SOME STUDY ON HIGHER DIMENSIONAL COMPLEX ANALYSIS AND GEOMETRY

THESIS SUBMITTED TO THE

JADAVPUR UNIVERSITY

FOR AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY (SCIENCE)

BY

TANDRA SARKAR

DEPARTMENT OF MATHEMATICS
JADAVPUR UNIVERSITY
KOLKATA-700032, INDIA
MARCH, 2025

JADAVPUR UNIVERSITY

DEPARTMENT OF MATHEMATICS

KOLKATA- 700032

CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled "**SOME STUDY ON HIGHER DIMENSIONAL COMPLEX ANALYSIS AND GEOMETRY**" submitted by Ms. **Tandra Sarkar** who got her name registered on **15th February, 2022 (Index no.: 34/22/Maths./27)** for the award of Ph. D. (Science) Degree of Jadavpur University, is absolutely based upon her own work under the supervision of **Prof. (Dr.) Arindam Bhattacharyya, Professor, Department of Mathematics, Jadavpur University, Kolkata-700032** and **Prof. (Dr.) Sanjib Kumar Datta, Professor, Department of Mathematics, University of Kalyani, Nadia-741235** and that neither this thesis nor any part of it has been submitted for either any degree / diploma or any other academic award anywhere before.

Arindam Bhattacharyya 10/3/25

Professor
DEPARTMENT OF MATHEMATICS
Jadavpur University
Kolkata - 700 032, West Bengal

Sanjib Kumar Datta 10.03.2025

(Signature of the Supervisor(s) date with official seal)

Prof. Sanjib Kumar Datta
Professor
Department of Mathematics
University of Kalyani
West Bengal-741 235, India

Dedicated to
my parents
Chandra Shekhar Sarkar
&
Tapati Mondal Sarkar
and my brother
Arijeet Sarkar
for their support, love and sacrifices.

Acknowledgement

It is a great pleasure for me to be able to express my deepest gratitude and appreciation to my supervisor Prof. (Dr.) Arindam Bhattacharyya, Professor, Department of Mathematics, Jadavpur University and co-supervisor Prof. (Dr.) Sanjib Kumar Datta, Professor, Department of Mathematics, University of Kalyani for their guidance to prepare the thesis without their supervision the research would not have seen the light of the day. I am indebted to them for sharing their valuable time from their hectic schedule with me and giving me valuable suggestions for the enrichment the thesis. I owe my success to them.

I would also like to extend my sincere appreciation to the infrastructural support of the Departments of Mathematics, Jadavpur University and University of Kalyani during my Ph.D. work. Moreover, I also appreciate the cooperation and technical suggestions of my fellow research scholars to the improvement of the thesis. My heartiest gratitude to the entire the faculty of the Department of Mathematics, Jadavpur University for their encouragement and moral support.

I am very grateful for the insightful works of numerous authors whose papers and books helped me to give shape to this thesis. I am also thankful to Dr. Mukul Sk, Dr. Md Sadikur Rahman, Paritosh Ghosh, Rajdip Biswas, Bidhan Mondal and Kushal Mondal, for the simulating discussion, patience, encouragement and support. I am thankful to those who lent their expertise and assistance in helping me to bring my research work to fruition.

Lastly, I cannot forget to thank my family for their love and support and for the sacrifices they have made throughout my Ph.D. journey. A few words are not enough to express how grateful I am to my parents Chandra Shekhar Sarkar and Tapati Mondal Sarkar, younger brother Arijeet Sarkar and my grand mother Rai Kishori Mondal their encouragement and understanding represent the world to me.

Finally , I take full responsibility for any mistake or inaccuracy that occurred in the thesis and apologize for that oversight.

Date: 10/03/2025
Place : Jadavpur

Tandra Sarkar
(Tandra Sarkar)

ABSTRACT

Title: SOME STUDY ON HIGHER DIMENSIONAL COMPLEX ANALYSIS AND GEOMETRY
Index No.: 34/22/Maths./27

The prime concern of the present thesis focuses on discussions and exploration covering different aspects of complex analysis and geometry of higher index.

The thesis consists of Eight Chapters.

Chapter One deals with the preliminaries of bicomplex numbers, p-adic numbers and complex manifold. Also, the introductory theories concerning the influence of higher dimensional complex valued functions in Nevanlinna's Value distribution theories mainly both p-adic analysis and bicomplex analysis are briefly discussed in this chapter.

The prime concern of **Chapter Two**, is to prove some common fixed point theorems for converse commuting self-maps for non-complete bicomplex valued metric spaces. Our results are the generalizations of the results of Chauhan & Sahper (2013) and Kumar et al. (2014). Moreover, we wish to investigate some common fixed point theorems in bicomplex valued b-metric spaces satisfying some rational inequalities for a pair of self contracting mappings and extend some theorems of Azam et. al. (2011) and Rouzkard & Imdad (2012) regarding common fixed point theorems in complex valued metric spaces.

The main purpose of **Chapter Three** is to establish another representation of order and lower order of $\mathsf{f} \in \mathfrak{A}(\mathcal{K})$ where $\mathcal{K}$ is a complete ultametric algebraically closed field and $\mathfrak{A}(\mathcal{K})$ is the $\mathcal{K}$-algebra of entire functions on $\mathcal{K}$. Here, we establish the integral representation of generalized (β, γ)-th relative type and generalized (β, γ)-th relative weak type of entire function f with respect to another entire function g, where $\mathsf{f}, \ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ and their equivalence relation under some certain conditions are also obtained.

The main goal of **Chapter Four** is to establish some results focusing on the comparative growth properties of composition of two k-valued entire algebroidal functions in terms of their relative growth indicators from the view point of slowly changing functions in the light of p-adic analysis. Although the main aim of this chapter is to find out the estimates of relative (β, γ)-th type, relative (β, γ)-th lower type and relative (β, γ)-th weak type under somewhat different conditions where p and q are any two positive integers via the concept of p-adic analysis of entire algebroidal functions.

The **Chapter Five** aims to extend the classical Enstr$\ddot{o}$m-Kakeya theorem for entire functions of non zero finite order having lacunary type power series expansion. A few examples with related figures are given here to justify the results obtained. We

investigate some common fixed point theorems in bicomplex valued metric space and find some sufficient conditions for the existence of common fixed points for a pair of self contracting mappings.

Chapter Six is focused on the derivation of some results related to the distribution of zeros of bicomplex valued entire functions in a certain domain. Some examples are provided to justify the results obtained.

The nucleus of **Chapter Seven** is to derive some results related to the Bicomplex manifold, almost bicomplex manifold, almost bicomplex Hermite manifold and to discuss some properties of Nijenhuis tensor, contravariant almost analytic vector fields etc.

Chapter Eight is mainly based on future prospects including further course of work and also their applications of the works as carried out in the thesis.

Tandra Sarkar 10/03/2025
(Tandra Sarkar)

Arindam Bhattacharyya 10/3/25
(Prof. Arindam Bhattacharyya)
Professor
DEPARTMENT OF MATHEMATICS
Jadavpur University
Kolkata - 700 032, West Bengal

Sanjib Kumar Datta 10.03.2025
(Prof. Sanjib Kumar Datta)
Prof. Sanjib Kumar Datta
Professor
Department of Mathematics
University of Kalyani
West Bengal-741 235, India

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LIST OF PUBLICATIONS

1. S. K. Datta, R. Sarkar, J. Saha and **T. Sarkar**: *Common fixed point theorems for converse commuting mappings in bicomplex valued metric spaces*,South East Asian Journal of Mathematics and Mathematical Science (SEAJMMS) (SCOPUS Indexed), Vol. 18, No. 1 (2021), pp. 205-214, ISSN (Online): 2582-0850, ISSN (Print): 0972-7752.

2. S. K. Datta, T. Molla, J. Saha and **T. Sarkar**: *On the distribution of zeros of bicomplex valued entire functions in a certain domain*, Scientific Studies and Research. Series Mathematics and Informatics (SSRSMI) (UGC CARE Listed), Vol. 30, No. 2, (2020), pp. 71-84, ISSN (Print): 2067-3566, ISSN (Online): 2457-497X.

3. A. Bandyopadhyay, **T. Sarkar**, S. K. Datta and A. Bhattacharyya: *Different representations of growth indicators of entire functions under the flavour of p-adic analysis*, Fuzzy Systems and Soft Computing (UGC CARE Listed), Vol. 19, No. 2(IV) (2024), ISSN: 1819-4362.

4. S. K. Datta, A. Biswas, M. Sk and **T. Sarkar**: *On the growths of entire algebroidal functions under the treatment of p-adic analysis*, Journal of Xi'an University of Architecture & Technology (JXAT), Vol. XIII, No. I (2021), pp. 412-425, ISSN: 1006-7930.

5. S. K. Datta, T. Molla, M. Sk and **T. Sarkar**: *On the generalization of Enström-Kakeya theorem for entire functions*, Journal of Xi'an University of Architecture & Technology (JXAT), Vol. XII, No. IX (2020), pp. 693-700, ISSN: 1006-7930.

6. S. K. Datta, D. Pal, R. Sarkar and **T. Sarkar**: *Few common fixed point theorems in bicomplex valued b-metric spaces*, Mathematical Forum (UGC CARE Listed), Accepted for publication and to appear, ISSN: 0972-9852.

7. **T. Sarkar**, A. Bhattacharyya and S. K. Datta: *A note on Bicomplex Manifold*, Communicated.

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Contents

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List of Mathematical Symbols

The below list describes several symbols that will be used later within the body of the thesis

$\aleph$	Complex valued metric
$\mathbb{BC}$	The set of all bicomplex numbers
$\mathbb{C}$	The set of all complex numbers
$\mathbb{N}$	The set of all natural numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}_{\geqslant 0}$	The set of all non negative real numbers
ϕ	The empty set
$\precsim$	Partial order relation on the set of complex numbers
$\precsim_{i_2}$	Partial order relation on the set of bicomplex numbers
Θ_n	The set of singular elements in $\mathbb{C}_n$ $(n = 0, 1, 2, \dots)$
Υ	Bicomplex valued metric
ϑ	Bicomplex valued b-metric
I	Identity transformation

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CHAPTER ONE

INTRODUCTION

Chapter 1

Introduction

1.1 Introduction to Preliminaries

In this section, a brief history with some notable properties of the bicomplex and p-adic numbers, the generalizaion of complex numbers are discussed. Further, the preliminaries concepts of complex valued metric space and complex manifold are presented here.

1.1.1 Bicomplex number

Bicomplex number is regarded as a well known extension of complex number which have been studied for quite a long time and a lot of work has been done in this area. In 1844, W. R. Hamilton {cf. [46] & [47]} introduced quaternions, an extension of complex numbers with three imaginary units $i,\ j,\ k$ that anti-commute with the property $ij = -ji = k$. Quaternions form a skew field with respect to usual operations. Although from the algebraic point of view, the lack of commutativity is not such a terrible problem. But this lack of commutativity opens a new direction of research where a four dimensional algebra containing $\mathbb{C}$ as a subalgebra is fornmed and it preserves the commutativity property. This can be done by considering two imaginary units $i,\ j$ with $ij = ji = k$.

In 1848, Cockle {cf. [18], [19], [20] & [21]} wrote a series of papers introducing a new algebra named as the algebra of tessarines. Influencing from the works of Cockle, Segre [87] introduced an infinite set of algebras and the concept of bicomplex numbers with their idempotent elements $\frac{1+ij}{2}$ and $\frac{1-ij}{2}$ which plays most important role in the theory of bicomplex analysis. Later Spampinato [88] and Dragoni [30] set the groundwork for function theory in the study of bicomplex numbers.

The next major breakthrough in bicomplex analysis came from Riley [79] in 1953 by developing the theory of functions of bicomplex variables. However, the most recommendable contribution, the theory of holomorphic functions of a bicomplex variable as well as multicomplex variables was done by Price [76].

Definition 1.1.1 *[73] The set of bicomplex numbers* $\mathbb{BC}$ *is defined by* $\mathbb{BC} = \{\mathfrak{Z} : \mathfrak{Z} = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ *or equivalently* $\mathbb{BC} = \{\mathfrak{z}_1 + j\mathfrak{z}_2 : \mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}\}$*, such that* $i^2 = j^2 = -k^2 = -1$ *and* $ij = ji = k$.

Definition 1.1.2 *[73] For any bicomplex number* $\mathfrak{Z} = \mathfrak{z}_1 + \mathfrak{z}_2\mathbf{j}$ *the conjugation is defined in the following way:*

$$\overline{\mathfrak{Z}}_i = \overline{\mathfrak{z}_1} + \overline{\mathfrak{z}_2}\mathbf{j}, \quad \overline{\mathfrak{Z}}_j = \mathfrak{z}_1 - \mathfrak{z}_2\mathbf{j}, \quad \overline{\mathfrak{Z}}_k = \overline{\mathfrak{z}_1} - \overline{\mathfrak{z}_2}\mathbf{j}.$$

1.1.1.1 Idempotent Representation.

One of the important features of a bicomplex number is its idempotent representation. The bicomplex numbers $e_1 := \frac{1+ij}{2}, e_2 := \frac{1-ij}{2}$ are linearly independent in the $\mathbb{C}$-linear space $\mathbb{BC}$ and $e_1 + e_2 = 1, e_1 - e_2 = ij, e_1.e_2 = 0, e_1^2 = e_1, e_2^2 = e_2$. Any $\mathfrak{Z} \in \mathbb{BC}$ can be uniquely expressed as $\mathfrak{Z} = (\mathfrak{z}_1 - i\mathfrak{z}_2)e_1 + (\mathfrak{z}_1 + i\mathfrak{z}_2)e_2$, which is known as the idempotent representation of $\mathfrak{Z}$.

Definition 1.1.3 *The norm* $\|\| : \mathbb{BC} \to \mathbb{R}_{\geqslant 0}$ *of* $\mathfrak{Z} = \mathfrak{z}_1 + j\mathfrak{z}_2 = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{BC}$ *is defined as follows:*

$$\|\mathfrak{Z}\| = \{|\mathfrak{z}_1|^2 + |\mathfrak{z}_2|^2\}^{\frac{1}{2}} = \left\{\frac{|\xi_1|^2 + |\xi_2|^2}{2}\right\}^{\frac{1}{2}}.$$

Definition 1.1.4 *The spaces* $\mathfrak{D}_1 = \{\mathfrak{z}_1 - i\mathfrak{z}_2 : \mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}\}$ *and* $\mathfrak{D}_2 = \{\mathfrak{z}_1 + i\mathfrak{z}_2 : \mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}\}$ *are said to be the auxiliary complex spaces. Each point* $\mathfrak{z}_1 + j\mathfrak{z}_2 = (\mathfrak{z}_1 - i\mathfrak{z}_2)e_1 + (\mathfrak{z}_1 + i\mathfrak{z}_2)e_2$ *in* $\mathbb{BC}$ *associates the points* $\mathfrak{z}_1 - i\mathfrak{z}_2 \in \mathfrak{D}_1$ *and* $\mathfrak{z}_1 + i\mathfrak{z}_2 \in \mathfrak{D}_2$*. Also to each pair of points* $(\mathfrak{z}_1 - i\mathfrak{z}_2,\ \mathfrak{z}_1 + i\mathfrak{z}_2) \in \mathfrak{D}_1 \times \mathfrak{D}_2$ *there is a unique point in* $\mathbb{BC}$.

Definition 1.1.5 *An open discus* $D(\xi; r_1, r_2)$ *with centre* $\xi = \xi_1 e_1 + \xi_2 e_2$ *and radii* $r_1 > 0, r_2 > 0$ *is defined by*

$$D(\xi; r_1, r_2) = \{\mathfrak{z}_1 e_1 + \mathfrak{z}_2 e_2 \in \mathbb{BC} : |\mathfrak{z}_1 - \xi_1| < r_1, |\mathfrak{z}_2 - \xi_2| < r_2\}.$$

Definition 1.1.6 *A closed discus* $\bar{D}(\xi; r_1, r_2)$ *with centre* $\xi = \xi_1 e_1 + \xi_2 e_2$ *and radii* $r_1 > 0, r_2 > 0$ *is defined as*

$$\bar{D}(\xi; r_1, r_2) = \{\mathfrak{z}_1 e_1 + \mathfrak{z}_2 e_2 \in \mathbb{BC} : |\mathfrak{z}_1 - \xi_1| \leqslant r_1, |\mathfrak{z}_2 - \xi_2| \leqslant r_2\}.$$

Definition 1.1.7 *If* $r_1 > 0$, $r_2 > 0$ *and* $r_1 = r_2 = r$*, then the discus is called a disc in* $\mathbb{BC}$ *and is denoted by* $D(\xi; r, r) = D(\xi; r)$.

1.1.2 Complex and bicomplex valued metric spaces

The Banach contraction principle [10], introduced by Banach in 1922, following Brouwer's foundational work in 1912 plays a significant role in fixed point theory. It states that "Let (χ, ν) be a complete metric space and $\mathcal{C} : \chi \to \chi$ be a contraction on χ, that is there is a constant $\lambda \in [0, 1)$ such that $\nu(\mathcal{C}x, \mathcal{C}y) \leqslant \lambda\nu(x, y)$ for all $x, y \in \chi$. Then $\mathcal{C}$ has a unique fixed point in χ."

Azam et al. [4] introduced the concept of a complex-valued metric space which is the generalization of the well known metric space. They done so to prove some fixed point results for contractive mappings with rational expression. Thereafter, Kakutani [59] and Kannan [60] extended the results of Brouwer and Banach respectively. Similar to the complex valued metric space, anthoer generalization of the cone metric space named as bicomplex-valued metric spaces were done $\{cf.$ [6], [7], [53], [65], [78], [86] & [90]$\}$.

Lii [68] proposed the idea of converse commuting maps and established some fixed point theorems for single valued maps in metric spaces. After that Lii & Hu [70] and Popa [77] derived some fixed point theorems for converse commuting multivalued mappings using their proposed concept on converse commuting multivalued mappings. Kumar et al. [64] made further contributions by obtaining several common fixed point theorems for converse commuting mappings under the ambiance of complex valued metric spaces.

Definition 1.1.8 *The partial order relation $\precsim$ on $\mathbb{C}$ is defined as follows*

$$\mathfrak{z}_1 \precsim \mathfrak{z}_2 \text{ if and only if } Re(\mathfrak{z}_1) \leqslant Re(\mathfrak{z}_2) \text{ and } Im(\mathfrak{z}_1) \leqslant Im(\mathfrak{z}_2).$$

Thus $\mathfrak{z}_1 \precsim \mathfrak{z}_2$ if one of the following conditions is satisfied

(i) $Re(\mathfrak{z}_1) = Re(\mathfrak{z}_2)$ and $Im(\mathfrak{z}_1) = Im(\mathfrak{z}_2)$,

(ii) $Re(\mathfrak{z}_1) < Re(\mathfrak{z}_2)$ and $Im(\mathfrak{z}_1) = Im(\mathfrak{z}_2)$,

(iii) $Re(\mathfrak{z}_1) = Re(\mathfrak{z}_2)$ and $Im(\mathfrak{z}_1) < Im(\mathfrak{z}_2)$,

(iv) $Re(\mathfrak{z}_1) < Re(\mathfrak{z}_2)$ and $Im(\mathfrak{z}_1) < Im(\mathfrak{z}_2)$.

We write $\mathfrak{z}_1 \precnsim \mathfrak{z}_2$ if $\mathfrak{z}_1 \precsim \mathfrak{z}_2$ and $\mathfrak{z}_1 \neq \mathfrak{z}_2$ i.e., one of (ii), (iii) and (iv) is satisfied and we write $\mathfrak{z}_1 \prec \mathfrak{z}_2$ if only (iv) is satisfied.

Taking this into account some fundamental properties of the partial order $\precsim$ on $\mathbb{C}$ is defined as follows:

(1) If $0 \precsim \mathfrak{z}_1 \precsim \mathfrak{z}_2$ then $|\mathfrak{z}_1| < |\mathfrak{z}_2|$,

(2) If $\mathfrak{z}_1 \precsim \mathfrak{z}_2$, $\mathfrak{z}_2 \precsim \mathfrak{z}_3$ then $\mathfrak{z}_1 \precsim \mathfrak{z}_3$ and

(3) If $\mathfrak{z}_1 \precsim \mathfrak{z}_2$ and $\lambda < 1$ is a non-negative real number then $\lambda\mathfrak{z}_1 \precsim \mathfrak{z}_2$.

Azam et. al. [4] defined the complex valued metric space in the following way:

Definition 1.1.9 *[4] Let $X \neq \phi$ and the mapping $\aleph : X \times X \to \mathbb{C}$, meets the following conditions:*

(d_1) $0 \precsim \aleph(x, y)$, $\forall x, y \in X$ and $\aleph(x, y) = 0$ if and only if $x = y$,

(d_2) $\aleph(x,y) = \aleph(y,x), \forall x, y \in X$ *and*
(d_3) $\aleph(x,y) \precsim \aleph(x,z) + \aleph(z,y),\ \forall x,y,z \in X.$

Then the pair $(X, \aleph)$ *is said to be a complex valued metric space and* $\aleph$ *is called a complex valued metric on* X*.*

Definition 1.1.10 *[26] The partial order relation* $\precsim_{i_2}$ *on* $\mathbb{BC}$ *is defined as follows*

$$\mathfrak{a} \precsim_{i_2} \mathfrak{b} \text{ if and only if } \mathfrak{a}_1 \precsim \mathfrak{a}_2 \text{ and } \mathfrak{b}_1 \precsim \mathfrak{b}_2,\ \text{where } \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathbb{C}$$

Definition 1.1.11 *Let* $\Upsilon : \mathcal{X} \times \mathcal{X} \to \mathbb{BC}$, $\mathcal{X} \neq \phi$ *be a mapping satisfies the following properties:*

(i) $0 \precsim_{i_2} \Upsilon(x,y)$, $\forall x,\ y \in \mathcal{X}$,
(ii) $\Upsilon(x,y) = 0$ *if and only if* $x = y$,
(iii) $\Upsilon(x,y) = \Upsilon(y,x)$, $\forall x,\ y \in \mathcal{X}$ *and*
(iv) $\Upsilon(x,y) \precsim_{i_2} \Upsilon(x,z) + \Upsilon(z,y)$, $\forall x,y,z \in \mathcal{X}$.

Then the pair $(\mathcal{X}, \Upsilon)$ *is called bicomplex valued metric space and* Υ *is said to be a bicomplex valued metric in* $\mathcal{X}$*.*

Definition 1.1.12 *Let* $(\mathcal{X}, \Upsilon)$ *be a bicomplex valued metric space.*

(i) A sequence $\{\wp_n\}$ *in* $\mathcal{X}$ *is called convergent and converges to* $\wp$ *if for any* $0 \prec_{i_2} r \in \mathbb{BC}$ *there exists* $n_0 \in \mathbb{N}$ *such that* $\Upsilon(\wp_n, \wp) \prec_{i_2} r$, $\forall n > n_0$ *and symbolically* $\lim\limits_{n\to\infty} \wp_n = \wp$ *or* $\wp_n \to \wp$ *as* $n \to \infty$.

(ii) A sequence $\{\wp_n\}$ *in* $\mathcal{X}$ *is called Cauchy sequence in* $(\mathcal{X}, \Upsilon)$ *if for any* $0 \prec_{i_2} r \in \mathbb{BC}$ *there exists* $k_0 \in \mathbb{N}$ *such that* $\Upsilon(\wp_n, \wp_{n+m}) \prec_{i_2} r$, $\forall m, n \in \mathbb{N}$ *and* $n > k_0$.

(iii) A bicomplex valued metric space $(\mathcal{X}, \Upsilon)$ *is said to be complete if every Cauchy sequence in* $\mathcal{X}$ *is convergent in* $\mathcal{X}$*.*

Definition 1.1.13 *[9] A point* $x \in \chi$ *is said to be commuting point of* $\mathcal{S}, \mathcal{T} : \chi \to \chi$ *if* $\mathcal{S}\mathcal{T}x = \mathcal{T}\mathcal{S}x$.

Definition 1.1.14 *[9] Two maps* $\mathcal{S}, \mathcal{T} : \chi \to \chi$ *are called converse commuting if* $\mathcal{S}\mathcal{T}x = \mathcal{T}\mathcal{S}x$ *implies* $\mathcal{S}x = \mathcal{T}x$*. The set of converse commuting points of* $\mathcal{S}$ *and* $\mathcal{T}$ *is denoted by* $C(\mathcal{S}, \mathcal{T})$*.*

Definition 1.1.15 *[29] The max function for the partial order* $\preccurlyeq_{i_2}$ *on* $\mathbb{BC}$ *is defined as follows:*

(i) $\max\{u,v\} = v,\ u \preccurlyeq_{i_2} v;$
(ii) $u \preccurlyeq_{i_2} \max\{u,v\}$ *implies* $u \preccurlyeq_{i_2} v$ *or* $\|u\| \leqslant \|v\|$;
(iii) $\max\{u,v\} = v$ *iff* $u \preccurlyeq_{i_2} v$ *or* $\|u\| \leqslant \|v\|$.

Definition 1.1.16 *Let* $\mathcal{X} \neq \phi$ *be a set and let* $s \geqslant 1$*. Suppose* $\vartheta : \mathcal{X} \times \mathcal{X} \to \mathbb{BC}$ *satisfies the following properties:*

(i) $0 \precsim_{i_2} \vartheta(x,y)$ $\forall x, y \in \mathcal{X}$,
(ii) $\vartheta(x,y) = 0$ *if and only if* $x = y$,

(iii) $\vartheta(x, y) = \vartheta(y, x)$, $\forall x, y \in \mathcal{X}$ *and*

(iv) $\vartheta(x, y) \precsim_{i_2} s\,[\vartheta(x, z) + \vartheta(z, y)]$, $\forall x, y, z \in \mathcal{X}$.

Then the pair $(\mathcal{X}, \vartheta)$ *is said to be bicomplex valued b-metric space and* ϑ *is called a bicomplex valued b-metric on* $\mathcal{X}$.

1.1.3 Complex Manifold

A complex structure[31] on a real vector space V of finite dimension is an endomorphism F such that

$$F(F(X)) = -X \quad , \forall X \in V$$
$$i.e., \quad F^2 = -I$$

where I represents identity linear map.

Now the pair (V, F) is called complex vector space if following condition holds:

$$(a + ib)X = aX + bF(X),$$

where $X \in V$ and $a, b \in \mathbb{R}$.
Clearly, the real dimension m of V must be even and $m/2$ is the complex dimension of V.

Definition 1.1.17 *[31] An almost complex structure on a real differentiable manifold* M *of dimension* $n(n = 2m, m$ *is a positive integer) is a tensor field* F *which is at every point* x *of* M*, an endomorphism of the tangent space* $T_x(M)$ *such that* $F^2 = -I$*, where* I *denotes the identity transformation of* $T_x(M)$*. A manifold with a fixed almost complex structure is said to be an almost complex manifold.*

Definition 1.1.18 *[31] A vector field is said to be contravariant almost analytic if*

$$(L_V F)(X) = 0 \quad ,$$

where L_V *denotes the Lie derivative with respect to the vector field* V.

Definition 1.1.19 *[31] A vector field is said to be strictly contravariant almost analytic if*

$$(L_V F)(X) = 0 \quad and$$
$$(L_{F(V)} F)(X) = 0.$$

Definition 1.1.20 *[31] An almost complex manifold endowed with a metric* g *such that*

$$g(F(X), F(Y)) = g(X, Y)$$

is said to be an almost Hermite manifold and (F, g) *is said to be an almost Hermite structure.*

1.1.4 p-adic number

In 1897, Kurt Hensel redefined the distance between two rational numbers by the notion of p-adic numbers where p is any prime number. Initially p-adic numbers were used to extend algebraic numbers in power series. Later, Kűrschák made a further generalization of p-adic numbers in 1913 . These numbers are useful because these provide another set of tools which is sometimes easier to deal with than the real numbers. Its have wide applications in number theory, analysis, algebra and many other fields, specifically Mahler's Theorem, Hensel's lemma, Monsky's Theorem etc.

The theory of complex-valued functions on p-adic numbers is an important topic in the area of locally compact groups. In the area of p-adic functional analysis and spectral theory, a number of developments have been done in implication point of view. However, p-adic analysis is less subtle than classical analysis due to its ultrametric inequality, e.g., the convergence of infinite series of p-adic numbers is much simpler. Topological vector spaces over p-adic fields show distinctive features; e.g., aspects relating to convexity and the Hahn–Banach theorem are different.

Definition 1.1.21 *An absolute value on $\mathcal{K}$ is a function $|.| : \mathcal{K} \times \mathcal{K} \to \mathbb{R}_{\geqslant 0}$ that satisfies the following conditions for all $\omega, \varpi \in \mathcal{K}$:*
(i) $|\omega| = 0$ *if and only if* $\omega = 0$,
(ii) $|\omega\varpi| = |\omega|\,|\varpi|$, *and*
(iii) $|\omega + \varpi| \leqslant |\omega| + |\varpi|$
Moreover, an absolute value is non-Archimedean if it also satisfies the following property:
(iv) $|\omega + \varpi| \leqslant \max\{|\omega|, |\varpi|\}$
An absolute value that does not satisfy property (iv) is Archimedean.

Definition 1.1.22 *A metric $d : \mathcal{K} \times \mathcal{K} \to \mathbb{R}_{\geqslant 0}$ is defined by the following distance function*

$$d(\omega, \varpi) = |\omega - \varpi| \text{ for all } \omega, \varpi \in \mathcal{K}.$$

A set on which a metric is defined is said to be a metric space. A set with a metric induced by a non-Archimedean absolute value is called an ultrametric space.

Definition 1.1.23 *[14] For $\varkappa \in [0, \infty)$ and $k \in \mathbb{N}$, the Neperian logarithm is defined as*

$$\log^{[k]} \varkappa = \log(\log^{[k-1]} \varkappa)$$

and

$$\exp^{[k]} \varkappa = \exp(\exp^{[k-1]} \varkappa),$$

We also denote $\log^{[0]} \varkappa = \varkappa$ *and* $\exp^{[0]} \varkappa = \varkappa$.

Definition 1.1.24 *[66] The order ϱ of an entire function $\mathsf{f}(\mathfrak{z})$ is defined as*

$$\varrho = \inf\{k > 0 : M_{\mathsf{f}}(r) < e^{r^k}, r > r_0\} \text{ where } M(r, \mathsf{f}) := M_{\mathsf{f}}(r) = \max_{|\mathfrak{z}|=r} |\mathsf{f}(\mathfrak{z})|.$$

Definition 1.1.24 can be alternatively stated as:

Definition 1.1.25 *[66] The order ϱ of an entire function $\mathsf{f}(\mathfrak{z})$ is defined as*

$$\varrho = \limsup_{r \longrightarrow \infty} \frac{\log^{[2]} M(r, \mathsf{f})}{\log r}.$$

If $\varrho < \infty$ then $\mathsf{f}(\mathfrak{z})$ called as of finite order. Also $\varrho = 0$ indicates $\mathsf{f}(\mathfrak{z})$ is of order zero.

In light of this the order and lower order of an entire function $\mathsf{f} \in \mathfrak{A}(\mathcal{K})$ is provided by {cf.[13], [14] & [41]}

$$\varrho(\mathsf{f}) = \lim_{r\to\infty} \sup \frac{\log^{[2]} |\mathsf{f}|(r)}{\log r}$$

and

$$\lambda(\mathsf{f}) = \lim_{r\to\infty} \inf \frac{\log^{[2]} |\mathsf{f}|(r)}{\log r}.$$

In addition, to compare the relative growth of two entire functions having the same non-zero finite order, Boussaf et al.[13] proposed another type of growth indicators called type $\sigma(\mathsf{f})$ and lower type $\overline{\sigma}(\mathsf{f})$ in the following manner,

$$\sigma(\mathsf{f}) = \lim_{r\to+\infty} \sup \frac{\log |\mathsf{f}|(r)}{r^{\varrho(\mathsf{f})}}$$

and

$$\overline{\sigma}(\mathsf{f}) = \lim_{r\to+\infty} \inf \frac{\log |\mathsf{f}|(r)}{r^{\varrho(\mathsf{f})}},$$

where $0 < \varrho(\mathsf{f}) < \infty$.

Analogously for $0 < \lambda(\mathsf{f}) < \infty$, we can draw another growth indicators namely weak type $\tau(\mathsf{f})$ and lower weak type $\overline{\tau}(\mathsf{f})$ as

$$\tau(\mathsf{f}) = \lim_{r\to+\infty} \inf \frac{\log |\mathsf{f}|(r)}{r^{\lambda(\mathsf{f})}}$$

and

$$\overline{\tau}(\mathsf{f}) = \lim_{r\to+\infty} \sup \frac{\log |\mathsf{f}|(r)}{r^{\lambda(\mathsf{f})}}.$$

Apart from the introductory part, this thesis consists of seven chapters which are summarized as follows:

In Chapter Two, we focus on the proving of some certain well-known fixed point theorems for converse commuting self-maps in the incomplete bicomplex valued metric spaces. Our results extend the results of Chauhan[25] and Kumar et al.[64]. We also wish to investigate a number of common fixed point theorems in the space of bicomplex valued b-metric that satisfy certain rational inequalities for a pair of self contracting mappings. Finally, we try to extend some of the theorems of Azam et al.[4] and Rouzkard[83] concerning common fixed point theorems in complex valued metric spaces.

