in V_1 is only adjacent to elements in V_2. The elements in V_2 are all mutually adjacent, except for themselves.

To calculate the Merrifield-Simmons index of the graph $\mathcal{G}$, we focus solely on the subset V_1. Thus,

$$i(\mathcal{G}) = \sum_{k=0}^{|V|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + \sum_{k=2}^{|V|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + \sum_{k=2}^{|V_1|} i_k(\mathcal{G})$$

$$= 1 + p^2 - 1 + \sum_{k=2}^{p^2-p} \binom{p^2-p}{k} = p^2 + 2^{(p^2-p)} - 1 - p^2 + p = 2^{(p^2-p)} + p - 1. \qquad \square$$

Lemma 6.2.2. *The Merrifield-Simmons index of the zero divisor graph* $\Gamma(\mathbb{Z}_{p^4})$ *is* $2^{(p^3-p^2)}(p^2-p+1)+p-1$.

Proof. Let $R = \mathbb{Z}_{p^4}$ and $\mathcal{G} = \Gamma(\mathbb{Z}_{p^4})$. Then the set of zero divisors of R is $V = Z^*(R) = \{p, 2p, 3p, ..., (p^3-1)p\}$ with cardinality $p^3 - 1$. We can divide the vertex set V of $\mathcal{G}$ into three disjoint subsets V_1, V_2, V_3, where

$$V_1 = \{k_1 p : k_1 = 1, 2, 3, ..., p^3 - 1 \text{ and } p \nmid k_1\}$$
$$V_2 = \{k_2 p^2 : k_2 = 1, 2, 3, ..., p^2 - 1 \text{ and } p \nmid k_2\}$$
$$V_3 = \{k_3 p^3 : k_3 = 1, 2, 3, ..., p - 1\}$$

with $|V_1| = p^3 - p^2$, $|V_2| = p^2 - p$, $|V_3| = p - 1$ such that $V = V_1 \cup V_2 \cup V_3$.

It is clear from the Figure 6.1 that no two elements in V_1 are adjacent, and each element in V_1 is only adjacent to elements in V_3. Similarly, each element in V_2 is adjacent to every other element in V_2 except itself, and is also adjacent to every element in V_3. Each element in V_3 is adjacent to all other elements in V_3 except itself, and to all elements in both V_2 and V_1. If we select one vertex from V_2 and one vertex from V_1, there are $\binom{|V_2|}{1} \times \binom{|V_1|}{1}$ distinct pairs of vertices that are non-adjacent. If we select one vertices from V_2 and select two vertices from V_1, there are $\binom{|V_2|}{1} \times \binom{|V_1|}{2}$ distinct 3-tuple pairs where the vertices are pairwise non-adjacent. Continuing this process, if we select one vertex from V_2 and choose all possible vertices from V_1, we obtain $\binom{|V_2|}{1} \sum_{k=1}^{|V_1|} \binom{|V_1|}{k}$, which gives the total number of all possible independent subsets formed by using one element from V_2 and all possible elements from V_1.

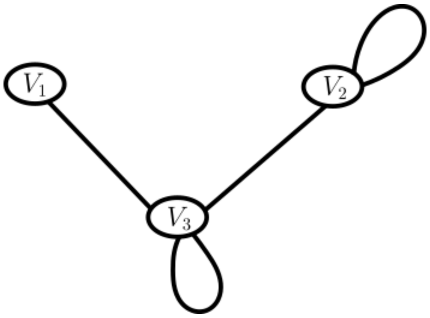

Figure 6.1: Adjacency structure(Vertex set) of the graph $\Gamma(\mathbb{Z}_{p^4})$

Now, to calculate Merrifield-Simmons index of the graph $\mathcal{G}$, we focus solely on the sets V_1 and V_2. Therefore,

$$\begin{aligned}
i(\mathcal{G}) &= \sum_{k=0}^{|V|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + \sum_{k=2}^{|V|} i_k(\mathcal{G}) \\
&= i_0(\mathcal{G}) + i_1(\mathcal{G}) + \sum_{k=2}^{|V_1|} i_k(\mathcal{G}) + \binom{|V_2|}{1} \sum_{k=1}^{|V_1|} \binom{|V_1|}{k} \\
&= 1 + p^3 - 1 + \sum_{k=2}^{p^3-p^2} \binom{p^3-p^2}{k} + \binom{p^2-p}{1} \sum_{k=1}^{p^3-p^2} \binom{p^3-p^2}{k} \\
&= p^3 + 2^{p^3-p^2} - (p^3-p^2) - 1 + (p^2-p)(2^{p^3-p^2}-1) \\
&= 2^{(p^3-p^2)}(p^2-p+1) + p - 1. \qquad \square
\end{aligned}$$

Lemma 6.2.3. *The Merrifield-Simmons index of the zero divisor graph $\Gamma(\mathbb{Z}_{p^5})$ is $2^{(p^4-p^2)} + (p^2-p)2^{(p^4-p^3)} + p - 1$.*

Proof. Let $R = \mathbb{Z}_{p^5}$ and $\mathcal{G} = \Gamma(\mathbb{Z}_{p^5})$. Then the set of zero divisors of R is $V = Z^*(R) = \{p, 2p, 3p, \cdots, (p^4-1)p\}$ with cardinality $p^4 - 1$. We can partition the vertex set V of $\mathcal{G}$ into four disjoint subsets: V_1, V_2, V_3, and V_4, where

$$\begin{aligned}
V_1 &= \{k_1 p : k_1 = 1, 2, 3, \cdots, p^4 - 1 \textit{ and } p \nmid k_1\} \\
V_2 &= \{k_2 p^2 : k_2 = 1, 2, 3, \cdots, p^3 - 1 \textit{ and } p \nmid k_2\} \\
V_3 &= \{k_3 p^3 : k_3 = 1, 2, 3, \cdots, p^2 - 1 \textit{ and } p \nmid k_3\} \\
V_4 &= \{k_4 p^4 : k_4 = 1, 2, 3, \cdots, p - 1\}
\end{aligned}$$

with $|V_1| = p^4 - p^3$, $|V_2| = p^3 - p^2$, $|V_3| = p^2 - p$, $|V_4| = p - 1$ such that $V = V_1 \cup V_2 \cup V_3 \cup V_4$.

It's easy from the Figure 6.2 to see that no two elements in V_1 are adjacent, and each element in V_1 is adjacent only to the elements in V_4. Similarly, no two elements in V_2 are adjacent, and each element in V_2 is adjacent only to the elements in V_3 and V_4. Each element in V_3 is adjacent to all other elements in V_3 except itself, as well as to all elements in V_2 and V_4. Finally, each element in V_4 is adjacent to all other elements in V_4 except itself, as well as to all elements in V_3, V_2, and V_1. In fact no two elements in $V_1 \cup V_2$ are adjacent.

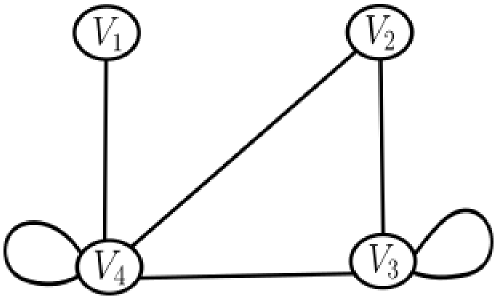

Figure 6.2: Adjacency structure(Vertex set) of the graph $\Gamma(\mathbb{Z}_{p^5})$

If we select one vertex from V_3 and one from V_1, there are $\binom{|V_3|}{1} \times \binom{|V_1|}{1}$ distinct pairs of vertices that are non-adjacent. If we select one vertex from V_3 and two vertices from V_1, there are $\binom{|V_3|}{1} \times \binom{|V_1|}{2}$ distinct 3-tuple pairs where the vertices are pairwise non-adjacent. Continuing in this manner, if we select one vertex from V_3 and all possible vertices from V_1, we obtain $\binom{|V_3|}{1}\sum_{k=1}^{|V_1|}\binom{|V_1|}{k}$, which gives the total number of all possible independent subsets formed by using one element from V_3 and all possible elements from V_1.

Therefore, the Merrifield-Simmons index of the graph $\mathcal{G}$ is calculated as

$$\begin{aligned}
i(\mathcal{G}) &= \sum_{k=0}^{|V|} i_k(\mathcal{G}) = i_0(\mathcal{G}) + i_1(\mathcal{G}) + \sum_{k=2}^{|V|} i_k(\mathcal{G}) \\
&= i_0(\mathcal{G}) + i_1(\mathcal{G}) + \sum_{k=2}^{|V_1 \cup V_2|} i_k(\mathcal{G}) + \binom{|V_3|}{1}\sum_{k=1}^{|V_1|} i_k(\mathcal{G}) \\
&= 1 + p^4 - 1 + \sum_{k=2}^{p^4-p^2}\binom{p^4-p^2}{k} + \binom{p^2-p}{1}\sum_{k=1}^{p^4-p^3}\binom{p^4-p^3}{k} \\
&= p^4 + 2^{(p^4-p^2)} - (p^4-p^2+1) + (p^2-p)(2^{(p^4-p^3)} - 1)
\end{aligned}$$

$= 2^{(p^4-p^2)} + (p^2-p)2^{(p^4-p^3)} + p - 1.$ $\square$

Theorem 6.2.4. *Let $R = \mathbb{Z}_{p^n}$ with p prime and $n(>3) \in \mathbb{N}$. Then the Merrifield-Simmons index of the zero divisor graph $\Gamma(\mathbb{Z}_{p^n})$ is $i(\Gamma(\mathbb{Z}_{p^n}))$*

$$= \begin{cases} (p^{\lceil \frac{n}{2} \rceil} - p^{\lceil \frac{n}{2} \rceil - 1} + 1)2^{p^{(n-1)} - p^{\lceil \frac{n}{2} \rceil}} + (p^{\lceil \frac{n}{2} \rceil - 1} - p^{\lceil \frac{n}{2} \rceil - 2})2^{p^{(n-1)} - p^{(\lceil \frac{n}{2} \rceil + 1)}} \\ +(p^{\lceil \frac{n}{2} \rceil - 2} - p^{\lceil \frac{n}{2} \rceil - 3})2^{p^{(n-1)} - p^{(\lceil \frac{n}{2} \rceil + 2)}} + \cdots + (p^2-p)2^{p^{(n-1)} - p^{(n-2)}} + p - 1, \quad \textit{if } n \textit{ is even} \\ 2^{p^{n-1} - p^{\lceil \frac{n}{2} \rceil - 1}} + (p^{\lceil \frac{n}{2} \rceil - 1} - p^{\lceil \frac{n}{2} \rceil - 2})2^{p^{n-1} - p^{\lceil \frac{n}{2} \rceil}} + (p^{\lceil \frac{n}{2} \rceil - 2} - p^{\lceil \frac{n}{2} \rceil - 3})2^{p^{n-1} - p^{\lceil \frac{n}{2} \rceil + 1}} \\ + \cdots + (p^2-p)2^{p^{n-1} - p^{n-2}} + p - 1, \quad \textit{if } n \textit{ is odd} \end{cases} .$$

Proof. Let $R = \mathbb{Z}_{p^n}$ with p prime and $n(>3) \in \mathbb{N}$. Then the set of non-zero zero divisors of R is $Z^*(R) = \{p, 2p, 3p, \cdots, (p^{n-1}-1)p\}$ with cardinality $p^{n-1} - 1$.

Now we rewrite $Z^*(R) = V_1 \cup V_2 \cup \cdots \cup V_{n-1}$, where

$V_1 = \{k_1 p : k_1 = 1, 2, 3, \cdots, p^{n-1} - 1 \textit{ and } p \nmid k_1\}$, with $|V_1| = p^{n-1} - p^{n-2}$

$V_2 = \{k_2 p^2 : k_2 = 1, 2, 3, \cdots, p^{n-2} - 1 \textit{ and } p \nmid k_2\}$, with $|V_2| = p^{n-2} - p^{n-3}$

$V_3 = \{k_3 p^3 : k_2 = 1, 2, 3, \cdots, p^{n-3} - 1 \textit{ and } p \nmid k_3\}$, with $|V_3| = p^{n-3} - p^{n-4}$

$\vdots$

$V_i = \{k_i p^i : k_i = 1, 2, 3, \cdots, p^{n-i} - 1 \textit{ and } p \nmid k_i\}$, with $|V_i| = p^{n-i} - p^{n-(i+1)}$

$\vdots$

$V_{n-1} = \{k_{n-1} p^{n-1} : k_{n-1} = 1, 2, 3, \cdots, p - 1\}$, with $|V_{n-1}| = p - 1$.

Case-1: Let n be an even number, i.e., $n = 2m$ for some $m \in \mathbb{N}$. Then, $\lceil \frac{n}{2} \rceil = m$.

Observations:

i) No two elements of V_1 are adjacent, and each element in V_1 is adjacent only to the elements in V_{n-1}.

ii) No two elements of V_2 are adjacent, and each element in V_2 is adjacent to the elements in V_{n-2} and V_{n-1}.

iii) No two elements of V_3 are adjacent, and each element in V_3 is adjacent to the elements of V_{n-3}, V_{n-2}, and V_{n-1}.

iv) In general, no two elements of V_i, where $i < m$, are adjacent, and each element in V_i is adjacent to the elements in V_{n-j}, where $j = 1, 2, 3, \cdots, i$.

v) Each element in V_k, where $k \geq m$, is adjacent to every element in V_k except itself, and also to every elements in V_{n-l}, where $l = 1, 2, 3, \cdots, m-1$, as well as to every element in V_{n-k}.

vi) No two elements of $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$ are adjacent.

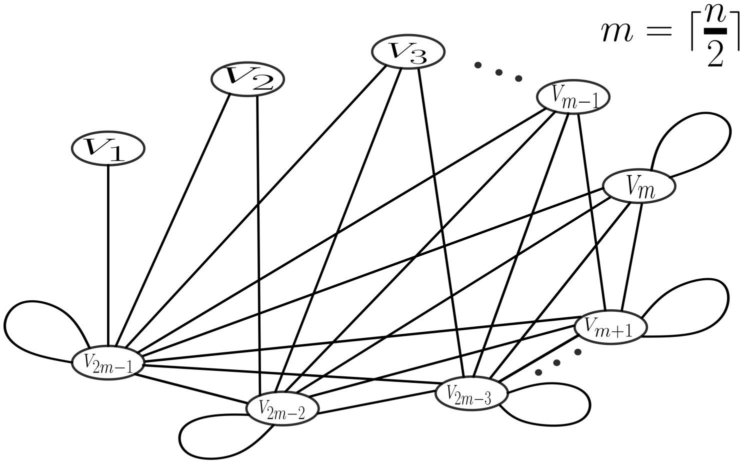

Figure 6.3: Adjacency structure(Vertex set) of the graph $\Gamma(\mathbb{Z}_{p^n})$, when n is even

vii) If we select one vertex from V_m and one vertex from $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$, we obtain $\binom{|V_m|}{1} \times \binom{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|}{1}$ distinct pairs of vertices that are non-adjacent. If we select one vertex from V_m and two vertices from $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$, we obtain $\binom{|V_m|}{1} \times \binom{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|}{2}$ distinct 3-tuples of pairwise non-adjacent vertices. Continuing in this manner, if we select one vertex from V_m and all possible vertices from $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$, we get $\binom{|V_m|}{1} \displaystyle\sum_{k=1}^{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|} \binom{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|}{k}$, which gives the total cardinality of all possible independent subsets formed by one element of V_m and all possible elements of $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$.

viii) Similarly, if we select one vertex from V_{m+1} and all possible vertices from $V_1 \cup V_2 \cup \cdots \cup V_{m-2}$, we obtain $\binom{|V_{m+1}|}{1} \displaystyle\sum_{k=1}^{|V_1 \cup V_2 \cup \cdots \cup V_{m-2}|} \binom{|V_1 \cup V_2 \cup \cdots \cup V_{m-2}|}{k}$, which gives the total cardinality of all possible independent subsets formed by one element of V_{m+1} and all possible elements of $V_1 \cup V_2 \cup \cdots \cup V_{m-2}$.

ix) Continuing this process, if we select one vertex from V_{2m-2} and all possible vertices from V_1, we get $\binom{|V_{2m-2}|}{1}\sum_{k=1}^{|V_1|}\binom{|V_1|}{k}$, which gives the total cardinality of all possible independent subsets formed by one element of V_{2m-2} and all possible elements of V_1.

x) Also, $|V_1 \cup V_2 \cup \cdots \cup V_{m-1}| = p^{2m-1} - p^m$ and $|V_m| = p^m - p^{(m-1)}$
$|V_1 \cup V_2 \cup \cdots \cup V_{m-2}| = p^{2m-1} - p^{(m+1)}$ and $|V_{m+1}| = p^{(m-1)} - p^{(m-2)}$
$\vdots$
$|V_1| = p^{2m-1} - p^{2m-2}$ and $|V_{2m-2}| = p^2 - p$

Therefore, the Merrifield-Simmons index of the graph $\Gamma(R)$ is computed as follows:

$$i(\Gamma(R)) = \sum_{k=0}^{|Z^*(R)|} i_k(\Gamma(R)) = i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|Z^*(R)|} i_k(\Gamma(R))$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V_1\cup V_2\cup\cdots\cup V_{m-1}|} i_k(\Gamma(R)) + \binom{|V_m|}{1}\sum_{k=1}^{|V_1\cup V_2\cup\cdots\cup V_{m-1}|} i_k(\Gamma(R))$$

$$+\binom{|V_{m+1}|}{1}\sum_{k=1}^{|V_1\cup V_2\cup\cdots\cup V_{m-2}|} i_k(\Gamma(R)) + \cdots + \binom{|V_{2m-2}|}{1}\sum_{k=1}^{|V_1|}\binom{|V_1|}{k}$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{(p^{(2m-1)}-p^m)}\binom{p^{(2m-1)}-p^m}{k} + \binom{p^m-p^{(m-1)}}{1}\sum_{k=1}^{(p^{(2m-1)}-p^m)}\binom{p^{(2m-1)}-p^m}{k}$$

$$+\binom{p^{(m-1)}-p^{(m-2)}}{1}\sum_{k=1}^{(p^{(2m-1)}-p^{(m+1)})}\binom{p^{(2m-1)}-p^{(m+1)}}{k} + \cdots$$

$$+\binom{p^2-p}{1}\sum_{k=1}^{p^{(2m-1)}-p^{(2m-2)}}\binom{p^{(2m-1)}-p^{(2m-2)}}{k}$$

$$= 1 + p^{2m-1} - 1 + (2^{p^{(2m-1)}-p^m} - p^{(2m-1)} + p^m - 1) + (p^m - p^{(m-1)})(2^{p^{(2m-1)}-p^m} - 1) +$$
$$(p^{(m-1)} - p^{(m-2)})(2^{p^{(2m-1)}-p^{(m+1)}} - 1) + \cdots + (p^2 - p)(2^{p^{(2m-1)}-p^{(2m-2)}} - 1)$$

$$= 2^{p^{(2m-1)}-p^m} + (p^m - p^{m-1})(2^{p^{(2m-1)}-p^m} - 1) + (p^{m-1} - p^{m-2})(2^{p^{(2m-1)}-p^{m+1}} - 1) + \cdots +$$
$$(p^2 - p)(2^{p^{(2m-1)}-p^{(2m-2)}} - 1) + p^{(2m-1)} + p^m - p^{(2m-1)} - 1$$

$$= (p^m - p^{m-1} + 1)2^{p^{(2m-1)}-p^m} + (p^{m-1} - p^{m-2})2^{p^{(2m-1)}-p^{(m+1)}} + (p^{m-2} - p^{m-3})2^{p^{(2m-1)}-p^{(m+2)}} +$$
$$\cdots + (p^2 - p)2^{p^{(2m-1)}-p^{(2m-2)}} + p - 1$$

$$= (p^{\lceil\frac{n}{2}\rceil} - p^{\lceil\frac{n}{2}\rceil-1} + 1)2^{p^{(n-1)}-p^{\lceil\frac{n}{2}\rceil}} + (p^{\lceil\frac{n}{2}\rceil-1} - p^{\lceil\frac{n}{2}\rceil-2})2^{p^{(n-1)}-p^{(\lceil\frac{n}{2}\rceil+1)}} +$$
$$(p^{\lceil\frac{n}{2}\rceil-2} - p^{\lceil\frac{n}{2}\rceil-3})2^{p^{(n-1)}-p^{(\lceil\frac{n}{2}\rceil+2)}} + \cdots + (p^2 - p)2^{p^{(n-1)}-p^{(n-2)}} + p - 1$$

Case-2: Let n be an odd number, i.e., $n = 2m+1$ for $m \in \mathbb{N}$. Then, $\lceil\frac{n}{2}\rceil = m+1$.

Using similar arguments as in Case-1, one observe that:

i) No two vertices in $V_1 \cup V_2 \cup \cdots \cup V_m$ are adjacent.

ii) If we select one vertex from V_{m+1} and one vertex from $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$, the number of distinct pairs of non-adjacent vertices is given by $\binom{|V_{m+1}|}{1} \times \binom{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|}{1}$. For choosing one vertex from V_{m+1} and two vertices from $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$, the number of 3-tuples of pairwise non-adjacent vertices is $\binom{|V_{m+1}|}{1} \times \binom{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|}{2}$. In general, choosing one vertex from V_{m+1} and any number of vertices from $V_1 \cup V_2 \cup \cdots \cup V_{m-1}$, the total number of independent subsets formed is

$$\binom{|V_{m+1}|}{1} \sum_{k=1}^{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|} \binom{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|}{k}.$$

iii) Similarly, selecting one vertex from V_{m+2} and any number of vertices from $V_1 \cup V_2 \cup \cdots \cup V_{m-2}$, the total number of independent subsets formed is

$$\binom{|V_{m+2}|}{1} \sum_{k=1}^{|V_1 \cup V_2 \cup \cdots \cup V_{m-2}|} \binom{|V_1 \cup V_2 \cup \cdots \cup V_{m-2}|}{k}.$$

iv) Continuing this process, choosing one vertex from V_{2m-1} and choose all possible vertices from V_1, the total number of independent subsets formed is $\binom{|V_{2m-1}|}{1} \sum_{k=1}^{|V_1|} \binom{|V_1|}{k}$.

v) Also, $|V_1 \cup V_2 \cup \cdots \cup V_m| = p^{2m} - p^m$ and $|V_m| = p^{m+1} - p^m$

$|V_1 \cup V_2 \cup \cdots \cup V_{m-1}| = p^{2m} - p^{(m+1)}$ and $|V_{m+1}| = p^m - p^{(m-1)}$

$|V_1 \cup V_2 \cup \cdots \cup V_{m-2}| = p^{2m} - p^{(m+2)}$ and $|V_{m+2}| = p^{(m-1)} - p^{(m-2)}$

$\vdots$

$|V_1| = p^{2m} - p^{2m-1}$ and $|V_{2m-1}| = p^2 - p$.

Therefore, the Merrifield-Simmons index of the graph $\Gamma(R)$ is calculated as follows:

$$i(\Gamma(R)) = \sum_{k=0}^{|Z^*(R)|} i_k(\Gamma(R)) = i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|Z^*(R)|} i_k(\Gamma(R))$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V_1 \cup V_2 \cup \cdots \cup V_m|} i_k(\Gamma(R)) + \binom{|V_{m+1}|}{1} \sum_{k=1}^{|V_1 \cup V_2 \cup \cdots \cup V_{m-1}|} i_k(\Gamma(R))$$

$$+ \binom{|V_{m+2}|}{1} \sum_{k=1}^{|V_1 \cup V_2 \cup \cdots \cup V_{m-2}|} i_k(\Gamma(R)) + \cdots + \binom{|V_{2m-1}|}{1} \sum_{k=1}^{|V_1|} \binom{|V_1|}{k}$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{(p^{2m}-p^m)} \binom{p^{2m}-p^m}{k} + \binom{p^m-p^{m-1}}{1} \sum_{k=1}^{(p^{2m}-p^{m+1})} \binom{p^{2m}-p^{m+1}}{k}$$

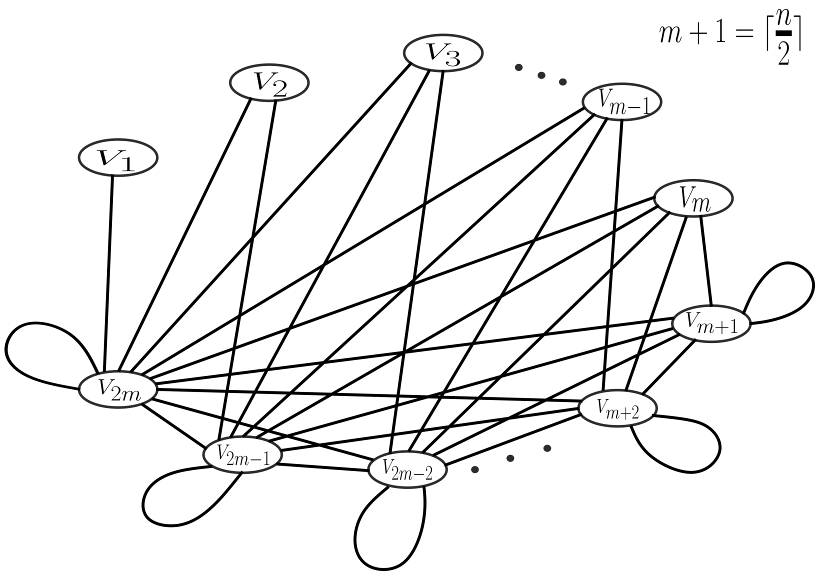

Figure 6.4: Adjacency structure(Vertex set) of the graph $\Gamma(\mathbb{Z}_{p^n})$, when n is odd

$$+\tbinom{p^{m-1}-p^{m-2}}{1}\sum_{k=1}^{(p^{2m}-p^{m+2})}\binom{p^{2m}-p^{m+2}}{k}+\cdots+\tbinom{p^2-p}{1}\sum_{k=1}^{p^{2m}-p^{2m-1}}\binom{p^{2m}-p^{2m-1})}{k}$$

$$=1+p^{2m}-1+(2^{p^{2m}-p^{m}}-p^{2m}+p^m-1)+(p^m-p^{m-1})(2^{p^{2m}-p^{m+1}}-1)$$

$$+(p^{m-1}-p^{m-2})(2^{p^{2m}-p^{m+2}}-1)+\cdots+(p^2-p)(2^{p^{2m}-p^{2m-1}}-1)$$

$$=2^{p^{2m}-p^{m}}+(p^m-p^{m-1})2^{p^{2m}-p^{m+1}}+(p^{m-1}-p^{m-2})2^{p^{2m}-p^{m+2}}+\cdots+(p^2-p)2^{p^{2m}-p^{2m-1}}+$$

$$p-1$$

$$=2^{p^{n-1}-p^{\lceil\frac{n}{2}\rceil-1}}+(p^{\lceil\frac{n}{2}\rceil-1}-p^{\lceil\frac{n}{2}\rceil-2})2^{p^{n-1}-p^{\lceil\frac{n}{2}\rceil}}+(p^{\lceil\frac{n}{2}\rceil-2}-p^{\lceil\frac{n}{2}\rceil-3})2^{p^{n-1}-p^{\lceil\frac{n}{2}\rceil+1}}+\cdots+$$

$$(p^2-p)2^{p^{n-1}-p^{n-2}}+p-1$$ □

Theorem 6.2.5. *The Merrifield-Simmons index of the zero divisor graph $\Gamma(\mathbb{Z}_{pq})$ is $2^{p-1}+2^{q-1}-1$, where p and q are distinct primes.*

Proof. Let $R=\mathbb{Z}_{pq}$, where p and q are distinct prime numbers. Then the set of non-zero zero divisors of R is given by $Z^*(R)=\{p,2p,3p,\cdots,(q-1)p,q,2q,3q,\cdots,(p-1)q\}$, with $Z^*(R)=p+q-2$.

