Characterization of some Ricci solitons admitting contact and para contact metrics and an application of Ricci flow in prey-predator model

Thesis

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By

Somnath Mondal

under the joint supervisons of

Prof.(Dr.)Ashis Kumar Sarkar,

Prof.(Dr.)Arindam Bhattacharyya

Professor, Department of Mathematics

Jadavpur University, Kolkata-700032

Department of Mathematics,

Jadavpur University,

Kolkata-700032, India.

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JADAVPUR UNIVERSITY
DEPARTMENT OF MATHEMATICS
KOLKATA - 700032

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This is to certify that thesis entitled "**Characterization of some Ricci solitons admitting contact and para contact metrics and an application of Ricci flow in prey-predator model**" submitted by **Somnath Mondal** whose name was registered vide INDEX NO.: 102/19/Maths./26 for the award of Doctor of Philosophy (Science) degree of Jadavpur University, Kolkata 700032, is absolutely based upon his own work under the supervision of **Prof. Ashis Kumar Sarkar** and co-supervision of **Prof. Arindam Bhattacharyya** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

10/07/2025
Prof. Ashis Kumar Sarakar
(Signature of the Supervisor, date with official seal)

Professor
DEPARTMENT OF MATHEMATICS
Jadavpur university
Kolkata - 700032, West Bengal

10/07/2025
Prof. Arindam Bhattacharyya
(Signature of the Co-Supervisor, date with official seal)

Professor
DEPARTMENT OF MATHEMATICS
Jadavpur university
Kolkata - 700032, West Bengal

This Thesis Dedicated to

My Beloved Parents

Sri Panchanan Mondal

and

Smt. Kanan Mondal

for their patience, support and love.

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Abstract

The aim of this doctoral thesis is to study Characterization of some Ricci solitons admitting contact and para contact metrics and an application of Ricci flow in prey-predator model. The thesis consists of **six chapters**. An introduction of the Differential Geometry, Ricci soliton and prey-predator model in **Chapter 1**.

In the **second chapter**, we initiate the study of almost η-Ricci-Yamabe soliton and gradient almost η-Ricci-Yamabe soliton within the framework of Kenmotsu manifold and obtain some characteristics of the manifold and the potential vector field. Finally we deliberate $*$-η-Ricci soliton admitting $(\kappa, \mu)'$-almost Kenmotsu manifold and proved that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{2n+1}(-1)$. Lastly we construct two examples.

In the **third chapter**, we establish some results regarding δ-Ricci-Yamabe almost soliton within the framework of paracontact metric manifolds. Here, we study δ-Ricci-Yamabe almost soliton and gradient δ-Ricci-Yamabe almost soliton on K-paracontact and para-Sasakian manifolds. Here, we prove that if K-paracontact metric g represents δ-Ricci-Yamabe almost soliton with the non-zero potential vector field V is parallel to ξ, then g being an Einstein with Einstein constant $-2n$. Later, we initiate that if g represents a gradient almost $*$-Ricci-Bourguinon soliton on a $(2n+1)$-dimensional η-Einstein para-Kenmotsu manifold then M^{2n+1} is either Einstein or there exists a vector field V is pointwise collinear with Reeb vector field ξ. Finally, we prove that if

the para-Sasakian metric is a $*$-Ricci Bourguinon soliton on a manifold, then M^{2n+1} is either $\mathcal{D}$-homothetic to an Einstein manifold, or the Ricci tensor of M^{2n+1} with respect to the canonical paracontact connection vanishes.

In **fourth chapter**, we demonstrate that if a Kenmotsu manifold admits an almost $*$-Ricci-Bourguinon soliton, then the manifold is η-Einstein. Next, we prove that if a $(\kappa, 2)'$-nullity distribution where $\kappa < -1$ admits an almost $*$-Ricci-Bourguinon soliton, then the manifold is Ricci flat. Further, we show that if a Kenmotsu manifold endows a gradient almost $*$-Ricci-Bourguignon soliton and ξ leaves the scalar curvature r invariant, then the manifold is an Einstein manifold with constant scalar curvature $r = n(1 - 2n)$. Later, we have studied on a Sasakian manifold if g represents an almost $*$-η-Ricci-Bourguignon soliton with potential vector field V_1 is pointwise collinear with ξ, then the manifold is an η-Einstein.

In the **fifth chapter**, we study W_2 - semisymmetric and W_2- pseudosymmeric trans-Sasakian space form,W_2-locally symmetric trans-Sasakian space form,W_2- locally ϕ- symmetric trans-Sasakian space form and W_2 -ϕ-recurrent trans-Sasakian space form. Later, we find some results on trans-Sasakian manifold which are conformal η-Einstein solitons where the Ricci tensor is cyclic parallel and Codazzi type. We also consider some curvature conditions with addition to conformal η-Einstein solitons on trans-Sasakian manifold. We also use torse-forming vector fields in addition to conformal η-Einstein solitons on trans-Sasakian manifold. Finally,we constructed an example.

In the **sixth chapter**, We consider a prey–predator model with Holling type III response function incorporating a prey refuge. The purpose of the work is to offer mathematical analysis of the model and to discuss some significant qualitative results that are expected to arise from the interplay of biological forces. Some numerical simulations are carried out. **The thesis contains the subject matter of the published/communicated papers whose titles, journal information and chapterwise distribution are given below:**

Authors	Title of paper and journal information	Chapter
Somnath Mondal, Santu Dey and Arindam Bhattacharyya	“ Characterization of almost η-Ricci-Yamabe soliton and gradient almost η-Ricci-Yamabe soliton on almost Kenmotsu manifolds”, **Acta Mathematica Sinica, English Serires**, Vol. 39, No. 4, pp 728-748 (2023).	2
Somnath Mondal, Santu Dey, Ali H. Alkhaldi, Ashis Kumar Sarkar and Arindam Bhattacharyya	“ Geometry of almost $*$-η-Ricci-Yamabe soliton on Kenmotsu manifold”, **Filomat**,Vol 38:24, PP 8525-8540, (2024).	2
Somnath Mondal, Santu Dey, Young Jin Suh and Arindam Bhattacharyya	“ The Geometry δ-Ricci-Yamabe almost soliton on Para-contact metric Manifolds”, **KYUNGPOOK Math. J.**, No. 63, 623-638 (2023).	3
Somnath Mondal, Santu Dey, Young Jin Suh and Ashis Kumar Sarkar	“ Some Characterizations of almost $*$-Ricci-Bourguignon solitons”, **Reviews in Mathematical Physics**, Accepted (2025).	3
Somnath Mondal, Santu Dey and Ashis Kumar Sarkar	“Characterizations of almost $*$-Ricci Boutrguignon soliton on Kenmotsu manifols”, Communicated.	4
Somnath Mondal, Santu Dey and Ashis Kumar Sarkar	“Sasakian metrics as an almost $*$-η-Ricci Bourguignon soliton”, Communicated.	4
Somnath Mondal and Arindam Bhattacharyya	“W_2-curvature tensor on trans-Sasakian space form”, **Bull. Cal. Math. Soc.**, 112, (5) 417-430 (2020).	5
Somnath Mondal, Yanlin Li, Santu Dey, Arindam Bhattacharyya and Akram Ali	“A study of Conformal η-Einstein solitons on trans-Sasakian 3-manifold”, **Journal of Nonlinear Mathematical Physics**, vol:30, 428-454 (2023).	5
Somnath Mondal, Arindam Bhattacharyya and Ashis Kumar Sarkar	“Generalised holling type-III prey-predator model with crowding effects of predator and its geometric comparisons with Ricci flow equation.”, Communicated.	6

Contents

1 Introduction

1.1 Introduction to differentiable manifolds

The concept of a manifold is central to many parts of geometry and modern mathematical physics because it allows more complicated structures to be described and understood in terms of the relatively well-understood properties of Euclidean spaces. A manifold is defined as below,

Definition 1.1.1 (Manifold)**.** *[48] A topological space M is said to be a n-dimensional manifold if it is Hausdorff, second countable and each point of M has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$ i.e., locally Euclidean of dimension n.*

1-dimensional manifolds include lines and circles. 2-dimensional manifolds are also called surfaces. The unit n-sphere, n-dimensional real projective space are some examples of n-dimensional manifolds.

A *chart* on a n-dimensional manifold M is a pair (U, φ), where the domain U is an open subset of M and φ is a homeomorphism from U to an open subset of $\mathbb{R}^n$. An *atlas* is a collection of charts whose domains cover M. Moreover, an atlas $\mathcal{A}$ is called *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible to each other. A maximal smooth atlas defines a *differentiable structure* or *smooth structure* on a manifold.

In mathematics, a differentiable manifold is a type of manifold that is locally similar enough to a linear space to allow one to do calculus. One may apply ideas from calculus while working within the individual charts, since image of each chart lies within a linear space to which the usual rules of calculus apply. If the charts are smoothly compatible, then computations done in one chart are valid in any other differentiable chart.

The metric which is a *Riemannian metric*, usually denoted by g, is a symmetric, smooth, covariant 2-tensor field on a smooth manifold M that is positive definite at each point. The pair (M, g) is called a *Riemannian manifold*. A Riemannian metric is not the same thing as a metric in the sense of metric spaces, although the two concepts are related.

If g is a metric on M, then for every point $p \in M$, the tensor g_p is an inner product on the tangent space of M at the point p, denoted by T_pM. We often define the real number $g_p(V_1', V_2')$ by $\langle V_1', V_2' \rangle_g$ for $V_1', V_2' \in T_pM$. In any smooth local coordinates (x^i), the Riemannian metric g can be expressed as

$$g = g_{ij} dx^i dx^j,$$

where (g_{ij}) is a symmetric, positive definite matrix of smooth functions. It is also well known theorem that every smooth $(i.e., C^\infty)$ manifold with or without boundary admits a Riemannian metric g. There may have enormous number of Riemannian metrics which can be defined in a manifold. For any point $p \in M$, we can define length or norm of a tangent vector and angle between two nonzero tangent vectors on T_pM using the Riemannian metric g.

In differential geometry, a *pseudo-Riemannian manifold* or a *semi-Riemannian manifold*, is a differentiable manifold M with a covariant 2-metric tensor g that is smooth, symmetric and everywhere nondegenerate. This is a generalization of a Riemannian manifold in which the requirement of positive-definiteness is relaxed. A well known result from linear algebra permits us to make a change of basis such that in the new base, g is represented by a diagonal matrix with -1 or 1 elements in the diagonal. If there are i, -1 elements and j, 1 elements in the diagonal, the tensor is said to have signature (i, j). The signature will be invariant in every connected component of M, but usually the restriction that it be a global invariant is added to the definition of a pseudo-Riemannian manifold. Unlike a Riemannian metric, some manifolds do not admit a pseudo-Riemannian metric.

Pseudo-Riemannian manifolds are crucial in Physics and in particular in General Theory of Relativity where space-time is modeled as a 4-pseudo Riemannian manifold with signature $(1, 3)$. Intuitively pseudo-Riemannian manifolds are generalizations of Minkowski's space just as a Riemannian manifold is a generalization of a vector space with a positive definite metric. The fundamental theorem of Riemannian geometry is true for pseudo-Riemannian manifolds as well. This allows one to speak of the Levi-Civita connection on a pseudo-Riemannian manifold along with the associated curvature tensor.

Definition 1.1.2 (Einstein manifold)**.** *[89] An* Einstein manifold *which is named after Albert Einstein, is a connected Riemannian manifold (M, g) whose Ricci tensor is proportional to the metric. In other language, a Riemannian manifold M is called an Einstein manifold if there exists some constant k such that $S = kg$, where the Ricci tensor of the manifold is denoted by S. Furthermore, if $k = 0$ then the manifold is called Ricci flat manifold.*

Definition 1.1.3 (Killing vector field)**.** *[89] A vector field V_1' on a Riemannian manifold (M, g) is said to be infinitesimal isometry or a Killing vector field if the Lie derivative with respect to V_1' of the metric g vanishes, i.e., $\mathcal{L}_{V_1'} g = 0$. Killing vector field is named after the German mathematician Wilhelm Karl Joseph Killing.*

Definition 1.1.4 (Koszul's formula)**.** *[20] If (M, g) is a Riemannian manifold (or pseudo-Riemannian manifold) with a Levi-Civita connection ∇, then for any vector fields V_1', V_2' and V_3' on the manifold M the Koszul's formula is defined by,*

$$\begin{aligned} 2g(\nabla_{V_1'} V_2', V_3') &= V_1' g(V_2', V_3') + V_2' g(V_3', V_1') - V_3' g(V_1', V_2') \\ &- g(V_1', [V_2', V_3']) - g(V_2', [V_1', V_3']) + g(V_3', [V_1', V_2']). \end{aligned} \tag{1.1.1}$$

We want to revisit some well known formulas from Yano[88] which are used extensively through out the entire thesis. On a Riemannian manifold (or semi-Riemannian manifold) (M, g) the following properties hold,

$$(\mathcal{L}_V \nabla)(V_1', V_2') = \mathcal{L}_V \nabla_{V_1'} V_2' - \nabla_{V_1'} \mathcal{L}_V V_2' - \nabla_{[V, V_1']} V_2', \tag{1.1.2}$$

$$(\mathcal{L}_V \nabla_{V_3'} g - \nabla_{V_3'} \mathcal{L}_V g - \nabla_{[V, V_3']} g)(V_1', V_2') = -g((\mathcal{L}_V \nabla)(V_1', V_3'), V_2') - g((\mathcal{L}_V \nabla)(V_2', V_3'), V_1'), \tag{1.1.3}$$

$$R(V_1', V_2') V_3' = \nabla_{V_1'} \nabla_{V_2'} V_3' - \nabla_{V_2'} \nabla_{V_1'} V_3' - \nabla_{[V_1', V_2']} V_3', \tag{1.1.4}$$

$$(\mathcal{L}_V R)(V_1', V_2') V_3' = (\nabla_{V_1'} \mathcal{L}_V \nabla)(V_2', V_3') - (\nabla_{V_2'} \mathcal{L}_V \nabla)(V_1', V_3'), \tag{1.1.5}$$

where $V_1', V_2', V_3' \in \chi(M)$ and R is the Riemannian curvature tensor.

Definition 1.1.5. *In [58] Pokhariyal and Mishra have defined the W_2-curvature tensor on a differential manifold of dimension $(2n+1)$ is given by*

$$W_2(V_1', V_2')V_3' = R(V_1', V_2')V_3' + \frac{1}{2n}\{g(V_1', V_3')QV_2' - g(V_2', V_3')QV_1'\}. \tag{1.1.6}$$

Here R and Q are the Riemannian curvature tensor and Ricci-operator of the Riemannian manifold respectively.

Now we revisit some of the important contact and para-contact manifolds which form the basis of this thesis. These manifolds have been used broadly throughout the thesis.

1.1.1 Contact manifold

Let M be a $(2n+1)$ dimensional smooth manifold and ϕ', ξ', η' be a tensor field of type $(1,1)$, a vector field, a 1-form on M respectively. If ϕ', ξ', η' satisfies the conditions

$$\phi'^2 V_1' = -V_1' + \eta'(V_1')\xi', \tag{1.1.7}$$

$$\eta'(\xi') = 1, \tag{1.1.8}$$

for any vector field V_1' on M, then M is said to have an almost contact structure (ϕ', ξ', η'). The manifold M equipped with the almost contact structure (ϕ', ξ', η') is called an almost contact manifold (for more details we refer to [9], [67]).

For a compatible Riemannian metric we have a relation

$$g(\phi' V_1', \phi' V_2') = g(V_1', V_2') - \eta'(V_1')\eta'(V_2'), \tag{1.1.9}$$

for arbitrary $V_1', V_2' \in \chi(M)$. A manifold having almost contact structure along with compatible Riemannian metric is called *almost contact metric manifold.*

An almost contact metric manifold $(M, \phi', \xi', \eta', g)$ has the following properties,

$$\phi'\xi' = 0, \tag{1.1.10}$$

$$\eta' \circ \phi' = 0, \tag{1.1.11}$$

$$g(V_1', \xi') = \eta'(V_1'), \tag{1.1.12}$$

$$g(\phi' V_1', V_2') = -g(V_1', \phi' V_2'), \tag{1.1.13}$$

for arbitrary $V_1', V_2' \in \chi(M)$.

A contact manifold is a $(2n+1)$-dimensional C^∞ manifold M [68] equipped with a global 1-form η' such that the volume form $\eta' \wedge (d\eta')^n$ is non-zero everywhere on M. This 1-form η' is called a contact form on M. For a contact form η' there exists a global vector field ξ' satisfying $d\eta'(\xi', V_1') = 0$ and $\eta'(\xi') = 1$. This vector field ξ' is called associated vector field to η'. Every contact manifold M with contact form η' such that

$$g(V_1', \phi' V_2') = d\eta'(V_1', V_2'), \tag{1.1.14}$$

hold $\forall$ V_1', $V_2' \in \chi(M)$.

The fundamental 2-form ϕ' is defined on an almost contact metric structure (ϕ', ξ', η', g) by $\phi'(V_1', V_2') = g(V_1', \phi' V_2')$. An almost contact structure constructed from a contact form η' is called contact metric structure associated to η' and a manifold with such a structure is called a contact metric manifold. An almost contact metric structure with $\phi' = d\eta'$ is a contact metric structure (for details see [39]).

A contact structure on M gives rise to an almost complex structure J on the product $M \times \mathbb{R}$. If the almost complex structure ϕ' on $M \times \mathbb{R}$ defined by

$$J(V_1', f\frac{d}{dt}) = (\phi' V_1' - f\xi', \eta'(V_1')\frac{d}{dt}),$$

where f is a real valued function. It is easy to verify that $J^2 = -I$. If J is integrable, we say that the almost contact structure is normal. The normality of an almost contact structure is equivalent with the vanishing of the tensor $N_\phi' = [\phi', \phi'] + 2d\eta' \otimes \xi'$, where $[\phi', \phi']$ is the Nijenhuis tensor of ϕ' (for more details we refer to [9]).

Sasakian manifold

Sasakian manifold is named after the great Japanese geometer Shigeo Sasaki and is defined as below.

Definition 1.1.6 (Sasakian manifold)**.** *If the contact structure of a differentiable manifold is normal, then the manifold is called Sasakian manifold or normal contact manifold.*

It is equivalent to say that, an almost contact metric manifold $M^{2n+1}(\phi', \xi', \eta', g)$ is a Sasakian manifold if and only if

$$(\nabla_{V_1'} \phi') V_2' = g(V_1', V_2')\xi' - \eta'(V_2')V_1', \tag{1.1.15}$$

holds for any vector fields V_1' and V_2' of $\chi(M)$.

A contact manifold is called a *K-contact manifold* if the characteristic vector field ξ' is Killing vector field. A Sasakian manifold is a K-contact manifold. The converse is also true but only for 3-dimensional manifold.

On a $(2n+1)$-dimensional Sasakian manifold the following relations hold ([2], [64], [31])

$$\nabla_{V_1'}\xi' = -\phi' V_1', \tag{1.1.16}$$

$$(\nabla_{V_1'}\eta')V_2' = g(V_1', V_2') - \eta'(V_1')\eta'(V_2'), \tag{1.1.17}$$

$$R(V_1', V_2')\xi' = \eta'(V_2')V_1' - \eta'(V_1')V_2', \tag{1.1.18}$$

$$R(V_1', \xi')V_2' = \eta'(V_2')V_1' - g(V_1', V_2')\xi', \tag{1.1.19}$$

$$Q\phi' = \phi' Q, \tag{1.1.20}$$

$$S(V_1', \xi') = 2n\eta'(V_1') \quad \Leftrightarrow \quad Q\xi' = 2n\xi', \tag{1.1.21}$$

$$S(\phi' V_1', \phi' V_2') = S(V_1', V_2') - 2n\eta'(V_1')\eta'(V_2'), \tag{1.1.22}$$

for all $V_1', V_2' \in \chi(M)$, where ∇, R, Q, S are Levi-Civita connection with respect to the metric g, Riemannian curvature tensor, Ricci operator and Ricci tensor respectively.

Kenmotsu manifold

In 1969, S. Tanno[78] classified the connected almost contact metric manifolds whose automorphism groups have maximal dimensions, as follows,

(i) Homogeneous normal contact Riemannian manifolds with constant ϕ'-holomorphic sectional curvature if $k(\xi', V_1') > 0$;

(ii) Global Riemannian product of a line or a circle and a Kählerian manifold with constant holomorphic sectional curvature if $k(\xi', V_1') = 0$;

(iii) A warped product space $\mathbb{R} \times_f N$, where $\mathbb{R}$ is the real line and N is a Kählerian manifold, if $k(\xi', V_1') < 0$;

here $k(\xi', V_1')$ denotes the sectional curvature of the plane section containing the characteristic vector field ξ' and an arbitrary vector field V_1'.

In 1972, K. Kenmotsu [46] derived certain tensorial equations characterizing the manifolds of the third class using the warping function $f(t) = ce^t$ on the interval $J = (-\epsilon, \epsilon)$. Since then, such manifolds have been known as Kenmotsu manifolds. Conversely, every point on a Kenmotsu manifold admits neighbourhood that is locally a warped product $J \times_f N$, where f is given by the expression above.

Definition 1.1.7 (Almost Kenmotsu manifold)**.** *An almost Kenmotsu manifold is a type of almost contact metric manifold characterized by the condition that the 1-form η' is closed, i.e., $d\eta' = 0$ and the fundamental 2-form ϕ' satisfies $d\phi' = 2\eta' \wedge \phi'$.*

Definition 1.1.8 (Kenmotsu manifold)**.** *A Kenmotsu manifold is a normal almost Kenmotsu manifold. Where the structure is normal, meaning the Nijenhuis torsion associated with ϕ' and η' vanishes.*

By [9], if in an almost contact metric manifold M the 1-form η' and the (1,1)-tensor field ϕ' satisfy the following condition for arbitrary $V_1', V_2' \in \chi(M)$

$$(\nabla_{V_1'}\phi')V_2' = g(\phi' V_1', V_2')\xi' - \eta'(V_2')\phi' V_1', \tag{1.1.23}$$

then the manifold M is called a Kenmotsu manifold. Moreover, it can be shown that the condition is equivalent to the normality of the structure, i.e., the vanishing of the Nijenhuis tensor associated with (ϕ', ξ', η').

In Kenmotsu manifold of dimension $(2n+1)$, the following relations hold,

$$\nabla_{V_1'}\xi' = V_1' - \eta'(V_1')\xi', \tag{1.1.24}$$

$$(\nabla_{V_1'}\eta')V_2' = g(V_1', V_2') - \eta'(V_1')\eta'(V_2'), \tag{1.1.25}$$

$$R(V_1', V_2')\xi' = \eta'(V_1')V_2' - \eta'(V_2')V_1', \tag{1.1.26}$$

$$S(V_1', \xi') = -2n\eta'(V_1') \Leftrightarrow Q\xi' = -2n\xi', \tag{1.1.27}$$

$$\mathcal{L}_\xi' g(V_1', V_2') = 2g(V_1', V_2') - 2\eta'(V_1')\eta'(V_2'), \tag{1.1.28}$$

for arbitrary $V_1', V_2' \in \chi(M)$, R is Riemannian curvature tensor, S is the Ricci tensor and $\mathcal{L}$ is the lie derivative operator.

Definition 1.1.9 (η'-Einstein manifold)**.** *[89] An almost contact (or almost para-contact) metric manifold M is said to be η'-Einstein manifold if there exists two constants a and b such that the Ricci tensor S satisfies the following relation,*

$$S(V_1', V_2') = ag(V_1', V_2') + b\eta'(V_1')\eta'(V_2'), \tag{1.1.29}$$

for all $V_1', V_2' \in \chi(M)$. Clearly, if $b = 0$, then an η'-Einstein manifold reduces to an Einstein manifold.

Now, considering $V_1' = \xi'$ in the η'-Einstein condition and using (1.1.27) we obtain $a + b = -2n$. Contracting equation (1.1.29) over V_1' and V_2', we get the scalar curvature $r = (2n+1)a + b$, where r denotes the scalar curvature of the manifold. Solving these two equation we obtain, $a = \frac{1}{2n}(2n+r)$ and $b = -\frac{1}{2n}\{2n(2n+1)+r\}$. Substituting these values into equation (1.1.29), we can rewrite it as

$$S(V_1', V_2') = \frac{1}{2n}(2n+r)g(V_1', V_2') - \frac{1}{2n}\{2n(2n+1)+r\}\eta'(V_1')\eta'(V_2'). \tag{1.1.30}$$

$(\kappa, \mu')'$ almost Kenmotsu manifold

On an almost contact manifold, we define two $(1,1)$-type tensor fields $h = \frac{1}{2}\mathcal{L}_{\xi}'\phi'$ and $h' = h \circ \phi'$ and an operator $\ell = R(., \xi')\xi'$, where $\mathcal{L}_{\xi}'\phi'$ is the Lie derivative of ϕ' along the Reeb vector field ξ'.

Some renowned mathematicians defined many nullity distributions on contact manifolds. In 1995, Blair et al. in [10] have defined (κ, μ')-nullity distribution on a contact metric manifold $M^{2n+1}(\phi', \xi'\eta', g)$, for two real numbers κ and μ', by

$$\begin{aligned} N(\kappa, \mu') : p \to N_p(\kappa, \mu') = \{V_3' \in T_pM | R(V_1', V_2')V_3' =& \kappa(g(V_2', V_3')V_1' - g(V_1', V_3')V_2') \\ &+ \mu'(g(V_2', V_3')hV_1' - g(V_1', V_3')hV_2')\}, \end{aligned} \tag{1.1.31}$$

for any given vector fields V_1' and V_2' on M. In [25], Dileo and Pastore introduced the notion of $(k, \mu_1')'$-nullity distribution, on an almost Kenmotsu manifold $(M, \phi', \xi', \eta', g)$. It is defined at any point $p \in M$, for real constants $k, \mu_1' \in \mathbb{R}$, as follows:

$$\begin{aligned} N_p(k, \mu')' &= \{V_3' \in T_p(M) : R(V_1', V_2')V_3' = k[g(V_2', V_3')V_1' - g(V_1', V_3')V_2'] \\ &+ \mu'[g(V_2', V_3')h'V_1' - g(V_1', V_3')h'V_2']\}, \end{aligned} \tag{1.1.32}$$

$\forall\ V_1', V_2'$ on $T_p(M)$. For a more comprehensive study on almost Kenmotsu manifolds, we refer the reader to [21, 25, 74, 82], and other references listed in the bibliography.

If the characteristic vector field ξ' belongs to the (κ, μ')-nullity distribution, then

$$R(V_1', V_2')\xi' = \kappa(\eta'(V_2')V_1' - \eta'(V_1')V_2') + \mu'(\eta'(V_2')hV_1' - \eta'(V_1')hV_2').$$

This nullity distribution is a generalization of κ-nullity distribution. If we consider $\mu' = 0$, then (κ, μ')-nullity distribution reduces to κ-nullity distribution. A contact metric manifold whose characteristic vector field ξ' belongs to κ-nullity distribution, i.e., the relation $R(V_1', V_2')\xi' = \kappa(\eta'(V_2')V_1' - \eta'(V_1')V_2')$ holds, is called a $N(\kappa)$-contact metric manifold. A $N(\kappa)$-contact metric manifold is Sasakian if and only if $\kappa = 1$.

In 2009, Dileo and Pastore [25] first considered some nullity distributions like (κ, μ')-nullity distribution, $(\kappa, \mu')'$-nullity distribution on almost Kenmotsu manifold.

The tensor fields h and h' play significant roles in the geometry of an almost Kenmotsu manifold. Both tensors are symmetric and satisfy the following relations,

$$\nabla_{V_1'}\xi' = V_1' - \eta'(V_1')\xi' + \phi' h V_1', \tag{1.1.33}$$

$$h\xi' = h'\xi' = 0, \tag{1.1.34}$$

$$h\phi' = -\phi' h, \tag{1.1.35}$$

$$tr(h) = tr(h') = 0, h = 0 \Leftrightarrow h' = 0 \tag{1.1.36}$$

for any $V_1', V_2' \in \chi(M)$, where ∇ is the Levi-Civita connection of the manifold M. In addition, the following curvature property is also satisfied,

$$R(V_1', V_2')\xi' = \eta'(V_1')(V_2' + h'V_2') - \eta'(V_2')(V_1' + h'V_1') + (\nabla_{V_1'}h')V_2' - (\nabla_{V_2'}h')V_1', \tag{1.1.37}$$

where R is the Riemannian curvature tensor of (M, g).

Definition 1.1.10 ($(\kappa, \mu')'$-almost Kenmotsu manifold)**.** *: An almost Kenmotsu manifold whose characteristic vector field ξ' satisfies the $(\kappa, \mu')'$-nullity distribution condition is defined by the relation:*

$$R(V_1', V_2')\xi' = \kappa(\eta'(V_2')V_1' - \eta'(V_1')V_2') + \mu'(\eta'(V_2')h'V_1' - \eta'(V_1')h'V_2'), \tag{1.1.38}$$

for any $V_1', V_2' \in \chi(M)$, where κ and μ' are real constants, is called $(\kappa, \mu')'$-almost Kenmotsu manifold.

On a $(\kappa, \mu')'$-almost Kenmotsu manifold M we have (see [25]),

$$h'^2(V_1') = -(\kappa + 1)[V_1' - \eta'(V_1')\xi'], \tag{1.1.39}$$

$$h^2(V_1') = -(\kappa + 1)[V_1' - \eta'(V_1')\xi'], \tag{1.1.40}$$

for $V_1' \in \chi(M)$. From the previous relation, it implies that $h' = 0$ if and only if $\kappa = -1$, and $h' \neq 0$. Let $V_1' \in Ker(\eta')$ be an eigenvector field of h' orthogonal to ξ', corresponding to the eigenvalue α'. Then, from equation (1.1.39), we obtain $\alpha'^2 = -(\kappa+1)$, which implies $\kappa \leq -1$. Dileo and Pastore proved that on a $(\kappa, \mu')'$-almost Kenmotsu manifold with $\kappa < -1$, the parameter $\mu' = -2$ (Proposition 4.1 of [25]). In what follows, we work with $(\kappa, -2)'$-almost Kenmotsu manifolds as a standard case.

We recall some useful results on a $(2n+1)$ dimensional $(\kappa, -2)'$-almost Kenmotsu manifold M with $\kappa < -1$, as follows,

$$
\begin{aligned}
R(\xi', V_1')V_2' &= \kappa(g(V_1', V_2')\xi' - \eta'(V_2')V_1') - 2(g(h'V_1', V_2')\xi' - \eta'(V_2')h'V_1'), && (1.1.41)\\
QV_1' &= -2nV_1' + 2n(\kappa+1)\eta'(V_1')\xi' - 2nh'(V_1'), && (1.1.42)\\
r &= 2n(\kappa - 2n), && (1.1.43)\\
(\nabla_{V_1'}\eta')V_2' &= g(V_1', V_2') - \eta'(V_1')\eta'(V_2') + g(h'V_1', V_2'), && (1.1.44)\\
(\nabla_{V_1'}h')V_2' &= -g(h'V_1' + h'^2V_1', V_2')\xi' - \eta'(V_2')(h'V_1' + h'^2V_1'), && (1.1.45)
\end{aligned}
$$

where $V_1', V_2' \in \chi(M)$, Q, r are the Ricci operator and scalar curvature of M respectively.

Trans-Sasakian manifold and space form

An almost contact metric manifold M is called a trans-Sasakian manifold if the product manifold $M \times \mathbb{R}$, equipped with the almost Hermitian structure (J, G), where G is the product metric on $M \times \mathbb{R}$, belongs to the Gray–Hervella class W_4 (see [38]).

Definition 1.1.11 (Trans-Sasakian manifold)**.** *An almost contact manifold $M(\phi', \xi', \eta', g)$ is called trans-Sasakian manifold of type (α', β') if there are smooth functions α', β' satisfying,*

$$(\nabla_{V_1'}\phi')V_2' = \alpha'[g(V_1', V_2')\xi' - \eta'(V_2')V_1'] + \beta'[g(\phi'V_1', V_2')\xi' - \eta'(V_2')\phi'V_1'], \quad (1.1.46)$$

where $V_1', V_2' \in \chi(M)$ are arbitrary.

The functions α' and β' are called structure functions of the manifold. Trans-Sasakian manifolds of type $(0,0)$, $(\alpha', 0)$ and $(0, \beta')$ are known as cosymplectic, α'-Sasakian, β'-Kenmotsu manifolds, respectively. Form equation (1.1.46) we can deduce that,

$$\nabla_{V_1'}\xi' = -\alpha'(\phi'V_1') + \beta'(V_1' - \eta'(V_1')\xi'). \quad (1.1.47)$$

$$(\nabla_{V_1'}\eta')V_2' = -\alpha' g(\phi' V_1', V_2') + \beta' g(\phi' V_1', \phi' V_2'). \tag{1.1.48}$$

The manifold $(M, \phi', \xi', \eta', g, \alpha', \beta')$ is said to be a trans-Sasakian manifold of type (α', β'). A Sasakian manifold appears as a special case of an α'-Sasakian manifold with $\alpha' = 1$ and $\beta' = 0$, while a Kenmotsu manifold corresponds to the case $\alpha' = 0$ and $\beta' = 1$. Furthermore Marrero [49] has shown that a trans-Sasakian manifold of dimension greater than 5 is necessarily either cosympletic, α'-Sasakian or β'-Kenmotsu.

Now we define a trans-Sasakian space form as follows [52].

Definition 1.1.12. *A trans-Sasakian manifold M^{2n+1} of constant ϕ'-sectional curvature c is called trans-Sasakian space form denoted by $M^{2n+1}(c)$ and its curvature tensor is given by*

$$\begin{aligned} R(V_1', V_2')V_3' &= \frac{\alpha'(c+3)+\beta'(c-3)}{4}[g(V_2', V_3')V_1' - g(V_1', V_3')V_2'] \\ &+ \frac{\alpha'(c-1)+\beta'(c+1)}{4}[\eta'(V_1')\eta'(V_3')V_2' - \eta'(V_2')\eta'(V_3')V_1' \\ &+ g(V_1', V_3')\eta'(V_2')\xi' - g(V_2', V_3')\eta'(V_1')\xi' + g(\phi' V_2', V_3')\phi' V_1' \\ &- g(\phi' V_1', V_3')\phi' V_2' + 2g(V_1', \phi' V_2')\phi' V_3', \end{aligned} \tag{1.1.49}$$

where α' and β' are smooth funtions on M.

The properties of trans-Sasakian space form are studied by Bhattacharyya et al [52]. In a trans-Sasakian space form the W_2-curvature tensor satisfies the condition

$$\eta'(W_2(V_1', V_2')V_3') = 0. \tag{1.1.50}$$

Again the following relations hold for a $(2n+1)$ dimentional trans-Sasakian space form:

$$\begin{aligned} S(V_1', V_2') &= \frac{1}{2}\{(3n-1)(\alpha'-\beta') + c(n+1)(\alpha'+\beta')\}g(V_1', V_2') \\ &- \frac{n+1}{2}\{c(\alpha'+\beta') - (\alpha'-\beta')\}\eta'(V_1')\eta'(V_2'), \end{aligned} \tag{1.1.51}$$

$$S(\phi' V_1', \phi' V_2') = S(V_1', V_2') - 2n(\alpha'-\beta')\eta'(V_1')\eta'(V_2'), \tag{1.1.52}$$

$$\begin{aligned} QV_1' &= \frac{1}{2}\{(3n-1)(\alpha'-\beta') + c(n+1)(\alpha'+\beta')\}V_1' \\ &- \frac{n+1}{2}\{c(\alpha'+\beta') - (\alpha'-\beta')\}\eta'(V_1')\xi', \end{aligned} \tag{1.1.53}$$

$$r = n(\alpha'-\beta')(3n+1) + nc(\alpha'+\beta')(n+1), \tag{1.1.54}$$

$$R(V_1', V_2')\xi' = (\alpha' - \beta')\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\}, \tag{1.1.55}$$

$$R(\xi', V_1')V_2' = (\alpha' - \beta')\{g(V_1', V_2')\xi' - \eta'(V_2')V_1'\}, \tag{1.1.56}$$

$$\eta'(R(V_1', V_2')V_3') = (\alpha' - \beta')\{g(V_2', V_3')\eta'(V_1') - g(V_1', V_3')\eta'(V_2')\}, \tag{1.1.57}$$

$$S(V_1', \xi') = 2n(\alpha' - \beta')\eta'(V_1'), \tag{1.1.58}$$

Here R, S, Q and r are the Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature tensor of trans-Sasakian space form respectively. (1.1.51)

Again, in a trans-Sasakian 3-manifold (M, g) the Ricci tensor is given by

$$\begin{aligned} S(V_1', V_2') &= [\frac{r}{2} + \xi'\beta' - (\alpha'^2 - \beta'^2)]g(V_1', V_2') - [\frac{r}{2} + \xi'\beta' - 3(\alpha'^2 - \beta'^2)]\eta'(V_1')\eta'(V_2') \\ &- [V_2'\beta' + \phi'(V_2')\alpha']\eta'(V_1') - [V_1'\beta' + \phi'(V_1')\alpha']\eta'(V_2'), \end{aligned} \tag{1.1.59}$$

$$R(V_1', V_2')\xi' = (\alpha'^2 - \beta'^2)[\eta'(V_2')V_1' - \eta'(V_1')V_2'], \tag{1.1.60}$$

$$R(\xi', V_1')V_2' = (\alpha'^2 - \beta'^2)[g(V_1', V_2')\xi' - \eta'(V_2')V_1'], \tag{1.1.61}$$

$$R(\xi', V_1')\xi' = (\alpha'^2 - \beta'^2)[\eta'(V_1')\xi' - V_1'], \tag{1.1.62}$$

$$S(V_1', V_2') = [\frac{r}{2} - (\alpha'^2 - \beta'^2)]g(V_1', V_2') - [\frac{r}{2} - 3(\alpha'^2 - \beta'^2)]\eta'(V_1')\eta'(V_2'), \tag{1.1.63}$$

$$S(V_1', \xi') = 2(\alpha'^2 - \beta'^2)\eta'(V_1'). \tag{1.1.64}$$

1.1.2 Para-contact manifold

The notion of an almost para-contact manifold was first introduced by Sato [69]. Subsequently, Kaneyuki and Williams [44] associated a pseudo-Riemannian metric with such manifolds, building upon the earlier work of Takahashi [77], who introduced pseudo-Riemannian metrics in the context of contact manifolds—particularly in Sasakian geometry.

A $(2n+1)$-dimensional smooth manifold M is said to admit an almost para-contact structure if there exist a vector field ξ', a (1,1)-tensor field ϕ', and a 1-form η' satisfying the following conditions

$$i)\ \phi'^2 = I - \eta' \otimes \xi', \tag{1.1.65}$$

$$ii)\ \eta'(\xi') = 1, \tag{1.1.66}$$

iii) The tensor field ϕ' induces on the 2n-dimensional distribution $\mathcal{D} \equiv ker(\eta')$, an almost paracomplex structure $\mathcal{P}$, that is, $\mathcal{P}^2 \equiv I_{\chi(M)}$. The eigen subbundles $\mathcal{D}^+$ and $\mathcal{D}^-$, corresponding to the eigenvalues $+1$ and -1 of $\mathcal{P}$, respectively, have equal dimensions n. Hence, the distribution decomposes as $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$.

Now follow this with identities that hold in almost para-contact manifolds, such as

$$\phi'\xi' = 0, \tag{1.1.67}$$

$$\eta' \circ \phi' = 0. \tag{1.1.68}$$

The tensor field ϕ' induces an almost paracomplex structure on each fibre of $Ker(\eta')$; that is, the eigen distributions corresponding to the eigenvalues 1 and -1 of ϕ' have the same dimension n.

Zamkovoy in [90] proved that any almost para-contact structure admits a pseudo-Riemannian metric. If a manifold with an almost para-contact structure (M, ϕ', ξ', η') admits a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi' V_1', \phi' V_2') = -g(V_1', V_2') + \eta'(V_1')\eta'(V_2'), \tag{1.1.69}$$

holds for any $V_1', V_2' \in \chi(M)$, then g is called compatible metric, and the manifold $(M, \phi', \xi', \eta', g)$ is called almost para-contact metric manifold.

The fundamental 2-form ϕ' on an almost para-contact metric manifold $(M, \phi', \xi', \eta', g)$ is defined by $\phi'(V_1', V_2') = g(V_1', \phi' V_2')$ for any vector fields V_1', $V_2' \in \chi(M)$. Clearly the skew-symmetry of the 2-form ϕ' follows directly from the properties of ϕ'. An almost para-contact metric manifold for which

$$\phi'(V_1', V_2') = d\eta'(V_1', V_2') = g(V_1', \phi' V_2'), \tag{1.1.70}$$

is said to be para-contact metric manifold. In this case, η' becomes a contact form i.e., $\eta' \wedge (d\eta')^n \neq 0$. On a para-contact metric manifold $M^{2n+1}(\phi', \xi', \eta', g)$ we consider a self-adjoint operator $h = \frac{1}{2}\mathcal{L}'_\xi \phi'$, where $\mathcal{L}'_\xi$ denotes the Lie derivative along the Reeb vector field ξ'. The operator h is symmetric and satisfies the following properties

$$h\phi' = -\phi' h, \tag{1.1.71}$$

$$h\xi' = 0, \tag{1.1.72}$$

$$\nabla_{V_1'}\xi' = -\phi' V_1' + \phi' h V_1', \tag{1.1.73}$$

here, ∇ denotes the Levi-Civita connection (i.e., the operator of covariant differentiation) with respect to the metric g. The normality of a para-contact metric manifold $(M, \phi', \xi', \eta', g)$ is equivalent to the vanishing of the $(1,2)$- type Nijenhuis torsion tensor defined by $N_{\phi'}(V_1', V_2') = [\phi', \phi'](V_1', V_2') - 2d\eta'(V_1', V_2')\xi'$, where $[\phi', \phi'](V_1', V_2') = \phi'^2[V_1', V_2'] + [\phi' V_1', \phi' V_2'] - \phi'[V_1', \phi' V_2'] - \phi'[\phi' V_1', V_2']$ for any $V_1', V_2' \in \chi(M)$.

Para-Kenmotsu manifold

By analogy with the Kenmotsu manifold, Welyczko [85] introduced the notion of a para-Kenmotsu manifold (also referred to as a p-Kenmotsu manifold).

Definition 1.1.13 (almost para-Kenmotsu manifold)**.** *If an almost para-contact metric manifold satisfies*

$$(\nabla_{V_1'}\phi')V_2' = g(\phi' V_1', V_2')\xi' - \eta'(V_2')\phi' V_1', \tag{1.1.74}$$

for arbitrary vector fields V_1' and V_2', then the manifold is called almost para-Kenmotsu manifold.

Definition 1.1.14 (para-Kenmotsu manifold)**.** *A normal almost para-Kenmotsu manifold is called para-Kenmotsu manifold.*

The following properties hold on a $(2n+1)$-dimensional para-Kenmotsu manifold

$$\nabla_{V_1'}\xi' = V_1' - \eta'(V_1')\xi', \tag{1.1.75}$$

$$(\nabla_{V_1'}\eta')V_2' = g(V_1', V_2') - \eta'(V_1')\eta'(V_2'), \tag{1.1.76}$$

$$Q\xi' = -2n\xi', \tag{1.1.77}$$

$$R(V_1', V_2')\xi' = \eta'(V_1')V_2' - \eta'(V_2')V_1', \tag{1.1.78}$$

$$R(V_1', \xi')\xi' = \eta'(V_1')\xi' - V_1', \tag{1.1.79}$$

$$R(V_1', \xi')V_2' = g(V_1', V_2')\xi' - \eta'(V_2')V_1', \tag{1.1.80}$$

$$(\mathcal{L}'_\xi g)(V_1', V_2') = 2[g(V_1', V_2') - \eta'(V_1')\eta'(V_2')], \tag{1.1.81}$$

for any $V_1', V_2' \in \chi(M)$ here, $\mathcal{L}$ and ∇ denote the Lie derivative and the Levi-Civita connection (i.e., covariant differentiation with respect to the metric g), respectively. The symbol Q denotes the Ricci operator associated with the Ricci tensor S, defined by $S(V_1', V_2') = g(QV_1', V_2')$, The symbol R denotes the Riemannian curvature tensor of the manifold (M, g).

From equation (1.1.29) and following Zamkovoy [91], proposition (4.1), it is shown that if M^{2n+1} is an η'-Einstein para-Kenmotsu manifold of dimension greater than 3, then we get

$$V_3'(b) - 2b\eta'(V_3') = 0$$

for any $V_3' \in TM$.

Para-Sasakian manifold

Sato and Matsumoto [70] defined and studied the concept of a para-Sasakian manifold (briefly, p-Sasakian manifold) as a special case of an almost paracontact manifold. Subsequently, Adati et al. [1] deduced several fundamental properties of para-Sasakian manifolds, laying the groundwork for further geometric and structural analysis.

Definition 1.1.15 (para-Sasakian manifold). *A normal para-contact metric manifold is called a para-Sasakian metric manifold.*

Equivalently, an almost para-contact metric manifold is called a para-Sasakian manifold if it satisfies the condition.

$$(\nabla_{V_1'}\phi')V_2' = -g(V_1', V_2')\xi' + \eta'(V_2')V_1', \tag{1.1.82}$$

for arbitrary $V_1', V_2' \in \chi(M)$. In a para-Sasakian manifold, the operator $h = \frac{1}{2}\mathcal{L}_{\xi'}\phi'$ vanishes identically i.e., $h = 0$. Moreover, the manifold satisfies the following identity:

$$\begin{aligned}
\nabla_{V_1'}\xi' &= -\phi' V_1', && (1.1.83)\\
R(V_1', V_2')\xi' &= \eta'(V_1')V_2' - \eta'(V_2')V_1', && (1.1.84)\\
S(V_1', \xi') = -2n\eta'(V_1'), \quad Q\xi' &= -2n\xi', && (1.1.85)
\end{aligned}$$

for all vector fields V_1' and V_2' on M and R, Q denote Riemannian curvature tensor and Ricci operator associated with the Ricci tensor S defined by $S(V_1', V_2') = g(QV_1', V_2')$.

Subsequent, we remind the following commutation formula for our later use (see [87])

$$\begin{aligned}
(\mathcal{L}_V\nabla_{V_3'}g - \nabla_{V_3'}\mathcal{L}_V g - \nabla_{[V,V_3']}g)(V_1', V_2') &= -g((\mathcal{L}_V\nabla)(V_3', V_1'), V_2')\\
&\quad -g((\mathcal{L}_V\nabla)(V_3', V_2'), V_1') \qquad (1.1.86)
\end{aligned}$$

for all vector fields V_1', V_2' on M^{2n+1} and ∇ is the metric connection. By virtue of parallelism of the pseudo-Riemannian metric $\nabla g = 0$, this formula abates to

$$(\nabla_{V_3'} \mathcal{L}_V g)(V_1', V_2') = g((\mathcal{L}_V \nabla)(V_3', V_1'), V_2') + g((\mathcal{L}_V \nabla)(V_3', V_2'), V_1') \tag{1.1.87}$$

for all vector fields V_1', V_2' on M.

Here, we want to evoke some useful important definitions,

Definition 1.1.16. *[Contact vector field]. [35] A vector field V_1' on a contact manifold is said to be a contact vector field if it preserve the contact form η' i.e., if there exist a smooth function $\nu : M \to \mathbb{R}$ such that*

$$\mathcal{L}_{V_1'} \eta' = \nu \eta', \tag{1.1.88}$$

if $\nu = 0$, then the vector field V_1' is called strict see in [8]. A vector field V on a contact metric manifold $(M, \phi', \xi', \eta', g)$ is said to be an infinitesimal automorphism if it preserves the structure tensors ϕ', ξ', η' and the metric g.

Definition 1.1.17 (Infinitesimal contact transformation)**.** *[89] In an almost contact (or almost para-contact) metric manifold M, a vector field V_1' is said to be infinitesimal contact transformation if $\mathcal{L}_{V_1'} \xi' = f \xi'$, for some smooth function $f : M \to R$. In particular, if $f = 0$, i.e., $\mathcal{L}_{V_1'} \xi' = 0$, then V_1' is called a strict infinitesimal contact transformation.*

Definition 1.1.18 (Torse forming vector field)**.** *A vector field ξ' on a manifold M is called torse forming vector field [87] if it satisfies the condition*

$$\nabla_{V_1'} \xi' = f V_1' + \gamma(V_1') \xi', \tag{1.1.89}$$

for all vector fields $V_1' \in \chi(M)$, where $f \in C^\infty(M)$ is a smooth function γ is 1-form γ on M. A torse forming vector field is called recurrent if $f = 0$; that is $\nabla_{V_1'} \xi' = \gamma(V_1') \xi'$.

Definition 1.1.19 (Conformal vector field)**.** *[89] On an almost contact (or almost para-contact) metric manifold M, a vector field V is said to be conformal Killing vector field (or simply conformal vector field) if there exists a smooth function $\rho \in C^\infty(M)$ such that*

$$\mathcal{L}_V g = \rho g.$$

where $\mathcal{L}_V g$ denotes the Lie derivative of the metric tensor g along V, and ρ is called the conformal coefficient. If the conformal coefficient ρ, then V becomes a Killing vector field, i.e., $\mathcal{L}_V g = 0$.

1.2 Introduction to Ricci solitons

The Ricci flow, a powerful tool in Riemannian geometry, was first introduced by Richard Hamilton in 1982 to study and eventually establish Thurston's Geometrization Conjecture. It's a partial differential equation that describes how the metric on a Riemannian manifold evolves over time, "smoothing out" the geometry by reducing the curvature. The differential equation that was to play a key role in solving the Poincaré conjecture is the Ricci flow equation. Any closed 3-manifold can be canonically decomposed into pieces in such a way that each admits a unique homogeneous geometry. Here the flow is studied to interpret the evolution of manifold with respect to time. Our aim is to study the proposed model and its geometric perspective under the Ricci flow. The Ricci flow equation is given by

$$\frac{dg_{ij}}{dt} = -2R_{ij}, \tag{1.2.90}$$

where R_{ij} denotes the components of the Ricci curvature tensor corresponding to the metric g, whose components are g_{ij}."Three manifolds with positive Ricci curvature closed 3-manifolds of positive Ricci curvature are topologically classified as spherical space form" until that time, most results relating the curvature of a 3-manifold to its topology involved the influence of curvature on the fundamental group.

A Riemannian manifold (or pseudo-Riemannian manifold) (M, g) is said to admit a Ricci soliton, which is a natural generalization of Einstein metric (i.e., when the Ricci tensor $S = ag$ for some constant a), if there exists a smooth non-zero vector field V and a constant λ' such that:

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda' g = 0, \tag{1.2.91}$$

where $\mathcal{L}_V$ denotes Lie derivative along the direction V, and S denotes the Ricci curvature tensor of the manifold. The vector field V is called potential vector field, and the constant λ' is called soliton constant.

The Ricci soliton is a self-similar solution to Hamilton's Ricci flow [42], which is defined by the evolution equation

$$\tfrac{\partial g(t)}{\partial t} = -2S(g(t)),$$

with initial condition $g(0) = g$, where $g(t)$ is a one-parameter family of Riemannian metrics on the manifold M, and $S(g(t))$ denotes the Ricci curvature tensor with respect to $g(t)$. The potential vector field V and the soliton constant λ' play crucial roles in determining the nature of the soliton. A soliton is classified as shrinking, steady or expanding according as $\lambda' < 0$, $\lambda' = 0$ or $\lambda' > 0$. If the vector field V is zero or Killing, then the Ricci soliton reduces to Einstein manifold, and in this case the soliton is called a trivial soliton.

If the potential vector field V is the gradient of a smooth function f, denoted by ∇f or Df, then the Ricci soliton is called a gradient Ricci soliton. In this case, the soliton equation reduces to:

$$Hess(f) + S + \lambda' g = 0, \tag{1.2.92}$$

where $Hess(f)$ or $\nabla^2 f$ denotes the Hessian of the function f, that is, the covariant derivative of the gradient vector field ∇f. Perelman [57] proved that a Ricci soliton on a compact manifold is necessarily a gradient Ricci soliton.

In 2014, Kaimakamis and Panagiotidou [43] introduced a modified definition of Ricci soliton by replacing the classical Ricci tensor S with the $*$-Ricci tensor S^*, a concept originally introduced by Tachibana [76] and further studied by Hamada [41]. The $*$-Ricci tensor S^* is defined by

$$S^*(V_1', V_2') = \frac{1}{2}(trace\{\phi'.R(V_1', \phi' V_2')\}),$$

$\forall$ vector fields V_1' and V_2' on M. They applied the concept of $*$-Ricci soliton within the framework of real hypersurfaces of a complex space form. In this setting, a pseudo-Riemannian metric g is said to define a $*$-Ricci soliton if there exists a smooth vector field V (called the potential vector field) and a constant λ' such that the following condition holds:

$$\mathcal{L}_V g + 2S^* + 2\lambda' g = 0. \tag{1.2.93}$$

Note that a $*$-Ricci soliton is said to be trivial if the potential vector field V is a Killing vector field. In this case, the soliton equation reduces to the condition that the $*$-Ricci tensor S^* is proportional to the metric tensor g, and hence the manifold becomes a $*$-Einstein manifold. By a $*$-Einstein manifold, we mean a manifold satisfying

$$S^* = cg$$

for some constant $c \in \mathbb{R}$. Thus, the concept of a $*$-Ricci soliton serves as a natural generalization of the $*$-Einstein metric. Furthermore, a $*$-Ricci soliton is called an almost

$*$-Ricci soliton if the soliton constant λ' is allowed to be a smooth function on the manifold M, rather than a constant. This generalization broadens the class of solutions and allows for richer geometric structures.

In [43], it has been studied that real hypersurfaces of a non-flat complex space form admitting a $*$-Ricci soliton with the structure vector field as the potential vector field exhibit strong geometric restrictions. In particular, it was proved that no real hypersurface in a complex projective space admits a $*$-Ricci soliton under this condition. Furthermore, it was shown that a real hypersurface of complex hyperbolic space admitting a $*$-Ricci soliton is locally congruent to a geodesic hypersphere.

In 2009, Cho and Kimura [16] introduced the notion of an η'-Ricci soliton, which serves as a generalization of the classical Ricci soliton. The defining equation for an η'-Ricci soliton is given by

$$\mathcal{L}_{\xi'} g + 2S + 2\lambda' g + 2\mu'\eta' \otimes \eta' = 0, \tag{1.2.94}$$

where μ' is a real constant, η' is an 1-form defined as $\eta'(V_1') = g(V_1', \xi')$ for any $V_1' \in \chi(M)$. Clearly, it can be observed that if $\mu' = 0$, then the η'-Ricci soliton reduces to classical Ricci soliton.

In 2020, S. Dey et al. [23], introduced the notion of a $*$-η'-Ricci soliton as

$$\mathcal{L}_{\xi'} g + 2S^* + 2\lambda' g + 2\mu'\eta' \otimes \eta' = 0.$$

The results concerning $*$-η'-Ricci solitons have been primarily studied under the assumption that the potential vector field V coincides with the characteristic vector field ξ'. Motivated by this observation, we extend the definition by considering the potential vector field V to be an arbitrary vector field. Accordingly, we define a $*$-η'-Ricci soliton on a Riemannian manifold (M, g) as a solution to the equation:

$$\mathcal{L}_V g + 2S^* + 2\lambda' g + 2\mu'\eta' \otimes \eta' = 0. \tag{1.2.95}$$

Now, if we consider the potential vector field V to be the gradient of a smooth function f, that is, $V = \nabla f$, then the $*$-η'-Ricci soliton equation can be rewritten as

$$Hess(f) + S^* + \lambda' g + \mu'\eta' \otimes \eta' = 0. \tag{1.2.96}$$

An n-dimensional Riemannian manifold (M, g), $n > 2$, is said to be an Einstein soliton [6] if there exists a vector field ξ' and a real constant λ' such that

$$\frac{1}{2}\mathcal{L}_{\xi'} g + S + (\lambda' - \frac{1}{2}r)g = 0,$$

and also the η'-Einstein soliton [5] on a Riemannian manifold (M, g) is given by,

$$\mathcal{L}_{\xi'} g + 2S + (2\lambda' - r)g + 2\mu'\eta' \otimes \eta' = 0, \tag{1.2.97}$$

where r is the scalar curvature of the metric g and λ' and μ' are constants. For $\mu' = 0$, from (1.2.97) the data (g, ξ', λ') is called Einstein soliton [14].

In [65], Roy, Dey and Bhattacharyya considered conformal Einstein soliton, defined by on an n-dimensional manifold:

$$\mathcal{L}_V g + 2S + \Big(2\lambda' - r + \Big(p + \frac{2}{n}\Big)\Big)g = 0,$$

where λ' is real constant, p is a scalar non-dynamical field, called conformal presure.

Moreover, an n-dimensional Riemannian manifold (M, g) is said to admit a conformal η'-Einstein soliton if there exists a vector field V, and real constants λ' and μ', such that the following conformal soliton equation holds:

$$\mathcal{L}_{\xi'} g + 2S + \Big(2\lambda' - r + \Big(p + \frac{2}{n}\Big)\Big)g + 2\mu'\eta' \otimes \eta' = 0, \tag{1.2.98}$$

as shown in [14]. Note that, the conformal η'-Einstein soliton becomes the Einstein soliton (g, ξ', λ').

The Ricci-Yamabe flow is a deformation equation in Riemannian geometry that evolves a metric over time, aiming to deform it to a metric with constant scalar curvature. It's a generalization of both the Ricci flow and the Yamabe flow. The equation is typically expressed as:

$$\frac{\partial g}{\partial t} = [\beta_2' r - 2\beta_1' S]g,$$

where β_1' and β_2' are scalars, and the sign of these scalars determines the nature of the flow (Riemannian, semi-Riemannian, or singular Riemannian).

A Ricci-Yamabe soliton (RYS) on a Riemannian manifold (M^n, g), $n > 2$, is a natural generalization of both Ricci soliton and Yamabe soliton. A metric g on M is said to admit a (α', β')-Ricci-Yamabe soliton if there exists, a smooth vector field V (called the potential vector field), a real constant λ' (called the soliton constant), and real constants α', β' (weights for Ricci and scalar curvature parts, respectively), such that the following Ricci-Yamabe soliton equation holds:

$$\mathcal{L}_V g + 2\alpha' S + [2\lambda' - \beta' r]g = 0, \tag{1.2.99}$$

where $\mathcal{L}_V g$ is the Lie derivative of the metric g along the vector field V, S is the Ricci tensor, r is the scalar curvature, α', β', λ' are real constants.

where $\mathcal{L}_V g$ denotes the Lie derivative of the metric g along the vector field V, S is the Ricci tensor, r is the scalar curvature and $\lambda', \alpha', \beta'$ are real scalars.
In that case, the Ricci-Yamabe soliton equation transforms into its gradient form, called the Gradient Ricci-Yamabe Soliton (GRYS) equation. This happens when the potential vector field V is the gradient of a smooth function f, i.e., $V = \nabla f$, then the equation (1.2.99) becomes

$$Hess(f) + \alpha' S + \left[\lambda' - \frac{1}{2}\beta' r\right] g = 0, \tag{1.2.100}$$

where $Hess(f)$ is the Hessian of the smooth function f.