The main objective of Chapter Three is to develop an alternative representation of the order and lower order of $\mathsf{f} \in \mathfrak{A}(\mathcal{K})$, where $\mathcal{K}$ is a complete ultametric algebraically closed field and $\mathfrak{A}(\mathcal{K})$ is the $\mathcal{K}$-algebra of entire functions on $\mathcal{K}$. Then, in this chapter, the integral representation of generalized (β, γ)-th relative type and generalized (β, γ)-th relative weak type of entire function f over another entire function g is established, where $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$.

Chapter Four aims to establish some results on the comparative growth features of composition of two k-valued entire algebroidal functions in terms of their relative growth indicators from the perspective of slowly changing functions in the light of p-adic analysis. However, the primary goal of this chapter is to use the idea of p-adic analysis of entire algebraic functions to determine the estimates of relative (β, γ)-th type, relative (β, γ)-th lower type, and relative (β, γ)-th weak type under somewhat different conditions, where β and γ are any two positive integers.

The famous Enström-Kakeya theorem for entire functions of non-zero finite order with lacunary type power series expansion is going to be extended in Chapter Five. In order to support the findings, few examples are provided along with relevant figures. For a pair of self contracting mappings, we find adequate criteria for the presence of common fixed points and study certain common fixed point theorems in bicomplex valued metric space.

The goal of Chapter Six is to derive some conclusions about the distribution of zeros for bicomplex valued entire functions in a certain domain. To justify the findings, a number of examples are provided here.

In Chapter Seven, we deduce some theorems of the bicomplex manifold, almost bicomplex manifold and almost bicomplex Hermite manifold. Also, in this chapter, we discuss the properties of the Nijenhuis tensor and contravariant almost analytic vector fields in a bicomplex field.

Chapter Eight is mostly concerned with prospects for future research work with its implications in different directions.

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CHAPTER TWO

A STUDY ON SOME COMMON FIXED POINT THEOREMS IN BICOMPLEX VALUED METRIC SPACES AND BICOMPLEX VALUED b-METRIC SPACES

Chapter 2

A study on some common fixed point theorems in bicomplex valued metric spaces and bicomplex valued b-metric spaces

2.1 Introduction.

This chapter focuses on proving some common fixed point theorems using converse commuting mappings on bicomplex-valued metric spaces. The results of this chapter are generaliZations of the results of Chauhan & Sahper [25] and Kumar et.al.[64] using the concepts from {cf. [57] & [58]}.

2.2 Lemmas.

This section provides essential lemmas that will play a crucial role in the study of the chapter.

Lemma 2.2.1 *[4] Let* $(\mathcal{X}, \Upsilon)$ *be a bicomplex valued metric space and a sequence* $\{\wp_n\}$ *in* $\mathcal{X}$ *is said to tend to a point* $\wp$ *iff* $\lim_{n\to\infty} \|\Upsilon(\wp_n, \wp)\| = 0$.

Lemma 2.2.2 *[4] Let* $(\mathcal{X}, \Upsilon)$ *be a bicomplex valued metric space and a sequence* $\{\wp_n\}$ *in* $\mathcal{X}$ *is said to be a Cauchy in* $\mathcal{X}$ *iff* $\lim_{n\to\infty} \|\Upsilon(\wp_n, \wp_{n+m})\| = 0$.

2.3 Main Results.

In the following section, converse commuting mappings are used to prove several common fixed point theorems.

Theorem 2.3.1 *Let* $\eth_1,\ \eth_2,\ \Lambda_1$ *and* $\Lambda_2 : \mathcal{X} \to \mathcal{X}$ *be four mappings in a bicomplex valued metric space* $(\mathcal{X}, \Upsilon)$ *such that the pairs* $(\eth_1, \Lambda_1)$ *&* $(\eth_2, \Lambda_2)$ *be conversely commuting and satisfying the condition*

$$\Upsilon(\eth_1 x, \eth_2 y) \preccurlyeq_{i_2} \lambda \max\{\Upsilon(\Lambda_1 x, \Lambda_2 y),\ \Upsilon(\eth_1 x, \Lambda_1 x),\ \Upsilon(\eth_2 y, \Lambda_2 y)\} \tag{2.1}$$

for all $x, y \in \mathcal{X}\ (\subseteq \mathbb{C}_2)$, $0 \leqslant \lambda < 1$. *If each pairs* $(\eth_1,\ \Lambda_1)$ *and* $(\eth_2,\ \Lambda_2)$ *have a commuting point, then* $\eth_1, \eth_2, \Lambda_1$ *and* Λ_2 *have a unique common fixed point in* $\mathcal{X}$.

Proof. *Let* u *and* v *be two commuting points of the pairs* $(\eth_1,\ \Lambda_1)$ *and* $(\eth_2,\ \Lambda_2)$ *respectively, then* $\eth_1\Lambda_1 u = \Lambda_1\eth_1 u$ *and* $\eth_2\Lambda_2 v = \Lambda_2\eth_2 v$. *Since the pairs* $(\eth_1,\ \Lambda_1)$ *and* $(\eth_2,\ \Lambda_2)$ *are conversely commuting, then* $\eth_1 u = \Lambda_1 u$ *and* $\eth_2 v = \Lambda_2 v$.
Therefore, $\eth_1\eth_1 u = \eth_1\Lambda_1 u = \Lambda_1\eth_1 u = \Lambda_1\Lambda_1 u$ *and* $\eth_2\eth_2 v = \eth_2\Lambda_2 v = \Lambda_2\eth_2 v = \Lambda_2\Lambda_2 v$.

Here we claim that $\eth_1 u = \eth_2 v$. *If not, then from the relation (2.1) we obtain that*

$$\begin{aligned}
\Upsilon(\eth_1 u, \eth_2 v) &\preccurlyeq_{i_2} \lambda \max\{\Upsilon(\Lambda_1 u, \Lambda_2 v),\ \Upsilon(\eth_1 u, \Lambda_1 u),\ \Upsilon(\eth_2 v, \Lambda_2 v)\} \\
&\preccurlyeq_{i_2} \lambda \max\{\Upsilon(\Lambda_1 u, \Lambda_2 v),\ 0,\ 0\}
\end{aligned}$$

$$i.e.,\quad \|\Upsilon(\eth_1 u, \eth_2 v)\| \leqslant \lambda \|\Upsilon(\eth_1 u, \eth_2 v)\|,$$

which implies that $\|\Upsilon(\eth_1 u, \eth_2 v)\| = 0$, *as* $\lambda < 1$.
Therefore, $\eth_1 u = \eth_2 v$ *i.e.,* $\eth_1 u = \Lambda_1 u = \eth_2 v = \Lambda_2 v$.

At first show that $\eth_1 u$ *is a fixed point of the mapping* $\eth_1$. *Now Putting* $x = \eth_1 u$ *and* $y = v$ *in (2.1) we get that*

$$\begin{aligned}
\Upsilon(\eth_1\eth_1 u, \eth_2 v) &\preccurlyeq_{i_2} \lambda \max\{\Upsilon(\Lambda_1\eth_1 u, \Lambda_2 v), \Upsilon(\eth_1\eth_1 u, \Lambda_1\eth_1 u), \Upsilon(\eth_2 v, \Lambda_2 v)\} \\
i.e.,\ \Upsilon(\eth_1\eth_1 u, \eth_1 u) &\preccurlyeq_{i_2} \lambda \max\{\Upsilon(\eth_1\eth_1 u, \eth_1 u), \Upsilon(\eth_1\eth_1 u, \eth_1\eth_1 u), \Upsilon(\eth_2 v, \eth_2 v)\} \\
i.e.,\ \Upsilon(\eth_1\eth_1 u, \eth_1 u) &\preccurlyeq_{i_2} \lambda \max\{\Upsilon(\eth_1\eth_1 u, \eth_1 u), 0, 0\},
\end{aligned}$$

which implies that $\|\Upsilon(\eth_1\eth_1 u, \eth_1 u)\| = 0$, *as* $\lambda < 1$.

Similarly putting $x = u$ *and* $y = \eth_2 v$ *in (2.1) we get that*

$$\Upsilon(\eth_2 v, \eth_2\eth_2 v) \preccurlyeq_{i_2} \lambda \max\{\Upsilon(\eth_2 v, \eth_2\eth_2 v),\ 0,\ 0\}.$$

Hence, $\|\Upsilon(\eth_2 v, \eth_2\eth_2 v)\| \leqslant \lambda \|\Upsilon(\eth_2 v, \eth_2\eth_2 v)\|$, *a contradiction.*

Therefore, $\|\Upsilon(\eth_2 v, \eth_2\eth_2 v)\| = 0$, *i.e.,* $\eth_2\eth_2 v = \eth_2 v$.

Thus we have $\eth_1 u = \eth_2 v = \eth_2\eth_2 v = \eth_2\eth_1 u$ *and hence* $\eth_1 u$ *is a fixed point of the mapping* $\eth_2$. *Again we have,* $\eth_1 u = \eth_1\eth_1 u = \eth_1\Lambda_1 u = \Lambda_1\eth_1 u$.

Hence $\eth_1 u = w$ *is a common fixed point of* $\eth_1,\ \eth_2,\ \Lambda_1$ *and* Λ_2 *in* $\mathcal{X}$.

Uniqueness:

If possible let w^* *be another common fixed point of the mappings* $\eth_1, \eth_2, \Lambda_1$ *and* Λ_2.
From (2.1) we obtain that

$$\begin{aligned}
\Upsilon(\eth_1 w, \eth_2 w^*) &\precsim_{i_2} \lambda \max\{\Upsilon(\Lambda_1 w, \Lambda_2 w^*), \Upsilon(\eth_1 w, \Lambda_1 w), \Upsilon(\eth_2 w^*, \Lambda_2 w^*)\} \\
i.e., \Upsilon(w, w^*) &\precsim_{i_2} \lambda \Upsilon(w, w^*),
\end{aligned}$$

which gives $\|\Upsilon(w, w^*)\| \leqslant \lambda \|\Upsilon(w, w^*)\|$, *i.e.,* $\|\Upsilon(w, w^*)\| = 0$ *implies* $w = w^*$.

Therefore, w *is a unique common fixed point of* $\eth_1, \eth_2, \Lambda_1$ *and* Λ_2.

This completes the proof of the theorem. ■

Corollary 2.3.1 *Let* $\eth_1$ *and* $\Lambda_1 : \mathcal{X} \to \mathcal{X}$ *be mappings in a bicomplex valued metric space* $(\mathcal{X}, \Upsilon)$ *such that the pair* $(\eth_1, \Lambda_1)$ *be conversely commuting and satisfying the following condition*

$$\Upsilon(\eth_1 x, \eth_1 y) \preccurlyeq_{i_2} \lambda \max\{\Upsilon(\Lambda_1 x, \Lambda_1 y), \ \Upsilon(\eth_1 x, \Lambda_1 x), \ \Upsilon(\eth_1 y, \Lambda_1 y)\}$$

for all $x, y \in \mathcal{X}$ *and* $0 \leqslant \lambda < 1$. *If* $\eth_1$ *and* Λ_1 *have a commuting point then* $\eth_1$ *and* Λ_1 *have a unique common fixed point in* $\mathcal{X}$ *where* $\mathcal{X} \subseteq \mathbb{BC}$.

Proof. *We can prove the result by considering* $\eth_1 = \eth_2$ *and* $\Lambda_1 = \Lambda_2$ *in the proof of the Theorem 2.3.1.* ■

Theorem 2.3.2 *Let* $\Im_1, \Im_2, \Im_3$ *and* $\Im_4 : \mathcal{X} \to \mathcal{X}$ *be self-mappings in a bicomplex valued metric space* $(\mathcal{X}, \Upsilon)$ *such that the pairs* $(\Im_1, \Im_4)$ *and* $(\Im_2, \Im_3)$ *be conversely commuting and satisfying*

$$\Upsilon(\Im_1 \mathfrak{z}, \Im_2 \mathfrak{z}') \precsim_{i_2} \lambda \max\left\{ \begin{array}{c} \frac{\Upsilon(\Im_4\mathfrak{z}, \Im_3\mathfrak{z}')+\Upsilon(\Im_4\mathfrak{z}, \Im_1\mathfrak{z})}{2}, \frac{\Upsilon(\Im_4\mathfrak{z}, \Im_1\mathfrak{z})+\Upsilon(\Im_3\mathfrak{z}', \Im_2\mathfrak{z}')}{2}, \\ \frac{\Upsilon(\Im_4\mathfrak{z}, \Im_2\mathfrak{z}')+\Upsilon(\Im_3\mathfrak{z}', \Im_1\mathfrak{z})}{2} \end{array} \right\} \tag{2.2}$$

for all $\mathfrak{z}, \mathfrak{z}' \in \mathcal{X}\ (\subseteq \mathbb{BC})$, $\lambda \in (0,1)$. *If the pairs* $(\Im_1, \Im_4)$ *and* $(\Im_2, \Im_3)$ *have a commuting point then* $\Im_1, \Im_2, \Im_3$ *and* $\Im_4$ *have a unique common fixed point in* $\mathcal{X}$.

Proof. Let $u \in C(\Im_1, \Im_4)$ and $v \in C(\Im_2, \Im_3)$, then we have $\Im_1\Im_4 u = \Im_4\Im_1 u$ and $\Im_2\Im_3 v = \Im_3\Im_2 v$ which implies $\Im_1 u = \Im_4 u$ and $\Im_2 v = \Im_3 v$.

First we prove that $\Im_1 u = \Im_2 v$. If not then putting $\mathfrak{z} = u, \mathfrak{z}' = v$ in (2.2), we obtain that

$$\begin{aligned} \Upsilon(\Im_1 u, \Im_2 v) &\precsim_{i_2} \lambda \max\left\{ \begin{array}{c} \frac{\Upsilon(\Im_4 u, \Im_3 v)+\Upsilon(\Im_4 u, \Im_1 u)}{2}, \frac{\Upsilon(\Im_4 u, \Im_1 u)+\Upsilon(\Im_3 v, \Im_2 v)}{2}, \\ \frac{\Upsilon(\Im_1 u, \Im_2 v)+\Upsilon(\Im_2 v, \Im_1 u)}{2} \end{array} \right\} \\ &\precsim_{i_2} \lambda \max\left\{ \frac{\Upsilon(\Im_1 u, \Im_2 v)}{2}, 0, \Upsilon(\Im_1 u, \Im_2 v) \right\}, \end{aligned}$$

which implies that

$$\Upsilon(\Im_1 u, \Im_2 v) \precsim_{i_2} \lambda \Upsilon(\Im_1 u, \Im_2 v)$$
$$i.e., \ \ \|\Upsilon(\Im_1 u, \Im_2 v)\| \leqslant \lambda \|\Upsilon(\Im_1 u, \Im_2 v)\|,$$

which leads to a contradiction. Therefore $\|\Upsilon(\Im_1 u, \Im_2 v)\| = 0$, *i.e.*, $\Im_1 u = \Im_2 v$.

Now we claim that $\Im_1^2 u = \Im_1 u$. If not then using(2.2) for $\mathfrak{z} = \Im_1 u, \mathfrak{z}' = v$ we have

$$\begin{aligned} \Upsilon(\Im_1^2 u, \Im_2 v) &\precsim_{i_2} \lambda \max\left\{ \begin{array}{c} \frac{\Upsilon(\Im_4\Im_1 u, \Im_3 v)+\Upsilon(\Im_4\Im_1 u, \Im_1\Im_1 u)}{2}, \frac{\Upsilon(\Im_4\Im_1 u, \Im_1\Im_1 u)+\Upsilon(\Im_3 v, \Im_2 v)}{2}, \\ \frac{\Upsilon(\Im_1\Im_4 u, \Im_2 v)+\Upsilon(\Im_2 v, \Im_1\Im_1 u)}{2} \end{array} \right\} \\ &\precsim_{i_2} \lambda \max\left\{ \begin{array}{c} \frac{\Upsilon(\Im_1\Im_1 u, \Im_2 v)+\Upsilon(\Im_1\Im_1 u, \Im_1\Im_1 u)}{2}, \frac{\Upsilon(\Im_1\Im_1 u, \Im_1\Im_1 u)+\Upsilon(\Im_2 v, \Im_2 v)}{2}, \\ \frac{\Upsilon(\Im_1\Im_1 u, \Im_2 v)+\Upsilon(\Im_2 v, \Im_1\Im_1 u)}{2} \end{array} \right\} \\ &\precsim_{i_2} \lambda \max\left\{ \frac{\Upsilon(\Im_1^2 u, \Im_1 u)}{2}, 0, \Upsilon(\Im_1^2 u, \Im_1 u) \right\}, \end{aligned}$$

which implies that

$$\Upsilon(\Im_1^2 u, \Im_1 u) \precsim \lambda \Upsilon(\Im_1^2 u, \Im_1 u)$$

i.e.

$$\left\|\Upsilon(\Im_1^2 u, \Im_1 u)\right\| \leqslant \lambda \left\|\Upsilon(\Im_1^2 u, \Im_1 u)\right\|,$$

which contradicts. Therefore $\|\Upsilon(\Im_1^2 u, \Im_1 u)\| = 0$, i.e., $\Im_1^2 u = \Im_1 u$. Similarly we get that

$$\begin{gathered}\Im_2^2 v = \Im_2 v,\\ \Im_1 u = \Im_1\Im_1 u = \Im_1\Im_4 u = \Im_4\Im_1 u,\\ \Im_2 v = \Im_2\Im_2 v = \Im_2\Im_3 v = \Im_3\Im_2 v.\end{gathered}$$

Hence $\Im_1, \Im_2, \Im_3$ and $\Im_4$ have a common fixed point in $\mathcal{X}$.
Let $w = \Im_1 u$ be a common fixed point of $\Im_1, \Im_2, \Im_3$ and $\Im_4$ in $\mathcal{X}$.

Now we prove that w is a unique common fixed point. If possible, let w' be another common fixed point in $\mathcal{X}$. Then using (2.2) we obtain that

$$\begin{aligned}\Upsilon(w, w') &= \Upsilon(\Im_1 w, \Im_2 w')\\ &\precsim_{i_2} \lambda \max\left\{\begin{array}{c}\frac{\Upsilon(\Im_4 w,\Im_3 w')+\Upsilon(\Im_4 w,\Im_1 w)}{2}, \frac{\Upsilon(\Im_4 w,\Im_1 w)+\Upsilon(\Im_3 w',\Im_2 w')}{2},\\ \frac{\Upsilon(\Im_4 w,\Im_2 w')+\Upsilon(\Im_3 w',\Im_1 w)}{2}\end{array}\right\}\\ &\precsim_{i_2} \lambda \max\left\{\begin{array}{c}\frac{\Upsilon(w,w')+\Upsilon(w,w)}{2}, \frac{\Upsilon(w,w)+\Upsilon(w',w')}{2},\\ \frac{\Upsilon(w,w')+\Upsilon(w',w)}{2}\end{array}\right\}\\ &\precsim_{i_2} \lambda \max\left\{\frac{\Upsilon(w,w')}{2}, 0, \Upsilon(w,w')\right\}\end{aligned}$$

which implies that

$$\Upsilon(w, w') \precsim_{i_2} \lambda \Upsilon(w, w')$$

$$i.e.,\ \|\Upsilon(w, w')\| \leqslant \lambda \|\Upsilon(w, w')\|,$$

which is a contradiction. Therefore $\|\Upsilon(w, w')\| = 0$, i.e., $w = w'$. Thus w is a unique common fixed point of $\Im_1, \Im_2, \Im_3$ and $\Im_4$. ■

Corollary 2.3.2 *Let $\Im_1$ and $\Im_4$ be self-mappings in a bicomplex valued metric space $(\mathcal{X}, \Upsilon)$ such that the pair $(\Im_1, \Im_4)$ be conversely commuting and satisfying*

$$\Upsilon(\Im_1 \mathfrak{z}, \Im_1 \mathfrak{z}') \precsim_{i_2} \lambda \max\left\{\begin{array}{c}\frac{\Upsilon(\Im_4\mathfrak{z},\Im_4\mathfrak{z}')+\Upsilon(\Im_4\mathfrak{z},\Im_1\mathfrak{z})}{2}, \frac{\Upsilon(\Im_4\mathfrak{z},\Im_1\mathfrak{z})+\Upsilon(\Im_4\mathfrak{z}',\Im_1\mathfrak{z}')}{2},\\ \frac{\Upsilon(\Im_4\mathfrak{z},\Im_1\mathfrak{z}')+\Upsilon(\Im_4\mathfrak{z}',\Im_1\mathfrak{z})}{2}\end{array}\right\}$$

for all $\mathfrak{z}, \mathfrak{z}' \in \mathcal{X}$, $\lambda \in (0, 1)$. If the pair $(\Im_1, \Im_4)$ have a commuting point then $(\Im_1, \Im_4)$ have a unique common fixed point in $\mathcal{X}$.
Proof. To prove the result we have to put $\Im_1 = \Im_2$ and $\Im_3 = \Im_4$ in the proof of Theorem 2.3.2. ■

Example 2.3.1 *Let* $\mathcal{X} = \{\xi = \mathfrak{z}_1 + i_2\mathfrak{z}_2 \in \mathbb{BC} : 0 \leqslant \|\mathfrak{z}_1\| < 1, \|\mathfrak{z}_2\| = 0\}$. *Let* $\Upsilon : \mathcal{X} \times \mathcal{X} \to \mathbb{BC}$ *be the metric, defined by*

$$\Upsilon(\xi_1, \xi_2) = \|x_1 - x_2\| + i_2 \|y_1 - y_2\|$$

for all $\xi_1 = x_1 + i_2 y_1, \xi_2 = x_2 + i_2 y_2, \mathfrak{z}_1 = x_1 + i_1 y_1, \xi_2 = x_2 + i_1 y_2$.

Define the maps $\Im_1, \Im_2, \Im_3$ *and* $\Im_4 : \mathcal{X} \to \mathcal{X}$ *for* $\mathfrak{z} = \frac{i_1 i_2}{n}, n \in \mathbb{N}$ *as follows:*

$$\Im_1(\xi) = \begin{cases} \frac{i_1 i_2}{n+3} \text{ if } n \text{ is odd} \\ \frac{i_1 i_2}{n+4} \text{if } n \text{ is even} \end{cases}, \qquad \Im_2(\xi) = \begin{cases} \frac{i_1 i_2}{n+4} \text{ if } n \text{ is odd} \\ \frac{i_1 i_2}{n+3} \text{if } n \text{ is even} \end{cases},$$

$$\Im_4(\xi) = \begin{cases} \frac{i_1 i_2}{n+2} \text{ if } n \text{ is odd} \\ \frac{i_1 i_2}{n+1} \text{if } n \text{ is even} \end{cases}, \qquad \Im_3(\xi) = \begin{cases} \frac{i_1 i_2}{n+1} \text{ if } n \text{ is odd} \\ \frac{i_1 i_2}{n+2} \text{if } n \text{ is even} \end{cases}.$$

The set of all conversely commuting points of the mappings $\Im_1$ *and* $\Im_4$ *are denoted by* $C(\Im_1, \Im_4) = \mathcal{X} - \left\{\frac{i_1 i_2}{n}, n \in \mathbb{N}\right\}$ *and all conversely commuting points of the mappings* $\Im_2$ *and* $\Im_3$ *are denoted by* $C(\Im_2, \Im_3) = \mathcal{X} - \left\{\frac{i_1 i_2}{n}, n \in \mathbb{N}\right\}$.

Therefore the condition (2.2) are satiesfied by $\Im_1, \Im_2, \Im_3$ *and* $\Im_4$.

Theorem 2.3.3 *Let* $(\mathcal{X}, \vartheta)$ *be a complete bicomplex valued b-metric spaces with the coefficient* $s \geqslant 1$ *and* $1 + \vartheta(x, y)$ *be degenerated for all* $x, y \in \mathcal{X}$. *Also let* Λ_1 *and* Λ_2 *be two self-mappings defined on* $\mathcal{X}$ *such that*

$$\vartheta(\Lambda_1 x, \Lambda_2 y) \precsim_{i_2} a\vartheta(x, y) + \frac{b\vartheta(x, \Lambda_1 x)\vartheta(y, \Lambda_2 y)}{1 + \vartheta(x, y)} \tag{2.1}$$

for all $x, y \in \mathcal{X}$, *where* a, b *are non-negative real numbers with* $sa + \sqrt{2}b < 1$. *Then* Λ_1 *and* Λ_2 *have a unique common fixed point in* $\mathcal{X}$.

Proof. Let $\{x_n\}$ be any sequence in $\mathcal{X}$ and x_0 be an arbitrary point in $\mathcal{X}$. We define

$$x_{2k+1} = \Lambda_1 x_{2k}, \quad x_{2k+2} = \Lambda_2 x_{2k+1}, \quad k = 0, 1, 2, ... \tag{2.2}$$

Now

$$\begin{aligned} \vartheta(x_{2k+1}, x_{2k+2}) &= \vartheta(\Lambda_1 x_{2k}, \Lambda_2 x_{2k+1}) \\ &\precsim_{i_2} a\vartheta(x_{2k}, x_{2k+1}) + \frac{b\vartheta(x_{2k}, \Lambda_1 x_{2k})\vartheta(x_{2k+1}, \Lambda_2 x_{2k+1})}{1 + \vartheta(x_{2k}, x_{2k+1})} \\ &\precsim_{i_2} a\vartheta(x_{2k}, x_{2k+1}) + \frac{b\vartheta(x_{2k}, x_{2k+1})\vartheta(x_{2k+1}, x_{2k+2})}{1 + \vartheta(x_{2k}, x_{2k+1})}. \end{aligned}$$

Therefore,

$$\|\vartheta(x_{2k+1}, x_{2k+2})\| \leqslant a\|\vartheta(x_{2k}, x_{2k+1})\| + \sqrt{2}b\frac{\|\vartheta(x_{2k}, x_{2k+1})\|}{\|1 + \vartheta(x_{2k}, x_{2k+1})\|}\|\vartheta(x_{2k+1}, x_{2k+2})\|.$$

Since $\|\vartheta(x_{2k}, x_{2k+1})\| \leqslant \|1 + \vartheta(x_{2k}, x_{2k+1})\|$, therefore we obtain that

$$\begin{aligned} \|\vartheta(x_{2k+1}, x_{2k+2})\| &\leqslant a\|\vartheta(x_{2k}, x_{2k+1})\| + \sqrt{2}b\|\vartheta(x_{2k+1}, x_{2k+2})\| \\ \textit{i.e.,}\ \left(1 - \sqrt{2}b\right)\|\vartheta(x_{2k+1}, x_{2k+2})\| &\leqslant a\|\vartheta(x_{2k}, x_{2k+1})\| \\ \textit{i.e.,}\ \|\vartheta(x_{2k+1}, x_{2k+2})\| &\leqslant \frac{a}{(1 - \sqrt{2}b)}\|\vartheta(x_{2k}, x_{2k+1})\|. \end{aligned}$$

Similarly,

$$\begin{aligned}
\vartheta\left(x_{2k+2}, x_{2k+3}\right) &= \vartheta\left(\Lambda_2 x_{2k+1}, \Lambda_1 x_{2k+2}\right) = \vartheta\left(\Lambda_1 x_{2k+2}, \Lambda_2 x_{2k+1}\right) \\
&\precsim_{i_2} a\vartheta\left(x_{2k+2}, x_{2k+1}\right) + \frac{b\vartheta\left(x_{2k+2}, \Lambda_1 x_{2k+2}\right)\vartheta\left(x_{2k+1}, \Lambda_2 x_{2k+1}\right)}{1+\vartheta\left(x_{2k+2}, x_{2k+1}\right)} \\
&\precsim_{i_2} a\vartheta\left(x_{2k+2}, x_{2k+1}\right) + \frac{b\vartheta\left(x_{2k+2}, x_{2k+3}\right)\vartheta\left(x_{2k+1}, x_{2k+2}\right)}{1+\vartheta\left(x_{2k+2}, x_{2k+1}\right)}.
\end{aligned}$$

Therefore,

$$\left\|\vartheta\left(x_{2k+2}, x_{2k+3}\right)\right\| \leqslant a\left\|\vartheta\left(x_{2k+2}, x_{2k+1}\right)\right\| + \sqrt{2}b\frac{\left\|\vartheta\left(x_{2k+1}, x_{2k+2}\right)\right\|}{\left\|1+\vartheta\left(x_{2k+1}, x_{2k+2}\right)\right\|}\left\|\vartheta\left(x_{2k+2}, x_{2k+3}\right)\right\|.$$

Again $\left\|\vartheta\left(x_{2k+1}, x_{2k+2}\right)\right\| \leqslant \left\|1+\vartheta\left(x_{2k+1}, x_{2k+2}\right)\right\|$ implies that

$$\begin{aligned}
\left\|\vartheta\left(x_{2k+2}, x_{2k+3}\right)\right\| &\leqslant a\left\|\vartheta\left(x_{2k+2}, x_{2k+1}\right)\right\| + \sqrt{2}b\left\|\vartheta\left(x_{2k+2}, x_{2k+3}\right)\right\| \\
i.e., \left(1-\sqrt{2}b\right)\left\|\vartheta\left(x_{2k+2}, x_{2k+3}\right)\right\| &\leqslant a\left\|\vartheta\left(x_{2k+2}, x_{2k+3}\right)\right\| \\
i.e., \left\|\vartheta\left(x_{2k+2}, x_{2k+3}\right)\right\| &\leqslant \frac{a}{\left(1-\sqrt{2}b\right)}\left\|\vartheta\left(x_{2k+2}, x_{2k+3}\right)\right\|.
\end{aligned}$$

Let us consider $\alpha = \frac{a}{1-\sqrt{2}b}$. Then $0 \leqslant \alpha < 1$ and

$$\begin{aligned}
\left\|\vartheta\left(x_{n+1}, x_{n+2}\right)\right\| &\leqslant \alpha\left\|\vartheta\left(x_n, x_{n+1}\right)\right\| \\
&\leqslant \alpha^2\left\|\vartheta\left(x_{n-1}, x_n\right)\right\| ..... \leqslant \alpha^{n+1}\left\|\vartheta\left(x_0, x_1\right)\right\|.
\end{aligned} \tag{2.3}$$

Thus for any two positive integers m, n with $m > n$ and $s\alpha = \frac{sa}{1-\sqrt{2}b} < 1$, we obtain that

$$\vartheta\left(x_n, x_m\right) \precsim_{i_2} s\left[\vartheta\left(x_n, x_{n+1}\right) + \vartheta\left(x_{n+1}, x_m\right)\right].$$

Therefore,

$$\begin{aligned}
\left\|\vartheta\left(x_n, x_m\right)\right\| &\leqslant s\left\|\vartheta\left(x_n, x_{n+1}\right)\right\| + s\left\|\vartheta\left(x_{n+1}, x_m\right)\right\| \\
&\leqslant s\left\|\vartheta\left(x_n, x_{n+1}\right)\right\| + s^2\left\|\vartheta\left(x_{n+1}, x_{n+2}\right)\right\| + s^2\left\|\vartheta\left(x_{n+2}, x_m\right)\right\| \\
&\leqslant \left\{\begin{array}{c} s\left\|\vartheta\left(x_n, x_{n+1}\right)\right\| + s^2\left\|\vartheta\left(x_{n+1}, x_{n+2}\right)\right\| + \\ s^3\left\|\vartheta\left(x_{n+2}, x_{n+3}\right)\right\| + s^3\left\|\vartheta\left(x_{n+3}, x_m\right)\right\| \end{array}\right\}.
\end{aligned}$$

$$\begin{aligned}
i.e., \quad \left\|\vartheta\left(x_n, x_m\right)\right\| &\leqslant \left\{\begin{array}{c} s\left\|\vartheta\left(x_n, x_{n+1}\right)\right\| + s^2\left\|\vartheta\left(x_{n+1}, x_{n+2}\right)\right\| + \\ s^3\left\|\vartheta\left(x_{n+2}, x_{n+3}\right)\right\| + ... + s^{m-n-1}\left\|\vartheta\left(x_{m-1}, x_m\right)\right\| \end{array}\right\} \\
i.e., \quad \left\|\vartheta\left(x_n, x_m\right)\right\| &\leqslant \left\{\begin{array}{c} s\left\|\vartheta\left(x_n, x_{n+1}\right)\right\| + s^2\left\|\vartheta\left(x_{n+1}, x_{n+2}\right)\right\| + \\ s^3\left\|\vartheta\left(x_{n+2}, x_{n+3}\right)\right\| + ... + s^{m-n}\left\|\vartheta\left(x_{m-1}, x_m\right)\right\| \end{array}\right\}, \text{ as } s \geqslant 1
\end{aligned}$$

Now by using(2.3) we get that

$$\left\|\vartheta\left(x_n, x_m\right)\right\| \leqslant \left\{\begin{array}{c} s\alpha^n\left\|\vartheta\left(x_0, x_1\right)\right\| + s^2\alpha^{n+1}\left\|\vartheta\left(x_0, x_1\right)\right\| + \\ s^3\alpha^{n+2}\left\|\vartheta\left(x_0, x_1\right)\right\| + ... + s^{m-n}\alpha^{m-1}\left\|\vartheta\left(x_0, x_1\right)\right\| \end{array}\right\}$$

$$i.e., \ \ \|\vartheta(x_n,x_m)\| \leqslant \sum_{i=1}^{m-n} s^i \alpha^{i+n-1} \|\vartheta(x_0,x_1)\|$$

$$i.e., \ \ \|\vartheta(x_n,x_m)\| \leqslant \sum_{i=1}^{m-n} s^{i+n-1} \alpha^{i+n-1} \|\vartheta(x_0,x_1)\|, \text{ as } s \geqslant 1$$

$$i.e., \ \ \|\vartheta(x_n,x_m)\| \leqslant \sum_{j=n}^{m-1} s^j \alpha^j \|\vartheta(x_0,x_1)\|$$

$$i.e., \ \ \|\vartheta(x_n,x_m)\| \leqslant \sum_{j=n}^{\infty} (s\alpha)^j \|\vartheta(x_0,x_1)\|$$

$$i.e., \ \ \|\vartheta(x_n,x_m)\| \leqslant \frac{(s\alpha)^n}{1-s\alpha} \|\vartheta(x_0,x_1)\|.$$

Since $\frac{(s\alpha)^n}{1-s\alpha} \longrightarrow 0$ as $n \longrightarrow \infty$, therefore for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\|\vartheta(x_n,x_m)\| < \varepsilon$ for all $m,n > n_0$. Hence $\{x_n\}$ is Cauchy in $\mathcal{X}$. Again since $\mathcal{X}$ is a complete bicomplex valued b-metric space, therefore there exists a $u \in \mathcal{X}$ such that $\lim_{n\longrightarrow\infty} x_n = u$.

