Study of Berezin number inequalities of operators on reproducing kernel Hilbert space

Anirban Sen

(Index No.: 35/22/Maths./27)

THIS THESIS IS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY IN SCIENCE

DEPARTMENT OF MATHEMATICS

JADAVPUR UNIVERSITY

KOLKATA-700032

INDIA

DECEMBER, 2024

Study of Berezin number inequalities of operators on reproducing kernel Hilbert space

Anirban Sen

(Index No.: 35/22/Maths./27)

THIS THESIS IS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY IN SCIENCE

DEPARTMENT OF MATHEMATICS

JADAVPUR UNIVERSITY

KOLKATA-700032

INDIA

DECEMBER, 2024

যাদবপুর বিশ্ববিদ্যালয়

FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS

JADAVPUR UNIVERSITY
KOLKATA-700 032, INDIA
TELEPHONE : 91 (33) 2414-6717

CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Study of Berezin number inequalities of operators on reproducing kernel Hilbert space**" submitted by **Anirban Sen** who got his name registered on 15/02/2022 (**Index No.: 35/22/Maths./27**) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own research work under the supervision of **Prof. Kallol Paul**, Department of Mathematics, Jadavpur University, Kolkata 700032, India and that neither this thesis nor any part of it has been submitted for either any degree / diploma or any other academic award anywhere before.

Professor
DEPARTMENT OF MATHEMATICS
Jadavpur University
Kolkata - 700 032, West Bengal

KPaul 04.12.2024

....

(Prof. Kallol Paul)
(Signature of the Supervisor
and date with official seal)

Dedicated to my parents

Mr. Nirmal Kumar Sen

and

Mrs. Shilpi Sen

ACKNOWLEDGMENTS

This thesis represents not only my work over the past years but also the guidance, support and encouragement of many people. It is a pleasure to convey my gratitude to all of them in this humble acknowledgement.

First and foremost, I would like to express my deepest gratitude to my supervisor, Professor Kallol Paul, Department of Mathematics, Jadavpur University, Kolkata, for his continuous support, guidance and encouragement throughout my PhD journey. His expertise, insightful feedback and commitment to my research have been invaluable. His guidance helps me to grow both professionally and personally. Without his support this work would not have been possible. I am truly fortunate to have had the opportunity to learn from such an inspiring mentor. Thank you for believing in me and for pushing me to achieve my best.

I am also grateful to the members of my research advisory committee, Head of the Department, Department of Mathematics, Jadavpur University, Kolkata, Professor Binayak S. Choudhury, Department of Mathematics, IIEST, Shibpur and Dr. Riddhick Birbonshi, Department of Mathematics, Jadavpur University, Kolkata, for their insightful comments and encouragements.

I would like to acknowledge the PROM Programme no. POWR.03.03.00-00PN13/18 to obtain materials for PhD thesis in the Department of Mathematics, Pedagogical University, Krakow, during 10.03.2023 to 30.03.2023.

I am extremely grateful to Dr. Pintu Bhunia, Department of Mathematics, Indian Institute of Science, Bengaluru, Karnataka, India, for his insightful advice, encouragement and support which significantly enhanced my research.

I would like to extend my heartfelt thanks to Dr. Riddhick Birbonshi, for his valuable feedback and discussions over the past few years.

I would like to thank Dr. Santanu Bag, Dr. Debmalya Sain, Dr. Puja Ghosh, Dr. Kalidas Mandal, Dr. Arpita Mal, Dr. Raj Kumar Nayak, Dr. Suvendu Jana, Mr. Souvik Ghosh, Mr. Shamim Sohel, Mr. Jayanta Manna, Miss Somdatta Barik and Miss Reepa Datta for sharing their knowledge, experience and companionship throughout this journey. The supportive environment in our department greatly facilitated my work.

I would like to thank the Council of Scientific and Industrial Research (CSIR), Government of India, New Delhi, for providing financial support for my research.

Last but not least, I would like to thank my family and friends for their support and encouragement throughout my life. This journey would not have been possible without their support.

Anirban Sen

Anirban Sen

CONTENTS

1 Introduction **1**

1.1 Motivation and historical backgrounds . . . 1

1.2 Preliminaries . . . 2

1.3 Outline of the thesis . . . 8

2 Development of the Berezin number inequalities **10**

2.1 Introduction . . . 10

2.2 Auxiliary results . . . 11

2.3 New upper bounds for single operator . . . 12

2.4 Development of upper bounds via Cartesian decomposition . . . 20

2.5 On the lower bounds of Berezin number . . . 22

3 Sharper bounds of the Berezin number for product operators **25**

3.1 Introduction . . . 25

3.2 Auxiliary results . . . 26

3.3 Inequalities for product of operators . . . 27

4 Berezin number and Berezin norm: inequalities and equalities **36**

4.1 Introduction . . . 36

4.2 Berezin number and Berezin norm inequalities . . . 37

4.3 Refined inequalities for sum of two operators . . . 48

5 On a new norm on reproducing kernel Hilbert space operators **53**

5.1 Introduction . . . 53

5.2 Introduction of the α-Berezin norm and its properties 54
5.3 α-Berezin norm and Berezin number inequalities 56

6 Berezin number and Berezin norm inequalities of operator matrices 67
6.1 Introduction 67
6.2 Berezin number and Berezin norm inequalities for $n \times n$ operator matrices 68
6.3 Berezin number inequalities for 2×2 operator matrices 75
6.4 Further Berezin number inequalities 80

7 Final Remarks and future prospect 87

References 90

CHAPTER 1

INTRODUCTION

1.1 Motivation and historical backgrounds

Inequalities are mathematical expressions that indicate the relationship between two values. It also indicates whether one quantity is less than, greater than, or equal to another. The concept of inequalities in mathematics has not only a rich historical background but also has been gradually developed alongside the development of mathematical theory. In ancient times, Euclid studied the geometric properties of triangle using inequalities. He proved that the sum of the lengths of any two sides of a triangle is greater than the length of the remaining side and this result is essentially an early version of the triangle inequality. Later, in the 17th century, Jacob and Johann Bernoulli proved a fundamental inequality which is known as the arithmetic mean-geometric mean (AM-GM) inequality. After that, in the 19th and 20th centuries, mathematicians like Augustin-Louis Cauchy, Karl Weierstrass, and David Hilbert formalized the concept of inequalities, and they proved several new inequalities, such as the Cauchy-Schwarz inequality, Hölder's inequality, and Minkowski inequality. These results established a link between the inner product and norms and also played an important role in generalizing the concept of inequalities and advancing mathematical analysis. In the early 20th century, G.H. Hardy, J.E. Littlewood and G. Pólya made their significant contributions to the theory of inequalities. Their book on inequalities was published in 1934, and this book [51] is the first classic textbook on inequalities. The second book on this topic [13] was written by E.F. Beckenbach and R. Bellman and published in 1961. This book is another significant contribution to the literature on inequalities. In recent years, matrix inequalities and numerical radius inequalities have become two important research topics. Researchers have developed new inequalities, refined existing bounds, and explored applications in operator theory, quantum information, and numerical analysis. For recent developments in these topics, we refer to the following books [17, 24]. Now, let us discuss the Berezin number which is a central concept in the title of this thesis, but first, we will say a few words about reproducing kernel Hilbert spaces. The concept of reproducing kernel Hilbert spaces began with David Hilbert's work in 1904. The modern theory of reproducing kernel Hilbert space was formally defined by N. Aronszajn in his seminal paper [4]. The concept of the Berezin transform was introduced by the Soviet mathematician F.A. Berezin in his papers [14, 15]. In [58, 60], M.T. Karaev introduced two new numerical characteristics associated with the Berezin symbol, which he called the Berezin range and the Berezin radius. Berezin range and Berezin radius are also known as Berezin set and Berezin number, respectively. These are very useful for studying the properties of bounded linear operators on reproducing kernel Hilbert spaces. The Berezin symbol and Berezin number have numerous applications in quantum mechanics, signal processing, and control theory, providing valuable information about the structure of operators in these domains. The Berezin number inequalities serve as a valuable tool for investigating various phenomena in these fields. The study of Berezin number inequalities has seen significant advancements, establishing new bounds and refining existing results. The exploration of these inequalities continues to be an active area of research, and in this dissertation, we further investigate the Berezin number inequalities of operators on reproducing kernel Hilbert spaces. Through detailed proofs and examples, our main aim is to contribute to understand the Berezin number inequalities and their significance in the broader context of operator theory.

1.2 Preliminaries

We begin with the definitions of norm and inner product, which generalize the concepts of length and dot product in arbitrary vector spaces.

Definition 1.1. *Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$, where $\mathbb{F}$ is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A norm on $\mathbb{V}$ is a function $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ with the following properties:*

$$(i)\quad \|x\| \geq 0 \ \text{ for all } x \in \mathbb{V} \ \text{ and } \ \|x\| = 0 \ \text{ if and only if } \ x = 0.$$

$$(ii)\quad \|\alpha x\| = |\alpha|\|x\| \quad \text{for all } x \in \mathbb{V} \text{ and } \alpha \in \mathbb{F}.$$

$$(iii)\quad \|x + y\| \leq \|x\| + \|y\| \ \text{ for all } x, y \in \mathbb{V}.$$

Here property (iii) is known as the triangle inequality. A vector space $\mathbb{V}$ equipped with a norm $\|\cdot\|$ is called normed linear space. The norm $\|\cdot\|$ defines a metric $d(\cdot,\cdot)$ on $\mathbb{V}$, and is given by $d(x,y) = \|x-y\|$. If the metric is complete then $(\mathbb{V}, \|\cdot\|)$ or simply $\mathbb{V}$ is called a Banach space. It follows from the triangle inequality that the norm is a continuous function on $\mathbb{V}$.

Definition 1.2. *An inner product on a complex vector space $\mathbb{V}$ is a mapping $\langle\cdot,\cdot\rangle : \mathbb{V}\times\mathbb{V}\to\mathbb{C}$ that satisfies the following properties:*

$$(i)\quad \langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle \quad \textit{for all } x,y,z\in\mathbb{V}.$$
$$(ii)\quad \langle \alpha x, y\rangle = \alpha\langle x,y\rangle \quad \textit{for all } x,y\in\mathbb{V} \textit{ and } \alpha\in\mathbb{C}.$$
$$(iii)\quad \langle x,y\rangle = \overline{\langle y,x\rangle} \quad \textit{for all } x,y\in\mathbb{V}.$$
$$(iv)\quad \langle x,x\rangle \geq 0 \quad \textit{for all } x\in\mathbb{V} \quad \textit{and} \quad \langle x,x\rangle = 0 \quad \textit{if and only if} \quad x=0.$$

An inner product space is a vector space $\mathbb{V}$ together with an inner product defined on $\mathbb{V}$. An inner product on $\mathbb{V}$ defines a norm on $\mathbb{V}$ given by $\|x\| = \langle x,x\rangle^{1/2}$, and a metric on $\mathbb{V}$ given by

$$d(x,y) = \|x-y\| = \langle x-y, x-y\rangle^{1/2}. \tag{1.1}$$

If the inner product space $\mathbb{V}$ with respect to the metric (1.1) is complete then it is called complex Hilbert space. The next result is known as the Reisz representation Theorem which is a fundamental result in functional analysis that characterizes all bounded linear functionals on a Hilbert space, see [67].

Theorem 1.1. *For every bounded linear functional f on a complex Hilbert space $\mathbb{H}$, there exists a unique element $z_f\in\mathbb{H}$ with $\|f\| = \|z_f\|$ such that*

$$f(x) = \langle x, z_f\rangle \quad \textit{for all } x\in\mathbb{H}.$$

A straightforward computation shows that the norm in any inner product space satisfies the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Furthermore, if x is orthogonal to y i.e. $\langle x,y\rangle = 0$ then the following equality holds:

$$\|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2.$$

This result is known as the Pythagorean Theorem. The norm and the inner product are related by the polarisation identity

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2). \tag{1.2}$$

The identity (1.2) also allows us to recover the inner product from the norm. An inner product and the norm induced by the inner product satisfy the Cauchy-Schwartz inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \tag{1.3}$$

and the equality holds if and only if x, y are linearly dependent. By applying (1.3) it is easy to verify that the inner product is a continuous function on $\mathbb{V} \times \mathbb{V}$. We refer to the readers to [16, 32, 67, 71, 89] for more about normed linear space and inner product space.

Let $\mathbb{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the norm induced by the inner product. Let $\mathbb{B}(\mathbb{H})$ be the C^*-algebra of all bounded linear operators on $\mathbb{H}$. An operator A is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{H}$, and we write $A \geq 0$. For $A \in \mathbb{B}(\mathbb{H})$, A^* denotes the adjoint of A and $|A|$ denotes the positive operator $(A^*A)^{1/2}$. Any $A \in \mathbb{B}(\mathbb{H})$ can be expressed uniquely as $A = \Re(A) + i\Im(A)$. Here $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$ are called real and imaginary parts of A, respectively. This decomposition is called the Cartesian decomposition of A. Recall that for $A \in \mathbb{B}(\mathbb{H})$ the operator norm of A is defined as $\|A\| = \sup\{\|Ax\| : x \in \mathbb{H}, \|x\| = 1\}$. The numerical range of A is denoted by $W(A)$ and is defined as

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{H}, \|x\| = 1\}.$$

Next we mention some basic properties of the numerical range of Hilbert space operators. The first result, known as the Ellipse Lemma, fully describes the numerical range of an operator on a two-dimensional Hilbert space, for the proof, we refer to [35].

Theorem 1.2. *Let A be an operator on a two-dimensional space $\mathbb{H}$, then $W(A)$ is an ellipse whose foci are the eigenvalues of A.*

The next result, known as the Toeplitz-Hausdorff Theorem, establishes one of the most important property of the numerical range—its convexity. In fact, the study of the numerical range originates from the discovery of this property by Toeplitz and Hausdorff. Originally, in [90] Toeplitz proved that the boundary of the numerical range is always a convex curve and later in [52] Hausdorff showed that $W(A)$ is simply connected. The Toeplitz-Hausdorff Theorem has numerous proofs and we refer to [47, 68].

Theorem 1.3. *The numerical range of an operator is convex.*

For further insights on the numerical range, readers can see [46, 49, 92]. One of the key numerical constants associated with the numerical range of a bounded linear operator is the numerical radius. For $A \in \mathbb{B}(\mathbb{H})$, the numerical radius of A, denoted by $w(A)$, is defined as the radius of the smallest circular disk centered at the origin of complex plane and containing the numerical range $W(A)$, i.e.

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, and is equivalent to the operator norm $\|\cdot\|$. In fact, every $A \in \mathbb{B}(\mathbb{H})$ satisfies the following inequality:

$$\frac{\|A\|}{2} \le w(A) \le \|A\|. \tag{1.4}$$

The first inequality becomes equality if $A^2 = 0$ and the second inequality becomes equality if A is a normal operator (see [65, Cor. 1] and [46, Th. 1.3-2]). Over the years, various refinements of the inequality (1.4) have been established, interested readers may see [2, 3, 19, 25, 26, 27, 28, 29, 30, 57, 64, 72, 74, 75, 76].

Now, we introduce the definition of a reproducing kernel Hilbert space, which is a special class of Hilbert spaces that has some additional structure, see [77]. Let Ω be a non-empty set and $\mathcal{F}(\Omega, \mathbb{C})$ be the set of all complex valued functions on Ω. Then $\mathcal{F}(\Omega, \mathbb{C})$ is a vector space over $\mathbb{C}$ with the operation of pointwise addition and scaler multiplication.

Definition 1.3. *For a non-empty set Ω, a subset $\mathcal{H}$ of $\mathcal{F}(\Omega, \mathbb{C})$ is said to be a reproducing kernel Hilbert space(RKHS in short) on Ω if $\mathcal{H}$ has the following properties:*

(*i*) *$\mathcal{H}$ is a vector subspace of $\mathcal{F}(\Omega, \mathbb{C})$.*

(*ii*) *$\mathcal{H}$ is endowed with an inner product $\langle \cdot, \cdot \rangle$, with respect to which $\mathcal{H}$ is a Hilbert space.*

(*iii*) *For every $\lambda \in \Omega$ the point evaluation functional $E_\lambda : \mathcal{H} \to \mathbb{C}$ defined by $E_\lambda(f) = f(\lambda)$, is bounded.*

If $\mathcal{H}$ is an RKHS on the set Ω then the Riesz representation Theorem shows that the linear evaluation functional is given by the inner product with a unique vector in $\mathcal{H}$. Thus for each $\lambda \in \Omega$, there exists unique $k_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle \quad \text{for all } f \in \mathcal{H}. \tag{1.5}$$

The identity (1.5) is called the reproducing property of $\mathcal{H}$. The collection of functions $\{k_\lambda : \lambda \in \Omega\}$ is said to be the set of all reproducing kernels of $\mathcal{H}$. The normalized reproducing kernel at

$\lambda \in \Omega$ is the function $\hat{k}_\lambda = k_\lambda/\|k_\lambda\|$. The collection of functions $\{\hat{k}_\lambda : \lambda \in \Omega\}$ is the set of all normalized reproducing kernels of $\mathcal{H}$. Before moving on, we give two will-known examples of reproducing kernel Hilbert space.

Example 1.4. *(i) The space of complex n-tuples $\mathbb{C}^n$ is a reproducing kernel Hilbert space on the set $\Omega = \{1, 2, \ldots, n\}$ with the usual inner product on $\mathbb{C}^n$ given by $\langle u, v\rangle = \sum_{i=1}^n u_i\overline{v_i}$ for $u = (u_1, u_2, \ldots u_n), v = (v_1, v_2, \ldots v_n) \in \mathbb{C}^n$. Let $\{e_j\}_{j=1}^n$ denote the canonical orthonormal basis for $\mathbb{C}^n$, i.e., $e_j(i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, then for every $u \in \mathbb{C}^n$ we have $u(j) = u_j = \langle u, e_j\rangle$. So, the canonical basis for $\mathbb{C}^n$ is precisely the set of kernel functions for point evaluations when we regard $\mathbb{C}^n$ as a space of functions.*

(ii) The Hardy-Hilbert space on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is denoted by $H^2(\mathbb{D})$, and is defined as

$$H^2(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) : f(z) = \sum_{n=0}^\infty a_n z^n, \sum_{n=0}^\infty |a_n|^2 < \infty \right\},$$

with the inner product

$$\langle f, g\rangle = \sum_{n=0}^\infty a_n \bar{b}_n,$$

where $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$. Here $Hol(\mathbb{D})$ is the collection of all holomorphic functions on $\mathbb{D}$. $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space on $\mathbb{D}$ with the kernel functions at $\lambda \in \mathbb{D}$ given by

$$k_\lambda(z) = \frac{1}{1 - \bar{\lambda} z} \quad \text{for all } z \in \mathbb{D}.$$

For more details about the Hardy-Hilbert space, we refer to the book [70].

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on the set Ω. For $A \in \mathbb{B}(\mathcal{H})$, the function $\widetilde{A} : \Omega \to \mathbb{C}$ defined by

$$\widetilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle \quad \text{for all } \lambda \in \Omega$$

is called Berezin symbol of A. This was first introduced by Berezin in [14, 15]. A reproducing kernel Hilbert space possesses "Ber property", if the Berezin symbol uniquely determines operator (see [59]). Here we note that the most familiar RKHS, including Hardy-Hilbert space and Bergman spaces have the "Ber property" (see [96]). For in-depth discussion of the Berezin range, refer to [85]. The Berezin set of A, denoted by $\mathbf{Ber}(A)$, introduced in [58] is defined as

$$\mathbf{Ber}(A) = \{\widetilde{A}(\lambda) : \lambda \in \Omega\}.$$

In [58] Karaev provided explicit geometric shape of the Berezin set of some operators acting on the Model space. It follows from the definition that for any bounded linear operator the Berezin set is a subset of the numerical range. The convexity of the numerical range is one of its most important property and motivated by this property in [33] Cowen and Felder arise the following question "Given a bounded linear operator A on a RKHS $\mathcal{H}$, is $\mathbf{Ber}(A)$ is convex? Conversely, if $\mathbf{Ber}(A)$ is convex, what can be said of A?". In [33] they characterizes the convexity of the Berezin range for matrices, multiplication operators and some class of composition operators acting on $H^2(\mathbb{D})$. Later the convexity of the Berezin range on Bergman space is studied in [5]. Karaev [60] first formally presented the Berezin number of A, which is denoted by $\mathbf{ber}(A)$ and is defined as

$$\mathbf{ber}(A) = \sup\{|\widetilde{A}(\lambda)| : \lambda \in \Omega\}.$$

In [7], Bakherad and Yamancı introduced the Berezin norm of A as

$$\|A\|_{ber} = \sup\{|\langle A\hat{k}_\lambda, \hat{k}_\mu\rangle| : \lambda, \mu \in \Omega\}.$$

The following result easily follows from the definitions of the Berezin number and the Berezin norm:

Proposition 1.1. *Let $A, B \in \mathbb{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}$. Then the following hold:*

(i) $\boldsymbol{ber}(\alpha A) = |\alpha|\boldsymbol{ber}(A)$.

(ii) $\boldsymbol{ber}(A + B) \leq \boldsymbol{ber}(A) + \boldsymbol{ber}(B)$.

(iii) $\boldsymbol{ber}(A) \leq w(A)$ *and* $\boldsymbol{ber}(A) \leq \|A\|_{ber} \leq \|A\|$.

(iv) $\boldsymbol{ber}(A) = \boldsymbol{ber}(A^*)$ *and* $\|A\|_{ber} = \|A^*\|_{ber}$.

Here we note that $\|\cdot\|_{ber}$ defines a norm on $\mathbb{B}(\mathcal{H})$ but $\mathbf{ber}(\cdot)$ does not generally defines a norm on $\mathbb{B}(\mathcal{H})$. Furthermore, if the RKHS $\mathcal{H}$ possesses the "Ber property" then $\mathbf{ber}(\cdot)$ becomes a norm.

Over the years, many mathematicians have studied inequalities of reproducing kernel Hilbert space operators. The major interest of researchers is to refine and generalize the existing inequalities. By examining these inequalities, researchers have gained insights into the relationships between the Berezin numbers, Berezin norms, operator norms and numerical radius, and we refer to see [9, 12, 20, 39, 41, 43, 45, 69, 79, 81, 87, 88, 93, 94, 95]. Next, we highlight some important results related to the Berezin number inequalities. In [8], Bakherad and Garayev proved the following upper bound of Berezin number, namely, for any $A \in \mathbb{B}(\mathcal{H})$

$$\mathbf{ber}^2(A) \leq \frac{1}{2}\mathbf{ber}(A^*A + AA^*). \tag{1.6}$$

The authors also obtained in [8] the following inequality for product of three operators acting on RKHS. They proved that for any $A, X, B \in \mathbb{B}(\mathcal{H})$

$$\mathbf{ber}(A^*XB) \leq \frac{1}{2}\mathbf{ber}(B^*|X|B + A^*|X^*|A). \tag{1.7}$$

Taghavi et al. [86] generalize both the inequalities (1.6) and (1.7). They shown that if $A, X, B \in \mathbb{B}(\mathcal{H})$, then for all $\alpha \in [0,1]$ and $r \geq 2\max\{\alpha, 1-\alpha\}$ the following inequalities hold:

$$\mathbf{ber}^r(A) \leq \mathbf{ber}\left(\alpha|A|^{\frac{r}{2\alpha}} + (1-\alpha)|A^*|^{\frac{r}{2(1-\alpha)}}\right) \tag{1.8}$$

and

$$\mathbf{ber}^r(A^*XB) \leq \mathbf{ber}\left(\alpha(B^*|X|B)^{\frac{r}{2\alpha}} + (1-\alpha)(A^*|X^*|A)^{\frac{r}{2(1-\alpha)}}\right). \tag{1.9}$$

Further, Hajmohamadi et al. [48] obtained another inequality, namely, for any $A, X, B \in \mathbb{B}(\mathcal{H})$,

$$\mathbf{ber}^r(A^*XB) \leq \|X\|^r\mathbf{ber}\left(\frac{1}{\alpha}|A|^{\alpha r} + \frac{1}{\beta}|B|^{\beta r}\right), \tag{1.10}$$

for $r \geq 0$, $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\alpha r, \beta r \geq 2$. The authors also proved that for all $\alpha \in [0,1]$,

$$\mathbf{ber}(A^*XB) \leq \frac{1}{2}\mathbf{ber}\left(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A\right). \tag{1.11}$$

Clearly, the inequality (1.11) is a generalized versions of (1.7).

In this thesis, we develop various new Berezin number inequalities of bounded linear operators defined on reproducing kernel Hilbert space $\mathcal{H}$, which serve to tighten the inequalities, specifically, (1.6)–(1.11). The next section provides a concise overview of the thesis.

1.3 Outline of the thesis

This thesis contains seven chapters including the introductory one. In Chapter 1 we have provided a brief history of the Berezin number inequalities of reproducing kernel Hilbert space operators along with some definitions, notations and preliminary results.

The purpose of Chapter 2 is to develop the Berezin number inequalities for bounded linear operators acting on reproducing kernel Hilbert spaces. By applying some auxiliary results and using the Cartesian decomposition of bounded linear operators, we derive upper and lower bounds for reproducing kernel Hilbert space operators, which improve the existing results.

In Chapter 3, we derive new bounds of the Berezin number specifically for the sum and product of operators defined on reproducing kernel Hilbert spaces. These bounds not only generalize the existing bounds but also offer sharper estimates compared to existing ones.

The main aim of Chapter 4 is to establish several inequalities involving the Berezin number and the Berezin norm of operators defined on reproducing kernel Hilbert spaces. As a special case, we prove that if A is a positive bounded linear operator on a reproducing kernel Hilbert space, then $\|A\|_{ber} = \mathbf{ber}(A)$. We also provide an example to show that this equality may not hold for general self-adjoint operators. Additionally, we derive upper bounds for the Berezin norm in the case of the sum of two positive operators and for the α-weighted arithmetic mean of two operators.

In Chapter 5, we present a new norm on the space of all bounded reproducing kernel Hilbert space operators. This new norm is called the α-Berezin norm, which generalizes the Berezin number and the Berezin norm. We study some basic properties of this new norm and develop various inequalities involving the α-Berezin norm and the Berezin number. Using these inequalities, we obtain improved bounds for the Berezin number of reproducing kernel Hilbert space operators.

The purpose of Chapter 6 is to establish upper bounds for the Berezin number and Berezin norm of $n \times n$ operator matrices which refines the existing bounds. Among other results, we prove that if $A = [A_{ij}]$ is an $n \times n$ operator matrix with A_{ij} are bounded linear operators on a reproducing kernel Hilbert space for $i, j = 1, 2, \ldots, n$, then $\|A\|_{ber} \leq \|[\|A_{ij}\|_{ber}]\|$ and $\mathbf{ber}(A) \leq w([a_{ij}])$, where $a_{ii} = \mathbf{ber}(A_{ii})$, $a_{ij} = \||A_{ij}| + |A_{ji}^*|\|_{ber}^{\frac{1}{2}} \||A_{ji}| + |A_{ij}^*|\|_{ber}^{\frac{1}{2}}$ if $i < j$ and $a_{ij} = 0$ if $i > j$. We provide some examples which illustrate these bounds for some concrete operators acting on the Hardy-Hilbert space.

Finally, we present the future prospects of our work followed by the bibliography.

In the following chapters, we describe in detail results on the Berezin number inequalities of operators on reproducing kernel Hilbert spaces. To make each chapter self-contained, we provide a brief motivation, along with the relevant notations and terminologies, for the convenience of the readers.

CHAPTER 2

DEVELOPMENT OF THE BEREZIN NUMBER INEQUALITIES

2.1 Introduction

The main focus of this chapter is to develop new upper and lower bounds of the Berezin number of bounded linear operators defined on reproducing kernel Hilbert space that refines the existing bounds. Let us now introduce the following notations and terminologies that are used in this chapter.

Let Ω be a non-empty set and $\mathcal{H}$ be a reproducing kernel Hilbert space on Ω. Let $\mathbb{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For $A \in \mathbb{B}(\mathcal{H})$, A^* denote the adjoint of A and $|A|$ denote the positive operator $(A^*A)^{1/2}$. The Cartesian decomposition of A is given

Content of this chapter is partially based on the following three papers:

- P. Bhunia, **A. Sen** and K. Paul, Development of the Berezin number inequalities, Acta Math. Sin. (Engl. Ser.), 39 (2023), no. 7, 1219–1228. https://doi.org/10.1007/s10114-023-2090-1
- **A. Sen**, P. Bhunia and K. Paul, Bounds for the Berezin number of reproducing kernel Hilbert space operators, Filomat, 37 (2023), no. 6, 1741–1749. https://doi.org/10.2298/FIL2306741S
- **A. Sen**, P. Bhunia and K. Paul, Berezin number inequalities of operators on reproducing kernel Hilbert spaces, Rocky Mountain J. Math., 52 (2022), no. 3, 1039–1046. DOI: 10.1216/rmj.2022.52.1039

by $A = \Re(A) + i\Im(A)$, where $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$. Let $\{\hat{k}_\lambda : \lambda \in \Omega\}$ be the set of normalized reproducing kernel of $\mathcal{H}$. The Berezin symbol of A is the function $\widetilde{A} : \Omega \to \mathbb{C}$ defined by $\widetilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle$ for all $\lambda \in \Omega$. The Berezin set and the Berezin number of A are defined respectively by $\mathbf{Ber}(A) = \{\widetilde{A}(\lambda) : \lambda \in \Omega\}$ and $\mathbf{ber}(A) = \sup_{\lambda\in\Omega} |\widetilde{A}(\lambda)|$. Likewise the Berezin number, the least Berezin number is defined as $c(A) = \inf_{\lambda\in\Omega} |\widetilde{A}(\lambda)|$.

2.2 Auxiliary results

We begin with the following sequence of lemmas, which will be used repeatedly to reach our goal in this present chapter.

Lemma 2.1. *[84, p. 20] Let $A \in \mathbb{B}(\mathcal{H})$ with $A \geq 0$ and $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle,$$

for all $r \geq 1$.

Lemma 2.2. *[38] Let $A \in \mathbb{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha} x, x\rangle \ \ \langle |A^*|^{2(1-\alpha)} y, y\rangle,$$

for all $0 \leq \alpha \leq 1$.

Lemma 2.3. *[49, pp. 75-76] Let $A \in \mathbb{B}(\mathcal{H})$ and let $x \in \mathcal{H}$. Then*

$$|\langle Ax, x\rangle| \leq \langle |A|x, x\rangle^{1/2} \ \ \langle |A^*|x, x\rangle^{1/2}.$$

Lemma 2.4. *[66] Let $A \in \mathbb{B}(\mathcal{H})$ be self-adjoint and $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$|\langle Ax, x\rangle| \leq \langle |A|x, x\rangle.$$

Lemma 2.5. *[50, P. 26] For $a, b \geq 0$, $0 < \alpha < 1$, and $r \neq 0$, let $M_r(a, b, \alpha) = (\alpha a^r + (1-\alpha)b^r)^{1/r}$ and $M_0(a, b, \alpha) = a^\alpha b^{1-\alpha}$. Then*

$$M_r(a, b, \alpha) \leq M_s(a, b, \alpha),$$

for $0 \leq r \leq s$.

Lemma 2.6. *[50, P. 28] For $a, b \geq 0$, and $r > 0$, let $N_r(a,b) = (a^r + b^r)^{1/r}$. Then*

$$N_s(a,b) \leq N_r(a,b),$$

for $s \geq r > 0$.

Lemma 2.7. *[31] Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}\left(\|x\|\|y\| + |\langle x, y\rangle|\right).$$

2.3 New upper bounds for single operator

In this section we present some upper bounds of the Berezin number of single operator on reproducing Kernel Hilbert space.

Theorem 2.1. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^{2r}(A) \leq \boldsymbol{ber}\left(\alpha|A|^{2r} + (1-\alpha)|A^*|^{2r}\right),$$

for all $r \geq 1$ and for all $\alpha \in [0,1]$.

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then for all $\alpha \in [0,1]$, we have

$$\begin{aligned} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| &= \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| + (1-\alpha)|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\ &\leq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| + (1-\alpha)\|A^*\hat{k}_\lambda\|. \end{aligned}$$

By convexity of $f(t) = t^{2r}$ $(r \geq 1)$, we get

$$\begin{aligned} &|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} \\ &\leq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} + (1-\alpha)\|A^*\hat{k}_\lambda\|^{2r} \\ &\leq \alpha\langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \ \ \langle |A^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle^r + (1-\alpha)\langle |A^*|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \ \ \big(\text{by Lemma } 2.3\big) \\ &\leq \alpha\langle |A|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle \ \ \langle |A^*|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \ \ \big(\text{by Lemma } 2.1\big) \\ &\leq \frac{\alpha}{2}\left(\langle |A|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle |A^*|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\right) + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &\leq \frac{\alpha}{2}\left(\langle |A|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle\right) + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &= \left\langle \left\{\frac{\alpha}{2}\left(|A|^{2r} + |A^*|^{2r}\right) + (1-\alpha)|A^*|^{2r}\right\}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\ &\leq \mathbf{ber}\left(\frac{\alpha}{2}|A|^{2r} + (1-\frac{\alpha}{2})|A^*|^{2r}\right). \end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we get

$$\mathbf{ber}^{2r}(A) \leq \mathbf{ber}\left(\frac{\alpha}{2}|A|^{2r} + (1-\frac{\alpha}{2})|A^*|^{2r}\right). \tag{2.1}$$

Replacing A by A^* in (2.1), we get

$$\mathbf{ber}^{2r}(A) \leq \mathbf{ber}\left((1-\frac{\alpha}{2})|A|^{2r} + \frac{\alpha}{2}|A^*|^{2r}\right). \tag{2.2}$$

Combining (2.1) and (2.2), we get

$$\mathbf{ber}^{2r}(A) \leq \mathbf{ber}\left(\alpha|A|^{2r} + (1-\alpha)|A^*|^{2r}\right).$$

□

Considering $r = 1$ in Theorem 2.1 we get the following corollary.

