Duality for Many-valued Modal Logic and a Basic Study of Coalgebraic Fuzzy Geometric Logic

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BY

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CERTIFICATE FROM THE SUPERVISOR(S)

This is to certify that the thesis entitled **"Duality for Many-valued Modal Logic and a Basic Study of Coalgebraic Fuzzy Geometric Logic"** submitted by **Mr. Litan Kumar Das** who got his name registered on **18th February, 2022 (Index No.: 56/22/Maths./27)** for the award of Ph. D. (Science) degree of Jadavpur University, Kolkata is absolutely based upon his own work under the supervision of **Prof. Prakash Chandra Mali, Dept. of Mathematics, Jadavpur University** and **Prof. Kumar Sankar Ray, ECSU, ISI Kolkata** and that neither this thesis nor any part of it has been submitted for either any degree / diploma or any other academic award anywhere before.

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The thesis is dedicated to my parents

Kanai Lal Das

&

Shefali Das

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List of Symbols

$\mathcal{L}$	Finite Heyting algebra
$\mathcal{L}$-**VL**	$\mathcal{L}$-valued logic
$\mathcal{L}$-**VL**-algebras	Algebras of Fitting's Heyting-valued logic
$\mathcal{VA}_{\mathcal{L}}$	Category of $\mathcal{L}$-**VL**-algebras
$\mathcal{L}$-**BS**	Category of $\mathcal{L}$-Boolean spaces
$\mathcal{L}$-**BSYM**	Category of $\mathcal{L}$-**Boolean systems**
$\mathcal{L}$-**ML**	Heyting-valued modal logic
$\mathcal{L}$-**ML**-algebras	Algebras of Fitting's Heyting-valued modal logic
$\mathcal{MA}_{\mathcal{L}}$	Category of $\mathcal{L}$-**ML**-algebras
$\mathcal{L}$-**RS**	Category of $\mathcal{L}$-Boolean spaces with relation
$\mathcal{L}$-**RSYM**	Category of $\mathcal{L}$-relational systems
$PBS_{\mathcal{L}}$	Category of $\mathcal{L}$-pairwise Boolean spaces
$\mathbb{ISP}(\mathcal{L})$	Class of all isomorphic copies of sub-algebras of direct powers of a finite algebra $\mathcal{L}$
$\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$	Class of all isomorphic copies of sub-algebras of intuitionistic power of a finite algebra $\mathcal{L}$
$Pspa$	Category of Priestley spaces
$Hspa$	Category of Esakia spaces
$V_{\mathcal{L}}^{bi}$	$\mathcal{L}$-biVietoris functor on $PBS_{\mathcal{L}}$
$COALG(V_{\mathcal{L}}^{bi})$	Category of coalgebras for $V_{\mathcal{L}}^{bi}$
$V_P(S)$	Pairwise Vietoris space of a pairwise space S
Fuzzy-Top	Category of fuzzy topological spaces
FS	Category of fuzzy sets
FRM	Category of frames
S-FRM	Category of spatial frames
SFuzzy-Top	Category of sober fuzzy topological spaces

Abstract

The thesis focuses on developing a duality for Fitting's many-valued modal logic in a bitopological framework and exploring modal fuzzy geometric logic using coalgebraic logic approaches. There are many applications of the notion of duality in several pure and applied sciences. For instance, there is a duality in logic between syntax and semantics, a duality in mathematics between spaces and algebra, and a duality in information science between systems and observable properties. The current thesis explores and articulates the structure of duality for many-valued logic and many-valued modal logic by drawing on category theory and universal algebra. Since categorical relationships between systems and algebras, also referred to as frames, already exist in the literature, it is expected that these relationships can be extended to many-valued contexts. This is the goal that this thesis pursues in the first step. However, the investigation of duality for many-valued logic and many-valued modal logic using the methods of bitopological spaces, has drawn greater attention from scholars recently due to the fact that it can offer a more comprehensive viewpoint in this context. In this thesis, natural duality theory and modal natural duality theory are generalized in a bitopological framework by studying bitopological duality theory for Fitting's many-valued logic and many-valued modal logic. Thus, a coalgebraic duality theory is explored for multi-valued modal logics to shed light on more subtle aspects of bitopological duality. Coalgebraic logic is a proven framework that facilitates the development of an extended version of modal logic. In light of this, the thesis investigates the connections between fuzzy geometric logic and coalgebraic logic.

The thesis is divided into seven main chapters, excluding the introduction and conclusion.

- Chapter 2 presents the idea of lattice-valued Boolean systems and examines the adjoint and co-adjoint properties of functors that are defined on them. Consequently, a duality for algebras of lattice-valued logic is obtained.

- Chapter 3 introduces the concept of lattice-valued relational systems, intending to demonstrate a duality between systems and algebras of Fitting's lattice-valued modal logic.

- Chapter 4 establishes a duality for algebras of Fitting's Heyting-valued logic within the scope of bitopological techniques. In actuality, it extends the natural duality theory in a bi-topological context.

- Chapter 5 focuses on the extension of the natural duality theory for $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$, the class of all isomorphic copies of sub-algebras of intuitionistic power of $\mathcal{L}$. Thus, an intuitionistic version of the natural duality theory is developed.

- Chapter 6 aims to develop a bitopological duality for algebras of Fitting's many-valued modal logic. This has led to an extension of the natural duality theory for modal algebras.

- Chapter 7 sheds light on a coalgebraic description of the bitopological duality for Fitting's many-valued modal logic. This yields a coalgebraic duality for Fitting's many-valued modal logic.

- In Chapter 8, we investigate modal fuzzy geometric logic by applying coalgebra theory. In other words, this chapter introduces modal operators to the language of fuzzy geometric logic using the methods of coalgebraic logic, to examine how these logics are interpreted in specific fuzzy topological coalgebras.

List of Papers included in the thesis

1. Das, Litan Kumar., Ray, Kumar Sankar.: Bitopological duality for algebras of Fitting's logic and natural duality extension. Acta Informatica 58, 571–584 (2020). https://doi.org/10.1007/s00236-020-00384-5
2. Ray, Kumar Sankar., Das, Litan Kumar.: Categorical study for algebras of Fitting's lattice-valued logic and lattice-valued modal logic. Ann Math Artif Intell , 89, 409–429 (2021). https://doi.org/10.1007/s10472-020-09707-1
3. Das, Litan Kumar., Ray, Kumar Sankar., Mali, Prakash Chandra.: Bisimulations for Fuzzy Geometric Models. In: Tiwari, R.K., Sahoo, G. (eds) Recent Trends in Artificial Intelligence and IoT. ICAII 2023. Communications in Computer and Information Science, vol 1822. Springer, Cham (2023). https://doi.org/10.1007/978-3-031-37303-9_12
4. Das, Litan Kumar., Ray, Kumar Sankar., Mali, Prakash Chandra.: Duality for Fitting's Heyting-valued modal logic via Bitopology and Bi-Vietoris coalgebra. Theoretical Computer Science, Elsevier (Under Review).
5. Das, Litan Kumar., Ray, Kumar Sankar., Mali, Prakash Chandra.: Coalgebraic Fuzzy geometric logic. International Journal of Information technology, Springer (accepted).

Contents

Chapter 1

Introduction

The main objective of this thesis is to develop duality theory for Fitting's many-valued modal logic and investigate coalgebraic fuzzy geometric logic. Category theory is useful in numerous domains of science beyond mathematics. Categorical duality is the main focus of this thesis, and it is certainly present outside of mathematics.

The initial spark of our interest comes from the informational duality between systems and observable properties. Vickers proposed the notion of topological system in his book "Topology via Logic" [98] and highlighted its relationship to geometric logic. To study geometric logic (topology through logic), it is crucial to understand the links between topological space, topological system, frame, and geometric logic. An extension of topological system to lattice-valued topological system was performed in [16, 17, 18]. Additionally, a categorical relationship between the spaces and systems has been studied. Thus, the issue emerges: is it possible to build a duality for many-valued logic by establishing a categorical relationship between algebras of many-valued logic and appropriate topological systems? This question drives our first study. As an initial step, we introduce some relevant topological systems and establish their interrelationship with appropriate topological spaces and algebraic structures. These relationships are investigated in a categorical framework. As such, the study of duality is an important aspect of this thesis.

In [21], the concept of lattice-valued Boolean spaces and lattice-valued Boolean spaces with a relation has been utilized to establish a duality for the algebras of Fitting's many-valued logic and many-valued modal logic, respectively. We have contributed to an area of considerable interest which is the categorical relationship between the categories of lattice-valued Boolean spaces, lattice-valued Boolean systems, and algebras of many-valued logic (see Chapters 2, 3).This leads to the presentation of yet another proof of the duality provided in [21].

The thesis has been pursuing another interesting duality theory for Fitting's many-valued modal logic in the context of bitopological languages. As a consequence, natural duality theory [13] has been extended for modal algebras in a bitopological context. In the framework of natural duality theory, possibly the most successful theory of dualities for finitely generated quasi-varieties of algebras, we look at a more nuanced duality mechanism. Natural duality theory extends Stone-Priestley-type dualities through universal algebra approaches. In the realm of Botopological techniques, the thesis introduces modal natural duality theory. Moreover, the thesis incorporates an intuitionistic interpretation of natural duality theory. The first step is to develop a duality for algebras of a version of Fitting's many-valued logic via bitopological techniques. A topological duality theorem is also derived for the class of all isomorphic copies of subalgebras of the intuitionistic power of Heyting algebra, leading to the development of an intuitionistic version of natural duality theory.
By establishing a concept of $PRBS_{\mathcal{L}}$ as a category of $\mathcal{L}$-valued pairwise Boolean spaces with a relation, we intend to achieve a bitopological duality for algebras of Fitting's Heyting-valued modal logic in the second place. So, in the setting of bitopological languages, the natural duality theory for modal algebras is extended. The main results are bitopological and coalgebraic dualities for Fitting's many-valued modal logic, where $\mathcal{L}$ is a semi-primal algebra having a bounded lattice reduct. Our general theory extends both the Jónsson-Tarski duality and the Abramsky-Kupke-Kurz-Venema coalgebraic duality [1, 55] in the setting of bitopological language. It also proposes a new coalgebraic duality for algebras of many-valued modal logics.

The thesis will also focus on the development of coalgebraic fuzzy geometric logic. Fuzzy geometric logic is presented in [77] as a logical progression of propositional geometric logic [98]. Propositional geometric logic developed from the interaction of (pointfree) topology, logic, and the logic of finite observations [4, 98]. The formulae of this logic are generated from a collection of proposition letters using propositional connectives: finite conjunctions and arbitrary disjunctions, that preserve the property of finite observability. It is important to emphasize that geometric logic has no universal quantifier, negation, implication. With the appropriate topological connection, formulas of geometric logic can be interpreted in an algebraic structure (frame) of open sets in a topological space. A topological system is defined as a triple $(X, \models, A)$, where X is a non-empty set, A is a frame and $\models$ is a satisfaction relation from X to A. Chakraborty et al. [77] generalized geometric logic to the many-valued context by extending the notion of satisfaction relation.

It came to light that when the satisfaction relation is fuzzy, the related consequence relation ($\vdash$) can be either crisp or fuzzy. As a result, fuzzy geometric logic and fuzzy geometric logic with graded consequences were introduced.

Within the context of coalgebraic logic, modal logics are produced parametrically in the signature of the language and through an endofunctor $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ on a base category $\mathcal{C}$. Coalgebraic logic for endofunctors on the category of sets has been thoroughly researched and remains an active research topic (e.g. see [66, 67]). Within this framework, the concept of relation lifting [86] or predicate lifting [87] can be applied to define modal operators. Coalgebraic logic in the category of Stone coalgebras has been dealt with in [55, 68, 69, 73]. Many studies have been conducted on the development of a coalgebraic modal logic based on the Stone-type duality (for example, [72, 71, 70]).

The thesis attempts to investigate some relationships between fuzzy geometric logic and coalgebraic logic. In other words, we incorporate modal operators into the language of fuzzy geometric logic using the methods of coalgebraic logic, intending to examine how these logics are interpreted in fuzzy topological coalgebras. So, the aim of this study is to develop a framework for coalgebraic fuzzy geometric logics arising from extending fuzzy geometric logic with modalities that are generated by suitable predicate liftings.

Motivation

We now give some background information on the motivation for studying bitopological duality of many-valued modal logic.

The Stone duality [51] between Boolean algebras and sets represents the syntax and semantics of a propositional logic. The algebras and coalgebras of the endofunctors define the syntax and semantics of the modal propositional logic. As an illustration, the modal logic K and Kripke semantics derive from the Stone duality by taking an endofunctor on sets. So, in acceptable circumstances, we can achieve duality between the relevant algebras and coalgebras. In addition to demonstrating the fact that the widely recognized Stone duality could be articulated in coalgebraic terms, Abramsky [1] also showed that a coalgebraic formulation could be provided for the Jónsson-Tarski duality between descriptive general Kripke frames and modal algebras (see also [55] for further information). In particular, the category of descriptive general Kripke frames is isomorphic to the category of Boolean spaces. Esakia [64] also noticed this connection. Therefore, coalgebras for the Vietoris functor on the category of Boolean spaces can represent sound and complete semantics for modal logic. In [65], the author showed that coalgebras of a Vietoris functor on the category of Priestley spaces, i.e., compact, totally ordered disconnected spaces, provide sound and complete semantics for positive modal logic. The objective of this study is to combine the idea that the semantics of Fitting's many-valued modal logic can be understood as coalgebras for the bi-Vietoris functor on the category $PBS_{\mathcal{L}}$ of $\mathcal{L}$-valued pairwise Boolean spaces and pairwise continuous maps.

An overview of the motivation for studying coalgebraic fuzzy geometric logic is given here.
An illustration of the requirement for generalization in the satisfiability relation of a topological system may be found in [83]. We look at Vicker's interpretation [98] of topological systems. Let M be a collection of computer programs that generate 0's and 1's, and A be the assertions about the sequence of bits produced by those computer programs. Consider an assertion $a =$ starts1010. Then, the assertion a is true if a computer program, say m, generates sequence of bits 101010101...... So, in this case, $m \models$ starts1010. Suppose that a computer program m_1 produces an infinite sequence of bits in which the initial four bits are similar to but not equal to 1010. In this case, $m_1 \models$ start1010 to some extent. To deal with this kind of situation, the fuzzy topological system notion is therefore essential.
Based on the definition of $\models$, we can consider if an assertion holds to some degree in a single state in M, we look at all states in M. As a result, an assertion a is true in $M \iff$ it is satisfiable at some state. So we can construct a model structure $W = (M, R, V)$, where M is a set of states i.e., each state is a computer program, R is relation on M and V is valuation map from $\Phi \times W$ to $[0, 1]$, Φ is a set of propositional variables. This fact leads us to believe that it would be beneficial to incorporate modal operators into the fuzzy geometric logic language. As a result, we develop modal fuzzy geometric logic using coalgebra theory, known as coalgebraic fuzzy geometric logic.

Preliminaries

Almost every ground idea that could be needed to make this thesis self-contained is covered in this section. We review the fundamental concepts of lattice, frame, Boolean algebra, category theory, topological spaces, and topological systems.

Lattice, Frame and Boolean Algebra

We refer the reader to [36] for lattice theory.

Definition 1.0.1. *A **partially ordered set** or **poset** is a tuple* $(S, \preceq)$ *where* $\preceq \subseteq S \times S$ *is a binary relation such that for any* $a, b, c \in S$

- $a \preceq a$ *(reflexivity);*
- $a \preceq b$ *and* $b \preceq c \implies a \preceq c$ *(transitivity);*
- $a \preceq b$ *and* $b \preceq a \implies a = b$ *(antisymmetry).*

The binary relation $\preceq$ on S is called a partial order relation. An element s of S is said to be a least upper bound (l.u.b) of $a, b \in S$ if and only if $a \preceq s$ and $b \preceq s$ and for any $t \in S$ if $a \preceq t$, $b \preceq t$ then $s \preceq t$. Note that the least upper bound for any two elements in a poset may or may not exist, and if it does, it will be unique. Similarly, an element $r \in S$ is called a greatest lower bound (g.l.b) of $a, b \in S$ if and only if $r \preceq a, r \preceq b$ and for any $t \in S$ if $t \preceq a, t \preceq b$ then $t \preceq r$. In this instance, the greatest lower bound for any two elements in a poset may or may not exist; if it does, it will be unique.
Any two elements $a, b \in S$ that have $l.u.b$ and $g.l.b$ are represented by $a \vee b$ (join) and $a \wedge b$ (meet), respectively. The representation of an arbitrary join (if it exists) for any subset R of S is $\bigvee R = \bigvee_{r \in R}\{r\}$, while the representation of an arbitrary meet (if it exists) is $\bigwedge R = \bigwedge_{r \in R}\{r\}$. Additionally, if arbitrary joins and arbitrary meets of any subset of S are exist then they are also unique.

Definition 1.0.2. *A poset* $(\mathfrak{L}, \preceq)$ *is said to be lattice if for any two elements* ℓ_1, ℓ_2 *of* $\mathfrak{L}$*,* $\ell_1 \vee \ell_2$ *(join) and* $\ell_1 \wedge \ell_2$ *(meet) exist.*

A lattice $\mathfrak{L}$ is said to be distributive if for any $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$, $\ell_1 \wedge (\ell_2 \vee \ell_3) = (\ell_1 \wedge \ell_2) \vee (\ell_1 \wedge \ell_3)$ or $\ell_1 \vee (\ell_2 \wedge \ell_3) = (\ell_1 \vee \ell_2) \wedge (\ell_1 \vee \ell_3)$ satisfies.
A lattice $\mathfrak{L}$ is said to be bounded if it has a greatest (top) and least (bottom) element, designated as $\top$, and $\bot$, respectively. A lattice in which all subsets have both a supremum (join) and an infimum (meet) is said to be complete lattice.

Definition 1.0.3. *A bounded lattice $\mathfrak{L}$ is said to be complemented if every element of it has a complement i.e., for each $\ell \in \mathfrak{L}$ $\exists$ an element $\ell^c \in \mathfrak{L}$ such that $\ell \vee \ell^c = \top$ and $\ell \wedge \ell^c = \bot$.*

Definition 1.0.4 ([98])**.** *A poset(partially ordered set) S is said to be a frame if and only if*

(i) any subset $\mathcal{X}$ of S has a supremum (join) i.e., $\bigvee \mathcal{X}$ exists,

(ii) any finite subset $\mathcal{X}'$ of S has an infimum (meet) i.e., $\bigwedge \mathcal{X}'$ exists,

(iii) meet distribute over arbitrary join i.e.,

$$s \wedge \bigvee \mathcal{X} = \bigvee \{s \wedge x : x \in \mathcal{X}\}.$$

A frame homomorphism is defined as follows:

Definition 1.0.5 ([98])**.** *A function f from a frame F_1 to a frame F_2 is said to be a frame homomorphism if the function f preserves finite meets and arbitrary joins.*

The collection of frames and frame homomorphisms forms a category, denoted by **FRM**.
Similar to other algebraic structures, frames may be presented by generators and relations $\langle G|R\rangle$, where G denotes the set of generators, and R is the set of relations between expressions generated by G. One can find a detailed description of frame presentations in [100].

Note 1.0.1 ([89])**.** *Consider a frame F_1. Now, $\langle G|R\rangle$ presents the frame F_1 if $\exists$ an assignment $h : G \to F_1^*$, where F_1^* denotes the underlying set of F_1, such that the following properties hold:*

(i) F_1 is generated by the set $\{h(s) : s \in G\}$;

h can be extended to an assignment $\hat{h}$ for any expression r that is generated by G.

(ii) If $r = r'*$ is a relation in R, then $\hat{h}(r*) = \hat{h}(r'*)$ in F_1;*

(iii) For a frame F_2 and an assignment $h' : G \to F_2^*$ that satisfies (ii) there is a unique frame homomorphism $g : F_1 \to F_2$ such that $g^* \circ h = h'$, where g^* is a mapping from F_1^* to F_2^*. So, the diagram shown in Fig. 1.1 commutes.

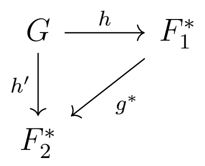

Figure 1.1: Illustration of frame presentation

Remark 1.0.1. *We can define a frame homomorphism $f : F_1 \to F_2$ from a frame F_1 to a frame F_2, where $\langle G|R \rangle$ presents the frame F_1. According to Note 1.0.1, it is sufficient to provide an assignment $\hat{f} : G \to F_2$ that satisfies the condition that if $r* = r'*$ is a relation in R, then $\hat{f}(r*) = \hat{f}(r'*)$ in F_2.*

A complemented distributive lattice is said to be a Boolean lattice.

Definition 1.0.6. *A Boolean algebra is defined by a structure $(\mathcal{B}, \vee, \wedge, 0, 1)$ such that the following conditions are met:*

(i) $(\mathcal{B}, \vee, \wedge)$ *is a distributive lattice;*

(ii) $a \vee 0 = a$ *and* $a \wedge 1 = a \ \forall a \in \mathcal{B}$*;*

(iii) $a \vee a^c = 1$ *and* $a \wedge a^c = 0 \ \forall a \in \mathcal{B}$*.*

A Boolean algebra homomorphism is a function between two Boolean algebras such that it preserves join($\vee$), meet($\wedge$) and complementation.
Arend Heyting introduced Heyting algebras in 1930 as a framework for intuitionistic logic.

Definition 1.0.7. *A Heyting lattice or Heyting algebra is a bounded distributive lattice H equipped with a binary operation $\to$ called implication, such that $c \leq (a \to b) \iff (a \wedge c) \leq b$.*

It is clear that any finite distributive lattice is a finite Heyting algebra. Heyting algebras that are complete as a lattice are called complete Heyting algebras.
It should be mentioned that a complete Heyting algebra is a frame.

Category Theory

For category theory, we refer the reader to [2, 3].

Definition 1.0.8 ([2]). *A category is a quadruple $C = (\mathcal{O}, HOM, ID, \circ)$ which includes the following:*

1. *a class $\mathcal{O}$, whose elements are called C-objects,*
2. *for each pair (P, Q) of C-objects, a set $HOM(P, Q)$, whose members are referred to as C-morphisms from P to Q (the sets $HOM(P, Q)$ are pairwise disjoint),*
3. *for any three C-objects P, Q, R, a map $\circ : HOM(P, Q) \times HOM(Q, R) \to HOM(P, R)$, called composition, is defined by $\circ(f, g) = g \circ f$, such that*
   - *(a) composition is associative i.e., $h \circ (g \circ f) = (h \circ g) \circ f$ for all morphisms $f \in HOM(P, Q)$, $g \in HOM(Q, R)$, and $h \in HOM(R, T)$,*
   - *(b) for each C-object P there exists $ID_P \in HOM(P, P)$, called C-identity on P, such that for a C-morphisms $f : P \to Q$, we have $ID_Q \circ f = f$ and $f \circ ID_P = f$*

Definition 1.0.9. *For a category $C = (\mathcal{O}, HOM, ID, \circ)$ the dual or opposite category of C is the category C^{op} with the same objects as C but for any C-morphisms $f : P \to Q$ in C, there is only one morphism $f^{op} : Q \to P$ and $f^{op} \circ g^{op} = (g \circ f)^{op}$, where $g \in HOM(Q, R)$.*

Definition 1.0.10. *Let C and D be categories. A functor $\mathcal{F} : C \to D$ is a function such that*

1. *F carries each C-object P to D-objects $F(P)$,*
2. *F carries each C-morphism $f \in HOM(P, Q)$ to D-morphisms $F(f) \in HOM(F(P), F(Q))$ such that*
   - *(i) $F(g \circ f) = F(g) \circ F(f)$ for all $f \in HOM(P, Q), g \in HOM(Q, R)$, i.e., F preserves compositions,*
   - *(ii) $F(ID_P) = ID_{F(P)}$ for all $P \in C$ i.e., F preserves identity morphisms.*

Definition 1.0.11. *Let F and G be functors from a category C to a category D. A natural transformation $\zeta : F \to G$ is a class of morphisms that satisfies the following condition:*

- *ζ must associates each C-object P, a D-morphism $\zeta_P : F(P) \to G(P)$ such that for every C-morphism $f : P \to Q$ we have $\zeta_Q \circ F(f) = G(f) \circ \zeta_P$ i.e., the following diagram commutes.*

$$\begin{array}{ccccc} P & \quad & F(P) & \xrightarrow{\zeta_P} & G(P) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ Q & & F(Q) & \xrightarrow[\zeta_Q]{} & G(Q) \end{array}$$

Figure 1.2: Representation of Natural transformation

Note that the D-morphism ζ_P is said to be component of ζ at P.

Definition 1.0.12. *Let $f : \boldsymbol{G} \longrightarrow \boldsymbol{H}$ be a functor, and H be a $\boldsymbol{H}$-object.*

(i) A f-structured arrow with domain H is a pair (g, G) consisting of a $\boldsymbol{G}$-object G and a $\boldsymbol{H}$-morphism $g : H \longrightarrow f(G)$.

(ii) A f-structured arrow with domain H is called a f-universal arrow for H provided that for each f-structured arrow (g', G') with domain H there exists a unique $\boldsymbol{G}$-morphism $\tilde{g} : G \longrightarrow G'$ with $g' = f(\tilde{g}) \circ g$. In other words the triangle as shown in Fig. 1.3 commutes.

$$\begin{array}{ccc} H & \xrightarrow{g} & f(G) \\ & {\scriptstyle g'}\searrow \quad \swarrow{\scriptstyle f(\tilde{g})} & \\ & f(G') & \end{array}$$

$$G \xrightarrow{\tilde{g}} G'$$

Figure 1.3: Representation of Universal arrow

(iii) A f-costructured arrow with codomain H is a pair (G, g) consisting of a $\boldsymbol{G}$-object G and a $\boldsymbol{H}$-morphism $g : f(G) \longrightarrow H$.

(iv) A f-costructured arrow (G, g) with codomain H is called a f-couniversal arrow for H provided that for each f-costructured arrow (G', g') with codomain H there exists a unique $\boldsymbol{G}$-morphism $\tilde{f} : G' \longrightarrow G$ such that $g' = g \circ f(\tilde{g})$.