We can partition the vertex set of $\Gamma(R)$ as $Z^*(R)=V_1\cup V_2$, where $V_1=\{p,2p,3p,\cdots,(q-1)p\}$ and $V_2=\{q,2q,3q,\cdots,(p-1)q\}$. Here, $|V_1|=q-1$ and $|V_2|=p-1$. In this graph, no two elements of V_1 are adjacent, and every element of V_1 is adjacent to every element of V_2. Similarly, no two elements of V_2 are adjacent, and every element of V_2 is adjacent to element of V_1.

Therefore, $i(\Gamma(R)) = \sum_{k=0}^{|Z^*(R)|} i_k(\Gamma(R)) = i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|Z^*(R)|} i_k(\Gamma(R))$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V_1|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_2|} i_k(\Gamma(R))$$
$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{q-1} i_k(\Gamma(R)) + \sum_{k=2}^{p-1} i_k(\Gamma(R))$$
$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{q-1} \binom{q-1}{k} + \sum_{k=2}^{p-1} \binom{p-1}{k}$$
$$= 1 + p + q - 2 + 2^{q-1} - (q-1) - 1 + 2^{p-1} - (p-1) - 1$$
$$= 2^{p-1} + 2^{q-1} - 1.$$
□

Theorem 6.2.6. *The Merrifield-Simmons index of the graph* $\Gamma(\mathbb{Z}_{p^2q})$ *is* $2^{pq+p^2-2p-q+1} + 2^{pq-p} - 2^{pq-p-q+1} + (p-1)2^{p^2-p}$, *where* p, q *are distinct prime numbers.*

Proof. Let $R = \mathbb{Z}_{p^2q}$, where p, q are distinct prime numbers. The set of non-zero zero divisors of R is given by $V = Z^*(R) = V_1 \cup V_2 \cup V_3 \cup V_4$, where:

$V_1 = \{k_1 p : k_1 = 1, 2, 3, \cdots, (pq-1) \textit{ and } p \nmid k_1,\ q \nmid k_1\}$, with $|V_1| = pq - p - q + 1$

$V_2 = \{k_2 p^2 : k_2 = 1, 2, 3, \cdots, q-1\}$, with $|V_2| = q - 1$

$V_3 = \{k_3 q : k_3 = 1, 2, 3, \cdots, p^2 - 1 \textit{ and } p \nmid k_3\}$, with $|V_3| = p^2 - p$

$V_4 = \{k_4 pq : k_4 = 1, 2, 3, \cdots, p-1\}$, with $|V_4| = p - 1$

The cardinality of V is $p^2q - \phi(p^2q) - 1 = pq + p^2 - p - 1$.

We observe the following:

i) No two elements of $V_1 \cup V_2$ are adjacent.

ii) No two elements of $V_1 \cup V_3$ are adjacent.

iii) For choosing one vertex from V_4 and one vertex from V_3, there are $\binom{|V_4|}{1} \times \binom{|V_3|}{1}$ distinct pairs of non-adjacent vertices. If we choose one vertex from V_4 and two vertices from V_3, there are $\binom{|V_4|}{1} \times \binom{|V_3|}{2}$ distinct 3-tuples of pairwise non-adjacent vertices. In general, for choosing one vertex from V_4 and any number of vertices from V_3, the total number of independent subsets is $\binom{|V_4|}{1} \sum_{k=1}^{|V_3|} \binom{|V_3|}{k}$.

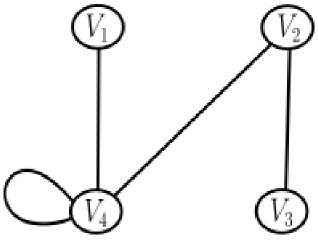

Figure 6.5: Adjacency structure(Vertex set) of the graph $\Gamma(\mathbb{Z}_{p^2q})$

Therefore, $i(\Gamma(R)) = \sum_{k=0}^{|V|} i_k(\Gamma(R))$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V|} i_k(\Gamma(R))$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V_1 \cup V_2|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_1 \cup V_3|} i_k(\Gamma(R)) - \sum_{k=2}^{|V_1|} i_k(\Gamma(R)) + \binom{|V_4|}{1} \sum_{k=1}^{|V_3|} \binom{|V_3|}{k}$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{pq-p} \binom{pq-p}{k} + \sum_{k=2}^{pq+p^2-2p-q+1} \binom{pq+p^2-2p-q+1}{k}$$

$$- \sum_{k=2}^{pq-p-q+1} \binom{pq-p-q+1}{k} + \binom{p-1}{1} \sum_{k=1}^{p^2-p} \binom{p^2-p}{k}$$

$$= 1 + pq + p^2 - p - 1 + (2^{pq-p} - pq + p - 1) + (2^{pq+p^2-2p-q+1} - pq - p^2 + 2p + q - 1 - 1) - (2^{pq-p-q+1} - pq + p + q - 1 - 1) + (p-1)(2^{p^2-p} - 1)$$

$$= 2^{pq+p^2-2p-q+1} + 2^{pq-p} - 2^{pq-p-q+1} + (p-1)2^{p^2-p}. \qquad \square$$

Theorem 6.2.7. *The Merrifield-Simmons index of the graph* $\Gamma(\mathbb{Z}_{p^2q^2})$ *is* $(pq-p-q+2)2^{p^2q+pq^2-2pq-p^2+p} + 2^{p^2q+pq^2-2pq-q^2+q} - 2^{(p+q)(pq-p-q+1)} + (pq-p-q+1)2^{p^2-p} + (q-1)2^{pq^2-pq} + (p-1)2^{p^2q-pq} - pq + p + q - 1$, *where* p, q *are distinct prime numbers.*

Proof. Let $R = \mathbb{Z}_{p^2q^2}$, where p, q are distinct prime numbers. Then the set of non-zero zero divisors of R is $V = Z^*(R) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7$, where

$V_1 = \{k_1p : k_1 = 1, 2, 3, \cdots, pq^2 - 1 \ and \ p \nmid k_1, \ q \nmid k_1\}$, with $|V_1| = pq^2 - pq - q^2 + q$

$V_2 = \{k_2p^2 : k_2 = 1, 2, 3, \cdots, q^2 - 1 \ and \ q \nmid k_2\}$, with $|V_2| = q^2 - q$

$V_3 = \{k_3q : k_3 = 1, 2, 3, \cdots, p^2q - 1 \ and \ p \nmid k_3, \ q \nmid k_3\}$, with $|V_3| = p^2q - pq - p^2 + p$

$V_4 = \{k_4q^2 : k_4 = 1, 2, 3, \cdots, p^2 - 1 \ and \ p \nmid k_4\}$, with $|V_4| = p^2 - p$

$V_5 = \{k_5pq : k_5 = 1, 2, 3, \cdots, pq - 1 \ and \ p \nmid k_5, q \nmid k_5\}$, with $|V_5| = pq - p - q + 1$

$V_6 = \{k_6p^2q : k_6 = 1, 2, 3, \cdots, q - 1\}$, with $|V_6| = q - 1$

$V_7 = \{k_7pq^2 : k_7 = 1, 2, 3, \cdots, p-1\}$, with $|V_7| = p-1$

The cardinality of V is $p^2q^2 - \phi(p^2q^2) - 1 = p^2q + pq^2 - pq - 1$.

We observe the following:

i) No two elements of $V_1 \cup V_2 \cup V_3$ and V_4 are adjacent.

ii) Every pair of elements within V_i $(i = 5, 6, 7)$ are adjacent.

iii) For selecting one vertex from V_4 and one vertex from $V_1 \cup V_3$, there are $\binom{|V_4|}{1} \times \binom{|V_1 \cup V_3|}{1}$ distinct pairs of non-adjacent vertices. For choosing two vertices from V_4 and one vertex from $V_1 \cup V_3$, or one vertex from V_4 and two vertices from $V_1 \cup V_3$, the number of 3-tuples of pairwise non-adjacent vertices is $\binom{|V_4|}{2} \times \binom{|V_1 \cup V_3|}{1} + \binom{|V_4|}{1} \times \binom{|V_1 \cup V_3|}{2}$. In general, the total number of independent subsets formed using vertices from V_4 and $V_1 \cup V_3$ is given by $\sum\limits_{k=1}^{|V_4|} \binom{|V_4|}{k} \sum\limits_{k=1}^{|V_1 \cup V_3|} \binom{|V_1 \cup V_3|}{k}$.

iv) For choosing one vertex from V_5 and any vertices from V_4, the total number of independent subsets is $\binom{|V_5|}{1} \sum\limits_{k=1}^{|V_4|} \binom{|V_4|}{k}$.

v) For choosing one vertex from V_5 and any vertices from $V_1 \cup V_2 \cup V_3$, the total number of independent subsets is $\binom{|V_5|}{1} \sum\limits_{k=1}^{|V_1 \cup V_2 \cup V_3|} \binom{|V_1 \cup V_2 \cup V_3|}{k}$.

vi) For choosing one vertex from V_6 and any vertices from $V_1 \cup V_2$, the total number of independent subsets is $\binom{|V_6|}{1} \sum\limits_{k=1}^{|V_1 \cup V_2|} \binom{|V_1 \cup V_2|}{k}$.

vii) For choosing one vertex from V_7 and any vertices from $V_3 \cup V_4$, the total number of independent subsets is $\binom{|V_7|}{1} \sum\limits_{k=1}^{|V_3 \cup V_4|} \binom{|V_3 \cup V_4|}{k}$.

Therefore, the Merrifield-Simmons index of the graph $\Gamma(R)$ is given by

$$i(\Gamma(R)) = \sum_{k=0}^{|V|} i_k(\Gamma(R)) = i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V|} i_k(\Gamma(R))$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V_1 \cup V_2 \cup V_3|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_4|} i_k(\Gamma(R)) + \sum_{k=1}^{|V_4|} \binom{|V_4|}{k} \sum_{k=1}^{|V_1 \cup V_3|} \binom{|V_1 \cup V_3|}{k}$$

$$+ \binom{|V_5|}{1} \left(\sum_{k=1}^{|V_4|} \binom{|V_4|}{k} + \sum_{k=1}^{|V_1 \cup V_2 \cup V_3|} \binom{|V_1 \cup V_2 \cup V_3|}{k} \right) + \binom{|V_6|}{1} \sum_{k=1}^{|V_1 \cup V_2|} \binom{|V_1 \cup V_2|}{k} +$$

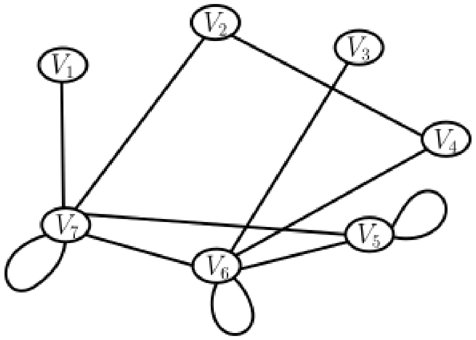

Figure 6.6: Adjacency structure(Vertex set) of the graph $\Gamma(\mathbb{Z}_{p^2q^2})$

$$\tbinom{|V_7|}{1}\sum_{k=1}^{|V_3\cup V_4|}\binom{|V_3\cup V_4|}{k}$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{p^2q+pq^2-2pq-p^2+p}\binom{p^2q+pq^2-2pq-p^2+p}{k} + \sum_{k=2}^{p^2-p}\binom{p^2-p}{k}$$

$$+\sum_{k=1}^{p^2-p}\binom{p^2-p}{k}\sum_{k=1}^{p^2q+pq^2-2pq-p^2-q^2+p+q}\binom{p^2q+pq^2-2pq-p^2-q^2+p+q}{k}$$

$$+\tbinom{pq-p-q+1}{1}(\sum_{1}^{p^2-p}\binom{p^2-p}{k} + \sum_{k=1}^{p^2q+pq^2-2pq-p^2+p}\binom{p^2q+pq^2-2pq-p^2+p}{k})$$

$$+\tbinom{q-1}{1}\sum_{k=1}^{pq^2-pq}\binom{pq^2-pq}{k}+\tbinom{p-1}{1}\sum_{k=1}^{p^2q-pq}\binom{p^2q-pq}{k}$$

$= 1 + p^2q + pq^2 - pq - 1 + [2^{p^2q+pq^2-2pq-p^2+p} - p^2q - pq^2 + 2pq + p^2 - p - 1] + [2^{p^2-p} - p^2 + p - 1] + [2^{p^2-p} - 1][2^{p^2q+pq^2-2pq-p^2-q^2+p+q} - 1] + (pq - p - q + 1)[2^{p^2-p} - 1 + 2^{p^2q+pq^2-2pq-p^2+p} - 1] + (q - 1)[2^{pq^2-pq} - 1] + (p - 1)[2^{p^2q-pq} - 1]$

$= (pq - p - q + 2)2^{p^2q+pq^2-2pq-p^2+p} + 2^{p^2q+pq^2-2pq-q^2+q} - 2^{(p+q)(pq-p-q+1)} + (pq - p - q + 1)2^{p^2-p} + (q - 1)2^{pq^2-pq} + (p - 1)2^{p^2q-pq} - pq + p + q - 1.$ □

Theorem 6.2.8. *Let $R = \mathbb{Z}_{pqr}$, where p, q, r are distinct odd prime numbers. Then the Merrifield-Simmons index of the graph $\Gamma(\mathbb{Z}_{pqr})$ is $2^{pq+pr-p-q-r+1}+2^{pr+qr-p-q-r+1}+2^{pq+qr-p-q-r+1} - 2^{qr-q-r+1} - 2^{pr-p-r+1} - 2^{pq-p-q+1} + 1$.*

Proof. Let $R = \mathbb{Z}_{pqr}$, where p, q, r are distinct prime numbers. Then the set of non-zero zero divisors of R is $V = Z^*(R) = \{p, 2p, 3p, \cdots, (qr-1)p, q, 2q, 3q, \cdots, (pr-1)q, r, 2r, 3r, \cdots, (pq-1)r\}$ with a cardinality of $pqr-\phi(pqr)-1 = pq+qr+pr-p-q-r$.

We can decompose the vertex set V of $\Gamma(R)$ into subsets as follows: $V = V_1 \cup V_2 \cup$

$V_3 \cup V_4 \cup V_5 \cup V_6$, where

$V_1 = \{k_1 p : k_1 = 1, 2, 3, \cdots, qr - 1 \text{ and } q \nmid k_1,\ r \nmid k_1\}$, with $|V_1| = qr - q - r + 1$

$V_2 = \{k_2 q : k_2 = 1, 2, 3, \cdots, pr - 1 \text{ and } r \nmid k_2,\ p \nmid k_2\}$, with $|V_2| = pr - p - r + 1$

$V_3 = \{k_3 r : k_3 = 1, 2, 3, \cdots, pq - 1 \text{ and } p \nmid k_3,\ q \nmid k_3\}$, with $|V_3| = pq - p - q + 1$

$V_4 = \{k_5 qr : k_5 = 1, 2, 3, \cdots, p - 1\}$, with $|V_4| = p - 1$

$V_5 = \{k_6 pr : k_6 = 1, 2, 3, \cdots, q - 1\}$, with $|V_5| = q - 1$

$V_6 = \{k_4 pq : k_4 = 1, 2, 3, \cdots, r - 1\}$, with $|V_6| = r - 1$

We observe the following:

i) No two elements within any single set V_i for $i = 1, 2, 3, 4, 5, 6$ are adjacent to each other.

ii) No two elements from $V_2 \cup V_3 \cup V_4$ are adjacent, No two elements from $V_1 \cup V_2 \cup V_6$ are adjacent and No two elements from $V_1 \cup V_3 \cup V_5$ are adjacent.

iii) Every element of V_1 is adjacent to every element of V_4. Similarly, every element of V_2 is adjacent to every element of V_5 and, every element of V_3 is adjacent to every element of V_6.

iv) Every element of V_4 is adjacent to each element of V_1, V_5, and V_6. Every element of V_5 is adjacent to each element of V_2, V_4, and V_6. Every element of V_6 is adjacent to each element of V_3, V_4, and V_5.

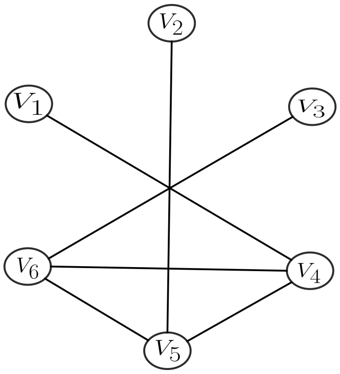

Figure 6.7: Adjacency structure(Vertex set) of the graph $\Gamma(\mathbb{Z}_{pqr})$

Therefore, the Merrifield-Simmons index of the graph $\Gamma(R)$ is

$$i(\Gamma(R)) = \sum_{k=0}^{|V|} i_k(\Gamma(R)) = i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V|} i_k(\Gamma(R))$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V_2 \cup V_3 \cup V_4|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_1 \cup V_2 \cup V_6|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_1 \cup V_3 \cup V_5|} i_k(\Gamma(R))$$

$$-\sum_{k=2}^{|V_1|} i_k(\Gamma(R)) - \sum_{k=2}^{|V_2|} i_k(\Gamma(R)) - \sum_{k=2}^{|V_3|} i_k(\Gamma(R))$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{pq+pr-p-q-r+1} \binom{pq+pr-p-q-r+1}{k} +$$

$$\sum_{k=2}^{pr+qr-p-q-r+1} \binom{pr+qr-p-q-r+1}{k} + \sum_{k=2}^{pq+qr-p-q-r+1} \binom{pq+qr-p-q-r+1}{k} -$$

$$\sum_{k=2}^{qr-q-r+1} \binom{qr-q-r+1}{k} - \sum_{k=2}^{pr-p-r+1} \binom{pr-p-r+1}{k} - \sum_{k=2}^{pq-p-q+1} \binom{pq-p-q+1}{k}$$

$$= 1+pq+qr+pr-p-q-r+[2^{pq+pr-p-q-r+1}-pq-pr+p+q+r-1-1]+[2^{pr+qr-p-q-r+1}-pr-qr+p+q+r-1-1]+[2^{pq+qr-p-q-r+1}-pq-qr+p+q+r-1-1]-[2^{qr-q-r+1}-qr+q+r-1-1]-[2^{pr-p-r+1}-pr+p+r-1-1]-[2^{pq-p-q+1}-pq+p+q-1-1]$$

$$= 2^{pq+pr-p-q-r+1}+2^{pr+qr-p-q-r+1}+2^{pq+qr-p-q-r+1}-2^{qr-q-r+1}-2^{pr-p-r+1}-2^{pq-p-q+1}+1.$$

□

Theorem 6.2.9. *Let* $R = \mathbb{Z}_{pq}[x]/ < x^2 >$ *with any prime* $2 < p < q$. *Then the Merrifield-Simmons index of the graph* $i(\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >))$ *is* $pq + (2^{(p-1)q(q-1)} - 1)(2^{p(p-1)(q-1)} + 2^{q(q-1)} + 2^{p(p-1)} + pq - p - 2) + (2^{p(p-1)(q-1)} - 1)(2^{q(q-1)} + 2^{p(p-1)} + pq - q - 1) + (2^{q(q-1)} - 1)(pq - p + 1) + (2^{p(p-1)} - 1)(pq - q + 1)$.

Proof. Let $R = \mathbb{Z}_{pq}[x]/ < x^2 >$, where $2 < p < q$ are distinct primes. The zero-divisor graph $\Gamma(R)$ has a vertex set consisting of the non-zero zero divisors of R. The number of vertices in $\Gamma(R)$ is $(pq^2 + p^2q - pq - 1)$, which corresponds to the cardinality of $(\mathbb{Z}_{pq}[x]/ < x^2 >) \setminus \{0\}$. Then the details of $\Gamma(R)$ as in [44] are as follows:

$V(\Gamma(R)) = \{p, 2p, \cdots, (q-1)p, q, 2q, \cdots, (p-1)q, x, 2x, \cdots, (pq-1)x, x+p, x+2p, \cdots, x+(q-1)p, x+q, x+2q, \cdots, x+(p-1)q, 2x+p, 2x+2p, \cdots, 2x+(q-1)p, 2x+q, 2x+2q, \cdots, 2x+(p-1)q, \cdots, (q-1)px+p.(q-1)px+2p, \cdots, (q-1)px+(q-1)p, (p-1)qx+q, (p-1)qx+2q, \cdots, (p-1)qx+(p-1)q\}$.

The vertex set V is divided into seven disjoint subsets $V_1, V_2, V_3, V_4, V_5, V_6, V_7$ such that:

$V_1 = \{kqx : k = 1, 2, \cdots, p-1\}$, with $|V_1| = p-1$

$V_2 = \{kx + mp : k = 1, 2, \cdots, pq-1, m = 1, 2, \cdots, q-1 \textit{ and } p \nmid k\}$, with $|V_2| = (p-1)q(q-1)$

$V_3 = \{kx + np : k = 1, 2, \cdots, pq-1, n = 1, 2, \cdots, p-1 \textit{ and } q \nmid k\}$, with $|V_3| = p(p-1)(q-1)$

$V_4 = \{lpx + mp : l = 0, 1, \cdots, q-1, m = 1, 2, \cdots, q-1\}$, with $|V_4| = q(q-1)$

$V_5 = \{lqx + nq : l = 0, 1, \cdots, p-1, n = 1, 2, \cdots, p-1\}$, with $|V_5| = p(p-1)$

$V_6 = \{mpx : m = 1, 2, \cdots, q-1\}$, with $|V_6| = q-1$

$V_7 = \{kx : k = 1, 2, \cdots, pq-1 \textit{ and } p, q \nmid k\}$, with $|V_7| = (p-1)(q-1)$

The adjacency structure of $\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >)$ is illustrated in Figure 6.8.

From Figure 8, it is clear to see that $a \longleftrightarrow b$ $\forall a, b \in V_1, V_6, V_7$ and $a \nleftrightarrow b$ $\forall a, b \in V_2, V_3, V_4, V_5$. Also, $a \longleftrightarrow b$ for $a \in V_1$ and $b \in V_2$. Similarly, $a \longleftrightarrow b$ for $a \in V_1$ and $b \in V_4$, $a \longleftrightarrow b$ for $a \in V_1$ and $b \in V_6$, $a \longleftrightarrow b$ for $a \in V_1$ and $b \in V_7$, $a \longleftrightarrow b$ for $a \in V_3$ and $b \in V_6$, $a \longleftrightarrow b$ for $a \in V_4$ and $b \in V_5$, $a \longleftrightarrow b$ for $a \in V_5$ and $b \in V_6$, $a \longleftrightarrow b$ for $a \in V_6$ and $b \in V_7$.

Also, $a \nleftrightarrow b$ for $a \in V_1$ and $b \in V_3$. Similarly, $a \nleftrightarrow b$ for $a \in V_1$ and $b \in V_5$, $a \nleftrightarrow b$ for $a \in V_2$ and $b \in V_3$, $a \nleftrightarrow b$ for $a \in V_2$ and $b \in V_4$, $a \nleftrightarrow b$ for $a \in V_2$ and $b \in V_5$, $a \nleftrightarrow b$ for $a \in V_2$ and $b \in V_6$, $a \nleftrightarrow b$ for $a \in V_2$ and $b \in V_7$, $a \nleftrightarrow b$ for $a \in V_3$ and $b \in V_4$, $a \nleftrightarrow b$ for $a \in V_3$ and $b \in V_5$, $a \nleftrightarrow b$ for $a \in V_3$ and $b \in V_7$, $a \nleftrightarrow b$ for $a \in V_4$ and $b \in V_6$, $a \nleftrightarrow b$ for $a \in V_4$ and $b \in V_7$, $a \nleftrightarrow b$ for $a \in V_5$ and $b \in V_7$.