Corollary 2.1. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned} \boldsymbol{ber}^2(A) &\leq \min_{\alpha\in[0,1]} \boldsymbol{ber}(\alpha A^*A + (1-\alpha)AA^*) & (2.3)\\ &\leq \frac{1}{2}\boldsymbol{ber}(A^*A + AA^*). & (2.4) \end{aligned}$$

Remark 2.2. *Clearly the inequallity (2.3) is better than the existing inequality $\boldsymbol{ber}^2(A) \leq \frac{1}{2}\boldsymbol{ber}(|A|^2 + |A^*|^2)$, obtained in [86, Cor. 3.5(i)].*

Our next result is

Theorem 2.3. *Let $A \in \mathbb{B}(\mathcal{H})$. Then for all $r \geq 1$ and for all $\alpha \in [0,1]$,*

$$\boldsymbol{ber}^{2r}(A) \leq \frac{\alpha}{2}\boldsymbol{ber}^r(A^2) + \boldsymbol{ber}\left(\frac{\alpha}{4}|A|^{2r} + \left(1-\frac{3}{4}\alpha\right)|A^*|^{2r}\right), \tag{2.5}$$

and

$$\boldsymbol{ber}^{2r}(A) \leq \frac{\alpha}{2}\boldsymbol{ber}^r(A^2) + \boldsymbol{ber}\left(\left(1-\frac{3}{4}\alpha\right)|A|^{2r} + \frac{\alpha}{4}|A^*|^{2r}\right). \tag{2.6}$$

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| &= \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| + (1-\alpha)|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\ &\leq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| + (1-\alpha)\|A^*\hat{k}_\lambda\|. \end{aligned}$$

By convexity of $f(t) = t^{2r}$ $(r \geq 1)$, we get

$$
\begin{aligned}
&|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} \\
&\leq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} + (1-\alpha)\|A^*\hat{k}_\lambda\|^{2r} \\
&\leq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \quad \big(\text{by Lemma 2.1}\big) \\
&\leq \frac{\alpha}{2}|\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{r} + \frac{\alpha}{4}\left\langle (|A|^{2r} + |A^*|^{2r})\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \big(\text{by Lemma 2.7}\big) \\
&= \frac{\alpha}{2}|\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{r} + \left\langle \left\{\frac{\alpha}{4}\left(|A|^{2r} + |A^*|^{2r}\right) + (1-\alpha)|A^*|^{2r}\right\}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&= \frac{\alpha}{2}|\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{r} + \left\langle \left\{\frac{\alpha}{4}|A|^{2r} + \left(1-\frac{3}{4}\alpha\right)|A^*|^{2r}\right\}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\leq \frac{\alpha}{2}\mathbf{ber}^r(A^2) + \mathbf{ber}\left(\frac{\alpha}{4}|A|^{2r} + \left(1-\frac{3}{4}\alpha\right)|A^*|^{2r}\right).
\end{aligned}
$$

Taking supremum over all $\lambda \in \Omega$, we get (2.5). Replacing A by A^* in (2.5), we get (2.6). This compleates our proof.

□

Considering $r = 1$ in (2.5) and (2.6) we get the following corollary.

Corollary 2.2. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^2(A) \leq \min\{\beta_1, \beta_2\}, \tag{2.7}$$

where

$$\beta_1 = \min_{0\leq\alpha\leq 1}\left[\frac{\alpha}{2}\boldsymbol{ber}^r(A^2) + \boldsymbol{ber}\left(\frac{\alpha}{4}|A|^{2r} + \left(1-\frac{3}{4}\alpha\right)|A^*|^{2r}\right)\right]$$

and

$$\beta_2 = \min_{0\leq\alpha\leq 1}\left[\frac{\alpha}{2}\boldsymbol{ber}^r(A^2) + \boldsymbol{ber}\left(\left(1-\frac{3}{4}\alpha\right)|A|^{2r} + \frac{\alpha}{4}|A^*|^{2r}\right)\right].$$

If we consider $\alpha = 1$ in (2.5) or (2.6), we get the following inequality.

Corollary 2.3. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^{2r}(A) \leq \frac{1}{2}\boldsymbol{ber}^r(A^2) + \frac{1}{4}\boldsymbol{ber}(|A|^{2r} + |A^*|^{2r}),$$

for all $r \geq 1$ and for all $\alpha \in [0,1]$.

Remark 2.4. *If $A^2 = 0$ then Corollary 2.3 gives, $\boldsymbol{ber}^{2r}(A) \leq \frac{1}{4}\boldsymbol{ber}(|A|^{2r} + |A^*|^{2r})$ for all $r \geq 1$. This is stronger than the existing inequality $\boldsymbol{ber}^{2r}(A) \leq \frac{1}{2}\boldsymbol{ber}(|A|^{2r} + |A^*|^{2r})$, obtained in [86, Cor. 3.5 (i)].*

Theorem 2.5. *Let $A \in \mathbb{B}(\mathcal{H})$. Then for all $r \geq 1$ and for all $\alpha \in [0,1]$, we have*

$$\boldsymbol{ber}^{2r}(A) \leq \boldsymbol{ber}\left(\alpha\left(\frac{|A|+|A^*|}{2}\right)^{2r} + (1-\alpha)\,|A^*|^{2r}\right) \tag{2.8}$$

and

$$\boldsymbol{ber}^{2r}(A) \leq \boldsymbol{ber}\left(\alpha\left(\frac{|A|+|A^*|}{2}\right)^{2r} + (1-\alpha)\,|A|^{2r}\right). \tag{2.9}$$

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then using Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| &= \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| + (1-\alpha)|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\ &\leq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| + (1-\alpha)\|A^*\hat{k}_\lambda\| \end{aligned}$$

By convexity of $f(t) = t^{2r}$ $(r \geq 1)$, we get

$$\begin{aligned} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} &\leq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} + (1-\alpha)\|A^*x\|^{2r} \\ &\leq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \quad \big(\text{by Lemma 2.1}\big) \\ &\leq \alpha\left(\langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle |A^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\right)^{2r} + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &\qquad \big(\text{by Lemma 2.3}\big) \\ &\leq \alpha\left(\frac{\langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |A^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^{2r} + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &= \alpha\left(\frac{\langle (|A|+|A^*|)\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^{2r} + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &\leq \alpha\left\langle \left(\frac{|A|+|A^*|}{2}\right)^{2r}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + (1-\alpha)\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \quad \big(\text{by Lemma 2.1}\big) \\ &= \left\langle \left\{\alpha\left(\frac{|A|+|A^*|}{2}\right)^{2r} + (1-\alpha)\,|A^*|^{2r}\right\}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\ &\leq \mathbf{ber}\left(\alpha\left(\frac{|A|+|A^*|}{2}\right)^{2r} + (1-\alpha)\,|A^*|^{2r}\right). \end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we get (2.8). Replacing A by A^* in (2.8), we get (2.9). □

Considering $r = 1$ in (2.8) and (2.9) we get the following corollary.

Corollary 2.4. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^2(A) \quad \leq \quad \min\{\gamma_1, \gamma_2\}, \tag{2.10}$$

$$\gamma_1 = \min_{0\leq\alpha\leq1} \boldsymbol{ber}\left(\alpha\left(\frac{|A|+|A^*|}{2}\right)^2 + (1-\alpha)\,|A^*|^2\right)$$

and

$$\gamma_2 = \min_{0\leq\alpha\leq1} \boldsymbol{ber}\left(\alpha\left(\frac{|A|+|A^*|}{2}\right)^2 + (1-\alpha)\,|A|^2\right).$$

Theorem 2.6. *Let $A \in \mathbb{B}(\mathcal{H})$. Then for all $r \geq 1$,*

$$\boldsymbol{ber}^{2r}(A) \leq \frac{1}{4}\boldsymbol{ber}\left(|A|^{2r} + |A^*|^{2r}\right) + \frac{1}{2}\boldsymbol{ber}\left(|A|^r|A^*|^r\right).$$

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then if follows from Lemma 2.3 and Lemma 2.1 that

$$|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} \leq \langle |A|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A^*|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle = \langle |A^*|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, |A|^r\hat{k}_\lambda\rangle.$$

From Lemma 2.7 we have,

$$\langle |A^*|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, |A|^r\hat{k}_\lambda\rangle \quad \leq \quad \frac{1}{2}\left\||A|^r\hat{k}_\lambda\right\|\left\||A^*|^r\hat{k}_\lambda\right\| + \frac{1}{2}\left|\langle |A^*|^r\hat{k}_\lambda, |A|^r\hat{k}_\lambda\rangle\right|.$$

Hence,

$$\begin{aligned}
|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} &\leq \frac{1}{4}\left(\left\||A|^r\hat{k}_\lambda\right\|^2 + \left\||A^*|^r\hat{k}_\lambda\right\|^2\right) + \frac{1}{2}\left|\langle |A|^r|A^*|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle\right| \\
&= \frac{1}{4}\left(\left\langle |A|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle |A^*|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) + \frac{1}{2}\left|\langle |A|^r|A^*|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle\right| \\
&= \frac{1}{4}\left\langle \left(|A|^{2r} + |A^*|^{2r}\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \frac{1}{2}\left|\langle |A|^r|A^*|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle\right| \\
&\leq \frac{1}{4}\mathbf{ber}\left(|A|^{2r} + |A^*|^{2r}\right) + \frac{1}{2}\mathbf{ber}\left(|A|^r|A^*|^r\right)
\end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we get

$$\mathbf{ber}^{2r}(A) \leq \frac{1}{4}\mathbf{ber}\left(|A|^{2r} + |A^*|^{2r}\right) + \frac{1}{2}\mathbf{ber}\left(|A|^r|A^*|^r\right).$$

□

Considering $r = 1$ in Theorem 2.6 we get the following corollary.

Corollary 2.5. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^2(A) \leq \frac{1}{4}\boldsymbol{ber}\left(|A|^2 + |A^*|^2\right) + \frac{1}{2}\boldsymbol{ber}\left(|A||A^*|\right).$$

Theorem 2.7. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^2(A) \leq \frac{1}{4}\boldsymbol{ber}(|A|^{4\alpha} + |A^*|^{4(1-\alpha)}) + \frac{1}{2}\boldsymbol{ber}(|A^*|^{2(1-\alpha)}|A|^{2\alpha}),$$

for all $\alpha \in [0,1]$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have,

$$\begin{aligned}
& |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 \\
\leq\ & \langle |A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A^*|^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle \quad \big(\text{by Lemma 2.2}\big) \\
=\ & \langle |A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, |A^*|^{2(1-\alpha)}\hat{k}_\lambda\rangle \\
\leq\ & \frac{1}{2}\||A|^{2\alpha}\hat{k}_\lambda\|\||A^*|^{2(1-\alpha)}\hat{k}_\lambda\|| + \frac{1}{2}|\langle |A|^{2\alpha}\hat{k}_\lambda, |A^*|^{2(1-\alpha)}\hat{k}_\lambda\rangle| \ \big(\text{by Lemma 2.7}\big) \\
=\ & \frac{1}{2}\langle |A|^{4\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle |A^*|^{4(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} + \frac{1}{2}|\langle |A^*|^{2(1-\alpha)}|A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\
\leq\ & \frac{1}{4}\langle (|A|^{4\alpha} + |A^*|^{4(1-\alpha)})\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{2}|\langle |A^*|^{2(1-\alpha)}|A|^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\
\leq\ & \frac{1}{4}\mathbf{ber}(|A|^{4\alpha} + |A^*|^{4(1-\alpha)}) + \frac{1}{2}\mathbf{ber}(|A^*|^{2(1-\alpha)}|A|^{2\alpha}).
\end{aligned}$$

Therefore, taking supremum over all $\lambda \in \Omega$, we get the desired inequality. □

Based on the above theorem we have the following upper bound for $\mathbf{ber}^2(A)$.

Corollary 2.6. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\boldsymbol{ber}^2(A) &\leq \min_{\alpha\in[0,1]}\left\{\frac{1}{4}\boldsymbol{ber}(|A|^{4\alpha} + |A^*|^{4(1-\alpha)}) + \frac{1}{2}\boldsymbol{ber}(|A^*|^{2(1-\alpha)}|A|^{2\alpha})\right\} \\
&\leq \frac{1}{4}\boldsymbol{ber}(|A|^2 + |A^*|^2) + \frac{1}{2}\boldsymbol{ber}(|A^*||A|).
\end{aligned}$$

Our next bound reads as follows.

Theorem 2.8. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^4(A) \leq \frac{1}{4}\Big(\boldsymbol{ber}^2(A^2) + \frac{1}{4}\boldsymbol{ber}(|A|^4 + |A^*|^4) + \frac{1}{2}\boldsymbol{ber}(A^*A^2A^*) + \boldsymbol{ber}(A^2)\boldsymbol{ber}(|A|^2 + |A^*|^2)\Big).$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have,

$$\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle$$

$$
\begin{aligned}
&= \langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, AA^*\hat{k}_\lambda\rangle \\
&\le \frac{1}{2}\|A^*A\hat{k}_\lambda\|\|AA^*\hat{k}_\lambda\| + \frac{1}{2}|\langle AA^*\hat{k}_\lambda, A^*A\hat{k}_\lambda\rangle| \quad \big(\text{by Lemma 2.7}\big) \\
&\le \frac{1}{4}(\|A^*A\hat{k}_\lambda\|^2 + \|AA^*\hat{k}_\lambda\|^2) + \frac{1}{2}|\langle A^*A^2A^*\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\
&= \frac{1}{4}\langle (|A|^4 + |A^*|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \frac{1}{2}|\langle A^*A^2A^*\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\
&\le \frac{1}{4}\mathbf{ber}(|A|^4 + |A^*|^4) + \frac{1}{2}\mathbf{ber}(A^*A^2A^*).
\end{aligned}
$$

Now,

$$
\begin{aligned}
& |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^4 \\
= \ & |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 \\
\le \ & \frac{1}{4}\Big(|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle| + \|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\Big)^2 \quad \big(\text{by Lemma 2.7}\big) \\
= \ & \frac{1}{4}\Big(|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 + \|A\hat{k}_\lambda\|^2\|A^*\hat{k}_\lambda\|^2 + 2|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\Big) \\
\le \ & \frac{1}{4}\Big(|\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + \langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|(\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle)\Big) \\
= \ & \frac{1}{4}\Big(|\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + \langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|A|^2 + |A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
\le \ & \frac{1}{4}\Big(\mathbf{ber}^2(A^2) + \frac{1}{4}\mathbf{ber}(|A|^4 + |A^*|^4) + \frac{1}{2}\mathbf{ber}(A^*A^2A^*) + \mathbf{ber}(A^2)\mathbf{ber}(|A|^2 + |A^*|^2)\Big).
\end{aligned}
$$

Therefore, by taking supremum over all $\lambda \in \Omega$, we get the desired result. □

Remark 2.9. *From [86, Cor. 3.5(i)] (for the case $r = 4$) we have that*

$$
\mathbf{ber}^4(A) \le \frac{1}{2}\mathbf{ber}(|A|^4 + |A^*|^4). \tag{2.11}
$$

If $A^2 = 0$, then it follows from Theorem 2.8 that

$$
\mathbf{ber}^4(A) \le \frac{1}{16}\mathbf{ber}(|A|^4 + |A^*|^4). \tag{2.12}
$$

Therefore, we would like to notice that for the case $A^2 = 0$, Theorem 2.8 gives stronger bound than that in (2.11).

Theorem 2.10. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$
\begin{aligned}
4\,\mathbf{ber}^4(A) \ \le \ & \mathbf{ber}(t|A|^2 + (1-t)|A^*|^2)\mathbf{ber}((1-t)|A|^2 + t|A^*|^2) + \mathbf{ber}^2(A^2) \\
& + \mathbf{ber}(A^2)\mathbf{ber}(|A|^2 + |A^*|^2),
\end{aligned}
$$

for all $t \in [0, 1]$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then

$$\begin{aligned}
&|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^4 \\
=\;& |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 \\
\leq\;& \frac{1}{4}\left(\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\| + |\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|\right)^2 \quad \text{(by Lemma 2.7)} \\
=\;& \frac{1}{4}\left(\|A\hat{k}_\lambda\|^2\|A^*\hat{k}_\lambda\|^2 + |\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 + 2|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\right) \\
\leq\;& \frac{1}{4}\Big(\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (A^*A + AA^*)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
=\;& \frac{1}{4}\Big(\langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^t\langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-t}\langle |A^*|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-t}\langle |A^*|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^t \\
&\qquad + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|A|^2 + |A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
\leq\;& \frac{1}{4}\Big(\langle (t|A|^2 + (1-t)|A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle ((1-t)|A|^2 + t|A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\qquad + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle|\langle (|A|^2 + |A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
\leq\;& \frac{1}{4}\Big(\mathbf{ber}(t|A|^2 + (1-t)|A^*|^2)\mathbf{ber}((1-t)|A|^2 + t|A^*|^2) \\
&\qquad + \mathbf{ber}^2(A^2) + \mathbf{ber}(A^2)\mathbf{ber}(|A|^2 + |A^*|^2)\Big).
\end{aligned}$$

Therefore, taking supremum over all $\lambda \in \Omega$, we have the desired inequality of the theorem. □

Based on the above theorem we have the following corollary.

Corollary 2.7. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\boldsymbol{ber}^4(A) &\leq \frac{1}{4}\Big(\min_{t\in[0,1]}\big\{\boldsymbol{ber}(t|A|^2 + (1-t)|A^*|^2)\boldsymbol{ber}((1-t)|A|^2 + t|A^*|^2)\big\} \\
&\qquad + \boldsymbol{ber}^2(A^2) + \boldsymbol{ber}(A^2)\boldsymbol{ber}(|A|^2 + |A^*|^2)\Big) \\
&\leq \frac{1}{4}\left(\frac{1}{4}\boldsymbol{ber}^2(|A|^2 + |A^*|^2) + \boldsymbol{ber}^2(A^2) + \boldsymbol{ber}(A^2)\boldsymbol{ber}(|A|^2 + |A^*|^2)\right).
\end{aligned}$$

Remark 2.11. *From [86, Cor. 3.5 (i)] (for the case $r = 2$) we have that*

$$\boldsymbol{ber}^2(A) \leq \frac{1}{2}\boldsymbol{ber}(|A|^2 + |A^*|^2). \tag{2.13}$$

If $A^2 = 0$, then it follows from Corollary 2.7 that

$$\boldsymbol{ber}^2(A) \leq \frac{1}{4}\boldsymbol{ber}(|A|^2 + |A^*|^2). \tag{2.14}$$

Therefore, we would like to remark that for the case $A^2 = 0$, Corollary 2.7 gives stronger bound than that in (2.13).

2.4 Development of upper bounds via Cartesian decomposition

In the following theorems we obtain upper bounds for the Berezin number of bounded linear operators by using the Cartesian decomposition. Recall that every $A \in \mathbb{B}(\mathcal{H})$ can be expressed uniquely as $A = \Re(A) + i\Im(A)$, where $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$. Here $\Re(A)$ and $\Im(A)$ are called real and imaginary parts of A, respectively. This decomposition is known as the Cartesian decomposition.

Theorem 2.12. *Let* $A \in \mathbb{B}(\mathcal{H})$. *Then*

$$\boldsymbol{ber}^r(A) \leq \boldsymbol{ber}\left(|\Re(A)|^r + |\Im(A)|^r\right),$$

for all $1 \leq r \leq 2$.

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then

$$\begin{aligned} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| &= \left(\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\right)^{1/2} \\ &\leq \left(|\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r + |\langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r\right)^{1/r} \quad \text{(by Lemma 2.6)} \\ &\leq \left(\langle |\Re(A)|\hat{k}_\lambda, \hat{k}_\lambda\rangle^r + \langle |\Im(A)|\hat{k}_\lambda, \hat{k}_\lambda\rangle^r\right)^{1/r} \quad \text{(by Lemma 2.4)} \\ &\leq \left(\langle |\Re(A)|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |\Im(A)|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^{1/r} \quad \text{(by Lemma 2.1)} \\ &\leq \left(\langle (|\Re(A)|^r + |\Im(A)|^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^{1/r}. \end{aligned}$$

So, we obtain

$$\begin{aligned} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r &\leq \langle |\Re(A)|^r + |\Im(A)|^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &\leq \mathbf{ber}\left(|\Re(A)|^r + |\Im(A)|^r\right). \end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we get

$$\mathbf{ber}^r(A) \leq \mathbf{ber}\left(|\Re(A)|^r + |\Im(A)|^r\right),$$

for all $1 \leq r \leq 2$. □

Theorem 2.13. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^r(A) \leq 2^{\frac{r}{2}-1}\boldsymbol{ber}\left(|\Re(A)|^r + |\Im(A)|^r\right),$$

for all $r \geq 2$.

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then we get

$$|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| = \left(\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\right)^{1/2}. \tag{2.15}$$

From the equation (2.15), we have

$$\begin{aligned}
\frac{|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|}{\sqrt{2}} &= \left(\frac{\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2}{2}\right)^{1/2} \\
&\leq \left(\frac{|\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r + |\langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r}{2}\right)^{1/r} \quad \text{(by Lemma 2.5)} \\
&\leq 2^{-1/r}\left(|\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r + |\langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r\right)^{1/r} \\
&\leq 2^{-1/r}\left(\langle |\Re(A)|\hat{k}_\lambda, \hat{k}_\lambda\rangle^r + \langle |\Im(A)|\hat{k}_\lambda, \hat{k}_\lambda\rangle^r\right)^{1/r} \quad \text{(by Lemma 2.4)} \\
&\leq 2^{-1/r}\left(\langle |\Re(A)|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |\Im(A)|^r\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^{1/r} \quad \text{(by Lemma 2.1)} \\
&\leq 2^{-1/r}\left(\langle (|\Re(A)|^r + |\Im(A)|^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^{1/r}
\end{aligned}$$

So, we get

$$\begin{aligned}
|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r &\leq 2^{\frac{r}{2}-1}\langle (|\Re(A)|^r + |\Im(A)|^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leq 2^{\frac{r}{2}-1}\mathbf{ber}\left(|\Re(A)|^r + |\Im(A)|^r\right)
\end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we get our desired result. □

Theorem 2.14. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^r(A) \leq \frac{1}{2}\boldsymbol{ber}\left(|\Re(A) + \Im(A)|^r + |\Re(A) - \Im(A)|^r\right),$$

for all $r \geq 2$.

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then

$$\begin{aligned}
&|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r \\
\leq\ &\left(\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\right)^{r/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq\ 2^{-r/2}\Big(\langle(\Re(A)+\Im(A))\hat{k}_\lambda,\hat{k}_\lambda\rangle^2+\langle(\Re(A)-\Im(A))\hat{k}_\lambda,\hat{k}_\lambda\rangle^2\Big)^{r/2}\\
&\leq\ 2^{-r/2}2^{r/2-1}\Big(|\langle(\Re(A)+\Im(A))\hat{k}_\lambda,\hat{k}_\lambda\rangle|^r+|\langle(\Re(A)-\Im(A))\hat{k}_\lambda,\hat{k}_\lambda\rangle|^r\Big)\\
&\qquad \big(\text{by the convexity of the function } f(t)=t^{r/2} \text{ on } [\,0,\infty\,)\big)\\
&\leq\ \frac{1}{2}\Big(\langle|\Re(A)+\Im(A)|\hat{k}_\lambda,\hat{k}_\lambda\rangle^r+\langle|\Re(A)-\Im(A)|\hat{k}_\lambda,\hat{k}_\lambda\rangle^r\Big)\quad\big(\text{by Lemma 2.4}\big)\\
&\leq\ \frac{1}{2}\Big(\langle|\Re(A)+\Im(A)|^r\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle|\Re(A)-\Im(A)|^r\hat{k}_\lambda,\hat{k}_\lambda\rangle\Big)\ \big(\text{by Lemma 2.1}\big)\\
&\leq\ \frac{1}{2}\Big(\langle(|\Re(A)+\Im(A)|^r+|\Re(A)-\Im(A)|^r)\hat{k}_\lambda,\hat{k}_\lambda\rangle\Big)\\
&\leq\ \frac{1}{2}\mathbf{ber}\left(|\Re(A)+\Im(A)|^r+|\Re(A)-\Im(A)|^r\right)
\end{aligned}
$$

Taking supremum over all $\lambda\in\Omega$, we get

$$\mathbf{ber}^r(A)\leq\frac{1}{2}\mathbf{ber}\left(|\Re(A)+\Im(A)|^r+|\Re(A)-\Im(A)|^r\right).$$

□

2.5 On the lower bounds of Berezin number

We begin this section with the elementary identity that $\max\{\beta,\gamma\}=\frac{1}{2}(\beta+\gamma)+\frac{1}{2}|\beta-\gamma|$ for all real numbers β and γ. Based on the above identity we obtain our first lower bound for the Berezin number of bounded linear operators on reproducing kernel Hilbert space $\mathcal{H}$.

Theorem 2.15. *Let $A\in\mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}(A)\geq\frac{1}{2}\boldsymbol{ber}\Big(\Re(A)\pm\Im(A)\Big)+\frac{1}{2}\big|\boldsymbol{ber}(\Re(A))-\boldsymbol{ber}(\Im(A))\big|.$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then by the Cartesian decomposition of A, we get

$$|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2=|\langle\Re(A)\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2+|\langle\Im(A)\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2.$$

From this we infer that

$$\mathbf{ber}(A)\geq\mathbf{ber}(\Re(A))\tag{2.16}$$

and

$$\mathbf{ber}(A)\geq\mathbf{ber}(\Im(A)).\tag{2.17}$$

Therefore, combining (2.16) together with (2.17) we have,

$$\begin{aligned}\mathbf{ber}(A) &\geq \max\{\mathbf{ber}(\Re(A)), \mathbf{ber}(\Im(A))\} \\ &= \frac{\mathbf{ber}(\Re(A)) + \mathbf{ber}(\Im(A))}{2} + \frac{|\mathbf{ber}(\Re(A)) - \mathbf{ber}(\Im(A))|}{2} \\ &\geq \frac{\mathbf{ber}(\Re(A) \pm \Im(A))}{2} + \frac{|\mathbf{ber}(\Re(A)) - \mathbf{ber}(\Im(A))|}{2},\end{aligned}$$

as required. □

Next inequality reads as:

Theorem 2.16. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}\boldsymbol{ber}^2(A) &\geq \frac{1}{4}\boldsymbol{ber}^2\Big(\Re(A) \pm \Im(A)\Big) + \frac{c^2(\Re(A)) + c^2(\Im(A))}{2} \\ &+ \left|\frac{\boldsymbol{ber}^2(\Re(A)) - \boldsymbol{ber}^2(\Im(A))}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right|.\end{aligned}$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then it follows from the Cartesian decomposition of A that

$$|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 = |\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2.$$

Therefore, we infer that

$$\mathbf{ber}^2(A) \geq \mathbf{ber}^2(\Re(A)) + c^2(\Im(A)) \tag{2.18}$$

and

$$\mathbf{ber}^2(A) \geq \mathbf{ber}^2(\Im(A)) + c^2(\Re(A)). \tag{2.19}$$

Now we have by combining (2.18) together with (2.19) that

$$\begin{aligned}\mathbf{ber}^2(A) &\geq \max\{\mathbf{ber}^2(\Re(A)) + c^2(\Im(A)), \mathbf{ber}^2(\Im(A)) + c^2(\Re(A))\} \\ &= \frac{\mathbf{ber}^2(\Re(A)) + c^2(\Im(A)) + \mathbf{ber}^2(\Im(A)) + c^2(\Re(A))}{2} \\ &\quad + \left|\frac{\mathbf{ber}^2(\Re(A)) + c^2(\Im(A)) - \mathbf{ber}^2(\Im(A)) - c^2(\Re(A))}{2}\right| \\ &= \frac{\mathbf{ber}^2(\Re(A)) + \mathbf{ber}^2(\Im(A))}{2} + \frac{c^2(\Im(A)) + c^2(\Re(A))}{2} \\ &\quad + \left|\frac{\mathbf{ber}^2(\Re(A)) - \mathbf{ber}^2(\Im(A))}{2} + \frac{c^2(\Im(A)) - c^2(\Re(A))}{2}\right|\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{4}\Big(\mathbf{ber}(\Re(A))+\mathbf{ber}(\Im(A))\Big)^2+\frac{c^2(\Im(A))+c^2(\Re(A))}{2}\\
&\quad+\left|\frac{\mathbf{ber}^2(\Re(A))-\mathbf{ber}^2(\Im(A))}{2}+\frac{c^2(\Im(A))-c^2(\Re(A))}{2}\right|\\
&\geq \frac{1}{4}\mathbf{ber}^2\Big(\Re(A)\pm\Im(A)\Big)+\frac{c^2(\Im(A))+c^2(\Re(A))}{2}\\
&\quad+\left|\frac{\mathbf{ber}^2(\Re(A))-\mathbf{ber}^2(\Im(A))}{2}+\frac{c^2(\Im(A))-c^2(\Re(A))}{2}\right|.
\end{aligned}$$

This completes the proof. □

Finally, we prove the following lower bound.

Theorem 2.17. *If* $A\in\mathbb{B}(\mathcal{H})$, *then* $\boldsymbol{ber}^2(A)\geq\max\{\beta,\gamma\}$, *where*

$$\beta=\frac{\boldsymbol{ber}^2\Big(\Re(A)+\Im(A)\Big)}{2}+\frac{c^2\Big(\Re(A)-\Im(A)\Big)}{2}$$

and

$$\gamma=\frac{\boldsymbol{ber}^2\Big(\Re(A)-\Im(A)\Big)}{2}+\frac{c^2\Big(\Re(A)+\Im(A)\Big)}{2}.$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then from the Cartesian decomposition of A, we get

$$\begin{aligned}
|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2 &= \langle\Re(A)\hat{k}_\lambda,\hat{k}_\lambda\rangle^2+\langle\Im(A)\hat{k}_\lambda,\hat{k}_\lambda\rangle^2\\
&=\frac{\langle(\Re(A)+\Im(A))\hat{k}_\lambda,\hat{k}_\lambda\rangle^2}{2}+\frac{\langle(\Re(A)-\Im(A))\hat{k}_\lambda,\hat{k}_\lambda\rangle^2}{2}.
\end{aligned}$$

Therefore, we have the following two inequalities:

$$\mathbf{ber}^2(A)\geq\frac{\mathbf{ber}^2(\Re(A)+\Im(A))}{2}+\frac{c^2(\Re(A)-\Im(A))}{2}\tag{2.20}$$

and

$$\mathbf{ber}^2(A)\geq\frac{\mathbf{ber}^2(\Re(A)-\Im(A))}{2}+\frac{c^2(\Re(A)+\Im(A))}{2}.\tag{2.21}$$

By combining (2.20) together with (2.21) we infer that the desired inequality. □

As an easy consequence of the above theorem we infer the following inequality.

Corollary 2.8. *If* $A\in\mathbb{B}(\mathcal{H})$, *then*

$$\boldsymbol{ber}(A)\ \geq\ \frac{1}{\sqrt{2}}\boldsymbol{ber}\Big(\Re(A)\pm\Im(A)\Big).$$

CHAPTER 3

SHARPER BOUNDS OF THE BEREZIN NUMBER FOR PRODUCT OPERATORS

3.1 Introduction

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on the non-empty set Ω. Let $\mathbb{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For $A \in \mathbb{B}(\mathcal{H})$, A^* denote the adjoint of A and $|A|$ denote the positive operator $(A^*A)^{1/2}$. Let $\{\hat{k}_\lambda : \lambda \in \Omega\}$ be the set of normalized reproducing kernels of $\mathcal{H}$. The Berezin set of A, denoted by $\mathbf{Ber}(A)$, and is defined as

$$\mathbf{Ber}(A) = \{\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle : \lambda \in \Omega\}.$$

Content of this chapter is partially based on the following three papers:

- P. Bhunia, **A. Sen** and K. Paul, Development of the Berezin number inequalities, Acta Math. Sin. (Engl. Ser.), 39 (2023), no. 7, 1219–1228. https://doi.org/10.1007/s10114-023-2090-1
- **A. Sen**, P. Bhunia and K. Paul, Bounds for the Berezin number of reproducing kernel Hilbert space operators, Filomat, 37 (2023), no. 6, 1741–1749. https://doi.org/10.2298/FIL2306741S
- **A. Sen**, P. Bhunia and K. Paul, Berezin number inequalities of operators on reproducing kernel Hilbert spaces, Rocky Mountain J. Math., 52 (2022), no. 3, 1039–1046. DOI: 10.1216/rmj.2022.52.1039

The Berezin number of A is defined as

$$\mathbf{ber}(A) = \sup_{\lambda \in \Omega} |\langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle|.$$

The least Berezin number of A is denoted by $c(A)$, and defined as $c(A) = \inf_{\lambda \in \Omega} |\langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle|$. In this chapter, we develop bounds of the Berezin number for the sum and product of operators acting on reproducing kernel Hilbert spaces.