Now we show that $u = \Lambda_1 u$. If not then there exists an $(0 \prec_{i_2}) \, \xi \in \mathbb{BC}$ such that $\vartheta(u, \Lambda_1 u) = \xi$.

Therefore,

$$\begin{aligned}
\xi &= \vartheta(u,\Lambda_1 u) \\
&\precsim_{i_2} s\vartheta(u,x_{2k+2}) + s\vartheta(x_{2k+2},\Lambda_1 u) \\
&\precsim_{i_2} s\vartheta(u,x_{2k+2}) + s\vartheta(\Lambda_2 x_{2k+1},\Lambda_1 u) \\
&\precsim_{i_2} s\vartheta(u,x_{2k+2}) + sa\vartheta(x_{2k+1},u) + \frac{sb\vartheta(x_{2k+1},\Lambda_2 x_{2k+1})\,\vartheta(u,\Lambda_1 u)}{1+\vartheta(x_{2k+1},u)} \\
i.e., \xi &\precsim_{i_2} s\vartheta(u,x_{2k+2}) + sa\vartheta(x_{2k+1},u) + \frac{b\vartheta(x_{2k+1},x_{2k+2})\,\xi}{1+\vartheta(x_{2k+1},u)}.
\end{aligned}$$

Thus

$$\|\xi\| \leqslant s\|\vartheta(u,x_{2k+2})\| + sa\|\vartheta(x_{2k+1},u)\| + \sqrt{2}\frac{sb\|\vartheta(x_{2k+1},x_{2k+2})\|\,\|\xi\|}{\|1+\vartheta(x_{2k+1},u)\|}.$$

Since $\lim_{n\longrightarrow\infty} x_n = u$, therefore we get that $\|\xi\| \leqslant 0$, which is a contradiction.
Hence $\|\xi\| = 0 \Rightarrow \|\vartheta(u,\Lambda_1 u)\| = 0 \Rightarrow u = \Lambda_1 u$. Similarly, we can show that $u = \Lambda_2 u$.

Therefore Λ_1 and Λ_2 have a common fixed point.

Now we show that Λ_1 and Λ_2 have a unique common fixed point. If possible let $u^* \in \mathcal{X}$ be another common fixed point of Λ_1 and Λ_2.

Then

$$\vartheta(u,u^*) = \vartheta(\Lambda_1 u, \Lambda_2 u^*) \precsim_{i_2} a\vartheta(u,u^*) + \frac{b\vartheta(u,\Lambda_1 u)\,\vartheta(u^*,\Lambda_2 u^*)}{1+\vartheta(u,u^*)}$$

$$\begin{aligned} i.e.,\ \|\vartheta(u,u^*)\| &\leqslant a\|\vartheta(u,u^*)\| + \sqrt{2}\frac{b\|\vartheta(u,\Lambda_1 u)\|\,\|\vartheta(u^*,\Lambda_2 u^*)\|}{\|1+\vartheta(u,u^*)\|} \\ i.e.,\ \|\vartheta(u,u^*)\| &\leqslant a\|\vartheta(u,u^*)\| \\ i.e.,\ \|\vartheta(u,u^*)\| &= 0 \\ i.e.,\ u &= u^*. \end{aligned}$$

This completes the proof of the theorem. ■

Corollary 2.3.3 *Let $(\mathcal{X},\vartheta)$ be a complete bicomplex valued b-metric space with the coefficient $s \geqslant 1$ and $1+\vartheta(x,y)$ be degenerated for all $x,y \in \mathcal{X}$. Let $\Lambda_1 : \mathcal{X} \to \mathcal{X}$ be any mapping satisfying the condition*

$$\vartheta(\Lambda_1 x, \Lambda_1 y) \precsim_{i_2} a\vartheta(x,y) + \frac{b\vartheta(x,\Lambda_1 x)\,\vartheta(y,\Lambda_1 y)}{1+\vartheta(x,y)}$$

for all $x,y \in \mathcal{X}$, where a,b are non-negative real numbers with $sa+\sqrt{2}b<1$. Then Λ_1 has a unique fixed point in $\mathcal{X}$.

Proof. Putting $\Lambda_2 = \Lambda_1$ in Theorem 2.3.3 the proof can be easily established. ■

Corollary 2.3.4 *Let $(\mathcal{X},\vartheta)$ be a complete bicomplex valued b-metric space with the coefficient $s \geqslant 1$ and $1+\vartheta(x,y)$ be degenerated for all $x,y \in \mathcal{X}$. Let $\Lambda_1 : \mathcal{X} \to \mathcal{X}$ be any mapping satisfying the condition*

$$\vartheta(\Lambda_1^n x, \Lambda_1^n y) \precsim_{i_2} a\vartheta(x,y) + \frac{b\vartheta(x,\Lambda_1^n x)\,\vartheta(y,\Lambda_1^n y)}{1+\vartheta(x,y)}$$

for all $x,y \in \mathcal{X}$, where a,b are non-negative real numbers with $sa+\sqrt{2}b<1$. Then Λ_1 has a unique fixed point in $\mathcal{X}$.

Proof. By Corollary 2.3.3, there exists a unique point $u \in \mathcal{X}$ such that $\Lambda_1^n u = u$. Therefore,

$$\vartheta(\Lambda_1 u, u) = \vartheta(\Lambda_1\Lambda_1^n u, \Lambda_1^n u) = \vartheta(\Lambda_1^n\Lambda_1 u, \Lambda_1^n u) \precsim_{i_2} a\vartheta(\Lambda_1 u, u) + \frac{b\vartheta(\Lambda_1 u, \Lambda_1^n\Lambda_1 u)\,\vartheta(u,\Lambda_1^n u)}{1+\vartheta(\Lambda_1 u, u)}.$$

$$\begin{aligned} i.e.,\ \vartheta(\Lambda_1 u, u) &\precsim_{i_2} a\vartheta(\Lambda_1 u, u) + \frac{b\vartheta(\Lambda_1 u, \Lambda_1^n\Lambda_1 u)\,\vartheta(u,u)}{1+\vartheta(\Lambda_1 u, u)} \\ i.e.,\ \vartheta(\Lambda_1 u, u) &\precsim_{i_2} a\vartheta(\Lambda_1 u, u) \\ i.e.,\ \|\vartheta(\Lambda_1 u, u)\| &\leqslant a\|\vartheta(\Lambda_1 u, u)\| \\ i.e.,\ \|\vartheta(\Lambda_1 u, u)\| &= 0 \\ i.e.,\ \Lambda_1 u &= u. \end{aligned}$$

This completes the proof of the theorem. ■

Theorem 2.3.4 *Let $(\mathcal{X}, \vartheta)$ be a complete bicomplex valued b-metric space with the coefficient $s \geqslant 1$ and $\Lambda_1, \Lambda_2 : \mathcal{X} \to \mathcal{X}$ be mappings satisfying the condition*

$$\vartheta(\Lambda_1 x, \Lambda_2 y) \precsim_{i_2} \frac{a[\vartheta(x, \Lambda_1 x)\vartheta(x, \Lambda_2 y) + \vartheta(y, \Lambda_2 y)\vartheta(y, \Lambda_1 x)]}{\vartheta(x, \Lambda_2 y) + \vartheta(y, \Lambda_1 x)} \tag{2.4}$$

for all $x, y \in \mathcal{X}$. If $\|\vartheta(x, \Lambda_2 y) + \vartheta(y, \Lambda_1 x)\| \neq 0$ and $\vartheta(x, \Lambda_2 y) + \vartheta(y, \Lambda_1 x)$ be degenerated, where a is non-negative real number with $0 \leqslant sa < 1$ then Λ_1 and Λ_2 have a unique common fixed point in $\mathcal{X}$.

Proof. Let x_0 be an arbitrary point in $\mathcal{X}$. We consider a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$x_{n+1} = \Lambda_1 x_n \text{ and } x_{n+2} = \Lambda_2 x_{n+1} \text{ for all } n = 0, 1, 2, ...$$

Then

$$\begin{aligned}
\vartheta(x_{n+1}, x_{n+2}) &= \vartheta(\Lambda_1 x_n, \Lambda_2 x_{n+1}) \\
&\precsim_{i_2} \frac{a[\vartheta(x_n, \Lambda_1 x_n)\vartheta(x_n, \Lambda_2 x_{n+1}) + \vartheta(x_{n+1}, \Lambda_2 x_{n+1})\vartheta(x_{n+1}, \Lambda_1 x_n)]}{\vartheta(x_n, \Lambda_2 x_{n+1}) + \vartheta(x_{n+1}, \Lambda_1 x_n)} \\
&\precsim_{i_2} \frac{a[\vartheta(x_n, x_{n+1})\vartheta(x_n, x_{n+2}) + \vartheta(x_{n+1}, x_{n+2})\vartheta(x_{n+1}, x_{n+1})]}{\vartheta(x_n, x_{n+2}) + \vartheta(x_{n+1}, x_{n+1})} \\
&\precsim_{i_2} \frac{a\vartheta(x_n, x_{n+1})\vartheta(x_n, x_{n+2})}{\vartheta(x_n, x_{n+2})} \\
&\precsim_{i_2} a\vartheta(x_n, x_{n+1}).
\end{aligned}$$

Therefore for all $n \geqslant 0$ we get that

$$\vartheta(x_{n+1}, x_{n+2}) \precsim_{i_2} a\vartheta(x_n, x_{n+1}) \precsim_{i_2} a^2\vartheta(x_{n-1}, x_n) \precsim_{i_2} .... \precsim_{i_2} a^{n+1}\vartheta(x_0, x_1). \tag{2.5}$$

Then for any two positive integers m, n with $m > n$ we obtain that

$$\vartheta(x_n, x_m) \precsim_{i_2} s[\vartheta(x_n, x_{n+1}) + \vartheta(x_{n+1}, x_m)].$$

Thus

$$\begin{aligned}
\|\vartheta(x_n, x_m)\| &\leqslant s\|\vartheta(x_n, x_{n+1})\| + s\|\vartheta(x_{n+1}, x_m)\| \\
&\leqslant s\|\vartheta(x_n, x_{n+1})\| + s^2\|\vartheta(x_{n+1}, x_{n+2})\| + s^2\|\vartheta(x_{n+2}, x_m)\| \\
&\leqslant s\|\vartheta(x_n, x_{n+1})\| + s^2\|\vartheta(x_{n+1}, x_{n+2})\| \\
&\quad + s^3\|\vartheta(x_{n+2}, x_{n+3})\| + s^3\|\vartheta(x_{n+3}, x_m)\| \\
i.e., \quad \|\vartheta(x_n, x_m)\| &\leqslant \left\{ \begin{array}{c} s\|\vartheta(x_n, x_{n+1})\| + s^2\|\vartheta(x_{n+1}, x_{n+2})\| + \\ s^3\|\vartheta(x_{n+2}, x_{n+3})\| + ... + s^{m-n-1}\|\vartheta(x_{m-1}, x_m)\| \end{array} \right\} \\
i.e., \quad \|\vartheta(x_n, x_m)\| &\leqslant \left\{ \begin{array}{c} s\|\vartheta(x_n, x_{n+1})\| + s^2\|\vartheta(x_{n+1}, x_{n+2})\| + \\ s^3\|\vartheta(x_{n+2}, x_{n+3})\| + ... + s^{m-n}\|\vartheta(x_{m-1}, x_m)\| \end{array} \right\}, \text{ as } s \geqslant 1
\end{aligned}$$

Therefore by using (2.5) we have

$$\begin{aligned}
\|\vartheta(x_n,x_m)\| &\leqslant sa^n\|\vartheta(x_0,x_1)\| + s^2a^{n+1}\|\vartheta(x_0,x_1)\| + s^3a^{n+2}\|\vartheta(x_0,x_1)\| \\
&\qquad + \ldots.. + s^{m-n}a^{m-1}\|\vartheta(x_0,x_1)\| \\
i.e.,\ \|\vartheta(x_n,x_m)\| &\leqslant \sum_{i=1}^{m-n} s^i a^{i+n-1}\|\vartheta(x_0,x_1)\| \\
i.e.,\ \|\vartheta(x_n,x_m)\| &\leqslant \sum_{i=1}^{m-n} s^{i+n-1} a^{i+n-1}\|\vartheta(x_0,x_1)\|,\ \text{as } s \geqslant 1 \\
i.e.,\ \|\vartheta(x_n,x_m)\| &\leqslant \sum_{j=n}^{m-1} s^j a^j\|\vartheta(x_0,x_1)\| \\
i.e.,\ \|\vartheta(x_n,x_m)\| &\leqslant \sum_{j=n}^{\infty} (sa)^j\|\vartheta(x_0,x_1)\| \\
i.e.,\ \|\vartheta(x_n,x_m)\| &\leqslant \frac{(sa)^n}{1-sa}\|\vartheta(x_0,x_1)\|,\ \text{as } 0 \leqslant sa < 1.
\end{aligned}$$

Since $\frac{(sa)^n}{1-sa} \longrightarrow 0$ as $n \longrightarrow \infty$, therefore for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\|\vartheta(x_n,x_m)\| < \varepsilon,\quad$ for all $m,n > n_0$. Hence $\{x_n\}$ is Cauchy in $\mathcal{X}$. Again since $\mathcal{X}$ is a complete bicomplex valued b-metric space, therefore there exists a $u \in \mathcal{X}$ such that $\lim_{n\longrightarrow\infty} x_n = u$.

Now we show that $u = \Lambda_1 u$. If not then there exists an $(0 \prec_{i_2})\,\xi \in \mathbb{BC}$ such that $\vartheta(u,\Lambda_1 u) = \xi$.Therefore,

$$\begin{aligned}
\xi &= \vartheta(u,\Lambda_1 u) \\
&\precsim_{i_2} s\vartheta(u,x_{n+2}) + s\vartheta(x_{n+2},\Lambda_1 u) \\
&\precsim_{i_2} s\vartheta(u,x_{n+2}) + s\vartheta(\Lambda_1 u,\Lambda_2 x_{n+1}) \\
&\precsim_{i_2} s\vartheta(u,x_{n+2}) + \frac{sa\left[\vartheta(u,\Lambda_1 u)\,\vartheta(u,\Lambda_2 x_{n+1}) + \vartheta(x_{n+1},\Lambda_2 x_{n+1})\,\vartheta(x_{n+1},\Lambda_1 u)\right]}{\vartheta(u,\Lambda_2 x_{n+1}) + \vartheta(x_{n+1},\Lambda_1 u)} \\
&\precsim_{i_2} s\vartheta(u,x_{n+2}) + \frac{sa\left[\xi\vartheta(u,x_{n+2}) + \vartheta(x_{n+1},x_{n+2})\,\vartheta(x_{n+1},\Lambda_1 u)\right]}{\vartheta(u,x_{n+2}) + \vartheta(x_{n+1},\Lambda_1 u)}.
\end{aligned}$$

Thus

$$\|\xi\| \leqslant \|\vartheta(u,x_{n+2})\| + \sqrt{2}\frac{sa\left[\|\xi\|\,\|\vartheta(u,x_{n+2})\| + \|\vartheta(x_{n+1},x_{n+2})\|\,\|\vartheta(x_{n+1},\Lambda_1 u)\|\right]}{\|\vartheta(u,x_{n+2}) + \vartheta(x_{n+1},\Lambda_1 u)\|}.$$

Taking limit as $n \to \infty$ we get that $\|\xi\| \leqslant 0$, which is a contradiction. Therefore, $\|\xi\| = 0 \Rightarrow \|\vartheta(u,\Lambda_1 u)\| = 0 \Rightarrow u = \Lambda_1 u$. Similarly, we can show that $u = \Lambda_2 u$.

Hence Λ_1 and Λ_2 have a common fixed point.

Now we show that Λ_1 and Λ_2 have a unique common fixed point in $\mathcal{X}$. If possible let $u^* \in \mathcal{X}$ be another common fixed point of Λ_1 and Λ_2.

Then

$$\begin{aligned}
\vartheta(u,u^*) = \vartheta(\Lambda_1 u, \Lambda_2 u^*) &\precsim_{i_2} \frac{a\left[\vartheta(u,\Lambda_1 u)\,\vartheta(u,\Lambda_2 u^*) + \vartheta(u^*,\Lambda_2 u^*)\,\vartheta(u^*,\Lambda_1 u)\right]}{\vartheta(u,\Lambda_2 u^*) + \vartheta(u^*,\Lambda_1 u)}, \\
i.e.,\ \|\vartheta(u,u^*)\| &\leqslant \sqrt{2}\frac{a\left[\|\vartheta(u,\Lambda_1 u)\|\,\|\vartheta(u,\Lambda_2 u^*)\| + \|\vartheta(u^*,\Lambda_2 u^*)\|\,\|\vartheta(u^*,\Lambda_1 u)\|\right]}{\|\vartheta(u,\Lambda_2 u^*) + \vartheta(u^*,\Lambda_1 u)\|} \\
i.e.,\ \|\vartheta(u,u^*)\| &\leqslant 0,
\end{aligned}$$

which is a contradiction. Therefore,

$$\begin{aligned}
\|\vartheta(u,u^*)\| &= 0, \\
i.e.,\ u &= u^*.
\end{aligned}$$

Thus the proof of the theorem is established. ■

By taking $\Lambda_2 = \Lambda_1$ and $\Lambda_2 = \Lambda_1 = \Lambda_1^n$ respectively in Theorem 2.3.4 we get the following corollaries:

Corollary 2.3.5 *Let $(\mathcal{X},\vartheta)$ be a complete bicomplex valued b-metric spaces with the coefficient $s \geqslant 1$ and $\Lambda_1 : \mathcal{X} \to \mathcal{X}$ be any mapping satisfying the condition*

$$\vartheta(\Lambda_1 x, \Lambda_1 y) \precsim_{i_2} \frac{a\left[\vartheta(x,\Lambda_1 x)\,\vartheta(x,\Lambda_1 y) + \vartheta(y,\Lambda_1 y)\,\vartheta(y,\Lambda_1 x)\right]}{\vartheta(x,\Lambda_1 y) + \vartheta(y,\Lambda_1 x)}$$

for all $x, y \in \mathcal{X}$. If $\|\vartheta(x,\Lambda_1 y) + \vartheta(y,\Lambda_1 x)\| \neq 0$ and $\vartheta(x,\Lambda_1 y) + \vartheta(y,\Lambda_1 x)$ is degenerated, where a is non-negative real number with $0 \leqslant sa < 1$ then Λ_1 has a unique fixed point in $\mathcal{X}$.

Corollary 2.3.6 *Let $(\mathcal{X},\vartheta)$ be a complete bicomplex valued b-metric spaces with the coefficient $s \geqslant 1$ and $\Lambda_1 : \mathcal{X} \to \mathcal{X}$ be any mapping satisfying the condition*

$$\vartheta(\Lambda_1^n x, \Lambda_1^n y) \precsim_{i_2} \frac{a\left[\vartheta(x,\Lambda_1^n x)\,\vartheta(x,\Lambda_1^n y) + \vartheta(y,\Lambda_1^n y)\,\vartheta(y,\Lambda_1^n x)\right]}{\vartheta(x,\Lambda_1^n y) + \vartheta(y,\Lambda_1^n x)}$$

for all $x, y \in \mathcal{X}$. If $\|\vartheta(x,\Lambda_1^n y) + \vartheta(y,\Lambda_1^n x)\| \neq 0$ and $\vartheta(x,\Lambda_1^n y) + \vartheta(y,\Lambda_1^n x)$ be degenerated, where a is non-negative real number with $0 \leqslant sa < 1$ then Λ_1 has a unique fixed point in $\mathcal{X}$.

Proof. By Corollary 2.3.5 there exists a unique point $u \in \mathcal{X}$ such that $\Lambda_1^n u = u$.

Therefore,

$$\begin{aligned}
\vartheta\left(\Lambda_1 u, u\right) = & \vartheta\left(\Lambda_1 \Lambda_1^n u, \Lambda_1^n u\right) \\
= & \vartheta\left(\Lambda_1^n \Lambda_1 u, \Lambda_1^n u\right) \\
\precsim_{i_2} & \frac{a\left[\vartheta\left(\Lambda_1 u, \Lambda_1^n \Lambda_1 u\right) \vartheta\left(\Lambda_1 u, \Lambda_1^n u\right)+\vartheta\left(u, \Lambda_1^n u\right) \vartheta\left(u, \Lambda_1^n \Lambda_1 u\right)\right]}{\vartheta\left(\Lambda_1 u, \Lambda_1^n u\right)+\vartheta\left(u, \Lambda_1^n \Lambda_1 u\right)}, \\
i.e., \ \vartheta\left(\Lambda_1 u, u\right) \precsim_{i_2} & \frac{a\left[\vartheta\left(\Lambda_1 u, \Lambda_1 \Lambda_1^n u\right) \vartheta\left(\Lambda_1 u, u\right)+\vartheta(u, u) \vartheta\left(u, \Lambda_1 \Lambda_1^n u\right)\right]}{\vartheta\left(\Lambda_1 u, u\right)+\vartheta\left(u, \Lambda_1 \Lambda_1^n u\right)}, \\
i.e., \ \vartheta\left(\Lambda_1 u, u\right) \precsim_{i_2} & \frac{a\left[\vartheta\left(\Lambda_1 u, \Lambda_1 u\right) \vartheta\left(\Lambda_1 u, u\right)+\vartheta(u, u) \vartheta\left(u, \Lambda_1 u\right)\right]}{\vartheta\left(\Lambda_1 u, u\right)+\vartheta\left(u, \Lambda_1 u\right)}, \\
i.e., \ \vartheta\left(\Lambda_1 u, u\right) \precsim_{i_2} & \ 0, \\
i.e., \ \left\|\vartheta\left(\Lambda_1 u, u\right)\right\| = & \ 0, \\
i.e., \ \Lambda_1 u = & \ u .
\end{aligned}$$

This completes the proof of the corollary. ■

2.4 Future Prospects

In the line of the works as carried out in the chapter one may think of the deduction of fixed point theorems using fuzzy metric, partial metric, quasi metric, probabilistic metric, p-adic metric (where p is a prime number), quasi semi metric, cone metric and other different types of metrics under the flavour of bicomplex analysis. This may be regarded as an active area of research to the future workers in this branch.

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Some portion of this chapter have been published in **South East Asian Journal of Mathematics and Mathematical Science (SEAJMMS) (SCOPUS Indexed), Vol. 18, No. 1 (2022), pp. 205-214, ISSN (Online): 2582-0850, ISSN (Print): 0972-7752** and the remaining portion of this chapter have been accepted for publication and to appear in **Mathematical Forum (UGC CARE Listed), ISSN: 0972-9852.**

CHAPTER THREE

ON DIFFERENT GROWTH INDICATORS OF ENTIRE FUNCTIONS UNDER THE TREATMENT OF p-ADIC ANALYSIS

Chapter 3

On different growth indicators of entire functions under the treatment of p-adic analysis

3.1 Introduction.

Let us consider $\mathcal{K}$ to be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|.|$ (example $\mathbb{C}_p$) and $\mathfrak{A}(\mathcal{K})$ represents the $\mathcal{K}$-algebra of analytic functions in $\mathcal{K}$ i.e. the set of power series with an infinite radius of convergence. During the several years, the idea of p-adic analysis have been studied from different aspects and we get many important results from {cf.[14], [15], [16], [75] & [35]}.

In this chapter we establish some representation of order and lower order of an entire function $\mathsf{f} \in \mathfrak{A}(\mathcal{K})$, is the $\mathcal{K}$-algebra of entire functions on $\mathcal{K}$ and $\mathcal{K}$ is a complete ultametric algebraically closed field. Also we establish the integral representation of generalized (β, γ)-th relative type and generalized (β, γ)-th relative weak type of entire function f with respect to another entire function g, where f, $\mathsf{g} \in \mathfrak{A}(\mathcal{K})$ and their equivalence relation under some certain conditions are also obtained.

Let $\mathsf{f} \in \mathfrak{A}(\mathcal{K})$ and r be a positive real number, then we denote by $|\mathsf{f}|(r)$ the number $\sup\{|\mathsf{f}(x)| : |x| = r\}$, where $|.|(r)$ is a multiplicative norm on $\mathfrak{A}(\mathcal{K})$. Moreover if f is not constant the $|\mathsf{f}|(r)$ is a strictly increasing function of r and tends to $+\infty$ with r. So there exists its inverse function $\widehat{|\mathsf{f}|} : (|\mathsf{f}(0)|, \infty) \to (0, \infty)$ with

$$\lim_{s\to\infty} \widehat{|\mathsf{f}|}(s) = \infty.$$

Therefore for any two entire functions $\mathsf{f} \in \mathfrak{A}(\mathcal{K})$ and $\mathsf{g} \in \mathfrak{A}(\mathcal{K})$ the ratio $\frac{|\mathsf{f}|(r)}{|\mathsf{g}|(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their multiplicative norm.

Now taking this into account let us define some basic definitions which will be needed in this sequel:

Definition 3.1.1 *[16] Let* $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$*. The relative* (β, γ)*-th order and* (β, γ)*-th lower order of entire function* f *with respect to another entire function* g *are defined as,*

$$\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \lim_{r\to\infty} \sup \frac{\log^{[\beta]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}| (r)\right)}{\log^{[\gamma]} r}$$

and

$$\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \lim_{r\to\infty} \inf \frac{\log^{[\beta]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}| (r)\right)}{\log^{[\gamma]} r}$$

where β, γ *are two positive integers. Further for any* $\mathsf{f} \in \mathfrak{A}(\mathcal{K})$ *for which* (β, γ)*-th relative order and* (β, γ)*-th relative lower order with respect to* $\mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *are the same is called a function of regular relative* (β, γ)*-th growth with respect to* g *otherwise* f *is said to be irregular relative* (β, γ)*-th growth with respect to* g.

Definition 3.1.2 *[16] Let* $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$*. The* (β, γ)*-th relative type of* f *with respect to* g *having finite positive* (β, γ)*-th relative order* $\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *is defined as*

$$\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \lim_{r\to+\infty} \sup \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}| (r)\right)}{\left(\log^{[\gamma-1]} r\right)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}}$$

where β, γ *are any two positive integers.*

Definition 3.1.3 *Let* $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *be any two entire functions having finite positive* (β, γ)*-th relative generalized order* $\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$, *where* β, γ *are any two positive integers. Then the* (β, γ)*-th relative generalized type* $\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *of entire function* f *with respect to the entire function* g *is defined as*

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}| (r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]} r\right)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, \;\; (r_0 > 0)$$

converges for $k > \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *and diverges for* $k < \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.

Definition 3.1.4 *[16] Let* $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$*. The* (β, γ)*-th relative generalized weak type* $\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *of entire function* f *with respect to the entire function* g *having finite positive* (β, γ)*-th relative lower order* $\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *is defined as*

$$\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \lim_{r\to+\infty} \inf \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}| (r)\right)}{\left(\log^{[\gamma-1]} r\right)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}}$$

where β, γ *are any two positive integers.*

Definition 3.1.5 *Let* $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *be any two entire functions having finite positive* (β, γ)*-th relative generalized lower order* $\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$, *where* $\beta,\ \gamma$ *are any two positive integers. Then the* (β, γ)*-th relative generalized weak type* $\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *of entire function* f *with respect to the entire function* g *is defined as*

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr,\ \ (r_0 > 0)$$

converges for $k > \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *and diverges for* $k < \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.

Definition 3.1.6 *[16] Let* $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$. *The* (β, γ)*-th relative type of* f *with respect to* g *having finite positive* (β, γ)*-th relative order* $\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *is defined as*

$$\overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \lim_{r\to+\infty} \inf \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}}$$

where $\beta,\ \gamma$ *are any two positive integers.*

Definition 3.1.7 *Let* $\mathsf{f},\ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *be any two entire functions having finite positive* (β, γ)*-th relative generalized order* $\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$, *where* $\beta,\ \gamma$ *are any two positive integers. Then* (β, γ)*-th relative generalized lower type* $\overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *of entire function* f *with respect to the entire function* g *is defined as*

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr,\ \ (r_0 > 0)$$

converges for $k > \overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *and diverges for* $k < \overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.

Definition 3.1.8 *[16] Let* $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$. *The growth indicator* $\overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *of entire function* f *with respect to the entire function* g *having finite positive* (β, γ)*-th relative lower order* $\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *is defined as*

$$\overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \lim_{r\to+\infty} \sup \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}}$$

where $\beta,\ \gamma$ *are any two positive integers.*

Definition 3.1.9 *Let* $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *be any two entire functions having finite positive* (β, γ)*-th relative generalized lower order* $\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$, *where* $\beta,\ \gamma$ *are any two positive integers. Then the growth indicator* $\overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *of entire function* f *with respect to the entire function* g *is defined as*

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr,\ \ (r_0 > 0)$$

converges for $k > \overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *and diverges for* $k < \overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.

3.2 Lemma.

In this section, we present a lemma which will be needed in the sequel.

Lemma 3.2.1 *Let* $\mathsf{f},\mathsf{g}\in\mathfrak{A}(\mathcal{K})$ *be any two entire functions and let the integral*

$$\int_{r_0}^{\infty}\frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r\right)^{\alpha}\right\}\right]^{k+1}}dr,\ \ (r_0>0)$$

converges where $0<\alpha<\infty$. *Then*

$$\lim_{r\to\infty}\frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r\right)^{\alpha}\right\}\right]^{k}}=0.$$

Proof. Since the integral

$$\int_{r_0}^{\infty}\frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r\right)^{\alpha}\right\}\right]^{k+1}}dr,\ \ (r_0>0)$$

converges then

$$\int_{r_0}^{\infty}\frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r\right)^{\alpha}\right\}\right]^{k+1}}dr<\varepsilon\ \ \text{if } r_0>R(\varepsilon).$$

Therefore,

$$\int_{r_0}^{\exp\left\{\left(\log^{[\gamma-1]}r_0\right)^{\alpha}\right\}+r_0}\frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r\right)^{\alpha}\right\}\right]^{k+1}}dr<\varepsilon.$$

Since $\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)$ increases with r, so

$$\int_{r_0}^{\exp\left\{\left(\log^{[\gamma-1]}r_0\right)^{\alpha}\right\}+r_0}\frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r\right)^{\alpha}\right\}\right]^{k+1}}dr$$

$$\geqslant\ \frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r_0)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r_0\right)^{\alpha}\right\}\right]^{k+1}}\cdot\exp\left\{\left(\log^{[\gamma-1]}r_0\right)^{\alpha}\right\}$$

i.e., for all large values of r,

$$\int_{r_0}^{\exp\left\{\left(\log^{[\gamma-1]}r_0\right)^{\alpha}\right\}+r_0}\frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r\right)^{\alpha}\right\}\right]^{k+1}}dr$$

$$\geqslant\ \frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r_0)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]}r_0\right)^{\alpha}\right\}\right]^{k}}$$

so that,

$$\frac{\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|\left(r_0\right)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r_0)^{\alpha}\right\}\right]^{k}} < \varepsilon \text{ if } r_0 > R\left(\varepsilon\right).$$

Therefore,

$$\lim_{r\to\infty}\frac{\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|\left(r\right)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\alpha}\right\}\right]^{k}} = 0.$$

This proves the lemma. ■

3.3 Main Results.

In this section, we establish the main results of the chapter.