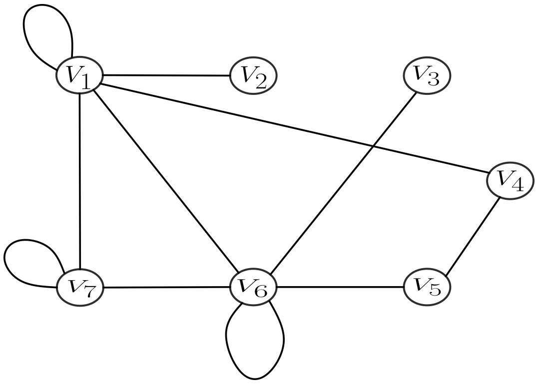

Figure 6.8: Adjacency structure(Vertex set) of the graph $\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >)$

Now, we choose all possible $n-tuple$ pair of vertices from V_i for $i = 1, 2, \cdots, 7$ for suitable n such that they are pairwise non-adjacent.Therefore, the Merrifield-Simmons index of the graph $\Gamma(R)$ is computed as follows:

$$i(\Gamma(R)) = \sum_{k=0}^{|V|} i_k(\Gamma(R)) = i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V|} i_k(\Gamma(R))$$

$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V_2|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_3|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_4|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_5|} i_k(\Gamma(R)) +$$

$$\tbinom{|V_1|}{1}\sum_{k=1}^{|V_3|} i_k(\Gamma(R)) + \tbinom{|V_1|}{1}\sum_{k=1}^{|V_5|} i_k(\Gamma(R)) + \sum_{k=1}^{|V_2|} i_k(\Gamma(R))\sum_{k=1}^{|V_3|} i_k(\Gamma(R)) +$$

$$\sum_{k=1}^{|V_2|} i_k(\Gamma(R))\sum_{k=1}^{|V_4|} i_k(\Gamma(R)) + \sum_{k=1}^{|V_2|} i_k(\Gamma(R))\sum_{k=1}^{|V_5|} i_k(\Gamma(R)) + \tbinom{|V_6|}{1}\sum_{k=1}^{|V_2|} i_k(\Gamma(R)) +$$

$$\tbinom{|V_7|}{1}\sum_{k=1}^{|V_2|} i_k(\Gamma(R)) + \sum_{k=1}^{|V_3|} i_k(\Gamma(R))\sum_{k=1}^{|V_4|} i_k(\Gamma(R)) + \sum_{k=1}^{|V_3|} i_k(\Gamma(R))\sum_{k=1}^{|V_5|} i_k(\Gamma(R)) +$$

$$\tbinom{|V_7|}{1}\sum_{k=1}^{|V_3|} i_k(\Gamma(R)) + \tbinom{|V_6|}{1}\sum_{k=1}^{|V_4|} i_k(\Gamma(R)) + \tbinom{|V_7|}{1}\sum_{k=1}^{|V_4|} i_k(\Gamma(R)) + \tbinom{|V_7|}{1}\sum_{k=1}^{|V_5|} i_k(\Gamma(R))$$

$$= i_0(\Gamma(R)) + |V_1| + |V_2| + |V_3| + |V_4| + |V_5| + |V_6| + |V_7| - |V_2| - |V_3| - |V_4| - |V_5| +$$

$$\sum_{k=1}^{|V_2|} i_k(\Gamma(R))(1 + \sum_{k=1}^{|V_3|} i_k(\Gamma(R)) + \sum_{k=1}^{|V_4|} i_k(\Gamma(R)) + \sum_{k=1}^{|V_5|} i_k(\Gamma(R)) + \tbinom{|V_6|}{1} + \tbinom{|V_7|}{1}) + \sum_{k=1}^{|V_3|} i_k(\Gamma(R))(1+$$

$$\sum_{k=1}^{|V_4|} i_k(\Gamma(R)) + \sum_{k=1}^{|V_5|} i_k(\Gamma(R)) + \tbinom{|V_1|}{1} + \tbinom{|V_7|}{1}) + \sum_{k=1}^{|V_4|} i_k(\Gamma(R))(1 + \tbinom{|V_6|}{1} + \tbinom{|V_7|}{1})$$

$$+ \sum_{k=1}^{|V_5|} i_k(\Gamma(R))(1 + \tbinom{|V_1|}{1} + \tbinom{|V_7|}{1})$$

$$= 1 + |V_1| + |V_6| + |V_7| + \sum_{k=1}^{(p-1)q(q-1)} \binom{(p-1)q(q-1)}{k} (1 + \sum_{k=1}^{p(p-1)(q-1)} \binom{p(p-1)(q-1)}{k} +$$

$$\sum_{k=1}^{q(q-1)} \binom{q(q-1)}{k} + \sum_{k=1}^{p(p-1)} \binom{p(p-1)}{k} + \tbinom{q-1}{1} + \tbinom{(p-1)(q-1)}{1}) +$$

$$\sum_{k=1}^{p(p-1)(q-1)} \binom{p(p-1)(q-1)}{k} (1 + \sum_{k=1}^{q(q-1)} \binom{q(q-1)}{k} + \sum_{k=1}^{p(p-1)} \binom{p(p-1)}{k} + \tbinom{p-1}{1} +$$

$$\tbinom{(p-1)(q-1)}{1}) + \sum_{k=1}^{q(q-1)} \binom{q(q-1)}{k} (1 + \tbinom{q-1}{1} + \tbinom{(p-1)(q-1)}{1})$$

$$+ \sum_{k=1}^{p(p-1)} \binom{p(p-1)}{k} (1 + \tbinom{p-1}{1} + \tbinom{(p-1)(q-1)}{1})$$

$$= 1 + p - 1 + q - 1 + (p-1)(q-1) + (2^{(p-1)q(q-1)} - 1)(1 + 2^{p(p-1)(q-1)} - 1 + 2^{q(q-1)} - 1 +$$

$2^{p(p-1)}-1+q-1+(p-1)(q-1))+(2^{p(p-1)(q-1)}-1)(1+2^{q(q-1)}-1+2^{p(p-1)}-1+p-1+(p-1)(q-1))+(2^{q(q-1)}-1)(1+q-1+(p-1)(q-1))+(2^{p(p-1)}-1)(1+p-1+(p-1)(q-1))$
$= pq + (2^{(p-1)q(q-1)} - 1)(2^{p(p-1)(q-1)} + 2^{q(q-1)} + 2^{p(p-1)} + pq - p - 2) + (2^{p(p-1)(q-1)} - 1)(2^{q(q-1)} + 2^{p(p-1)} + pq - q - 1) + (2^{q(q-1)} - 1)(pq - p + 1) + (2^{p(p-1)} - 1)(pq - q + 1)$. □

6.3 Merrifield-Simmons index of Zero-divisor graphs of small finite commutative rings

In this section, we mainly computed the Merrifield-Simmons index of zero divisor graphs of some small finite commutative rings. In [45], the details of zero divisor graphs of small order finite commutative rings are described.

Example 6.3.1. *In the complete graph K_n, for $n \in \mathbb{N}$, every pair of vertices are adjacent to each other. Now, Merrifield-Simmons inx of the graph K_n is*

$$i(K_n) = \sum_{k=0}^{|V|} i_k(\Gamma(R)) = i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V|} i_k(\Gamma(R)) = 1 + n + 0 = n + 1.$$

Example 6.3.2. *For the complete bipartite graph $K_{m,n}$ for $m, n \in \mathbb{N}$, let V be the vertex set of $K_{m,n}$. Now, the vertex set V has been divided into two disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$ with $|V_1| = m$, and $|V_2| = n$. Then one can visualize that no two elements of V_1 and V_2 are adjacent but each element of V_1 is adjacent to every element of V_2.*

Now, Merrifield-Simmons index of $K_{m,n}$ is

$$i(K_{m,n}) = \sum_{k=0}^{|V|} i_k(\Gamma(R)) = i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V|} i_k(\Gamma(R))$$
$$= i_0(\Gamma(R)) + i_1(\Gamma(R)) + \sum_{k=2}^{|V_1|} i_k(\Gamma(R)) + \sum_{k=2}^{|V_2|} i_k(\Gamma(R))$$
$$= 1 + m + n + 2^m - m - 1 + 2^n - n - 1 = 2^m + 2^n - 1.$$

List of figures of Zero divisor graph of small finite commutative rings

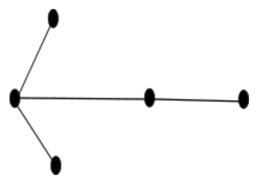

Figure 6.9: $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_4)/\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2[X]/(X^2))$

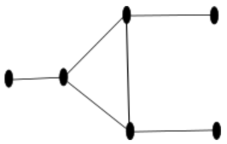

Figure 6.10: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

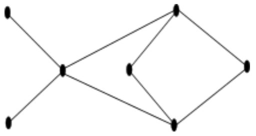

Figure 6.11: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2))$ or $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_4)$

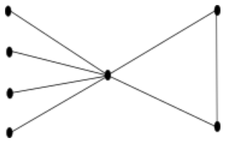

Figure 6.12: $\Gamma(\mathbb{Z}_{16})$ or $\Gamma(\mathbb{Z}_2[X]/(X^4))$ or $\Gamma(\mathbb{Z}_4[X]/(X^2+2))$ or $\Gamma(\mathbb{Z}_4[X]/(X^2+3X))$ or $\Gamma(\mathbb{Z}_4[X]/(x^3-2,2X^2,2X))$

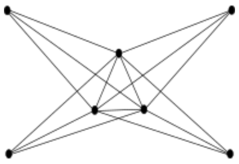

Figure 6.13: $\Gamma(\mathbb{Z}_2[X,Y]/(X^3,XY,Y^2))$ or $\Gamma(\mathbb{Z}_8[X]/(2X,X^2))$ or $\Gamma(\mathbb{Z}_4[X,Y]/(X^3,2X^2,2X))$

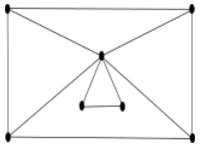

Figure 6.14: $\Gamma(\mathbb{Z}_4[X]/(X^2+2X))$ or $\Gamma(\mathbb{Z}_8[X]/(2X,x^2+4))$ or $\Gamma(\mathbb{Z}_2[X,Y]/(X^2,Y^2-XY))$ or $\Gamma(\mathbb{Z}_4[X,Y]/(X^2,Y^2-XY,XY-2,2X,2Y))$

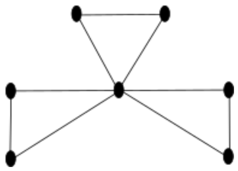

Figure 6.15: $\Gamma(\mathbb{Z}_4[X,Y]/(X^2,Y^2,XY-2,2X,2Y))$ or $\Gamma(\mathbb{Z}_2[X,Y]/(X^2,Y^2))$ or $\Gamma(\mathbb{Z}_4[X]/(X^2))$

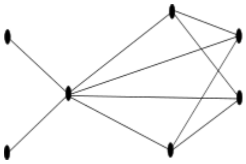

Figure 6.16: $\Gamma(\mathbb{Z}_4[X]/(X^3-X^2-2,2X^2,2X))$

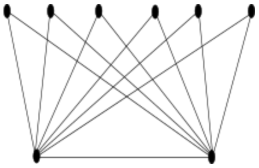

Figure 6.17: $\Gamma(\mathbb{Z}_{27})$ or $\Gamma(\mathbb{Z}_9[X]/(3X,X^2-3))$ or $\Gamma(\mathbb{Z}_9[X]/(3X,X^2-6))$ or $\Gamma(\mathbb{Z}_3[X]/(X^3))$

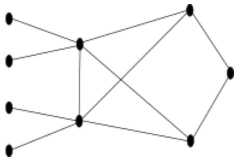

Figure 6.18: $\Gamma(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3)$

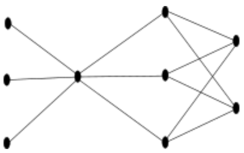

Figure 6.19: $\Gamma(\mathbb{Z}_4\times\mathbb{F}_4)$ or $\Gamma(\mathbb{Z}_2[X]/(X^2)\times\mathbb{F}_4)$

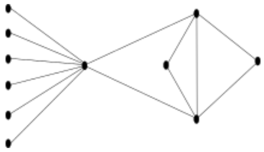

Figure 6.20: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_9)$ or $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2))$

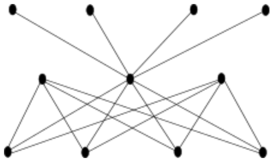

Figure 6.21: $\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_4)$ or $\Gamma(\mathbb{Z}_5 \times \mathbb{Z}_2[X]/(X^2))$

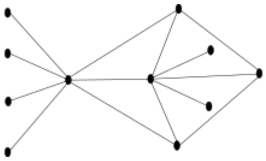

Figure 6.22: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_8)$ or $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3))$ or $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2-2))$

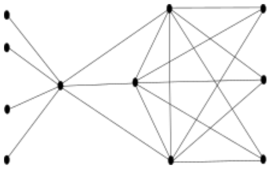

Figure 6.23: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X,Y)^2)$ or $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2,X)^2)$

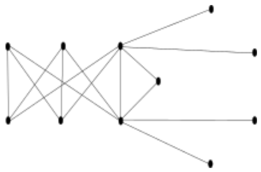

Figure 6.24: $\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_4)$ or $\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2))$ or $\Gamma(\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2))$

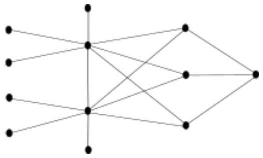

Figure 6.25: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4)$

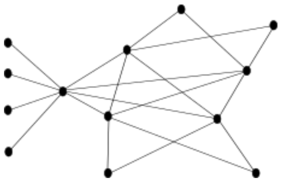

Figure 6.26: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$

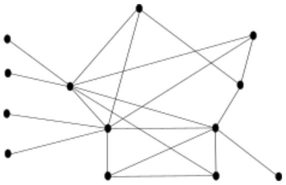

Figure 6.27: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4)$ or $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2))$

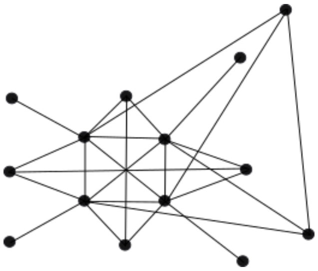

Figure 6.28: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

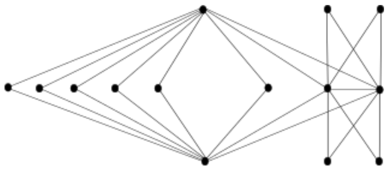

Figure 6.29: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_9)$ or $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2))$

In this context, let R represent a finite commutative ring, $|V|$ the total number of vertices, $\Gamma(R)$ the zero-divisor graph corresponding to R, and $i(\Gamma(R))$ the Merrifield-Simmons index of this zero-divisor graph

Merrifield-Simmons index of Zero-divisor graphs of small finite commutative rings

$\|V\|$	R	$\Gamma(R)$	$i(\Gamma(R))$
3	$\mathbb{Z}_6$	$K_{1,2}$	5
	$\mathbb{Z}_8$	$K_{1,2}$	5
	$\mathbb{Z}_2[X]/(X^3)$	$K_{1,2}$	5
	$\mathbb{Z}_4[X]/(2X, X^2-2)$	$K_{1,2}$	5
	$\mathbb{Z}_2[X,Y]/(X,Y)^2$	K_3	4
	$\mathbb{Z}_4[X]/(2,X)^2$	K_3	4
	$\mathbb{F}_4[X]/(X^2)$	K_3	4
	$\mathbb{Z}_4[X]/(X^2+X+1)$	K_3	4
4	$\mathbb{Z}_2 \times \mathbb{F}_4$	$K_{1,3}$	9
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$K_{2,2}$	7
	$\mathbb{Z}_{25}$	K_4	5
	$\mathbb{Z}_5[X]/(X^2)$	K_4	5
5	$\mathbb{Z}_2 \times \mathbb{Z}_5$	$K_{1,4}$	17
	$\mathbb{Z}_3 \times \mathbb{F}_4$	$K_{2,3}$	11
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Figure 6.9	14
	$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.9	14
6	$\mathbb{Z}_3 \times \mathbb{Z}_5$	$K_{2,4}$	19
	$\mathbb{F}_4 \times \mathbb{F}_4$	$K_{3,3}$	15
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Figure 6.10	20
	$\mathbb{Z}_{49}$	K_6	7
	$\mathbb{Z}_7[X]/(X^2)$	K_6	7
7	$\mathbb{Z}_2 \times \mathbb{Z}_7$	$K_{1,6}$	65
	$\mathbb{F}_4 \times \mathbb{Z}_5$	$K_{3,4}$	23
	$\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.11	32
	$\mathbb{Z}_3 \times \mathbb{Z}_4$	Figure 6.11	32
	$\mathbb{Z}_{16}$	Figure 6.12	49
	$\mathbb{Z}_2[X]/(X^4)$	Figure 6.12	49
	$\mathbb{Z}_4[X]/(X^2+2)$	Figure 6.12	49
	$\mathbb{Z}_4[X]/(X^2+3X)$	Figure 6.12	49
	$\mathbb{Z}_4[X]/(x^3-2, 2X^2, 2X)$	Figure 6.12	49

$\lvert V\rvert$	R	$\Gamma(R)$	$i(\Gamma(R))$
7	$\mathbb{Z}_2[X,Y]/(X^3,XY,Y^2)$	Figure 6.13	19
	$\mathbb{Z}_8[X]/(2X,X^2)$	Figure 6.13	19
	$\mathbb{Z}_4[X,Y]/(X^3,2X^2,2X)$	Figure 6.13	19
	$\mathbb{Z}_4[X]/(X^2+2X)$	Figure 6.14	20
	$\mathbb{Z}_8[X]/(2X,x^2+4)$	Figure 6.14	20
	$\mathbb{Z}_2[X,Y]/(X^2,Y^2-XY)$	Figure 6.14	20
	$\mathbb{Z}_4[X,Y]/(X^2,Y^2-XY,XY-2,2X,2Y)$	Figure 6.14	20
	$\mathbb{Z}_4[X,Y]/(X^2,Y^2,XY-2,2X,2Y)$	Figure 6.15	28
	$\mathbb{Z}_2[X,Y]/(X^2,Y^2)$	Figure 6.15	28
	$\mathbb{Z}_4[X]/(X^2)$	Figure 6.15	28
	$\mathbb{Z}_4[X]/(X^3-X^2-2,2X^2,2X)$	Figure 6.16	22
	$\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2$	K_7	129
	$\mathbb{Z}_4[X,Y]/(X^2,Y^2,XY,2X,2Y)$	K_7	129
	$\mathbb{F}_8[X]/(X^2)$	K_7	129
	$\mathbb{Z}_4[X]/(X^3+X+1)$	K_7	129
8	$\mathbb{Z}_2\times\mathbb{F}_8$	$K_{1,7}$	129
	$\mathbb{Z}_3\times\mathbb{Z}_7$	$K_{2,6}$	67
	$\mathbb{Z}_5\times\mathbb{Z}_5$	$K_{2,6}$	67
	$\mathbb{Z}_{27}$	Figure 6.17	66
	$\mathbb{Z}_9[X]/(3X,X^2-3)$	Figure 6.17	66
	$\mathbb{Z}_9[X]/(3X,X^2-6)$	Figure 6.17	66
	$\mathbb{Z}_3[X]/(X^3)$	Figure 6.17	66
	$\mathbb{Z}_3[X,Y]/(X,Y)^2$	K_8	257
	$\mathbb{Z}_9[X]/(3,X)^2$	K_8	257
	$\mathbb{F}_9[X]/(X^2)$	K_8	257
	$\mathbb{Z}_9[X]/(X^2+1)$	K_8	257
9	$\mathbb{Z}_2\times\mathbb{F}_9$	$K_{1,8}$	257
	$\mathbb{Z}_3\times\mathbb{F}_8$	$K_{2,7}$	131
	$\mathbb{F}_4\times\mathbb{Z}_7$	$K_{3,6}$	71
	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	Figure 6.18	71
	$\mathbb{Z}_4\times\mathbb{F}_4$	Figure 6.19	72
	$\mathbb{Z}_2[X]/(X^2)\times\mathbb{F}_4$	Figure 6.19	72
10	$\mathbb{Z}_3\times\mathbb{F}_9$	$K_{2,8}$	259
	$\mathbb{F}_4\times\mathbb{F}_8$	$K_{3,7}$	135
	$\mathbb{Z}_5\times\mathbb{Z}_7$	$K_{4,6}$	79
	$\mathbb{Z}_{121}$	K_{10}	11
	$\mathbb{Z}_{11}[X]/(X^2)$	K_{10}	11

$\|V\|$	R	$\Gamma(R)$	$i(\Gamma(R))$
11	$\mathbb{Z}_2 \times \mathbb{Z}_{11}$	$K_{1,10}$	1025
	$\mathbb{F}_4 \times \mathbb{F}_9$	$K_{3,8}$	263
	$\mathbb{Z}_5 \times \mathbb{F}_8$	$K_{4,7}$	143
	$\mathbb{Z}_2 \times \mathbb{Z}_9$	Figure 6.20	388
	$\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$	Figure 6.20	388
	$\mathbb{Z}_5 \times \mathbb{Z}_4$	Figure 6.21	308
	$\mathbb{Z}_5 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.21	308
	$\mathbb{Z}_2 \times \mathbb{Z}_8$	Figure 6.22	294
	$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	Figure 6.22	294
	$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2-2)$	Figure 6.22	294
	$\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X,Y)^2$	Figure 6.23	180
	$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2,X)^2$	Figure 6.23	180
	$\mathbb{Z}_4 \times \mathbb{Z}_4$	Figure 6.24	284
	$\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.24	284
	$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.24	284
12	$\mathbb{Z}_3 \times \mathbb{Z}_{11}$	$K_{2,10}$	1027
	$\mathbb{Z}_5 \times \mathbb{F}_9$	$K_{4,8}$	271
	$\mathbb{Z}_7 \times \mathbb{Z}_7$	$K_{6,6}$	127
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$	Figure 6.25	580
	$\mathbb{Z}_{169}$	K_{12}	13
	$\mathbb{Z}_{13}[X]/(X^2)$	K_{12}	13
13	$\mathbb{Z}_2 \times \mathbb{Z}_{13}$	$K_{1,12}$	4095
	$\mathbb{F}_4 \times \mathbb{Z}_{11}$	$K_{3,10}$	1031
	$\mathbb{Z}_7 \times \mathbb{F}_8$	$K_{6,7}$	191
	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	Figure 6.26	605
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	Figure 6.27	507
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.27	507
14	$\mathbb{Z}_3 \times \mathbb{Z}_{13}$	$K_{2,12}$	4099
	$\mathbb{Z}_5 \times \mathbb{Z}_{11}$	$K_{4,10}$	1039
	$\mathbb{Z}_7 \times \mathbb{F}_9$	$K_{6,8}$	319
	$\mathbb{F}_8 \times \mathbb{F}_8$	$K_{7,7}$	255
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Figure 6.28	630
	$\mathbb{Z}_3 \times \mathbb{Z}_9$	Figure 6.29	1162
	$\mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2)$	Figure 6.29	1162

Chapter 7

Sum and Product connectivity Gourava index of zero divisor graph over finite commutative ring

Chapter 7

Sum and Product connectivity Gourava index of $\Gamma(R)$

7.1 Introduction

Over the past two decades, research on the zero divisor graph of a commutative ring has yielded intriguing discoveries and raised many issues. There are several articles on assigning a graph to a ring, including [5], [17] and [7]. Refer to [44] and [45] for further concepts discussed in this paper. Our study focuses on zero divisor graphs, specifically those of $\mathbb{Z}_n$. For your convenience, we provide a working introduction to the concepts involved.

The zero-divisor graph of a commutative ring R with identity is a graph with vertex set consists of non-zero zero-divisors set $Z^*(R)$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. $\Gamma(R)$ is the zero-divisor graph associated with R. In chapter 6, we studied about the Merrifield-Simmons index of zero divisor graph over finite commutative ring. In the preliminaries section, we studied about the definitions and connections between first and second Zagreb indices, the first and second Gourava index, the Sum connectivity index, Randic connectivity index of a graph $\mathcal{G}$.

Motivated by the definitions of the first Gourava index and the sum connectivity index, Kulli introduced the sum connectivity Gourava index, which is defined as follows:

Definition 7.1.1. *Let $\mathcal{G}$ be a graph, then the sum connectivity Gourava index, denoted as $SGO(\mathcal{G})$ in [33], for the graph $\mathcal{G}$, is defined as*

$$SGO(\mathcal{G}) = \sum_{uv \in E(\mathcal{G})} \frac{1}{\sqrt{d_{\mathcal{G}}(u) + d_{\mathcal{G}}(v) + d_{\mathcal{G}}(u)d_{\mathcal{G}}(v)}}$$

In [35], Kulli introduced the product connectivity Gourava index of a graph in the following manner, which is inspired by the definition of the Randić connectivity index: The product connectivity Gourava index, is defined as follows:

Definition 7.1.2. *Let $\mathcal{G}$ be a graph. Then the product connectivity Gourava index $PGO(\mathcal{G})$ of the graph $\mathcal{G}$ is defined as*

$$PGO(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{(d_G(u) + d_G(v))d_G(u)d_G(v)}}$$

In this chapter, we compute the sum-connectivity Gourava index and the product-connectivity Gourava index of the zero-divisor graph over the ring $\mathbb{Z}_n$, for $n = p^m, pq, p^2q, p^2q^2, pqr$, where p, q are distinct prime numbers. Furthermore, we compute the sum and product connectivity Gourava index for the zero-divisor graph $\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >)$. In the final section, we compute the Sum and Product connectivity Gourava index of zero divisor graphs from several small finite commutative rings.

7.2 Sum and product connectivity Gourava index of zero-divisor graph over ring $\mathbb{Z}_n$.

We define $E_{i,j}$ is the set of all edges from V_i to V_j. $|E_{i,j}|$ = total number of edges from V_i to V_j which is the same number of edges in $K_{|V_i|,|V_j|}$, and $|E_{i,j}| = |V_i||V_j|$. Also, $E_{i,i}$ is the set of all edges from V_i to V_i except itself. $|E_{i,i}|$ = total number of edges from V_i to V_i which is the same number of edges in $K_{|V_i|}$, and $|E_{i,i}| = \binom{|V_i|}{2}$.

Example 7.2.1. *In the complete graph K_n, for $n \in \mathbb{N}$, every pair of vertices are adjacent to each other. So, degree of each vertex is $n-1$. Therefore,*

$$SGO(K_n) = \sum_{uv \in E(K_n)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} = \frac{n(n-1)}{2\sqrt{n-1+n-1+(n-1)^2}} = \frac{n\sqrt{n-1}}{2\sqrt{n+1}}.$$

Moreover,

$$PGO(K_n) = \sum_{uv \in E(K_n)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} = \frac{n(n-1)}{2\sqrt{(n-1+n-1)(n-1)^2}} = \frac{n}{2\sqrt{2(n-1)}}.$$

Example 7.2.2. *For the complete bipartite graph $K_{m,n}$ for $m, n \in \mathbb{N}$, let V be the vertex set of $K_{m,n}$. Now, the vertex set V has been divided into two disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$ with $|V_1| = m$, and $|V_2| = n$. Then one can visualize that no two elements of V_1 and V_2 are adjacent but each element of V_1 is adjacent to every element of V_2. So, $d(x) = n \ \forall\ x \in\ V_1$ and $d(x) = m \ \forall\ x \in\ V_2$.*

Therefore, the sum-connectivity Gourava index of $K_{m,n}$ is

$$SGO(K_{m,n}) = \sum_{uv \in E(K_n)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} = \frac{mn}{\sqrt{n+m+mn}}.$$

Moreover, the product-connectivity Gourava index of $K_{m,n}$ is

$$PGO(K_{m,n}) = \sum_{uv \in E(K_n)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} = \frac{mn}{\sqrt{(n+m)mn}}.$$

Lemma 7.2.3. *For any prime number p, the Sum and Product connectivity Gourava index of zero-divisor graph $\Gamma(\mathbb{Z}_{p^3})$ are*

$$SGO(\Gamma(\mathbb{Z}_{p^3})) = \frac{(p-1)(p^2-2)}{\sqrt{p^3-p-1}} + \frac{(p^2-2)(p^3-3)}{2p\sqrt{p^2-2}}$$

$$PGO(\Gamma(\mathbb{Z}_{p^3})) = \frac{\sqrt{(p-1)(p^2-2)}}{\sqrt{p^2+p-3}} + \frac{p^2-3}{2\sqrt{2(p^2-2)}}$$

Proof. Let $R = \mathbb{Z}_{p^3}$ and $G = \Gamma(\mathbb{Z}_{p^3})$. The set of zero divisors of R is $V = Z^*(R) = \{p, 2p, 3p, ..., (p^2-1)p\}$, with cardinality p^2-1. Divide the vertex set V of G into two disjoint subsets: V_1 and V_2, where

$$V_1 = \{k_1 p : k_1 = 1, 2, 3, ..., p^2-1 \text{ and } p \nmid k_1\}$$

$$V_2 = \{k_2 p^2 : k_2 = 1, 2, 3, ..., p-1\}$$

with $|V_1| = p^2 - p$, $|V_2| = p - 1$.