3.2 Auxiliary results

We start with a series of lemmas that will be repeatedly applied to achieve the objectives of this chapter.

Lemma 3.1. *[84] Let $A \in \mathbb{B}(\mathcal{H})$ with $A \geq 0$ and $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$(a)\ \langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \quad for \quad r \geq 1,$$
$$(b)\ \langle A^r x, x\rangle \leq \langle Ax, x\rangle^r \quad for \quad 0 < r \leq 1.$$

Lemma 3.2. *[66, Th. 5] Let $A \in \mathbb{B}(\mathcal{H})$. Let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|,$$

for all $x, y \in \mathcal{H}$.

Lemma 3.3. *[38] Let $A \in \mathbb{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha} x, x\rangle \quad \langle |A^*|^{2(1-\alpha)} y, y\rangle,$$

for all $0 \leq \alpha \leq 1$.

Lemma 3.4. *[53] If $a, b \geq 0$ and $0 \leq \alpha \leq 1$, then*

$$(a^\alpha b^{1-\alpha})^2 \leq (\alpha a + (1-\alpha)b)^2 - r_0^2 (a-b)^2, \ where\ r_0 = min\{\alpha, 1-\alpha\}.$$

Lemma 3.5. *[63, Th. 2.1] For $a, b \geq 0$ and $0 \leq \nu \leq 1$,*

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b - r_0 \left(a^{\frac{1}{2}} - b^{\frac{1}{2}}\right)^2,$$

where $r_0 = min\{\nu, 1-\nu\}$.

Lemma 3.6. *[91] For* $i = 1, 2, \ldots, n$, *let* a_i *be a positive real number. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \le n^{r-1}\sum_{i=1}^{n} a_i^r,$$

for all $r \ge 1$.

3.3 Inequalities for product of operators

Our first theorem provides upper bounds for the Berezin numbers of A^*XB, in particular, we find bounds for $\mathbf{ber}^r(A^*XB)$ and $\mathbf{ber}^2(A^*XB)$, where $A, X, B \in \mathbb{B}(\mathcal{H})$.

Theorem 3.1. *Let* $A, B, X \in \mathbb{B}(\mathcal{H})$. *Then*

(i) $\mathbf{ber}^r(A^*XB) \le \dfrac{\|X\|^r}{2}\sqrt{\mathbf{ber}^2(|A|^{2r}+|B|^{2r}) - c^2(|A|^{2r}-|B|^{2r})}$ *for all* $r \ge 1$.

(ii) $\mathbf{ber}^2(A^*XB) \le \dfrac{1}{4}\Big(\mathbf{ber}^2(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A) - c^2(B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A)\Big),$

for every $\alpha \in [0, 1]$.

Proof. (i) Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have,

$$\begin{aligned}
& |\langle (A^*XB)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r} \\
= \; & |\langle XB\hat{k}_\lambda, A\hat{k}_\lambda\rangle|^{2r} \\
\le \; & \|X\|^{2r}\|A\hat{k}_\lambda\|^{2r}\|B\hat{k}_\lambda\|^{2r} \quad \big(\text{using Cauchy-Schwarz inequality}\big) \\
= \; & \|X\|^{2r}\langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \langle |B|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \\
\le \; & \|X\|^{2r}\langle |A|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \langle |B|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle \; \big(\text{using Lemma 3.1}(a)\big) \\
= \; & \frac{\|X\|^{2r}}{4}\Big(\langle (|A|^{2r}+|B|^{2r})\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 - \langle (|A|^{2r}-|B|^{2r})\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\Big) \\
\le \; & \frac{\|X\|^{2r}}{4}\left(\mathbf{ber}^2(|A|^{2r}+|B|^{2r}) - c^2(|A|^{2r}-|B|^{2r})\right).
\end{aligned}$$

Therefore, by taking supremum over all $\lambda \in \Omega$, we get the result as desired.
(ii) Again,

$$\begin{aligned}
& |\langle (A^*XB)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 \\
= \; & |\langle XB\hat{k}_\lambda, A\hat{k}_\lambda\rangle|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \langle |X|^{2\alpha} B\hat{k}_\lambda, B\hat{k}_\lambda\rangle \langle |X^*|^{2(1-\alpha)} A\hat{k}_\lambda, A\hat{k}_\lambda\rangle \; \big(\text{by Lemma 3.3}\big) \\
&= \langle B^*|X|^{2\alpha} B\hat{k}_\lambda, \hat{k}_\lambda\rangle \langle A^*|X^*|^{2(1-\alpha)} A\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \frac{1}{4}\Big(\langle (B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 - \langle (B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\Big) \\
&\leq \frac{1}{4}\Big(\mathbf{ber}^2(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A) - c^2(B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A)\Big).
\end{aligned}
$$

Therefore, by considering supremum over all $\lambda \in \Omega$, we get the desired inequality. □

Remark 3.2. *Following [48, Th. 2.5(i)] (for the case $p = q = 2$) we have,*

$$\mathbf{ber}^r(A^*XB) \leq \frac{\|X\|^r}{2}\mathbf{ber}(|A|^{2r} + |B|^{2r}). \tag{3.1}$$

It is clear that the inequality obtained in Theorem 3.1(i) is better than that in (3.1).
We also remark that the inequality obtained in Theorem 3.1(ii) improves on the inequality [48, Th. 2.5(ii)], namely,

$$\mathbf{ber}(A^*XB) \leq \frac{1}{2}\mathbf{ber}(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A), \;\; \forall \alpha \in [0,1].$$

In particular, considering $X = I$ in Theorem 3.1(i) we get the following corollary.

Corollary 3.1. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$\mathbf{ber}^r(A^*B) \leq \frac{1}{2}\sqrt{\mathbf{ber}^2(|A|^{2r} + |B|^{2r}) - c^2(|A|^{2r} - |B|^{2r})}.$$

Remark 3.3. *Following [86, Cor. 3.7(ii)] (for the case $\nu = \frac{1}{2}$) we get,*

$$\mathbf{ber}^r(A^*B) \leq \frac{1}{2}\mathbf{ber}(|A|^{2r} + |B|^{2r}). \tag{3.2}$$

Clearly, Corollary 3.1 is sharper than that in (3.2).

Our next result reads as follows.

Theorem 3.4. *If $A, B, X \in \mathbb{B}(\mathcal{H})$ with A, B are positive, then for all $r \geq 1$ and for all $\alpha \in [0,1]$,*

$$\mathbf{ber}^{2r}(A^\alpha XB^{1-\alpha}) \leq \|X\|^{2r}\sqrt{\mathbf{ber}^2(\alpha A^{2r} + (1-\alpha)B^{2r}) - {r_0}^2c^2(A^{2r} - B^{2r})},$$

where $r_0 = min\{\alpha, 1-\alpha\}$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have

$$|\langle (A^\alpha XB^{1-\alpha})\hat{k}_\lambda, \hat{k}_\lambda\rangle|^{2r}$$

$$\begin{aligned}
&= |\langle XB^{1-\alpha}\hat{k}_\lambda, A^\alpha\hat{k}_\lambda\rangle|^{2r} \\
&\leq \|X\|^{2r}\|B^{1-\alpha}\hat{k}_\lambda\|^{2r}\|A^\alpha\hat{k}_\lambda\|^{2r} \quad \big(\text{by Cauchy-Schwarz inequality}\big) \\
&= \|X\|^{2r}\langle B^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \langle A^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \\
&\leq \|X\|^{2r}\langle A^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle^\alpha \langle B^{2r}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{(1-\alpha)} \quad \big(\text{by Lemma 3.1}(a)\big) \\
&\leq \|X\|^{2r}\sqrt{\langle(\alpha A^{2r}+(1-\alpha)B^{2r})\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 - {r_0}^2\langle(A^{2r}-B^{2r})\hat{k}_\lambda, \hat{k}_\lambda\rangle^2} \quad \big(\text{by Lemma 3.4}\big) \\
&\leq \|X\|^{2r}\sqrt{\mathbf{ber}^2(\alpha A^{2r}+(1-\alpha)B^{2r}) - {r_0}^2 c^2(A^{2r}-B^{2r})}.
\end{aligned}$$

Hence, by taking supremum over all $\lambda \in \Omega$, we get the desired result. $\square$

In particular, considering $X = I$ in Theorem 3.4 we get the following corollary.

Corollary 3.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive. Then for all $r \geq 1$ and $\alpha \in [0,1]$,*

$$\boldsymbol{ber}^{2r}(A^\alpha B^{1-\alpha}) \leq \sqrt{\boldsymbol{ber}^2(\alpha A^{2r}+(1-\alpha)B^{2r}) - {r_0}^2c^2(A^{2r}-B^{2r})},$$

where $r_0 = min\{\alpha, 1-\alpha\}$. In particular, if $AB = BA$, then

$$\boldsymbol{ber}(\sqrt{AB}) \leq \frac{1}{\sqrt{2}}\left(\boldsymbol{ber}^2(A^2+B^2) - c^2(A^2-B^2)\right)^{1/4}.$$

Next, we develop the following upper bound.

Theorem 3.5. *Let $A_i, B_i, X_i \in \mathbb{B}(\mathcal{H}), i = 1, 2, \ldots, n$. Let f and g be two nonnegative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then for $\nu \in [0,1]$ and for all $r \geq 2\ max\{\nu, 1-\nu\}$,*

$$\begin{aligned}
&\boldsymbol{ber}^r\left(\sum_{i=1}^n A_i^*X_iB_i\right) \leq \\
&\sqrt{2}n^{r-1}\boldsymbol{ber}\left(\sum_{i=1}^n\left(\nu\left[B_i^*f^2(|X_i|)B_i\right]^{\frac{r}{2\nu}} + \mathrm{i}(1-\nu)\left[A_i^*g^2(|X_i^*|)A_i\right]^{\frac{r}{2(1-\nu)}}\right)\right).
\end{aligned}$$

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then we have

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^n A_i^*X_iB_i\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \\
&= \left|\sum_{i=1}^n\left\langle A_i^*X_iB_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \\
&\leq \left(\sum_{i=1}^n\left|\left\langle A_i^*X_iB_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right)^r
\end{aligned}$$

$$
\begin{aligned}
&\leq n^{r-1}\left(\sum_{i=1}^{n}\left|\left\langle A_i^*X_iB_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\right) \quad \text{(by Lemma 3.6)}\\
&= n^{r-1}\left(\sum_{i=1}^{n}\left|\left\langle X_iB_i\hat{k}_\lambda,A_i\hat{k}_\lambda\right\rangle\right|^r\right)\\
&\leq n^{r-1}\left(\sum_{i=1}^{n}\left\|f(|X_i|)B_i\hat{k}_\lambda\right\|^r\left\|g(|X_i^*|)A_i\hat{k}_\lambda\right\|^r\right) \quad \text{(by Lemma 3.2)}\\
&= n^{r-1}\left(\sum_{i=1}^{n}\left\langle f^2(|X_i|)B_i\hat{k}_\lambda,B_i\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\left\langle g^2(|X_i^*|)A_i\hat{k}_\lambda,A_i\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\right)\\
&= n^{r-1}\left(\sum_{i=1}^{n}\left\langle B_i^*f^2(|X_i|)B_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\left\langle A_i^*g^2(|X_i^*|)A_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\right)\\
&\leq n^{r-1}\left(\sum_{i=1}^{n}\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^{\frac{r}{2\nu}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\nu}\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^{\frac{r}{2(1-\nu)}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{1-\nu}\right)\\
&\hspace{22em}\text{(by Lemma 3.1}(a))\\
&\leq n^{r-1}\Big(\sum_{i=1}^{n}\Big(\nu\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^{\frac{r}{2\nu}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+(1-\nu)\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^{\frac{r}{2(1-\nu)}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\Big)\Big)\\
&\hspace{16em}\left(\text{as } a^\nu b^{1-\nu}\leq\nu a+(1-\nu)b\ \forall\ a,b\geq 0\right)\\
&\leq \sqrt{2}n^{r-1}\Big(\Big|\nu\sum_{i=1}^{n}\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^{\frac{r}{2\nu}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\\
&+ \mathrm{i}(1-\nu)\sum_{i=1}^{n}\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^{\frac{r}{2(1-\nu)}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\Big|\Big) \quad \left(\text{as } |a+b|\leq\sqrt{2}|a+\mathrm{i}b|\ \forall\ a,b\in\mathbb{R}\right)\\
&= \sqrt{2}n^{r-1}\Big|\Big\langle\Big(\sum_{i=1}^{n}\Big(\nu\left[B_i^*f^2(|X_i|)B_i\right]^{\frac{r}{2\nu}}\mathrm{i}(1-\nu)\left[A_i^*g^2(|X_i^*|)A_i\right]^{\frac{r}{2(1-\nu)}}\Big)\Big)\hat{k}_\lambda,\hat{k}_\lambda\Big\rangle\Big|\\
&\leq \sqrt{2}n^{r-1}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\nu\left[B_i^*f^2(|X_i|)B_i\right]^{\frac{r}{2\nu}}+\mathrm{i}(1-\nu)\left[A_i^*g^2(|X_i^*|)A_i\right]^{\frac{r}{2(1-\nu)}}\right)\right).
\end{aligned}
$$

Taking supremum over all $\lambda\in\Omega$, we get

$$
\begin{aligned}
&\mathbf{ber}^r\left(\sum_{i=1}^{n}A_i^*X_iB_i\right)\\
&\leq\sqrt{2}n^{r-1}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\nu\left[B_i^*f^2(|X_i|)B_i\right]^{\frac{r}{2\nu}}+\mathrm{i}(1-\nu)\left[A_i^*g^2(|X_i^*|)A_i\right]^{\frac{r}{2(1-\nu)}}\right)\right).
\end{aligned}
$$

□

The following corollaries now follow easily from Theorem 3.5.

Corollary 3.3. *Let $A_i, B_i, X_i \in \mathbb{B}(\mathcal{H}), i = 1, 2, \ldots, n$. Let f and g be two nonnegative functions on $[0,\infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then for* *all $r \geq 1$ and $0 \leq \alpha \leq 1$, we have*

$$(i)\ \boldsymbol{ber}^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{\sqrt{2}}\boldsymbol{ber}\left(\sum_{i=1}^n \left(\left(B_i^* f^2(|X_i|)B_i\right)^r + \mathrm{i}\left(A_i^* g^2(|X_i^*|)A_i\right)^r\right)\right),$$

$$(ii)\ \boldsymbol{ber}^r\left(\sum_{i=1}^n X_i\right) \leq \frac{n^{r-1}}{\sqrt{2}}\boldsymbol{ber}\left(\sum_{i=1}^n \left(f^{2r}(|X_i|) + \mathrm{i}\ g^{2r}(|X_i^*|)\right)\right),$$

$$(iii)\ \boldsymbol{ber}^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{\sqrt{2}}\boldsymbol{ber}\left(\sum_{i=1}^n \left(\left(B_i^* |X_i|^{2\alpha} B_i\right)^r + \mathrm{i}\left(A_i^* |X_i^*|^{2(1-\alpha)}A_i\right)^r\right)\right).$$

Corollary 3.4. *Let $A, B, X \in \mathbb{B}(\mathcal{H})$. Then for all $r \geq 1$,*

$$(i)\ \boldsymbol{ber}(A^*XB) \leq \frac{1}{\sqrt{2}}\boldsymbol{ber}\left(\left(B^*|X|^{2\alpha}B\right) + \mathrm{i}(A^*|X^*|^{2(1-\alpha)}A)\right).$$

$$(ii)\ \boldsymbol{ber}(A^*XB) \leq \frac{1}{\sqrt{2}}\boldsymbol{ber}\left((B^*|X|B) + \mathrm{i}(A^*|X^*|A)\right).$$

Corollary 3.5. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$(i)\ \boldsymbol{ber}(A) \leq \frac{1}{\sqrt{2}}\boldsymbol{ber}(|A| + \mathrm{i}|A^*|).$$

$$(ii)\ \boldsymbol{ber}^r(A^*B) \leq \frac{1}{\sqrt{2}}\boldsymbol{ber}\left(|B|^{2r} + \mathrm{i}|A|^{2r}\right) \text{ for all } r \geq 1.$$

Remark 3.6. *It is easy to observe that $\boldsymbol{ber}^2(|A| + \mathrm{i}|A^*|) \leq \boldsymbol{ber}(|A|^2 + |A^*|^2)$. Thus, Corollary 3.5(i) refines the existing bound [86, Cor. 3.5(i)], namely, $\boldsymbol{ber}^2(A) \leq \frac{1}{2}\boldsymbol{ber}(|A|^2 + |A^*|^2)$.*

Next result reads as follows:

Theorem 3.7. *Let $A_i, B_i, X_i \in \mathbb{B}(\mathcal{H})$, for $i = 1, 2, \ldots, n$. Then*

$$\boldsymbol{ber}\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leq \frac{1}{2}\boldsymbol{ber}\left(\sum_{i=1}^n (B_i^*|X_i|^{2\alpha}B_i + A_i^*|X_i^*|^{2(1-\alpha)}A_i)\right) - \frac{1}{4}\sum_{i=1}^n \left(\frac{c^2\left(B_i^*|X_i|^{2\alpha}B_i - A_i^*|X_i^*|^{2(1-\alpha)}A_i\right)}{\boldsymbol{ber}\left(B_i^*|X_i|^{2\alpha}B_i + A_i^*|X_i^*|^{2(1-\alpha)}A_i\right)}\right),$$

for every $\alpha \in [0, 1]$.

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then, we have

$$
\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^n A_i^* X_i B_i\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\
&\leq \sum_{i=1}^n |\langle A_i^* X_i B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle| \\
&\leq \sum_{i=1}^n |\langle X_i B_i \hat{k}_\lambda, A_i \hat{k}_\lambda\rangle| \\
&\leq \sum_{i=1}^n \langle |X_i|^{2\alpha} B_i \hat{k}_\lambda, B_i \hat{k}_\lambda\rangle^{1/2} \langle |X_i^*|^{2(1-\alpha)} A_i \hat{k}_\lambda, A_i \hat{k}_\lambda\rangle^{1/2} \quad \text{(by Lemma 3.3)} \\
&\leq \sum_{i=1}^n \langle B_i^* |X_i|^{2\alpha} B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} \langle A_i^* |X_i^*|^{2(1-\alpha)} A_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} \\
&\leq \sum_{i=1}^n \frac{1}{2}\left(\langle B_i^* |X_i|^{2\alpha} B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle A_i^* |X_i^*|^{2(1-\alpha)} A_i \hat{k}_\lambda, \hat{k}_\lambda\rangle\right) \\
&\quad - \sum_{i=1}^n \frac{1}{2}\left(\langle B_i^* |X_i|^{2\alpha} B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} - \langle A_i^* |X_i^*|^{2(1-\alpha)} A_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}}\right)^2 \quad \text{(by Lemma 3.5)} \\
&= \frac{1}{2}\sum_{i=1}^n \langle (B_i^* |X_i|^{2\alpha} B_i + A_i^* |X_i^*|^{2(1-\alpha)} A_i)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\quad - \frac{1}{2}\sum_{i=1}^n \left(\frac{\langle B_i^* |X_i|^{2\alpha} B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle - \langle A_i^* |X_i^*|^{2(1-\alpha)} A_i \hat{k}_\lambda, \hat{k}_\lambda\rangle}{\langle B_i^* |X_i|^{2\alpha} B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} + \langle A_i^* |X_i^*|^{2(1-\alpha)} A_i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}}}\right)^2 \\
&\leq \frac{1}{2}\langle \sum_{i=1}^n (B_i^* |X_i|^{2\alpha} B_i + A_i^* |X_i^*|^{2(1-\alpha)} A_i)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\quad - \frac{1}{4}\sum_{i=1}^n \frac{\langle (B_i^* |X_i|^{2\alpha} B_i - A_i^* |X_i^*|^{2(1-\alpha)} A_i)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2}{\langle B_i^* |X_i|^{2\alpha} B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle A_i^* |X_i^*|^{2(1-\alpha)} A_i \hat{k}_\lambda, \hat{k}_\lambda\rangle} \quad \text{(by convexity of } f(t) = t^2 \text{)} \\
&\leq \frac{1}{2}\mathbf{ber}\left(\sum_{i=1}^n (B_i^* |X_i|^{2\alpha} B_i + A_i^* |X_i^*|^{2(1-\alpha)} A_i)\right) \\
&\quad - \frac{1}{4}\sum_{i=1}^n \left(\frac{c^2\left(B_i^* |X_i|^{2\alpha} B_i - A_i^* |X_i^*|^{2(1-\alpha)} A_i\right)}{\mathbf{ber}\left(B_i^* |X_i|^{2\alpha} B_i + A_i^* |X_i^*|^{2(1-\alpha)} A_i\right)}\right).
\end{aligned}
$$

Taking supremum over all $\lambda \in \Omega$, we get the desired inequality of the theorem. □

Considering $n = 2$, we get the following corollary.

Corollary 3.6. *Let* $A, B, X, Y \in \mathbb{B}(\mathcal{H})$. *Then*

(i) $$\boldsymbol{ber}(A^* X B + B^* Y A) \leq \frac{1}{2}\boldsymbol{ber}\left(B^*|X|^{2\alpha} B + A^*|X^*|^{2(1-\alpha)} A + A^*|Y|^{2\alpha} A + B^*|Y^*|^{2(1-\alpha)} B\right)$$

$$- \quad \frac{1}{4}\left(\frac{c^2\left(B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A\right)}{\boldsymbol{ber}\left(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A\right)} + \frac{c^2\left(A^*|Y|^{2\alpha}A - B^*|Y^*|^{2(1-\alpha)}B\right)}{\boldsymbol{ber}\left(A^*|Y|^{2\alpha}A + B^*|Y^*|^{2(1-\alpha)}B\right)}\right),$$

for every $\alpha \in [0,1]$.

(ii) $\boldsymbol{ber}(A^*XB + B^*YA) \leq$

$$\frac{1}{2}\boldsymbol{ber}\left(B^*|X|B + A^*|X^*|A + A^*|Y|A + B^*|Y^*|B\right)$$
$$- \quad \frac{1}{4}\left(\frac{c^2\left(B^*|X|B - A^*|X^*|A\right)}{\boldsymbol{ber}\left(B^*|X|B + A^*|X^*|A\right)} + \frac{c^2\left(A^*|Y|A - B^*|Y^*|B\right)}{\boldsymbol{ber}\left(A^*|Y|A + B^*|Y^*|B\right)}\right).$$

(iii) $\boldsymbol{ber}(A^*B + B^*A) \leq$

$$\frac{1}{2}\boldsymbol{ber}\left(B^*B + A^*A + A^*A + B^*B\right)$$
$$- \quad \frac{1}{4}\left(\frac{c^2\left(B^*B - A^*A\right)}{\boldsymbol{ber}\left(B^*B + A^*A\right)} + \frac{c^2\left(A^*A - B^*B\right)}{\boldsymbol{ber}\left(A^*A + B^*B\right)}\right).$$

Remark 3.8. *For every* $\alpha \in [0,1]$, *Corollary 3.6(i) gives stronger inequality than the existing inequality [48, Th. 2.6], namely,*

$$\boldsymbol{ber}(A^*XB + B^*YA) \leq$$
$$\frac{1}{2}\boldsymbol{ber}\left(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A + A^*|Y|^{2\alpha}A + B^*|Y^*|^{2(1-\alpha)}B\right),$$

for every $\alpha \in [0,1]$.

Considering $n = 1$ in Theorem 3.7, we get the following inequalities.

Corollary 3.7. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Then*

(i) $\boldsymbol{ber}(A^*XB) \leq$

$$\frac{1}{2}\boldsymbol{ber}\left(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A\right) - \frac{1}{4}\frac{c^2\left(B^*|X|^{2\alpha}B - A^*|X^*|^{2(1-\alpha)}A\right)}{\boldsymbol{ber}\left(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A\right)},$$

for every $\alpha \in [0,1]$.

(ii) $\boldsymbol{ber}(A^*XB) \leq \frac{1}{2}\boldsymbol{ber}\left(B^*|X|B + A^*|X^*|A\right) - \frac{1}{4}\frac{c^2\left(B^*|X|B - A^*|X^*|A\right)}{\boldsymbol{ber}\left(B^*|X|B + A^*|X^*|A\right)}.$

(iii) $\boldsymbol{ber}(A^*B) \leq \frac{1}{2}\boldsymbol{ber}\left(B^*B + A^*A\right) - \frac{1}{4}\frac{c^2\left(B^*B - A^*A\right)}{\boldsymbol{ber}\left(B^*B + A^*A\right)}.$

(iv) $\boldsymbol{ber}(A) \leq \frac{1}{2}\boldsymbol{ber}\left(I + AA^*\right) - \frac{1}{4}\frac{c^2\left(I - AA^*\right)}{\boldsymbol{ber}\left(I + AA^*\right)}.$

Remark 3.9. *For every* $\alpha \in [0,1]$, *Corollary 3.7(i) gives stronger inequality than the existing inequality [48, Th. 2.5(ii)], namely*

$$\boldsymbol{ber}(A^*XB) \leq \frac{1}{2}\boldsymbol{ber}\left(B^*|X|^{2\alpha}B + A^*|X^*|^{2(1-\alpha)}A\right),$$

for every $\alpha \in [0,1]$.

Next we present an upper bound for the Berezin number of the product operators.

Theorem 3.10. *Let* $A, B, X \in \mathbb{B}(\mathcal{H})$ *such that* $A, B \geq 0$. *Then*

$$\boldsymbol{ber}^r(A^\alpha X B^{1-\alpha}) \leq \|X\|^r \left(\boldsymbol{ber}(\alpha A^r + (1-\alpha)B^r) - \frac{r_0}{2} \frac{c^2(A^r - B^r)}{\boldsymbol{ber}(A^r + B^r)} \right),$$

where $r_0 = min\{\alpha, 1-\alpha\}, r \geq 2$ *and* $\alpha \in [0,1]$.

Proof. Let $\hat{k}_\lambda$ be a normalised reproducing kernel of $\mathcal{H}$. Then

$$\begin{aligned}
& |\langle (A^\alpha X B^{1-\alpha})\hat{k}_\lambda, \hat{k}_\lambda\rangle|^r \\
= \; & |\langle X B^{1-\alpha}\hat{k}_\lambda, A^\alpha \hat{k}_\lambda\rangle|^r \\
\leq \; & \|X\|^r \|B^{1-\alpha}\hat{k}_\lambda\|^r \|A^\alpha \hat{k}_\lambda\|^r \quad \big(\text{by the Cauchy-Schwarz inequality}\big) \\
= \; & \|X\|^r \langle B^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{r/2} \langle A^\alpha \hat{k}_\lambda, \hat{k}_\lambda\rangle^{r/2} \\
\leq \; & \|X\|^r \langle B^r \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1-\alpha} \langle A^r \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\alpha} \quad \big(\text{by Lemma 3.1}(b)\big) \\
\leq \; & \|X\|^r \Big(\langle (\alpha A^r + (1-\alpha)B^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle - r_0 (\langle A^r \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} - \langle B^r \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2})^2 \Big) \\
& \hfill \big(\text{by Lemma 3.5}\big) \\
= \; & \|X\|^r \left(\langle (\alpha A^r + (1-\alpha)B^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle - r_0 \left(\frac{\langle (A^r - B^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle}{\langle A^r \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} + \langle B^r \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}} \right)^2 \right) \\
\leq \; & \|X\|^r \left(\langle (\alpha A^r + (1-\alpha)B^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle - \frac{r_0}{2} \frac{\langle (A^r - B^r)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2}{\langle A^r \hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle B^r \hat{k}_\lambda, \hat{k}_\lambda\rangle} \right) \\
& \hfill \big(\text{ by using convexity of the function } f(t) = t^2\big) \\
\leq \; & \|X\|^r \left(\mathbf{ber}(\alpha A^r + (1-\alpha)B^r) - \frac{r_0}{2} \frac{c^2(A^r - B^r)}{\mathbf{ber}(A^r + B^r)} \right).
\end{aligned}$$

Taking supremum over all $\lambda \in \Omega$, we get

$$\mathbf{ber}^r(A^\alpha X B^{1-\alpha}) \leq \|X\|^r \left(\mathbf{ber}(\alpha A^r + (1-\alpha)B^r) - \frac{r_0}{2} \frac{c^2(A^r - B^r)}{\mathbf{ber}(A^r + B^r)} \right).$$

□

Taking $X = I$ in the above theorem we get the following corollary.

Corollary 3.8. *Let* $A, B \in \mathbb{B}(\mathcal{H})$ *such that* $A, B \geq 0$. *Then*
(i) For all $\alpha \in [0,1]$,

$$\boldsymbol{ber}^r(A^\alpha B^{1-\alpha}) \leq \left(\boldsymbol{ber}(\alpha A^r + (1-\alpha)B^r) - \frac{r_0}{2} \frac{c^2(A^r - B^r)}{\boldsymbol{ber}(A^r + B^r)} \right),$$

where $r_0 = min\{\alpha, 1-\alpha\}$ *and* $r \geq 2$.

(ii) Morever, if $AB = BA$*, then*

$$\boldsymbol{ber}(\sqrt{AB}) \leq \left(\frac{1}{2}\boldsymbol{ber}(A^2+B^2) - \frac{1}{4}\frac{c^2(A^2-B^2)}{\boldsymbol{ber}(A^2+B^2)}\right)^{1/2}.$$

CHAPTER 4

BEREZIN NUMBER AND BEREZIN NORM: INEQUALITIES AND EQUALITIES

4.1 Introduction

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space (RKHS in short) over a non-empty set Ω. Let $\mathbb{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. Let $\{\hat{k}_\lambda = k_\lambda/\|k_\lambda\| : \lambda \in \Omega\}$ be the set of all normalized reproducing kernels of $\mathcal{H}$. Recall that the Berezin number and the Berezin norm of $A \in \mathbb{B}(\mathcal{H})$ are defined as $\mathbf{ber}(A) = \sup_{\lambda\in\Omega} \left|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|$ and $\|A\|_{ber} = \sup_{\lambda,\mu\in\Omega} \left|\langle A\hat{k}_\lambda, \hat{k}_\mu\rangle\right|$, respectively. In this chapter, we obtain inequalities involving the Berezin norm and the Berezin number of bounded linear operators defined on a reproducing kernel Hilbert space $\mathcal{H}$. As a special cases we derive several inequalities that refine the existing results.

Content of this chapter is partially based on the following two papers:

- P. Bhunia, K. Paul and **A. Sen**, Inequalities involving Berezin norm and Berezin number, Complex Anal. Oper. Theory, 17 (2023), no. 1, Paper No. 7, 15 pp. https://doi.org/10.1007/s11785-022-01305-9
- **A. Sen** and K. Paul, Berezin number and numerical radius inequalities, Vietnam J. Math., (2023). https://doi.org/10.1007/s10013-023-00658-8

Among many inequalities we prove that for any positive bounded linear operator on $\mathcal{H}$, the Berezin number and the Berezin norm are equal. We also provide an example to show that the equality does not hold for general self-adjoint operators.

4.2 Berezin number and Berezin norm inequalities

We start with the following lemmas that will be used to develop the Berezin number and Berezin norm inequalities.

Lemma 4.1. *[84] Let $A \in \mathbb{B}(\mathcal{H})$ be positive, and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \quad \textit{for all} \quad r \geq 1.$$

Lemma 4.2. *[38] Let $A \in \mathbb{B}(\mathcal{H})$, and let $x, y \in \mathcal{H}$. Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha} x, x\rangle \langle |A^*|^{2(1-\alpha)} y, y\rangle \quad \textit{for all } \alpha \in [0,1].$$

In particular, for $\alpha = 1/2$ the above inequality becomes

$$\langle Ax, y\rangle| \leq \langle |A|x, x\rangle^{1/2} \langle |A^*|y, y\rangle^{1/2}.$$

Lemma 4.3. *[50] For $a, b \geq 0$, $0 < \alpha < 1$ and $r \neq 0$, let $M_r(a, b, \alpha) = (\alpha a^r + (1-\alpha)b^r)^{1/r}$ and $M_0(a, b, \alpha) = a^\alpha b^{1-\alpha}$. Then*

$$M_r(a, b, \alpha) \leq M_s(a, b, \alpha) \quad \textit{for } r \leq s.$$

Now, we are in a position to prove a general inequality involving the Berezin norm and Berezin number, which leads to several inequalities as special cases.

Theorem 4.1. *Let $A, B, C, D, X, Y \in \mathbb{B}(\mathcal{H})$, and let $\alpha \in [0,1]$. Then*

$$\begin{aligned}
&\left\| \frac{A^*XB + C^*YD}{2} \right\|_{ber}^2 \\
&\leq \boldsymbol{ber}^{1/r}\left(\frac{(B^*|X|^{2\alpha}B)^r + (D^*|Y|^{2\alpha}D)^r}{2} \right) \boldsymbol{ber}^{1/s}\left(\frac{(A^*|X^*|^{2(1-\alpha)}A)^s + (C^*|Y^*|^{2(1-\alpha)}C)^s}{2} \right),
\end{aligned}$$

for all $r, s \geq 1$.