In 2020, Siddiqi and Akyol [74] introduced a new generalization of Ricci-Yamabe soliton, namely η'-Ricci-Yamabe soliton, which is given by

$$\mathcal{L}_V g + 2\alpha' S + [2\lambda' - \beta' r] g + 2\mu' \eta' \otimes \eta' = 0, \tag{1.2.101}$$

where μ' is a constant and η' is a 1-form on M.
Now, if the scalars λ' and μ' are smooth functions, then η'-Ricci-Yamabe soliton becomes an almost η'-Ricci-Yamabe soliton. So, we define the almost η'-Ricci-Yamabe soliton in the following way. A Riemannian manifold (M^n, g) $n > 2$ is said to be an almost η'-Ricci-Yamabe soliton or almost η'-(α', β')-Ricci-Yamabe soliton $(g, V, \lambda', \mu', \alpha', \beta')$ if

$$\mathcal{L}_V g + 2\alpha' S + 2\mu' \eta' \otimes \eta' = (2\lambda' - \beta' r) g, \tag{1.2.102}$$

where λ' and μ' are smooth functions. If V is a gradient of some smooth function f on M, then the above expression is called gradient almost η'-Ricci-Yamabe soliton and then (1.2.102) reduces to

$$\nabla^2 f + \alpha' S + \mu' \eta' \otimes \eta' = (\lambda' - \frac{1}{2}\beta' r) g, \tag{1.2.103}$$

where $\nabla^2 f$ is the Hessian of f.

In, 2021, Dey et al.[66] developed the notion $*$-η'-Ricci-Yamabe soliton(in short $*$-η'-RYS) as

$$\mathcal{L}_{\xi'} g + 2\rho S^* + [2\lambda' - q r^*] g + 2\mu' \eta' \otimes \eta' = 0,$$

where μ' and λ' are constants and η' is a 1-form on M of dimension $(2n+1)$.

As per the authors' knowledge, the results concerning $*$-η'-Ricci-Yamabe solitons ($*$-η'-RYS) have been primarily studied in the context where the potential vector field V is chosen as the characteristic (Reeb) vector field ξ'. Motivated by the desire to study a broader class of solitons and to capture more general geometric behavior, we now generalize this concept by allowing the potential vector field V to be an arbitrary smooth vector field on the manifold. Accordingly, we define:

$$\mathcal{L}_V g + 2\rho S^* + [2\lambda' - qr^*]g + 2\mu'\eta' \otimes \eta' = 0. \tag{1.2.104}$$

Now, if the scalar functions λ' and μ' are smooth functions on the manifold M, then the $*$-η'-Ricci-Yamabe soliton is referred to as an almost $*$-η'-Ricci-Yamabe soliton (almost $*$-η'-RYS). Furthermore, if the potential vector field V is chosen as the gradient of a smooth function $f \in C^\infty(M)$ Thus, the equation (1.2.104) becomes:

$$Hess(f) + \rho S^* + (\lambda' - \frac{qr^*}{2})g + \mu'\eta' \otimes \eta' = 0, \tag{1.2.105}$$

for a gradient almost $*$-η'-RYS means a gradient $*$-η'-RYS, where λ' and μ' are smooth functions.

In [22], Dey et al. have defined a new notation δ-Ricci-Yamabe soliton (in short δ-RYS). A complete Riemannian manifold (M, g) is said to be a δ-Ricci-Yamabe almost soliton, denoted by $(M, g, V, \delta, \lambda')$, if there exists smooth vector field V on M, a soliton function $\lambda' \in C^\infty(M)$ and a non-zero real valued function δ on M such that

$$\delta\mathcal{L}_V g + 2\alpha' S + (2\lambda' - \beta' r)g = 0. \tag{1.2.106}$$

This soliton is called shrinking, steady and expanding according as λ' is negative, zero and positive respectively. If the potential vector field V can be written as a gradient of a smooth function u on M, i.e., $V = \nabla u$, then the δ-Ricci-Yamabe almost soliton is called a gradient δ-Ricci-Yamabe almost soliton. Then (1.2.106) can be expressed as

$$\delta\nabla^2 u + \alpha' S + (\lambda' - \frac{1}{2}\beta' r)g = 0, \tag{1.2.107}$$

where $\nabla^2 u$ be the Hessian of u. We denote this soliton by the quadruple (M, g, Du, λ'). Now, the identity (1.2.107) can be written as

$$\delta Hess(f) + \alpha' S + (\lambda' - \frac{1}{2}\beta' r)g = 0. \tag{1.2.108}$$

A family of metrics $g(t)$ on a n-dimensional Riemannian manifold (M, g) is said to evolve by the Ricci-Bourguignon flow (RB flow for short) if $g(t)$ satisfies the following evolution equation,

$$\frac{\partial g}{\partial t} = -2(S - \rho r g), \tag{1.2.109}$$

where S is the Ricci tensor of the metric, r is the scalar curvature and $\rho \in \mathbb{R}$ is a constant.

From the above definition we can easily say that if $\rho = 0$ in (1.2.109), then it becomes Ricci flow. Now, from [27], we get different tensor like the Einstein tensor, traceless Ricci tensor, Schouten tensor and Ricci tensor for different values of $\rho = \frac{1}{2}$, $\rho = \frac{1}{n}$, $\rho = \frac{1}{2(n-1)}$ and $\rho = 0$.

A Riemannian manifold (M, g), $n \geq 3$, is said to be an almost Ricci-Bourguinon soliton (or almost RBS) if there is a smooth vector field V and a smooth function λ' on M satisfying

$$\tfrac{1}{2}(\mathcal{L}_V g)(V_1', V_2') + S(V_1', V_2') - (\lambda' + \rho r)g(V_1', V_2') = 0,$$

where S is the Ricci tensor of g, r is the scalar curvature and $\mathcal{L}_V g$ is the Lie derivative of g in the V direction.

A Riemannian manifold (M, g), $n \geq 3$, is said to be an almost $*$-Ricci-Bourguinon soliton (or almost $*$-RBS) if there is a potential vector field V and a smooth function λ' on M satisfying

$$\frac{1}{2}(\mathcal{L}_V g)(V_1', V_2') + S^*(V_1', V_2') - (\lambda' + \rho r^*)g(V_1', V_2') = 0, \tag{1.2.110}$$

where S^* is the $*$-Ricci tensor of g, r^* is the $*$-scalar curvature and $\mathcal{L}_V g$ is the Lie derivative of g in the V direction. An almost $*$-Ricci-Bourguignon soliton is trivial if the vector field V is a Killing vector field, that is, $\mathcal{L}_V g = 0$; in this case, the manifold is $*$-Einstein.

It is said to be expanding, steady or shrinking if $\lambda' > 0$, $\lambda' = 0$ or $\lambda' < 0$ respectively. Now, if we consider the potential vector field V as the gradient of a smooth function f, then the $*$-Ricci-Bourguignon soliton equation can be written as

$$Hess(f)(V_1', V_2') + S^*(V_1', V_2') - (\lambda' + \rho r^*)g(V_1', V_2') = 0. \tag{1.2.111}$$

A Riemannian manifold (M, g), with dimension $n \geq 1$ is said to admit an almost η'-Ricci-Bourguignon soliton (η'-RB soliton) if there exist, a smooth vector field V on M (called the potential vector field), such that the following soliton equation is satisfied:

$$\mathcal{L}_V g(V_1', V_2') + 2S(V_1', V_2') = 2[\lambda' + \rho r]g + 2\mu'\eta'(V_1', V_2')\eta'(V_1', V_2'), \tag{1.2.112}$$

where $\mathcal{L}_V g$ denotes the Lie derivative of the metric g with respect to the vector field V and λ', μ' are smooth functions. If λ' and μ' are real constants, then an almost η'-Ricci-Bourguignon soliton becomes an η'-Ricci-Bourguignon soliton. Also it is said to be expanding, steady or shrinking according as $\lambda' < 0$, $\lambda' = 0$ and $\lambda' > 0$ respectively. If V is a gradient of some smooth function f on M, then the above expression is called gradient almost η'-Ricci-Bourguignon soliton and then (1.2.112) reduces to

$$\nabla^2 f + S = [\lambda' + \rho r]g + \mu'\eta' \otimes \eta', \tag{1.2.113}$$

where $\nabla^2 f$ is the Hessian of f.

A Riemannian manifold (M, g), $n \geq 1$, is said to be an almost $*$-Ricci-Bourguignon soliton if there is a smooth vector field V called potential vector field and a smooth function λ' on M satisfying

$$S^*(V_1', V_2') + \frac{1}{2}\mathcal{L}_V g(V_1', V_2') = (\lambda' + \rho r^*)g(V_1', V_2').$$

An almost $*$-Ricci-Bourguignon soliton is trivial if the vector field V is a Killing vector field, that is, $\mathcal{L}_V g = 0$; in this case, the manifold is $*$-Einstein.

In 2014, Kaimakamis and Panagiotidou [43] introduced a modification of the Ricci soliton by replacing the classical Ricci tensor S with the $*$-Ricci tensor S^*, leading to the concept of $*$-Ricci soliton, which is viewed as a natural generalization of a $*$-Einstein metric. Motivated by the above studies, Dey et al. [24] have defined $*$-η'-Ricci-Bourguignon soliton by replacing the Ricci tensor S and the scalar curvature r with the $*$-Ricci tensor S^* and the $*$-scalar curvature r^*, respectively, in η'-Ricci-Bourguignon soliton equation (1.2.112).

A Riemannian manifold (M, g), with dimension $n \geq 1$ is said to admit a $*$-η'-Ricci-Bourguignon soliton ($*$-η'-RBS soliton, for short) if there exists, a smooth vector field V on M (called the soliton vector field), such that the following differential equation holds:

$$2S^*(V_1', V_2') + \mathcal{L}_V g(V_1', V_2') = 2(\lambda' + \rho r^*)g(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2'). \tag{1.2.114}$$

Now, if the scalars λ' and μ' are smooth functions on the manifold M, then $*$-η'-Ricci-Bourguignon soliton becomes an almost $*$-η'-Ricci-Bourguignon soliton. Furthermore, if the potential vector field V is the gradient of a smooth function f, i.e., $V = \nabla f$, then the $*$-η'-Ricci-Bourguignon soliton equation can be rewritten as

$$\nabla^2 f + S^* = (\lambda' + \rho r^*)g + \mu'\eta' \otimes \eta', \tag{1.2.115}$$

by gradient almost $*$-η'-Ricci-Bourguignon soliton, we mean a gradient $*$-η'-Ricci-Bourguignon soliton, where we consider λ' and μ' are smooth functions.

All the solitons related to η'-Ricci soliton are called almost solitons if we consider λ' and μ' to be smooth functions.

1.3 Prey-Predator model

Community ecology deals with population and its interaction with other populations and ecosystem ecology, is mainly concerned with the functioning of the over all system composed of biological organisms and their abiotic environment. In the development of predator prey theory different formulations of the prey growth rate in absence of predator have been proposed in the literature .The study of non-linear Mathematical models based on realistic phenomenon in Mathematical biology is a reflection of their usage for the purpose of umderstanding the dynamical process involve in many such areas of predator prey competitive interaction, renewable resource management evaluation of pesticide resistant strains, ecological control of pest, plant herbirore system and so on. Mathematical ecology is a field of mathematical study of Mathematical models dealing with the relation between the living organism with respect to each other and their natural environment predator prey interaction is the fundamental structure in population dynamics understanding the dynamics of prey models becomes very helpful for investigating multiple specis interaction.

The fundamental response, is the rate at which each predator capture prey,following easily pioneering work of Holling (1959a,1959b) have been widely used to model the rate of consumptions of indivisual consumers with respect to the density of food resource. There are three type of Holling type functional response. Here we consider the uptake function of predator prey model of generalised **Holling type-III**(sigmoid). $\phi'(N(t))=\frac{\alpha'(N(t))}{1+bN(t)+aN^2(t)}$ where N(t) is prey density, parameter α' the percapita rate of the consumption of prey by predator population, a prey saturation constant b, predator interference, either $b > 0$ or $b < 0$ ($b^2 - 4a < 0$ when $b < 0$). Keeping in view of literature and using center manifold theorem, we investigate the stability predator prey system with crowding effect of predator where the predator is partially dependent on prey. In prey-predator models, a **crowding effect** refers to the impact of high population density on the growth rate of individuals within a population. It's a form of self-limitation where, as the number of individuals increases, the rate of increase per individual decreases due to factors like limited resources, competition, or stress. This effect can stabilize the system and prevent oscillations in the prey and predator populations.

Among the six chapters of this thesis, this first chapter consists an introduction to different types of smooth manifolds, solitons and pre-predator model.

In the **second chapter**, we investigate almost η'-Ricci-Yamabe soliton and gradient almost η'-Ricci-Yamabe solitons within the framework of almost Kenmotsu manifolds. It is shown that a normal almost Kenmotsu manifold admitting an almost η'-Ricci-Yamabe soliton (or its gradient counterpart) is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Furthermore, prove that if a (κ, μ') almost Kenmotsu manifold admits an almost η'-Ricci-Yamabe soliton, then it must be an η'-Einstein. In addition we derive the condition under which non-normal almost Kenmotsu manifolds admit a gradient almost η'-Ricci-Yamabe soliton. We also analyze the properties of such solitons on on $(\kappa, \mu')'$-almost Kenmotsu manifold.

Next, we condider an almost $*$-η'-Ricci-Yamabe soliton within the framework of Kenmotsu manifolds. It is shown that if a Kenmotsu manifold admits a $*$-η'-Ricci-Yamabe soliton, then the manifold is η'-Einstein. Moreover we prove that if a $(\kappa, -2)'$-nullity distribution with $\kappa < -1$ admits a $*$-η'-Ricci-Yamabe soliton, then the manifold is Ricci flat. If the metric g defines a gradient almost $*$-η'-Ricci-Yamabe soliton and the characteristic vector field ξ' leaves the scalar curvature r invariant, then the manifold is again η'-Einstein. Furthermore, we have show that if a Kenmotsu manifold admits an almost $*$-η'-Ricci-Yamabe soliton with potential vector field V is pointwise collinear with ξ', then the manifold is η'-Einstein. Finally, we construct two illustrative example: one of a gradient almost $*$-η'-Ricci-Yamabe soliton on a 5-dimensional Kenmotsu manifold, and another of a gradient almost η'-Ricci-Yamabe soliton on a 3-dimensional Kenmotsu manifold.

In the **third chapter**, first we study δ-Ricci-Yamabe almost soliton and gradient δ-Ricci-Yamabe almost soliton on K-paracontact and para-Sasakian manifolds. We prove that if K-paracontact metric g admits a δ-Ricci-Yamabe almost soliton with a non-zero potential vector field V that is parallel to the Reeb vector field ξ', then g is an Einstein with Einstein constant $-2n$. Furthermore, we establish that no para-Sasakian manifold can admit a gradient δ-Ricci-Yamabe almost soliton. Additionally, we explore δ-Ricci-

Yamabe almost solitons within the framework of (κ, μ')-paracontact manifolds, providing structural insights and curvature properties associated with such manifolds.

Next, we consider almost $*$-Ricci-Bourguignon solitons in the context of paracontact geometry. It is shown that if the metric g of an η'-Einstein para-Kenmotsu manifold of dimension greater than three admits an almost $*$-Ricci-Bourguignon soliton, then the manifold M^{2n+1} is Einstein. Furthermore, if g represents a gradient almost $*$-Ricci-Bourguignon soliton on a $(2n+1)$-dimensional η'-Einstein para-Kenmotsu manifold, then M^{2n+1} is either Einstein, or the potential vector field V is pointwise collinear with the Reeb vector field ξ'. In the three-dimensional case, if a para-Kenmotsu manifold admits a $*$-Ricci-Bourguignon soliton, then it is of constant curvature -1. Finally, we prove that if a para-Sasakian manifold admits a $*$-Ricci-Bourguignon soliton, then M^{2n+1} is either $\mathcal{D}$-homothetic to an Einstein manifold, or the Ricci tensor with respect to the canonical paracontact connection vanishes.

In **fourth chapter**, first, we study almost $*$-Ricci-Bourguignon solitons on Kenmotsu manifolds. It is shown that if a Kenmotsu manifold admits an almost $*$-Ricci-Bourguignon soliton, then the manifold is necessarily an η'-Einstein manifold. Moreover, we prove that if a Kenmotsu manifold admitting a $(\kappa, 2)'$-nullity distribution with $\kappa < 1$ also admits an almost $*$-Ricci-Bourguignon soliton, then the manifold is Ricci flat. Furthermore, we demonstrate that if a Kenmotsu manifold admits a gradient almost $*$-Ricci-Bourguignon soliton and the characteristic vector field ξ' leaves the scalar curvature r invariant, then the manifold is an Einstein manifold with constant scalar curvature $r = n(1-2n)$, where n is the dimension of the manifold's contact distribution.

Next, we consider almost $*$-η'-Ricci-Bourguignon solitons within the framework of Sasakian manifolds. It is established that if a Sasakian manifold admits an almost $*$-η'-Ricci-Bourguignon soliton, then the manifold is necessarily η'-Einstein. Furthermore, if the metric g represents a gradient almost $*$-η'-Ricci-Bourguignon soliton and the Reeb vector field ξ' preserves the scalar curvature r, then the manifold is again η'-Einstein. Additionally, we show that if a Sasakian manifold admits an almost $*$-η'-Ricci-Bourguignon soliton for which the potential vector field V is pointwise collinear with ξ', then the manifold is η'-Einstein.

In the **fifth chapter**, first we characterize trans-Sasakian space form satisfying certain curvature conditions on W_2 - curvature tensor. We study W_2 - semisymmetric and W_2-

pseudosymmeric trans-Sasakian space form,W_2-locally symmetric trans-Sasakian space form,W_2- locally ϕ'- symmetric trans-Sasakian space form and W_2 -ϕ'-recurrent trans-Sasakian space form. Some of these results are in the form of necessary and sufficient conditions.

Next, we study conformal η'-Einstein solitons on the framework of trans-Sasakian manifold in dimension three. Existence of conformal η'-Einstein solitons on trans-Sasakian manifold is discussed. Then we find some results on trans-Sasakian manifold which are conformal η'-Einstein solitons where the Ricci tensor is cyclic parallel and Codazzi type. We also consider some curvature conditions with addition to conformal η'-Einstein solitons on trans-Sasakian manifold. We also use torse-forming vector fields in addition to conformal η'-Einstein solitons on trans-Sasakian manifold. Finally, an example of conformal η'-Einstein solitons on trans-Sasakian manifold is constructed.

In the **sixth chapter**, We consider a prey–predator model with Holling type III response function incorporating a crowding effects of predator. The purpose of the work is to offer mathematical analysis of the model and to discuss some significant qualitative results that are expected to arise from the interplay of biological forces. Some numerical simulations are carried out. Here the flow is studied to interpret the evolution of manifold with respect to time. Our aim is to study the proposed model and its geometric perspective under the Ricci flow.

2

Solitons on Kenmotsu Monifold

2.1 Introduction

This chapter is divided into five sections. Fist two sections consist of introduction and preliminaries.

In the third section, the primary objective of this article is to study an almost η'-Ricci-Yamabe soliton and gradient almost η'-Ricci-Yamabe soliton within the framework of almost Kenmotsu manifolds. It is shown that normal almost Kenmotsu manifold admitting an almost η'-Ricci-Yamabe soliton or gradient η'-Ricci-Yamabe soliton is locally isometric to hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Besides, we find the condition for non-normal almost Kenmotsu manifolds acknowledging gradient almost η'-Ricci-Yamabe soliton. Finally, we present an example to verify our findings.

In next section, we initiate the study of $*$-η'-Ricci soliton within the framework of Kenmotsu manifold as a characterization of Einstein metric. We prove that if a (κ, μ') almost Kenmotsu manifold admits an almost η'-Ricci-Yamabe soliton, then the manifold is η'-Einstein. We prove that if a $(\kappa, -2)'$-nullity distribution, where $\kappa < -1$ acknowledges a $*$-η'-Ricci-Yamabe soliton, then the manifold is Ricci flat. We include an example to illustrate our findings.

In the last section, if g represents a gradient almost $*$-η'-Ricci-Yamabe soliton and ξ' leaves the scalar curvature r invariant on a Kenmotsu manifold, then the manifold is an η'-Einstein. Further, we have studied on a Kenmotsu manifold if g represents an almost $*$-η'-Ricci-Yamabe soliton with potential vector field V is pointwise collinear with ξ', then the manifold is an η'-Einstein.

2.2 Preliminaries

In the introductory chapter, definitions and some fundamental properties of Kenmotsu manifold and $(\kappa, -2)'$-almost Kenmotsu manifold are given. An almost Kenmotsu manifold M^{2n+1} is defined as a generalized (κ, μ')-almost Kenmotsu manifold if ξ' belongs to the generalized (κ, μ')-nullity distribution and from (1.1.31), we get

$$R(V_1', V_2')\xi' = \kappa[\eta'(V_2')V_1' - \eta'(V_1')V_2'] + \mu'[\eta'(V_2')hV_1' - \eta'(V_1')hV_2'], \tag{2.2.1}$$

$\forall$ vector fields V_1', V_2' on M, where κ, μ' are smooth functions on M. An almost Kenmotsu manifold $M^{2n+1}(g, \phi', \xi', \eta')$ is said to be a generalized $(\kappa, \mu')'$-nullity distribution and from (1.1.32), we have

$$R(V_1', V_2')\xi' = \kappa[\eta'(V_2')V_1' - \eta'(V_1')V_2'] + \mu'[\eta'(V_2')h'V_1' - \eta'(V_1')h'V_2'], \tag{2.2.2}$$

for all vector fields V_1', V_2' on M, where κ, μ' are smooth functions on M. Moreover if both κ and μ' are constants in (2.2.2), then M is called a $(\kappa, \mu')'$-almost Kenmotsu manifold (see [25, 53, 83]). On generalized (κ, μ') or $(\kappa, \mu')'$-almost Kenmotsu manifold with $h \neq 0$ (equivalently, $h' \neq 0$), the following relations hold [25]

$$h'^2 = (\kappa + 1)\phi'^2 \quad (\Leftrightarrow h^2 = (\kappa + 1)\phi'^2), \tag{2.2.3}$$

$$Q\xi' = 2n\kappa\xi'. \tag{2.2.4}$$

It can be deduced from (2.2.3) that $\kappa \leq -1$ and $v = \pm\sqrt{-(\kappa + 1)}$, where v is an eigenvalue associated to eigenvector $X \in \mathcal{D}(\mathcal{D} = ker(\eta'))$ of h'. The equation holds true if and only if $h = 0$ (equivalently, $h' = 0$). Thus $h' \neq 0$ if and only if $\kappa < -1$.

We now refer to several key results that form the basis of our subsequent analysis.

Lemma 2.2.1. *[83] Let $M^{2n+1}(g, \xi', \eta', \phi')$ be a generalized $(\kappa, \mu')'$-almost Kenmotsu manifold with $h' \neq 0$, for $n > 1$. Then the Ricci operator Q of M may be formulated as*

$$QV_1' = -2nV_1' + 2n(\kappa + 1)\eta'(V_1')\xi' - \{\mu' - 2(n - 1)\}h'V_1',$$

for arbitrary vector field V_1' on M. In addition, if κ and μ' are constants and $n \geq 1$, then $\mu' = -2$ and It follows that

$$QV_1' = -2nV_1' + 2n(\kappa + 1)\eta'(V_1')\xi' + 2nh'V_1', \tag{2.2.5}$$

for any vector field V_1' on M. In both cases, the scalar curvature of M is $2n(\kappa - 2n)$.

Lemma 2.2.2. *[83] Let $M^{2n+1}(g, \xi', \eta', \phi')$ be a generalized (κ, μ')-almost Kenmotsu manifold with $h' \neq 0$, for $n > 1$. Then the Ricci operator Q of M may be formulated as*

$$QV_1' = -2nV_1' + 2n(\kappa+1)\eta'(V_1')\xi' - 2(n-1)h'V_1' + \mu' hV_1',$$

for vector field V_1' on M. Also, the scalar curvature of M is $2n(\kappa - 2n)$.

Lemma 2.2.3. *[80] The Ricci operator Q on a $(2n+1)$-dimensional Kenmotsu manifold satisfies*

$$(\nabla_{V_1'} Q)\xi' = -QV_1' - 2nV_1', \tag{2.2.6}$$

$$(\nabla_{\xi'} Q)V_1' = -2QV_1' - 4nV_1', \tag{2.2.7}$$

for an arbitrary vector field V_1' on the manifold M.

Lemma 2.2.4. *[80] The $*$-Ricci tensor S^* on a $(2n+1)$-dimensional Kenmotsu manifold is given by*

$$S^*(V_1', V_2') = S(V_1', V_2') + (2n-1)g(V_1', V_2') + \eta'(V_1')\eta'(V_2'), \tag{2.2.8}$$

for arbitrary vector fields V_1' and V_2' on the manifold and the respective $*$-scalar curvature is given in the form $r^* = r + 4n^2$.

Lemma 2.2.5. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Kenmotsu manifold admits a $*$-η'-RYS. Then the Lie derivative of the curvature tensor R with the soliton vector V are given by the expression*

$$\begin{aligned}(\mathcal{L}_V R)(V_1', \xi')\xi' &= 4n(1-\rho)\{\eta'(V_1')\xi' + V_1'\} + q\{V_1'(r)\xi' - V_1'(Dr)\} \\ &+ q\{\xi'(Dr) - \xi'(r)\xi' - Dr\}.\end{aligned}$$

Proof. Assume the metric g of Kenmotsu manifold $M^{2n+1}(\phi', \xi', \eta', g)$ represents a $*$-η'-RYS. It follows that both equations (1.2.104) and (2.2.8) are satisfied. From these two equations, we derive

$$\begin{aligned}(\mathcal{L}_V g)(V_1', V_2') &= -2\rho S(V_1', V_2') - \{2\rho(2n-1) + 2\lambda' - q(r+4n^2)\}g(V_1', V_2') \\ &\quad -2(\mu' + \rho)\eta'(V_1')\eta'(V_2'). \end{aligned} \tag{2.2.9}$$

We compute the covariant derivative with respect to an arbitrary vector field V_3' making use of the identity (1.1.25) to obtain

$$\begin{aligned}&(\nabla_{V_3'} \mathcal{L}_V g)(V_1', V_2') = -2\rho(\nabla_{V_3'} S)(V_1', V_2') + qV_3'(r)g(V_1', V_2') \\ &-2(\mu' + \rho)\{g(V_1', V_3')\eta'(V_2') + g(V_2', V_3')\eta'(V_1') - 2\eta'(V_1')\eta'(V_2')\eta'(V_3')\}, \end{aligned} \tag{2.2.10}$$

for all V_1' V_2' $V_3' \in \chi(M)$. Combining (2.2.10) and (1.1.87) and by a straightforward combinatorial computation and applying the symmetry of $(\mathcal{L}_V\nabla)$, (1.1.87) implies

$$\begin{aligned} g((\mathcal{L}_V\nabla)(V_1', V_2'), V_3') &= \rho\{(\nabla_{V_3'}S)(V_1', V_2') - (\nabla_{V_1'}S)(V_2', V_3') - (\nabla_{V_2'}S)(V_3', V_1')\} \\ &- qV_3'(r)g(V_1', V_2') + qV_1'(r)g(V_2', V_3') + qV_2'(r)g(V_1', V_3') \\ &- 2(\mu' + \rho)\{g(V_1', V_2')\eta'(V_3') - \eta'(V_1')\eta'(V_2')\eta'(V_3')\}, \end{aligned} \quad (2.2.11)$$

for arbitrary vector fields V_1', V_2' and V_3' on M. Using (2.2.6) and (2.2.7), the foregoing equation yields

$$(\mathcal{L}_V\nabla)(V_1', \xi') = 2\rho QV_1' + \{4n + q\xi'(r)\}V_1' + qV_1'(r)\xi' - q\eta'(V_1')Dr, \quad (2.2.12)$$

for all $V_1' \in \chi(M)$. Now, we differentiate covariantly this with respect to arbitrary vector field V_2' and using (1.1.24), we achieve

$$\begin{aligned} (\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', \xi') &= 2\rho(\nabla_{V_2'}Q)V_1' - (\mathcal{L}_V\nabla)(V_1', V_2') + \eta'(V_2')(\mathcal{L}_V\nabla)(V_1', \xi'). \\ &+ qV_2'(\xi'(r))V_1' + qg(V_1', \nabla_{V_2'}Dr) - qV_1'(r)\phi'^2V_2' \\ &- q\eta'(V_1')\nabla_{V_2'}Dr - qg(V_1', V_2')Dr + \eta'(V_1')\eta'(V_2')Dr. \end{aligned} \quad (2.2.13)$$

It is well known that, $(\mathcal{L}_VR)(V_1', V_2')V_3' = (\nabla_{V_1'}\mathcal{L}_V\nabla)(V_2', V_3') - (\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', V_3')$. In view of (2.2.13) it follows from the previous relation that

$$\begin{aligned} (\mathcal{L}_VR)(V_1', V_2')\xi' &= 2\rho(\nabla_{V_1'}Q)V_2' - 2\rho(\nabla_{V_2'}Q)V_1' + \eta'(V_1')(\mathcal{L}_V\nabla)(V_2', \xi') \\ &- \eta'(V_2')(\mathcal{L}_V\nabla)(V_1', \xi') + q\{V_1'(\xi'(r))V_2' - V_2'(\xi'(r))V_1'\} \\ &+ q\{V_1'(r)\phi'^2V_2' - V_2'(r)\phi'^2V_1'\} \\ &+ q\{\eta'(V_1')\nabla_{V_2'}Dr - \eta'(V_2')\nabla_{V_1'}Dr\}, \end{aligned} \quad (2.2.14)$$

for arbitrary vector fields V_1' and V_2' on M and where we have used that $\mathcal{L}_V\nabla$ $Hess_r$ are symmetric. In view of (1.1.24) Putting $V_2' = \xi'$ in (2.2.14) and using (1.1.27), (2.2.6) and (2.2.7), we get

$$\begin{aligned} (\mathcal{L}_VR)(V_1', \xi')\xi' &= 4n(1 - \rho)\{\eta'(V_1')\xi' + V_1'\} + q\{V_1'(r)\xi' - V_1'(Dr)\} \\ &+ q\{\xi'(Dr) - \xi'(r)\xi' - Dr\}. \end{aligned} \quad (2.2.15)$$

Thus, we end our proof. □

Lemma 2.2.6. *[18] On a $(\kappa, \mu')'$-almost Kenmotsu manifold with $\kappa < -1$ the $*$-Ricci tensor is given by*

$$S^*(V_1', V_2') = -(\kappa + 2)\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}, \tag{2.2.16}$$

for any pair of vector fields V_1' and V_2' on M.

2.3 Normal and Non-normal almost Kenmotsu manifold admitting almost η'-Ricci-Yamabe soliton

This section is devoted to the study of normal almost Kenmotsu manifolds that admit an almost η'-Ricci-Yamabe soliton as well as a gradient almost η'-Ricci-Yamabe soliton.

Theorem 2.3.1. *If the metric of a Kenmotsu manifold M^{2n+1} admits a gradient almost η'-Ricci-Yamabe soliton with $\alpha' \neq 0$, then it is η'-Einstein. Moreover, if M is complete and ξ' leaves the scalar curvature invariant, then it is locally isometric to Hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Also λ' can be expressed locally λ'=Asinh t+Bcosh t$+\mu' - n\{\beta'(2n+1) + 2\alpha'\}$ where A, B are constants on M.*

Proof. Suppose the metric g of Kenmotsu manifold $M^{2n+1}(\phi', \xi', \eta', g)$ satisfies a gradient η'-Ricci-Yamabe soliton. Then from (1.2.103), we obtain

$$\nabla_{V_1'} Df = \sigma V_1' - \alpha' Q V_1' - \mu'\eta'(V_1')\xi', \tag{2.3.1}$$

for an arbitrary vector field V_1' on M and $\sigma = \lambda' - \frac{1}{2}\beta' r$ is a smooth function and also μ' is a smooth function.

Now, we take an inner product of the identity (2.3.1) along arbitrary vector field V_2', and using the equation (1.1.24) to yield

$$\begin{aligned} \nabla_{V_2'}\nabla_{V_1'} Df &= (V_2'\sigma)V_1' - \sigma(\nabla_{V_2'}V_1') - \alpha'(\nabla_{V_2'}Q)V_1' - \alpha' Q(\nabla_{V_2'}V_1') \\ &- (V_2'\mu')\eta'(V_1')\xi' - \mu'\{(\nabla_{V_2'}\eta')(V_1') + \eta'(V_1')V_2' - \eta'(V_1')\eta'(V_2')\}. \end{aligned} \tag{2.3.2}$$

Invoking (2.3.2) by applying the well-known identity $R(V_1', V_2')Df = \nabla_{V_1'}\nabla_{V_2'}Df - \nabla_{V_2'}\nabla_{V_1'}Df - \nabla_{[V_1', V_2']}Df$ provides

$$\begin{aligned} R(V_1', V_2')Df &= (V_1'\sigma)V_2' - (V_2'\sigma)V_1' - \alpha'[(\nabla_{V_1'}Q)V_2' - (\nabla_{V_2'}Q)V_1'] - \{(V_1'\mu')\eta'(V_2')\xi' \\ &- (V_2'\mu')\eta'(V_1')\xi'\} - \mu'\{(\nabla_{V_1'}\eta')(V_2') - (\nabla_{V_2'}\eta')(V_1') \\ &+ \eta'(V_2')V_1' - \eta'(V_1')V_2'\}. \end{aligned} \tag{2.3.3}$$

Now, we take a covariant derivative of the identity (1.1.27) and utilize the equation (1.1.24) to deduce $(\nabla_{V_1'}Q)\xi' = -2n(V_1' - \eta'(V_1')\xi')$. We take the inner product of the identity (2.3.3) with the vector field ξ' to derive

$$g(R(V_1', V_2')Df, \xi') = V_1'(\sigma - \mu')\eta'(V_2') - V_2'(\sigma - \mu')\eta'(V_1'). \tag{2.3.4}$$

Now, we grasp an inner product of (1.1.26) with Df to obtain

$$g(R(V_1', V_2')\xi', Df) = (V_2'f)\eta'(V_1') - (V_1'f)\eta'(V_2'). \tag{2.3.5}$$

We combine the identities (2.3.4) and (2.3.5) and replacing V_2' by ξ' to infer

$$d(\sigma - \mu' - f) = \xi'(\sigma - \mu' - f)\eta', \tag{2.3.6}$$

where d is the exterior derivative ($df(V_1') = V_1'f$ for smooth function f). So, $(\sigma - f - \mu')$ is invariant along the distribution $\mathcal{D}$ (i.e. $\mathcal{D} = ker\eta'$), hence $(\sigma - f - \mu')$ is constant for all $V_1' \in \mathcal{D}$.

We contract the equation (2.3.3) to get

$$S(V_2', Df) = -2n(V_2'\sigma) + \frac{\alpha'}{2}(V_2'r) + (V_2'\mu'), \tag{2.3.7}$$

given any vector field V_2' on M. We replace V_2' by ξ' into (2.3.3) and by taking the inner product with V_2' yields

$$\begin{aligned} g(R(V_1', \xi')Df, V_2') &= V_1'(\sigma - \mu')\eta'(V_2') - \xi'(\sigma - \mu')g(V_1', V_2') \\ &- \alpha' S(V_1', V_2') + 2n\alpha' g(V_1', V_2'). \end{aligned} \tag{2.3.8}$$

In light of the two equations (1.1.24) and (1.1.26) into the identity (2.3.8), we achieve

$$[(V_1'f) - (V_1'(\sigma - \mu'))]\eta'(V_2') + \xi'(\sigma - \mu' - f)g(V_1', V_2') + \alpha' S(V_1', V_2') + 2n\alpha' g(V_1', V_2') = 0. \tag{2.3.9}$$

We proceed by contracting (2.3.9) over V_1' leads

$$2n\xi'(\sigma - \mu' - f) + \alpha'[r + 2n(2n+1)] = 0. \tag{2.3.10}$$

We displace V_2' by ξ' into (2.3.7) and exploiting the identities (2.3.10) and (1.1.27), it follows that $\xi' r + 2\{r + 2n(2n+1)\} = 0$, for $\alpha' \neq 0$. Now, we feed the equation (2.3.10) into (2.3.6) to arrive

$$d(\sigma - \mu' - f) = -\frac{\alpha'}{2n}\{2n(2n+1) + r\}\eta'. \tag{2.3.11}$$

Applying Poincare lemma and using the fact $d\eta' = 0$ into (2.3.11), we get $-\alpha' dr \wedge \eta' = 0$ and relying of the value of $\xi' r$, we have

$$Dr = -2(r + 2n(2n+1))\xi'. \tag{2.3.12}$$

If we lay hold of inner product of the identity (2.3.11) with vector field V_1', by inserting it along with (2.3.10) into (2.3.9) to yield

$$S(V_1', V_2') = (\frac{r}{2n} + 1)g(V_1', V_2') - (\frac{r}{2n} + 2n + 1)\eta'(V_1')\eta'(V_2'), \tag{2.3.13}$$

given any vector field V_1' on M. Therefore M is η'-Einstein.

Assume that $\xi' r = 0$, i.e. ξ' leaves the scalar curvature invariant. Inconsequence of this, as a result, we find that $r = -2n(2n+1)$, a constant. We insert this into (2.3.13) to arrive $S(V_1', V_2') = -2ng(V_1', V_2')$ i.e., M is Einstein. Next, we assume that M is complete.. As r is constant, (2.3.11) gives $Df = D\lambda'$. As a consequence of this, (2.3.1) becomes

$$\nabla_{V_1'} D\lambda' = (\lambda' + k)V_1' - \mu'\eta'(V_1')\xi', \tag{2.3.14}$$

where $k = n(2\alpha' + \beta'(2n+1))$. Now, we apply Tashiro's theorem [79] to demonstrate that the manifold is locally isometric to hyperbolic space $\mathbb{H}^{2n+1}(-1)$. We supersede V_1' by ξ' and taking an inner product with ξ', (2.3.14) provides $\xi'(\xi'\lambda') = \lambda' + k - \mu'$. But since it is well known that a Kenmotsu manifold is locally isometric to a warped product $(-\epsilon, \epsilon) \times_{ce^t} N$, where N is a Kählar manifold of dimension $2n$ and $(-\epsilon, \epsilon)$ is an open interval, using the local parametrization, $\xi' = \frac{\partial}{\partial t}$ (where t is the coordinate on $(-\epsilon, \epsilon)$), we obtain from the identity (2.3.14)

$$\frac{\partial^2 \lambda'}{\partial t^2} = \lambda' + 2n\alpha' + n\beta'(2n+1) - \mu'.$$

The explicit form of the solution is λ'=Asinh t+Bcosh t+$\mu' - n\{\beta'(2n+1) + 2\alpha'\}$, where A, B are constants on M. Thus, we end the proof. □

Next, we proof the following proposition.

Proposition 2.3.1. *If the metric of a Kenmotsu manifold $M^{2n+1}(\phi', \xi', \eta', g)(n > 1)$ admits an almost η'-Ricci-Yamabe soliton, then*

$$\xi'(\xi'\lambda') + \xi'(\xi'\mu') + \xi'\lambda' + \xi'\mu' = 2\{2n\alpha' + \mu' + \lambda' + n\beta'(2n+1)\}.$$

Proof. We take the covariant derivative of (1.2.102) along the arbitrary vector field V_1' to yield

$$\begin{aligned}(\nabla_{V_1'}\mathcal{L}_V g)(V_2', V_3') &= 2(V_1'\sigma)g(V_2', V_3') - 2\alpha'(\nabla_{V_1'}S)(V_2', V_3') - 2(V_1'\mu')\eta'(V_2')\eta'(V_3') \\ &- 2\mu'\{\eta'(V_2')(\nabla_{V_1'}\eta')(V_3') + \eta'(V_3')(\nabla_{V_1'}\eta')(V_2')\}, \qquad (2.3.15)\end{aligned}$$

where $\sigma = \lambda' - \frac{\beta' r}{2}$ and μ' are smooth function. From known results, it follows that $\mathcal{L}_V\nabla$ is a symmetric tensor of type $(1,2)$ and it then follows from (1.1.87) that

$$2g((\mathcal{L}_V\nabla)(V_1', V_2'), V_3') = (\nabla_{V_1'}\mathcal{L}_V g)(V_2', V_3') + (\nabla_{V_2'}\mathcal{L}_V g)(V_3', V_1') - (\nabla_{V_3'}\mathcal{L}_V g)(V_1', V_3'). \qquad (2.3.16)$$

We insert the equation (2.3.15) into (2.3.16) and also displacing V_2' by ξ' to arrive

$$(\mathcal{L}_V\nabla)(V_1', \xi') = 2\alpha' QV_1' + (4n\alpha' + \xi'(\sigma - \mu'))V_1' + g(V_1', D(\sigma - \mu'))\xi' - \eta'(V_1')D(\sigma - \mu'). \qquad (2.3.17)$$

We again take the covariant derivative of (2.3.17) with respect to an arbitrary vector field V_2' leads

$$\begin{aligned}(\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', \xi') &+ (\mathcal{L}_V\nabla)(V_1', V_2') - \eta'(V_2')(\mathcal{L}_V\nabla)(V_1', \xi') \\ = 2\alpha'(\nabla_{V_2'}Q)V_1' &+ V_2'(\xi'(\sigma - \mu'))V_1' + g(V_1', \nabla_{V_2'}D(\sigma - \mu'))\xi' \\ -g(V_1', D(\sigma - \mu'))\phi'^2 V_2' &+ g(V_1', \phi'^2 V_2')D(\sigma - \mu') - \eta'(V_1')(\nabla_{V_2'}D(\sigma - \mu')) \qquad (2.3.18)\end{aligned}$$

Incorporating this into the formula (see [88])

$$(\mathcal{L}_V R)(V_1', V_2')V_3' = (\nabla_{V_1'}\mathcal{L}_V\nabla)(V_2', V_3') - (\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', V_3'),$$

we get

$$\begin{aligned}(\mathcal{L}_V R)(V_1', V_2')\xi' &= 2\alpha'\{(\nabla_{V_1'}Q)V_2' - (\nabla_{V_2'}Q)V_1'\} + V_1'(\xi'(\sigma - \mu'))V_2' - V_2'(\xi'(\sigma \\ &- \mu'))V_1' + g(V_2', D(\sigma - \mu'))V_1' - g(V_1', D(\sigma - \mu'))V_2' \\ &+ \eta'(V_1')\nabla_{V_2'}D(\sigma - \mu') - \eta'(V_2')\nabla_{V_1'}D(\sigma - \mu') + 2\alpha'\{\eta'(V_1')QV_2' \\ &- \eta'(V_2')QV_1'\} + (4\alpha' n + \xi'(\sigma - \mu'))\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\}. \qquad (2.3.19)\end{aligned}$$

Now, we differentiate $\xi'(\sigma - \mu') = g(\xi', D(\sigma - \mu'))$ along the vector field V_1' and with the help of the identity (1.1.24) to obtain

$$V_1'(\xi'(\sigma - \mu')) = g(V_1', D(\sigma - \mu')) - \xi'(\sigma - \mu')\eta'(V_1') + g(\nabla_{V_1'}D(\sigma - \mu'), \xi'). \qquad (2.3.20)$$

We again replace V_2' by ξ' into (1.2.102) and invoking it in the Lie-derivative of (1.1.26) to deduce

$$(\mathcal{L}_V R)(V_1', V_2')\xi' + R(V_1', V_2')\mathcal{L}_V\xi' = g(V_1', \mathcal{L}_V\xi')V_2' - g(V_2', \mathcal{L}_V\xi')V_1'$$
$$+2(\sigma + 2\alpha' n - \mu')\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\}. \qquad (2.3.21)$$

We combine the identities (2.3.19), (2.3.20) and (2.3.21) to get

$$g(V_1', \mathcal{L}_V\xi')V_2' - g(V_2', \mathcal{L}_V\xi')V_1' - R(V_1', V_2')\mathcal{L}_V\xi' = 2\alpha'\{\eta'(V_1')QV_2' - \eta'(V_2')QV_1'\}$$
$$+2\alpha'\{\eta'(V_1')QV_2' - \eta'(V_2')QV_1'\} + g(\nabla_{V_1'}D(\sigma - \mu'), \xi')V_2' - g(\nabla_{V_2'}D(\sigma - \mu'), \xi')V_1'$$
$$+\eta'(V_1')\nabla_{V_2'}D(\sigma - \mu') - \eta'(V_2')\nabla_{V_1'}D(\sigma - \mu') - 2\sigma\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\}$$
$$+2\mu'\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\}. \qquad (2.3.22)$$

We displace vector fields V_1' and V_2' by $\phi' V_1'$ and $\phi' V_2'$ into identity (2.3.22) and contracting the obtained equation and invoking Lemma 4.2 (see [34]) results in

$$S(V_2', \mathcal{L}_V\xi') + 2ng(V_2', \mathcal{L}_V\xi') = \alpha'(V_2' r) + 2\alpha'(r + 4n^2 + 2n)\eta'(V_2') - g(\nabla_{\xi'}D\lambda', \phi'^2 V_2').$$

We contract the identity (2.3.22) and using this together with the preceding equation provides

$$2(n-1)\{g(\nabla_{\xi'}D\sigma, V_2') - g(\nabla_{\xi'}D\mu', V_2')\} + \{\xi'(\xi'\sigma) - \xi'(\xi'\mu')\}\eta'(V_2')$$
$$+\eta'(V_2')\{divD\sigma - divD\mu'\} - 4n(2n\alpha' - \mu' + \sigma)\eta'(V_2') = 0. \qquad (2.3.23)$$

We replace V_2' by ξ' in (2.3.23) to deduce

$$(2n-1)\{\xi'(\xi'\sigma) - \xi'(\xi'\mu')\} + divD\sigma - divD\mu' = 4n(\sigma - \mu' + 2n\alpha').$$

In view of this in (2.3.23) to infer $g(\nabla_{\xi'}D\sigma, V_1') - g(\nabla_{\xi'}D\mu', V_1') = \{\xi'(\xi'\sigma) - \xi'(\xi'\mu')\}\eta'(V_1')$ for $n > 1$. Let us consider ξ' in place of V_2' in (2.3.22) and utilizing the preceding relation, it follows that

$$\nabla_{V_1'}D\sigma - \nabla_{V_1'}D\mu' + 2(\sigma - \mu' + 2n\alpha')\phi'^2 V_1' = \{\xi'(\xi'\sigma) - \xi'(\xi'\mu')\}\{\phi'^2 V_1' + \eta'(V_1')\xi'\}. \quad (2.3.24)$$

As a consequence of (2.3.24), the curvature tensor takes the form:

$$R(V_1', V_2')D(\sigma - \mu') = 2\{(V_2'\sigma) - (V_2'\mu')\}\phi'^2 V_1' - 2\{(V_1'\sigma) - (V_1'\mu')\}\phi'^2 V_2'$$
$$+V_2'\{\xi'(\xi'\sigma) - \xi'(\xi'\mu')\}V_1' - V_1'\{\xi'(\xi'\sigma) - \xi'(\xi'\mu')\}V_2'$$
$$+2\{\xi'(\xi'\sigma) - \xi'(\xi'\mu') + \mu' - \sigma - 2n\alpha'\}\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\}. \qquad (2.3.25)$$

Replacing V_2' by ξ' in (2.3.25), substituting the obtained equation back into (2.3.25) produces

$$
\begin{aligned}
&R(V_1',V_2')D(\sigma-\mu') = (V_1'\sigma)V_2' - (V_1'\mu')V_2' - (V_2'\sigma)V_1' + (V_2'\mu')V_1' \\
&-2[\{(V_1'\sigma)-(V_1'\mu')\}\eta'(V_2')\xi' - \{(V_2'\sigma)-(V_2'\mu')\}\eta'(V_1')\xi'] \\
&+[\xi'\{\xi'(\xi'\sigma-\xi'\mu')\} - \xi'\sigma + \xi'\mu'][\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\} \\
&+2\{\xi'(\xi'\sigma-\xi'\mu') + \mu' - \sigma - 2n\alpha'\}\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\}. \qquad (2.3.26)
\end{aligned}
$$

We displace the vector fields V_1' and V_2' by $\phi' V_1'$ and $\phi' V_2'$ in (2.3.26) then contracting the obtained result and together with the observation that $\sigma = \lambda' - \frac{\beta' r}{2}$ yields

$$S(V_2', D\mu') = S(V_2', D(\lambda' - \tfrac{\beta' r}{2})) + 2ng(V_2', D(\lambda' - \tfrac{\beta' r}{2})) - 2ng(V_2', D\mu').$$

Consequently, the contraction of (2.3.26) and followed by replacing V_2' by $\phi' V_2'$ in the resulting expression yields $\phi'\{D(\lambda' - \frac{\beta' r}{2})\} = \phi'(D\mu')$. Differentiating this along vector field V_1' and inserting it in (2.3.24) and (1.1.24) gives

$$\xi'(\xi'\lambda') + \xi'(\xi'\mu') + \xi'\lambda' + \xi'\mu' = 2\{2n\alpha' + \mu' + \lambda' + n\beta'(2n+1)\}. \qquad (2.3.27)$$

Hence, the proof is complete. $\square$

Theorem 2.3.2. *If M^{2n+1} $(g, \xi', \lambda', \mu')$ be a $(\kappa, \mu')'$-almost Kenmotsu manifold with $h' \neq 0$ admitting gradient almost η'-Ricci-Yamabe soliton, then for $\kappa + 2 = 0$ either M is locally isometric to $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$ or potential vector field is pointwise collinear with the Reeb vector field.*

Proof. Let us suppose that $(\kappa, \mu')'$-almost Kenmotsu manifold satisfies gradient almost η'-Ricci-Yamabe soliton. Then equations (2.3.1), (2.3.2) and (2.3.3) are valid. We take an inner product of (2.3.3) with ξ' and inserting Lemma (2.2.1) to yield

$$
\begin{aligned}
g(R(V_1',V_2')Df,\xi') &= (V_1'(\lambda' - \frac{\beta' r}{2} - \mu'))\eta'(V_2') - (V_2'(\lambda' - \frac{\beta' r}{2} - \mu'))\eta'(V_1') \\
&- \alpha'[g(Qh'V_2', V_1') - g(Qh'V_1', V_2')] \\
&- \mu'\{\eta'(g(V_1' + h'V_1', \phi' V_2')) - \eta'(g(V_2' + h'V_2', \phi' V_1'))\}. \qquad (2.3.28)
\end{aligned}
$$

We again take an inner product of (2.2.2) with Df and inserting it into (2.3.28) and replacing V_1' by ξ' reads

$$D\lambda' - D\mu' - (\xi'(\lambda' - \mu'))\xi' = \kappa\{(\xi' f)\xi' - Df\} - \mu' h' Df. \qquad (2.3.29)$$

Now, we contract the identity (2.3.3) over V_1' to obtain $QDf = 2nD\mu' - 2nD\lambda'$. As a consequence of this in Lemma (2.2.1) provides

$$2D\lambda' - 2D\mu' - 2Df + 2(\kappa+1)(\xi' f)\xi' = -\mu' h' Df. \tag{2.3.30}$$

We combine the identities (2.3.29) and (2.3.30) to get

$$D\lambda' - D\mu' + (\xi'\lambda')\xi' - (\xi'\mu')\xi' - (\kappa+2)Df + (3\kappa+2)(\xi' f)\xi' = 0. \tag{2.3.31}$$

We operate the forgoing equation by ϕ' gives

$$\phi'(D\lambda' - D\mu') = (\kappa+2)\phi' Df,$$

this shows that

$$D\lambda' - D\mu' - (\kappa+2)Df \in \mathbb{R}\xi'.$$

We may therefore write $D\lambda' = D\mu' + (\kappa+2)Df + t\xi'$, where t is a smooth function. In light of this in (2.3.29) infer

$$2(\kappa+1)Df + (t + \xi'\mu' - \xi'\lambda' - \kappa\xi' f)\xi' = 2\mu' h' Df. \tag{2.3.32}$$

Operating (2.3.32) by h' by inserting the resulting expression into (2.3.32) yields

$$(\kappa+2)\phi' Df = 0.$$

Therefore, it follows that either $\kappa + 2 = 0$ or $Df = (\xi' f)\xi'$.
Let us assume that $\kappa + 2 = 0$, hence, without loss of generality, we assume that $v = 1$. Therefore we get from Theorem (5.1) of [53]

$$R(V'_{1v}, V'_{2v})V'_{3v} = -4[g(V'_{2v}, V'_{3v})V'_{1v} - g(V'_{1v}, V'_{3v})V'_{2v}],$$

$$R(V'_{1-v}, V'_{2-v})V'_{3-v} = 0$$

for any $V'_{1v}, V'_{2v}, V'_{3v} \in [v]'$ and $V'_{1-v}, V'_{2-v}, V'_{3-v} \in [-v]'$. As a consequence of this in Proposition 4.1 and Proposition 4.3 of [53] together with $v = 1$, this shows that the manifold is locally isometric to $\mathbb{H}^{2n+1}(-4) \times \mathbb{R}^n$. That completes the proof. □

Corollary 2.3.1. *Let $(M, g, \phi', \xi', \eta')$ be a non-Kenmotsu $(\kappa, \mu')'$-almost Kenmotsu manifold with $\kappa + 2 \neq 0$ admitting a gradient almost η'-Ricci-Yamabe soliton. Then smooth functions are related by $\lambda' = \mu' + \frac{\beta' r}{2}$.*

Proof. Suppose $\kappa+2 \neq 0$. Then making use of Theorem (2.3.2), we get $V = Df = (\xi' f)\xi'$. Take $F = \xi' f$ and taking the covariant derivative of $V = F\xi'$ with respect to an arbitrary vector field V_1', we obtain

$$\nabla_{V_1'} V = (V_1' F)\xi' + F(-\phi'^2 V_1' + h' V_1').$$

Now, we exploit this into (1.2.102) to deduce

$$(V_1'F)\eta'(V_2') + (V_2'F)\eta'(V_1') + 2Fg(V_1', V_2') - 2F\eta'(V_1')\eta'(V_2')$$
$$+2Fg(h'V_1', V_2') = (2\lambda' - 2\mu' - \beta' r)g(V_1', V_2') - 2\alpha' S(V_1', V_2'). \qquad (2.3.33)$$

We displace V_2' by ξ' into (2.3.33) to yield

$$V_1'F = (2\lambda' - 2\mu' - \beta' r - 4n\alpha'\kappa - \xi' F)\eta'(V_1'), \qquad (2.3.34)$$

for any vector field V_1' on M. We contract the identity (2.3.33) and then insert it into (2.3.34) and again replace V_1' by ξ' into the resulting expression to achieve

$$F = \lambda' - \mu' - \frac{\beta' r}{2} + 2n\alpha'. \qquad (2.3.35)$$

We insert the identity (2.3.34) into (2.3.33) and compare it with Lemma (2.2.1) to yield $(F - 2n\alpha')(\kappa+1)\phi'^2 V_1' = 0$ for any V_1' on M. As $\kappa < -1$, we see that $F = 2n\alpha'$, in view of this equation (2.3.35) implies $\lambda' = \mu' + \frac{\beta' r}{2}$. □

Theorem 2.3.3. *For a generalized (κ, μ')-almost Kenmotsu manifold $M^{2n+1}(g, \phi', \eta', \xi')$ $(n > 1)$ with $h' \neq 0$, there does not exist a gradient almost η'-Ricci-Yamabe soliton and also have the relation*

i) $\lambda' - \mu' = \frac{\beta' r}{2} - 2\alpha'$,

ii) $\xi' f = 2\alpha'(n-1)$ *and* $\xi'(\xi' f) = 0$.

Proof. We replace V_1' by $\phi' V_1'$ and V_2' by $\phi' V_2'$ into (2.3.28) to get

$$\alpha'\{g(Qh\phi'^2 V_2', \phi' V_1') - g(Qh\phi'^2 V_1', \phi' V_2')\} = 0. \qquad (2.3.36)$$

Now, we use the Lemma (2.2.2) into (2.3.36) to yield $\alpha'\mu' h^2 \phi'^2 V_1' = 0$ for an arbitrary vector field V_1' on M. From the last equation, it follows that $\alpha'(\kappa+1)\mu' h^2 V_1' = 0$. As $h' \neq 0$, $\kappa < -1$ it follows that $\mu'\alpha' = 0$. For $\mu' = 0$, equation (2.3.29) becomes

$$D\lambda' - D\mu' + (\xi'\mu')\xi' - (\xi'\lambda')\xi' = \kappa\{(\xi' f) - Df\}. \qquad (2.3.37)$$

We insert Lemma (2.2.2) into $QDf = 2nD\mu' - 2nD\lambda'$ and then operate it by ϕ' to obtain

$$2n\phi' D(\lambda' - \tfrac{\beta' r}{2}) - 2n\phi' D\mu' + 2n\phi' Df = 2(1-n)\phi' h' Df,$$

We operate (2.3.37) by ϕ' and then insert in the previous expression to yield

$$2n(\kappa+1)\phi' Df + 2(1-n)hDf = 0. \tag{2.3.38}$$

We again utilize by $h'\phi'$ into (2.3.38) and then plugging it into (2.3.38) to arrive

$$[n^2(\kappa+1) + (1-n)^2]\phi'^2 Df = 0.$$

The possible cases are as follows:

Case-I: $\phi'^2 Df \neq 0$ then $\kappa = -1 - (\frac{n-1}{n})^2$ i.e., a constant. As a result of Proposition in 3.2 [40], we have $\kappa = -1$, a contradiction.