It's evident that no two elements in V_1 are adjacent, and each element in V_1 is only adjacent to elements in V_2. The elements in V_2 are all mutually adjacent, except for themselves. The following graph in Figure 7.1 shows the adjacency property of the zero divisor graph $\Gamma(\mathbb{Z}_{p^3})$.

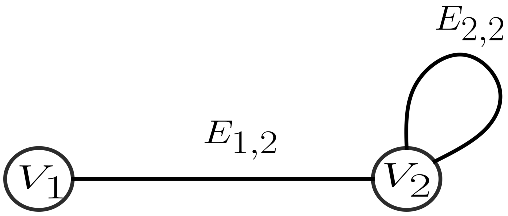

Figure 7.1: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{p^3})$

Now one can see that degrees of each V_i for $i = 1, 2$ are in the following.

$$d(x) = |V_2| = p - 1 \ \forall x \in V_1.$$
$$d(x) = |V_2| - 1 + |V_1| = p^2 - 2 \ \forall x \in V_2.$$

Now, we divide edge set E of $\Gamma(\mathbb{Z}_{p^3})$ as $E = E_{1,2} \cup E_{2,2}$. Therefore, the Sum-Connectivity Gourava index of $\Gamma(\mathbb{Z}_{p^3})$ is

$$SGO(\Gamma(\mathbb{Z}_{p^3})) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u) + d(v) + d(u)d(v)}}$$
$$= \sum_{uv \in E_{1,2}} \frac{1}{\sqrt{d(u) + d(v) + d(u)d(v)}} + \sum_{uv \in E_{2,2}} \frac{1}{\sqrt{d(u) + d(v) + d(u)d(v)}}$$
$$= \frac{|V_1||V_2|}{\sqrt{(p-1) + (p^2-2) + (p-1)(p^2-2)}} + \frac{\binom{|V_2|}{2}}{\sqrt{2(p^2-2) + (p^2-2)^2}}$$
$$= \frac{(p-1)(p^2-2)}{\sqrt{p^3 - p - 1}} + \frac{(p^2-2)(p^3-3)}{2p\sqrt{p^2-2}}.$$

Moreover, the Product-Connectivity Gourava index of $\Gamma(\mathbb{Z}_{p^3})$ is

$$PGO(\Gamma(\mathbb{Z}_{p^3})) = \sum_{uv \in E(G)} \frac{1}{\sqrt{(d(u) + d(v))d(u)d(v)}}$$
$$= \sum_{uv \in E_{1,2}} \frac{1}{\sqrt{(d(u) + d(v))d(u)d(v)}} + \sum_{uv \in E_{2,2}} \frac{1}{\sqrt{(d(u) + d(v))d(u)d(v)}}$$
$$= \frac{|V_1||V_2|}{\sqrt{(p-1+p^2-2)(p-1)(p^2-2)}} + \frac{\binom{|V_2|}{2}}{\sqrt{2(p^2-2)(p^2-2)^2}}$$

$$= \frac{(p-1)(p^2-2)}{\sqrt{(p-1)(p^2-2)(p^2+p-3)}} + \frac{(p^2-2)(p^2-3)}{2(p^2-2)\sqrt{2(p^2-2)}}$$
$$= \frac{\sqrt{(p-1)(p^2-2)}}{\sqrt{p^2+p-3}} + \frac{p^2-3}{2\sqrt{2(p^2-2)}}. \qquad \square$$

Lemma 7.2.4. *For any prime p, the Sum and Product Connectivity Gourava index of the zero divisor graph $\Gamma(\mathbb{Z}_{p^4})$ are*

$$SGO(\Gamma(\mathbb{Z}_{p^4})) = \frac{p^2(p-1)^2}{\sqrt{p^4-p-1}} + \frac{p(p-1)^2}{\sqrt{p^5-p^3-p^2}} + \frac{(p-1)(p^2-p-1)}{2\sqrt{p^2-2}} + \frac{(p-1)(p-2)}{2p\sqrt{p(p^3-2)}}.$$

$$PGO(\Gamma(\mathbb{Z}_{p^4})) = \frac{p^2(p-1)^{\frac{3}{2}}}{\sqrt{(p^3+p-3)(p^2-2)}} + \frac{p(p-1)^2}{\sqrt{(p^3+p^2-4)(p^3-2)(p^2-2)}} +$$
$$\frac{p(p-1)(p^2-p-1)}{2(p^2-2)\sqrt{2(p^2-2)}} + \frac{(p-1)(p-2)}{2(p^3-2)\sqrt{2(p^3-2)}}$$

Proof. Let $R = \mathbb{Z}_{p^4}$ and $G = \Gamma(\mathbb{Z}_{p^4})$. Then the set of zero divisors of R is $V = Z^*(R) = \{p, 2p, 3p, ..., (p^3-1)p\}$ with cardinality p^3-1. Now, let the vertex set V of G be divided into disjoint subsets V_1, V_2, V_3, where

$$V_1 = \{k_1p : k_1 = 1, 2, 3, ..., p^3-1 \textit{ and } p \nmid k_1\}$$
$$V_2 = \{k_2p^2 : k_2 = 1, 2, 3, ..., p^2-1 \textit{ and } p \nmid k_2\}$$
$$V_3 = \{k_3p^3 : k_3 = 1, 2, 3, ..., p-1\}$$

with $|V_1| = p^3-p^2$, $|V_2| = p^2-p$, $|V_3| = p-1$ such that $V = V_1 \cup V_2 \cup V_3$.

Now if there is an edge between V_i and V_j for some choice of i and j, it means each element of V_i is adjacent with every element of V_j. Furthermore if there is a self-loop at V_k for some choice of k, it means every element of V_k is adjacent with every element of V_k except itself. For our case the following graph in Figure 7.2 shows the adjacency property of the zero divisor graph $\Gamma(\mathbb{Z}_{p^4})$.

Now one can see that degrees of each V_i for $i = 1, 2, 3$ are in the following.

$$d(x) = |V_3| = p-1 \ \forall x \in V_1.$$
$$d(x) = |V_2| - 1 + |V_3| = p^2-2 \ \forall x \in V_2.$$
$$d(x) = |V_1| + |V_2| + |V_3| - 1 = p^3-2 \ \forall x \in V_3.$$

Now, we divide edge set E of $\Gamma(\mathbb{Z}_{p^4})$ as $E = E_{1,3} \cup E_{2,3} \cup E_{2,2} \cup E_{3,3}$.

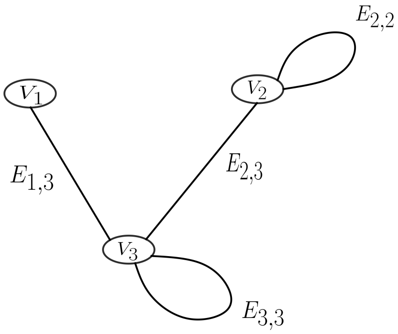

Figure 7.2: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{p^4})$

Therefore, the sum-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^4})$ is calculated as

$$SGO(\Gamma(\mathbb{Z}_{p^4})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \sum_{uv\in E_{1,3}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} + \sum_{uv\in E_{2,3}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$+ \sum_{uv\in E_{2,2}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} + \sum_{uv\in E_{3,3}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \frac{(p^3-p^2)(p-1)}{\sqrt{p-1+p^3-2+(p-1)(p^3-2)}} + \frac{(p^2-p)(p-1)}{\sqrt{p^2-2+p^3-2+(p^2-2)(p^3-2)}}$$

$$+ \frac{(p^2-p)(p^2-p-1)}{2\sqrt{p^2-2+p^2-2+(p^2-2)^2}} + \frac{(p-1)(p-2)}{2\sqrt{p^3-2+p^3-2+(p^3-2)^2}}$$

$$= \frac{p^2(p-1)^2}{\sqrt{p^4-p-1}} + \frac{p(p-1)^2}{\sqrt{p^5-p^3-p^2}} + \frac{(p-1)(p^2-p-1)}{2\sqrt{p^2-2}} + \frac{(p-1)(p-2)}{2p\sqrt{p(p^3-2)}}.$$

Moreover, the Product-Connectivity Gourava index of $\Gamma(\mathbb{Z}_{p^4})$ is

$$PGO(\Gamma(\mathbb{Z}_{p^4})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= \sum_{uv\in E_{1,3}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} + \sum_{uv\in E_{2,3}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$+ \sum_{uv\in E_{2,2}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} + \sum_{uv\in E_{3,3}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= \frac{|V_1||V_3|}{\sqrt{(p-1+p^3-2)(p-1)(p^3-2)}} + \frac{|V_2||V_3|}{\sqrt{(p^2-2+p^3-2)(p^2-2)(p^3-2)}} +$$

$$\frac{\binom{|V_2|}{2}}{2\sqrt{2(p^2-2)^3}} + \frac{\binom{|V_3|}{2}}{2\sqrt{2(p^3-2)^3}}$$

$$= \frac{(p^3-p^2)(p-1)}{\sqrt{(p-1+p^3-2)(p-1)(p^3-2)}} + \frac{(p^2-p)(p-1)}{\sqrt{(p^2-2+p^3-2)(p^2-2)(p^3-2)}} +$$

$$\frac{(p^2-p)(p^2-p-1)}{2\sqrt{2(p^2-2)^3}}+\frac{(p-1)(p-2)}{2\sqrt{2(p^3-2)^3}}$$

$$=\frac{p^2(p-1)^{\frac{3}{2}}}{\sqrt{(p^3+p-3)(p^2-2)}}+\frac{p(p-1)^2}{\sqrt{(p^3+p^2-4)(p^3-2)(p^2-2)}}+\frac{p(p-1)(p^2-p-1)}{2(p^2-2)\sqrt{2(p^2-2)}}+\frac{(p-1)(p-2)}{2(p^3-2)\sqrt{2(p^3-2)}}.$$

□

Lemma 7.2.5. *For any prime p, the Sum and Product Connectivity Gourava index of the zero divisor graph $\Gamma(\mathbb{Z}_{p^5})$ are*

$$SGO(\Gamma(\mathbb{Z}_{p^5}))=\frac{p^3(p-1)^2}{\sqrt{p^5-p-1}}+\frac{p^3(p-1)^2}{\sqrt{p^5-p^2-1}}+\frac{p^2(p-1)^2}{\sqrt{p^6-p^2-1}}+\frac{(p-1)^2}{\sqrt{p^5-p^2-p}}+\frac{(p-1)(p^2-p-1)}{2\sqrt{p(p^3-2)}}+\frac{(p-1)(p-2)}{2\sqrt{p^2-1}}$$

$$PGO(\Gamma(\mathbb{Z}_{p^5}))=\frac{p^3(p-1)^{\frac{3}{2}}}{\sqrt{(p^4+p-3)(p^4-2)}}+\frac{p^3(p-1)^{\frac{3}{2}}}{\sqrt{(p^3+p^2-3)(p^3-2)(p+1)}}+\frac{p^2(p-1)^{\frac{3}{2}}}{\sqrt{(p^4+p^2-3)(p^4-2)(p+1)}}+\frac{p(p-1)^2}{\sqrt{(p^4+p^3-4)(p^4-2)(p^3-2)}}+\frac{(p^2-p)(p^2-p-1)}{2(p^3-2)\sqrt{2(p^3-2)}}+\frac{p-2}{2\sqrt{2(p-1)}}.$$

Proof. Let $R=\mathbb{Z}_{p^5}$ and $G=\Gamma(\mathbb{Z}_{p^5})$. Then the set of zero divisors of R is $V=Z^*(R)=\{p,2p,3p,\cdots,(p^4-1)p\}$ with cardinality p^4-1. Now, let the vertex set V of G be divided into disjoint subsets V_1,V_2,V_3,V_4, where

$$V_1=\{k_1p : k_1=1,2,3,\cdots,p^4-1 \text{ } and \text{ } p\nmid k_1\}$$
$$V_2=\{k_2p^2 : k_2=1,2,3,\cdots,p^3-1 \text{ } and \text{ } p\nmid k_2\}$$
$$V_3=\{k_3p^3 : k_3=1,2,3,\cdots,p^2-1 \text{ } and \text{ } p\nmid k_3\}$$
$$V_4=\{k_4p^4 : k_4=1,2,3,\cdots,p-1\}$$

with $|V_1|=p^4-p^3$, $|V_2|=p^3-p^2$, $|V_3|=p^2-p$, $|V_4|=p-1$.

Now, similar hypothesis as discussed in 7.2.4, the adjacency property of the zero divisor graph $\Gamma(\mathbb{Z}_{p^5})$ is shown in Figure 7.3.

Now one can see that degrees of each V_i for $i=1,2,3,4$ are in the following.

$$d(x)=|V_4|=p-1 \ \forall x\in V_1.$$
$$d(x)=|V_3|+|V_4|=p^2-1 \ \forall x\in V_2.$$
$$d(x)=|V_2|+|V_3|-1+|V_4|=p^3-2 \ \forall x\in V_3.$$
$$d(x)=|V_1|+|V_2|+|V_3|+|V_4|-1=p^4-2 \ \forall x\in V_4.$$

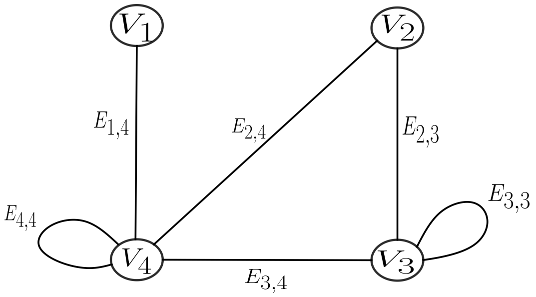

Figure 7.3: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{p^5})$

Now, we divide edge set E of $\Gamma(\mathbb{Z}_{p^5})$ as $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$.

Therefore, the sum-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^5})$ is calculated as

$$SGO(\Gamma(\mathbb{Z}_{p^5})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \sum_{uv\in E_{1,4}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} + \sum_{uv\in E_{2,3}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} +$$

$$\sum_{uv\in E_{2,4}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} + \sum_{uv\in E_{3,4}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} +$$

$$\sum_{uv\in E_{3,3}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} + \sum_{uv\in E_{4,4}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \frac{|V_1||V_4|}{\sqrt{p-1+p^4-2+(p-1)(p^4-2)}} + \frac{|V_2||V_3|}{\sqrt{p^2-1+p^3-2+(p^2-1)(p^3-2)}} +$$

$$\frac{|V_2||V_4|}{\sqrt{p^2-1+p^4-2+(p^2-1)(p^4-2)}} + \frac{|V_3||V_4|}{\sqrt{p^3-2+p^4-2+(p^3-2)(p^4-2)}} +$$

$$\frac{\binom{|V_3|}{2}}{\sqrt{(p^3-2)+(p^3-2)+(p^3-2)^2}} + \frac{\binom{|V_4|}{2}}{\sqrt{p-1+p-1+(p-1)^2}}$$

$$= \frac{(p^4-p^3)(p-1)}{\sqrt{p-1+p^4-2+(p-1)(p^4-2)}} + \frac{(p^3-p^2)(p^2-p)}{\sqrt{p^2-1+p^3-2+(p^2-1)(p^3-2)}} +$$

$$\frac{(p^3-p^2)(p-1)}{\sqrt{p^2-1+p^4-2+(p^2-1)(p^4-2)}} + \frac{(p^2-p)(p-1)}{\sqrt{p^3-2+p^4-2+(p^3-2)(p^4-2)}} +$$

$$\frac{(p^2-p)(p^2-p-1)}{2\sqrt{(p^3-2)+(p^3-2)+(p^3-2)^2}} + \frac{(p-1)(p-2)}{2\sqrt{p-1+p-1+(p-1)^2}}$$

$$= \frac{p^3(p-1)^2}{\sqrt{p^5-p-1}} + \frac{p^3(p-1)^2}{\sqrt{p^5-p^2-1}} + \frac{p^2(p-1)^2}{\sqrt{p^6-p^2-1}} + \frac{(p-1)^2}{\sqrt{p^5-p^2-p}} + \frac{(p-1)(p^2-p-1)}{2\sqrt{p(p^3-2)}} +$$

$$\frac{(p-1)(p-2)}{2\sqrt{p^2-1}}.$$

Moreover, the Product-Connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^5})$ is

$$PGO(\Gamma(\mathbb{Z}_{p^5})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= \sum_{uv\in E_{1,4}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} + \sum_{uv\in E_{2,3}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} +$$
$$\sum_{uv\in E_{2,4}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} + \sum_{uv\in E_{3,4}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} +$$
$$\sum_{uv\in E_{3,3}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} + \sum_{uv\in E_{4,4}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= \frac{|V_1||V_4|}{\sqrt{(p-1+p^4-2)(p-1)(p^4-2)}} + \frac{|V_2||V_3|}{\sqrt{(p^2-1+p^3-2)(p^2-1)(p^3-2)}} +$$
$$\frac{|V_2||V_4|}{\sqrt{(p^2-1+p^4-2)(p^2-1)(p^4-2)}} + \frac{|V_3||V_4|}{\sqrt{(p^3-2+p^4-2)(p^3-2)(p^4-2)}} +$$
$$\frac{\binom{|V_3|}{2}}{\sqrt{2(p^3-2)^3}} + \frac{\binom{|V_4|}{2}}{\sqrt{2(p-1)^3}}$$

$$= \frac{(p^4-p^3)(p-1)}{\sqrt{(p-1+p^4-2)(p-1)(p^4-2)}} + \frac{(p^3-p^2)(p^2-p)}{\sqrt{(p^2-1+p^3-2)(p^2-1)(p^3-2)}} +$$
$$\frac{(p^3-p^2)(p-1)}{\sqrt{(p^2-1+p^4-2)(p^2-1)(p^4-2)}} + \frac{(p^2-p)(p-1)}{\sqrt{(p^3-2+p^4-2)(p^3-2)(p^4-2)}} +$$
$$\frac{(p^2-p)(p^2-p-1)}{2\sqrt{2(p^3-2)^3}} + \frac{(p-1)(p-2)}{2\sqrt{2(p-1)^3}}$$

$$= \frac{p^3(p-1)^{\frac{3}{2}}}{\sqrt{(p^4+p-3)(p^4-2)}} + \frac{p^3(p-1)^{\frac{3}{2}}}{\sqrt{(p^3+p^2-3)(p^3-2)(p+1)}} +$$
$$\frac{p^2(p-1)^{\frac{3}{2}}}{\sqrt{(p^4+p^2-3)(p^4-2)(p+1)}} + \frac{p(p-1)^2}{\sqrt{(p^4+p^3-4)(p^4-2)(p^3-2)}} +$$
$$\frac{(p^2-p)(p^2-p-1)}{2(p^3-2)\sqrt{2(p^3-2)}} + \frac{p-2}{2\sqrt{2(p-1)}}. \qquad \square$$

Theorem 7.2.6. *For any prime p, the Sum and Product Connectivity Gourava index of the zero divisor graph $\Gamma(\mathbb{Z}_{p^n})$ with $n \in \mathbb{N}$ are*

If n is even and $m = \lceil \frac{n}{2} \rceil$, then $SGO(\Gamma(\mathbb{Z}_{p^n})) =$

$$\frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m}-p-1}} + (\frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+1}-p^2-1}} + \frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m}-p^2-1}}) + (\frac{p^{2m-4}(p-1)^2}{\sqrt{p^{2m+2}-p^3-1}} +$$
$$\frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+1}-p^3-1}} + \frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m}-p^3-1}}) + \cdots + (\frac{p^{m}(p-1)^2}{\sqrt{p^{3m-2}-p^{m-1}-1}} + \cdots +$$
$$\frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m}-p^{m-1}-1}}) + (\frac{p^{m-1}(p-1)^2}{\sqrt{p^{3m-1}-p^{2m-1}-p^m}} + \cdots + \frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+1}-p^{m+1}-p^m}}) +$$
$$(\frac{p^{m-2}(p-1)^2}{\sqrt{p^{3m}-p^{2m-1}-p^{m+1}}} + \cdots + \frac{p^{2m-5}(p-1)^2}{\sqrt{p^{2m+3}-p^{m+2}-p^{m+1}}}) + \cdots + (\frac{p^{2}(p-1)^2}{\sqrt{p^{4m-4}-p^{2m-1}-p^{2m-3}}}$$

$$+\frac{p^3(p-1)^2}{\sqrt{p^{4m-5}-p^{2m-2}-p^{2m-3}}})+\frac{p(p-1)^2}{\sqrt{p^{4m-3}-p^{2m-1}-p^{2m-2}}}+(\frac{(p^m-p^{m-1})(p^m-p^{m-1}-1)}{2\sqrt{p^m(p^m-2)}}$$
$$+\cdots+\frac{(p-1)(p-2)}{2\sqrt{p^{2m-1}(p^{2m-1}-2)}})$$

If n *is odd and* $m+1=\lceil\frac{n}{2}\rceil$, *then* $SGO(\Gamma(\mathbb{Z}_{p^n}))=$

$$\frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p-1}}+(\frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m+2}-p^2-1}}+\frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p^2-1}})+(\frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+3}-p^3-1}}+$$
$$\frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m+2}-p^3-1}}+\frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p^3-1}})+\cdots+(\frac{p^{m+1}(p-1)^2}{\sqrt{p^{3m-1}-p^{m-1}-1}}+\cdots+$$
$$\frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p^{m-1}-1}})+(\frac{p^m(p-1)^2}{\sqrt{p^{3m}-p^{2m}-p^m}}+\cdots+\frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p^{m+1}-p^m}})+$$
$$(\frac{p^{m-1}(p-1)^2}{\sqrt{p^{3m+1}-p^{2m}-p^{m+1}}}+\cdots+\frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+3}-p^{m+2}-p^{m+1}}})+\cdots+(\frac{p^2(p-1)^2}{\sqrt{p^{4m-2}-p^{2m}-p^{2m-2}}}+$$
$$\frac{p^3(p-1)^2}{\sqrt{p^{4m-3}-p^{2m-1}-p^{2m-2}}})+\frac{p(p-1)^2}{\sqrt{p^{4m-1}-p^{2m}-p^{2m-1}}}+(\frac{(p^m-p^{m-1})(p^m-p^{m-1}-1)}{2\sqrt{p^{m+1}(p^{m+1}-2)}}+$$
$$\cdots+\frac{(p-1)(p-2)}{2\sqrt{p^{2m}(p^{2m}-2)}})$$

If n *is even and* $m=\lceil\frac{n}{2}\rceil$, *then* $PGO(\Gamma(\mathbb{Z}_{p^n}))=$

$$\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m-1}+p-3)(p-1)(p^{2m-1}-2)}}+(\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{2m-1}+p^2-3)(p^2-1)(p^{2m-1}-2)}}+$$
$$\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m-2}+p^2-3)(p^2-1)(p^{2m-2}-2)}})+(\frac{p^{2m-4}(p-1)^2}{\sqrt{(p^{2m-1}+p^3-3)(p^3-1)(p^{2m-1}-2)}}+$$
$$\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{2m-2}+p^3-3)(p^3-1)(p^{2m-2}-2)}}+\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m-3}+p^3-3)(p^3-1)(p^{2m-3}-2)}})+\cdots+$$

$$(\frac{p^m(p-1)^2}{\sqrt{(p^{2m-1}+p^{m-1}-3)(p^{m-1}-1)(p^{2m-1}-2)}}+\cdots+$$
$$\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{m+1}+p^{m-1}-3)(p^{m-1}-1)(p^{m+1}-2)}})+(\frac{p^{m-1}(p-1)^2}{\sqrt{(p^{2m-1}+p^m-4)(p^m-2)(p^{2m-1}-2)}}+$$
$$\cdots+\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{m+1}+p^m-4)(p^m-2)(p^{m+1}-2)}})+(\frac{p^{m-2}(p-1)^2}{\sqrt{(p^{2m-1}+p^{m+1}-4)(p^{m+1}-2)(p^{2m-1}-2)}}$$
$$+\cdots+\frac{p^{2m-5}(p-1)^2}{\sqrt{(p^{m+2}+p^{m+1}-4)(p^{m+1}-2)(p^{m+2}-2)}})+\cdots+$$
$$(\frac{p^2(p-1)^2}{\sqrt{(p^{2m-1}+p^{2m-3}-4)(p^{2m-3}-2)(p^{2m-1}-2)}}+$$
$$\frac{p^3(p-1)^2}{\sqrt{(p^{2m-2}+p^{2m-3}-4)(p^{2m-3}-2)(p^{2m-2}-2)}})+$$

$$\frac{p(p-1)^2}{\sqrt{(p^{2m-1}+p^{2m-2}-4)(p^{2m-2}-2)(p^{2m-1}-2)}}+(\frac{(p^m-p^{m-1})(p^m-p^{m-1}-1)}{2\sqrt{2(p^m-2)^3}}+\cdots+$$
$$\frac{(p-1)(p-2)}{2\sqrt{2(p^{2m-1}-2)^3}})$$

If n is odd and $m+1=\lceil\frac{n}{2}\rceil$, then $PGO(\Gamma(\mathbb{Z}_{p^n}))=$

$$\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{2m}+p-3)(p-1)(p^{2m}-2)}}+(\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m}+p^2-3)(p^2-1)(p^{2m}-2)}}+$$
$$\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{2m-1}+p^2-3)(p^2-1)(p^{2m-1}-2)}})+(\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{2m}+p^3-3)(p^3-1)(p^{2m}-2)}}+$$
$$\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m-1}+p^3-3)(p^3-1)(p^{2m-1}-2)}}+\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{2m-2}+p^3-3)(p^3-1)(p^{2m-2}-2)}})+\cdots+$$

$$(\frac{p^{m+1}(p-1)^2}{\sqrt{(p^{2m}+p^{m-1}-3)(p^{m-1}-1)(p^{2m}-2)}}+\cdots+\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{m+2}+p^{m-1}-3)(p^{m-1}-1)(p^{m+2}-2)}})$$
$$+(\frac{p^{m}(p-1)^2}{\sqrt{(p^{2m}+p^{m}-4)(p^{m}-2)(p^{2m}-2)}}+\cdots+\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{m+1}+p^{m}-4)(p^{m}-2)(p^{m+1}-2)}})+$$
$$(\frac{p^{m-1}(p-1)^2}{\sqrt{(p^{2m}+p^{m+1}-4)(p^{m+1}-2)(p^{2m}-2)}}+\cdots+\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{m+2}+p^{m+1}-4)(p^{m+1}-2)(p^{m+2}-2)}})$$
$$+\cdots+(\frac{p^2(p-1)^2}{\sqrt{(p^{2m}+p^{2m-2}-4)(p^{2m-2}-2)(p^{2m}-2)}}+$$
$$\frac{p^3(p-1)^2}{\sqrt{(p^{2m-1}+p^{2m-2}-4)(p^{2m-2}-2)(p^{2m-1}-2)}})+\frac{p(p-1)^2}{\sqrt{(p^{2m}+p^{2m-1}-4)(p^{2m-1}-2)(p^{2m}-2)}}$$
$$+(\frac{(p^m-p^{m-1})(p^m-p^{m-1}-1)}{2\sqrt{2(p^{m+1}-2)^3}}+\cdots+\frac{(p-1)(p-2)}{2\sqrt{2(p^{2m}-2)^3}})$$

Proof. Let $R=\mathbb{Z}_{p^n}$ with p prime and $n\in\mathbb{N}$. Then the set of non-zero zero divisors of R is $Z^*(R)=\{p,2p,3p,\cdots,(p^{n-1}-1)p\}$ with cardinality $p^{n-1}-1$.