Proof. Let $\hat{k}_\lambda$ and $\hat{k}_\mu$ be two normalized reproducing kernels of $\mathcal{H}$. Then,

$$
\begin{aligned}
&\frac{1}{4}|\langle (A^*XB + C^*YD)\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 \\
&\le \frac{1}{4}(|\langle A^*XB\hat{k}_\lambda, \hat{k}_\mu\rangle| + |\langle C^*YD\hat{k}_\lambda, \hat{k}_\mu\rangle|)^2 \\
&= \frac{1}{4}(|\langle XB\hat{k}_\lambda, A\hat{k}_\mu\rangle| + |\langle YD\hat{k}_\lambda, C\hat{k}_\mu\rangle|)^2 \\
&\le \frac{1}{4}\Big(\langle |X|^{2\alpha}B\hat{k}_\lambda, B\hat{k}_\lambda\rangle^{1/2}\langle |X^*|^{2(1-\alpha)}A\hat{k}_\mu, A\hat{k}_\mu\rangle^{1/2} \\
&\qquad + \langle |Y|^{2\alpha}D\hat{k}_\lambda, D\hat{k}_\lambda\rangle^{1/2}\langle |Y^*|^{2(1-\alpha)}C\hat{k}_\mu, C\hat{k}_\mu\rangle^{1/2}\Big)^2 \qquad \text{(by Lemma 4.2)} \\
&= \frac{1}{4}\Big(\langle B^*|X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle A^*|X^*|^{2(1-\alpha)}A\hat{k}_\mu, \hat{k}_\mu\rangle^{1/2} \\
&\qquad + \langle D^*|Y|^{2\alpha}D\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle C^*|Y^*|^{2(1-\alpha)}C\hat{k}_\mu, \hat{k}_\mu\rangle^{1/2}\Big)^2 \\
&\le \frac{1}{4}\Big(\langle B^*|X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle D^*|Y|^{2\alpha}D\hat{k}_\lambda, \hat{k}_\lambda\rangle\Big) \\
&\qquad \times \Big(\langle A^*|X^*|^{2(1-\alpha)}A\hat{k}_\mu, \hat{k}_\mu\rangle + \langle C^*|Y^*|^{2(1-\alpha)}C\hat{k}_\mu, \hat{k}_\mu\rangle\Big) \\
&\Big(\text{using the inequality } (ab+cd)^2 \le (a^2+c^2)(b^2+d^2) \text{ for all } a,b,c,d \in \mathbb{R}\Big) \\
&= \left(\frac{\langle B^*|X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle D^*|Y|^{2\alpha}D\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right) \\
&\qquad \times \left(\frac{\langle A^*|X^*|^{2(1-\alpha)}A\hat{k}_\mu, \hat{k}_\mu\rangle + \langle C^*|Y^*|^{2(1-\alpha)}C\hat{k}_\mu, \hat{k}_\mu\rangle}{2}\right) \\
&\le \left(\frac{\langle B^*|X|^{2\alpha}B\hat{k}_\lambda, \hat{k}_\lambda\rangle^r + \langle D^*|Y|^{2\alpha}D\hat{k}_\lambda, \hat{k}_\lambda\rangle^r}{2}\right)^{1/r} \\
&\qquad \times \left(\frac{\langle A^*|X^*|^{2(1-\alpha)}A\hat{k}_\mu, \hat{k}_\mu\rangle^s + \langle C^*|Y^*|^{2(1-\alpha)}C\hat{k}_\mu, \hat{k}_\mu\rangle^s}{2}\right)^{1/s} \qquad \text{(by Lemma 4.3)} \\
&\le \left(\frac{\langle (B^*|X|^{2\alpha}B)^r\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle (D^*|Y|^{2\alpha}D)^r\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^{1/r} \\
&\qquad \times \left(\frac{\langle (A^*|X^*|^{2(1-\alpha)}A)^s\hat{k}_\mu, \hat{k}_\mu\rangle + \langle (C^*|Y^*|^{2(1-\alpha)}C)^s\hat{k}_\mu, \hat{k}_\mu\rangle}{2}\right)^{1/s} \qquad \text{(by Lemma 4.1)} \\
&= \left\langle \left(\frac{(B^*|X|^{2\alpha}B)^r + (D^*|Y|^{2\alpha}D)^r}{2}\right)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{1/r} \\
&\qquad \times \left\langle \left(\frac{(A^*|X^*|^{2(1-\alpha)}A)^s + (C^*|Y^*|^{2(1-\alpha)}C)^s}{2}\right)\hat{k}_\mu, \hat{k}_\mu \right\rangle^{1/s} \\
&\le \mathbf{ber}^{1/r}\left(\frac{(B^*|X|^{2\alpha}B)^r + (D^*|Y|^{2\alpha}D)^r}{2}\right)\mathbf{ber}^{1/s}\left(\frac{(A^*|X^*|^{2(1-\alpha)}A)^s + (C^*|Y^*|^{2(1-\alpha)}C)^s}{2}\right).
\end{aligned}
$$

Therefore, taking the supremum over all $\lambda, \mu \in \Omega$, we get the desired inequality.

□

Example 4.2. *Consider $\mathbb{C}^3$ as a RKHS on the set $\Omega = \{1, 2, 3\}$, see in [77, pp. 4-5]. Then the standard orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$ is precisely the set of normalized reproducing kernel of $\mathbb{C}^3$. If we take* $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ *and* $X = Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *then from Theorem 4.1 (for $r = 1, s = 2$) we have,* $\left\|\frac{A^*XB+C^*YD}{2}\right\|_{ber}^2 \leq 27\sqrt{10}/2$.

The following corollary follows easily from Theorem 4.1 by taking $A = B = C = D = I$, $X = A$ and $Y = B$.

Corollary 4.1. *Let $A, B \in \mathbb{B}(\mathcal{H})$, and let $\alpha \in [0, 1]$. Then*

$$\left\|\frac{A+B}{2}\right\|_{ber}^2 \leq \boldsymbol{ber}^{1/r}\left(\frac{|A|^{2\alpha r} + |B|^{2\alpha r}}{2}\right) \boldsymbol{ber}^{1/s}\left(\frac{|A^*|^{2(1-\alpha)s} + |B^*|^{2(1-\alpha)s}}{2}\right),$$

for all $r, s \geq 1$.

In particular, for $s = r$

$$\|A+B\|_{ber}^r \leq 2^{r-1}\boldsymbol{ber}^{1/2}\left(|A|^{2\alpha r} + |B|^{2\alpha r}\right) \boldsymbol{ber}^{1/2}\left(|A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r}\right), \tag{4.1}$$

for all $r \geq 1$.

Remark 4.3. *It was proved in [83, Th. 2.17.] that if $A, B \in \mathbb{B}(\mathcal{H})$, then*

$$\|A+B\|_{ber}^r \leq 2^{r-2}\left(\boldsymbol{ber}(|A|^{2\alpha r} + |B|^{2\alpha r}) + \boldsymbol{ber}(|A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r})\right), \tag{4.2}$$

for $0 < \alpha < 1$ and for all $r \geq 1$. Clearly, by applying AM-GM inequality, we infer that

$$\begin{aligned} &\boldsymbol{ber}^{1/2}\left(|A|^{2\alpha r} + |B|^{2\alpha r}\right) \boldsymbol{ber}^{1/2}\left(|A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r}\right) \\ &\leq \frac{1}{2}\left(\boldsymbol{ber}(|A|^{2\alpha r} + |B|^{2\alpha r}) + \boldsymbol{ber}(|A^*|^{2(1-\alpha)r} + |B^*|^{2(1-\alpha)r})\right). \end{aligned}$$

Thus, the inequality (4.1) is sharper than the inequality (4.2).

Again, by considering $X = Y = I$ in Theorem 4.1, we get the following inequality.

Corollary 4.2. *Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then*

$$\left\| \frac{A^*B + C^*D}{2} \right\|_{ber}^2 \leq \boldsymbol{ber}^{1/r}\left(\frac{|A|^{2r} + |C|^{2r}}{2}\right) \boldsymbol{ber}^{1/s}\left(\frac{|B|^{2s} + |D|^{2s}}{2}\right), \tag{4.3}$$

for all $r, s \geq 1$.

Remark 4.4. *It was proved in [42, Th. 3.5.] that if $A, B, C, D \in \mathbb{B}(\mathcal{H})$, then*

$$\boldsymbol{ber}^2\left(\frac{A^*B + C^*D}{2}\right) \leq \boldsymbol{ber}^{1/r}\left(\frac{|A|^{2r} + |C|^{2r}}{2}\right) \boldsymbol{ber}^{1/s}\left(\frac{|B|^{2s} + |D|^{2s}}{2}\right), \tag{4.4}$$

for all $r, s \geq 1$.

Since $\boldsymbol{ber}(A^*B + C^*D) \leq \|A^*B + C^*D\|_{ber}$, the inequality (4.4) follows from (4.3).

Next inequality follows from Corollary 4.2 by taking $s = r$.

Corollary 4.3. *Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then*

$$\left\| \frac{A^*B + C^*D}{2} \right\|_{ber}^{2r} \leq \boldsymbol{ber}\left(\frac{|A|^{2r} + |C|^{2r}}{2}\right) \boldsymbol{ber}\left(\frac{|B|^{2r} + |D|^{2r}}{2}\right),$$

for all $r \geq 1$.

Now, we prove an interesting equality for positive operators.

Proposition 4.1. *If $A \in \mathbb{B}(\mathcal{H})$ is positive (i.e., $A \geq 0$), then*

$$\|A\|_{ber} = \boldsymbol{ber}(A).$$

Proof. Putting $A = B = X$, $C = D = 0$ and $r = 1$ in Corollary 4.3, we have

$$\|X^*X\|_{ber} \leq \mathbf{ber}(X^*X) \text{ for every } X \in \mathbb{B}(\mathcal{H}).$$

Therefore, for every $X \in \mathbb{B}(\mathcal{H})$

$$\|X^*X\|_{ber} = \mathbf{ber}(X^*X). \tag{4.5}$$

Since A is positive, there exists a $Y \in \mathbb{B}(\mathcal{H})$ such that $A = Y^*Y$. This argument together with (4.5) gives that $\|A\|_{ber} = \mathbf{ber}(A)$. □

In the following example we show that the above proposition may not be true for general selfadjoint operators.

Example 4.5. *Consider $\mathbb{C}^{2n}$ as a RKHS on the set $\Omega = \{1, 2, \ldots, 2n\}$ (see in [77, pp. 4-5]). Let $\{e_1, e_2, \ldots, e_{2n}\}$ be the standard orthonormal basis for $\mathbb{C}^{2n}$, i.e., e_i be the function defined by*

$$e_i(j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

for $i, j \in \{1, 2, \ldots, 2n\}$. Then $\{e_1, e_2, \ldots, e_{2n}\}$ is the set of all normalized reproducing kernel functions for $\mathbb{C}^{2n}$. Consider a self-adjoint (not positive) operator A on the RKHS $\mathbb{C}^{2n}$ defined as the matrix

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & & & & \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}_{2n \times 2n}.$$

Then, we have

$$\boldsymbol{ber}(A) = \sup\{|\langle Ae_i, e_i\rangle| \ : \ i \in \{1, 2, \ldots, 2n\}\} = 0$$

and

$$\|A\|_{ber} = \sup\{|\langle Ae_i, e_j\rangle| \ : \ i, j \in \{1, 2, \ldots, 2n\}\} = 1.$$

Therefore, $\|A\|_{ber} \neq \boldsymbol{ber}(A)$.

Next inequality follows by taking $C = B$ and $D = A$ in Corollary 4.2.

Corollary 4.4. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

(i) $$\left\| \frac{A^*B + B^*A}{2} \right\|_{ber}^2 \leq \boldsymbol{ber}^{1/r}\left(\frac{|A|^{2r} + |B|^{2r}}{2}\right) \boldsymbol{ber}^{1/s}\left(\frac{|A|^{2s} + |B|^{2s}}{2}\right),$$

for all $r, s \geq 1$.

(ii) $$\left\| \frac{A^*B + B^*A}{2} \right\|_{ber}^r \leq \boldsymbol{ber}\left(\frac{|A|^{2r} + |B|^{2r}}{2}\right),$$

for all $r \geq 1$.

In particular, for $r = 1$

$$\|A^*B + B^*A\|_{ber} \leq \boldsymbol{ber}(A^*A + B^*B) = \|A^*A + B^*B\|_{ber}. \tag{4.6}$$

Now, taking $A = C = I, B = A$ and $D = B$ in Corollary 4.2 we get the following corollary.

Corollary 4.5. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$\left\| \frac{A+B}{2} \right\|_{ber}^{2r} \leq \boldsymbol{ber}\left(\frac{|A|^{2r} + |B|^{2r}}{2} \right), \tag{4.7}$$

for all $r \geq 1$.

In particular, for $r = 1$

$$\|A+B\|_{ber}^2 \leq 2\, \boldsymbol{ber}(A^*A + B^*B) = 2\|A^*A + B^*B\|_{ber} \tag{4.8}$$

If we take $A = \Re(A)$ and $B = i\Im(A)$ in (4.7), then we get

$$\|A\|_{ber}^{2r} \leq 2^{2r-1}\mathbf{ber}\left(\Re(A)^{2r} + \Im(A)^{2r}\right), \tag{4.9}$$

for all $r \geq 1$.

Also, if we take $A = A$ and $B = A^*$ in (4.7), then we get

$$\|\Re(A)\|_{ber}^{2r} \leq \mathbf{ber}\left(\frac{|A|^{2r} + |A^*|^{2r}}{2} \right) \text{ for all } r \geq 1 \tag{4.10}$$

and if we take $A = A$ and $B = -A^*$ in (4.7), then we get

$$\|\Im(A)\|_{ber}^{2r} \leq \mathbf{ber}\left(\frac{|A|^{2r} + |A^*|^{2r}}{2} \right) \text{ for all } r \geq 1. \tag{4.11}$$

The following result follows from Corollary 4.2 by considering $A = A^*, B = A, C = B^*$ and $D = B$.

Corollary 4.6. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

(i) $$\left\| \frac{A^2+B^2}{2} \right\|_{ber}^{2} \leq \boldsymbol{ber}^{1/r}\left(\frac{|A|^{2r} + |B|^{2r}}{2} \right) \boldsymbol{ber}^{1/s}\left(\frac{|A^*|^{2s} + |B^*|^{2s}}{2} \right),$$

for all $r, s \geq 1$.

(ii) $$\left\| \frac{A^2+B^2}{2} \right\|_{ber}^{2r} \leq \boldsymbol{ber}\left(\frac{|A|^{2r} + |B|^{2r}}{2} \right) \boldsymbol{ber}\left(\frac{|A^*|^{2r} + |B^*|^{2r}}{2} \right),$$

for all $r \geq 1$.

In particular, for $r = 1$

$$\left\|A^2 + B^2\right\|_{ber}^2 \leq \boldsymbol{ber}(A^*A + B^*B)\boldsymbol{ber}(AA^* + BB^*) \tag{4.12}$$

$$= \|A^*A + B^*B\|_{ber} \|AA^* + BB^*\|_{ber}.$$

Also, the following inequality follows from Corollary 4.2 by choosing $A = I, D = I, C = B^*$ and $B = A$.

Corollary 4.7. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$\left\| \frac{A+B}{2} \right\|_{ber}^2 \leq \boldsymbol{ber}^{1/r} \left(\frac{|A|^{2r} + I}{2} \right) \boldsymbol{ber}^{1/s} \left(\frac{|B^*|^{2s} + I}{2} \right), \tag{4.13}$$

for all $r, s \geq 1$.

In particular, for $B = A$

$$\|A\|_{ber}^2 \leq \boldsymbol{ber}^{1/r} \left(\frac{|A|^{2r} + I}{2} \right) \boldsymbol{ber}^{1/s} \left(\frac{|A^*|^{2s} + I}{2} \right), \tag{4.14}$$

for all $r, s \geq 1$.

Moreover, for $s = r$

$$\|A\|_{ber}^{2r} \leq \boldsymbol{ber} \left(\frac{|A|^{2r} + I}{2} \right) \boldsymbol{ber} \left(\frac{|A^*|^{2r} + I}{2} \right), \tag{4.15}$$

for all $r \geq 1$.

Now, taking $A = A^*$ and $C = D = 0$ in Corollary 4.2 we get the following result.

Corollary 4.8. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

(i) $\|AB\|_{ber}^2 \leq 2^{2-1/r-1/s} \boldsymbol{ber}^{1/r} \left(|A^*|^{2r} \right) \boldsymbol{ber}^{1/s} \left(|B|^{2s} \right)$ *for all $r, s \geq 1$.*

(ii) $\|AB\|_{ber}^{2r} \leq 2^{2r-2} \boldsymbol{ber} \left(|A^*|^{2r} \right) \boldsymbol{ber} \left(|B|^{2r} \right)$ *for all $r \geq 1$.*

In particular, for $r = 1$

$$\|AB\|_{ber} \leq \boldsymbol{ber}^{1/2} (AA^*) \, \boldsymbol{ber}^{1/2} (B^*B) = \|AA^*\|_{ber}^{1/2} \|B^*B\|_{ber}^{1/2}. \tag{4.16}$$

Also, the next result follows from Corollary 4.2 by taking $A = A^*, B = B, C = \pm B^*$ and $D = A$.

Corollary 4.9. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

(i) $$\left\| \frac{AB \pm BA}{2} \right\|_{ber}^2 \leq \boldsymbol{ber}^{1/r} \left(\frac{|A|^{2r} + |B|^{2r}}{2} \right) \boldsymbol{ber}^{1/s} \left(\frac{|A^*|^{2s} + |B^*|^{2s}}{2} \right),$$

for all $r, s \geq 1$.

$$(ii)\quad \left\|\frac{AB \pm BA}{2}\right\|_{ber}^{2r} \leq \boldsymbol{ber}\left(\frac{|A|^{2r}+|B|^{2r}}{2}\right)\boldsymbol{ber}\left(\frac{|A^*|^{2r}+|B^*|^{2r}}{2}\right),$$

for all $r \geq 1$.

In particular, for $B = A^*$ in Corollary 4.9(ii), we have

$$\|AA^* \pm A^*A\|_{ber}^{r} \leq 2^{r-1}\mathbf{ber}\left((A^*A)^r + (AA^*)^r\right) \text{ for all } r \geq 1. \tag{4.17}$$

Moreover, for $r = 1$ in (4.17), we have

$$\|AA^* - A^*A\|_{ber} \leq \mathbf{ber}\left(A^*A + AA^*\right) = \|AA^* + A^*A\|_{ber}. \tag{4.18}$$

Now, we prove the following result.

Theorem 4.6. *Let* $A, B \in \mathbb{B}(\mathcal{H})$. *Then*

$$\left\|\frac{|A|^2+|B|^2}{2}\right\|_{ber} \leq \boldsymbol{ber}(\Re(B^*A)) + \frac{1}{2}\left\||A-B|^2\right\|_{ber}.$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we have

$$\begin{aligned}\left\||A-B|^2\right\|_{ber} &\geq \langle |A-B|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle \\ &= \langle (A-B)\hat{k}_\lambda, (A-B)\hat{k}_\lambda\rangle \\ &= \langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |B|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle - 2\Re\langle A\hat{k}_\lambda, B\hat{k}_\lambda\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle\left(|A|^2+|B|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle &\leq 2\Re\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle + \left\||A-B|^2\right\|_{ber} \\ &\leq 2\mathbf{ber}(\Re(B^*A)) + \left\||A-B|^2\right\|_{ber}.\end{aligned}$$

So, taking the supremum over all $\lambda \in \Omega$, we get the desired result. □

Remark 4.7. *Since* $\left\||A-B|^2\right\|_{ber} \leq \|A-B\|^2$ *so the bound of Theorem 4.6 refines the existing bound obtained in [40, Th. 5.1], namely*

$$\frac{1}{2}\boldsymbol{ber}(|A|^2+|B|^2) \leq \boldsymbol{ber}(B^*A) + \frac{1}{2}\|A-B\|^2.$$

To show that the refinement is proper, we consider the reproducing kernel Hilbert space $\mathcal{H} = \mathbb{C}^2$,

$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ *then we have*

$$\left\||A-B|^2\right\|_{ber} = 5 < 9 = \|A-B\|^2.$$

The following two inequalities are immediate from Theorem 4.6 by taking $B = A^*$ and $B = I$, respectively.

Corollary 4.10. *Let* $A \in \mathbb{B}(\mathcal{H})$. *Then*

$$(i) \quad \left\|\frac{|A|^2+|A^*|^2}{2}\right\|_{ber} - \frac{1}{2}\||A-A^*|^2\|_{ber} \leq \boldsymbol{ber}(\Re(A^2)).$$
$$(ii) \quad \left\|\frac{|A|^2+I}{2}\right\|_{ber} - \frac{1}{2}\||A-I|^2\|_{ber} \leq \boldsymbol{ber}(\Re(A)).$$

Next result reads as follows:

Theorem 4.8. *Let* $A, B \in \mathbb{B}(\mathcal{H})$. *Then*

$$\boldsymbol{ber}(A)\boldsymbol{ber}(B) \leq \frac{1}{2}\boldsymbol{ber}(B^*A) + \frac{1}{4}\||A|^2+|B|^2\|_{ber}.$$

Proof. To prove this result, we first introduce the following inequality [36]:

$$|\langle x,z\rangle||\langle y,z\rangle| \leq |\langle x,y\rangle|\|z\|^2 + \sqrt{\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2}\sqrt{\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2}, \tag{4.19}$$

where $x, y \in \mathcal{H}$.
Let $\hat{k}_\lambda$ be a normalized reproducing kernels of $\mathcal{H}$. Putting $x = A\hat{k}_\lambda$, $y = B\hat{k}_\lambda$ and $z = \hat{k}_\lambda$ in (4.19), we get

$$\begin{aligned}
&|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle||\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle| \\
&\leq |\langle A\hat{k}_\lambda, B\hat{k}_\lambda\rangle| + \sqrt{\|A\hat{k}_\lambda\|^2 - |\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2}\sqrt{\|B\hat{k}_\lambda\|^2 - |\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2} \\
&\leq |\langle B^*A\hat{k}_\lambda,\hat{k}_\lambda\rangle| + \frac{1}{2}\left(\|A\hat{k}_\lambda\|^2 + \|B\hat{k}_\lambda\|^2 - |\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2 - |\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2\right) \\
&\leq |\langle B^*A\hat{k}_\lambda,\hat{k}_\lambda\rangle| + \frac{1}{2}\langle(|A|^2+|B|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle - |\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle||\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle|.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle||\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle| &\leq \frac{1}{2}|\langle B^*A\hat{k}_\lambda,\hat{k}_\lambda\rangle| + \frac{1}{4}\langle(|A|^2+|B|^2)\hat{k}_\lambda,\hat{k}_\lambda\rangle \\
&\leq \frac{1}{2}\mathbf{ber}(B^*A) + \frac{1}{4}\||A|^2+|B|^2\|_{ber}.
\end{aligned}$$

So, taking supremum over all $\lambda \in \Omega$, we get the required result. ☐

Taking $B = A^*$ in Theorem 4.8 we get the following corollary.

Corollary 4.11. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\textbf{ber}^2(A) \leq \frac{1}{2}\|A^2\|_{ber} + \frac{1}{4}\||A|^2 + |A^*|^2\|_{ber}.$$

Remark 4.9. *If $A^2 = 0$, then it follows from Corollary 4.11 that*

$$\textbf{ber}^2\,(A) \leq \frac{1}{4}\||A|^2 + |A^*|^2\|_{ber}.$$

Therefore, we remark that for $A^2 = 0$, Corollary 4.11 gives sharper bound than the existing bound obtained in [86, Cor. 3.5(i), for $r = 2$], namely

$$\textbf{ber}^2\,(A) \leq \frac{1}{2}\textbf{ber}\left(|A|^2 + |A^*|^2\right).$$

Next we obtain the following result.

Theorem 4.10. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$\textbf{ber}^2(A) + \textbf{ber}^2(B) \leq \sqrt{\||A|^4 + |B|^4\|_{ber} + \textbf{ber}^2(B^*A)}. \tag{4.20}$$

Proof. We use the following inequality [37, p. 148]:

$$|\langle x, z\rangle|^2 + |\langle z, y\rangle|^2 \leq \sqrt{\langle x, x\rangle^2 + \langle y, y\rangle^2 + |\langle x, y\rangle|^2}, \tag{4.21}$$

for every $x, y, z \in \mathcal{H}$ with $\|z\| = 1$.
Let $\hat{k}_\lambda$ be a normalized reproducing kernels of $\mathcal{H}$. Putting $x = A\hat{k}_\lambda$, $y = B\hat{k}_\lambda$ and $z = \hat{k}_\lambda$ in (4.21), we get

$$\begin{aligned}
|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle \hat{k}_\lambda, B\hat{k}_\lambda\rangle|^2 &\leq \sqrt{\langle A\hat{k}_\lambda, A\hat{k}_\lambda\rangle^2 + \langle B\hat{k}_\lambda, B\hat{k}_\lambda\rangle^2 + |\langle A\hat{k}_\lambda, B\hat{k}_\lambda\rangle|^2} \\
&= \sqrt{\langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle |B|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + |\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2} \\
&\leq \sqrt{\langle (|A|^4 + |B|^4)\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2} \quad \text{(by Lemma 4.1)} \\
&\leq \sqrt{\||A|^4 + |B|^4\|_{ber} + \mathbf{ber}^2(B^*A)}.
\end{aligned}$$

Therefore, the desired result follows by taking supremum over all $\lambda \in \Omega$. ☐

Considering $B = A^*$ in Theorem 4.10, we get the following corollary.

Corollary 4.12. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^4(A) \leq \frac{1}{4}\|A^2\|_{ber}^2 + \frac{1}{4}\||A|^4 + |A^*|^4\|_{ber}.$$

Remark 4.11. *If $A^2 = 0$, then it follows from Corollary 4.12 that*

$$\boldsymbol{ber}^4\left(A\right) \leq \frac{1}{4}\||A|^4 + |A^*|^4\|_{ber}.$$

Therefore, we remark that for $A^2 = 0$, Corollary 4.12 gives sharper bound than the existing bound obtained in [86, Cor. 3.5(i), for $r = 4$], namely

$$\boldsymbol{ber}^4\left(A\right) \leq \frac{1}{2}\boldsymbol{ber}\left(|A|^4 + |A^*|^4\right).$$

Next inequality reads as follows:

Theorem 4.12. *Let $A, B \in \mathbb{B}(\mathcal{H})$ and $\alpha(\neq 0) \in \mathbb{C}$. Then for $\lambda \in \Omega$,*

$$\left|\widetilde{A}(\lambda)\widetilde{B}(\lambda)\right| \leq \frac{\max\{1, |\alpha - 1|\}}{2|\alpha|}\||A|^2 + |B|^2\|_{ber} + \frac{1}{|\alpha|}\boldsymbol{ber}(B^*A).$$

Proof. To prove this result, we need the following inequality which follows from [73, Cor. 2.5]:

$$|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{|\alpha|}\left(\max\{1, |\alpha - 1|\}\|x\|\|y\| + |\langle x, y\rangle|\right), \tag{4.22}$$

for all $x, y, z \in \mathcal{H}$ with $\|z\| = 1$ and $\alpha(\neq 0) \in \mathbb{C}$.
Let $\lambda \in \Omega$ and $\hat{k}_\lambda$ be the corresponding normalized reproducing kernels of $\mathcal{H}$. Putting $x = A\hat{k}_\lambda$, $y = B\hat{k}_\lambda$ and $z = \hat{k}_\lambda$ in (4.22), we get

$$\begin{aligned}
\left|\widetilde{A}(\lambda)\widetilde{B}(\lambda)\right| &\leq \frac{1}{|\alpha|}\left(\max\{1, |\alpha - 1|\}\|A\hat{k}_\lambda\|\|B\hat{k}_\lambda\| + |\langle A\hat{k}_\lambda, B\hat{k}_\lambda\rangle|\right) \\
&\leq \frac{1}{|\alpha|}\left(\frac{\max\{1, |\alpha - 1|\}}{2}\langle(|A|^2 + |B|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle|\right) \\
&\leq \frac{1}{|\alpha|}\left(\frac{\max\{1, |\alpha - 1|\}}{2}\||A|^2 + |B|^2\|_{ber} + \mathbf{ber}(B^*A)\right),
\end{aligned}$$

as desired. □

By choosing $B = A^*$ in Theorem 4.12, we obtain the following upper bound of Berezin number.

Corollary 4.13. *Let $A \in \mathbb{B}(\mathcal{H})$ and $\alpha(\neq 0) \in \mathbb{C}$. Then*

$$\boldsymbol{ber}^2(A) \leq \frac{\max\{1, |\alpha - 1|\}}{2|\alpha|}\||A|^2 + |A^*|^2\|_{ber} + \frac{1}{|\alpha|}\|A^2\|_{ber}.$$

In particular,

$$\boldsymbol{ber}^2(A) \leq \frac{1}{4}\||A|^2 + |A^*|^2\|_{ber} + \frac{1}{2}\|A^2\|_{ber}. \tag{4.23}$$

Remark 4.13. *If $A^2 = 0$, then the inequality (4.23) becomes*

$$\boldsymbol{ber}^2\,(A) \leq \frac{1}{4}\||A|^2 + |A^*|^2\|_{ber}.$$

So, for $A^2 = 0$ the bound given by (4.23) is better than the existing bound

$$\boldsymbol{ber}^2\,(A) \leq \frac{1}{2}\boldsymbol{ber}\left(|A|^2 + |A^*|^2\right),$$

obtained in [86, Cor. 3.5(i), for $r = 2$].

4.3 Refined inequalities for sum of two operators

In the following theorem we obtain an upper bound for the Berezin norm of the sum of the product of two positive operators.

Theorem 4.14. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive. Then*

$$\left\|\frac{A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha}{2}\right\|_{ber}^2 \leq \boldsymbol{ber}^{1/r}\left(\frac{A^{2\alpha r} + A^{2(1-\alpha)r}}{2}\right)\boldsymbol{ber}^{1/s}\left(\frac{B^{2\alpha s} + B^{2(1-\alpha)s}}{2}\right),$$

for all $r, s \geq 1$ and for all $\alpha \in [0, 1]$.

Proof. Suppose that $\hat{k}_\lambda$ and $\hat{k}_\mu$ are two normalized reproducing kernels of $\mathcal{H}$. Then

$$\begin{aligned}
& |\langle (A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha)\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 \\
\leq\ & (|\langle A^\alpha B^{1-\alpha}\hat{k}_\lambda, \hat{k}_\mu\rangle| + |\langle A^{1-\alpha}B^\alpha\hat{k}_\lambda, \hat{k}_\mu\rangle|)^2 \\
=\ & (|\langle B^{1-\alpha}\hat{k}_\lambda, A^\alpha\hat{k}_\mu\rangle| + |\langle B^\alpha\hat{k}_\lambda, A^{1-\alpha}\hat{k}_\mu\rangle|)^2 \\
\leq\ & (\|B^{1-\alpha}\hat{k}_\lambda\|\|A^\alpha\hat{k}_\mu\| + \|B^\alpha\hat{k}_\lambda\|\|A^{1-\alpha}\hat{k}_\mu\|)^2 \\
=\ & \left(\langle A^{2\alpha}\hat{k}_\mu, \hat{k}_\mu\rangle^{1/2}\langle B^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} + \langle A^{2(1-\alpha)}\hat{k}_\mu, \hat{k}_\mu\rangle^{1/2}\langle B^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\right)^2 \\
\leq\ & 4\left(\frac{\langle A^{2\alpha}\hat{k}_\mu, \hat{k}_\mu\rangle + \langle A^{2(1-\alpha)}\hat{k}_\mu, \hat{k}_\mu\rangle}{2}\right)\left(\frac{\langle B^{2\alpha}\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle B^{2(1-\alpha)}\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right) \\
& \big(\text{using the inequality } (ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2) \text{ for all } a, b, c, d \in \mathbb{R}\big)
\end{aligned}$$

$$
\begin{aligned}
&\leq 4\left(\frac{\langle A^{2\alpha}\hat{k}_\mu,\hat{k}_\mu\rangle^r+\langle A^{2(1-\alpha)}\hat{k}_\mu,\hat{k}_\mu\rangle^r}{2}\right)^{1/r}\left(\frac{\langle B^{2\alpha}\hat{k}_\lambda,\hat{k}_\lambda\rangle^s+\langle B^{2(1-\alpha)}\hat{k}_\lambda,\hat{k}_\lambda\rangle^s}{2}\right)^{1/s}\\
&\qquad\text{(by Lemma 4.3)}\\
&\leq 4\left(\frac{\langle A^{2\alpha r}\hat{k}_\mu,\hat{k}_\mu\rangle+\langle A^{2(1-\alpha)r}\hat{k}_\mu,\hat{k}_\mu\rangle}{2}\right)^{1/r}\left(\frac{\langle B^{2\alpha s}\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle B^{2(1-\alpha)s}\hat{k}_\lambda,\hat{k}_\lambda\rangle}{2}\right)^{1/s}\\
&\qquad\text{(by Lemma 4.1)}\\
&= 4\left\langle\left(\frac{A^{2\alpha r}+A^{2(1-\alpha)r}}{2}\right)\hat{k}_\mu,\hat{k}_\mu\right\rangle^{1/r}\left\langle\left(\frac{B^{2\alpha s}+B^{2(1-\alpha)s}}{2}\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{1/s}\\
&\leq 4\,\mathbf{ber}^{1/r}\left(\frac{A^{2\alpha r}+A^{2(1-\alpha)r}}{2}\right)\mathbf{ber}^{1/s}\left(\frac{B^{2\alpha s}+B^{2(1-\alpha)s}}{2}\right).
\end{aligned}
$$

Therefore, taking the supremum over all $\lambda,\mu\in\Omega$, we get the desired inequality. □

Example 4.15. *Consider $\mathbb{C}^3$ as a RKHS on the set $\Omega=\{1,2,3\}$. Then the standard orthonormal basis $\{e_1,e_2,e_3\}$ of $\mathbb{C}^3$ is precisely the set of all normalized reproducing kernel of $\mathbb{C}^3$. If we consider $A=\begin{pmatrix}2&0&0\\0&3&0\\0&0&1\end{pmatrix}$ and $B=\begin{pmatrix}4&0&0\\0&1&0\\0&0&6\end{pmatrix}$, then from Theorem 4.14 (for $\alpha=1/3, r=3, s=3/2$) we have, $\left\|\frac{A^\alpha B^{1-\alpha}+A^{1-\alpha}B^\alpha}{2}\right\|_{ber}^2\leq 3^{4/3}5^{1/3}7^{2/3}$.*

Considering $s=r$ in Theorem 4.14, we infer the following corollary.