Case-II: $\phi'^2 Df = 0$ shows that $Df = (\xi' f)\xi' = 0$. We take the covariant derivative of the preceding equation, and upon substitution (2.3.1) and (1.1.24) we evaluate,

$$(\lambda' - \mu' - \frac{\beta' r}{2})V_1' - V_1'(\xi' f)\xi' + (\xi' f)(\phi'^2 V_1' - h' V_1') - \alpha' Q V_1' = 0. \tag{2.3.39}$$

We combine (2.3.39) with Lemma (2.2.2) and contracting it to give

$$(2n+1)(\lambda' - \mu' - \frac{\beta' r}{2} + 2n\alpha') - \xi'(\xi' f) - 2n(\xi' f) = 2n\alpha'(\kappa+1). \tag{2.3.40}$$

We replace V_1' by ξ' and then take an inner product with ξ' to obtain $\xi'(\xi' f) = \lambda' - \mu' + 2n\alpha'\kappa - \frac{\beta' r}{2}$. Based on this, in equation (2.3.40) infer

$$\xi' f = \lambda' - \mu' + 2n\alpha' - \frac{\beta' r}{2}. \tag{2.3.41}$$

Now, we employ (2.3.39) by ϕ' and using the identity (2.3.41) to yield

$$(\lambda' - \mu' + 2\alpha' - \frac{\beta' r}{2})\phi' h' V_1' = 0. \tag{2.3.42}$$

We again utilize the above equation by h and in view of the fact that $\kappa < -1$ to deduce $\lambda' - \mu' = \frac{\beta' r}{2} - 2\alpha'$. As a consequence, equation (2.3.41), we observe that $\xi' f = 2\alpha'(n-1)$ i.e., a constant. Therefore $\xi'(\xi' f) = 0$ which shows that $2\alpha'(n\kappa + 1) = 0$, as $\alpha' \neq 0$, we get $\kappa = -\frac{1}{n}$, a contradiction. This concludes the proof. □

2.4 On (κ, μ') and $(\kappa, \mu')'$ almost Kenmotsu manifold

In this section, we investigate that the metric g of a $(2n+1)$-dimensional Kenmotsu manifold, where the characteristic vector field ξ' satisfies $(\kappa, -2)'$ nullity distribution. We study an almost η'-Ricci-Yamabe soliton on (κ, μ') almost Kenmotsu manifolds with some curvature properties. From (1.1.31)

$$R(V_1', V_2')\xi' = \kappa[\eta'(V_2')V_1' - \eta'(V_1')V_2'] + \mu'[\eta'(V_2')hV_1' - \eta'(V_1')hV_2']. \tag{2.4.1}$$

In [25], Dileo and Pastore established that for a (κ, μ')-almost Kenmotsu manifold, $\kappa = -1$ and $h = 0$. Therefore from (2.4.1), the result is as follows

$$R(V_1', V_2')\xi' = \eta'(V_1')V_2' - \eta'(V_2')V_1', \tag{2.4.2}$$

$$R(\xi', V_1')V_2' = -g(V_1', V_2')\xi' + \eta'(V_2')V_1', \tag{2.4.3}$$

$$S(V_1', \xi') = -2n\eta'(V_1') \quad and \quad Q\xi' = -2n\xi'. \tag{2.4.4}$$

We also have from (1.1.33), we conclude that

$$\nabla_{V_1'}\xi' = V_1' - \eta'(V_1')\xi'. \tag{2.4.5}$$

Once more in [25], it has been proven that in an almost Kenmotsu manifold with ξ' contained in the (κ, μ')-nullity distribution, the sectional curvature $\kappa(V_1', \xi') = -1$ from this, we compute $r = -4n^2 - 2n$.

Theorem 2.4.1. *If $(g, \xi', \lambda', \alpha', \beta')$ is an almost η'-Ricci-Yamabe soliton on a $(2n+1)$-dimensional (κ, μ')-almost Kenmotsu manifold M, then the manifold M is η'-Einstein.*

Proof. We consider $V = \xi'$ into (1.2.102) to obtain

$$(\mathcal{L}_{\xi'}g)(V_1', V_2') + 2\alpha' S(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2') = (2\lambda' - \beta' r)g(V_1', V_2'). \tag{2.4.6}$$

Now, we know that

$$(\mathcal{L}_{\xi'}g) = g(\nabla_{V_1'}\xi', V_2') + g(\nabla_{V_2'}\xi', V_1').$$

In light of (2.4.5) in the forgoing equation yields

$$(\mathcal{L}_{\xi'}g)(V_1', V_2') = 2g(V_1', V_2') - 2\eta'(V_1')\eta'(V_2'). \tag{2.4.7}$$

We substitute the identity (2.4.7) into (2.4.6) and using $r = -4n^2 - 2n$ to deduce

$$S(V_1', V_2') = ag(V_1', V_2') + b\eta'(V_1')\eta'(V_2'), \tag{2.4.8}$$

where $a = \frac{1}{\alpha'}[(\lambda' - 1) + \beta'(2n^2 + n)]$ and $b = \frac{1}{\alpha'}(1 - \mu')$ are smooth functions. This shows that the manifold M is η'-Einstein. This ends the proof. $\square$

Now from (2.4.4), we have

$$S(\xi', \xi') = -2n.$$

We set $V_1' = \xi'$ and $V_2' = \xi'$ into identity (2.4.8) and also using the value of a and b to arrive

$$S(\xi', \xi') = \frac{1}{\alpha'}[\lambda' + \beta'(2n^2 + n) - \mu']. \tag{2.4.9}$$

Now, we equate these two values of $S(\xi', \xi')$ to achieve

$$\lambda' + \beta'(2n^2 + n) = \mu' - 2n\alpha'. \tag{2.4.10}$$

With the help of (2.4.10) into (2.4.8) to yield

$$S(V_1', V_2') = \frac{1}{\alpha'}[\mu' - 2n\alpha' - 1]g(V_1', V_2') + \frac{1}{\alpha'}(1 - \mu')\eta'(V_1')\eta'(V_2'), \tag{2.4.11}$$

which implies

$$QV_1' = \frac{1}{\alpha'}(\mu' - 2n\alpha' - 1)V_1' + \frac{1}{\alpha'}(1 - \mu')\eta'(V_1')\xi'. \tag{2.4.12}$$

Next, we consider an almost η'-Ricci-Yamabe soliton on a $(2n+1)$-dimensional (κ, μ')-almost Kenmotsu manifold endowed with a structure satisfying the curvature condition $Q \cdot P = 0$, where P is the projective curvature tensor defined for a Riemannian manifold as

$$P(V_1', V_2')V_3' = R(V_1', V_2')V_3' - \frac{1}{2n}[S(V_2', V_3')V_1' - S(V_1', V_3')V_2']. \tag{2.4.13}$$

Theorem 2.4.2. *If $(g, \xi', \lambda', \alpha', \beta')$ is an almost η'-Ricci-Yamabe soliton on a $(2n+1)$-dimensional (κ, μ')-almost Kenmotsu manifold M satisfying the curvature property $Q \cdot P = 0$, then $r = -2n$ and the manifold M is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof. Let us suppose that the curvature property $Q \cdot P = 0$ holds on M. Then for any vector fields V_1', V_2', V_3' on M, we have

$$Q(P(V_1', V_2')V_3') - P(QV_1', V_2')V_3' - P(V_1', QV_2')V_3' - P(V_1', V_2')QV_3' = 0.$$

Applying (2.4.12) in the forgoing equation leads to

$$\frac{1}{\alpha'}(1-\mu')[\eta'(P(V_1',V_2')V_3')\xi' - \eta'(V_1')P(\xi',V_2')V_3' - \eta'(V_2')P(V_1',\xi')V_3' - \eta'(V_3')P(V_1',V_2')\xi'] + \frac{2}{\alpha'}(2n\alpha'+1-\mu')P(V_1',V_2')V_3' = 0. \tag{2.4.14}$$

Now in light of the identities (2.4.2), (2.4.3) and (2.4.4), we get the following:

$$P(\xi',V_2')V_3' = -\{g(V_2',V_3')\xi' + \frac{1}{2n}S(V_2',V_3')\xi'\}, \tag{2.4.15}$$

$$P(V_1',\xi')V_3' = g(V_1',V_3')\xi' + \frac{1}{2n}S(V_1',V_3')\xi', \tag{2.4.16}$$

$$P(V_1',V_2')\xi' = 0, \tag{2.4.17}$$

$$\eta'(P(V_1',V_2')V_3') = \eta'(V_2')g(V_1',V_3') - \eta'(V_1')g(V_2',V_3') - \frac{1}{2n}[S(V_2',V_3')\eta'(V_1') - S(V_1',V_3')\eta'(V_2')]. \tag{2.4.18}$$

We substitute the equations from (2.4.15) to (2.4.18) into (2.4.14) to yield

$$\tfrac{2}{\alpha'}(2n\alpha'+1-\mu')P(V_1',V_2')V_3' = 0,$$

from which it follows that either $2n\alpha'+1-\mu' = 0$ or $P(V_1',V_2')V_3' = 0$.
If $2n\alpha'+1-\mu' = 0$, then from (2.4.11), we get

$$S(V_1',V_2') = \tfrac{1}{\alpha'}(1-\mu')\eta'(V_1')\eta'(V_2') = -2n\eta'(V_1')\eta'(V_2'),$$

implying that $r = -2n$, this contradicts our hypothesis that $r = -2n(2n+1)$.
Hence, $P(V_1',V_2')V_3' = 0$ then, (2.4.13) shows that

$$R(V_1',V_2')V_3' = \frac{1}{2n}[S(V_2',V_3')V_1' - S(V_1',V_3')V_2']. \tag{2.4.19}$$

We take an inner product of (2.4.19) with V and then contracting V_2' and V_3' to acquire

$$S(V_1',V) = -2ng(V_1',V). \tag{2.4.20}$$

We use the identity (2.4.20) into (2.4.19) to find

$$R(V_1',V_2')V_3' = -[g(V_2',V_3')V_1' - g(V_1',V_3')V_2'].$$

Thus, we complete our proof. □

Next, we establish the non-existence of such a curvature condition $Q \cdot R = 0$ on (κ,μ')-almost Kenmotsu manifold admitting an almost η'-Ricci-Yamabe soliton.

Theorem 2.4.3. *If M is a $(2n+1)$-dimensional (κ, μ')-almost Kenmotsu manifold admitting an almost η'-Ricci-Yamabe soliton, then the curvature property $Q \cdot R = 0$ does not hold for the relation $\frac{1-\mu'}{2n\alpha'+1-\mu'} = 1$ on M.*

Proof. Let us assume that curvature property $Q \cdot R = 0$ holds on M. Consequently, for all vector fields V_1', V_2' and V_3' on M, it follows that

$$Q(R(V_1', V_2')V_3') - R(QV_1', V_2')V_3' - R(V_1', QV_2')V_3' - R(V_1', V_2')QV_3' = 0. \tag{2.4.21}$$

Making use of (2.4.12) into identity (2.4.21) to deduce

$$2(2n\alpha' + 1 - \mu')R(V_1', V_2')V_3' + (1 - \mu')[\eta'(R(V_1', V_2')V_3')\xi' - \eta'(V_1')R(\xi', V_2')V_3' \\ -\eta'(V_2')R(V_1', \xi')V_3' - \eta'(V_3')R(V_1', V_2')\xi'] = 0. \tag{2.4.22}$$

In light of (2.4.2), we get

$$\eta'(R(V_1', V_2')V_3') = g(V_1', V_3')\eta'(V_2') - g(V_2', V_3')\eta'(V_1'). \tag{2.4.23}$$

Now, we feed the identities (2.4.3) and (2.4.23) into (2.4.22) to yield

$$(2n\alpha' + 1 - \mu')R(V_1', V_2')V_3' = (1 - \mu')\eta'(V_3')[\eta'(V_1')V_2' - \eta'(V_2')V_1']. \tag{2.4.24}$$

Then, we plug $V_3' = \xi'$ into (2.4.24) to obtain

$$R(V_1', V_2')\xi' = \frac{1 - \mu'}{2n\alpha' + 1 - \mu'}[\eta'(V_1')V_2' - \eta'(V_2')V_1']. \tag{2.4.25}$$

We compare the equation (2.4.2) with (2.4.25) to obtain $\frac{1-\mu'}{2n\alpha'+1-\mu'} = 1$, which implies $\alpha' = 0$, a contradiction. Thus we finish our proof. □

Theorem 2.4.4. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be an almost Kenmotsu manifold such that ξ' belongs to $(\kappa, -2)'$-nullity distribution where $\kappa < -1$. If the metric g represents a $*$-η'-RYS satisfying*

$$\lambda' + \mu' \neq \tfrac{q}{2}(r + 4n^2) + \tfrac{q}{2}\{\xi'(\xi'(r)) - \xi'(Dr)\},$$

then M is Ricci-flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Proof. Comparing (1.2.104) and (2.2.16) and using $r^* = r + 4n^2$, we get

$$(\mathcal{L}_V g)(V_1', V_2') = \{2\rho(\kappa + 2) - 2\lambda' + q(r + 4n^2)\}g(V_1', V_2') - 2\{\rho(\kappa + 2) + \mu'\}\eta'(V_1')\eta'(V_2'), \tag{2.4.26}$$

for all vector fields V_1' and V_2' on M. Now taking covariant derivative of (2.4.26) along the arbitrary vector field V_3' and using (1.1.12) we have

$$(\nabla_{V_3'}\mathcal{L}_V g)(V_1', V_2') = qV_3'(r)g(V_1', V_2') - 2\{\rho(\kappa+2) + \mu'\}$$
$$[\eta'(V_2')g(V_1', V_3') + \eta'(V_1')g(V_2', V_3') + \eta'(V_2')g(h'V_3', V_1') + \eta'(V_1')g(h'V_3', V_2')$$
$$-2\eta'(V_1')\eta'(V_2')\eta'(V_3')]. \tag{2.4.27}$$

Then using (1.1.87) and by the symmetry of $(\mathcal{L}_V\nabla)$ from the above equation (2.4.27) we have

$$\begin{aligned}(\mathcal{L}_V\nabla)(V_1', V_2') &= -2\{\rho(\kappa+2) + \mu'\}[g(V_1', V_2') + g(h'V_1', V_2') - \eta'(V_1')\eta'(V_2')] \\ &- q\{g(V_1', V_2')Dr - V_1'(r)V_2' - V_2'(r)V_1'\},\end{aligned} \tag{2.4.28}$$

for all V_1', $V_2' \in \chi(M)$. Putting $V_2' = \xi'$ and using (1.1.8), (1.1.12) and (1.1.34), we acquire

$$(\mathcal{L}_V\nabla)(V_1', \xi') = -q\{\eta'(V_1')Dr - V_1'(r)\xi' - \xi'(r)V_1'\}, \tag{2.4.29}$$

for any given vector V_1' on M. Now taking differentiation (2.4.29) covariantly along arbitrary vector field V_2' and using (1.1.33) and (2.4.28) into account we can get

$$\begin{aligned}(\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', \xi') &= 2\{\rho(\kappa+2) + \mu'\}[g(V_1', V_2') + g(h'V_1', V_2') - \eta'(V_1')\eta'(V_2')]\xi' \\ &- q\{(\nabla_{V_2'}\eta')V_1'Dr + \eta'(V_1')V_2'(Dr) - V_1'(r)\nabla_{V_2'}\xi' - V_2'(\xi'(r))V_1'\},\end{aligned} \tag{2.4.30}$$

for any vector fields V_1' and V_2' on M. Again Yano demonstrates that the well-known curvature property, $(\mathcal{L}_V R)(V_1', V_2')V_3' = (\nabla_{V_1'}\mathcal{L}_V\nabla)(V_2', V_3') - (\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', V_3')$. Using (1.1.33), (1.1.44) and setting $V_3' = \xi'$ then using (2.4.30), we obtain

$$(\mathcal{L}_V R)(V_1', V_2')\xi' = q\{\eta'(V_2')V_1'(Dr) - \eta'(V_1')V_2'(Dr) + V_2'(\xi'(r))V_1' - V_1'(\xi'(r)V_2'\}, \tag{2.4.31}$$

$\forall$ V_1', $V_2' \in \chi(M)$. Now taking Lie derivative of (1.1.38) with respect to the potential vector field V and also using (1.1.8) and (1.1.34), we get

$$(\mathcal{L}_V R)(V_1', \xi')\xi' = \kappa[g(V_1', \mathcal{L}_V\xi')\xi' - 2\eta'(\mathcal{L}_V\xi')V_1' - ((\mathcal{L}_V\eta')V_1')\xi']$$
$$+2[2\eta'(\mathcal{L}_V\xi')h'V_1' - \eta'(V_1')(h'(\mathcal{L}_V\xi')) - g(h'V_1', \mathcal{L}_V\xi')\xi' - (\mathcal{L}_V h')V_1'], \tag{2.4.32}$$

for any $V_1' \in \chi(M)$. Putting $V_2' = \xi'$ in (2.4.26) we infer

$$(\mathcal{L}_V\eta')V_1' - g(V_1', \mathcal{L}_V\xi') = \{-2\lambda' + q(r + 4n^2) - 2\mu'\}\eta'(V_1'), \tag{2.4.33}$$

for any $V_1' \in \chi(M)$. Next we put $V_1' = \xi'$ into (2.4.33) to yield

$$\eta'(\mathcal{L}_V\xi') = \lambda' - \frac{q}{2}(r + 4n^2) + \mu'. \tag{2.4.34}$$

By the help of (2.4.31), (2.4.33) and (2.4.34), we can write the equation (2.4.32) as

$$\kappa\{-2\lambda' + q(r + 4n^2) - 2\mu' + q\xi'(\xi'(r)) - q\xi'(Dr)\}(V_1' - \eta'(V_1')\xi') + 2\{2\lambda' - q(r + 4n^2) + 2\mu'\}h'V_1' - 2\eta'(V_1')h'(\mathcal{L}_V\xi') - 2g(h'V_1', \mathcal{L}_V\xi')\xi' - 2(\mathcal{L}_V h')V_1' = 0. \tag{2.4.35}$$

We take an inner product of (2.4.35) with respect to the arbitrary vector field V_2' on M to obtain

$$\{-2\lambda' + q(r + 4n^2) - 2\mu' + q\xi'(\xi'(r)) - q\xi'(Dr)\}[\kappa\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\} -2g(h'V_1', V_2')] - 2\eta'(V_1')g(h'(\mathcal{L}_V\xi'), V_2') - 2g(h'V_1', \mathcal{L}_V\xi')\eta'(V_2') -2g((\mathcal{L}_V h')V_1', V_2') = 0. \tag{2.4.36}$$

As the above equation (2.4.36) is true for any given vector fields V_1' and V_2' on M. We replacing V_1' by $\phi'V_1'$ and V_2' by $\phi'V_2'$ and employing (1.1.11) taking into account we obtain as

$$\{-2\lambda' + q(r + 4n^2) - 2\mu' + q\xi'(\xi'(r)) - q\xi'(Dr)\}[\kappa g(\phi'V_1', \phi'V_2') -2g(h'\phi'V_1', \phi'V_2')] - 2g((\mathcal{L}_V h')\phi'V_1', \phi'V_2') = 0, \tag{2.4.37}$$

for all given vector fields V_1' and V_2' on M. As $spec(h') = \{0, \alpha', -\alpha'\}$, let V_1' and V belong to the eigen spaces of $-\alpha'$ and α' represented by $[-\alpha']'$ and $[\alpha']'$ respectively. Then $\phi'V_1' \in [\alpha']'$ (see [25]). Hence (2.4.37) can be expressed as

$$\{-2\lambda' + q(r + 4n^2) - 2\mu' + q\xi'(\xi'(r)) - q\xi'(Dr)\}(\kappa - 2)g(\phi'V_1', \phi'V_2') -2g((\mathcal{L}_V h')\phi'V_1', \phi'V_2') = 0, \tag{2.4.38}$$

for all given vector fields on M. We now compute the value of $g((\mathcal{L}_V h')\phi'V_1', \phi'V_2')$. To obtain this value, we establish a more general result. In a $(\kappa, \mu')'$-almost Kenmotsu manifold $(\mathcal{L}_{V_1'} h')V_2' = 0$, where the vector fields V_1' and V_2' lie in the same eigen spaces.

It suffices to assume that the vector fields V_1', $V_2' \in [\alpha']'$, where $spec(h') = \{0, -\alpha', \alpha'\}$. Let us consider a local orthonormal ϕ'-basis as $\{\xi', e_i, \phi'e_i\}$, $i = 1, 2, 3, ..., n$ it follows that

$$\nabla_{V_1'}V_2' = \sum_{i=1}^{n} g(\nabla_{V_1'}V_2', e_i)e_i - (\alpha' + 1)g(V_1', V_2')\xi', \tag{2.4.39}$$

and also

$$\begin{aligned}(\mathcal{L}_{V_1'}h')V_2' &= \mathcal{L}_{V_1'}(h'V_2') - h'(\mathcal{L}_{V_1'}V_2')\\ &= \alpha'(\mathcal{L}_{V_1'}V_2') - h'(\mathcal{L}_{V_1'}V_2')\\ &= \alpha'(\nabla_{V_1'}V_2' - \nabla_{V_2'}V_1') - h'(\nabla_{V_1'}V_2' - \nabla_{V_2'}V_1')\\ &= \alpha'(\alpha'+1)g(V_1',V_2')\xi' - \alpha'(\alpha'+1)g(V_1',V_2')\xi'\\ &= 0. \qquad (2.4.40)\end{aligned}$$

Similarly, it can be shown that the above results remain valid if V_1', $V_2' \in [-\alpha']'$. For more details see [25]. Now from (2.4.38), we get

$$\{-2\lambda' + q(r+4n^2) - 2\mu' + q\xi'(\xi'(r)) - q\xi'(Dr)\}(\kappa-2)g(\phi'V_1', \phi'V_2') = 0, \qquad (2.4.41)$$

for all vector fields V_1' and V_2' on M. As $g(\phi'V_1', \phi'V_2') \neq 0$ then from the for going equation we have either $\lambda' + \mu' = \frac{q}{2}(r+4n^2) + \frac{q}{2}\{\xi'(\xi'(r)) - \xi'(Dr)\}$ or $\kappa = 2$.

Now, for $\lambda' + \mu' \neq \frac{q}{2}(r+4n^2) + \frac{q}{2}\{\xi'(\xi'(r)) - \xi'(Dr)\}$ from the equation (2.4.41) we infer that $\kappa = 2$, (2.2.16) implies

$$S^*(V_1', V_2') = -4\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}. \qquad (2.4.42)$$

Again from (2.4.42) and the proposition 4.1 of [25]. In the final step we conclude that M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, where $\mathbb{H}^{n+1}(-4)$ is the hyperbolic space of constant curvature -4.

Also, for any vector fields V_1' and V_2' on M. By hypothesis $\lambda' + \mu' \neq \frac{q}{2}(r+4n^2) + \frac{q}{2}\{\xi'(\xi'(r)) - \xi'(Dr)\}$, from the equation (2.4.41) it can be deduced that $\kappa = 2\alpha'$. Again from ${\alpha'}^2 = -(\kappa+1)$, we get $\alpha' = -1$ and $\kappa = -2$, putting the value of κ in (2.2.16) $S^*(V_1', V_2') = 0$ i.e. the manifold is $*$-Ricci flat. This completes the proof. □

Let $V_1' \in D$ be the eigen vector of h' associated to the eigen values δ. Then it can be seen that $\delta^2 = -(\kappa+1)$, a constant. In [25], we have that in a $(2n+1)$-dimensional $(\kappa, \mu')'$-almost Kenmotsu manifold M with $h' \neq 0$, $\kappa < -1$, $\mu' = -2$ and $Spec(h') = 0, \delta, -\delta$, with 0 as simple eigen value and $\delta = \sqrt{-(\kappa+1)}$. In [81], Wang and Liu are shown that for a $2n+1$-dimensional $(\kappa, \mu')'$-almost Kenmotsu manifold M with $h' \neq 0$, the Ricci operator Q of M is given by

$$QV_1' = -2nV_1' + 2n(\kappa+1)\eta'(V_1')\xi' - 2nh'V_1'. \qquad (2.4.43)$$

Furthermore, the scalar curvature of M is $r = 2n(\kappa - 2n)$. From (1.1.32), it yields

$$R(V_1', V_2')\xi' = \kappa[\eta'(V_2')V_1' - \eta'(V_1')V_2'] - 2[\eta'(V_2')h'V_1' - \eta'(V_1')h'V_2'], \quad (2.4.44)$$

where κ, $\mu' \in \mathbb{R}$. It also follows from (2.4.44), we derive

$$R(\xi', V_1')V_2' = \kappa[g(V_1', V_2')\xi' - \eta'(V_2')V_1'] - 2[g(h'V_1', V_2')\xi' - \eta'(V_2')h'V_1']. \quad (2.4.45)$$

Again from (2.4.43), as a result, we find

$$S(V_2', \xi') = g(QV_2', \xi') = 2n\kappa\eta'(V_2'). \quad (2.4.46)$$

Applying (1.1.33), as a result, we can deduce

$$(\nabla_{V_1'}\eta')V_2' = g(V_1', V_2') - \eta'(V_1')\eta'(V_2') + g(h'V_1', V_2'). \quad (2.4.47)$$

Here, we consider the notion of almost gradient η'-Ricci-Yamabe soliton in the framework of $(\kappa, \mu')'$-almost Kenmotsu manifold and extend the preceding by considering V as a gradient vector field. In this regard, the following theorem is proved.

Theorem 2.4.5. *If $(g, V, \lambda', \alpha', \beta')$ is an almost gradient η'-Ricci-Yamabe soliton on a $(2n+1)$-dimensional $(\kappa, \mu')'$-almost Kenmotsu manifold M, then either $2\kappa - 1 = 0$ or, $D(\sigma - \mu') = \xi'(\sigma - \mu') + \kappa(\xi' f - 1)$ where $\sigma = \lambda' - \frac{\beta' r}{2}$, and $V = Df$.*

Proof. Assume that V is the gradient of a non-zero smooth function $f : M \to \mathbb{R}$, which means $V = Df$, where D is the gradient operator. It then follows from (1.2.103), we express this as

$$\nabla_{V_1'} Df = (\lambda' - \frac{1}{2}\beta' r)V_1' - \alpha' QV_1' - \mu'\eta'(V_1')\xi'. \quad (2.4.48)$$

We differentiate this covariantly with respect to any vector field V_2' to achieve

$$\begin{aligned} \nabla_{V_2'}\nabla_{V_1'} Df &= (V_2'\sigma)V_1' + \sigma(\nabla_{V_2'}V_1') - \alpha'\nabla_{V_2'}QV_1' - (V_2'\mu')\eta'(V_1')\xi' \\ &- \mu'\{(\nabla_{V_2'}\eta')(V_1')\xi' + \eta'(V_1')\nabla_{V_2'}\xi'\}, \end{aligned} \quad (2.4.49)$$

where $\sigma = \lambda' - \frac{1}{2}\beta' r$. Interchanging V_1' by V_2' in (2.4.49) yields

$$\begin{aligned} \nabla_{V_1'}\nabla_{V_2'} Df &= (V_1'\sigma)V_2' + \sigma(\nabla_{V_1'}V_2') - \alpha'\nabla_{V_1'}QV_2' - (V_1'\mu')\eta'(V_2')\xi' \\ &- \mu'\{(\nabla_{V_1'}\eta')(V_2')\xi' + \eta'(V_2')\nabla_{V_1'}\xi'\}. \end{aligned} \quad (2.4.50)$$

From (2.4.48), we get

$$\nabla_{[V_1', V_2']} Df = \sigma(\nabla_{V_1'}V_2' - \nabla_{V_2'}V_1') - \alpha' Q(\nabla_{V_1'}V_2' - \nabla_{V_2'}V_1') - \mu'\eta'([V_1', V_2'])\xi'. \quad (2.4.51)$$

It is well known that

$$R(V_1', V_2')Df = \nabla_{V_1'}\nabla_{V_2'}Df - \nabla_{V_2'}\nabla_{V_1'}Df - \nabla_{[V_1', V_2']}Df.$$

We now substitute identities (2.4.49), (2.4.50) and (2.4.51) into previous equation and then using (1.1.33) to get

$$\begin{aligned} &R(V_1', V_2')Df = (V_1'\sigma)V_2' - (V_2'\sigma)V_1' - (V_1'\mu')\eta'(V_2')\xi' + (V_2'\mu')\eta'(V_1')\xi' \\ &+\alpha'[(\nabla_{V_2'}Q)V_1' - (\nabla_{V_1'}Q)V_2'] - \mu'[\eta'(V_2')(V_1' - \phi' hV_1') \\ &-\eta'(V_1')(V_2' - \phi' hV_2')]. \end{aligned} \tag{2.4.52}$$

Making use of (2.4.43), we obtain

$$\begin{aligned} (\nabla_{V_2'}Q)V_1' &= \nabla_{V_2'}QV_1' - Q(\nabla_{V_2'}V_1') \\ &= 2n(\kappa+1)(g(V_1', V_2') - \eta'(V_1')\eta'(V_2') + g(hV_1', V_2'))\xi' \\ &+ 2n(\kappa+1)\eta'(V_1')(V_2' - \eta'(V_2')\xi' - \phi' hV_2') \\ &+ 2ng(hV_2' + h^2V_2', V_1')\xi' + 2n\eta'(V_1')(hV_2' + h^2V_2'). \end{aligned}$$

Now, we interchange V_1' and V_2' in the forgoing equation to yield $(\nabla_{V_1'}Q)V_2'$. Now, we set this two values in (2.4.52) and using (1.1.40) to deduce

$$\begin{aligned} R(V_1', V_2')V_3' &= (V_1'\sigma)V_2' - (V_2'\sigma)V_1' - (V_1'\mu')\eta'(V_2')\xi' + (V_2'\mu')\eta'(V_1')\xi' \\ &+ 2n\alpha'(\kappa+2)[\eta'(V_1')hV_2' - \eta'(V_2')h] \\ &- [\eta'(V_2')(V_1' - \phi' hV_1') - \eta'(V_2')(V_2' - \phi' hV_2')]. \end{aligned} \tag{2.4.53}$$

We put $V_1' = \xi'$ into (2.4.53) and we take the inner product with V_1' to simplify the expression

$$\begin{aligned} g((\xi', V_2')Df, V_1') &= (\xi'\sigma + \mu')g(V_1', V_2') - V_2'(\sigma - \mu')\eta'(V_1') + 2n\alpha'(\kappa+2)g(hV_2', V_1') \\ &- (\xi'\mu' + \mu')\eta'(V_1')\eta'(V_2') + \mu' g(\phi' V_1', hV_2'). \end{aligned} \tag{2.4.54}$$

Again using (2.4.45), we have

$$\begin{aligned} g(R(\xi', V_2')Df, V_1') &= -g(R(\xi', V_2')V_1', Df) \\ &= -\kappa g(V_1', V_2')(\xi' f) + \kappa\eta'(V_1')\eta'(V_2') \\ &\quad +2g(hV_1', V_2')(\xi' f) - 2\eta'(V_1')((h'V_2')f). \end{aligned} \tag{2.4.55}$$

Now, we equate (2.4.54) and (2.4.55) and then antisymmetrizing to find out

$$(\xi'\sigma + \mu' + \kappa(\xi' f))g(V_1', V_2') - ((\xi'\mu') + \mu' + \kappa)\eta'(V_1')\eta'(V_2') - \{V_2'(\sigma - \mu') - 2(h'V_2')f\}\eta'(V_1') + 2n\alpha'(\kappa + 2)g(hV_1', V_2') + (\mu' - 2(\xi' f))g(hV_1', V_2') = 0.$$

We replace V_1' by ξ' in the above equation to obtain

$$(\xi'(\sigma - \mu') + \kappa(\xi' f - 1))\eta'(V_2') - V_2'(\sigma - \mu') + 2(hV_2')f = 0,$$

which implies

$$\{\xi'(\sigma - \mu') + \kappa(\xi' f - 1)\}\xi' - D(\sigma - \mu') + 2h(Df) = 0. \tag{2.4.56}$$

We operate h into (2.4.56) and using (1.1.40) to yield

$$h(Df) = -2(\kappa + 1)[\{\xi'(\sigma - \mu') + \kappa(\xi' f - 1)\}\xi' - D(\sigma - \mu')]. \tag{2.4.57}$$

Substituting (2.4.57) in (2.4.56), we obtain

$$(2\kappa - 1)[\xi'(\sigma - \mu') + \kappa(\xi' f - 1) - D(\sigma - \mu')] = 0, \tag{2.4.58}$$

which implies either $2\kappa - 1 = 0$ or $D(\sigma - \mu') = \xi'(\sigma - \mu') + \kappa(\xi' f - 1)$, where $\sigma = \lambda' - \frac{\beta' r}{2}$. This completes the proof. □

Example 2.4.1. *We consider the three-dimensional manifold* $M = \{(V_1', V_2', V_3') \in \mathbb{R}^3, (V_1', V_2', V_3') \neq (0, 0, 0)\}$, *where* (V_1', V_2', V_3') *are standard coordinates in* $\mathbb{R}^3$. *The vector fields*

$$e_1 = V_3' \frac{\partial}{\partial V_1'}, \quad e_2 = V_3' \frac{\partial}{\partial V_2'}, \quad e_3 = -V_3' \frac{\partial}{\partial V_3'}$$

are linearly independent at each point of M. *Let* g *be the Riemannian metric defined by*

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η' *be the 1-form defined by* $\eta'(V_3') = g(V_3', e_3)$, *for any* $V_3' \in \chi(M)$,*where* $\chi(M)$ *is the set of all differentiable vector fields on* M *and* ϕ' *be the (1, 1)-tensor field defined by*

$$\phi' e_1 = -e_2, \quad \phi' e_2 = e_1, \quad \phi' e_3 = 0.$$

Then using the linearity of ϕ' *and* g, *we have,*

$$\eta'(e_3) = 1, \quad {\phi'}^2 V_3' = -V_3' + \eta'(V_3')e_3, \quad g(\phi' V_3', \phi' W) = g(V_3', W) - \eta'(V_3')\eta'(W),$$

for any $V_3', W \in \chi(M)$. *Thus for* $e_3 = \xi'$, (ϕ', ξ', η', g) *defines an almost contact metric structure on* M.
Let ∇ *be the Levi-Civita connection with respect to the Riemannian metric* g. *Then we have*

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

The connection ∇ *of the metric* g *is given by*

$$\begin{aligned} 2g(\nabla_{V_1'} V_2', V_3') &= V_1' g(V_2', V_3') + V_2' g(V_3', V_1') - V_3' g(V_1', V_2') \\ &- g(V_1', [V_2', V_3']) - g(V_2', [V_1', V_3']) + g(V_3', [V_1', V_2']), \end{aligned}$$

which is known as Koszul's formula.
Using Koszul's formula, we can easily calculate

$$\begin{gathered} \nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1, \\ \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2, \\ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \end{gathered} \tag{2.4.59}$$

From the above it follows that the manifold satisfies $\nabla_{V_1'} \xi' = V_1' - \eta'(V_1')\xi'$, *for* $\xi' = e_3$. *Hence, the manifold is a Kenmotsu manifold.*
Also, the Riemannian curvature tensor R *is given by*

$$R(V_1', V_2')V_3' = \nabla_{V_1'} \nabla_{V_2'} V_3' - \nabla_{V_2'} \nabla_{V_1'} V_3' - \nabla_{[V_1', V_2']} V_3'.$$

Hence,

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_1)e_1 = -e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_3, e_2)e_2 = -e_3,$$

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_1 = 0, \quad R(e_3, e_1)e_2 = 0.$$

Then, the Ricci tensor S *is given by*

$$S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2. \tag{2.4.60}$$

and therefore

$$Ric(V_1', V_2') = -2g(V_1', V_2'), \quad \text{for all } \mathrm{V}_1', \mathrm{V}_2' \in \chi(\mathrm{M}^5). \tag{2.4.61}$$

Also the scalar curvature becomes

$$r = \sum_{i=1}^{3} S(e_i, e_i) = -6. \tag{2.4.62}$$

Then $(\pounds_V g)(e_1, e_1) = -2g(\pounds_V e_1, e_1) = 2.$
Similarly, $(\pounds_V g)(e_2, e_2) = 2, \quad (\pounds_V g)(e_3, e_3) = 0.$
Let $f : M \to \mathbb{R}$ *be a smooth function defined by*

$$f(V_1', V_2', V_3') = V_1'^2 + V_2'^2 + \frac{V_3'^2}{2}. \tag{2.4.63}$$

Then the gradient of f*,* Df *is given by,*

$$Df = 2V_1' \frac{\partial}{\partial V_1'} + 2V_2' \frac{\partial}{\partial V_2'} + V_3' \frac{\partial}{\partial V_3'}. \tag{2.4.64}$$

Then with the help of (2.4.59) *we can show that*

$$(\mathcal{L}_{Df} g)(V_1', V_2') = 2\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\} \tag{2.4.65}$$

for all V_1'*,* $V_2' \in \chi(M^5)$*. So, combining* (2.4.61) *and* (2.4.62)*, we observe that soliton Eq.* (1.2.102) *holds for* $\lambda' = 2\alpha' - 3\beta' - 1$ *and* $\mu' = 1$ *i.e., the metric* g *is a gradient almost* η'*-Ricci-Yamabe soliton with this potential vector field* $V = Df$ *and the smooth functions* $\lambda' = 2\alpha' - 3\beta' - 1$ *and* $\mu' = 1$*. Here, we see that* $\xi' r = 0$ *i.e.,* ξ' *leaves the scalar curvature invariant. Also, we see that* $r = -6 = -2n(2n+1)$ *and the manifold becomes Einstein. So, Theorem* (2.3.1) *is verified.*

2.5 $*$-η'-Ricci-Yamabe soliton on Kenmotsu manifold

In this section we consider the manifold as a $(2n+1)$-dimensional Kenmotsu manifold admitting a $*$-η'-Ricci-Yamabe soliton.

Theorem 2.5.1. *Let* $M^{2n+1}(\phi', \xi', \eta', g)$ *be a Kenmotsu manifold represents a* $*$-η'*-RYS. Therefore the manifold is an* η'*-Einstein.*

Proof. We take a Lie derivative of $g(\xi', \xi') = 1$ with respect to the potential vector field V, in account of (2.2.9) to derive

$$\eta'(\mathcal{L}_V \xi') = \lambda' + \mu' - \frac{q}{2}(r + 4n^2). \tag{2.5.1}$$

Putting $V_2' = \xi'$ and following (1.1.8) and (1.1.12), the equation (2.2.9) provides

$$(\mathcal{L}_V \eta')V_1' - g(V_1', \mathcal{L}_V \xi') = -\{2\lambda' + 2\mu' - q(r + 4n^2)\}\eta'(V_1'), \tag{2.5.2}$$

for arbitrary given vector field V_1' on M. From (1.1.26) we get $R(V_1', \xi')\xi' = \eta'(V_1')\xi' - V_1'$. Taking Lie derivative along the potential vector field V and using (2.5.1) and (2.5.2), this becomes

$$(\mathcal{L}_V R)(V_1', \xi')\xi' = \{2\lambda' + 2\mu' - q(r + 4n^2)\}(V_1' - \eta'(V_1')\xi'), \tag{2.5.3}$$

for all $V_1' \in \chi(M)$. Then from (2.2.15), we get

$$S(V_1', V_2') = ag(V_1', V_2') + b\eta'(V_1')\eta'(V_2'),$$

where $a = \frac{1}{2(\rho-1)}\{2\lambda' + 2\mu' - q(r+4n^2) - 4n(\rho-1)\}$ and $b = -\frac{1}{2(\rho-1)}\{2\lambda' + 2\mu' - q(r+4n^2)\}$, for all V_1', $V_2' \in \chi(M)$.
Thus, we complete the theorem. □

Theorem 2.5.2. *If a Kenmotsu manifold $M^{2n+1}(\phi', \xi', \eta', g)$ admits a gradient almost $*$-η'-RYS and ξ' leaves the scalar curvature r invariant, then (M, g) is an η'-Einstein manifold with constant scalar curvature $r = -2n(2n+1)$.*

Proof. The gradient of the soliton equation (1.2.105) can be expressed for any V_1' belongs to $\chi(M)$ as

$$\nabla_{V_1'} Df + \rho Q V_1' + \{\rho(2n-1) + \lambda' - \frac{q}{2}(r + 4n^2)\}V_1' + (\mu' + \rho)\eta'(V_1')\xi' = 0. \tag{2.5.4}$$

Then, by applying the expression for the Riemannian curvature tensor $R(V_1', V_2')Df = \nabla_{V_1'}\nabla_{V_2'}Df - \nabla_{V_2'}\nabla_{V_1'}Df - \nabla_{[V_1', V_2']}Df$, we obtain the desired identity

$$\begin{aligned} R(V_1', V_2')Df &= \rho(\nabla_{V_2'} Q)V_1' - \rho(\nabla_{V_1'} Q)V_2' + V_2'(\sigma)V_1' - V_1'(\sigma)V_2' \\ &+ V_2'(\mu')\eta'(V_1')\xi' - V_1'(\mu')\eta'(V_2')\xi' + (\mu' + \rho)\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\}, \end{aligned} \tag{2.5.5}$$

for all V_1', $V_2' \in \chi(M)$ and $\sigma = \rho(2n+1) + \lambda' - \frac{q}{2}(r + 4n^2)$, a smooth function as λ' is a smooth function. Now putting $V_2' = \xi'$ in (2.5.5) and using (2.2.6) and (2.2.7), we get

$$\begin{aligned} R(V_1', \xi')Df &= -\rho Q V_1' - 2n\rho V_1' + \xi'(\sigma)V_1' - V_1'(\sigma)\xi' + \xi'(\mu')\eta'(V_1')\xi' \\ &\quad - V_1'(\mu')\xi' + (\mu' + \rho)(V_1' - \eta'(V_1')\xi'), \end{aligned} \tag{2.5.6}$$

for any $V_1' \in \chi(M)$. By virtue of (1.1.26), equation (2.5.6) reduces to

$$\begin{aligned} V_1'(\sigma + \mu' + f)\xi' &= -\rho Q V_1' + \{\xi'(\sigma + f) + \mu' + \rho - 2\rho n\} V_1' \\ &\quad + \{\xi'(\mu') - \mu' - \rho\}\eta'(V_1')\xi', \end{aligned} \tag{2.5.7}$$

for any $V_1' \in \chi(M)$. Now, we take an inner product of (2.5.7) with ξ' and applying (1.1.26), we get $V_1'(\sigma + \mu' + f) = \xi'(\sigma + \mu' + f)\eta'(V_1')$. Substituting this into (2.5.7), we obtain

$$QV_1' = \frac{1}{\rho}\{\xi'(\sigma + f) + \mu' + \rho - 2\rho n\}V_1' - \frac{1}{\rho}\{\xi'(\sigma + f) + \mu' + \rho\}\eta'(V_1')\xi', \tag{2.5.8}$$

for any $V_1' \in \chi(M)$. It can be deduced that the manifold (M, g) is an η'-Einstein manifold. Now contracting (2.5.5) over V_1' with respect to an orthonormal basis $\{e_i\}$, $1 \leq i \leq 2n+1$, we compute

$$\begin{aligned} S(V_2', Df) &= -\rho \sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)V_2', e_i) + V_2'(r) + 2nV_2'(\sigma) \\ &\quad + V_2'(\mu') - \eta'(V_2')\xi'(\mu') + 2n(\mu' + \rho)\eta'(V_2'). \end{aligned} \tag{2.5.9}$$

Now, using the formula for the Riemannian manifold which is well known:

$$\begin{aligned} trac_g\{V_1' \to (\nabla_{V_1'} Q)V_2'\} &= \frac{1}{2}V_2'(r) \\ \sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)V_2', e_i) &= 1. \end{aligned} \tag{2.5.10}$$

Then from (2.5.9) and (2.5.10), we get

$$S(V_2', Df) = \frac{1}{2}V_2'(r) + 2nV_2'(\sigma) + V_2'(\mu') - \eta'(V_2')\xi'(\mu') + 2n(\mu' + \rho)\eta'(V_2'), \tag{2.5.11}$$

for any $V_1' \in \chi(M)$. From (1.1.26), we can evaluate $S(\xi', Df) = -2n\xi'(f)$, putting this into (2.5.11) to compute $\xi'(r) = -4n\{\xi'(\sigma + f) + \mu' + \rho\}$. Applying this result in the trace of (2.2.7), we get $\xi'(\sigma + f) = (2n + 1) - \mu' - \rho + \frac{r}{2n}$. Using this result, equation (2.5.8) becomes

$$QV_1' = \frac{1}{2n\rho}\{r + 4n^2 + +2n - 4n^2\rho\}V_1' - \frac{1}{2n\rho}\{r + 4n^2 + 2n\}\eta'(V_1')\xi', \tag{2.5.12}$$

for any $V_1' \in \chi(M)$. According to our assumption, $\xi'(r) = 0$, the trace of (2.2.7) gives $r = -2n(2n + 1)$. Therefore, from (2.5.12) Therefore, the result is established. $\square$

In the next part, we consider a Kenmotsu metric as an almost $*$-η'-RYS, where the non-zero potential vector field V is pointwise collinear with the Reeb vector field ξ'. We extend the previous result in the form of the following theorem.

Theorem 2.5.3. *If a Kenmotsu manifold $M^{2n+1}(\phi', \xi', \eta', g)$ admits an almost $*$-η'-RYS with non-zero potential vector field V that is collinear to the Reeb vector field ξ', then manifold is η'-Eistein. Moreover, if ξ' leaves the scalar curvature r invariant, then (M, g) is an Einstein manifold with $\tau + \lambda' = \rho + \frac{q(r+4n^2)}{2}$.*

Proof. It follows from the fact that $V = \tau\xi'$, for some smooth function τ on M, it shows that

$$(\mathcal{L}_V g)(V_1', V_2') = V_1'(r)\eta'(V_2') + V_2'(r)\eta'(V_1') + 2\tau\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}, \qquad (2.5.13)$$

for any given vector fields V_1' and $V_2' \in \chi(M)$. In the presence of this condition the soliton equation (1.2.104) transforms into

$$\begin{aligned} &2\rho S(V_1', V_2') + V_1'(\tau)\eta'(V_2') + V_2'(\tau)\eta'(V_1') + \{2\lambda' - q(r + 4n^2) \\ &+2\rho(2n-1) + 2\lambda'\}g(V_1', V_2') = 2(\tau - \mu' - \rho)\eta'(V_1')\eta'(V_2'), \qquad (2.5.14)\end{aligned}$$

for any given vector field V_1', $V_2' \in \chi(M)$. Now putting $V_1' = \xi'$ and $V_2' = \xi'$ in (2.5.14) and using (1.1.27), we obtain $\xi'(\tau) = \frac{q}{2}(r+4n^2) - \mu' - \lambda'$. Thus putting in (2.5.14) yields $V_1'(\tau) = \{\frac{q}{2}(r+4n^2) - \mu' - \lambda'\}\eta'(V_1')$, similarly $V_2'(\tau) = \{\frac{q}{2}(r+4n^2) - \mu' - \lambda'\}\eta'(V_2')$. Using these two values, (2.5.14) implies that

$$\begin{aligned} S(V_1', V_2') &= \frac{1}{\rho}\{\frac{q}{2}(r + 4n^2) - \lambda' - \rho(2n-1) - \tau\}g(V_1', V_2') \\ &+ \frac{1}{\rho}\{\tau + \lambda' - \rho - \frac{q}{2}(r + 4n^2)\}\eta'(V_1')\eta'(V_2'). \qquad (2.5.15)\end{aligned}$$

Hence, (2.5.15) reduces that (M, g) is η'-Einstein manifold. Also, if ξ' leaves the scalar curvature r invariant, i.e., $\xi'(r) = 0$, again tracing (2.2.7) gives $r = -2n(2n+1)$. Applying this in the trace of (2.5.15) yields $\tau + \lambda' = \rho + \frac{q(r+4n^2)}{2}$. By (2.5.15), $S(V_1', V_2') = -2ng(V_1', V_2')$. Thus (M, g) is an Einstein manifold, which completes the proof. □

Theorem 2.5.4. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Kenmotsu manifold satisfies a gradient almost $*$-η'-RYS. Then either M is an η'-Einstein when $r = -\frac{1}{\rho}\{4n(1-\rho) + 2n\rho\} - 4n^2$ or there exists an open set where the potential vector field V is pointwise collinear with ξ' when $r \neq -\frac{1}{\rho}\{4n(1-\rho) + 2n\rho\} - 4n^2$.*

Proof. In view of (2.2.8) and from the definition of gradient almost $*$-η'-RYS given by the equation (1.2.105), we get

$$\nabla_{V_1'} Df = -\rho Q V_1' - \{\lambda' - \frac{q}{2}(r + 4n^2) + \rho(2n-1)\}V_1' - (\mu' + \rho)\eta'(V_1')\xi', \qquad (2.5.16)$$

for any given vector field V_1' on M. We take a covariant derivative with the arbitrary vector field V_2' and then using (1.1.24) and (1.1.25) we have

$$\begin{aligned} \nabla_{V_2'}\nabla_{V_1'} Df &= -\rho(\nabla_{V_2'}Q)V_1' - \rho Q(\nabla_{V_2'}V_1') - V_2'(\sigma)V_1' - \sigma(\nabla_{V_2'}V_1') \\ &\quad -V_2'(\mu')\eta'(V_1')\xi' - (\mu'+\rho)\{g(V_1', V_2')\xi' - 2\eta'(V_1')\eta'(V_2')\xi' \\ &\quad +\eta'(V_1')V_2' + \eta'(\nabla_{V_2'}V_1')\xi'\}, \end{aligned} \qquad (2.5.17)$$

where $\sigma = \lambda' - \frac{q}{2}(r + 4n^2) + \rho(2n-1)$. Now, we interchange V_1' and V_2' in the previous equation to find

$$\begin{aligned} \nabla_{V_1'}\nabla_{V_2'} Df &= -\rho(\nabla_{V_1'}Q)V_2' - \rho Q(\nabla_{V_1'}V_2') - V_1'(\sigma)V_2' - \sigma(\nabla_{V_1'}V_2') \\ &\quad -V_1'(\mu')\eta'(V_2')\xi' - (\mu'+\rho)\{g(V_1', V_2')\xi' - 2\eta'(V_1')\eta'(V_2')\xi' \\ &\quad +\eta'(V_2')V_1' + \eta'(\nabla_{V_1'}V_2')\xi'\}. \end{aligned} \qquad (2.5.18)$$

Then, applying the expression for the Riemannian curvature tensor in equation

$$R(V_1', V_2')Df = \nabla_{V_1'}\nabla_{V_2'}Df - \nabla_{V_2'}\nabla_{V_1'}Df - \nabla_{[V_1', V_2']Df},$$

we obtain the following result

$$\begin{aligned} R(V_1', V_2')Df &= \rho(\nabla_{V_2'}Q)V_1' - \rho(\nabla_{V_1'}Q)V_2' + V_2'(\sigma)V_1' - V_1'(\sigma)V_2' + V_2'(\mu')\eta'(V_1')\xi' \\ &\quad -V_1'(\mu')\eta'(V_2')\xi' - (\mu'+\rho)\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\}. \end{aligned} \qquad (2.5.19)$$

Then we take an inner product w.r.t ξ' and use of (2.2.6) and (2.2.7) to yield

$$g(R(V_1', V_2')Df, \xi') = V_2'(\sigma)\eta'(V_1') - V_1'(\sigma)\eta'(V_2') + V_2'(\mu')\eta'(V_1') - V_1'(\mu')\eta'(V_2'), \quad (2.5.20)$$

for any vector fields V_1' and $V_2' \in \chi(M)$. Moreover taking inner product of (1.1.26) with the potential vector field Df provides

$$g(R(V_1', V_2')\xi', Df) = \eta'(V_1')g(V_2', Df) - \eta'(V_2')g(V_1', Df), \qquad (2.5.21)$$

for arbitrary vector fields V_1' and V_2' on M. Comparing (2.5.20) and (2.5.21) and putting $V_2' = \xi'$ in (2.5.19), we get $V_1'(\sigma + f + \mu') = \xi'(f + \sigma + \mu')\eta'(V_1')$, where $\sigma = \lambda' - \frac{q}{2}(r + 4n^2) + \rho(2n-1)$. From this we get

$$d(\sigma + f + \mu') = \xi'(f + \sigma + \mu')\eta'. \qquad (2.5.22)$$

This shows that $\sigma + f + \mu'$ is invariant along the distribution $Ker(\eta')$ that means if $V_1' \in \chi(M)$ then $V_1'(\sigma + f + \mu') = d(\sigma + f + \mu')V_1' = 0$.

Now, we taking the inner product with respect to arbitrary vector field V_3' after putting $V_1' = \xi'$ in (2.5.19) and using (2.2.6) and (2.2.7), we get

$$\begin{aligned} g(R(\xi', V_2')Df, V_3') &= \rho S(V_2', V_3') + \{2n\rho - \xi'(\sigma) + \mu' + \rho\}g(V_2', V_3') \\ &+ V_2'(\sigma + \mu')\eta'(V_3') - \{\xi'(\mu') + \mu' + \rho\}\eta'(V_2')\eta'(V_3'). \quad (2.5.23) \end{aligned}$$

Also, from (1.1.26), we can easily deduce for arbitrary vector fields V_2' and V_3' on M

$$g(R(\xi', V_2')Df, V_3') = \xi'(f)g(V_2', V_3') - V_2'(f)\eta'(V_3'). \quad (2.5.24)$$

Comparing the equations (2.5.23) and (2.5.24) and applying (2.5.22), we can write

$$\begin{aligned} S(V_2', V_3') &= \frac{1}{\rho}\{\xi'(\sigma + f) - 2n\rho - \mu' - \rho\}g(V_2', V_3') \\ &+\frac{1}{\rho}\{\mu' + \rho - \xi'(\sigma + f)\}\eta'(V_2')\eta'(V_3'). \quad (2.5.25) \end{aligned}$$

As the equation (2.5.25) holds for arbitrary vector fields V_2' and V_3', so the manifold is an η'-Einstein. Now contracting (2.5.25), we get

$$\xi'(f + \sigma) = \frac{\rho r}{2n} + 2n\rho + \mu' + 2\rho. \quad (2.5.26)$$

Putting this value in (2.5.25), we acquire

$$S(V_2', V_3') = \tfrac{1}{2n}(r + 2n)g(V_2', V_3') - \tfrac{1}{2n}(r + 4n^2 + 2n)\eta'(V_2')\eta'(V_3'),$$

this shows that for arbitrary vector fields V_2' and V_3' on M which is same as (1.1.30). Now contracting (2.5.19) with respect to V_1' reduces to

$$S(V_2', Df) = \frac{\rho}{2}V_2'(r) + 2nV_2'(\sigma) - 2n(\mu' + \rho)\eta'(V_2'), \quad (2.5.27)$$

which is satisfied for any $V_2' \in \chi(M)$. Now, taking into with (1.1.30), we compute

$$\begin{aligned} &(r + 2n)V_2'(f) - (r + 2n + 4n^2)\eta'(V_2')\xi'(f) - n\rho V_2'(r) \\ &-4n^2V_2'(\sigma) - 4n^2(\mu' + \rho)\eta'(V_2') = 0, \quad (2.5.28) \end{aligned}$$

for all $V_2' \in \chi(M)$. Now, putting $V_2' = \xi'$ and then from (2.5.26), we can easily find the relation

$$\xi'(r) = -2\rho(r + 2n + 4n^2). \quad (2.5.29)$$

As $d^2 = 0$ and $d\eta' = 0$ from (2.5.22) we have $dr \wedge \eta' = 0$ that is, $dr(V_1')\eta'(V_2') - dr(V_2')\eta'(V_1') = 0$ for arbitrary V_1', $V_2' \in \chi(M)$. After substituting $V_2' = \xi'$ and then using (2.5.29) this becomes $V_1'(r) = -2\rho(r + 2n + 4n^2)\xi'$. Since V_1' is an arbitrary vector field so, we can say that

$$Dr = -2\rho(r + 2n + 4n^2)\xi'. \tag{2.5.30}$$

Let V_2' be a vector field of the distribution $Ker(\eta')$. Then (2.5.28) provides

$$\rho(r + 2n)V_2'(f) - 4n^2 V_2'(\sigma) = 0.$$

Now using (2.5.22) and (2.5.26) we obtain $\{\rho(r + 2n) + 4n^2\}V_2'(f) = 0$, from this and using the relation $r^* = r + 4n^2$ we can write

$$\{(r + 4n^2)\rho + 4n(1 - \rho) + 2n\rho\}(Df - \xi'(f)\xi') = 0.$$

If $r = -\frac{1}{\rho}\{4n(1 - \rho) + 2n\rho\} - 4n^2$, then from (1.1.30) we obtain that the manifold is η'-Einstein.

If $r \neq -\frac{1}{\rho}\{4n(1 - \rho) + 2n\rho\} - 4n^2$, on some open set Q of M, then $Df = \xi'(f)\xi'$ on that open set that is, the potential vector field is pointwise collinear with the characteristic vector field ξ'. This completes the proof. □

Example 2.5.1. *Let us consider the set* $M = \{(v_1', v_2', v_3', u, v) \in \mathbb{R}^5\}$ *as our manifold where* (v_1', v_2', v_3', u, v) *are the standard coordinates in* $\mathbb{R}^5$. *The vector fields defined below:*

$$e'_1 = e'^{-v}\frac{\partial}{\partial v_1'}, \quad e'_2 = e'^{-v}\frac{\partial}{\partial v_2'}, \quad e'_3 = e'^{-v}\frac{\partial}{\partial v_3'}, \quad e'_4 = e'^{-v}\frac{\partial}{\partial u}, \quad e'_5 = \frac{\partial}{\partial v}$$

are linearly independent at each point of M. *We define the metric* g *as*

$$g(e'_i, e'_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 3, 4, 5\} \\ 0, & \text{otherwise.} \end{cases}$$

Let η' *be a 1-form defined by* $\eta'(V_1') = g(V_1', e'_5)$, *for arbitrary* $V_1' \in \chi(M)$. *Let us define (1,1)-tensor field* ϕ' *as:*

$$\phi'(e'_1) = e'_3, \quad \phi'(e'_2) = e'_4, \quad \phi'(e'_3) = -e'_1, \quad \phi'(e'_4) = -e'_2, \quad \phi'(e'_5) = 0.$$

Then it satisfy the relations $\eta'(\xi') = 1$, $\phi'^2(V_1') = -V_1' + \eta'(V_1')\xi'$ *and* $g(\phi' V_1', \phi' V_2') = g(V_1', V_2') - \eta'(V_1')\eta'(V_2')$, *where* $\xi' = e'_5$ *and* V_1', V_2' *are arbitrary given vector field on* M. *So,* $(M, \phi', \xi', \eta', g)$ *defines an almost contact structure on* M.