Now we rewrite $Z^*(R)=V_1\cup V_2\cup\cdots\cup V_{n-1}$, where

$V_1=\{k_1p:k_1=1,2,3,\cdots,p^{n-1}-1 \textit{ and } p\nmid k_1\}$

$V_2=\{k_2p^2:k_2=1,2,3,\cdots,p^{n-2}-1 \textit{ and } p\nmid k_2\}$

$V_3=\{k_3p^3:k_2=1,2,3,\cdots,p^{n-3}-1 \textit{ and } p\nmid k_3\}$

$\vdots$

$V_i=\{k_ip^i:k_i=1,2,3,\cdots,p^{n-i}-1 \textit{ and } p\nmid k_i\}$

$\vdots$

$V_{n-1}=\{k_{n-1}p^{n-1}:k_{n-1}=1,2,3,\cdots,p-1\}$

with $|V_1|=p^{n-1}-p^{n-2},|V_2|=p^{n-2}-p^{n-3},|V_3|=p^{n-3}-p^{n-4},\cdots,$ $|V_i|=p^{n-i}-p^{n-(i+1)},\cdots,|V_{n-1}|=p-1$.

Case-1: Let n be an even number, i.e., $n = 2m$ for some $m \in \mathbb{N}$. Then, $\lceil \frac{n}{2} \rceil = m$. Then one can visualize the adjacency structure of $\Gamma(\mathbb{Z}_{p^n})$ in the Figure 7.4.

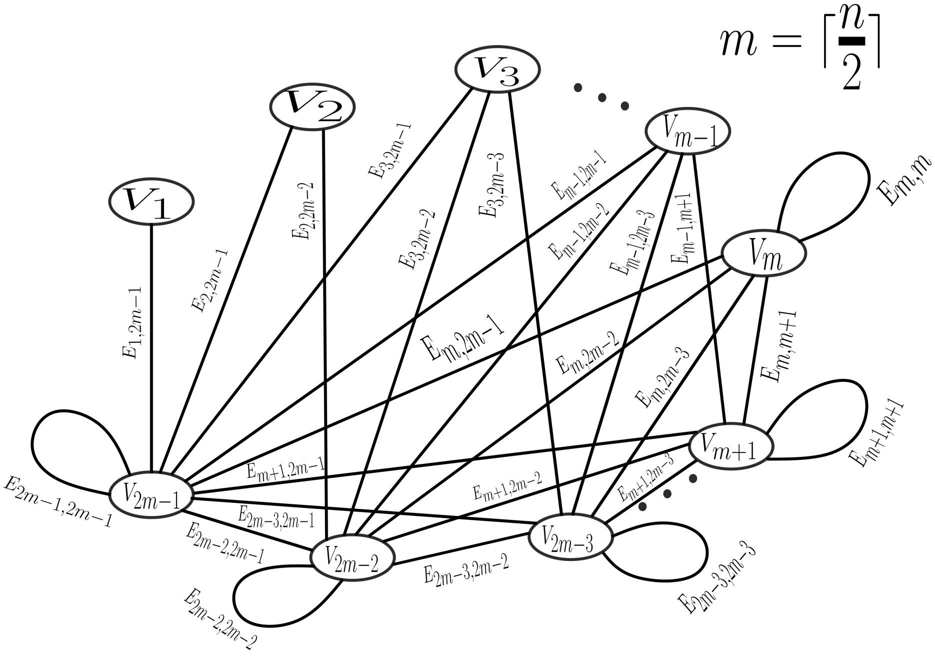

Figure 7.4: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{p^n})$, when n is even.

Now one can see that degrees of each V_i for $i = 1, 2, 3, 4$ are in the following.

$d(x) = |V_{2m-1}| = p - 1 \ \forall x \in V_1$.

$d(x) = |V_{2m-1}| + |V_{2m-2}| = p^2 - 1 \ \forall x \in V_2$.

$\vdots$

$d(x) = |V_{2m-1}| + |V_{2m-2}| + \cdots + |V_{m+1}| = p^{m-1} - 1 \ \forall x \in V_{m-1}$.

$d(x) = |V_{2m-1}| + |V_{2m-2}| + \cdots + |V_{m+1}| + |V_m| - 1 = p^m - 2 \ \forall x \in V_m$

$d(x) = |V_{2m-1}| + |V_{2m-2}| + \cdots + |V_m| + |V_{m+1}| - 1 = p^{m+1} - 2 \ \forall x \in V_{m+1}$

$\vdots$

$d(x) = |V_{2m-1}| + |V_{2m-2}| + |V_{2m-3}| - 1 + \cdots + |V_3| = p^{2m-3} - 2 \ \forall x \in V_{2m-3}$

$d(x) = |V_{2m-1}| + |V_{2m-2}| - 1 + |V_{2m-3}| + \cdots + |V_2| = p^{2m-2} - 2 \ \forall x \in V_{2m-2}$

$d(x) = |V_{2m-1}| - 1 + |V_{2m-2}| + |V_{2m-3}| + \cdots + |V_1| = p^{2m-1} - 2 \ \forall x \in V_{2m-1}$.

Now, we divide the edge set of $\Gamma(\mathbb{Z}_{p^n})$ as

$E = E_{1,2m-1} \cup (E_{2,2m-1} \cup E_{2,2m-2}) \cup (E_{3,2m-1} \cup E_{3,2m-2} \cup E_{3,2m-3}) \cup \cdots \cup (E_{m-1,2m-1} \cup \cdots \cup E_{m-1,m+1}) \cup (E_{m,2m-1} \cup \cdots \cup E_{m,m+1}) \cup (E_{m+1,2m-1} \cup \cdots \cup E_{m+1,m+2}) \cup \cdots \cup$

$(E_{2m-3,2m-1} \cup E_{2m-3,2m-2}) \cup E_{2m-2,2m-1} \cup (E_{m,m} \cup \cdots \cup E_{2m-1,2m-1})$.

Therefore, the Sum-Connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^n})$ is computed as follows:

$$SGO(\Gamma(\mathbb{Z}_{p^n})) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= [\sum_{uv\in E_{1,2m-1}} + (\sum_{uv\in E_{2,2m-1}} + \sum_{uv\in E_{2,2m-2}}) + (\sum_{uv\in E_{3,2m-1}} + \sum_{uv\in E_{3,2m-2}} + \sum_{uv\in E_{3,2m-3}}) + \cdots + (\sum_{uv\in E_{m-1,2m-1}} + \cdots + \sum_{uv\in E_{m-1,m+1}}) + (\sum_{uv\in E_{m,2m-1}} + \cdots + \sum_{uv\in E_{m,m+1}}) + (\sum_{uv\in E_{m+1,2m-1}} + \cdots + \sum_{uv\in E_{m+1,m+2}}) + \cdots + (\sum_{uv\in E_{2m-3,2m-1}} + \sum_{uv\in E_{2m-3,2m-2}}) + \sum_{uv\in E_{2m-2,2m-1}} + (\sum_{uv\in E_{m,m}} + \cdots + \sum_{uv\in E_{2m-1,2m-1}})] \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \frac{|V_1||V_{2m-1}|}{\sqrt{p-1+p^{2m-1}-2+(p-1)(p^{2m-1}-2)}} + (\frac{|V_2||V_{2m-1}|}{\sqrt{p^2-1+p^{2m-1}-2+(p^2-1)(p^{2m-1}-2)}} + \frac{|V_2||V_{2m-2}|}{\sqrt{p^2-1+p^{2m-2}-2+(p^2-1)(p^{2m-2}-2)}}) + (\frac{|V_3||V_{2m-1}|}{\sqrt{p^3-1+p^{2m-1}-2+(p^3-1)(p^{2m-1}-2)}} + \frac{|V_3||V_{2m-2}|}{\sqrt{p^3-1+p^{2m-2}-2+(p^3-1)(p^{2m-2}-2)}} + \frac{|V_3||V_{2m-3}|}{\sqrt{p^3-1+p^{2m-3}-2+(p^3-1)(p^{2m-3}-2)}})$$

$$+ \cdots + (\frac{|V_{m-1}||V_{2m-1}|}{\sqrt{p^{m-1}-1+p^{2m-1}-2+(p^{m-1}-1)(p^{2m-1}-2)}} + \cdots + \frac{|V_{m-1}||V_{m+1}|}{\sqrt{p^{m-1}-1+p^{m+1}-2+(p^{m-1}-1)(p^{m+1}-2)}}) +$$

$$(\frac{|V_m||V_{2m-1}|}{\sqrt{p^m-2+p^{2m-1}-2+(p^m-2)(p^{2m-1}-2)}} + \cdots + \frac{|V_m||V_{m+1}|}{\sqrt{p^m-2+p^{m+1}-2+(p^m-2)(p^{m+1}-2)}}) +$$

$$(\frac{|V_{m+1}||V_{2m-1}|}{\sqrt{p^{m+1}-2+p^{2m-1}-2+(p^{m+1}-2)(p^{2m-1}-2)}} + \cdots + \frac{|V_{m+1}||V_{m+2}|}{\sqrt{p^{m+1}-2+p^{m+2}-2+(p^{m+1}-2)(p^{m+2}-2)}}) + \cdots +$$

$$(\frac{|V_{2m-3}||V_{2m-1}|}{\sqrt{p^{2m-3}-2+p^{2m-1}-2+(p^{2m-3}-2)(p^{2m-1}-2)}} + \frac{|V_{2m-3}||V_{2m-2}|}{\sqrt{p^{2m-3}-2+p^{2m-2}-2+(p^{2m-3}-2)(p^{2m-2}-2)}}) +$$

$$\frac{|V_{2m-2}||V_{2m-1}|}{\sqrt{p^{2m-2}-2+p^{2m-1}-2+(p^{2m-2}-2)(p^{2m-1}-2)}} + (\frac{\binom{|V_m|}{2}}{\sqrt{2(p^m-2)+(p^m-2)^2}} + \cdots +$$

$$\frac{\binom{|V_{2m-1}|}{2}}{\sqrt{2(p^{2m-1}-2)+(p^{2m-1}-2)^2}})$$

$$= \frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m}-p-1}} + (\frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+1}-p^2-1}} + \frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m}-p^2-1}}) + (\frac{p^{2m-4}(p-1)^2}{\sqrt{p^{2m+2}-p^3-1}} +$$

$$\frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+1}-p^3-1}} + \frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m}-p^3-1}}) + \cdots + (\frac{p^{m}(p-1)^2}{\sqrt{p^{3m-2}-p^{m-1}-1}} + \cdots +$$

$$\frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m}-p^{m-1}-1}}) + (\frac{p^{m-1}(p-1)^2}{\sqrt{p^{3m-1}-p^{2m-1}-p^m}} + \cdots + \frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+1}-p^{m+1}-p^m}}) +$$

$$(\frac{p^{m-2}(p-1)^2}{\sqrt{p^{3m}-p^{2m-1}-p^{m+1}}} + \cdots + \frac{p^{2m-5}(p-1)^2}{\sqrt{p^{2m+3}-p^{m+2}-p^{m+1}}}) + \cdots + (\frac{p^{2}(p-1)^2}{\sqrt{p^{4m-4}-p^{2m-1}-p^{2m-3}}} +$$

$$\frac{p^{3}(p-1)^2}{\sqrt{p^{4m-5}-p^{2m-2}-p^{2m-3}}}) + \frac{p(p-1)^2}{\sqrt{p^{4m-3}-p^{2m-1}-p^{2m-2}}} + (\frac{(p^m-p^{m-1})(p^m-p^{m-1}-1)}{2\sqrt{p^m(p^m-2)}} +$$

$$\cdots + \frac{(p-1)(p-2)}{2\sqrt{p^{2m-1}(p^{2m-1}-2)}}).$$

Moreover, the Product-Connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^n})$ is

$$PGO(\Gamma(\mathbb{Z}_n)) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= [\sum_{uv\in E_{1,2m-1}} + (\sum_{uv\in E_{2,2m-1}} + \sum_{uv\in E_{2,2m-2}}) + (\sum_{uv\in E_{3,2m-1}} + \sum_{uv\in E_{3,2m-2}} + \sum_{uv\in E_{3,2m-3}}) + \cdots +$$

$$(\sum_{uv\in E_{m-1,2m-1}} + \cdots + \sum_{uv\in E_{m-1,m+1}}) + (\sum_{uv\in E_{m,2m-1}} + \cdots + \sum_{uv\in E_{m,m+1}}) + (\sum_{uv\in E_{m+1,2m-1}} + \cdots +$$

$$\sum_{uv\in E_{m+1,m+2}}) + \cdots + (\sum_{uv\in E_{2m-3,2m-1}} + \sum_{uv\in E_{2m-3,2m-2}}) + \sum_{uv\in E_{2m-2,2m-1}} + (\sum_{uv\in E_{m,m}} + \cdots +$$

$$\sum_{uv\in E_{2m-1,2m-1}})] \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= \frac{|V_1||V_{2m-1}|}{\sqrt{(p-1+p^{2m-1}-2)(p-1)(p^{2m-1}-2)}} + (\frac{|V_2||V_{2m-1}|}{\sqrt{(p^2-1+p^{2m-1}-2)(p^2-1)(p^{2m-1}-2)}} +$$

$$\frac{|V_2||V_{2m-2}|}{\sqrt{(p^2-1+p^{2m-2}-2)(p^2-1)(p^{2m-2}-2)}}) + (\frac{|V_3||V_{2m-1}|}{\sqrt{(p^3-1+p^{2m-1}-2)(p^3-1)(p^{2m-1}-2)}} +$$

$$\frac{|V_3||V_{2m-2}|}{\sqrt{(p^3-1+p^{2m-2}-2)(p^3-1)(p^{2m-2}-2)}} + \frac{|V_3||V_{2m-3}|}{\sqrt{(p^3-1+p^{2m-3}-2)(p^3-1)(p^{2m-3}-2)}}) +$$

$$\cdots + (\frac{|V_{m-1}||V_{2m-1}|}{\sqrt{(p^{m-1}-1+p^{2m-1}-2)(p^{m-1}-1)(p^{2m-1}-2)}} + \cdots +$$

$$\frac{|V_{m-1}||V_{m+1}|}{\sqrt{(p^{m-1}-1+p^{m+1}-2)(p^{m-1}-1)(p^{m+1}-2)}}) +$$

$$(\frac{|V_m||V_{2m-1}|}{\sqrt{(p^m-2+p^{2m-1}-2)(p^m-2)(p^{2m-1}-2)}} + \cdots +$$

$$\frac{|V_m||V_{m+1}|}{\sqrt{(p^m-2+p^{m+1}-2)(p^m-2)(p^{m+1}-2)}}) +$$

$$(\frac{|V_{m+1}||V_{2m-1}|}{\sqrt{(p^{m+1}-2+p^{2m-1}-2)(p^{m+1}-2)(p^{2m-1}-2)}}+\cdots+$$

$$\frac{|V_{m+1}||V_{m+2}|}{\sqrt{(p^{m+1}-2+p^{m+2}-2)(p^{m+1}-2)(p^{m+2}-2)}})+\cdots+$$

$$(\frac{|V_{2m-3}||V_{2m-1}|}{\sqrt{(p^{2m-3}-2+p^{2m-1}-2)(p^{2m-3}-2)(p^{2m-1}-2)}}+$$

$$\frac{|V_{2m-3}||V_{2m-2}|}{\sqrt{(p^{2m-3}-2+p^{2m-2}-2)(p^{2m-3}-2)(p^{2m-2}-2)}})+$$

$$\frac{|V_{2m-2}||V_{2m-1}|}{\sqrt{(p^{2m-2}-2+p^{2m-1}-2)(p^{2m-2}-2)(p^{2m-1}-2)}}+$$

$$(\frac{\binom{|V_m|}{2}}{\sqrt{2(p^m-2)(p^m-2)^2}}+\cdots+\frac{\binom{|V_{2m-1}|}{2}}{\sqrt{2(p^{2m-1}-2)(p^{2m-1}-2)^2}})$$

$$=\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m-1}+p-3)(p-1)(p^{2m-1}-2)}}+(\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{2m-1}+p^2-3)(p^2-1)(p^{2m-1}-2)}}+$$

$$\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m-2}+p^2-3)(p^2-1)(p^{2m-2}-2)}})+(\frac{p^{2m-4}(p-1)^2}{\sqrt{(p^{2m-1}+p^3-3)(p^3-1)(p^{2m-1}-2)}}+$$

$$\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{2m-2}+p^3-3)(p^3-1)(p^{2m-2}-2)}}+\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m-3}+p^3-3)(p^3-1)(p^{2m-3}-2)}})$$

$$+\cdots+(\frac{p^m(p-1)^2}{\sqrt{(p^{2m-1}+p^{m-1}-3)(p^{m-1}-1)(p^{2m-1}-2)}}+\cdots+$$

$$\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{m+1}+p^{m-1}-3)(p^{m-1}-1)(p^{m+1}-2)}})+(\frac{p^{m-1}(p-1)^2}{\sqrt{(p^{2m-1}+p^m-4)(p^m-2)(p^{2m-1}-2)}}+$$

$$\cdots+\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{m+1}+p^m-4)(p^m-2)(p^{m+1}-2)}})+(\frac{p^{m-2}(p-1)^2}{\sqrt{(p^{2m-1}+p^{m+1}-4)(p^{m+1}-2)(p^{2m-1}-2)}}$$

$$+\cdots+\frac{p^{2m-5}(p-1)^2}{\sqrt{(p^{m+2}+p^{m+1}-4)(p^{m+1}-2)(p^{m+2}-2)}})+\cdots+$$

$$(\frac{p^2(p-1)^2}{\sqrt{(p^{2m-1}+p^{2m-3}-4)(p^{2m-3}-2)(p^{2m-1}-2)}}+$$

$$\frac{p^3(p-1)^2}{\sqrt{(p^{2m-2}+p^{2m-3}-4)(p^{2m-3}-2)(p^{2m-2}-2)}})+$$

$$\frac{p(p-1)^2}{\sqrt{(p^{2m-1}+p^{2m-2}-4)(p^{2m-2}-2)(p^{2m-1}-2)}}+$$

$$(\frac{(p^m-p^{m-1})(p^m-p^{m-1}-1)}{2\sqrt{2(p^m-2)^3}}+\cdots+\frac{(p-1)(p-2)}{2\sqrt{2(p^{2m-1}-2)^3}})$$

Case-2: Let n be an odd number, i.e., $n=2m+1$ for $m\in\mathbb{N}$. Then, $\lceil\frac{n}{2}\rceil=m+1$. Then one can visualize the adjacency structure of $\Gamma(\mathbb{Z}_{p^n})$ in the Figure 7.5.

Now one can see that degrees of each V_i for $i=1,2,3,4$ are in the following.

$d(x)=|V_{2m}|=p-1\ \forall x\in V_1$.

$d(x)=|V_{2m}|+|V_{2m-1}|=p^2-1\ \forall x\in V_2$.

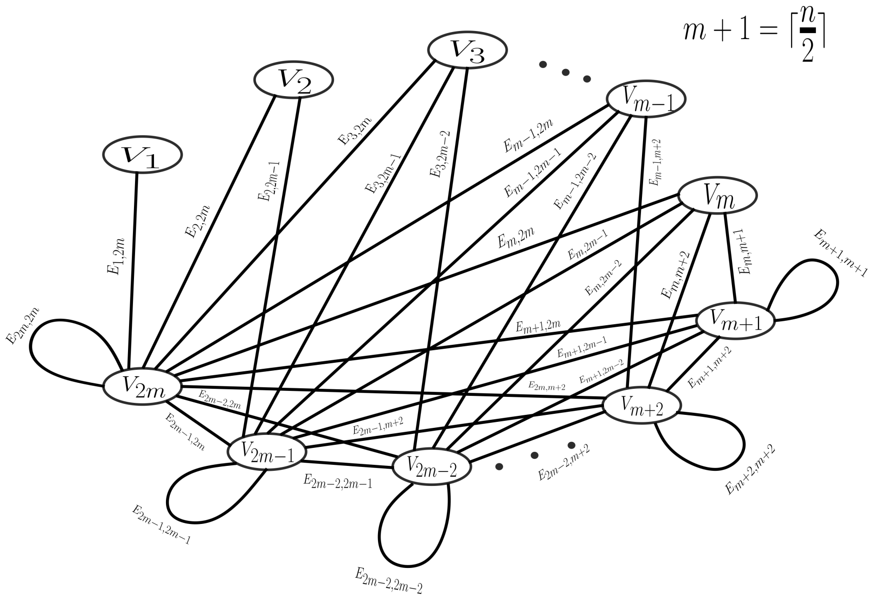

Figure 7.5: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{p^n})$, when n is odd.

$\vdots$

$d(x) = |V_{2m}| + |V_{2m-1}| + \cdots + |V_{m+2}| = p^{m-1} - 1 \; \forall x \in V_{m-1}.$

$d(x) = |V_{2m}| + |V_{2m-1}| + \cdots + |V_{m+1}| = p^{m} - 1 \; \forall x \in V_{m}$

$d(x) = |V_{2m}| + |V_{2m-1}| + \cdots + |V_{m}| + |V_{m+1}| - 1 = p^{m+1} - 2 \; \forall x \in V_{m+1}$

$\vdots$

$d(x) = |V_{2m}| + |V_{2m-1}| + |V_{2m-2}| - 1 + \cdots + |V_{3}| = p^{2m-2} - 2 \; \forall x \in V_{2m-2}$

$d(x) = |V_{2m}| + |V_{2m-1}| - 1 + |V_{2m-2}| + \cdots + |V_{2}| = p^{2m-1} - 2 \; \forall x \in V_{2m-1}$

$d(x) = |V_{2m}| - 1 + |V_{2m-1}| + |V_{2m-2}| + \cdots + |V_{1}| = p^{2m} - 2 \; \forall x \in V_{2m}.$

Now, we divide the edge set of $\Gamma(\mathbb{Z}_{p^n})$ as

$E = E_{1,2m} \cup (E_{2,2m} \cup E_{2,2m-1}) \cup (E_{3,2m} \cup E_{3,2m-1} \cup E_{3,2m-2}) \cup \cdots \cup (E_{m-1,2m} \cup \cdots \cup E_{m-1,m+2}) \cup (E_{m,2m} \cup \cdots \cup E_{m,m+1}) \cup (E_{m+1,2m} \cup \cdots \cup E_{m+1,m+2}) \cup \cdots \cup (E_{2m-2,2m} \cup E_{2m-2,2m-1}) \cup E_{2m-1,2m} \cup (E_{m+1,m+1} \cup \cdots \cup E_{2m,2m}).$

Therefore, the Sum-Connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^n})$ is calculated as follows:

$$SGO(\Gamma(\mathbb{Z}_{p^n})) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u) + d(v) + d(u)d(v)}}$$

$$= [\sum_{uv\in E_{1,2m}} + (\sum_{uv\in E_{2,2m}} + \sum_{uv\in E_{2,2m-1}}) + (\sum_{uv\in E_{3,2m}} + \sum_{uv\in E_{3,2m-1}} + \sum_{uv\in E_{3,2m-2}}) + \cdots + (\sum_{uv\in E_{m-1,2m}} +$$

$$\cdots + \sum_{uv\in E_{m-1,m+2}}) + (\sum_{uv\in E_{m,2m}} + \cdots + \sum_{uv\in E_{m,m+1}}) + (\sum_{uv\in E_{m+1,2m}} + \cdots + \sum_{uv\in E_{m+1,m+2}}) +$$

$$\cdots + (\sum_{uv\in E_{2m-2,2m}} + \sum_{uv\in E_{2m-2,2m-1}}) + \sum_{uv\in E_{2m-1,2m}} + (\sum_{uv\in E_{m+1,m+1}} + \cdots +$$

$$\sum_{uv\in E_{2m,2m}})]\frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \frac{|V_1||V_{2m}|}{\sqrt{p-1+p^{2m}-2+(p-1)(p^{2m}-2)}} + (\frac{|V_2||V_{2m}|}{\sqrt{p^2-1+p^{2m}-2+(p^2-1)(p^{2m}-2)}} +$$

$$\frac{|V_2||V_{2m-1}|}{\sqrt{p^2-1+p^{2m-1}-2+(p^2-1)(p^{2m-1}-2)}}) + (\frac{|V_3||V_{2m}|}{\sqrt{p^3-1+p^{2m}-2+(p^3-1)(p^{2m}-2)}} +$$

$$\frac{|V_3||V_{2m-1}|}{\sqrt{p^3-1+p^{2m-1}-2+(p^3-1)(p^{2m-1}-2)}} + \frac{|V_3||V_{2m-2}|}{\sqrt{p^3-1+p^{2m-2}-2+(p^3-1)(p^{2m-2}-2)}})$$

$$+ \cdots + (\frac{|V_{m-1}||V_{2m}|}{\sqrt{p^{m-1}-1+p^{2m}-2+(p^{m-1}-1)(p^{2m}-2)}} + \cdots +$$

$$\frac{|V_{m-1}||V_{m+2}|}{\sqrt{p^{m-1}-1+p^{m+2}-2+(p^{m-1}-1)(p^{m+2}-2)}}) +$$

$$(\frac{|V_m||V_{2m}|}{\sqrt{p^m-2+p^{2m}-2+(p^m-2)(p^{2m}-2)}} + \cdots +$$

$$\frac{|V_m||V_{m+1}|}{\sqrt{p^m-2+p^{m+1}-2+(p^m-2)(p^{m+1}-2)}}) +$$

$$(\frac{|V_{m+1}||V_{2m}|}{\sqrt{p^{m+1}-2+p^{2m}-2+(p^{m+1}-2)(p^{2m}-2)}} + \cdots +$$

$$\frac{|V_{m+1}||V_{m+2}|}{\sqrt{p^{m+1}-2+p^{m+2}-2+(p^{m+1}-2)(p^{m+2}-2)}}) + \cdots +$$

$$(\frac{|V_{2m-2}||V_{2m}|}{\sqrt{p^{2m-2}-2+p^{2m}-2+(p^{2m-2}-2)(p^{2m}-2)}} +$$

$$\frac{|V_{2m-2}||V_{2m-1}|}{\sqrt{p^{2m-2}-2+p^{2m-1}-2+(p^{2m-2}-2)(p^{2m-1}-2)}}) +$$

$$\frac{|V_{2m-1}||V_{2m}|}{\sqrt{p^{2m-1}-2+p^{2m}-2+(p^{2m-1}-2)(p^{2m}-2)}} + (\frac{\binom{|V_{m+1}|}{2}}{\sqrt{2(p^{m+1}-2)+(p^{m+1}-2)^2}}$$

$$+ \cdots + \frac{\binom{|V_{2m}|}{2}}{\sqrt{2(p^{2m}-2)+(p^{2m}-2)^2}})$$

$$= \frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p-1}} + (\frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m+2}-p^2-1}} + \frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p^2-1}}) + (\frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+3}-p^3-1}}$$

$$+ \frac{p^{2m-2}(p-1)^2}{\sqrt{p^{2m+2}-p^3-1}} + \frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p^3-1}}) + \cdots + (\frac{p^{m+1}(p-1)^2}{\sqrt{p^{3m-1}-p^{m-1}-1}} + \cdots +$$

$$\frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p^{m-1}-1}}) + (\frac{p^m(p-1)^2}{\sqrt{p^{3m}-p^{2m}-p^m}} + \cdots + \frac{p^{2m-1}(p-1)^2}{\sqrt{p^{2m+1}-p^{m+1}-p^m}}) +$$

$$(\frac{p^{m-1}(p-1)^2}{\sqrt{p^{3m+1}-p^{2m}-p^{m+1}}}+\cdots+\frac{p^{2m-3}(p-1)^2}{\sqrt{p^{2m+3}-p^{m+2}-p^{m+1}}})+\cdots+(\frac{p^2(p-1)^2}{\sqrt{p^{4m-2}-p^{2m}-p^{2m-2}}}$$
$$+\frac{p^3(p-1)^2}{\sqrt{p^{4m-3}-p^{2m-1}-p^{2m-2}}})+\frac{p(p-1)^2}{\sqrt{p^{4m-1}-p^{2m}-p^{2m-1}}}+(\frac{(p^m-p^{m-1})(p^m-p^{m-1}-1)}{2\sqrt{p^{m+1}(p^{m+1}-2)}}$$
$$+\cdots+\frac{(p-1)(p-2)}{2\sqrt{p^{2m}(p^{2m}-2)}}).$$