Theorem 3.3.1 *If $f \in \mathfrak{A}(\mathcal{K})$ then*

$$\varrho(\mathsf{f}) = \sup\left\{s \in\,]0,\infty[\ |\ \limsup_{r\to\infty}\frac{\log|\mathsf{f}|\left(r\right)}{r^s} > 0\right\}.$$

Proof. Let

$$P = \sup\left\{s \in\,]0,\infty[\ |\ \limsup_{r\to\infty}\frac{\log|\mathsf{f}|\left(r\right)}{r^s} > 0\right\}.$$

Let us suppose that for some $s > 0$, we have

$$\limsup_{r\to\infty}\frac{\log|\mathsf{f}|\left(r\right)}{r^s} = b > 0.$$

From the definition of supremum we have for arbitrary ε and for all large values of r,

$$\begin{aligned}
\frac{\log|\mathsf{f}|\left(r\right)}{r^s} &\leqslant b+\varepsilon \\
i.e., \log\left(\log|\mathsf{f}|\left(r\right)\right) &\leqslant s\log r + \log\left(b+\varepsilon\right)
\end{aligned}$$

$$i.e., \frac{\log\left(\log|\mathsf{f}|\left(r\right)\right)}{\log r} \leqslant s + \frac{\log\left(b+\varepsilon\right)}{\log r}. \tag{3.1}$$

Again for a sequence of values of r tending to ∞,

$$\frac{\log|\mathsf{f}|\left(r\right)}{r^s} \geqslant b-\varepsilon$$

$$i.e., \frac{\log\left(\log|f|\left(r\right)\right)}{\log r} \geqslant s + \frac{\log\left(b-\varepsilon\right)}{\log r}. \tag{3.2}$$

Combining Equation (3.1) and Equation (3.2) we get that

$$s+\frac{\log(b-\varepsilon)}{\log r}\leqslant\frac{\log(\log|f|(r))}{\log r}\leqslant s+\frac{\log(b+\varepsilon)}{\log r}.$$

Since $\varepsilon>0$ is arbitrary, we get that

$$\begin{aligned}\lim_{r\to\infty}\sup\frac{\log(\log|f|(r))}{\log r} &= P\\ i.e., \varrho(f) &= P.\end{aligned}$$

This proves the theorem. ■

Theorem 3.3.2 *If* $\mathsf{f}\in\mathfrak{A}(\mathcal{K})$ *then*

$$\lambda(\mathsf{f})=\inf\left\{s\in\,]0,\infty[\;|\;\lim_{r\to\infty}\inf\frac{\log|\mathsf{f}|(r)}{r^s}>0\right\}.$$

Proof. Let

$$Q=\inf\left\{s\in\,]0,\infty[\;|\;\lim_{r\to\infty}\inf\frac{\log|\mathsf{f}|(r)}{r^s}>0\right\}.$$

Let us suppose that for some $s>0$ we have

$$\lim_{r\to\infty}\inf\frac{\log|\mathsf{f}|(r)}{r^s}=d>0.$$

From the definition of supremum we have for arbitrary ε and for all large values of r

$$\frac{\log|\mathsf{f}|(r)}{r^s}\geqslant d-\varepsilon$$

$$i.e., \frac{\log(\log|\mathsf{f}|(r))}{\log r}\geqslant s+\frac{\log(d-\varepsilon)}{\log r}. \tag{3.3}$$

Again for a sequence of values of r tending to ∞

$$\frac{\log|\mathsf{f}|(r)}{r^s}\leqslant d+\varepsilon$$

$$i.e., \frac{\log(\log|\mathsf{f}|(r))}{\log r}\leqslant s+\frac{\log(d+\varepsilon)}{\log r}. \tag{3.4}$$

Combining Equation (3.3) and Equation (3.4) we get that

$$s+\frac{\log(d-\varepsilon)}{\log r}\leqslant\frac{\log(\log|\mathsf{f}|(r))}{\log r}\leqslant s+\frac{\log(d+\varepsilon)}{\log r}.$$

Since $\varepsilon > 0$ is arbitrary we obtain that

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log\left(\log|\mathsf{f}|(r)\right)}{\log r} &= Q \\ i.e., \lambda(\mathsf{f}) &= Q, \end{aligned}$$

which proves the theorem. ■

Theorem 3.3.3 *Let* $\mathsf{f},\ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *be any two entire functions having finite positive* (β,γ)*-th relative generalized order* $\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$ *and* (β,γ)*-th relative generalized type* $\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, *where* $\beta,\ \gamma$ *are any two positive integers. Then Definition 3.1.2 and Definition 3.1.3 are equivalent.*

Proof.

Let $\mathsf{f},\ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ be any two entire functions such that $\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$ exists , where $\beta,\ \gamma$ are any two positive integers.

Case I Let

$$\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \infty.$$

Definition 3.1.2 $\Longrightarrow$ Definition 3.1.3.

As $\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \infty$, from Definition 3.1.2 we have for an arbitrary $G > 0$ and a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[\beta-1]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right) &> G \cdot (\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})} \\ i.e., \log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right) &> \left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G}. \quad (3.5) \end{aligned}$$

If possible let the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G+1}} dr,\ (r_0 > 0)$$

be converge. Then by Lemma 3.2.1, we have

$$\limsup_{r\to\infty} \frac{\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G}} = 0.$$

So for all sufficiently large values of r,

$$\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right) < \left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G}. \quad (3.6)$$

Therefore, by Equation (3.5) and Equation (3.6) we arrive at a contradiction. Hence

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G+1}} dr, \ (r_0 > 0)$$

diverges where $G > 0$ is finite, which is Definition 3.1.3.

Now we show Definition 3.1.3 $\Longrightarrow$ Definition 3.1.2.

Let G be any positive number. Since

$$\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \infty,$$

from Definition 3.1.3 the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G+1}} dr, \ (r_0 > 0)$$

gives an arbitrary positive ε and for a sequence of values of r tending to infinity

$$\begin{aligned} \log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) &> \left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G-\varepsilon} \\ i.e., \log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) &> (G-\varepsilon)\,(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}. \end{aligned}$$

which implies that

$$\frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} > G - \varepsilon.$$

Since $G > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} = \infty$$

$$i.e., \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \infty.$$

Thus Definition 3.1.2 follows.

Case II Let

$$0 \leqslant \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty.$$

First we show that Definition 3.1.2 $\Longrightarrow$ Definition 3.1.3.

Sub case (A) Let $\mathsf{f}, \ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ be any two entire functions such that

$$0 < \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty$$

exists for positive integers $\beta, \ \gamma$. Then according to Definition 3.1.2 for any arbitrary positive ε and for large value of r we get that

$$\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) < \left(\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) + \varepsilon\right) \left\{\log^{[\gamma-1]} r\right\}^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}$$

$$i.e., \log^{[\beta-2]} |\widehat{\mathsf{g}}| (|f| (r)) < \left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\left(\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon\right)},$$

$$i.e., \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k}} < \frac{1}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k-\left(\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon\right)}}.$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, \ (r_0 > 0)$$

converges for $k > \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.

Again by Definition 3.1.2 we obtain for a sequence of values of r tending to infinity that

$$\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) > \left(\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) - \varepsilon\right) \left\{\log^{[\gamma-1]} r\right\}^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}$$

$$i.e., \log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) > \left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\left(\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon\right)}, \tag{3.7}$$

so for $k < \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, we get from Equation (3.7) that

$$\frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k}} > \frac{1}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k-\left(\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon\right)}}.$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, \ (r_0 > 0)$$

diverges for $k < \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$. Hence

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, \ (r_0 > 0)$$

converges for $k > \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ and diverges for $k < \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.

Sub case (B) Let

$$\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = 0.$$

When $\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = 0$ for positive integer $\beta, \ \gamma$ Definition 3.1.2 gives for all sufficiently large values of r that

$$\frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} < \varepsilon.$$

Then similar as before we get that

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr,\ (r_0 > 0)$$

converges for $k > 0$ and diverges for $k < 0$. Thus combining Subcase (A) and Subcase (B) Definition 3.1.3 follows.

Now we show Definition 3.1.3 $\Longrightarrow$ Definition 3.1.2.

From Definition 3.1.3 and arbitrary positive ε, the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon+1}} dr,\ (r_0 > 0)$$

converges . Then by Lemma 3.2.1 we obtain that

$$\frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon}} = 0.$$

So, we obtain for all sufficiently large values of r that

$$\frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon}} < \varepsilon$$

$$i.e., \limsup_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \leqslant \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \leqslant \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.8}$$

On the other hand the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon+1}} dr,\ (r_0 > 0)$$

implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon+1}} > \frac{1}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{1+\varepsilon}},$$

$$i.e., \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} > \left(\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) - 2\varepsilon\right).$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\lim_{r\to\infty} \sup \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \geqslant \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.9}$$

So from Equation (3.8) and Equation (3.9) we get that

$$\lim_{r\to\infty} \sup \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} = \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}).$$

This proves the theorem. ■

Remark 3.3.1 *We give an example below which validates Theorem 3.3.3.*

Example 3.3.1 *Let* $\mathsf{f}(\mathfrak{z}) = \mathfrak{z}$, $\mathsf{g}(\mathfrak{z}) = \log \mathfrak{z}$, $\beta = 3$ *and* $\gamma = 2$.*So* $\widehat{\mathsf{g}}(\mathfrak{z}) = \exp(\mathfrak{z})$. *Then*

$$\begin{aligned}
\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) &= \lim_{r\to\infty} \sup \frac{\log^{[\beta]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma]} r)} \\
&= \lim_{r\to\infty} \sup \frac{\log^{[3]} \exp(r)}{\log^{[2]} r} \\
&= 1.
\end{aligned}$$

Again

$$\begin{aligned}
\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) &= \lim_{r\to\infty} \sup \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \lim_{r\to\infty} \sup \frac{\log^{[2]} \exp(r)}{\log^{[1]} r} \\
&= 1.
\end{aligned}$$

Next if we take $k = 2$, *that is* $k > \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ *we see that the value of the integral for* $r_0 > 0$,

$$\begin{aligned}
&\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, \\
&= \int_{r_0}^{\infty} \frac{\log \exp(r)}{\left[\exp(\log r)^1\right]^{2+1}} dr \\
&= \int_{r_0}^{\infty} \frac{1}{r^2} dr \\
&= \frac{1}{r_0},
\end{aligned}$$

which converges . Next if we take $k = 0$, that is $k < \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ we see that the value of the integral for $r_0 > 0$,

$$\begin{aligned}
&\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr \\
= &\int_{r_0}^{\infty} \frac{\log \exp (r)}{\left[\exp (\log r)^{1}\right]^{0+1}} dr \\
= &\int_{r_0}^{\infty} dr \\
= &\ \infty,
\end{aligned}$$

which diverges.

Theorem 3.3.4 *Let $\mathsf{f}, \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ be any two entire functions having finite positive (β,γ)-th relative generalized lower order $\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$ and (β,γ)-th relative generalized weak type $\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, where β, γ are any two positive integers. Then Definition 3.1.4 and Definition 3.1.5 are equivalent.*

Proof. Case I Let

$$\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \infty.$$

Definition 3.1.4 $\Longrightarrow$ Definition 3.1.5.

As $\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \infty$, from Definition 3.1.4 we get for an arbitrary positive G and for all sufficiently large values of r that

$$\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) > G \cdot (\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}$$

$$i.e., \log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) > \left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G}. \tag{3.10}$$

If possible, let the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G+1}} dr, \ (r_0 > 0)$$

be convergent. Then by Lemma 3.2.1

$$\liminf_{r\to\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G}} = 0.$$

So, for a sequence of values of r tending to infinity, we get that

$$\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) < \left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G}. \tag{3.11}$$

Therefore from Equation (3.10) and Equation (3.11) we arrive at a contradiction. Hence,

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G+1}} dr, \ (r_0 > 0)$$

diverges, whenever G is finite which is Definition 3.1.5.

Now we show that Definition 3.1.5 $\Longrightarrow$ Definition 3.1.4.

Let G be any positive number. Since

$$\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \infty,$$

from Definition 3.1.3 the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G+1}} dr, \ (r_0 > 0)$$

gives an arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[\beta-2]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r)) \quad > \quad \left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{G-\varepsilon}$$

$$i.e., \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} > G - \varepsilon.$$

$$i.e., \liminf_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \geqslant G - \varepsilon$$

Since $G > 0$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| (|\mathsf{f}| (r))}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \quad = \quad \infty$$
$$i.e., \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \infty.$$

Thus Definition 3.1.4 follows.

Case II Let

$$0 \leqslant \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty.$$

First we show that Definition 3.1.4 $\Longrightarrow$ Definition 3.1.5.

Sub case (A_1) Let $\mathsf{f},\ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ be any two entire functions such that

$$0 < \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty$$

exists for positive integers $\beta,\ \gamma$.
Then according to Definition 3.1.4 for any arbitrary positive ε and for large value of r we get that

$$\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right) < \left(\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon\right)\left\{\log^{[\gamma-1]} r\right\}^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}$$

$$i.e., \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k}} < \frac{1}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k-\left(\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon\right)}}.$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr,\ \ (r_0 > 0)$$

converges for $k > \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.

Again by Definition 3.1.4 we obtain for all sufficiently large values of r that

$$\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right) > \left(\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon\right)\left\{\log^{[\gamma-1]} r\right\}^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}$$

$$i.e., \log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right) > \left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\left(\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon\right)}. \tag{3.12}$$

So, for $k < \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, we get from Equation (3.12) that

$$\frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k}} > \frac{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\left(\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon\right)}}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k}}$$

$$i.e., \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k}} > \frac{1}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k-\left(\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon\right)}}.$$

Therefore,

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr,\ \ (r_0 > 0)$$

diverges for $k < \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.
Hence

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr,\ \ (r_0 > 0)$$

converges for $k > \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ and diverges for $k < \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.

Subcase (B_1) Let

$$\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = 0.$$

When $\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = 0$, for any positive integer $\beta,\ \gamma$, Definition 3.1.4 gives for a sequence of values of r tending to infinity that

$$\frac{\log^{[\beta-1]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} < \varepsilon.$$

Then similar as before we get that

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr,\ \ (r_0 > 0)$$

converges for $k > 0$ and diverges for $k < 0$.Thus combining Subcase (A_1) and Subcase (B_1) Definition 3.1.5 follows.

Now we show that Definition 3.1.5 $\Longrightarrow$ Definition 3.1.4.

From Definition 3.1.5 and arbitrary positive ε, the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon+1}} dr,\ \ (r_0 > 0)$$

converges . Then by Lemma 3.2.1, we get that

$$\liminf_{r\to\infty} \frac{\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})+\varepsilon}} < \varepsilon$$

$$i.e., \log^{[\beta-1]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right) < \log\varepsilon + \left(\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) + \varepsilon\right)\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\},$$

$$i.e., \liminf_{r\to\infty} \frac{\log^{[\beta-1]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \leqslant \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log^{[\beta-1]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \leqslant \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.13}$$

On the other hand, the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]}|\widehat{\mathsf{g}}|\left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon+1}} dr,\ \ (r_0 > 0)$$

implies for all sufficiently large values of r that

$$\frac{\log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left[\exp\left\{\left(\log^{[\gamma-1]} r\right)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-\varepsilon+1}} > \frac{1}{\left[\exp\left\{\left(\log^{[\gamma-1]} r\right)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{1+\varepsilon}},$$

$$i.e., \ \log^{[\beta-2]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right) > \left[\exp\left\{\left(\log^{[\gamma-1]} r\right)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})-2\varepsilon}$$

$$i.e., \frac{\log^{[\beta-1]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left(\log^{[\gamma-1]} r\right)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} > \left(\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) - 2\varepsilon\right).$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[\beta-1]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left(\log^{[\gamma-1]} r\right)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \geqslant \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.14}$$

So, from Equation (3.13) and Equation (3.14) we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[\beta-1]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left(\log^{[\gamma-1]} r\right)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} = \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}).$$

This proves the theorem. ■

Remark 3.3.2 *We give an example below, which validates the Theorem 3.3.4.*

Example 3.3.2 *Let* $f(\mathfrak{z}) = \mathfrak{z}$, $\mathsf{g}(\mathfrak{z}) = \log \mathfrak{z}, (\mathfrak{z} > 0)$, $\beta = 3$ *and* $\gamma = 2$.*So* $\widehat{\mathsf{g}}(\mathfrak{z}) = \exp(\mathfrak{z})$.

Then

$$\begin{aligned}
\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) &= \liminf_{r\to\infty} \frac{\log^{[\beta]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left(\log^{[\gamma]} r\right)} \\
&= \liminf_{r\to\infty} \frac{\log^{[3]}\exp(r)}{\log^{[2]} r} \\
&= 1.
\end{aligned}$$

Again

$$\begin{aligned}
\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) &= \liminf_{r\to\infty} \frac{\log^{[\beta-1]}\left|\widehat{\mathsf{g}}\right|\left(\left|\mathsf{f}\right|(r)\right)}{\left(\log^{[\gamma-1]} r\right)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \liminf_{r\to\infty} \frac{\log^{[2]}\exp(r)}{\log^{[1]} r} \\
&= 1.
\end{aligned}$$

Next if we take $k = 2$, that is $k > \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ we see that the value of the integral for $r_0 > 0$,

$$\begin{aligned}
&\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|\,(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, \\
= &\int_{r_0}^{\infty} \frac{\log \exp\,(r)}{\left[\exp\,(\log r)^1\right]^{2+1}} dr \\
= &\int_{r_0}^{\infty} r^{-2} dr \\
= &\frac{1}{r_0},
\end{aligned}$$

which converges. Next if we take $k = 0$, that is $k < \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ we see that the value of the integral for $r_0 > 0$,

$$\begin{aligned}
&\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|\,(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr \\
= &\int_{r_0}^{\infty} \frac{\log \exp\,(r)}{\left[\exp\,(\log r)^1\right]^{0+1}} dr \\
= &\int_{r_0}^{\infty} dr \\
= &\infty,
\end{aligned}$$

which diverges.

Theorem 3.3.5 *Let $\mathsf{f},\ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ be any two entire functions having finite positive (β,γ)-th relative generalized order $\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$ and (β,γ)-th relative generalized lower type $\overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$,where β,γ are any two positive integers. Then Definition 3.1.6 and Definition 3.1.7 are equivalent.*

Proof. With the help of Lemma 3.2.1 and similar to the proof of Theorem 3.3.1 we can prove the above theorem. ■

Theorem 3.3.6 *Let $\mathsf{f},\ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ be any two entire functions having finite positive (β,γ)-th relative generalized lower order $\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, $\left(0 < \lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right)$ and the growth indicator $\overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$, where $\beta,\ \gamma$ are any two positive integers. Then Definition 3.1.8 and Definition 3.1.9 are equivalent.*

Proof. With the help of Lemma 3.2.1 and similar to the proof of Theorem 3.3.2 we can prove the above theorem. ■

Theorem 3.3.7 *Let* $\mathsf{f}, \ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *be any two entire functions such that* f *is regular* (β, γ)*-th relative generalized growth with respect to* g *,i.e,*

$$\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}), \left(0 < \lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) < \infty\right),$$

where $\beta, \ \gamma$ *are any two positive integers. Then the following quantities*

$$(i) \ \ \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) \ \ , (ii) \ \ \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}), \ \ (iii) \ \ \overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) \ \ , (iv) \ \ \overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$$

are all equivalent.

Proof. From Definition 3.1.3 it follows that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, \ \ (r_0 > 0),$$

converges for $k > \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ and diverges for $k < \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$. On the other hand Definition 3.1.3 implies that the integral

$$\int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, \ \ (r_0 > 0),$$

converges for $k > \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$ and diverges for $k < \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})$.
We show that (i) $\Longrightarrow$ (ii).

Now it is obvious that all the quantities in the expression

$$\left[\frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} - \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}}\right]$$

are non negative type. So,

$$\int_{r_0}^{\infty} \left[\frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} - \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}}\right] dr \geqslant 0, (r_0 > 0),$$

$$i.e., \int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr \geqslant \int_{r_0}^{\infty} \frac{\log^{[\beta-2]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{\left[\exp\left\{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}\right\}\right]^{k+1}} dr, (r_0 > 0),$$

$$i.e., \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) \geqslant \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.15}$$

Further, f is of regular (β,γ)-th relative generalized growth with respect to g. So, we get that

$$\begin{aligned}
\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) &= \limsup_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&\geqslant \liminf_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \liminf_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}),
\end{aligned}$$

$$i.e., \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) \geqslant \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.16}$$

Combining Equation (3.15) and Equation (3.16) we get that

$$\sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.17}$$

Now we show (ii) $\Longrightarrow$ (iii).

Since f is of regular (β,γ)-th relative generalized growth with respect to g. So, we get that

$$\begin{aligned}
\tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) &= \liminf_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \liminf_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}),
\end{aligned}$$

$$i.e., \ \tau_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.18}$$

Next we show (iii) $\Longrightarrow$ (iv).

From Equation (3.17), Equation (3.18) and the condition

$$\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}),$$

it follows that

$$\overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.19}$$

So,

$$\begin{aligned}
\overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) &= \limsup_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|\mathsf{f}|(r)\right)}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \limsup_{r\to\infty} \frac{\log^{[\beta-1]} |\widehat{\mathsf{g}}| \left(|f|(r)\right)}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}),
\end{aligned}$$

$$i.e., \overline{\sigma}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}).$$

Now we show (iv) $\Longrightarrow$ (i).

Since f is of regular (β,γ)-th relative generalized growth with respect to g, we get that

$$\begin{aligned}
\overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) &= \limsup_{r\to\infty} \frac{\log^{[\beta-1]}|\widehat{\mathsf{g}}|(|\mathsf{f}|(r))}{(\log^{[\gamma-1]} r)^{\lambda_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \limsup_{r\to\infty} \frac{\log^{[\beta-1]}|\widehat{\mathsf{g}}|(|\mathsf{f}|(r))}{(\log^{[\gamma-1]} r)^{\varrho_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f})}} \\
&= \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}),
\end{aligned}$$

$$i.e., \overline{\tau}_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}) = \sigma_{\mathsf{g}}^{(\beta,\gamma)}(\mathsf{f}). \tag{3.20}$$

Thus the theorem follows. ■

3.4 Future Prospects.

In the same vein as the work carried out in the chapter it is possible to think of finding out integral representation of relative $(\beta,\gamma,t)\,L^{th}$ Ψ-growth and $(\beta,\gamma,t)\,L^{*}$ Ψ-growth of entire and meromorphic function with respect to another one and this treatment can be done under the flavour of bicomplex analysis. As a consequence the derivation of relevant results is still open to the future workers of this branch.

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The works of this chapter have been published in **Fuzzy Systems and Soft Computing, Vol. 19, No. 2(IV) (2024), pp. 1-15, ISSN: 1819-4362.**

CHAPTER FOUR

ON THE GROWTHS OF ENTIRE ALGEBROIDAL FUNCTIONS UNDER THE FLAVOUR OF p-ADIC ANALYSIS

Chapter 4

On the growths of entire algebroidal functions under the flavour of p-adic analysis

4.1 Introduction.

The motivation of this work comes from the following literature cf. [11], [12], [17], [36] ,[40] [37], [38], [39] & [75] where a closed investigation was carried out on the properties of relative order of entire functions under the umbrella of p-adic analysis.

In this chapter we establish some results focusing on the comparative growth properties of composition of k-valued entire algebroidal functions in terms of their relative growth indicators from the view point of slowly changing functions in the light of p-adic analysis. Although the main aim of this chapter is to find out the estimates of relative (β, γ)-th type, relative (β, γ)-th lower type and relative (β, γ)-th weak type under some different conditions where β and γ are any two positive integers from the view point of p-adic analysis of entire algebroidal functions.

4.2 Lemmas.

This section contains some important lemmas that will be required for the sequel.

Lemma 4.2.1 *[11] Let* $\mathsf{f},\ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *be transcendental. If* $\varrho(\mathsf{f}) \neq 0$, *then* $\varrho(\mathsf{f} \circ \mathsf{g}) = +\infty$. *If* $\varrho(\mathsf{f}) = 0$ *then*

$$\varrho(\mathsf{f} \circ \mathsf{g}) \geqslant \varrho(\mathsf{g}).$$

Lemma 4.2.2 *[17] Let* $\mathsf{f},\ \mathsf{g} \in \mathfrak{A}(\mathcal{K})$ *be two entire functions such that* $0 < \varrho^{(\delta,\eta)}(\mathsf{g}) <$

$\lambda^{(\beta,\gamma)}(\mathsf{f}) \leqslant \varrho^{(\beta,\gamma)}(\mathsf{f}) < \infty,$

$$\begin{aligned} \lim_{r\to\infty} \frac{\log^{[\beta]} |\mathsf{f}\circ\mathsf{g}| \left(\exp^{[\eta+1]} r\right)}{\log^{[\beta]} |\mathsf{f}| \left[\exp^{[\gamma]} r^{\alpha}\right]} &= \infty \ if \quad \gamma = \delta \\ \text{and} \ \lim_{r\to\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}|}\, |\mathsf{f}\circ\mathsf{g}| \left(\exp^{[\gamma+\eta-\delta+1]} r\right)}{\log^{[\beta]} |\mathsf{f}| \left[\exp^{[\gamma]} r^{\alpha}\right]} &= \infty \ if \quad \gamma > \delta \end{aligned}$$

where $\alpha > 0$ is a constant.

Lemma 4.2.3 *[17] Let f, g, $\mathsf{h} \in \mathfrak{A}(\mathcal{K})$ be three entire functions such that f and g have finite* relative (β,γ)-th L-order and lower order *with respect to h respectively then*

$$\begin{aligned} \overline{\lim_{r\to\infty}} \frac{\log^{[\beta]} \widehat{|h|}\, |\mathsf{f}\circ\mathsf{g}| \left(\exp^{[\eta+1]} r\right)}{\log^{[\beta]} \widehat{|\mathsf{h}|} \left(|\mathsf{f}| \left[\exp^{[\gamma]} r^{\alpha}\right]\right)} &= \infty \ if \quad \gamma = \delta \\ \text{and} \ \overline{\lim_{r\to\infty}} \frac{\log^{[\beta]} \widehat{|\mathsf{h}|}\, |\mathsf{f}\circ\mathsf{g}| \left(\exp^{[\gamma+\eta-\delta+1]} r\right)}{\log^{[\beta]} \widehat{|\mathsf{h}|} \left(|\mathsf{f}| \left[\exp^{[\gamma]} r^{\alpha}\right]\right)} &= \infty \ if \quad \gamma > \delta \end{aligned}$$

where $\alpha > 0$ is a constant.

Lemma 4.2.4 *Let $\mathfrak{F}$ and $\mathfrak{G}$ be two k-valued p-adic entire algebroidal functions. Then for all sufficiently large positive numbers of r,*

$$|\mathfrak{F}\circ\mathfrak{G}|(r) = |\mathfrak{F}|\left(|\mathfrak{G}|(r)\right).$$

4.3 Main Results.

This section deals with the main findings of this chapter.

Theorem 4.3.1 *Let $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ be all k-valued p-adic entire algebroidal functions defined by the following irreducible equations:*

$$\mathsf{f}_k\mathfrak{F}^k + \mathsf{f}_{k-1}\mathfrak{F}^{k-1} + \mathsf{f}_{k-2}\mathfrak{F}^{k-2} + \cdots + \mathsf{f}_0 = 0 \qquad \text{(i)}$$

$$\mathsf{g}_k\mathfrak{G}^k + \mathsf{g}_{k-1}\mathfrak{G}^{k-1} + \mathsf{g}_{k-2}\mathfrak{G}^{k-2} + \cdots + \mathsf{g}_0 = 0 \qquad \text{(ii)}$$

$$\mathsf{h}_k\mathfrak{H}^k + \mathsf{h}_{k-1}\mathfrak{H}^{k-1} + \mathsf{h}_{k-2}\mathfrak{H}^{k-2} + \cdots + \mathsf{h}_0 = 0 \qquad \text{(iii)}$$

where f_i, g_i and h_i be entire functions belonging to $\mathfrak{A}(\mathcal{K})$ having no common zeros $(i = 0,1,2,\ldots,k)$ such that $0 < \lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) \leqslant \varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) < \infty$ and $\varrho^{(\delta,\eta)L}(\mathfrak{G}) > 0$, where β, γ, δ, η are positive integers. Then for every positive constant α, we have

$$(a) \limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\, |\mathfrak{F}\circ\mathfrak{G}| \left(\exp^{[\eta+1]} r\right)}{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}| \left[\exp^{[\gamma]} (rL(r))^{\alpha}\right]\right)} = \infty \ if \quad \gamma = \delta,$$

$$(b)\limsup_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\gamma+\eta+1-\delta]}r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])}=\infty\ if\quad\gamma>\delta$$

$$(c)\limsup_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta-1]}r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])}=\infty\ if\quad\gamma\leqslant\delta-1\ and\ \varrho^{(\delta,\eta)L}(\mathfrak{G})>\alpha\ .$$

Proof. For all sufficiently large values of r it is clear that $|\mathfrak{F}\circ\mathfrak{G}|\,(r)=|\mathfrak{F}|\,(|\mathfrak{G}|\,(r))$.

Now from the definition of $\varrho_{h_i}^{(\beta,\gamma)L}(\mathfrak{F})$, for any $\varepsilon(>0)$ and for all sufficiently large values of r we have

$$\log^{[\beta]}\widehat{|h_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}(rL(r))^{\alpha}\right]\right)\leqslant\left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\varepsilon\right)(rL(r))^{\alpha}\,. \tag{4.1}$$

Also we get for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned}
\log^{[\delta]}|\mathfrak{G}|\,(\exp^{[\gamma+\eta+1-\delta]}r) &\geqslant \left(\varrho^{(\delta,\eta)L}(\mathfrak{G})-\epsilon\right)\log^{[\eta]}(\exp^{[\gamma+\eta+1-\delta]}r)\\
\text{i.e.,}\quad \log^{[\gamma-\delta]}\log^{[\delta]}|\mathfrak{G}|\,(\exp^{[\gamma+\eta+1-\delta]}r) &\geqslant \log^{[\gamma-\delta]}\left(\varrho^{(\delta,\eta)L}(\mathfrak{G})-\epsilon\right)\exp^{[\gamma+1-\delta]}r\\
\text{i.e.,}\ \log^{[\gamma]}|\mathfrak{G}|\,(\exp^{[\gamma+\eta+1-\delta]}r) &\geqslant \exp r+O(1), \qquad (4.2)
\end{aligned}$$

and

$$\begin{aligned}
\log^{[\delta]}|\mathfrak{G}|\,(\exp^{[\eta-1]}r) &\geqslant \left(\varrho^{(\delta,\eta)L}(\mathfrak{G})-\epsilon\right)\log^{[\eta]}(\exp^{[\eta-1]}r)\\
i.e.,\ \log^{[\delta-1]}|\mathfrak{G}|\,(\exp^{[\eta-1]}r) &\geqslant r^{\left(\varrho^{(\delta,\eta)L}(\mathfrak{G})-\epsilon\right)}. \qquad (4.3)
\end{aligned}$$

Since $\widehat{|\mathsf{h}_i|}(r)$ is an increasing function of r, for all sufficiently large positive values of r we have

$$\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(r)\geqslant\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right)\log^{[\gamma]}(|\mathfrak{G}|\,r). \tag{4.4}$$

Case I First let $\gamma=\delta$ then it follows from Equation (4.4) for a sequence of positive numbers of r $(\to\infty)$ that

$$\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta+1]}r)\geqslant\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right)\left(\varrho^{(\delta,\eta)L}(\mathfrak{G})-\epsilon\right)\exp r. \tag{4.5}$$

Case II Let $\gamma>\delta$. Then from Equation (4.2) and Equation (4.4), for a sequence of positive numbers of r $(\to\infty)$ we get that

$$\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\gamma+\eta+1-\delta]}r)\geqslant\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right)\exp r+O(1). \tag{4.6}$$

Case III Again let $\gamma\leqslant\delta-1$. Then we have from Equation (4.3) and Equation (4.4) for a sequence of positive numbers of r $(\to\infty)$ that

$$\begin{aligned}
\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta-1]}r) &\geqslant \left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right)\log^{[\gamma]}|\mathfrak{G}|\,(\exp^{[\eta-1]}r)\\
i.e.,\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta-1]}r) &\geqslant \left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right)\log^{[\delta-1]}|\mathfrak{G}|\,(\exp^{[\eta-1]}r)\\
i.e.,\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta-1]}r) &\geqslant \left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right)r^{\left(\varrho^{(\delta,\eta)L}(\mathfrak{G})-\epsilon\right)}. \qquad (4.7)
\end{aligned}$$

Now combining Equation (4.1) and Equation (4.5) we obtain that

$$\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta+1]} r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])} \geqslant \frac{\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right)\left(\varrho^{(\delta,\eta)L}(\mathfrak{G})-\epsilon\right)\exp r}{\left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\varepsilon\right)(rL(r))^{\alpha}}.$$

As $\frac{\exp r}{(rL(r))^{\alpha}} \to \infty$ as $r \to \infty$, then we get that

$$\limsup_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta+1]} r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])} = \infty.$$

Hence the first part of the theorem follows.

Again combining Equation (4.1) and Equation (4.6) it follows that

$$\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\gamma+\eta+1-\delta]} r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])} \geqslant \frac{\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right)\exp r + O(1)}{\left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\varepsilon\right)(rL(r))^{\alpha}}$$

$$\limsup_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\gamma+\eta+1-\delta]} r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])} = \infty.$$

Thus the second part of the theorem.

Also from Equation (4.1) and Equation (4.7) we obtain that

$$\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta-1]} r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])} \geqslant \frac{\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\varepsilon\right) r^{\left(\varrho^{(\delta,\eta)L}(\mathfrak{G})-\epsilon\right)}}{\left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\varepsilon\right)(rL(r))^{\alpha}}.$$

As $\varrho^{(\delta,\eta)L}(\mathfrak{G}) > \alpha$, therefore from above it follows that

$$\limsup_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta-1]} r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])} = \infty.$$

This validates the third part of the theorem.

Hence this completes the proof. ■

In the context of Theorem 4.3.1 the subsequent theorem is feasible to be derived.