We can now deduce that,

$$
\begin{array}{llll}
[e'_1, e'_2] = 0 & [e'_1, e'_3] = 0 & [e'_1, e'_4] = 0 & [e'_1, e'_5] = e'_1 \\
[e'_2, e'_1] = 0 & [e'_2, e'_3] = 0 & [e'_2, e'_4] = 0 & [e'_2, e'_5] = e'_2 \\
[e'_3, e'_1] = 0 & [e'_3, e'_2] = 0 & [e'_3, e'_4] = 0 & [e'_3, e'_5] = e'_3 \\
[e'_4, e'_1] = 0 & [e'_4, e'_2] = 0 & [e'_4, e'_3] = 0 & [e'_4, e'_5] = e'_4 \\
[e'_5, e'_1] = -e'_1 & [e'_5, e'_2] = -e'_2 & [e'_5, e'_3] = -e'_3 & [e'_5, e'_4] = -e'_4.
\end{array}
$$

Let ∇ *be the Levi-Civita connection of* g. *Then from* $Koszul'sformula$ *for arbitrary* $V'_1, V'_2, V'_3 \in \chi(M)$ *given by:*

$$
\begin{aligned}
2g(\nabla_{V'_1} V'_2, V'_3) &= V'_1 g(V'_2, V'_3) + V'_2 g(V'_3, V'_1) - V'_3 g(V'_1, V'_2) - g(V'_1, [V'_2, V'_3]) \\
&- g(V'_2, [V'_1, V'_3]) + g(V'_3, [V'_1, V'_2]),
\end{aligned}
$$

we can have:

$$
\begin{array}{lllll}
\nabla_{e'_1} e'_1 = -e'_5 & \nabla_{e'_1} e'_2 = 0 & \nabla_{e'_1} e'_3 = 0 & \nabla_{e'_1} e'_4 = 0 & \nabla_{e'_1} e'_5 = e'_1 \\
\nabla_{e'_2} e'_1 = 0 & \nabla_{e'_2} e'_2 = -e'_5 & \nabla_{e'_2} e'_3 = 0 & \nabla_{e'_2} e'_4 = 0 & \nabla_{e'_2} e'_5 = e'_2 \\
\nabla_{e'_3} e'_1 = 0 & \nabla_{e'_3} e'_2 = 0 & \nabla_{e'_3} e'_3 = -e'_5 & \nabla_{e'_3} e'_4 = 0 & \nabla_{e'_3} e'_5 = e'_3 \\
\nabla_{e'_4} e'_1 = 0 & \nabla_{e'_4} e'_2 = 0 & \nabla_{e'_4} e'_3 = 0 & \nabla_{e'_4} e'_4 = -e'_5 & \nabla_{e'_4} e'_5 = e'_4 \\
\nabla_{e'_5} e'_1 = 0 & \nabla_{e'_5} e'_2 = 0 & \nabla_{e'_5} e'_3 = 0 & \nabla_{e'_5} e'_4 = 0 & \nabla_{e'_5} e'_5 = 0.
\end{array}
$$

Therefore $(\nabla_{V'_1} \phi') V'_2 = g(\phi' V'_1, V'_2)\xi' - \eta'(V'_2)\phi' V'_1$ *is satisfied for arbitrary* $V'_1, V'_2 \in \chi(M)$. *So* $(M, \phi', \xi', \eta', g)$ *becomes a Kenmotsu manifold.*
The non-vanishing components of curvature tensor are:

$$
\begin{array}{lll}
R(e'_1, e'_2)e'_2 = -e'_1 & R(e'_1, e'_3)e'_3 = -e'_1 & R(e'_1, e'_4)e'_4 = -e'_1 \\
R(e'_1, e'_5)e'_5 = -e'_1 & R(e'_1, e'_2)e'_1 = e'_2 & R(e'_1, e'_3)e'_1 = e'_3 \\
R(e'_1, e'_4)e'_1 = e'_4 & R(e'_1, e'_5)e'_1 = e'_5 & R(e'_2, e'_3)e'_2 = e'_3 \\
R(e'_2, e'_4)e'_2 = e'_4 & R(e'_2, e'_5)e'_2 = e'_5 & R(e'_2, e'_3)e'_3 = -e'_2 \\
R(e'_2, e'_4)e'_4 = -e'_2 & R(e'_2, e'_5)e'_5 = -e'_2 & R(e'_3, e'_4)e'_3 = e'_4 \\
R(e'_3, e'_5)e'_3 = e'_5 & R(e'_3, e'_4)e'_4 = -e'_3 & R(e'_4, e'_5)e'_4 = e'_5 \\
R(e'_5, e'_3)e'_5 = e'_3 & R(e'_5, e'_4)e'_5 = e'_4. &
\end{array}
$$

Now from the above results we have, $S(e'_i, e'_i) = -4$ *for* $i = 1, 2, 3, 4, 5$ *and*

$$S(V_1', V_2') = -4g(V_1', V_2') \quad \forall V_1', V_2' \in \chi(M). \tag{2.5.31}$$

Contracting this we have $r = \sum_{i=1}^{5} S(e'_i, e'_i) = -20 = -2n(2n+1)$ *where dimension of the manifold* $2n + 1 = 5$*. Also, we have*

$$S^*(e'_i, e'_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4 \\ 0, & \text{if } i = 5. \end{cases}$$

and $r^* = r + 4n^2 = -20 + 16 = -4$*. So*

$$S^*(V_1', V_2') = -g(V_1', V_2') + \eta'(V_1')\eta'(V_2') \quad \forall V_1', V_2' \in \chi(M). \tag{2.5.32}$$

Now, we consider a vector field V *as*

$$V = v_1' \frac{\partial}{\partial v_1'} + v_2' \frac{\partial}{\partial v_2'} + v_3' \frac{\partial}{\partial v_3'} + u \frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{2.5.33}$$

Then from the above results we can justify that

$$(\mathcal{L}_V g)(V_1', V_2') = 4\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}, \tag{2.5.34}$$

which holds for all $V_1', V_2' \in \chi(M)$*. From (2.5.32) and (2.5.34), we can conclude that* g *represents a* $*$*-*η'*-RYS i.e., it satisfies (1.2.104) for potential vector field* V *defined by (2.5.33) for* $\lambda' = \rho - 2q - 2$ *and* $\mu' = 2$*. Here, we see that* $\xi' r = 0$ *implies* $\xi' r^* = 0$ *i.e.,* ξ' *leaves the scalar curvature invariant and from the identity* (2.5.32)*, we get that the manifold becomes* $*$*-*η'*-Einstein and* $r^* = -4 = -2n$ *i.e.,* $r = -20 = -2n(2n+1)$*. Therefore, Theorem* (2.5.2) *is verified from this example.*

3

Some solitons on Para-contact geometry

3.1 Introduction

This chapter consists of six sections. Fist two sections contain introduction and preliminaries, respectively.

In the third section, we study δ-Ricci-Yamabe almost solitons within the framework of paracontact metric manifolds. Specifically, we consider both δ-Ricci-Yamabe almost solitons and gradient δ-Ricci-Yamabe almost solitons on K-paracontact and para-Sasakian manifolds. We prove that if a K-paracontact metric g represents a δ-Ricci-Yamabe almost soliton with a non-zero potential vector field V parallel to the Reeb vector field ξ', then g is Einstein with Einstein constant $-2n$. Later we also demonstrate that if a para-Sasakian manifold admits a gradient δ-Ricci-Yamabe almost soliton, then there does not exist such manifold. We also demonstrate δ-Ricci-Yamabe almost soliton on (κ, μ')-paracontact manifold.

In the later section we initiate is to study almost $*$-Ricci-Bourguignon soliton on paracontact geometry. It is shown that if the metric g of η'-Einstein para-Kenmotsu manifold (dim > 3) is almost $*$-Ricci-Bourguinon soliton, then M^{2n+1} is Einstein. Later, if g represents a gradient almost $*$-Ricci-Bourguinon soliton on a $(2n+1)$-dimensional η'-Einstein para-Kenmotsu manifold then M^{2n+1} is either Einstein or there exists a vector field V is pointwise collinear with Reeb vector field ξ'. Next, if a three-dimensional para-Kenmotsu manifold admits a $*$-Ricci-Bourguinon soliton, then it is of constant curvature -1. Finally, we prove that if the para-Sasakian metric is a $*$-Ricci Bourguinon soliton on a manifold, then M^{2n+1} is either $\mathcal{D}$-homothetic to an Einstein manifold, or the Ricci tensor of M^{2n+1} with respect to the canonical paracontact connection vanishes.

3.2 Preliminaries

Definitions and some properties of para-contact structure manifold, para-Sasakian manifold, para-Kenmotsu manifold and (κ, μ')-para-contact manifold are given in this section. Here we want to recall some results on these manifolds proved by some eminent mathematicians which are useful to get the results stated in this chapter. We establish some sufficient conditions under which a paracontact metric manifold admits a δ-Ricci-Yamabe almost soliton or a gradient δ-Ricci-Yamabe almost soliton is Einstein (trivial). In this article, we have proved some results in the following.

Lemma 3.2.1. *If a K-paracontact metric g is a δ-Ricci-Yamabe almost soliton, then*

$$(\mathcal{L}_V\eta')(\xi') = -\eta'(\mathcal{L}_V\xi') = \frac{1}{\delta}\{4n\alpha' - (2\lambda' - \beta' r)\}. \tag{3.2.1}$$

Proof. *In light of the identity* (1.1.85) *and the soliton equation* (1.2.106) *gives*

$$(\mathcal{L}_V g)(V_1', \xi') = \frac{1}{\delta}\{4n\alpha' - (2\lambda' - \beta' r)\}\eta'(V_1'). \tag{3.2.2}$$

Now, taking the Lie differentiation of $\eta'(V_1') = g(V_1', \xi')$ with respect to the vector field V, we achieve $(\mathcal{L}_V\eta')(V_1') - g(\mathcal{L}_V\xi', V_1') = (\mathcal{L}_V g)(V_1', \xi')$. By using (3.2.2)*, we derive*

$$(\mathcal{L}_V\eta')(V_1') - g(\mathcal{L}_V\xi', V_1') = \frac{1}{\delta}\{4n\alpha' - (2\lambda' - \beta' r)\}\eta'(V_1'). \tag{3.2.3}$$

Next, we utilize the identity $g(\xi', \xi') = 1$ and using (3.2.3) *to acquire the needful result.* □

Lemma 3.2.2. *[55] On a K-paracontact manifold $M^{2n+1}(\phi', \xi', \eta', g)$, we have*

$$(i)(\nabla_{V_1'}Q)\xi' = Q\phi' V_1' + 2n\phi' V_1',$$

$$(ii)(\nabla_{\xi'}Q)V_1' = Q\phi' V_1' - \phi' Q V_1',$$

for vector field V_1' on $M^{2n+1}(\phi', \xi', \eta', g)$.

Proposition 3.2.1. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a para-Sasakian manifold. If the metric g defines a δ-Ricci-Yamabe almost soliton with the potential vector field V, then the following relation holds:*

$$\begin{aligned}(\nabla_{\xi'}\mathcal{L}_V\nabla)(\xi', \xi') &= (\beta' r - 2\lambda' + 4n\alpha')\eta'(\nabla_{\xi'}D\delta) + \beta'\{\eta'(\nabla_{\xi'}Dr) + \xi'(\xi'(r))\xi' \\ &- \xi'(r)D\delta - \nabla_{\xi'}Dr\} - 2\{\eta'(\nabla_{\xi'}D\lambda') + \xi'(\xi'(\lambda'))\xi' - \nabla_{\xi'}D\lambda' \\ &- \xi'(\lambda')D\delta\} + (2\lambda' - \beta' r - 4n\alpha')\nabla_{\xi'}D\delta.\end{aligned}$$

Proof. *We use two identities* (1.2.106) *and* (1.1.87) *to acquire*

$$g((\mathcal{L}_V\nabla)(V_3', V_1'), V_2') + g((\mathcal{L}_V\nabla)(V_3', V_2'), V_1') = -\frac{1}{\delta}[V_3'(\delta)(\mathcal{L}_V g)(V_1', V_2')$$
$$+2\alpha'(\nabla_{V_3'}S)(V_1', V_2') - \{2V_3'(\lambda') - \beta' V_3'(r)\}g(V_1', V_2')],$$

$\forall$ *vector fields* V_1', V_2', V_3' *on* M^{2n+1}. *Now interchanging cyclically the roles of* V_1', V_2' *and* V_3' *in the above relation and through a straightforward computation, we obtain*

$$\begin{aligned} g((\mathcal{L}_V\nabla)(V_1', V_2'), V_3') &= -\frac{1}{\delta}[2\alpha'\{(\nabla_{V_1'}S)(V_2', V_3') + (\nabla_{V_2'}S)(V_1', V_3') \\ &- (\nabla_{V_3'}S)(V_1', V_2')\} + V_1'(\delta)(\mathcal{L}_V g)(V_2', V_3') \\ &+ V_2'(\delta)(\mathcal{L}_V g)(V_1', V_3') - V_3'(\delta)(\mathcal{L}_V g)(V_1', V_2') \\ &+ \{2V_3'(\lambda') - \beta' V_3'(r)\}g(V_1', V_2') - \{2V_1'(\lambda') \\ &- \beta' V_1'(r)\}g(V_2', V_3') - \{2V_2'(\lambda') - \beta' V_2'(r)\}g(V_1', V_3')], \end{aligned} \quad (3.2.4)$$

$\forall$ V_1', V_2', V_3' *on* M^{2n+1}. *Now recalling the results in [90] lemma* 3.15*:*

$$\begin{aligned} (\nabla_{V_3'}S)(V_1', V_2') &= (\nabla_{V_1'}S)(V_2', V_3') - (\nabla_{\phi' V_2'}S)(\phi' V_1', V_3') \\ &- \eta'(V_1')S(V_2', V_3') - 2\eta'(V_2')S(\phi' V_1', V_3') \\ &- 2n\eta'(V_1')g(\phi' V_2', V_3') - 4n\eta'(V_2')g(\phi' V_1', V_3'). \end{aligned} \quad (3.2.5)$$

Making use of $(\nabla_{V_3'}S)(V_1', V_2') = g((\nabla_{V_3'}Q)V_1', V_2')$ *and the identity* (1.1.66),(1.1.67)*, Lemma* (3.2.2)*, we have* $\nabla_{\xi'}Q = Q\phi' - \phi' Q = 2n\eta' \otimes \xi'$ *after inserting* $V_3' = \xi'$ *into* (3.2.5). *Taking this into account, we proceed with the Lemma* (3.2.2) *and replacing* V_2' *by* ξ' *in* (3.2.5), *we can receive*

$$\begin{aligned} (\mathcal{L}_V\nabla)(V_1', \xi') &= -\frac{2\alpha'}{\delta}(2n\eta'(V_1') + 4n\phi' V_1') + \{\beta' V_1'(r) - 2V_1'(\lambda')\}\xi' \\ &- \{(2\lambda' - \beta' r)\xi' - 4n\alpha'\}V_1'(\delta) + \{\beta'\xi'(r) - 2\xi'(\lambda')\}V_1' \\ &- \{2\alpha' Q V_1' + (2\lambda' - \beta' r)V_1'\}\xi'(\delta) + \{(2\lambda' - \beta' r - 4n\alpha')D\delta \\ &+ 2D\lambda' - \beta' Dr\}\eta'(V_1'), \end{aligned} \quad (3.2.6)$$

$\forall$ V_1' *on* M. *Now, we take the covariant differentiation of* (3.2.6) *by a vector field* V_2' *on*

M^{2n+1} and aplying (1.1.83), (1.1.76), (1.1.85) *and* (1.1.82) *we obtain*

$$
\begin{aligned}
(\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1',\xi') &+ (\mathcal{L}_V\nabla)(V_1',V_2') - \eta'(V_2')(\mathcal{L}_V\nabla)(V_1',\xi') \\
&= -\frac{2\alpha'}{\delta}\{2n(\nabla_{V_2'}\eta')V_1' + 4n(\nabla_{V_2'}\phi')V_1'\} \\
&+ 2\alpha' V_2'(\delta)(2Q\phi' V_1' + 4n\phi' V_1') + \{\beta' g(V_1', \nabla_{V_2'}Dr) \\
&- 2g(V_1', \nabla_{V_2'}D\lambda')\} - \{\beta' V_1'(r) - 2V_1'(\lambda')\}\phi'^2 V_2' \\
&+ V_1'(\delta)\{2\lambda' - \beta' r - 4n\alpha'\}\phi'^2 V_2' - (2\lambda' - \beta' r - 4n\alpha') \\
&\quad g(V_1', \nabla_{V_2'}D\delta) + \{\beta' V_2'(\xi'(r)) - 2V_2'(\xi'(\lambda'))\}V_1' \\
&+ \{(2V_2'(\lambda') - \beta' V_2'(r))D\delta + (2\lambda' - \beta' r - 4n\alpha')\nabla_{V_2'}D\delta \\
&+ 2\nabla_{V_2'}D\lambda' - \beta'\nabla_{V_2'}Dr\}\eta'(V_1') + \{(2\lambda' - \beta' r - 4n\alpha')D\delta \\
&+ 2D\lambda' - \beta' Dr\}(g(V_1', V_2') - \eta'(V_1')\eta'(V_2')).
\end{aligned} \tag{3.2.7}
$$

We proceed by inserting $V_1' = \xi'$, $V_2' = \xi'$ in the following equation (3.2.7) *then using* (1.1.66), (1.1.67), (1.1.68), (1.1.83), (1.1.76) *and* (1.1.82), *so that we reach*

$$
\begin{aligned}
(\nabla_{\xi'}\mathcal{L}_V\nabla)(\xi',\xi') &= (\beta' r - 2\lambda' + 4n\alpha')\eta'(\nabla_{\xi'}D\delta) + \beta'\{\eta'(\nabla_{\xi'}Dr) + \xi'(\xi'(r))\xi' \\
&- \xi'(r)D\delta - \nabla_{\xi'}Dr\} - 2\{\eta'(\nabla_{\xi'}D\lambda') + \xi'(\xi'(\lambda'))\xi' - \nabla_{\xi'}D\lambda' \\
&- \xi'(\lambda')D\delta\} + (2\lambda' - \beta' r - 4n\alpha')\nabla_{\xi'}D\delta.
\end{aligned}
$$

Thus, the result follows. □

Lemma 3.2.3. *(see [13]). In any (κ,μ')-paracontact manifold $M^{2n+1}(\phi',\xi',\eta',g)$, the Ricci operator Q of M can be written as*

$$
\begin{aligned}
QV_1' &= [2(1-n) + n\mu']V_1' + [2(n-1) + \mu']hV_1' \\
&+ [2(n-1) + n(2\kappa - \mu')]\eta'(V_1')\xi', \qquad \kappa > -1
\end{aligned} \tag{3.2.8}
$$

for given vector field V_1' on M^{2n+1}. Moreover, the scalar curvature of M is $2n(2(1-n) + \kappa + n\mu')$.

Lemma 3.2.4. *(see [13]). On a (κ,μ')-paracontact manifold $M^{2n+1}(\phi',\xi',\eta',g)$, we have*

$$
(\nabla_{\xi'}h)V_1' = -\mu'\phi' hV_1', \tag{3.2.9}
$$

for any given vector field V_1' in M^{2n+1}.

Lemma 3.2.5. *If a (κ, μ')-paracontact manifold of dimension $(2n+1)$ with $\kappa > -1$ admits a gradient δ-Ricci-Yamabe almost soliton, then*

$$\kappa(2-\mu') = \mu'(n+1). \tag{3.2.10}$$

Proof. *First, we take the covariant derivative of* (2.2.4) *along an arbitrary vector field V_1' on M and conducting* (1.1.73) *it deduces that*

$$(\nabla_{V_1'}Q)\xi' = Q(\phi' - \phi' h)V_1' - 2n\kappa(\phi' - \phi' h)V_1'. \tag{3.2.11}$$

Now, from (1.2.107) *we can exhibited as*

$$\delta\nabla_{V_2'}Du + \alpha' QV_2' + (\lambda' - \frac{\beta' r}{2})g = 0, \tag{3.2.12}$$

$\forall$ vector fields V_2' on M^{2n+1}. Using the preceding equation in the standard expression for the Riemannian curvature tensor $R(V_1', V_2') = [\nabla_{V_1'}, \nabla_{V_2'}] - \nabla_{[V_1', V_2']}$, we obtain

$$\delta R(V_1', V_2')Du = \alpha'\{(\nabla_{V_2'}Q)V_1' - (\nabla_{V_2'}Q)V_1'\}, \tag{3.2.13}$$

$\forall$ V_1' and V_2' on M. Using this approach, the scalar product of (3.2.13) *with ξ' and through the application of* (2.2.3) *and* (3.2.11) *it follows that*

$$\begin{aligned} \delta g(R(V_1', V_2')Du, \xi') &= \alpha'\{g((Q\phi' + \phi' Q)V_2', V_1') \\ &- g((Q\phi' h + h\phi' Q)V_2', V_1') - 4n\kappa g(\phi' V_2', V_1')\}. \end{aligned} \tag{3.2.14}$$

Replacing V_1' by $\phi' V_1'$ and V_2' by $\phi' V_2'$ in (3.2.14) *and noting that $R(\phi' V_1', \phi' V_2')\xi' = 0$ (from* (3.4.1)*) and* (1.1.66), *we attained*

$$Q\phi' V_1' + \phi' QV_1' + \phi' QhV_1' + hQ\phi' V_1' - 4n\kappa\phi' V_1' = 0. \tag{3.2.15}$$

Now, we put $V_1' = \phi' V_1'$ into (3.2.8) *and using the fact $\phi'\xi' = 0$ to obtain*

$$Q\phi' V_1' = [2(1-n) + n\mu']\phi' V_1' + [2(n-1) + \mu']h\phi' V_1'.$$

Elsewhere, by acting h on the last equation and making conduct of (1.1.65), (1.1.66), (**??**) *and $h\xi' = 0$ leaves*

$$hQ\phi' V_1' = [2(1-n) + n\mu']h\phi' V_1' + (\kappa+1)[2(n-1) + \mu']\phi' V_1'.$$

In addition, operating ϕ' on (3.2.8) *and using $\phi'\xi' = 0$, we get*

$$\phi' Q V_1' = [2(1-n) + n\mu']\phi' V_1' + [2(n-1) + \mu']\phi' h V_1'.$$

We now substitute V_1' *by* hV_1' *into the foregoing equation and using* (2.2.3) *to yield*

$$\phi' Q h V_1' = [2(1-n) + n\mu']\phi' h V_1' + (\kappa+1)[2(n-1) + \mu']\phi' V_1'.$$

Conducting the last four equation in (3.2.13) *and also using* $\phi' h = -h\phi'$ *we obtain* (3.2.10). *This concludes the proof.* □

Lemma 3.2.6. *In [80] a* $*$*-Ricci tensor on a* $(2n+1)$*-dimensional para-Kenmotsu manifold* $M^{2n+1}(\phi', \xi', \eta', g)$ *is given by*

$$S(V_1', V_2') = -S(V_1', V_2') - (2n-1)g(V_1', V_2') - \eta'(V_1', V_2'), \tag{3.2.16}$$

for any given vector fields V_1', V_2' *on* $M^{2n+1}(\phi', \xi', \eta', g)$.

Accordingly, the associated $*$-scalar curvature is expressed as $r^* = -(r + 4n^2)$.

Lemma 3.2.7. *Let* $M^{2n+1}(\phi', \xi', \eta', g)$ *be a para-Sasakian manifold. Then [62]*

$$(i).\ (\nabla_{V_1'} Q)\xi' = Q\phi' V_1' + 2n\phi' V_1' \qquad and \qquad (ii).\ \nabla_{\xi'} Q = -\phi'.$$

Lemma 3.2.8. *In [62] the* $*$*-Ricci tensor on a* $(2n+1)$*-dimensional para-Sasakian manifold* $M^{2n+1}(\phi', \xi', \eta', g)$ *is given by*

$$S^*(V_1', V_2') = -S(V_1', V_2') - (2n-1)g(V_1', V_2') - \eta'(V_1')\eta'(V_2'), \tag{3.2.17}$$

$\forall$ *vector fields* V_1', V_2' *on* M^{2n+1}.

Now, we prove following two lemmas using above two lemmas.

Lemma 3.2.9. *For a para-Sasakian manifold, we have the following relation*

$$(\mathcal{L}_V \eta')(\xi') = -\eta'(\mathcal{L}_V \xi') = \lambda' + \rho r^*. \tag{3.2.18}$$

Proof. *By Virtue of the equation* (3.2.17) *of lemma* (3.2.8) *can be expressed as*

$$(\mathcal{L}_V g)(V_1', V_2') = 2S(V_1', V_2') + 2(2n - 1 + \lambda' + \rho r^*)g(V_1', V_2') + 2\eta'(V_1')\eta'(V_2'). \tag{3.2.19}$$

Plugging ξ' *for* V_2' *in* (3.2.19) *and using* (1.1.85), *it follows that* $(\mathcal{L}_V g)(V_1', \xi') = 2(\lambda' + \rho r^*)\eta'(V_1')$. *Taking Lie derivative of the equation* $g(V_1', \xi') = \eta'(V_1')$ *along* V *and virtue of the last equation, we gain*

$$(\mathcal{L}_V \eta')(V_1') - g(\mathcal{L}_V \xi', V_1') - 2(\lambda' + \rho r^*)\eta'(V_1') = 0. \tag{3.2.20}$$

Next, Lie derivative of $g(\xi', \xi') = 1$ *along* V *and using equation* (3.2.20) *completes the proof.* □

Lemma 3.2.10. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a $(2n+1)$-dimensional para-Sasakian manifold. If g is a $*$-Ricci-Bourguignon solitons, then M^{2n+1} is an η'-Einstein manifold and the Ricci tensor can be expressed as*

$$S(V_1', V_2') = -\{2n - 1 + \frac{\lambda' + \rho r^*}{2}\}g(V_1', V_2') + \{\frac{\lambda' + \rho r^*}{2} - 1\}\eta'(V_1')\eta'(V_2'). \qquad (3.2.21)$$

Proof. *First, taking covariant differentiation of* (3.2.19) *along an arbitrary vector field V_3', we get*

$$(\nabla_{V_3'}\mathcal{L}_V g)(V_1', V_2') = 2\{(\nabla_{V_3'}S)(V_1', V_2') - g(V_1', \phi' V_3')\eta'(V_2') - g(V_2', \phi' V_3')\eta'(V_1')\}. \quad (3.2.22)$$

Comparing (3.2.22) *and* (1.1.87), *we achieve*

$$\begin{aligned} &g((\mathcal{L}_V\nabla)(V_3', V_1'), V_2') + g((\mathcal{L}_V\nabla)(V_3', V_2'), V_1') \\ &= 2\{(\nabla_{V_3'}S)(V_1', V_2') - g(V_1', \phi' V_3')\eta'(V_2') - g(V_2', \phi' V_3')\eta'(V_1')\}. \end{aligned} \qquad (3.2.23)$$

By a straightforward combinatorial combination of (3.2.23) *gives*

$$\begin{aligned} g((\mathcal{L}_V\nabla)(V_1', V_2'), V_3') \;=\; & -(\nabla_{V_3'}S)(V_1', V_2') + (\nabla_{V_1'}S)(V_2', V_3') \\ & +(\nabla_{V_2'}S)(V_3', V_1') + 2g(V_1', \phi' V_3')\eta'(V_2') \\ & +2g(V_2', \phi' V_3')\eta'(V_1'). \end{aligned} \qquad (3.2.24)$$

We plug $V_2' = \xi'$ into (3.2.24) *and using lemma* (3.2.7) *to read*

$$(\mathcal{L}_V\nabla)(V_1', \xi') = 2(2n)\phi' V_1' + 2Q\phi' V_1'. \qquad (3.2.25)$$

Further, differentiating (3.2.25) *covariantly with respect to the vector field V_2' on M^{2n+1} and then applying the relations* (1.1.85) *and* (1.1.82), *we get*

$$\begin{aligned} &(\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', \xi') + (\mathcal{L}_V\nabla)(V_1', \phi' V_2') \\ &= 2\{(\nabla_{V_2'}Q)\phi' V_1' + \eta'(V_1')QV_2' + 2n\eta'(V_1')V_2' + g(V_1', V_2')\xi'\}. \end{aligned} \qquad (3.2.26)$$

Again, according to Yano [88] we have the following commutation formula

$$(\mathcal{L}_V R)(V_1', V_2')V_3' = (\nabla_{V_1'}\mathcal{L}_V\nabla)(V_2', V_3') - (\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', V_3'). \qquad (3.2.27)$$

Setting $V_3' = \xi'$ in (3.2.27) *and taking into account of* (3.2.26), *we obtain*

$$\begin{aligned} &(\mathcal{L}_V R)(V_1', V_2')\xi' + (\mathcal{L}_V\nabla)(V_2', \phi' V_1') - (\mathcal{L}_V\nabla)(V_1', \phi' V_2') \\ &= 2\{(\nabla_{V_1'}Q)\phi' V_2' - (\nabla_{V_2'}Q)\phi' V_1' + \eta'(V_2')QV_1' - \eta'(V_1')QV_2' \\ &+2n(\eta'(V_2')V_1' - \eta'(V_1')V_2')\}. \end{aligned} \qquad (3.2.28)$$

Taking $V_2' = \xi'$ *in* (3.2.28)*, then using* (1.1.85)*,* (3.2.25) *and lemma* (3.2.7)*, we have*

$$(\mathcal{L}_V R)(V_1', \xi')\xi' = 4\{QV_1' + 2nV_1' + \eta'(V_1')\xi'\}. \tag{3.2.29}$$

Next, putting $V_2' = \xi'$ *in* (1.1.84) *then taking Lie-derivative of the resultant along* V *and taking into account of* (1.1.84) *and* (3.2.18) *one can get*

$$(\mathcal{L}_V R)(V_1', \xi')\xi' = (\mathcal{L}_V \eta')(V_1')\xi' - g(\mathcal{L}_V V_1', \xi') - 2(\lambda' + \rho r^*)V_1'. \tag{3.2.30}$$

Comparing (3.2.29) *with* (3.2.30)*, and making use of* (3.2.20)*, we get the required result.*
□

Lemma 3.2.11. *In [62] any* $(2n+1)$*-dimensional* η'*-Einstein para-Sasakian manifold with scalar curvature not equal to* $2n$ *is* $\mathcal{D}$*-homothetic to an Einstein manifold.*

3.3 Para-contact Geometry and Gradient Almost δ-RYS

In this section, we prove the results stated regarding δ-Ricci-Yamabe almost soliton on K-paracontact and para-Sasakian manifolds and also we take into account the gradient almost δ-Ricci-Yamabe solitons on paracontact metric manifolds.

Theorem 3.3.1. *If* K*-paracontact metric* g *admits a* δ*-Ricci-Yamabe almost soliton whose non-zero potential vector field* V *is parallel to* ξ'*, then the manifold is an Einstein manifold with Einstein constant* $-2n$*. Moreover,* V *is a constant multiple of* ξ'*.*

Proof. Since the potential vector field V is parallel to ξ', i.e., $V = \sigma\xi'$ for a non-zero smooth function σ on M, then it follows that $\nabla_{V_1'} V = V_1'(\sigma)\xi' - \sigma(\phi' V_1')$ by the derivative of $V = \sigma\xi'$ covariantly by $V_1' \in \chi(M)$ and applying the identity (1.1.83). Thus, the equation (1.2.106), becomes to

$$\delta\{V_1'(\sigma)\eta'(V_2') + V_2'(\sigma)\eta'(V_1')\} + 2\alpha' S(V_1', V_2') + (2\lambda' - \beta' r)g(V_1', V_2') = 0, \tag{3.3.1}$$

for all $V_1', V_2' \in \chi(M)$. Now, we introduce $V_1' = V_2' = \xi'$ in (3.3.1) and invoking the fact (1.1.85) to yield $\xi'(\sigma) = \frac{1}{2\delta}\{4n\alpha' - (2\lambda' - \beta' r)\}$. Taking account of this, placing $V_2' = \xi'$ in (3.3.1) and recalling (1.1.85), we achieve

$$V_1'(\sigma) = \xi'(\sigma)\eta'(V_1'), \qquad V_1' \in \chi(M)$$

and therefore, it follows from (1.1.83)

$$Hess_\sigma(V_1', V_2') = V_1'(\xi'(\sigma))\eta'(V_2') - \xi'(\sigma)g(\phi'V_1', V_2'), \quad V_1', V_2' \in \chi(M). \tag{3.3.2}$$

Since $Hess_\sigma$ is symmetric and ϕ' is skew-symmetric, by (1.1.65) and (3.3.2), we get

$$\xi'(\sigma)d\eta'(V_1', V_2') = 0 \quad \forall V_1', V_2' \perp \xi',$$

as $d\eta'(V_1', V_2') = g(V_1', \phi'V_2')$. This exposes that $\xi'(\sigma) = 0$, as $d\eta'$ is a non-zero on M, hence, $\nabla\sigma = 0$. Hence, σ is constant on M. This simplifies the equation (3.3.2) to

$$2\alpha' S(V_1', V_2') = -(2\lambda' - \beta' r)g(V_1', V_2') = -4n\alpha' g(V_1', V_2'), \quad V_1', V_2' \in \chi(M),$$

using $Q\xi' = -2n\xi'$ and hence (M, g) is an Einstein with Einstein constant $-2n$. This finishes the proof. □

Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a paracontact metric manifold admitting a gradient almost δ-Ricci-Yamabe soliton. Then the soliton equation (1.2.107) can be expressed as follows:

$$\delta\nabla_{V_1'}\nabla u + \alpha' QV_1' + (\lambda' - \frac{\beta' r}{2})V_1' = 0, \tag{3.3.3}$$

for all $V_1' \in \chi(M)$ and hence, the curvature tensor obtained from (3.3.3) and (1.1.4) fulfills

$$\begin{aligned} \delta R(V_1', V_2')\nabla u &= \alpha'\{(\nabla_{V_2'}Q)V_1' - (\nabla_{V_1'}Q)V_2'\} + V_1'(\lambda')V_2' \\ &- V_2'(\lambda')V_1' - \frac{\beta'}{2}\{V_1'(r)V_2' - V_2'(r)V_1'\}. \end{aligned} \tag{3.3.4}$$

Theorem 3.3.2. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a K-paracontact manifold. If the metric g represents a gradient δ-Ricci-Yamabe almost soliton, then M^{2n+1} fulfills either*

$$(\nabla_{\xi'}Q)V_1' + 2\phi' QV_1' + 4n\phi' V_1' = 0 \tag{3.3.5}$$

or $\alpha' = 0$, that is, it reduces δ-Ricci-Yamabe almost soliton, on the condition that $\beta' = 2$.

Proof. First, we take the covariant differentiation of (1.1.85) through the vector field $V_1' \in \chi(M)$, and conducting (1.1.83) to yield

$$(\nabla_{V_1'}Q)\xi' = Q\phi' V_1' + 2n\phi' V_1'. \tag{3.3.6}$$

Since ξ' is Killing, then we have

$$\begin{aligned} 0 &= (\mathcal{L}_{\xi'}Q)V_1' \\ &= \mathcal{L}_{\xi'}(QV_1') - Q(\mathcal{L}_{\xi'}V_1') \\ &= [\xi', QV_1'] - Q([\xi', V_1']) \\ &= \nabla_{\xi'}(QV_1') - \nabla_{QV_1'}\xi' - Q(\nabla_{\xi'}V_1' - \nabla_{V_1'}\xi') \\ &= (\nabla_{\xi'}Q)V_1' - \nabla_{QV_1'}\xi' + Q(\nabla_{V_1'}\xi'). \end{aligned}$$

It ensures since (1.1.83) that $\nabla_{\xi'}Q = Q\phi' - \phi'Q$. Now, we replace V_1' with ξ' into identity (3.3.4) and then adopting the scalar product with $V_1' \in \chi(M)$ to get

$$\begin{aligned} \delta g(R(\xi', V_2')\nabla u, V_1') &= \alpha'\{g(\phi'QV_2', V_1') + 2ng(\phi'V_2', V_1') + 2n\eta'(V_1')\eta'(V_2')\} \\ &+ \{\xi'(\lambda') - \frac{\beta'}{2}\xi'(r)\}g(V_1', V_2') \\ &- \{V_2'(\lambda') - \frac{\beta'}{2}V_2'(r)\}\eta'(V_1'). \end{aligned} \tag{3.3.7}$$

Now, we utilize the identity (3.3.6) and using the equation (1.1.83) to derive

$$g((\nabla_{V_1'}\phi')V_2', V_3') - g((\nabla_{V_2'}\phi')V_1', V_3') = g(R(V_1', V_2')V_3', \xi').$$

Then by introducing the Bianchis's first identity, we achieve

$$g(R(\xi', V_3')V_2', V_1') = g((\nabla_{V_3'}\phi')V_2', V_1'), \quad V_1', V_2', V_3' \in \chi(M).$$

We now apply the above identity in (3.3.7) and $\xi'(r) = 0$ (pursues since $\nabla_{\xi'}Q = Q\phi' - \phi'Q$) to get

$$\begin{aligned} &\delta g((\nabla_{V_2'}\phi')V_1', \nabla u) + \alpha'\{g(\phi'QV_2', V_1') + 2ng(\phi'V_2', V_1') + 2n\eta'(V_1')\eta'(V_2')\} \\ &+\xi'(\lambda')g(V_1', V_2') - \{V_2'(\lambda') - \frac{\beta'}{2}V_2'(r)\}\eta'(V_1') = 0. \end{aligned} \tag{3.3.8}$$

Now setting $V_1' = \phi'V_1'$, and $V_2' = \phi'V_2'$ in (3.3.8) and eliminating (3.3.8) from obtaining expression, inflicts

$$\begin{aligned} &\delta\{g((\nabla_{\phi'V_2'}\phi')\phi'V_1', \nabla u) - g((\nabla_{V_2'}\phi')V_1', \nabla u)\} - \alpha' g(Q\phi'V_2' + \phi'QV_2', V_1') \\ &-2\xi'(\lambda')g(V_1', V_2') + V_2'(\lambda' - \frac{\beta'}{2}r)\eta'(V_1') + \xi'(\lambda')\eta'(V_1')\eta'(V_2') \\ &-4n\alpha' g(\phi'V_2', V_1') = 0, \end{aligned} \tag{3.3.9}$$

otherwise we have used (1.1.85). Next repeating the formula on paracontact metric manifold (cf. [90], lemma 2.7), we get

$$(\nabla_{\phi' V_2'}\phi')\phi' V_1' - (\nabla_{V_2'}\phi')V_1' = 2g(V_1', V_2')\xi' - \eta'(V_1')\{V_2' - hV_2' + \eta'(V_2')\xi'\}. \tag{3.3.10}$$

By the help of (3.3.9) and (3.3.10), it may be inferred that

$$\begin{aligned} &2\xi'(\delta u - \lambda')g(V_1', V_2') + V_2'(\lambda' - \frac{\beta' r}{2} - \delta u)\eta'(V_1') \\ &-\xi'(\delta u - \lambda')\eta'(V_1')\eta'(V_2') = \alpha' g(Q\phi' V_2' + \phi' Q V_2', V_1') + 4n\alpha' g(\phi' V_2', V_1'), \end{aligned} \tag{3.3.11}$$

since $h = 0$ for K-paracontact manifold. At this point, placing the value of V_2' by ξ' in (3.3.4), we gain

$$\begin{aligned} \delta R(V_1', \xi')\nabla u &= \alpha'\{(\nabla_{\xi'} Q)V_1' - (\nabla_{V_1'} Q)\xi'\} + V_1'(\lambda')\xi' \\ &- \xi'(\lambda')V_1' - \frac{\beta'}{2}\{V_1'(r)\xi' - \xi'(r)V_1'\}, \end{aligned}$$

acceptance the scalar product of the above result with ξ' and conducting (1.1.79), (3.3.6) outturn

$$V_1'(\lambda' - u\delta - \frac{\beta' r}{2}) = \xi'(\lambda' - u\delta)\eta'(V_1'), \tag{3.3.12}$$

by conducting $\nabla_{\xi'} Q = Q\phi' - \phi' Q$. Let $\sigma = \lambda' - u\delta - \frac{\beta' r}{2}$, hence the equation (3.3.12) can be constructed as for $V_1'(\sigma) = \xi'(\sigma)\eta'(V_1')$, for $V_1' \in \chi(M)$ as $\xi'(r) = 0$. In this manner by preceding argument in section 3.3, we decide that $\sigma = \lambda' - u\delta - \frac{\beta' r}{2}$ is constant on M, then using $\nabla_{\xi'} Q = Q\phi' - \phi' Q$ and follows from (3.3.11), we get

$$\alpha'\{g((\nabla_{\xi'} Q)V_2', V_1') + 2g(\phi' Q V_2', V_1') + 4ng(\phi' V_2', V_1')\} = 0,$$

which implies that $\alpha'\{g((\nabla_{\xi'} Q)V_2' + 2\phi' Q V_2' + 4n\phi' V_2', V_1')\} = 0$, then either $\alpha' = 0$ or $g((\nabla_{\xi'} Q)V_2' + 2\phi' Q V_2' + 4n\phi' V_2', V_1') = 0$, which completes the proof. □

In [33], Ghosh proved that if a K-contact manifold admits a gradient Ricci almost soliton, then it has constant scalar curvature. More recently, Patra [54] generalized this result by showing that if a K-contact manifold admits a non-trivial gradient Ricci almost soliton, then the manifold is Einstein with constant scalar curvature $2n(2n+1)$. In this work we prove the nonexistence of a para-Sasakian metric g admiting a gradient Ricci–Yamabe almost soliton under the addition assumption that the Ricci operator Q, which commutes with a paracontact metric structure ϕ'. Now, we get the following result.

Theorem 3.3.3. *A gradient δ-Ricci-Yamabe almost soliton cannot exist on any para-Sasakian manifold $M^{2n+1}(\phi', \xi', \eta', g)$.*

Proof. The Ricci operator property satisfies in a para-Sasakian manifold (see in [90] lemma 3.15)

$$QV_1' = \phi' Q\phi' V_1' - 2n\eta'(V_1')\xi', \quad V_1' \in \chi(M). \tag{3.3.13}$$

Operating (3.3.13) by ϕ' and conducting (1.1.65) and (1.1.66) we examine the behavior of the Ricci operator Q with respect to the paracontact structure ϕ'.

It is well known that a para-Sasakian structure implies a K-paracontact structure. So it holds $\nabla_{\xi'} Q = Q\phi' - \phi' Q = 2n\eta' \otimes \xi'$. Then it implies $\eta' \otimes \xi' = 0$, that inflicts a inconsistency. The result follows. □

By virtue of this formula $\nabla_{\xi'} Q = Q\phi' - \phi' Q = 2n\eta' \otimes \xi'$, Theorem (4.3.3) gives a non-existence theorem.
As every para-Sasakian manifold is always K-paracontact, so, this theorem holds for K-paracontact manifold also.

3.4 δ-Ricci-Yamabe Almost Solitons on (κ, μ')- Para-contact Manifold

This article is devoted to the study of nullity distributions on paracontact geometry is a most interesting parts in the paracontact geometry. In [13], Cappelletti-Montano et al. introduced the notion of (κ, μ')-paracontact structures. According to them a (κ, μ')-paracontact manifold is a paracontact metric manifold $M^{2n+1}(\phi', \xi', \eta', g)$ whose curvature tensor satisfies

$$R(V_1', V_2')\xi' = \kappa\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\} + \mu'\{\eta'(V_2')hV_1' - \eta'(V_1')hV_2'\}, \tag{3.4.1}$$

$\forall\ V_1', V_2' \in \chi(M)$ and for some real numbers (κ, μ'). Equivalently, this equation can be written as

$$R(V_1', \xi')V_2' = \kappa\{\eta'(V_2')V_1' - g(V_1', V_2')\xi'\} + \mu'\{\eta'(V_2')hV_1' - g(hV_1', V_2')\xi'\}, \tag{3.4.2}$$

$\forall\ V_1', V_2' \in \chi(M)$. Further, most of geometers have well-acquainted (κ, μ')-paracontact manifold and delivered many important properties on these types of manifolds (instant see in [47, 83, 61, 13]).

Theorem 3.4.1. *If a (κ, μ')-paracontact manifold of dimension $(2n+1)$ with $\kappa > -1$ admits a gradient δ-Ricci-Yamabe almost soliton, then the manifold is locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature -4.*

Proof. First, we substitute ξ' instead of V_1' into (3.2.14) and using the identity (2.2.4) and $h\xi' = \phi'\xi' = 0$ to acquire $\delta g(R(\xi', V_2')\xi', Du) = 0$. In this sight, from (2.2.3) we earned

$$\kappa\{Du - (\xi' u)\xi'\} + \mu' hDu = 0, \tag{3.4.3}$$

where we conduct $g(V_1', Du) = V_1' u$. We now apply covariant differentiation to the equation (3.4.3) by ξ' and conducting the relation (3.2.9), $\nabla_{\xi'}\xi' = 0$ to achieve

$$\kappa\{\nabla_{\xi'} Du - \xi'(\xi' u)\xi'\} + \mu'\{\mu' h\phi' Du + h(\nabla_{\xi'} Du)\} = 0. \tag{3.4.4}$$

Since the equations (3.2.8) and (3.2.12), we have

$$\delta\nabla_{\xi'} D(\lambda' - \frac{\beta' r}{2}) = (2n\kappa\alpha' + \lambda' - \frac{\beta' r}{2})\xi' \tag{3.4.5}$$

, and also

$$\delta\xi'(\xi' u) = 2n\kappa\alpha' + (\lambda' - \frac{\beta' r}{2}). \tag{3.4.6}$$

Making use of (3.4.5) and (3.4.6) in (3.4.4) and using $\phi'\xi' = 0$, we have $\mu'^2 h\phi' Du = 0$. Acting this by ϕ' and conducting (1.1.65) and (1.1.66), we get $\mu'^2 hDu = 0$. By the operation of h and the use of (1.1.66) and (2.2.3), yields

$$\mu'^2(\kappa + 1)(Du - (\xi' u)\xi') = 0.$$

For $\kappa > -1$, then either (i) $\mu' = 0$ or (ii) $\mu' \neq 0$.

Case (i). In this case, when $\kappa > -1$ it follows from (3.2.10) that $\kappa = 0$. Hence $R(V_1', V_2')\xi' = 0$ for any vector fields $V_1', V_1' \in \chi(M)$, and therefore M is the product of a flat $(n+1)$-dimensional manifold of negative constant curvature -4 (see [91] Theorem 3.3).

Case (ii). This case yields $Du = (\xi' u)\xi'$. We differentiate this along with an arbitary vector field V_1' together with (1.1.65) and (1.1.66) to acquire

$$\nabla_{V_1'} Du = V_1'(\xi' u)\xi' - (\xi' u)(\phi' V_1' - \phi' h V_1').$$

As $g(\nabla_{V_1'} D(\lambda' - \frac{\beta' r}{2}), V_2') = g(\nabla_{V_2'} D(\lambda' - \frac{\beta' r}{2}), V_1')$, the last equation provides

$$V_1'(\xi'u)\eta'(V_2') - V_2'(\xi'u)\eta'(V_1') + (\xi'u)d\eta'(V_1', V_2') = 0.$$

Replacing V_1' by $\phi'V_1'$ and V_2' by $\phi'V_2'$ and applying $\phi'\xi' = 0$, we get $\xi'u = 0$, that we conduct $d\eta' \neq 0$ on M^{2n+1}. Then, $Du = 0$, i.e., u is constant and therefore (3.2.12), (3.4.5) and (3.4.6) yields $S = (\lambda' - \frac{\beta' r}{2})g = 2n\kappa\alpha'$, i.e., M^{2n+1} is an Einstein. This gives $r = 2n\kappa\alpha'(2n+1)$. In addition Lemma 3 yields $r = 2n\{2(1-n) + \kappa\alpha' + n\mu'\}$. Combining both results, we obtain

$$n\mu' = 2(n\kappa\alpha' + n - 1). \tag{3.4.7}$$

Now, conducting (3.4.7) and $S = 2n\kappa\alpha' g$ in (3.2.8) we gain $2(n-1) + \mu' = 0$. Thus (3.2.10) outurns $\kappa = \frac{1-n^2}{n}$, a inconsistency. This finishes the proof. □

3.5 ∗-Ricci-Bourguignon soliton on Para-Kenmotsu and Para-Sasakian Manifolds

In this section, we investigate that the metric g of a $(2n+1)$-dimensional para-Kenmotsu and Para-Sasakian manifolds admitting a ∗-Ricci-Bourguignon soliton.

Theorem 3.5.1. *If the metric g of η'-Einstein para-Kenmotsu manifold of dimension > 3 is a ∗-Ricci-Bourguignon soliton, then it is Einstein manifold when $r = -2n(2n+1)$ otherwise $\mathcal{L}_V\xi' = 2\rho r^*\xi'$ (where $r^* = -(r + 4n^2)$).*

Proof. As $M^{2n+1}(\phi', \xi', \eta', g)$ is η'-Einstein, taking $V_2' = \xi'$ in (**??**) and making use of (1.1.85), we have

$$a + b = -2n. \tag{3.5.1}$$

We contract the identity (1.1.29) to yield the scalar curvature $r = (2n+1)a + b$. Combining this with (3.5.1) yields $a = (1 + \frac{r}{2n})$ and $b = -\{(2n+1) + \frac{r}{2n}\}$. Thus equation (1.1.29) takes the form

$$S(V_1', V_2') = (1 + \frac{r}{2n})(V_1', V_2') - \{(2n+1) + \frac{r}{2n}\}\eta'(V_1')\eta'(V_1'). \tag{3.5.2}$$

In view of (3.2.16), (3.5.2) and using the relation $r^* = -(r + 4n^2)$, equation (1.2.110) can be written as

$$(\mathcal{L}_V g)(V_2', V_3') = \{2(2n+\lambda') + (\frac{1}{n} - 2\rho)r - 8n^2\rho\}g(V_2', V_3') - \{\frac{r}{n} + 4n\}\eta'(V_2')\eta'(V_3'). \tag{3.5.3}$$

We differentiate (3.5.3) along an arbitrary vector field V_1' and using (1.1.76) to acquire

$$\begin{aligned}(\nabla_{V_1'}\mathcal{L}_V g)(V_2', V_3') &= (\frac{1}{n} - 2\rho)(V_1'r)g(V_1', V_2') - \frac{(V_1'r)}{n}\eta'(V_2')\eta'(V_3') \\ &- \{\frac{r}{n} + 4n\}\{g(V_1', V_2')\eta'(V_3') + g(V_1', V_3')\eta'(V_2') \\ &- 2\eta'(V_1')\eta'(V_2')\eta'(V_3')\}. \end{aligned} \tag{3.5.4}$$

Given that $\mathcal{L}_V\nabla$ is a symmetric $(1,2)$-type tensor and therefore it follows from (1.1.87) that

$$\begin{aligned} g((\mathcal{L}_V\nabla)(V_1', V_2'), V_3') &= \frac{1}{2}\{(\nabla_{V_1'}\mathcal{L}_V g)(V_2', V_3') + (\nabla_{V_2'}\mathcal{L}_V g)(V_3', V_1') \\ &+ (\nabla_{V_3'}\mathcal{L}_V g)(V_1', V_2')\}. \end{aligned} \tag{3.5.5}$$

By some computation and keeping in mind that $\mathcal{L}_V\nabla$ is a symmetric operator, the foregoing equations yields

$$\begin{aligned} 2(\mathcal{L}_V\nabla)(V_1', V_2') &= (\frac{1}{n} - 2\rho)(V_1'r)V_2' - \frac{1}{n}(V_1'r)\eta'(V_2')\xi' \\ &+ (\frac{1}{n} - 2\rho)(V_2'r)V_1' - \frac{1}{n}(V_2'r)\eta'(V_1')\xi' \\ &- (\frac{1}{n} - 2\rho)g(V_1', V_2')Dr - \frac{1}{n}\eta'(V_1')\eta'(V_2')Dr \\ &- 2(\frac{r}{n} + 4n)\{g(V_1', V_2')\xi' - \eta'(V_1')\eta'(V_2')\xi'\}, \end{aligned} \tag{3.5.6}$$

for all vector fields V_1', $V_2' \in M^{2n+1}$, where D is the gradient operator of g. We insert $V_1' = V_2' = e_i$(where $\{e_i : i = 1, 2, 3, ..., 2n+1\}$ is an orthonormal frame) into (3.5.6) and summing over i to find

$$n\sum_{i=1}^{2n+1}(\mathcal{L}_V\nabla)(e_i, e_i) = \{\rho(2n+1) - 1\}Dr - (\xi'r)\xi' - 2n(r + 4n^2)\xi'. \tag{3.5.7}$$

Now, we take the covariant differentiation of $*$-Ricci-Bourguignon soliton equation (1.2.110) along a vector field V_1' to achieve

$$(\nabla_{V_1'}\mathcal{L}_V g)(V_2', V_3') = -2(\nabla_{V_1'}S^*)(V_2', V_3') - 2\rho(V_1'r)g(V_2', V_3'),$$

where $r^* = -(r + 4n^2)$. We displace this into (3.5.5) to gain

$$\begin{aligned} g((\mathcal{L}_V\nabla)(V_1', V_2'), V_3') &= (\nabla_{V_3'}S^*)(V_1', V_2') - (\nabla_{V_1'}S^*)(V_2', V_3') \\ &\quad -(\nabla_{V_2'}S^*)(V_3', V_1') + \rho\{(V_3'r)g(V_1', V_2') \\ &\quad -(V_1'r)g(V_2', V_3') - (V_2'r)g(V_1', V_3')\}. \end{aligned} \tag{3.5.8}$$

Again, we take covariant differentiation of (3.2.16) with respect to V_3' and then using (1.1.76) to obtain

$$\begin{aligned}(\nabla_{V_3'}S^*)(V_1',V_2') &= (\nabla_{V_3'}S)(V_1',V_2') - \{g(V_3',V_1')\eta'(V_2')\\ &+ g(V_1',V_2')\eta'(V_3') - 2\eta'(V_1')\eta'(V_2')\eta'(V_3')\}.\end{aligned} \tag{3.5.9}$$

We combine two identities (3.5.8) and (3.5.9) to yield

$$\begin{aligned}g((\mathcal{L}_V\nabla)(V_1',V_2'),V_3') &= -(\nabla_{V_3'}S)(V_1',V_2') + (\nabla_{V_1'}S)(V_2',V_3')\\ &+(\nabla_{V_2'}S)(V_3',V_1') + 2\{g(V_1',V_2')\eta'(V_3')\\ &-\eta'(V_1')\eta'(V_2')\eta'(V_3')\} + \rho\{(V_3'r)g(V_1',V_2')\\ &-(V_1'r)g(V_2',V_3') - (V_2'r)g(V_1',V_3')\}.\end{aligned} \tag{3.5.10}$$

We replace V_1' and V_2' by e_i in (3.5.10) and taking the sum over i, to find

$$\sum_{i=1}^{2n+1}(\mathcal{L}_V\nabla)(e_i,e_i) = 4n\xi' + \rho(2n+1)Dr. \tag{3.5.11}$$

As a consequence of (3.5.11) and (3.5.7), it follows directly that

$$\{\rho(2n+1)(n-1)+1\}Dr + (\xi'r)\xi' + 2n\{2n(2n+1)+r\}\xi' = 0. \tag{3.5.12}$$

We take the inner product of (3.5.12) with respect to ξ' yields $(\xi'r)+2\{2n(2n+1)+r\} = 0$. Utilizing this in (3.5.12) provides $\{\rho(2n+1)(n-1)+1\}Dr = (n-1)(\xi'r)\xi'$, where as $n > 1$. Next putting $V_2' = \xi'$ in (3.5.6), it follows that

$$2n(\mathcal{L}_V\nabla)(V_1',\xi') = -2\rho n\{(V_1'r)\xi' + (\xi'r)V_1' - \eta'(V_1')Dr\} + \xi'(r)\phi'^2V_1'. \tag{3.5.13}$$

Differentiating (3.5.13) along the vector field V_2' and using (1.1.83) and (3.5.13), we get

$$\begin{aligned}&2n(\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1',\xi') + 2n(\mathcal{L}_V\nabla)(V_1',V_2') = -2\rho n\{(\nabla_{V_2'}(V_1'r))\xi'\\ &+(V_1'r)\nabla_{V_2'}\xi' + V_2'(\xi'r)V_1' + (\xi'r)\nabla_{V_2'}V_1' - (\nabla_{V_2'}\eta')V_1'Dr\\ &-\eta'(V_1')V_2'(Dr)\} + V_2'(\xi'r)\phi'^2V_1' - (\xi'r)\{g(V_1',V_2')\xi'\\ &+\eta'(V_1')V_2' - \eta'(V_2')V_1' - \eta'(V_1')\eta'(V_2')\xi'\}.\end{aligned} \tag{3.5.14}$$

Replacing V_3' by ξ' in (3.2.27) and taking into account of (3.5.14) [88], we gain

$$\begin{aligned}2n(\mathcal{L}_VR)(V_1',V_2')\xi' &= -2\rho n\{(\nabla_{V_1'}(V_2'r))\xi' - (\nabla_{V_2'}(V_1'r))\xi'\\ &+ V_1'(\xi'r)V_2' - V_2'(\xi'r)V_1' + \xi'(r)(\nabla_{V_1'}V_2' - \nabla_{V_2'}V_1')\\ &- \eta'(V_2')V_1'(Dr) + \eta'(V_1')V_2'(Dr)\} + (V_1'(\xi'r))\phi'^2V_2'\\ &- (V_2'(\xi'r))\phi'^2V_1' + 2\xi'(r)\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\}.\end{aligned} \tag{3.5.15}$$

Now, we contract the previous identity along with V_1' and making use of $\{\rho(2n+1)(n-1)+1\}Dr = (n-1)(\xi' r)\xi'$ to acquire $(\mathcal{L}_V S)(V_2', \xi') = 0$. Next, we take the Lie derivative of (1.1.85) along with V with the help of the identity (3.5.2) to obtain

$$(1+\frac{r}{2n})(V_2', \mathcal{L}_V \xi') - \{(2n+1)+\frac{r}{2n}\}\eta'(V_2')\eta'(\mathcal{L}_V \xi')$$
$$= -4n(\lambda' + \rho r^*)\eta'(V_2') - 2ng(V_2', \mathcal{L}_V \xi'). \tag{3.5.16}$$

Substituting ξ' for V_2' in (3.5.16), we get $\lambda' + \rho r^* = 0$, Furthermore, putting $V_1' = V_2' = \xi'$ in (3.5.3) gives $\eta'(\mathcal{L}_V \xi') = 2\rho r^*$, where $r^* = -(r+4n^2)$. Thus making use this results, equation (3.5.16) becomes

$$(2n+1+\frac{r}{2n})(\mathcal{L}_V \xi' - 2\rho r^* \xi') = 0. \tag{3.5.17}$$

Now, if $r = -2n(2n+1)$, then it follows from (3.5.2) that M^{2n+1} is Einstein.

Let us assume that $r \neq -2n(2n+1)$ in some open set $\mathcal{N}$, $\mathcal{L}_V \xi' = 2\rho r^* \xi'$. This together with (1.1.83) yields

$$\nabla_{\xi'} V = V - \eta'(V)\xi'.$$

Putting $V_2' = \xi'$ in (3.5.3) and using $\lambda' + \rho r^* = 0$, we have $(\mathcal{L}_V g)(V_1', \xi') = 0$. From this we have $\mathcal{L}_V \eta' = 0$. Putting $V_2' = \xi'$ in the well-known formula (see [88]).

$$(\mathcal{L}_V \nabla)(V_1', V_2') = \mathcal{L}_V \nabla_{V_1'} V_2' - \nabla_{V_1'} \mathcal{L}_V V_2' - \nabla_{[V, V_1']} V_2'$$

and by virtue of (1.1.83), (3.5.13) $\mathcal{L}_V \xi' = 2\rho r^* \xi'$ (where $r^* = -(r+4n^2)$) and $\mathcal{L}_V \eta' = 0$, we obtain $(\xi' r) = 0$. Since $\{\rho(2n+1)(n-1)+1\}Dr = (n-1)(\xi' r)\xi'$, we see that r is constant. Thus, (3.5.12) implies that $r = -2n(2n+1)$ on $\mathcal{N}$. Thus, a contradiction is obtained, and the proof is established. □

Let us now consider gradient almost $*$-Ricci-Bourguignon soliton in para-Kenmotsu manifold and prove the following theorem.