Definition 1.0.13. *A functor $f\colon$ $\boldsymbol{G} \longrightarrow \boldsymbol{H}$ is said to be adjoint if for every $\boldsymbol{H}$-object H there exists a f-universal arrow with domain H. Consequently, there exists a natural transformation, called the unit(see Fig. 1.4) $\eta_H : ID_H(H) \longrightarrow ff_1(H)$, where ID_H is an identity morphism from H to H and $f_1 : \boldsymbol{H} \longrightarrow \boldsymbol{G}$ is a functor. More precisely, for a given morphism $g : H \longrightarrow f(G)$ there is a unique $\boldsymbol{G}$-morphism $\tilde{g} : f_1(H) \longrightarrow G$ such that the triangle of Fig. 1.4 commutes i.e., $g = f(\tilde{g}) \circ \eta_H$.*

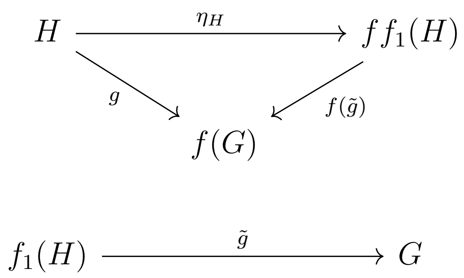

Figure 1.4: Illustration of the unit

Definition 1.0.14. *A functor $f\colon$ $\boldsymbol{G} \longrightarrow \boldsymbol{H}$ is said to be co-adjoint if for every $\boldsymbol{H}$-object H there exists a f-couniversal arrow with codomain H. As a result, there exists a natural transformation, called the counit(see Fig. 1.5) $\xi_G : f_1 \circ f(G) \longrightarrow ID_G(G)$, where ID_G is an identity morphism from G to G, and $f_1 : \boldsymbol{H} \longrightarrow \boldsymbol{G}$ is a functor. More precisely, for a given morphism $g : f_1(H) \longrightarrow G$, there is a unique $\boldsymbol{H}$-morphism $\tilde{g} : H \longrightarrow f(G)$ such that the triangle of Fig. 1.5 commutes. In other words $g = \xi_G \circ f_1(\tilde{g})$.*

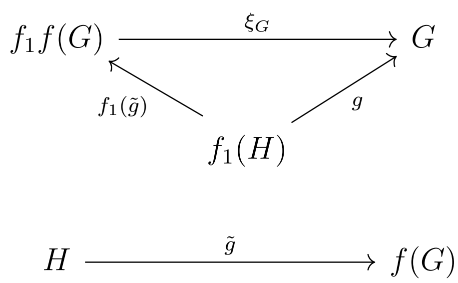

Figure 1.5: Illustration of the counit

Topological Spaces

We refer to [8] for general topology.

Definition 1.0.15. *Let X be a set. A collection τ^X of some subsets of X is said to be a topology on X if and only if*

1. $\emptyset, X \in \tau^X$,
2. τ^X *is closed under arbitrary union,*
3. τ^X *is closed under finite intersection.*

If τ^X is a topology on X then the pair (X, τ^X) is called a topological space. The members of τ^X are called open sets.

Note 1.0.2. *For a topological space (X, τ^X), $(\tau^X, \subseteq)$ forms a frame.*

Definition 1.0.16. *Let τ^X and τ^Y be two topologies on X and Y, respectively. A mapping $f : X \to Y$ is said to be continuous if and only if for every open set $U_Y \in \tau^Y$, $f^{-1}(U_Y) \in \tau^X$*

The concept of lattice-valued topological spaces was presented in [17]. Let us go over the concept of lattice-valued topology.
Let $\mathfrak{L}$ be a lattice. A lattice-valued topology, $\mathfrak{L}$-**TOP**, on X is a collection $\mathcal{T} \subseteq \mathfrak{L}^X$ such that $\mathcal{T}$ is closed under arbitrary join ($\bigvee$) and finite meet ($\wedge$). Then $(X, \mathcal{T})$ is said to be $\mathfrak{L}$-topological space.

Definition 1.0.17 ([18])**.** *Let S_1 and S_2 be two sets, and $\mathfrak{L}$ be a lattice. For a function $\psi : S_1 \longrightarrow S_2$, the Zadeh image operator $\psi_{\mathfrak{L}} : \mathfrak{L}^{S_1} \longrightarrow \mathfrak{L}^{S_2}$ and inverse image operator $\psi_{\mathfrak{L}}^{-1} : \mathfrak{L}^{S_2} \longrightarrow \mathfrak{L}^{S_1}$ are defined by $\psi_{\mathfrak{L}}(\sigma)(s_2) = \bigvee\{\sigma(s') : s' \in \psi^{-1}(\{s_2\})\}$, $\psi_{\mathfrak{L}}^{-1}(\varphi) = \varphi \circ \psi$.*

Definition 1.0.18. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two $\mathfrak{L}$-topological spaces. A mapping $f : X \to Y$ is said to be $\mathfrak{L}$-continuous if and only if for every $v \in \mathcal{T}_2$, $f^{-1}(v) = v \circ f \in \mathcal{T}_1$.*

Topological Systems

Vickers developed the notion of topological systems in his work on topology via logic [98]. A Topological system is defined by a mathematical structure as $(S, \mathcal{F}, \models)$, where S is a non-empty set, $\mathcal{F}$ is a frame, and $\models$ is a binary relation from S to $\mathcal{F}$ such that

1. for any finite subset X of $\mathcal{F}$, $s \models \bigwedge X \iff s \models x, \forall x \in X$;
2. for any subset X of $\mathcal{F}$, $s \models \bigvee X \iff s \models x,$ for some $x \in X$

We write $s \models x$ for $(s, x) \in \models$ and call it as s satisfies x. Thus the relation $\models$ satisfies both join and finite meet interchange laws.
The set S can be understood as the collection of objects, and the set $\mathcal{F}$ as the collection of properties. Then $\models$ states which properties are satisfied by which object. It should be noted that $\bigwedge X = \top$ if $X = \emptyset$.

Observation 1.0.1. *1. $s \models \top$, $\forall s \in S$.*

2. $s \models \bot$, for no $s \in S$.

3. if $s \models g$ and $g \leq h$ then $s \models h$.

Proposition 1.0.1. *Let (S, τ^S) be a topological space. Then $(S, \models, \tau^S)$ is a topological system, where the satisfaction relation $\models$ is defined as $s \models U \iff s \in U$, $s \in S$ and $U \in \tau^S$.*

Example 1.0.1. *Let $\mathcal{A}$ be a frame. Then it can be shown that $(HOM(\mathcal{A}, \{0,1\}), \models, \mathcal{A})$ is a topological systems, where $HOM(\mathcal{A}, \{0,1\})$ is the set of all frame homomorphisms from $\mathcal{A}$ to $\{0,1\}$ and $\psi \models a \iff \psi(a) = 1$, $\psi \in HOM(\mathcal{A}, \{0,1\})$.*

We now review the concept of extent in a topological systems.

Definition 1.0.19. *Let $(S, \models, \mathcal{F})$ be a topological system and $g \in \mathcal{F}$. The extent of g, denoted as $ext(g)$, is defined by $ext(g) = \{s \in S : s \models g\}$.*

Thus, $ext(\mathcal{F}) = \{ext(g) : g \in \mathcal{F}\}$

Proposition 1.0.2. *Let $(S, \models, \mathcal{F})$ be a topological system. Then $ext(\mathcal{F})$ forms a topology on S i.e., $(S, ext(\mathcal{F}))$ is a topological space.*

Proof. Since $\mathcal{F}$ is a frame, so $\top, \bot \in \mathcal{F}$. Now, $ext(\top) = \{s \in S : s \models \top\} = S \in ext(\mathcal{F})$, and $ext(\bot) = \{s \in S : s \models \bot\} = \emptyset \in ext(\mathcal{F})$. Let $\{ext(g_\lambda) : \lambda \in \Lambda\}$ be an arbitrary collection of elements in $ext(\mathcal{F})$. Then, $\bigcup\limits_{\lambda} ext(g_\lambda) = \bigcup\limits_{\lambda}\{s \in S : s \models g_\lambda\} = \{s \in S : s \models \bigvee\limits_{\lambda} g_\lambda\} = ext(\bigvee\limits_{\lambda} g_\lambda) \in ext(\mathcal{F})$. Similarly, if $ext(g_1), ext(g_2) \in ext(\mathcal{F})$ then $ext(g_1 \wedge g_2) \in ext(\mathcal{F})$. Thus, $ext(\mathcal{F})$ is closed under arbitrary join and finite meet. $\square$

Definition 1.0.20. *A continuous map* Φ *from a topological system* $(S, \mathcal{F}, \models)$ *to a topological system* $(S', \mathcal{F}', \models')$ *is defined by a pair of maps* (f, g) *such that*

- $f : S \to S'$ *is a set function,*
- g *is a frame homomorphism from* $\mathcal{F}'$ *to* $\mathcal{F}$ *satisfying* $s \models g(t) \iff f(s) \models' t$ *for any* $s \in S$ *and* $t \in \mathcal{F}'$.

Chapter 2

Category of lattice-valued Boolean systems

2.1 Introduction

This chapter explores categorical interconnections between lattice-valued Boolean systems and algebras of Fitting's lattice-valued logic. After introducing lattice-valued Boolean systems, we discuss the adjointness and co-adjointness of the functors defined on these systems.

In the study of geometric logic, Vickers [98] proposed the idea of topological systems, which was later explored in [100]. A topological system is a mathematical structure $(S, \mathcal{F}, \models)$, where S is a non-empty set, $\mathcal{F}$ is a frame, and $\models$ is a satisfaction relation on $S \times \mathcal{F}$. We read $s \models g$ as "s satisfies g". Denniston et al. [16] established the concept of lattice-valued topological systems by extending the satisfaction relation to a lattice-valued satisfaction relation. Furthermore, as noted by Vickers, topological spaces constitute a special kind of topological system; this was also reported in [16] for lattice-valued topological spaces.

To establish a categorical relationship between the systems and spaces, the authors in [16, 17, 18] make use of the concept of lattice-valued topological systems. Furthermore, as an additional generalization of lattice-valued topological systems, variable-basis topological systems were presented in [19]. These systems were then examined from a different angle in [22].

The outcomes of this chapter appear in [57] **Ray, Kumar Sankar., Das, Litan Kumar.**: **Categorical study for Algebras of lattice-valued logic and lattice-valued modal logic. Annals of Mathematics and Artificial Intelligence, Springer, 89, 409-429 (2021).**

From an additional perspective, it is also important to generalize topological systems to lattice-valued topological systems. It is conscious that semantic-consequence relation in first-order logic is defined in the context of satisfaction relation. The associated consequence relation may be conventional or many-valued when the satisfaction relation is many-valued. Regarding the nature of logical consequence, there are numerous many-valued logics. Thus, it is possible to think of lattice-valued topological systems as a generalization of many-valued logics. However, we will not go into detail on this topic.

For a finite distributive lattice $\mathcal{L}$, Fitting presented the concept of $\mathcal{L}$-valued logic and $\mathcal{L}$-valued modal logic in [23], where the elements of $\mathcal{L}$ are regarded as truth constants. Fitting's logic has been the subject of numerous research (e.g., [24, 25, 28, 30]). However, this chapter will not address $\mathcal{L}$-valued modal logic.

Maruyama [20] defined $\mathcal{L}$-**VL**-algebras as an algebraic structure of Fitting's $\mathcal{L}$-valued logic and thus established a duality in [21] between the category of $\mathcal{L}$-**VL**-algebras and homomorphisms of $\mathcal{L}$-**VL**-algebras and the category $\mathcal{L}$-**BS** of lattice-valued versions of Boolean spaces by employing the theory of natural dualities [13].

Our motivation to consider if there exist systems that are categorically connected with algebras of Fitting's multi-valued logic comes from the work of [17]. Our goal is to characterize such systems and prove that they are categorically equivalent to the lattice-valued Boolean spaces. This will lead to the establishment of duality between the category of $\mathcal{L}$-**VL**-algebras and the category of lattice-valued Boolean spaces. This outcome provides an additional evidence for the duality established in [21]. The idea of lattice-valued Boolean systems is helpful in obtaining the conclusions presented in this chapter.

2.2 $\mathcal{L}$-VL-algebras, $\mathcal{L}$-Boolean Spaces, $\mathcal{L}$-Boolean Systems, and Categorical interconnections

Throughout this section, $\mathcal{L}$ denotes a finite distributive lattice with top element 1 and bottom element 0 ($1 \neq 0$). Consequently, $\mathcal{L}$ forms a complete Heyting algebra. Let $a \to b$ represent the pseudo-complement of a with respect to b for all $a, b \in \mathcal{L}$.

Definition 2.2.1. *For all $\ell \in \mathcal{L}$, the unary operation $T_\ell : \mathcal{L} \to \mathcal{L}$ equipped with $\mathcal{L}$ is defined as* $T_\ell(x) = \begin{cases} 1 & \text{if } x = \ell \\ 0 & \text{if } x \neq \ell \end{cases}$

2.2.1 $\mathcal{L}$-VL-algebras

A $\mathcal{L}$-valued logic $\mathcal{L}$-**VL** is basically a many-valued logic and the operations of $\mathcal{L}$-**VL** are $\vee, \wedge, \rightarrow, 0, 1$ and $T_L(L \in \mathcal{L})$, where $\wedge, \vee, \rightarrow$ are binary operations, $0, 1$ are nullary operations and for each $L \in \mathcal{L}$, T_L is a unary operation.

The concept of $\mathcal{L}$-**VL**-algebras, initially laid out in [20], provides a sound and complete algebraic semantics for $\mathcal{L}$-valued logic $\mathcal{L}$-**VL**.
Now let us review the notion of $\mathcal{L}$-**VL**-algebra. Let $a \leq b$ denote $a \wedge b = a$ and $a \leftrightarrow b$ denote $(a \rightarrow b) \wedge (b \rightarrow a)$.

Definition 2.2.2 ([20]). *An algebraic system $(\mathcal{A}, \wedge, \vee, \rightarrow, T_L(L \in \mathcal{L}), 0, 1)$ forms a $\mathcal{L}$-**VL**-algebra if and only if for any $L_1, L_2 \in \mathcal{L}$, and $a, b \in \mathcal{A}$ the following axioms hold:*

(i) *the algebraic structure $(\mathcal{A}, \wedge, \vee, \rightarrow, 0, 1)$ forms a Heyting algebra;*

(ii) $T_{L_1}(a) \wedge T_{L_2}(b) \leq T_{L_1 \rightarrow L_2}(a \rightarrow b) \wedge T_{L_1 \wedge L_2}(a \wedge b) \wedge T_{L_1 \vee L_2}(a \vee b)$;
$T_{L_2}(a) \leq T_{T_{L_1}(L_2)}(T_{L_1}(a))$;

(iii) $T_0(0) = 1$; $T_L(0) = 0$ $(L \neq 0)$, $T_1(1) = 1$, $T_L(1) = 0, L \neq 1$;

(iv) $\bigvee\{T_L(a) : L \in \mathcal{L}\} = 1$, $T_{L_1}(a) \vee (T_{L_2}(a) \rightarrow 0) = 1$;
$T_{L_1}(a) \wedge T_{L_2}(a) = 0$ $(L_1 \neq L_2)$;

(v) $T_1(T_L(a)) = T_L(a)$, $T_0(T_L(a)) = T_L(a) \rightarrow 0$, $T_{L_2}(T_{L_1}(a)) = 0$ $(L_2 \neq 0, 1)$;

(vi) $T_1(a) \leq a$, $T_1(a \wedge b) = T_1(a) \wedge T_1(b)$;

(vii) $\bigwedge_{L \in \mathcal{L}} (T_L(a) \leftrightarrow T_L(b)) \leq (a \leftrightarrow b)$.

Definition 2.2.3 ([20]). *A $\mathcal{L}$-**VL**-algebras homomorphism is a mapping f between $\mathcal{L}$-**VL**-algebras which preserves the operations $\vee, \wedge, \rightarrow, T_L(L \in \mathcal{L}), 0, 1$.*

Definition 2.2.4 ([21]). *Let $\mathcal{A}$ be a $\mathcal{L}$-**VL**-algebra. A non-empty subset F of $\mathcal{A}$ is called a $\mathcal{L}$-filter iff F is a filter of lattices which is closed under T_1. Let $\mathcal{P}$ be a proper $\mathcal{L}$-filter of $\mathcal{A}$. Then*

(i) *$\mathcal{P}$ is a prime $\mathcal{L}$-filter of $\mathcal{A}$ iff for any $L \in \mathcal{L}$, $T_L(x \vee y) \in \mathcal{P}$, then there exist $L_1, L_2 \in \mathcal{L}$ with $L_1 \vee L_2 = L$ such that $T_{L_1}(x) \in \mathcal{P}$ and $T_{L_2}(y) \in \mathcal{P}$.*

(ii) *$\mathcal{P}$ is an ultra $\mathcal{L}$-filter of $\mathcal{A}$ iff $\forall x \in \mathcal{A}$, $\exists L \in \mathcal{L}$ such that $T_L(x) \in \mathcal{P}$.*

(iii) $\mathcal{P}$ *is a maximal* $\mathcal{L}$*-filter iff* $\mathcal{P}$ *is maximal with respect to inclusion.*

Proposition 2.2.1 ([21])**.** *(a) Let* $\mathcal{A}$ *be a* $\mathcal{L}$*-**VL**-algebra. For any two distinct members* x, y *of* $\mathcal{A}$ *,there exist* $L \in \mathcal{L}$ *and a prime* $\mathcal{L}$*-filter* $\mathcal{P}$ *of* $\mathcal{A}$ *such that* $T_L(x) \in \mathcal{P}$ *and* $T_L(y) \notin \mathcal{P}$*.*

(b) For a prime $\mathcal{L}$*-filter* $\mathcal{P}$ *of a* $\mathcal{L}$*-**VL**-algebra* $\mathcal{A}$*, define* $\Theta_{\mathcal{P}} : \mathcal{A} \longrightarrow \mathcal{L}$ *by* $\Theta_{\mathcal{P}}(x) = q \Leftrightarrow T_q(x) \in \mathcal{P}$*. Then,* $\Theta_{\mathcal{P}}$ *is a homomorphism of* $\mathcal{L}$*-**VL**-algebras.*

(c) Let $\mathcal{A}$ *be a* $\mathcal{L}$*-**VL**-algebra. A bijective mapping exists from the set of all prime* $\mathcal{L}$*-filters of* $\mathcal{A}$ *to the set of all homomorphisms from* $\mathcal{A}$ *to* $\mathcal{L}$*.*

Definition 2.2.5 ([21])**.** *For a* $\mathcal{L}$*-**VL**-algebra* $\mathcal{A}$*, define* $\mathfrak{B}(\mathcal{A}) = \{a \in \mathcal{A} : T_1(a) = a\}$*. Then* $\mathfrak{B}(\mathcal{A})$ *is a Boolean algebra.*

The spectrum of a $\mathcal{L}$-**VL**-algebra $\mathcal{A}$ is designated by $Spec_{\mathcal{L}}(\mathcal{A})$.

Definition 2.2.6 ([21])**.** *Let* $\mathcal{A}$ *be a* $\mathcal{L}$*-**VL**-algebra. If* K *is a sub-algebra of* $\mathcal{L}$*, then* $Spec_K(\mathcal{A}) = \{f : \mathcal{A} \longrightarrow K |\ f$ *is a* $\mathcal{L}$*-**VL**-algebras homomorphism* $\}$*.*

Remark 2.2.1. $\mathcal{L}$ *and* $\mathcal{L}$*-**VL**-algebra are frames.*

Category $\mathcal{VA}_{\mathcal{L}}$

Definition 2.2.7 ([21])**.** $\mathcal{L}$*-**VL**-algebras together with* $\mathcal{L}$*-**VL**-algebras homomorphisms form the category* $\mathcal{VA}_{\mathcal{L}}$*.*

Definition 2.2.8. *The opposite category of the category* $\mathcal{VA}_{\mathcal{L}}$ *is denoted by* $(\mathcal{VA}_{\mathcal{L}})^{op}$*, and which is defined as follows:*

(i) objects in $(\mathcal{VA}_{\mathcal{L}})^{op}$ *are objects in* $\mathcal{VA}_{\mathcal{L}}$*;*

(ii) arrows in $(\mathcal{VA}_{\mathcal{L}})^{op}$ *are arrows in* $\mathcal{VA}_{\mathcal{L}}$ *but acting in reverse direction.*

2.2.2 Lattice-valued Boolean spaces

The concept of lattice-valued topological space can be found in [17]. For a lattice-valued topological space (E, τ), $Cont(E, \tau)$ is taken as the collection of all continuous functions from E to $\mathcal{L}$.

Definition 2.2.9 ([27])**.** *A lattice-valued topological space* (E, τ) *is said to be Kolmogorov* $\Longleftrightarrow$ *for any* $e_1, e_2 \in E$ *with* $e_1 \neq e_2$*, there exists an open* $\mathcal{L}$*-valued map* $\mu : E \to \mathcal{L}$ *such that* $\mu(e_1) \neq \mu(e_2)$*.*

Definition 2.2.10 ([27]). *A lattice-valued topological space (E, τ) is said to be Hausdorff $\iff$ for any $e_1, e_2 \in E$ with $e_1 \neq e_2$, there are $\ell \in \mathcal{L}$ and an open $\mathcal{L}$-valued maps μ_1 and μ_2 on E such that $\mu_1(e_1) \geq \ell$, $\mu_2(e_2) \geq \ell$ and $\mu_1 \wedge \mu_2 < \ell$.*

Definition 2.2.11 ([27]). *A lattice-valued topological space (E, τ) is said to be compact $\iff$ $1_E = \bigvee_{\lambda \in \Lambda} u_\lambda$, where each μ_λ is an open $\mathcal{L}$-valued map on E, then there exists a finite collection Λ^* of Λ such that $1_E = \bigvee_{\lambda \in \Lambda^*} \mu_\lambda$, 1_E is a constant map on E that maps each element of E to 1.*

Definition 2.2.12. *A lattice-valued topological space (E, τ) is said to be zero-dimensional $\iff$ $Cont(E, \tau)$ forms a clopen basis of (E, τ).*

Definition 2.2.13. *A lattice-valued topological space (E, τ) is said to be lattice-valued Boolean space denoted by $\mathcal{L}$-**Boolean space** $\iff$ (E, τ) is compact, zero-dimensional and Hausdorff.*

If $\mathcal{B}$ is a $\mathcal{L}$-**Boolean space**, then the collection of all closed subspaces of $\mathcal{B}$ is denoted by $\Omega(\mathcal{B})$. Now it is easy to follow that each member of $\Omega(\mathcal{B})$ is also a $\mathcal{L}$-**Boolean space**.
Let the subalgebras of $\mathcal{L}$ be denoted by $Subalg(\mathcal{L})$.
Now, we define the category $\mathcal{L}$-**BS** of $\mathcal{L}$-**Boolean spaces**.

Definition 2.2.14. *(a) An objects in $\mathcal{L}$-**BS** is defined by a tuple (X, β), where X is a $\mathcal{L}$-**Boolean space** and $\beta : Subalg(\mathcal{L}) \longrightarrow \Omega(X)$ is a mapping which has the following properties:*

(i) $\beta(\mathcal{L}) = X$;

(ii) $\beta(\mathcal{L}_1) \subset \beta(\mathcal{L}_2)$ whenever $\mathcal{L}_1, \mathcal{L}_2 \in Subalg(\mathcal{L})$ and $\mathcal{L}_1$ is a subalgebra of $\mathcal{L}_2$;

(iii) $\beta(\mathcal{L}_3) = \beta(\mathcal{L}_1) \cap \beta(\mathcal{L}_2)$ whenever $\mathcal{L}_3 = \mathcal{L}_1 \cap \mathcal{L}_2$.

*(b) An arrow in $\mathcal{L}$-**BS** is defied by a map $\psi : (X_1, \beta_1) \longrightarrow (X_2, \beta_2)$ such that $\psi : X_1 \longrightarrow X_2$ is a $\mathcal{L}$-valued continuous map and for each member K of $Subalg(\mathcal{L})$, if $s_1 \in \beta_1(K)$ then $\psi(s_1) \in \beta_2(K)$, in other words, ψ preserves corresponding subspaces.*

Remark 2.2.2. *Here $\mathcal{L}$ and subalgebras of $\mathcal{L}$ are taken with discrete topology. Then $\mathcal{L}$ with discrete topology forms a Boolean space and hence $(\mathcal{L}, \phi_\mathcal{L})$ is an object in the category $\mathcal{L}$-**BS**, where $\phi_\mathcal{L} : Subalg(\mathcal{L}) \longrightarrow \Omega(\mathcal{L})$ is a mapping defined by $\phi_\mathcal{L}(K) = K$, $K \in Subalg(\mathcal{L})$.*

Definition 2.2.15. *Let $(\mathcal{O}, \phi)$ be an object in the category $\mathcal{L}$-**BS**. $Cont(\mathcal{O}, \phi)$ is the collection of all continuous functions $\psi : (\mathcal{O}, \phi) \longrightarrow (\mathcal{L}, \beta)$ which preserve subspaces.*

Remark 2.2.3. *The algebraic structure $(Cont(\mathcal{O}, \phi), \wedge, \vee, \rightarrow, T_L(L \in \mathcal{L}), 0, 1)$ forms a $\mathcal{L}$-**VL**-algebra. Operations are defined as follows:*
Let $\xi_1, \xi_2 \in Cont(\mathcal{O}, \phi)$. Then $(\xi_1 * \xi_2)(O) = \xi_1(O) * \xi_2(O)$, $* = \wedge, \vee, \rightarrow$, and $(T_L(\xi_1))(O) = T_L(\xi_1(O))$.

2.2.3 $\mathcal{L}$-Boolean systems

We now introduce the notion of **$\mathcal{L}$-Boolean systems** in Definition 2.2.16.

Definition 2.2.16. *Let E be a non-empty set, and let $\mathcal{A}$ be a $\mathcal{L}$-**VL**-algebra. A **$\mathcal{L}$-Boolean system** is defined by a mathematical structure $(E, \mathcal{A}, \models_{(E\times\mathcal{A})})$, where $\models_{(E\times\mathcal{A})}$ is a $\mathcal{L}$-valued satisfaction relation on $(E, \mathcal{A})$, which satisfies the following conditions:*

(i) if $\{a_\lambda\}_{\lambda\in J}$(J is an index set) be a collection of members of $\mathcal{A}$, then $\models_{(E\times\mathcal{A})} (e, \bigvee_{\lambda\in J} a_\lambda) = \bigvee_{\lambda\in J} \models_{(E\times\mathcal{A})} (e, a_\lambda)$;
if a_1, a_2 be any two members of $\mathcal{A}$, then $\models_{(E\times\mathcal{A})} (e, a_1 \wedge a_2) = \models_{(E\times\mathcal{A})} (e, a_1) \wedge \models_{(E\times\mathcal{A})} (e, a_2)$;

(ii) if $e_1 \neq e_2$ in E then there is $a \in \mathcal{A}$, such that $\models_{(E\times\mathcal{A})} (e_1, a) \neq \models_{(E\times\mathcal{A})} (e_2, a)$;

(iii) $\models_{(E\times\mathcal{A})} (e, a_1 \rightarrow a_2) = \models_{(E\times\mathcal{A})} (e, a_1) \rightarrow \models_{(E\times\mathcal{A})} (e, a_2)$;

(iv) $\models_{(E\times\mathcal{A})} (e, T_s(a)) = T_s(\models_{(E\times\mathcal{A})} (e, a))$, for $a \in \mathcal{A}$ and $s \in \mathcal{L}$;

(v) $\models_{(E\times\mathcal{A})} (e, 0) = 0$, $\models_{(E\times\mathcal{A})} (e, 1) = 1$.