Moreover, the Product-Connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^n})$ is

$$PGO(\Gamma(\mathbb{Z}_n)) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$
$$= [\sum_{uv\in E_{1,2m}} +(\sum_{uv\in E_{2,2m}} + \sum_{uv\in E_{2,2m-1}})+(\sum_{uv\in E_{3,2m}} + \sum_{uv\in E_{3,2m-1}} + \sum_{uv\in E_{3,2m-2}})+\cdots+(\sum_{uv\in E_{m-1,2m}} +$$
$$\cdots + \sum_{uv\in E_{m-1,m+2}}) + (\sum_{uv\in E_{m,2m}} + \cdots + \sum_{uv\in E_{m,m+1}}) + (\sum_{uv\in E_{m+1,2m}} + \cdots + \sum_{uv\in E_{m+1,m+2}}) +$$
$$\cdots + (\sum_{uv\in E_{2m-2,2m}} + \sum_{uv\in E_{2m-2,2m-1}}) + \sum_{uv\in E_{2m-1,2m}} + (\sum_{uv\in E_{m+1,m+1}} + \cdots +$$
$$\sum_{uv\in E_{2m,2m}})]\frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$
$$= \frac{|V_1||V_{2m}|}{\sqrt{(p-1+p^{2m}-2)(p-1)(p^{2m}-2)}} + (\frac{|V_2||V_{2m}|}{\sqrt{(p^2-1+p^{2m}-2)(p^2-1)(p^{2m}-2)}} +$$
$$\frac{|V_2||V_{2m-1}|}{\sqrt{(p^2-1+p^{2m-1}-2)(p^2-1)(p^{2m-1}-2)}})+(\frac{|V_3||V_{2m}|}{\sqrt{(p^3-1+p^{2m}-2)(p^3-1)(p^{2m}-2)}}+$$
$$\frac{|V_3||V_{2m-1}|}{\sqrt{(p^3-1+p^{2m-1}-2)(p^3-1)(p^{2m-1}-2)}}+\frac{|V_3||V_{2m-2}|}{\sqrt{(p^3-1+p^{2m-2}-2)(p^3-1)(p^{2m-2}-2)}})$$
$$+\cdots+(\frac{|V_{m-1}||V_{2m}|}{\sqrt{(p^{m-1}-1+p^{2m}-2)(p^{m-1}-1)(p^{2m}-2)}}+\cdots+$$
$$\frac{|V_{m-1}||V_{m+2}|}{\sqrt{(p^{m-1}-1+p^{m+2}-2)(p^{m-1}-1)(p^{m+2}-2)}}) +$$
$$(\frac{|V_m||V_{2m}|}{\sqrt{(p^m-2+p^{2m}-2)(p^m-2)(p^{2m}-2)}} + \cdots +$$
$$\frac{|V_m||V_{m+1}|}{\sqrt{(p^m-2+p^{m+1}-2)(p^m-2)(p^{m+1}-2)}}) +$$
$$(\frac{|V_{m+1}||V_{2m}|}{\sqrt{(p^{m+1}-2+p^{2m}-2)(p^{m+1}-2)(p^{2m}-2)}} + \cdots +$$
$$\frac{|V_{m+1}||V_{m+2}|}{\sqrt{(p^{m+1}-2+p^{m+2}-2)(p^{m+1}-2)(p^{m+2}-2)}}) + \cdots +$$
$$(\frac{|V_{2m-2}||V_{2m}|}{\sqrt{(p^{2m-2}-2+p^{2m}-2)(p^{2m-2}-2)(p^{2m}-2)}} +$$
$$\frac{|V_{2m-2}||V_{2m-1}|}{\sqrt{(p^{2m-2}-2+p^{2m-1}-2)(p^{2m-2}-2)(p^{2m-1}-2)}}) +$$

$$\frac{|V_{2m-1}||V_{2m}|}{\sqrt{(p^{2m-1}-2+p^{2m}-2)(p^{2m-1}-2)(p^{2m}-2)}}+(\frac{\binom{|V_{m+1}|}{2}}{\sqrt{2(p^{m+1}-2)(p^{m+1}-2)^2}}$$
$$+\cdots+\frac{\binom{|V_{2m}|}{2}}{\sqrt{2(p^{2m}-2)(p^{2m}-2)^2}})$$
$$=\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{2m}+p-3)(p-1)(p^{2m}-2)}}+(\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m}+p^2-3)(p^2-1)(p^{2m}-2)}}+$$
$$\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{2m-1}+p^2-3)(p^2-1)(p^{2m-1}-2)}})+(\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{2m}+p^3-3)(p^3-1)(p^{2m}-2)}}+$$
$$\frac{p^{2m-2}(p-1)^2}{\sqrt{(p^{2m-1}+p^3-3)(p^3-1)(p^{2m-1}-2)}}+\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{2m-2}+p^3-3)(p^3-1)(p^{2m-2}-2)}})$$
$$+\cdots+(\frac{p^{m+1}(p-1)^2}{\sqrt{(p^{2m}+p^{m-1}-3)(p^{m-1}-1)(p^{2m}-2)}}+\cdots+$$
$$\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{m+2}+p^{m-1}-3)(p^{m-1}-1)(p^{m+2}-2)}})+(\frac{p^{m}(p-1)^2}{\sqrt{(p^{2m}+p^{m}-4)(p^{m}-2)(p^{2m}-2)}}+$$
$$\cdots+\frac{p^{2m-1}(p-1)^2}{\sqrt{(p^{m+1}+p^{m}-4)(p^{m}-2)(p^{m+1}-2)}})+(\frac{p^{m-1}(p-1)^2}{\sqrt{(p^{2m}+p^{m+1}-4)(p^{m+1}-2)(p^{2m}-2)}}$$
$$+\cdots+\frac{p^{2m-3}(p-1)^2}{\sqrt{(p^{m+2}+p^{m+1}-4)(p^{m+1}-2)(p^{m+2}-2)}})+\cdots+$$
$$(\frac{p^2(p-1)^2}{\sqrt{(p^{2m}+p^{2m-2}-4)(p^{2m-2}-2)(p^{2m}-2)}}+$$
$$\frac{p^3(p-1)^2}{\sqrt{(p^{2m-1}+p^{2m-2}-4)(p^{2m-2}-2)(p^{2m-1}-2)}})+$$
$$\frac{p(p-1)^2}{\sqrt{(p^{2m}+p^{2m-1}-4)(p^{2m-1}-2)(p^{2m}-2)}}+$$
$$(\frac{(p^m-p^{m-1})(p^m-p^{m-1}-1)}{2\sqrt{2(p^{m+1}-2)^3}}+\cdots+\frac{(p-1)(p-2)}{2\sqrt{2(p^{2m}-2)^3}}).\qquad\square$$

Theorem 7.2.7. *The Sum and Product Connectivity Gourava index of the zero divisor graph $\Gamma(\mathbb{Z}_{pq})$ are*

$$SGO(\Gamma(\mathbb{Z}_{pq}))=\frac{(p-1)(q-1)}{\sqrt{pq-1}}\ \textit{and}\ PGO(\Gamma(\mathbb{Z}_{pq}))=\frac{\sqrt{(p-1)(q-1)}}{\sqrt{(p+q-2)}}$$

where p and q are distinct odd primes.

Proof. Let $R=\mathbb{Z}_{pq}$, where p and q are distinct odd prime numbers. Then the set of non-zero zero divisors of R is $Z^*(R)=\{p,2p,3p,\cdots,(q-1)p,q,2q,3q,\cdots,(p-1)q\}$ with cardinality $p+q-2$.

Now we can rewrite the vertex set of $\Gamma(R)$ as $Z^*(R)=V_1\cup V_2$, where $V_1=\{p,2p,3p,\cdots,(q-1)p\}$ and $V_2=\{q,2q,3q,\cdots,(p-1)q\}$ with $|V_1|=q-1$ and

$|V_2| = p-1$. Since no two elements of V_1 are adjacent and every element of V_1 is adjacent with the elements of V_2. Also no two elements of V_2 are adjacent and every element of V_2 is adjacent with the elements of V_1. So, one can say that $\Gamma(\mathbb{Z}_{pq})$ is graph isomorphic to $K_{|V_1|,|V_2|}$.

Figure 7.6: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{pq})$

Therefore, the sum-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{pq})$ is

$$SGO(\Gamma(\mathbb{Z}_{pq})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= SGO(K_{|V_1|,|V_2|}) = SGO(K_{(q-1),(p-1)})$$

$$= \frac{(p-1)(q-1)}{\sqrt{p+q-2+(p-1)(q-1)}} = \frac{(p-1)(q-1)}{\sqrt{pq-1}}.$$

Moreover, the Product-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{pq})$ is

$$PGO(\Gamma(\mathbb{Z}_{pq})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= PGO(K_{|V_1|,|V_2|}) = PGO(K_{(q-1),(p-1)})$$

$$= \frac{(p-1)(q-1)}{\sqrt{(p+q-2)(p-1)(q-1)}} = \frac{\sqrt{(p-1)(q-1)}}{\sqrt{(p+q-2)}}. \qquad \square$$

Theorem 7.2.8. *The sum and product connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^2q})$ are*

$$SGO(\Gamma(\mathbb{Z}_{p^2q})) = \frac{(p-1)^2(q-1)}{\sqrt{p^2q-p-1}}+\frac{p(p-1)(q-1)}{\sqrt{p^2q-1}}+\frac{(p-1)(q-1)}{\sqrt{p^3q-p^2-1}}+\frac{(p-1)(p-2)}{2\sqrt{pq(pq-2)}}$$

$$PGO(\Gamma(\mathbb{Z}_{p^2q})) = \frac{(p-1)^{\frac{3}{2}}(q-1)}{\sqrt{(pq+p-3)(pq-2)}} + \frac{p\sqrt{(p-1)(q-1)}}{\sqrt{(p^2+q-2)(p+1)}} + \frac{\sqrt{(p-1)}(q-1)}{\sqrt{(p^2+pq-3)(p+1)(pq-2)}} + \frac{(p-1)(p-2)}{2(pq-2)\sqrt{2(pq-2)}}$$

where p, q are distinct odd prime numbers.

Proof. Let $R = \mathbb{Z}_{p^2q}$, where p, q are distinct prime numbers. Then the set of non-zero zero divisors of R is $V = Z^*(R) = V_1 \cup V_2 \cup V_3 \cup V_4$, where

$V_1 = \{k_1p : k_1 = 1,2,3,\cdots,(pq-1)\ \textit{and}\ p \nmid k_1,\ q \nmid k_1\}$, with $|V_1| = pq-p-q+1$

$V_2 = \{k_2p^2 : k_2 = 1,2,3,\cdots,q-1\}$, with $|V_2| = q-1$

$V_3 = \{k_3q : k_3 = 1,2,3,\cdots,p^2-1\ \textit{and}\ p \nmid k_3\}$, with $|V_3| = p^2-p$

$V_4 = \{k_4pq : k_4 = 1, 2, 3, \cdots, p-1\}$, with $|V_4| = p-1$
Also the cardinality of V is $p^2q - \phi(p^2q) - 1 = pq + p^2 - p - 1$.

By the similar way which we discussed in 7.2.4, the adjacency property of the zero divisor graph $\Gamma(\mathbb{Z}_{p^2q})$ is shown in Figure 7.7.

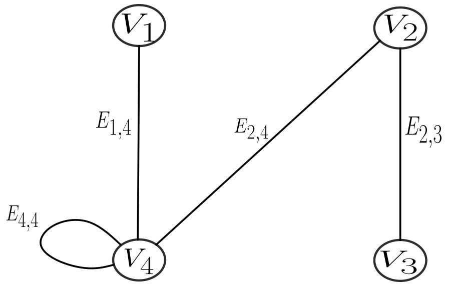

Figure 7.7: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{p^2q})$

Now one can see that degrees of each V_i for $i = 1, 2, 3, 4$ are in the following.

$$d(x) = |V_4| = p - 1 \ \forall x \in V_1.$$

$d(x) = |V_3| + |V_4| = p^2 - 1 \ \forall x \in V_2.$

$d(x) = |V_2| = q - 1 \ \forall x \in V_3.$

$d(x) = |V_1| + |V_2| + |V_4| - 1 = pq - 2 \ \forall x \in V_4.$

Now, we divide edge set E of $\Gamma(\mathbb{Z}_{p^2q})$ as $E = E_{1,4} \cup E_{2,3} \cup E_{2,4} \cup E_{4,4}$.

Therefore, the sum-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^2q})$ is

$$SGO(\Gamma(\mathbb{Z}_{p^2q})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \sum_{uv\in E_{1,4}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} + \sum_{uv\in E_{2,3}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$+ \sum_{uv\in E_{2,4}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}} + \sum_{uv\in E_{4,4}} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \frac{(p-1)^2(q-1)}{\sqrt{p-1+pq-2+(p-1)(pq-2)}} + \frac{p(p-1)(q-1)}{\sqrt{p^2-1+q-1+(p^2-1)(q-1)}} +$$

$$\frac{(p-1)(q-1)}{\sqrt{p^2-1+pq-2+(p^2-1)(pq-2)}} + \frac{(p-1)(p-2)}{2\sqrt{(pq-2)^2+2(pq-2)}}$$

$$= \frac{(p-1)^2(q-1)}{\sqrt{p^2q-p-1}} + \frac{p(p-1)(q-1)}{\sqrt{p^2q-1}} + \frac{(p-1)(q-1)}{\sqrt{p^3q-p^2-1}} + \frac{(p-1)(p-2)}{2\sqrt{pq(pq-2)}}.$$

Moreover, the product-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^2q})$ is

$$PGO(\Gamma(\mathbb{Z}_{p^2q})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$
$$= \sum_{uv\in E_{1,4}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} + \sum_{uv\in E_{2,3}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$
$$+ \sum_{uv\in E_{2,4}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}} + \sum_{uv\in E_{4,4}} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$
$$= \frac{(p-1)^2(q-1)}{\sqrt{(p-1+pq-2)(p-1)(pq-2)}} + \frac{p(p-1)(q-1)}{\sqrt{(p^2-1+q-1)(p^2-1)(q-1)}} +$$
$$\frac{(p-1)(q-1)}{\sqrt{(p^2-1+pq-2)(p^2-1)(pq-2)}} + \frac{(p-1)(p-2)}{2\sqrt{2(pq-2)(pq-2)^2}}$$
$$= \frac{(p-1)^{\frac{3}{2}}(q-1)}{\sqrt{(pq+p-3)(pq-2)}} + \frac{p\sqrt{(p-1)(q-1)}}{\sqrt{(p^2+q-2)(p+1)}} + \frac{\sqrt{(p-1)}(q-1)}{\sqrt{(p^2+pq-3)(p+1)(pq-2)}} +$$
$$\frac{(p-1)(p-2)}{2(pq-2)\sqrt{2(pq-2)}}.$$

□

Theorem 7.2.9. *The sum and product connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^2q^2})$ are*

$$SGO(\Gamma(\mathbb{Z}_{p^2q^2})) = \frac{q(p-1)^2(q-1)}{\sqrt{p^2q^2-p-1}} + \frac{pq(p-1)(q-1)}{\sqrt{p^2q^2-1}} + \frac{q(p-1)(q-1)}{\sqrt{p^3q^2-p^2-1}} +$$
$$\frac{p(p-1)^2(q-1)}{\sqrt{p^2q^2-q-1}} + \frac{p(p-1)(q-1)}{\sqrt{p^2q^3-q^2-1}} + \frac{(p-1)(q-1)^2}{\sqrt{p^3q^2-p^2q-pq}} + \frac{(p-1)^2(q-1)}{\sqrt{p^2q^3-pq^2-pq}} +$$
$$\frac{(p-1)(q-1)}{\sqrt{p^3q^3-p^2q-pq^2}} + \frac{(p-1)(q-1)(pq-p-q)}{2\sqrt{pq(pq-2)}} + \frac{(q-1)(q-2)}{2\sqrt{p^2q(p^2q-2)}} + \frac{(p-1)(p-2)}{2\sqrt{pq^2(pq^2-2)}}.$$

$$PGO(\Gamma(\mathbb{Z}_{p^2q^2})) = \frac{q(p-1)^{\frac{3}{2}}(q-1)}{\sqrt{(pq^2+p-3)(pq^2-2)}} + \frac{pq\sqrt{(p-1)(q-1)}}{\sqrt{(p^2+q^2-2)(p+1)(q+1)}} +$$
$$\frac{q\sqrt{p-1}(q-1)}{\sqrt{(pq^2+p^2-3)(p+1)pq^2-2}} + \frac{p(p-1)^2\sqrt{q-1}}{\sqrt{(p^2q+q-3)(p^2q-2)}} +$$
$$\frac{p(p-1)\sqrt{q-1}}{\sqrt{(p^2q+q^2-3)(q+1)(p^2q-2)}} + \frac{(p-1)(q-1)^2}{\sqrt{(p^2q+pq-4)(p^2q-2)(pq-2)}} +$$
$$\frac{(p-1)^2(q-1)}{\sqrt{(pq^2+pq-4)(pq^2-2)(pq-2)}} + \frac{(p-1)(q-1)}{\sqrt{(p^2q+pq^2-4)(p^2q-2)(pq^2-2)}} +$$
$$\frac{(p-1)(q-1)(pq-p-q)}{2(pq-2)\sqrt{(pq-2)}} + \frac{(q-1)(q-2)}{2(p^2q-2)\sqrt{(p^2q-2)}} + \frac{(p-1)(p-2)}{2(pq^2-2)\sqrt{(pq^2-2)}}.$$

where p, q are distinct odd prime numbers.

Proof. Let $R = \mathbb{Z}_{p^2q^2}$, where p, q are distinct prime numbers. Then the set of non-zero zero divisors of R is $V = Z^*(R) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6 \cup V_7$, where

$V_1 = \{k_1p : k_1 = 1, 2, 3, \cdots, pq^2-1 \text{ } and \text{ } p \nmid k_1, \ q \nmid k_1\}$, with $|V_1| = q(p-1)(q-1)$

$V_2 = \{k_2p^2 : k_2 = 1, 2, 3, \cdots, q^2-1 \text{ } and \text{ } q \nmid k_2\}$, with $|V_2| = q(q-1)$

$V_3 = \{k_3 q : k_3 = 1, 2, 3, \cdots, p^2q - 1 \ and \ p \nmid k_3,\ q \nmid k_3\}$, with $|V_3| = p(p-1)(q-1)$

$V_4 = \{k_4 q^2 : k_4 = 1, 2, 3, \cdots, p^2 - 1 \ and \ p \nmid k_4\}$, with $|V_4| = p(p-1)$

$V_5 = \{k_5 pq : k_5 = 1, 2, 3, \cdots, pq - 1 \ and \ p \nmid k_5, q \nmid k_5\}$, with $|V_5| = (p-1)(q-1)$

$V_6 = \{k_6 p^2 q : k_6 = 1, 2, 3, \cdots, q - 1\}$, with $|V_6| = q - 1$

$V_7 = \{k_7 pq^2 : k_7 = 1, 2, 3, \cdots, p - 1\}$, with $|V_7| = p - 1$

Also the cardinality of V is $p^2q^2 - \phi(p^2q^2) - 1 = p^2q + pq^2 - pq - 1$.

By the similar way which we discussed in 7.2.4, the adjacency property of the zero divisor graph $\Gamma(\mathbb{Z}_{p^2q^2})$ is shown in Figure 7.8.

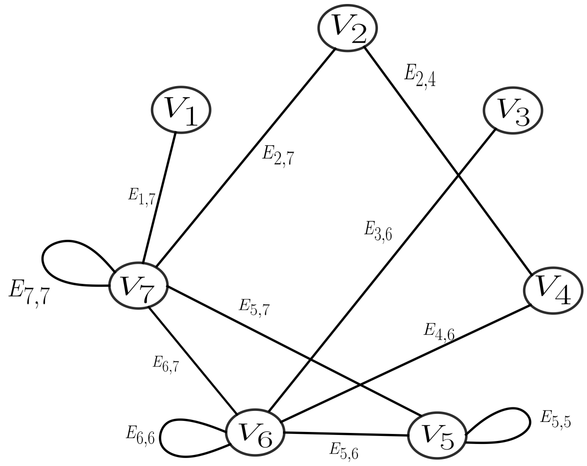

Figure 7.8: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{p^2q^2})$

Now one can see that degrees of each V_i for $i = 1, 2, 3, 4, 5, 6, 7$ are in the following.

$d(x) = |V_7| = p - 1 \ \forall x \in V_1$.

$d(x) = |V_4| + |V_7| = p^2 - 1 \ \forall x \in V_2$.

$d(x) = |V_6| = q - 1 \ \forall x \in V_3$.

$d(x) = |V_2| + |V_6| = q^2 - 1 \ \forall x \in V_4$.

$d(x) = |V_5| - 1 + |V_6| + |V_7| = pq - 2 \ \forall x \in V_5$.

$d(x) = |V_3| + |V_4| + |V_5| + |V_6| - 1 + |V_7| = p^2q - 2 \ \forall x \in V_6$.

$d(x) = |V_1| + |V_2| + |V_5| + |V_6| + |V_7| - 1 = pq^2 - 2 \ \forall x \in V_7$.

Now, we divide edge set E of $\Gamma(\mathbb{Z}_{p^2q^2})$ as $E = E_{1,7} \cup E_{2,4} \cup E_{2,7} \cup E_{3,6} \cup E_{4,6} \cup E_{5,6} \cup E_{5,7} \cup E_{6,7} \cup E_{5,5} \cup E_{6,6} \cup E_{7,7}$.