Theorem 4.3.2 *Let $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ be three k-valued p-adic entire algebroidal functions as defined in Theorem 4.3.1 by the Equations (i), (ii) and (iii) where f_i, g_i and $\mathsf{h}_i \in \mathfrak{A}(\mathcal{K})$ are entire functions having no common zeros $(i = 0, 1, 2, \ldots, k)$ be such that $0 < \lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) \leqslant \varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) < \infty$ and $\lambda^{(\delta,\eta)L}(\mathfrak{G}) > 0$. Then for every positive constant α we obtain that*

$$(a)\ \lim_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\eta+1]} r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])} = \infty \ if \quad \gamma = \delta$$

$$(b)\ \lim_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\,(\exp^{[\gamma+\eta+1-\delta]} r)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\,(|\mathfrak{F}|\,[\exp^{[\gamma]}(rL(r))^{\alpha}])} = \infty \ if \quad \gamma > \delta.$$

Theorem 4.3.3 *Let $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ be three k-valued p-adic entire algebroidal functions defined by the irreducible Equations (i), (ii) and (iii) as in Theorem 4.3.1, where f_i, g_i and h_i are entire functions belonging to $\mathfrak{A}(\mathcal{K})$ having no common zeros ($i = 0, 1, 2, \ldots, k$) be such that $0 < \lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) \leqslant \varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) < \infty$ and $\varrho^{(\delta,\eta)L}(\mathfrak{G}) > 0$. Also let W be another k-valued p-adic entire algebroidal functions defined by the irreducible equation:*

$$\mathsf{w}_k W^k + \mathsf{w}_{k-1} W^{k-1} + \mathsf{w}_{k-2} W^{k-2} + \cdots + \mathsf{w}_0 = 0$$

where $\mathsf{w}_i (\in \mathfrak{A}(\mathcal{K}))$ are entire functions such that $\varrho_{\mathsf{w}_i}^{(\mu,\eta)L}(\mathfrak{G}) < \infty$ and β, γ, δ, η, μ are positive integers. Then for every positive constant α we obtain the followings

$$(a) \limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \, |\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta+1]} r\right)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(|\mathfrak{G}| \left[\exp^{[\eta]} \left(rL(r)\right)^{\alpha}\right]\right)} = \infty \ if \quad \gamma = \delta.$$

$$(b) \limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \, |\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\gamma+\eta+1-\delta]} r\right)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(|\mathfrak{G}| \left[\exp^{[\eta]} \left(rL(r)\right)^{\alpha}\right]\right)} = \infty \ if \quad \gamma > \delta.$$

$$(c) \limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \, |\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta-1]} r\right)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(|\mathfrak{G}| \left[\exp^{[\eta]} \left(rL(r)\right)^{\alpha}\right]\right)} = \infty \ if \quad \gamma \leqslant \delta - 1 \ and \ \varrho^{(\delta,\eta)L}(\mathfrak{G}) > \alpha.$$

Proof. Let us first consider α_0 be a positive constant be such that

$$\alpha < \alpha_0 < \varrho^{(\delta,\eta)L}(\mathfrak{G}). \tag{4.8}$$

Case I Let $\gamma = \delta$. Then in the light of the first part of Theorem 4.3.1, for any arbitrary $\varepsilon (> 0)$ we have for a sequence of positive integers of r ($\to \infty$) and a integer $R > 1$,

$$\log^{[\beta]} \widehat{|\mathsf{h}_i|} \, |\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta+1]} r\right) > R \left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) - \varepsilon\right) \left(rL(r)\right)^{\alpha_0}. \tag{4.9}$$

Case II Let $\gamma > \delta$. Then as per the second part of Theorem 4.3.1 for a sequence positive numbers of r ($\to \infty$) and a integer $R > 1$,

$$\log^{[\beta]} \widehat{|\mathsf{h}_i|} \, |\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\gamma+\eta+1-\delta]} r\right) > R \left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) - \varepsilon\right) \left(rL(r)\right)^{\alpha_0}. \tag{4.10}$$

Case III Again let $\gamma \leqslant \delta - 1$. Then we obtain from the third part of the Theorem 4.3.1 for a sequence of positive numbers of r ($\to \infty$) and a integer $R > 1$,

$$\log^{[\beta]} \widehat{|\mathsf{h}_i|} \, |\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta-1]} r\right) > R \left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) - \varepsilon\right) \left(rL(r)\right)^{\alpha_0}. \tag{4.11}$$

Now from the definition of $\varrho_{\mathsf{h}_i}^{(\beta,\eta)L}(\mathfrak{G})$, we have for all sufficiently large values of r that

$$\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(|\mathfrak{G}| \left[\exp^{[\eta]} \left(rL(r)\right)^{\alpha}\right]\right) \leqslant \left(\varrho_{\mathsf{w}_i}^{(\mu,\eta)L}(\mathfrak{G}) + \varepsilon\right) \left(rL(r)\right)^{\alpha}. \tag{4.12}$$

Now combining Equation (4.9) and Equation (4.12) it follows that

$$\frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left|\mathfrak{F} \circ \mathfrak{G}\right| (\exp^{[\eta+1]} r)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(\left|\mathfrak{G}\right| \left[\exp^{[\eta]} (rL(r))^{\alpha}\right]\right)} > \frac{R\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) - \varepsilon\right)(rL(r))^{\alpha_0}}{\left(\varrho_{\mathsf{w}_i}^{(\mu,\eta)L}(\mathfrak{G}) + \varepsilon\right)(rL(r))^{\alpha}}. \tag{4.13}$$

Since $\alpha < \alpha_0$, we have from Equation (4.13)

$$\limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left|\mathfrak{F} \circ \mathfrak{G}\right| (\exp^{[\eta+1]} r)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(\left|\mathfrak{G}\right| \left[\exp^{[\eta]} (rL(r))^{\alpha}\right]\right)} = \infty.$$

Hence the first part of the theorem follows.

Again for $\alpha < \alpha_0$, it follows from the Equation (4.10) and (4.12) for all sufficiently large values of r $(\to \infty)$ that

$$\frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left|\mathfrak{F} \circ \mathfrak{G}\right| (\exp^{[\gamma+\eta+1-\delta]} r)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(\left|\mathfrak{G}\right| \left[\exp^{[\eta]} (rL(r))^{\alpha}\right]\right)} > \frac{R\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) - \varepsilon\right)(rL(r))^{\alpha_0}}{\left(\varrho_{\mathsf{w}_i}^{(\mu,\eta)L}(\mathfrak{G}) + \varepsilon\right)(rL(r))^{\alpha}}.$$

Thus from above we obtain that

$$\limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left|\mathfrak{F} \circ \mathfrak{G}\right| (\exp^{[\gamma+\eta+1-\delta]} r)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(\left|\mathfrak{G}\right| \left[\exp^{[\eta]} (rL(r))^{\alpha}\right]\right)} = \infty.$$

This implies the second part of the theorem.

By similar arguments it follows from Equation (4.11) and Equation (4.12) for sequence of positive values of r tending to infinity that

$$\frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left|\mathfrak{F} \circ \mathfrak{G}\right| (\exp^{[\eta-1]} r)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(\left|\mathfrak{G}\right| \left[\exp^{[\eta]} (rL(r))^{\alpha}\right]\right)} > \frac{R\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) - \varepsilon\right)(rL(r))^{\alpha_0}}{\left(\varrho_{\mathsf{w}_i}^{(\mu,\eta)L}(\mathfrak{G}) + \varepsilon\right)(rL(r))^{\alpha}}.$$

Therefore in view of Equation (4.8) and above we get that

$$\limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left|\mathfrak{F} \circ \mathfrak{G}\right| (\exp^{[\eta-1]} r)}{\log^{[\mu]} \widehat{|\mathsf{w}_i|} \left(\left|\mathfrak{G}\right| \left[\exp^{[\eta]} (rL(r))^{\alpha}\right]\right)} = \infty.$$

This leads to the third part of the theorem.

Hence the theorem follows. ■

Theorem 4.3.4 *Let $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ be three k-valued p-adic entire algebroidal functions defined by the irreducible Equations (i), (ii), (iii) as in Theorem 4.3.1 where* f_i, g_i *and* $\mathsf{h}_i \in \mathfrak{A}(\mathcal{K})$ *are entire functions having no common zeros* $(i = 0, 1, 2, \ldots, k)$. *Also let*

$0 < \lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) \leqslant \varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) < \infty$ *and* $\lambda^{(\delta,\eta)L}(\mathfrak{G}) > 0$ *and* $\varrho_{k_i}^{(\mu,\eta)L}(\mathfrak{G}) < \infty$. *Then for every positive constant* α,

$$(a) \lim_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\, |\mathfrak{F}\circ\mathfrak{G}| \left(\exp^{[\eta+1]} r\right)}{\log^{[\mu]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{G}| \left[\exp^{[\eta]} \left(rL(r)\right)^{\alpha}\right]\right)} = \infty \ if \quad \gamma = \delta$$

$$(b) \lim_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\, |\mathfrak{F}\circ\mathfrak{G}| \left(\exp^{[\gamma+\eta+1-\delta]} r\right)}{\log^{[\mu]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{G}| \left[\exp^{[\eta]} \left(rL(r)\right)^{\alpha}\right]\right)} = \infty \ if \quad \gamma > \delta.$$

The proof of the theorem follows from Theorem 4.3.2 and Theorem 4.3.3.

Theorem 4.3.5 *Let* $\mathfrak{F}$, $\mathfrak{G}$ *and* $\mathfrak{H}$ *be three k-valued p-adic entire algebroidal functions defined by the irreducible Equations (i), (ii), (iii) as in Theorem 4.3.1 where* f_i, g_i *and* $\mathsf{h}_i \in \mathfrak{A}(\mathcal{K})$ *are entire functions having no common zeros* $(i = 0, 1, 2, \ldots, k)$. *Also let* $0 < \lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) \leqslant \varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) < \infty$ *and* $0 < \lambda^{(\delta,\eta)L}(\mathfrak{G}) < \infty$. *Then for a positive constant* α *we have*

$$(a) \limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}| \left[\exp^{[\gamma]} \left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}\circ\mathfrak{G}| \left(\exp^{[\eta]} r\right)\right)} = \infty \ if \quad \gamma \geqslant \delta \ and \ \alpha > 1,$$

$$(b) \limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}| \left[\exp^{[\gamma]} \left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}\circ\mathfrak{G}| \left(\exp^{[\eta-1]} r\right)\right)} = \infty$$

$$if \quad \gamma \geqslant \delta \ or \ \gamma = \delta - 1 \quad with \quad \delta \neq 1 \ and \ \lambda^{(\delta,\eta)L}(\mathfrak{G}) < \alpha,$$

$$(c) \limsup_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}| \left[\exp^{[\gamma]} \left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta+\delta-\gamma-1]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}\circ\mathfrak{G}| \left(\exp^{[\eta-1]} r\right)\right)} = \infty \ if \quad \delta > \gamma+1 \ and \ \alpha > \lambda^{(\delta,\eta)L}(\mathfrak{G}).$$

Proof. By the definition of $\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})$, for any $\varepsilon\,(> 0)$ and for all sufficiently large values of r we get that

$$\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}| \left[\exp^{[\gamma]} \left(rL(r)\right)^{\alpha}\right]\right) \geqslant \left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) - \varepsilon\right) \left(rL(r)\right)^{\alpha}. \tag{4.14}$$

Also we have for a sequence of positive numbers of r tending to infinity that

$$\begin{aligned} \log^{[\delta]} |\mathfrak{G}| \left(\exp^{[\eta-1]} r\right) &\leqslant \left(\lambda^{(\delta,\eta)L}(\mathfrak{G}) + \epsilon\right) \log r \\ i.e., \log^{[\delta]} |\mathfrak{G}| \left(\exp^{[\eta-1]} r\right) &\leqslant \log r^{\left(\lambda^{(\delta,\eta)L}(\mathfrak{G})+\epsilon\right)} && (4.15) \\ i.e., \log^{[\delta-1]} |\mathfrak{G}| \left(\exp^{[\eta-1]} r\right) &\leqslant r^{\left(\lambda^{(\delta,\eta)L}(\mathfrak{G})+\epsilon\right)}. && (4.16) \end{aligned}$$

Since $\widehat{|\mathsf{h}_i|}(r)$ is an increasing function of r then for all sufficiently large values of r we get from Lemma 4.2.1 that

$$\log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F}\circ\mathfrak{G}| (r)\right) \leqslant \left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) + \epsilon\right) \log^{[\gamma]} \left(|\mathfrak{G}| (r)\right). \tag{4.17}$$

Case I First let $\gamma \geqslant \delta$. We see from the Equation (4.17) for a sequence of positive numbers of r $(\rightarrow \infty)$ that

$$\begin{aligned} \log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta]} r\right)\right) &\leqslant \left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\epsilon\right)\log^{[\delta]}\left(\left|\mathfrak{G}\right|\left(\exp^{[\eta]} r\right)\right) \\ i.e.,\ \log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta]} r\right)\right) &\leqslant \left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\epsilon\right)\left(\lambda^{(\delta,\eta)L}\left(\mathfrak{G}\right)+\epsilon\right)\log r. \qquad (4.18) \end{aligned}$$

Case II Next let $\gamma \geqslant \gamma\delta$ or $\gamma = \delta - 1$ with $\delta \neq 1$. Then it follows from Equation (4.16) and Equation (4.17) for all sufficiently large values of r that

$$\begin{aligned} \log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta-1]} r\right)\right) &\leqslant \left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\epsilon\right)\log^{[\delta-1]}\left(\left|\mathfrak{G}\right|\left(\exp^{[\eta-1]} r\right)\right) \\ i.e.,\ \log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta-1]} r\right)\right) &\leqslant \left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\epsilon\right) r^{\left(\lambda^{(\delta,\eta)L}(\mathfrak{G})+\epsilon\right)}. \qquad (4.19) \end{aligned}$$

Case III Let $\delta > \gamma + 1$. Then we get from Equation (4.15) and Equation (4.17) we get that

$$\begin{aligned} \log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta-1]} r\right)\right) &\leqslant \left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\epsilon\right)\log^{[\gamma-\delta]}\log^{[\delta]}\left(\left|\mathfrak{G}\right|\left(\exp^{[\eta-1]} r\right)\right) \\ i.e.,\ \log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta-1]} r\right)\right) &\leqslant \left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\epsilon\right)\exp^{[\delta-\gamma-1]} r^{\left(\lambda^{(\delta,\eta)L}(\mathfrak{G})+\epsilon\right)} \\ i.e.,\ \log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta-1]} r\right)\right) &\leqslant r^{\left(\lambda^{(\delta,\eta)L}(\mathfrak{G})+\epsilon\right)} + O(1). \qquad (4.20) \end{aligned}$$

When $\gamma \geqslant \delta$ and $\alpha > 1$ we see from Equation (4.14) and Equation (4.18) for all sufficiently large values of r

$$\begin{aligned} \frac{\log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\right|\left[\exp^{[\gamma]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta]} r\right)\right)} &\geqslant \frac{\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\epsilon\right)\left(rL(r)\right)^{\alpha}}{\left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\epsilon\right)\left(\lambda^{(\delta,\eta)L}\left(\mathfrak{G}\right)+\epsilon\right) r} \\ i.e.,\quad \limsup_{r\rightarrow+\infty}\frac{\log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\right|\left[\exp^{[\gamma]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta]} r\right)\right)} &= \infty. \end{aligned}$$

This proves the first part of the theorem.

Again if $\gamma \geqslant \delta$ or $\gamma = \delta(\neq 1) - 1$ it follows from Equation (4.14) and Equation (4.19) we obtain that

$$\frac{\log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\right|\left[\exp^{[\gamma]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta]}\left|\widehat{\mathsf{h}_i}\right|\left(\left|\mathfrak{F}\circ\mathfrak{G}\right|\left(\exp^{[\eta-1]} r\right)\right)} \geqslant \frac{\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\epsilon\right)\left(rL(r)\right)^{\alpha}}{\left(\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})+\epsilon\right) r^{\left(\lambda^{(\delta,\eta)L}(\mathfrak{G})+\epsilon\right)}}. \qquad (4.21)$$

As $\alpha > \lambda^{(\delta,\eta)L}\left(\mathfrak{G}\right)$ we consider an arbitrary $\epsilon > 0$ in such a manner that

$$\lambda^{(\delta,\eta)L}\left(\mathfrak{G}\right)+\epsilon < \alpha. \qquad (4.22)$$

Combining Equation (4.21) and Equation (4.22) we get that

$$\limsup_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]}r\right)\right)}=\infty.$$

Hence the second part of the theorem follows.

When $\delta>\gamma+1$, we obtain from Equation (4.14) and Equation (4.20) for all sufficiently large values of r that

$$\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta+\delta-\gamma-1]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]}r\right)\right)}\geqslant\frac{\left(\lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})-\epsilon\right)\left(rL(r)\right)^{\alpha}}{r^{\left(\lambda^{(\delta,\eta)L}(\mathfrak{G})+\epsilon\right)}+O(1)}. \tag{4.23}$$

Combining Equation (4.22) and Equation (4.23) we have

$$\limsup_{r\to+\infty}\frac{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta+\delta-\gamma-1]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]}r\right)\right)}=\infty.$$

Hence, it completes the proof of the third part of the theorem.

This establishes the theorem. ■

Theorem 4.3.6 *Let four k-valued p-adic entire algebroidal functions $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ and W where f_i, g_i, h_i and $\mathsf{w}_i\in\mathfrak{A}(\mathcal{K})$ be entire functions having no common zeros ($i=0,1,2,\ldots,k$). Also let $\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})$, $\lambda^{(\delta,\eta)L}(\mathfrak{G})$ and $\lambda_{\mathsf{w}_i}^{(\mu,\eta)L}(\mathfrak{G})$ be finite, where β, γ, δ, η, μ are positive integers. Then for every positive constant α,*

$$(a)\ \limsup_{r\to+\infty}\frac{\log^{[\mu]}\widehat{|\mathsf{w}_i|}\left(|\mathfrak{G}|\left[\exp^{[\eta]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta]}r\right)\right)}=\infty\ if\quad\gamma\geqslant\delta\ and\ \alpha>1,$$

$$(b)\ \limsup_{r\to+\infty}\frac{\log^{[\mu]}\widehat{|\mathsf{w}_i|}\left(|\mathfrak{G}|\left[\exp^{[\eta]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]}r\right)\right)}\quad=\quad\infty$$
$$if\quad\gamma\geqslant\delta\ or\ \gamma=\ \delta-1\quad with\ \delta\neq1\ and\ \lambda^{(\delta,\eta)L}(\mathfrak{G})<\alpha,$$

$$(c)\ \limsup_{r\to+\infty}\frac{\log^{[\mu]}\widehat{|\mathsf{w}_i|}\left(|\mathfrak{G}|\left[\exp^{[\eta]}\left(rL(r)\right)^{\alpha}\right]\right)}{\log^{[\beta+\delta-\gamma-1]}\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]}r\right)\right)}=\infty\ if\quad\delta>\gamma+1\ and\ \alpha>\lambda^{(\delta,\eta)L}(\mathfrak{G}).$$

The proof of this theorem is omitted as it follows from Theorem 4.3.5 and Theorem 4.3.4.

Theorem 4.3.7 *Let us suppose that $\mathfrak{F}$,$\mathfrak{G}$ and $\mathfrak{H}$ be three k-valued p-adic entire algebroidal functions as defined in Theorem 4.3.1 by the Equations (i), (ii) and (iii) where f_i , g_i and $\mathsf{h}_i\in\mathfrak{A}(\mathcal{K})$ are entire functions having no common zeros ($i=0,1,2,\ldots,k$)*

with finite relative (β, γ)-th L-order and finite relative (β, γ)-th L-lower order of $\mathfrak{F}$ with respect to another entire function h_i . Also let $\varrho^{(\delta,\eta)L}(\mathfrak{G}) < \infty$. Then

$$(a) \lim_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}(rL(r))^{\alpha}\right]\right)}{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta]} r\right)\right)} = \infty \; if \quad \gamma \geqslant \delta \; and \; \alpha > 1,$$

$$(b) \lim_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}(rL(r))^{\alpha}\right]\right)}{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]} r\right)\right)} = \infty$$

$$if \quad \gamma \geqslant \delta \; or \; \gamma = \delta - 1 \quad with \; \delta \neq 1 \; and \; \varrho^{(\delta,\eta)L}(\mathfrak{G}) < \alpha,$$

$$(c) \lim_{r\to+\infty} \frac{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}(rL(r))^{\alpha}\right]\right)}{\log^{[\beta+\delta-\gamma-1]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]} r\right)\right)} = \infty \; if \quad \delta > \gamma + 1 \; and \; \varrho^{(\delta,\eta)L}(\mathfrak{G}) < \alpha.$$

Theorem 4.3.8 *Let us consider four k-valued p-adic entire algebroidal functions $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ and W where f_i, g_i , h_i and $\mathsf{w}_i \in \mathfrak{A}(\mathcal{K})$ be entire functions having no common zeros $(i = 0, 1, 2, \ldots, k)$. Also let $\varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F})$, $\varrho^{(\delta,\eta)L}(\mathfrak{G})$ and $\lambda_{\mathsf{w}_i}^{(\mu,\eta)L}(\mathfrak{G})$ be finite, where β, γ, δ, η, μ are positive integers. Then for every positive constant α,*

$$(a) \lim_{r\to+\infty} \frac{\log^{[\mu]} \widehat{|\mathsf{w}_i|}\left(|\mathfrak{G}|\left[\exp^{[\eta]}(rL(r))^{\alpha}\right]\right)}{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta]} r\right)\right)} = \infty \; if \quad \gamma \geqslant \delta \; and \; \alpha > 1,$$

$$(b) \lim_{r\to+\infty} \frac{\log^{[\mu]} \widehat{|\mathsf{w}_i|}\left(|\mathfrak{G}|\left[\exp^{[\eta]}(rL(r))^{\alpha}\right]\right)}{\log^{[\beta]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]} r\right)\right)} = \infty$$

$$if \quad \gamma \geqslant \delta \; or \; \gamma = \delta - 1 \quad with \; \delta \neq 1 \; and \; \varrho^{(\delta,\eta)L}(\mathfrak{G}) < \alpha,$$

$$(c) \lim_{r\to+\infty} \frac{\log^{[\mu]} \widehat{|\mathsf{w}_i|}\left(|\mathfrak{G}|\left[\exp^{[\eta]}(rL(r))^{\alpha}\right]\right)}{\log^{[\beta+\delta-\gamma-1]} \widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]} r\right)\right)} = \infty \; if \quad \delta > \gamma + 1 \; and \; \alpha > \varrho^{(\delta,\eta)L}(\mathfrak{G}).$$

The proofs of Theorem 4.3.7 and Theorem 4.3.8 are omitted as those can be carried out in the line of Theorem 4.3.5 and Theorem 4.3.6 respectively.

In view of Theorem 4.3.1 and Theorem 4.3.5 we have an application which is as follows.

Theorem 4.3.9 *Let $\mathfrak{F}$,$\mathfrak{G}$ and $\mathfrak{H}$ be three k-valued p-adic entire algebroidal functions defined by the irreducible Equations (i), (ii), (iii) as in Theorem 4.3.1 where f_i, g_i and $\mathsf{h}_i \in \mathfrak{A}(\mathcal{K})$ are entire functions having no common zeros $(i = 0, 1, 2, \ldots, k)$. Also let $0 < \lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) \leqslant \varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) < \infty$ and $0 < \lambda^{(\delta,\eta)L}(\mathfrak{G}) < \alpha < \varrho^{(\delta,\eta)L}(\mathfrak{G})$. Then for $\delta \neq 1$ and $\gamma = \delta - 1$,*

$$\liminf_{r\to+\infty} \frac{\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]} r\right)}{\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}(rL(r))^{\alpha}\right]\right)} \leqslant 1 \leqslant \limsup_{r\to+\infty} \frac{\widehat{|\mathsf{h}_i|}\,|\mathfrak{F}\circ\mathfrak{G}|\left(\exp^{[\eta-1]} r\right)}{\widehat{|\mathsf{h}_i|}\left(|\mathfrak{F}|\left[\exp^{[\gamma]}(rL(r))^{\alpha}\right]\right)}$$

Proof. By Theorem 4.3.1, for a sequence of positive real numbers of r tends to infinity we obtain that

$$\begin{aligned} \log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta-1]} r\right)\right) &\geqslant \log^{[\beta]} \widehat{|\mathsf{h}_i|}\, |\mathfrak{F}| \left[\exp^{[\gamma]} \left(rL(r)\right)^{\alpha}\right] \\ i.e., \limsup_{r\to+\infty} \frac{\widehat{|\mathsf{h}_i|} \left(|\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta-1]} r\right)\right)}{\widehat{|\mathsf{h}_i|}\, |\mathfrak{F}| \left[\exp^{[\gamma]} \left(rL(r)\right)^{\alpha}\right]} &\geqslant 1. \end{aligned} \tag{4.24}$$

Again in view of Theorem 4.3.5 for all sufficiently large values of r we get that

$$\begin{aligned} \log^{[\beta]} \widehat{|\mathsf{h}_i|}\, |\mathfrak{F}| \left[\exp^{[\gamma]} \left(rL(r)\right)^{\alpha}\right] &\geqslant \log^{[\beta]} \widehat{|\mathsf{h}_i|} \left(|\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta-1]} r\right)\right) \\ i.e., \liminf_{r\to+\infty} \frac{\widehat{|\mathsf{h}_i|} \left(|\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta-1]} r\right)\right)}{\widehat{|\mathsf{h}_i|}\, |\mathfrak{F}| \left[\exp^{[\gamma]} \left(rL(r)\right)^{\alpha}\right]} &\leqslant 1. \end{aligned} \tag{4.25}$$

Combining Equation (4.24) and Equation (4.25) we get the required result.

This completes the proof. ∎

Theorem 4.3.10 *Let us consider four k-valued p-adic entire algebroidal functions $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ and W where f_i, g_i, h_i and $\mathsf{w}_i \in \mathfrak{A}(\mathcal{K})$ are entire functions having no common zeros $(i = 0, 1, 2, \ldots, k)$. Also let $0 < \lambda_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) \leqslant \varrho_{\mathsf{h}_i}^{(\beta,\gamma)L}(\mathfrak{F}) < \infty$, $0 < \lambda_{\mathsf{w}_i}^{(\beta,\eta)L}(\mathfrak{G}) \leqslant \varrho_{k_i}^{(\beta,\eta)L}(\mathfrak{G}) < \infty$ and $0 < \lambda^{(\delta,\eta)L}(\mathfrak{G}) < \alpha < \varrho^{(\delta,\eta)L}(\mathfrak{G})$. Then we obtain for $\delta \neq 1$ and $\gamma = \delta - 1$ that*

$$\liminf_{r\to+\infty} \frac{\widehat{|\mathsf{h}_i|}\, |\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta-1]} r\right)}{\widehat{|\mathsf{w}_i|} \left(|\mathfrak{G}| \left[\exp^{[\eta]} \left(rL(r)\right)^{\alpha}\right]\right)} \leqslant 1 \leqslant \limsup_{r\to+\infty} \frac{\widehat{|\mathsf{h}_i|}\, |\mathfrak{F} \circ \mathfrak{G}| \left(\exp^{[\eta-1]} r\right)}{\widehat{|\mathsf{w}_i|} \left(|\mathfrak{G}| \left[\exp^{[\eta]} \left(rL(r)\right)^{\alpha}\right]\right)}.$$

The proof of this theorem is omitted as it can be carried out by the similar arguments of Theorem 4.3.9.

4.4 Future Prospects.

The main aim of this chapter is actually to extend and to modify some relative growth properties of entire algebroidal functions where as the coefficients are all entire functions belonging to $\mathfrak{A}(\mathcal{K})$ from the view point of p -adic analysis, where p is any prime. We also modify the results in the context of slowly changing function for more generalization. Further the concept of relative index-pair of an entire function belonging to $\mathfrak{A}(\mathcal{K})$ have been deeply studied here with more extension.

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The works of this chapter have been published in **Journal of Xi'an University of Architecture & Technology (JXAT), Vol. XIII, No. I (2021), pp. 412-425, ISSN: 1006-7930.**

CHAPTER FIVE

ON THE GENERALIZATION OF ENSTRÖM- KAKEYA THEOREM FOR ENTIRE FUNCTIONS

Chapter 5

On the generalization of Enstr$\ddot{o}$m-Kakeya theorem for entire functions

5.1 Introduction.

The classical Enström-Kakeya theorem {cf.[45]} states that if $\mathfrak{p}(\mathfrak{z}) = \sum_{j=0}^{n} a_j \mathfrak{z}^j$ is a polynomial of degree n with real coefficients satisfying $0 \leqslant a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$, then all the zeros of $\mathfrak{p}(\mathfrak{z})$ lie in the unit disk $|\mathfrak{z}| \leqslant 1$ in the complex plane $\mathbb{C}$. Many results on generalization of Enstr$\ddot{o}$m-Kakeya theorem by putting various conditions on the coefficients of the polynomials exist {cf.[1], [23], [42], [45], [54] & [55]}. Aziz and Zargar {cf.[1]} first generalized Ensröm-Kakeya theorem in a different direction by considering the even indexed and odd indexed coefficients separately {cf.[1]} and consequently a lot of improvements of their results for polynomials and analytic functions can be found in the literature {cf. [45], [56] & [69]}.

The aim of this chapter is to derive zero free region for some entire functions of non-zero finite order with lacunary type power series expansion putting various conditions on the coefficients.

5.2 Lemma.

In this section, we present a lemma which will be needed in the sequel.

Lemma 5.2.1 *[44] Let $\{\mathsf{f}_n(\mathfrak{z})\}$, be a sequence of functions that are analytic in a region D and that converge uniformly to a function $\mathsf{f}(\mathfrak{z})$ in every closed sub-region of D. Let $\mathfrak{z}_0$ be an interior point of D. If $\mathfrak{z}_0$ is a limit point of the zeros of $\mathsf{f}_n(\mathfrak{z})$, then $\mathfrak{z}_0$ is a zero of $\mathsf{f}(\mathfrak{z})$. Conversely, if $\mathfrak{z}_0$ is an m-fold zero of $\mathsf{f}(\mathfrak{z})$, every sufficiently small neighborhood of $\mathfrak{z}_0$ contains exactly m zeros (counted with their multiplicities) of each f_n with $n > N$ for a sufficiently large integer N.*

Remark 5.2.1 *Lemma 5.2.1 is known as Hurwitz theorem in $\mathbb{C}$.*

5.3 Main Results.

In this section, we highlight the key findings of this chapter.

Theorem 5.3.1 *Let $\mathfrak{p}(\mathfrak{z}) = a_0 + a_{n_1}\mathfrak{z}^{n_1} + a_{n_2}\mathfrak{z}^{n_2} + \dots$ be an entire function of non zero finite order ϱ with $a_0 \neq 0$ and $n_1, n_2, \dots$ be positive integers such that $1 \leqslant n_1 < n_2 < \dots$ Also, let*

$$\varrho^{n_1}|a_{n_1}| \geqslant \varrho^{n_2}|a_{n_2}| \geqslant \dots$$

Then no zeros of $\mathfrak{p}(\mathfrak{z})$ lie in

$$|\mathfrak{z}| < \frac{\varrho|a_0|}{|a_0| + \varrho^{n_1}|a_{n_1}|} \ .$$

Proof. Let $\mathfrak{p}_k(\mathfrak{z}) = a_0 + a_{n_1}\mathfrak{z}^{n_1} + a_{n_2}\mathfrak{z}^{n_2} + \dots + a_{n_k}\mathfrak{z}^{n_k}$ and $\mathcal{P}(\mathfrak{z}) = \mathfrak{z}^{n_k}\mathfrak{p}_k(\frac{\varrho}{\mathfrak{z}})$. Then

$$\begin{aligned} |\mathcal{P}(\mathfrak{z})| =& |a_0\mathfrak{z}^{n_k} + a_{n_1}\varrho^{n_1}\mathfrak{z}^{n_k-n_1} + a_{n_2}\varrho^{n_2}\mathfrak{z}^{n_k-n_2} + \dots + a_{n_k}\varrho^{n_k}| \\ \text{i.e., } |\mathcal{P}(\mathfrak{z})| \geqslant& |a_0||\mathfrak{z}|^{n_k} - |a_{n_1}\varrho^{n_1}\mathfrak{z}^{n_k-n_1} + a_{n_2}\varrho^{n_2}\mathfrak{z}^{n_k-n_2} + \dots + a_{n_k}\varrho^{n_k}| \ . \end{aligned} \tag{5.1}$$

Now for $|\mathfrak{z}| = R(> 1)$, it follows that

$$\begin{aligned} &|a_{n_1}\varrho^{n_1}\mathfrak{z}^{n_k-n_1} + a_{n_2}\varrho^{n_2}\mathfrak{z}^{n_k-n_2} + \dots + a_{n_k}\varrho^{n_k}| \\ \leqslant& |a_{n_1}|\varrho^{n_1}R^{n_k}\sum_{j=1}^{\infty}\frac{1}{R^j} \\ =& |a_{n_1}|\varrho^{n_1}R^{n_k}\frac{1}{R-1} \ . \end{aligned}$$

Hence we get from (5.1) for $|\mathfrak{z}| = R(> 1)$ that

$$|\mathcal{P}(\mathfrak{z})| \geqslant |a_0|R^{n_k} - |a_{n_1}|\varrho^{n_1}R^{n_k}\frac{1}{R-1} > 0 \text{ if } R > \frac{|a_0| + |a_{n_1}|\varrho^{n_1}}{|a_0|}$$

$$\text{i.e., } |\mathcal{P}(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| > \frac{|a_0| + |a_{n_1}|\varrho^{n_1}}{|a_0|} \ .$$

Consequently,

$$|\mathfrak{p}_k(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| < \frac{|a_0|\varrho}{|a_0| + |a_{n_1}|\varrho^{n_1}} \ .$$

Hence by Lemma 5.2.1, it follows that

$$|\mathfrak{p}(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| < \frac{|a_0|\varrho}{|a_0| + |a_{n_1}|\varrho^{n_1}} \ .$$

This proves the theorem. ■

Remark 5.3.1 *To validate Theorem 5.3.1 following example is considered which is justified by relevant figure.*

Example 5.3.1 *Let* $\mathfrak{p}(\mathfrak{z}) = \cos\mathfrak{z} + 1$.