As it develops, a $\mathcal{L}$- **Boolean system** is essentially a $\mathcal{L}$- **topological system** with certain further conditions.

Definition 2.2.17. *The category $\mathcal{L}$-**BSYM** of $\mathcal{L}$-**Boolean systems** is defined as follows:*

*(i) An object in $\mathcal{L}$-**BSYM** is a $\mathcal{L}$-**Boolean systems** $(E, \mathcal{A}, \models_{(E\times\mathcal{A})})$;*

*(ii) An arrow in $\mathcal{L}$-**BSYM** is a continuous map $(\psi_1, \psi_2) : (E_1, \mathcal{A}, \models_{(E_1\times\mathcal{A})}) \longrightarrow (E_2, \mathcal{B}, \models_{(E_2\times\mathcal{B})})$ between any two objects in $\mathcal{L}$-**BSYM**, where*

(a) $\psi_1 : E_1 \longrightarrow E_2$ *is a set map;*

(b) $\psi_2 : \mathcal{B} \longrightarrow \mathcal{A}$ *is a* $\mathcal{L}$-***VL****-algebras homomorphism;*

(c) $\models_{(E_1\times\mathcal{A})} (e_1, \psi_2(y)) = \models_{(E_2\times\mathcal{B})} (\psi_1(e_1), y)$, *for* $e_1 \in E_1$ *and* $y \in \mathcal{B}$.

(iii) *For each object* $P = (E, \mathcal{A}, \models_{(E\times\mathcal{A})})$, *the identity arrow* $I_P : P \longrightarrow P$ *is defined as* (I'_P, I''_P), *where* $I'_P : E \longrightarrow E$ *is an identity mapping;*
$I''_P : \mathcal{A} \longrightarrow \mathcal{A}$ *is an identity mapping that is* $\mathcal{L}$-***VL****-algebras homomorphism .*

(iv) *For the given objects* $P' = (E_1, \mathcal{A}, \models_{(E_1\times\mathcal{A})})$, $Q' = (E_2, \mathcal{B}, \models_{(E_2\times\mathcal{B})})$ *and* $R' = (E_3, \mathcal{C}, \models_{(E_3\times\mathcal{C})})$ *in* $\mathcal{L}$-***BSYM*** *, let us take two arrows* $(\psi_1, \psi_2) : P' \to Q'$ *and* $(\phi_1, \phi_2) : Q' \to R'$. *The composition of these two arrows is defined as* $(\phi_1, \phi_2) \circ (\psi_1, \psi_2) : P' \to R'$ *such that*

$$\phi_1 \circ \psi_1 : E_1 \longrightarrow E_3;$$
$$\psi_2 \circ \phi_2 : \mathcal{C} \longrightarrow \mathcal{A}.$$

Definition 2.2.18. *We now introduce the notion of extent in the category* $\mathcal{L}$-***BSYM***. *If* $P = (E, \mathcal{A}, \models_{(E\times\mathcal{A})})$ *is an object in* $\mathcal{L}$-***BSYM***, *then for each* x *in* $\mathcal{A}$, *its extent in* P *is a function* $ext_{\mathcal{L}}(x) : E \to \mathcal{L}$ *defined by* $ext_{\mathcal{L}}(x)(e) = \models_{(E\times\mathcal{A})} (e, x)$. *Thus,* $ext_{\mathcal{L}}(\mathcal{A}) = \{ext_{\mathcal{L}}(x) : x \in \mathcal{A}\}$. *On the set* $ext_{\mathcal{L}}(\mathcal{A})$, *the operations* $(\wedge, \vee, \rightarrow, T_L(L \in \mathcal{L}), 0, 1)$ *are defined pointwise. Therefore,* $ext_{\mathcal{L}} : \mathcal{A} \longrightarrow \mathcal{L}^E$ *is a* $\mathcal{L}$-***VL****-algebras homomorphism.*

Definition 2.2.19. *A continuous map* $\psi = (\psi_1, \psi_2) : E' = (E_1, \mathcal{A}, \models_{(E_1\times\mathcal{A})}) \longrightarrow E'' = (E_2, \mathcal{B}, \models_{(E_2\times\mathcal{B})})$ *is called a homeomorphism if and only if there exists an arrow* $\tilde{\psi} = (\psi'_1, \psi'_2) : E'' \longrightarrow E'$ *such that* $\tilde{\psi} \circ \psi = Id_{E'}$, *and* $\psi \circ \tilde{\psi} = Id_{E''}$.
E' *and* E'' *are said to be homeomorphic if there exists a homeomorphism between* E' *and* E''.

Remark 2.2.4. *If the systems* E' *and* E'' *are homeomorphic then the systems are structurally equivalent, i.e.,*

(a) *there exists a bijective mapping between* E_1 *and* E_2;

(b) $\mathcal{L}$-***VL****-algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *are isomorphic;*

(c) $\models_{(E_1\times\mathcal{A})} (e_1, \psi_2(b)) = \models_{(E_2\times\mathcal{B})} (\psi_1(e_1), b)$.

Theorem 2.2.1. *If* $(E, \mathcal{A}, \models_{(E\times\mathcal{A})})$ *is a* $\mathcal{L}$-***Boolean systems*** *then* $(E, ext_{\mathcal{L}}(\mathcal{A}))$ *is a lattice-valued Boolean space, where* $\mathcal{A}$ *is a* $\mathcal{L}$-***VL****-algebra.*

Proof. We shall show that $(E, ext_{\mathcal{L}}(\mathcal{A}))$ is compact, zero-dimensional and Hausdorff space.
First, we show that $(E, ext_{\mathcal{L}}(\mathcal{A}))$ is compact.
Let $1_E = \bigvee_{\lambda\in\Lambda} ext_{\mathcal{L}}(a_\lambda)$, where $a_\lambda \in \mathcal{A}$ and 1_E is a constant map on E whose value is always 1. Now, $1 = T_1 \circ 1_E = T_1 \circ \bigvee_{\lambda\in\Lambda} ext_{\mathcal{L}}(a_\lambda) = \bigvee_{\lambda\in\Lambda} T_1 \circ ext_{\mathcal{L}}(a_\lambda) = \bigvee_{\lambda\in\Lambda} ext_{\mathcal{L}}(T_1(a_\lambda))$.
Thus, $0_E = (\bigvee_{\lambda\in\Lambda} ext_{\mathcal{L}}(T_1(a_\lambda)))^{\perp} = \bigwedge_{\lambda\in\Lambda} ext_{\mathcal{L}}((T_1(a_\lambda))^{\perp})$. Therefore

$$0 = 0_E(x) = (\bigwedge_{\lambda\in\Lambda} ext_{\mathcal{L}}((T_1(a_\lambda))^{\perp}))(x)$$

Thus for a fixed $x \in E$, we have

$$0 = (\bigwedge_{\lambda\in\Lambda} ext_{\mathcal{L}}((T_1(a_\lambda))^{\perp}))(x) = \bigwedge_{\lambda\in\Lambda} ext_{\mathcal{L}}((T_1(a_\lambda))^{\perp})(x) = \bigwedge_{\lambda\in\Lambda} \models_{E\times\mathcal{A}} (x, (T_1(a_\lambda))^{\perp}). \tag{2.1}$$

Let there exist a $\mathcal{L}$-**VL**-algebras homomorphism $v : \mathcal{A} \to \mathcal{L}$ defined by $v((T_1(a_\lambda))^{\perp}) = 1$, for all $\lambda \in \Lambda$. Then

$$\models_{E\times\mathcal{A}} (x, (T_1(a_\lambda))^{\perp}) = 1 = v((T_1(a_\lambda))^{\perp}).$$

As a result, $\bigwedge_{\lambda\in\Lambda} \models_{E\times\mathcal{A}} (x, v((T_1(a_\lambda))^{\perp})) = 1$, which contradict 2.1. Thus, there does not exist a $\mathcal{L}$-**VL**-algebras homomorphism $v : \mathcal{A} \to \mathcal{L}$ such that $v((T_1(a_\lambda))^{\perp}) = 1$, for all $\lambda \in \Lambda$. Then by Proposition 2.2.1 there is no prime $\mathcal{L}$-filter of $\mathcal{A}$ which contains $\{(T_1(a_\lambda))^{\perp} : \lambda \in \Lambda\}$. So the collection $\{(T_1(a_\lambda))^{\perp} : \lambda \in \Lambda\}$ does not have finite intersection property with respect to $\wedge$ (meet) otherwise we get a contradiction. Therefore, there exists a finite collection $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of Λ such that

$$(T_1(a_{\lambda_1}))^{\perp} \wedge (T_1(a_{\lambda_2}))^{\perp} \wedge (T_1(a_{\lambda_3}))^{\perp} \wedge \cdots \wedge (T_1(a_{\lambda_n}))^{\perp} = 0$$

Thus,

$$T_1(a_{\lambda_1}) \vee T_1(a_{\lambda_2}) \vee T_1(a_{\lambda_3}) \vee \cdots \vee T_1(a_{\lambda_n}) = 1 \text{ i.e., } T_1(a_{\lambda_1} \vee a_{\lambda_2} \vee \cdots \vee a_{\lambda_n}) = 1$$

So, $a_{\lambda_1} \vee a_{\lambda_2} \vee \cdots \vee a_{\lambda_n} = 1$. Therefore, $ext_{\mathcal{L}}(a_{\lambda_1} \vee a_{\lambda_2} \vee \cdots \vee a_{\lambda_n}) = 1$.
Second, we show that $(E, ext_{\mathcal{L}}(\mathcal{A}))$ is zero-dimensional.
We shall show that $Cont(E, ext_{\mathcal{L}}(\mathcal{A}))$ forms a clopen basis of $(E, ext_{\mathcal{L}}(\mathcal{A}))$. It is

already defined that for $\mu_1, \mu_2 \in Cont(E, ext_{\mathcal{L}}(\mathcal{A}))$, $(\mu_1 \wedge \mu_2)(e) = \mu_1(e) \wedge \mu_2(e)$. Thus, in order to prove that $Cont(E, ext_{\mathcal{L}}(\mathcal{A}))$ forms a clopen basis of $(E, ext_{\mathcal{L}}(\mathcal{A}))$, it is sufficient to show that, for every $a \in \mathcal{A}$, $ext_{\mathcal{L}}(a) \in Cont(E, ext_{\mathcal{L}}(\mathcal{A}))$. Let $\mu : \mathcal{L} \to \mathcal{L}$ be an open continuous map. Now $ext_{\mathcal{L}}(a)^{-1}(\mu)(e) = \mu \circ ext_{\mathcal{L}}(a)(e) = \mu(ext_{\mathcal{L}}(a)(e)) = \bigvee_{L \in \mathcal{L}} T_{\mu(L)}(ext_{\mathcal{L}}(a)(e)) = \bigvee_{L \in \mathcal{L}} T_{\mu(L)}(\models (e, a)) = \bigvee_{L \in \mathcal{L}} (\models (e, T_{\mu(L)}(a)) = \bigvee_{L \in \mathcal{L}} ext_{\mathcal{L}}(T_{\mu(L)}(a))(e) = ext_{\mathcal{L}}(\bigvee_{L \in \mathcal{L}} T_{\mu(L)}(a))(e)$. Therefore, $ext_{\mathcal{L}}(a)^{-1}(\mu) = ext_{\mathcal{L}}(\bigvee_{L \in \mathcal{L}} T_{\mu(L)}(a)) \in ext_{\mathcal{L}}(\mathcal{A})$. Hence, $ext_{\mathcal{L}}(a) \in Cont(E, ext_{\mathcal{L}}(\mathcal{A}))$.

Lastly, we demonstrate that $(E, ext_{\mathcal{L}}(\mathcal{A}))$ is Hausdorff. Since $(E, ext_{\mathcal{L}}(\mathcal{A}))$ is zero-dimensional, it is sufficient to demonstrate that $(E, ext_{\mathcal{L}}(\mathcal{A}))$ is Kolmogorov.

Consider $e_1, e_2 \in E$ such that $e_1 \neq e_2$. Then $\exists a \in \mathcal{A}$ such that $\models_{E \times \mathcal{A}} (e_1, a) \neq \models_{E \times \mathcal{A}} (e_2, a)$. Consequently, $ext_{\mathcal{L}}(a)(e_1) \neq ext_{\mathcal{L}}(a)(e_2)$.

□

2.2.4 Functorial relationships

We shall now explore functorial relationships between the categories $\mathcal{L}$-**BSYM**, $\mathcal{L}$-**BS** and $\mathcal{VA}_{\mathcal{L}}$.

Definition 2.2.20. *We define a functor $Ext_{\mathcal{L}}$: $\mathcal{L}$-**BSYM**$\longrightarrow$ $\mathcal{L}$-**BS** as follows:*

*(i) $Ext_{\mathcal{L}}(S, \mathcal{A}_1, \models_{(S \times \mathcal{A}_1)}) = ((S, ext_{\mathcal{L}}(\mathcal{A}_1)), \beta)$, where $(S, \mathcal{A}_1, \models_{(S \times \mathcal{A}_1)})$ is an object in $\mathcal{L}$-**BSYM** and the mapping $\beta : Subalg(\mathcal{L}) \longrightarrow \Omega((S, ext_{\mathcal{L}}(\mathcal{A}_1))$ is defined by $\beta(K) = (S, ext_K(\mathcal{A}_1))$, where $K \in Subalg(\mathcal{L})$;*

*(ii) $Ext_{\mathcal{L}}(\phi_1, \phi_2) = \phi_1$, where $(\phi_1, \phi_2) : (S', \mathcal{A}_1, \models_{(S' \times \mathcal{A}_1)}) \longrightarrow (S'', \mathcal{A}_2, \models_{(S'' \times \mathcal{A}_2)})$ is an arrow in $\mathcal{L}$-**BSYM**, and $\phi_1 : ((S', ext_{\mathcal{L}}(\mathcal{A}_1)), \beta_1) \longrightarrow ((S'', ext_{\mathcal{L}}(\mathcal{A}_2)), \beta_2)$ is a $\mathcal{L}$-valued continuous map which preserves subspaces.*

The well-definedness of the functor $Ext_{\mathcal{L}}$ is shown by Theorem 2.2.1 and the following Proposition 2.2.2

Proposition 2.2.2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be an objects in $\mathcal{VA}_{\mathcal{L}}$ and $(\phi_1, \phi_2) : (S', \mathcal{A}_1, \models_{(S' \times \mathcal{A}_1)}) \longrightarrow (S'', \mathcal{A}_2, \models_{(S'' \times \mathcal{A}_2)})$ is an arrow in $\mathcal{L}$-**BSYM**. Then $Ext_{\mathcal{L}}(\phi_1, \phi_2)$ is an arrow in $\mathcal{L}$-**BS**.*

Proof. $\phi_1 : ((S', ext_{\mathcal{L}}(\mathcal{A}_1)), \beta_1) \longrightarrow ((S'', ext_{\mathcal{L}}(\mathcal{A}_2), \beta_2)$ is indeed a $\mathcal{L}$-valued continuous map, since $\phi_1^{-1}(ext_{\mathcal{L}}(y))(s_1) = ext_{\mathcal{L}}(y)\phi_1(s_1) =\models_{(S''\times\mathcal{A}_2)} (\phi_1(s_1), y) =\models_{(S'\times\mathcal{A}_1)} (s_1, \phi_2(y)) = ext_{\mathcal{L}}(\phi_2(y))(s_1)$, $y \in \mathcal{A}_2$ and $s_1 \in S'$. Therefore $\phi_1^{-1}(ext_{\mathcal{L}}(y)) = ext_{\mathcal{L}}(\phi_2(y)) \in ext_{\mathcal{L}}(\mathcal{A}_1)$. It is easy to follow that ϕ_1 is a subspace-preserving map. □

Lemma 2.2.1. *For an object $(\mathcal{R}, \beta)$ in $\mathcal{L}$-**BS**, $(\mathcal{R}, Cont(\mathcal{R}, \beta), \models)$ is an object in $\mathcal{L}$-**BSYM**.*

Proof. Define $\models (t, \psi) = \psi(t)$, where $\psi \in Cont(\mathcal{R}, \beta)$. Now we verify that $Cont(\mathcal{R}, \beta)$ is an object in $\mathcal{L}$-**BSYM**.

(i) For a collection $\{u_\lambda\}_{\lambda\in J}$ (J is an index set) of $Cont(R, \beta)$, we have $\models (t, \vee_{\lambda\in J} u_\lambda) = (\vee_{\lambda\in J} u_\lambda)(t) = \vee_{\lambda\in J} u_\lambda(t) = \vee_{\lambda\in J} \models (t, u_\lambda)$.
For any $\psi_1, \psi_2 \in Cont(\mathcal{R}, \beta)$, $\models (t, \psi_1 \wedge \psi_2) = (\psi_1 \wedge \psi_2)(t) = \psi_1(t) \wedge \psi_2(t) =\models (t, \psi_1) \wedge \models (t, \psi_2)$.

(ii) As $\mathcal{R}$ is a **$\mathcal{L}$-Boolean space** i.e., a zero-dimensional and Hausdorff space and hence Kolmogorov, we have for $t_1 \neq t_2$ in $\mathcal{R}$ there exists $\psi \in Cont(\mathcal{R}, \beta)$ for which $\psi(t_1) \neq \psi(t_2)$. So $\models (t_1, \psi) \neq \models (t_2, \psi)$.

(iii) $T_L(\models (t, \psi)) = T_L(\psi(t)) = T_L(\psi)(t) =\models (t, T_L(\psi))$, $L \in \mathcal{L}$.

(iv) $\models (t, \psi \to \psi') = (\psi \to \psi')(t) = \psi(t) \to \psi'(t)$.

□

Definition 2.2.21. *A functor $G : \mathcal{L}$-**BS** $\longrightarrow \mathcal{L}$-**BSYM** is defined as follows:*

*(a) $G(P, \beta) = (P, Cont(P, \beta), \models)$, where (P, β) is an object in $\mathcal{L}$-**BS**;*

*(b) For an arrow $\tilde{h} : (P_1, \beta_1) \longrightarrow (P_2, \beta_2)$ in $\mathcal{L}$-**BS**, $G(\tilde{h}) = (\tilde{h}, \tilde{h}^{-1}) : (P_1, Cont(P_1, \beta_1), \models_1) \longrightarrow (P_2, Cont(P_2, \beta_2), \models_2)$, where*

(i) $\tilde{h} : P_1 \longrightarrow P_2$, a set function;

*(ii) $\tilde{h}^{-1} : Cont(P_2, \beta_2) \longrightarrow Cont(P_1, \beta_1)$ is a $\mathcal{L}$-**VL**-algebras homomorphism, and which is defined by $\tilde{h}^{-1}(g) = g \circ \tilde{h}$, $g \in Cont(P_2, \beta_2)$.*

Proposition 2.2.3. *$G(\tilde{h}) = (\tilde{h}, \tilde{h}^{-1})$ is an arrow in $\mathcal{L}$-**BSYM**, whenever $\tilde{h}$ is an arrow in the category $\mathcal{L}$-**BS**.*

Proof. Here $\tilde{h} : P_1 \longrightarrow P_2$ is a set function and the mapping $\tilde{h}^{-1} : Cont(P_2, \beta_2) \longrightarrow Cont(P_1, \beta_1)$ is a $\mathcal{L}$-**VL**-algebras homomorphism defined by $\tilde{h}^{-1}(g) = g \circ \tilde{h}$. Now we observe that $\models_2 (\tilde{h}(p_1), g) = g(\tilde{h}(p_1)) = \tilde{h}^{-1}(g)(p_1) =\models_1 (p_1, \tilde{h}^{-1}(g))$. So $(\tilde{h}, \tilde{h}^{-1})$ is a $\mathcal{L}$-valued continuous map and hence an arrow in $\mathcal{L}$-**BSYM**. □

So the functor G is well-defined by Lemma 2.2.1 and Proposition 2.2.3.

Definition 2.2.22. *A functor* $H : \mathcal{L}\text{-}\boldsymbol{BSYM} \longrightarrow (\mathcal{VA}_{\mathcal{L}})^{op}$ *is defined as follows:*

(i) $H(S, \mathcal{B}, \models_{(S\times\mathcal{B})}) = \mathcal{B}$*, where* $(S, \mathcal{B}, \models_{(S\times\mathcal{B})})$ *is an object in* $\mathcal{L}$-***BSYM*** *and* $\mathcal{B}$ *is a* $\mathcal{L}$-***VL****-algebra;*

(ii) $H(\tilde{g}_1, \tilde{g}_2) = \tilde{g}_2^{op} : \mathcal{A} \longrightarrow \mathcal{B}$*, where* $(\tilde{g}_1, \tilde{g}_2) : (S_1, \mathcal{A}, \models_{(S_1\times\mathcal{A})}) \longrightarrow (S_2, \mathcal{B}, \models_{(S_2\times\mathcal{B})})$ *is an arrow in* $\mathcal{L}$-***BSYM*** *and* $\tilde{g}_2^{op}$ *is a* $\mathcal{L}$-***VL****-algebras homomorphism in* $(\mathcal{VA}_{\mathcal{L}})^{op}$

It is easy to follow that the functor H is well-defined.

Definition 2.2.23. *A functor* $R : (\mathcal{VA}_{\mathcal{L}})^{op} \longrightarrow \mathcal{L}\text{-}\boldsymbol{BSYM}$ *is defined as follows:*

(i) $R(\mathcal{A}) = (Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{A}, \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})})$*, where* $\mathcal{A}$ *is an object in* $(\mathcal{VA}_{\mathcal{L}})^{op}$*;*

(ii) $R(f) = (f^{-1}, f^{op})$*, where* $f : \mathcal{A} \longrightarrow \mathcal{B}$ *is an arrow in* $(\mathcal{VA}_{\mathcal{L}})^{op}$*.*

The well-definedness of the functor R is shown by Proposition 2.2.4 and Proposition 2.2.5.

Proposition 2.2.4. *Let* $\mathcal{A}$ *be a* $\mathcal{L}$-***VL****-algebra. Then* $(Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{A}, \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})})$ *is an object in the category* $\mathcal{L}$-***BSYM****.*

Proof. Here $Spec_{\mathcal{L}}(\mathcal{A})$ is a set. For some member $\tilde{s}$ of $Spec_{\mathcal{L}}(\mathcal{A})$, we define $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, b) = \tilde{s}(b)$. Now we verify the following:

(i) For a collection $\{b_j\}_{j\in J}$ of elements of $\mathcal{A}$, where J is an index set, we have $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, \bigvee_{j\in J} b_j) = \tilde{s}(\bigvee_{j\in J} b_j) = \bigvee_{j\in J} \tilde{s}(b_j) = \bigvee_{j\in J} \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, b_j)$.
For any two elements $b_1, b_2 \in \mathcal{A}$, we get $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, b_1 \wedge b_2) = \tilde{s}(b_1 \wedge b_2) = \tilde{s}(b_1) \wedge \tilde{s}(b_2) =\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, b_1)\wedge \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, b_2)$.

(ii) For $p \in \mathcal{L}$, $T_p(\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, b)) = T_p(\tilde{s}(b)) = \tilde{s}(T_p(b))$, and $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, T_p(b)) = \tilde{s}(T_p(b))$. Therefore, $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, T_p(b)) = T_p(\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\tilde{s}, b))$.
Others properties can be verified easily.

□

Proposition 2.2.5. *For an arrow $f : \mathcal{A} \longrightarrow \mathcal{B}$ in $(\mathcal{VA_L})^{op}$, $R(f)$ is an arrow in* $\mathcal{L}$-***BSYM***.

Proof. Recall that $R(f) = (f^{-1}, f^{op})$, where f^{op} is a $\mathcal{L}$-**VL**-algebras homomorphism and f^{-1} is a mapping from $Spec_{\mathcal{L}}(\mathcal{A})$ to $Spec_{\mathcal{L}}(\mathcal{B})$ defined by $f^{-1}(v) = v \circ f^{op}$, where $v \in Spec_{\mathcal{L}}(\mathcal{A})$. Now we see that $\models_{(Spec_{\mathcal{L}}(\mathcal{B})\times\mathcal{B})} (f^{-1}(v), b) = f^{-1}(v)(b) = v \circ f^{op}(b) = v(f^{op}(b)) = \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (v, f^{op}(b))$. It shows that $R(f)$ is an arrow in $\mathcal{L}$-**BSYM**. □

Theorem 2.2.2. *$Ext_{\mathcal{L}}$ is a co-adjoint to the functor G.*

Proof. We prove the theorem by presenting the co-unit of the adjunction. Figure 2.1 illustrates the counit.

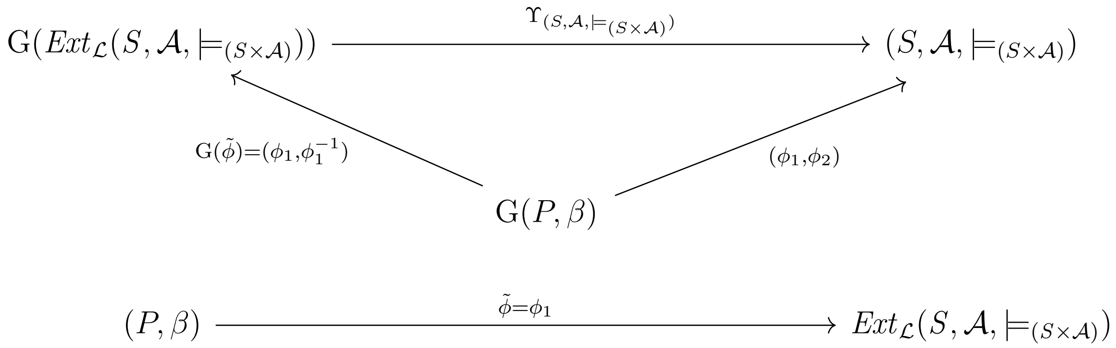

Figure 2.1: Illustration of the counit

Recall that $G(P, \beta) = (P, Cont(P, \beta), \models)$ and $Ext_{\mathcal{L}}(S, \mathcal{A}, \models_{(S\times\mathcal{A})}) = (S, ext_{\mathcal{L}}(\mathcal{A}), \beta)$. Hence $G(Ext_{\mathcal{L}}(S, \mathcal{A}, \models_{(S\times\mathcal{A})})) = G((S, ext_{\mathcal{L}}(\mathcal{A})), \beta) = (S, Cont((S, ext_{\mathcal{L}}(\mathcal{A})), \beta), \models)$. Here Counit is taken by Υ and defined by $\Upsilon_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (ID_S, ext_{\mathcal{L}}) : G(Ext_{\mathcal{L}}(S, \mathcal{A}, \models_{(S\times\mathcal{A})})) \longrightarrow (S, \mathcal{A}, \models_{(S\times\mathcal{A})})$, where

(i) $ID_S : S \longrightarrow S$;

(ii) $ext_{\mathcal{L}} : \mathcal{A} \longrightarrow Cont((S, ext_{\mathcal{L}}(\mathcal{A})), \beta)$.