Theorem 3.5.2. *Let M^{2n+1} be a $(2n+1)$-dimensional η'-Einstein para-Kenmotsu manifold. If g represents a gradient almost $*$-Ricci-Bourguignon soliton, then either M^{2n+1} is Einstein or the potential vector field is pointwise collinear with the Reeb vector field ξ'.*

Proof. If the metric g of a η'-Einstein para-Kenmotsu manifold is a gradient almost $*$-Ricci-Bourguignon soliton, then we obtain using the identities (3.2.16) and (1.2.111)

$$\nabla_{V_1'} Df = QV_1' + (2n-1+\lambda'+\rho r^*)V_1' + \eta'(V_1')\xi', \tag{3.5.18}$$

for any vector field V_1' on M^{2n+1}. Taking covariant differentiation of (3.5.18) in the direction of an arbitrary vector field V_2' on M^{2n+1}, yields

$$\begin{aligned}\nabla_{V_2'}\nabla_{V_1'}Df &= (\nabla_{V_2'}Q)V_1' + Q(\nabla_{V_2'}V_1') + (2n-1+\lambda'+\rho r^*)\nabla_{V_2'}V_1' \\ &+(V_2'\lambda')V_1' + \rho(V_2'r^*)V_1' + (\nabla_{V_2'}\eta')(V_1')\xi' \\ &+\eta'(\nabla_{V_2'}V_1')\xi' + \eta'(V_1')\nabla_{V_2'}\xi'. \end{aligned} \tag{3.5.19}$$

Making use of (3.5.18) and (3.5.19) in the well-known result of curvature tensor $R(V_1', V_2') = \nabla_{V_1'}\nabla_{V_2'} - \nabla_{V_2'}\nabla_{V_1'} - \nabla_{[V_1',V_2']}$, we deduce that

$$\begin{aligned}R(V_1', V_2')Df &= (\nabla_{V_1'}Q)V_2' - (\nabla_{V_2'}Q)V_1' + (V_1'\lambda')V_2' - (V_2'\lambda')V_1' \\ &+ \rho(V_1'r^*)V_2' - \rho(V_2'r^*)V_1' + \eta'(V_2')V_1' - \eta'(V_1')V_2'. \end{aligned} \tag{3.5.20}$$

In view of (3.5.2), we have

$$QV_1' = (1+\frac{r}{2n})V_1' - \{(2n+1)+\frac{r}{2n}\}\eta'(V_1')\xi'. \tag{3.5.21}$$

Differentiating the foregoing Eq. along an arbitrary vector field V_2' and using (1.1.76), we get

$$(\nabla_{V_2'}Q)V_1' = \frac{1}{2n}(V_2'r)(V_1' - \eta'(V_1')\xi') - \{(2n+1)+\frac{r}{2n}\}[g(V_1', V_2')\xi' - \eta'(V_1')V_2']. \tag{3.5.22}$$

In view of (3.5.22), we get from (3.5.20)

$$\begin{aligned}R(V_1', V_2')Df &= \frac{1}{2n}(V_1'r)(V_2' - \eta'(V_2')\xi') - \frac{1}{2n}(V_2'r)(V_1' - \eta'(V_1')\xi') \\ &+ \{(2n+2)+\frac{r}{2n}\}\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\} + (V_1'\lambda')V_2' \\ &- (V_2'\lambda')V_1' + \rho\{(V_1'r^*)V_2' - (V_2'r^*)V_1'\}. \end{aligned} \tag{3.5.23}$$

By virtue of the above Eq., we can easily see that

$$g(R(V_1', V_2')Df, \xi') = (V_1'\lambda')\eta'(V_2') - (V_2'\lambda')\eta'(V_1') + \rho\{(V_1'r^*)\eta'(V_2') - (V_2'r^*)\eta'(V_1')\}. \tag{3.5.24}$$

Also, we have from (1.1.78) that

$$g(R(V_1', V_2')Df, \xi') = (V_2'f)\eta'(V_1') - (V_1'f)\eta'(V_2'). \tag{3.5.25}$$

Comparing (3.5.24) and (3.5.25) and putting $V_2' = \xi'$ in the resulting equation, we achieve

$$d(\lambda' + \rho r^* - f) = \xi'(\lambda' + \rho r^* - f)\eta', \tag{3.5.26}$$

where d is the exterior derivative. This means that $\lambda' + \rho r^* - f$ is invariant along the distribution $\mathcal{D}$ (where $\mathcal{D}$ is $ker\eta'$); that is $\lambda' + \rho r^* - f$is constant for all vector field $V_1' \in \mathcal{D}$.

Contracting (3.5.20) over V_2', we obtain

$$S(V_1', Df) = -\frac{1}{2n}(V_1'r) - 2n(V_1'\lambda') - 2n\rho(V_1'r^*) + 2n\eta'(V_1'). \tag{3.5.27}$$

In view of (3.5.21), the foregoing Equation yields

$$\begin{aligned}&(1 + \frac{r}{2n})(V_1'f) - \{(2n+1) + \frac{r}{2n}\}\eta'(V_1')(\xi'f) + \frac{1}{2n}(V_1'r)\\ &+2n(V_1'\lambda') + 2n\rho(V_1'r^*) - 2n\eta'(V_1') = 0,\end{aligned} \tag{3.5.28}$$

for all $V_1' \in TM$. Replacing $V_1' = \xi'$ in the above equation, we get,

$$2n\xi'\{\lambda' + \rho r^* - f\} + \frac{1}{2}(\xi'r) - 2n = 0. \tag{3.5.29}$$

Now, we plug $V_2' = \xi'$ into (3.5.23) and taking inner product with V_2' to yield

$$\begin{aligned} g(R(V_1', \xi')Df, V_2') &= -\frac{1}{2n}(\xi'r)\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\} + (V_1'\lambda')\eta'(V_2')\\ &+ \{(2n+2) + \frac{r}{2n}\}\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}\\ &- (\xi'\lambda')g(V_1', V_2') + \rho\{(V_1'r^*)\eta'(V_2') - (\xi'r^*)g(V_1', V_2')\}. \end{aligned} \tag{3.5.30}$$

By virtue of (1.1.81)

$$g(R(V_1', \xi')V_2', Df) = (\xi'f)g(V_1', V_2') - (V_1'f)\eta'(V_2'). \tag{3.5.31}$$

Comparing (3.5.30) with (3.5.31), we can achieve

$$\begin{aligned}&-\frac{1}{2n}(\xi'r)\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}\\ &+\{(2n+2) + \frac{r}{2n}\}\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}\\ &+V_1'(\lambda' + \rho r^* - f)\eta'(V_2') - \xi'(\lambda' + \rho r^* - f)g(V_1', V_2') = 0.\end{aligned} \tag{3.5.32}$$

Taking the trace of the above equation, we get

$$-\frac{1}{2n}(\xi'r) + (2n+2) + \frac{r}{2n} - \xi'(\lambda' + \rho r^* - f = 0. \tag{3.5.33}$$

Making use of (3.5.33) in (3.5.29), we can easily gain

$$(\xi'r) = 2\{2n(2n+1) + r\}. \tag{3.5.34}$$

By virtue of (3.5.34) and (3.5.33), from this, we conclude that

$$\xi'(\lambda' + \rho r^* - f) = -(\frac{r}{2n} + 2n). \tag{3.5.35}$$

With the help of the identity (3.5.35) into (3.5.26), we can obtain

$$d(\lambda' + \rho r^* - f) = -(\frac{r}{2n} + 2n)\eta'. \tag{3.5.36}$$

Using the Poincaré Lemma together with the condition $d\eta' = 0$, we derive the following from the previous equation, we get $-dr \wedge \eta' = 0$, and making use of (3.5.34), we have

$$Dr = \frac{2(n-1)\{r + 2n(2n+1)\}\xi'}{\rho(2n+1)(n+1)+1}. \tag{3.5.37}$$

Suppose that V_1' in (3.5.28) is orthogonal to ξ'. Keeping in mind that $(\lambda' + \rho r^* - f)$ is constant along $\mathcal{D}$ and making use of (3.5.36) and (3.5.37), then we have

$$\tfrac{(n-1)\{r+2n(2n+1)\}\xi'}{\rho(2n+1)(n+1)+1}(V_1'f) = 0,$$

for all $V_1' \in \mathcal{D}$. This implies that

$$\frac{(n-1)\{r + 2n(2n+1)\}\xi'}{\rho(2n+1)(n+1)+1}(Df - (\xi'f)\xi') = 0. \tag{3.5.38}$$

Since $n - 1 \neq 0$(as dimension > 3), we may consider two cases from (3.5.38).

Case I. Suppose $r = -2n(2n+1)$, then from the relation (3.5.21), we see that $QV_1' = -2nV_1'$, and hence M^{2n+1} is an Einstein manifold.

Case II. If $r \neq -2n(2n+1)$, then we have from (3.5.38) $Df = (\xi'r)\xi'$. This shows that potential vector field is everywhere collinear with the Reeb vector field ξ'. This concludes the proof. □

Next, we study $*$-Ricci-Bourguignon soliton in three-dimensional para-Kenmotsu manifold. So, we state the following theorem.

Theorem 3.5.3. *If the metric g of three-dimensional para-Kenmotsu manifold is a $*$-Ricci-Bourguignon soliton, consequently, the manifold is of constant curvature -1.*

Proof. It is a classical result that in a three-dimensional pseudo-Riemannian manifold, the following relation holds

$$\begin{aligned} R(V_1', V_2')V_3' &= g(V_2', V_3')QV_1' - g(V_1', V_3')QV_2' + S(V_2', V_3')V_1' \\ &\quad -S(V_1', V_3')V_2' - \frac{1}{2}[g(V_2', V_3')V_1' - g(V_1', V_3')V_2']. \end{aligned} \tag{3.5.39}$$

Now, we insert $V_1' = V_2' = \xi'$ in the above relation and making use of (1.1.78) and (1.1.77) to yield

$$QV_1' = (1 + \frac{r}{2})V_1' - (3 + \frac{r}{2})\eta'(V_1')\xi', \tag{3.5.40}$$

which is equivalent to

$$R(V_1', V_2') = (1 + \frac{r}{2})g(V_1', V_2') - (3 + \frac{r}{2})\eta'(V_1')\eta'(V_2'). \tag{3.5.41}$$

Proceeding in the similar manner as in the proof of Theorem (3.5.1). In dimension 3, that is, for $n = 1$ all equation (3.5.3)-(3.5.15) holds true. Thus, (3.5.12) becomes

$$\begin{aligned}
2(\mathcal{L}_V R)(V_1', V_2')\xi' &= -2\rho\{(\nabla_{V_1'}(V_2'r))\xi' - (\nabla_{V_2'}(V_1'r))\xi' + V_1'(\xi'r)V_2' \\
&- V_2'(\xi'r)V_1' + \xi'(r)(\nabla_{V_1'}V_2' - \nabla_{V_2'}V_1') - \eta'(V_2')V_1'(Dr) \\
&+ \eta'(V_1')V_2'(Dr)\} + V_1'(\xi'r)\phi'^2V_2' - V_2'(\xi'r)\phi'^2V_1' \\
&+ 2\xi'(r)\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\}. \qquad (3.5.42)
\end{aligned}$$

Taking Lie derivative of (1.1.78) along V and making use of (3.5.3) yields

$$\begin{aligned}
(\mathcal{L}_V R)(V_1', V_2')\xi' + R(V_1', V_2')\mathcal{L}_V\xi' &= 2(\lambda' + \rho r^*)\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\} \\
&+ g(V_1', \mathcal{L}_V\xi')V_2' - g(V_2', \mathcal{L}_V\xi')V_1'. \qquad (3.5.43)
\end{aligned}$$

In view of (3.5.42) and (3.5.43), we have

$$-2\rho\{(\nabla_{V_1'}(V_2'r))\xi' - (\nabla_{V_2'}(V_1'r))\xi' + V_1'(\xi'r)V_2' - V_2'(\xi'r)V_1' + \xi'(r)(\nabla_{V_1'}V_2' - \nabla_{V_2'}V_1') - \eta'(V_2')V_1'(Dr) + \eta'(V_1')V_2'(Dr)\} + V_1'(\xi'r)\phi'^2V_2' - V_2'(\xi'r)\phi'^2V_1' + 2\xi'(r)\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\} + 2R(V_1', V_2')\mathcal{L}_V\xi' = 4(\lambda' + \rho r^*)\{\eta'(V_1')V_2' - \eta'(V_2')V_1'\} + 2\{g(V_1', \mathcal{L}_V\xi')V_2' - g(V_2', \mathcal{L}_V\xi')V_1'\}.$$

By contracting the above equation over V_1' and using (3.5.41), we get

$$\begin{aligned}
&(r + 6)g(V_2', \mathcal{L}_V\xi') - (r + 6)\eta'(V_2')\eta'(\mathcal{L}_V\xi') \\
&= V_2'(\xi'r) + \{\xi'(\xi'r) + 4(\xi'r) - 8(\lambda' + \rho r^*)\}\eta'(V_2'). \qquad (3.5.44)
\end{aligned}$$

Setting $V_2' = \xi'$ into (3.5.44) and using (3.5.12), we have $\lambda' + \rho r^* = 0$. In view of (3.5.3), we obtain $(\mathcal{L}_V g)(V_2', \xi') = 0$, which implies $\eta'(\mathcal{L}_V\xi') = 0$. Thus, using $\lambda' + \rho r^* = 0$, $\eta'(\mathcal{L}_V\xi') = 0$ and (3.5.12), equation (3.5.44), reduces to

$$(r + 6)g(V_2', \mathcal{L}_V\xi') = -2\{(V_2'r) - (\xi'r)\eta'(V_2')\}. \tag{3.5.45}$$

Suppose that $r = -6$, then from (3.5.41), we can see that it is Einstein and $QV_1' = -2V_1'$. This together with (3.5.40) gives:

$$R(V_1', V_2')V_3' = g(V_1', V_3')V_2' - g(V_2', V_3')V_1', \tag{3.5.46}$$

showing that M^{2n+1} is of constant curvature -1. □

The last part of this paper is to study $*$-Ricci-Bourguignon soliton within the framework of para-Sasakian manifold. Here, we study canonical paracontact connection on a paracontact manifold, which has been defined by [90]. Next theorem deals with canonical paracontact connection $\widetilde{\nabla}$ on para-Sasakian manifold, that has been deeply discussed in (equation (1.1.82), (3.2.16) and (3.5.1) of [62]). We proceed to state and prove the following theorem.

Theorem 3.5.4. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a para-Sasakian manifold. If g is a $*$-Ricci-Bourguignon soliton on M^{2n+1}, then either M^{2n+1} is $\mathcal{D}$-homothetic to an Einstein manifold, or the Ricci tensor of M^{2n+1} with respect to canonical paracontact connection vanishes. In the first case, the soliton vector field is Killing and in the second case, the soliton vector field leaves ϕ' invariant.*

Proof. Making use of (3.2.21), the soliton equation (3.2.19) takes the form

$$(\mathcal{L}_V g)(V_1', V_2') = (\lambda' + \rho r^*)\{g(V_1', V_2') + \eta'(V_1')\eta'(V_2')\}. \tag{3.5.47}$$

Now, we take Lie-derivative of (3.2.21) along with the vector field V and using (3.2.19) to yield

$$\begin{aligned}(\mathcal{L}_V S)(V_1', V_2') = \frac{1}{2}(\lambda' + \rho r^* - 2)\{\eta'(V_2')(\mathcal{L}_V \eta')(V_1') + \eta'(V_1')(\mathcal{L}_V \eta')(V_2')\} \\ -\frac{1}{2}(4n - 2 + \lambda' + \rho r^*)(\lambda' + \rho r^*)\{g(V_1', V_2') + \eta'(V_1')\eta'(V_2')\}.\end{aligned} \tag{3.5.48}$$

Again, we differentiate the equation (3.2.21) covariantly along with arbitrary vector field V_3' on M^{2n+1} and then using (1.1.83) to obtain

$$(\nabla_{V_3'} S)(V_1', V_2') = \frac{1}{2}(2 - \lambda' - \rho r^*)\{g(V_1', \phi' V_3')\eta'(V_2') + g(V_2', \phi' V_3')\eta'(V_1')\}. \tag{3.5.49}$$

Now, in light of the identity (3.5.49), the equation (3.2.24), becomes

$$(\mathcal{L}_V \nabla)(V_1', V_2') = -(\lambda' + \rho r^*)\{\eta'(V_2')\phi' V_1' + \eta'(V_1')\phi' V_2'\}. \tag{3.5.50}$$

Again, we differentiate (3.5.50) covariantly along an arbitrary vector field V_3' on M^{2n+1} and using the identities (1.1.83) and (1.1.82) to acquire

$$\begin{aligned}(\nabla_{V_3'}\mathcal{L}_V\nabla)(V_1',V_2') &= (\lambda'+\rho r^*)\{g(V_2',\phi'V_3')\phi'V_1'+g(V_1',\phi'V_3')\phi'V_2'\\ &+ g(V_1',V_3')\eta'(V_2')\xi'+g(V_2',V_3')\eta'(V_1')\xi'\\ &- 2\eta'(V_1')\eta'(V_2')V_3'\}. \end{aligned}\tag{3.5.51}$$

We use the previous identity (3.5.51) into commutation formula (3.2.27) and making use of (1.1.83) to find

$$\begin{aligned}(\mathcal{L}_V R)(V_1',V_2')V_3' &= (\lambda'+\rho r^*)\{g(\phi'V_1',V_3')\phi'V_2'-g(\phi'V_2',V_3')\phi'V_1'\\ &+ 2g(\phi'V_1',V_2')\phi'V_3'+g(V_1',V_3')\eta'(V_2')\xi'-g(V_2',V_3')\eta'(V_1')\xi'\\ &- 2\eta'(V_2')\eta'(V_3')V_1'+2\eta'(V_1')\eta'(V_3')V_2'\}. \end{aligned}\tag{3.5.52}$$

We contract the foregoing equation (3.5.52) over V_3' to achieve

$$(\mathcal{L}_V S)(V_2',V_3') = 2(\lambda'+\rho r^*)\{g(V_2',V_3')-(2n+1)\eta'(V_2')\eta'(V_3')\}. \tag{3.5.53}$$

Now, we compare the two identities (3.5.48) and (3.5.53) to yield

$$\begin{aligned}&\frac{1}{2}(\lambda'+\rho r^*-2)\{\eta'(V_2')(\mathcal{L}_V\eta')(V_3')+\eta'(V_3')(\mathcal{L}_V\eta')(V_2')\}\\ &-\frac{1}{2}(4n-2+\lambda'+\rho r^*)(\lambda'+\rho r^*)\{g(V_2',V_3')+\eta'(V_2')\eta'(V_3')\}\\ &= 2(\lambda'+\rho r^*)\{g(V_2',V_3')-(2n+1)\eta'(V_2')\eta'(V_3')\}. \end{aligned}\tag{3.5.54}$$

We insert $V_2' = \phi'^2 V_2'$ into (3.5.54) and also using the identites (1.1.65), (1.1.66) and (3.2.18) to get

$$\begin{aligned}\frac{1}{2}(\lambda'+\rho r^*-2)(\mathcal{L}_V\eta')(V_2')\eta'(V_3') &= \frac{1}{2}(\lambda'+\rho r^*)(4n+2+\lambda'+\rho r^*)g(V_2',V_3')\\ &- 2n(\lambda'+\rho r^*)\eta'(V_2')\eta'(V_3'). \end{aligned}\tag{3.5.55}$$

Making use of the identity (3.5.55) into (3.5.54) with displacing V_3' by $\phi'V_3'$, we find

$$\frac{1}{2}(\lambda'+\rho r^*)(4n+2+\lambda'+\rho r^*)g(V_2',\phi'V_3') = 0. \tag{3.5.56}$$

As $\phi'(V_2',V_3') = g(V_2',\phi'V_3') \neq 0$ on M^{2n+1}, we acquire either $(\lambda'+\rho r^*) = 0$ or $(4n+2+\lambda'+\rho r^*) = 0$. Now, we consider two cases as

Case I: If $\lambda' + \rho r^* = 0$, from (3.5.47) we see that $\mathcal{L}_V g = 0$, i. e., V is Killing. From the identity (3.2.21) we obtain

$$S(V_1', V_2') = -(2n-1)g(V_1', V_2') - \eta'(V_1')\eta'(V_2'). \tag{3.5.57}$$

By contracting the equation (3.5.57) to yield $r = -4n^2$, where r is the scalar curvature of the manifold M^{2n+1}. This shows that M^{2n+1} is an η'-Einstein manifold with scalar curvature $r \neq 2n$. Hence, we conclude that M^{2n+1} is $\mathcal{D}$-homothetic to an Einstein manifold. **Case II:** If $4n + 2 + \lambda' + \rho r^* = 0$, then we set $V_3' = \xi'$ into (3.5.55) and insert $V_2' = \phi' V_2'$ of the resulting equation to achieve $\frac{1}{2}(\lambda' + \rho r^* - 2)(\mathcal{L}_V \eta')(\phi' V_2') = 0$. Since $4n + 2 + \lambda' + \rho r^* = 0$, we have $\lambda' + \rho r^* \neq 2$. Thus we have $(\mathcal{L}_V \eta')(\phi' V_2') = 0$. We plug $V_2' = \phi' V_2'$ into the foregoing equation and making use of (1.1.65), (1.1.66), lemma (3.2.9) to yield

$$(\mathcal{L}_V \eta')(V_2') = -2(2n+1)\eta'(V_2'). \tag{3.5.58}$$

Next, we take exterior differentiation d on (3.5.58) to obtain

$$(\mathcal{L}_V d\eta')(V_1', V_2') = -2(2n+1)g(V_1', \phi' V_2'), \tag{3.5.59}$$

noting that d commutes with $\mathcal{L}_V$. Also, we take the Lie-derivative of the well-known equation $d\eta'(V_1', V_2') = g(V_1', \phi' V_2')$ along the given soliton vector field V to get

$$(\mathcal{L}_V d\eta')(V_1', V_2') = (\mathcal{L}_V g)(V_1', \phi' V_2') + g(V_1', (\mathcal{L}_V \phi')V_2'). \tag{3.5.60}$$

From (3.5.47), we also deduce that

$$(\mathcal{L}_V g)(V_1', \phi' V_2') = -2(2n+1)g(V_1', \phi' V_2'). \tag{3.5.61}$$

Making use of the identities (3.5.59) and (3.5.61) into (3.5.60), we obtain $\mathcal{L}_V \phi' = 0$. Hence, the soliton vector field V leaves ϕ' invariant.

Again, we use the relation $4n + 2 + \lambda' + \rho r^* = 0$ into (3.2.21) to yield

$$S(V_1', V_2') = 2g(V_1', V_2') - (2n+2)\eta'(V_1')\eta'(V_2'). \tag{3.5.62}$$

We contract the foregoing equation to obtain $r = 2n$(i.e., the manifold M^{2n+1} cannot be $\mathcal{D}$-homothetic to an Eintein manifold). Then, by taking account of (3.5.62) in [62] of equation 17 we obtain $\widetilde{S}(V_1', V_2') = 0$. That is, the Ricci tensor with respect to the connection $\widetilde{\nabla}$ vanishes. Therefore, the desired result follows. □

4

Almost $*$-Ricci Bourguignon soliton on Kenmotsu and Sasakian Manifolds

4.1 Introduction

This chapter contains six sections of which the first two sections are introduction and preliminaries.

In the third and fourth section, we study an almost $*$-Ricci-Bourguignon soliton on Kenmotsu manifold. It has been shown that if a Kenmotsu manifold admits an almost $*$-Ricci-Bourguinon soliton, then the manifold is η'-Einstein. Next, we prove that if a $(\kappa, 2)'$-nullity distribution where $\kappa < -1$ admits an almost $*$-Ricci-Bourguinon soliton, then the manifold is Ricci flat. Further, we show that if a Kenmotsu manifold endows a gradient almost $*$-Ricci-Bourguignon soliton and ξ' leaves the scalar curvature r invariant, then the manifold is an Einstein manifold with constant scalar curvature $r = n(1 - 2n)$.

In the later section, we deliberate the study of almost almost $*$-η'-Ricci-Bourguignon soliton within the framework of Sasakian manifolds. It is shown that if a Sasakian manifold admits a almost $*$-η'-Ricci-Bourguignon soliton, then it is η'-Einstein. Next, if g represents a gradient almost almost $*$-η'-Ricci-Bourguignon soliton and ξ' leaves the scalar curvature r invariant on a Sasakian manifold, then the manifold is an η'-Einstein. Further, we have studied on a Sasakian manifold if g represents an almost $*$-η'-Ricci-Bourguignon soliton with potential vector field V_1' is pointwise collinear with ξ', then the manifold is an η'-Einstein.

In the final section, we evolve the physical applications and conclusion.

4.2 Preliminaries

This section is devoted to the study of $(2n+1)$-dimensional Kenmotsu manifold whose metric g admits both an almost $*$-Ricci-Bourguignon soliton as well as a gradient almost $*$-Ricci-Bourguignon soliton. We begin by recalling several lemmas that are essential for our study.

Lemma 4.2.1. *[80] On a $(2n+1)$-dimensional Kenmotsu manifold, the Ricci operator Q satisfies*

$$(\nabla_{V_1'}Q)\xi' = -QV_1' - 2nV_1', \tag{4.2.1}$$

$$(\nabla_{\xi'}Q)V_1' = -2QV_1' - 4nV_1', \tag{4.2.2}$$

the relation holds for all vector field V_1' on the manifold.

Lemma 4.2.2. *[80] on a $(2n+1)$-dimensional Kenmotsu manifold the $*$-Ricci tensor $\mathcal{S}^*$ is expressed as follows*

$$\mathcal{S}^*(X,Y) = \mathcal{S}(V_1', V_2') + (2n-1)g(V_1', V_2') + \eta'(V_1')\eta'(V_2'), \tag{4.2.3}$$

the relation holds for all vector fields V_1' and $V_2' \in \chi(M)$ and the associated $*$-scalar curvature is given by the equation $r^* = r + 4n^2$.

Lemma 4.2.3. *Let a metric g of a Kenmotsu manifold $M^{2n+1}(\phi', \xi', \eta', g)$ admits an almost $*$-Ricci-Bourguignon soliton (or almost $*$-RB soliton), then we have the following*

$$(\mathcal{L}_V R)(V_1', \xi') = 2(\lambda' + \rho(r+4n^2))\{\eta'(V_1')\xi' - V_1'\}.$$

Proof. If a metric g of a Kenmotsu manifold M^{2n+1} admits almost $*$-RB soliton, then by $*$-Ricci tensor expression of (4.2.3), an almost $*$-RB soliton (1.2.110), becomes

$$(\mathcal{L}_V g)(V_1', V_2') + 2\mathcal{S}(V_1', V_2') = 2\{(\lambda' + \rho r^*) - (2n-1)\}g(V_1', V_2') - 2\eta'(V_1')\eta'(V_2'), \tag{4.2.4}$$

for all $V_1', V_2' \in \chi(M)$. We take the Lie derivative of the expression $R(V_1', \xi')\xi' = \eta'(V_1')\xi' - V_1'$ (follows from (1.1.26)) with respect to the vector field V and applying (1.1.26), we derive

$$(\mathcal{L}_V R)(V_1', \xi')\xi' + R(V_1', \xi')\mathcal{L}_V\xi' + \eta'(\mathcal{L}_V\xi')V_1' + (\mathcal{L}_V g)(V_1', \xi')\xi' + g(V_1', \mathcal{L}_V\xi')\xi' = 0, \tag{4.2.5}$$

for all $V_1' \in \chi(M)$. Using $Q\xi' = -2n\xi'$ follows from (1.1.27) and (4.2.4), we have

$$(\mathcal{L}_V g)(V_1', \xi') = 2(\lambda' + \rho r^*)\eta'(V_1'). \tag{4.2.6}$$

Now taking the Lie-derivative of $\eta'(V_1') = g(V_1', \xi')$ and $\eta'(\xi') = 1$ we get $(\mathcal{L}_V \eta')(V_1') = (\mathcal{L}_V g)(V_1', \xi')$ and $(\mathcal{L}_V \eta')(\xi') + \eta'(\mathcal{L}_V \xi') = 0$ respectively then using (4.2.6), we can compute $(\mathcal{L}_V \eta')(\xi') = \lambda' + \rho r^*$ and $\eta'(\mathcal{L}_V \xi') = -\lambda' - \rho r^*$. Thus by the virtue of (1.1.26), lemma (4.2.2) and equation (4.2.5), we get

$$(\mathcal{L}_V R)(V_1', \xi') = 2(\lambda' + \rho(r + 4n^2))\{\eta'(V_1')\xi' - V_1'\}.$$

This completes the proof. □

Proposition 4.2.1. *Let a metric g of a Kenmotsu manifold $M^{2n+1}(\phi', \xi', \eta', g)$ satisfies an almost $*$-RB soliton, then the following results hold*

$$(\mathcal{L}_V \nabla)(V_1', \xi') = 2(2n-1)\phi' V_1' - 2\phi' Q V_1' + V_1'(\lambda' + \rho(r + 4n^2))\xi' + \{\xi'(\lambda' + \rho(r + 4n^2)) - 2\}V_1' + \eta'(V_1')\{\nabla(\lambda' + \rho(r + 4n^2)) - 2\xi'\}.$$

Proof. First, we take the covariant differentiation of (4.2.4) along an arbitrary vector field $V_3' \in \chi(M)$ and using (1.1.24) and (1.1.25) to yield

$$\begin{aligned}(\nabla_{V_3'} \mathcal{L}_V g)(V_1', V_2') + 2(\nabla_{V_3'} \mathcal{S})(V_1', V_2') &= 2V_3'(\lambda' + \rho r^*)g(V_1', V_2') \\ -2\{g(V_1', V_3')\eta'(V_2') + g(V_2', V_3')\eta'(V_1') &- 2\eta'(V_1')\eta'(V_2')\eta'(V_3')\},\end{aligned} \tag{4.2.7}$$

for all $V_1', V_2', V_3' \in \chi(M)$.

Now, we look back on the following formula (see in [88])

$$(\mathcal{L}_V V_3' g - \nabla_{V_3'} \mathcal{L}_V g - \nabla_{[V, V_3']})(V_1', V_2') = -g((\mathcal{L}_V \nabla)(V_3', V_1'), V_2') - g((\mathcal{L}_V \nabla)(V_3', V_2'), V_1'),$$

for all $V_1', V_2', V_3' \in \chi(M)$. As the Riemannian metric g is parallel, Inserting (4.2.7) into the above formula yields the following

$$\begin{aligned}&g((\mathcal{L}_V \nabla)(V_3', V_1'), V_2') + g((\mathcal{L}_V \nabla)(V_3', V_2'), V_1') + 2(\nabla_{V_3'} \mathcal{S})(V_1', V_2') \\ &= 2\{V_3'(\lambda' + \rho r^*)g(V_1', V_2') - g(V_1', V_3')\eta'(V_2') - g(V_2', V_3')\eta'(V_1') \\ &+2\eta'(V_1')\eta'(V_2')\eta'(V_3')\}.\end{aligned}$$

In out look of symmetry $(\mathcal{L}_V\nabla)(V_1', V_2') = (\mathcal{L}_V\nabla)(V_2', V_1')$ of the $(1,2)$-type tensor field $\mathcal{L}_V\nabla$, by cyclically interchanging the roles of V_1', V_2', V_3' in the forgoing equations, we attain

$$\begin{aligned}
g((\mathcal{L}_V\nabla)(V_1', V_2'), V_3') &= (\nabla_{V_3'}\mathcal{S})(V_1', V_2') - (\nabla_{V_1'}\mathcal{S})(V_2', V_3') - (\nabla_{V_2'}\mathcal{S})(V_3', V_1') \\
&+V_1'(\lambda' + \rho r^*)g(V_2', V_3') + V_2'(\lambda' + \rho r^*)g(V_3', V_1') - V_3'(\lambda' + \rho r^*)g(V_1', V_2') \\
&-2\{\eta'(V_1')g(V_2', V_3') + \eta'(V_2')g(V_1', V_3') - 2\eta'(V_1')\eta'(V_2')\eta'(V_3')\}.
\end{aligned} \tag{4.2.8}$$

Now, taking the covariant derivative of (1.1.27) along the vector field $V_1' \in \chi(M)$ and also using (1.1.24) we find

$$(\nabla_{V_1'}Q)\xi' = Q\phi'V_1' - 2n\phi'V_1'. \tag{4.2.9}$$

It is known that the Ricci operator is compatible with the contact metric structure ϕ' commute on Kenmotsu manifold [9]. So, we have

$$\nabla_{\xi'}Q = Q\phi' - \phi'Q. \tag{4.2.10}$$

Lastly, we insert $V_2' = \xi'$ into (4.2.8) and using the identities (4.2.9) and (4.2.10) and also applying the symmetry of Q and using lemma (4.2.2) and lemma (4.2.3) to yield

$$\begin{aligned}
(\mathcal{L}_V\nabla)(V_1', \xi') = 2(2n-1)\phi'V_1' - 2\phi'QV_1' + V_1'(\lambda' + \rho(r + 4n^2))\xi' + \{\xi'(\lambda' + \rho(r + 4n^2)) - \\
2\}V_1' + \eta'(V_1')\{\nabla(\lambda' + \rho(r + 4n^2)) - 2\xi'\}.
\end{aligned}$$

This completes the proof. □

Lemma 4.2.4. *[18] The $*$-Ricci tensor on a $(\kappa, \mu')'$-almost Kenmotsu manifold with $\kappa < -1$ is expressed as follows*

$$\mathcal{S}^*(V_1', V_2') = -(\kappa + 2)\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}, \tag{4.2.11}$$

the relation holds for all vector fields V_1' and V_2' on M.

Lemma 4.2.5. *[36] Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Sasakian manifold and $\{E_i\}_{1\leq i\leq 2n+1}$ is a local orthonormal frame on M^{2n+1}. Then for $V_2' \in \chi(M)$, we have*

$$\sum_i g((\nabla_{\phi'V_2'}Q)\phi'E_i, E_i) = 0, \qquad \sum_i g((\nabla_{\phi'E_i}Q)\phi'V_2', E_i) = -\frac{1}{2}V_1'(r).$$

Lemma 4.2.6. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Sasakian manifold and f be a smooth function on M^{2n+1}. If $\{E_i\}_{1\leq i\leq 2n+1}$ is a local orthonormal frame on M^{2n+1}, then for $V_2' \in \chi(M)$, we have*

$$\begin{aligned}
&\sum_i g(V_2', \nabla_{\phi' E_i} \nabla f) g(\xi', E_i) = 0, \\
&\sum_i g(\xi', \nabla_{E_i} \nabla f) g(\phi' V_2', E_i) = g(\phi' V_2', \nabla_{\xi'} \nabla f), \\
&\sum_i g(\phi' V_2', \nabla_{E_i} \nabla f) g(\xi', E_i) = g(\xi', \nabla_{\phi' V_2'} \nabla f), \\
&\sum_i g(\xi', R(E_i, V_2') \nabla f) g(\xi', E_i) = V_2'(f) - \eta'(\nabla f)\eta'(V_2').
\end{aligned}$$

Applying the symmetry of $Hess_f$, curvature properties and (1.1.18), the above formulas can be proved straightforwardly. Let us recall the formula for the $*$-Ricci tensor on a Sasakian manifold.

Lemma 4.2.7. *In [32] the expression of $*$-Ricci tensor S^* on a Sasakian manifold $M^{2n+1}(\phi', \xi', \eta', g)$ is*

$$S^*(V_1', V_2') = S(V_1', V_2') - (2n-1)g(X, Y) - \eta'(X)\eta'(Y), \quad V_1', V_2' \in \chi(M). \tag{4.2.12}$$

Lemma 4.2.8. *If a Sasakian metric g admits an almost $*$-η'-Ricci-Bourguignon soliton, then we have*

$$(\mathcal{L}_V R)(V_1', \xi')\xi' = 2(\lambda' + \rho r^* + \mu')(V_1' - \eta'(V_1')\xi'), \quad V_1' \in \chi(M). \tag{4.2.13}$$

Proof. By $*$-Ricci tensor expression (4.2.12), the almost $*$-η'-Ricci-Bourguignon soliton equation (1.2.113), becomes

$$(\mathcal{L}_V g)(V_1', V_2') + 2S(V_1', V_2') = 2(\lambda' + \rho r^* + 2n - 1)g(V_1', V_2') + 2(\mu' + 1)\eta'(V_1')\eta'(V_2'), \tag{4.2.14}$$

for all $V_1', V_2' \in \chi(M)$. We take the Lie-derivative of the equality $R(V_1', \xi')\xi' = V_1' - \eta'(V_1')\xi'$ (follows from (1.1.18)) along V and using (1.1.18), we get

$$(\mathcal{L}_V R)(V_1', \xi')\xi' + R(V_1', \xi')\mathcal{L}_V \xi' + \eta'(\mathcal{L}_V \xi')V_1' + (\mathcal{L}_V g)(V_1', \xi')\xi' + g(V_1', \mathcal{L}_V \xi')\xi' = 0, \tag{4.2.15}$$

for all $V_1' \in \chi(M)$. Using (1.1.21) and (4.2.14), we find that

$$(\mathcal{L}_V g)(V_1', \xi') = 2(\lambda' + \rho r^* + \mu')\eta'(V_1'), \quad V_1' \in \chi(M).$$

Employing this expression in the Lie derivative of the formulas $\eta'(V_1') = g(V_1', \xi')$ and $\eta'(\xi') = 1$, one can deduce $\eta'(\mathcal{L}_V \xi') = -(\lambda' + \rho r^* + \mu')$ and $(\mathcal{L}_V \eta')(\xi') = \lambda' + \rho r^* + \mu'$. Thus, by virtue of (1.1.18), equation (4.2.15) gives the required result. This completes the proof. □

Proposition 4.2.2. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Sasakian manifold admits a $*$-η'-Ricci-Bourguignon soliton. Then the soliton vector V are given by the expression*

$$(\mathcal{L}_V \nabla)(V_1', \xi') = -2QV_1' + \{4n + \xi'(\lambda' + \rho r^*)\}V_1' + V_1'(\lambda' + \rho r^*)\xi' - \nabla(\lambda' + \rho r^*)\eta'(V_1'). \quad (4.2.16)$$

Proof. First of all, taking a covariant derivative (4.2.14) along to an arbitrary vector field V_3' and applying the identity (1.1.16) to obtain

$$\begin{aligned}(\nabla_{V_3'} \mathcal{L}_V g)(V_1', V_2') &= -2(\nabla_{V_3'} S)(V_1', V_2') + 2V_3'(\lambda' + \rho r^*)g(V_1', V_2') \\ &+ 2V_3'(\mu')\eta'(V_1')\eta'(V_2') + 2(\mu' + 1)\{g(V_1', V_3')\eta'(V_2') \\ &+ g(V_2', V_3')\eta'(V_1') - 2\eta'(V_1')\eta'(V_2')\eta'(V_3')\}, \quad (4.2.17)\end{aligned}$$

for all V_1' V_2' $V_3' \in \chi(M)$. Combining (4.2.17) and (1.1.87) and using a straightforward combinatorial argument and the symmetry of $(\mathcal{L}_V \nabla)$ then (1.1.87) implies

$$\begin{aligned}&g((\mathcal{L}_V \nabla)(V_1', V_2'), V_3') = (\nabla_{V_3'} S)(V_1', V_2') - (\nabla_{V_1'} S)(V_2', V_3') - (\nabla_{V_2'} S)(V_3', V_1') \\ &-V_3'(\lambda' + \rho r^*)g(V_1', V_2') + V_1'(\lambda' + \rho r^*)g(V_2', V_3') + V_2'(\lambda' + \rho r^*)g(V_1', V_3') \\ &+2(\mu' + 1)\{g(V_1', V_2')\eta'(V_3') - \eta'(V_1')\eta'(V_2')\eta'(V_3')\}, \quad (4.2.18)\end{aligned}$$

the relation holds for given arbitrary vector fields V_1', V_2' and V_3' on M^{2n+1}. Moreover, recalling (1.1.16) and taking the covariant derivative of (1.1.21) along $V_1' \in \chi(M)$, one obtains

$$(\nabla_{V_1'} Q)\xi' = Q\phi' V_1' - 2n\phi' V_1'. \quad (4.2.19)$$

Using (4.2.19), the forgoing equation yields

$$(\mathcal{L}_V \nabla)(V_1', \xi') = -2QV_1' + \{4n + \xi'(\lambda' + \rho r^*)\}V_1' + V_1'(\lambda' + \rho r^*)\xi' - \nabla(\lambda' + \rho r^*)\eta'(V_1'), \quad (4.2.20)$$

for all $V_1' \in \chi(M)$. Therefore, the theorem is proved. □

4.3 Almost $*$-Ricci-Bourguignon soliton on Kenmotsu manifold

Theorem 4.3.1. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Kenmotsu manifold, if the metric g represents an almost $*$-Ricci-Bourguignon soliton, then the manifold is η'-Einstein.*

Proof. Taking the covariant derivative of (4.2.4) with respect to an arbitrary vector field V_3' and using (1.1.25), we get

$$(\nabla_{V_3'}\mathcal{L}_V g)(V_1', V_2') + 2(\nabla_{V_3'}\mathcal{S})(V_1', V_2') = 2V_3'(\lambda' + \rho r^*)g(V_1', V_2')$$
$$-2\{g(V_1', V_3')\eta'(V_2') + g(V_2', V_3')\eta'(V_1') - 2\eta'(V_1')\eta'(V_2')\eta'(V_3')\}, \qquad (4.3.1)$$

for $V_1', V_2', V_3' \in \chi(M)$. Combining (4.3.1) and (1.1.87) and by a straight forward combinatorial calculation and also using the symmetry of $(\mathcal{L}_V\nabla)$, then (1.1.87) implies

$$g((\mathcal{L}_V\nabla)(V_1', V_2'), V_3') = 2\{(\nabla_{V_3'}\mathcal{S})(V_1', V_2') - (\nabla_{V_1'}\mathcal{S})(V_2', V_3')$$
$$-(\nabla_{V_2'}\mathcal{S})(V_3', V_1')\} - 2\{(\eta'(V_3') - V_3'(\lambda' + \rho r^*))g(V_1', V_2')$$
$$-\eta'(V_1')\eta'(V_2')\eta'(V_3')\}, \qquad (4.3.2)$$

for given vector fields V_1', V_2' and V_3' on M. Using (4.2.1) and (4.2.2), also putting $V_2' = \xi'$ the forgoing equation yields

$$(\mathcal{L}_V\nabla)(V_1', \xi') = 2QV_1' + 4nV_1', \qquad (4.3.3)$$

for all $V_1' \in \chi(M)$. We now take the covariant derivative of this expression with respect to an arbitrary vector field V_2', we obtain

$$(\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1', \xi') = 2(\nabla_{V_2'}Q)V_1' - (\mathcal{L}_V\nabla)(V_1', V_2') + \eta'(V_2')(2QV_1' + 4nV_1'). \qquad (4.3.4)$$

Again from (3.2.27) and in view of (4.3.4) we acquire

$$(\mathcal{L}_V R)(V_1', V_2')\xi' = 2(\nabla_{V_1'}Q)V_2' - 2(\nabla_{V_2'}Q)V_1'$$
$$+2\eta'(V_1')\{QV_2' + 2nV_2'\} - 2\eta'(V_2')\{QV_1' + 2nV_1'\}, \qquad (4.3.5)$$

for any given arbitrary vector fields V_1' and V_2' on M^{2n+1}. Putting $V_2' = \xi'$ in (4.3.5) and using (1.1.27), (4.2.1) and (4.2.2) we have

$$(\mathcal{L}_V R)(V_1', \xi')\xi' = 2QV_1' + 4nV_1'. \qquad (4.3.6)$$

Now, taking the Lie derivative of $g(\xi', \xi')$ along the potential vector field V, in account of (4.2.4)

$$\eta'(\mathcal{L}_V \xi') = \lambda' + \rho r^*. \tag{4.3.7}$$

Putting $V_2' = \xi'$ in (4.2.4) and following (1.1.8) and (1.1.12) provides

$$(\mathcal{L}_V \eta')V_1' - g(V_1', \mathcal{L}_{\xi'}) = 2(\lambda' + \rho r^*)\eta'(V_1'), \tag{4.3.8}$$

for arbitrary vector field V_1' on M^{2n+1}. From (1.1.26) we get $R(V_1', \xi')\xi' = \eta'(V_1')\xi' - V_1'$. We take the Lie derivative of this with respect to the potential vector field V and using (4.3.7), (4.3.8) and lemma (4.2.2), this reduces to

$$(\mathcal{L}_V R)(V_1', \xi')\xi' = 2(\lambda' + \rho(r + 4n^2))(V_1' - \eta'(V_1')\xi'), \tag{4.3.9}$$

for all $V_1' \in \chi(M)$. Then from (4.3.6) we get

$$\mathcal{S}(V_1', V_2') = \{(\lambda' + \rho(r + 4n^2)) - 2n\}g(V_1', V_2') - (\lambda' + \rho(r + 4n^2))\eta'(V_1')\eta'(V_2'),$$

for all $V_1', V_2' \in \chi(M)$. Which is an η'-Einstein manifold and the theorem is proved. □

We now examine the case where a Kenmotsu manifold admits an almost $*$-Ricci-Bourguignon soliton with a non-zero potential vector field V is pointwise collinear to the Reeb vector field ξ'.

Theorem 4.3.2. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Kenmotsu manifold admitting an almost $*$-Ricci-Bourguignon soliton with non-zero potential vector field V, which is collinear to the Reeb vector field ξ'. If ξ' preserves the scalar curvature $r = -2(8n^2 + 3n - 1)$ invariant, then (M, g) is an η'-Einstein manifold with $\lambda' = 6n\{\rho(2n+1) - 1\} + 2\rho$.*

Proof. Since the potential vector field V is parallel to the Reeb vector field ξ', then $V = \alpha'\xi'$ for some smooth function α', from (1.1.24), it follows that

$$(\mathcal{L}_V g)(V_1', V_2') = V_1'(\alpha')\eta'(V_2') + V_2'(\alpha')\eta'(V_1'), \tag{4.3.10}$$

for any vector fields V_1' and $V_2' \in \chi(M)$. By applying the anti-symmetry of ϕ', then the equation (4.2.4) implies

$$\begin{aligned} &V_1'(\alpha')\eta'(V_2') + V_2'(\alpha')\eta'(V_1') + 2\mathcal{S}(V_1', V_2') \\ &= 2\{(\lambda' + \rho r^*) - (2n-1)\}g(V_1', V_2') - 2\eta'(V_1')\eta'(V_2'). \end{aligned} \tag{4.3.11}$$

Now setting $V_1' = \xi'$ and $V_2' = \xi'$ in (4.3.11) and using (1.1.27) gives $\xi'(\alpha') = \lambda' + \rho r^* - 3n$. Similarly plugging $V_2' = \xi'$ in (4.3.11) and also using (1.1.27), we have

$$\begin{aligned} V_1'(\alpha') &= \{2(\lambda' + \rho r^*) - \xi'(\alpha')\}\eta'(V_1') \\ &= \{\xi'(\alpha') + 6n\}, \end{aligned} \tag{4.3.12}$$

for all $V_1' \in \chi(M)$. We take its covariant derivative by $V_2' \in \chi(M)$, and applying (1.1.24), we obtain

$$g(\nabla_{V_2'}\nabla\alpha', V_1') = V_2'(\xi'(\alpha'))\eta'(V_1' + (\xi'(\alpha'))\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}.$$

Since $Hess_{\alpha'}$ is symmetry, it follows that

$$V_1'(\xi'(\alpha'))\eta'(V_2') - V_2'(\xi'(\alpha'))\eta'(V_1') = 2\{\xi'(\alpha') + 6n\}\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\},$$

using (1.1.25), which yields that

$$\{\xi'(\alpha') + 6n\}(\nabla_{V_1'}\eta')V_2' = 0, \qquad \forall \;\; V_1', V_2' \perp \xi'.$$

Since $(\nabla_{V_1'}\eta')V_2' \neq 0$ on M, the above expression implies $\xi'(\alpha') = -6n$ and consequently $\nabla\alpha' = -6n$ on M. It follows that α' is not constant on M, and hence, (4.3.10) shows that V is a Killing vector field. Thus (M, g) is *-Einstein (trivial). Furthermore, from (4.3.12) it concludes that $\lambda' + \rho r^* = -6n$ and (4.3.11) becomes

$$\mathcal{S}(V_1', V_2') = (1 - 8n)g(V_1', V_2') + \eta'(V_1')\eta'(V_2'). \tag{4.3.13}$$

Hence, using (4.3.13) in the equation (4.2.3), we get $S = -6ng + \eta' \otimes \eta'$ and corresponding $r^* = -6n(2n + 1) + 2$, moreover, the scalar curvature is $r = -2(8n^2 + 3n - 1)$ and $\lambda' = 6n\{\rho(2n + 1) - 1\} + 2\rho$, which finishes the proof. □

Next, we consider a gradient almost $*$-Ricci-Bourguignon soliton on a Kenmotsu manifold. It is known that an almost $*$-Ricci-Bourguignon soliton satisfying equation (1.2.110) for some smooth functions λ', is a generalization Ricci-Yamabe soliton. In [33], Ghosh studied Ricci almost soliton on a Kenmotsu manifold and proved that if a Kenmotsu metric admits a gradient Ricci almost soliton and the Reeb vector field ξ' leaves the scalar curvature r invariant, then the manifold is Einstein. To generalized this results, we consider gradient almost $*$-Ricci-Bourguignon soliton on Kenmotsu manifold and establish following theorem.

Theorem 4.3.3. *If a Kenmotsu manifold $M^{2n+1}(\phi', \xi', \eta', g)$ admits a gradient almost $*$-Ricci-Bourguignon soliton and the Reeb vector field ξ' leaves the scalar curvature r invariant, then (M, g) is an Einstein manifold with constant scalar curvature $r = n(1 - 2n)$.*

Proof. For any vector field V_1' belongs to $\chi(M)$ the gradient form of the soliton equation (1.2.111) is given by

$$\nabla_{V_1'} Df + QV_1' + \{(2n-1) - (\lambda' + \rho r^*)\}V_1' + \eta'(V_1')\xi' = 0. \tag{4.3.14}$$

By applying the standard formula for the Riemannian curvature tensor $R(V_1', V_2')Df = \nabla_{V_1'}\nabla_{V_2'}Df - \nabla_{V_2'}\nabla_{V_1'}Df - \nabla_{[V_1', V_2']}Df$, we obtain

$$\begin{aligned} R(V_1', V_2')Df &= (\nabla_{V_2'}Q)V_1' - (\nabla_{V_1'}Q)V_2' - V_2'(\lambda' + \rho r^*)V_1' \\ &+ V_1'(\lambda' + \rho r^*)V_2' + \{\eta'(V_2')V_1' - \eta'(V_1')V_2'\}, \end{aligned} \tag{4.3.15}$$

for all V_1', $V_2' \in \chi(M)$. Now putting $V_2' = \xi'$ in (4.3.15) and using (4.2.1) and (4.2.2), we get

$$R(V_1', \xi')Df = -QV_1' - 2nV_1' - \xi'(\lambda' + \rho r^*)V_1' + V_1'(\lambda' + \rho r^*)\xi' + (V_1' - \eta'(V_1')\xi'), \tag{4.3.16}$$

for any $V_1' \in \chi(M)$. By virtue of (1.1.26), equation (4.3.16) reduces to

$$V_1'(f - (\lambda' + \rho r^*))\xi' = -QV_1' + \{\xi'(f - (\lambda' + \rho r^*)) - 2n + 1\}V_1' - \eta'(V_1')\xi', \tag{4.3.17}$$

for any $V_1' \in \chi(M)$. Now, taking an inner product of (4.3.17) with ξ' and using (1.1.26), we get $V_1'(f - (\lambda' + \rho r^*)) = \xi'(f - (\lambda' + \rho r^*))\eta'(V_1')$. Putting this into (4.3.17), we obtain

$$QV_1' = \{\xi'(f - (\lambda' + \rho r^*)) - 2n + 1\}V_1' + \{\xi'(f - (\lambda' + \rho r^*)) - 1\}\eta'(V_1')\xi', \tag{4.3.18}$$

for any $V_1' \in \chi(M)$. This follows that the manifold (M, g) is an η'-Einstein manifold. Now contracting (4.3.15) over V_1' along an orthonormal basis $\{e_i\}$, $1 \le i \le 2n+1$, we compute

$$S(V_2', Df) = -\sum_{i=1}^{2n+1} g((\nabla_{e_i}Q)V_2', e_i) + V_2'(r) - 2nV_2'(\lambda' + \rho r^*) + 2n\eta'(V_2'). \tag{4.3.19}$$

Now, using the formula (2.5.10) for the Riemannian manifold and from (4.3.19), we get

$$S(V_2', Df) = \frac{1}{2}V_2'(r) - 2nV_2'(\lambda' + \rho r^*) + 2n\eta'(V_2'), \tag{4.3.20}$$

for any $V_2' \in \chi(M)$. From (1.1.26), we can calculate $S(\xi', Df) = -2n\xi'(f)$, putting this in (4.3.20), we get $\xi'(r) = 4n\{\xi'(\lambda' + \rho r^* - f) - 1\}$. Using this in the trace of (4.2.2), we get $\xi'(f - (\lambda' + \rho r^*)) = \frac{r}{n} + 2n$. By this result, equation (4.3.18) reduces to

$$QV_1' = (\frac{r}{n} + 1)V_1' + \{\frac{r}{n} + 2n - 1\}\eta'(V_1')\xi', \tag{4.3.21}$$

for any $V_1' \in \chi(M)$. By our assumption, $\xi'(r) = 0$, the trace of (4.2.2) gives $r = n(1-2n)$. Thus, from (4.3.21) the required result follows as claimed. □

4.4 Almost $*$-Ricci Bourguignon soliton on $(\kappa, \mu')'$-almost Kenmotsu manifold with $\kappa < -1$

In this section, we consider a $(2n+1)$-dimensional almost Kenmotsu manifold in which the characteristic vector field ξ' belongs to the $(\kappa, -2)'$-nullity distribution. We then assume that the metric g defines an almost $*$-Ricci-Bourguignon soliton.

Theorem 4.4.1. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be an almost Kenmotsu manifold such that the Reeb vector field ξ' belongs to $(\kappa, -2)'$-nullity distribution, where $\kappa < -1$. If the metric g represents an almost $*$-Ricci-Bourguignon soliton satisfying $\lambda' \neq -\rho(r + 4n^2) - \frac{1}{2}(D\lambda' + \rho Dr)$, then M is Ricci-flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Proof. In light of the identities (1.2.110) and (4.2.11) and with the help of $r^* = r + 4n^2$ we have

$$(\mathcal{L}_V g)(V_1', V_2') = 2\{(\kappa + 2) + \lambda' + \rho(r + 4n^2)\}g(V_1', V_2') - 2(\kappa + 2)\eta'(V_1')\eta'(V_2'), \tag{4.4.1}$$

for all vector fields V_1' and V_2' on M. Now, we take a covariant derivative of (4.4.1) along the arbitrary vector field V_3' and using (1.1.12) to yield

$$\begin{aligned}&(\nabla_{V_3'}\mathcal{L}_V g)(V_1', V_2') = \{V_3'(\lambda') + \rho V_3'(r)\}g(V_1', V_2') - 2(\kappa + 2)[\eta'(V_2')g(V_1', V_3')\\&+\eta'(V_1')g(V_2', V_3') + \eta'(V_2')g(h'V_3', V_1') + \eta'(V_1')g(h'V_3', V_2')\\&-2\eta'(V_1')\eta'(V_2')\eta'(V_3')].\end{aligned} \tag{4.4.2}$$

Then using (1.1.87) and by the symmetry of $(\mathcal{L}_V \nabla)$ from the equation (4.4.2), we obtain

$$\begin{aligned}(\mathcal{L}_V \nabla)(V_1', V_2') &= -2(\kappa + 2)[g(V_1', V_2') + g(h'V_1', V_2') - \eta'(V_1')\eta'(V_2')]\\&- \{(D\lambda' + \rho Dr)g(V_1', V_2') - (V_1'(\lambda') + V_1'(r))V_2'\\&- (V_2'(\lambda') + V_2'(r))V_1'\},\end{aligned} \tag{4.4.3}$$

for all V_1', $V_2' \in \chi(M)$. We insert $V_2' = \xi'$ and with the help of the identities (1.1.8), (1.1.12) and (1.1.34) to achieve

$$(\mathcal{L}_V\nabla)(V_1',\xi') = -\{(D\lambda' + \rho Dr)\eta'(V_1') - (V_1'(\lambda') + V_1'(r))\xi' - (\xi'(\lambda') + \xi'(r))V_1'\}, \quad (4.4.4)$$

for arbitrary vector V_1' on M. Now taking differentiation (4.4.4) covariantly along arbitrary vector field V_2' and using (1.1.33) and (4.4.3) into account we can get

$$\begin{aligned}(\nabla_{V_2'}\mathcal{L}_V\nabla)(V_1',\xi') &= 2(\kappa+2)[g(V_1',V_2') + g(h'V_1',V_2') - \eta'(V_1')\eta'(V_2')]\xi' \\ &- \{(D\lambda' + \rho Dr)(\nabla_{V_2'}\eta')V_1' + \eta'(V_1')V_2'(D\lambda' + \rho Dr)\} \\ &+ \{V_1'(\lambda') + \rho V_1'(r)\}\nabla_{V_2'}\xi' + V_2'\{\xi'(\lambda') + \rho\xi'(r)\}V_1', \quad (4.4.5)\end{aligned}$$

for any vector fields V_1' and V_2' on M. Again from the expression (3.2.27), substituting $V_3' = \xi'$ then applying (4.4.5), we get

$$\begin{aligned}(\mathcal{L}_V R)(V_1',V_2')\xi' &= V_1'(D\lambda' + \rho Dr)\eta'(V_2') - V_2'(D\lambda' + \rho Dr)\eta'(V_1') \\ &+ V_2'(\xi'(\lambda') + \rho\xi'(r))V_1' - V_1'(\xi'(\lambda') + \rho\xi'(r))V_2', \quad (4.4.6)\end{aligned}$$

for any given vector field V_1' and V_2' on M. Next, we consider the Lie derivative of (1.1.38) by the potential vector field V and also making use of (1.1.8) and (1.1.34) to yield

$$\begin{aligned}&(\mathcal{L}_V R)(V_1',\xi')\xi' = \kappa[g(V_1',\mathcal{L}_V\xi')\xi' - 2\eta'(\mathcal{L}_V\xi')V_1' - ((\mathcal{L}_V\eta')V_1')\xi'] \\ &+2[2\eta'(\mathcal{L}_V\xi')h'V_1' - \eta'(V_1')(h'(\mathcal{L}_V\xi')) - g(h'V_1',\mathcal{L}_V\xi')\xi' \\ &-(\mathcal{L}_V h')V_1'], \quad (4.4.7)\end{aligned}$$

for any $V_1' \in \chi(M)$. We insert $V_2' = \xi'$ into (4.4.1) to yield

$$(\mathcal{L}_V\eta')V_1' - g(V_1',\mathcal{L}_V\xi') = 2\{\lambda' + \rho(r + 4n^2)\}\eta'(V_1'), \quad (4.4.8)$$

for any $V_1' \in \chi(M)$. Now putting $V_1' = \xi'$ in (4.4.8) we get

$$\eta'(\mathcal{L}_V\xi') = -2\{\lambda' + \rho(r + 4n^2)\}. \quad (4.4.9)$$

By the help of (4.4.6), (4.4.8) and (4.4.9), we can write the equation (4.4.7) as

$$\begin{aligned}&2\kappa\{\lambda' + \rho(r + 4n^2) + \frac{1}{2}(D\lambda' + \rho Dr)\}(V_1' - \eta'(V_1')\xi') + 4\{\lambda' + \rho(r + 4n^2)\}h'V_1' \\ &-2\eta'(V_1')h'(\mathcal{L}_V\xi') - 2g(h'V_1',\mathcal{L}_V\xi')\xi' - 2(\mathcal{L}_V h')V_1' = 0. \quad (4.4.10)\end{aligned}$$

Taking an inner product of (4.4.10) with respect to the arbitrary vector field V_2' on M, we obtain

$$2\{\lambda' + \rho(r + 4n^2) + \frac{1}{2}(D\lambda' + \rho Dr)\}[\kappa\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\} - 2g(h'V_1', V_2')]$$
$$-2\eta'(V_1')g(h'(\mathcal{L}_V\xi'), V_2') - 2g(h'V_1', \mathcal{L}_V\xi')\eta'(V_2') - 2g((\mathcal{L}_V h')V_1', V_2') = 0. \quad (4.4.11)$$

As the above equation (4.4.11) is true for all vector fields V_1' and V_2' on M, replacing V_1' by $\phi'(V_1')$ and V_2' by $\phi'(V_2')$ and taking (1.1.11) into account we get as

$$2\{\lambda' + \rho(r + 4n^2) + \frac{1}{2}(D\lambda' + \rho Dr)\}[\kappa g(\phi'V_1', \phi'V_2') - 2g(h'\phi'V_1', \phi'V_2')]$$
$$-2g((\mathcal{L}_V h')\phi'V_1', \phi'V_2') = 0, \quad (4.4.12)$$

for all given vector fields V_1' and V_2' on M. Since $spec(h') = \{0, \alpha', -\alpha'\}$, let V_1' and V belong to the eigenspaces of $-\alpha'$ and α' denoted by $[-\alpha']'$ and $[\alpha']'$ respectively. Then $\phi'V_1' \in [\alpha']'$ (see [25]). Then (4.4.12) can be rewritten as

$$2\{\lambda'+\rho(r+4n^2)+\frac{1}{2}(D\lambda'+\rho Dr)\}(\kappa-2)g(\phi'V_1', \phi'V_2')-2g((\mathcal{L}_V h')\phi'V_1', \phi'V_2') = 0, \quad (4.4.13)$$

for all vector fields on M. Now from (2.4.39), (2.4.40) and (4.4.13), we get

$$\{\lambda' + \rho(r + 4n^2) + \frac{1}{2}(D\lambda' + \rho Dr)\}(\kappa - 2)g(\phi'V_1', \phi'V_2') = 0, \quad (4.4.14)$$

for all vector fields V_1' and V_2' on M. As $g(\phi'V_1', \phi'V_2') \neq 0$ then from the for going equation we have either $\lambda' = -\rho(r + 4n^2) - \frac{1}{2}(D\lambda' + \rho Dr)$ or $\kappa = 2$.