Corollary 4.14. *Let $A,B\in\mathbb{B}(\mathcal{H})$ be positive. Then*

$$\left\|A^\alpha B^{1-\alpha}+A^{1-\alpha}B^\alpha\right\|_{ber}^{2r}\leq 2^{2r-2}\,\textbf{\textit{ber}}\left(A^{2\alpha r}+A^{2(1-\alpha)r}\right)\textbf{\textit{ber}}\left(B^{2\alpha r}+B^{2(1-\alpha)r}\right), \tag{4.24}$$

for all $r\geq 1$ and for all $\alpha\in[0,1]$.

In particular, for $\alpha=\frac{1}{2}$ and $r=1$, we have

$$\left\|A^{1/2}B^{1/2}\right\|_{ber}\leq \textbf{\textit{ber}}^{1/2}(A)\,\textbf{\textit{ber}}^{1/2}(B). \tag{4.25}$$

Moreover, if $AB=BA$, then

$$\left\|\sqrt{AB}\right\|_{ber}\leq\sqrt{\textbf{\textit{ber}}(A)\,\textbf{\textit{ber}}(B)}. \tag{4.26}$$

Following Proposition 4.1, since $\mathbf{ber}(\sqrt{AB})=\|\sqrt{AB}\|_{ber}$, the inequality (4.26) is same as the existing inequality [86, Cor. 2.10.], namely, $\mathbf{ber}(\sqrt{AB})\leq\sqrt{\mathbf{ber}(A)\mathbf{ber}(B)}$, where $A,B\in\mathbb{B}(\mathcal{H})$ with $A,B\geq 0$ and $AB=BA$.

For $A, B \in \mathbb{B}(\mathcal{H})$ and $\alpha \in [0,1]$, the α-weighted arithmetic mean of A and B is given by $\alpha A + (1-\alpha)B$. Now, in the following theorem we obtain an upper bound for the Berezin norm for α-weighted arithmetic mean of two opetators.

Theorem 4.16. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$\|\alpha A + (1-\alpha)B\|_{ber}^2 \leq \boldsymbol{ber}\left(\alpha^2|A|^2 + (1-\alpha)^2|B|^2\right) + 2\alpha(1-\alpha)\boldsymbol{ber}(B^*A),$$

for all $\alpha \in [0,1]$.

Proof. Let $\hat{k}_\lambda$ and $\hat{k}_\mu$ be two normalized reproducing kernels of $\mathcal{H}$. Then

$$\begin{aligned}
& |\langle(\alpha A + (1-\alpha)B)\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 \\
\leq\ & \|(\alpha A + (1-\alpha)B)\hat{k}_\lambda\|^2 \\
=\ & \langle(\alpha A + (1-\alpha)B)\hat{k}_\lambda, (\alpha A + (1-\alpha)B)\hat{k}_\lambda\rangle \\
=\ & \alpha^2\langle A\hat{k}_\lambda, A\hat{k}_\lambda\rangle + (1-\alpha)^2\langle B\hat{k}_\lambda, B\hat{k}_\lambda\rangle + 2\alpha(1-\alpha)Re\langle A\hat{k}_\lambda, B\hat{k}_\lambda\rangle \\
\leq\ & \alpha^2\langle|A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\alpha)^2\langle|B|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + 2\alpha(1-\alpha)|\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\
\leq\ & \mathbf{ber}\left(\alpha^2|A|^2 + (1-\alpha)^2|B|^2\right) + 2\alpha(1-\alpha)\mathbf{ber}(B^*A).
\end{aligned}$$

Therefore, taking the supremum over all $\lambda, \mu \in \Omega$, we get

$$\|\alpha A + (1-\alpha)B\|_{ber}^2 \leq \mathbf{ber}\left(\alpha^2|A|^2 + (1-\alpha)^2|B|^2\right) + 2\alpha(1-\alpha)\mathbf{ber}(B^*A).$$

□

Example 4.17. *Consider $\mathbb{C}^3$ as a RKHS on the set $\Omega = \{1,2,3\}$. Then the standard orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$ is precisely the set of all normalized reproducing kernel of $\mathbb{C}^3$. If we consider the matrices $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then from Theorem 4.16 (for $\alpha = 1/3$) we have, $\|\alpha A + (1-\alpha)B\|_{ber}^2 \leq 13/9$.*

Putting $\alpha = \frac{1}{2}$ in Theorem 4.16, we get the following corollary which presents upper bound for the Berezin norm of the sum of two operators.

Corollary 4.15. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$\|A + B\|_{ber}^2 \leq \boldsymbol{ber}\left(|A|^2 + |B|^2\right) + 2\,\boldsymbol{ber}(B^*A).$$

Next we obtain an upper bound of the Berezin norm for the sum of two operators.

Theorem 4.18. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$\|A+B\|_{ber}^2 \leq \boldsymbol{ber}(|A|+i|B|)\boldsymbol{ber}(|A^*|+i|B^*|) + \frac{1}{2}\||A|+|B|\|_{ber}\||A^*|+|B^*|\|_{ber}.$$

Proof. Let $\hat{k}_\lambda$ and $\hat{k}_\mu$ be two normalized reproducing kernels of $\mathcal{H}$. Then we have

$$\begin{aligned}
&|\langle (A+B)\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 \\
&\leq |\langle A\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 + |\langle B\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 + 2|\langle A\hat{k}_\lambda, \hat{k}_\mu\rangle||\langle B\hat{k}_\lambda, \hat{k}_\mu\rangle| \\
&\leq |\langle A\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 + |\langle B\hat{k}_\lambda, \hat{k}_\mu\rangle|^2 + \frac{1}{2}\Big(|\langle A\hat{k}_\lambda, \hat{k}_\mu\rangle| + |\langle B\hat{k}_\lambda, \hat{k}_\mu\rangle|\Big)^2 \\
&\leq \langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A^*|\hat{k}_\mu, \hat{k}_\mu\rangle + \langle |B|\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |B^*|\hat{k}_\mu, \hat{k}_\mu\rangle \\
&+ \frac{1}{2}\Big(\langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle |A^*|\hat{k}_\mu, \hat{k}_\mu\rangle^{1/2} + \langle |B|\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle |B^*|\hat{k}_\mu, \hat{k}_\mu\rangle^{1/2}\Big)^2 \quad \text{(by Lemma 4.2)} \\
&\leq \Big(\langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle |B|\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\Big)^{1/2}\Big(\langle |A^*|\hat{k}_\mu, \hat{k}_\mu\rangle^2 + \langle |B^*|\hat{k}_\mu, \hat{k}_\mu\rangle^2\Big)^{1/2} \\
&+ \frac{1}{2}\langle (|A|+|B|)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle (|A^*|+|B^*|)\hat{k}_\mu, \hat{k}_\mu\rangle \\
&\big(\text{by the inequality } (ab+cd)^2 \leq (a^2+c^2)(b^2+d^2) \text{ for all } a,b,c,d \in \mathbb{R}\big) \\
&= |\langle (|A|+i|B|)\hat{k}_\lambda, \hat{k}_\lambda\rangle||\langle (|A^*|+i|B^*|)\hat{k}_\mu, \hat{k}_\mu\rangle| \\
&+ \frac{1}{2}\langle (|A|+|B|)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle (|A^*|+|B^*|)\hat{k}_\mu, \hat{k}_\mu\rangle \\
&\leq \mathbf{ber}(|A|+i|B|)\mathbf{ber}(|A^*|+i|B^*|) + \frac{1}{2}\||A|+|B|\|_{ber}\||A^*|+|B^*|\|_{ber}.
\end{aligned}$$

Taking supremum over all $\lambda, \mu \in \Omega$, we get the desired result. □

Remark 4.19. *Consider the Hardy-Hilbert space $H^2(\mathbb{D})$ of the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, which is defined as the Hilbert space of all square summable analytic functions on the unit disk (see [77]). The Hardy-Hilbert space is a RKHS and for any $\lambda \in \mathbb{D}$, the corresponding reproducing kernel is $k_\lambda(z) = \sum_{n=0}^{\infty} \overline{\lambda}^n z^n$. Consider the operators $P_{\mathbb{C}}$ and M_z on $H^2(\mathbb{D})$, respectively defined as $P_{\mathbb{C}}f(w) = \langle f(w), 1\rangle$ and $M_z f(w) = wf(w)$ ($f \in H^2(\mathbb{D})$, $w \in \mathbb{D}$). Let $A = P_{\mathbb{C}}$ and $B = M_z$. Then it is easy to observe that $\boldsymbol{ber}(|A|+i|B|)\boldsymbol{ber}(|A^*|+i|B^*|)+\frac{1}{2}\||A|+|B|\|_{ber}\||A^*|+|B^*|\|_{ber} = 1+\sqrt{2} < 4 = (\|A\|_{ber} + \|B\|_{ber})^2$. This shows that the inequality obtained in Theorem 4.18 is better than the triangle inequality.*

In the following theorem we present an upper bound of the Berezin number for the sum of two operators.

Theorem 4.20. *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^2(A+B) \leq \boldsymbol{ber}(|A|+i|B^*|)\boldsymbol{ber}(|B|+i|A^*|) + \frac{1}{2}\||A|+|B^*|\|_{ber}\||A^*|+|B|\|_{ber}.$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then we get

$$
\begin{aligned}
&|\langle (A+B)\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 \\
&\leq |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + 2|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle||\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\
&\leq |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + \frac{1}{2}\left(|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| + |\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle|\right)^2 \\
&\leq \langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |B|\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |B^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&+ \frac{1}{2}\left(\langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle |A^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2} + \langle |B|\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle |B^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\right)^2 \quad \text{(by Lemma 4.2)} \\
&\leq \left(\langle |A|\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle |B^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\right)^{1/2}\left(\langle |A^*|\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle |B|\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\right)^{1/2} \\
&+ \frac{1}{2}\langle (|A| + |B^*|)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle (|A^*| + |B|)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\big(\text{by the inequality } (ab+cd)^2 \leq (a^2+c^2)(b^2+d^2) \text{ for all } a,b,c,d \in \mathbb{R}\big) \\
&= |\langle (|A| + i|B^*|)\hat{k}_\lambda, \hat{k}_\lambda\rangle||\langle (|B| + i|A^*|)\hat{k}_\lambda, \hat{k}_\lambda\rangle| + \frac{1}{2}\langle (|A| + |B^*|)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle (|A^*| + |B|)\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leq \mathbf{ber}(|A| + i|B^*|)\mathbf{ber}(|B| + i|A^*|) + \frac{1}{2}\||A| + |B^*|\|_{ber}\||A^*| + |B|\|_{ber}.
\end{aligned}
$$

Taking supremum over all $\lambda \in \Omega$, we obtain the desired result. □

The next corollary follows from Theorem 4.20 by choosing $B = A$.

Corollary 4.16. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^2(A) \leq \frac{1}{4}\boldsymbol{ber}^2(|A| + i|A^*|) + \frac{1}{8}\||A| + |A^*|\|_{ber}^2.$$

Remark 4.21. *It is easy to observe that $\||A| + |A^*|\|_{ber} \leq \sqrt{2}\boldsymbol{ber}(|A| + i|A^*|)$ and $\boldsymbol{ber}^2(|A| + i|A^*|) \leq \||A|^2 + |A^*|^2\|_{ber}$. Therefore, the bound given in Corollary 4.16 is sharper than the existing bounds obtained in [86, Cor. 3.5(i), for $r = 2$] and [23, Cor. 2.21(i)], namely $\boldsymbol{ber}^2(A) \leq \frac{1}{2}\boldsymbol{ber}(|A|^2 + |A^*|^2)$ and $\boldsymbol{ber}(A) \leq \frac{1}{\sqrt{2}}\boldsymbol{ber}(|A| + i|A^*|)$, respectively.*

CHAPTER 5

ON A NEW NORM ON REPRODUCING KERNEL HILBERT SPACE OPERATORS

5.1 Introduction

In this chapter, we provide a new norm(α-Berezin norm) on the space of all bounded linear operators defined on reproducing kernel Hilbert space. The α-Berezin norm generalizes the Berezin norm and the Berezin number. We study some basic properties of this norm and develop various inequalities involving the α-Berezin norm and the Berezin number. By using the these inequalities we obtain various improved bounds of the Berezin number which improves earlier existing bounds.

In this chapter we work on reproducing kernel Hilbert space $\mathcal{H}$ (RKHS) on a non-empty set Ω. Let $\mathbb{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operator on $\mathcal{H}$. Let $\hat{k}_\lambda$ be the normalized reproducing kernel at $\lambda \in \Omega$. For $A \in \mathbb{B}(\mathcal{H})$, the function defined on Ω as $\widetilde{A}(\lambda) = \langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle$ is called the Berezin symbol of A. The Berezin set and the Berezin number of A are defined by $\mathbf{Ber}(A) = \{\widetilde{A}(\lambda) : \lambda \in \Omega\}$ and $\mathbf{ber} = \sup_{\lambda\in\Omega} |\widetilde{A}(\lambda)|$, respectively. Throughout this chapter

Content of this chapter is partially based on the following paper:

- P. Bhunia, M. Gurdal, K. Paul, **A. Sen** and R. Tapdigoglu, On a new norm on the space of reproducing kernel Hilbert space operators and Berezin radius inequalities, Numer. Funct. Anal. Optim., 44 (2023), no. 9, 970–986. https://doi.org/10.1080/01630563.2023.2221857 the Berezin norm of A is defined by $\|A\|_{ber} = \sup_{\lambda\in\Omega} \|A\hat{k}_\lambda\|$. It is easy to observe that $\|\cdot\|_{ber}$ defines a norm on $\mathcal{B}(\mathcal{H})$.

5.2 Introduction of the α-Berezin norm and its properties

In [80], Sain et al. introduced (α,β)-norm on the space of all bounded linear operators defined on a complex Hilbert space. Motivated by this idea for each $\alpha \in [0,1]$, here we introduce the α-Berezin norm on $\mathbb{B}(\mathcal{H})$.

Definition 5.1. *For $A \in \mathbb{B}(\mathcal{H})$ and $\alpha \in [0,1]$, the α-Berezin norm of A is denoted by $\|A\|_{\alpha-\text{ber}}$ and is defined as*

$$\|A\|_{\alpha-\text{ber}} = \sup_{\lambda\in\Omega} \sqrt{\alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + (1-\alpha)\|A\hat{k}_\lambda\|^2}.$$

It is easy to verify that for $\alpha \in [0,1)$, the α-Berezin norm is also a norm on $\mathbb{B}(\mathcal{H})$, and for $\alpha = 1$ it is norm if the reproducing kernel Hilbert space has the "Ber property". Therefore for all $\alpha \in [0,1]$ the α-Berezin norm is a norm in the most familiar reproducing kernel Hilbert spaces, including Hardy-Hilbert spaces and Bergman spaces. It is clear form the definition that the α-Berezin norm satisfies the following inequality:

$$\mathbf{ber}(A) \le \|A\|_{\alpha-\text{ber}} \le \|A\|_{\text{ber}} \text{ for all } \alpha \in [0,1]. \tag{5.1}$$

Note that the α-Berezin norm generalizes both the Berezin number and the Berezin norm. We observe that there exists operator $A \in \mathbb{B}(\mathcal{H})$ for which $\|A\|_{\alpha-\text{ber}} \ne \|A^*\|_{\alpha-\text{ber}}$ for $\alpha \ne 1$. For example, if we consider the reproducing kernel Hilbert space $\mathbb{C}^2$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then $\|A\|_{\alpha-\text{ber}} = 1 \ne \sqrt{2-\alpha} = \|A^*\|_{\alpha-\text{ber}}$ for every $\alpha \in [0,1)$. Next, we give a computational example on a most familiar reproducing kernel Hilbert space, Hardy-Hilbert space. The Hardy-Hilbert space $H^2(\mathbb{D})$ is a Hilbert space consists of all analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ on the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. The reproducing kernel in $H^2(\mathbb{D})$ corresponding to $\lambda \in \mathbb{D}$ is given by $k_\lambda(z) = \sum_{n=0}^{\infty} \overline{\lambda^n} z^n = 1/(1-\overline{\lambda}z)$. We consider the operator M on $H^2(\mathbb{D})$ defined as $Mf(w) = \langle f(w), w^2\rangle w^2$, $f \in H^2(\mathbb{D})$. Then by simple calculations we have $\mathbf{ber}(M) = \frac{4}{27}$, $\|M\|_{ber} = \frac{2}{\sqrt{27}}$ and $\|M\|_{\alpha-\text{ber}} = \frac{2\sqrt{27-23\alpha}}{27}$. Clearly, $\mathbf{ber}(M) < \|M\|_{\alpha-\text{ber}} < \|M\|_{ber}$ for all $\alpha \in (0,1)$.

Here we study the basic properties of the α-Berezin norm and develop various bounds for the α-Berezin norm to derive new upper bounds for the Berezin number of bounded linear operators. First result reads as follows.

Proposition 5.1. *The α-Berezin norm $\|\cdot\|_{\alpha-\text{ber}}$ defines a norm on $\mathbb{B}(\mathcal{H})$ for each $\alpha \in (0,1]$ and it is equivalent to the Berezin norm $\|\cdot\|_{\text{ber}}$. Moreover,*

$$\max\left\{\mathbf{ber}(A), \sqrt{1-\alpha}\|A\|_{\text{ber}}\right\} \leq \|A\|_{\alpha-\text{ber}} \leq \|A\|_{\text{ber}}.$$

Proof. The proof follows from the definition of the α-Berezin norm. □

Next proposition reads as follows.

Proposition 5.2. *Let $A \in \mathbb{B}(\mathcal{H})$ and let $\alpha \in (0,1)$. Then the following two conditions are equivalent:*

(i) $\|A\|_{\alpha-\text{ber}} = \sqrt{\alpha\mathbf{ber}^2(A) + (1-\alpha)\|A\|_{ber}^2}$.

(ii) There exists a sequence $\{\lambda_n\}$ of Ω such that $\lim_{n\to\infty} |\langle A\hat{k}_{\lambda_n}, \hat{k}_{\lambda_n}\rangle| = \mathbf{ber}(A)$, *and* $\lim_{n\to\infty} \|A\hat{k}_{\lambda_n}\| = \|A\|_{ber}$.

Proof. We first prove that (i) implies (ii). It is clear that there exists a sequence $\{\lambda_n\}$ of Ω such that

$$\|A\|_{\alpha-\text{ber}}^2 = \alpha \lim_{n\to\infty} |\langle A\hat{k}_{\lambda_n}, \hat{k}_{\lambda_n}\rangle|^2 + (1-\alpha) \lim_{n\to\infty} \|A\hat{k}_{\lambda_n}\|^2.$$

Since $\{|\langle A\hat{k}_{\lambda_n}, \hat{k}_{\lambda_n}\rangle|\}$ and $\{\|A\hat{k}_{\lambda_n}\|\}$ are both bounded sequence of real numbers, there exists a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that both $\{|\langle A\hat{k}_{\lambda_{n_k}}, \hat{k}_{\lambda_{n_k}}\rangle|\}$ and $\{\|A\hat{k}_{\lambda_{n_k}}\|\}$ are convergent. Therefore, we get

$$\begin{aligned}\alpha\mathbf{ber}^2(A) + (1-\alpha)\|A\|_{ber}^2 &= \|A\|_{\alpha-\text{ber}}^2 \\ &= \alpha \lim_{k\to\infty} |\langle A\hat{k}_{\lambda_{n_k}}, \hat{k}_{\lambda_{n_k}}\rangle|^2 + (1-\alpha) \lim_{k\to\infty} \|A\hat{k}_{\lambda_{n_k}}\|^2 \\ &\leq \alpha\mathbf{ber}^2(A) + (1-\alpha)\|A\|_{ber}^2.\end{aligned}$$

This implies that

$$\lim_{k\to\infty} |\langle A\hat{k}_{\lambda_{n_k}}, \hat{k}_{\lambda_{n_k}}\rangle| = \mathbf{ber}(A) \quad \text{and} \quad \lim_{k\to\infty} \|A\hat{k}_{\lambda_{n_k}}\| = \|A\|_{ber},$$

as desired.

Now we prove that (ii) implies (i). We have

$$\begin{aligned}\|A\|^2_{\alpha-\mathrm{ber}} &= \sup_{\lambda\in\Omega}\left\{\alpha|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2+(1-\alpha)\|A\hat{k}_\lambda\|^2\right\}\\ &\geq \lim_{n\to\infty}\left\{\alpha|\langle A\hat{k}_{\lambda_n},\hat{k}_{\lambda_n}\rangle|^2+(1-\alpha)\|A\hat{k}_{\lambda_n}\|^2\right\}\\ &= \alpha\mathbf{ber}^2(A)+(1-\alpha)\|A\|^2_{ber}.\end{aligned}$$

This completes the proof. □

Remark 5.1. *For $A\in\mathbb{B}(\mathcal{H})$, it is easy to observe that $\mathbf{ber}(A)=\|A\|_{ber}$ implies that there exists a sequence $\{\lambda_n\}$ of Ω such that $\lim_{n\to\infty}|\langle A\hat{k}_{\lambda_n},\hat{k}_{\lambda_n}\rangle|=\mathbf{ber}(A)$ and $\lim_{n\to\infty}\|A\hat{k}_{\lambda_n}\|=\|A\|_{ber}$. But, in general, the converse is not true. As for example, we consider the reproducing kernel Hilbert space $\mathbb{C}^2$ on the set $\Omega=\{1,2\}$. Then the set of all normalized reproducing kernel is $\{\hat{k}_1=e_1,\hat{k}_2=e_2\}$, where e_1 and e_2 are the standard orthonormal basis of $\mathbb{C}^2$. If we take $A=\begin{pmatrix}1&0\\1&0\end{pmatrix}$, then there exists a sequence $\{\lambda_n=1\}$ such that $\mathbf{ber}(A)=\langle Te_1,e_1\rangle$ and $\|A\|_{ber}=\|Te_1\|$. However, $\mathbf{ber}(A)=1\neq\sqrt{2}=\|A\|_{ber}$.*

5.3 α-Berezin norm and Berezin number inequalities

Here we note necessary known lemmas to study our results in this section. For a bounded linear operator A, we denote $|A|=(A^*A)^{1/2}$ and $|A^*|=(AA^*)^{1/2}$. The Cartesian decomposition of A is $A=\Re(A)+i\Im(A)$, where $\Re(A)=\frac{1}{2}(A+A^*)$ and $\Im(A)=\frac{1}{2i}(A-A^*)$.

Lemma 5.1. *[61] If $A\in\mathbb{B}(\mathcal{H})$ and $0\leq\alpha\leq 1$, then*

$$|\langle Ax,y\rangle|^2\leq\langle|A|^{2\alpha}x,x\rangle\langle|A^*|^{2(1-\alpha)}y,y\rangle\ \forall x,y\in\mathcal{H}.$$

Especially for $\alpha=1/2$,

$$|\langle Ax,y\rangle|^2\leq\langle|A|\,x,x\rangle\,\langle|A^*|\,y,y\rangle\ \ \forall x,y\in\mathcal{H}. \tag{5.2}$$

Lemma 5.2. *[66, Th. 1] If $f,g:[0,\infty)\to[0,\infty)$ are continuous functions satisfying $f(t)\,g(t)=t$ for all $t\geq 0$, then*

$$|\langle Ax,y\rangle|^2\leq\left\langle f^2(|A|)\,x,x\right\rangle\left\langle g^2(|A^*|)\,y,y\right\rangle,\ \forall x,y\in\mathcal{H}.$$

Lemma 5.3. *[84, p. 20] If $A \in \mathbb{B}(\mathcal{H})$ is positive, then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle, \ \forall r \geq 1, \ \forall x \in \mathcal{H}, \|x\| = 1. \tag{5.3}$$

The inequality (5.3) is reversed when $0 \leq r \leq 1$.

Lemma 5.4. *[31] If $x, y, e \in \mathcal{H}$ with $\|e\| = 1$, then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}\left(\|x\|\|y\| + |\langle x, y\rangle|\right).$$

Now, in the following theorem we obtain a lower bound for the α-Berezin norm.

Theorem 5.2. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\|A\|_{\alpha-\mathrm{ber}}^2 \geq \max\left\{\alpha \mathbf{ber}^2(A) + (1-\alpha)c(A^*A), \alpha c^2(A) + (1-\alpha)\|A\|_{\mathrm{ber}}^2\right\},$$

where $c(A) = \inf_{\lambda\in\Omega} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}\|A\|_{\alpha-\mathrm{ber}}^2 &\geq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + (1-\alpha)\|A\hat{k}_\lambda\|^2\\ &= \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + (1-\alpha)\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\\ &\geq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + (1-\alpha)c(A^*A).\end{aligned}$$

Taking the supremum over $\lambda \in \Omega$, we get

$$\|A\|_{\alpha-\mathrm{ber}}^2 \geq \alpha \mathbf{ber}^2(A) + (1-\alpha)c(A^*A). \tag{5.4}$$

Also, $\|A\|_{\alpha-\mathrm{ber}}^2 \geq \alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + \alpha\|A\hat{k}_\lambda\|^2 \geq \alpha c^2(A) + (1-\alpha)\|A\hat{k}_\lambda\|^2$. Taking the supremum over $\lambda \in \Omega$, we get

$$\|A\|_{\alpha-\mathrm{ber}}^2 \geq \alpha c^2(A) + \alpha\|A\|_{\mathrm{ber}}^2. \tag{5.5}$$

Combining (5.4) and (5.5), we get the required result. □

Now, in the following theorem we obtain an upper bound for the α-Berezin norm.

Theorem 5.3. *Let $A \in \mathbb{B}(\mathcal{H})$ and $f, g : [0,\infty) \to [0,\infty)$ be two continuous functions such that $f(t)g(t) = t$ for all $t \geq 0$. Then*

$$\begin{aligned}&\|A\|_{\alpha-\mathrm{ber}}^2\\ &\leq \mathbf{ber}\left(\frac{\alpha}{4}\left(f^4(|A|) + g^4(|A^*|)\right) + (1-\alpha)|A|^2\right) + \frac{\alpha}{2}\mathbf{ber}\left(\Re\left(f^2(|A|)g^2(|A^*|)\right)\right).\end{aligned}$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then, it follows Lemma 5.2 and Lemma 5.3 that

$$\begin{aligned}
\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &\leq \left\langle f^2\left(|A|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \left\langle g^2\left(|A^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&= \left(\left\langle f^2\left(|A|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{1/2} \left\langle g^2\left(|A^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{1/2}\right)^2 \\
&\leq \frac{1}{4}\left(\left\langle f^2\left(|A|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle g^2\left(|A^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^2 \\
&= \frac{1}{4}\left\langle \left(f^2\left(|A|\right) + g^2\left(|A^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^2 \\
&\leq \frac{1}{4}\left\langle \left(f^2\left(|A|\right) + g^2\left(|A^*|\right)\right)^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&= \frac{1}{4}\left\langle \left(f^4\left(|A|\right) + g^4\left(|A^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \frac{1}{2}\left\langle \Re\left(f^2\left(|A|\right)g^2\left(|A^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\|A\|^2_{\alpha-\text{ber}} &= \sup_{\lambda\in\Omega}\left\{\alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + (1-\alpha)\|A\hat{k}_\lambda\|^2\right\} \\
&\leq \sup_{\lambda\in\Omega}\left\langle\left(\frac{\alpha}{4}\left(f^4\left(|A|\right) + g^4\left(|A^*|\right)\right) + (1-\alpha)|T|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&+ \frac{\alpha}{2}\sup_{\lambda\in\Omega}\left|\left\langle \Re\left(f^2\left(|A|\right)g^2\left(|A^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\
&= \mathbf{ber}\left(\frac{\alpha}{4}\left(f^4\left(|A|\right) + g^4\left(|A^*|\right)\right) + (1-\alpha)|A|^2\right) \\
&+ \frac{\alpha}{2}\mathbf{ber}\left(\Re\left(f^2\left(|A|\right)g^2\left(|A^*|\right)\right)\right),
\end{aligned}$$

as desired. □

Now, taking $f(t) = g(t) = t^{1/2}$ in Theorem 5.3, we get the following corollary.

Corollary 5.1. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\|A\|^2_{\alpha-\text{ber}} \leq \mathbf{ber}\left(\left(1 - \frac{3\alpha}{4}\right)|A|^2 + \frac{\alpha}{4}|A^*|^2\right) + \frac{\alpha}{2}\mathbf{ber}\left(\Re\left(|A|\,|A^*|\right)\right).$$

From the first inequality in (5.1) together with Corollary 5.1, we get the following upper bound for the Berezin number.

Corollary 5.2. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}
\mathbf{ber}^2(A) &\leq \inf_{\alpha\in[0,1]}\left\{\mathbf{ber}\left(\left(1 - \frac{3\alpha}{4}\right)|A|^2 + \frac{\alpha}{4}|A^*|^2\right) + \frac{\alpha}{2}\mathbf{ber}\left(\Re\left(|A|\,|A^*|\right)\right)\right\} \\
&\leq \frac{1}{4}\mathbf{ber}\left(|A|^2 + |A^*|^2\right) + \frac{1}{2}\mathbf{ber}\left(\Re\left(|A|\,|A^*|\right)\right).
\end{aligned}$$

Remark 5.4. *Since* $\mathbf{ber}\left(\Re\left(|A|\,|A^*|\right)\right) \leq \frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right)$*, Corollary 5.2 refines the existing bound* $\mathbf{ber}^2(A) \leq \frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right)$*, which was given in [86, Cor. 3.5(i) for $r = 2$].*

Another upper bound for the α-Berezin norm reads as follows.

Theorem 5.5. *Let* $A \in \mathbb{B}(\mathcal{H})$*. Then*

$$\|A\|_{\alpha-\mathrm{ber}}^2 \leq \mathbf{ber}\left(\alpha\,|A|^2 + (1-\alpha)\,|A^*|^2\right).$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. By using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|A\|_{\alpha-\mathrm{ber}}^2 &= \sup_{\lambda\in\Omega}\left\{\alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + (1-\alpha)\,\|A\hat{k}_\lambda\|^2\right\} \\
&= \sup_{\lambda\in\Omega}\left\{\alpha|\langle \hat{k}_\lambda, A^*\hat{k}_\lambda\rangle|^2 + (1-\alpha)\,\|A\hat{k}_\lambda\|^2\right\} \\
&\leq \sup_{\lambda\in\Omega}\left\{\alpha\|A^*\hat{k}_\lambda\|^2 + (1-\alpha)\,\|A\hat{k}_\lambda\|^2\right\} \\
&= \sup_{\lambda\in\Omega}\left\{\alpha\langle |A^*|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle + (1-\alpha)\,\langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\rangle\right\} \\
&= \sup_{\lambda\in\Omega}\left\langle \left(\alpha\,|A^*|^2 + (1-\alpha)\,|A|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&= \mathbf{ber}\left(\alpha\,|A|^2 + (1-\alpha)\,|A^*|^2\right),
\end{aligned}$$

as desired. □

The following upper bound for the Berezin number follows easily from Theorem 5.5 together with the first inequality of (5.1).

Corollary 5.3. *Let* $A \in \mathbb{B}(\mathcal{H})$*. Then*

$$\mathbf{ber}^2(A) \leq \inf_{\alpha\in(0,1]}\mathbf{ber}\left(\alpha\,|A|^2 + (1-\alpha)\,|A^*|^2\right) \leq \frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right).$$

To show the second inequality of Corollary 5.3 is proper, we consider the reproducing kernel Hilbert space $\mathbb{C}^3$ and $A = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\inf_{\alpha\in(0,1]}\mathbf{ber}\left(\alpha\,|A|^2 + (1-\alpha)\,|A^*|^2\right) = \frac{256}{31}$ and $\frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right) = 8.5$. Hence for that matrix A,

$$\inf_{\alpha\in(0,1]}\mathbf{ber}\left(\alpha\,|A|^2 + (1-\alpha)\,|A^*|^2\right) < \frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right).$$

Note that the first inequality in Corollary 5.3 also follows from [78, Th. 3.11]. Next, we prove the following theorem.