Let $\sigma : \mathcal{L} \longrightarrow \mathcal{L}$ be an open continuous map in $(\mathcal{L}, \beta_{\mathcal{L}})$. We show that $ext_{\mathcal{L}}(a) \in Cont((S, ext_{\mathcal{L}}(\mathcal{A})), \beta)$. Now $ext_{\mathcal{L}}(a)^{-1}(\sigma)(s) = \sigma \circ ext_{\mathcal{L}}(a)(s) = \sigma(ext_{\mathcal{L}}(a)(s)) = \bigvee_{L\in\mathcal{L}} T_{\sigma(L)}(ext_{\mathcal{L}}(a)(s)) = \bigvee_{L\in\mathcal{L}} T_{\sigma(L)}(\models (s, a)) = \bigvee_{L\in\mathcal{L}} (\models (s, T_{\sigma(L)}(a)) =$

$\bigvee_{L\in\mathcal{L}} ext_{\mathcal{L}}(T_{\sigma(L)}(a))(s) = ext_{\mathcal{L}}(\bigvee_{L\in\mathcal{L}} T_{\sigma(L)}(a))(s)$. Therefore $ext_{\mathcal{L}}(a)^{-1}(\sigma) = ext_{\mathcal{L}}(\bigvee_{L\in\mathcal{L}} T_{\sigma(L)}(a)) \in ext_{\mathcal{L}}(\mathcal{A})$. Also $ext_{\mathcal{L}}(a)$ is a subspace preserving mapping. Now we claim that $(ID_S, ext_{\mathcal{L}})$ is a continuous map in $\mathcal{L}$-**BSYM**. To establish the claim, it is necessary to observe that $\models (s, ext_{\mathcal{L}}(a)) = ext_{\mathcal{L}}(a)(s) = \models_{(S\times\mathcal{A})} (s,a) = \models_{(S\times\mathcal{A})} (ID_S(s), a)$. For a given arrow $(\phi_1,\phi_2) : \mathrm{G}(P,\beta) \longrightarrow (S,\mathcal{A},\models_{(S\times\mathcal{A})})$ in $\mathcal{L}$-**BSYM** there exists an arrow, which we define $\tilde{\phi} = \phi_1 : (P,\beta) \longrightarrow Ext_{\mathcal{L}}(S,\mathcal{A},\models_{(S\times\mathcal{A})})$ in $\mathcal{L}$-**BS**. We now show that the triangle of Figure 2.1 commutes i.e., $(\phi_1,\phi_2) = (ID_S, ext_{\mathcal{L}}) \circ (\phi_1, \phi_1^{-1})$. We see that $ID_S \circ \phi_1 = \phi_1$. Now we have to prove that $\phi_1^{-1} \circ ext_{\mathcal{L}} = \phi_2$.
As $(ID_S, ext_{\mathcal{L}})$ is continuous, we get $ext_{\mathcal{L}}(a) = a$. Now, for each $a \in \mathcal{A}$, $\phi_1^{-1} \circ (ext_{\mathcal{L}}(a)) = \phi_1^{-1}(a)$. Since (ϕ_1,ϕ_2) is continuous, we have for any $p \in P$, and $a \in \mathcal{A}$, $\models_{(S\times\mathcal{A})} (\phi_1(p), a) = \models (p, \phi_2(a))$ i.e., $\phi_1^{-1}(a) = \phi_2(a)$. Henceforth, we get $\phi_1^{-1} \circ (ext_{\mathcal{L}}(a)) = \phi_2(a)$. Therefore, $\phi_1^{-1} \circ ext_{\mathcal{L}} = \phi_2$. Hence, $\Upsilon_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (ID_S, ext_{\mathcal{L}})$ is the counit and as a result $Ext_{\mathcal{L}}$ is a co-adjoint to the functor G.

□

Also, G is an adjoint to the functor $Ext_{\mathcal{L}}$. Unit of the adjunction is shown in Figure 2.2.

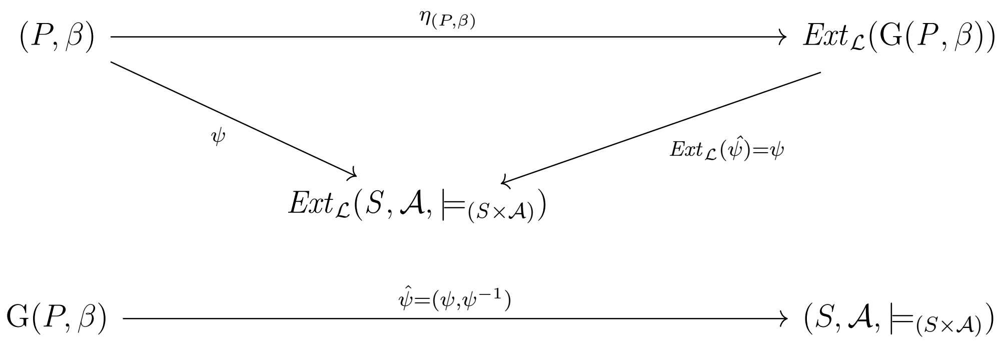

Figure 2.2: Illustration of the unit

Theorem 2.2.3. *H is an adjoint to the functor R.*

Proof. We prove the theorem by presenting unit of the adjunction. Figure 2.3 illustrates the unit.

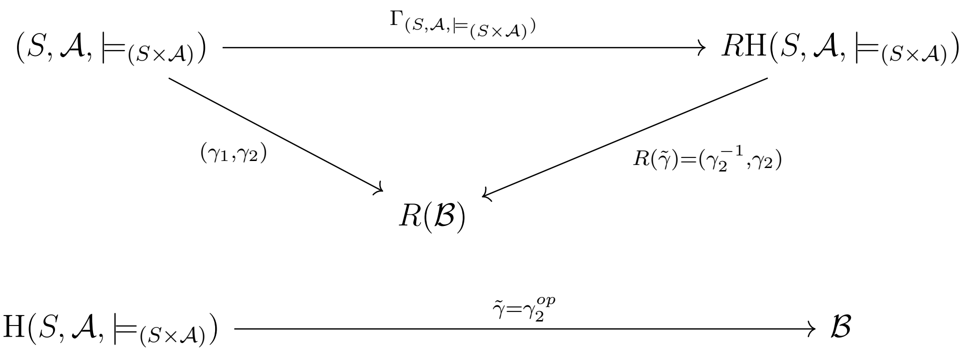

Figure 2.3: Illustration of the unit

We recall that $R(\mathcal{B}) = (Spec_{\mathcal{L}}(\mathcal{B}), \mathcal{B}, \models_{(Spec_{\mathcal{L}}(\mathcal{B})\times\mathcal{B})})$, where $\models_{(Spec_{\mathcal{L}}(\mathcal{B})\times\mathcal{B})} (\varphi, b) = \varphi(b)$. So, $R\mathrm{H}(S, \mathcal{A}, \models_{(S\times\mathcal{A})}) = R(\mathcal{A}) = (Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{A}, \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})})$. Unit is taken by Γ and defined as $\Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (\gamma, ID_{\mathcal{A}}) : (S, \mathcal{A}, \models_{(S\times\mathcal{A})}) \longrightarrow R\mathrm{H}(S, \mathcal{A}, \models_{(S\times\mathcal{A})})$, where the mapping $\gamma : S \longrightarrow Spec_{\mathcal{L}}(\mathcal{A})$ is defined by $\gamma(s) = \gamma_s$, for each $s \in S$, $\gamma_s : \mathcal{A} \longrightarrow \mathcal{L}$ is defined by $\gamma_s(a) =\models_{(S\times\mathcal{A})} (s, a)$. We claim that for each $s \in S$, γ_s is a $\mathcal{L}$-**VL**-algebras homomorphism. Now for any $a, b \in \mathcal{A}$, $\gamma_s(a \vee b) =\models_{(S\times\mathcal{A})} (s, a \vee b) =\models_{(S\times\mathcal{A})} (s, a)\vee \models_{(S\times\mathcal{A})} (s, b) = \gamma_s(a) \vee \gamma_s(b)$. Also $\gamma_s(a \wedge b) = \gamma_s(a) \wedge \gamma_s(b)$ and $\gamma_s(a \rightarrow b) = \gamma_s(a) \rightarrow \gamma_s(b)$. We observe that $\gamma_s(T_L(a)) =\models_{(S\times\mathcal{A})} (s, T_L(a)) = T_L(\models_{(S\times\mathcal{A})} (s, a)) = T_L(\gamma_s(a))$, where $L \in \mathcal{L}$. Therefore γ_s is a $\mathcal{L}$-**VL**-algebras homomorphism. The unit $\Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (\gamma, ID_{\mathcal{A}})$ is a continuous map in $\mathcal{L}$-**BSYM**, since $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\gamma(s), a) = \gamma_s(a) =\models_{(S\times\mathcal{A})} (s, a) =\models_{(S\times\mathcal{A})} (s, ID_{\mathcal{A}}(a))$. For a given arrow $(\gamma_1, \gamma_2) : (S, \mathcal{A}, \models_{(S\times\mathcal{A})}) \longrightarrow R(\mathcal{B})$, we define $\tilde{\gamma} = (\gamma_2)^{op}$ in $(\mathcal{L}\text{-VA})^{op}$. Now we show that the triangle of Figure 2.3 commutes i.e., $(\gamma_1, \gamma_2) = R(\tilde{\gamma}) \circ \Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (\gamma_2^{-1}, \gamma_2) \circ (\gamma, ID_{\mathcal{A}}) = (\gamma_2^{-1} \circ \gamma, ID_{\mathcal{A}} \circ \gamma_2)$. It clearly shows that $ID_{\mathcal{A}} \circ \gamma_2 = \gamma_2$. Now we are to show that $\gamma_1 = \gamma_2^{-1} \circ \gamma$. For each $s \in S$, $\gamma_1(s) = \gamma_2^{-1} \circ \gamma(s) = \gamma_2^{-1} \circ \gamma_s = \gamma_s \circ \gamma_2$, and for all $b \in \mathcal{B}$, $(\gamma_s \circ \gamma_2)(b) = \gamma_s(\gamma_2(b)) =\models_{(S\times\mathcal{A})} (s, \gamma_2(b)) =\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\gamma_1(s), b) = \gamma_1(s)(b)$. Therefore $\gamma_s \circ \gamma_2 = \gamma_1(s)$. Hence $\gamma_2^{-1} \circ \gamma = \gamma_1$, and as a result $\Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is the unit. Therefore H is an adjoint to the functor R.

□

Theorem 2.2.4. *R is a co-adjoint to the functor H*

Proof. It is also possible to prove the theorem by counit of the adjunction. The counit is illustrated in Figure 2.4.

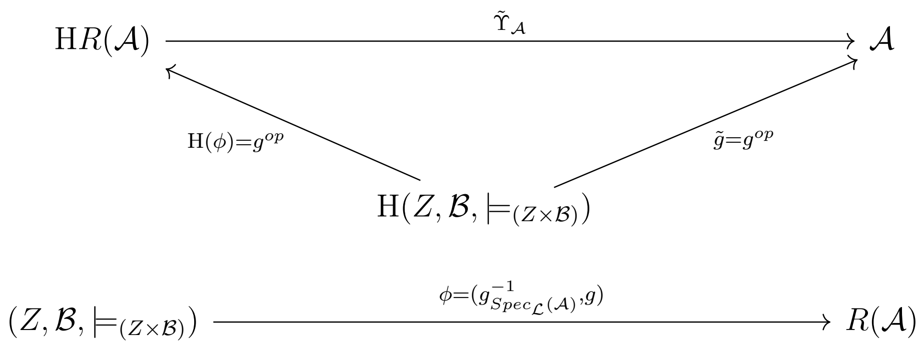

Figure 2.4: Illustration of the counit

The counit is taken by $\tilde{\Upsilon}$ and defined as $\tilde{\Upsilon}_{\mathcal{A}} = ID_{\mathcal{A}}$. For a given arrow $\tilde{g} = g^{op}$ in $(\mathcal{VA}_{\mathcal{L}})^{op}$, we define $\phi = (g^{-1}_{Spec_{\mathcal{L}}(\mathcal{A})}, g)$ in $\mathcal{L}$-**BSYM**, where $g^{-1}_{Spec_{\mathcal{L}}(\mathcal{A})} : Z \longrightarrow Spec_{\mathcal{L}}(\mathcal{A})$ and $g : \mathcal{A} \longrightarrow \mathcal{B}$. Now we define the map $g^{-1}_{Spec_{\mathcal{L}}(\mathcal{A})}$. For each $z \in Z$, $\zeta_z : \mathcal{A} \longrightarrow \mathcal{L}$ is defined by $\zeta_z(a) =\models_{(Z\times\mathcal{A})} (z, a)$ and henceforth $g^{-1}_{Spec_{\mathcal{L}}(\mathcal{A})}(\zeta_z) = \zeta_z \circ g^{op}$, where $\zeta_z \in Spec_{\mathcal{L}}(\mathcal{A})$, and g^{op} is a $\mathcal{L}$-**VL**-algebras homomorphism from $\mathcal{B}$ to $\mathcal{A}$ in $(\mathcal{VA}_{\mathcal{L}})^{op}$. Thus, it is simple to show that the triangle of Figure 2.4 commutes i.e., $\tilde{\Upsilon}_{\mathcal{A}} \circ \mathrm{H}(\phi) = \tilde{g}$.

□

Theorem 2.2.5. *The categories $\mathcal{L}$-**BS** and $\mathcal{L}$-**BSYM** are equivalent.*

Proof. We choose two identity functors $ID_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ and $ID_{(P,\beta)}$ on $\mathcal{L}$-**BSYM** and $\mathcal{L}$-**BS**, respectively. We get two natural transformations Υ and η such that $\Upsilon : \mathrm{G} \circ Ext_{\mathcal{L}} \longrightarrow ID_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ and $\eta : ID_{(P,\beta)} \longrightarrow Ext_{\mathcal{L}} \circ \mathrm{G}$. We show that $\Upsilon_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} : \mathrm{G}(Ext_{\mathcal{L}}(S, \mathcal{A}, \models_{(S\times\mathcal{A})})) \longrightarrow (S, \mathcal{A}, \models_{(S\times\mathcal{A})})$ is a natural isomorphism. We recall that $\mathrm{G}(Ext_{\mathcal{L}}(S, \mathcal{A}, \models_{(S\times\mathcal{A})})) = (S, Cont((S, ext_{\mathcal{L}}(\mathcal{A})), \beta), \models)$ and $\Upsilon_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (ID_S, ext_{\mathcal{L}})$. We show that $\Upsilon_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is a homeomorphism.
Now, $ext_{\mathcal{L}} : \mathcal{A} \longrightarrow Cont((S, ext_{\mathcal{L}}(\mathcal{A})), \beta)$ is a $\mathcal{L}$-**VL**-algebras homomorphism. The mapping ID_S is definitely both injective and surjective. The only part we have to show is that $ext_{\mathcal{L}}$ is an isomorphism. Let $p_1, p_2 \in \mathcal{A}$, and $p_1 \neq p_2$. We show that $ext_{\mathcal{L}}(p_1) \neq ext_{\mathcal{L}}(p_2)$. Suppose $ext_{\mathcal{L}}(p_1) = ext_{\mathcal{L}}(p_2)$. As $p_1 \neq p_2$, then $p_1 \leftrightarrow p_2 \neq 1$. So by Definition 2.2.2, we have $\bigwedge_{L\in\mathcal{L}} (T_L(p_1) \leftrightarrow T_L(p_2)) \neq 1$, and then there exists $k \in \mathcal{L}$ such that $T_k(p_1) \neq T_k(p_2)$. Now for each $s \in S$, $ext_{\mathcal{L}}(p_1)(s) = ext_{\mathcal{L}}(p_2)(s)$, and hence $\models_{(S\times\mathcal{A})} (s, p_1) =\models_{(S\times\mathcal{A})} (s, p_2)$. So $T_k(\models_{(S\times\mathcal{A})} (s, p_1)) = T_k(\models_{(S\times\mathcal{A})} (s, p_2))$, and by Definition 2.2.16 we have $\models_{(S\times\mathcal{A})} (s, T_k(p_1)) =\models_{(S\times\mathcal{A})} (s, T_k(p_2))$. It shows that $T_k(p_1) = T_k(p_2)$, and which contradicts the assumption that $T_k(p_1) \neq T_k(p_2)$.

Therefore $ext_{\mathcal{L}}(p_1) \neq ext_{\mathcal{L}}(p_2)$, and hence $ext_{\mathcal{L}}$ is injective. Clearly $ext_{\mathcal{L}}$ is surjective. Finally $\Upsilon_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ satisfies the continuity condition, since $\models (s, ext_{\mathcal{L}}(a)) = ext_{\mathcal{L}}(a)(s) = \models_{(S\times\mathcal{A})} (s,a) = \models_{(S\times\mathcal{A})} (ID_S(s),a)$. Therefore $\Upsilon_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is an isomorphism. As a result Υ is a natural isomorphism.
Now we shall show that η is a natural isomorphism.
We recall that $Ext_{\mathcal{L}}(\mathrm{G}(P,\beta)) = (P, ext_{\mathcal{L}}(Cont(P,\beta)), \beta')$, where the function $\beta' : Subalg(\mathcal{L}) \longrightarrow \Omega(P, ext_{\mathcal{L}}(Cont(P,\beta))$ is defined by $\beta'(\mathcal{M}) = (P, ext_{\mathcal{M}}(Cont(P,\beta)))$ for $\mathcal{M} \in Subalg(\mathcal{L})$. Define $\eta_{(P,\beta)} : (P,\beta) \longrightarrow Ext_{\mathcal{L}} \circ \mathrm{G}(P,\beta)$ by $\eta_{(P,\beta)}(p)(\psi) = \psi(p)$, where $p \in P$ and $\psi \in Cont(P,\beta)$. It is easy to verify that η is a homeomorphism. Also, η satisfies the naturality condition. Hence, η is a natural isomorphism.

□

Theorem 2.2.6. $(\mathcal{VA}_{\mathcal{L}})^{op}$ *is equivalent to* $\mathcal{L}$-***BSYM****.*

Proof. We have two natural transformations Γ and $\tilde{\Upsilon}$ such that $\Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} : (S,\mathcal{A},\models_{(S\times\mathcal{A})}) \longrightarrow (Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{A}, \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})})$ and $\tilde{\Upsilon}_{\mathcal{A}} = ID : \mathrm{H}R(\mathcal{A}) \longrightarrow \mathcal{A}$. It is clear that $\tilde{\Upsilon}$ is a natural isomorphism. We show that $\Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is a natural isomorphism between objects in $\mathcal{L}$-**BSYM**. We define $\Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (\gamma, ID_{\mathcal{A}})$ such that

- $\gamma : S \longrightarrow Spec_{\mathcal{L}}(\mathcal{A})$ is a mapping between sets;
- $ID_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ is a $\mathcal{L}$-**VL**-algebras homomorphism.

We have to show that $\Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is a homeomorphism. First, we show that γ is bijective. Claim: γ is injective and surjective. Let $s_1 \neq s_2$ in S. Then by Definition 2.2.16 we have $\models_{(S\times\mathcal{A})} (s_1,a) \neq \models_{(S\times\mathcal{A})} (s_2,a)$, for some $a \in \mathcal{A}$. Therefore $\gamma(s_1)(a) \neq \gamma(s_2)(a)$, for some $a \in \mathcal{A}$. As a result γ is injective. The mapping γ is already defined in the proof of Theorem 2.2.3, and we can say that γ is also surjective. Hence our claim is now established.
Finally, we observe that $\gamma(s)(a) = \gamma_s(a) = \models_{(S\times\mathcal{A})} (s,a)$ and $\gamma_s(a) = \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\gamma(s),a)$. Therefore $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (\gamma(s),a) = \models_{(S\times\mathcal{A})} (s, ID_{\mathcal{A}}(a))$. Hence $\Gamma_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is an isomorphism and therefore $\mathcal{VA}_{\mathcal{L}}$ is dually equivalent to $\mathcal{L}$-**BSYM**. □

Ultimately, we arrive at the following outcome:

Theorem 2.2.7. $(\mathcal{VA}_{\mathcal{L}})^{op}$ *is equivalent to* $\mathcal{L}$-***BS****.*

Proof. As adjunctions can be composed, hence the composition of equivalences of Theorems 2.2.5 and 2.2.6 shows the result. □

Remark 2.2.5. *The duality discovered in [21] is also shown in Theorem 2.2.7; however, our methodology is not the same as that of [21].*

2.3 Conclusion

In this chapter, we have introduced the idea of lattice-valued Boolean systems, which are represented by the notation $\mathcal{L}$-**Boolean systems**, where $\mathcal{L}$ is a finite distributive lattice. In this context, the concept of lattice-valued topological systems gives rise to lattice-valued Boolean systems, which are useful for proving duality between algebras of Heyting-valued logic and systems. We have considered algebras of Fitting's style many-valued logic. A thorough analysis of the categorical relationships among $\mathcal{L}$-**BSYM**, $\mathcal{L}$-**BS**, and $\mathcal{VA}_{\mathcal{L}}$ has been accomplished. We have created a duality for Fitting's multi-valued logic in light of Vickers' work on the "logic of finite observations" [98], as well as the work of Denniston et al. [17].

Chapter 3

Category of $\mathcal{L}$-relational systems

3.1 Introduction

Maruyama [20] defined $\mathcal{L}$-**ML**-algebras as an algebraic structure of Fitting's $\mathcal{L}$-valued modal logic for a finite distributive lattice $\mathcal{L}$. Subsequently, in [21], a duality for the algebras of Fitting's $\mathcal{L}$-valued modal logic was found, which generalizes Jónsson-Tarski duality for modal algebras (e.g., [35, 15, 54, 29]). This chapter introduces the concept of $\mathcal{L}$-**relational systems**, building upon the idea of $\mathcal{L}$-**Boolean systems** (see Chapter 2) to establish a duality between systems and algebras for Fitting's $\mathcal{L}$-valued modal logic. Furthermore, it will be demonstrated that the category of $\mathcal{L}$-relational systems is equivalent to the category $\mathcal{L}$-**RS** of $\mathcal{L}$-**relational spaces**. This leads to the demonstration of the duality between the category $\mathcal{L}$-**RS** and the category of $\mathcal{L}$-**ML**-algebras. This outcome provides an alternative demonstration of the duality established in [21].

3.2 $\mathcal{L}$-ML-algebras, $\mathcal{L}$-relational systems, $\mathcal{L}$-relational spaces and their Categorical interconnections

Throughout this section $\mathcal{L}$ denotes a finite distributive lattice. Thus $\mathcal{L}$ is a finite Heyting algebra.

The outcomes of this chapter can be found in [57] **Ray, Kumar Sankar., Das, Litan Kumar.: Categorical study for Algebras of lattice-valued logic and lattice-valued modal logic. Annals of Mathematics and Artificial Intelligence, Springer, 89, 409-429 (2021).**

3.2.1 $\mathcal{L}$-ML-algebras

$\mathcal{L}$-**ML** denotes the $\mathcal{L}$-valued modal logic, which is defined by $\mathcal{L}$-valued Kripke semantics. The set of all formulas of $\mathcal{L}$-valued modal logic is denoted by $FORM_{\Box}$. We now introduce the notion of $\mathcal{L}$-valued Kripke model from [20].

Definition 3.2.1 ([20])**.** *Let $(Z, \mathcal{W})$ be a Kripke frame. Then q is a Kripke $\mathcal{L}$-valuation on $(Z, \mathcal{W})$ iff $q : Z \times FORM_{\Box} \longrightarrow \mathcal{L}$ is a function such that for any $z \in Z$ and $x \in FORM_{\Box}$ satisfies the following conditions:*

(i) $q(z, \Box x) = \bigwedge\{q(z', x) : z\mathcal{W}z'\}$;

(ii) $q(z, T_a(x)) = T_a(q(z, x))$;

(iii) $q(z, x \vee y) = q(z, x) \vee q(z, y)$, $q(z, x \wedge y) = q(z, x) \wedge q(z, y)$, $q(z, x \to y) = q(z, x) \to q(z, y)$;

(iv) $q(z, t) = t$ *where* $t = 0, 1$.

Then $(Z, \mathcal{W}, q)$ is called a $\mathcal{L}$-valued Kripke model.

We now recall the notion of $\mathcal{L}$-**ML**-algebras, which provides a sound and complete algebraic semantics for $\mathcal{L}$-valued modal logic $\mathcal{L}$-**ML**.

Definition 3.2.2 ([20])**.** *An algebraic system $(\mathcal{A}, \wedge, \vee, \to, T_L(L \in \mathcal{L}), \Box, 0, 1)$ is said to be a $\mathcal{L}$-**ML**-algebra iff it satisfies the following conditions:*

(i) $(\mathcal{A}, \wedge, \vee, \to, T_L(L \in \mathcal{L}), 0, 1)$ *is a $\mathcal{L}$-**VL**-algebra;*

(ii) $\Box(a_1 \wedge a_2) = \Box a_1 \wedge \Box a_2$ *and* $\Box 1 = 1$;

(iii) for all $L \in \mathcal{L}$, $U_L(\Box a) = \Box U_L(a)$, where $U_L(a) = \bigvee\{T_{L_1}(a) | L \leq L_1\}$.

Definition 3.2.3 ([21])**.** *A $\mathcal{L}$-**ML**-algebras homomorphism is a homomorphism of $\mathcal{L}$-**VL**-algebras which also preserves the unary operation $\Box$.*

Definition 3.2.4 ([21])**.** *Let $\mathcal{A}$ be a $\mathcal{L}$-**ML**-algebra. A binary relation $\mathcal{W}$ on $Spec_{\mathcal{L}}(\mathcal{A})$ is defined as follows:*
$f\mathcal{W}g \Leftrightarrow \forall L \in \mathcal{L}, \forall a \in \mathcal{A}, f(\Box a) \geq L \Rightarrow g(a) \geq L$. Then $(Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{W}, q)$ is a $\mathcal{L}$-valued canonical model of $\mathcal{A}$, where q is a Kripke $\mathcal{L}$-valuation on $(Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{W})$ defined as $q(f, a) = f(a)$, $\forall f \in Spec_{\mathcal{L}}(\mathcal{A})$.