Now, for $\lambda' \neq -\rho(r + 4n^2) - \frac{1}{2}(D\lambda' + \rho Dr)$ from the equation (4.4.14) we infer that $\kappa = 2$, (4.2.11) implies

$$S^*(V_1', V_2') = -4\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}. \quad (4.4.15)$$

Thus the $*$-Ricci tensor is η'-Einstein manifold.

Again from (4.4.15) and the proposition 4.1 of [25], lastly we conclude that M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, where $\mathbb{H}^{n+1}(-4)$ is the hyperbolic space of constant curvature -4.

By hypothesis $\lambda' \neq -\rho(r+4n^2)-\frac{1}{2}(D\lambda'+\rho Dr)$, from the equation (4.4.14) therefore, it can be deduced that $\kappa = 2\alpha'$. Again from $\alpha'^2 = -(\kappa + 1)$ we get $\alpha' = -1$ and $\kappa = -2$, putting the value of κ in (4.2.11) $S^*(V_1', V_2') = 0$ i.e. $*$-Ricci flat. Thus, the theorem is completed. □

4.5 Sasakian metrics as an almost $*$-η'-Ricci Bourguignon soliton

Further, Sharma [73] studied the K-contact manifold as a gradient Ricci soliton and a Ricci soliton with the potential vector field V point-wise collinear with ξ' and the manifold becomes Einstein. After some years, Ghosh [33] proved that *if a K-contact manifold admits a gradient Ricci almost soliton, then the scalar curvature is constant. Moreover, if M is compact, then it is Einstein, Sasakian and isometric to a unit sphere* Very recently, Patra [56] proved the result for completeness instead of compactness. Recently, Patra et al. [63] proved that *A complete Sasakian manifold admitting a gradient almost $*$-RB soliton structure is $*$-Ricci flat (trivial or $*$-Einstein), compact positive-Sasakian.* An almost $*$-η'-Ricci-Bourguignon soliton is a generalization of $*$-Ricci-Bourguignon soliton and $*$-Einstein manifold. Based on the above facts and discussions in the research of contact geometry, a natural question arises.

Is a complete Sasakian metric as a gradient almost $*$-η'-Ricci-Bourguignon soliton Einstein or not?

Here, we will answer this question affirmatively using some different techniques, several conditions and prove the following.

Theorem 4.5.1. *A complete Sasakian manifold $M^{2n+1}(\phi', \xi', \eta', g)$ admitting a gradient almost $*$-η'-Ricci-Bourguignon soliton and ξ' leaves the scalar curvature r invariant, then $M^{2n+1}(\phi', \xi', \eta', g)$ is a compact positive Sasakian and an η'-Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Proof. For any vector field V_1' belongs to $\chi(M)$ the gradient form of the soliton equation (1.2.114) is given by

$$\nabla_{V_1'} Df + QV_1' = \{\lambda' + \rho r^* + 2n - 1\}V_1' + (\mu' + 1)\eta'(V_1')\xi'. \qquad (4.5.1)$$

By applying the standard formula for the Riemannian curvature tensor $R(V_1', V_2')Df = \nabla_{V_1'}\nabla_{V_2'}Df - \nabla_{V_2'}\nabla_{V_1'}Df - \nabla_{[V_1',V_2']}Df$, we get

$$\begin{aligned} R(V_1', V_2')Df &= (\nabla_{V_2'}Q)V_1' - (\nabla_{V_1'}Q)V_2' - V_2'(\sigma)V_1' + V_1'(\sigma)V_2' \\ &-V_2'(\mu')\eta'(V_1')\xi' + V_1'(\mu')\eta'(V_2')\xi' - (\mu' + 1)\{\eta'(V_2')V_1' - \eta'(V_1')V_2'\}, \end{aligned} \qquad (4.5.2)$$

for all V_1', $V_2' \in \chi(M)$ and $\sigma = \lambda' + \rho r^* + 2n - 1$, a smooth function as λ' is a smooth function. Now putting $V_2' = \xi'$ in (4.5.2) and using (4.2.19), also using the relation $\nabla_{\xi'} Q = Q\phi' - \phi' Q = 0$ in [37], we get

$$\begin{aligned} R(V_1', \xi')Df &= -QV_1' - 2nV_1' - \xi'(\sigma)V_1' - V_1'(\sigma)\xi' - \xi'(\mu')\eta'(V_1')\xi' \\ &\quad + V_1'(\mu')\xi' - (\mu' + 1)(V_1' - \eta'(V_1')\xi'), \end{aligned} \tag{4.5.3}$$

for any $V_1' \in \chi(M)$. By virtue of (1.1.18), equation (4.5.3) reduces to

$$\begin{aligned} V_1'(\sigma + \mu' + f)\xi' &= -QV_1' + \{\xi'(\sigma + f) + \mu' + 1 - 2n\}V_1' \\ &\quad + \{\xi'(\mu') - \mu' - 1\}\eta'(V_1')\xi', \end{aligned} \tag{4.5.4}$$

for any $V_1' \in \chi(M)$. Now, taking an inner product of (4.5.4) with ξ' and using (1.1.18), we get $V_1'(\sigma + \mu' + f) = \xi'(\sigma + \mu' + f)\eta'(V_1')$. Putting this into (4.5.4), we obtain

$$QV_1' = \{\xi'(\sigma + f) + \mu' + 1 - 2n\}V_1' - \{\xi'(\sigma + f) + \mu' + 1\}\eta'(V_1')\xi', \tag{4.5.5}$$

for any $V_1' \in \chi(M)$. Now contracting (4.5.2) over V_1' with respect to an orthonormal basis $\{e_i\}$, $1 \le i \le 2n + 1$, we compute

$$\begin{aligned} S(V_2', Df) &= -\sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)Y, e_i) + Y(r) + 2nV_2'(\sigma) \\ &\quad + V_2'(\mu') - \eta'(V_2')\xi'(\mu') + 2n(\mu' + 1)\eta'(V_2'). \end{aligned} \tag{4.5.6}$$

Then from (2.5.10) and (4.5.6), we get

$$S(V_2', Df) = \frac{1}{2}V_2'(r) + 2nV_2'(\sigma) + V_2'(\mu') - \eta'(V_2')\xi'(\mu') + 2n(\mu' + 1)\eta'(V_2'), \tag{4.5.7}$$

for any $V_1' \in \chi(M)$. From (1.1.18), we can easily compute that $S(\xi', Df) = -2n\xi'(f)$, putting this into (4.5.7) to get $\xi'(r) = -4n\{\xi'(\sigma + f) + \mu' + 1\}$. Using this in the trace of (4.5.2), we get $\xi'(\sigma + f) = (2n + 1) - \mu' - 1 + \frac{r}{2n}$. By this result, equation (4.5.5) reduces to

$$QV_1' = \frac{1}{2n}\{r + 2n\}V_1' - \frac{1}{2n}\{r + 4n^2 + 2n\}\eta'(V_1')\xi', \tag{4.5.8}$$

for any $V_1' \in \chi(M)$. By our assumption, $\xi'(r) = 0$, the trace of (4.5.2) gives $r = -2n(2n + 1)$. Thus, from (4.5.8) follows that (M^{2n+1}, g) is an η'-Einstein manifold. Since M^{2n+1} is complete, equation (4.5.8) follows that M^{2n+1} is compact and positive Sasakian. This completes the proof. □

As an almost $*$-Ricci-Bourguignon soliton postulates $*$-Ricci-Bourguignon soliton and $*$-Einstein manifold, so we have the following question:

When or under what conditions an almost $*$-Ricci-Bourguignon soliton becomes a $*$-Ricci-Bourguignon soliton?

In [63], Rovenski et al. answered the above question for almost $*$-Ricci soliton on Sasakian manifold. This answer is also true for almost $*$-Ricci-Bourguignon soliton as it is a generalization of almost $*$-Ricci soliton. Bescause, in the definition 1.1, if we replace the smooth function $\lambda' + \rho r$ by smooth function λ' (say), then it becomes almost $*$-Ricci soliton. Now, we are engrossed to study the another question:

Under what condition an almost $*$-η'-Ricci-Bourguignon soliton becomes a $*$-η'-Ricci-Bourguignon soliton?

In the succeeding theorem, we take the condition on the potential vector field to prove it.

Theorem 4.5.2. *If a Sasakian manifold $M^{2n+1}(\phi', \xi', \eta', g)$ has an almost $*$-η'-Ricci-Bourguignon soliton structure $(g, \lambda', \mu', V_1')$ such that V_1' is a Jacobi field on the ξ'-integral curves, then $\lambda' + \rho r^*$ is constant on M^{2n+1}.*

Proof. Applying the equation (4.2.16) of proposition (4.2.2) to the very familiar formula:

$$\nabla_{V_2'}\nabla_{V_3'}V_1' - \nabla_{\nabla_{V_2'}V_3'}V_1' - R(V_2', V_1')V_3' = (\mathcal{L}_{V_1'}\nabla)(V_2', V_3'), \tag{4.5.9}$$

see [88], we acquire

$$\begin{aligned} &\nabla_{V_2'}\nabla_{\xi'}V_1' - \nabla_{\nabla_{V_2'}\xi'}V_1' - R(V_2', V_1')\xi' \\ &= -2Q\phi' V_2' + 4n\phi' V_2' + \xi'(\lambda' + \rho r^*)V_2' + V_2'(\lambda' + \rho r^*)\xi' - \nabla(\lambda' + \rho r^*)\eta'(V_2'). \end{aligned} \tag{4.5.10}$$

By conditions, V_1' is a Jacobi vector field on the ξ'-integral curves, see [84], i.e.,

$$\nabla_{\xi'}\nabla_{\xi'}V_1' + R(V_1', \xi')\xi' = 0. \tag{4.5.11}$$

Using $V_2' = \xi'$ in (4.5.10) and (1.1.16) ($\nabla_{\xi'}\xi' = 0$ that is the consequence of (1.1.16) by putting $V_1' = \xi'$), we achieve the equality

$$\nabla(\lambda' + \rho r^*) = 2\xi'(\lambda' + \rho r^*)\xi', \tag{4.5.12}$$

or, using the exterior derivative,

$$d(\lambda' + \rho r^*) = 2\xi'(\lambda' + \rho r^*)\eta'. \tag{4.5.13}$$

Applying exterior derivative, the Poincaré lemma ($d^2 = 0$), and the wedge product with η', we acquire $\xi'(\lambda' + \rho r^*)\eta' \wedge d\eta' = 0$; thus $\xi'(\lambda' + \rho r^*) = 0$, as $\eta' \wedge d\eta'$ is nowhere zero on contact manifold. Thus, $d(\lambda' + \rho r^*) = 0$, i.e., $\lambda' + \rho r^*$ is constant on M^{2n+1}. □

In the following theorem, we consider non-gradient almost $*$-η'-Ricci-Bourguignon soliton in the case of Sasakian manifold, and we are looking for an sufficient condition on the potential vector field V under which Sasakian manifold having an almost $*$-η'-Ricci-Bourguignon soliton structure ia $*$-Ricci flat.

Theorem 4.5.3. *Suppose a Sasakian manifold $M^{2n+1}(\phi', \xi', \eta', g)$ admits an almost $*$-η'-Ricci-Bourguignon soliton, where the potential vector field V is parallel to the Reeb vector field ξ'. Then the associated vector field V_1' is a Killing vector field, the manifold $M^{2n+1}(\phi', \xi', \eta', g)$ is $*$-Ricci flat and has constant scalar curvature $4n^2$, moreover, the soliton is steady for $\mu' = 0$ and any λ'.*

Proof. Since, the potential vector field V is parallel to the Reeb vector field ξ', then we can write $V = \beta'\xi'$, from (1.1.16), it follows that

$$(\mathcal{L}_V g)(V_1', V_2') = V_1'(\beta')\eta'(V_2') + V_2'(\beta')\eta'(V_1'), \quad V_1', V_2' \in \chi(M), \tag{4.5.14}$$

owing to the anti-symmetry of ϕ', equation (4.2.14) implies

$$\begin{aligned} V_1'(\beta')\eta'(V_2') + V_2'(\beta')\eta'(V_1') + 2S(V_1', V_2') &= 2\{\lambda' + \rho r^* + 2n - 1\}g(V_1', V_2') \\ &+ 2(\mu' + 1)\eta'(V_1')\eta'(V_2'). \end{aligned} \tag{4.5.15}$$

Now, inserting $V_1' = V_2' = \xi'$ in (4.5.15) and using (1.1.21) gives $\xi'(\beta') = \lambda' + \rho r^* + \mu'$. Similarly, substituting ξ' for V_2' in (4.5.15) and using (1.1.21), we get

$$\begin{aligned} V_1'(\beta') &= \{2(\lambda' + \rho r^* + \mu') - \xi'(\beta')\}\eta'(V_1') \\ &= \xi'(\beta')\eta'(V_1'), \quad V_1' \in \chi(M). \end{aligned} \tag{4.5.16}$$

By computing the covariant derivative along $V_2' \in \chi(M)$ and using (1.1.16), we find

$$g(\nabla_{V_2'}\nabla_{\beta'}, V_1') = V_2'(\xi'(\beta'))\eta'(V_1') + \xi'(\beta')g(\phi' V_1', V_2').$$

Since $Hess_{\beta'}$ is symmetric, it shows that

$$V_1'(\xi'(\beta'))\eta'(V_2') - V_2'(\xi'(\beta'))\eta'(V_1') = \xi'(\beta')g(\phi' V_1', V_2'),$$

which yields that

$$\xi'(\beta')d\eta'(V_1', V_2') = 0, \quad \forall \quad V_1', V_2' \perp \xi',$$

by using (1.1.14). As $d\eta'$ is non-zero on M^{2n+1}, the above expression implies $\xi'(\beta') = 0$ and consequently, $\nabla\beta' = 0$ on M^{2n+1}. Therefore, we conclude that β' is constant on M^{2n+1}, and therefore, (4.5.14) implies that V is Killing vector field. Thus (M^{2n+1}, g) is $*$-Einstein(trivial). Furthermore, from (4.5.16) it follows that $\lambda' + \rho r^* + \mu' = 0$ and (4.5.15) reduces to

$$S(V_1', V_2') = (2n-1)g(V_1', V_2') + \eta'(V_1')\eta'(V_2'). \tag{4.5.17}$$

Hence, (M^{2n+1}, g) is $*$-Ricci-flat and $*$-scalar curvature $r^* = 0$, by using (4.2.12), moreover, the scalar curvature $r = 4n^2$ and $\lambda' = -\mu'$, which finishes the proof. □

In [28], Patra et al. studied compact Sasakian manifold admitting an almost $*$-RB soliton is $*$-Ricci flat under some restriction on the potential vector field V. Here, we proved this result on almost $*$-η'-Ricci-Bourguignon soliton and using one more suffient condition for g assuming compactness of the manifold i.e., potential vector field V is an infinitesimal contact transformation.

Theorem 4.5.4. *Suppose a compact Sasakian manifold $M^{2n+1}(\phi', \xi', \eta', g)$ admits an almost $*$-η'-Ricci-Bourguignon soliton, where the potential vector field V is an infinitesimal contact transformation. Then M^{2n+1} is $*$-Ricci-flat and has constant scalar curvature $4n^2$. Moreover, V is an infinitesimal automorphism and the soliton is shrinking, steady and expanding according as $\mu' < 0, \mu' = 0, \mu' > 0$ respectively for any ρ.*

Proof. First of all, plugging $V_2' = \xi'$ in (4.2.14) and then using (1.1.21), it follows that

$$(\mathcal{L}_V g)(V_1', \xi') = 2(\lambda' + \rho r^* + \mu')\eta'(V_1').$$

Using this in the expression for the Lie derivative of $\eta'(\xi') = 1$, this leads us to compute $(\mathcal{L}_V\eta')(\xi') = -\eta'(\mathcal{L}_V\xi') = \lambda' + \rho r^* + \mu'$, hence (1.1.88) leads to $\nu = \lambda' + \rho r^* + \mu'$. In light of this and (1.1.88), the Lie-derivative of $\eta'(V_1') = g(V_1', \xi')$ in the direction of V gives

$$\begin{aligned} g(\mathcal{L}_V\xi', V_1') &= (\nu - 2(\lambda' + \rho r^* + \mu'))\eta'(V_1') \\ &= -(\lambda' + \rho r^* + \mu')\eta'(V_1'), \end{aligned} \tag{4.5.18}$$

for all $V_1' \in \chi(M)$. Applying the exterior derivative of (1.1.88) to take

$$\begin{aligned}(\mathcal{L}_V d\eta')(V_1', V_2') &= d(\mathcal{L}_V \eta')(V_1', V_2') \\ &= \frac{1}{2}\{V_1'(\nu)\eta'(V_2') - V_2'(\nu)\eta'(V_1')\} + \nu d\eta'(V_1', V_2'),\end{aligned} \tag{4.5.19}$$

for all $V_1', V_2' \in \chi(M)$. Now, consider the Lie derivative of (1.1.14) in direction of V and introducing (1.1.88), (4.2.14) and (4.5.19), we obtain

$$2(\mathcal{L}_V\phi')(V_1')+2\{2(\lambda'+\rho r^*+\mu'+2n-1)-\nu\}\phi' V_1' = 4\phi' Q V_1' - V_1'(\nu)\xi' + \eta'(V_1')\nabla\nu. \tag{4.5.20}$$

Using $\phi'\xi' = 0$, it can be verified that $(\mathcal{L}_V\phi')(\xi') = 0$ and hence, substituting V_1' by ξ' in (4.5.20) and applying (1.1.16) and $\phi'(\xi') = 0$, we obtain that $\nabla\nu = \xi'(\nu)\xi'$. Therefore, from the preceding argument, ν must be constant on M^{2n+1}. Therefore, employing $\nu = \lambda' + \rho r^* + \mu'$, (1.1.7), (1.1.12) and (1.1.21), from (4.5.20), we gain

$$\begin{aligned}(\mathcal{L}_V\phi')(\phi' V_1') &= \phi'(\mathcal{L}_V\phi') \\ &= -2QV_1' + (\lambda' + \rho r^* + \mu' + 2(2n-1))V_1' \\ &- (\lambda' + \rho r^* + \mu' - 2)\eta'(V_1')\xi',\end{aligned} \tag{4.5.21}$$

by using the property $Q\phi' = \phi' Q$ of a Sasakian manifold (see [9]), In addition, Lie derivative of the first equality of (1.1.7) in the direction V yields that

$$(\mathcal{L}_V\phi')(\phi' V_1') + \phi'(\mathcal{L}_V\phi')(V_1') = (\mathcal{L}_V\eta')(V_1')\xi' + \eta'(V_1')\mathcal{L}_V\xi'. \tag{4.5.22}$$

Now, it suffices to combine (1.1.88), (4.5.18), (4.5.21) and (4.5.22) to arrive at

$$2S(V_1', V_2') = (\lambda'+\rho r^*+\mu'+2(2n-1))g(V_1', V_2') - (\lambda'+\rho r^*+\mu'-2)\eta'(V_1')\eta'(V_2'), \tag{4.5.23}$$

by using the relation $\nu = \lambda' + \rho r^* + \mu'$, and invoking equation (4.5.23) into (4.5.20), we obtain $\mathcal{L}_V\phi' = 0$. Thus, the vector field V leaves ϕ' invariant. Let λ' be the volume form of a contact metric manifold. Then $\lambda' = \eta' \wedge (d\eta')^n \neq 0$ and taking Lie derivative in the direction V together with the identity (1.1.88) yields

$$\mathcal{L}_V\lambda' = (n+1)\nu\lambda'.$$

Next, invoking the result $\mathcal{L}_V\lambda' = (divV)\lambda'$, it follows that $divV = (n+1)\nu$. Integrating this identity over compact manifold M^{2n+1} and applying the divergence theorem yields

$\nu = 0$, and consequently, $\lambda' + \rho r^* + \mu' = 0$. From equations (4.5.23) and (4.5.17), we then obtain the scalar curvature $r = 4n^2$. Thus, by equation (4.2.12), it follows that (M^{2n+1}, g) is $*$-Ricci flat, and the $*$-scalar curvature $r^* = 0$. In addition, from the soliton equation (1.2.114), it follows that V is Killing vector field. We also observe that $\lambda' + \mu' = \rho r^* = 0$, which implies that $\lambda' = -\mu'$. Hence, the soliton is shrinking, steady and expanding according as $\mu' < 0$, $\mu' = 0$, $\mu' > 0$ respectively. Moreover, from equations (4.5.18) and (1.1.88), it follows that V leaves both η' and ξ' invariant. This completes the proof of the required results. □

In the next theorem, we not only establish a sufficient condition for a manifold to be Einstein or η'-Einstein, but also provided a characterization of a non-zero potential vector field V that is collinear with the Reeb vector field ξ' on a Sasakian manifold admitting almost $*$-η'-Ricci-Bourguignon soliton. Furthermore, we study both the almost $*$-η'-Ricci-Bourguignon soliton and gradient almost $*$-η'-Ricci-Bourguignon soliton within the framework of Sasakian Geometry. It is known that an almost $*$-η'-Ricci-Bourguignon soliton satisfying equation (1.2.114) for some smooth functions λ' and μ', serves as a generalization of Ricci-Yamabe soliton. In [32], Ghosh investigated Ricci almost solitons and proved that if a Kenmotsu metric admits a gradient Ricci almost soliton and the Reeb vector field ξ' leaves the scalar curvature r invariant, then the manifold is Einstein. To generalized these findings, we consider gradient almost $*$-η'-Ricci-Bourguignon soliton on Sasakian manifold and prove the following theorem.

Theorem 4.5.5. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Sasakian manifold admitting an almost $*$-η'-Ricci-Bourguignon soliton with a non-zero potential vector field V that is pointwise collinear to the Reeb vector field ξ'. Then manifold is an η'-Einstein manifold. Moreover, if ξ' leaves the scalar curvature r invariant, then (M, g) is an Einstein manifold with $\tau = 1 + (\lambda' + \rho r^*)$.*

Proof. Since $V = \tau\xi'$, for some smooth function τ on M, it follows that

$$(\mathcal{L}_V g)(V_1', V_2') = V_1'(r)\eta'(V_2') + V_2'(r)\eta'(V_1') + 2\tau\{g(V_1', V_2') - \eta'(V_1')\eta'(V_2')\}, \tag{4.5.24}$$

for any given vector fields V_1' and $V_2' \in \chi(M)$. For this condition the soliton equation (1.2.114) transforms into

$$\begin{aligned} &2S(X, Y) + V_1'(\tau)\eta'(V_2') + V_2'(\tau)\eta'(V_1') + \{-2(\lambda' + \rho r^*) \\ &+2(2n-1) - 2\lambda'\}g(V_1', V_2') = 2(\tau + \mu' - 1)\eta'(V_1')\eta'(V_2'), \end{aligned} \tag{4.5.25}$$

for any vector field V_1', $V_2' \in \chi(M)$. Now putting $V_1' = \xi'$ and $V_2' = \xi'$ in (4.5.25) and using (1.1.21), we obtain $\xi'(\tau) = \lambda' + \rho r^* + \mu'$. Thus putting in (4.5.25) yields $V_1'(\tau) = \{\lambda' + \rho r^* + \mu'\}\eta'(V_1')$, similarly $V_2'(\tau) = \{\lambda' + \rho r^* + \mu'\}\eta'(V_2')$. Using these two values, (4.5.25) implies that

$$\begin{aligned} S(V_1', V_2') &= \{\lambda' + \rho r^* - (2n-1) - \tau\}g(V_1', V_2') \\ &+ \{\tau - (\lambda' + \rho r^*) - 1\}\eta'(V_1')\eta'(V_2'). \end{aligned} \tag{4.5.26}$$

Hence, equation (4.5.26) follows that (M, g) is η'-Einstein manifold. Moreover, if the Reeb vector field ξ' leaves the scalar curvature r invariant, i.e., $\xi'(r) = 0$, then we recall that on a Sasakian manifold, the Ricci operator and contact metric structure ϕ' commute [37]. Therefore, we obtain the relation

$$\nabla_{\xi'} Q = Q\phi' - \phi' Q.$$

Applying this identity in the trace of (4.2.19) yields

$$\tau = 1 + (\lambda' + \rho r^*).$$

Also, from equation (4.5.26), we have

$$S(V_1', V_2') = -2ng(V_1', V_2').$$

Which shows that (M, g) is an Einstein manifold, which completes the theorem. □

Recently, Ghosh et al. [31] proved that if a Sasakian manifold of dimension $(2n + 1)$ admits a $*$-Ricci soliton, then the manifold is either positive Sasakian, or null-Sasakian. In the former case, the soliton vector field is Killing, while in the latter, the soliton vector field preserves the tensor field ϕ'. In this work, we extend and define this result in the context of an almost $*$-η'-Ricci-Bourguignon soliton, providing a deeper interpretation of the role of the potential vector field V. We now establish the following result.

Theorem 4.5.6. *Let $M^{2n+1}(\phi', \xi', \eta', g)$ be a Sasakian manifold admitting a non-trivial almost $*$-η'-Ricci-Bourguignon soliton with the potential vector field V is a jacobi field along trajectories of the Reeb vector field ξ' and V leaves the structure tensor ϕ' invariant. Moreover, the soliton is either expanding if $\varepsilon - \mu' - \rho r^* > 0$ or shrinking if $\varepsilon - \mu' - \rho r^* < 0$.*

Proof. Firstly, we remind the very well-known formula, see in [88]:

$$\nabla_{V_2'}\nabla_{V_1'}V - \nabla_{V_1'}\nabla_{V_2'}V + R(V, V_2')V_1' = (\mathcal{L}_V\nabla)(V_2', V_1'), \quad V_1', V_2' \in \chi(M).$$

Then, plugging $V_1' = V_2' = \xi'$ in the preceding formula, by invoking (4.5.1) (the gradient of the soliton), we obtain

$$\begin{aligned} \nabla_{\xi'}\nabla_{\xi'}V + R(V,\xi')\xi' &= (\mathcal{L}_V\nabla)(\xi',\xi') \\ &= 2\xi'(\lambda' + \rho r^* + \mu') - \nabla(\lambda' + \rho r^* + \mu'), \end{aligned} \tag{4.5.27}$$

where we have also used (1.1.21) and $\nabla_{\xi'}\xi' = 0$(that formula follows from (1.1.16)). By hypothesis, V is a jacobi field along trajectories of ξ', i.e.,

$$\nabla_{\xi'}\nabla_{\xi'}V + R(V,\xi')\xi' = 0.$$

Let $\varepsilon = \lambda' + \rho r^* + \mu'$. By substituting the preceding equation into (4.5.27) yields $2\xi'(\varepsilon)\xi' = \nabla\varepsilon$. It is also worth noting that, from (1.1.16), we have

$$2\{V_1'(\xi'(\varepsilon))\eta'(V_2') - \xi'(\varepsilon)g(\phi'V_1', V_2')\} = g(\nabla_{V_1'}\nabla\varepsilon, V_2'), \quad V_1', V_2' \in \chi(M). \tag{4.5.28}$$

Since $Hess_\varepsilon$ is symmetric and ϕ' is skew-symmetric, by (1.1.14) and (4.5.28), we attain

$$\xi'(\varepsilon)d\eta'(V_1', V_2') = 0, \quad \forall V_1', V_2' \perp \xi'.$$

Since $d\eta'$ non-zero on M^{2n+1}, it follows that $\xi'(\varepsilon) = 0$ and consequently, $\nabla\varepsilon = 0$; hence $\varepsilon = \lambda' + \rho r^* + \mu'$ is constant on M^{2n+1}. Thus (M^{2n+1}, g) admits a $*$-Ricci soliton, This completes the proof. □

4.6 Conclusion and physical application

The concept of an almost $*$-Ricci–Bourguignon soliton is a newly emerging geometric notion that has recently attracted attention due to its relevance not only in the study of differentiable manifolds but also in the broader context of mathematical physics. It holds particular significance in fields such as quantum cosmology, quantum gravity, and black hole physics. This structure encapsulates both geometric and physical interpretations, including applications to relativistic viscous fluid spacetimes that admit heat flux and stress, as well as models involving dark matter, dust-filled universes, and the radiation-dominated era in general relativity. The framework of almost $*$-Ricci–Bourguignon solitons and their gradient forms offers valuable insight into the renormalization group flow of mass in two-dimensional spacetime models. These solitons are of particular importance because they provide a geometric perspective on physical quantities such as energy and entropy in the context of general relativity. Their behavior closely resembles that of the classical heat equation, which governs the evolution of temperature in an isolated system tending toward thermal equilibrium through heat dissipation.

Solitons are actually waves that physically cultivate with some energy loss and maintain their speed and shape after shattering with another wave of a similar kind. In this article, we have employed techniques from local Riemannian and semi-Riemannian geometry to investigate solutions of equation (1.2.113) and to construct Einstein metrics within a broad class of metrics characterized by almost $*$-η'-Ricci-Bourguignon solitons on contact geometric structures, with a particular focus on Kenmotsu manifolds. The findings presented herein contribute meaningfully to the field of differential geometry and enrich the understanding of soliton structures in contact geometry. Moreover, both almost $*$-η'-Ricci-Bourguignon solitons and their gradient counterparts have significant implications in mathematical physics, especially within the frameworks of general relativity and quantum cosmology, and offer potential directions for further exploration in complex geometry. The primary objective of this work is to explore the geometric properties and structures associated with these solitons on contact Riemannian manifolds.

By examining the kinetic and potential characteristics of relativistic spacetime, we construct physical models—namely shrinking, steady, and expanding—within the framework of perfect and dust fluid solutions of almost $*$-Ricci–Bourguignon soliton spacetimes, offering valuable applications to cosmology and general relativity. In the first case, the soliton is *shrinking* ($\lambda' < 0$) existing on a minimal time interval $-1 < t < b$ where $b < 1$; in the *steady* case ($\lambda' = 0$), the solution exists for all time; and in the *expanding* case ($\lambda' > 0$), it exists on a maximal time interval $a < t < 1$, where $a > -1$. These three classes give an example of ancient, eternal and immortal solutions, respectively. By [26, 86] (briefly discussed in the above) we can think more about physical applications of *almost $*$-Ricci-Bourguignon soliton* and *almost $*$-η'-Ricci-Bourguignon soliton.* There are some questions arises from our article to study in further research:
(i) Is the **Theorem** (4.3.2) true if we consider a non-zero potential vector field V is not collinear to the Reeb vector field ξ'?
(ii) Is the **Theorem** (4.3.3) true if the scalar curvature r is not invariant?
(iii) Is the **Theorem** (4.5.1) true if ξ' leaves the scalar curvature r not invariant?

(iv) Is the **Theorem**(4.5.2) true if V_1' is not a jacobi field on the ξ'-integral curves?

(v) Is the **Theorem**(4.5.4) true if the potential vector field V is not an infinitesimal contact transformation?

(vi) Is the **Theorem**(4.5.5) true if we consider a non-zero potential vector field V is not collinear to the Reeb vector field ξ'?

(vii) Which of the results are also true for nearly Kenmotsu manifolds, f-Kenmotsu manifolds or Kähler manifolds?

5

A Study of Trans-Sasakian manifold

5.1 Introduction

The objective of this chapter is to characterize trans-Sasakian space forms that satisfy specific curvature conditions involving different curvature tensor. Firstly, we study $\mathcal{W}_2$ - semisymmetric and $\mathcal{W}_2$- pseudosymmeric trans-Sasakian space form, $\mathcal{W}_2$-locally symmetric trans-Sasakian space form, $\mathcal{W}_2$- locally ϕ'- symmetric trans-Sasakian space form and $\mathcal{W}_2$ -ϕ'-recurrent trans-Sasakian space form. Some of these results are in the form of necessary and sufficient conditions.

Next, we study conformal η'-Einstein solitons within the framework of three-dimensional trans-Sasakian manifolds with parallel structure function L. We explore the existence of such solitons and derive several results concerning trans-Sasakian manifolds admitting conformal η'-Einstein solitons where the Ricci tensor is cyclic parallel or of Codazzi type. Furthermore, we investigate the impact of various curvature conditions, including the $\mathcal{M}$-projective curvature condition, $\mathcal{W}_2$ curvature condition, and the $\mathcal{C}$-Bochner curvature condition, in the presence of conformal η'-Einstein solitons. Additionally, we examine the role of torse-forming vector fields in this context. Finally, we provide an explicit example of a conformal η'-Einstein soliton on a trans-Sasakian manifold to illustrate our theoretical results.

5.2 Preliminaries

Definition 5.2.1. *A* $(2n+1)$ *dimensional Riemannian manifold* M^{2n+1} *is said to be pseudosymmetric, if [72]*

$$(R(V_1', V_2').R)(U, V)W = L_R\{((V_1' \wedge V_2').R)(U, V)W\}. \tag{5.2.1}$$

Where L_R *is some smooth function on* $U_R = \{V_1' \in M^{2n+1} | R - \frac{r}{n(n-1)}G \neq 0 \text{ at } V_1'\}$, *where* G *is the* $(0,4)$*-tensor defined by* $G(V_1', V_2', V_3', V_4') = g((V_1' \wedge V_2')V_3', V_4')$ *and* $(V_1' \wedge V_2')V_3'$ *is the endomorphism and it is defined by*

$$(V_1' \wedge V_2')V_3' = g(V_2', V_3')V_1' - g(V_1', V_3')V_2'. \tag{5.2.2}$$

A trans-Sasakian 3-manifold (M, g) is said to be Einstein semi-symmetric [75] if the Riemann curvature tensor R satisfies the condition:

$$R \cdot E = 0,$$

where E denotes the Einstein tensor given by

$$E(V_1', V_2') = S(V_1', V_2') - \frac{r}{2}g(V_1', V_2'), \tag{5.2.3}$$

for all vector fields $V_1', V_2' \in TM$ and r is the scalar curvature of the manifold.

Lemma 5.2.1. *A* 3*-dimensional Einstein semi-symmetric trans-Sasakian manifold is necessarily* η'*-Einstein manifold.*

Proof. Let $(M^3, \phi', \xi', \eta', g)$ be a 3-dimensional trans-Sasakian manifold. Suppose M^3 is Einstein semi-symmetric, i.e., the curvature condition $R \cdot E = 0$. Then for all the vector fields $V_1', V_2', V_3', V_4' \in TM$, we can write

$$E(R(V_1', V_2')V_3', V_4') + E(V_3', R(V_1', V_2')V_4') = 0. \tag{5.2.4}$$

In view of (5.2.3), the equation (5.2.4) becomes

$$S(R(V_1', V_2')V_3', V_4') + S(V_3', R(V_1', V_2')V_4') = \frac{r}{2}[g(R(V_1', V_2')V_3', V_4') + g(V_3', R(V_1', V_2')V_4')]. \tag{5.2.5}$$

Replacing $V_1' = V_3' = \xi'$ in the above equation (5.2.5) and then using (1.1.61), (1.1.62), we arrive at

$$(\alpha'^2 - \beta'^2)S(V_2', V_4') = (\alpha'^2 - \beta'^2)[\eta'(V_2')S(\xi', V_4') + \eta'(V_4')S(\xi', V_2') - g(V_2', V_4')S(\xi', \xi')]. \tag{5.2.6}$$

Equation (5.2.6) along with (1.1.64) implies that

$$S(V_2', V_4') = -2(\alpha'^2 - \beta'^2)g(V_2', V_4') + 4(\alpha'^2 - \beta'^2)\eta'(V_2')\eta'(V_4'), \tag{5.2.7}$$

for all $V_2', V_4' \in TM$. This implies that tha manifold is an η'-Einstein manifold. □

5.3 $\mathcal{W}_2$-curvature tensor on Trans-Sasakian space form

Theorem 5.3.1. *A $(2n+1)$ dimensional $(n > 1)$ trans-Sasakian space form satisfies $R.\mathcal{W}_2 = 0$ if and only if $\alpha' = \beta'$ or $\mathcal{W}_2 = 0$.*

Proof. A $(2n+1)$ dimensional trans-Sasakian space form is called $\mathcal{W}_2$-semisymmetric [19] if it is satisfies $R.\mathcal{W}_2 = 0$, where R is the Riemannian curvature tensor of the space form

$$i.e. \ \ (R(V_1', V_2').\mathcal{W}_2)(U, V)V_3' = 0. \tag{5.3.1}$$

Putting $V_1' = \xi'$ in (5.3.1) it can be written as

$$\begin{aligned} &R(\xi', V_2')\mathcal{W}_2(U, V)V_3' - \mathcal{W}_2(R(\xi', V_2')U, V)V_3' \\ &-\mathcal{W}_2(U, R(\xi', V_2')V)V_3' - \mathcal{W}_2(U, V)R(\xi', V_2')V_3' = 0. \end{aligned} \tag{5.3.2}$$

In the view of (1.1.56) the foregoing expression becomes

$$\begin{aligned} &(\alpha' - \beta')\{g(V_2', \mathcal{W}_2(U, V)V_3')\xi' - \eta'(\mathcal{W}_2(U, V)V_3')V_2' \\ -\ & g(V_2', U)\mathcal{W}_2(\xi', V)V_3' + \eta'(U)\mathcal{W}_2(V_2', V)V_3' \\ -\ & g(V_2', V)\mathcal{W}_2(U, \xi')V_3' + \eta'(V)\mathcal{W}_2(U, V_2')V_3' \\ -\ & g(V_2', V_3')\mathcal{W}_2(U, V)\xi' + \eta'(V_3')\mathcal{W}_2(U, V)V_2'\} = 0. \end{aligned} \tag{5.3.3}$$

Now, we take an inner product of the foregoing equation with ξ' and using equation (1.1.8), (1.1.10), (1.1.12) and (1.1.11), we get

$$\begin{aligned} &(\alpha' - \beta')\{\mathcal{W}_2(U, V, V_3', V_2') - \eta'(\mathcal{W}_2(U, V)V_3')\eta'(V_2') \\ -\ & g(V_2', U)\eta'(\mathcal{W}_2(\xi', V)V_3') + \eta'(U)\eta'(\mathcal{W}_2(V_2', V)V_3') \\ -\ & g(V_2', V)\eta'(\mathcal{W}_2(U, \xi')V_3') + \eta'(V)\eta'(\mathcal{W}_2(U, V_2')V_3') \\ -\ & g(V_2', V_3')\eta'(\mathcal{W}_2(U, V)\xi') + \eta'(V_3')\eta'(\mathcal{W}_2(U, V)V_2')\} = 0, \end{aligned} \tag{5.3.4}$$

where $\mathcal{W}_2(U,V,V_3',V_2') = g(V_2', \mathcal{W}_2(U,V)V_3')$ and using (1.1.50), we get

$$(\alpha' - \beta')\mathcal{W}_2(U,V,V_3',V_2') = 0. \tag{5.3.5}$$

This shows that either $\alpha' = \beta'$ or $\mathcal{W}_2(U,V,V_3',V_2') = 0$. □

Converse part is obvious from equation (5.3.1) and (5.3.5).

Every pseudosymmetric manifold is semisymmetric but semisymmetric manifold need not be pseudosymmetric.

Theorem 5.3.2. *If M^{2n+1} is $\mathcal{W}_2$-pseudosymmetric trans-Sasakian space form then M^{2n+1} is either $\mathcal{W}_2$-flat or $L_{\mathcal{W}_2} = \alpha' - \beta'$ if $\alpha' \neq \beta'$.*

Proof. A $(2n+1)$ dimensional trans-Sasakian space form M^{2n+1}, where $n > 1$, is said to be $\mathcal{W}_2$-pseudosymmetric, if

$$(R(V_1',V_2').\mathcal{W}_2)(U,V)W = L_{\mathcal{W}_2}\{((V_1' \wedge V_2').\mathcal{W}_2)(U,V)W\} \tag{5.3.6}$$

holds on the set $U_{\mathcal{W}_2} = \{V_1' \in M^{2n+1} | \mathcal{W}_2 \neq 0 \text{ at } V_1'\}$, where $L_{\mathcal{W}_2}$ is some function on $U_{\mathcal{W}_2}$.

Suppose that trans-Sasakian space form is $\mathcal{W}_2$-pseudosymmetric. Now, the left hand side of (5.3.6) with putting $V_1' = \xi'$, it reduces to

$$\begin{aligned} &R(\xi',V_2')\mathcal{W}_2(U,V)V_3' - \mathcal{W}_2(R(\xi',V_2')U,V)V_3' \\ &-\mathcal{W}_2(U,R(\xi',V_2')V)V_3' - \mathcal{W}_2(U,V)R(\xi',V_2')V_3'. \end{aligned} \tag{5.3.7}$$

In the view of (1.1.56) the above expression becomes

$$\begin{aligned} &(\alpha' - \beta')\{g(V_2', \mathcal{W}_2(U,V)V_3')\xi' - \eta'(\mathcal{W}_2(U,V)V_3')V_2' \\ - \ &g(V_2',U)\mathcal{W}_2(\xi',V)V_3' + \eta'(U)\mathcal{W}_2(V_2',V)V_3' \\ - \ &g(V_2',V)\mathcal{W}_2(U,\xi')V_3' + \eta'(V)\mathcal{W}_2(U,V_2')V_3' \\ - \ &g(V_2',V_3')\mathcal{W}_2(U,V)\xi' + \eta'(V_3')\mathcal{W}_2(U,V)V_2'\} = 0. \end{aligned} \tag{5.3.8}$$

Next putting $V_1' = \xi'$ in the right hand side of (5.3.6), we get

$$\begin{aligned} &L_{\mathcal{W}_2}(\xi' \wedge V_2')\mathcal{W}_2(U,V)V_3' - \mathcal{W}_2(L_{\mathcal{W}_2}(\xi' \wedge V_2')U,V)V_3' \\ &-\mathcal{W}_2(U,L_{\mathcal{W}_2}(\xi' \wedge V_2')V)V_3' - \mathcal{W}_2(U,V)L_{\mathcal{W}_2}(\xi' \wedge V_2')V_3'. \end{aligned} \tag{5.3.9}$$

By virtue of (5.2.2), the equation (5.3.9) becomes

$$\begin{aligned} & L_{\mathcal{W}_2}\{g(V_2', \mathcal{W}_2(U,V)V_3')\xi' - \eta'(\mathcal{W}_2(U,V)V_3')V_2' \\ - & g(V_2', U)\mathcal{W}_2(\xi', V)V_3' + \eta'(U)\mathcal{W}_2(V_2', V)V_3' \\ - & g(V_2', V)\mathcal{W}_2(U, \xi')V_3' + \eta'(V)\mathcal{W}_2(U, V_2')V_3' \\ - & g(V_2', V_3')\mathcal{W}_2(U,V)\xi' + \eta'(V_3')\mathcal{W}_2(U,V)V_2'\} = 0. \end{aligned} \tag{5.3.10}$$

Using the expression (5.3.8) and (5.3.10) in (5.3.6) and taking inner product with ξ', we obtain

$$\begin{aligned} & \{L_{\mathcal{W}_2} - (\alpha' - \beta')\}[g(V_2', \mathcal{W}_2(U,V)V_3') - \eta'(\mathcal{W}_2(U,V)V_3')\eta'(V_2') \\ - & g(V_2', U)\eta'(\mathcal{W}_2(\xi', V)V_3') + \eta'(U)\eta'(\mathcal{W}_2(V_2', V)V_3') \\ - & g(V_2', V)\eta'(\mathcal{W}_2(U, \xi')V_3') + \eta'(V)\eta'(\mathcal{W}_2(U, V_2')V_3') \\ - & g(V_2', V_3')\eta'(\mathcal{W}_2(U,V)\xi') + \eta'(V_3')\eta'(\mathcal{W}_2(U,V)V_2')] = 0. \end{aligned} \tag{5.3.11}$$

Using (1.1.50) as $\eta'(W_2(V_1', V_2')V_3') = 0$, we get $\{L_{\mathcal{W}_2} - (\alpha' - \beta')\}\mathcal{W}_2(U, V, V_3', V_2') = 0$, where $\mathcal{W}_2(U, V, V_3', V_2') = g(V_2', \mathcal{W}_2(U,V)V_3')$, this implies either $L_{\mathcal{W}_2} = \alpha' - \beta'$ or $\mathcal{W}_2(U, V, V_3', V_2') = 0$. This shows that M^{2n+1} is either $\mathcal{W}_2$-flat or $L_{\mathcal{W}_2} = \alpha' - \beta'$ if $\alpha' \neq \beta'$. □

Theorem 5.3.3. *A $(2n+1)$ dimensional $(n > 1)$ trans-Sasakian space form satisfying $\mathcal{W}_2 = 0$ is an η'-Einstein manifold.*

Proof. Suppose $\mathcal{W}_2 = 0$ on trans-Sasakian space form, then from equation (1.1.6) we have

$$\begin{aligned} R(V_1', V_2')V_3' &= -\frac{1}{2n}[\frac{1}{2}\{(3n-1)(\alpha' - \beta') + c(n+1)(\alpha' + \beta')\}\{g(V_1', V_3')V_2' - g(V_2', V_3')V_1'\} \\ &- \frac{n+1}{2}\{c(\alpha' + \beta') - (\alpha' - \beta')\}\{(g(V_1', V_3')\eta'(V_2') - g(V_2', V_3')\eta'(V_1'))\xi'\}]. \end{aligned} \tag{5.3.12}$$

Taking inner product of the above equation with U, we get

$$\begin{aligned} g(R(V_1', V_2')V_3', U) &= -\frac{1}{2n}[\frac{1}{2}\{(3n-1)(\alpha' - \beta') + c(n+1)(\alpha' + \beta')\}\{g(V_1', V_3')g(V_2', U) \\ &- g(V_2', V_3')g(V_1', U)\} - \frac{n+1}{2}\{c(\alpha' + \beta') - (\alpha' - \beta')\} \\ & \{g(V_1', V_3')\eta'(V_2')\eta'(U) - g(V_2', V_3')\eta'(V_1')\eta'(U)\}]. \end{aligned} \tag{5.3.13}$$

Putting $V_2' = V_3' = e_i$ in the above equation and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned} S(V_1', U) &= \frac{1}{2}\{(3n-1)(\alpha' - \beta') + c(n+1)(\alpha' + \beta')\}g(V_1', U) \\ &+ \frac{n+1}{2}\{-c(\alpha' + \beta') + (\alpha'\beta')\}\eta'(V_1')\eta'(U) \\ &= ag(V_1', U) + b\eta'(V_1')\eta'(U), \end{aligned} \tag{5.3.14}$$

where $a = \frac{1}{2}\{(3n-1)(\alpha' - \beta') + c(n+1)(\alpha' + \beta')\}$ and $b = \frac{n+1}{2}\{-c(\alpha' + \beta') + (\alpha' - \beta')\}$ are smooth functions. Which shows that M^{2n+1} is an η'-Einstein manifold. This completes the proof. □

Theorem 5.3.4. *If M^{2n+1} is $\mathcal{W}_2$-pseudosymmetric trans-Sasakian space form then M^{2n+1} is either η'-Einstein manifold or $L_{\mathcal{W}_2} = \alpha' - \beta'$ if $\alpha' \neq \beta'$.*

Theorem 5.3.5. *A $(2n+1)$ dimensional $(n > 1)$ trans-Sasakian space form is $\mathcal{W}_2$-locally symmetric.*

Proof. A $(2n+1)$ dimensional $(n > 1)$ trans-Sasakian space form is called $\mathcal{W}_2$-locally symmetric, that is, In differential geometry, W_2-locally symmetric manifolds are a generalization of locally symmetric spaces. A Riemannian manifold (M, g) is said to be W_2-locally symmetric if it satisfies [72]

$$(\nabla_W \mathcal{W}_2)(V_1', V_2')V_3' = 0, \tag{5.3.15}$$

for all vector fields V_1', V_2', V_3' orthogonal to ξ' and for an arbitrary vector field W, where ∇ is the Levi-Civita connection associated with the metric g. This condition is weaker than local symmetry, which requires $\nabla R = 0$. In W_2-local symmetry, the combination of covariant derivatives in the above form vanishes, allowing more flexibility in the curvature behavior. The concept of W_2-symmetry was introduced as part of a classification of Riemannian manifolds by curvature conditions that is, weaker than the classical local symmetry.