Therefore, the sum-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^2q^2})$ is

$$SGO(\Gamma(\mathbb{Z}_{p^2q^2})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= [\sum_{uv\in E_{1,7}} + \sum_{uv\in E_{2,4}} + \sum_{uv\in E_{2,7}} + \sum_{uv\in E_{3,6}} + \sum_{uv\in E_{4,6}} + \sum_{uv\in E_{5,6}} + \sum_{uv\in E_{5,7}} + \sum_{uv\in E_{6,7}} + \sum_{uv\in E_{5,5}} + \sum_{uv\in E_{6,6}} + \sum_{uv\in E_{7,7}}]\frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \frac{|V_1||V_7|}{\sqrt{p-1+pq^2-2+(p-1)(pq^2-2)}} + \frac{|V_2||V_4|}{\sqrt{p^2-1+q^2-1+(p^2-1)(q^2-1)}} + \frac{|V_2||V_7|}{\sqrt{p^2-1+pq^2-2+(p^2-1)(pq^2-2)}} + \frac{|V_3||V_6|}{\sqrt{q-1+p^2q-2+(q-1)(p^2q-2)}} + \frac{|V_4||V_6|}{\sqrt{q^2-1+p^2q-2+(q^2-1)(p^2q-2)}} + \frac{|V_5||V_6|}{\sqrt{pq-2+p^2q-2+(pq-2)(p^2q-2)}} + \frac{|V_5||V_7|}{\sqrt{pq-2+pq^2-2+(pq-2)(pq^2-2)}} + \frac{|V_6||V_7|}{\sqrt{p^2q-2+pq^2-2+(p^2q-2)(pq^2-2)}} + \frac{\binom{|V_5|}{2}}{\sqrt{2pq-4+(pq-2)^2}} + \frac{\binom{|V_6|}{2}}{\sqrt{2p^2q-4+(p^2q-2)^2}} + \frac{\binom{|V_7|}{2}}{\sqrt{2pq^2-4+(pq^2-2)^2}}$$

$$= \frac{q(p-1)^2(q-1)}{\sqrt{p^2q^2-p-1}} + \frac{pq(p-1)(q-1)}{\sqrt{p^2q^2-1}} + \frac{q(p-1)(q-1)}{\sqrt{p^3q^2-p^2-1}} + \frac{p(p-1)^2(q-1)}{\sqrt{p^2q^2-q-1}} + \frac{p(p-1)(q-1)}{\sqrt{p^2q^3-q^2-1}} + \frac{(p-1)(q-1)^2}{\sqrt{p^3q^2-p^2q-pq}} + \frac{(p-1)^2(q-1)}{\sqrt{p^2q^3-pq^2-pq}} + \frac{(p-1)(q-1)}{\sqrt{p^3q^3-p^2q-pq^2}} + \frac{(p-1)(q-1)(pq-p-q)}{2\sqrt{pq(pq-2)}} + \frac{(q-1)(q-2)}{2\sqrt{p^2q(p^2q-2)}} + \frac{(p-1)(p-2)}{2\sqrt{pq^2(pq^2-2)}}.$$

Moreover, the product-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{p^2q^2})$ is

$$PGO(\Gamma(\mathbb{Z}_{p^2q^2})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= [\sum_{uv\in E_{1,7}} + \sum_{uv\in E_{2,4}} + \sum_{uv\in E_{2,7}} + \sum_{uv\in E_{3,6}} + \sum_{uv\in E_{4,6}} + \sum_{uv\in E_{5,6}} + \sum_{uv\in E_{5,7}} + \sum_{uv\in E_{6,7}} + \sum_{uv\in E_{5,5}} + \sum_{uv\in E_{6,6}} + \sum_{uv\in E_{7,7}}]\frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= \frac{|V_1||V_7|}{\sqrt{(p-1+pq^2-2)(p-1)(pq^2-2)}} + \frac{|V_2||V_4|}{\sqrt{(p^2-1+q^2-1)(p^2-1)(q^2-1)}} + \frac{|V_2||V_7|}{\sqrt{(p^2-1+pq^2-2)(p^2-1)(pq^2-2)}} + \frac{|V_3||V_6|}{\sqrt{(q-1+p^2q-2)(q-1)(p^2q-2)}} + \frac{|V_4||V_6|}{\sqrt{(q^2-1+p^2q-2)(q^2-1)(p^2q-2)}} + \frac{|V_5||V_6|}{\sqrt{(pq-2+p^2q-2)(pq-2)(p^2q-2)}} + \frac{|V_5||V_7|}{\sqrt{(pq-2+pq^2-2)(pq-2)(pq^2-2)}} + \frac{|V_6||V_7|}{\sqrt{(p^2q-2+pq^2-2)(p^2q-2)(pq^2-2)}} + \frac{\binom{|V_5|}{2}}{\sqrt{2(pq-2)^3}} + \frac{\binom{|V_6|}{2}}{\sqrt{2(p^2q-2)^3}} + \frac{\binom{|V_7|}{2}}{\sqrt{2(pq^2-2)^3}}$$

$$= \frac{q(p-1)^{\frac{3}{2}}(q-1)}{\sqrt{(pq^2+p-3)(pq^2-2)}} + \frac{pq\sqrt{(p-1)(q-1)}}{\sqrt{(p^2+q^2-2)(p+1)(q+1)}} + \frac{q\sqrt{p-1}(q-1)}{\sqrt{(pq^2+p^2-3)(p+1)pq^2-2}} + \frac{p(p-1)^2\sqrt{q-1}}{\sqrt{(p^2q+q-3)(p^2q-2)}} + \frac{p(p-1)\sqrt{q-1}}{\sqrt{(p^2q+q^2-3)(q+1)(p^2q-2)}} + \frac{(p-1)(q-1)^2}{\sqrt{(p^2q+pq-4)(p^2q-2)(pq-2)}} + \frac{(p-1)^2(q-1)}{\sqrt{(pq^2+pq-4)(pq^2-2)(pq-2)}} + \frac{(p-1)(q-1)}{\sqrt{(p^2q+pq^2-4)(p^2q-2)(pq^2-2)}} + \frac{(p-1)(q-1)(pq-p-q)}{2(pq-2)\sqrt{(pq-2)}} + \frac{(q-1)(q-2)}{2(p^2q-2)\sqrt{(p^2q-2)}} + \frac{(p-1)(p-2)}{2(pq^2-2)\sqrt{(pq^2-2)}}. \qquad \square$$

Theorem 7.2.10. *The sum and product connectivity Gourava index of the graph* $\Gamma(\mathbb{Z}_{pqr})$ *are*

$$SGO(\Gamma(\mathbb{Z}_{pqr})) = \frac{3(p-1)(q-1)(r-1)}{\sqrt{pqr-1}} + \frac{(p-1)(q-1)}{\sqrt{pqr^2-1}} + \frac{(p-1)(r-1)}{\sqrt{pq^2r-1}} + \frac{(q-1)(r-1)}{\sqrt{p^2qr-1}}$$

$$PGO(\Gamma(\mathbb{Z}_{pqr})) = \frac{\sqrt{(p-1)(q-1)(r-1)}}{\sqrt{(p+qr-2)(qr-1)}} + \frac{(p-1)\sqrt{(q-1)(r-1)}}{\sqrt{(q+pr-2)(pr-1)}} + \frac{(p-1)(q-1)\sqrt{(r-1)}}{\sqrt{(r+pq-2)(pq-1)}} + \frac{(p-1)(q-1)}{\sqrt{(qr+pr-2)(qr-1)(pr-1)}} + \frac{(p-1)(r-1)}{\sqrt{(pq+qr-2)(pq-1)(qr-1)}} + \frac{(q-1)(r-1)}{\sqrt{(pq+pr-2)(pq-1)(pr-1)}},$$

where p, q, r *are distinct odd prime numbers.*

Proof. Let $R = \mathbb{Z}_{pqr}$, where p, q, r are distinct prime numbers. Then the set of non-zero zero divisors of R is $V = Z^*(R) = \{p, 2p, 3p, \cdots, (qr-1)p, q, 2q, 3q, \cdots, (pr-1)q, r, 2r, 3r, \cdots, (pq-1)r\}$ with cardinality $pqr - \phi(pqr) - 1 = pq + qr + pr - p - q - r$.

We can decompose the vertex set V of $\Gamma(R)$ into subsets as follows: $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$, where

$V_1 = \{k_1 p : k_1 = 1, 2, 3, \cdots, qr-1 \textit{ and } q \nmid k_1,\ r \nmid k_1\}$, with $|V_1| = (q-1)(r-1)$

$V_2 = \{k_2 q : k_2 = 1, 2, 3, \cdots, pr-1 \textit{ and } r \nmid k_2,\ p \nmid k_2\}$, with $|V_2| = (p-1)(r-1)$

$V_3 = \{k_3 r : k_3 = 1, 2, 3, \cdots, pq-1 \textit{ and } p \nmid k_3,\ q \nmid k_3\}$, with $|V_3| = (p-1)(q-1)$

$V_4 = \{k_5 qr : k_5 = 1, 2, 3, \cdots, p-1\}$, with $|V_4| = p-1$

$V_5 = \{k_6 pr : k_6 = 1, 2, 3, \cdots, q-1\}$, with $|V_5| = q-1$

$V_6 = \{k_4 pq : k_4 = 1, 2, 3, \cdots, r-1\}$, with $|V_6| = r-1$

One can easily understand the adjacency structure of the graph $\Gamma(\mathbb{Z}_{pqr})$ in Figure 7.9.

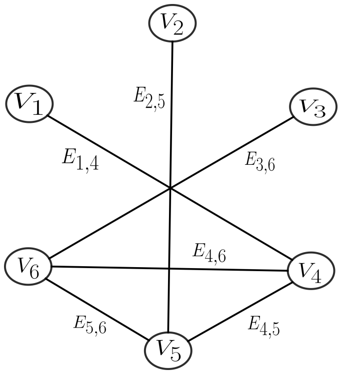

Figure 7.9: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{pqr})$

From the Figure 7.9, degrees of each V_i for $i = 1, 2, 3, 4, 5, 6$ are in the following:

$$\begin{aligned}
d(x) &= |V_4| = p - 1 \ \forall x \in V_1 \\
d(x) &= |V_5| = q - 1 \ \forall x \in V_2 \\
d(x) &= |V_6| = r - 1 \ \forall x \in V_3 \\
d(x) &= |V_1| + |V_5| + |V_6| = qr - 1 \ \forall x \in V_4 \\
d(x) &= |V_2| + |V_4| + |V_6| = pr - 1 \ \forall x \in V_5 \\
d(x) &= |V_3| + |V_4| + |V_5| = pq - 1 \ \forall x \in V_6
\end{aligned}$$

Now, we divide the edge set E of $\Gamma(\mathbb{Z}_{pqr})$ as $E = E_{1,4} \cup E_{2,5} \cup E_{3,6} \cup E_{4,5} \cup E_{4,6} \cup E_{5,6}$.

Therefore, the Sum-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{pqr})$ is

$$SGO(\Gamma(\mathbb{Z}_{pqr})) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u) + d(v) + d(u)d(v)}}$$

$$= [\sum_{uv \in E_{1,4}} + \sum_{uv \in E_{2,5}} + \sum_{uv \in E_{3,6}} + \sum_{uv \in E_{4,5}} + \sum_{uv \in E_{4,6}} + \sum_{uv \in E_{5,6}}] \frac{1}{\sqrt{d(u) + d(v) + d(u)d(v)}}$$

$$= \frac{|V_1||V_4|}{\sqrt{p - 1 + qr - 1 + (p - 1)(qr - 1)}} + \frac{|V_2||V_5|}{\sqrt{q - 1 + pr - 1 + (q - 1)(pr - 1)}} +$$

$$\frac{|V_3||V_6|}{\sqrt{r - 1 + pq - 1 + (r - 1)(pq - 1)}} + \frac{|V_4||V_5|}{\sqrt{qr - 1 + pr - 1 + (qr - 1)(pr - 1)}} +$$

$$\frac{|V_4||V_6|}{\sqrt{qr - 1 + pq - 1 + (qr - 1)(pq - 1)}} + \frac{|V_5|||V_6}{\sqrt{pr - 1 + pq - 1 + (pr - 1)(pq - 1)}}$$

$$= \frac{(p - 1)(q - 1)(r - 1)}{\sqrt{pqr - 1}} + \frac{(p - 1)(q - 1)(r - 1)}{\sqrt{pqr - 1}} + \frac{(p - 1)(q - 1)(r - 1)}{\sqrt{pqr - 1}} +$$

$$\frac{(p - 1)(q - 1)}{\sqrt{pqr^2 - 1}} + \frac{(p - 1)(r - 1)}{\sqrt{pq^2r - 1}} + \frac{(q - 1)(r - 1)}{\sqrt{p^2qr - 1}}$$

$$= \frac{3(p-1)(q-1)(r-1)}{\sqrt{pqr-1}} + \frac{(p-1)(q-1)}{\sqrt{pqr^2-1}} + \frac{(p-1)(r-1)}{\sqrt{pq^2r-1}} + \frac{(q-1)(r-1)}{\sqrt{p^2qr-1}}.$$

Moreover, the Product-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{pqr})$ is

$$PGO(\Gamma(\mathbb{Z}_{pqr})) = \sum_{uv\in E(G)} \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= [\sum_{uv\in E_{1,4}} + \sum_{uv\in E_{2,5}} + \sum_{uv\in E_{3,6}} + \sum_{uv\in E_{4,5}} + \sum_{uv\in E_{4,6}} + \sum_{uv\in E_{5,6}}] \frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$= \frac{|V_1||V_4|}{\sqrt{(p-1+qr-1)(p-1)(qr-1)}} + \frac{|V_2||V_5|}{\sqrt{(q-1+pr-1)(q-1)(pr-1)}} + \frac{|V_3||V_6|}{\sqrt{(r-1+pq-1)(r-1)(pq-1)}} + \frac{|V_4||V_5|}{\sqrt{(qr-1+pr-1)(qr-1)(pr-1)}} + \frac{|V_4||V_6|}{\sqrt{(qr-1+pq-1)(qr-1)(pq-1)}} + \frac{|V_5|||V_6}{\sqrt{(pr-1+pq-1)(pr-1)(pq-1)}}$$

$$= \frac{(p-1)(q-1)(r-1)}{\sqrt{(p+qr-2)(p-1)(qr-1)}} + \frac{(p-1)(q-1)(r-1)}{\sqrt{(q+pr-2)(q-1)(pr-1)}} + \frac{(p-1)(q-1)(r-1)}{\sqrt{(r+pq-2)(r-1)(pq-1)}} + \frac{(p-1)(q-1)}{\sqrt{(qr+pr-2)(qr-1)(pr-1)}} + \frac{(p-1)(r-1)}{\sqrt{(pq+qr-2)(pq-1)(qr-1)}} + \frac{(q-1)(r-1)}{\sqrt{(pq+pr-2)(pq-1)(pr-1)}}$$

$$= \frac{\sqrt{(p-1)}(q-1)(r-1)}{\sqrt{(p+qr-2)(qr-1)}} + \frac{(p-1)\sqrt{(q-1)}(r-1)}{\sqrt{(q+pr-2)(pr-1)}} + \frac{(p-1)(q-1)\sqrt{(r-1)}}{\sqrt{(r+pq-2)(pq-1)}} + \frac{(p-1)(q-1)}{\sqrt{(qr+pr-2)(qr-1)(pr-1)}} + \frac{(p-1)(r-1)}{\sqrt{(pq+qr-2)(pq-1)(qr-1)}} + \frac{(q-1)(r-1)}{\sqrt{(pq+pr-2)(pq-1)(pr-1)}}. \qquad \square$$

Theorem 7.2.11. *Let $R = \mathbb{Z}_{pq}[x]/ < x^2 >$ with any prime $2 < p < q$. Then the sum and product Gourava index of the graph $\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >)$ are*

$$SGO(\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >)) = \frac{(p-1)^2q(q-1)}{\sqrt{p^2q^2-p-1}} + \frac{(p-1)q(q-1)}{\sqrt{p^3q^2-p^2-1}} + \frac{(p-1)(q-1)}{\sqrt{p^3q^3-p^2q-pq^2}} + \frac{(p-1)^2(q-1)}{\sqrt{p^2q^3-pq^2-pq}} + \frac{p(p-1)(q-1)^2}{\sqrt{p^2q^2-q-1}} + \frac{pq(p-1)(q-1)}{\sqrt{p^2q^2-1}} + \frac{p(p-1)(q-1)}{\sqrt{p^2q^3-q^2-1}} + \frac{(p-1)(q-1)^2}{\sqrt{p^3q^2-p^2q-pq}} + \frac{(p-1)(p-2)}{2q\sqrt{p(pq^2-2)}} + \frac{(q-1)(q-2)}{2p\sqrt{q(p^2q-2)}} + \frac{(p-1)(q-1)(pq-p-q)}{2\sqrt{pq(pq-2)}}$$

$$PGO(\Gamma(\mathbb{Z}_{pq}[x]/ < x^2 >)) = \frac{(p-1)^{\frac{3}{2}}q(q-1)}{\sqrt{(pq^2+p-3)(pq^2-2)}} + \frac{\sqrt{p-1}q(q-1)}{\sqrt{(pq^2+p^2-3)(pq^2-2)(p+1)}} + \frac{(p-1)(q-1)}{\sqrt{(p^2q+pq^2-4)(p^2q-2)(pq^2-2)}} + \frac{(p-1)^2(q-1)}{\sqrt{(pq^2+pq-4)(pq^2-2)(pq-2)}} + \frac{p(p-1)(q-1)^{\frac{3}{2}}}{\sqrt{(p^2q+q-3)(p^2q-2)(q+1)}} + \frac{pq\sqrt{(p-1)(q-1)}}{\sqrt{(p^2+q^2-2)(p+1)(q+1)}} +$$

$$\frac{p(p-1)\sqrt{q-1}}{\sqrt{(p^2q+q^2-3)(p^2q-2)(q+1)}}+\frac{(p-1)(q-1)^2}{\sqrt{(p^2q+pq-4)(p^2q-2)(pq-2)}}+$$
$$\frac{(p-1)(p-2)}{2(pq^2-2)\sqrt{2(pq^2-2)}}+\frac{(q-1)(q-2)}{2(p^2q-2)\sqrt{2(p^2q-2)}}+\frac{(p-1)(q-1)(pq-p-q)}{2(pq-2)\sqrt{2(pq-2)}}.$$

Proof. Let $R=\mathbb{Z}_{pq}[x]/<x^2>$, where $2<p<q$ are distinct primes. The zero-divisor graph $\Gamma(R)$ has a vertex set consisting of the non-zero zero divisors of R. The number of vertices in $\Gamma(R)$ is $(pq^2+p^2q-pq-1)$, which corresponds to the cardinality of $(\mathbb{Z}_{pq}[x]/<x^2>)\setminus\{0\}$. Then the details of $\Gamma(R)$ as in [44] are as follows:

$V(\Gamma(R))=\{p,2p,\cdots,(q-1)p,q,2q,\cdots,(p-1)q,x,2x,\cdots,(pq-1)x,x+p,x+2p,\cdots,x+(q-1)p,x+q,x+2q,\cdots,x+(p-1)q,2x+p,2x+2p,\cdots,2x+(q-1)p,2x+q,2x+2q,\cdots,2x+(p-1)q,\cdots,(q-1)px+p.(q-1)px+2p,\cdots,(q-1)px+(q-1)p,(p-1)qx+q,(p-1)qx+2q,\cdots,(p-1)qx+(p-1)q\}$.

The vertex set V is divided into seven disjoint subsets $V_1,V_2,V_3,V_4,V_5,V_6,V_7$ such that:

$V_1=\{kqx:k=1,2,\cdots,p-1\}$, with $|V_1|=p-1$

$V_2=\{kx+mp:k=1,2,\cdots,pq-1,m=1,2,\cdots,q-1\ and\ p\nmid k\}$, with $|V_2|=(p-1)q(q-1)$

$V_3=\{kx+np:k=1,2,\cdots,pq-1,n=1,2,\cdots,p-1\ and\ q\nmid k\}$, with $|V_3|=p(p-1)(q-1)$

$V_4=\{lpx+mp:l=0,1,\cdots,q-1,m=1,2,\cdots,q-1\}$, with $|V_4|=q(q-1)$

$V_5=\{lqx+nq:l=0,1,\cdots,p-1,n=1,2,\cdots,p-1\}$, with $|V_5|=p(p-1)$

$V_6=\{mpx:m=1,2,\cdots,q-1\}$, with $|V_6|=q-1$

$V_7=\{kx:k=1,2,\cdots,pq-1\ and\ p,q\nmid k\}$, with $|V_7|=(p-1)(q-1)$

The adjacency structure of $\Gamma(\mathbb{Z}_{pq}[x]/<x^2>)$ is illustrated in Figure 7.10.

Also, degrees of V_i for $i=1,2,3,4,5,6,7$ are in the following:

$d(x)=|V_1|-1+|V_2|+|V_4|+|V_6|+|V_7|=pq^2-2\ \forall x\in V_1$

$d(x)=|V_1|=p-1\ \forall x\in V_2$

$d(x)=|V_6|=q-1\ \forall x\in V_3$

$d(x)=|V_1|+|V_5|=p^2-1\ \forall x\in V_4$

$d(x)=|V_4|+|V_6|=q^2-1\ \forall x\in V_5$

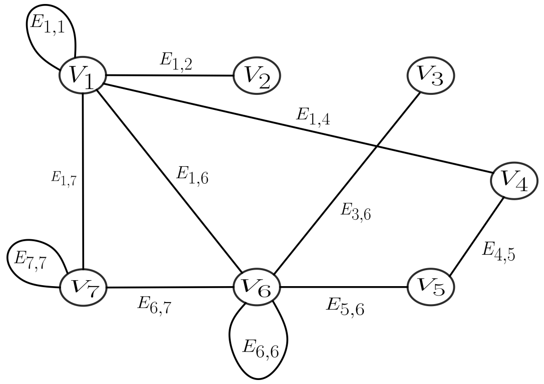

Figure 7.10: Adjacency structure(Edge set) of the graph $\Gamma(\mathbb{Z}_{pq}[x]/<x^2>)$

$d(x) = |V_1| + |V_3| + |V_5| + |V_6| - 1 + |V_7| = p^2q - 2 \ \forall x \in V_6$

$d(x) = |V_1| + |V_6| + |V_7| - 1 = pq - 2 \ \forall x \in V_7$

Now, we divide the edge set E of $\Gamma(\mathbb{Z}_{pq}[x]/<x^2>)$ as $E = E_{1,2} \cup E_{1,4} \cup E_{1,6} \cup E_{1,7} \cup E_{3,6} \cup E_{4,5} \cup E_{5,6} \cup E_{6,7} \cup E_{1,1} \cup E_{6,6} \cup E_{7,7}$

Therefore, the Sum-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{pq}[x]/<x^2>)$ is computed as follows:

$$SGO(\Gamma(\mathbb{Z}_{pq}[x]/<x^2>)) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= [\sum_{uv\in E_{1,2}} + \sum_{uv\in E_{1,4}} + \sum_{uv\in E_{1,6}} + \sum_{uv\in E_{1,7}} + \sum_{uv\in E_{3,6}} + \sum_{uv\in E_{4,5}} + \sum_{uv\in E_{5,6}} + \sum_{uv\in E_{6,7}} + \sum_{uv\in E_{1,1}} + \sum_{uv\in E_{6,6}} + \sum_{uv\in E_{7,7}}]\frac{1}{\sqrt{d(u)+d(v)+d(u)d(v)}}$$

$$= \frac{|V_1||V_2|}{\sqrt{pq^2-2+p-1+(pq^2-2)(p-1)}} + \frac{|V_1||V_4|}{\sqrt{pq^2-2+p^2-1+(pq^2-2)(p^2-1)}} + \frac{|V_1||V_6|}{\sqrt{pq^2-2+p^2q-2+(pq^2-2)(p^2q-2)}} + \frac{|V_1||V_7|}{\sqrt{pq^2-2+pq-2+(pq^2-2)(pq-2)}} + \frac{|V_3||V_6|}{\sqrt{q-1+p^2q-2+(q-1)(p^2q-2)}} + \frac{|V_4||V_5|}{\sqrt{p^2-1+q^2-1+(p^2-1)(q^2-1)}} + \frac{|V_5||V_6|}{\sqrt{q^2-1+p^2q-2+(q^2-1)(p^2q-2)}} + \frac{|V_6||V_7|}{\sqrt{p^2q-2+pq-2+(p^2q-2)(pq-2)}} + \frac{\binom{|V_1|}{2}}{\sqrt{(pq^2-2)^2+2(pq^2-2)}} + \frac{\binom{|V_6|}{2}}{\sqrt{(p^2q-2)^2+2(p^2q-2)}} + \frac{\binom{|V_7|}{2}}{\sqrt{(pq-2)^2+2(pq-2)}}$$

$$= \frac{(p-1)^2q(q-1)}{\sqrt{p^2q^2-p-1}} + \frac{(p-1)q(q-1)}{\sqrt{p^3q^2-p^2-1}} + \frac{(p-1)(q-1)}{\sqrt{p^3q^3-p^2q-pq^2}} + \frac{(p-1)^2(q-1)}{\sqrt{p^2q^3-pq^2-pq}} +$$

$$\frac{p(p-1)(q-1)^2}{\sqrt{p^2q^2-q-1}}+\frac{pq(p-1)(q-1)}{\sqrt{p^2q^2-1}}+\frac{p(p-1)(q-1)}{\sqrt{p^2q^3-q^2-1}}+\frac{(p-1)(q-1)^2}{\sqrt{p^3q^2-p^2q-pq}}+$$
$$\frac{(p-1)(p-2)}{2q\sqrt{p(pq^2-2)}}+\frac{(q-1)(q-2)}{2p\sqrt{q(p^2q-2)}}+\frac{(p-1)(q-1)(pq-p-q)}{2\sqrt{pq(pq-2)}}.$$

Morover, the Product-connectivity Gourava index of the graph $\Gamma(\mathbb{Z}_{pq}[x]/<x^2>)$ is computed as follows:

$$PGO(\Gamma(\mathbb{Z}_{pq}[x]/<x^2>))=\sum_{uv\in E(G)}\frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$=[\sum_{uv\in E_{1,2}}+\sum_{uv\in E_{1,4}}+\sum_{uv\in E_{1,6}}+\sum_{uv\in E_{1,7}}+\sum_{uv\in E_{3,6}}+\sum_{uv\in E_{4,5}}+\sum_{uv\in E_{5,6}}+\sum_{uv\in E_{6,7}}+\sum_{uv\in E_{1,1}}+$$
$$\sum_{uv\in E_{6,6}}+\sum_{uv\in E_{7,7}}]\frac{1}{\sqrt{(d(u)+d(v))d(u)d(v)}}$$

$$=\frac{|V_1||V_2|}{\sqrt{(pq^2-2+p-1)(pq^2-2)(p-1)}}+\frac{|V_1||V_4|}{\sqrt{(pq^2-2+p^2-1)(pq^2-2)(p^2-1)}}+$$
$$\frac{|V_1||V_6|}{\sqrt{(pq^2-2+p^2q-2)(pq^2-2)(p^2q-2)}}+\frac{|V_1||V_7|}{\sqrt{(pq^2-2+pq-2)(pq^2-2)(pq-2)}}+$$
$$\frac{|V_3||V_6|}{\sqrt{(q-1+p^2q-2)(q-1)(p^2q-2)}}+\frac{|V_4||V_5|}{\sqrt{(p^2-1+q^2-1)(p^2-1)(q^2-1)}}+$$
$$\frac{|V_5||V_6|}{\sqrt{(q^2-1+p^2q-2)(q^2-1)(p^2q-2)}}+\frac{|V_6||V_7|}{\sqrt{(p^2q-2+pq-2)(p^2q-2)(pq-2)}}+$$
$$\frac{\binom{|V_1|}{2}}{\sqrt{2(pq^2-2)^3}}+\frac{\binom{|V_6|}{2}}{\sqrt{2(p^2q-2)^3}}+\frac{\binom{|V_7|}{2}}{\sqrt{2(pq-2)^3}}$$

$$=\frac{(p-1)^{\frac{3}{2}}q(q-1)}{\sqrt{(pq^2+p-3)(pq^2-2)}}+\frac{\sqrt{p-1}q(q-1)}{\sqrt{(pq^2+p^2-3)(pq^2-2)(p+1)}}+$$
$$\frac{(p-1)(q-1)}{\sqrt{(p^2q+pq^2-4)(p^2q-2)(pq^2-2)}}+\frac{(p-1)^2(q-1)}{\sqrt{(pq^2+pq-4)(pq^2-2)(pq-2)}}+$$
$$\frac{p(p-1)(q-1)^{\frac{3}{2}}}{\sqrt{(p^2q+q-3)(p^2q-2)(q+1)}}+\frac{pq\sqrt{(p-1)(q-1)}}{\sqrt{(p^2+q^2-2)(p+1)(q+1)}}+$$
$$\frac{p(p-1)\sqrt{q-1}}{\sqrt{(p^2q+q^2-3)(p^2q-2)(q+1)}}+\frac{(p-1)(q-1)^2}{\sqrt{(p^2q+pq-4)(p^2q-2)(pq-2)}}+$$
$$\frac{(p-1)(p-2)}{2(pq^2-2)\sqrt{2(pq^2-2)}}+\frac{(q-1)(q-2)}{2(p^2q-2)\sqrt{2(p^2q-2)}}+\frac{(p-1)(q-1)(pq-p-q)}{2(pq-2)\sqrt{2(pq-2)}}.\qquad\square$$

7.3 Sum and product connectivity Gourava index of Zero-divisor graphs of small finite commutative rings.

In this section, we mainly computed the sum and product connectivity Gourava index of zero divisor graphs of some small finite commutative rings. In [45], the details of zero divisor graphs of small order finite commutative rings are described.