Now the Taylor's series expansion of $\mathfrak{p}(\mathfrak{z})$ *is*

$$\mathfrak{p}(\mathfrak{z}) = 2 - \frac{\mathfrak{z}^2}{2!} + \frac{\mathfrak{z}^4}{4!} - \dots$$

Here, $\varrho = 1, a_0 = 2$ *and* $a_{n_1} = -\frac{1}{2}$.
Also, it follows that

$$\varrho^{n_1}|a_{n_1}| \geqslant \varrho^{n_2}|a_{n_2}| \geqslant \dots$$

Hence by Theorem 5.3.1, no zeros of $\mathfrak{p}(\mathfrak{z})$ *lie in* $|\mathfrak{z}| < 0.8$.

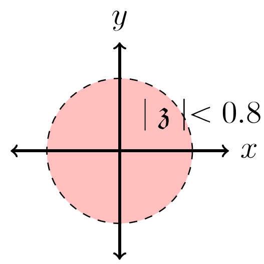

Figure 5.1: Zero free region of $\mathfrak{p}(\mathfrak{z}) = \cos\mathfrak{z} + 1$

Theorem 5.3.2 *Let* $\mathfrak{p}(\mathfrak{z}) = a_0 + a_{n_1}\mathfrak{z}^{n_1} + \cdots + a_{n_l}\mathfrak{z}^{n_l} + a_{n_m}\mathfrak{z}^{n_m} + \dots$ *be an entire function of finite order* $\varrho(> 0)$ *with* $a_0 \neq 0$ *and* $n_1, n_2, \dots, n_l, n_m, \dots$ *be positive integers such that* $1 \leqslant n_1 < \cdots < n_l < n_m < \dots$ *Also let*

$$\varrho^{n_l}|a_{n_l}| \geqslant \varrho^{n_m}|a_{n_m}| \geqslant \dots$$

Then no zeros of $\mathfrak{p}(\mathfrak{z})$ *lie in*

$$|\mathfrak{z}| < \frac{|a_0|\varrho}{|a_0| + M}$$

where $M = \max\{\varrho^{n_1}|a_{n_1}|, \varrho^{n_2}|a_{n_2}|, \dots, \varrho^{n_l}|a_{n_l}|\}$.

Proof. Let $\mathfrak{p}_k(\mathfrak{z}) = a_0 + a_{n_1}\mathfrak{z}^{n_1} + \cdots + a_{n_l}\mathfrak{z}^{n_l} + a_{n_m}\mathfrak{z}^{n_m} + \cdots + a_{n_k}\mathfrak{z}^{n_k}$ and $\mathcal{P}(\mathfrak{z}) = \mathfrak{z}^{n_k}\mathfrak{p}_k(\frac{\varrho}{\mathfrak{z}})$
i.e, $\mathcal{P}(\mathfrak{z}) = a_0\mathfrak{z}^{n_k} + a_{n_1}\varrho^{n_1}\mathfrak{z}^{n_k-n_1} + \cdots + a_{n_l}\varrho^{n_l}\mathfrak{z}^{n_k-n_l} + a_{n_m}\varrho^{n_m}\mathfrak{z}^{n_k-n_m} + \cdots + a_{n_k}\varrho^{n_k}$.

Now for $|\mathfrak{z}| = R(> 1)$, we get that

$$\begin{aligned}
&|a_{n_1}\varrho^{n_1}\mathfrak{z}^{n_k-n_1} + a_{n_2}\varrho^{n_2}\mathfrak{z}^{n_k-n_2} + \cdots + a_{n_l}\varrho^{n_l}\mathfrak{z}^{n_k-n_l} + a_{n_m}\varrho^{n_m}\mathfrak{z}^{n_k-n_m} + \cdots + a_{n_k}\varrho^{n_k}| \\
&\leqslant MR^{n_k}\sum_{j=1}^{\infty}\frac{1}{R^j}, \text{ where } M = \max\{\varrho^{n_1}|a_{n_1}|, \varrho^{n_2}|a_{n_2}|, \dots, \varrho^{n_l}|a_{n_l}|\} \\
&= MR^{n_k}\frac{1}{R-1} .
\end{aligned}$$

Hence for $|\mathfrak{z}| = R(> 1)$, it follows that

$$|\mathcal{P}(\mathfrak{z})| \geqslant |a_0|R^{n_k} - MR^{n_k}\frac{1}{R-1} > 0 \text{ if } R > \frac{|a_0| + M}{|a_0|} \; .$$

$$\text{i.e., } |\mathcal{P}(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| > \frac{|a_0| + M}{|a_0|} \; .$$

Consequently,

$$|\mathfrak{p}_k(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| < \frac{|a_0|\varrho}{|a_0| + M} \; .$$

Finally by Lemma 5.2.1, we have

$$|\mathfrak{p}(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| < \frac{|a_0|\varrho}{|a_0| + M} \; .$$

This completes the proof. ■

Remark 5.3.2 *To validate Theorem 5.3.2 following example is considered which is justified by relevant figure.*

Example 5.3.2 *Let* $\mathfrak{p}(\mathfrak{z}) = \cos\mathfrak{z} + 1 + \mathfrak{z} + \mathfrak{z}^3$.

Then the Taylor's series expansion of $\mathfrak{p}(\mathfrak{z})$ *is*

$$\mathfrak{p}(\mathfrak{z}) = 2 + \mathfrak{z} - \frac{\mathfrak{z}^2}{2!} + \mathfrak{z}^3 + \frac{\mathfrak{z}^4}{4!} - \ldots$$

Here, $\varrho = 1, a_0 = 2, a_{n_1} = 1, a_{n_2} = -\frac{1}{2}, a_{n_3} = 1$
Also, it follows that

$$\varrho^{n_3}|a_{n_3}| \geqslant \varrho^{n_4}|a_{n_4}| \geqslant \ldots$$

Hence by Theorem 5.3.2, no zeros of $\mathfrak{p}(\mathfrak{z})$ *lie in* $|\mathfrak{z}| < 0.67$ *.*

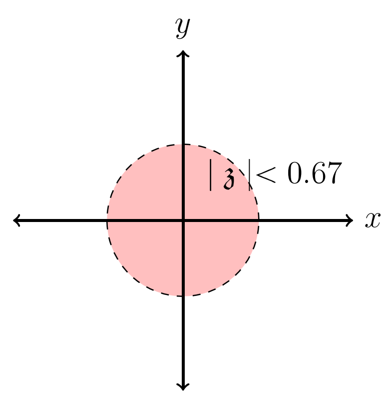

Figure 5.2: Zero free region of $\mathfrak{p}(\mathfrak{z}) = \cos\mathfrak{z} + 1 + \mathfrak{z} + \mathfrak{z}^3$

Theorem 5.3.3 *Let $\mathfrak{p}(\mathfrak{z}) = a_0 + a_1\mathfrak{z} + a_2\mathfrak{z}^2 + \dots$ be an entire function of non zero finite order ϱ with $a_0 \neq 0$. Also, let*

$$|a_0| \geqslant \varrho^2|a_2| \geqslant \varrho^4|a_4| \geqslant \dots \quad \text{and} \quad \varrho|a_1| \geqslant \varrho^3|a_3| \geqslant \varrho^5|a_5| \geqslant \dots$$

Then $\mathfrak{p}(\mathfrak{z})$ does not vanish in

$$|\mathfrak{z}| < \frac{\varrho|a_0|}{|a_0| + 2M}$$

where $M = \max\{|a_0|, |a_1|\varrho\}$.

Proof. Let $f_n(\mathfrak{z}) = a_0 + a_1\mathfrak{z} + a_2\mathfrak{z}^2 + \cdots + a_n\mathfrak{z}^n$ and $\mathcal{P}(\mathfrak{z}) = \mathfrak{z}^n f_n(\frac{\varrho}{\mathfrak{z}})$.
Again, let

$$\begin{aligned}
&\mathcal{Q}(\mathfrak{z}) = (\mathfrak{z}^2 - 1)\mathcal{P}(\mathfrak{z}) \\
\text{i.e, } &\mathcal{Q}(\mathfrak{z}) = a_0\mathfrak{z}^{n+2} + a_1\varrho\mathfrak{z}^{n+1} + (a_2\varrho^2 - a_0)\mathfrak{z}^n + \cdots - a_{n-1}\varrho^{n-1}\mathfrak{z} - a_n\varrho^n \\
\text{i.e, } &\mathcal{Q}(\mathfrak{z}) = a_0\mathfrak{z}^{n+2} + P(\mathfrak{z}) \ . \qquad (5.2)
\end{aligned}$$

Now for $|\mathfrak{z}| = R(> 1)$, we get that

$$\begin{aligned}
|P(\mathfrak{z})| &= |a_1\varrho\mathfrak{z}^{n+1} + (a_2\varrho^2 - a_0)\mathfrak{z}^n + (a_3\varrho^3 - a_1\varrho)\mathfrak{z}^{n-1} + \cdots - a_{n-1}\varrho^{n-1}\mathfrak{z} - a_n\varrho^n| \\
&\leqslant 2MR^{n+2}\sum_{k=1}^{\infty}\frac{1}{R^k}, \text{ where } M = \max\{|a_0|, |a_1|\varrho\} \\
&= 2MR^{n+2}\frac{1}{R-1} \ .
\end{aligned}$$

Hence for $|\mathfrak{z}| = R(> 1)$, it follows from (5.2) that

$$|\mathcal{Q}(\mathfrak{z})| \geqslant |a_0|R^{n+2} - 2MR^{n+2}.\frac{1}{R-1} > 0 \text{ if } R > \frac{|a_0| + 2M}{|a_0|}$$

$$\text{i.e, } |\mathcal{Q}(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| > \frac{|a_0| + 2M}{|a_0|} \ .$$

Therefore,

$$\begin{aligned}
&|\mathcal{P}(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| > \frac{|a_0| + 2M}{|a_0|} \\
\text{i.e, } &|f_n(\frac{\varrho}{\mathfrak{z}})| > 0 \text{ if } |\mathfrak{z}| > \frac{|a_0| + 2M}{|a_0|} \\
\text{i.e, } &|f_n(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| < \frac{|a_0|\varrho}{|a_0| + 2M} \ .
\end{aligned}$$

Thus, it follows by Lemma 5.2.1 that

$$|\mathfrak{p}(\mathfrak{z})| > 0 \text{ if } |\mathfrak{z}| < \frac{|a_0|\varrho}{|a_0| + 2M} \ .$$

This completes the proof of the theorem. ∎

Remark 5.3.3 *To validate Theorem 5.3.3 following example is considered which is justified by relevant figure.*

Example 5.3.3 *Let* $\mathfrak{p}(\mathfrak{z}) = \sin 2z + \cos \mathfrak{z}$.
Then the Taylor's series expansion of $\mathfrak{p}(\mathfrak{z})$ *is*

$$\mathfrak{p}(\mathfrak{z}) = 1 + 2z - \frac{\mathfrak{z}^2}{2} - \frac{4}{3}\mathfrak{z}^3 + \frac{\mathfrak{z}^4}{24} + \frac{4}{15}\mathfrak{z}^5 - \dots$$

Here, $\varrho = 1, a_0 = 1$ *and* $a_1 = 2$. *Also, it follows that*

$$|a_0| \geqslant \varrho^2|a_2| \geqslant \varrho^4|a_4| \geqslant \dots \quad \textit{and} \quad \varrho|a_1| \geqslant \varrho^3|a_3| \geqslant \varrho^5|a_5| \geqslant \dots$$

Hence by Theorem 5.3.3, $\mathfrak{p}(\mathfrak{z})$ *does not vanish in* $|\mathfrak{z}| < 0.2$.

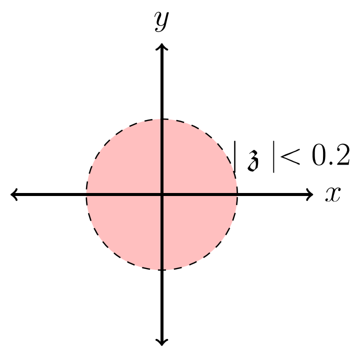

Figure 5.3: Zero free region of $\mathfrak{p}(\mathfrak{z}) = \sin 2z + \cos \mathfrak{z}$

Theorem 5.3.4 *Let* $\mathfrak{p}(\mathfrak{z}) = a_0 + a_1\mathfrak{z} + a_2\mathfrak{z}^2 + \dots$ *be an entire function of finite order* $\varrho(> 0)$ *with* $a_0 \neq 0$. *Also, let for some positive integer* l

$$\varrho^{2l}|a_{2l}| \geqslant \varrho^{2l+2}|a_{2l+2}| \geqslant \varrho^{2l+4}|a_{2l+4}| \geqslant \dots$$
$$\textit{and } \varrho^{2l+1}|a_{2l+1}| \geqslant \varrho^{2l+3}|a_{2l+3}| \geqslant \varrho^{2l+5}|a_{2l+5}| \geqslant \dots$$

Then $\mathfrak{p}(\mathfrak{z})$ *does not vanish in*

$$|\mathfrak{z}| < \frac{|a_0|\varrho}{|a_0| + 2M}$$

where $M = \max\{|a_0|, \varrho|a_1|, \dots, \varrho^{2l}|a_{2l}|, \varrho^{2l+1}|a_{2l+1}|\}$.

The proof is similar as of Theorem 5.3.3 and therefore it is omitted.

Remark 5.3.4 *The following example with a related figure justifies the validity of Theorem 5.3.4.*

Example 5.3.4 *Let* $\mathfrak{p}(\mathfrak{z}) = \sin 2z + \cos \mathfrak{z} + 1 - 2z^2 - \mathfrak{z}^3$.

Now the Taylor's series expansion of $\mathfrak{p}(\mathfrak{z})$ *is*

$$\mathfrak{p}(\mathfrak{z}) = 2 + 2z - \frac{5}{2}\mathfrak{z}^2 - \frac{7}{3}\mathfrak{z}^3 + \frac{\mathfrak{z}^4}{24} + \frac{4}{15}\mathfrak{z}^5 - \dots$$

Here, $\varrho = 1, a_0 = 2, a_1 = 2, a_2 = -\frac{5}{2}$ *and* $a_3 = -\frac{7}{3}$. *Again, it follows that*

$$\varrho^2|a_2| \geqslant \varrho^4|a_4| \geqslant \varrho^6|a_6| \geqslant \dots \quad \text{and} \quad \varrho^3|a_3| \geqslant \varrho^5|a_5| \geqslant \varrho^7|a_7| \geqslant \dots$$

Hence by Theorem 5.3.4, $\mathfrak{p}(\mathfrak{z})$ *does not vanish in* $|\mathfrak{z}| < 0.29$.

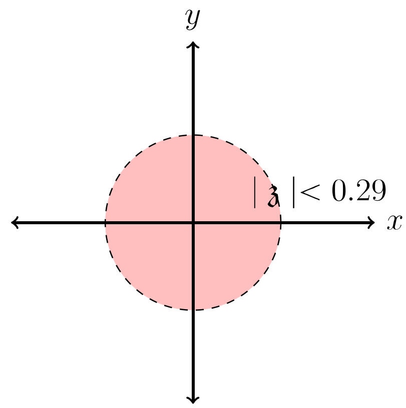

Figure 5.4: Zero free region of $\mathfrak{p}(\mathfrak{z}) = \sin 2z + \cos \mathfrak{z} + 1 - 2z^2 - \mathfrak{z}^3$

5.4 Future Prospects.

In the line of the finds accomplished in this chapter, one may try to generalization the Enström-Kakeya theorem for entire functions with infinite order.

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The works of this chapter have been published in **Journal of Xi'an University of Architecture & Technology (JXAT), Vol. XII, No. IX (2020), pp. 693-700, ISSN: 1006-7930.**

CHAPTER SIX

ON THE DISTRIBUTION OF ZEROS OF BICOMPLEX VALUED ENTIRE FUNCTIONS IN A CERTAIN DOMAIN

Chapter 6

On the distribution of zeros of bicomplex valued entire functions in a certain domain

6.1 Introduction.

Bicomplex numbers which are the commutative generalization of complex numbers were first introduced by Segre {cf.[87]}. Standard definitions, notations and many more properties of bicomplex numbers are available in {cf. [72] & [73]}. A bicomplex entire function $\mathcal{F}(\mathfrak{Z})$ is also represented by an everywhere convergent power series as $\mathcal{F}(\mathfrak{Z}) = \sum_{j=0}^{\infty} ß_j \mathfrak{Z}^j$, where $ß_j$'s and $\mathfrak{Z}$ are bicomplex numbers. Thus, bicomplex entire functions can be thought of the natural generalization of bicomplex polynomials.

The aim of the chapter is to establish some results concerning the distribution of zeros of bicomplex entire functions in a certain domain.

6.2 Lemmas.

The following lemmas are presented here which will be required to prove our main results.

Lemma 6.2.1 *[2] If* $\mathsf{f}(\mathfrak{z})$ *is holomorphic in* $\mid \mathfrak{z} \mid \leqslant R$ *in* $\mathbb{C}, f(0) = 0, f'(0) = b,$ *and* $\mid \mathsf{f}(\mathfrak{z}) \mid \leqslant M$ *for* $\mid \mathfrak{z} \mid = R$*, then for* $\mid \mathfrak{z} \mid \leqslant R$*,*

$$\mid \mathsf{f}(\mathfrak{z}) \mid \leqslant \frac{M \mid \mathfrak{z} \mid}{R^2} . \frac{M \mid \mathfrak{z} \mid + R^2 \mid b \mid}{M + \mid b \mid\mid \mathfrak{z} \mid}.$$

The following lemma is due to Schwarz in $\mathbb{C}$.

Lemma 6.2.2 *[22] If* $\mathsf{g}(\mathfrak{z})$ *is holomorphic in* $\mid \mathfrak{z} \mid \leqslant R$ *in* $\mathbb{C}, g(0) = 0$ *and* $\mid \mathsf{g}(\mathfrak{z}) \mid \leqslant M$ *for* $\mid \mathfrak{z} \mid = R$*, then*

$$|\mathsf{g}(\mathfrak{z})| \leqslant \frac{M|\mathfrak{z}|}{R}.$$

Lemma 6.2.3 *[49] Let* $\mathsf{f}(\mathfrak{z})$ *be holomorphic for* $|\ \mathfrak{z}\ | < R$ *in* $\mathbb{C}$. *Suppose* $\mathfrak{f}(0) \neq 0$ *and let* $\{r_n\}$ *be a non decreasing sequence of the moduli of zeros of* $\mathsf{f}(\mathfrak{z})$ *in* $|\ \mathfrak{z}\ | < R$. *If* $r_n \leqslant r \leqslant r_{n+1}$, *then*

$$\log \frac{r^n \mid \mathsf{f}(0) \mid}{r_1 r_2 \ldots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log \mid \mathsf{f}(re^{i\theta}) \mid d\theta,$$

where a zero of order m *is counted* m *times.*

Remark 6.2.1 *Lemma 6.2.3 is known as Jensen's Theorem in* $\mathbb{C}$.

Lemma 6.2.4 *[76] Let* $\mathfrak{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 := \{\xi_1 e_1 + \xi_2 e_2 : \xi_1 \in \mathcal{X}_1, \xi_2 \in \mathcal{X}_2\}$ *be a domain in* $\mathbb{BC}$. *A bicomplex function* $\mathcal{F} = \mathcal{L}_1 e_1 + \mathcal{L}_2 e_2 : \mathfrak{X} \to \mathbb{BC}$ *is holomorphic if and only if both the component function* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are holomorphic in* $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *respectively .*

Lemma 6.2.5 *[76] Let* $\mathcal{F}$ *be a bicomplex holomorphic function defined in a domain* $\mathfrak{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 := \{\xi_1 e_1 + \xi_2 e_2 : \xi_1 \in \mathcal{X}_1, \xi_2 \in \mathcal{X}_2\}$ *such that* $\mathcal{F}(\mathfrak{Z}) = \mathcal{L}_1(\xi_1) e_1 + \mathcal{L}_2(\xi_2) e_2$, *for all* $\mathfrak{Z} = \xi_1 e_1 + \xi_2 e_2 \in \mathfrak{X}$. *Then,* $\mathcal{F}(\mathfrak{Z})$ *has zero in* $\mathfrak{X}$ *if and only if* $\mathcal{L}_1(\xi_1)$ *and* $\mathcal{L}_2(\xi_2)$ *both have zero at* ξ_1 *in* $\mathcal{X}_1$ *and at* ξ_2 *in* $\mathcal{X}_2$ *respectively.*

Lemma 6.2.6 *Let* $\mathcal{F}(\mathfrak{Z}) = \mathcal{L}_1(\xi_1) e_1 + \mathcal{L}_2(\xi_2) e_2$ *be a bicomplex holomorphic function with* $\|\mathcal{F}(\mathbf{0})\| \neq 0$ *and* $\|\mathcal{F}(\mathfrak{Z})\| \leqslant M(r_1, r_2)$ *for all* $\mathfrak{Z} \in \bar{D}(0; r_1, r_2)$. *Then the number of zeros* $N_1(\frac{r_1}{2})$ *of* $\mathcal{L}_1(\xi_1)$ *in the domain* $\{\xi_1 \in \mathfrak{D}_1 : |\xi_1| < \frac{r_1}{2}\}$ *and* $N_2(\frac{r_1}{2})$ *of* $\mathcal{L}_2(\xi_2)$ *in* $\{\xi_2 \in \mathfrak{D}_2 : |\xi_2| < \frac{r_2}{2}\}$ *do not exceed respectively*

$$\frac{1}{\log 2} \left\{ \log \frac{\sqrt{2} M(r_1, r_2)}{|\mathcal{L}_1(0)|} \right\} \ \textit{and} \ \frac{1}{\log 2} \left\{ \log \frac{\sqrt{2} M(r_1, r_2)}{|\mathcal{L}_2(0)|} \right\}.$$

Proof. In view of Lemma 6.2.4, $\mathcal{L}_1(\xi_1)$ and $\mathcal{L}_2(\xi_2)$ are holomorphic respectively in $\mathcal{X}_1 = \{\xi_1 \in \mathfrak{D}_1 : |\xi_1| \leqslant r_1\}$ and $\mathcal{X}_2 = \{\xi_2 \in \mathfrak{D}_2 : |\xi_2| \leqslant r_2\}$.

Since $\frac{|\mathcal{L}_1(\xi_1)|}{\sqrt{2}} \leqslant \left\{ \frac{|\mathcal{L}_1(\xi_1)|^2 + |\mathcal{L}_2(\xi_2)|^2}{2} \right\}^{\frac{1}{2}} = \|\mathcal{F}(\mathfrak{Z})\| \leqslant M(r_1, r_2)$ for $\mathfrak{Z} = \xi_1 e_1 + \xi_2 e_2 \in \bar{D}(0; r_1, r_2)$, $|\ \mathcal{L}_1(\xi_1)\ | \leqslant \sqrt{2} M(r_1, r_2)$ for $\xi_1 \in \mathcal{X}_1$.

Let $\xi_{11}, \xi_{12}, \ldots, \xi_{1n}$ be n zeros of $\mathcal{L}_1(\xi_1)$ such that $|\xi_{11}| \leqslant |\xi_{12}| \leqslant \ldots \leqslant |\xi_{1n}| < r_1$.
Then by Lemma 6.2.3,

$$N_1(\frac{r_1}{2}) \log 2 \leqslant \sum_{i=1}^{N_1(\frac{r_1}{2})} \log \frac{r_1}{\mid \xi_{1i} \mid} \leqslant \sum_{i=1}^{N_1(r_1)} \log \frac{r_1}{\mid \xi_{1i} \mid} \leqslant \log \sqrt{2} M(r_1, r_2) - \log |\mathcal{L}_1(0)|,$$

$$\text{i.e, } N_1(\frac{r_1}{2}) \leqslant \frac{1}{\log 2} \left\{ \log \frac{\sqrt{2} M(r_1, r_2)}{|\mathcal{L}_1(0)|} \right\}.$$

Similarly,

$N_2(\frac{r_2}{2}) \leqslant \frac{1}{\log 2} \left\{ \log \frac{\sqrt{2} M(r_1, r_2)}{|\mathcal{L}_2(0)|} \right\}$.

This completes the proof of the lemma. ■

6.3 Main Results.

The main results of this chapter are depicted in the following section:

Theorem 6.3.1 *Let $\mathcal{F}(\mathfrak{Z}) = \sum_{k=0}^{\infty} \beta_k \mathfrak{Z}^k$ be an entire function with $\| \mathcal{F}(\mathbf{0}) \| \neq 0$, $\beta_k = a_k e_1 + b_k e_2, k = 0, 1, 2, \ldots$ and $\mathfrak{Z} = \xi_1 e_1 + \xi_2 e_2$. Let N_1, N_2 be the respective highest positive integers less than or equal to $N_1(r_1)$ in $\{\xi_1 \in \mathfrak{D}_1 :| \xi_1 |< r_1\}$ and $N_2(r_2)$ in $\{\xi_2 \in \mathfrak{D}_2 :| \xi_2 |< r_2\}$ such that $a_{N_1} \neq 0$, $a_{N_2} \neq 0$. Then within the open discus $D(0; r_1, r_2)$, $\mathcal{F}(\mathfrak{Z})$ does not vanish in the open discus $D(0; t_1, t_2)$ where t_1, t_2 are respectively the least positive roots of the equations*

$$g_1(t) \equiv | a_0 | r_1^{N_1+1} - (| a_0 | + \sqrt{2} N_1 M(r_1, r_2)) r_1^{N_1} t + \sqrt{2} N_1 M(r_1, r_2) r_1^{N_1-1} t^2 \\ - \sqrt{2} M(r_1, r_2) t^{N_1+1} = 0$$

and

$$g_2(t) \equiv | b_0 | r_2^{N_2+1} - (| b_0 | + \sqrt{2} N_2 M(r_1, r_2)) r_2^{N_2} t + \sqrt{2} N_2 M(r_1, r_2) r_2^{N_2-1} t^2 \\ - \sqrt{2} M(r_1, r_2) t^{N_2+1} = 0,$$

$$\max_{\mathfrak{Z} \in \bar{D}(0; r_1, r_2)} \| \mathcal{F}(\mathfrak{Z}) \| \leqslant M(r_1, r_2).$$

Proof. As $\mathcal{F}(\mathfrak{Z})$ can be written as

$$\mathcal{F}(\mathfrak{Z}) = \sum_{k=0}^{\infty} a_k \xi_1^k e_1 + \sum_{k=0}^{\infty} b_k \xi_2^k e_2 = \mathsf{f}_1(\xi_1) e_1 + \mathsf{f}_2(\xi_2) e_2,$$

clearly $\mathcal{F}(\mathfrak{Z})$ is holomorphic in the closed discus $\bar{D}(0; r_1, r_2)$, $0 < r_1, r_2 < \infty$. Hence in view of Lemma 6.2.4, $\mathsf{f}_1(\xi_1)$ and $\mathsf{f}_2(\xi_1)$ are holomorphic respectively in $\mathcal{X}_1 = \{\xi_1 \in \mathfrak{D}_1 :| \xi_1 |\leqslant r_1\} \subset \mathbb{C}$ and $\mathcal{X}_2 = \{\xi_2 \in \mathfrak{D}_2 :| \xi_2 |\leqslant r_2\} \subset \mathbb{C}$.
Now for $| \xi_1 |< r_1$,

$$\mathsf{f}_1(\xi_1) = \sum_{k=0}^{\infty} a_k \xi_1^k = a_0 + G(\xi_1) + H(\xi_1), \tag{6.1}$$

where $G(\xi_1) = a_1 \xi_1 + a_2 \xi_1^2 + \ldots + a_{N_1} \xi_1^{N_1}$ and $H(\xi_1) = \sum_{k=N_1+1}^{\infty} a_k \xi_1^k$.

Since $\| \mathcal{F}(\mathfrak{Z}) \| = \left\{ \frac{|\mathsf{f}_1(\xi_1)|^2 + |\mathsf{f}_2(\xi_2)|^2}{2} \right\}^{\frac{1}{2}} \geqslant \frac{|\mathsf{f}_1(\xi_1)|}{\sqrt{2}}$, we have for the coefficients of the power series $\sum_{k=0}^{\infty} a_k \xi_1^k$ in $| \xi_1 |\leqslant r_1$,

$$| a_k | \leqslant \frac{\max_{|\xi_1|=r_1} | \mathsf{f}_1(\xi_1) |}{r_1^k} \leqslant \frac{\sqrt{2} M(r_1, r_2)}{r_1^k}. \tag{6.2}$$

Hence for $| \xi_1 |= r_1$, by using (6.2) it follows that

$$\begin{aligned} | G(\xi_1) | &\leqslant | a_1 || \xi_1 | + | a_2 || \xi_1 |^2 + \cdots + | a_{N_1} || \xi_1 |^{N_1} \\ &\leqslant | a_1 | r_1 + | a_2 | r_1^2 + \ldots + | a_{N_1} | r_1^{N_1} \\ &\leqslant \sqrt{2} N_1 M(r_1, r_2). \end{aligned}$$

Since $G(\xi_1)$ is holomorphic in $|\ \xi_1\ |\leqslant r_1$, $G(0)=0$ and $|\ G(\xi_1)\ |\leqslant \sqrt{2}N_1M(r_1,r_2)$ for $|\ \xi_1\ |= r_1$, by Lemma 6.2.2, we get that

$$|\ G(\xi_1)\ |\leqslant \frac{\sqrt{2}N_1M(r_1,r_2)\ |\ \xi_1\ |}{r_1}. \tag{6.3}$$

Also for $|\ \xi_1\ |< r_1$, by using (6.2) we have

$$\begin{aligned} |\ H(\xi_1)\ | &= |\sum_{k=N_1+1}^{\infty} a_k\xi_1^k\ | \\ &\leqslant \sqrt{2}M(r_1,r_2)\sum_{k=N_1+1}^{\infty}\left(\frac{|\ \xi_1\ |}{r_1}\right)^k \\ &= \frac{\sqrt{2}M(r_1,r_2)\ |\ \xi_1\ |^{N_1+1}}{r_1^{N_1}(r_1-|\ \xi_1\ |)}. \end{aligned} \tag{6.4}$$

Hence for $|\ \xi_1\ |< r_1$, from (6.1) it follows that

$$\begin{aligned} \mathsf{f}_1(\xi_1) &\geqslant |\ a_0\ | - |\ G(\xi_1)\ | - |\ H(\xi_1)\ | \\ &\geqslant |\ a_0\ | - \frac{\sqrt{2}N_1M(r_1,r_2)\ |\ \xi_1\ |}{r_1} - \frac{\sqrt{2}M(r_1,r_2)\ |\ \xi_1\ |^{N_1+1}}{r_1^{N_1}(r_1-|\ \xi_1\ |)} \\ &= \Big[\ |\ a_0\ |\ r_1^{N_1+1} - (|\ a_0\ | + \sqrt{2}N_1M(r_1,r_2))r_1^{N_1}\ |\ \xi_1\ | + \sqrt{2}N_1M(r_1,r_2)r_1^{N_1-1}\ |\ \xi_1\ |^2 \\ &\qquad - \sqrt{2}M(r_1,r_2)\ |\ \xi_1\ |^{N_1+1}\Big]/r_1^{N_1}(r_1-|\ \xi_1\ |). \end{aligned}$$

Let

$$\begin{aligned} g_1(t) \equiv |\ a_0\ |\ r_1^{N_1+1} - (|\ a_0\ | + \sqrt{2}N_1M(r_1,r_2))r_1^{N_1}t + \sqrt{2}N_1M(r_1,r_2)r_1^{N_1-1}t^2 \\ - \sqrt{2}M(r_1,r_2)t^{N_1+1}. \end{aligned}$$

According to the Descarte's rule of sign, the equation $g_1(t)=0$ has either three or one positive real roots.

Let us consider t_1 be the least positive root of the equation $g_1(t)=0$.
Since $g_1(0) = |\ a_0\ |\ r^{N_1+1} > 0$ and $g_1(\infty) = -\infty < 0$, $g_1(t) > 0$ if $t < t_1$, otherwise there will be another positive root in $(0,t_1)$, which makes a contradiction.
Hence for $|\ \xi_1\ |< r_1$, $|\ \mathsf{f}_1(\xi_1)\ |> 0$ if $|\ \xi_1\ |< t_1$.

Similarly for $|\ \xi_2\ |< r_2$, $|\ \mathsf{f}_2(\xi_2)\ |> 0$ if $|\ \xi_2\ |< t_2$ where t_2 is the least positive root of the equation

$$\begin{aligned} g_2(t) \equiv |\ b_0\ |\ r_2^{N_2+1} - (|\ b_0\ | + \sqrt{2}N_2M(r_1,r_2))r_2^{N_2}t + \sqrt{2}N_2M(r_1,r_2)r_2^{N_2-1}t^2 \\ - \sqrt{2}M(r_1,r_2)t^{N_2+1} = 0. \end{aligned}$$

Therefore both $\mathsf{f}_1(\xi_1)$ and $\mathsf{f}_2(\xi_2)$ have no zeros respectively in $\mathcal{X}_1' = \{\xi_1 \in \mathcal{X}_1 :| \xi_1 |< t_1\}$ and $\mathcal{X}_2' = \{\xi_2 \in \mathcal{X}_2 :| \xi_2 |< t_2\}$.