Theorem 5.6. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\|A\|_{\alpha-\mathrm{ber}}^2 \leq \frac{\alpha}{4}\mathbf{ber}^2\left(|A| + i\,|T^*|\right) + \frac{\alpha}{4}\mathbf{ber}\left(|A|\,|A^*|\right) + \mathbf{ber}\left(\left(1-\frac{7\alpha}{8}\right)|A|^2 + \frac{\alpha}{8}\,|A^*|^2\right).$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. By applying the inequality (5.2), we get

$$\begin{aligned}
&|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 \\
&\leq \langle |A|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leq \frac{1}{4}\left(\langle |A|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^2 \\
&= \frac{1}{4}\left(\langle |A|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle |A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\right) + \frac{1}{2}\langle |A|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&= \frac{1}{4}\left|\langle |A|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle + i\langle |A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^2 + \frac{1}{2}\langle |A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle \hat{k}_\lambda, |A|\,\hat{k}_\lambda\rangle \\
&\leq \frac{1}{4}\left|\langle (|A| + i\,|A^*|)\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^2 + \frac{1}{4}\|\,|A^*|\,\hat{k}_\lambda\|\|\,|A|\,\hat{k}_\lambda\| + \frac{1}{4}\left|\langle |A^*|\,\hat{k}_\lambda, |A|\,\hat{k}_\lambda\rangle\right| \;\; \text{(by Lemma 5.4)} \\
&\leq \frac{1}{4}\left|\langle (|A| + i\,|A^*|)\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^2 + \frac{1}{8}\left(\|\,|A^*|\,\hat{k}_\lambda\|^2 + \|\,|A|\,\hat{k}_\lambda\|^2\right) + \frac{1}{4}\left|\langle |A|\,|A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|A\|_{\alpha-\mathrm{ber}}^2 &= \sup_{\lambda\in\Omega}\left\{\alpha\left|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^2 + (1-\alpha)\,\|A\hat{k}_\lambda\|^2\right\} \\
&\leq \frac{\alpha}{4}\sup_{\lambda\in\Omega}\left|\langle (|A| + i\,|A^*|)\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^2 + \frac{\alpha}{4}\sup_{\lambda\in\Omega}\left|\langle |A|\,|A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\right| \\
&+ \sup_{\lambda\in\Omega}\left\langle\left(\left(1-\frac{7\alpha}{8}\right)|A|^2 + \frac{\alpha}{8}\,|A^*|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&= \frac{\alpha}{4}\mathbf{ber}^2\left(|A| + i\,|A^*|\right) + \frac{\alpha}{4}\mathbf{ber}\left(|A|\,|A^*|\right) + \mathbf{ber}\left(\left(1-\frac{7\alpha}{8}\right)|A|^2 + \frac{\alpha}{8}\,|A^*|^2\right).
\end{aligned}$$

This completes the proof. □

From Theorem 5.6 together with the first inequality in (5.1), we get the following new upper bound for the Berezin number.

Corollary 5.4. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}
&\mathbf{ber}^2(A) \\
&\leq \inf_{\alpha\in[0,1]}\left\{\frac{\alpha}{4}\mathbf{ber}^2\left(|A| + i\,|A^*|\right) + \frac{\alpha}{4}\mathbf{ber}\left(|A|\,|T^*|\right) + \mathbf{ber}\left(\left(1-\frac{7\alpha}{8}\right)|A|^2 + \frac{\alpha}{8}\,|A^*|^2\right)\right\}
\end{aligned}$$

$$\leq \frac{1}{4}\mathbf{ber}^2\left(|A| + i\,|A^*|\right) + \frac{1}{4}\mathbf{ber}\left(|A|\,|A^*|\right) + \frac{1}{8}\mathbf{ber}\left(|A|^2 + |A^*|^2\right).$$

Remark 5.7. *Since* $\mathbf{ber}^2\left(|A| + i\,|A^*|\right) \leq \mathbf{ber}\left(|A|^2 + |A^*|^2\right)$ *and* $\mathbf{ber}\left(|A|\,|A^*|\right) \leq \frac{1}{2}\mathbf{ber}(|A|^2 + |A^*|^2)$*, Corollary 5.4 improves the bound* $\mathbf{ber}^2\left(A\right) \leq \frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right)$*, which was given in [86, Cor. 3.5(i) for* $r = 2$*].*

Now, in the following theorem we obtain an upper bound for the α-Berezin norm.

Theorem 5.8. *Let* $A \in \mathbb{B}\left(\mathcal{H}\right)$*. Then*

$$\|A\|^2_{\alpha-\mathrm{ber}} \leq \mathbf{ber}\left(\left(1 - \frac{\alpha}{2}\right)|A|^2 + \frac{\alpha}{2}|A^*|^2\right).$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then

$$\begin{aligned}
&\alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + (1-\alpha)\|A\hat{k}_\lambda\|^2\\
&= \alpha\left(\langle\Re\left(A\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2 + \langle\Im\left(A\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle^2\right) + (1-\alpha)\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
&\leq \alpha\left(\|\Re\left(A\right)\hat{k}_\lambda\|^2 + \|\Im\left(A\right)\hat{k}_\lambda\|^2\right) + (1-\alpha)\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
&= \alpha\left(\langle\Re^2(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle\Im^2\left(A\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right) + (1-\alpha)\langle A^*T\hat{k}_\lambda, \hat{k}_\lambda\rangle\\
&= \left\langle\left(\alpha\left(\Re^2\left(A\right) + \Im^2\left(A\right)\right) + (1-\alpha)A^*A\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\\
&= \left\langle\left(\frac{\alpha}{2}\left(|A|^2 + |A^*|^2\right) + (1-\alpha)|A|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\\
&\leq \mathbf{ber}\left(\frac{\alpha}{2}\left(|A|^2 + |A^*|^2\right) + (1-\alpha)|A|^2\right).
\end{aligned}$$

Taking the supremum over $\lambda \in \Omega$, we get the desired inequality. □

The following corollary follows easily from Theorem 5.8 together with the first inequality of (5.1).

Corollary 5.5. *Let* $A \in \mathbb{B}\left(\mathcal{H}\right)$*. Then*

$$\mathbf{ber}^2\left(A\right) \leq \inf_{\alpha\in[0,1]}\mathbf{ber}\left(\left(1 - \frac{\alpha}{2}\right)|A|^2 + \frac{\alpha}{2}|A^*|^2\right) \leq \frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right).$$

Remark 5.9. *If we consider the reproducing kernel Hilbert space* $\mathbb{C}^3$ *and* $A = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}$*, then simple calculation shows that* $\inf_{\alpha\in[0,1]}\mathbf{ber}\left(\left(1 - \frac{\alpha}{2}\right)|A|^2 + \frac{\alpha}{2}|A^*|^2\right) = \frac{16}{7}$ *and* $\frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right) =$

2.5. *Therefore for this example*

$$\inf_{\alpha\in[0,1]} \mathbf{ber}\left(\left(1-\frac{\alpha}{2}\right)|A|^2+\frac{\alpha}{2}|A^*|^2\right) < \frac{1}{2}\mathbf{ber}\left(|A|^2+|A^*|^2\right).$$

Now, in the following theorem we obtain another upper bound for the α-Berezin norm and Berezin number.

Theorem 5.10. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}
(i)\quad & \|A\|^2_{\alpha-\text{ber}} \le \frac{\alpha}{2}\mathbf{ber}\left(A^2\right)+\mathbf{ber}\left(\frac{\alpha}{4}|A^*|^2+\left(1-\frac{3\alpha}{4}\right)|A|^2\right),\\
(ii)\quad & \mathbf{ber}^2(A) \le \inf_{\alpha\in[0,1]}\left\{\frac{\alpha}{2}\mathbf{ber}\left(A^2\right)+\mathbf{ber}\left(\frac{\alpha}{4}|A^*|^2+\left(1-\frac{3\alpha}{4}\right)|A|^2\right)\right\}\\
& \qquad\le \frac{1}{2}\mathbf{ber}\left(A^2\right)+\frac{1}{4}\mathbf{ber}\left(|A|^2+|A^*|^2\right).
\end{aligned}$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. By using Lemma 5.4, we get

$$\begin{aligned}
|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2 &\le \frac{1}{2}\left(\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|+|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2\right)\\
&= \frac{1}{2}|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2+\frac{1}{2}\langle A^*A\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}}\langle AA^*\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{1}{2}}\\
&\le \frac{1}{2}|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2+\frac{1}{4}\left(\langle A^*A\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle AA^*\hat{k}_\lambda,\hat{k}_\lambda\rangle\right)\\
&= \frac{1}{2}|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2+\frac{1}{4}\langle\left(A^*A+AA^*\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\alpha|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle|^2+(1-\alpha)\|A\hat{k}_\lambda\|^2\\
&\le \frac{\alpha}{2}|\langle A^2\hat{k}_\lambda,\hat{k}_\lambda\rangle|+\left\langle\left(\frac{\alpha}{4}\left(A^*A+AA^*\right)+(1-\alpha)T^*T\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\\
&\le \frac{\alpha}{2}\mathbf{ber}\left(A^2\right)+\mathbf{ber}\left(\frac{\alpha}{4}\left(A^*A+AA^*\right)+(1-\alpha)A^*A\right).
\end{aligned}$$

Taking the supremum over $\lambda\in\Omega$ in the above inequality, we get

$$\|A\|^2_{\alpha-\text{ber}} \le \frac{\alpha}{2}\mathbf{ber}\left(A^2\right)+\mathbf{ber}\left(\frac{\alpha}{4}\left(A^*A+AA^*\right)+(1-\alpha)A^*A\right).$$

Hence, we get the desired inequality (i). Again, using the first inequality of (5.1) and taking the infimum over $\alpha\in[0,1]$, the first inequality of (ii) follows. The second inequality of (ii) follows by taking $\alpha = 1$. □

Remark 5.11. *If $A^2 = 0$, then it follows from Theorem 5.10(ii) that*

$$\mathbf{ber}^2(A) \leq \frac{1}{4}\mathbf{ber}\left(|A|^2 + |A^*|^2\right).$$

Therefore, we remark that for $A^2 = 0$, Theorem 5.10(ii) gives better bound than the existing bound [86, Cor. 3.5(i)], namely,

$$\mathbf{ber}^2(A) \leq \frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right).$$

Next bounds for the α-Berezin norm and Berezin number of a bounded linear operator A involve the real and imaginary parts of A.

Theorem 5.12. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*
(i) $\|A\|^2_{\alpha-\text{ber}} \leq \mathbf{ber}\left(\alpha\left(|\Re(A)| + |\Im(A)|\right)^2 + (1-\alpha)|A|^2\right)$,
(ii) $\mathbf{ber}^2(A) \leq \inf_{\alpha\in[0,1]} \mathbf{ber}\left(\alpha\left(|\Re(A)| + |\Im(A)|\right)^2 + (1-\alpha)|A|^2\right)$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}
&\alpha|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + (1-\alpha)\|A\hat{k}_\lambda\|^2 \\
&= \alpha\left|\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle + i\langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^2 + (1-\alpha)\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leq \alpha\left(|\langle \Re(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle| + |\langle \Im(A)\hat{k}_\lambda, \hat{k}_\lambda\rangle|\right)^2 + (1-\alpha)\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leq \alpha\left(\langle |\Re(A)|\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |\Im(A)|\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)^2 + (1-\alpha)\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\rangle \\
&\leq \left\langle \left(\alpha\left(|\Re(A)| + |\Im(A)|\right)^2 + (1-\alpha)A^*A\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\leq \mathbf{ber}\left(\alpha\left(|\Re(A)| + |\Im(A)|\right)^2 + (1-\alpha)A^*A\right).
\end{aligned}$$

Taking the supremum over $\lambda \in \Omega$, we get the desired inequality of (i). By using the first inequality of (5.1) and taking the infimum over $\alpha \in [0,1]$, we get the desired result (ii). □

Next inequality reads as follows:

Theorem 5.13. *Let $A, B \in \mathbb{B}(\mathcal{H})$, and $0 \leq p \leq 1$, $0 \leq q \leq 1$. Then for $\lambda \in \Omega$, we have*

$$\begin{aligned}
&|\widetilde{A}(\lambda)\widetilde{B}(\lambda)| \\
&\leq \frac{1}{4}\mathbf{ber}\left(p\,|A|^2 + (1-p)\,|A^*|^2 + q\,|B|^2 + (1-q)\,|B^*|^2\right) + \frac{1}{8}\mathbf{ber}\left(|A|^2 + |B^*|^2\right) + \frac{1}{4}\mathbf{ber}(BA).
\end{aligned}$$

Proof. Let $\lambda \in \Omega$. Then, we get

$$|\widetilde{A}(\lambda)\widetilde{B}(\lambda)|$$

$$
\begin{aligned}
&\le \frac{1}{4}\left(|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle| + |\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle|\right)^2 \\
&= \frac{1}{4}\left(|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + |\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle|^2 + 2|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle||\langle B\hat{k}_\lambda, \hat{k}_\lambda\rangle|\right) \\
&\le \frac{1}{4}\left(\langle |A|^{2p}\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |A^*|^{2(1-p)}\,\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle |B|^{2q}\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle |B^*|^{2(1-q)}\,\hat{k}_\lambda, \hat{k}_\lambda\rangle \right.\\
&\left. + 2|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle||\langle \hat{k}_\lambda, B^*\hat{k}_\lambda\rangle|\right) \ \text{(by Lemma 5.1)} \\
&\le \frac{1}{4}\left(\langle |A|^{2}\,\hat{k}_\lambda, \hat{k}_\lambda\rangle^p\langle |A^*|^{2}\,\hat{k}_\lambda, \hat{k}_\lambda\rangle^{(1-p)} + \langle |B|^{2}\,\hat{k}_\lambda, \hat{k}_\lambda\rangle^q\langle |B^*|^{2}\,\hat{k}_\lambda, \hat{k}_\lambda\rangle^{(1-q)} \right.\\
&\left. + \|A\hat{k}_\lambda\|\|B^*\hat{k}_\lambda\| + |\langle A\hat{k}_\lambda, B^*\hat{k}_\lambda\rangle|\right) \ \text{(by Lemma 5.4)} \\
&\le \frac{1}{4}\left\langle \left(p\,|A|^2 + (1-p)\,|A^*|^2 + q\,|B|^2 + (1-q)\,|B^*|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&+ \frac{1}{4}\left(\frac{1}{2}\langle\left(|A|^2 + |B^*|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + |\langle BA\hat{k}_\lambda, \hat{k}_\lambda\rangle|\right) \\
&\le \frac{1}{4}\mathbf{ber}\left(p\,|A|^2 + (1-p)\,|A^*|^2 + q\,|B|^2 + (1-q)\,|B^*|^2\right) + \frac{1}{8}\mathbf{ber}\left(|A|^2 + |B^*|^2\right) \\
&+ \frac{1}{4}\mathbf{ber}(BA),
\end{aligned}
$$

as desired. □

Taking $B = A$ in Theorem 5.13, we obtain the following corollary.

Corollary 5.6. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$
\begin{aligned}
\mathbf{ber}^2(A) &\le \frac{1}{4}\inf_{\alpha\in[0,2]}\mathbf{ber}\left(\alpha\,|A|^2 + (2-\alpha)\,|A^*|^2\right) + \frac{1}{8}\mathbf{ber}\left(|A|^2 + |A^*|^2\right) + \frac{1}{4}\mathbf{ber}(A^2) \\
&\le \frac{3}{8}\mathbf{ber}\left(|A|^2 + |A^*|^2\right) + \frac{1}{4}\mathbf{ber}(A^2).
\end{aligned}
$$

Remark 5.14. *If $T^2 = 0$, then it follows from Corollary 5.6 that*

$$
\mathbf{ber}^2(A) \le \frac{3}{8}\mathbf{ber}\left(|A|^2 + |A^*|^2\right).
$$

Thus, we would like to remark that for the operator T with $T^2 = 0$, Corollary 5.6 gives stronger bound than the earlier bound [86, Cor. 3.5(i)], namely,

$$
\mathbf{ber}^2(A) \le \frac{1}{2}\mathbf{ber}\left(|A|^2 + |A^*|^2\right).
$$

To prove our next theorem, we need the following lemmas.

Lemma 5.5. *[91] For each $i = 1, 2, \ldots, n$, let a_i be a positive real number. Then*

$$
\left(\sum_{i=1}^{n} a_i\right)^{\nu} \le n^{\nu-1}\sum_{i=1}^{n} a_i^{\nu}
$$

for all $\nu \geq 1$.

Lemma 5.6. *[66] Let* $A, B \in \mathbb{B}(\mathcal{H})$ *with* $|A|B = B^*|A|$. *Let* $f, g : [0, \infty] \to [0, \infty]$ *are continuous functions satisfying* $f(t)\, g(t) = t$ *for all* $t \geq 0$. *Then*

$$|\langle ABx, y\rangle|^2 \leq r(B)\|f^2(|A|)\, x\|\|g^2(|A^*|)\, y\|$$

for all $x, y \in \mathcal{H}$, *and* $r(B)$ *is the spectral radius of* B.

Theorem 5.15. *Let* $A_i, B_i \in \mathbb{B}(\mathcal{H})$ *be such that* $|A_i|\, B_i = B_i^*\, |A_i|$ *for* $i = 1, 2, \cdots, n$. *Then*

$$\mathrm{ber}^{\nu}\left(\sum_{i=1}^{n} A_i B_i\right) \leq \frac{n^{\nu-1}}{\sqrt{2}} \mathbf{ber}\left(\sum_{i=1}^{n} r^{\nu}(B_i)\left(f^{2\nu}(|A_i|) + i g^{2\nu}(|A_i^*|)\right)\right)$$

for all $\nu \geq 1$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then, we get

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n} A_i B_i\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^{\nu} \\
&= \left|\sum_{i=1}^{n}\langle A_i B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^{\nu} \\
&\leq \left(\sum_{i=1}^{n}\left|\langle A_i B_i \hat{k}_\lambda, \hat{k}_\lambda\rangle\right|\right)^{\nu} \\
&\leq \left(\sum_{i=1}^{n} r(B_i)\, \|f(|A_i|)\, \hat{k}_\lambda\|\|g(|A_i^*|)\, \hat{k}_\lambda\|\right)^{\nu} \quad \text{(by Lemma 5.6)} \\
&= \left(\sum_{i=1}^{n} r(B_i)\, \langle f^2(|A_i|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} \langle g^2(|A_i^*|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}}\right)^{\nu} \\
&\leq \left(\sum_{i=1}^{n} r(B_i)\, \frac{\langle f^2(|A_i|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^2(|A_i^*|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^{\nu} \\
&\leq n^{\nu-1}\sum_{i=1}^{n} r^{\nu}(B_i)\left(\frac{\langle f^2(|A_i|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^2(|A_i^*|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^{\nu} \quad \text{(by Lemma 5.5)} \\
&\leq \frac{n^{\nu-1}}{2}\sum_{i=1}^{n} r^{\nu}(B_i)\left(\langle f^2(|A_i|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\nu} + \langle g^2(|A_i^*|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle^{p}\right) \\
&\qquad\qquad \left(\text{by convexity of } f(t) = t^{\nu} \text{ for } \nu \geq 1\right) \\
&\leq \frac{n^{\nu-1}}{2}\sum_{i=1}^{n} r^{\nu}(B_i)\left(\langle f^{2\nu}(|A_i|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^{2\nu}(|A_i^*|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle\right) \quad \text{(by Lemma 5.3)} \\
&\leq \frac{n^{\nu-1}}{\sqrt{2}}\left|\sum_{i=1}^{n} r^{\nu}(B_i)\left(\langle f^{2\nu}(|A_i|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle + i\langle g^{2\nu}(|A_i^*|)\, \hat{k}_\lambda, \hat{k}_\lambda\rangle\right)\right|
\end{aligned}$$

$$(\text{as } |a+b| \leq \sqrt{2}|a+ib| \text{ for all } a,b \in \nu)$$

$$\leq \frac{n^{\nu-1}}{\sqrt{2}} \left| \sum_{i=1}^{n} r^{\nu}(B_i) \left\langle \left(f^{2\nu}(|A_i|) + ig^{2\nu}(|A_i^*|)\right) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|$$

$$\leq \frac{n^{\nu-1}}{\sqrt{2}} \mathbf{ber} \left(\sum_{i=1}^{n} r^{\nu}(B_i) \left(f^{2\nu}(|A_i|) + ig^{2\nu}(|A_i^*|)\right) \right).$$

Now, taking the supremum over all $\lambda \in \Omega$, we obtain the required result. □

Note that when $\nu = 1$, the inequality in Theorem 5.15 does not depend on the quantity n of summands. Considering $\nu = n = 1$, $A_1 = A$, $B_1 = B$, and $f(t) = g(t) = \sqrt{t}$ in Theorem 5.15, we obtain the following bound for product of two operators.

Corollary 5.7. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be such that $|A|\, B = B^*\, |A|$. Then, we have*

(i) $\mathbf{ber}(AB) \leq \frac{1}{\sqrt{2}} r(B)\, \mathbf{ber}(|A| + i\,|A^*|)$.

(ii) $\mathbf{ber}(A) \leq \frac{1}{\sqrt{2}} \mathbf{ber}(|A| + i\,|A^*|)$.

Note that Corollary 5.7(ii) is also obtained in [56] and [23, Cor. 2.21(i)]. In particular, if $B_i = I$ for each $i = 1, 2, \ldots, n$, Theorem 5.15 leads to the following inequality for the sum of operators.

Corollary 5.8. *Let $A_i \in \mathbb{B}(\mathcal{H})$ for $i = 1, 2, \cdots, n$, and let f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ for all $t \geq 0$. Then*

$$\mathbf{ber}^{\nu} \left(\sum_{i=1}^{n} A_i \right) \leq \frac{n^{\nu-1}}{\sqrt{2}} \mathbf{ber} \left(\sum_{i=1}^{n} \left(f^{2\nu}(|A_i|) + ig^{2\nu}(|A_i^*|)\right) \right),$$

for all $\nu \geq 1$.

It should be noted that the bound of Corollary 5.8 is same as the one obtained in [23, Cor. 2.19(ii)]. Finally, taking $n = 1$ and $f(t) = g(t) = \sqrt{t}$ in Corollary 5.8, we obtain the following upper bound for the Berezin number.

Corollary 5.9. *Let $A \in \mathbb{B}(\mathcal{H})$. Then for all $\nu \geq 1$,*

$$\mathbf{ber}^{\nu}(A) \leq \frac{1}{\sqrt{2}} \mathbf{ber}\left(|A|^{\nu} + i\,|A^*|^{\nu}\right).$$

CHAPTER 6

BEREZIN NUMBER AND BEREZIN NORM INEQUALITIES OF OPERATOR MATRICES

6.1 Introduction

In this chapter, we present new upper bounds for the Berezin number and Berezin norm of $n \times n$ operator matrices. For the special case of 2×2 operator matrices, the bounds obtained here refine the existing results. Furthermore, we develop several upper bounds for single and product operators. We also provide some computational examples to estimate upper bounds for the Berezin number and the Berezin norm of some concrete operators acting on the Hardy-Hilbert space.

Let $\mathcal{H}$ be a reproducing kernel Hilbert space (RKHS in short) on a non-empty set Ω. Let $\mathbb{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. Let $\{\hat{k}_\lambda : \lambda \in \Omega\}$ be the set of all normalized reproducing kernel of $\mathcal{H}$. For $A \in \mathbb{B}(\mathcal{H})$, the Berezin number and Berezin norm of

Content of this chapter is partially based on the following paper:

- P. Bhunia, **A. Sen**, S. Barik and K. Paul, Berezin number and Berezin norm inequalities for operator matrices, Linear Multilinear Algebra, 72 (2024), no.16, 2749–2768. https://doi.org/10.1080/03081087.2023.2299388

A are defined as $\mathbf{ber}(A) = \sup_{\lambda\in\Omega} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|$ and $\|A\|_{ber} = \sup_{\lambda,\mu\in\Omega} |\langle A\hat{k}_\lambda, \hat{k}_\mu\rangle|$, respectively. Let $\{\Omega_i : i = 1, 2, \ldots, n\}$ be a collection of non-empty sets and $\mathcal{H}_i = \mathcal{H}(\Omega_i)$ be RKHS on Ω_i for each i. Let us consider the direct sum $\mathcal{H} = \oplus_{i=1}^{n} \mathcal{H}_i$. Then it is easy to observe that $\mathcal{H}$ is an RKHS on the nonempty set $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$. Every operator $A \in \mathbb{B}(\mathcal{H})$ has an $n \times n$ operator matrix representation $A = [A_{ij}]_{n\times n}$, where $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$. Here $\mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}_j$ to $\mathcal{H}_i$. In recent years many mathematicians have developed bounds involving the numerical radius and operator norm of operator matrices on complex Hilbert spaces, see [1, 11, 18, 55]. Motivated by this we study the Berezin number and Berezin norm inequalities for operator matrices on reproducing kernel Hilbert spaces.

6.2 Berezin number and Berezin norm inequalities for $n \times n$ operator matrices

We begin this chapter with the following lemmas, which will be used often to prove the desired results.

Lemma 6.1. *[84, p. 20] Let $A \in \mathbb{B}(\mathcal{H})$ be positive and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle \quad \text{for all} \quad r \geq 1.$$

Lemma 6.2. *[49, pp. 75-76] Let $A \in \mathbb{B}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|x, x\rangle\langle |A^*|y, y\rangle.$$

Lemma 6.3. *[66, Th. 5] Let $A \in \mathbb{B}(\mathcal{H})$. Let f and g be continuous functions on $[0, \infty)$, which satisfy the relation $f(\lambda)g(\lambda) = \lambda$ for all $\lambda \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|$$

for all $x, y \in \mathcal{H}$.

Lemma 6.4. *[54, p. 44] If $A = [a_{ij}]$ is an $n \times n$ complex matrix such that $a_{ij} \geq 0$ for all $i, j = 1, 2, ..., n$, then $w(A) = w\left(\frac{A+A^*}{2}\right) = r\left(\frac{A+A^*}{2}\right)$, where $r(\cdot)$ denotes the spectral radius.*

Lemma 6.5. *[62, Cor. 2.5] Let $x_1, x_2, e \in \mathcal{H}$ with $\|e\| = 1$ and $\alpha \in \mathbb{C}\backslash\{0\}$. Then*

$$|\langle x_1, e\rangle\langle e, x_2\rangle| \leq \frac{1}{|\alpha|}\big(\max\{1, |\alpha - 1|\}\|x_1\|\|x_2\| + |\langle x_1, x_2\rangle|\big).$$

In particular for $\alpha = 2$, *the above inequality becomes Buzano's inequality* [31]

$$|\langle x_1, e\rangle\langle e, x_2\rangle| \leq \frac{1}{2}(\|x_1\|\|x_2\| + |\langle x_1, x_2\rangle|).$$

Lemma 6.6. *[10, p. 1001] Let* $A \in \mathbb{B}(\mathcal{H}_1), B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1), C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $D \in \mathbb{B}(\mathcal{H}_2)$. *Then the following inequalities hold:*

$$(i) \qquad \boldsymbol{ber}\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \leq \max\Big\{\boldsymbol{ber}(A), \boldsymbol{ber}(D)\Big\},$$

$$(ii) \qquad \boldsymbol{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \frac{1}{2}\big(\|B\| + \|C\|\big).$$

In particular, if $\mathcal{H}_1 = \mathcal{H}_2$, *then* $\boldsymbol{ber}\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) \leq \|B\|.$

Now, we are in a position to prove our first result which provides an upper bound for the Berezin number of $n \times n$ operator matrices.

Theorem 6.1. *Let* $A = [A_{ij}]$ *be an* $n \times n$ *operator matrix, where* $A_{ij} \in \mathbb{B}(\mathcal{H})$ *for all* $i, j = 1, 2, \ldots, n$. *If* $f, g : [0, \infty) \to [0, \infty)$ *are continuous functions satisfying* $f(\lambda)g(\lambda) = \lambda$ *for all* $\lambda \in [0, \infty)$, *then*

$$\boldsymbol{ber}\left(\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}\right) \leq w\left(\begin{bmatrix} \boldsymbol{ber}(A_{11}) & a_{12} & \cdots & a_{1n} \\ 0 & \boldsymbol{ber}(A_{22}) & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{ber}(A_{nn}) \end{bmatrix}\right),$$

where $a_{ij} = \big\|f^2(|A_{ij}|) + g^2(|A_{ji}^*|)\big\|_{ber}^{\frac{1}{2}} \big\|f^2(|A_{ji}|) + g^2(|A_{ij}^*|)\big\|_{ber}^{\frac{1}{2}}$.

Proof. For $(\lambda_1, ..., \lambda_n) \in \Omega^n = \Omega \times \cdots \times \Omega$ (n-copies), let $\hat{k}_{(\lambda_1,...,\lambda_n)}$ be the corresponding normalized reproducing kernel of $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n-copies). Then

$$\begin{aligned} &|\langle A\hat{k}_{(\lambda_1,...,\lambda_n)}, \hat{k}_{(\lambda_1,...,\lambda_n)}\rangle| \\ =& \left|\sum_{i,j=1}^{n} \langle A_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle\right| \\ \leq& \sum_{i,j=1}^{n} \big|\langle A_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle\big| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |\langle A_{ii}k_{\lambda_i}, k_{\lambda_i}\rangle| + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left|\langle A_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle\right| \\
&= \sum_{i=1}^{n} |\langle A_{ii}k_{\lambda_i}, k_{\lambda_i}\rangle| + \sum_{\substack{i,j=1\\ i<j}}^{n} \Big(\left|\langle A_{ij}k_{\lambda_j}, k_{\lambda_i}\rangle\right| + \left|\langle A_{ji}k_{\lambda_i}, k_{\lambda_j}\rangle\right| \Big) \\
&\leq \sum_{i=1}^{n} |\langle A_{ii}k_{\lambda_i}, k_{\lambda_i}\rangle| \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{n} \Big(\|f(|A_{ij}|)k_{\lambda_j}\| \|g(|A_{ij}^*|)k_{\lambda_i}\| + \|f(|A_{ji}|)k_{\lambda_i}\| \|g(|A_{ji}^*|)k_{\lambda_j}\| \Big) \text{(by Lemma 6.3)} \\
&\leq \sum_{i=1}^{n} |\langle A_{ii}k_{\lambda_i}, k_{\lambda_i}\rangle| \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{n} \Big(\|f(|A_{ij}|)k_{\lambda_j}\|^2 + \|g(|A_{ji}^*|)k_{\lambda_j}\|^2 \Big)^{\frac{1}{2}} \Big(\|f(|A_{ji}|)k_{\lambda_i}\|^2 + \|g(|A_{ij}^*|)k_{\lambda_i}\|^2 \Big)^{\frac{1}{2}} \\
&= \sum_{i=1}^{n} |\langle A_{ii}k_{\lambda_i}, k_{\lambda_i}\rangle| \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{n} \langle (f^2(|A_{ij}|) + g^2(|A_{ji}^*|))k_{\lambda_j}, k_{\lambda_j}\rangle^{\frac{1}{2}} \langle (f^2(|A_{ji}|) + g^2(|A_{ij}^*|))k_{\lambda_i}, k_{\lambda_i}\rangle^{\frac{1}{2}} \\
&\leq \sum_{i=1}^{n} \mathbf{ber}(A_{ii}) \|k_{\lambda_i}\|^2 \\
&\quad + \sum_{\substack{i,j=1\\ i<j}}^{n} \left\|f^2(|A_{ij}|) + g^2(|A_{ji}^*|)\right\|_{ber}^{\frac{1}{2}} \left\|f^2(|A_{ji}|) + g^2(|A_{ij}^*|)\right\|_{ber}^{\frac{1}{2}} \|k_{\lambda_i}\| \|k_{\lambda_j}\| \\
&= \langle \hat{A}y, y\rangle,
\end{aligned}
$$

where $y = \begin{bmatrix} \|k_{\lambda_1}\| \\ \|k_{\lambda_2}\| \\ \vdots \\ \|k_{\lambda_n}\| \end{bmatrix} \in \mathbb{C}^n$ is an unit vector and $\hat{A} = [a_{ij}]$ is an $n \times n$ complex matrix, with

$$
a_{ij} = \begin{cases} \mathbf{ber}(A_{ii}) & \text{when } i = j \\ \left\|f^2(|A_{ij}|) + g^2(|A_{ji}^*|)\right\|_{ber}^{\frac{1}{2}} \left\|f^2(|A_{ji}|) + g^2(|A_{ij}^*|)\right\|_{ber}^{\frac{1}{2}} & \text{when } i < j \\ 0 & \text{otherwise.} \end{cases}
$$

Since $\|y\| = 1$, we have

$$
|\langle A\hat{k}_{(\lambda_1,\ldots,\lambda_n)}, \hat{k}_{(\lambda_1,\ldots,\lambda_n)}\rangle| \leq w(\hat{A}).
$$

Taking the supremum over all $(\lambda_1, ..., \lambda_n) \in \Omega^n$, we get $\mathbf{ber}(A) \leq w(\hat{A})$. □

Considering $n = 2$ and $f(\lambda) = g(\lambda) = \lambda^{1/2}$ in Theorem 6.1, we obtain the following upper bound of the Berezin number for 2×2 operator matrices.