Proposition 3.2.1 ([21])**.** *The $\mathcal{L}$-valued canonical model $(Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{W}, q)$ of $\mathcal{A}$ is a $\mathcal{L}$-valued Kripke model. In other words, $q(f, \Box a) = f(\Box a) = \bigwedge\{g(a) | f\mathcal{W}g\}$.*

Proposition 3.2.2 ([21])**.** *The Boolean algebra $\mathfrak{B}(\mathcal{A})$ is a modal algebra, whenever $\mathcal{A}$ is a $\mathcal{L}$-**ML**-algebra.*

The category $\mathcal{MA}_{\mathcal{L}}$

The category $\mathcal{MA}_{\mathcal{L}}$ of $\mathcal{L}$-**ML**-algebras is defined as follows.

Definition 3.2.5 ([21]). *$\mathcal{L}$-**ML**-algebras together with $\mathcal{L}$-**ML**-algebras homomorphisms form the category $\mathcal{MA}_{\mathcal{L}}$.*

$(\mathcal{MA}_{\mathcal{L}})^{op}$ is the opposite category of the category $\mathcal{MA}_{\mathcal{L}}$.

3.2.2 $\mathcal{L}$-relational spaces

The fundamental structure of $\mathcal{L}$-**relational spaces** is a $\mathcal{L}$-**Boolean spaces** with a relation defined on it that satisfies specific axioms.

The category $\mathcal{L}$-RS

First, let us review the definition below.

Definition 3.2.6 ([21]). *Let $(Z, \mathcal{W})$ be a Kripke frame and $\psi \in \mathcal{L}^Z$. Then a unary operation $\Box_{\mathcal{W}}$ on $\mathcal{L}^Z$ is defined as follows:*
$\Box_{\mathcal{W}}\psi : Z \longrightarrow \mathcal{L}$ is defined by $(\Box_{\mathcal{W}}\psi)(z) = \bigwedge\{\psi(z') : z\mathcal{W}z'\}$

Let $(Z, \mathcal{W})$ be a Kripke frame. Then for $z \in Z$, $\mathcal{W}[z] = \{z' \in Z : z\mathcal{W}z'\}$. For a subset $X \subset Z$, $\mathcal{W}^{-1}[X] = \{z \in Z : \exists z' \in Xz\mathcal{W}z'\}$.

Definition 3.2.7. *The category $\mathcal{L}$-**RS** is defined as follows:*

*(a) Objects: An object in $\mathcal{L}$-**RS** is defined by $(Z, \beta, \mathcal{W})$, where (Z, β) is an object in $\mathcal{L}$-**BS** and $\mathcal{W}$ is a binary relation on Z which has the following properities:*

  (i) if $\forall f \in Cont(Z, \beta)$, $(\Box_{\mathcal{W}}f)(z) = 1 \Rightarrow f(z') = 1$ then $(z, z') \in \mathcal{W}$;

  (ii) if Z' is a clopen subset of Z then $\mathcal{W}^{-1}[Z']$ is a clopen subset of Z;

  (iii) Let $\mathcal{L}' \in Subalg(\mathcal{L})$. If $z \in \beta(\mathcal{L}')$ then $\mathcal{W}[z] \subset \beta(\mathcal{L}')$;

*(b) Arrows: An arrow $f : ((Z_1, \beta_1), \mathcal{W}_1) \longrightarrow ((Z_2, \beta_2), \mathcal{W}_2)$ in $\mathcal{L}$-**RS** is an arrow $f : (Z_1, \beta_1) \longrightarrow (Z_2, \beta_2)$ in $\mathcal{L}$-**BS** which has the following properties;*

  (i) if $z\mathcal{W}_1t$ then $f(z)\mathcal{W}_2f(t)$;

  (ii) if $f(z_1)\mathcal{W}_2z_2$ then there exists a $t_1 \in Z_1$ such that $z_1\mathcal{W}_1t_1$ and $f(t_1) = z_2$.

3.2.3 $\mathcal{L}$-relational systems

We now introduce the notion of $\mathcal{L}$-relational systems.

Definition 3.2.8. *A $\mathcal{L}$-relational systems is a triple $(Z, \mathcal{A}, \models_{(Z\times\mathcal{A})})$ where Z is a nonempty set, $\mathcal{A}$ is a $\mathcal{L}$-**ML**-algebra and $\models_{(Z\times\mathcal{A})}$ is a $\mathcal{L}$-valued satisfaction relation from Z to $\mathcal{A}$ such that the following hold:*

(i) $\models_{(Z\times\mathcal{A})}(z, \bigvee_{r\in J} a_r) = \bigvee_{r\in J} \models_{(Z\times\mathcal{A})}(z, a_r)$, *$J$ is an index set;*

$\models_{(Z\times\mathcal{A})}(z, \bigwedge_{\lambda\in J} a_\lambda) = \bigwedge_{\lambda\in J} \models_{(Z\times\mathcal{A})}(z, a_\lambda)$;

(ii) $\models_{(Z\times\mathcal{A})}(z, \Box a) = \bigwedge\{\models_{(Z\times\mathcal{A})}(z', a) | z\mathcal{W}_0 z'\}$, *where $\mathcal{W}_0$ is a binary relation on Z;*

(iii) $\models_{(Z\times\mathcal{A})}(z, T_L(a)) = T_L(\models(z, a))$;

(iv) $\models_{(Z\times\mathcal{A})}(z, 0) = 0$, $\models_{(Z\times\mathcal{A})}(z, 1) = 1$;

(v) $\models_{(Z\times\mathcal{A})}(z, a \to b) = \models_{(Z\times\mathcal{A})}(z, a) \to \models_{(Z\times\mathcal{A})}(z, b)$.

We construct a category $\mathcal{L}$-**RSYM** of $\mathcal{L}$-relational systems, in accordance with the Definition 2.2.17.

Definition 3.2.9. *We define the category $\mathcal{L}$-**RSYM** as follows:*

1. *Object: An object in $\mathcal{L}$-**RSYM** is a $\mathcal{L}$-relational systems $(Z, \mathcal{A}, \models_{(Z\times\mathcal{A})})$.*
2. *Arrow: An arrow $(\psi_1, \psi_2) : (Z_1, \mathcal{A}, \models_{(Z_1\times\mathcal{A})}) \longrightarrow (Z_2, \mathcal{B}, \models_{(Z_2\times\mathcal{B})})$ in $\mathcal{L}$-**RSYM** is a continuous map between any two objects, where*
   - *(i)* $\psi_1 : Z_1 \longrightarrow Z_2$ *is a set map;*
   - *(ii)* $\psi_2 : \mathcal{B} \longrightarrow \mathcal{A}$ *is a $\mathcal{L}$-**ML**-algebras homomorphism;*
   - *(iii)* $\models_{(Z_1\times\mathcal{A})}(z_1, \psi_2(b)) = \models_{(Z_2\times\mathcal{B})}(\psi_1(z_1), b)$, *for $z_1 \in Z_1$ and $b \in \mathcal{B}$.*

3.2.4 Functorial relationships

In this section we shall explore functorial relationships between the categories $\mathcal{L}$-**RSYM**, $\mathcal{L}$-**RS** and $\mathcal{MA}_\mathcal{L}$.

Definition 3.2.10. *A binary relation $\mathcal{W}_\Box$ on $(Z, ext_\mathcal{L}(\mathcal{A}))$ is defined as follows:*
$z\mathcal{W}_\Box w \Leftrightarrow \forall L \in \mathcal{L}, \forall a \in \mathcal{A}, ext_\mathcal{L}(\Box a)(z) \geq L \Rightarrow ext_\mathcal{L}(a)(w) \geq L.$

Definition 3.2.11. *A functor $Ext^*_\mathcal{L} : \mathcal{L}$-**RSYM** $\longrightarrow \mathcal{L}$-**RS** is defined as follows:*

(i) $Ext^*_\mathcal{L}(Z, \mathcal{A}, \models_{(Z\times\mathcal{A})}) = ((Z, ext_\mathcal{L}(\mathcal{A})), \beta, \mathcal{W}_\Box)$*, where* $(Z, \mathcal{A}, \models_{(Z\times\mathcal{A})})$ *is an object in $\mathcal{L}$-**RSYM**;*

(ii) $Ext^*_\mathcal{L}(\phi_1, \phi_2) = \phi_1$*, where* $(\phi_1, \phi_2) : (Z, \mathcal{A}, \models_{(Z\times\mathcal{A})}) \longrightarrow (W, \mathcal{B}, \models_{(W\times\mathcal{B})})$ *is an arrow in $\mathcal{L}$-**RSYM**.*

The well-definedness of the functor $Ext^*_\mathcal{L}$ is shown by the Lemma 3.2.1 and Lemma 3.2.2.

Lemma 3.2.1. $((Z, ext_\mathcal{L}(\mathcal{A})), \beta, \mathcal{W}_\Box)$ *is an object in $\mathcal{L}$-**RS**.*

Proof. We verify the first condition in the object section of Definition 3.2.7. More precisely, if for all $h \in Cont((Z, ext_\mathcal{L}(\mathcal{A})), \beta), (\Box_\mathcal{W} h)(z) = 1 \Rightarrow h(z') = 1$, then $z\mathcal{W}_\Box z'$. We prove the contrapositive statement. Suppose $(z, z') \notin \mathcal{W}_\Box$. Then there exists $L \in \mathcal{L}$ and $a \in \mathcal{A}$ such that $ext_\mathcal{L}(\Box a)(z) \geq L \Rightarrow ext_\mathcal{L}(a)(z') \ngeq L$. Now $U_L(ext_\mathcal{L}(\Box a)(z)) = 1 \Rightarrow ext_\mathcal{L}(U_L(\Box a))(z) = 1$, but $ext_\mathcal{L}(U_L(a))(z') \neq 1$. Define $h : ((Z, ext_\mathcal{L}(\mathcal{A}), \beta) \longrightarrow (\mathcal{L}, \beta_\mathcal{L})$ by $h(z) = ext_\mathcal{L}(U_L(a))(z)$. Then we have $(\Box_\mathcal{W} h)(z) = \bigwedge\{h(y) : z\mathcal{W}_\Box y\} = \bigwedge\{ext_\mathcal{L}(U_L(a))(y) : z\mathcal{W}_\Box y\} = ext_\mathcal{L}(\Box U_L(a))(z) = 1$, but $h(z') = ext_\mathcal{L}(U_L(a))(z') \neq 1$. As we know $ext_\mathcal{L}(a) \in Cont((Z, ext_\mathcal{L}(\mathcal{A})), \beta)$, so by definition of h, we have $h \in Cont((Z, ext_\mathcal{L}(\mathcal{A})), \beta)$.
Now we verify the second condition in the object section of Definition 3.2.7.
For each $L \in \mathcal{L}$, $(ext_\mathcal{L}(a))^{-1}(\{L\}) = (T_L \circ ext_\mathcal{L}(a))^{-1}(\{1\})$ is a clopen set i.e., both open and closed (since $T_L \circ ext_\mathcal{L}(a) \in Cont((Z, ext_\mathcal{L}(\mathcal{A})), \beta)$). Now we are to show that $\mathcal{W}_\Box^{-1}[(ext_\mathcal{L}(a)^{-1}(\{L\}))] = \mathcal{W}_\Box^{-1}[(T_L \circ ext_\mathcal{L}(a))^{-1}(\{1\})]$ is clopen in Z. It suffices to show that $\mathcal{W}_\Box^{-1}[(ext_\mathcal{L}(a)^{-1}(\{1\}))]$ is clopen in Z. We claim that $\mathcal{W}_\Box^{-1}[(ext_\mathcal{L}(a)^{-1}(\{1\}))] = ext_\mathcal{L}(\neg\Box\neg T_1(a))^{-1}(\{1\})$. It is clear that $ext_\mathcal{L}(\neg\Box\neg T_1(a))^{-1}(\{1\})$ is clopen in Z. Now assume that $z \in ext_\mathcal{L}(\neg\Box\neg T_1(a))^{-1}(\{1\})$. Then $ext_\mathcal{L}(\neg\Box\neg T_1(a))(z) = 1$ and hence $ext_\mathcal{L}(\Box\neg T_1(a))(z) = 0$. Now

$$\begin{aligned}
0 &= ext_\mathcal{L}(\Box\neg T_1(a))(z) \\
&= \models_{(Z\times\mathcal{A})} (z, \Box\neg T_1(a)) \\
&= \bigwedge\{\models_{(Z\times\mathcal{A})} (z', \neg T_1(a)) : z\mathcal{W}_\Box z'\} \text{ [By Definition 3.2.8]} \\
&= \bigwedge\{ext_\mathcal{L}(\neg T_1(a))(z') : z\mathcal{W}_\Box z'\}
\end{aligned}$$

Since $ext_{\mathcal{L}}(\neg T_1(a))(z')$ is either 0 or 1, therefore there exists $w \in Z$ and $z\mathcal{W}_{\Box}w$ such that $ext_{\mathcal{L}}(\neg T_1(a))(w) = 0$. Now $ext_{\mathcal{L}}(\neg T_1(a))(w) = 0 \Rightarrow ext_{\mathcal{L}}(T_1(a))(w) = 1$. Henceforth, $ext_{\mathcal{L}}(a)(w) = 1$. Therefore $z \in \mathcal{W}_{\Box}^{-1}[(ext_{\mathcal{L}}(a)^{-1}(\{1\}))]$. Similarly it can be proved the converse part.

After this, we verify the third condition in the object section of Definition 3.2.7. Here $\beta(\mathcal{L}') = (Z, ext_{\mathcal{L}'}(\mathcal{A}))$, $\mathcal{L}'$ is a subalgebra of $\mathcal{L}$. Let $z \in (Z, ext_{\mathcal{L}'}(\mathcal{A}))$ and $\mathcal{W}_{\Box}[z] - \beta(\mathcal{L}') \neq \phi$. Then for any $w \in \mathcal{W}_{\Box}[z] - \beta(\mathcal{L}')$, we have $ext_{\mathcal{L}'}(a)(w) \notin \mathcal{L}'$. Define $ext_{\mathcal{L}'}(a)(w) = L$. For $w' \in (Z, ext_{\mathcal{L}}(\mathcal{A}))$,

$$ext_{\mathcal{L}}(T_L(a) \to a)(w') = \begin{cases} 1, & \text{if } ext_{\mathcal{L}}(a)(w') \neq L \\ L, & \text{if } ext_{\mathcal{L}}(a)(w') = L \end{cases}$$

Now $ext_{\mathcal{L}'}(\Box(T_L(a) \to a))(z) = \models (z, \Box(T_L(a) \to a)) = \bigwedge\{\models (w', T_L(a) \to a)|z\mathcal{W}_{\Box}w'\} = \bigwedge\{ext_{\mathcal{L}}(T_L(a) \to a)(w')|z\mathcal{W}_{\Box}w'\} = L$. But this contradicts our assumption that $ext_{\mathcal{L}'}(\Box(T_L(a) \to a))(z) \in \mathcal{L}'$. Therefore, if $z \in \beta(\mathcal{L}')$ then $\mathcal{W}_{\Box}[z] \subset \beta(\mathcal{L}')$.

□

Lemma 3.2.2. *For an arrow $(\phi_1, \phi_2) : (Z, \mathcal{A}, \models_{(Z\times\mathcal{A})}) \longrightarrow (W, \mathcal{B}, \models_{(W\times\mathcal{B})})$ in $\mathcal{L}$-**RSYM**, $Ext^*_{\mathcal{L}}(\phi_1, \phi_2)$ is an arrow in $\mathcal{L}$-**RS**.*

Proof. We verify the first condition in the arrow section of Definition 3.2.7. In particular, $Ext^*_{\mathcal{L}}(\phi_1, \phi_2) = \phi_1 : ((Z, ext_{\mathcal{L}}(\mathcal{A})), \beta_1, \mathcal{W}_{1_\Box}) \longrightarrow ((W, ext_{\mathcal{L}}(\mathcal{B})), \beta_2, \mathcal{W}_{2_\Box})$ is an arrow in $\mathcal{L}$-**RS**. Here we note that $\phi_1 : ((Z, ext_{\mathcal{L}}(\mathcal{A})), \beta_1) \longrightarrow ((W, ext_{\mathcal{L}}(\mathcal{B})), \beta_2)$ is an arrow in $\mathcal{L}$-**BS**. Assume $z\mathcal{W}_{1_\Box}w$. Claim: $\phi_1(z)\mathcal{W}_{2_\Box}\phi_1(w)$. By Definition 3.2.10, we have $ext_{\mathcal{L}}(\Box a)(z) \geq L \Rightarrow ext_{\mathcal{L}}(a)(w) \geq L$. Now if for all $b \in B$ and $L_1 \in \mathcal{L}$, $ext_{\mathcal{L}}(\Box b)\phi_1(z) \geq L_1$, then $\models_{(W\times\mathcal{B})} (\phi_1(z), \Box b) \geq L_1$.
By Definition 3.2.8, we have $\models_{(W\times\mathcal{B})} (\phi_1(z), \Box b) = \bigwedge\{\models_{(W\times\mathcal{B})} (\phi_1(y), b)|z\mathcal{W}_{1_\Box}y\} \geq L_1$. This shows that $ext_{\mathcal{L}}(b)(\phi_1(w)) \geq L_1$ and hence $\phi_1(z)\mathcal{W}_{2_\Box}\phi_1(w)$.

We next verify that $Ext^*_{\mathcal{L}}(\phi_1, \phi_2)$ satisfies the second condition in the arrow section of Definition 3.2.7. Assume $\phi_1(z_1)\mathcal{W}_{2_\Box}w$. Define $Ext_2(\phi_1^*, \phi_2^*) : (Z, ext_2(\mathfrak{B}(\mathcal{A}_1)), \beta_1^*, \mathcal{W}^*_{1_\Box}) \longrightarrow (W, ext_2(\mathfrak{B}(\mathcal{A}_2)), \beta_2^*, \mathcal{W}^*_{1_\Box})$ by $Ext_2(\phi_1^*, \phi_2^*) = \phi_1^*$, where $\phi_1^*(z) = \phi_1(z)$ for $z \in (Z, ext_2(\mathfrak{B}(\mathcal{A}_1)))$. So $Ext_2(\phi_1^*, \phi_2^*)$ is an arrow in $\mathcal{L}''$-**RS** ($\mathcal{L}''$ is a set of two elements $\{0, 1\}$), and if $\phi_1^*(z_1)\mathcal{W}^*_{2_\Box}w$ then there is z in $(Z, ext_2(\mathfrak{B}(\mathcal{A}_1)))$ such that $z_1\mathcal{W}^*_{1_\Box}z$ and $\phi_1^*(z) = w$. Now $ext_{\mathcal{L}}(a_1)(z) = L \Leftrightarrow ext_2(T_L(a_1))(z) = 1$. We claim $z_1\mathcal{W}_{1_\Box}z$ and $\phi_1(z) = w$. If $ext_{\mathcal{L}}(\Box a)(z_1) \geq L$ then $T_1 \circ (ext_{\mathcal{L}}(\Box U_L(a)))(z_1) = 1$. Therefore $ext_{\mathcal{L}}(\Box T_1(U_L(a))(z_1) = 1$. Since $z_1\mathcal{W}^*_{1_\Box}z$, we have $ext_2(U_L(a))(z) = 1 \Rightarrow ext_{\mathcal{L}}(a)(z) \geq L$. Therefore $z_1\mathcal{W}_{1_\Box}z$. Let

$ext_{\mathcal{L}}(b)(\phi_1(z)) = L$.
Then $ext_2(T_L(b))(\phi_1^*(z)) = 1 \Rightarrow ext_2(T_L(b))(w) = 1$. Hence $ext_{\mathcal{L}}(b)(w) = L$. Now

$$ext_{\mathcal{L}}(b)(w) = ext_{\mathcal{L}}(b)(\phi_1(z)) \Rightarrow \models_{(W\times\mathcal{B})} (w,b) = \models_{(W\times\mathcal{B})} (\phi_1(z), b) \Rightarrow \phi_1(z) = w$$

So the claim is now established.

$\square$

Definition 3.2.12. *A functor $G^* : \mathcal{L}$-**RS** $\longrightarrow \mathcal{L}$-**RSYM** is defined as follows:*

*(i) $G^*(P, \beta, \mathcal{W}_\square) = (P, (Cont(P, \beta), \square_{\mathcal{W}}), \models)$, where $(P, \beta, \mathcal{W}_\square)$ is an object in $\mathcal{L}$-**RS**. For $p \in P$, and $\Theta \in Cont(P, \beta)$, define $\models (p, \Theta) = \Theta(p)$.*

*(ii) For an arrow $\phi : (P_1, \beta_1, \mathcal{W}_{1_\square}) \longrightarrow (P_2, \beta_2, \mathcal{W}_{2_\square})$ in $\mathcal{L}$-**RS**, $G^*(\phi) = (\phi, \phi^{-1})$,*

where

- *$\phi : P_1 \longrightarrow P_2$ is a set function;*
- *$\phi^{-1} : (Cont(P_2, \beta_2), \square_{\mathcal{W}_2}) \longrightarrow (Cont(P_1, \beta_1), \square_{\mathcal{W}_1})$ is a $\mathcal{L}$-**ML**-algebras homomorphism, which is defined by $\phi^{-1}(\Theta) = \Theta \circ \phi$, $\Theta \in Cont(P_2, \beta_2)$.*

Note 3.2.1. *Remark 2.2.3 indicates that $(Cont(P, \beta), \wedge, \vee, \rightarrow, T_\ell(\ell \in \mathcal{L}), 0, 1)$ is a $\mathcal{L}$-**VL**-algebra. Thus, it is easy to follow that $(Cont(P, \beta), \wedge, \vee, \rightarrow, T_\ell(\ell \in \mathcal{L}), \square_{\mathcal{W}}, 0, 1)$ is a $\mathcal{L}$-**ML**-algebra.*

The well-definedness of the functor G* is shown by the Lemma 3.2.3 and Lemma 3.2.4.

Lemma 3.2.3. *Let $(P, \beta, \mathcal{W}_\square)$ be an object in $\mathcal{L}$-**RS**. Then, $G^*(P, \beta, \mathcal{W}_\square)$ is an object in $\mathcal{L}$-**RSYM**.*

Proof. We recall that G$^*(P, \beta, \mathcal{W}_\square) = (P, (Cont(P, \beta), \square_{\mathcal{W}}), \models)$ and $\models (p, \Theta) = \Theta(p)$, where $\Theta \in (Cont(P, \beta), \square_{\mathcal{W}})$. In order to prove $(P, (Cont(P, \beta), \square_{\mathcal{W}}), \models)$ is an object in $\mathcal{L}$-**RSYM**, we verify the conditions given in Definition 3.2.8.

(i) $\models (p, \bigvee_{r\in J} \Theta_r) = (\bigvee_{r\in J} \Theta_r)(p) = \bigvee_{r\in J} \Theta_r(p)$, where $p \in P$, and for each $r \in J$ (where J is an index set) $\Theta_r \in (Cont(P, \beta), \square_{\mathcal{W}})$. Also we observe that $\models (p, \bigwedge_{r\in J'} \Theta_r) = (\bigwedge_{r\in J'} \Theta_r)(p) = \bigwedge_{r\in J'} \Theta_r(p)$, where J' is a finite set of natural numbers.

(ii) $\models (p, \Box_{\mathcal{W}}\Theta) = (\Box_{\mathcal{W}}\Theta)(p)$. Now

$$\begin{aligned}(\Box_{\mathcal{W}}\Theta)(p) &= \bigwedge\{\Theta(p') : p\mathcal{W}p'\}(\text{ by Definition 3.2.6 }) \\ &= \bigwedge\{\models (p', \Theta) : p\mathcal{W}p'\}\end{aligned}$$

(iii) For $p \in P$ and $L \in \mathcal{L}, \models (p, T_L(\Theta)) = T_L(\Theta)(p) = T_L(\Theta(p)) = T_L(\models (p, \Theta))$.

(iv) It is easy to observe that $\models (p, 0) = 0, \models (p, 1) = 1$, where 0 and 1 are the constant functions whose values are always 0 and 1, respectively.

(v) $\models (p, f \to g) = (f \to g)(p) = f(p) \to g(p) = \models (p, f) \to \models (p, g)$.

□

Lemma 3.2.4. *Let $\phi : (P_1, \beta_1, \mathcal{W}_{1_\Box}) \longrightarrow (P_2, \beta_2, \mathcal{W}_{2_\Box})$ be an arrow in $\mathcal{L}$-**RS**. Then $G^*(\phi)$ is an arrow in $\mathcal{L}$-**RSYM**.*

Proof. Recall that $\mathrm{G}^*(\phi) = (\phi, \phi^{-1})$, where $\phi : P_1 \longrightarrow P_2$ is a set function and $\phi^{-1} : (Cont(P_2, \beta_2), \Box_{\mathcal{W}_2}) \longrightarrow (Cont(P_1, \beta_1), \Box_{\mathcal{W}_1})$ is a $\mathcal{L}$-**ML**-algebras homomorphism, which is defined by $\phi^{-1}(\Theta) = \Theta \circ \phi$, $\Theta \in Cont(P_2, \beta_2)$. Now we observe that

$$\begin{aligned}\models (p_1, \phi^{-1}(\Theta)) &= (\Theta \circ \phi)(p_1) \\ &= \Theta(\phi(p_1)) \\ &= \models (\phi(p_1), \Theta)\end{aligned}$$

Therefore (ϕ, ϕ^{-1}) is a continuous map in $\mathcal{L}$-**RSYM**, and hence $\mathrm{G}^*(\phi)$ is an arrow in $\mathcal{L}$-**RSYM**.

□

Definition 3.2.13. *A functor $H^* : \mathcal{L}$-**RSYM** $\longrightarrow (\mathcal{MA_L})^{op}$ is defined as follows:*

*(i) $H^*(S, \mathcal{A}, \models_{(S\times\mathcal{A})}) = \mathcal{A}$, where $(S, \mathcal{A}, \models_{(S\times\mathcal{A})})$ is an object in $\mathcal{L}$-**RSYM**;*

*(ii) $H^*(g_1, g_2) = g_2^{op}$, where $(g_1, g_2) : (S_1, \mathcal{A}, \models_{(S_1\times\mathcal{A})}) \longrightarrow (S_2, \mathcal{B}, \models_{(S_2\times\mathcal{B})})$ is an arrow in $\mathcal{L}$-**RSYM**.*

It is easy to observe that the functor H^* is well-defined.

Definition 3.2.14. *A functor $R^* : (\mathcal{MA_L})^{op} \longrightarrow \mathcal{L}$-**RSYM** is defined as follows:*

*(i) $R^*(\mathcal{A}) = (Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{A}, \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})})$, where $\mathcal{A}$ is an object in $(\mathcal{L}$-**MA**$)^{op}$;*

(ii) $R^*(\psi) = (\psi^{-1}, \psi^{op})$, $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ *is an arrow in* ($\mathcal{L}$-$\boldsymbol{MA})^{op}$.