Hence by Lemma 6.2.5, $\mathcal{F}(\mathfrak{z})$ has no zero in $\mathcal{X}_1'e_1 + \mathcal{X}_2'e_2 = D(0; t_1, t_2)$, which proves the theorem.

■

Remark 6.3.1 *To validate Theorem 6.3.1 following example is considered which is justified by relevant figure.*

Example 6.3.1 *Let* $\mathcal{F}(\mathfrak{z}) = cosz = cos(\xi_1)e_1 + cos(\xi_2)e_2$.

Here, $\mathsf{f}_1(\xi_1) = cos(\xi_1) = 1 - \frac{1}{2!}\xi_1^2 + \frac{1}{4!}\xi_1^4 - \dots$ *and* $\mathsf{f}_2(\xi_2) = cos(\xi_2) = 1 - \frac{1}{2!}\xi_2^2 + \frac{1}{4!}\xi_2^4 - \dots$

For $r_1 = r_2 = 1$*, by Lemma 6.2.6,* $N_1(r_1) \leqslant 2.41, N_2(r_2) \leqslant 2.41.$
Since $a_2 \neq 0, b_2 \neq 0$ *, we may take* $N_1 = N_2 = 2.$

Now, $\|\mathcal{F}(\mathfrak{z})\| = \left\{\frac{|cos(\xi_1)|^2+|cos(\xi_2)|^2}{2}\right\}^{\frac{1}{2}} \leqslant \frac{e+e^{-1}}{2}$*, for all* $\mathfrak{z} \in \bar{D}(0; 1, 1)$.

Hence,

$$\begin{aligned} g_1(t) \equiv &| a_0 | r_1^{N_1+1} - (| a_0 | + \sqrt{2}N_1M(r_1, r_2))r_1^{N_1}t + \sqrt{2}N_1M(r_1, r_2)r_1^{N_1-1}t^2 \\ &- \sqrt{2}M(r_1, r_2)t^{N_1+1} \\ = &1 - (1 + \sqrt{2}(e + e^{-1}))t + \sqrt{2}(e + e^{-1})t^2 - \sqrt{2}.\frac{e + e^{-1}}{2}t^3 \end{aligned}$$

and

$$\begin{aligned} g_2(t) \equiv &| b_0 | r_2^{N_2+1} - (| b_0 | + \sqrt{2}N_2M(r_1, r_2))r_2^{N_2}t + \sqrt{2}N_2M(r_1, r_2)r_2^{N_2-1}t^2 \\ &- \sqrt{2}M(r_1, r_2)t^{N_2+1} \\ = &1 - (1 + \sqrt{2}(e + e^{-1}))t + \sqrt{2}(e + e^{-1})t^2 - \sqrt{2}.\frac{e + e^{-1}}{2}t^3. \end{aligned}$$

We see that $g_1(.22) > 0$*,* $g_1(.23) < 0$ *and* $g_1(t) > 0$ *for* $0 \leqslant t \leqslant .22$*. Hence the positive root of* $g_1(t)$ *lies between* .22 *and* .23 .

Similarly, one can obtain the range of the least positive zero of $g_2(t)$ *as* $.22 \leqslant t \leqslant .23$.

Thus by Theorem 6.3.1, $\mathcal{F}(\mathfrak{z})$ *has no zeros in* $\bar{D}(0; .22, .22)$ *within the open discus* $D(0; 1, 1)$.

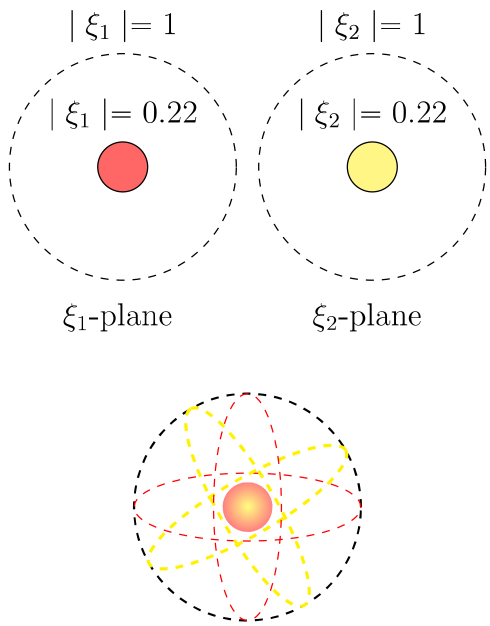

Figure 6.1: Zero free region of $\mathcal{F}(\mathfrak{z}) = cosz$ in $D(0;1,1)$

Theorem 6.3.2 *Let $\mathcal{F}(\mathfrak{z}) = \sum_{k=0}^{\infty} \text{ß}_k \mathfrak{z}^k$ be an entire function with $\|\mathcal{F}(\mathbf{0})\| \neq 0, \text{ß}_k = a_k e_1 + b_k e_2, k = 0, 1, 2, ..$ and $\mathfrak{z} = \xi_1 e_1 + \xi_2 e_2$. Then within any open discus $D(0; r_1, r_2)$, $\mathcal{F}(\mathfrak{z})$ has no zero in the open discus $D(0; t_1, t_2)$ where t_1 and t_2 are respectively the positive roots of the equation*

$$|a_0|(|a_0| + 2B)r_1^2 - 2B|a_0 - r_1 a_1|t - (|a_0| + 2B)^2 t^2 = 0$$
$$\textit{and } |b_0|(|b_0| + 2C)r_2^2 - 2C|b_0 - r_2 b_1|t - (|b_0| + 2C)^2 t^2 = 0,$$

where $B = \sum_{k=1}^{\infty} |a_k| r_1^k$, $C = \sum_{k=1}^{\infty} |b_k| r_2^k$

Proof. As $\mathcal{F}(\mathfrak{z})$ can be expressed as

$$\mathcal{F}(\mathfrak{z}) = \sum_{k=0}^{\infty} a_k \xi_1^k e_1 + \sum_{k=0}^{\infty} b_k \xi_2^k e_2 = \mathsf{f}_1(\xi_1)e_1 + \mathsf{f}_2(\xi_2)e_2,$$

$\mathcal{F}(\mathfrak{z})$ being holomorphic in $\mathbb{BC}$, by Lemma 6.2.4, $\mathsf{f}_1(\xi_1)$ and $\mathsf{f}_2(\xi_2)$ both are holomorphic respectively in $\mathcal{X}_1 = \{\xi_1 \in \mathfrak{D}_1 :| \xi_1 |\leqslant r_1\} \subset \mathbb{C}$ and $\mathcal{X}_2 = \{\xi_2 \in \mathfrak{D}_2 :| \xi_2 |\leqslant r_2\} \subset \mathbb{C}$.

Clearly, $\lim\limits_{k\to\infty} a_k r_1^k = 0$ and $\lim\limits_{k\to\infty} b_k r_2^k = 0$.

Let us consider
$\Gamma(\xi_1) = (\xi_1 - r_1)\mathsf{f}_1(\xi_1) = -a_0 r_1 + \sum_{k=1}^{\infty}(a_{k-1} - r_1 a_k)\xi_1^k = -a_0 r_1 + G(\xi_1)$.
Then for $|\xi_1| < r_1$,

$$|\Gamma(\xi_1)| \geqslant |a_0| r_1 - |G(\xi_1)|. \tag{6.5}$$

For $|\xi_1| = r_1$ and as because the series $\sum_{k=1}^{\infty}|a_k|r_1^k$ converges for $|\xi_1| \leqslant r_1$ we have

$$
\begin{aligned}
|G(\xi_1)| &= |\textstyle\sum_{k=1}^{\infty}(a_{k-1} - r_1 a_k)\xi_1^k| \\
&\leqslant \textstyle\sum_{k=1}^{\infty}(|a_{k-1}| + r_1|a_k|)r_1^k \\
&= |a_0|r_1 + 2r_1\textstyle\sum_{k=1}^{\infty}|a_k|r_1^k \\
&= (|a_0| + 2B)r_1 \text{ , where } B = \textstyle\sum_{k=1}^{\infty}|a_k|r_1^k \text{ .}
\end{aligned}
$$

Since $G(\xi_1)$ is analytic in $|\ \xi_1\ | \leqslant r_1$, $G(0) = 0, G'(0) = (a_0 - r_1 a_1)$ and $|\ G(\xi_1)\ | \leqslant (|a_0| + 2B)r_1$ for $|\ \xi_1\ | = r_1$ and in view of Lemma 6.2.1, it follows for $|\xi_1| \leqslant r_1$ that

$$
\begin{aligned}
|\ G(\xi_1)\ | &\leqslant \frac{(|a_0| + 2B)r_1|\xi_1|}{r_1^2} \cdot \frac{(|a_0| + 2B)r_1|\xi_1| + r_1^2|a_0 - r_1 a_1|}{(|a_0| + 2B)r_1 + |a_0 - r_1 a_1||\xi_1|} \\
&= \frac{(|a_0| + 2B)|\xi_1|\{(|a_0| + 2B)|\xi_1| + r_1|a_0 - r_1 a_1|\}}{(|a_0| + 2B)r_1 + |a_0 - r_1 a_1||\xi_1|} \text{ .}
\end{aligned}
$$

Hence for $|\xi_1| < r_1$, we obtain from (6.1) that

$$
\begin{aligned}
|\Gamma(\xi_1)| &\geqslant |a_0|r_1 - \frac{(|a_0| + 2B)|\xi_1|\{(|a_0| + 2B)|\xi_1| + r_1|a_0 - r_1 a_1|\}}{(|a_0| + 2B)r_1 + |a_0 - r_1 a_1||\xi_1|} \\
&= \frac{|a_0|(|a_0| + 2B)r_1^2 - 2B|a_0 - r_1 a_1||\xi_1| - (|a_0| + 2B)^2|\xi_1|^2}{(|a_0| + 2B)r_1 + |a_0 - r_1 a_1||\xi_1|} \text{ .}
\end{aligned}
$$

Clearly the equation $g(t) \equiv |a_0|(|a_0| + 2B)r_1^2 - 2B|a_0 - r_1 a_1|t - (|a_0| + 2B)^2 t^2 = 0$ has exactly one posite root. Let the positive root of the equation be t_1.

Since $g(0) = |a_0|(|a_0| + 2B)r_1^2 > 0$, $g(t) > 0$ for $t < t_1$.

Hence for $|\xi_1| < r_1$, $|\Gamma(\xi_1)| > 0$, i.e. $|\mathsf{f}_1(\xi_1)| > 0$ if $|\xi_1| < t_1$.

Similarly for $|\xi_2| < r_2$, $|\mathsf{f}_2(\xi_2)| > 0$ if $|\xi_2| < t_2$ where t_2 is the positive root of the equation $|b_0|(|b_0| + 2C)r_2^2 - 2C|b_0 - r_2 b_1|t - (|b_0| + 2C)^2 t^2 = 0$ where $C = \sum_{k=1}^{\infty}|b_k|r_2^k$.

Therefore both $\mathsf{f}_1(\xi_1)$ and $\mathsf{f}_2(\xi_2)$ have no zeros respectively in $\mathcal{X}_1' = \{\xi_1 \in \mathcal{X}_1 :|\ \xi_1\ | < t_1\} \subset \mathfrak{D}_1$ and $\mathcal{X}_2' = \{\xi_2 \in \mathcal{X}_2 :|\ \xi_2\ | < t_2\} \subset \mathfrak{D}_2$.

Consequently by Lemma 6.2.5, $\digamma(\mathfrak{Z})$ has no zeros in $\mathcal{X}_1' e_1 + \mathcal{X}_2' e_2 = D(0; t_1, t_2)$.
Hence the theorem is obtained. ■

Remark 6.3.2 *To validate Theorem 6.3.2 following example is considered which is justified by relevant figure.*

Example 6.3.2 *Let* $\digamma(\mathfrak{Z}) = -(1 - j) + \mathfrak{Z} + (2 + ji)\mathfrak{Z}^2$.
Here, $ß_0 = -(1 - j) = (-1 - i)e_1 + (-1 + i)e_2 = a_0 e_1 + b_0 e_2$, $ß_1 = 1 = 1e_1 + 1e_2 = a_1 e_1 + b_1 e_2$, $ß_2 = (2 + ji) = (2 - i.i)e_1 + (2 + i.i)e_2 = 3e_1 + 1e_2 = a_2 e_1 + b_2 e_2$.

Now, $\mathcal{F}(\mathfrak{z})$ *can be written as*

$$\mathcal{F}(\mathfrak{z}) = (3\xi_1^2 + \xi_1 - 1 - i)e_1 + (\xi_2^2 + \xi_2 - 1 + i)e_2$$

For $r_1 = r_2 = 1$*, we get* $B = 4, C = 2$*.*

Hence the equation $|a_0|(|a_0| + 2B)r_1^2 - 2B|a_0 - r_1 a_1|t - (|a_0| + 2B)^2 t^2 = 0$ *becomes* $\sqrt{2}(\sqrt{2} + 8) - 8\sqrt{5}t - (\sqrt{2} + 8)^2 t^2 = 0$ *and the positive root* $t_1 \approx 0.30$*.*

Again the equation $|b_0|(|b_0| + 2C)r_2^2 - 2C|b_0 - r_2 b_1|t - (|b_0| + 2C)^2 t^2 = 0$ *reduces to* $\sqrt{2}(\sqrt{2} + 4) - 4\sqrt{5}t - (\sqrt{2} + 4)^2 t^2 = 0$ *and the positive root* $t_2 \approx 0.38$*.*

Hence by Theorem 6.3.2, $\mathcal{F}(\mathfrak{z})$ *has no zeros in* $D(0; .30, .38)$ *within the open discus* $D(0; 1, 1)$ *.*

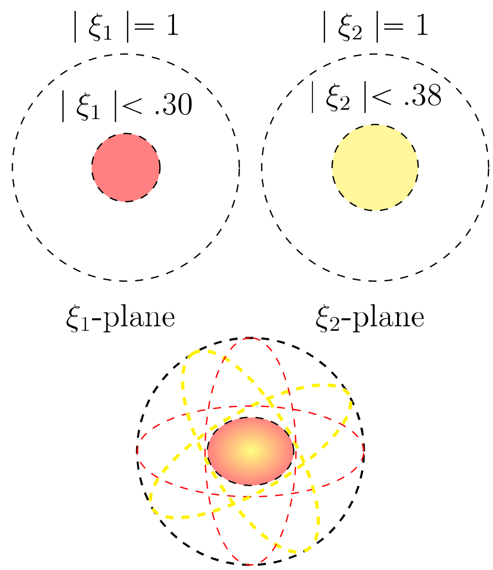

Figure 6.2: Zero free region of $\mathcal{F}(\mathfrak{z}) = -(1 - j) + \mathfrak{z} + (2 + ji)\mathfrak{z}^2$ in $D(0; 1, 1)$

6.4 Future Prospects.

For future research, one may extend the results obtained dealing with n-dimensional bicomplex numbers with the help of the idempotents $0, 1, \frac{1 \pm i_1 i_2}{2}, \frac{1 \pm i_2 i_3}{2}, \ldots, \frac{1 \pm i_{n-1} i_n}{2}$ in $\mathbb{C}_n$. As a consequence, the problem of taking the coeffiecients of the power series in $\mathbb{C}_n$ is still virgin and may be considered as an open problem to the future workers of this branch.

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The works of this chapter have been published in **Scientific Studies and Research. Series Mathematics and Informatics (SSRSMI) (UGC CARE Listed), Volume 30, No. 2, pp. 71-84, 2020, Print ISSN No. 2067-3566, Online ISSN: 2457-497X.**

CHAPTER SEVEN

A NOTE ON BICOMPLEX MANIFOLD

Chapter 7

A note on bicomplex manifold

7.1 Introduction.

Almost complex manifold and complex manifold have wide range of applications in Mathematical Physics specially in String Theory, Fluid mechanics etc. Motivated from the works done on complex manifold we have introduced almost bicomplex and bicomplex manifold.

In this chapter we wish to establish some results and discuss some properties of Bicomplex manifold, almost bicomplex manifold, almost bicomplex Hermite manifold, Nijenhuis tensor, contravariant almost analytic vector fields etc. Before proving our main results, we need the following definitions.

Definition 7.1.1 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two structures on a finite dimensional real vector space V are endomorphism $\mathcal{F}_1$ and $\mathcal{F}_2$ such that*

$$\mathcal{F}_1(\mathcal{F}_1(X)) = -X,\ \mathcal{F}_2(\mathcal{F}_2(X)) = -X \text{ and } \mathcal{F}_1(\mathcal{F}_2(X)) = \mathcal{F}_2(\mathcal{F}_1(X) = X,\ \forall X \in V$$

i.e., $\mathcal{F}_1^2 = -I, \quad \mathcal{F}_2^2 = -I \quad$ *and* $\quad \mathcal{F}_1\mathcal{F}_2 = \mathcal{F}_2\mathcal{F}_1 = I.$

A real vector space V with complex structures $\mathcal{F}_1$ and $\mathcal{F}_2$ can be turned into a bicomplex vector space by defining scalar multiplication by bicomplex number as follows:

$$(a + i_1b + i_2c + i_1i_2d)X = aX + b\mathcal{F}_1(X) + c\mathcal{F}_2(X) + d\mathcal{F}_1(\mathcal{F}_2(X)),$$

for $X \in V$ and $a, b, c, d \in \mathbb{R}$.

Definition 7.1.2 *An almost bicomplex structure on a real differentiable manifold M of dimension $n(n = 4m, m \in N)$ is a tensor field $\mathcal{F}_1$ and $\mathcal{F}_2$ which is at every point x of M, an endomorphism of the tangent space $T_x(M)$ such that $\mathcal{F}_1^2 = -I$ and $\mathcal{F}_2^2 = -I$ but $\mathcal{F}_1 \neq \mathcal{F}_2$, where I denotes the identity transformation of $T_x(M)$. A manifold with a fixed almost bicomplex structure is called an almost bicomplex manifold.*

Definition 7.1.3 *A bi-complex manifold is a Riemannian manifold endowed with integrable almost complex structures $\mathcal{F}_1$ and $\mathcal{F}_2$ with respect to the Riemannian metric and satisfy the relations $\mathcal{F}_1^2 = \mathcal{F}_2^2 = -I$. $(M, g, \mathcal{F}_1, \mathcal{F}_2)$ is a bi-complex manifold, then the tangent space $T_x(M)$ is a bicomplex vector space for each point x of M, i.e. it is isomorphic to H^n for some integer n, where H is the algebra of bicomplex.*

The compact symplectic group $S_p(n)$ can be considered as the group of orthogonal transformations of which are linear with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$. From this, it follows that the holonomy group of the Riemannian manifold (M, g) is contained in $S_p(n)$. Conversely, if the holonomy group of a Riemannian manifold (M, g) of dimension $4n$ is contained in $S_p(n)$, choose complex structures $\mathcal{F}_1$ and $\mathcal{F}_2$ on T_xM which make T_xM into a bicomplex vector space. Parallel transport of these complex structures gives the required complex structures on M making into a bicomplex manifold.

Definition 7.1.4 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two almost complex structures in a bicomplex manifold $M^n (n = 2m$, where m is a positive integer). Nijenhuis tensor with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$ is a vector valued bilinear function N defined by*

$$N(X, Y) = [\mathcal{F}_1(X), \mathcal{F}_2(Y)] + [X, Y] - \mathcal{F}_1[\mathcal{F}_2(X), Y] - \mathcal{F}_2[X, \mathcal{F}_1(Y)]$$

Definition 7.1.5 *An almost bicomplex Hermite manifold endowed with a metric g such that*

$$g(\mathcal{F}_1(X), \mathcal{F}_2(Y)) = g(X, Y)$$

where $g(\mathcal{F}_1(X), \mathcal{F}_1(Y)) = -g(X, Y)$ and $g(\mathcal{F}_2(X), \mathcal{F}_2(Y)) = -g(X, Y)$ is called an almost bicomplex Hermite manifold and $(\mathcal{F}_1, \mathcal{F}_2, g)$ is called an almost bicomplex Hermite structure.

Definition 7.1.6 *A vector field is said to be contravariant almost bi-complex analytic if*

$$(L_V\mathcal{F}_1)(X) = 0 \quad \text{and} \quad (L_V\mathcal{F}_2)(X) = 0$$

where L_V denotes the Lie derivative with respect to the vector field V.

Definition 7.1.7 *A vector field is said to be strictly contravariant almost bi-complex analytic if*

$$\begin{aligned} (L_V\mathcal{F}_1)(X) &= 0, \qquad (L_{\mathcal{F}_1(V)}\mathcal{F}_1)(X) = 0, \\ \text{and } (L_V\mathcal{F}_2)(X) &= 0, \qquad (L_{\mathcal{F}_2(V)}\mathcal{F}_2)(X) = 0 \end{aligned}$$

where L_V denotes the Lie derivative with respect to the vector field V.

Definition 7.1.8 *Let M^n be bi-complex manifold with bi-complex structures $\mathcal{F}_1$ and $\mathcal{F}_2$. The fundamental 2-form $\tilde{\mathcal{F}}_1$ and $\hat{\mathcal{F}}_2$ and M^n is defined by*

$$\tilde{\mathcal{F}}_1(X, Y) = g(\mathcal{F}_1(X), Y) \text{ and } \hat{\mathcal{F}}_2(X, Y) = g(X, \mathcal{F}_2(Y))$$

7.2 Lemmas.

Some important lemmas which are required in this chapter are presented here.

Lemma 7.2.1 *In an almost complex manifold $M^n (n = 2m)$, the almost complex structure F has m eigen values i and m eigen values $-i$.*

Lemma 7.2.2 *In an almost complex manifold a vector field is contravariant almost analytic if and only if*

$$L_V F(X) = F(L_V X), \qquad or \qquad F(L_V F(X)) + L_V X = 0.$$

Lemma 7.2.3 *If a vector field V in an almost complex manifold is strictly contravariant almost analytic, then $N(V, Y) = 0$ for every vector field X.*

7.3 Main results.

This section describes the main findings of this chapter.

Theorem 7.3.1 *Let $(\mathcal{F}_1, \mathcal{F}_2, g)$ be an almost bicomplex Hermite manifold. Then*

$$g(\mathcal{F}_1(X), \mathcal{F}_2(Y)) = g(\mathcal{F}_2(X), \mathcal{F}_1(Y)).$$

Proof. Here

$$\begin{aligned} g(\mathcal{F}_1(X), \mathcal{F}_2(Y)) =& g(-\mathcal{F}_2(X), \mathcal{F}_2(Y)) \\ =& g(-\mathcal{F}_2(X), -\mathcal{F}_1(Y)) \\ =& g(\mathcal{F}_2(X), \mathcal{F}_1(Y)). \end{aligned}$$

Therefore, $g(\mathcal{F}_1(X), \mathcal{F}_2(Y)) = g(\mathcal{F}_2(X), \mathcal{F}_1(Y))$. ∎

Theorem 7.3.2 *In an almost bicomplex manifold $M^n (n = 4m)$*

(a) $N(X, \mathcal{F}_1(X)) = \mathcal{F}_1[\mathcal{F}_1(X), \mathcal{F}_2(X)]$,

(b) $N(\mathcal{F}_2(X), X) = \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_2(X)]$,

(c) $N(\mathcal{F}_1(X), X) = -N(X, \mathcal{F}_2(X)) = [\mathcal{F}_1(X), X] + [\mathcal{F}_2(X), X]$,

(d) $N(\mathcal{F}_1(X), Y) = [\mathcal{F}_1(X), Y] - [X, \mathcal{F}_2(Y)] - \mathcal{F}_1[X, Y] - \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_1(Y)]$,

(e) $N(X, \mathcal{F}_1(Y)) = [\mathcal{F}_1(X), Y] + [X, \mathcal{F}_1(Y)] - \mathcal{F}_1[\mathcal{F}_2(X), \mathcal{F}_1(Y)] - \mathcal{F}_2[X, Y]$,

(f) $N(\mathcal{F}_1(X), \mathcal{F}_2(Y)) = [X, Y] + [\mathcal{F}_1(X), \mathcal{F}_2(Y)] - \mathcal{F}_1[X, \mathcal{F}_2(Y)]$
$- \mathcal{F}_2[\mathcal{F}_1(X), Y]$,

(g) $\mathcal{F}_1(N(X,Y)) = \mathcal{F}_1[\mathcal{F}_1(X), \mathcal{F}_2(Y)] + \mathcal{F}_1[X,Y] + [\mathcal{F}_2(X), Y] - [X, \mathcal{F}_1(Y)]$,

(h) $\mathcal{F}_2(N(X,Y)) = \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_2(Y)] + \mathcal{F}_2[X,Y] - [\mathcal{F}_2(X), Y] + [X, \mathcal{F}_1(Y)]$,

(i) $\mathcal{F}_2(N(\mathcal{F}_1(X), \mathcal{F}_2(Y))) = \mathcal{F}_2[X,Y] + \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_2(Y)] - [X, \mathcal{F}_2(Y)] + [\mathcal{F}_1(X), Y]$.

Proof. (a): We have

$$\begin{aligned}
N(X, \mathcal{F}_1(X)) &= [\mathcal{F}_1(X), \mathcal{F}_2\mathcal{F}_1(X)] + [X, \mathcal{F}_1(X)] - \mathcal{F}_1[\mathcal{F}_2(X), \mathcal{F}_1(X)] \\
&\qquad - \mathcal{F}_2[X, \mathcal{F}_1^2(X)] \\
&= [\mathcal{F}_1(X), X] + [X, \mathcal{F}_1(X)] - \mathcal{F}_1[\mathcal{F}_2(X), \mathcal{F}_1(X)] - \mathcal{F}_2[X,X] \\
&= \mathcal{F}_1[\mathcal{F}_1(X), \mathcal{F}_2(X)].
\end{aligned}$$

This completes the proof of (a).
(b): We have

$$\begin{aligned}
N(\mathcal{F}_2(X), X) &= [\mathcal{F}_1\mathcal{F}_2(X), \mathcal{F}_2(X)] + [\mathcal{F}_2(X), X] - \mathcal{F}_1[\mathcal{F}_2^2(X), X] \\
&\qquad - \mathcal{F}_2[\mathcal{F}_2(X), \mathcal{F}_1(X)] \\
&= [X, \mathcal{F}_2(X)] + [\mathcal{F}_2(X), X] + \mathcal{F}_1[X,X] - \mathcal{F}_2[\mathcal{F}_2(X), \mathcal{F}_1(X)] \\
&= \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_2(X)].
\end{aligned}$$

This completes the proof of (b).
(c):

$$\begin{aligned}
N(X, \mathcal{F}_2(X)) &= [\mathcal{F}_1(X), \mathcal{F}_2^2(X)] + [X, \mathcal{F}_2(X)] - \mathcal{F}_1[\mathcal{F}_2(X), \mathcal{F}_2(X)] \\
&\qquad - \mathcal{F}_2[X, \mathcal{F}_1\mathcal{F}_2(X)] \\
&= -[\mathcal{F}_1(X), X] + [X, \mathcal{F}_2(X)] - \mathcal{F}_1[\mathcal{F}_2(X), \mathcal{F}_2(X)] + \mathcal{F}_2[X,X] \\
&= [X, \mathcal{F}_1(X)] + [X, \mathcal{F}_2(X)]
\end{aligned}$$

and

$$\begin{aligned}
N(\mathcal{F}_1(X), X) &= [\mathcal{F}_1^2(X), \mathcal{F}_2(X)] + [\mathcal{F}_1(X), X] - \mathcal{F}_1[\mathcal{F}_2\mathcal{F}_1(X), X] \\
&\qquad - \mathcal{F}_2[\mathcal{F}_1 X, \mathcal{F}_1(X)] \\
&= -[X, \mathcal{F}_2(X)] + [\mathcal{F}_1(X), X] - \mathcal{F}_1[\mathcal{F}_2\mathcal{F}_1(X), X] \\
&\qquad - \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_1(X)] \\
&= -[X, \mathcal{F}_1(X)] - [X, \mathcal{F}_2(X)] \\
&= -N(X, \mathcal{F}_2(X)).
\end{aligned}$$

This completes the proof of (c).
(d):

$$\begin{aligned}
N(\mathcal{F}_1(X), Y) &= [\mathcal{F}_1^2(X), \mathcal{F}_2(Y)] + [\mathcal{F}_1(X), Y] - \mathcal{F}_1[\mathcal{F}_2\mathcal{F}_1(X), Y] \\
&\qquad - \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_1(Y)] \\
&= [\mathcal{F}_1(X), Y] - [X, \mathcal{F}_2(Y)] - \mathcal{F}_1[X,Y] - \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_1(Y)].
\end{aligned}$$

This completes the proof of (d).
(e):

$$\begin{aligned} N(X, \mathcal{F}_1(Y)) &= [\mathcal{F}_1(X), \mathcal{F}_2\mathcal{F}_1(Y)] + [X, \mathcal{F}_1(Y)] - \mathcal{F}_1[\mathcal{F}_2(X), \mathcal{F}_1(Y)] \\ &\qquad - \mathcal{F}_2[X, \mathcal{F}_1^2(Y)] \\ &= [\mathcal{F}_1(X), Y] + [X, \mathcal{F}_1(Y)] - \mathcal{F}_1[\mathcal{F}_2(X), \mathcal{F}_1(Y)] - \mathcal{F}_2[X, Y]. \end{aligned}$$

This completes the proof of (e).
(f):

$$\begin{aligned} N(\mathcal{F}_1(X), \mathcal{F}_2(Y)) &= [\mathcal{F}_1^2(X), \mathcal{F}_2^2(Y)] + [\mathcal{F}_1(X), \mathcal{F}_2(Y)] \\ &\qquad - \mathcal{F}_1[\mathcal{F}_2\mathcal{F}_1(X), \mathcal{F}_2(Y)] - \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_1\mathcal{F}_2(Y)] \\ &= [X, Y] + [\mathcal{F}_1(X), \mathcal{F}_2(Y)] - \mathcal{F}_1[X, \mathcal{F}_2(Y)] - \mathcal{F}_2[\mathcal{F}_1(X), Y]. \end{aligned}$$

This completes the proof of (f).
(g):

$$\begin{aligned} \mathcal{F}_1(N(X, Y)) &= \mathcal{F}_1[\mathcal{F}_1(X), \mathcal{F}_2(Y)] + \mathcal{F}_1[X, Y] - \mathcal{F}_1^2[\mathcal{F}_2(X), Y] \\ &\qquad - \mathcal{F}_1\mathcal{F}_2[X, \mathcal{F}_1(Y)] \\ &= \mathcal{F}_1[\mathcal{F}_1(X), \mathcal{F}_2(Y)] + \mathcal{F}_1[X, Y] + [\mathcal{F}_2(X), Y] - [X, \mathcal{F}_1(Y)]. \end{aligned}$$

This completes the proof of (g).
(h) :

$$\begin{aligned} \mathcal{F}_2(N(X, Y)) &= \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_2(Y)] + \mathcal{F}_2[X, Y] - \mathcal{F}_1\mathcal{F}_1[\mathcal{F}_2(X), Y] \\ &\qquad - \mathcal{F}_2^2[X, \mathcal{F}_1(Y)] \\ &= \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_2(Y)] + \mathcal{F}_2[X, Y] - [\mathcal{F}_2(X), Y] + [X, \mathcal{F}_1(Y)]. \end{aligned}$$

This completes the proof of (h).
(i):

$$\begin{aligned} \mathcal{F}_2(N(\mathcal{F}_1(X), \mathcal{F}_2(Y))) &= \mathcal{F}_2[\mathcal{F}_1\mathcal{F}_1(X), \mathcal{F}_2\mathcal{F}_2(Y)] + \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_2(Y)] \\ &\qquad - \mathcal{F}_2\mathcal{F}_1[\mathcal{F}_2\mathcal{F}_1(X), \mathcal{F}_2(Y)] - \mathcal{F}_2^2[\mathcal{F}_1(X), \mathcal{F}_1\mathcal{F}_2(Y)] \\ &= \mathcal{F}_2[X, Y] + \mathcal{F}_2[\mathcal{F}_1(X), \mathcal{F}_2(Y)] - [X, \mathcal{F}_2(Y)] \\ &\qquad + [\mathcal{F}_1(X), Y]. \end{aligned}$$

This completes the proof of (i). ■

Theorem 7.3.3 *In an almost bi-complex manifold $M^n(n = 2m)$, the complex structures $\mathcal{F}_1$ has $m/2$ eigen values i and $m/2$ eigen values $-i$ and $\mathcal{F}_2$ has $m/2$ eigen values i and $m/2$ eigen values $-i$.*

Proof. Let $M^n(n = 2m)$ be an almost bi-complex manifold with complex structures $\mathcal{F}_1$ and $\mathcal{F}_2$. That is, $\mathcal{F}_1$ and $\mathcal{F}_2$ are vector valued real linear functions on M such that

$$\mathcal{F}_1(X) = \overline{X},\ \mathcal{F}_2(X) = \overline{X} \text{ and } \mathcal{F}_1^2 = -X,\ \mathcal{F}_2^2 = -X \text{ but } \mathcal{F}_1 \neq \mathcal{F}_2.$$