From (1.1.6) and (1.1.49), we have

$$
\begin{aligned}
\mathcal{W}_2(V_1',V_2')V_3' &= \frac{\alpha'(c+3)+\beta'(c-3)}{4}[g(V_2',V_3')V_1'-g(V_1',V_3')V_2'] \\
&+ \frac{\alpha'(c-1)+\beta'(c+1)}{4}[\eta'(V_1')\eta'(V_3')V_2'-\eta'(V_2')\eta'(V_3')V_1' \\
&+ g(V_1',V_3')\eta'(V_2')\xi'-g(V_2',V_3')\eta'(V_1')\xi'+g(\phi'V_2',V_3')\phi'V_1' \\
&- g(\phi'V_1',V_3')\phi'V_2'+2g(V_1',\phi'V_2')\phi'V_3'] \\
&+ \frac{1}{2n}\{g(V_1',V_3')QV_2'-g(V_2',V_3')QV_1'\}. \qquad (5.3.16)
\end{aligned}
$$

Taking covariant differentiation of (5.3.16) in the direction of the vector field W,we get

$$
\begin{aligned}
(\nabla_W\mathcal{W}_2)(V_1',V_2')V_3' &= \frac{d\alpha'(c+3)+d\beta'(c-3)}{4}(W)\{g(V_2',V_3')V_1'-g(V_1',V_3')V_2'\} \\
&+ \frac{d\alpha'(c-1)+d\beta'(c+1)}{4}(W)\{\eta'(V_1')\eta'(V_3')V_2'-\eta'(V_2')\eta'(V_3')V_1' \\
&+ g(V_1',V_3')\eta'(V_2')\xi'-g(V_2',V_3')\eta'(V_1')\xi' \\
&+ g(\phi'V_2',V_3')\phi'V_1'-g(\phi'V_1',V_3')\phi'V_2'+2g(V_1',\phi'V_2')\phi'V_3'\} \\
&+ \frac{\alpha'(c-1)+\beta'(c+1)}{4}\{(\nabla_W\eta')(V_1')\eta'(V_3')V_2'+\eta'(V_1')(\nabla_W\eta')(V_3')V_2' \\
&- (\nabla_W\eta')(V_2')\eta'(V_3')V_1'-\eta'(V_2')(\nabla_W\eta')(V_3')V_1' \\
&+ g(V_1',V_3')(\nabla_W\eta')(V_2')\xi'+g(V_1',V_3')\eta'(V_2')\nabla_W\xi' \\
&- g(V_2',V_3')(\nabla_W\eta')(V_1')\xi'-g(V_2',V_3')\eta'(V_1')\nabla_W\xi' \\
&+ g(\phi'V_2',V_3')(\nabla_W\phi')V_1'+g((\nabla_W\phi')V_2',V_3')\phi'X \\
&- g(\phi'V_1',V_3')(\nabla_W\phi')V_2'-g((\nabla_W\phi')V_1',V_3')\phi'V_2' \\
&+ 2g(V_1',\phi'V_2')(\nabla_W\phi')V_3'+2g(V_1',(\nabla_W\phi')V_2')\phi'V_3'\} \\
&+ \frac{1}{2n}\{g(V_1',V_3')(\nabla_WQ)(V_2')-g(V_2',V_3')(\nabla_WQ)(V_1')\}, \qquad (5.3.17)
\end{aligned}
$$

where ∇ denotes the Riemannian connection on the manifold. Differentiating (1.1.53) covariantly with respect to W, one can get

$$
\begin{aligned}
(\nabla_WQ)(V_1') &= \frac{1}{2}\{(3n-1)d(\alpha'-\beta')+c(n+1)d(\alpha'+\beta')\}(W)V_1' \\
&- \frac{n+1}{2}\{cd(\alpha'+\beta')-d(\alpha'-\beta')\}(W)\eta'(V_1')\xi' \\
&- \frac{n+1}{2}\{c(\alpha'+\beta')-(\alpha'-\beta')\}\{(\nabla_W\eta')(V_1')\xi'+\eta'(V_1')(\nabla_W\xi')\}.
\end{aligned}
$$
(5.3.18)

In the view of (5.3.18) and (5.3.17) it follows that

$$
\begin{aligned}
(\nabla_W \mathcal{W}_2)(V_1', V_2')V_3' &= \frac{d\alpha'(c+3) + d\beta'(c-3)}{4}(W)\{g(V_2', V_3')V_1' - g(V_1', V_3')V_2'\} \\
&+ \frac{d\alpha'(c-1) + d\beta'(c+1)}{4}(W)\{\eta'(V_1')\eta'(V_3')V_2' - \eta'(V_2')\eta'(V_3')V_1' \\
&+ g(V_1', V_3')\eta'(V_2')\xi' - g(V_2', V_3')\eta'(V_1')\xi' \\
&+ g(\phi' V_2', V_3')\phi' V_1' - g(\phi' V_1', V_3')\phi' V_2' + 2g(V_1', \phi' V_2')\phi' V_3'\} \\
&+ \frac{\alpha'(c-1) + \beta'(c+1)}{4}\{(\nabla_W \eta')(V_1')\eta'(V_3')V_2' + \eta'(V_1')(\nabla_W \eta')(V_3')V_2' \\
&- (\nabla_W \eta')(V_2')\eta'(V_3')V_1' - \eta'(V_2')(\nabla_W \eta')(V_3')V_1' \\
&+ g(V_1', V_3')(\nabla_W \eta')(V_2')\xi' + g(V_1', V_3')\eta'(V_2')\nabla_W \xi' \\
&- g(V_2', V_3')(\nabla_W \eta')(V_1')\xi' - g(V_2', V_3')\eta'(V_1')\nabla_W \xi' \\
&+ g(\phi' V_2', V_3')(\nabla_W \phi')V_1' + g((\nabla_W \phi')V_2', V_3')\phi' V_1' \\
&- g(\phi' V_1', V_3')(\nabla_W \phi')V_2' - g((\nabla_W \phi')V_1', V_3')\phi' V_2' \\
&+ 2g(V_1', \phi' V_2')(\nabla_W \phi')V_3' + 2g(V_1', (\nabla_W \phi')V_2')\phi' V_3'\} \\
&+ \frac{1}{2n}[g(V_1', V_3')\{\frac{1}{2}\{(3n-1)d(\alpha' - \beta') + c(n+1)d(\alpha' + \beta')\}(W)V_2' \\
&- \frac{n+1}{2}\{cd(\alpha' + \beta') - d(\alpha' - \beta')\}(W)\eta'(V_2')\xi' \\
&- \frac{n+1}{2}\{c(\alpha' + \beta') - (\alpha' - \beta')\}\{(\nabla_W \eta')(V_2')\xi' + \eta'(V_2')(\nabla_W \xi')\}\} \\
&- g(V_2', V_3')\{\frac{1}{2}\{(3n-1)d(\alpha' - \beta') + c(n+1)d(\alpha' + \beta')\}(W)V_1' \\
&- \frac{n+1}{2}\{cd(\alpha' + \beta') - d(\alpha' - \beta')\}(W)\eta'(V_1')\xi' \\
&- \frac{n+1}{2}\{c(\alpha' + \beta') - (\alpha' - \beta')\}\{(\nabla_W \eta')(V_1')\xi' + \eta'(V_1')(\nabla_W \xi')\}\}].
\end{aligned}
\tag{5.3.19}
$$

Taking V_1', V_2', V_3' orthogonal to ξ' in (5.3.19) and then taking the inner product of the resultant equation with V, followed by setting $V = V_3' = e_i$ in the the above equation, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and take summation over i, where $i = 1, 2, ..., 2n+1$, we get

$$\{\alpha'(c-1) + \beta'(c+1)\}2g(V_1', \phi' V_2')g((\nabla_W \phi')e_i, e_i). \tag{5.3.20}$$

For Levi-Civita connection ∇,

$$(\nabla_W g)(V_1', V_2') = 0$$

which gives

$$\nabla_W g(V_1', V_2') - g(\nabla_W V_1', V_2') - g(V_1', \nabla_W V_2') = 0.$$

Putting $V_1' = e_i$ and $V_2' = \phi' e_i$ in the above equation, we obtain

$$-g(\nabla_W e_i, \phi' e_i) - g(e_i, \nabla_W \phi' e_i) = 0,$$

which can be written as

$$g(e_i, \phi'(\nabla_W e_i)) - g(e_i, (\nabla_W \phi') e_i) = 0.$$

Thus, we have

$$\begin{aligned} g(e_i, \phi'(\nabla_W e_i) - \nabla_W \phi' e_i) &= 0 \\ \Rightarrow g(e_i, (\nabla_W \phi') e_i) &= 0. \end{aligned} \quad (5.3.21)$$

By the virtue of (5.3.20) and (5.3.21) takes the form

$$(\nabla_W \mathcal{W}_2)(V_1', V_2')V_3' = 0.$$

Therefore from (5.3.15), we say that the manifold is $\mathcal{W}_2$-locally symmetric. Therefore, the claim is proved. □

Theorem 5.3.6. *A $(2n+1)$ dimensional $n > 1$ trans-Sasakian space form M^{2n+1} is $\mathcal{W}_2$-locally ϕ'-symmetric if and only if $\alpha' = \beta'$ and $c = 0$.*

Proof. A trans-Sasakian space form M^{2n+1} of dimension greater than three is called W_2-locally ϕ'-symmetric if it satisfies

$$\phi'^2((\nabla_W(\mathcal{W}_2))(V_1', V_2')V_3') = 0, \quad (5.3.22)$$

for all vector fields V_1', V_2', V_3' orthogonal to ξ' on M^{2n+1}. Let us consider a $\mathcal{W}_2$-locally ϕ'-symmetric trans-Sasakian space form of dimension greater than three. Then from the definition of (1.1.7), we have

$$- \quad (\nabla_W \mathcal{W}_2)(V_1', V_2')V_3' + \eta'((\nabla_W \mathcal{W}_2)(V_1', V_2')V_3')\xi' = 0. \quad (5.3.23)$$

Taking the g-inner product of both sides of the above equation with U, we get

$$- \quad g((\nabla_W \mathcal{W}_2)(V_1', V_2')V_3', U) + \eta'((\nabla_W \mathcal{W}_2)(V_1', V_2')V_3')\eta'(U) = 0. \quad (5.3.24)$$

If we take U orthogonal to ξ', then the above equation yields

$$g(((\nabla_W \mathcal{W}_2)(V_1', V_2')V_3'), U) = 0. \tag{5.3.25}$$

The above equation is true for all U orthogonal to ξ', if we choose $U \neq 0$ and not orthogonal to $(\nabla_W \mathcal{W}_2)(V_1', V_2')V_3'$ then it follows that

$$(\nabla_W \mathcal{W}_2)(V_1', V_2')V_3' \;=\; 0. \tag{5.3.26}$$

Hence the manifold is $\mathcal{W}_2$-locally symmetric.
Again for V_1', V_2', V_3' orthogonal to ξ' and applying ϕ'^2 on both side to equation (5.3.19) and using (1.1.46), one can get

$$\begin{aligned}
\phi'^2(\nabla_W \mathcal{W}_2)(V_1', V_2')V_3' \;&=\; \frac{d\alpha'(c-1) + d\beta'(c+1)}{4}(W)\{g(\phi' V_1', V_3')\phi' V_2' \\
&-\; g(\phi' V_2', V_3')\phi' V_1' - 2g(V_1', \phi' V_2')\phi' V_3'\} \\
&+\; [\frac{d\alpha'(c+3) + d\beta'(c-3)}{4} - \frac{1}{4n}\{(3n-1)d(\alpha' - \beta') \\
&+\; c(n+1)d(\alpha' + \beta')\}](W)\{g(V_1', V_3')V_2' - g(V_2', V_3')V_1'\}.
\end{aligned} \tag{5.3.27}$$

Hence for $\alpha' = \beta'$ and $c = 0$, the above equation yields

$$\phi'^2(\nabla_W \mathcal{W}_2)(V_1', V_2')V_3' = 0,$$

for all V_1', V_2', V_3' orthogonal to ξ'. Therefore the manifold is $\mathcal{W}_2$-locally ϕ'-symmetric. □

Theorem 5.3.7. *If trans-Sasakian space form in M^{2n+1} is $\mathcal{W}_2$-ϕ'-recurrent, then it is an Einstein manifold provided ($\alpha' \neq \beta'$).*

Proof. A trans-Sasakian space form is said to be ϕ'-recurrent provided there exists a non-zero 1-form A for which [71]

$$\phi'^2((\nabla_W R)(V_1', V_2')V_3') = A(W)R(V_1', V_2')V_3', \tag{5.3.28}$$

for arbitrary vector fields V_1', V_2', V_3', W If the 1-form A vanishes, then the manifold becomes a ϕ'- locally symmetric manifold.
According to the definition of ϕ'-recurrent trans-Sasakian space form, we define $\mathcal{W}_2$-ϕ'-recurrent trans-Sasakian space form by

$$\phi'^2((\nabla_W \mathcal{W}_2)(V_1', V_2')V_3') \;=\; A(W)\mathcal{W}_2(V_1', V_2')V_3'. \tag{5.3.29}$$

Then by (1.1.7) and (5.3.29), we have

$$-(\nabla_W \mathcal{W}_2)(V_1', V_2')V_3' + \eta'((\nabla_W \mathcal{W}_2)(V_1', V_2')V_3')\xi' = A(W)\mathcal{W}_2(V_1', V_2')V_3', \quad (5.3.30)$$

for arbitrary vector fields V_1', V_2', V_3', W. From the above equation it follows that

$$\begin{aligned} & -g((\nabla_W \mathcal{W}_2)(V_1', V_2')V_3', U) + \eta'((\nabla_W \mathcal{W}_2)(V_1', V_2')V_3')\eta'(U) \\ = \ & A(W)g(\mathcal{W}_2(V_1', V_2')V_3', U). \end{aligned} \quad (5.3.31)$$

Let $\{e_i\}$,$i = 1, 2, ...2n + 1$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $V_1' = U = e_i$ in (5.3.31) and taking summation over i, $1 \leq i \leq 2n + 1$, we get

$$\begin{aligned} & - \frac{1}{2n}(2n+1)(\nabla_W S)(V_2', V_3') + \frac{1}{2n}g(V_2', V_3')dr(W) \\ & + \sum_{i=1}^{2n+1} \eta'((\nabla_W \mathcal{W}_2)(e_i, V_2')V_3')\eta'(e_i) \\ & = A(W)\frac{1}{2n}\{(2n+1)S(V_2', V_3') - g(V_2', V_3')r\}. \end{aligned} \quad (5.3.32)$$

Setting $V_3' = \xi'$ in (5.3.32) and then using (1.1.47), (1.1.48) and (1.1.53) then replacing V_2' by $\phi' V_2'$ in (5.3.32), we get

$$S(Y, W) = 2n(\alpha' - \beta')g(Y, W). \quad (5.3.33)$$

□

5.4 Conformal η'-Einstein soliton on Trans-Sasakian 3-manifolds with L parallel and cyclic parallel Ricci tensor

Theorem 5.4.1. *Let $(M, g, \phi', \eta', \xi', \alpha', \beta')$ be a trans-Sasakian manifold, dimM $= 3$ with α', β' constant ($\beta' \neq 0$). If the symmetric $(0, 2)$ tensor field L satisfying the condition $\beta' L(V_1', V_2') - \frac{\alpha'}{2}[L(\phi' V_1', V_2') + L(V_1', \phi' V_2')] = \mathcal{L}_{\xi'} g(V_1', V_2') + 2S(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2') - rg(V_1', V_2') + \{2\lambda' + (p + \frac{2}{3})\}g(V_1', V_2')$ is parallel with respect to the Levi-Civita connection associated to g, then $(g, \xi', \lambda', \mu')$ becomes a conformal η'-Einstein soliton.*

Proof. To investigate the existence of conformal η'-Einstein solitons on trans-Sasakian manifolds, we begin by considering a symmetric tensor field L that is parallel ($\nabla L = 0$). It then follows that.

$$L(R(V_1', V_2')V_3', V_4') + L(V_3', R(V_1', V_2')V_4') = 0, \tag{5.4.1}$$

for an arbitrary vector field V_1', V_2', V_3', V_4' on M. Using $V_3' = V_4' = \xi'$, we obtain

$$L(R(V_1', V_2')\xi', \xi') = 0, \tag{5.4.2}$$

for any $V_1', V_2' \in TM$. Using (1.1.60) and replacing V_1' by ξ', we get

$$L(V_2', \xi') = g(V_2', \xi')L(\xi', \xi'), \tag{5.4.3}$$

for any $V_2' \in TM$. By applying the covariant derivative in the direction of the vector field $V_1' \in TM$ to equation (5.4.3), we acquire

$$L(\nabla_{V_1'}V_2', \xi') + L(V_2', \nabla_{V_1'}\xi') = g(\nabla_{V_1'}V_2', \xi')L(\xi', \xi') + g(V_2', \nabla_{V_1'}\xi')L(\xi', \xi'). \tag{5.4.4}$$

Using the equation (1.1.47), we have

$$\beta' L(V_1', V_2') - \alpha' L(\phi' V_1', V_2') = -\alpha' g(\phi' V_1', V_2')L(\xi', \xi') + \beta' L(\xi', \xi')g(V_1', V_2'). \tag{5.4.5}$$

Now, we interchange V_1' by V_2' in above equation to yield

$$\beta' L(V_1', V_2') - \alpha' L(V_1', \phi' V_2') = -\alpha' g(V_1', \phi' V_2')L(\xi', \xi') + \beta' L(\xi', \xi')g(V_1', V_2'). \tag{5.4.6}$$

Then we add the above two equations (5.4.5) and (5.4.6) to achieve

$$\beta' L(V_1', V_2') - \frac{\alpha'}{2}[L(\phi' V_1', V_2') + L(V_1', \phi' V_2')] = \beta' L(\xi', \xi')g(V_1', V_2'). \tag{5.4.7}$$

We see that $\beta' L(V_1', V_2') - \frac{\alpha'}{2}[L(\phi' V_1', V_2') + L(V_1', \phi' V_2')]$ is a symmetric tensor of type $(0, 2)$. Let $\beta' L(V_1', V_2') - \frac{\alpha'}{2}[L(\phi' V_1', V_2') + L(V_1', \phi' V_2')] = \mathcal{L}_{\xi'} g(V_1', V_2') + 2S(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2') - rg(V_1', V_2') + \{2\lambda' + (p + \frac{2}{3})\}g(V_1', V_2')$.
Then, we compute

$\beta' L(\xi', \xi')g(V_1', V_2') = \mathcal{L}_{\xi'} g(V_1', V_2') + 2S(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2') - rg(V_1', V_2') + \{2\lambda' + (p + \frac{2}{3})\}g(V_1', Y)$. As L is parallel so, $L(\xi', \xi')$ is constant. It follows that we can write $L(\xi', \xi') = -\frac{1}{\beta'}(2\lambda' + (p + \frac{2}{3}))$ where β' is constant and $\beta' \neq 0$. Therefore $\mathcal{L}_{\xi'} g(V_1', V_2') + 2S(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2') - rg(V_1', V_2') = -(2\lambda' + p + \frac{2}{3})g(V_1', V_2')$ and so $(g, \xi', \lambda', \mu')$ reduces to a conformal η'-Einstein soliton. □

Corollary 5.4.1. *Let $(M, g, \phi', \eta', \xi', \alpha', \beta')$ be a trans-Sasakian manifold, dim$M = 3$ with α', β' constant $(\beta' \neq 0)$. If the symmetric $(0, 2)$ tensor field L under the condition that $\beta' L(V_1', V_2') - \frac{\alpha'}{2}[L(\phi' V_1', V_2') + L(V_1', \phi' V_2')] = \mathcal{L}_{\xi'} g(V_1', V_2') + 2S(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2')$ is parallel with respect to the Levi-Civita connection related to g, then (g, ξ', μ') reduces to an η'-Ricci soliton.*

Next we derive some results for 3-dimensional trans-Sasakian manifolds admitting a conformal η'-Einstein soliton, under the assumption that the manifold is Ricci-symmetric and has an η'-recurrent Ricci curvature tensor.

Theorem 5.4.2. *Let (M, g) be a trans-Sasakian manifold, dim$M = 3$ with α', β' constant $(\beta' \neq 0)$ admitting conformal η'-Einstein soliton.*
(i) If the manifold (M, g) is Ricci symmetric (i.e., $\nabla S = 0$) then $\mu' = \beta'$.
(ii) If the Ricci tensor is η'-recurrent (i.e., $\nabla S = \eta' \otimes S$) then $\mu' = 2\beta' - \frac{\alpha'^2}{\beta'}$.

Proof. From the equation (1.2.98), we get

$$2S(V_1', V_2') = -g(\nabla_{V_1'}\xi', V_2') - g(V_1', \nabla_{V_2'}\xi') - [2\lambda' - r + (p + \frac{2}{3})]g(V_1', V_2') - 2\mu'\eta'(V_1')\eta'(V_2'). \tag{5.4.8}$$

Now, we use the equation (1.1.47) into the identity (5.4.8) to yield

$$S(V_1', V_2') = [\frac{r}{2} - \lambda' - \beta' - (\frac{p}{2} + \frac{1}{3})]g(V_1', V_2') + (\beta' - \mu')\eta'(V_1')\eta'(V_2') \tag{5.4.9}$$

and

$$S(V_1', \xi') = \{\frac{r}{2} - \lambda' - (\frac{p}{2} + \frac{1}{3}) - \mu'\}\eta'(V_1'). \tag{5.4.10}$$

Also employing the identity (1.1.64) to (5.4.10), we obtain

$$\frac{r}{2} - \lambda' - (\frac{p}{2} + \frac{1}{3}) - \mu' = 2(\alpha'^2 - \beta'^2). \tag{5.4.11}$$

The Ricci operator Q is defined by the relation $g(QV_1', V_2') = S(V_1', V_2')$. Then, we get

$$QV_1' = \{\mu' - \beta' + 2(\alpha'^2 - \beta'^2)\}V_1' + (\beta' - \mu')\eta'(V_1')\xi'. \tag{5.4.12}$$

(i) Suppose that the manifold (M, g) is Ricci symmetric i.e.,

$$\nabla S = 0. \tag{5.4.13}$$

At this point, we obtain

$$\nabla_{V_1'} S(V_2', V_3') = V_1' S(V_2', V_3') - S(\nabla_{V_1'} V_2', V_3') - S(\nabla_{V_1'} V_3', V_2').$$

Using the equation (5.4.9) and (5.4.13), we obtain

$$(\beta' - \mu')[-\alpha'\{g(\phi' V_1', V_2') + g(\phi' V_1', V_3')\} + \beta'\{g(V_1', V_2')\eta'(V_3') - g(V_1', V_3')\eta'(V_2')\} - 2\beta'\eta'(V_1')\eta'(V_2')\eta'(V_3')] = 0.$$

By putting $V_2' = V_3' = \xi'$, the preceding expression implies $\mu' = \beta'$.
(ii) Suppose that the Ricci tensor is η'-recurrent, i.e.,

$$\nabla S = \eta' \otimes S. \tag{5.4.14}$$

Now, we have

$$\nabla_{V_1'} S(V_2', V_3') = \eta'(V_1') S(V_2', V_3'), \tag{5.4.15}$$

for all vector fields V_1', V_2', V_3'. Using the equations (5.4.9) and (5.4.15), we obtain $\mu' = 2\beta' - \frac{\alpha'^2}{\beta'}$. Hence, we complete the proof. □

Next, we investigate conformal η'-Einstein solitons on 3-dimensional trans-Sasakian manifolds whose Ricci tensor satisfies certain special conditions.

Theorem 5.4.3. *If a trans-Sasakian 3-manifold (M, g) represents a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$ then the manifold (M, g) reduces to an η'-Einstein manifold of constant scalar curvature $r = 6(\frac{p}{2} + \frac{1}{3}) + 6\lambda' + 4\beta' + 2\mu'$. Moreover, the soliton is shrinking, steady or expanding according as $\alpha'^2 < \beta'(\beta' + 1) + (\frac{p}{2} + \frac{1}{3})$, $\alpha'^2 = \beta'(\beta' + 1) + (\frac{p}{2} + \frac{1}{3})$, $\alpha'^2 > \beta'(\beta' + 1) + (\frac{p}{2} + \frac{1}{3})$ respectively.*

Proof. Let us assume that a trans-Sasakian 3-manifold (M, g) representing a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$ then from the equation (1.2.98), we can derive

$$(\mathcal{L}_{\xi'} g)(V_1', V_2') + 2S(V_1', V_2') + [2\lambda' - r + (p + \frac{2}{3})]g(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2') = 0, \tag{5.4.16}$$

for all $V_1', V_2' \in TM$.
Using $(\mathcal{L}_{\xi'} g)(V_1', V_2') = g(\nabla_{V_1'}\xi', V_2') + g(\nabla_{V_2'}\xi', V_1')$ and equation (1.1.47), we get

$$(\mathcal{L}_{\xi'} g)(V_1', V_2') = 2\beta'[g(V_1', V_2') - \eta'(V_1')\eta'(V_2')]. \tag{5.4.17}$$

Using equations (5.4.16) and (5.4.17), we achieve

$$S(V_1', V_2') = [\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta']g(V_1', V_2') + (\beta' - \mu')\eta'(V_1')\eta'(V_2'). \tag{5.4.18}$$

It follows that, (M, g) is an η'-Einstein manifold.
We now insert $V_2' = \xi'$ into (5.4.18) to find

$$S(V_1', \xi') = [\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \mu']\eta'(V_1'). \tag{5.4.19}$$

Comparing the above equation (5.4.19) with the identity (1.1.64), we obtain

$$r = 4(\alpha'^2 - \beta'^2) + 2(\frac{p}{2} + \frac{1}{3}) + 2\lambda' + 2\mu'. \tag{5.4.20}$$

Taking an orthonormal basis $\{e_1, e_2, e_3\}$ of (M, g) and then substituting $V_1' = V_2' = e_i$ in the equation (5.4.18) and summation over i, we obtain

$$r = 6(\frac{p}{2} + \frac{1}{3}) + 6\lambda' + 4\beta' + 2\mu'. \tag{5.4.21}$$

Finally combining equation (5.4.20) and (5.4.21), we arrive at

$$\lambda' = (\alpha'^2 - \beta'^2) - \beta' - (\frac{p}{2} + \frac{1}{3}). \tag{5.4.22}$$

This completes the proof. □

Theorem 5.4.4. *Suppose a trans-Sasakian 3-manifold (M, g) admits a conformal η'-Einstein soliton $(g, \mathcal{V}, \lambda', \mu')$, where $\mathcal{V}$ is pointwise collinear with ξ', then $\mathcal{V}$ must be a constant multiple of ξ' and (M, g) is an η'-Einstein manifold of constant scalar curvature.* $r = 2\lambda' + 2(\frac{p}{2} + \frac{1}{3}) + 2\mu' + 4(\alpha'^2 - \beta'^2)$.

Proof. Considering a trans-Sasakian 3-manifold (M, g) that represents a conformal η'-Einstein soliton $(g, \mathcal{V}, \lambda', \mu')$ where $\mathcal{V}$ is parallel to ξ', i.e. $\mathcal{V} = b\xi'$, for some function b, and using equation (1.2.98), implies that

$$\begin{aligned} bg(\nabla_{V_1'}\xi', V_2') \quad &+ \quad (V_1'b)\eta'(V_2') + bg(\nabla_{V_2'}\xi', V_1') + (V_2'b)\eta'(V_1') \\ +2S(V_1', V_2') \quad &+ \quad [2\lambda' - r + 2(\frac{p}{2} + \frac{1}{3})]g(V_1', V_2') + 2\mu'\eta'(V_1')\eta'(V_2') = 0. \end{aligned} \tag{5.4.23}$$

Then we utilize the identity (1.1.47) in the above equation (5.4.23) to get

$$\begin{aligned} [2b\beta' + 2\lambda' - r \quad &+ \quad 2(\frac{p}{2} + \frac{1}{3})]g(V_1', V_2') + (V_1'b)\eta'(V_2') \\ +(V_2'b)\eta'(V_1') \quad &+ \quad 2S(V_1', V_2') + 2(\mu' - b\beta')\eta'(V_1')\eta'(V_2') = 0. \end{aligned} \tag{5.4.24}$$

Now, we insert $V_2' = \xi'$ into the identity (5.4.24) to yield

$$[2\lambda' - r + 2(\frac{p}{2} + \frac{1}{3}) + 2\mu']\eta'(V_1') + (V_1'b) + (\xi'b)\eta'(V_1') + 2S(V_1', \xi') = 0. \tag{5.4.25}$$

Again taking $V_1' = \xi'$ in the above equation (5.4.25) and by virtue of (1.1.64), we acquire

$$2(\xi'b) = (r - 2\lambda' - 2(\frac{p}{2} + \frac{1}{3}) - 2\mu') - 4(\alpha'^2 - \beta'^2). \tag{5.4.26}$$

Using the value from (5.4.26) in the equation (5.4.25) and recalling (1.1.64), we may express this as

$$db = [\frac{r}{2} - \lambda' - (\frac{p}{2} + \frac{1}{3}) - \mu' - 2(\alpha'^2 - \beta'^2)]\eta'. \tag{5.4.27}$$

Now, applying the exterior derivative to both sides of equation (5.4.27), we get

$$r = 2\lambda' + 2(\frac{p}{2} + \frac{1}{3}) + 2\mu' + 4(\alpha'^2 - \beta'^2). \tag{5.4.28}$$

In view of the above identity (5.4.28), the equation (5.4.27) gives $db = 0$ i.e., the function b is constant. Hence the equation (5.4.24) becomes

$$S(V_1', V_2') = [\frac{r}{2} - \lambda' - b\beta' - (\frac{p}{2} + \frac{1}{3})]g(V_1', V_2') + (b\beta' - \mu')\eta'(V_1')\eta'(V_2'), \tag{5.4.29}$$

for all $V_1', V_2' \in TM$. This completes the proof. □

Theorem 5.4.5. *Let (M, g) be a trans-Sasakian 3-manifold admitting a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$. If the Ricci tensor of the manifold is of Codazzi type then the manifold becomes a β'-Kenmotsu manifold provided $\mu' \neq \beta'$.*

Proof. A Ricci tensor S is said to be of Codazzi type on a trans-Sasakian 3-manifold if it is non-zero and satisfies the following condition

$$(\nabla_{V_1'} S)(V_2', V_3') = (\nabla_{V_2'} S)(V_1', V_3'), \qquad \forall V_1', V_2', V_3' \in TM. \tag{5.4.30}$$

We consider a trans-Sasakian 3-manifold that has Codazzi type Ricci tensor and admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, then equation (5.4.18) holds. Taking covariant derivative in equation (5.4.18) and using (1.1.48), we conclude

$$\begin{aligned}(\nabla_{V_1'} S)(V_2', V_3') &= (\beta' - \mu')[\eta'(V_3')(-\alpha' g(\phi' V_1', V_2') + \beta' g(\phi' V_1', \phi' V_2')) \\ &+ \eta'(V_2')(-\alpha' g(\phi' V_1', V_3') + \beta' g(\phi' V_1', \phi' V_3'))].\end{aligned} \tag{5.4.31}$$

Also, we have

$$\begin{aligned}(\nabla_{V_2'} S)(V_1', V_3') &= (\beta' - \mu')[\eta'(V_3')(-\alpha' g(\phi' V_2', V_1') + \beta' g(\phi' V_2', \phi' V_1')) \\ &+ \eta'(V_1')(-\alpha' g(\phi' V_2', V_3') + \beta' g(\phi' V_2', \phi' V_3'))].\end{aligned} \tag{5.4.32}$$

Given that the Ricci tensor is of Codazzi type, we use (5.4.31) and (5.4.32) in the equation (5.4.30) and then recalling (1.1.9), to derive the following

$$\begin{aligned}&(\beta' - \mu')[\eta'(V_2')(-\alpha' g(\phi' V_1', V_3') + \beta' g(V_1', V_3')) - \eta'(V_1')(-\alpha' g(\phi' V_2', V_3') \\ &+\beta' g(V_2', V_3')) - 2\alpha'\eta'(V_3')g(\phi' V_1', V_2')] = 0.\end{aligned} \tag{5.4.33}$$

Now, we put $V_3' = \xi'$ in the foregoing equation (5.4.33) and view of (5.4.28) to derive

$$2\alpha'(\beta' - \mu')g(\phi' V_1', V_2') = 0, \tag{5.4.34}$$

for all $V_1', V_2' \in TM$. Therefore from (5.4.34) we can conclude that either $\alpha' = 0$ or $\beta' = \mu'$. □

We use $\alpha' = 0$ in the equation (5.4.22), and get $\lambda' = -\{\beta'(\beta' + 1) + (\frac{p}{2} + \frac{1}{3})\}$. This leads to the following conclusion:

Corollary 5.4.2. *Let (M, g) be a trans-Sasakian 3-manifold admitting a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$ with $\beta' \neq \mu'$. If the Ricci tensor of the manifold is of Codazzi type, then the soliton is shrinking if $\beta'(\beta'+1)+(\frac{p}{2}+\frac{1}{3}) > 0$, steady if $\beta'(\beta'+1)+(\frac{p}{2}+\frac{1}{3}) = 0$ and expanding if $\beta'(\beta' + 1) + (\frac{p}{2} + \frac{1}{3}) < 0$ respectively.*

Once again, from equation (5.4.33), it follows that $\mu' = \beta'$ if $\alpha' \neq 0$. Substituting this into equation (5.4.18) we obtain

$$S(V_1', V_2') = [\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta']g(V_1', V_2'), \tag{5.4.35}$$

for all $V_1', V_2' \in TM$. Then, by contracting equation (5.4.34) we obtain $r = 6(\frac{p}{2} + \frac{1}{3}) + 6\lambda' + 6\beta'$. Therefore, combining this result with equation (5.4.34), we obtain the following theorem:

Theorem 5.4.6. *Let (M,g) be a trans-Sasakian 3-manifold possessing a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$. If the Ricci tensor of the manifold is of Codazzi type then the manifold becomes an Einstein manifold of constant scalar curvature $r = 6(\frac{p}{2}+\frac{1}{3})+6\lambda'+6\beta'$ provided $\alpha' \neq 0$.*

Theorem 5.4.7. *Suppose (M, g) be a trans-Sasakian 3-manifold that admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$. If the Ricci tensor of M is cyclic parallel, then M is an α'-Sasakian manifold, provided $\mu' \neq \beta'$.*

Proof. A trans-Sasakian 3-manifold is said to possess a cyclic parallel Ricci tensor if its Ricci tensor S is non-zero and satisfies the following relation

$$(\nabla_{V_1'} S)(V_2', V_3') + (\nabla_{V_2'} S)(V_1', V_3') + (\nabla_{V_3'} S)(V_1', V_2') = 0, \forall V_1', V_2', V_3' \in TM. \tag{5.4.36}$$

Let us consider a trans-Sasakian 3-manifold endowed with a cyclic parallel Ricci tensor and admitting a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, then equation (5.4.18) holds. On taking covariant derivative in equation (5.4.18) and using equation (1.1.48), we obtain relations (5.4.31) and (5.4.32). Similarly we have

$$\begin{aligned}(\nabla_{V_3'} S)(V_1', V_2') &= (\beta' - \mu')[\eta'(V_1')(-\alpha' g(\phi' V_3', V_2') + \beta' g(\phi' V_3', \phi' V_2')) \\ &+ \eta'(V_2')(-\alpha' g(\phi' V_3', V_1') + \beta' g(\phi' V_3', \phi' V_1'))]. \qquad (5.4.37)\end{aligned}$$

Given that the Ricci tensor is cyclic parallel on using equations (5.4.31), (5.4.32) and (5.4.37) in the equation (5.4.36) and then making use of (1.1.9), we conclude

$$2\beta'(\beta' - \mu')[\eta'(V_1')g(\phi' V_2', \phi' V_3') + \eta'(V_2')g(\phi' V_3', \phi' V_1') + \eta'(V_3')g(\phi' V_1', \phi' V_2')] = 0. \quad (5.4.38)$$

Taking $V_3' = \xi'$ in equation (5.4.38), we obtain

$$2\beta'(\beta' - \mu')g(\phi' V_1', \phi' V_2') = 0, \quad (5.4.39)$$

for all $V_1', V_2' \in TM$. Since $g(\phi' V_1', \phi' V_2') \neq 0$, equation (5.4.39) it follows that either $\beta' = 0$ or $\mu' = \beta'$. This completes the proof. □

On using $\beta' = 0$ in equation (5.4.22), we obtain $\lambda' = {\alpha'}^2 - (\frac{p}{2} + \frac{1}{3})$. Consequently, we find

Corollary 5.4.3. *Suppose (M, g) is a trans-Sasakian 3-manifold that admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$ with $\beta' \neq \mu'$, and has a cyclic parallel Ricci tensor. Then the soliton is shrinking if ${\alpha'}^2 < (\frac{p}{2}+\frac{1}{3})$ steady if ${\alpha'}^2 = (\frac{p}{2}+\frac{1}{3})$ and expanding if ${\alpha'}^2 > (\frac{p}{2}+\frac{1}{3})$*

Also if $\beta' \neq 0$ then using (5.4.39)we have $\mu' = \beta'$. Hence, following a similar computation as in equation (5.4.35), we arrive at the following

Theorem 5.4.8. *Suppose (M, g) is a trans-Sasakian 3-manifold that admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, and has a cyclic parallel Ricci tensor. Then M is an Einstein manifold with constant scalar curvature $r = 6(\frac{p}{2} + \frac{1}{3}) + 6\lambda' + 6\beta'$ provided $\beta' \neq 0$.*

Theorem 5.4.9. *Suppose (M, g) is a trans-Sasakian 3-manifold that admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$. If M is Einstein semi-symmetric, then it is an η'-Einstein manifold with constant scalar curvature $r = 2(\frac{p}{2} + \frac{1}{3}) + 2\lambda' + \mu' + \beta'$ and the soliton is shrinking, steady or expanding as $(\frac{p}{2} + \frac{1}{3}) + \mu' + 3\beta' > 0$, $(\frac{p}{2} + \frac{1}{3}) + \mu' + 3\beta' = 0$ or, $(\frac{p}{2} + \frac{1}{3}) + \mu' + 3\beta' < 0$ respectively.*

Proof. Assume that the Einstein semi-symmetric trans-Sasakian 3-manifold (M, g) admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$. Then using equation (5.4.18) together with equation (5.2.7), we deduce

$$r = 2(\frac{p}{2} + \frac{1}{3}) + 2\lambda' + \mu' + \beta'. \tag{5.4.40}$$

Substituting equation (5.4.21) into equation (5.4.40), we derive

$$\lambda' = -\frac{1}{4}\{(\frac{p}{2} + \frac{1}{3}) + \mu' + 3\beta'\}. \tag{5.4.41}$$

This shows that the theorem is proved. □

5.5 Conformal η'-Einstein solitons on trans-Sasakian 3-manifolds satisfying $R(\xi', V_1')\cdot S = 0$ and $\mathcal{W}_2(\xi', V_1')\cdot S = 0$

Theorem 5.5.1. *Suppose (M, g) is a trans-Sasakian 3-manifold that admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, and satisfies the curvature condition $R(\xi', V_1') \cdot S = 0$. Then M is an Einstein manifold with constant scalar curvature $r = 6(\frac{p}{2} + \frac{1}{3}) + 6\lambda' + 6\beta'$.*

Proof. Let us consider a trans-Sasakian 3-manifold admitting a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, and assume that the manifold satisfies the curvature condition $R(\xi', V_1')\cdot S = 0$ then, we can write

$$S(R(\xi', V_1')V_2', V_3') + S(V_2', R(\xi', V_1')V_3') = 0. \tag{5.5.1}$$

Now using the equation (5.4.18) into (5.5.1), we compute

$$\begin{aligned} & (\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta')g(R(\xi', V_1')V_2', V_3') + (\beta' - \mu')\eta'(R(\xi', V_1')V_2')\eta'(V_3') \\ + & (\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta')g(V_2', R(\xi', V_1')V_3') \\ + & (\beta' - \mu')\eta'(R(\xi', V_1')V_3')\eta'(V_2') = 0. \end{aligned} \tag{5.5.2}$$

Using (1.1.61) in the previous equation, we obtain

$$({\alpha'}^2 - {\beta'}^2)(\beta' - \mu')[g(V_1', V_2')\eta'(V_3') + g(V_1', V_3')\eta'(V_2') - 2\eta'(V_1')\eta'(V_2')\eta'(V_3')] = 0. \tag{5.5.3}$$

By taking $V_3' = \xi'$ in equation (5.5.3) and using (1.1.9), we get

$$(\alpha'^2 - \beta'^2)(\beta' - \mu')g(\phi' V_1', \phi' V_2') = 0, \tag{5.5.4}$$

for all $V_1', V_2' \in TM$. As $g(\phi' V_1', \phi' V_2') \neq 0$ and for non-trivial case $\alpha'^2 \neq \beta'^2$, from equation (5.5.4) we deduce that $\mu' = \beta'$. Thus, using the expression (5.4.18), we have

$$S(V_1', V_2') = [\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta']g(V_1', V_2'), \tag{5.5.5}$$

for all $V_1', V_2' \in TM$. On contracting equation (5.5.5), we have $r = 6(\frac{p}{2} + \frac{1}{3}) + 6\lambda' + 6\beta'$. Plugging this information together with equation (5.5.5) we have proved the theorem. □

Our next result of this section involes $\mathcal{W}_2$-Curvature tensor that, which was introduced by Pokhariyal and Mishra in 1970 [58].

The $\mathcal{W}_2$-curvature tensor in a trans-Sasakian 3-manifold (M, g) from (1.1.6) by putting $n = 1$ is defined as

$$\mathcal{W}_2(V_1', V_2')V_3' = R(V_1', V_2')V_3' + \frac{1}{2}\{g(V_1', V_3')QV_2' - g(V_2', V_3')QV_1'\}. \tag{5.5.6}$$

Theorem 5.5.2. *Suppose (M, g) is a trans-Sasakian 3-manifold that admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, and satisfies the curvature condition $\mathcal{W}_2(\xi', V_1') \cdot S = 0$. Then the manifold is either Einstein manifold or has constant scalar curvature $r = 2(\frac{p}{2} + \frac{1}{3}) + 2\lambda' + 2\beta'$.*

Proof. Assume that (M, g) is a trans-Sasakian 3-manifold admitting a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, and suppose that it satisfies the curvature condition $\mathcal{W}_2(\xi', V_1') \cdot S = 0$. Thus, we can express this as

$$S(\mathcal{W}_2(\xi', V_1')V_2', V_3') + S(V_2', \mathcal{W}_2(\xi', V_1')V_3') = 0, \quad \forall \quad V_1', V_2' \in TM. \tag{5.5.7}$$

Using (5.4.18) the equation (5.5.7) takes the form

$$\begin{aligned} &(\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta')[g(\mathcal{W}_2(\xi', V_1')V_2', V_3') + g(\mathcal{W}_2(\xi', V_1')V_3', V_2')] \\ +\ &(\beta' - \mu')[\eta'(\mathcal{W}_2(\xi', V_1')V_2')\eta'(V_3') + \eta'(\mathcal{W}_2(\xi', V_1')V_3')\eta'(V_2')] = 0. \end{aligned} \tag{5.5.8}$$

Also, equation (5.4.18) implies

$$QV_1' = (\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta')V_1' + (\beta' - \mu')\eta'(V_1')\xi', \tag{5.5.9}$$

that is,

$$Q\xi' = (\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \mu')\xi'. \tag{5.5.10}$$

Now, we replace $V_1' = \xi'$ into (5.5.6) and then using equations (1.1.61), (5.5.9) and (5.5.10) to obtain

$$\mathcal{W}_2(\xi', V_2')V_3' = N'g(V_2', V_3')\xi' - M'\eta'(V_3')V_2' + (M' - N')\eta'(V_2')\eta'(V_3'), \tag{5.5.11}$$

where $M' = (\alpha'^2 - \beta'^2) - \frac{1}{2}(\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta')$ and $N' = (\alpha'^2 - \beta'^2) - \frac{1}{2}(\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \mu')$. On taking the inner product in equation (5.5.11) along ξ' yields

$$\eta'(\mathcal{W}_2(\xi', V_2')V_3') = N'[g(V_2', V_3') - \eta'(V_2')\eta'(V_3')]. \tag{5.5.12}$$

Using (5.5.11) and (5.5.12) in the equation (5.5.8) and then taking $V_3' = \xi'$, we arrive at

$$(N' - M')[2N' - (\tfrac{r}{2} - (\tfrac{p}{2} + \tfrac{1}{3}) - \lambda' - \beta')][g(V_1', V_2') - \eta'(V_1')\eta'(V_2')] = 0,$$

which in view of (1.1.9) implies

$$(N' - M')[2N' - (\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta')]g(\phi' V_1', \phi' V_2') = 0, \tag{5.5.13}$$

for all $V_1', V_2' \in TM$. As $g(\phi' V_1', \phi' V_2') \neq 0$, we can conclude from the equation (5.5.13) that either $M' = N'$ or $2N' = \frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta'$. Hence, using the expressions for M' and N', we conclude that either $\mu' = \beta'$ or

$$r - 2(\frac{p}{2} + \frac{1}{3}) - 2\lambda' - \mu' - \beta' = 2(\alpha'^2 - \beta'^2). \tag{5.5.14}$$

Now for $\beta' = \mu'$, in a similar way as in (5.5.5) Thus, the manifold reduces to an Einstein manifold. On using equation (5.5.14) with (5.4.20), we have

$$r = 2(\frac{p}{2} + \frac{1}{3}) + 2\lambda' + 2\beta'. \tag{5.5.15}$$

This completes the proof. □

Again in view of equation (5.4.21), the equation (5.5.15) implies $\lambda' = -\frac{1}{2}\{2(\frac{p}{2} + \frac{1}{3}) + \mu' + \beta'\}$. Thus, we get

Corollary 5.5.1. *Suppose (M, g) is a trans-Sasakian 3-manifold that admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$ with $\beta' \neq \mu'$, and satisfies the curvature condition $\mathcal{W}_2(\xi', V_1') \cdot S = 0$. Then the soliton is shrinking if $\mu' > -\{2(\frac{p}{2} + \frac{1}{3}) + \beta'\}$, steady if $\mu' = -\{2(\frac{p}{2} + \frac{1}{3}) + \beta'\}$ and expanding if $\mu' < -\{2(\frac{p}{2} + \frac{1}{3}) + \beta'\}$.*

5.6 Conformal η'-Einstein solitons on trans-Sasakian 3-manifolds satisfying $\mathcal{B}(\xi', V_1')\cdot S = 0$, $\mathcal{M}(\xi', V_1')\cdot S = 0$ and $S(\xi', V_1')\cdot \mathcal{M} = 0$

Recall that Bochner curvature tensor [11] was introduced as complex analogue of conformal curvature tensor. Its geometric significance, however, was later revealed in [7] through the Boothby-Wang fibration. The concept of C-Bochner curvature tensor in the context of Sasakian manifolds was introduced by M. Matsumoto, G. Chuman [50] in 1969. The $\mathcal{C}$-Bochner curvature tensor for a trans-Sasakian 3-manifold (M, g) is given by

$$\begin{aligned}
\mathcal{B}(V_1', V_2')V_3' &= R(V_1', V_2')V_3' + \frac{1}{6}[g(V_1', V_3')QV_2' - S(V_2', V_3')V_1' - g(V_2', V_3')QV_1' \\
&+ S(V_1', V_3')V_2' + g(\phi'V_1', V_3')Q\phi'V_2' - S(\phi'V_2', V_3')\phi'V_1' \\
&- g(\phi'V_2', V_3')Q\phi'V_1' + S(\phi'V_1', V_3')\phi'V_2' + 2S(\phi'V_1', V_2')\phi'V_3' \\
&+ 2g(\phi'V_1', V_2')Q\phi'V_3' + \eta'(V_2')\eta'(V_3')QV_1' - \eta'(V_2')S(V_1', V_3')\xi' \\
&+ \eta'(V_1')S(V_2', V_3')\xi' - \eta'(V_1')\eta'(V_3')QV_2'] - \frac{D+2}{6}[g(\phi'V_1', V_3')\phi'V_2' \\
&- g(\phi'V_2', V_3')\phi'V_1' + 2g(\phi'V_1', V_2')\phi'V_3'] + \frac{D}{6}[\eta'(V_2')g(V_1', V_3')\xi' \\
&- \eta'(V_2')\eta'(V_3')V_1' + \eta'(V_1')\eta'(V_3')V_2' - \eta'(V_1')g(V_2', V_3')\xi'] \\
&- \frac{D-4}{6}[g(V_1', V_3')V_2' - g(V_2', V_3')V_1'], \qquad (5.6.1)
\end{aligned}$$

where $D = \frac{r+2}{4}$.

Theorem 5.6.1. *Suppose (M, g) is a trans-Sasakian 3-manifold admitting a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, and assume that the manifold satisfies the curvature condition $\mathcal{B}(\xi', V_1') \cdot S = 0$, where $\mathcal{B}$ denotes $\mathcal{C}$-Bochner curvature tensor. Then M is either an Einstein manifold or a manifold of constant scalar curvature $r = 14\lambda' + 14(\frac{p}{2} + \frac{1}{3}) + 12\beta' + 2\mu' - 8$.*

Proof. Let (M, g) be a trans-Sasakian 3-manifold admitting a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, and assume it satisfies the curvature condition $B(\xi', V_1') \cdot S = 0$. So $\forall$ $V_1', V_2', V_3' \in TM$, we have

$$S(\mathcal{B}(\xi', V_1')V_2', V_3') + S(V_2', \mathcal{B}(\xi', V_1')V_3') = 0. \qquad (5.6.2)$$

Now using (5.4.18) in (5.6.2), we get

$$\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\beta'\}[g(\mathcal{B}(\xi',V_1')V_2',V_3')+g(\mathcal{B}(\xi',V_1')V_3',V_2')]$$
$$+(\beta'-\mu')[\eta'(\mathcal{B}(\xi',V_1')V_2')\eta'(V_3')+\eta'(\mathcal{B}(\xi',V_1')V_3')\eta'(V_2')]=0. \quad (5.6.3)$$

Moreover, using equation (5.4.18), we obtain

$$QV_1'=[\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\beta']V_1'+(\beta'-\mu')\eta'(V_1')\xi', \quad (5.6.4)$$

that is,

$$Q\xi'=[\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\mu']\xi'. \quad (5.6.5)$$

Thus, on taking $V_1'=\xi'$ in (5.6.1), we get

$$\begin{aligned}\mathcal{B}(\xi',V_2')V_3' &= R(\xi',V_2')V_3'+\frac{1}{6}[S(\xi',V_3')V_2'-g(V_2',V_3')Q\xi'+\eta'(V_2')\eta'(V_3')Q\xi'\\ &- \eta'(V_2')S(\xi',V_3')\xi']+\frac{4}{6}[\eta'(V_3')V_2'-g(V_2',V_3')\xi']. \quad (5.6.6)\end{aligned}$$

We insert the equations (1.1.61), (5.4.19) and (5.6.5) in the equation (5.6.6) to yield

$$\mathcal{B}(\xi',V_2')V_3'=[(\alpha'^2-\beta'^2)-\frac{1}{6}\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\mu'\}-\frac{4}{6}][g(V_2',V_3')\xi'-\eta'(V_3')V_2']. \quad (5.6.7)$$

In view of (5.6.7) the equation (5.6.3), becomes

$$\begin{aligned}[(\alpha'^2-\beta'^2) &- \frac{1}{6}\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\mu'\}-\frac{4}{6}](\beta'-\mu')[g(V_1',V_2')\eta'(V_3')\\ &+ g(V_1',V_3')\eta'(V_2')-2\eta'(V_1')\eta'(V_2')\eta'(V_3')]=0. \quad (5.6.8)\end{aligned}$$

We now substitute $V_3'=\xi'$ in the preceding equation (5.6.8) and recalling (1.1.9) to arrive

$$[(\alpha'^2-\beta'^2)-\frac{1}{6}\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\mu'\}-\frac{4}{6}](\beta'-\mu')g(\phi'V_1',\phi'V_2')=0, \quad (5.6.9)$$

for all vector fields $V_1',V_2'\in TM$ and $g(\phi'V_1',\phi'V_2')\neq 0$, hence from (5.6.9), we can conclude that either

$$[(\alpha'^2-\beta'^2)-\frac{1}{6}\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\mu'\}-\frac{4}{6}]=0, \quad (5.6.10)$$

or, $\beta'=\mu'$. Also for $\beta'=\mu'$ with a similar way as in equation (5.4.35), We conclude that the manifold is an Einstein manifold. Assuming $\beta'\neq\mu'$, and using equation (5.4.22) in the equation (5.6.10), we have

$$r=14\lambda'+14(\frac{p}{2}+\frac{1}{3})+12\beta'+2\mu'-8. \quad (5.6.11)$$

Hence, the scalar curvature is constant throughout the manifold. So, the theorem is proved. □

In the case where $\beta' \neq \mu'$, substituting equation (5.4.21) into (5.6.11) we obtain $\lambda' = 1 - (\frac{p}{2} + \frac{1}{3} + \beta')$.

Hence, we have the following

Corollary 5.6.1. *Suppose (M, g) is a trans-Sasakian 3-manifold that admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$ with $\mu' \neq \beta'$, and assume that the manifold satisfies the curvature condition $\mathcal{B}(\xi', V_1') \cdot S = 0$. Then the soliton is shrinking, steady or expanding according as $\frac{p}{2} + \frac{1}{3} + \beta' > 1$, $\frac{p}{2} + \frac{1}{3} + \beta' = 1$ or $\frac{p}{2} + \frac{1}{3} + \beta' < 1$ respectively.*

Next, we consider (M, g) be a trans-Sasakian 3-manifold. Then $\mathcal{M}$-projective curvature tensor of M is defined by [59]

$$\begin{aligned}\mathcal{M}(V_1', V_2')V_3' &= R(V_1', V_2')V_3' - \frac{1}{4}[S(V_2', V_3')V_1' - S(V_1', V_3')V_2' + g(V_2', V_3')QV_1' \\ &- g(V_1', V_3')QV_2'], \end{aligned} \tag{5.6.12}$$

for any $V_1', V_2', V_3' \in TM$.

Theorem 5.6.2. *If a 3-dimensional trans-Sasakian manifold $(M, g, \phi', \eta', \xi', \alpha', \beta')$, with α', β' constants, admits a conformal η'-Einstein soliton $(g, \xi', \lambda', \mu')$, satisfies the condition $\mathcal{M}(\xi', V_1') \cdot S = 0$ then $\mu' = \beta'$, $\lambda' = 2(\beta'^2 - \alpha'^2) - \beta' - (\frac{p}{2} + \frac{1}{3}) + \frac{r}{2}$.*

Proof. Let us consider a 3-dimensional trans-sasakian manifold with conformal η'-Einstein solitons $(g, \xi', \lambda', \mu')$ satisfy the condition

$$\mathcal{M}(\xi', V_1') \cdot S = 0.$$

Then, we have

$$(\mathcal{M}(\xi', V_1')V_2', V_3') + S(V_2', \mathcal{M}(\xi', V_1')V_3') = 0, \tag{5.6.13}$$

for any $V_1', V_2', V_3' \in TM$. Now from (1.1.64) and (5.4.19), we have

$$2(\beta'^2 - \alpha'^2) = [\lambda' + \mu' + (\frac{p}{2} + \frac{1}{3}) - \frac{r}{2}]. \tag{5.6.14}$$

Next, we can use the equations (1.1.61), (5.4.18), (5.4.19), (5.6.14) and (5.6.12) into the identity (5.6.13) to get

$$
\begin{aligned}
&[\frac{1}{2}(\alpha'^2-\beta'^2)\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\mu'\}+\frac{1}{4}\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\beta'\}\\
&+(\alpha'^2-\beta'^2)\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}+\frac{1}{4}\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}\\
&\{\lambda'+\mu'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}-\frac{1}{4}\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}\\
&\{\beta'-\mu'+2(\alpha'^2-\beta'^2)\}][g(V_1',V_2'V_1')\eta'(V_3')+g(V_1',V_3')\eta'(V_2')]+[-(\alpha'^2-\beta'^2)(\beta'-\mu')\\
&+\frac{(\beta'-\mu')}{4}\{2(\alpha'^2-\beta'^2)+\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\beta'\}]\eta'(V_1')\eta'(V_2')\eta'(V_3')=0.
\end{aligned}
$$

By putting $V_3'=\xi'$ in the foregoing equation, we find

$$
\begin{aligned}
&[\frac{1}{2}(\alpha'^2-\beta'^2)\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\mu'\}+\frac{1}{4}\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\beta'\}\\
&+(\alpha'^2-\beta'^2)\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}+\frac{1}{4}\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}\\
&\{\lambda'+\mu'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}-\frac{1}{4}\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}\\
&\{\beta'-\mu'+2(\alpha'^2-\beta'^2)\}][g(V_1',V_2')+\eta'(V_1')\eta'(V_2')]+[-(\alpha'^2-\beta'^2)(\beta'-\mu')\\
&+\frac{(\beta'-\mu')}{4}\{2(\alpha'^2-\beta'^2)+\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\beta'\}]\eta'(V_1')\eta'(V_2')=0.
\end{aligned}
$$

Now, we set $V_1'=\phi'V_1'$ and $V_2'=\phi'V_2'$ in the previous equation to arrive

$$
\begin{aligned}
&[\frac{1}{2}(\alpha'^2-\beta'^2)\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\mu'\}+\frac{1}{4}\{\frac{r}{2}-(\frac{p}{2}+\frac{1}{3})-\lambda'-\beta'\}\\
&+(\alpha'^2-\beta'^2)\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}+\frac{1}{4}\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}\\
&\{\lambda'+\mu'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}-\frac{1}{4}\{\lambda'+\beta'+(\frac{p}{2}+\frac{1}{3})-\frac{r}{2}\}\\
&\{\beta'-\mu'+2(\alpha'^2-\beta'^2)\}]g(\phi'V_1',\phi'V_2')=0. \qquad (5.6.15)
\end{aligned}
$$

Again using the equation (5.6.14), we have

$$
\mu'=\beta', \quad \lambda'=2(\beta'^2-\alpha'^2)-\beta'-(\frac{p}{2}+\frac{1}{3})+\frac{r}{2}. \qquad (5.6.16)
$$

□

Corollary 5.6.2. *If a 3-dimensional trans-Sasakian manifold (M,g) with α', β' constants satisfies the condition $\mathcal{M}(\xi',V_1')\cdot S=0$, then there does not exist a Ricci soliton M with the potential vector field ξ'.*

Theorem 5.6.3. *If a 3-dimensional trans-Sasakian manifold $(M,g,\phi',\eta',\xi',\alpha',\beta')$, with α', β' constants, admits a conformal η'-Einstein soliton (g,ξ',λ',μ'), and satisfies the condition $S(\xi',V_1')\cdot\mathcal{M}=0$ then*

$$\mu' = \beta',\ \lambda' = 2(\beta'^2 - \alpha'^2) - \beta' - (\tfrac{p}{2} + \tfrac{1}{3})$$

$$or \qquad \lambda' = 2(\alpha'^2 - \beta'^2) - (\tfrac{p}{2} + \tfrac{1}{3}) - \beta',\ \mu' = -4(\alpha'^2 - \beta'^2) + \beta'.$$

Proof. Suppose that 3-dimensional trans-Sasakian manifolds with conformal η'-Einstein soliton satisfy the condition

$$S(\xi', V_1') \cdot \mathcal{M} = 0.$$

So, we have

$$\begin{aligned}
&S(V_1', \mathcal{M}(V_2', V_3')\mathcal{V})\xi' - S(\xi', \mathcal{M}(V_2', V_3')\mathcal{V})V_1' + S(V_1', V_2')\mathcal{M}(\xi', V_3')\mathcal{V} \\
&-S(\xi', V_2')\mathcal{M}(V_1', V_3')\mathcal{V} + S(V_1', V_3')\mathcal{M}(V_2', \xi')\mathcal{V} - S(\xi', V_3')\mathcal{M}(V_2', V_1')\mathcal{V} \\
&+S(V_1', \mathcal{V})\mathcal{M}(V_2', V_3')\xi' - S(\xi', \mathcal{V})\mathcal{M}(V_2', V_3')V_1' = 0.
\end{aligned}$$

By taking the inner product with ξ', the above equation reduces to

$$\begin{aligned}
&S(V_1', \mathcal{M}(V_2', V_3')\mathcal{V}) - S(\xi', \mathcal{M}(V_2', V_3')\mathcal{V})\eta'(V_1') + S(V_1', V_2')\eta'(\mathcal{M}(\xi', V_3')\mathcal{V}) \\
&-S(\xi', V_2')\eta'(\mathcal{M}(V_1', V_3')\mathcal{V}) + S(V_1', V_3')\eta'(\mathcal{M}(V_2', \xi')) - S(\xi', V_3')\eta'(\mathcal{M}(V_2', V_1')\mathcal{V}) \\
&+S(V_1', \mathcal{V})\eta'(\mathcal{M}(V_2', V_3')\xi') - S(\xi', \mathcal{V})\eta'(\mathcal{M}(V_2', V_3')V_1') = 0. \qquad (5.6.17)
\end{aligned}$$

By letting $\mathcal{V} = \xi'$ and using the equation (1.1.47), (1.1.61), (5.4.18), (5.4.19), (5.6.12) and (5.6.14) the equation (5.6.17), becomes

$$\begin{aligned}
&[(2\lambda' + p + \frac{2}{3} + \mu' + \beta')(\alpha'^2 - \beta'^2) + \frac{1}{4}(2\lambda' + p + \frac{2}{3} + \mu' + \beta')^2 \\
&+(2\lambda' + p + \frac{2}{3} + \mu' + \beta')\{(\alpha'^2 - \beta'^2) + \frac{1}{4}(2\lambda' + p + \frac{2}{3} + \mu' + \beta')\}] \\
&\{g(V_1', V_3')\eta'(V_2') - g(V_1', V_2')\eta'(V_3')\} = 0. \qquad (5.6.18)
\end{aligned}$$

Now as $g(QV_1', V_2') = S(V_1', V_2')$, then from (5.4.18), we have

$$QV_1' = \{\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \beta'\}V_1' + (\beta' - \mu')\eta'(V_1')\xi'. \qquad (5.6.19)$$

Using the equation (5.6.19), we have

$$\mu' = \beta',\ \lambda' = 2(\beta'^2 - \alpha'^2) - \beta' - (\tfrac{p}{2} + \tfrac{1}{3})$$

$$\text{or} \qquad \lambda' = 2(\alpha'^2 - \beta'^2) - (\tfrac{p}{2} + \tfrac{1}{3}) - \beta',\ \mu' = -4(\alpha'^2 - \beta'^2) + \beta'.$$

□

Corollary 5.6.3. *Suppose (M, g) is a 3-dimensional trans-Sasakian manifold with constant α', β'. If the manifold satisfies the condition $S(\xi', V_1') \cdot \mathcal{M} = 0$, then it does not admits a Ricci soliton with the potential vector field ξ'.*

5.7 Conformal η'-Einstein solitons on trans-Sasakian 3-manifolds with torse-forming vector field

We examine the behavior of conformal η'-Einstein solitons on trans-Sasakian 3-manifolds when the potential vector field is torse-forming.

Theorem 5.7.1. *Suppose (M, g) is a trans-Sasakian 3-manifold admitting a conformal η'-Einstein soliton, where ξ' is a torse-forming vector field. Then the manifold becomes an η'-Einstein manifold, and the soliton is shrinking, steady or expanding according as $f > (\alpha'^2 - \beta'^2) - (\frac{p}{2} + \frac{1}{3})$, $f = (\alpha'^2 - \beta'^2) - (\frac{p}{2} + \frac{1}{3})$ or $f < (\alpha'^2 - \beta'^2) - (\frac{p}{2} + \frac{1}{3})$.*

Proof. Assume that $(g, \xi', \lambda', \mu')$ is a conformal η'-Einstein soliton on a trans-Sasakian 3-manifold (M, g), and that the Reeb vector field ξ' is torse-forming. Taking the inner product of equation (1.1.47) with ξ', we obtain

$$g(\nabla_{V_1'}\xi', \xi') = (\beta' - 1)\eta'(V_1'). \tag{5.7.1}$$

Also taking the inner product in equation (1.1.89) in the direction ξ', we get

$$g(\nabla_{V_1'}\xi', \xi') = f\eta'(V_1') + \gamma(V_1'). \tag{5.7.2}$$

Combining (5.7.1) and (5.7.2), we obtain $\gamma = (\beta' - 1 - f)\eta'$. Thus from equation (1.1.89) it follows that for a torse forming vector field ξ' on a trans-Sasakian 3-manifold, we have

$$\nabla_{V_1'}\xi' = f(V_1' - \eta'(V_1')\xi') + (\beta' - 1)\eta'(V_1')\xi'. \tag{5.7.3}$$

Now from the formula of Lie differentiation and using (5.7.3) yields

$$\begin{aligned}(\mathcal{L}_{\xi'}g)(V_1', V_2') &= g(\nabla_{V_1'}\xi', V_2') + g(\nabla_{V_2'}\xi', V_1') \\ &= 2f[g(V_1', V_2') - \eta'(V_1')\eta'(V_2')] + 2(\beta' - 1)\eta'(V_1')\eta'(V_2'). \end{aligned} \tag{5.7.4}$$

Since $(g, \xi', \lambda', \mu')$ is a conformal η'-Einstein soliton, equation (1.2.98) is satisfied. In light of equation (5.7.4), therefore the equation (1.2.98) reduces to

$$S(V_1', V_2') = [\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - f]g(V_1', V_2') + (f - \mu' - \beta' + 1)\eta'(V_1')\eta'(V_2'). \tag{5.7.5}$$

Therefore, the manifold reduces to an η'-Einstein manifold. On letting $V_2' = \xi'$ in (5.7.5), it follows that

$$S(V_1', \xi') = (\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \mu' - \beta' + 1)\eta'(V_1'). \tag{5.7.6}$$

Combining (5.7.6) with the equation (1.1.64) implies

$$2(\alpha'^2 - \beta'^2) = (\frac{r}{2} - (\frac{p}{2} + \frac{1}{3}) - \lambda' - \mu' - \beta' + 1). \tag{5.7.7}$$

On taking trace in equation (5.7.5), we obtain

$$r = 6\lambda' + 6(\frac{p}{2} + \frac{1}{3}) + 2\mu' + 4f + 2\beta' - 2. \tag{5.7.8}$$

Using equation (5.7.8) in (5.7.7), we get $\lambda' = (\alpha'^2 - \beta'^2) - f - (\frac{p}{2} + \frac{1}{3})$. This completes the proof. □

5.8 Example of conformal η'-Einstein solitons on trans-Sasakian 3-manifold

Finally, we construct an example to illustrate and support the results obtained.