Corollary 6.1. *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$. *Then*

$$\begin{aligned} \boldsymbol{ber}(A) \quad &\leq \quad \frac{1}{2}\Bigg(\boldsymbol{ber}(A_{11}) + \boldsymbol{ber}(A_{22}) \\ &\quad + \sqrt{\big(\boldsymbol{ber}(A_{11}) - \boldsymbol{ber}(A_{22})\big)^2 + \big\||A_{12}| + |A_{21}^*|\big\|_{ber}\big\||A_{21}| + |A_{12}^*|\big\|_{ber}}\Bigg). \end{aligned}$$

Proof. Taking $n = 2$ in Theorem 6.1, we get the inequality

$$\begin{aligned} \mathbf{ber}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \quad &\leq \quad w\left(\begin{bmatrix} \mathbf{ber}(A_{11}) & a_{12} \\ 0 & \mathbf{ber}(A_{22}) \end{bmatrix}\right) \\ &= \quad w\left(\begin{bmatrix} \mathbf{ber}(A_{11}) & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & \mathbf{ber}(A_{22}) \end{bmatrix}\right) \text{(by Lemma 6.4)} \\ &= \quad r\left(\begin{bmatrix} \mathbf{ber}(A_{11}) & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & \mathbf{ber}(A_{22}) \end{bmatrix}\right) \\ &= \quad \frac{1}{2}\Bigg(\mathbf{ber}(A_{11}) + \mathbf{ber}(A_{22}) \\ &\quad + \sqrt{\big(\mathbf{ber}(A_{11}) - \mathbf{ber}(A_{22})\big)^2 + a_{12}^2}\Bigg), \end{aligned}$$

where $a_{12} = \big\||A_{12}| + |A_{21}^*|\big\|_{ber}^{\frac{1}{2}}\big\||A_{21}| + |A_{12}^*|\big\|_{ber}^{\frac{1}{2}}$.

□

In particular, if $A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}$, then

$$\mathbf{ber}\left(\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}\right) \leq \frac{1}{2}\big\||A_{12}| + |A_{21}^*|\big\|_{ber}^{\frac{1}{2}}\,\big\||A_{21}| + |A_{12}^*|\big\|_{ber}^{\frac{1}{2}}.$$

In the following example we compute an upper bound for the Berezin number of a 2×2 operator matrix by applying Corollary 6.1.

Example 6.2. *Suppose* $\mathbb{M} = \begin{bmatrix} M_z & P_{\mathbb{C}} \\ P_z & M_{z^2} \end{bmatrix} \in \mathbb{B}(H^2(\mathbb{D}) \oplus H^2(\mathbb{D}))$, *where* $P_{\mathbb{C}}, P_z, M_z$ *and* M_{z^2} *are respectively defined as* $P_{\mathbb{C}}(f) = \langle f, \phi_0 \rangle, P_z(f) = \langle f, \phi_1 \rangle \phi_1, M_z(f) = \phi_1 \cdot f$ *and* $M_{z^2}(f) = \phi_2 \cdot f$ *(*$f \in H^2(\mathbb{D})$ *and* $\phi_i(z) = z^i$ *for all* $z \in \mathbb{D}$*) for* $i = 0, 1, 2$. *Then a simple computation shows that* $\textbf{\textit{ber}}(M_z) = \textbf{\textit{ber}}(M_{z^2}) = 1$ *and* $\||P_{\mathbb{C}}| + |P_z^*|\|_{ber} = \||P_z| + |P_{\mathbb{C}}^*|\|_{ber} = 1$. *From Corollary 6.1, it follows that*

$$\begin{aligned} \textbf{\textit{ber}}(\mathbb{M}) &\leq \frac{1}{2}\Bigg(\textbf{\textit{ber}}(M_z) + \textbf{\textit{ber}}(M_{z^2}) \\ &\quad + \sqrt{\big(\textbf{\textit{ber}}(M_z) - \textbf{\textit{ber}}(M_{z^2})\big)^2 + \||P_{\mathbb{C}}| + |P_z^*|\|_{ber} \||P_z| + |P_{\mathbb{C}}^*|\|_{ber}} \Bigg) \\ &= 1.5. \end{aligned}$$

Remark 6.3. *(i) In [10, Cor. 2.2], Bakherad obtained the bound, namely,*

$$\begin{aligned} \textbf{\textit{ber}}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) &\leq \frac{1}{2}\Bigg(\textbf{\textit{ber}}(A_{11}) + \textbf{\textit{ber}}(A_{22}) \\ &\quad + \sqrt{\big(\textbf{\textit{ber}}(A_{11}) - \textbf{\textit{ber}}(A_{22})\big)^2 + (\|A_{12}\| + \|A_{21}\|)^2} \Bigg). \end{aligned} \tag{6.1}$$

If we consider $\mathcal{H} = \mathbb{C}^2, A_{11} = A_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $A_{12} = A_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, *then from the bound in Corollary 6.1 we get* $\textbf{\textit{ber}}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) \leq 1.5$, *whereas the inequality (6.1) gives* $\textbf{\textit{ber}}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) \leq 2$. *Therefore for this example, the bound of Corollary 6.1 is better than the existing bound* (6.1).

(ii) The upper bound

$$\begin{aligned} \textbf{\textit{ber}}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) &\leq \frac{1}{2}\Bigg(\textbf{\textit{ber}}(A_{11}) + \textbf{\textit{ber}}(A_{22}) \\ &\quad + \sqrt{\big(\textbf{\textit{ber}}(A_{11}) - \textbf{\textit{ber}}(A_{22})\big)^2 + 4w^2\left(\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} \right)} \Bigg) \end{aligned} \tag{6.2}$$

was obtained in [78, Cor. 3.2]. Consider $A_{11} = A_{22} = 0$, $A_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $A_{21} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Then it is easy to observe that Corollary 6.1 gives $\textbf{ber}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \leq 1$, *whereas* (6.2) *gives* $\textbf{ber}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \leq \sqrt{2}$. *Hence the bound obtained in Corollary 6.1 is better than that given in* (6.2).

Our next theorem yields an upper bound for the Berezin norm of $n \times n$ operator matrices.

Theorem 6.4. *Let $A = [A_{ij}]$ be an $n \times n$ operator matrix with $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$, $1 \leq i, j \leq n$. Then*

$$\|A\|_{ber} \leq \|[\|A_{ij}\|_{ber}]\|, 1 \leq i, j \leq n.$$

Proof. Let $\mathcal{H} = \oplus_{i=1}^{n} \mathcal{H}_i$. For $(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n) \in \Omega_1 \times \cdots \times \Omega_n$, let $\hat{k}_{(\lambda_1,\ldots,\lambda_n)} = \begin{bmatrix} k_{\lambda_1} \\ \vdots \\ k_{\lambda_n} \end{bmatrix}$ and $\hat{k}_{(\mu_1,\ldots,\mu_n)} = \begin{bmatrix} k_{\mu_1} \\ \vdots \\ k_{\mu_n} \end{bmatrix}$ be the corresponding normalized reproducing kernels of $\mathcal{H}$. Then

$$\begin{aligned} |\langle A\hat{k}_{(\lambda_1,\ldots,\lambda_n)}, \hat{k}_{(\mu_1,\ldots,\mu_n)}\rangle| &= \left| \sum_{i,j=1}^{n} \langle A_{ij} k_{\lambda_j}, k_{\mu_i}\rangle \right| \\ &\leq \sum_{i,j=1}^{n} \left|\langle A_{ij} k_{\lambda_j}, k_{\mu_i}\rangle\right| \\ &\leq \sum_{i,j=1}^{n} \|A_{ij}\|_{ber} \|k_{\lambda_j}\| \|k_{\mu_i}\| = \langle [\|A_{ij}\|_{ber}] x, y\rangle, \end{aligned}$$

where $x = \begin{bmatrix} \|k_{\lambda_1}\| \\ \vdots \\ \|k_{\lambda_n}\| \end{bmatrix}$ and $y = \begin{bmatrix} \|k_{\mu_1}\| \\ \vdots \\ \|k_{\mu_n}\| \end{bmatrix}$. Since $\|x\| = \|y\| = 1$, we have

$$|\langle A\hat{k}_{(\lambda_1,\ldots,\lambda_n)}, \hat{k}_{(\mu_1,\ldots,\mu_n)}\rangle| \leq \|[\|A_{ij}\|_{ber}]\|.$$

Therefore, taking the supremum over all $(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n) \in \Omega_1 \times \cdots \times \Omega_n$, we get the desired result. □

Now, we give a computational example for an upper bound of the Berezin norm for a 2×2 operator matrix by applying Theorem 6.4.

Example 6.5. *Let* $\begin{bmatrix} P_{\mathbb{C}} & P_z \\ P_{z^2} & P_{z^3} \end{bmatrix} \in \mathbb{B}(H^2(\mathbb{D}) \oplus H^2(\mathbb{D}))$, *where* $P_{\mathbb{C}}(f) = \langle f, \phi_0 \rangle$ *and* $P_{z^i}(f) = \langle f, \phi_i \rangle \phi_i$ $(f \in H^2(\mathbb{D})$ *and* $\phi_i(z) = z^i$ *for all* $z \in \mathbb{D})$ *for* $i = 0, 1, 2, 3$. *A short computation shows that* $\|P_{\mathbb{C}}\|_{ber} = 1$, $\|P_z\|_{ber} = 1/4$, $\|P_{z^2}\|_{ber} = 4/27$ *and* $\|P_{z^3}\|_{ber} = 27/256$. *From Theorem 6.4, it follows that* $\left\| \begin{bmatrix} P_{\mathbb{C}} & P_z \\ P_{z^2} & P_{z^3} \end{bmatrix} \right\|_{ber} \leq \left\| \begin{bmatrix} 1 & 1/4 \\ 4/27 & 27/256 \end{bmatrix} \right\| \approx 1.045.$

The following inequalities concerning 2×2 operator matrices follow immediately from Theorem 6.4.

Corollary 6.2. *Let* $A \in \mathbb{B}(\mathcal{H}_1), B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1), C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $D \in \mathbb{B}(\mathcal{H}_2)$. *Then*

$$(i) \quad \left\| \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right\|_{ber} \leq \max\left\{ \|A\|_{ber}, \|D\|_{ber} \right\},$$

$$(ii) \quad \left\| \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\|_{ber} \leq \max\left\{ \|B\|_{ber}, \|C\|_{ber} \right\}.$$

Example 6.6. *Let* $\mathbb{P} = \begin{bmatrix} P_{\mathbb{C}} & 0 \\ 0 & P_z \end{bmatrix} \in \mathbb{B}(H^2(\mathbb{D}) \oplus H^2(\mathbb{D}))$, *where* $P_{\mathbb{C}}$ *and* P_z *on* $H^2(\mathbb{D})$, *respectively defined as* $P_{\mathbb{C}}(f) = \langle f, \phi_0 \rangle$ *and* $P_z(f) = \langle f, \phi_1 \rangle \phi_1$ $(f \in H^2(\mathbb{D})$ *and* $\phi_i(z) = z^i$ *for all* $z \in \mathbb{D})$ *for* $i = 0, 1$. *Then by simple computation it is easy to observe that* $\|\mathbb{P}\|_{ber} = 0.536$, $\|P_{\mathbb{C}}\|_{ber} = 1$ *and* $\|P_z\|_{ber} = 1/4$. *Therefore for this example, we have* $\left\| \begin{bmatrix} P_{\mathbb{C}} & 0 \\ 0 & P_z \end{bmatrix} \right\|_{ber} = 0.536 < 1 = \max\{\|P_{\mathbb{C}}\|_{ber}, \|P_z\|_{ber}\}$.

Remark 6.7. *By using the fact* $\boldsymbol{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \left\| \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\|_{ber}$, *from Corollary 6.2(ii), we have*

$$\boldsymbol{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \max\left\{ \|B\|_{ber}, \|C\|_{ber} \right\}. \tag{6.3}$$

Now, we give an example to show that the bounds in (6.3) and Lemma 6.6(ii) are not comparable, in general. If we consider $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, *then from (6.3), we get* $\boldsymbol{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq 1$, *whereas from Lemma 6.6(ii), we get* $\boldsymbol{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right) \leq \sqrt{2}$. *Again,* *if we consider* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, *then the inequality (6.3) gives* $\boldsymbol{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq 2$, *whereas Lemma 6.6(ii) gives* $\boldsymbol{ber}\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq 1.5$.

6.3 Berezin number inequalities for 2×2 operator matrices

The next result gives an estimate for the Berezin number of 2×2 off diagonal operator matrices.

Theorem 6.8. *Let* $A \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1), B \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $\alpha \in \mathbb{C}\backslash\{0\}$. *Then*

$$\begin{aligned} \boldsymbol{ber}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) &\leq \frac{1}{|\alpha|}\max\Big\{\boldsymbol{ber}(AB), \boldsymbol{ber}(BA)\Big\} \\ &+ \frac{\max\{1, |\alpha-1|\}}{2|\alpha|}\max\Big\{\|AA^* + B^*B\|_{ber}, \|BB^* + A^*A\|_{ber}\Big\}. \end{aligned}$$

Proof. Let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. For $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{k}_{(\lambda_1,\lambda_2)}$ be a normalized reproducing kernel of $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then

$$\begin{aligned} & |\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|^2 \\ = \ & |\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle\langle \hat{k}_{(\lambda_1,\lambda_2)}, T^*\hat{k}_{(\lambda_1,\lambda_2)}\rangle| \\ \leq \ & \frac{1}{|\alpha|}\Big(|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, T^*\hat{k}_{(\lambda_1,\lambda_2)}\rangle| + \max\{1, |\alpha-1|\}\|T\hat{k}_{(\lambda_1,\lambda_2)}\|\|T^*\hat{k}_{(\lambda_1,\lambda_2)}\|\Big) \\ & \hspace{6cm} \big(\text{by Lemma 6.5}\big) \\ \leq \ & \frac{1}{|\alpha|}\left(|\langle T^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle| + \frac{\max\{1, |\alpha-1|\}}{2}\Big(\|T\hat{k}_{(\lambda_1,\lambda_2)}\|^2 + \|T^*\hat{k}_{(\lambda_1,\lambda_2)}\|^2\Big)\right) \\ = \ & \frac{1}{|\alpha|}\left(|\langle T^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle| + \frac{\max\{1, |\alpha-1|\}}{2}\langle (T^*T + TT^*)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle\right) \\ \leq \ & \frac{1}{|\alpha|}\left(\mathbf{ber}\left(\begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix}\right) + \frac{\max\{1, |\alpha-1|\}}{2}\mathbf{ber}\left(\begin{bmatrix} AA^* + B^*B & 0 \\ 0 & A^*A + BB^* \end{bmatrix}\right)\right) \\ \leq \ & \frac{1}{|\alpha|}\max\Big\{\mathbf{ber}(AB), \mathbf{ber}(BA)\Big\} \\ & + \frac{\max\{1, |\alpha-1|\}}{2|\alpha|}\max\Big\{\|AA^* + B^*B\|_{ber}, \|BB^* + A^*A\|_{ber}\Big\} \ \big(\text{by Lemma 6.6}(i)\big). \end{aligned}$$

Taking the supremum over all $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, we get the desired result. □

The following corollary is obvious from Theorem 6.8.

Corollary 6.3. *If $\mathcal{H}_1 = \mathcal{H}_2$, then from Theorem 6.8, we have*

$$\boldsymbol{ber}^2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) \leq \frac{1}{2|\alpha|}\max\{1, |\alpha - 1|\}\|AA^* + A^*A\|_{ber} + \frac{1}{|\alpha|}\boldsymbol{ber}(A^2). \tag{6.4}$$

Considering $\alpha = n > 1$ $(n \in \mathbb{N})$ in (6.4), we get

$$\boldsymbol{ber}^2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) \leq \frac{n-1}{2n}\|AA^* + A^*A\|_{ber} + \frac{1}{n}\boldsymbol{ber}(A^2).$$

Now, taking $n \to \infty$, we have

$$\boldsymbol{ber}^2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) \leq \frac{1}{2}\|AA^* + A^*A\|_{ber} \leq \|A\|^2. \tag{6.5}$$

Therefore, the bound (6.5) refines the existing bound in Lemma 6.6(ii).

In particular, choosing $\alpha = n > 1$ $(n \in \mathbb{N})$ and taking $n \to \infty$ in Theorem 6.8, we get the following corollary.

Corollary 6.4. *Let $A \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $B \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then*

$$\boldsymbol{ber}^2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{1}{2}\max\Big\{\|AA^* + B^*B\|_{ber}, \|BB^* + A^*A\|_{ber}\Big\}.$$

Next, we develop the following upper bound for 2×2 operator matrices.

Theorem 6.9. *Let $A \in \mathbb{B}(\mathcal{H}_1), B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1), C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2), D \in \mathbb{B}(\mathcal{H}_2)$ and $\alpha \in \mathbb{C}\backslash\{0\}$. Then*

$$\begin{aligned}\boldsymbol{ber}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq\ & \max\Big\{\boldsymbol{ber}^2(A), \boldsymbol{ber}^2(D)\Big\} + \max\Big\{\|B\|_{ber}^2, \|C\|_{ber}^2\Big\} \\ & + \frac{\max\{1, |\alpha - 1|\}}{|\alpha|}\max\Big\{\|A^*A + BB^*\|_{ber}, \|CC^* + D^*D\|_{ber}\Big\} \\ & + \frac{2}{|\alpha|}\max\Big\{\|BD\|_{ber}, \|CA\|_{ber}\Big\}.\end{aligned}$$

Proof. Let $T = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$. For $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, let $\hat{k}_{(\lambda_1,\lambda_2)}$ be a normalized reproducing kernel of $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then

$$
\begin{aligned}
&\left| \left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right|^2 \\
= \;& |\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle + \langle S\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|^2 \\
\leq \;& \big(|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle| + |\langle S\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|\big)^2 \\
= \;& |\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|^2 + |\langle S\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle|^2 \\
& + 2|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle||\langle S\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle| \\
\leq \;& \mathbf{ber}^2(T) + \mathbf{ber}^2(S) + 2|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle\langle \hat{k}_{(\lambda_1,\lambda_2)}, S^*\hat{k}_{(\lambda_1,\lambda_2)}\rangle| \\
\leq \;& \mathbf{ber}^2(T) + \mathbf{ber}^2(S) + \frac{2}{|\alpha|}\max\{1, |\alpha-1|\}\|T\hat{k}_{(\lambda_1,\lambda_2)}\|\|S^*\hat{k}_{(\lambda_1,\lambda_2)}\| \\
& + \frac{2}{|\alpha|}|\langle T\hat{k}_{(\lambda_1,\lambda_2)}, S^*\hat{k}_{(\lambda_1,\lambda_2)}\rangle| \qquad \big(\text{by Lemma 6.5}\big) \\
\leq \;& \mathbf{ber}^2(T) + \mathbf{ber}^2(S) + \frac{1}{|\alpha|}\max\{1, |\alpha-1|\}\big(\|T\hat{k}_{(\lambda_1,\lambda_2)}\|^2 + \|S^*\hat{k}_{(\lambda_1,\lambda_2)}\|^2\big) \\
& + \frac{2}{|\alpha|}|\langle ST\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle| \\
= \;& \mathbf{ber}^2(T) + \mathbf{ber}^2(S) + \frac{1}{|\alpha|}\max\{1, |\alpha-1|\}\langle (T^*T + SS^*)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle \\
& + \frac{2}{|\alpha|}|\langle ST\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)}\rangle| \\
\leq \;& \mathbf{ber}^2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) + \mathbf{ber}^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + \frac{\max\{1, |\alpha-1|\}}{|\alpha|} \\
& \mathbf{ber}\left(\begin{bmatrix} A^*A + BB^* & 0 \\ 0 & CC^* + D^*D \end{bmatrix}\right) + \frac{2}{|\alpha|}\mathbf{ber}\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right) \\
\leq \;& \max\left\{\mathbf{ber}^2(A), \mathbf{ber}^2(D)\right\} + \max\left\{\|B\|_{ber}^2, \|C\|_{ber}^2\right\} \\
& + \frac{\max\{1, |\alpha-1|\}}{|\alpha|}\max\left\{\|A^*A + BB^*\|_{ber}, \|CC^* + D^*D\|_{ber}\right\} \\
& + \frac{2}{|\alpha|}\max\left\{\|BD\|_{ber}, \|CA\|_{ber}\right\} \quad \big(\text{by Lemma 6.6}(i) \text{ and Corollary 6.2}(ii)\big).
\end{aligned}
$$

Taking the supremum over all $(\lambda_1, \lambda_2) \in \Omega_1 \times \Omega_2$, we get the desired result. □

For $\alpha = n > 1$ $(n \in \mathbb{N})$ and taking $n \to \infty$ in Theorem 6.9, we obtain the following bound.

Corollary 6.5. *Let $A \in \mathbb{B}(\mathcal{H}_1), B \in \mathbb{B}(\mathcal{H}_2, \mathcal{H}_1), C \in \mathbb{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in \mathbb{B}(\mathcal{H}_2)$. Then*

$$\boldsymbol{ber}^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \max\left\{\boldsymbol{ber}^2(A), \boldsymbol{ber}^2(D)\right\} + \max\left\{\|B\|_{ber}^2, \|C\|_{ber}^2\right\} + \max\left\{\|A^*A + BB^*\|_{ber}, \|CC^* + D^*D\|_{ber}\right\}.$$

Remark 6.10. *If $\mathcal{H}_1 = \mathcal{H}_2$, then from Corollary 6.5, we have*

$$\boldsymbol{ber}\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leq \sqrt{\|B\|_{ber}^2 + \boldsymbol{ber}^2(A) + \|A^*A + BB^*\|_{ber}}. \tag{6.6}$$

In [48, Cor. 3.5], Hajmohamadi et al. obtained the upper bound, namely,

$$\boldsymbol{ber}\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leq \frac{1}{2}\big(\boldsymbol{ber}(|A| + |A^*|) + \boldsymbol{ber}(|B| + |B^*|)\big). \tag{6.7}$$

If we consider $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then (6.6) gives $\boldsymbol{ber}\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leq$ 1.41, whereas (6.7) gives $\boldsymbol{ber}\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) \leq 1.5$. Hence for this example the bound of (6.6) is finer than the existing bound (6.7).

Our next result reads as follows.

Theorem 6.11. *Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned} &\left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\|_{ber}^2 \\ \leq\ & \max\left\{\boldsymbol{ber}(|A| + i|C|), \boldsymbol{ber}(|D| + i|B|)\right\} \max\left\{\boldsymbol{ber}(|A^*| + i|B^*|), \boldsymbol{ber}(|D^*| + i|C^*|)\right\} \\ &+ \frac{1}{2}\max\left\{\||A|^2 + |C|^2\|_{ber}, \||B|^2 + |D|^2\|_{ber}\right\} + \max\left\{\|C^*D\|_{ber}, \|B^*A\|_{ber}\right\}. \end{aligned}$$

Proof. Let $M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, $N = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, $P = \begin{bmatrix} |A| & 0 \\ 0 & |D| \end{bmatrix}$, $Q = \begin{bmatrix} |A^*| & 0 \\ 0 & |D^*| \end{bmatrix}$, $R = \begin{bmatrix} |C| & 0 \\ 0 & |B| \end{bmatrix}$, $S = \begin{bmatrix} |B^*| & 0 \\ 0 & |C^*| \end{bmatrix}$ and $T = \begin{bmatrix} 0 & C^*D \\ B^*A & 0 \end{bmatrix}$. For $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \Omega^2$, let $\hat{k}_{(\lambda_1,\lambda_2)}$ and $\hat{k}_{(\mu_1,\mu_2)}$ be two normalized reproducing kernels of $\mathcal{H} \oplus \mathcal{H}$. Therefore $P = |M|, Q = |M^*|, R =$

$|N|, S = |N^*|$ and $P^2 + R^2 = \begin{bmatrix} |A|^2 + |C|^2 & 0 \\ 0 & |D|^2 + |B|^2 \end{bmatrix}$, $P + iR = \begin{bmatrix} |A| + i|C| & 0 \\ 0 & |D| + i|B| \end{bmatrix}$.
Then

$$\begin{aligned}
& \left| \left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} \hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right|^2 \\
= \ & \left| \left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle + \left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right|^2 \\
\leq \ & \left(\left| \left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right| + \left| \left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right| \right)^2 \\
= \ & \left| \left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right|^2 + \left| \left\langle N\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right|^2 \\
& +2 \left| \left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right| \left| \left\langle \hat{k}_{(\mu_1,\mu_2)}, N\hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right| \\
\leq \ & \left\langle |M|\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle |M^*|\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \\
& + \left\langle |N|\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle |N^*|\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle + \left\| M\hat{k}_{(\lambda_1,\lambda_2)} \right\| \left\| N\hat{k}_{(\lambda_1,\lambda_2)} \right\| \\
& + \left| \left\langle M\hat{k}_{(\lambda_1,\lambda_2)}, N\hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right| \qquad \text{(by Lemma 6.5 and Lemma 6.2)} \\
= \ & \left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle Q\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle + \left\langle R\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \left\langle S\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \\
& + \left\langle P^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle^{\frac{1}{2}} \left\langle R^2\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle^{\frac{1}{2}} + \left| \left\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right| \\
\leq \ & \left(\left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle^2 + \left\langle R\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle^2 \right)^{\frac{1}{2}} \left(\left\langle Q\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle^2 \right. \\
& \left. + \left\langle S\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle^2 \right)^{\frac{1}{2}} + \frac{1}{2} \left\langle (P^2+R^2)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle + \left| \left\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right| \\
= \ & \left| \left\langle P\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle + i \left\langle R\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right| \left| \left\langle Q\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle + i \left\langle S\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right| \\
& + \frac{1}{2} \left\langle (P^2+R^2)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle + \left| \left\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right| \\
= \ & \left| \left\langle (P+iR)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right| \left| \left\langle (Q+iS)\hat{k}_{(\mu_1,\mu_2)}, \hat{k}_{(\mu_1,\mu_2)} \right\rangle \right| \\
& + \frac{1}{2} \left\langle (P^2+R^2)\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle + \left| \left\langle T\hat{k}_{(\lambda_1,\lambda_2)}, \hat{k}_{(\lambda_1,\lambda_2)} \right\rangle \right|.
\end{aligned}$$

Taking the supremum over all $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \Omega^2$, we get

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\|_{ber}^2 \leq \mathbf{ber}(P+iR)\mathbf{ber}(Q+iS) + \frac{1}{2}\mathbf{ber}(P^2+R^2) + \mathbf{ber}(T). \tag{6.8}$$

By using Lemma 6.6(i) and Corollary 6.2(ii) in (6.8), we get the desired inequality. □

Remark 6.12. *Following [6, Th. 3.6] for the case $\alpha = \frac{1}{2}$, we have*

$$\begin{aligned} \textbf{\textit{ber}}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &\leq \frac{1}{2}\textbf{\textit{ber}}(D) + \textbf{\textit{ber}}(A) \\ &+\frac{1}{2}\sqrt{\frac{1}{4}\textbf{\textit{ber}}^2(D) + \|C\|^2} + \frac{1}{2}\sqrt{\frac{1}{4}\textbf{\textit{ber}}^2(D) + \|B\|^2}. \end{aligned} \tag{6.9}$$

If we take $\mathcal{H} = \mathbb{C}^2, A = C^* = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = D^* = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ then by simple computation from (6.9), we have $\textbf{\textit{ber}}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq 3$, whereas from Theorem 6.11, we get $\textbf{\textit{ber}}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq 2.4$. Hence for this example the bound obtained in Theorem 6.11 is better than the existing bound (6.9).

6.4 Further Berezin number inequalities

We need the following lemmas to prove our next result.

Lemma 6.7. *[91] For $i = 1, 2, \dots, n$, let a_i be a positive real number. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r$$

for all $r \geq 1$.

Lemma 6.8. *[10, Lemma 2.10] Let $A, B \in \mathbb{B}(\mathcal{H})$. If $\textbf{\textit{ber}}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) \leq \textbf{\textit{ber}}\left(\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}\right)$, then $\textbf{\textit{ber}}(A) \leq \textbf{\textit{ber}}(B)$.*

Lemma 6.9. *Let $A_i \in \mathbb{B}(\mathcal{H})$ be positive operators for all $i = 1, 2, \dots, n$. Then*

$$\left\|\sum_{i=1}^{n} A_i\right\|_{ber}^r \leq n^{r-1}\left\|\sum_{i=1}^{n} A_i^r\right\|_{ber}$$

for all $r \geq 1$.

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then using Lemma 6.1 and Lemma 6.7, we obtain

$$\left\langle\sum_{i=1}^{n} A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^r \leq n^{r-1}\sum_{i=1}^{n}\langle A_i\hat{k}_\lambda, \hat{k}_\lambda\rangle^r \leq n^{r-1}\sum_{i=1}^{n}\langle A_i^r\hat{k}_\lambda, \hat{k}_\lambda\rangle \leq n^{r-1}\left\|\sum_{i=1}^{n} A_i^r\right\|_{ber}.$$

So, taking the supremum over all $\lambda \in \Omega$, we get the desired result. $\square$

Now, we are in a position to prove the following result.

Theorem 6.13. *Let $A_i, B_i, X_i \in \mathbb{B}(\mathcal{H})$ for all $i = 1, 2, \ldots, n$. Then*

$$\boldsymbol{ber}^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2^r}\left(\max_{1\leq i\leq n}\|X_i\|^r\right)\left\|\sum_{i=1}^n (A_i^*A_i + B_i^*B_i)^r\right\|_{ber}$$

for all $r \geq 1$.

Proof. Consider the operator matrices

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix},\ B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n & 0 & \cdots & 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix}.$$

Then it is easy to verify that $A^*XB = \begin{bmatrix} \sum_{i=1}^n A_i^*X_iB_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$.

Therefore, we have

$$\begin{aligned}
&\mathbf{ber}\left(\begin{bmatrix} \sum_{i=1}^n A_i^*X_iB_i & 0 \\ 0 & 0 \end{bmatrix}\right) \\
&= \mathbf{ber}(A^*XB) \\
&= \sup\{|\langle A^*XB\hat{k}_{(\lambda_1,\ldots,\lambda_n)}, \hat{k}_{(\lambda_1,\ldots,\lambda_n)}\rangle| : (\lambda_1, \ldots, \lambda_n) \in \Omega^n\} \\
&= \sup\{|\langle XB\hat{k}_{(\lambda_1,\ldots,\lambda_n)}, A\hat{k}_{(\lambda_1,\ldots,\lambda_n)}\rangle| : (\lambda_1, \ldots, \lambda_n) \in \Omega^n\} \\
&\leq \|X\| \sup\{\|B\hat{k}_{(\lambda_1,\ldots,\lambda_n)}\|\|A\hat{k}_{(\lambda_1,\ldots,\lambda_n)}\| : (\lambda_1, \ldots, \lambda_n) \in \Omega^n\} \\
&\leq \frac{\|X\|}{2} \sup\{\|B\hat{k}_{(\lambda_1,\ldots,\lambda_n)}\|^2 + \|A\hat{k}_{(\lambda_1,\ldots,\lambda_n)}\|^2 : (\lambda_1, \ldots, \lambda_n) \in \Omega^n\} \\
&= \frac{\|X\|}{2} \sup\{\langle (A^*A + B^*B)\hat{k}_{(\lambda_1,\ldots,\lambda_n)}, \hat{k}_{(\lambda_1,\ldots,\lambda_n)}\rangle : (\lambda_1, \ldots, \lambda_n) \in \Omega^n\} \\
&= \frac{1}{2}\left(\max_{1\leq i\leq n}\|X_i\|\right)\mathbf{ber}(A^*A + B^*B) \\
&= \frac{1}{2}\left(\max_{1\leq i\leq n}\|X_i\|\right)\mathbf{ber}\left(\begin{bmatrix} \sum_{i=1}^n (A_i^*A_i + B_i^*B_i) & 0 \\ 0 & 0 \end{bmatrix}\right).
\end{aligned}$$

By using Lemma 6.8, we have

$$\mathbf{ber}\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{1}{2}\left(\max_{1\leq i\leq n} \|X_i\|\right) \mathbf{ber}\left(\sum_{i=1}^{n}(A_i^* A_i + B_i^* B_i)\right).$$

Now, by using Lemma 6.9, we obtain

$$\begin{aligned}\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) &\leq \frac{1}{2^r}\left(\max_{1\leq i\leq n} \|X_i\|^r\right)\left\|\sum_{i=1}^{n}(A_i^* A_i + B_i^* B_i)\right\|_{ber}^r \\ &\leq \frac{n^{r-1}}{2^r}\left(\max_{1\leq i\leq n} \|X_i\|^r\right)\left\|\sum_{i=1}^{n}(A_i^* A_i + B_i^* B_i)^r\right\|_{ber},\end{aligned}$$

as desired. □

The following corollaries are immediate from Theorem 6.13.