Also, let ρ_1 and ρ_2 be eigen values of $\mathcal{F}_1$ and $\mathcal{F}_2$ corresponding to the eigen vectors Z and W respectively. Then we get

$$\mathcal{F}_1(Z) = \rho_1(z) \text{ and } \mathcal{F}_2(W) = \rho_2(z).$$

Therefore, $-Z = \mathcal{F}_1^2(Z) = \rho_1(\mathcal{F}(Z)) = \rho_1^2(Z), i.e., (\rho_1^2+1)Z = 0$ and $-W = \mathcal{F}_2^2(W) = \rho_2(\mathcal{F}(W)) = \rho_2^2(W), i.e., (\rho_2^2+1)W = 0$. Hence $\rho_1^2 = -1$ and $\rho_2^2 = -1$.
Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are real valued linear function and of rank $2m$, therefore there are $m/2$ pairs of eigen values i and $-i$ of $\mathcal{F}_1$ and $m/2$ pairs of eigen values i and $-i$ of $\mathcal{F}_2$.
This completes the proof. ■

Theorem 7.3.4 *In an almost bi-complex manifold a vector field is contravariant almost bicomplex analytic if and only if*

$$\begin{aligned} &L_V\mathcal{F}_1(X) = \mathcal{F}_1(L_VX), && \textit{or} && \mathcal{F}_1(L_V\mathcal{F}_1(X)) + L_VX = 0,\\ &L_V\mathcal{F}_2(X) = 0 && && \mathcal{F}_2(L_V\mathcal{F}_2(X)) + L_VX = 0.\end{aligned}$$

Proof. We have

$$\begin{aligned} (L_V\mathcal{F}_1)(X) &= L_V\mathcal{F}_1(X) - \mathcal{F}_1(L_VX)\\ \textit{and } (L_V\mathcal{F}_1)(X) &= L_V\mathcal{F}_1(X) - \mathcal{F}_1(L_VX).\end{aligned}$$

If V is contravariant almost bicomplex analytic, then

$$(L_V\mathcal{F}_1)(X) = 0 \text{ and } (L_V\mathcal{F}_2)(X) = 0.$$

So

$$\begin{gathered} (L_V\mathcal{F}_1)(X) = \mathcal{F}_1(L_VX)\\ or,\ \mathcal{F}_1(L_V\mathcal{F}_1(X)) = \mathcal{F}_1^2(L_VX) = -L_VX\\ \text{and}\\ (L_V\mathcal{F}_2) = \mathcal{F}_2(L_VX)\\ or,\ \mathcal{F}_2(L_V\mathcal{F}_2(X)) = \mathcal{F}_2^2(L_VX) = -L_VX,\\ i.e.,\ \mathcal{F}_1(L_V\mathcal{F}_1(X)) + L_VX = 0 \text{ and } \mathcal{F}_2(L_V\mathcal{F}_2(X)) + L_VX = 0.\end{gathered}$$

Conversely, let $L_V\mathcal{F}_1(X) = \mathcal{F}_1(L_VX)$ and $L_V\mathcal{F}_2(X) = \mathcal{F}_2(L_VX)$.
Then

$$\begin{aligned} (L_V\mathcal{F}_1)(X) &= L_V\mathcal{F}_1(X) - \mathcal{F}_1(L_VX) = 0\\ \text{and } (L_V\mathcal{F}_2)(X) &= L_V\mathcal{F}_2(X) - \mathcal{F}_2(L_VX) = 0\end{aligned}$$

So, V is contravariant almost bicomplex analytic. ■

Theorem 7.3.5 *If a vector field V in an almost bi-complex manifold is strictly contravariant almost bicomplex analytic, then $N(V,Y)=0$ for every vector field X.*

Proof. Let V be strictly contravariant almost bicomplex analytic.
Then

$$\begin{aligned}(L_V\mathcal{F}_1)(X)=0,\ (L_V\mathcal{F}_2)(X)=0\\ \text{and } (L_{\mathcal{F}_1(V)}\mathcal{F})(X)=0,\ (L_{\mathcal{F}_2(V)}\mathcal{F})(X)=0\end{aligned}$$

From $L_V\mathcal{F}_1(X)=\mathcal{F}_1(L_VX)$ and $L_V\mathcal{F}_2(X)=\mathcal{F}_2(L_VX)$, we get that

$$\begin{aligned}[V,\mathcal{F}_1(X)]&=\mathcal{F}_1([V,X]), & [V,\mathcal{F}_2(X)]&=\mathcal{F}_2([V,X])\\ \text{and } [\mathcal{F}_1(V),\mathcal{F}_1(X)]&=\mathcal{F}_1([\mathcal{F}_1(V),X]), & [\mathcal{F}_2(V),\mathcal{F}_2(X)]&=\mathcal{F}_2([\mathcal{F}_2(V),X])\end{aligned}$$

Now

$$\begin{aligned}N(V,X)=&[\mathcal{F}_1(V),\mathcal{F}_2(X)]+[V,X]-\mathcal{F}_1[\mathcal{F}_2(V),X]-\mathcal{F}_2[V,\mathcal{F}_1(X)]\\ =&\mathcal{F}_2[\mathcal{F}_1(V),(X)]+[V,X]-\mathcal{F}_1[\mathcal{F}_2(V),X]-\mathcal{F}_2\mathcal{F}_1[V,X]\\ =&-\mathcal{F}_2[X,\mathcal{F}_1(V)]+[V,X]+\mathcal{F}_1[X,\mathcal{F}_2(V)]-[V,X]\\ =&-\mathcal{F}_2\mathcal{F}_1[X,V]+\mathcal{F}_1\mathcal{F}_2[X,V]\\ =&-[X,V]+[X,V]=0.\end{aligned}$$

This proves the theorem. ■

Theorem 7.3.6 *In an almost bicomplex Hermite manifold the following relations hold:*

(a) $\tilde{\mathcal{F}}_1(X,Y)=-\hat{\mathcal{F}}_2(X,Y)$,

(b) $\tilde{\mathcal{F}}_1(\mathcal{F}_1(X),\mathcal{F}_2(Y))=-\hat{\mathcal{F}}_2(X,Y)$,

(c) $\tilde{\mathcal{F}}_1(\mathcal{F}_2(X),\mathcal{F}_1(Y))=-\tilde{\mathcal{F}}_1(Y,X)$,

(d) $\hat{\mathcal{F}}_2(\mathcal{F}_1(X),\mathcal{F}_2(Y))=-\tilde{\mathcal{F}}_1(X,Y)$,

(e) $\hat{\mathcal{F}}_2(\mathcal{F}_2(X),\mathcal{F}_1(Y))=-\hat{\mathcal{F}}_2(X,Y)$.

Proof. (a): We know that $g(\mathcal{F}_1(X),\mathcal{F}_2(Y))=g(X,Y)$.
Putting $X=\mathcal{F}_1(X)$, we get that

$$\begin{aligned}g(\mathcal{F}_1^2(X),\mathcal{F}_2(Y))&=g(\mathcal{F}_1(X),Y)\\ \textit{or, }-g(X,\mathcal{F}_2(Y))&=g(\mathcal{F}_1(X),Y)\\ \textit{or, }-\hat{\mathcal{F}}_2(X,Y)&=\tilde{\mathcal{F}}_1(X,Y)\\ \textit{or, }\tilde{\mathcal{F}}_1(X,Y)&=-\hat{\mathcal{F}}_2(X,Y)\end{aligned}$$

This completes the proof of (a).
(b): Putting $X = \mathcal{F}_1(X)$ and $Y = \mathcal{F}_2(Y)$ in $\tilde{\mathcal{F}}_1(X,Y) = g(\mathcal{F}_1(X), Y)$, we obtain that

$$\begin{aligned}
\tilde{\mathcal{F}}_1(\mathcal{F}_1(X), \mathcal{F}_2(Y)) &= g(-X, \mathcal{F}_2(Y)) \\
&= -g(X, \mathcal{F}_2(Y)) \\
&= -\hat{\mathcal{F}}_2(X, Y).
\end{aligned}$$

Therefore, $\tilde{\mathcal{F}}_1(\mathcal{F}_1((X), \mathcal{F}_2(Y)) = -\hat{\mathcal{F}}_2(X, Y)$.
This completes the proof of (b).
(c): Putting $X = \mathcal{F}_2(X)$ and $Y = \mathcal{F}_1(Y)$ in $\tilde{\mathcal{F}}_1(X,Y) = g(\mathcal{F}_1(X), Y)$

$$\begin{aligned}
\tilde{\mathcal{F}}_1(\mathcal{F}_2(X), \mathcal{F}_1(Y)) &= g(\mathcal{F}_1\mathcal{F}_2(X), \mathcal{F}_1(Y)) \\
&= g(X, \mathcal{F}_1(Y)) \\
&= -g(\mathcal{F}_1(Y), X) \\
&= -\tilde{\mathcal{F}}_1(Y, X).
\end{aligned}$$

Therefore, $\tilde{\mathcal{F}}_1(\mathcal{F}_1((X), \mathcal{F}_2(Y)) = -\hat{\mathcal{F}}_2(X, Y)$.
This completes the proof of (c).
(d): Putting $X = \mathcal{F}_1(X)$ and $Y = \mathcal{F}_2(Y)$ in $\hat{\mathcal{F}}_2(X,Y) = g(X, \mathcal{F}_2(Y))$

$$\begin{aligned}
\hat{\mathcal{F}}_2(\mathcal{F}_1(X), \mathcal{F}_2(Y)) &= g(\mathcal{F}_1(X), \mathcal{F}_2^2(Y)) \\
&= -g(\mathcal{F}_1(X), Y) \\
&= -\tilde{\mathcal{F}}_1(X, Y).
\end{aligned}$$

Therefore, $\hat{\mathcal{F}}_2(\mathcal{F}_1(X), \mathcal{F}_2(Y)) = -\tilde{\mathcal{F}}_1(X, Y)$.
This completes the proof of (d).
(e): Putting $X = \mathcal{F}_2(X)$ and $Y = \mathcal{F}_1(Y)$ in $\hat{\mathcal{F}}_2(X,Y) = g(X, \mathcal{F}_2(Y))$

$$\begin{aligned}
\hat{\mathcal{F}}_2(\mathcal{F}_2(X), \mathcal{F}_1(Y)) &= g(\mathcal{F}_2(X), \mathcal{F}_2\mathcal{F}_1(Y)) \\
&= g(\mathcal{F}_2(X), Y) \\
&= -g(X, \mathcal{F}_2(Y)) \\
&= -\hat{\mathcal{F}}_2(X, Y).
\end{aligned}$$

Therefore, $\hat{\mathcal{F}}_2(\mathcal{F}_2(X), \mathcal{F}_1(Y)) = -\hat{\mathcal{F}}_2(X, Y)$.
This completes the proof of (e). ■

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The works of this chapter have been communicated.

CHAPTER EIGHT

FUTURE COURSE OF WORK

Chapter 8

Future course of work

8.1 Introduction.

The spaces $\mathbb{C}_0,\ \mathbb{C}_1,\ \mathbb{C}_2$ i.e., space of real numbers, complex numbers and bicomplex numbers are very well known to us. Now we consider the space $\mathbb{C}_n$ in which elements are linear combination of elements in $\mathbb{C}_0,\ \mathbb{C}_1,\ ...,\ \mathbb{C}_{n-1}$. Corresponding to every representation of the elements in $\mathbb{C}_n$, these are the linear space where $\mathbb{C}_n$ can be embedded. It is not just a replacement of 2 by n, these are richous in general theory and many novel relationships come to high in $\mathbb{C}_n$.
In this chapter, we prove the maximum and minimum value theorem in terms of multicomplex valued function.

Definition 8.1.1 *(The Space $\mathbb{C}_n$)*
$\mathbb{C}_n = \{\zeta_n : \zeta_n = \zeta_{n-1,1} + i_n\zeta_{n-1,2};\ \zeta_{n-1,1}, \zeta_{n-1,2} \in \mathbb{C}_{n-1}\}$ *where* $i_n^2 = -1$. *Elements of* $\mathbb{C}_n$ *are known as multicomplex numbers.*
In $\mathbb{C}_n$, *addition and multiplication are defined as follows:*

$$\begin{aligned}\zeta_n^1 + \zeta_n^2 =& (\zeta_{n-1,1}^1 + i_n\zeta_{n-1,2}^1) + (\zeta_{n-1,1}^2 + i_n\zeta_{n-1,2}^2)\\ =& (\zeta_{n-1,1}^1 + \zeta_{n-1,1}^2) + i_n(\zeta_{n-1,2}^1 + \zeta_{n-1,2}^2)\end{aligned}$$

$$\begin{aligned}\zeta_n^1\zeta_n^2 =& (\zeta_{n-1,1}^1\zeta_{n-1,1}^2 - \zeta_{n-1,2}^1\zeta_{n-1,2}^2)\\ &+ i_n(\zeta_{n-1,1}^1 + \zeta_{n-1,2}^2 + \zeta_{n-1,2}^1 + \zeta_{n-1,1}^2).\end{aligned}$$

In $\mathbb{C}_n$ norm is defined as follows

$$\|\zeta_n\|_n^2 = \|\zeta_{n-1,1}\|_{n-1}^2 + \|\zeta_{n-1,2}\|_{n-1}^2.$$

For convenience in notation any element ζ in $\mathbb{C}_n$ can be written in the form $\zeta = \zeta_1 + i_n\zeta_2$ where ζ_1 and $\zeta_2 \in \mathbb{C}_{n-1}$ and $i_n^2 = -1$.
The system $(\mathbb{C}_n, \|.\|_n, +, .)$ forms a Banach space.

Definition 8.1.2 *(Idempotent elements in $\mathbb{C}_n$) We know that $\frac{1+i_1i_2}{2}$ and $\frac{1-i_1i_2}{2}$ are idempotent elements in $\mathbb{C}_2$ and any element of $\mathbb{C}_2$ can be written as a linear combination of these idempotent elements.*
Similarly, it is easy to verify that the following are idempotent elements in $\mathbb{C}_n$

$$0,\ 1,\ \frac{1+i_1i_2}{2},\ \frac{1-i_1i_2}{2},\ \frac{1+i_1i_3}{2},\ \frac{1-i_1i_3}{2},\ \frac{1+i_2i_3}{2},$$
$$\frac{1-i_2i_3}{2},\ \ldots,\ \frac{1+i_{n-1}i_n}{2},\ \frac{1-i_{n-1}i_n}{2}.$$

For convenience in notations, define the symbol $e(i_ri_s)$ and $e(-i_ri_s)$ as follows

$$e(i_ri_s) = \frac{1+i_ri_s}{2},\ \ e(-i_ri_s) = \frac{1-i_ri_s}{2},\ \ \text{where } r, s \in \mathbb{N}.$$

Definition 8.1.3 *(Idempotent Representation) Let ζ be an element in $\mathbb{C}_n$, and let $\zeta = \zeta_1 + i_n\zeta_2$ with ζ_1 and ζ_2 in $\mathbb{C}_{n-1}$. Then*

$$\zeta = (\zeta_1 - i_{n-1}\zeta_2)\ e(i_{n-1}i_n) + (\zeta_1 + i_{n-1}\zeta_2)\ e(-i_{n-1}i_n).$$

Let ζ be an element in $\mathbb{C}_n$ and let $\zeta = \zeta_1 + i_n\zeta_2$, where ζ_1 and $\zeta_2 \in \mathbb{C}_{n-1}$, then

$$\zeta_n = \sqrt{\frac{\zeta_1 - i_{n-1}\zeta_2{}_{n-1}^{2} + \zeta_1 + i_{n-1}\zeta_2{}_{n-1}^{2}}{2}}.$$

Definition 8.1.4 *(Singular Element) An element $\zeta \in \mathbb{C}_n$ is non-singular iff three exist a unique $\mathcal{J} \in \mathbb{C}_n$ such that $\zeta\mathcal{J} = 1$ and ζ is singular iff it is non-singular. Θ_n $(n = 0, 1, 2, \ldots)$ is used to denote the set of singular elements in $\mathbb{C}_n$.*

The element $\zeta_1 + i_n\zeta_2 \in \mathbb{C}_n$ is non-singular if and only if $\zeta_1 - i_{n-1}\zeta_2$ and $\zeta + i_{n-1}\zeta_2$ are non-singular, the element $\zeta_1 + i_n\zeta_2$ is singular if at least one of the elements $\zeta_1 - i_{n-1}\zeta_2$ and $\zeta + i_{n-1}\zeta_2$ is singular.

Definition 8.1.5 *(Neighbourhood in $\mathbb{C}_n$) Suppose that $X_1 = \{\zeta_1 - i_{n-1}\zeta_2 : \zeta_1, \zeta_2 \in \mathbb{C}_{n-1}\}$ and $X_2 = \{\zeta_1 + i_{n-1}\zeta_2 : \zeta_1, \zeta_2 \in \mathbb{C}_{n-1}\}$ can be written in the form*

$$X = \{\zeta_1 + i_n\zeta_2 : \zeta_1 - i_{n-1}\zeta_2 \in X_1, \zeta_1 + i_{n-1}\zeta_2 \in X_2\}$$

then neighbourhood $N(a + i_nb) \subseteq X$ is defined as

$$N(a + i_nb, \epsilon) = \{\zeta_1 + \imath_n\zeta_2 \in \mathbb{C}_n : \|(\zeta_1 + i_n\zeta_2) - (a + i_nb)\|_n < \epsilon\}.$$

Another way it can be written in the form

$$\{\zeta_1 + i_n\zeta_2 \in \mathbb{C}_n : \|(\zeta_1 - i_{n-1}\zeta_2) - (a - i_{n-1}b)\|_{n-1} < \epsilon,$$
$$\|(\zeta_1 + i_{n-1}\zeta_2) - (a + i_{n-1}b)\|_{n-1} < \epsilon\}$$

Definition 8.1.6 *Let $f : X(\subseteq \mathbb{C}_n) \to \mathbb{C}_n,\ n \geqslant 1$ be a multicomplex valued function and $\zeta_0 \in X$. If f satisfies the strong Stolz condition at ζ_0, i.e., for all $n \geqslant 1$, $\exists$ a constant d (which depends on ζ_0) in $\mathbb{C}_n$ and a function $r(f;\zeta_0,\zeta)$ defined in as neighbourhood of ζ_0 and with the values in $\mathbb{C}_n$, such that*
(i) $f(\zeta) - f(\zeta_0) = d(\zeta - \zeta_0) + r(f;\zeta_0,\zeta) = 0$
(ii) $\lim_{\zeta \to \zeta_0} r(f;\zeta_0,\zeta) = 0$
then f is differentiable at ζ_0.
If f is differentiable in a neighbourhood of each point in domain $X(\subseteq \mathbb{C}_n)$ then f is called holomorphic multicomplex valued function.

Definition 8.1.7 *(Idempotent Representation of Holomorphic multicomplex valued function): Let $f : X(\subseteq \mathbb{C}_n) \to \mathbb{C}_n,\ n \geqslant 1$ be a holomorphic function then there exists holomorphic functions $f_1 : X_1 \to \mathbb{C}_{n-1}$ and $f_2 : X_2 \to \mathbb{C}_{n-1}$ such that*

$$f(\zeta_1 + i_n\zeta_2) = f_1(\zeta_1 - i_{n-1}\zeta_2)\ e(i_{n-1}i_n) + f_2(\zeta_n + i_{n-1}\zeta_2)\ e(-i_{n-1}i_n).$$

8.2 Lemmas.

Lemma 8.2.1 *Let f_1 and f_2 be two elements in $\mathbb{C}_n$ then*

$$\zeta_1.\zeta_{2n} \leqslant 2^{(n-1)/2}\zeta_{1n}\zeta_{2n}.$$

Lemma 8.2.2 *X is a domain in $\mathbb{C}_n,\ n \leqslant 2$ if and only if X_1 and X_2 are domains in $\mathbb{C}_{n-1}$.*

Lemma 8.2.3 *$\zeta_1 + i_n\zeta_2 \in \mathbb{C}_n$. $n \geqslant 2$ is non-singular if and only if $\zeta_1 - i_{n-1}\zeta_2$ and $\zeta_1 + i_{n-1}\zeta_2$ are non-singular and $\zeta_1 + i_n\zeta_2 \in \mathbb{C}_n$ is singular if at least one of elements $\zeta_1 - i_{n-1}\zeta_2$ and $\zeta_1 + i_{n-1}\zeta_2$ is singular.*

Lemma 8.2.4 *(Cauchy's Integral Formula): If $f : X \to \mathbb{C}_n$ satisfies strong Stolz condition in X and if ζ_0 is a point in the interior of $\boldsymbol{C}$, where $\boldsymbol{C}$ in a curve in X, then*

$$f(\zeta_0) = \frac{1}{2\pi i_1}\int_{\boldsymbol{C}} \frac{f(\zeta)}{\zeta - \zeta_0} d\zeta.$$

8.3 Theorems.

Theorem 8.3.1 *Let $f : X(\subseteq \mathbb{C}_n) \to \mathbb{C}_n,\ n \geqslant 1$ be a multicomplex valued function which satisfy strong Stolz condition in X. Let $a \in X$. Define a subset Y of neighbourhood of a, $N(a,d)$ as $Y = \{\zeta \in N(a,d) : \zeta = a + (x_1 + i_1x_2), x_1 + i_1x_2 \in \mathbb{C}\}$. Consider a circle $\mathbf{C}$ whose equation is $\zeta = a + r(\cos\theta + i_1\sin\theta), 0 \leqslant \theta \leqslant 2\pi$ is contained in Y. Also, let ζ_0 be interior point of $\mathbf{C}$, then*

$$\int_{\mathbf{C}} \frac{f(\zeta)}{(\zeta - \zeta_0)^{k+1}} d\zeta = \frac{2\pi i_1}{k!} f^k(\zeta_0).$$

Proof. We prove this result by induction on k. For $k = 0$, the above result is the same as Lemma 8.2.4 which is known as Cauchy's integral formula for multicomplex valued function.
Let us assume that the result is true for $k = p - 1$. Let us choose $h \in \mathbb{C}_n$ so small such that $(\zeta_0 + h)$ is inside $\mathbf{C}$.
Therefore, we can write

$$\begin{aligned}
&\frac{f^{(p-1)}(\zeta_0 + h) - f^{(p-1)}(\zeta_0)}{h} \\
=&\frac{(p-1)!}{2\pi i_1}\int_{\mathbf{C}}\frac{f(\zeta)}{h}\left(\frac{1}{(\zeta-\zeta_0-h)^p} - \frac{1}{(\zeta-\zeta_0)^p}\right)d\zeta \\
=&\frac{(p-1)!}{2\pi i_1}\int_{\mathbf{C}}\left[\frac{1}{(\zeta-(\zeta_0+h))^p(\zeta-\zeta_0)} + \dots + \frac{1}{(\zeta-(\zeta_0+h))(\zeta-\zeta_0)^p}\right]f(\zeta)d\zeta.
\end{aligned}$$

f satisfy strong Stolz condition

$$f^{(p-1)}(\zeta_0 + h) - f^{(p-1)}(\zeta_0) = f^p(\zeta_0)h + r(f; \zeta_0 + h, \zeta_0)h$$

$$\lim_{h \to 0} r(f; \zeta + h, \zeta) = 0.$$

Therefore,

$$\begin{aligned}
f^p(\zeta_0) &= \frac{(p-1)!}{2\pi i_1}\int_{\mathbf{C}}\frac{pf(\zeta)}{(\zeta-\zeta_0)^{p+1}}d\zeta \\
i.e., \int_{\mathbf{C}}\frac{f(\zeta)}{(\zeta-\zeta_0)^{p+1}} &= \frac{2\pi i_1}{p!}f^p(\zeta_0).
\end{aligned}$$

By the principle of induction, the result is true for all $k \in \mathbb{N}$. Thus,

$$\int_{\mathbf{C}}\frac{f(\zeta)}{(\zeta-\zeta_0)^{k+1}}d\zeta = \frac{2\pi i_1}{k!}f^k(\zeta_0).$$

This completes the proof. ∎

Remark 8.3.1 *This theorem can be identified as multicomplex version of Cauchy's Integral Formula for higher derivatives.*

Theorem 8.3.2 *Let $f : X(\subseteq \mathbb{C}_n) \to \mathbb{C}_n$, $n \geqslant 1$ be a multicomplex valued function which satisfy strong Stolz condition in X, where X is open. Also let $a \in X$ and $N(a, d)$ be a neighbourhood of a in X. Suppose Y be a subset of $N(a, d)$ defined by $Y = \{\zeta \in N(a, d) : \zeta = x_1 + i_1x_2, x_1 + i_1x_2 \in \mathbb{C}\}$, then Y is the portion of a 2-dimensional plane in $\mathbb{C}_n$ contained in $N(a, d)$. If $0 \leqslant r \leqslant d$ then the circle $\boldsymbol{C} = \{\zeta \in \mathbb{C} : \zeta = a + r(\cos\theta + i_1 \sin\theta), 0 \leqslant \theta \leqslant 2\pi\}$ is subset of Y. Also, if $\|f(\zeta)\| \leqslant M$, $\forall\zeta$ on $\boldsymbol{C}$, then*

$$\frac{f^k(a)}{k!} \leqslant \frac{M}{r^k}.$$

Proof. From the previous theorem, we have

$$f^k(a) = \frac{k!}{2\pi i_1} \int_{\mathbf{C}} \frac{f(\zeta)}{(\zeta - a)^{k+1}} d\zeta$$

Now,

$$\begin{aligned} \frac{f^k(a)}{k!} &\leqslant \frac{1}{2\pi} \int_{\mathbf{C}} \frac{f(\zeta)}{(\zeta - a)^{k+1}} d\zeta \\ &\leqslant \frac{1}{2\pi} . \frac{M}{r^{k+1}} . 2\pi r \\ &= \frac{M}{r^k}. \end{aligned}$$

This completes the proof. ■

Remark 8.3.2 *This theorem can be identified as multicomplex version of Cauchy's Inequality.*

Theorem 8.3.3 *Let X be a domain in $\mathbb{C}_n$, $n \geqslant 2$. $F : X \to \mathbb{C}_n$ and $G : X \to \mathbb{C}_n$ satisfies strong Stolz condition in X. Also*

$$F(\zeta_1 + i_n\zeta_2) = f_1(\zeta_1 - i_{n-1}\zeta_2)\ e(i_{n-1}i_n) + f_2(\zeta_1 + i_{n-1}\zeta_2)\ e(-i_{n-1}i_n)$$

and

$$G(\zeta_1 + i_n\zeta_2) = g_1(\zeta_1 - i_{n-1}\zeta_2)\ e(i_{n-1}i_n) + g_2(\zeta_1 + i_{n-1}\zeta_2)\ e(-i_{n-1}i_n).$$

If (i) $\|F(\zeta)\|_n < \|G(\zeta)\|_n$, $\forall \zeta \in X$,
(ii) $G(\zeta)$ having at least one zero in X, then $H(\zeta) = F(\zeta) + G(\zeta)$ has the same number of zeros in X as $G(\zeta)$ does.

Proof. We prove this result by induction on n.
For $n = 2$, the above result is true for bicomplex valued function.
Let us assume that the result is true for $n = 1$.
Let $ß_1 + i_nß_2 = (ß_1 - i_{n-1}ß_2)\ e(i_{n-1}i_n) + (ß_1 + i_{n-1}ß_2)\ e(-i_{n-1}i_n) \in X$ such that $G(ß) = 0$ implies

$$g_1(ß_1 - i_{n-1}ß_2) = 0,\ g_1(ß_1 + i_{n-1}ß_2) = 0.$$

If $g_1(ß_1 - i_{n-1}ß_2) = 0$, by (i)

$$\begin{aligned} &F(ß)_n \leqslant G(ß)_n \\ i.e., &\sqrt{\frac{f_1(ß_1 - i_{n-1}ß_2)_{n-1}^2 + f_2(ß_1 + i_{n-1}ß_2)_{n-1}^2}{2}} \\ &< \sqrt{\frac{g_1(ß_1 - i_{n-1}ß_2)_{n-1}^2 + g_2(ß_1 + i_{n-1}ß_2)_{n-1}^2}{2}} \\ &= \sqrt{\frac{g_1(ß_1 - i_{n-1}ß_2)_{n-1}^2}{2}} \\ i.e., &\ f_1(ß_1 - i_{n-1}ß_2)_{n-1} < g_1(ß_1 - i_{n-1}ß_2)_{n-1}. \end{aligned}$$

By induction hypothesis, $f_1(ß_1 - i_{n-1}ß_2) + g_1(ß_1 - i_{n-1}ß_2)$ has the same number of zeros as $g_1(ß_1 - i_{n-1}ß_2)$ does. Now, if $g_2(ß_1 + i_{n-1}ß_2) = 0$, by similar arguments we can also show that

$$f_2(ß_1 + i_{n-1}ß_2)_{n-1} < g_2(ß_1 + i_{n-1}ß_2)_{n-1}.$$

By induction hypothesis, $f_2(ß_1 + i_{n-1}ß_2) + g_2(ß_1 + i_{n-1}ß_2)$ has the same number of zeros as $g_2(ß_1 + i_{n-1}ß_2)$ does.
Therefore,

$$\begin{aligned} H(ß) =& F(ß) + G(ß) \\ =& [f_1(ß_1 - i_{n-1}ß_2)\ e(i_{n-1}i_n) + f_2(ß_1 + i_{n-1}ß_2)\ e(-i_{n-1}i_n)] \\ &+ [g_1(ß_1 - i_{n-1}ß_2)]\ e(i_{n-1}i_n) + g_2(ß_1 + i_{n-1}ß_2)\ e(-i_{n-1}i_n)] \\ =& [f_1(ß_1 - i_{n-1}ß_2) + g_1(ß_1 - i_{n-1}ß_2)]\ e(i_{n-1}i_n) \\ &+ [f_2(ß_1 + i_{n-1}ß_2) + g_2(ß_1 + i_{n-1}ß_2)]\ e(-i_{n-1}i_n) \end{aligned}$$

has same number of zeros in X as $G(ß)$ does. ∎

Remark 8.3.3 *This theorem can be identified as multicomplex version of Rouche's Theorem.*

Theorem 8.3.4 *Let X be a domain in $\mathbb{C}_n$, $n \geqslant 2$. $F : X \to \mathbb{C}_n$ and $G : X \to \mathbb{C}_n$ satisfies strong Stolz condition in X. Also*

$$F(\zeta_1 + i_{n-1}\zeta_2) = f_1(\zeta_1 - i_{n-1}\zeta_2)\ e(i_{n-1}i_n) + f_2(\zeta_1 + i_{n-1}\zeta_2)\ e(-i_{n-1}i_n)$$

and

$$G(\zeta_1 + i_{n-1}\zeta_2) = g_1(\zeta_1 - i_{n-1}\zeta_2)\ e(i_{n-1}i_n) + g_2(\zeta_1 + i_{n-1}\zeta_2)\ e(-i_{n-1}i_n).$$

If (i) $\|F(\zeta)\|_n < \|G(\zeta)\|_n$, $\forall \zeta \in \partial X$,
(ii) $G(\zeta)$ having at least one zero in X, then $H(\zeta) = F(\zeta) + G(\zeta)$ has the same number of zeros in X as $G(\zeta)$ does.

Proof. We prove this result by induction on n.
For $n = 2$, the above result is true for bicomplex valued function.
Let us assume that the result is true for $n = 1$.
Let $ß_1 + i_n ß_2 = (ß_1 - i_{n-1}ß_2)\ e(i_{n-1}i_n) + (ß_1 + i_{n-1}ß_2)\ e(-i_{n-1}i_n) \in X$ such that $H(ß) = 0$.

$$\begin{aligned}
&i.e.,\ F(ß) + G(ß) = 0\\
&i.e.,\ [f_1(ß_1 - i_{n-1}ß_2) + g_1(ß_1 - i_{n-1}ß_2)]\ e(i_{n-1}i_n)\\
&\qquad + [f_2(ß_1 + i_{n-1}ß_2) + g_2(ß_1 + i_{n-1}ß_2)]\ e(-i_{n-1}i_n) = 0\\
&i.e.,\ f_1(ß_1 - i_{n-1}ß_2) + g_1(ß_1 - i_{n-1}ß_2) = 0\\
&\qquad f_2(ß_1 + i_{n-1}ß_2) + g_2(ß_1 + i_{n-1}ß_2) = 0
\end{aligned}$$

and therefore

$$f_1(ß_1 - i_{n-1}ß_2)_{n-1} = g_1(ß_1 - i_{n-1}ß_2)_{n-1}, \tag{8.1}$$

$$f_2(ß_1 + i_{n-1}ß_2)_{n-1} = g_2(ß_1 + i_{n-1}ß_2)_{n-1} \tag{8.2}$$

$$\because F(ß)_n < G(ß)_n$$

$$f_1(ß_1 - i_{n-1}ß_2)^2_{n-1} + f_2(ß_1 + i_{n-1}ß_2)^2_{n-1} < g_1(ß_1 - i_{n-1}ß_2)^2_{n-1} + g_2(ß_1 + i_{n-1}ß_2)^2_{n-1}. \tag{8.3}$$

If Equation (8.1) happens then from Equation (8.3), we get that

$$f_2(ß_1 + i_{n-1}ß_2)_{n-1} < g_2(ß_1 + i_{n-1}ß_2)_{n-1}.$$

By previous theorem $g_2(ß_1 + i_{n-1}ß_2)$ has at least a zero in X.
Similarly if Equation (8.2) happens then from Equation (8.3), we obtain that

$$f_1(ß_1 - i_{n-1}ß_2)_{n-1} < g_1(ß_1 - i_{n-1}ß_2)_{n-1}.$$

By previous theorem $g_1(ß_1 - i_{n-1}ß_2)$ has at least a zero in X.
Hence combining these two result $G(ß)$ has at least one zero in X, which contradicts the fact that $G(\zeta)$ has no zero in X.
Hence, $H(\zeta) = (F + G)(\zeta)$ has no zero in X. ■

Conclusion: The research works in previous chapters may be explored with the space $\mathbb{C}_n$.

****

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