Example 5.8.1. *We consider* $M = \{(V_1', V_2', V_3') \in \mathbb{R}^3 : V_2' \neq 0\}$, *where* (V_1', V_2', V_3') *are the standard coordinates on the Euclidean space. Since* M *is an open subset of* $\mathbb{R}^3$, *it is a* 3-*dimensional smooth manifold. We define the following vector fields on* M*:*

$$e'_1 = e'^{2V_3'} \frac{\partial}{\partial V_1'}, \qquad e'_2 = e'^{2V_3'} \frac{\partial}{\partial V_2'}, \qquad e'_3 = \frac{\partial}{\partial V_3'}$$

are orthogonal with respect to the Riemannian metric g *is defined by:*

$$g_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\xi' = e'_3$. *Then the 1-form* η' *is defined by* $\eta'(V_3') = g(V_3', e'_3)$, *for arbitrary* $V_3' \in \chi(M)$, *then we have the following relations:*

$$\eta'(e'_1) = \eta'(e'_2) = 0, \qquad \eta'(e'_3) = 1.$$

Let us define the (1,1)-tensor field ϕ' *as*

$$\phi' e'_1 = e'_2, \qquad \phi' e'_2 = -e'_1, \qquad \phi' e'_3 = 0,$$

then it satisfies,

$$\begin{aligned} \phi'^2(V_3') &= -V_3' + \eta'(V_3')e'_3, \\ g(\phi' V_3', \phi' V_4') &= g(V_3', V_4') - \eta'(V_3')\eta'(V_4') \end{aligned}$$

for arbitrary $V_3', V_4' \in \chi(M)$. *Thus* (ϕ', ξ', η', g) *defines an almost contact metric structure on* M. *We can now easily conclude:*

$$[e'_1, e'_2] = 0, \qquad [e'_2, e'_3] = -2e'_2, \qquad [e'_1, e'_3] = -2e'_1.$$

Let ∇ *be the Levi-Civita connection of* g. *Then from* $Koszul's formula$ *for arbitrary* $V_1', V_2', V_3' \in \chi(M)$ *given by:*

$$\begin{aligned} 2g(\nabla_{V_1'} V_2', V_3') &= V_1' g(V_2', V_3') + V_2' g(V_3', V_1') - V_3' g(V_1', V_2') - g(V_1', [V_2', V_3']) \\ &- g(V_2', [V_1', V_3']) + g(V_3', [V_1', V_2']), \end{aligned}$$

we can have:

$$\begin{aligned} &\nabla_{e'_1} e'_1 = 2e'_3, && \nabla_{e'_1} e'_2 = 0, && \nabla_{e'_1} e'_3 = -2e'_1, \\ &\nabla_{e'_2} e'_1 = 0, && \nabla_{e'_2} e'_2 = 2e'_3, && \nabla_{e'_2} e'_3 = -2e'_2, \\ &\nabla_{e'_3} e'_1 = 0, && \nabla_{e'_3} e'_2 = 0, && \nabla_{e'_3} e'_3 = 0. \end{aligned}$$

From here we can easily verify that the relations (1.1.47) and (1.1.48) are satisfied. Hence the considered manifold is trans-Sasakian manifold of type $(0, -2)$. *The components of Riemannian curvature tensor are given by*

$$\begin{aligned} &R(e'_1, e'_2)e'_1 = -4e'_3, && R(e'_1, e'_2)e'_2 = -4e'_1, && R(e'_1, e'_2)e'_3 = 0, \\ &R(e'_1, e'_3)e'_1 = 4e'_2, && R(e'_1, e'_3)e'_2 = 0, && R(e'_1, e'_3)e'_3 = -4e'_1, \\ &R(e'_2, e'_3)e'_1 = 0, && R(e'_2, e'_3)e'_2 = -4e'_2, && R(e'_2, e'_3)e'_3 = -4e'_2. \end{aligned}$$

And the components of Ricci tensor and $*$*-Ricci tensor are given by:*

$$S(e'_1, e'_1) = 0, \qquad S(e'_2, e'_2) = 0, \qquad S(e'_3, e'_3) = -8.$$

From here, we can easily deduce that the scalar curvature of the manifold $r = \sum_{i=1}^{3} S(e'_i, e'_i) = -8$. *Let us define a vector field by,* $V = \xi'$. *Then we can obtain:*

$$(\mathcal{L}_V g)(e'_1, e'_1) = -4, \qquad (\mathcal{L}_V g)(e'_2, e'_2) = -4, \qquad (\mathcal{L}_V g)(e'_3, e'_3) = 0.$$

Contracting (1.2.98) and using the result $r = -8$, *we deduce* $\lambda' = -\frac{(3p+2)}{6} - \frac{\mu'}{3}$. *Now using the identity* (5.4.11), *we get* $\mu' = 6$. *Then* $\lambda' = -\left(\frac{p}{2} + \frac{7}{3}\right)$. *The value of* λ' *and* μ' *satisfies the relation* (5.4.21) *and* (5.4.22). *So,* g *defines a conformal* η'*-Einstein solitons on trans-Sasakian 3-manifold for* $\lambda' = -\left(\frac{p}{2} + \frac{7}{3}\right)$ *and* $\mu' = 6$. *Conformal* η'*-Einstein soliton is shrinking i.e.,* $\lambda' > 0$ *if* $\left(\frac{p}{2} + \frac{7}{3}\right) < 0$, *steady i.e.,* $\lambda' = 0$ *if* $\left(\frac{p}{2} + \frac{7}{3}\right) = 0$ *and expanding i.e.,* $\lambda' < 0$ *if* $\left(\frac{p}{2} + \frac{7}{3}\right) > 0$.

5.9 Geometrical and Physical Motivation and Conclusion

The study of conformal η'-Einstein solitons on Riemannian and pseudo-Riemannian manifolds holds substantial importance in differential geometry, particularly within Riemannian geometry and special relativistic physics. In the context of general relativity, there exist physical models of perfect fluid spacetimes that admit conformal η'-Einstein solitons and exhibit curvature inheritance symmetries. Such models provide both physical and geometric insight into the nature of conformal η'-Einstein solitons. Therefore, investigating perfect fluid spacetimes that admit these solitons helps highlight their physical significance and deepens our understanding of their role in geometric analysis and spacetime structure.

The mathematical concept of an almost conformal η'-Einstein soliton should not be confused with soliton solutions that arise in various areas of mathematical and theoretical physics. Rather, it represents a geometric structure with significant physical applications, particularly in general relativistic spacetimes involving viscous fluids with heat flux and stress, dark and dust-filled universes, and radiation-dominated cosmological models. Conformal Einstein solitons and conformal η'-Einstein solitons also play a role in the study of the renormalization group flow in nonlinear sigma models [86]. Specifically, the concept of a conformal η'-Einstein soliton can be examined in the context of renormalization group flow of mass in two-dimensional field theories. Moreover, general relativistic spacetime models admitting such solitons are of considerable interest across several fields, including astrophysics [12, 15], plasma physics [4], string theory, and nuclear physics [60].

In this work, we have employed techniques from local Riemannian and semi-Riemannian geometry to interpret solutions of equation (1.2.98), and to realize Einstein metrics within a broad class of conformal η'-Einstein soliton metrics on contact geometric structures—particularly on trans-Sasakian manifolds. Moreover, our results contribute meaningfully to the field of differential geometry by providing foundational tools and perspectives that may inspire further exploration. Conformal η'-Einstein solitons also play a substantial and motivating role in mathematical physics, general relativity, quantum cosmology, and the study of complex geometry. Their physical relevance and theoretical importance can be seen in works such as [26, 86].

Several intriguing questions arise from the present study, which may serve as the basis for future research in this evolving area.

1. Is Theorem (5.4.4) still true without assuming the vector field $\mathcal{V}$ is collinear with ξ'?

2. Is Theorem (5.4.9) still true if we do not assume that the manifold is Einstein semi-symmetric?

3. If the vector field is not torse-forming, does the corresponding theorem (5.7.1) is true?

4. Whether the results concerning conformal η'-Einstein solitons on trans-Sasakian 3-manifolds extend or remain valid in other geometric settings such as nearly Kenmotsu, paracontact manifolds and f-cosymplectic manifolds?

6

Generalised Holling Type-III Prey-Predator Model with Crowding Effects of Predator and its Geometric Comparison with the Ricci Flow Equation

6.1 Introduction

We consider a generalised Holling Type-III Prey-Predator Model with Crowding Effects of Predator and Holling type III response function incorporating a prey refuge. The type III functional response is similar to type II in that at high levels of prey density, saturation occurs. At low prey density levels, the graphical relationship of number of prey consumed and the density of the prey population is a super-linearly increasing function of prey consumed by predators, $f(r) = \frac{ar^k}{1+ahr^k}$, $k > 1$. This accelerating function was originally formulated in analogy with of the kinetics of an enzyme with two binding sites for $k = 2$. More generally, if a prey type is only accepted after every k encounters and rejected the $k - 1$ times in between, which mimics learning, the general form above is found.

Differential equation models describing interactions between species represent one of the classical applications of mathematics to biology. Advances in analytical techniques, coupled with the increasing power of computational tools, have significantly enhanced our understanding of such models over time. In this chapter we analyze a Lotka–Volterra type predator–prey model with Michaelis–Menten type functional response. In this particular model the population density of the prey is resource limited and each predator's functional response to the prey approaches a constant as the prey population increases (i.e. a type III response according to Holling). In addition [45], a spatial refuge protects a constant proportion of prey from predation. The purpose of the work is to offer mathematical analysis of the model and to discuss some significant qualitative results that are expected to arise from the interplay of biological forces. The mathematical analysis includes the existence of different feasible equilibria and their local as well as global stability. It is shown that there exists a stable limit cycle around the interior equilibrium point. Lastly we solve the geometrical equation with respect to Ricci flow and also draw some geometrical plot corresponding to the solution.

6.2 Model formulation and boundedness of solution

We consider the following basic model for the predator prey system.

$$\begin{aligned} \frac{du}{dT} &= ru(1-\frac{u}{k}) - \frac{au^2v}{b+u^2} \\ \frac{dv}{dT} &= -cv + \frac{eau^2v}{b+u^2}, \end{aligned} \tag{6.2.1}$$

where $u(T)$ represents the prey population(density), $v(T)$ represents the predator population, r intrinsic growth rate, k carrying capacity and the all positive parameters are a, b, c and e. Here we proposed the generalised Holling type-III response functions. It is usually describes the uptake of subtrate by micro organism in microbial dynamics kinetics.

For mathematics simplicity we consider the following transformation

$$x = \frac{u}{k}\ ,\ y = \frac{v}{ke}\ ,\ t = rT.$$

Therefore the transformation equation is

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{\epsilon x^2y}{\alpha+x^2} \\ \frac{dy}{dt} &= -\beta y + \frac{\epsilon x^2y}{\alpha+x^2}, \end{aligned} \tag{6.2.2}$$

where $\epsilon = \frac{ae}{r}$ $\alpha = \frac{b}{k^2}$ and $\beta = \frac{c}{r}$ and also $x(0) > 0$, $y(0) > 0$ and α, β and ϵ are all positive. The boundedness of solution of the system (6.2.1) is proved in the following lemma.

Lemma 6.2.1. *All solution of the system* (6.2.1) *that initiate in* ${\mathbb{R}_+}^2$, *are eventually uniformly bounded and enter into a region* B *defined by* $B = \{(x,y) : w = \frac{\mu}{v} + \xi$, *for any* $\xi > 0\}$.

Proof. *We define the function w=x+y, therefore the time derivative*

$$\frac{dw}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = x(1-x) - \frac{\epsilon x^2 y}{\alpha + x^2} - \beta y + \frac{\epsilon x^2 y}{\alpha + x^2}$$

For each $v > 0$ *,we have*

$$\frac{dw}{dt} + vw \leq \frac{1}{4}(v+1)^2 - (\beta - v).$$

Now we choose $v < \beta$*, then the righthand is bounded for all* $(x, y) \in \mathbb{R}^2_+$*.Thus we choose a* $\mu > 0$*, such that*

$$\frac{dw}{dt} + vw < 0.$$

Applying the theory of differential inequality [3] we obtain $0 < w(x, y) < \frac{\mu}{v}(1 - e^{-vt}) + w(x(0), y(0))e^{-vt}$*, which upon letting* $t \to \infty$*, yields* $0 < w < \frac{\mu}{v}$*, so we have that all the solutions of system* (6.2.1) *that start in* ${\mathbb{R}_+}^2$ *are confined to the region B, where* $B = \{(x, y) : w = \frac{\mu}{v} + \xi$*, for any* $\xi > 0\}$*.*

6.3 Basic Results

To, ensure the existence and uniqueness of system (6.2.1), we seek the solution in $\mathbb{R}^2_+ = \{(x, y) : x > 0, y > 0\}$ so that all the standard results of existence, uniqueness and continuous dependence on initial condition are evidently satisfies.

6.3.1 Equilibria

We now study the existence of equilibria of system (6.2.1) particular we are interested in the interior or positive equilibrium.

To begin with we list all possible equilibria.

(i) The trivial equilibrium $E_0(0, 0)$.

(ii) Equilibrium in the absence of predator $(y = 0)$ $E_1(1, 0)$

. (iii) The interior(positive) equilibrium $E_2(x^*, y^*)$, where $x^* = \sqrt{\frac{\alpha\beta}{\epsilon - \beta}}$ and $y^* = \frac{1}{\beta}\frac{\sqrt{\alpha\beta(\epsilon-\beta)} - \alpha\beta}{\epsilon - \beta}$.

For the equilibrium $E_2(x^*, y^*)$ to be positive, we first need $\epsilon - \beta > 0$.

For y^* to be positive, $\epsilon > \beta(\alpha + 1)$.

Thus for the existence of the positive equilibrium both $\epsilon > \beta$ and $\epsilon > \beta(\alpha + 1)$ must hold.

6.3.2 Dynamic Behaviour

In this subsection we shall discuss [51] the stability properties of the eqquilibrium E_0 E_1 and E_2. The Jacobian matrix of the system evaluated at the equilibrium point is given by:

$$V(x,y) = \begin{pmatrix} 1 - 2x - \frac{2\epsilon\alpha xy}{(\alpha+x^2)^2} & -\frac{\epsilon x^2}{\alpha+x^2} \\ \frac{2\epsilon\alpha xy}{(\alpha+x^2)^2} & -\beta + \frac{\epsilon x^2}{\alpha+x^2} \end{pmatrix}. \tag{6.3.3}$$

Now for the equilibrium point $E_0(0,0)$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -\beta \end{pmatrix}.$$

Hence the eigen values of this system are 1, $-\beta$.
Therefore $E_0(0,0)$ is a saddle point.
Jacobian matrix for the equilibrium $E_1(1,0)$ is given

$$\begin{pmatrix} -1 & -\frac{\epsilon}{\alpha+1} \\ 0 & -\beta + \frac{\epsilon}{\alpha+1} \end{pmatrix}.$$

The eigen values of the matrix are -1, $-\beta + \frac{\epsilon}{\alpha+1}$. Hence $E_1(1,0)$ is locally asymtotically stable when $\epsilon < \beta(\alpha+1)$ and unstable(saddle) when $\epsilon \geq \beta(\alpha+1)$. When both $E_0(0,0)$ and $E_1(1,0)$ are stable according to the theorem (3.1) of [29], system is persistent. It is observed that when E_2 exists, E_1 is unstable(saddle).
The jacobian about the equilibrium E_2 is given by

$$\begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix}$$

where

$$X = 1 - 2x - \frac{2\epsilon\alpha xy}{(\alpha+x^2)^2}, \tag{6.3.4}$$

$$Y = -\frac{\epsilon x^2}{\alpha+x^2}, \tag{6.3.5}$$

$$Z = \frac{2\epsilon\alpha xy}{(\alpha+x^2)^2}. \tag{6.3.6}$$

The characteristic equation is $\lambda^2 - \lambda X + YZ = 0$. The sum of the roots is equal to X and product of the roots is equal to YZ which is always positive as $\epsilon > \beta(\alpha+1)$.

Now X will be negative if $\frac{\sqrt{\epsilon-\beta}(\epsilon-2\beta)+2\beta\sqrt{\alpha\beta}}{\epsilon\sqrt{\epsilon-\beta}} > 0$.
From the above it is clear that E_2 is locally asymtotically stable.
Now if $\frac{\sqrt{\epsilon-\beta}(\epsilon-2\beta)+2\beta\sqrt{\alpha\beta}}{\epsilon\sqrt{\epsilon-\beta}} < 0$, then E_2 is locally unstable in the xy plane.
If $\frac{\sqrt{\epsilon-\beta}(\epsilon-2\beta)+2\beta\sqrt{\alpha\beta}}{\epsilon\sqrt{\epsilon-\beta}} = 0$, then the system (6.2.1) enter into Hopf type small amplitude periodic solution (limit cycle) near E_2.

6.3.3 Existence of Limit Cycles

In two dimensions it is well known that there can be no limit cycles in models of competitive or cooperative system. Further it is known for predator prey systems that the existence and stability of a limit cycle is related to the existence and stability for positive equilibrium. We assume that a positive equilibrium exists, for otherwise the predator population tends to extinction [30], if the equilibrium is asymtotically stable, there may exists limit cycles. The innermost of which must be unstable from the inside and the out most of which must be unstable. From the outside, if the limit cycle do not exist in this case, the equilibrium is globally asymtotically stable. If the positive equilibrium exists and is unstable, there must occur at least one limit cycle.

In the present subsection we shall prove the system (6.2.1) has unique stable limit cycle, when E_2 becomes locally stable.
Let us consider the system (6.2.1) in the form

$$\begin{aligned}\frac{dx}{dt} &= xg(x) - yp(x)\\ \frac{dy}{dt} &= y(-\beta + p(x)),\end{aligned} \tag{6.3.7}$$

where $g(x) = (1-x)$, $p(x) = \frac{\epsilon x^2}{\alpha + x^2}$.
We have the following theorem regarded uniqueness of limit cycles of system (6.3.7)

Theorem 6.3.1. *Suppose in the system* (6.3.7),

$$\frac{d}{dx}\left[\frac{xg'(x)+g(x)-xg(x)\frac{p'(x)}{p(x)}}{-\lambda+p(x)}\right]$$

in $0 \leq x < x^*$ *and* $x^* < x < 1$, *then system* (6.3.7) *has exactly one limit cycle which is globally asymtotically stable with respect to the set* $\{(x,y) : x > 0, y > 0\} - E_2(x^*, y^*)$.

By employing theorem (6.3.1), we can prove easily the following theorem.

Theorem 6.3.2. *If* $\frac{\sqrt{\epsilon-\beta}(\epsilon-2\beta)+2\beta\sqrt{\alpha\beta}}{\epsilon\sqrt{\epsilon-\beta}} \le 0$, *then system* (6.2.1) *has exactly one limit cycle which is globally stable with respect to the set* $\{(x,y): x>0, y>0\} - E_2(x^*, y^*)$.

Proof. This will be equivalent to proving

$$\frac{d}{dx}[\frac{(1-2x)-(1-x)\frac{2\alpha}{\alpha+x^2}}{-\beta+\frac{\epsilon x^2}{\alpha+x^2}}] \le 0$$

or,

$$\frac{d}{dx}[\frac{2x^3-x^2+\alpha}{x^2-\lambda}] \ge 0,$$

where $\lambda = \frac{\alpha\beta}{\epsilon-\beta}$ it is equivalent to proving

$$\frac{2x}{(x^2-\lambda)^2}[x^3-3x\lambda+(\lambda-\alpha)] \ge 0,$$

or,

$$\frac{2\beta-\epsilon}{2\beta} \ge \lambda,$$

which implies that,

$$\frac{\sqrt{\epsilon-\beta}(\epsilon-2\beta)+2\beta\sqrt{\alpha\beta}}{\epsilon\sqrt{\epsilon-\beta}} \le 0.$$

The equality holds if and only if

$$\frac{\sqrt{\epsilon-\beta}(\epsilon-2\beta)+2\beta\sqrt{\alpha\beta}}{\epsilon\sqrt{\epsilon-\beta}} = 0.$$

This completes the proof. □

6.4 Numerical Simulation and Discussions

To understand the dynamics of the qualitative analysis, numerical simulations have been carried out using MATLAB-R2016a and all the analytical results have been verified as shown in the figures. We have done the simulations using the standard MATLAB differential equations integrator for Runge-Kutta method, i.e. MATLAB routine ODE 45. In this numerical illustration of the system (6.2.2), we use different admissible values of the system parameters to ensure our theoretical results. Keeping the feasibility condition of equilibrium points in mind, we have chosen a set of parameter values such as $\beta = 0.2$, $\epsilon = 0.3$. Also we have chosen the values of α by using the following conditions:

$$(i) \quad \alpha > \frac{(\epsilon - \beta)(2\beta - \epsilon)^2}{4\beta^3} = \alpha^{**}$$

$$(ii) \quad \alpha < \alpha^{**}$$

$$(iii) \quad \alpha = \alpha^{**}$$

For the set of parameter values, the system (6.2.2) possesses a unique equilibrium point, i.e. $E_2(x^*, y^*)$. Through the numerical simulation it is shown that the parameter α plays an important role to shape the dynamics of the detritus based prey-predator system.

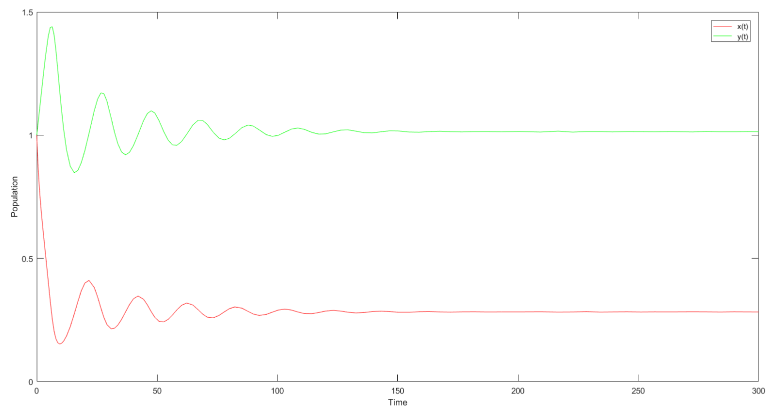

Figure 6.1: solution curves of the model for same initial values for $\beta = 0.2$, $\epsilon = 0.3$ when $\alpha = 0.04$.

In Figure 6.1, we set the value of $\alpha = 0.04$ and we have shown the steady state of the plant litter biomass and also observed that at first the population of microorganism pool

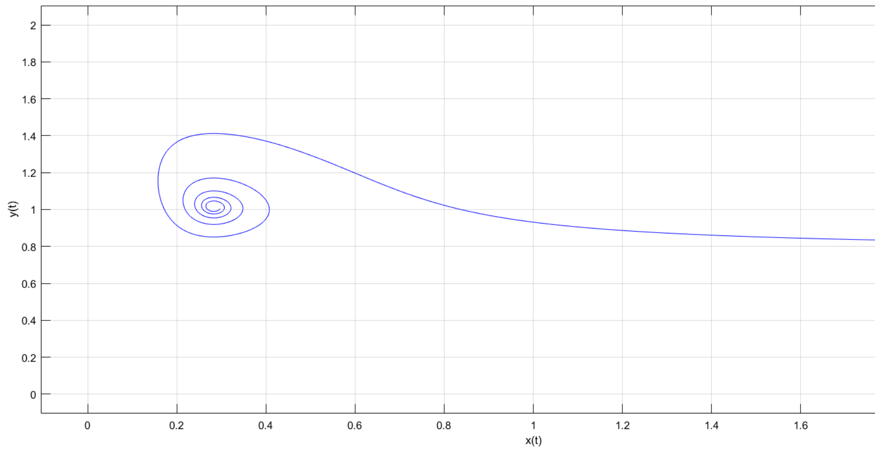

Figure 6.2: phase plane trajectories of the system for different initial values for $\beta = 0.2$, $\epsilon = 0.3$ when $\alpha = 0.04$.

oscillates with small amplitude and the oscillation gradually decreases as the time increases and the prey population attains its steady state value and the same picture is seen for predator population. So, through this figure, it is seen that the system (6.2.2) is globally asymptotically stable which proved the theoretical finding of global stability analysis. The corresponding phase portrait of global asymptotic stability of the equilibrium point (x^*, y^*, z^*) has been shown in the Figure 6.2, at the same parameter values.

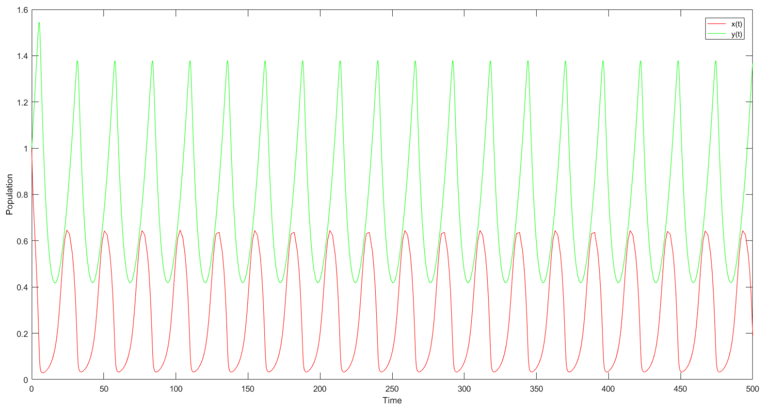

Figure 6.3: solution curves of the model for same initial values for $\beta = 0.2$, $\epsilon = 0.3$ when $\alpha = 0.009$.

Now in Figure 6.3, we choose the value of $\alpha = 0.009$. This figure shows the instability

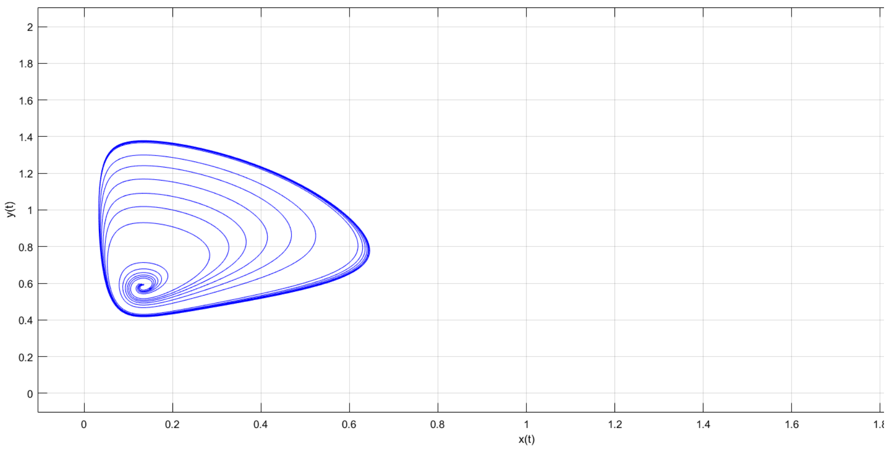

Figure 6.4: phase plane trajectories of the system for different initial values for $\beta = 0.2$, $\epsilon = 0.3$ when $\alpha = 0.009$.

of the system (6.2.2). In this figure the plant litter population maintains its steady state but it is observed from the figure that there is a large amplitude oscillation with increasing time for the prey and predator population which leads to limit cycle. The corresponding phase portrait of stable limit cycle of the system has been shown in the Figure 6.4, for the same parameter values.

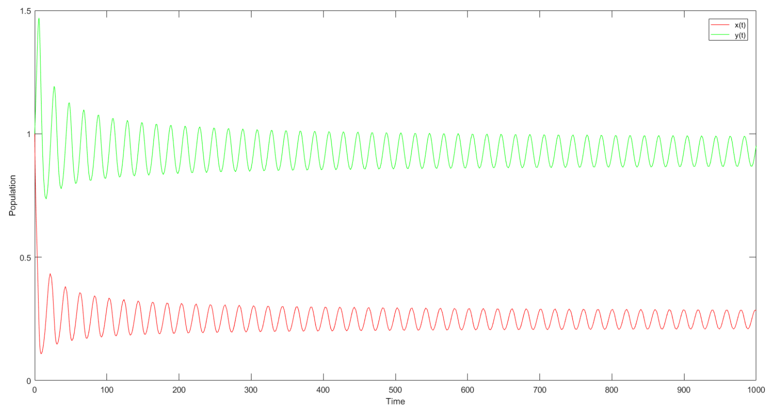

Figure 6.5: solution curves of the model for same initial values for $\beta = 0.2$, $\epsilon = 0.3$ when $\alpha = 0.0299$.

In Figure 6.5, we set the value of $\alpha = 0.0299$ and the figure shows the existence of small amplitude periodic oscillation which leads to Hopf-bifurcation, whereas the Figure

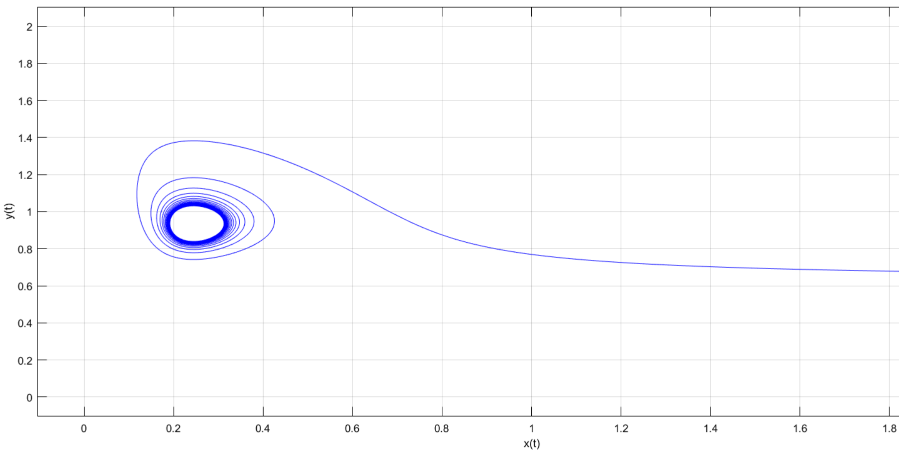

Figure 6.6: phase plane trajectories of the system for different initial values for $\beta = 0.2$, $\epsilon = 0.3$ when $\alpha = 0.0299$.

6.6 demonstrates Hopf-bifurcation with small amplitude periodic oscillation around the interior equilibrium point for the same parameter values.

6.5 Geometric Overview of These Solutions with respect to Ricci flow

Now from the equation (6.3.3) and using the relation $\beta = \frac{\epsilon x^2}{\alpha + x^2}$ from the equilibrium we have the matrix

$$\begin{pmatrix} 1-2x-\frac{2\alpha\beta^2 y}{\epsilon x^2} & -\beta \\ \frac{2\alpha\beta^2 y}{\epsilon x^2} & 0 \end{pmatrix}.$$

If we consider

$$\begin{aligned} g_{11} &= 1-2x-\frac{2\alpha\beta^2 y}{\epsilon x^3} \\ g_{12} &= -\beta \\ g_{21} &= \frac{2\alpha\beta^2 y}{\epsilon x^3} \\ g_{22} &= 0. \end{aligned} \tag{6.5.8}$$

Corresponding of the these values, we have

$$\begin{aligned} R_{11} &= -\frac{2\epsilon x^4 - \epsilon x^3 + 2\alpha\beta^2 y}{4\alpha\beta^2 y^3} \\ R_{12} &= \frac{1}{2y^2} \\ R_{21} &= \frac{1}{2y^2} \\ R_{22} &= 0. \end{aligned}$$

Using Ricci flow equation (1.2.90) corresponding to the above values we take first

$$\frac{dg_{11}}{dt} = -2R_{11}.$$

Putting the values of g_{11} and R_{11}, using the relation $y = \frac{x(1-x)}{\beta}$ from the equilibrium we get

$$\frac{2\alpha(1-x)^2\{2\epsilon x^3 + 2\alpha\beta(x-2)\}}{2\beta\epsilon^2 x^4 - \beta\epsilon^2 x^3 + 2\alpha\beta^2\epsilon x(1-x)}dx = -dt. \tag{6.5.9}$$

For simplicity we take the stability value as $\alpha = 0.4$, $\beta = 0.2$ and $\epsilon = 1$ (using different values of ϵ) then from the equation (6.5.9), we get

$$\frac{4(1-x)^2\{5x^3 + 4x - 8\}}{2x^4 - x^3 + 0.08x(1-x)}dx = -dt. \tag{6.5.10}$$

Then the general solution of the differential equation is

$$\frac{x^3}{3} - \frac{5x^3}{4} + \frac{15x}{4} - \frac{1}{4}\sum_{R} \frac{(77R^2 - 70R - 42)ln(x - R)}{6R^2 - 2R - 2} - 2ln(x) = t + A, \tag{6.5.11}$$

where R =root of $(2z^3 - z^2 + 2 - 2z)$ and A is a constant. Now initially $x = 1$, $t = 0$ and take $R = -2$ we get, $A = \frac{17}{6}$. Therefore the actual solution is

$$\frac{x^3}{3} - \frac{5x^3}{4} + \frac{15x}{4} - \frac{1}{4}\sum_{R} \frac{(77R^2 - 70R - 42)ln(x - R)}{6R^2 - 2R - 2} - 2ln(x) = t + \frac{17}{6}, \tag{6.5.12}$$

where R =root of the equation $(2z^3 - z^2 + 2 - 2z)$ and the corresponding plot of this solution is given below (using mapple).

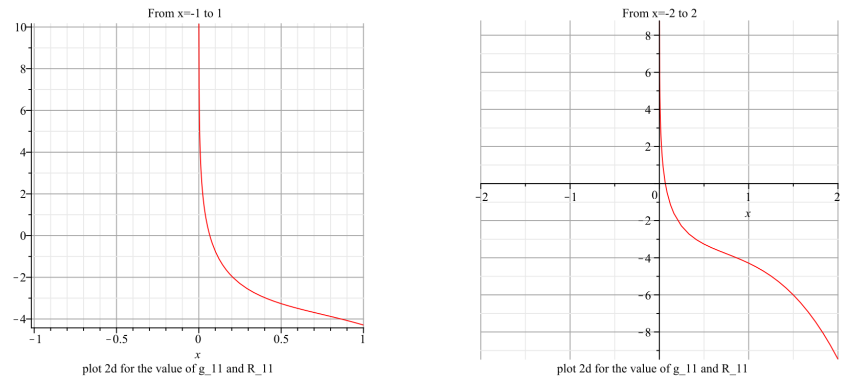

Figure 6.7: plot 2d for x=-1 to 1 and for x=-2 to 2, when g_{11}, R_{11} are given

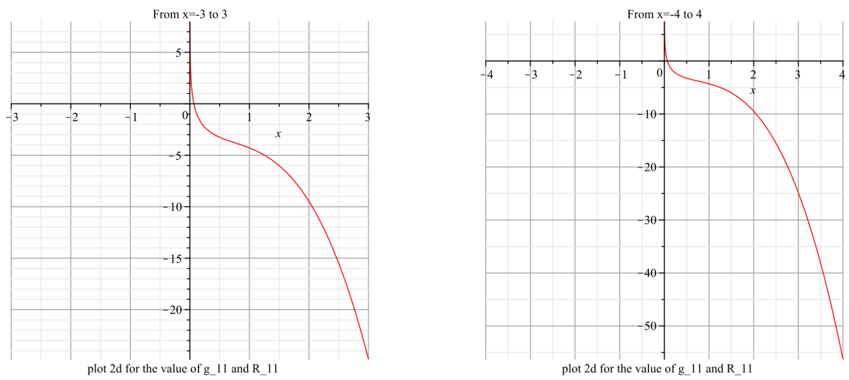

Figure 6.8: plot 2d for x=-3 to 3 and for x=-4 to 4, when g_{11}, R_{11} are given

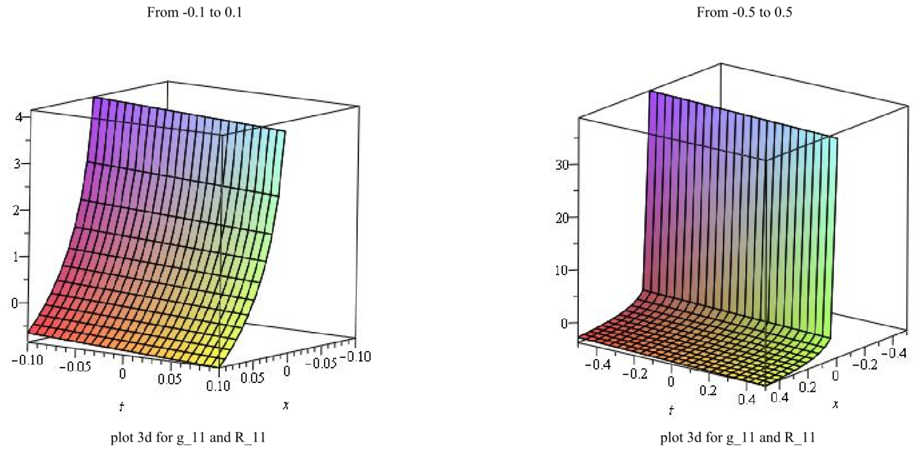

Figure 6.9: plot 3d for x, t=-0.1 to 0.1 and for x, t=-0.5 to 0.5, when g_{11}, R_{11} are given

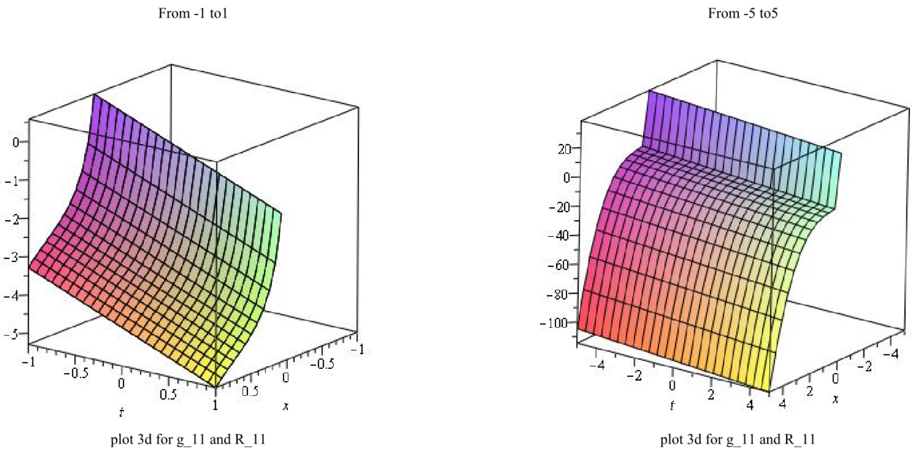

Figure 6.10: plot 3d for x, t=-1 to 1 and for x, t=-5 to 5, when g_{11}, R_{11} are given

Next we consider $g_{12} = -\beta$, $R_{12} = \frac{1}{2y^2}$ which implies from (1.2.90) we get, $y = 0$, which is trivial and the corresponding values of $x = 0, 1$. Therefore the solution represented on the x axis from 0 to 1.

Again $g_{21} = \frac{2\alpha^2\beta y}{\epsilon x^3}$, $R_{21} = \frac{1}{2y^2}$. Then by taking the stability values of α, β, ϵ and from the Ricci flow equation (1.2.90) we have the equation

$$\frac{2(1-x^2)(x-2)}{x}dx = -dt. \tag{6.5.13}$$

Therefore the general solution is given by

$$\frac{2}{3}x - 2x^2 - 2x + 4ln(x) = -t + A. \tag{6.5.14}$$

Initially $t = 0$, $x = 1$, from (6.5.14) implies $A = -\frac{10}{3}$, therefore the complete solution is

$$\frac{2}{3}x - 2x^2 - 2x + 4ln(x) + \frac{10}{3} = -t. \qquad (6.5.15)$$

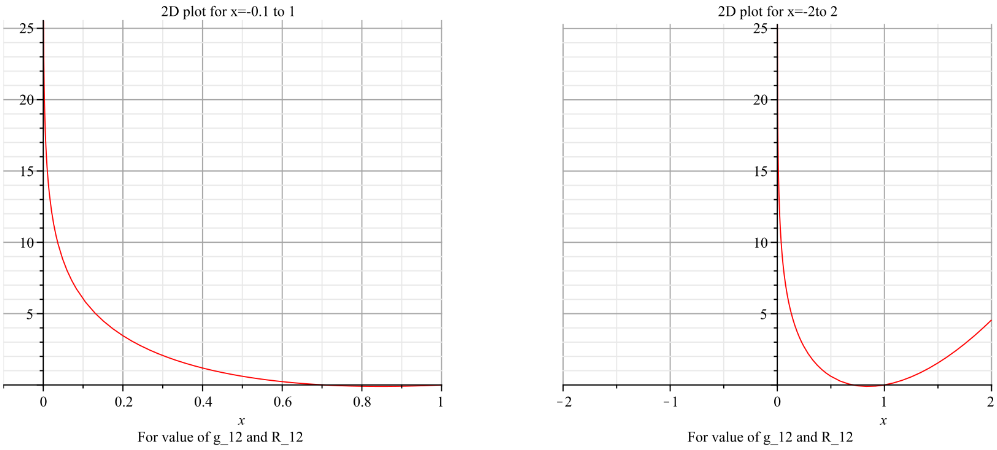

Figure 6.11: plot 2d for x=-0.1 to 1 and for x=-2 to 2, when g_{12}, R_{12} are given

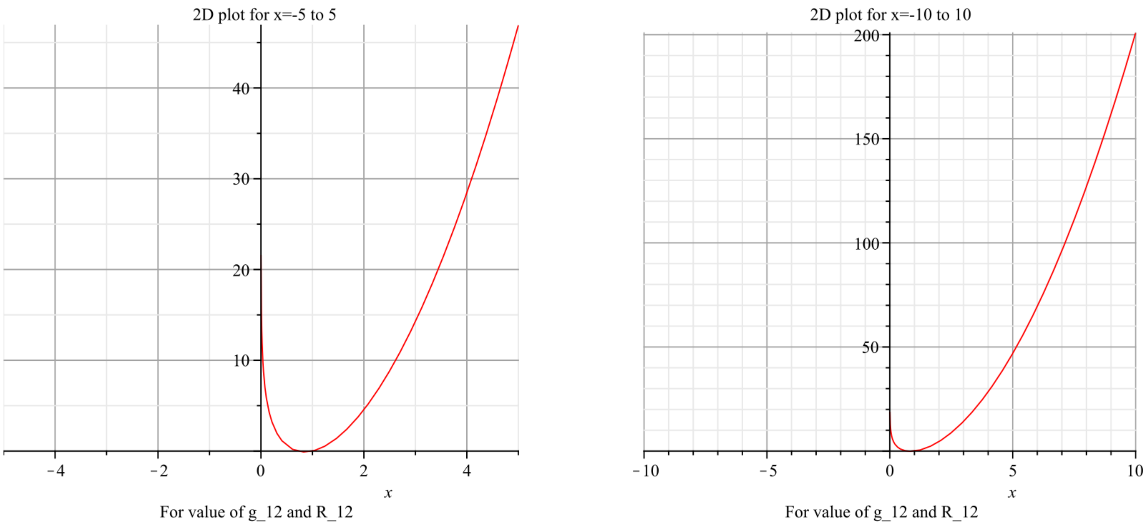

Figure 6.12: plot 2d for x=-5 to 5 and for x=-10 to 10, when g_{12}, R_{12} are given

For $g_{22} = 0$ and $R_{22} = 0$, there is no solution.

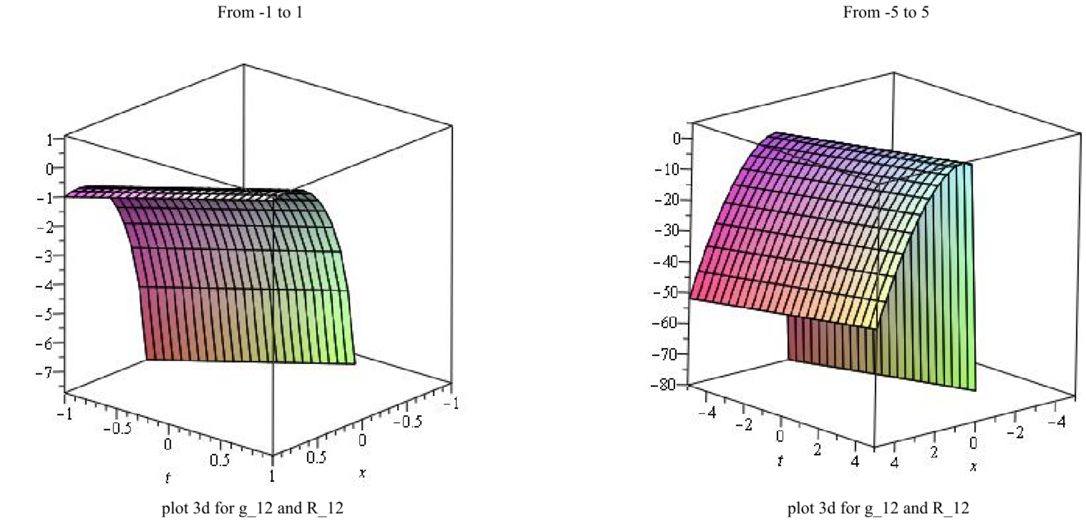

Figure 6.13: plot 3d for x, t=-1 to 1 and for x, t=-5 to 5, when g_{12}, R_{12} are given

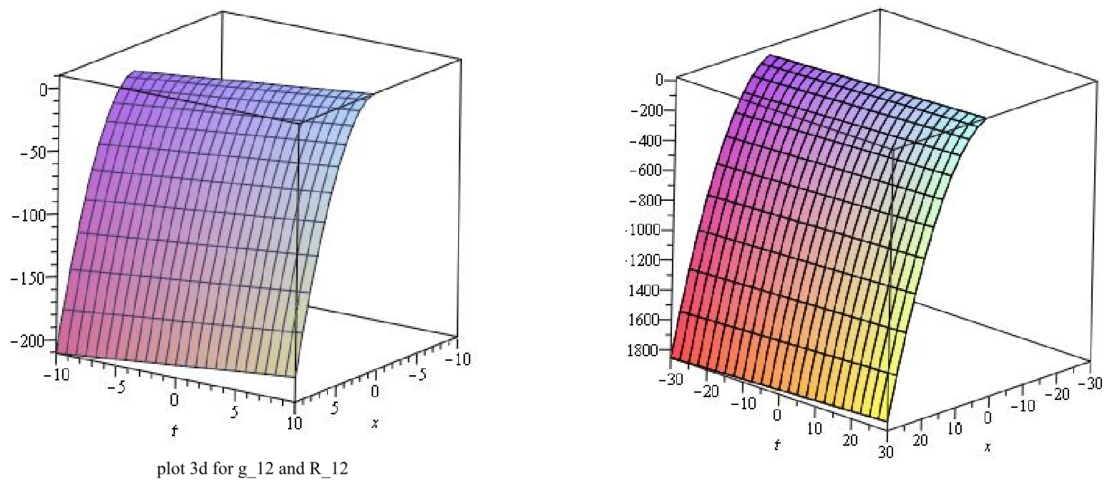

Figure 6.14: plot 3d for x, t=-10 to 10 and for x, t=-30 to 30, when g_{12}, R_{12} are given

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1. W_2-curvature tensor on trans-Sasakian space form, Somnath Mondal and Arindam Bhattacharyya, **Bull. Cal. Math. Soc.**, 112, (5) 417-430 (2020).

2. Characterization of almost η-Ricci-Yamabe soliton and gradient almost η-Ricci-Yamabe soliton on almost Kenmotsu manifolds, Somnath Mondal, Santu Dey and Arindam Bhattacharyya, **Acta Mathematica Sinica, English Serires**, Vol. 39, No. 4, pp 728-748 (2023).

3. The Geometry δ-Ricci-Yamabe almost soliton on Para-contact metric Manifolds, Somnath Mondal, Santu Dey, Young Jin Suh and Arindam Bhattacharyya, **KYUNGPOOK Math. J.**, No. 63, 623-638 (2023).

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W_2-CURVATURE TENSOR ON TRANS-SASAKIAN SPACE FORM

SOMNATH MONDAL[1], ARINDAM BHATTACHARYYA[2]

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Abstract. The objective of the present paper is to characterize trans-Sasakian space form satisfying certain curvature conditions on W_2-curvature tensor. In this paper we study W_2-semisymmetric and W_2- pseudosymmeric trans-Sasakian space form,W_2-locally symmetric trans-Sasakian space form,W_2-locally ϕ-symmetric trans-Sasakian space form and W_2-ϕ-recurrent trans-Sasakian space form. Some of these results are in the form of necessary and sufficient conditions.

Keywords: W_2-curvature tensor, trans-Sasakian space form, pseudosymmetric manifold, locally ϕ-symmetric manifold, ϕ-recurrent manifold .

Mathematics Subject Classification 2010: 53B15, 53B20, 53C15, 53C25.

1. Introduction. A Riemannian manifold with constant sectional curvature c is known as a real space and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$$

for all $X, Y, Z \in \chi(M)$. Representation for these space are hyperbolic spaces $(c < 0)$ spheres $(c > 0)$ and Euclidean spaces $(c = 0)$.

The concept of local symmetricness of a Riemannian manifold has been studied by many authors in several way to a different extend such has (Alegre, Blair and Carriazo, 2004, Alegre and Carriazo, 2008 and Kim, 2006). The locally ϕ-symmetricness of Sasakian manifold was introduced by Takahashi in (Takahashi, 1997). De and et al generalized this to the notion of ϕ-symmetricness and then introduced the notion of ϕ-recurrent Sasakian manifold (Shaikh, Sudipta and De, 2009). Futher ϕ-recurrent condition was studied on Kenmotsu (De, Yaliniz and Yildiz, 2009) LP-Sasakian (Bagewadi, Pradeep Kumar and Venkatesha, 2011) and $(LCS)_n$ manifold (Basu, Eyasmin and Shaikh, 2008).

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Characterization of Almost η-Ricci–Yamabe Soliton and Gradient Almost η-Ricci–Yamabe Soliton on Almost Kenmotsu Manifolds

Somnath MONDAL
Department of Mathematics, Jadavpur University, Kolkata-700032*, India*
E-mail: *somnathmondal.math@gmail.com*

Santu DEY[1)]
Department of Mathematics, Bidhan Chandra College
Asansol-4*, West Bengal*-713304*, India*
E-mail: *santu.mathju@gmail.com*

Arindam BHATTACHARYYA
Department of Mathematics, Jadavpur University, Kolkata-700032*, India*
Email: *bhattachar*1968*@yahoo.co.in*

Abstract The prime object in this article is to study an almost η-Ricci–Yamabe soliton and gradient almost η-Ricci–Yamabe soliton within the framework of almost Kenmotsu manifolds. It is shown that normal almost Kenmotsu manifold admitting an almost η-Ricci–Yamabe soliton or gradient η-Ricci–Yamabe soliton is locally isometric to hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Next, we prove that if a (κ,μ) almost Kenmotsu manifold admits an almost η-Ricci–Yamabe soliton, then the manifold is η-Einstein. Besides, we find the condition for non-normal almost Kenmotsu manifolds acknowledging gradient almost η-Ricci–Yamabe soliton. Moreover, an almost η-Ricci–Yamabe soliton on $(\kappa,\mu)'$-almost Kenmotsu manifold has been studied. Lastly, we construct an example of a gradient almost η-Ricci–Yamabe soliton on a 3-dimensional Kenmotsu manifold.

Keywords Ricci soliton, (κ,μ)-almost Kenmotsu manifold, $(\kappa,\mu)'$-almost Kenmotsu manifold, η-Ricci–Yamabe soliton

MR(2010) Subject Classification 53D15, 53C15, 53C25

1 Introduction and Motivations

Contact geometry methods play an important role in modern mathematics. Contact geometry has evolved from the mathematical formalism of classical mechanics. In 1969, Tanno [46] classified the connected almost contact metric manifolds whose automorphism groups have maximal dimensions as follows:

(a) Homogeneous normal contact Riemannian manifolds with constant ϕ-holomorphic sectional curvature if $k(\xi, X) > 0$;

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1) Corresponding author

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RESEARCH ARTICLE

A Study of Conformal η-Einstein Solitons on Trans-Sasakian 3-Manifold

Yanlin Li[1] · Somnath Mondal[2] · Santu Dey[3] · Arindam Bhattacharyya[2] · Akram Ali[4]

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Abstract

We study conformal η-Einstein solitons on the framework of trans-Sasakian manifold in dimension three. Existence of conformal η-Einstein solitons on trans-Sasakian manifold is discussed. Then we find some results on trans-Sasakian manifold which are conformal η-Einstein solitons where the Ricci tensor is cyclic parallel and Codazzi type. We also consider some curvature conditions with addition to conformal η-Einstein solitons on trans-Sasakian manifold. We also use torse-forming vector fields in addition to conformal η-Einstein solitons on trans-Sasakian manifold. Finally, an example of conformal η-Einstein solitons on trans-Sasakian manifold is constructed.

Keywords Trans-Sasakian manifold · Einstein soliton · Conformal η-Einstein soliton · Codazzi type Ricci tensor · $\mathcal{C}$-Bochner curvature tensor · $\mathcal{W}_2$ curvature tensor · $\mathcal{M}$-projective curvature tensor

1 Introduction

The Ricci flow on a smooth manifold M with Riemannian metric $g(t)$ is given by

$$\frac{\partial}{\partial t}g(t) = -2Ric,$$

where Ric is the Ricci tensor of the metric $g(t)$. A Ricci soliton is a solution of Ricci flow (see details [24, 25, 57]), defined on a pseudo-Riemannian manifold (M, g) by

Yanlin Li, Somnath Mondal, Santu Dey, Arindam Bhattacharyya and Akram Ali have contributed equally to this work.

✉ Yanlin Li
liyl@hznu.edu.cn

Extended author information available on the last page of the article

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The Geometry of δ-Ricci-Yamabe Almost Solitons on Paracontact Metric Manifolds

Somnath Mondal
Department of Mathematics, Jadavpur University, Kolkata-700032, India
e-mail : somnathmondal.math@gmail.com

Santu Dey
Department of Mathematics, Bidhan Chandra College, Asansol-4, West Bengal-713304, India
e-mail : santu.mathju@gmail.com or santu@bccollegeasansol.ac.in

Young Jin Suh*
Department of Mathematics, Kyungpook National University, Daegu 41566, Republic of Korea
e-mail : yjsuh@knu.ac.kr

Arindam Bhattacharyya
Department of Mathematics, Jadavpur University, Kolkata-700032, India
e-mail : bhattachar1968@yahoo.co.in

Abstract. In this article we study a δ-Ricci-Yamabe almost soliton within the framework of paracontact metric manifolds. In particular we study δ-Ricci-Yamabe almost soliton and gradient δ-Ricci-Yamabe almost soliton on K-paracontact and para-Sasakian manifolds. We prove that if a K-paracontact metric g represents a δ-Ricci-Yamabe almost soliton with the non-zero potential vector field V parallel to ξ, then g is Einstein with Einstein constant $-2n$. We also show that there are no para-Sasakian manifolds that admit a gradient δ-Ricci-Yamabe almost soliton. We demonstrate a δ-Ricci-Yamabe almost soliton on a (κ, μ)-paracontact manifold.

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Geometry of almost $*$-η-Ricci-Yamabe soliton on Kenmotsu manifolds

Somnath Mondal[a], Santu Dey[b,*], Ali H. Alkhaldi[c], Ashis Kumar Sarkar[a], Arindam Bhattacharyya[a]

[a] *Department of Mathematics, Jadavpur University, Kolkata-700032, India*
[b] *Department of Mathematics, Bidhan Chandra College, Asansol-4, West Bengal-713304, India*
[c] *Department of Mathematics, College of Science, King Khalid University, 9004 Abha, Saudi Arabia*

Abstract. The goal of the present object is to study almost $*$-η-Ricci-Yamabe soliton within the framework of Kenmotsu manifolds. It is shown that if a Kenmotsu manifold admits a $*$-η-Ricci-Yamabe soliton, then it is η-Einstein. Next, we prove that if a $(\kappa, -2)'$-nullity distribution, where $\kappa < -1$ acknowledges a $*$-η-Ricci-Yamabe soliton, then the manifold is Ricci flat. Later, if g represents a gradient almost $*$-η-Ricci-Yamabe soliton and ξ leaves the scalar curvature r invariant on a Kenmotsu manifold, then the manifold is an η-Einstein. Further, we have studied on a Kenmotsu manifold if g represents an almost $*$-η-Ricci-Yamabe soliton with potential vector field V is pointwise collinear with ξ, then the manifold is an η-Einstein. Lastly, we give an example of a gradient almost $*$-η-Ricci-Yamabe soliton on a 5-dimensional Kenmotsu manifold..

1. Introduction

In modern mathematics, contact geometry methods be involved in important role to the field of differential geometry. Contact geometry has enlarged from the mathematical formalism of classical mechanics. The concept of Ricci flow, which is an evolution equation for metrics defined over the connected almost contact metric manifolds whose automorphism groups have maximal dimensions.
In 1972, Kenmotsu [21] obtained some tensor equations to characterize the manifolds of the third class. Since then the manifolds of the third class were called Kenmotsu manifolds.

Very recently in 2019, Güler and Crasmareanu introduced a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map [15]. This flow is also known as Ricci-Yamabe flow of the type (ρ, q) [15].

A soliton to the Ricci-Yamabe flow is called Ricci-Yamabe soliton(RYS in short) if it moves only by one parameter group of diffeomorphism and scaling. The metric of the Riemannain manifold (M^n, g), $n > 2$ is said to admit (ρ, q)-Ricci-Yamabe soliton or simply Ricci-Yamabe soliton (RYS) (g, V, Ω, ρ, q) if it satisfies the equation:

$$\mathcal{L}_V g + 2\rho S + [2\Omega - qr]g = 0,$$

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* Corresponding author: Santu Dey
Email addresses: somnathmondal.math@gmail.com (Somnath Mondal), santu.mathju@gmail.com (Santu Dey), ahalkhaldi@kku.edu.sa (Ali H. Alkhaldi), aksarkar@gmail.com (Ashis Kumar Sarkar), bhattachar1968@yahoo.co.in (Arindam Bhattacharyya)