In this context, let R represent a finite commutative ring, $|V|$ the total number of vertices, $\Gamma(R)$ the zero-divisor graph corresponding to R, $SGO(\Gamma(R))$ the sum-connectivity Gourava index of this zero-divisor graph and $PGO(\Gamma(R))$ the product-connectivity Gourava index of this zero-divisor graph.

Sum and product connectivity Gourava index of Zero-divisor graphs of small finite commutative rings.

$\|V\|$	R	$\Gamma(R)$	$SGO(\Gamma(R))$	$PGO(\Gamma(R))$
3	$\mathbb{Z}_6$	$K_{1,2}$	$\frac{2}{\sqrt{5}}$	$\sqrt{\frac{2}{3}}$
	$\mathbb{Z}_8$	$K_{1,2}$	$\frac{2}{\sqrt{5}}$	$\sqrt{\frac{2}{3}}$
	$\mathbb{Z}_2[X]/(X^3)$	$K_{1,2}$	$\frac{2}{\sqrt{5}}$	$\sqrt{\frac{2}{3}}$
	$\mathbb{Z}_4[X]/(2X, X^2-2)$	$K_{1,2}$	$\frac{2}{\sqrt{5}}$	$\sqrt{\frac{2}{3}}$
	$\mathbb{Z}_2[X,Y]/(X,Y)^2$	K_3	$\frac{3}{2\sqrt{2}}$	$\frac{3}{4}$
	$\mathbb{Z}_4[X]/(2,X)^2$	K_3	$\frac{3}{2\sqrt{2}}$	$\frac{3}{4}$
	$\mathbb{F}_4[X]/(X^2)$	K_3	$\frac{3}{2\sqrt{2}}$	$\frac{3}{4}$
	$\mathbb{Z}_4[X]/(X^2+X+1)$	K_3	$\frac{3}{2\sqrt{2}}$	$\frac{3}{4}$
4	$\mathbb{Z}_2 \times \mathbb{F}_4$	$K_{1,3}$	$\frac{3}{\sqrt{7}}$	$\frac{\sqrt{3}}{2}$
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$K_{2,2}$	$\sqrt{2}$	1
	$\mathbb{Z}_{25}$	K_4	$\frac{2\sqrt{3}}{\sqrt{5}}$	$\sqrt{\frac{2}{3}}$
	$\mathbb{Z}_5[X]/(X^2)$	K_4	$\frac{2\sqrt{3}}{\sqrt{5}}$	$\sqrt{\frac{2}{3}}$

$\|V\|$	R	$\Gamma(R)$	$SGO(\Gamma(R))$	$PGO(\Gamma(R))$
5	$\mathbb{Z}_2 \times \mathbb{Z}_5$	$K_{1,4}$	$\frac{4}{3}$	$\frac{2}{\sqrt{5}}$
	$\mathbb{Z}_3 \times \mathbb{F}_4$	$K_{2,3}$	$\frac{6}{\sqrt{11}}$	$\sqrt{\frac{6}{5}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Figure 6.9	$\frac{2}{\sqrt{7}} + \frac{1}{\sqrt{11}} + \frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{6}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.9	$\frac{2}{\sqrt{7}} + \frac{1}{\sqrt{11}} + \frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{6}}$
6	$\mathbb{Z}_3 \times \mathbb{Z}_5$	$K_{2,4}$	$4\sqrt{\frac{2}{7}}$	$\sqrt{\frac{4}{3}}$
	$\mathbb{F}_4 \times \mathbb{F}_4$	$K_{3,3}$	$3\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{2}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Figure 6.10	$\frac{3}{\sqrt{7}} + \sqrt{\frac{3}{5}}$	$\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{6}}$
	$\mathbb{Z}_{49}$	K_6	$3\sqrt{\frac{5}{7}}$	$\frac{3}{\sqrt{10}}$
	$\mathbb{Z}_7[X]/(X^2)$	K_6	$3\sqrt{\frac{5}{7}}$	$\frac{3}{\sqrt{10}}$
7	$\mathbb{Z}_2 \times \mathbb{Z}_7$	$K_{1,6}$	$\frac{6}{\sqrt{13}}$	$\sqrt{\frac{6}{7}}$
	$\mathbb{F}_4 \times \mathbb{Z}_5$	$K_{3,4}$	$\frac{12}{\sqrt{19}}$	$\sqrt{\frac{12}{7}}$
	$\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.11	$\frac{2}{3} + \frac{2}{\sqrt{19}} + \frac{4}{\sqrt{11}}$	$\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{21}} + 2\sqrt{\frac{2}{15}}$
	$\mathbb{Z}_3 \times \mathbb{Z}_4$	Figure 6.11	$\frac{2}{3} + \frac{2}{\sqrt{19}} + \frac{4}{\sqrt{11}}$	$\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{21}} + 2\sqrt{\frac{2}{15}}$
	$\mathbb{Z}_{16}$	Figure 6.12	$\frac{4}{\sqrt{13}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$2\sqrt{\frac{2}{21}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$
	$\mathbb{Z}_2[X]/(X^4)$	Figure 6.12	$\frac{4}{\sqrt{13}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$2\sqrt{\frac{2}{21}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$
	$\mathbb{Z}_4[X]/(X^2+2)$	Figure 6.12	$\frac{4}{\sqrt{13}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$2\sqrt{\frac{2}{21}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$
	$\mathbb{Z}_4[X]/(X^2+3X)$	Figure 6.12	$\frac{4}{\sqrt{13}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$2\sqrt{\frac{2}{21}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$
	$\mathbb{Z}_4[X]/(x^3-2, 2X^2, 2X)$	Figure 6.12	$\frac{4}{\sqrt{13}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$2\sqrt{\frac{2}{21}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$
	$\mathbb{Z}_2[X,Y]/(X^3, XY, Y^2)$	Figure 6.13	$\frac{19}{4\sqrt{3}}$	$\frac{2\sqrt{2}}{3} + \frac{1}{4\sqrt{3}}$
	$\mathbb{Z}_8[X]/(2X, X^2)$	Figure 6.13	$\frac{19}{4\sqrt{3}}$	$\frac{2\sqrt{2}}{3} + \frac{1}{4\sqrt{3}}$
	$\mathbb{Z}_4[X,Y]/(X^3, 2X^2, 2X)$	Figure 6.13	$\frac{19}{4\sqrt{3}}$	$\frac{2\sqrt{2}}{3} + \frac{1}{4\sqrt{3}}$
	$\mathbb{Z}_4[X]/(X^2+2X)$	Figure 6.14	$\frac{4}{3\sqrt{3}} + \frac{4}{\sqrt{15}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$\frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{3\sqrt{3}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$
	$\mathbb{Z}_8[X]/(2X, x^2+4)$	Figure 6.14	$\frac{4}{3\sqrt{3}} + \frac{4}{\sqrt{15}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$\frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{3\sqrt{3}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$
	$\mathbb{Z}_2[X,Y]/(X^2, Y^2-XY)$	Figure 6.14	$\frac{4}{3\sqrt{3}} + \frac{4}{\sqrt{15}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$\frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{3\sqrt{3}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$
	$\mathbb{Z}_4[X,Y]/(X^2, Y^2-XY, XY-2, 2X, 2Y)$	Figure 6.14	$\frac{4}{3\sqrt{3}} + \frac{4}{\sqrt{15}} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}}$	$\frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{3\sqrt{3}} + \frac{1}{2\sqrt{6}} + \frac{1}{4}$

$\lvert V\rvert$	R	$\Gamma(R)$	$SGO(\Gamma(R))$	$PGO(\Gamma(R))$
7	$\mathbb{Z}_4[X,Y]/(X^2,Y^2,XY-2,2X,2Y)$	Figure 6.15	$\frac{3}{2\sqrt{5}}+\frac{3}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2\sqrt{2}}+\frac{3}{4}$
	$\mathbb{Z}_2[X,Y]/(X^2,Y^2)$	Figure 6.15	$\frac{3}{2\sqrt{5}}+\frac{3}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2\sqrt{2}}+\frac{3}{4}$
	$\mathbb{Z}_4[X]/(X^2)$	Figure 6.15	$\frac{3}{2\sqrt{5}}+\frac{3}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2\sqrt{2}}+\frac{3}{4}$
	$\mathbb{Z}_4[X]/(X^3-X^2-2,2X^2,2X)$	Figure 6.16	$\frac{2}{\sqrt{13}}+\frac{4}{3\sqrt{3}}+\frac{4}{\sqrt{15}}$	$\frac{\sqrt{2}}{\sqrt{21}}+\frac{\sqrt{2}}{9}+\frac{4}{9}$
	$\mathbb{Z}_2[X,Y,Z]/(X,Y,Z)^2$	K_7	$\frac{7\sqrt{3}}{4}$	$\frac{7}{4\sqrt{3}}$
	$\mathbb{Z}_4[X,Y]/(X^2,Y^2,XY,2X,2Y)$	K_7	$\frac{7\sqrt{3}}{4}$	$\frac{7}{4\sqrt{3}}$
	$\mathbb{F}_8[X]/(X^2)$	K_7	$\frac{7\sqrt{3}}{4}$	$\frac{7}{4\sqrt{3}}$
	$\mathbb{Z}_4[X]/(X^3+X+1)$	K_7	$\frac{7\sqrt{3}}{4}$	$\frac{7}{4\sqrt{3}}$
8	$\mathbb{Z}_2\times\mathbb{F}_8$	$K_{1,7}$	$\frac{7}{\sqrt{15}}$	$\frac{1}{2}\sqrt{\frac{7}{2}}$
	$\mathbb{Z}_3\times\mathbb{Z}_7$	$K_{2,6}$	$\frac{6}{\sqrt{5}}$	$\sqrt{\frac{3}{2}}$
	$\mathbb{Z}_5\times\mathbb{Z}_5$	$K_{2,6}$	$\frac{6}{\sqrt{5}}$	$\sqrt{\frac{3}{2}}$
	$\mathbb{Z}_{27}$	Figure 6.17	$\frac{12}{\sqrt{23}}+\frac{1}{3\sqrt{7}}$	$\frac{2\sqrt{2}}{\sqrt{7}}+\frac{1}{7\sqrt{14}}$
	$\mathbb{Z}_9[X]/(3X,X^2-3)$	Figure 6.17	$\frac{12}{\sqrt{23}}+\frac{1}{3\sqrt{7}}$	$\frac{2\sqrt{2}}{\sqrt{7}}+\frac{1}{7\sqrt{14}}$
	$\mathbb{Z}_9[X]/(3X,X^2-6)$	Figure 6.17	$\frac{12}{\sqrt{23}}+\frac{1}{3\sqrt{7}}$	$\frac{2\sqrt{2}}{\sqrt{7}}+\frac{1}{7\sqrt{14}}$
	$\mathbb{Z}_3[X]/(X^3)$	Figure 6.17	$\frac{12}{\sqrt{23}}+\frac{1}{3\sqrt{7}}$	$\frac{2\sqrt{2}}{\sqrt{7}}+\frac{1}{7\sqrt{14}}$
	$\mathbb{Z}_3[X,Y]/(X,Y)^2$	K_8	$\frac{4\sqrt{7}}{3}$	$\sqrt{\frac{2}{7}}$
	$\mathbb{Z}_9[X]/(3,X)^2$	K_8	$\frac{4\sqrt{7}}{3}$	$\sqrt{\frac{2}{7}}$
	$\mathbb{F}_9[X]/(X^2)$	K_8	$\frac{4\sqrt{7}}{3}$	$\sqrt{\frac{2}{7}}$
	$\mathbb{Z}_9[X]/(X^2+1)$	K_8	$\frac{4\sqrt{7}}{3}$	$\sqrt{\frac{2}{7}}$
9	$\mathbb{Z}_2\times\mathbb{F}_9$	$K_{1,8}$	$\frac{8}{\sqrt{17}}$	$\frac{2\sqrt{2}}{3}$
	$\mathbb{Z}_3\times\mathbb{F}_8$	$K_{2,7}$	$\frac{14}{\sqrt{23}}$	$\frac{\sqrt{14}}{3}$
	$\mathbb{F}_4\times\mathbb{Z}_7$	$K_{3,6}$	$2\sqrt{3}$	$\sqrt{2}$
	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	Figure 6.18	$\frac{6}{\sqrt{11}}+\frac{4}{\sqrt{23}}+\frac{1}{\sqrt{35}}$	$\frac{4\sqrt{2}}{\sqrt{15}}+\frac{1}{5\sqrt{10}}$
	$\mathbb{Z}_4\times\mathbb{F}_4$	Figure 6.19	$\frac{3}{\sqrt{13}}+\frac{1}{\sqrt{3}}+\frac{2\sqrt{3}}{\sqrt{5}}$	$\sqrt{\frac{3}{14}}+\frac{1}{3\sqrt{2}}+\sqrt{\frac{2}{5}}$
	$\mathbb{Z}_2[X]/(X^2)\times\mathbb{F}_4$	Figure 6.19	$\frac{3}{\sqrt{13}}+\frac{1}{\sqrt{3}}+\frac{2\sqrt{3}}{\sqrt{5}}$	$\sqrt{\frac{3}{14}}+\frac{1}{3\sqrt{2}}+\sqrt{\frac{2}{5}}$

$\lvert V\rvert$	R	$\Gamma(R)$	$SGO(\Gamma(R))$	$PGO(\Gamma(R))$
10	$\mathbb{Z}_3 \times \mathbb{F}_9$	$K_{2,8}$	$8\sqrt{\frac{2}{13}}$	$2\sqrt{\frac{2}{5}}$
	$\mathbb{F}_4 \times \mathbb{F}_8$	$K_{3,7}$	$\frac{21}{\sqrt{31}}$	$\sqrt{\frac{21}{10}}$
	$\mathbb{Z}_5 \times \mathbb{Z}_7$	$K_{4,6}$	$12\sqrt{\frac{2}{17}}$	$2\sqrt{\frac{3}{5}}$
	$\mathbb{Z}_{121}$	K_{10}	$\frac{15}{\sqrt{11}}$	$\frac{5}{3\sqrt{2}}$
	$\mathbb{Z}_{11}[X]/(X^2)$	K_{10}	$\frac{15}{\sqrt{11}}$	$\frac{5}{3\sqrt{2}}$
11	$\mathbb{Z}_2 \times \mathbb{Z}_{11}$	$K_{1,10}$	$\frac{10}{\sqrt{21}}$	$\sqrt{\frac{10}{11}}$
	$\mathbb{F}_4 \times \mathbb{F}_9$	$K_{3,8}$	$\frac{24}{\sqrt{35}}$	$\sqrt{\frac{24}{11}}$
	$\mathbb{Z}_5 \times \mathbb{F}_8$	$K_{4,7}$	$\frac{28}{\sqrt{39}}$	$\sqrt{\frac{28}{11}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_9$	Figure 6.20	$\frac{6}{\sqrt{17}}+\frac{1}{\sqrt{11}}+\frac{2\sqrt{2}}{\sqrt{7}}+\frac{1}{2\sqrt{6}}$	$\frac{1}{\sqrt{2}}+\frac{1}{4\sqrt{6}}+\frac{1}{\sqrt{3}}+\frac{1}{8\sqrt{2}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$	Figure 6.20	$\frac{6}{\sqrt{17}}+\frac{1}{\sqrt{11}}+\frac{2\sqrt{2}}{\sqrt{7}}+\frac{1}{2\sqrt{6}}$	$\frac{1}{\sqrt{2}}+\frac{1}{4\sqrt{6}}+\frac{1}{\sqrt{3}}+\frac{1}{8\sqrt{2}}$
	$\mathbb{Z}_5 \times \mathbb{Z}_4$	Figure 6.21	$\frac{4}{\sqrt{17}}+\frac{4}{\sqrt{35}}+\frac{8}{\sqrt{19}}$	$\frac{\sqrt{2}}{3}+\sqrt{\frac{2}{33}}+\frac{4}{\sqrt{21}}$
	$\mathbb{Z}_5 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.21	$\frac{4}{\sqrt{17}}+\frac{4}{\sqrt{35}}+\frac{8}{\sqrt{19}}$	$\frac{\sqrt{2}}{3}+\sqrt{\frac{2}{33}}+\frac{4}{\sqrt{21}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_8$	Figure 6.22	$\frac{2\sqrt{3}}{\sqrt{5}}+\frac{2}{\sqrt{31}}+\frac{1}{\sqrt{55}}+\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{13}}$	$\sqrt{\frac{2}{7}}+\sqrt{\frac{2}{105}}+\frac{1}{\sqrt{546}}+\frac{1}{3\sqrt{2}}+\frac{\sqrt{2}}{3\sqrt{3}}+\sqrt{\frac{2}{21}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$	Figure 6.22	$\frac{2\sqrt{3}}{\sqrt{5}}+\frac{2}{\sqrt{31}}+\frac{1}{\sqrt{55}}+\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{13}}$	$\sqrt{\frac{2}{7}}+\sqrt{\frac{2}{105}}+\frac{1}{\sqrt{546}}+\frac{1}{3\sqrt{2}}+\frac{\sqrt{2}}{3\sqrt{3}}+\sqrt{\frac{2}{21}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2-2)$	Figure 6.22	$\frac{2\sqrt{3}}{\sqrt{5}}+\frac{2}{\sqrt{31}}+\frac{1}{\sqrt{55}}+\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{13}}$	$\sqrt{\frac{2}{7}}+\sqrt{\frac{2}{105}}+\frac{1}{\sqrt{546}}+\frac{1}{3\sqrt{2}}+\frac{\sqrt{2}}{3\sqrt{3}}+\sqrt{\frac{2}{21}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2[X,Y]/(X,Y)^2$	Figure 6.23	$\frac{4}{\sqrt{15}}+\frac{3}{\sqrt{55}}+\frac{\sqrt{3}}{4}+\sqrt{3}$	$\sqrt{\frac{2}{7}}+\sqrt{\frac{3}{182}}+\frac{1}{4\sqrt{3}}+\frac{1}{\sqrt{2}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2,X)^2$	Figure 6.23	$\frac{4}{\sqrt{15}}+\frac{3}{\sqrt{55}}+\frac{\sqrt{3}}{4}+\sqrt{3}$	$\sqrt{\frac{2}{7}}+\sqrt{\frac{3}{182}}+\frac{1}{4\sqrt{3}}+\frac{1}{\sqrt{2}}$
	$\mathbb{Z}_4 \times \mathbb{Z}_4$	Figure 6.24	$\frac{4}{\sqrt{13}}+\frac{1}{\sqrt{5}}+\frac{19}{12\sqrt{3}}+\frac{4}{\sqrt{15}}$	$\frac{2\sqrt{2}}{\sqrt{21}}+\frac{1}{2\sqrt{6}}+\frac{1}{12\sqrt{3}}+\frac{2\sqrt{2}}{9}+\frac{2\sqrt{2}}{3\sqrt{3}}$
	$\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.24	$\frac{4}{\sqrt{13}}+\frac{1}{\sqrt{5}}+\frac{19}{12\sqrt{3}}+\frac{4}{\sqrt{15}}$	$\frac{2\sqrt{2}}{\sqrt{21}}+\frac{1}{2\sqrt{6}}+\frac{1}{12\sqrt{3}}+\frac{2\sqrt{2}}{9}+\frac{2\sqrt{2}}{3\sqrt{3}}$
	$\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.24	$\frac{4}{\sqrt{13}}+\frac{1}{\sqrt{5}}+\frac{19}{12\sqrt{3}}+\frac{4}{\sqrt{15}}$	$\frac{2\sqrt{2}}{\sqrt{21}}+\frac{1}{2\sqrt{6}}+\frac{1}{12\sqrt{3}}+\frac{2\sqrt{2}}{9}+\frac{2\sqrt{2}}{3\sqrt{3}}$

$\lvert V\rvert$	R	$\Gamma(R)$	$SGO(\Gamma(R))$	$PGO(\Gamma(R))$
12	$\mathbb{Z}_3 \times \mathbb{Z}_{11}$	$K_{2,10}$	$\frac{5}{\sqrt{2}}$	$\sqrt{\frac{5}{3}}$
	$\mathbb{Z}_5 \times \mathbb{F}_9$	$K_{4,8}$	$\frac{16}{\sqrt{11}}$	$2\sqrt{\frac{2}{3}}$
	$\mathbb{Z}_7 \times \mathbb{Z}_7$	$K_{6,6}$	$3\sqrt{3}$	$\sqrt{3}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$	Figure 6.25	$\frac{2\sqrt{3}}{\sqrt{5}} + \sqrt{\frac{6}{7}} + \sqrt{\frac{3}{5}} + \frac{1}{3\sqrt{7}}$	$\frac{3}{\sqrt{14}} + \sqrt{\frac{6}{35}} + \frac{1}{\sqrt{6}} + \frac{1}{7\sqrt{14}}$
	$\mathbb{Z}_{169}$	K_{12}	$6\sqrt{\frac{11}{13}}$	$3\sqrt{\frac{2}{11}}$
	$\mathbb{Z}_{13}[X]/(X^2)$	K_{12}	$6\sqrt{\frac{11}{13}}$	$3\sqrt{\frac{2}{11}}$
13	$\mathbb{Z}_2 \times \mathbb{Z}_{13}$	$K_{1,12}$	$\frac{12}{5}$	$2\sqrt{\frac{3}{13}}$
	$\mathbb{F}_4 \times \mathbb{Z}_{11}$	$K_{3,10}$	$\frac{30}{\sqrt{43}}$	$\sqrt{\frac{30}{13}}$
	$\mathbb{Z}_7 \times \mathbb{F}_8$	$K_{6,7}$	$\frac{42}{\sqrt{55}}$	$\sqrt{\frac{42}{13}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	Figure 6.26	$\frac{4}{\sqrt{17}} + \frac{4}{\sqrt{53}} + \frac{8}{\sqrt{17}} + \frac{4}{\sqrt{35}}$	$\frac{\sqrt{2}}{3} + \sqrt{\frac{2}{65}} + \frac{4\sqrt{2}}{\sqrt{35}} + \frac{2\sqrt{2}}{5\sqrt{5}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	Figure 6.27	$\frac{4}{\sqrt{15}} + \frac{1}{\sqrt{13}} + \frac{6}{\sqrt{31}} + \frac{1}{3\sqrt{7}} + \frac{2}{\sqrt{55}} + \sqrt{\frac{3}{5}} + \frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{7}} + \frac{1}{\sqrt{42}} + \sqrt{\frac{6}{35}} + \frac{1}{7\sqrt{14}} + \sqrt{\frac{2}{273}} + \frac{1}{\sqrt{6}} + \frac{1}{3\sqrt{2}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$	Figure 6.27	$\frac{4}{\sqrt{15}} + \frac{1}{\sqrt{13}} + \frac{6}{\sqrt{31}} + \frac{1}{3\sqrt{7}} + \frac{2}{\sqrt{55}} + \sqrt{\frac{3}{5}} + \frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{7}} + \frac{1}{\sqrt{42}} + \sqrt{\frac{6}{35}} + \frac{1}{7\sqrt{14}} + \sqrt{\frac{2}{273}} + \frac{1}{\sqrt{6}} + \frac{1}{3\sqrt{2}}$
14	$\mathbb{Z}_3 \times \mathbb{Z}_{13}$	$K_{2,12}$	$12\sqrt{\frac{2}{19}}$	$2\sqrt{\frac{3}{7}}$
	$\mathbb{Z}_5 \times \mathbb{Z}_{11}$	$K_{4,10}$	$\frac{20}{3}\sqrt{\frac{2}{3}}$	$2\sqrt{\frac{5}{7}}$
	$\mathbb{Z}_7 \times \mathbb{F}_9$	$K_{6,8}$	$24\sqrt{\frac{2}{31}}$	$2\sqrt{\frac{3}{7}}$
	$\mathbb{F}_8 \times \mathbb{F}_8$	$K_{7,7}$	$\frac{7\sqrt{7}}{3}$	$\sqrt{\frac{7}{2}}$
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Figure 6.28	$\frac{4}{\sqrt{15}} + \frac{12}{\sqrt{31}} + \sqrt{\frac{3}{5}} + \frac{2}{\sqrt{7}}$	$\sqrt{\frac{2}{7}} + \frac{2\sqrt{6}}{\sqrt{35}} + \frac{1}{\sqrt{6}} + \frac{3\sqrt{2}}{7\sqrt{7}}$
	$\mathbb{Z}_3 \times \mathbb{Z}_9$	Figure 6.29	$\frac{6\sqrt{2}}{\sqrt{13}} + \frac{4}{\sqrt{71}} + \frac{8}{\sqrt{23}} + \frac{1}{3\sqrt{7}}$	$\frac{3}{\sqrt{10}} + \sqrt{\frac{2}{105}} + \frac{4\sqrt{2}}{3\sqrt{7}} + \frac{1}{7\sqrt{14}}$
	$\mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2)$	Figure 6.29	$\frac{6\sqrt{2}}{\sqrt{13}} + \frac{4}{\sqrt{71}} + \frac{8}{\sqrt{23}} + \frac{1}{3\sqrt{7}}$	$\frac{3}{\sqrt{10}} + \sqrt{\frac{2}{105}} + \frac{4\sqrt{2}}{3\sqrt{7}} + \frac{1}{7\sqrt{14}}$

List of published/ Communicated papers based on this thesis

1) **S. Pradhan**, S. Kar and B. Biswas, Order two element graph over a group, Discrete Mathematics, Algorithms and Applications, 15(6) (2023), 2250136.

2) **S. Pradhan**, S. Kar and B. Biswas, Independence and domination number of order two element graph over a group, Discrete Mathematics, Algorithms and Applications, (2024).

3) **S. Pradhan**, S. Kar and B. Biswas, Topological indices of order two element graph over a group (communicated).

4) **S. Pradhan**, S. Kar and B. Biswas, Merrifield-Simmons index of Co-maximal graph over commutative ring (communicated).

5) **S. Pradhan**, S. Kar and B. Biswas, Merrifield-Simmons index of Zero-divisor graph over commutative ring (communicated).

6) **S. Pradhan**, S. Kar and B. Biswas, Sum and Product connectivity Gourava indices of Zero-divisor graph over commutative ring (communicated).

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