Corollary 6.6. *Let $A_i, B_i \in \mathbb{B}(\mathcal{H})$ for all $i = 1, 2, \ldots, n$. Then*

$$(i)\quad \boldsymbol{ber}^r\left(\sum_{i=1}^{n} A_i B_i\right) \leq \frac{n^{r-1}}{2^r}\left\|\sum_{i=1}^{n}(A_i A_i^* + B_i^* B_i)^r\right\|_{ber} \quad \textit{for all } r \geq 1,$$

$$(ii)\quad \boldsymbol{ber}^r\left(\sum_{i=1}^{n} A_i\right) \leq \frac{n^{r-1}}{2^r}\min\left\{\left\|\sum_{i=1}^{n}(A_i A_i^* + I)^r\right\|_{ber}, \left\|\sum_{i=1}^{n}(A_i^* A_i + I)^r\right\|_{ber}\right\}$$

for all $r \geq 1$,

$$(iii)\quad \boldsymbol{ber}\left(\sum_{i=1}^{n} A_i B_i\right) \leq \frac{1}{2}\left\|\sum_{i=1}^{n}(A_i A_i^* + B_i^* B_i)\right\|_{ber},$$

$$(iv)\quad \boldsymbol{ber}(A_1 B_1 + A_2 B_2) \leq \frac{1}{2}\|A_1 A_1^* + B_1^* B_1 + A_2 A_2^* + B_2^* B_2\|_{ber}.$$

Corollary 6.7. *Let $A, B, X \in \mathbb{B}(\mathcal{H})$. Then*

$$(i)\quad \boldsymbol{ber}^r(A^* X B) \leq \frac{\|X\|^r}{2^r}\|(A^* A + B^* B)^r\|_{ber} \quad \textit{for all } r \geq 1,$$

$$(ii)\quad \boldsymbol{ber}^r(A^* B) \leq \frac{\|(A^* A + B^* B)^r\|_{ber}}{2^r} \quad \textit{for all } r \geq 1.$$

Corollary 6.8. *Let $A, B, X \in \mathbb{B}(\mathcal{H})$ where A and B are positive. Then for all $r \geq 1$ and $\alpha \in [0, 1]$*

$$(i)\quad \boldsymbol{ber}^r\left(A^\alpha X B^{1-\alpha}\right) \leq \frac{\|X\|^r}{2^r}\|(A^{2\alpha} + B^{2(1-\alpha)})^r\|_{ber},$$

$$(ii)\quad \boldsymbol{ber}^r\left(A^\alpha B^{1-\alpha}\right) \leq \frac{\|(A^{2\alpha} + B^{2(1-\alpha)})^r\|_{ber}}{2^r}.$$

In particular, if $AB = BA$, then

$$\left\|\sqrt{AB}\right\|_{ber}^{r} \leq \left\|\left(\frac{A+B}{2}\right)^{r}\right\|_{ber}.$$

Remark 6.14. *(i) Following [23, Cor. 2.19(ii)] for the case $f(t) = g(t) = t^{1/2}$ and $r = n = 2$, we get*

$$\boldsymbol{ber}^2(A_1 + A_2) \leq \sqrt{2}\boldsymbol{ber}\left(|A_1|^2 + |A_2|^2 + i(|A_1^*|^2 + |A_2^*|^2)\right). \tag{6.10}$$

If we consider $\mathcal{H} = \mathbb{C}^2$, $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then from (6.10), we have $\boldsymbol{ber}^2(A_1 + A_2) \leq \sqrt{10}$. Again Corollary 6.6(ii) gives $\boldsymbol{ber}^2(A_1 + A_2) \leq 2.5$ for the case $n = r = 2$. Therefore for this example, the bound obtained in Corollary 6.6(i) is sharper than the existing bound given by (6.10).

(ii) The following result was obtained in [82, Cor. 2.7], namely,

$$\boldsymbol{ber}^r(A^*B) \leq \frac{1}{2}\sqrt{\boldsymbol{ber}^2(|A|^{2r} + |B|^{2r}) - c^2(|A|^{2r} - |B|^{2r})}, \tag{6.11}$$

where $c(A) = \inf_{\lambda\in\Omega} |\widetilde{A}(\lambda)|$. Considering $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then from Corollary 6.7 (ii) (for $r = 2$) we get $\boldsymbol{ber}^2(A^*B) \leq 1.25$, whereas for $r = 2$ the inequality (6.11) gives $\boldsymbol{ber}^2(A^*B) \leq 1.41$. Hence for this example, Corollary 6.7(ii) is better than the existing result given by (6.11).

(iii) For positive operators A and B, the following upper bound was obtained in [82, Cor. 2.10],

$$\boldsymbol{ber}^4(A^{1/2}B^{1/2}) \leq \frac{1}{4}\left(\|A^2 + B^2\|_{ber} - c^2(A^2 - B^2)\right). \tag{6.12}$$

If we take $A = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $B = \begin{bmatrix} 1/4 & 0 \\ 0 & 0 \end{bmatrix}$, then Corollary 6.8(ii) (for $r = 4, \alpha = \frac{1}{2}$) gives $\boldsymbol{ber}^2(A^{1/2}B^{1/2}) \leq 1/2^8$, whereas the inequality (6.12) gives $\boldsymbol{ber}^2(A^{1/2}B^{1/2}) \leq 1/2^6$. Therefore for this example, we can conclude that the bound of Corollary 6.8(ii) is sharper than the existing bound (6.12).

To prove our next theorem, we use the following lemma, which is a generalization of Lemma 6.5.

Lemma 6.10. *If $x_1, x_2, \ldots, x_n, e \in \mathcal{H}$ with $\|e\| = 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$, then*

$$\left|\prod_{i=1}^{n}\langle x_i, e\rangle\right| \leq \frac{1}{|\alpha|}\left(\left|\langle x_1, x_2\rangle \prod_{i=3}^{n}\langle x_i, e\rangle\right| + \max\{1, |\alpha - 1|\} \prod_{i=1}^{n}\|x_i\|\right).$$

Proof. Substituting x_2 by $\prod_{i=3}^{n}\langle x_i, e\rangle x_2$ and applying $\left|\prod_{i=3}^{n}\langle x_i, e\rangle\right| \leq \prod_{i=3}^{n}\|x_i\|$ in Lemma 6.5, we get the desired result. □

Note that for the case $\alpha = 2$, the inequality in Lemma 6.10 was observed in [21].

Theorem 6.15. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}\boldsymbol{ber}^3(A) \quad \leq \quad & \inf_{\alpha,\beta \in \mathbb{C}\setminus\{0\}}\left\{\frac{1}{|\alpha||\beta|}\boldsymbol{ber}(A^3) + \left(\frac{\max\{1, |\beta - 1|\}}{|\alpha||\beta|}\left\|A^{*2}A^2\right\|_{ber}^{\frac{1}{2}}\right.\right.\\ & \left.\left. + \frac{\max\{1, |\alpha - 1|\}}{2|\alpha|}\left\||A|^2 + |A^*|^2\right\|_{ber}\right)\|AA^*\|_{ber}^{\frac{1}{2}}\right\}.\end{aligned}$$

Proof. Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Then using Lemma 6.10, we get

$$\begin{aligned}|\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^3 &= |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle A^*\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle A^*\hat{k}_\lambda, \hat{k}_\lambda\rangle| \\ &\leq \frac{1}{|\alpha|}\left(|\langle A\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle\langle A^*\hat{k}_\lambda, \hat{k}_\lambda\rangle| + \max\{1, |\alpha - 1|\}\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|^2\right) \\ &\leq \frac{1}{|\alpha|}\left(\frac{1}{|\beta|}\left(|\langle A^2\hat{k}_\lambda, A^*\hat{k}_\lambda\rangle| + \max\{1, |\beta - 1|\}\|A^2\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\right)\right.\\ &\quad \left. + \max\{1, |\alpha - 1|\}\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|^2\right) \\ &= \frac{1}{|\alpha|}\left(\frac{1}{|\beta|}\left(|\langle A^3\hat{k}_\lambda, \hat{k}_\lambda\rangle| + \max\{1, |\beta - 1|\}\|A^2\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|\right)\right.\\ &\quad \left. + \max\{1, |\alpha - 1|\}\|A\hat{k}_\lambda\|\|A^*\hat{k}_\lambda\|^2\right) \\ &\leq \frac{1}{|\alpha||\beta|}|\langle A^3\hat{k}_\lambda, \hat{k}_\lambda\rangle| + \left(\frac{\max\{1, |\beta - 1|\}}{|\alpha||\beta|}\|A^2\hat{k}_\lambda\|\right.\\ &\quad \left. + \frac{\max\{1, |\alpha - 1|\}}{2|\alpha|}\left(\|A\hat{k}_\lambda\|^2 + \|A^*\hat{k}_\lambda\|^2\right)\right)\|A^*\hat{k}_\lambda\| \\ &= \frac{1}{|\alpha||\beta|}|\langle A^3\hat{k}_\lambda, \hat{k}_\lambda\rangle| + \left(\frac{\max\{1, |\beta - 1|\}}{|\alpha||\beta|}\langle A^{*2}A^2\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}}\right.\\ &\quad \left. + \frac{\max\{1, |\alpha - 1|\}}{2|\alpha|}\langle(|A|^2 + |A^*|^2)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)\langle AA^*\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}}\end{aligned}$$

$$\leq \frac{1}{|\alpha||\beta|}\mathbf{ber}(A^3) + \left(\frac{\max\{1,|\beta-1|\}}{|\alpha||\beta|}\left\|A^{*2}A^2\right\|_{ber}^{\frac{1}{2}} + \frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\left\||A|^2+|A^*|^2\right\|_{ber}\right)\|AA^*\|_{ber}^{\frac{1}{2}}.$$

Taking the supremum over all $\lambda \in \Omega$, we obtain

$$\mathbf{ber}^3(A) \leq \frac{1}{|\alpha||\beta|}\mathbf{ber}(A^3) + \left(\frac{\max\{1,|\beta-1|\}}{|\alpha||\beta|}\left\|A^{*2}A^2\right\|_{ber}^{\frac{1}{2}} + \frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\left\||A|^2+|A^*|^2\right\|_{ber}\right)\|AA^*\|_{ber}^{\frac{1}{2}}. \tag{6.13}$$

The desired result follows from (6.13) by taking infimum over all $\alpha, \beta \in \mathbb{C}\setminus\{0\}$. □

Considering $\alpha = \beta = 2$ in Theorem 6.15, we get the following corollary.

Corollary 6.9. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^3(A) \leq \frac{1}{4}\boldsymbol{ber}(A^3) + \frac{1}{4}\left(\left\|A^{*2}A^2\right\|_{ber}^{\frac{1}{2}} + \left\||A|^2+|A^*|^2\right\|_{ber}\right)\|AA^*\|_{ber}^{\frac{1}{2}}.$$

Remark 6.16. *In [86, Cor. 3.5(i)], Taghavi et al. obtained the following bound:*

$$\boldsymbol{ber}^r(A) \leq \frac{1}{2}\boldsymbol{ber}(|A|^r + |A^*|^r) \quad \text{for all } r \geq 1\ . \tag{6.14}$$

If we consider $\mathcal{H} = \mathbb{C}^3$ and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then by simple computation from Corollary 6.9, we get $\boldsymbol{ber}^3(A) \leq 0.75$, whereas for $r = 3$ the inequality (6.14) gives $\boldsymbol{ber}^3(A) \leq 1$. Therefore for this example, the bound obtained in Corollary 6.9 is better than the existing bound (6.14).

Finally, we obtain the following general power inequality of Berezin number.

Theorem 6.17. *Let $A \in \mathbb{B}(\mathcal{H})$. Then*

$$\boldsymbol{ber}^n(A) \leq \frac{1}{2^{n-1}}\boldsymbol{ber}(A^n) + \sum_{i=1}^{n-1}\frac{1}{2^i}\|A^{*i}A^i\|_{ber}^{1/2}\|AA^*\|_{ber}^{\frac{n-i}{2}}$$

for every positive integers $n \geq 2$.

Proof. For every positive integer $n \geq 2$ and for all $x \in \mathcal{H}$ with $\|x\| = 1$, the following inequality

([21, Th. 3.1]) holds

$$|\langle Ax, x\rangle|^n \leq \frac{1}{2^{n-1}}|\langle A^n x, x\rangle| + \sum_{i=1}^{n-1} \frac{1}{2^i}\|A^i x\|\|A^* x\|^{n-i}. \tag{6.15}$$

Let $\hat{k}_\lambda$ be a normalized reproducing kernel of $\mathcal{H}$. Now, putting $x = \hat{k}_\lambda$ in (6.15), we get

$$\begin{aligned} |\langle A\hat{k}_\lambda, \hat{k}_\lambda\rangle|^n &\leq \frac{1}{2^{n-1}}|\langle A^n \hat{k}_\lambda, \hat{k}_\lambda\rangle| + \sum_{i=1}^{n-1} \frac{1}{2^i}\|A^i \hat{k}_\lambda\|\|A^* \hat{k}_\lambda\|^{n-i} \\ &= \frac{1}{2^{n-1}}|\langle A^n \hat{k}_\lambda, \hat{k}_\lambda\rangle| + \sum_{i=1}^{n-1} \frac{1}{2^i}\langle A^{*i}A^i \hat{k}_\lambda, \hat{k}_\lambda\rangle^{1/2}\langle AA^* \hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{n-i}{2}} \\ &\leq \frac{1}{2^{n-1}}\mathbf{ber}(A^n) + \sum_{i=1}^{n-1} \frac{1}{2^i}\|A^{*i}A^i\|_{ber}^{1/2}\|AA^*\|_{ber}^{\frac{n-i}{2}}. \end{aligned}$$

Taking the supremum over all $\lambda \in \Omega$, we get the desired result. □

CHAPTER 7

FINAL REMARKS AND FUTURE PROSPECT

Throughout this dissertation, we develop new Berezin number inequalities of operators on reproducing kernel Hilbert spaces. We generalize and refine the existing inequalities and this work contributes to a deeper understanding of the properties of operators acting on reproducing kernel Hilbert spaces. There is a high potential for further work in this area. Specifically, the refinement and generalization of these inequalities seems still possible. The Berezin number and the Berezin norm can be generalized broader classes of operators and they can be extended to study more complex structures in higher dimensional settings. Its extensions to multivariable operators might provide additional insights into multivariable operator theory. Finally, we conclude with several questions and remarks followed by some examples, which could inspire future research in these directions. Some of these include:

If we consider $A \in \mathbb{B}(H^2(\mathbb{D}))$ defined by $Af(w) = \langle f(w), w^2 \rangle w^2$ $(f \in H^2(\mathbb{D}),\ w \in \mathbb{D})$, we have $\langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle = \langle A\hat{k}_\lambda(w), \hat{k}_\lambda(w) \rangle = |\lambda|^4(1-|\lambda|^2)$. Thus we get $\mathbf{Ber}(A) = \{|\lambda|^4(1-|\lambda|^2) : \lambda \in \mathbb{D}\} = [0, 4/27] \subsetneq [0,1] = W(A)$, and so $\mathbf{ber}(A) = 4/27 < 1 = w(A)$. Now, considering the operator $M_z \in \mathbb{B}(H^2(\mathbb{D}))$ defined by $M_z f(w) = wf(w)$ $(f \in H^2(\mathbb{D}),\ w \in \mathbb{D})$, we have $\mathbf{ber}(M_z) = 1 = w(M_z)$. This example motivates to raise the following question.

Question 7.1. *For a bounded linear operator A on a reproducing kernel Hilbert space $\mathcal{H}$, when $\mathbf{ber}(A) = w(A)$?*

Here we note that if there exits a sequence $\{\lambda_n\}$ in Ω be such that $\langle A\hat{k}_\lambda, \hat{k}_\lambda \rangle = w(A)$ then $\mathbf{ber}(A) = w(A)$.

In [22, Prop. 2.11], we prove that if A is a positive bounded linear operator on a reproducing kernel Hilbert space $\mathcal{H}$, then $\mathbf{ber}(A) = \|A\|_{ber}$. We also provide an example of an self-adjoint operator on $\mathbb{C}^{2n} (n \in \mathbb{N})$ for which $\mathbf{ber}(A) \neq \|A\|_{ber}$. This observation leads to the following two questions:

Question 7.2. *Does the equality $\boldsymbol{ber}(A) = \|A\|_{ber}$ holds for general self-adjoint operators acting on $\mathbb{C}^{2n+1} (n \in \mathbb{N})$?*

Question 7.3. *Does the equality $\boldsymbol{ber}(A) = \|A\|_{ber}$ holds for general self-adjoint operators acting on reproducing kernel Hilbert space which possesses "Ber property"?*

In [34], Crouzeix proved the following inequality: Let $\mathbb{H}$ be a complex Hilbert space. There exits a constant Q such that

$$\|p(A)\| \leq Q \sup \{|p(\lambda)| : \lambda \in W(A)\}$$

holds for all $A \in \mathbb{B}(\mathbb{H})$ and all polynomials $p : \mathbb{C} \to \mathbb{C}$.

Now, we raise the following question for the operators acting on reproducing kernel Hilbert space $\mathcal{H}$.

Question 7.4. *Let $A \in \mathbb{B}(\mathcal{H})$. Does there exists a constant M and N such that the following inequalities*

$$(i) \quad \boldsymbol{ber}(p(A)) \leq M \sup \{|p(\lambda)| : \lambda \in \boldsymbol{Ber}(A)\},$$
$$(ii) \quad \|p(A)\|_{ber} \leq N \sup \{|p(\lambda)| : \lambda \in \boldsymbol{Ber}(A)\},$$

holds for all $A \in \mathbb{B}(\mathcal{H})$ and all polynomials $p : \mathbb{C} \to \mathbb{C}$.

Recall that for $A \in \mathbb{B}(\mathbb{H})$ the spectrum of A is denoted by $\sigma(A)$, and is defined as $\sigma(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \text{ is not invertible}\}$. The following result is a fundamental result in functional analysis that relates the spectrum of a function of an operator with the spectrum of the operator, see [32, Chapter VII, Section 4].

Theorem 7.5. *If $A \in \mathbb{B}(\mathbb{H})$ and f is function analytic on a neighborhood of $\sigma(A)$, then $\sigma(f(A)) = f(\sigma(A))$.*

Now, the question is can we have similar result for Berezin set? The answer is no. If we consider the reproducing kernel Hilbert space $\mathbb{C}^2$, $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $f(z) = z^2$, then $\mathbf{Ber}(f(A)) = \{2, 3\}$ and $f(\mathbf{Ber}(A)) = \{1\}$. As $\mathbb{C}^2$ does not have "Ber property" so the following question is still valid.

Question 7.6. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space that possesses "Ber property" and $A \in \mathbb{B}(\mathcal{H})$. Does the relation $\boldsymbol{Ber}(f(A)) = f(\boldsymbol{Ber}(A))$ holds for all functions f that are analytic on a neighborhood of $\boldsymbol{Ber}(A)$?*

For $A \in \mathbb{B}(\mathbb{H})$, the numerical radius of A satisfies the inequality:

$$w(A^n) \leq w^n(A)$$

for all $n \in \mathbb{N}$. This result is known as power inequality, see [46, Th. 2.1-1]. In [44], Garayev et al. asked the following question

Question 7.7. *For a bounded linear operator on a reproducing kernel Hilbert space $\mathcal{H}$, when does the inequality*

$$\boldsymbol{ber}(A^n) \leq \boldsymbol{ber}^n(A)$$

hold for all $n \in \mathbb{N}$.

Here, we note that the question was partially solved in [40], although it remains unsolved in the general case. The above highlighted questions are promising for further investigation about the Berezin number inequalities of reproducing kernel Hilbert space operators. These interesting questions could deepen the understanding of operators and their applications in various fields. As researchers explore these challenges, they may develop new generalizations, refine existing inequalities, and discover important informations about the operators, thereby advancing both theoretical insights and practical applications in operator theory.

REFERENCES

[1] A. Abu-Omar and F. Kittaneh, Numerical radius inequalities for $n \times n$ operator matrices, Linear Algebra Appl., 468 (2015), 18–26.

[2] S. Abu Sammour, F. Kittaneh and M. Sababheh, A geometric approach to numerical radius inequalities, Linear Algebra Appl., 652 (2022), 1–17.

[3] S. Aici, A. Frakis and F. Kittaneh, Refinements of some numerical radius inequalities for operators, Rend. Circ. Mat. Palermo, 72(2) (2023), no. 8, 3815–3828.

[4] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc., 68 (1950), 337–404.

[5] A. Augustine, M. Garayev and P. Shankar, Composition operators, convexity of their Berezin range and related questions, Complex Anal. Oper. Theory, 17 (2023), no. 8, Paper No. 126, 22 pp.

[6] M. Bakherad, M. Hajmohamadi, R. Lashkaripour and S. Sahoo, Some extensions of Berezin number inequalities on operators, Rocky Mountain J. Math., 51 (2021), no. 6, 1941–1951.

[7] M. Bakherad and U. Yamancı, New estimations for the Berezin number inequality, J. Inequal. Appl., (2020), Paper No. 40, 9 pp.

[8] M. Bakherad and M.T. Garayev, Berezin number inequalities for Hilbert space operators, Concr. Oper., 6 (2019), 33–43.

[9] M. Bakherad, R. Lashkaripour, M Hajmohamadi and U. Yamancı, Complete refinements of the Berezin number inequalities, J. Math. Inequal., 13(4) (2019), 1117–1128.

[10] M. Bakherad, Some Berezin number inequalities for operator matrices, Czechoslovak Math. J., 68 (2018), 997–1009.

[11] W. Bani-Domi and F. Kittaneh, Numerical radius inequalities for operator matrices, Linear Multilinear Algebra, 57 (2009), 421–427.

[12] H. Başaran , M.B. Huban and M. Gürdal, Inequalities related to Berezin norm and Berezin number of operators, Bull. Math. Anal. and App., 14(2) (2022), 1–11.

[13] E.F. Beckenbach and R. Bellman, Inequalities, Springer, (1961).

[14] F.A. Berezin, Quantization, Math. USSR-Izv., 8 (1974), 1109–1163.

[15] F.A. Berezin, Covariant and contravariant symbols for operators, Math. USSR-Izv., 6 (1972), 1117–1151.

[16] R. Bhatia, Notes on functional analysis, Texts Read. Math., 50 Hindustan Book Agency, New Delhi, (2009).

[17] R. Bhatia, Matrix analysis, Grad. Texts in Math., 169, Springer-Verlag, New York, (1997).

[18] P. Bhunia, Sharper bounds for the numerical radius of $n \times n$ operator matrices, Arch. Math. (Basel), 123 (2024), no. 2, 173–183.

[19] P. Bhunia, Improved bounds for the numerical radius via polar decomposition of operators, Linear Algebra Appl., 683 (2024), 31–45.

[20] P. Bhunia, M.T. Garayev, K. Paul and R. Tapdigoglu, Some new applications of Berezin symbols, Complex Anal. Oper. Theory, 17 (2023), no. 6, Paper No. 96, 15 pp.

[21] P. Bhunia, Power numerical radius inequalities from an extension of Buzano's inequality, (2023). https://doi.org/10.48550/arXiv.2305.17657

[22] P. Bhunia, K. Paul and A. Sen, Inequalities involving Berezin norm and Berezin number, Complex Anal. Oper. Theory, 17 (2023), no. 1, Paper No. 7, 15 pp.

[23] P. Bhunia, A. Sen and K. Paul, Development of the Berezin number inequalities, Acta Math. Sin. (Engl. Ser.), 39 (2023), no. 7, 1219–1228.

[24] P. Bhunia, S.S. Dragomir, M.S. Moslehian and K. Paul, Lectures on numerical radius inequalities, Infosys Sci. Found. Ser. Math. Sci., Springer, Cham, (2022).

[25] P. Bhunia and K. Paul, Development of inequalities and characterization of equality conditions for the numerical radius, Linear Algebra Appl., 630 (2021), 306–315.

[26] P. Bhunia and K. Paul, Furtherance of numerical radius inequalities of Hilbert space operators, Arch. Math. (Basel), 117 (2021), no. 5, 537–546.

[27] P. Bhunia and K. Paul, Refinements of norm and numerical radius inequalities, Rocky Mountain J. Math., 51 (2021), no. 6, 1953–1965.

[28] P. Bhunia and K. Paul, Proper improvement of well-known numerical radius inequalities and their applications, Results Math., 76 (2021), no. 4, Paper No. 177, 12 pp.

[29] P. Bhunia, K. Paul and R.K. Nayak, Sharp inequalities for the numerical radius of Hilbert space operators and operator matrices, Math. Inequal. Appl., 24 (2021), no. 1, 167–183.

[30] P. Bhunia and K. Paul, New upper bounds for the numerical radius of Hilbert space operators, Bull. Sci. Math., 167 (2021), Paper No. 102959, 11 pp.

[31] M.L. Buzano, Generalizzatione della disuguaglianza di Cauchy-Schwarz, Rend. Semin. Mat. Univ. Politech. Torino, 31(1971/73) (1974), 405–409.

[32] J.B. Conway, A course in functional analysis, 2nd ed., Springer, New York, (1990).

[33] C.C. Cowen and C. Felder, Convexity of the Berezin range, Linear Algebra Appl., 647 (2022), 47–63.

[34] M. Crouzeix, Numerical range and functional calculus in Hilbert space. J. Funct. Anal., 244 (2), (2007), 668–690.

[35] W. F. Donoghue, On the Numerical Range of a Bounded Operator, Michigan Math. J., 4 (1957), 261–263.

[36] S.S. Dragomir, A note on numerical radius and the Krein-Lin inequality, Mathematica, 60 (83) (2018), no. 2, 149–151.

[37] S.S. Dragomir, Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces, Nova Science Publishers, New York, (2007).

[38] T. Furuta, A simplified proof of Heinz inequality and scrutiny of its equality, Proc. Amer. Math. Soc., 97 (1986), 751–753.

[39] M.T. Garayev and M.W. Alomari, Inequalities for the Berezin number of operators and related questions, Complex Anal. Oper. Theory, 15 (2021), no. 2, Paper no. 30, 30 pp.

[40] M.T. Garayev, H. Guediri, M. Gürdal and G. M. Alsahli, On some problems for operators on the reproducing kernel Hilbert space, Linear Multilinear Algebra, 69 (2021), no. 11, 2059–2077.

[41] M. Garayev, S. Saltan, F. Bouzeffour and B. Aktan, Some inequalities involving Berezin symbols of operator means and related questions, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 114 (2020), no. 2, Paper No. 85, 17 pp.

[42] M.T. Garayev and U. Yamancı, Čebyšev's type inequalities and power inequalities for the Berezin number of operators, Filomat, 33 (2019), no. 8, 2307–2316.

[43] M.T. Garayev, M. Gürdal and S. Saltan, Hardy type inequaltiy for reproducing kernel Hilbert space operators and related problems, Positivity, 21 (2017), 1615–1623.

[44] M.T. Garayev, M. Gürdal and A. Okudan, Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators, Math. Inequal. Appl., 19 (2016), 883–891.

[45] M. Guesba and F. Kittaneh, Berezin number inequalities for sums and products of operators and applications, Lobachevskii J. Math., 45 (2024), no. 3, 1086–1096.

[46] K.E. Gustafson and D.K.M. Rao, Numerical Range. The Field of Values of Linear Operators and Matrices, Springer-Verlag, New York, (1997).

[47] K.E. Gustafson, The Toeplitz-Hausdorff Theorem for Linear Operators, Proc. Amer. Math. Soc., 25 (1970), 203–204.

[48] M. Hajmohamadi, R. Lashkaripour and M. Bakherad, Improvements of Berezin number inequalities, Linear Multilinear Algebra, 68 (2020), no. 6, 1218–1229.

[49] P.R. Halmos, A Hilbert space problems book, Springer Verlag, New York, (1982).

[50] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities, 2nd ed., Cambridge Univ. Press, Cambridge, (1988).

[51] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities, Cambridge Mathematical Library, (1934).

[52] F. Hausdorff, Der Wertvorrat einer Bilinearform, Math. Z., 3 (1919), 314–316.

[53] O. Hirzallah and F. Kittaneh, Matrix Young inequalities for the Hilbert-Schmidt norm, Linear Algebra Appl., 308 (2000), 77–84.

[54] R.A. Horn and C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, (1991).

[55] J.C. Hou and H.K. Du, Norm inequalities of positive operator matrices, Integral Equations Operator Theory, 22 (1995), 281–294.

[56] M.B. Huban, H. Başaran and M. Gürdal, New upper bounds related to the Berezin number inequalities, J. Inequal. Spec. Funct., 12(3) (2021), 1–12.

[57] S. Jana, P. Bhunia and K. Paul, Numerical radius inequalities and estimation of zeros of polynomials, Georgian Math. J., 30 (2023), no. 5, 671–682.

[58] M.T. Karaev, Reproducing kernels and Berezin symbols techniques in various questions of operator theory, Complex Anal. Oper. Theory, 7(4) (2013), 983–1018.

[59] M.T. Karaev and N. Sh. Iskenderov, Berezin number of operators and related questions, Methods Funct. Anal. Topology, 19 (2013), no. 1, 68–72.

[60] M.T. Karaev, Berezin symbol and invertibility of operators on the functional Hilbert spaces, J. Funct. Anal., 238 (2006), 181–192.

[61] T. Kato, Notes on some inequalities for linear operators, Math. Ann., 125 (1952), 208–212.

[62] M. Khosravi, R. Drnovsek and M.S. Moslehian, A commutator approach to Buzano's inequality, Filomat, 26 (2012), no. 4, 827–832.

[63] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrices, J. Math. Anal. Appl., 361 (2010), 262–269.

[64] F. Kittaneh, Numerical radius inequalities for Hilbert spaces operators, Studia Math., 168 (2005), 73–80.

[65] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Studia Math., 158 (2003), no. 1, 11–17.

[66] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publ. Res. Inst. Math. Sci., 24 (1988), no. 2, 283–293.

[67] E. Kreyszig, Introductory functional analysis with applications, John Wiley & Sons, New York-London-Sydney, (1978).

[68] C.K. Li, C-Numerical Ranges and C-Numerical Radii, Linear Multilinear Algebra, 37 (1994), 51–82.

[69] S. Majee, A. Maji and A. Manna, Numerical radius and Berezin number inequality, J. Math. Anal. Appl., 517 (2023), no. 1, Paper No. 126566, 20 pp.

[70] A.R. Martínez-Avendaño and P. Rosenthal, An introduction to operators on the Hardy-Hilbert space, Grad. Texts in Math., 237, Springer, New York, (2007).

[71] R.E. Megginson, An Introduction to Banach Space Theory, Graduate Texts in Mathematics, 183, Springer-Verlag, New York, (1998).

[72] H. Moradi and M. Sababheh, New estimates for the numerical radius, Filomat, 35 (2021), no. 14, 4957–4962.

[73] M.S. Moslehian, M. Khosravi and R. Drnovšek, A commutator approach to Buzano's inequality, Filomat, 26 (2012), no. 4, 827–832.

[74] H. Najafi, Some numerical radius inequality for several operators, Linear Algebra Appl., 588 (2020), 489–496.

[75] M.E. Omidvar and H.R. Moradi, New estimates for the numerical radius of Hilbert space operators, Linear Multilinear Algebra, 69 (2021), no. 5, 946–956.

[76] M.E. Omidvar and H.R. Moradi, Better bounds on the numerical radii of Hilbert space operators, Linear Algebra Appl., 604 (2020), 265–277.

[77] V.I. Paulsen and M. Raghupati, An introduction to the theory of reproducing kernel Hilbert spaces, Cambridge Univ. Press, (2016).

[78] S. Sahoo, N. Das and N.C. Rout, On Berezin number inequalities for operator matrices, Acta Math. Sin. (Engl. Ser.), 37 (2021), no. 6, 873–892.

[79] S, Sahoo, N. Das and D. Mishra, Berezin number and numerical radius inequalities for operators on Hilbert spaces, Adv. Oper. Theory, 5 (2020), no. 3, 714–727.

[80] D. Sain, P. Bhunia, A. Bhanja and K. Paul, On a new norm on $\mathcal{B}(\mathcal{H})$ and its application to numerical radius inequalities, Ann. Funct. Anal., 12 (2021), no. 4, Paper No. 51, 25 pp.

[81] S. Saltan, R. Tapdigoglu and İ. Çalışır, Some new relations between the Berezin number and the Berezin norm of Operators, Rocky Mountain J. Math., 52 (2022), no.5, 1767–1774.

[82] A. Sen, P. Bhunia and K. Paul, Bounds for the Berezin number of reproducing kernel Hilbert space operators, Filomat, 37 (2023), no. 6, 1741–1749.

[83] A. Sen, P. Bhunia and K. Paul, Berezin number inequalities of operators on reproducing kernel Hilbert spaces, Rocky Mountain J. Math., 52 (2022), no. 3, 1039–1046.

[84] B. Simon, Trace ideals and their applications, Cambridge Univ. Press, (1979).

[85] K. Stroethoff, The Berezin transform and operators on spaces of analytic functions, in: Linear Operators, Warsaw, 1994, in: Banach Center Publ., vol.38, Polish Acad. Sci. Inst. Math, Warsaw, (1997), pp.361–380.

[86] A. Taghavi, T.A. Roushan and V. Darvish, Some upper bounds for the Berezin number of Hilbert space operators, Filomat, 33 (2019), no. 14, 4353–4360.

[87] R. Tapdigoglu, N. Altwaijry and M. Garayev, New inequalities via Berezin symbols and related questions, Korean J. Math., 31 (2023), no. 1, 109–120.

[88] R. Tapdigoglu, New Berezin symbol inequalities for operators on the reproducing kernel Hilbert space, Oper. Matrices, 15 (2021), no. 3, 1031–1043.

[89] A.E. Taylor and D.C. Lay, Introduction to Functional Analysis, 2nd ed., Wiley, New York, (1980).

[90] O. Toeplitz, Das algebraische Analogon zu einem Satz von Fejer, Math. Z., 2 (1918), 187–197.

[91] M.P. Vasić and D.J. Keĉkić, Some inequalities for complex numbers, Math. Balkanica, 1 (1971), 282–286.

[92] P.Y. Wu and H.-L. Gau, Numerical ranges of Hilbert space operators, Encyclopedia Math. Appl., vol.179, Cambridge University Press, Cambridge, (2021).

[93] U. Yamancı and İ.M. Karlı, Further refinements of the Berezin number inequalities on operators, Linear Multilinear Algebra, 70 (2022), no. 20, 5237–5246.

[94] U. Yamancı and M. Garayev, Some results related to the Berezin number inequalities, Turkish J. Math., 43 (2019), no. 4, 1940–1952.

[95] U. Yamancı and M. Gürdal, On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space, New York J. Math., 23 (2017), 1531–1537.

[96] K. Zhu, Operator Theory in Function Spaces, 2nd ed., Mathematical Surveys and Monographs, vol. 138, Amer. Math. Soc., Providence, RI, (2007).

Anirban Sen