The well-definedness of the functor R^* is shown by the Lemma 3.2.5 and Lemma 3.2.6.

Lemma 3.2.5. *Let* $\mathcal{A}$ *be a* $\mathcal{L}$-$\boldsymbol{ML}$*-algebra. Then* $(Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{A}, \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})})$ *is an object in* $\mathcal{L}$-$\boldsymbol{RSYM}$.

Proof. We define $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, a) = h(a)$, $h \in Spec_{\mathcal{L}}(\mathcal{A})$. Clearly $Spec_{\mathcal{L}}(\mathcal{A})$ is a set. Next we show that

(i) $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, \bigvee_{j\in J} a_j) = h(\bigvee_{j\in J} a_j) = \bigvee_{j\in J} h(a_j) = \bigvee_{j\in J} \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, a_j)$, ($J$ is an index set). Also $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, a_1 \wedge a_2) = h(a_1) \wedge h(a_2) = \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, a_1) \wedge \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, a_2)$.

(ii) We observe that

$$\begin{aligned}
\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, \Box a) &= h(\Box a) \\
&= \bigwedge\{h_1(a) : h\mathcal{W}h_1\}(\text{ using Proposition 3.2.1}) \\
&= \bigwedge\{\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h_1, a) : h\mathcal{W}h_1\}
\end{aligned}$$

(iii) $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, T_L(a)) = h(T_L(a)) = T_L(h(a)) = T_L(\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, a))$.

(iv) It is clear that $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, 0) = 0$ and $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, 1) = 1$

(v) $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, a \rightarrow b) = h(a \rightarrow b) = h(a) \rightarrow h(b) = \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, a) \rightarrow \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, b)$.

Therefore $(Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{A}, \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})})$ is an object in $\mathcal{L}$-**RSYM** □

Lemma 3.2.6. (ψ^{-1}, ψ^{op}) *is a continuous map in* $\mathcal{L}$-$\boldsymbol{RSYM}$*, whenever* ψ *is a* $\mathcal{L}$-$\boldsymbol{ML}$*-algebras homomorphism.*

Proof. Here $\psi^{-1} : Spec_{\mathcal{L}}(\mathcal{A}) \longrightarrow Spec_{\mathcal{L}}(\mathcal{B})$ is a set map, and $\psi^{op} : \mathcal{B} \longrightarrow \mathcal{A}$ is a $\mathcal{L}$-**ML**-algebras homomorphism.
Now, $\models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})} (h, \psi^{op}(b)) = h(\psi^{op}(b)) = (h \circ \psi^{op})(b) = \psi^{-1}(h)(b) = \models_{(Spec_{\mathcal{L}}(\mathcal{B})\times\mathcal{B})} (\psi^{-1}(h), b)$. Thus, (ψ^{-1}, ψ^{op}) is a continuous map in $\mathcal{L}$-**RSYM**. □

Theorem 3.2.1. $Ext^*_{\mathcal{L}}$ *is a co-adjoint to the functor* G^*.

Proof. We first define the counit of the adjunction. Figure 3.1 illustrates the counit.

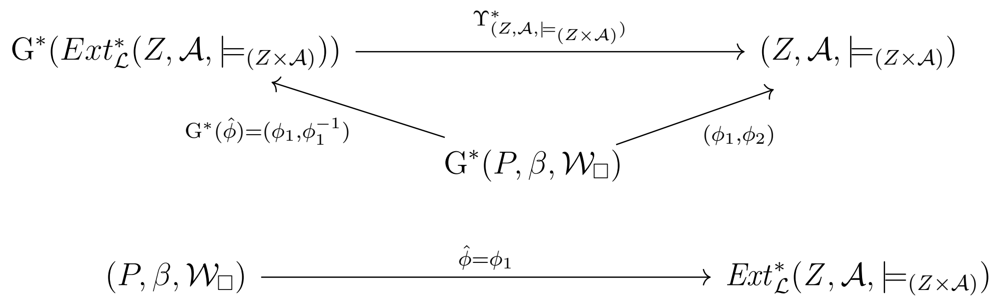

Figure 3.1: Illustration of the counit

Recall that $G^*(P,\beta,\mathcal{W}_\Box) = (P,(Cont(P,\beta),\Box_{\mathcal{W}}),\models)$ and $Ext^*_{\mathcal{L}}(Z,\mathcal{A},\models_{(Z\times\mathcal{A})}) = (((Z,ext_{\mathcal{L}}(\mathcal{A})),\beta),\mathcal{W}_\Box)$. So $G^*(Ext^*_{\mathcal{L}}(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})) = (Z,(Cont(Z,ext_{\mathcal{L}}(\mathcal{A}),\beta),\Box_{\mathcal{W}}),\models)$.

We show that the counit $\Upsilon^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})} = (ID^*_Z, ext^*_{\mathcal{L}}) : G^*(Ext^*_{\mathcal{L}}(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})) \longrightarrow (Z,\mathcal{A},\models_{(Z\times\mathcal{A})})$ is a continuous map in **$\mathcal{L}$-RSYM**, where

(i) $ID^*_Z : Z \longrightarrow Z$ is a set function;

(ii) $ext^*_{\mathcal{L}} : \mathcal{A} \longrightarrow (Cont(Z,ext_{\mathcal{L}}(\mathcal{A}),\beta),\Box_{\mathcal{W}})$ is a **$\mathcal{L}$-ML**-algebras homomorphism, where $ext^*_{\mathcal{L}}(a) = ext_{\mathcal{L}}(a)$, $\forall a \in \mathcal{A}$.

It is known that $ext^*_{\mathcal{L}}$ is a **$\mathcal{L}$-VL**-algebras homomorphism. We have to show that it preserves the unary operation $\Box$ i.e., $ext^*_{\mathcal{L}}(\Box a) = \Box(ext^*_{\mathcal{L}}(a))$. Now $ext^*_{\mathcal{L}}(\Box a)(z) = \models_{(Z\times\mathcal{A})} (z,\Box a) = \bigwedge\{\models_{(Z\times\mathcal{A})} (z',a) : z\mathcal{W}_\Box z'\} = \bigwedge\{ext_{\mathcal{L}}(a)(z') : z\mathcal{W}_\Box z'\}$. Using Definition 3.2.6, we have $\Box(ext^*_{\mathcal{L}}(a))(z) = \bigwedge\{ext_{\mathcal{L}}(a)(z') : z\mathcal{W}_\Box z'\}$. Therefore, $ext^*_{\mathcal{L}}(\Box a) = \Box(ext^*_{\mathcal{L}}(a))$. So $ext^*_{\mathcal{L}}$ is a **$\mathcal{L}$-ML**-algebras homomorphism.
To prove the continuity of $\Upsilon^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})}$, it is enough to show that $\models_{(Z\times\mathcal{A})} (ID^*_Z(z),a) = \models (z, ext^*_{\mathcal{L}}(a))$. We see that $\models_{(Z\times\mathcal{A})} (ID^*_Z(z),a) = \models_{(Z\times\mathcal{A})} (z,a) = ext_{\mathcal{L}}(a)(z) = \models (z, ext_{\mathcal{L}}(a)) = \models (z, ext^*_{\mathcal{L}}(a))$.
Next we prove that the triangle of Figure 3.1 commutes i.e., for a given arrow $(\phi_1,\phi_2) : G^*(P,\beta,\mathcal{W}_\Box) \longrightarrow (Z,\mathcal{A},\models_{(Z\times\mathcal{A})})$ in **$\mathcal{L}$-RSYM** there is an arrow, which we take $\hat{\phi} = \phi_1$ in **$\mathcal{L}$-RS** such that $(\phi_1,\phi_2) = \Upsilon^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})} \circ G^*(\hat{\phi})$.
Now

$$\begin{aligned}(\phi_1,\phi_2) &= (ID^*_Z, ext^*_{\mathcal{L}}) \circ (\phi_1,\phi_1^{-1})\\ &= (ID^*_Z \circ \phi_1, \phi_1^{-1} \circ ext^*_{\mathcal{L}})\end{aligned}$$

It is clear that $ID^*_Z \circ \phi_1 = \phi_1$. The only part we have to show is that $\phi_2 = \phi_1^{-1} \circ ext^*_{\mathcal{L}}$. Now as $(ID^*_Z, ext^*_{\mathcal{L}})$ is continuous, so $\models_{(Z\times\mathcal{A})} (ID^*_Z(z), a) = \models (z, ext^*_{\mathcal{L}}(a))$. Therefore $ext^*_{\mathcal{L}}(a) = a$.

We observe that for each $a \in \mathcal{A}$,

$$\begin{aligned} \phi_1^{-1} \circ ext^*_{\mathcal{L}}(a) &= \phi_1^{-1}(a) \\ &= \phi_2(a) \text{ (as } (\phi_1, \phi_2) \text{ is continuous)} \end{aligned}$$

Hence $\Upsilon^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})}$ is the counit and as a result $Ext^*_{\mathcal{L}}$ is a co-adjoint to the functor G^*.

□

Theorem 3.2.2. *G^* is an adjoint to the functor $Ext^*_{\mathcal{L}}$*

Proof. It is also possible to prove the theorem by unit of the adjunction. Figure 3.2 illustrates the unit.

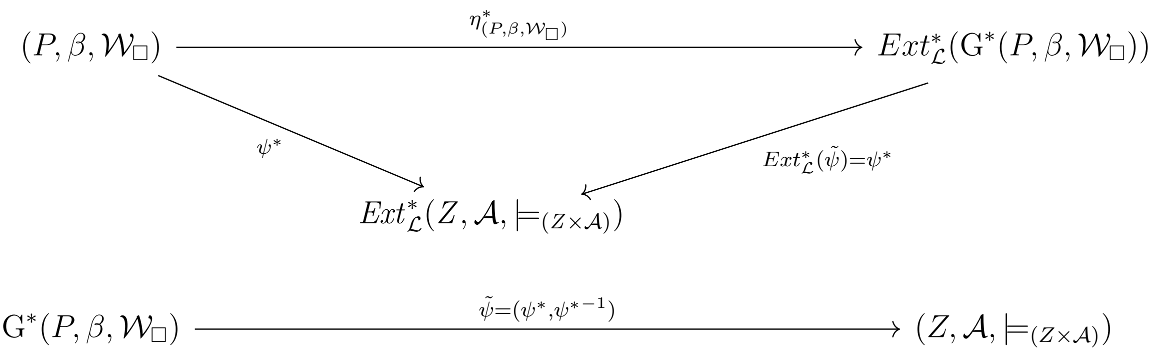

Figure 3.2: Illustration of the unit

For a given arrow $\psi^* : (P, \beta, \mathcal{W}_\Box) \longrightarrow Ext^*_{\mathcal{L}}(Z, \mathcal{A}, \models_{(Z\times\mathcal{A})})$ there is an arrow, which we take $\tilde{\psi} : \mathrm{G}^*(P, \beta, \mathcal{W}_\Box) \longrightarrow (Z, \mathcal{A}, \models_{(Z\times\mathcal{A})})$ such that $Ext^*_{\mathcal{L}}(\tilde{\psi}) = \psi^*$. It can be shown that the triangle of Figure 3.2 commutes i.e., $Ext^*_{\mathcal{L}}(\tilde{\psi}) \circ \eta^*_{(P,\beta,\mathcal{W}_\Box)} = \psi^*$.

□

Theorem 3.2.3. *The category $\mathcal{L}$-**RSYM** is equivalent to the category $\mathcal{L}$-**RS**.*

Proof. Let $ID^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})}$ and ID^* be two identity functors on $\mathcal{L}$-**RSYM** and $\mathcal{L}$-**RS**, respectively. Υ^* and η^* are two natural transformations such that $\Upsilon^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})} : \mathrm{G}^*(Ext^*_{\mathcal{L}}(Z, \mathcal{A}, \models_{(Z\times\mathcal{A})})) \longrightarrow (Z, \mathcal{A}, \models_{(Z\times\mathcal{A})})$ and $\eta^*_{(P,\beta,\mathcal{W}_\Box)} : (P, \beta, \mathcal{W}_\Box) \longrightarrow Ext^*_{\mathcal{L}}(\mathrm{G}^*(P, \beta, \mathcal{W}_\Box))$. Now we show that Υ^* and η^* are natural isomorphism.

$\Upsilon^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})}$ is a natural transformation between objects in $\mathcal{L}$-**RSYM**. We note that $\Upsilon^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})}$ is similar to $\Upsilon_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})}$, as defined in the proof of Theorem 2.2.5. So, by Theorem 2.2.5, $\Upsilon^*_{(Z,\mathcal{A},\models_{(Z\times\mathcal{A})})}$ is an isomorphism and hence Υ^* is a natural isomorphism.
Now we prove that η^* is a natural isomorphism.
Here $\eta^*_{(P,\beta,\mathcal{W}_\Box)} : (P,\beta,\mathcal{W}_\Box) \longrightarrow ((P, ext^*_{\mathcal{L}}(Cont(P,\beta),\Box_{\mathcal{W}})),\beta',\mathcal{W}'_\Box)$
($\mathcal{W}'_\Box$ is a binary relation on $(P, ext^*_{\mathcal{L}}(Cont(P,\beta),\Box_{\mathcal{W}})))$, and is defined by $\eta^*_{(P,\beta,\mathcal{W}_\Box)}(p)(\psi) = \psi(p)$, $\psi \in (Cont(P,\beta),\Box_{\mathcal{W}})$.
Define $\beta' : Subalg(\mathcal{L}) \longrightarrow \Omega(P, ext_{\mathcal{L}}(Cont(P,\beta),\Box_{\mathcal{W}}))$ by $\beta'(\mathcal{M}) = (P, ext_{\mathcal{M}}(Cont(P,\beta),\Box_{\mathcal{W}}))$. Since $ext^*_{\mathcal{L}}(\psi)(p) = \psi(p)$, so $\eta^*_{(P,\beta,\mathcal{W}_\Box)}$ is well-defined. Here $\eta^*_{(P,\beta,\mathcal{W}_\Box)}$ is very similar to $\eta_{(P,\beta)}$, which is defined in the proof of Theorem 2.2.5. So by Theorem 2.2.5, $\eta^*_{(P,\beta,\mathcal{W}_\Box)}$ is an isomorphism between objects in $\mathcal{L}$-**BS**. We have to show that $\eta^*_{(P,\beta,\mathcal{W}_\Box)}$ and $\eta^{*-1}_{(P,\beta,\mathcal{W}_\Box)}$ satisfy the first and second conditions in the arrow section of Definition 3.2.7. Assume for any $p_1,p_2 \in P$, $p_1\mathcal{W}_\Box p_2$. Then for any $L \in \mathcal{L}$, and $\psi \in Cont(P,\beta)$, $\eta^*_{(P,\beta,\mathcal{W}_\Box)}(p_1)(\Box_{\mathcal{W}}\psi) = ext^*_{\mathcal{L}}(\Box_{\mathcal{W}}\psi)(p_1) \geq L \Rightarrow (\Box_{\mathcal{W}}\psi)(p_1) \geq L$. Now $(\Box_{\mathcal{W}}\psi)(p_1) = \bigwedge\{\psi(p') : p_1\mathcal{W}_\Box p'\}$. Since $p_1\mathcal{W}_\Box p_2$, we have $\psi(p_2) \geq L$.
Therefore $ext_{\mathcal{L}}(\psi)(p_2) \geq L$ and hence $\eta^*_{(P,\beta,\mathcal{W}_\Box)}(p_1)\mathcal{W}'_\Box\eta^*_{(P,\beta,\mathcal{W}_\Box)}(p_2)$. Again we observe that if $(p_1,p_2) \notin \mathcal{W}_\Box$ then by the first condition in the object section of Definition 3.2.7 there exists $\psi^* \in Cont(P,\beta)$ such that $(\Box_{\mathcal{W}}\psi^*)(p_1) = 1$ but $\psi^*(p_2) \neq 1$. Therefore $ext^*_{\mathcal{L}}(\Box_{\mathcal{W}}\psi^*)(p_1) = 1$ and $ext^*_{\mathcal{L}}(\psi^*)(p_2) \neq 1$. Therefore $(\eta^*_{(P,\beta,\mathcal{W}_\Box)}(p_1),\eta^*_{(P,\beta,\mathcal{W}_\Box)}(p_2)) \notin \mathcal{W}'_\Box$. So we get for any $p_1,p_2 \in P$, $p_1\mathcal{W}_\Box p_2$ iff $\eta^*_{(P,\beta,\mathcal{W}_\Box)}\mathcal{W}'_\Box\eta^*_{(P,\beta,\mathcal{W}_\Box)}$.
Now we verify the second condition in the arrow section of Definition 3.2.7. Suppose $\eta^*_{(P,\beta,\mathcal{W}_\Box)}(p)\mathcal{W}'_\Box t$. Since $\eta^*_{(P,\beta,\mathcal{W}_\Box)}$ is surjective, there is $t_1 \in P$ such that $\eta^*_{(P,\beta,\mathcal{W}_\Box)}(t_1) = t$ and $p\mathcal{W}_\Box t_1$. Analogously we can verify for $\eta^{*-1}_{(P,\beta,\mathcal{W}_\Box)}$, and hence η^* is a natural isomorphism.

□

Theorem 3.2.4. *H^* is an adjoint to the functor R^*.*

Proof. We define the unit of the adjunction. Figure 3.3 illustrates the unit.

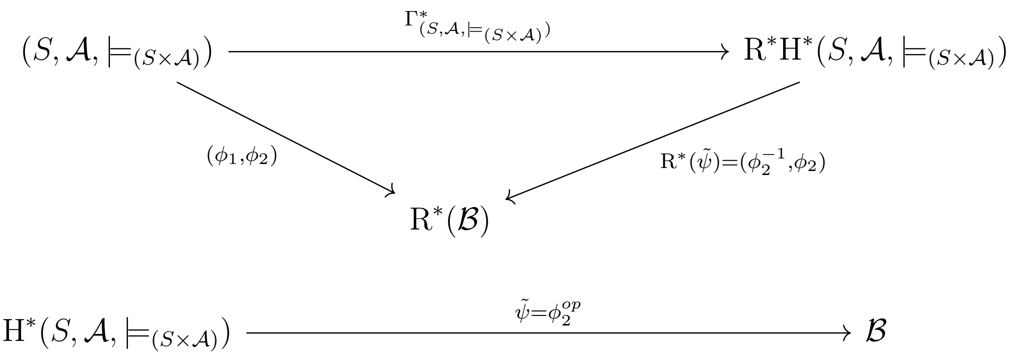

Figure 3.3: Illustration of the unit

Now $\mathrm{R}^*(\mathcal{B}) = (Spec_{\mathcal{L}}(\mathcal{B}), \mathcal{B}, \models_{(Spec_{\mathcal{L}}(\mathcal{B})\times\mathcal{B})})$, and $\models_{(Spec_{\mathcal{L}}(\mathcal{B})\times\mathcal{B})} (\varphi, b) = \varphi(b)$. So $\mathrm{R}^*\mathrm{H}^*(S, \mathcal{A}, \models_{(S\times\mathcal{A})}) = \mathrm{R}^*(\mathcal{A}) = (Spec_{\mathcal{L}}(\mathcal{A}), \mathcal{A}, \models_{(Spec_{\mathcal{L}}(\mathcal{A})\times\mathcal{A})})$.
Here the unit is taken by Γ^*. For an object $(S, \mathcal{A}, \models_{(S\times\mathcal{A})})$ in $\mathcal{L}$-**RSYM**, define $\Gamma^*_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} : (S, \mathcal{A}, \models_{(S\times\mathcal{A})}) \longrightarrow \mathrm{R}^*\mathrm{H}^*(S, \mathcal{A}, \models_{(S\times\mathcal{A})})$ by $\Gamma^*_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (g, ID_{\mathcal{A}})$, where

(i) $g : S \longrightarrow Spec_{\mathcal{L}}(\mathcal{A})$ is a set map. For each $s \in S$, define $g(s) = g_s$, where $g_s : \mathcal{A} \longrightarrow \mathcal{L}$ is defined by $g_s(a) = \models_{(S\times\mathcal{A})} (s, a)$;

(ii) $ID_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ is a $\mathcal{L}$-**ML**-algebras homomorphism.

It is already known that for each $s \in S$, g_s is a $\mathcal{L}$-**VL**-algebras homomorphism. From the proof of Theorem 2.2.3, we observe that $(g, ID_{\mathcal{A}})$ is a continuous map in $\mathcal{L}$-**RSYM**. Now we shall show that the triangle of Figure 3.3 commutes i.e., for a given arrow $(\phi_1, \phi_2) : (S, \mathcal{A}, \models_{(S\times\mathcal{A})}) \longrightarrow \mathrm{R}^*(\mathcal{B})$ there is an arrow $\tilde{\psi}$, which we define $\tilde{\psi} = \phi_2^{op} : \mathrm{H}^*(S, \mathcal{A}, \models_{(S\times\mathcal{A})}) \longrightarrow \mathcal{B}$ such that $(\phi_1, \phi_2) = \mathrm{R}^*(\tilde{\psi}) \circ \Gamma^*_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$. Now $\mathrm{R}^*(\tilde{\psi}) = \mathrm{R}^*(\phi_2^{op}) = (\phi_2^{-1}, \phi_2)$. It is clear that $\phi_2 = ID_{\mathcal{A}} \circ \phi_2$. Claim: $\phi_1 = \phi_2^{-1} \circ g$. For each $s \in S$, $\phi_1(s) = \phi_2^{-1} \circ g(s) = \phi_2^{-1} \circ g_s = g_s \circ \phi_2$. Now for each $b \in \mathcal{B}$, we have $g_s \circ \phi_2(b) = g_s(\phi_2(b)) = \models_{(S\times\mathcal{A})} (s, \phi_2(b))$. As (ϕ_1, ϕ_2) is continuous in $\mathcal{L}$-**RSYM**, so $\models_{(S\times\mathcal{A})} (s, \phi_2(b)) = \models_{(Spec_{\mathcal{L}}(\mathcal{B})\times\mathcal{B})} (\phi_1(s), b) = \phi_1(s)(b)$. Therefore $\phi_1 = \phi_2^{-1} \circ g$. Hence the theorem is proved.

□

Theorem 3.2.5. *R^* is a co-adjoint to the functor H^*.*

Proof. It is also possible to prove the theorem by counit of the adjunction. Figure 3.4 illustrates the counit.

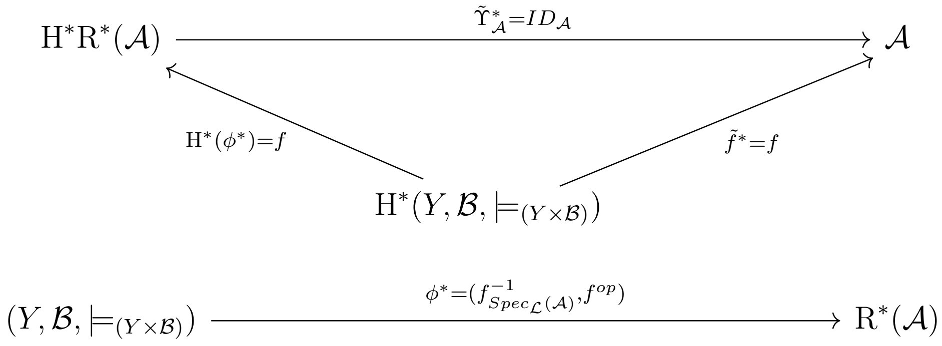

Figure 3.4: Illustration of the counit

Here the counit $\tilde{\Upsilon}^*$ is defined by $\tilde{\Upsilon}^*_{\mathcal{A}} = ID_{\mathcal{A}}$. For a given arrow $\tilde{f}^*$ in $(\mathcal{MA}_{\mathcal{L}})^{op}$ there is an arrow ϕ^* in $\mathcal{L}$-**RSYM**, which is defined by $\phi^* = (f^{-1}_{Spec_{\mathcal{L}}(\mathcal{A})}, f^{op})$ such that $H^*(\phi^*) = f$. It is now easy to see that the triangle of Figure 3.4 commutes i.e., $\tilde{\Upsilon}^*_{\mathcal{A}} \circ H^*(\phi^*) = \tilde{f}^*$.

□

Theorem 3.2.6. *The category $(\mathcal{MA}_{\mathcal{L}})^{op}$ is equivalent to the category $\mathcal{L}$-**RSYM**.*

Proof. We get two natural transformations $\tilde{\Upsilon}^*$ and Γ^* such that $\tilde{\Upsilon}^*_{\mathcal{A}} = ID_{\mathcal{A}} : H^*R^*(\mathcal{A}) \longrightarrow \mathcal{A}$ and $\Gamma^*_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} : (S,\mathcal{A},\models_{(S\times\mathcal{A})}) \longrightarrow R^*H^*(S,\mathcal{A},\models_{(S\times\mathcal{A})})$. Here $\tilde{\Upsilon}^*_{\mathcal{A}}$ is of course a natural isomorphism. We have to show that $\Gamma^*_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is a natural isomorphism between two objects in $\mathcal{L}$-**RSYM**.
Here $\Gamma^*_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})} = (g, ID_{\mathcal{A}})$. Using the proof of Theorem 2.2.6, we can say that $\Gamma^*_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is a homeomorphism and hence $\Gamma^*_{(S,\mathcal{A},\models_{(S\times\mathcal{A})})}$ is a natural isomorphism. Therefore $(\mathcal{MA}_{\mathcal{L}})^{op}$ is equivalent to the category $\mathcal{L}$-**RSYM**. Consequently, $\mathcal{MA}_{\mathcal{L}}$ is dually equivalent to the category $\mathcal{L}$-**RSYM**. □

Theorem 3.2.7. $(\mathcal{MA}_{\mathcal{L}})^{op}$ *is equivalent to $\mathcal{L}$-**RS**.*

Proof. The result can be obtained as the composition of equivalences of Theorem 3.2.3 and Theorem 3.2.6. Hence $\mathcal{MA}_{\mathcal{L}}$ is dually equivalent to $\mathcal{L}$-**RS**. □

Remark 3.2.1. *The same process can also be used to develop a duality for $\mathcal{L}$-**ML**-algebras with truth constants. The idea of $\mathcal{L}$-**ML**-algebras with truth constants appears in [21].*

3.3 Conclusion

We have seen in Chapter 2 that the concept of lattice-valued Boolean systems, which originates from lattice-valued topological systems, plays an essential role in proving duality between systems and algebras of multi-valued logic. The approach has then been extended to algebras of multi-valued modal logic. Introducing the notion of lattice-valued relational systems, we have found the categorical equivalence between systems and algebras of Fitting's multi-valued modal logic. This in turn establishes the duality between $\mathcal{L}$-**RS** and $\mathcal{MA}_{\mathcal{L}}$.

Chapter 4

Bitopological Duality for multi-valued logic

4.1 Introduction

The groundwork for duality theory was laid in 1937 by Stone [51], who demonstrated the dual equivalence between the categories of Boolean algebras and homomorphisms and the category of Stone spaces (compact, zero-dimensional and Hausdorff spaces) and continuous mappings. Furthermore, Stone developed a general work for the category of bounded distributive lattices in 1937 [43]. With the aid of ordered Stone spaces (also known as Priestley spaces), Priestley explored a different duality for the category of bounded distributive lattices in 1970 [50], resolving issues in Stone's work [43]. Esakia [10] discovered a duality for Heyting algebras, which is a limitation of Priestley duality.

From a logical perspective, topological dualities have been used to establish a relationship between syntax and semantic of a propositional logic. Several authors have approached the development of topological duality from various perspectives (e.g., [21, 53, 29]).

From a computer science perspective, topological dualities serve as the foundation for semantics of programming language (e.g., [14, 32, 33]). Abramsky [14] extended Smyth's concepts [34] by developing programming logic from denotational semantics. Stone-type dualities played crucial roles in Abramsky's ground breaking

The results of this chapter can be found in [58] **Das, Litan Kumar., Ray, Kumar Sankar.**: **Bitopological duality for algebras of Fitting's logic and natural duality extension. Acta Informatica, 58(5), 571-584 (2021).**

work [14] to derive the relationship between program logic and denotational semantics.
The concept of bi-topological spaces was introduced in [38]. Bitopological spaces may be employed to represent distributive lattices, as demonstrated by Jung and Moshier in [48]. As a result, the authors investigated a different explanation of Esakia duality in a bi-topological context in [40]. The objective of this chapter is to use bi-topological techniques to construct a duality for algebras of Fitting's multi-valued logic. In actuality, it extends the natural duality theory in a bi-topological context. We shall introduce a category $PBS_{\mathcal{L}}$ of lattice-valued pairwise Boolean spaces, and relate it to the category $\mathcal{VA}_{\mathcal{L}}$ of algebras of Fitting's multi-valued logic using appropriate functors. This leads us to propose a duality for Fitting's multi-valued logic in a bitopological setting.

4.2 $\mathcal{L}$-VL-algebras, $\mathcal{L}$-pairwise Boolean spaces and their categorical interconnections

Throughout this section $\mathcal{L}$ denotes a finite distributive lattice. Henceforth, $\mathcal{L}$ is a finite Heyting algebra.

4.2.1 $\mathcal{L}$-VL-algebras

In order to obtain algebraic axiomatization of Fitting's Heyting valued logic, Maruyama in [20] modified Fitting's $\mathcal{L}$-valued logic by removing fuzzy truth constants (except bottom and top elements 0, 1 respectively) and adding a new unary operation $T_\ell(-)$. From the logical point of view $T_\ell(p)$ means the truth value of a proposition p is exactly ℓ. Such operations $T_\ell(-)$ were introduced with reference to the Post algebras [37].

Definition 4.2.1. *For each $\ell \in \mathcal{L}$, the mapping $T_\ell : \mathcal{L} \longrightarrow \mathcal{L}$ is defined by*

$$T_\ell(r) = \begin{cases} 1 & r = \ell \\ 0 & r \neq \ell \end{cases}$$

Now let us review the algebraic structure of Fitting's Heyting-valued logic.

Definition 4.2.2 ([20]). $(\mathcal{A}, \wedge, \vee, \to, T_L(L \in \mathcal{L}), 0, 1)$ *forms a $\mathcal{L}$-**VL**-algebra if and only if for any $L_1, L_2 \in \mathcal{L}$, and $a, b \in \mathcal{A}$, it satisfies the following axioms:*

(i) the algebraic structure $(\mathcal{A}, \wedge, \vee, \rightarrow, T_L(L \in \mathcal{L}), 0, 1)$ is a Heyting algebra;

(ii) $T_{L_1}(a) \wedge T_{L_2}(b) \leq T_{L_1 \rightarrow L_2}(a \rightarrow b) \wedge T_{L_1 \wedge L_2}(a \wedge b) \wedge T_{L_1 \vee L_2}(a \vee b)$;
$T_{L_2}(a) \leq T_{T_{L_1}(L_2)}(T_{L_1}(a))$;

(iii) $T_0(0) = 1$; $T_L(0) = 0$ $(L \neq 0)$; $T_1(1) = 1$; $T_L(1) = 0$, if $L \neq 1$;

(iv) $\bigvee\{T_L(a) : L \in \mathcal{L}\} = 1$; $T_{L_1}(a) \vee (T_{L_2}(a) \rightarrow 0) = 1$;
$T_{L_1}(a) \wedge T_{L_2}(a) = 0$, $(L_1 \neq L_2)$;

(v) $T_1(T_L(a)) = T_L(a)$, $T_0(T_L(a)) = T_L(a) \rightarrow 0$, $T_{L_2}(T_{L_1}(a)) = 0$, $(L_2 \neq 0, 1)$;

(vi) $T_1(a) \leq a$, $T_1(a \wedge b) = T_1(a) \wedge T_1(b)$;

(vii) $\bigwedge_{L \in \mathcal{L}} (T_L(a) \leftrightarrow T_L(b)) \leq (a \leftrightarrow b)$.

Definition 4.2.3 ([20])**.** *A $\mathcal{L}$-**VL**-algebras homomorphism is a mapping f between two $\mathcal{L}$-**VL**-algebras such that the mapping f preserves the operations $\wedge, \vee, \rightarrow, T_L(L \in \mathcal{L}), 0, 1$.*

Definition 4.2.4 ([20])**.** *A non-empty subset $\mathcal{S}$ of $\mathcal{A}$, where $\mathcal{A}$ is a $\mathcal{L}$-**VL**-algebra, is said to be a $\mathcal{L}$-valued filter if the following hold:*

(i) if $s \in \mathcal{S}$ and $s \leq t$, then $t \in \mathcal{S}$;

(ii) $s \wedge t \in \mathcal{S}$ whenever $s, t \in \mathcal{S}$;

(iii) $T_1(s) \in \mathcal{S}$ whenever $s \in \mathcal{S}$.

Definition 4.2.5 ([20])**.** *A non-empty subset $\mathcal{S}$ of $\mathcal{A}$, where $\mathcal{A}$ is a $\mathcal{L}$-**VL**-algebra, is said to be a prime $\mathcal{L}$-valued filter if the following hold:*

(i) $\mathcal{S} \neq \mathcal{A}$;

(ii) if for any $r \in \mathcal{L}$, $T_r(s \vee t) \in \mathcal{S}$ then there exists $r_1, r_2 \in \mathcal{L}$ with $r_1 \vee r_2 = r$ such that $T_{r_1}(s) \in \mathcal{S}$ and $T_{r_2}(t) \in \mathcal{S}$.

Theorem 4.2.1 ([21])**.** *Let a and b be any two different elements of a $\mathcal{L}$-**VL**-algebra $\mathcal{A}$. Then, there exits $r \in \mathcal{L}$ and a prime $\mathcal{L}$-valued filter $\mathcal{S}$ of $\mathcal{A}$ such that $T_r(a) \in \mathcal{S}$ but $T_r(b) \notin \mathcal{S}$.*

Proposition 4.2.1 ([21])**.** *For each prime $\mathcal{L}$-valued filter $\mathcal{S}$ of a $\mathcal{L}$-**VL**-algebra $\mathcal{A}$, there is a homomorphism $h_{\mathcal{S}} : \mathcal{A} \longrightarrow \mathcal{L}$ defined by $h_{\mathcal{S}}(a) = r \iff T_r(a) \in \mathcal{S}$.*

The category $\mathcal{VA}_{\mathcal{L}}$

Definition 4.2.6. *$\mathcal{L}$-**VL**-algebras together with $\mathcal{L}$-**VL**-algebras homomorphisms form a category $\mathcal{VA}_{\mathcal{L}}$.*

4.2.2 $\mathcal{L}$-pairwise Boolean spaces

A bitopological space is defined by a triple (X, τ_1, τ_2), where X is a set and τ_1, τ_2 are two topologies on X. We now review the several key ideas about bitopological spaces pertinent to our work.

Definition 4.2.7 ([39])**.** *A bitopological space $(\mathcal{S}, \tau_1^{\mathcal{S}}, \tau_2^{\mathcal{S}})$ is said to be pairwise Hausdorff if for any two different points s_1, s_2 of $\mathcal{S}$ there exist a disjoint open sets $\mathcal{O}_1 \in \tau_1^{\mathcal{S}}$, and $\mathcal{O}_2 \in \tau_2^{\mathcal{S}}$ containing s_1 and s_2, respectively.*

Definition 4.2.8 ([39])**.** *A bitopological space $(\mathcal{S}, \tau_1^{\mathcal{S}}, \tau_2^{\mathcal{S}})$ is said to be pairwise zero-dimensional if the collection $B_1^{\mathcal{S}}$ of $\tau_1^{\mathcal{S}}$-open sets which are $\tau_2^{\mathcal{S}}$-closed, is a basis for the topology $\tau_1^{\mathcal{S}}$, and the collection $B_2^{\mathcal{S}}$ of $\tau_2^{\mathcal{S}}$-open sets which are $\tau_1^{\mathcal{S}}$-closed, is a basis for the topology $\tau_2^{\mathcal{S}}$, i.e., we can write $B_1^{\mathcal{S}} = \tau_1^{\mathcal{S}} \cap \varrho_2$, and $B_2^{\mathcal{S}} = \tau_2^{\mathcal{S}} \cap \varrho_1$. Here we designate ϱ_1, and ϱ_2 as the collections of $\tau_1^{\mathcal{S}}$-closed sets, and $\tau_2^{\mathcal{S}}$-closed sets, respectively.*

Definition 4.2.9 ([39])**.** *A bitopological space $(\mathcal{S}, \tau_1^{\mathcal{S}}, \tau_2^{\mathcal{S}})$ is said to be pairwise compact if every open cover $\{\mathcal{O}_i : i \in J, \mathcal{O}_i \in \tau_1^{\mathcal{S}} \cup \tau_2^{\mathcal{S}}\}$ of $\mathcal{S}$ has a finite sub-cover.*

Proposition 4.2.2 ([40])**.** *A bitopological space $(\mathcal{S}, \tau_1^{\mathcal{S}}, \tau_2^{\mathcal{S}})$ is pairwise compact if and only if $\varrho_1 \subset \Upsilon_2$ and $\varrho_2 \subset \Upsilon_1$, where Υ_1, and Υ_2 are denote respectively the set of all compact subsets of $(\mathcal{S}, \tau_1^{\mathcal{S}})$, and $(\mathcal{S}, \tau_2^{\mathcal{S}})$.*

Definition 4.2.10. *If a bitopological space is pairwise compact, pairwise Hausdorff, and pairwise zero-dimensional, then it is called pairwise Boolean space.*

For a pairwise Boolean space $\mathcal{B}$, we denote the set of all pairwise closed subspaces of $\mathcal{B}$ by $\Omega_{\mathcal{B}}$. As a pairwise closed subset of a pairwise compact space is also a pairwise compact [62], so each member of $\Omega_{\mathcal{B}}$ is a pairwise Boolean space. Let $\Sigma_{\mathcal{L}}$ represent the set of all sub-algebras of $\mathcal{L}$.

Definition 4.2.11. *A pairwise Boolean space together with a mapping α from $\Sigma_{\mathcal{L}}$ to $\Omega_{\mathcal{B}}$ that meets certain conditions form a $\mathcal{L}$-valued pairwise Boolean space, denoted as $\mathcal{L}$-pairwise Boolean space.*

Definition 4.2.12. *We define a category $PBS_{\mathcal{L}}$ of $\mathcal{L}$-pairwise Boolean spaces as follows:*

1. *An object in $PBS_{\mathcal{L}}$ is defined by a tuple $(\mathcal{B}, \alpha_{\mathcal{B}})$, where $\mathcal{B}$ is a pairwise Boolean space, and $\alpha_{\mathcal{B}}$ is a mapping from $\Sigma_{\mathcal{L}}$ to $\Omega_{\mathcal{B}}$ which satisfies the following conditions:*

   (i) $\alpha_{\mathcal{B}}(\mathcal{L}) = \mathcal{B}$;

   (ii) for any $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in \Sigma_{\mathcal{L}}$, if $\mathcal{L}_1 = \mathcal{L}_2 \wedge \mathcal{L}_3$, then $\alpha_{\mathcal{B}}(\mathcal{L}_1) = \alpha_{\mathcal{B}}(\mathcal{L}_2) \cap \alpha_{\mathcal{B}}(\mathcal{L}_3)$.

2. *A morphism $f : (\mathcal{B}, \alpha_{\mathcal{B}}) \longrightarrow (\mathcal{B}', \alpha_{\mathcal{B}'})$ in $PBS_{\mathcal{L}}$ is a pairwise continuous map $f : \mathcal{B} \longrightarrow \mathcal{B}'$ which satisfies;*

   (i) if $x \in \alpha_{\mathcal{B}}(\mathcal{L}')$, $\mathcal{L}' \in \Sigma_{\mathcal{L}}$, then $f(x) \in \alpha_{\mathcal{B}'}(\mathcal{L}')$.

Remark 4.2.1. *We consider a bitopological space $(\mathcal{L}, \tau, \tau)$, where τ is the discrete topology, and consequently, $(\mathcal{L}, \alpha_{\mathcal{L}})$, where the mapping $\alpha_{\mathcal{L}} : \Sigma_{\mathcal{L}} \longrightarrow \Omega_{\mathcal{L}}$ is defined by $\alpha_{\mathcal{L}}(\mathcal{L}') = \mathcal{L}'$, is an object in $PBS_{\mathcal{L}}$.*

4.2.3 Functorial relationships

In order to determine the functorial relationships between the categories $PBS_{\mathcal{L}}$ and $\mathcal{VA}_{\mathcal{L}}$, we will need to construct two functors: F from the category $PBS_{\mathcal{L}}$ to the category $\mathcal{VA}_{\mathcal{L}}$ of $\mathcal{L}$-**VL**-algebras, as well as G from the category $\mathcal{VA}_{\mathcal{L}}$ to the category $PBS_{\mathcal{L}}$.

Definition 4.2.13. *We define a functor $F : PBS_{\mathcal{L}} \longrightarrow \mathcal{VA}_{\mathcal{L}}$ as follows:*

(i) F acts on an object $(\mathcal{B}, \alpha_{\mathcal{B}})$ in $PBS_{\mathcal{L}}$ as $F(\mathcal{B}, \alpha_{\mathcal{B}}) = (Hom_{PBS_{\mathcal{L}}}((\mathcal{B}, \alpha_{\mathcal{B}}), (\mathcal{L}, \alpha_{\mathcal{L}})), \vee, \wedge, \rightarrow, T_p(p \in \mathcal{L}), 0, 1)$, where $(\mathcal{B}, \alpha_{\mathcal{B}})$ is an object in $PBS_{\mathcal{L}}$. The operations $\vee, \wedge, \rightarrow, T_p(p \in \mathcal{L}), 0, 1$ on the set $Hom_{PBS_{\mathcal{L}}}((\mathcal{B}, \alpha_{\mathcal{B}}), (\mathcal{L}, \alpha_{\mathcal{L}}))$ are defined pointwise i.e., $(\phi \vee \eta)(b) = \phi(b) \vee \eta(b)$, $(\phi \wedge \eta)(b) = \phi(b) \wedge \eta(b)$, $(\phi \rightarrow \eta)(b) = \phi(b) \rightarrow \eta(b)$, $T_p(\phi)(b) = T_p(\phi(b))$, and the operations $0, 1$ are considered to be constant functions, whose values are zero and one, respectively.

*(ii) F acts on an arrow $\phi : (\mathcal{B}, \alpha_{\mathcal{B}}) \longrightarrow (\mathcal{B}', \alpha_{\mathcal{B}'})$ in $PBS_{\mathcal{L}}$ as follows:
$F(\phi) : F(\mathcal{B}', \alpha_{\mathcal{B}'}) \longrightarrow F(\mathcal{B}, \alpha_{\mathcal{B}})$ defined by $F(\phi)(\eta) = \eta \circ \phi$, where $\eta \in Hom_{PBS_{\mathcal{L}}}((\mathcal{B}', \alpha_{\mathcal{B}'}), (\mathcal{L}, \alpha_{\mathcal{L}}))$.*

The well-definedness of the functor F is shown by Proposition 4.2.3 and Proposition 4.2.4.

Proposition 4.2.3. *For an object $(\mathcal{B}, \alpha_{\mathcal{B}})$ in $PBS_{\mathcal{L}}$, $F(\mathcal{B}, \alpha_{\mathcal{B}})$ is an object in $\mathcal{VA}_{\mathcal{L}}$.*

Proof. If $\phi, \eta : ((\mathcal{B}, \tau_1^{\mathcal{B}}, \tau_2^{\mathcal{B}}), \alpha_{\mathcal{B}}) \longrightarrow ((\mathcal{L}, \tau, \tau), \alpha_{\mathcal{L}})$, where τ is the discrete topology on $\mathcal{L}$, are both pairwise continuous maps, then $\phi \vee \eta, \phi \wedge \eta, \phi \rightarrow \eta, T_p(\phi)$ are also pairwise continuous maps. As a result, it can be shown that if $\phi, \eta \in Hom_{PBS_{\mathcal{L}}}((\mathcal{B}, \alpha_{\mathcal{B}}), (\mathcal{L}, \alpha_{\mathcal{L}}))$, then $\phi \vee \eta, \phi \wedge \eta, \phi \rightarrow \eta, T_p(\phi) \in Hom_{PBS_{\mathcal{L}}}((\mathcal{B}, \alpha_{\mathcal{B}}), (\mathcal{L}, \alpha_{\mathcal{L}}))$. Now it is being observed that $(Hom_{PBS_{\mathcal{L}}}((\mathcal{B}, \alpha_{\mathcal{B}}), (\mathcal{L}, \alpha_{\mathcal{L}})), \vee, \wedge, \rightarrow, T_p(p \in \mathcal{L}), 0, 1)$ is a $\mathcal{L}$-**VL**-algebra. □

Proposition 4.2.4. *For an arrow $\phi : (\mathcal{B}, \alpha_{\mathcal{B}}) \longrightarrow (\mathcal{B}', \alpha_{\mathcal{B}'})$ in $PBS_{\mathcal{L}}$, $F(\phi)$ is an arrow in $\mathcal{VA}_{\mathcal{L}}$.*

Proof. We recall that $F(\phi) : F(\mathcal{B}', \alpha_{\mathcal{B}'}) \longrightarrow F(\mathcal{B}, \alpha_{\mathcal{B}})$ is defined by $F(\phi)(\eta) = \eta \circ \phi$, where $\eta \in Hom_{PBS_{\mathcal{L}}}((\mathcal{B}', \alpha_{\mathcal{B}'}), (\mathcal{L}, \alpha_{\mathcal{L}}))$. Now $F(\phi)(\eta_1 \vee \eta_2) = (\eta_1 \vee \eta_2) \circ \phi = (\eta_1 \circ \phi) \vee (\eta_2 \circ \phi) = F(\phi)(\eta_1) \vee F(\phi)(\eta_2)$. Similarly, $F(\phi)(\eta_1 \wedge \eta_2) = F(\phi)(\eta_1) \wedge F(\phi)(\eta_2)$, $F(\phi)(\eta_1 \rightarrow \eta_2) = F(\phi)(\eta_1) \rightarrow F(\phi)(\eta_2)$, $F(\phi)(T_p(\eta)) = T_p(\eta) \circ \phi = T_p(\eta \circ \phi) = T_p(F(\phi)(\eta))$, $F(\phi)(0) = 0$, $F(\phi)(1) = 1$. Therefore, $F(\phi)$ preserves all the operations $\wedge, \vee, \rightarrow, T_p(p \in \mathcal{L}), 0, 1$. Henceforth, $F(\phi)$ is an arrow in $\mathcal{VA}_{\mathcal{L}}$.

□

Definition 4.2.14. *We define a functor $G : \mathcal{VA}_{\mathcal{L}} \longrightarrow PBS_{\mathcal{L}}$ as follows:*

(i) $G(\mathcal{A}) = (Hom_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2, \alpha_{\mathcal{A}})$, where $\mathcal{A}$ is an object in $\mathcal{VA}_{\mathcal{L}}$. The mapping $\alpha_{\mathcal{A}} : \Sigma_{\mathcal{L}} \longrightarrow \Omega_{Hom_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A},\mathcal{L})}$ is defined by $\alpha_{\mathcal{A}}(\mathcal{L}') = Hom_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}')$.

(ii) For an arrow $g : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ in $\mathcal{VA}_{\mathcal{L}}$, define $G(g) : G(\mathcal{A}_2) \longrightarrow G(\mathcal{A}_1)$ by $G(g)(\mu) = \mu \circ g$, where $\mu \in G(\mathcal{A}_2)$.

Remark 4.2.2. *In the first part of the above Definition 4.2.14, we take $\alpha_{\mathcal{A}}(\mathcal{L}') = Hom_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}')$, where $\mathcal{L}'$ is a sub-algebra of $\mathcal{L}$. We note that the subset $Hom_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}')$ of $Hom_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L})$ is σ_1-closed and σ_2-closed i.e., pairwise closed, where the topologies σ_1, and σ_2 are generated by the bases $\{\langle x \rangle : x \in \mathcal{A}\}$ and $\{\langle T_1(x) \rightarrow 0 \rangle : x \in \mathcal{A}\}$, respectively.*

Remark 4.2.3. *For a $\mathcal{L}$-**VL**-algebra $\mathcal{A}$, $(Hom_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2)$ is a bitopological space. The topologies σ_1 and σ_2 are generated by the bases $B^{\sigma_1} = \{\langle a \rangle : a \in \mathcal{A}\}$, where $\langle a \rangle = \{v \in Hom_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}) : v(a) = 1\}$, and $B^{\sigma_2} = \{\mathcal{O}^c : \mathcal{O} \in B^{\sigma_1}\}$, respectively.*

Well-definedness of the functor G is shown by Proposition 4.2.5 and Proposition 4.2.6.

Proposition 4.2.5. *For an object $\mathcal{A}$ in $\mathcal{VA_L}$, $G(\mathcal{A})$ is an object in $PBS_{\mathcal{L}}$.*

Proof. We recall from Definition 4.2.14 that $G(\mathcal{A}) = (Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2, \alpha_{\mathcal{A}})$.
First, we show that $(Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2)$ is a pairwise Hausdorff space. Let $v_1, v_2 \in Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L})$ such that $v_1 \neq v_2$. Then there exists an element $a \in \mathcal{A}$ such that $v_1 \in \langle a \rangle$, but $v_2 \notin \langle a \rangle$, and thus there exists disjoint open sets $U \in \sigma_1$, $V \in \sigma_2$ such that $v_1 \in U$, and $v_2 \in V$.
Second, we show that $(Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2)$ is pairwise compact. In this instance, we see that $\sigma_1 \cup \sigma_2 \subset \sigma_1$. Since, $(Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1)$ is compact, it follows that $(Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2)$ is pairwise compact.
Finally, we prove that $(Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2)$ is a pairwise zero-dimensional. To prove that $(Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2)$ is a pairwise zero-dimensional, we shall show that $B^{\sigma_1} = \sigma_1 \cap \varrho_2$, and $B^{\sigma_2} = \sigma_2 \cap \varrho_1$. We find that if $u \in B^{\sigma_1}$, then $u \in \sigma_1$. Since $u \in B^{\sigma_1}$, we have $u = \langle a \rangle$, for some $a \in \mathcal{A}$. Now $u^c = \langle T_1(a) \to 0 \rangle$, and hence $u^c \in B^{\sigma_2}$. As a result, $u \in \varrho_2$. Therefore, we have $u \in \sigma_1 \cap \varrho_2$. Next we take $u \in \sigma_1 \cap \varrho_2$, and prove that $u \in B^{\sigma_1}$. Since B^{σ_1} is the basis for the topology σ_1, so u can be expressed as the union of the members of B^{σ_1}. As $(Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2)$ is pairwise compact, so by Proposition 4.2.2, we have u is compact. Therefore, u can be covered by the finite collection of that members of B^{σ_1}. As the finite union of the members of B^{σ_1} is also in B^{σ_1}, hence $u \in B^{\sigma_1}$. Consequently, $B^{\sigma_1} = \sigma_1 \cap \varrho_2$. Analogously, we can show that $B^{\sigma_2} = \sigma_2 \cap \varrho_1$. Therefore, we can conclude that $(Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \sigma_1, \sigma_2)$ is a pairwise Boolean space.
It is easy to follow that the mapping $\alpha_{\mathcal{A}} : \Sigma_{\mathcal{L}} \longrightarrow \Omega_{Hom_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L})}$ satisfies the conditions given in the object part of Definition 4.2.12. Henceforth, $G(\mathcal{A})$ is an object in $PBS_{\mathcal{L}}$.

□

Proposition 4.2.6. *For an arrow $g : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ in $\mathcal{VA_L}$, $G(g) : G(\mathcal{A}_2) \longrightarrow G(\mathcal{A}_1)$ is an arrow in $PBS_{\mathcal{L}}$.*

Proof. For a basis open set $\langle x \rangle$, where $x \in \mathcal{A}_1$, in the topology $\sigma_1^{\mathcal{A}_1}$ on $G(\mathcal{A}_1)$, we get

$$\begin{aligned} G(g)^{-1}(\langle x \rangle) &= \{\phi \in Hom_{\mathcal{VA_L}}(\mathcal{A}_2, \mathcal{L}) : G(g)(\phi) \in \langle x \rangle\} \\ &= \{\phi \in Hom_{\mathcal{VA_L}}(\mathcal{A}_2, \mathcal{L}) : \phi \circ g \in \langle x \rangle\} \\ &= \langle g(x) \rangle \in \sigma_1^{\mathcal{A}_2}. \end{aligned}$$

Next we see that for a basis open set $\langle x\rangle^c$, where $x \in \mathcal{A}_1$, in the topology $\sigma_2^{\mathcal{A}_1}$ on $G(\mathcal{A}_1)$

$$\begin{aligned} G(g)^{-1}(\langle x\rangle^c) &= \{\phi \in Hom_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}_2, \mathcal{L}) : G(g)(\phi) \in \langle x\rangle^c\} \\ &= \{\phi \in Hom_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}_2, \mathcal{L}) : \phi \circ g \in \langle T_1(x) \to 0\rangle\} \\ &= \langle T_1(g(x)) \to 0\rangle \in \sigma_2^{\mathcal{A}_2}. \end{aligned}$$

Therefore, the mapping $G(g)$ is pairwise continuous. We also observe that, if $\xi \in \alpha_{\mathcal{A}_2}(\mathcal{L}')$, then $G(g)(\xi) \in \alpha_{\mathcal{A}_1}(\mathcal{L}')$. As a result, $G(g)$ is an arrow in $PBS_\mathcal{L}$.

□

4.3 Bitopological duality

We now establish a duality for algebras of Fitting's many-valued logic in a bitopological setting.

Theorem 4.3.1. *The category $\mathcal{VA}_\mathcal{L}$ is dually equivalent to the category $PBS_\mathcal{L}$.*

Proof. We shall prove this theorem by defining two natural isomorphisms $\beta : Id_\mathcal{A} \longrightarrow F \circ G$ and $\zeta : Id_{PBS_\mathcal{L}} \longrightarrow G \circ F$, where $Id_\mathcal{A}$ and $Id_{PBS_\mathcal{L}}$ are respectively the identity functors on the categories $\mathcal{VA}_\mathcal{L}$, and $PBS_\mathcal{L}$. Now for a $\mathcal{L}$-$\mathcal{VL}$ algebra $\mathcal{A}$, define $\beta^\mathcal{A} : \mathcal{A} \longrightarrow F \circ G(\mathcal{A})$ by $\beta^\mathcal{A}(a)(\phi) = \phi(a)$, where $a \in \mathcal{A}$ and $\phi \in G(\mathcal{A}) = Hom_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}, \mathcal{L})$. It is straightforward to demonstrate that $\beta^\mathcal{A}$ is a homomorphism. Using Theorem 4.2.1 and Proposition 4.2.3, we can establish that $\beta^\mathcal{A}$ is one-one and onto. Consequently, $\beta^\mathcal{A}$ is an isomorphism. The fact that β is a natural transformation is easily verified. Consequently, β is a natural isomorphism. Again, for an object $(\mathcal{S}, \alpha_\mathcal{S})$ in $PBS_\mathcal{L}$, define $\zeta_{(\mathcal{S},\alpha_\mathcal{S})} : (\mathcal{S}, \alpha_\mathcal{S}) \longrightarrow G \circ F(\mathcal{S}, \alpha_\mathcal{S})$ by $\zeta_{(\mathcal{S},\alpha_\mathcal{S})}(s)(\psi) = \psi(s)$, where $s \in \mathcal{S}$ and $\psi \in F(\mathcal{S}, \alpha_\mathcal{S}) = (Hom_{PBS_\mathcal{L}}((\mathcal{S}, \alpha_\mathcal{S}), (\mathcal{L}, \alpha_\mathcal{L})), \vee, \wedge, \to, T_p(p \in \mathcal{L}), 0, 1)$. We shall show that $\zeta_{(\mathcal{S},\alpha_\mathcal{S})}$ is a bi-homeomorphism. As $\psi \in Hom_{PBS_\mathcal{L}}((\mathcal{S}, \alpha_\mathcal{S}), (\mathcal{L}, \alpha_\mathcal{L}))$, so for each $s \in \mathcal{S}$, $\zeta_{(\mathcal{S},\alpha_\mathcal{S})}(s)$ is a $\mathcal{L}$-$\mathcal{VL}$-algebras homomorphism. Henceforth, $\zeta_{(\mathcal{S},\alpha_\mathcal{S})}$ is well-defined.
To prove the pairwise continuity of $\zeta_{(\mathcal{S},\alpha_\mathcal{S})}$, we show that $\zeta^{-1}_{(\mathcal{S},\alpha_\mathcal{S})}(\langle v\rangle)$, where $v \in Hom_{PBS_\mathcal{L}}((\mathcal{S}, \alpha_\mathcal{S}), (\mathcal{L}, \alpha_\mathcal{L}))$ is $\tau_1^\mathcal{S}$-open and $\zeta^{-1}_{(\mathcal{S},\alpha_\mathcal{S})}(\langle v\rangle^c)$, $v \in Hom_{PBS_\mathcal{L}}((\mathcal{S}, \alpha_\mathcal{S}), (\mathcal{L}, \alpha_\mathcal{L}))$ is $\tau_2^\mathcal{S}$-open.

Now

$$\begin{aligned}\zeta^{-1}_{(\mathcal{S},\alpha_{\mathcal{S}})}(\langle v\rangle) &= \{s\in\mathcal{S}:\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)\in\langle v\rangle\}\\ &= \{s\in\mathcal{S}:\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)(v)=1\}\\ &= v^{-1}(\{1\}).\end{aligned}$$

As $v^{-1}(\{1\})$ is $\tau_1^{\mathcal{S}}$-open, so $\zeta^{-1}_{(\mathcal{S},\alpha_{\mathcal{S}})}(\langle v\rangle)$ is $\tau_1^{\mathcal{S}}$-open.
Also we get,

$$\begin{aligned}\zeta^{-1}_{(\mathcal{S},\alpha_{\mathcal{S}})}(\langle v\rangle^c) &= \{s\in\mathcal{S}:\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)\in\langle T_1(v)\to 0\rangle\}\\ &= \{s\in\mathcal{S}:\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)(T_1(v)\to 0)=1\}\\ &= (T_1(v)\to 0)^{-1}(\{1\}).\end{aligned}$$

Since $(T_1(v)\to 0)^{-1}(\{1\})$ is $\tau_2^{\mathcal{S}}$-open, henceforth $\zeta^{-1}_{(\mathcal{S},\alpha_{\mathcal{S}})}(\langle v\rangle^c)$ is $\tau_2^{\mathcal{S}}$-open.

Now we show that $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}$ is injective. For any two points $s,s'\in\mathcal{S}$, let $s\neq s'$. Then, $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s),\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s')\in G\circ F(\mathcal{S},\alpha_{\mathcal{S}})$. We shall show that $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)\neq\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s')$. Suppose for contradiction, we take $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)=\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s')$. Now $\{\langle v\rangle : v\in Hom_{PBS_{\mathcal{L}}}((\mathcal{S},\alpha_{\mathcal{S}}),(\mathcal{L},\alpha_{\mathcal{L}}))\}$, and $\{\langle v\rangle^c : v\in Hom_{PBS_{\mathcal{L}}}((\mathcal{S},\alpha_{\mathcal{S}}),(\mathcal{L},\alpha_{\mathcal{L}}))\}$ are the bases for the topologies σ_1 and σ_2 on $G\circ F(\mathcal{S},\alpha_{\mathcal{S}})$, respectively. So, $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)$ can be expressed by σ_1-basis open sets. Also, it can be expressed by σ_2-basis open sets. Let $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)\in\langle v\rangle$. Then, we get $v(s)=v(s')=1$. Since $\mathcal{S}$ is pairwise Hausdorff, we get $s=s'$. Similarly, if $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s)\in\langle v\rangle^c$, then we get $s=s'$. Consequently, $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}$ is injective.
Next we show that $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}$ is onto. Let $\phi\in G\circ F(\mathcal{S},\alpha_{\mathcal{S}})$. Define $S_1=\{v^{-1}(\{1\})\in\beta_1^{\mathcal{S}} : \phi(v)=1\}$ and $S_2=\{u\in\beta_2^{\mathcal{S}} : u^c\notin S_1\}$, where $\beta_1^{\mathcal{S}}$, and $\beta_2^{\mathcal{S}}$ are the bases for the topologies $\tau_1^{\mathcal{S}}$ and $\tau_2^{\mathcal{S}}$, respectively. We show that $S_1\cup S_2$ has the finite intersection property. Since, $(S_1\cup S_2)\cap(S_1'\cup S_2')=(S_1\cap S_1')\cup(S_2\cap S_2')$, and $v^{-1}(\{1\})\cap v'^{-1}(\{1\})=(v\wedge v')^{-1}(\{1\})$, we have to show that $v^{-1}(\{1\})\neq\emptyset$ under the condition that $\phi(v)=1$. Suppose $\phi(v)=1$ but $v^{-1}(\{1\})=\emptyset$. Then we have $T_1(v)=0$, and henceforth $T_1(\phi(v))=\phi(T_1(v))=0\Rightarrow\phi(v)=0$, which contradicts the assumption that $\phi(v)=1$. Thus, $S_1\cup S_2$ has the finite intersection property.

As $\mathcal{S}$ is pairwise compact and pairwise Hausdorff, thus there exists $s\in\mathcal{S}$ such that $\{s\}=\bigcap(S_1\cup S_2)$. If $s\in S_1$, then we get $v(s)=1$, whenever $\phi(v)=1$. Moreover, we can get if $v(s)=1$, then $\phi(v)=1$. For contradiction we take $\phi(v)\neq 1$. Then $T_1(\phi(v))=0$, and hence $\phi(T_1(v))=0$. Now $\phi(T_1(v))\to 0=1$, and thus we get

$\phi(T_1(v) \to 0) = 1$. Since, $\phi(T_1(v) \to 0) = 1$, according to definition of S_1, we have $(T_1(v) \to 0)(s) = 1$. As a result, we get $T_1(v(s)) = 0$. Now $T_1(v(s)) = 0 \Rightarrow v(s) \neq 1$. Hence, for any $v \in Hom_{PBS_{\mathcal{L}}}((\mathcal{S}, \alpha_{\mathcal{S}}), (\mathcal{L}, \alpha_{\mathcal{L}}))$, $\phi(v) = 1 \Leftrightarrow v(s) = 1$. Thus, we have $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s) = \phi$. Similarly, if $s \in S_2$, then we can also get $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(s) = \phi$. Therefore, $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}$ is onto.
Finally, we show that $\zeta^{-1}_{(\mathcal{S},\alpha_{\mathcal{S}})}$ is pairwise continuous. It can be shown by verifying that $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}$ is a bi-closed map. Let U be a $\tau_1^{\mathcal{S}}$-closed set. Since $(\mathcal{S}, \tau_1^{\mathcal{S}}, \tau_2^{\mathcal{S}})$ is a pairwise Boolean space, both $(\mathcal{S}, \tau_1^{\mathcal{S}})$ and $(\mathcal{S}, \tau_2^{\mathcal{S}})$ are compact. Consequently, $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(U)$ is compact in $Hom_{\mathcal{VA}_{\mathcal{L}}}(Hom_{PBS_{\mathcal{L}}}((\mathcal{S}, \alpha_{\mathcal{S}}), (\mathcal{L}, \alpha_{\mathcal{L}})), \mathcal{L})$. We observe that the topological space $(Hom_{\mathcal{VA}_{\mathcal{L}}}(Hom_{PBS_{\mathcal{L}}}((\mathcal{S}, \alpha_{\mathcal{S}}), (\mathcal{L}, \alpha_{\mathcal{L}})), \mathcal{L}), \sigma_1)$ with basis $\{\langle v \rangle : v \in Hom_{PBS_{\mathcal{L}}}((\mathcal{S}, \alpha_{\mathcal{S}}), (\mathcal{L}, \alpha_{\mathcal{L}}))\}$ is itself a Hausdorff space, and thus $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(U)$ is closed. The topological space $(Hom_{\mathcal{VA}_{\mathcal{L}}}(Hom_{PBS_{\mathcal{L}}}((\mathcal{S}, \alpha_{\mathcal{S}}), (\mathcal{L}, \alpha_{\mathcal{L}})), \mathcal{L}), \sigma_2)$ with basis $\{\langle T_1(v) \to 0 \rangle : v \in Hom_{PBS_{\mathcal{L}}}((\mathcal{S}, \alpha_{\mathcal{S}}), (\mathcal{L}, \alpha_{\mathcal{L}}))\}$ is itself a Hausdorff space, so for a τ_2-closed set U', $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}(U')$ is closed. Therefore, $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}$ is a bi-homeomorphism. It is simple to verify that $\zeta_{(\mathcal{S},\alpha_{\mathcal{S}})}$, and $\zeta^{-1}_{(\mathcal{S},\alpha_{\mathcal{S}})}$ satisfy the condition given in item 2 of Definition 4.2.12. The verification that ζ is a natural transformation is simple. Hence, ζ is a natural isomorphism.

□

4.4 Conclusion

The primary outcome of this chapter is a duality for Fitting's Heyting-valued logic, which is obtained using an expanded form of the theory of natural dualities based on the theory of bitopology. We have proposed the category $PBS_{\mathcal{L}}$ of $\mathcal{L}$-pairwise Boolean spaces and linked it to the category $\mathcal{VA}_{\mathcal{L}}$ of algebras of Fitting's Heyting-valued logic via suitable functors. We have therefore discovered a duality for Fitting's Heyting-valued logic. Throughout this chapter, we have tried to show how logic, algebra, and bitopology are conceptually and technically related through the extended version of Stone-type duality in a bitopological context.

Chapter 5

Intuitionistic version of Natural Duality Theory

5.1 Introduction

The primary objective of this chapter is to extend the natural duality theory for $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$, the class of all isomorphic copies of sub-algebras of intuitionistic power of finite algebra $\mathcal{L}$. This will allow the theory of natural dualities to incorporate Esakia duality for Heyting algebras [64]. Let us begin by discussing an aspect of natural duality theory and the difficulties of incorporating it into the Esakia duality for the class of all Heyting algebras. We will then see how $\mathbb{ISP}_{\mathbb{I}}$ can help us address the challenge.

The natural duality theory [13] offers a potent comprehensive explanation of Stone-Priestley-type dualities based on the methods of universal algebra. It primarily covers duality theory of $\mathbb{ISP}(\mathcal{L})$. This approach proves beneficial in discovering novel dualities. It encompasses several previously established dualities, such as Stone duality for Boolean algebras [51], Priestley duality for distributed lattices [56], among others (see [45, 46, 47] for additional examples). However, it fails to incorporate Esakia duality for the class of all Heyting algebras. Although algebras of $\mathcal{L}$-valued logic can be loosely characterized as $\mathbb{ISP}(\mathcal{L})$, it is not able to represent the class of all Heyting algebras for any finite algebra $\mathcal{L}$. It is important to point out that the implication operation of a Heyting algebra cannot be defined pointwise on the

The results of this chapter can be found in [58] **Das, Litan Kumar., Ray, Kumar Sankar.**: **Bitopological duality for algebras of Fitting's logic and natural duality extension. Acta Informatica, 58(5), 571-584 (2021).**

topological space of prime filters of the Heyting algebra. For that same reason, it is not possible to describe the class of all Heyting algebras as $\mathbb{ISP}(\mathcal{L})$. Maruyama [41] also brought attention to this issue and proposed the concept of $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$.
Maruyama [41] utilized Hu-duality [44] to develop a duality for $\mathbb{ISP}_{\mathbb{M}}(\mathcal{L})$, a modalization of the notion of $\mathbb{ISP}(\mathcal{L})$. This allowed for the successfully unification of Jónsson-Tarski duality(e.g., [35, 54, 29]) and Abramsky-Kupke-Kurz-venema duality(e.g., [1, 55]). Furthermore, he proposed the ideas of $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathcal{L})$ and $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$, which reflected two alternative viewpoints on intuitionistic logic: the former, residuation-based, and the latter, Kripke semantic-based. Maruyama [52] obtained a duality for $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathcal{L})$, and incorporated Esakia duality into natural duality theory. In this chapter, we consider the notion of $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$ as a means of extending the theory of natural dualities, which in turn incorporates the Esakia duality. A noteworthy observation is that $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$ coincides with the class of all Heyting algebras if $\mathcal{L}$ is the two-element distributive lattice. In order to develop a duality for $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$, we first set up a duality for $\mathbb{ISP}(\mathcal{L})$.

5.2 The concept of $\mathbb{ISP}_{\mathbb{I}}$

Throughout this section, let $\mathcal{L}$ refer to a finite algebra with a bounded lattice reduct. Logically, one would anticipate that there is a bounded lattice reduct as most logics are endowed with the lattice connectives meet($\wedge$) and join($\vee$), along with the truth constants 0 and 1. From the viewpoint of logic, we may perceive $\mathcal{L}$ as an algebra of truth values. Let $\mathbb{ISP}(\mathcal{L})$ represent the class of all isomorphic copies of sub-algebras of direct powers of finite single algebra $\mathcal{L}$. For the 2-element distributive lattice $\{0, 1\}$, $\mathbb{ISP}(\{0, 1\})$ coincides with the class of all distributive lattices.

The category $ISP(\mathcal{L})$ is defined as follows:

Definition 5.2.1. *(i) Objects: objects in $ISP(\mathcal{L})$ are algebras in $\mathbb{ISP}(\mathcal{L})$;*

(ii) Arrows: arrows in $ISP(\mathcal{L})$ are homomorphisms, where a homomorphism is defined by a function between the objects, which preserve the operations defined on $\mathcal{L}$.

An intuitionistic Kripke frame can be defined by the tuple (W, R), where W is a non-empty set and R is a partial order relation on it. Consider $(\mathcal{L}, \leq)$ is a poset such that $\ell_1 \leq \ell_2$ iff $\ell_1 \vee \ell_2 = \ell_2$, equivalently, $\ell_1 \wedge \ell_2 = \ell_1$. For any two elements

$r_1, r_2 \in \mathcal{L}$, $r_1 \to r_2 = \bigvee\{\ell \in \mathcal{L} : r_1 \wedge \ell \leq r_2\}$.
We now define the notion of intuitionistic power.

Definition 5.2.2 ([41]). *The intuitionistic power of $\mathcal{L}$ with respect to an intuitionistic frame (W, R) is defined as $\mathcal{L}^W \in \mathbb{ISP}(\mathcal{L})$ equipped with the binary operation $\to$ (intuitionistic implication) on $\mathcal{L}^W$ defined as $(f \to g)(w) = \bigwedge\{f(w') \to g(w') : wRw'\}$, where $f, g \in \mathcal{L}^W$.*

The concept of $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$ is given in the following definition.

Definition 5.2.3. *$\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$ represents the class of all isomorphic copies of subalgebras of intuitionistic power of $\mathcal{L}$.*

The category $ISP_I(\mathcal{L})$ is defined as follows.

Definition 5.2.4. *$ISP_I(\mathcal{L})$ denotes the category of algebras in $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$ and homomorphisms between algebras, where a homomorphism is defined by a function that preserves the implication operation $\to$ and all the other operations of $\mathcal{L}$.*

For an object $\mathcal{A}$ in $ISP(\mathcal{L})$, $HOM_{ISP(\mathcal{L})}(\mathcal{A}, \mathcal{L})$ denotes the set of all homomorphisms between algebras A and $\mathcal{L}$.

Definition 5.2.5. *We define an order relation R on $Hom_{ISP(\mathcal{L})}(\mathcal{A}, \mathcal{L})$ as follows: for any $v_1, v_2 \in Hom_{ISP(\mathcal{L})}(\mathcal{A}, \mathcal{L})$, $v_1 R v_2$ iff $v_1(x) \leq v_2(x)$, for all $x \in \mathcal{A}$. Then $(Hom_{ISP(\mathcal{L})}(\mathcal{A}, \mathcal{L}), R)$ is a poset.*

Definition 5.2.6. *For any object $(\mathcal{A}, \to)$ of $ISP_I(\mathcal{L})$, and $v \in Hom_{ISP(\mathcal{L})}(\mathcal{A}, \mathcal{L})$, $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$ satisfies the intuitionistic Kripke model condition iff $v(x \to y) = \bigwedge\{w(x) \to_{\mathcal{L}} w(y) : vRw\}$.*

A zero-dimensional compact Hausdorff space is called a Stone space [7]. An ordered topological space is defined by a triple (X, τ, R), where the tuple (X, τ) is a topological-space and (X, R) is a partially ordered set. For an ordered set (X, R), we have $R(x) = \{y \in X : xRy\}$ and $R^{-1}(X_0) = \{y \in X : yRx, \text{for some } x \in X_0\}$, where $X_0 \subset X$. Then $R(x)$ is an up-set, and $R^{-1}(X_0)$ is a down-set.

5.3 Duality for $\mathbb{ISP}(\mathcal{L})$

In this section, we shall establish a duality for $\mathbb{ISP}(\mathcal{L})$. To establish a duality for $\mathbb{ISP}(\mathcal{L})$, we consider some term functions. It is important to note that $\mathcal{L}$ is dualizable in terms of discrete topology. The term function (also known as a polynomial function) is defined according to [5]. We now recall the concept of topological dualizability from [52].

Definition 5.3.1 ([52]). *Let $\mathcal{F}$ be a finite algebra. Then $\mathcal{F}$ is said to be dualizable in respect of a topology defined on $\mathcal{F}$ if and only if $\forall n \in \omega$, $Tf(\mathcal{F}) = C(\mathcal{F}^n, \mathcal{F})$, where $Tf(\mathcal{F})$, and $C(\mathcal{F}^n, \mathcal{F})$ are denote the set of all n-ary term functions on $\mathcal{F}$, and the set of all continuous functions from $\mathcal{F}^n$ to $\mathcal{F}$, respectively.*

Definition 5.3.2. *For each $\ell \in \mathcal{L}$, we consider a term function $T_\ell : \mathcal{L} \longrightarrow \mathcal{L}$ defined by*

$$T_\ell(s) = \begin{cases} 1 & s = \ell \\ 0 & s \neq \ell \end{cases}$$

From a logical perspective, $T_\ell(s)$ suggests that the truth value of a proposition s is exactly ℓ, where ℓ is an element of $\mathcal{L}$ which is the algebra of truth values.

Definition 5.3.3. *For each $\ell \in \mathcal{L}$, we consider a term function $\chi_\ell : \mathcal{L} \longrightarrow \mathcal{L}$ defined by*

$$\chi_\ell(s) = \begin{cases} \ell & s = 1 \\ 0 & otherwise \end{cases}$$

It should be emphasized that homomorphism commutes with the term functions.

Category:$Pspa$

Definition 5.3.4. *We consider a category PSpa as follows:*

1. *Object: An object in PSpa is a triple (X, τ, R), where (X, τ) is a compact space, and R is a partial order relation on X such that the following condition hold:*

   (i) if $x \not R y$, then for some clopen up-set W of X such that $x \in W$ but $y \notin W$.

2. *Arrow: An arrow $\psi : (X, \tau_1, R_1) \longrightarrow (Y, \tau_2, R_2)$ in PSpa is a continuous map $\psi : (X, \tau_1) \longrightarrow (Y, \tau_2)$, which is order preserving i.e., for any $x, y \in X$, if xR_1y then $\psi(x)R_2\psi(y)$.*

For $A \in \mathbb{ISP}(\mathcal{L})$, $(HOM_{ISP(\mathcal{L})}(A, \mathcal{L}), \tau, R)$ is an ordered topological space, where the topology τ is generated by $\{\langle a \rangle : a \in A\}$, $\langle a \rangle = \{v \in HOM_{ISP(\mathcal{L})}(A, \mathcal{L}) : v(a) = 1\}$. Then for each $a \in A$, the set $\langle a \rangle$ is a clopen up-set.

Functors: $\mathcal{G}$ and $\mathcal{C}$

Definition 5.3.5. *We define a functor $\mathcal{G} : ISP(\mathcal{L}) \longrightarrow PSpa$ as follows:*

- *$\mathcal{G}$ acts on an object A in $ISP(\mathcal{L})$ as $\mathcal{G}(A) = (HOM_{ISP(\mathcal{L})}(A, \mathcal{L}), \tau, R_A)$.*
- *$\mathcal{G}$ acts on an arrow $f : A \longrightarrow B$ in $ISP(\mathcal{L})$ as $\mathcal{G}(f) : \mathcal{G}(B) \longrightarrow \mathcal{G}(A)$ defined by $\mathcal{G}(f)(\phi) = \phi \circ f$, $\phi \in \mathcal{G}(B)$.*

We shall now verify the well-definedness of the functor $\mathcal{G}$.

Lemma 5.3.1. *For an object A in $ISP(\mathcal{L})$, $(HOM_{ISP(\mathcal{L})}(A, \mathcal{L}), R_A)$ is an object in $PSpa$.*

Proof. $HOM_{ISP(\mathcal{L})}(A, \mathcal{L})$ is a compact set as $\mathcal{L}^A$ with the product topology is compact, and $HOM_{ISP(\mathcal{L})}(A, \mathcal{L})$ is closed in the defined topology τ, which can be induced by the product topology on $\mathcal{L}^A$.
Now, if $v \not R_A w$, then there exists an element $a \in A$ such that $v(a) = 1$ and $w(a) \neq 1$. Thus $v \in \langle a \rangle$, and $w \in \langle a \rangle^c$. □

Lemma 5.3.2. *For an arrow f in $ISP(\mathcal{L})$, $\mathcal{G}(f)$ is an arrow in $PSpa$.*

Proof. For a given arrow $f : A \longrightarrow B$ in $ISP(\mathcal{L})$, $\mathcal{G}$ acts on f as $\mathcal{G}(f) : \mathcal{G}(B) \longrightarrow \mathcal{G}(A)$ defined by $\mathcal{G}(f)(\phi) = \phi \circ f$. It is observed that for each $a \in A$, $\mathcal{G}(f)^{-1}(\langle a \rangle) = \{v \in \mathcal{G}(B) : v \circ f(a) = 1\} = \langle f(a) \rangle$. Thus, $\mathcal{G}(f)$ is a continuous map.
Now for any two members v, w of $\mathcal{G}(B)$, if $vR_B w$, then we have $v(b) \leq w(b)$, $\forall b \in B$. Henceforth, $v(f(a)) \leq w(f(a))$, $\forall a \in A$, and thus $\mathcal{G}(f)(v) R_A \mathcal{G}(f)(w)$. □

Therefore, the functor $\mathcal{G}$ is well-defined by Lemma 5.3.1 and Lemma 5.3.2.

Definition 5.3.6. *We define a functor $\mathcal{C} : PSpa \longrightarrow ISP(\mathcal{L})$ as follows:*

- *$\mathcal{C}$ acts on an object (S, R) in $PSpa$ as $\mathcal{C}(S, R) = HOM_{PSpa}((S, R), (\mathcal{L}, \leq))$.*
- *$\mathcal{C}$ acts on an arrow $f : (S_1, R_1) \longrightarrow (S_2, R_2)$ in $PSpa$ as $\mathcal{C}(f) : \mathcal{C}(S_2, R_2) \longrightarrow \mathcal{C}(S_1, R_1)$ defined by $\mathcal{C}(f)(\phi) = \phi \circ f$, $\phi \in \mathcal{C}(S_2, R_2)$.*

Note 5.3.1. *For each object (S, R) in PSpa, the set $HOM_{PSpa}((S, R), (\mathcal{L}, \leq))$ is endowed with operations $\vee, \wedge, 0, 1$ which are defined pointwise i.e., for any $f, g \in HOM_{PSpa}((S, R), (\mathcal{L}, \leq))$, $(f \vee g)(s) = f(s) \vee g(s)$, $(f \wedge g)(s) = f(s) \wedge g(s)$ and the operations $0, 1$ are treated as constant functions with values 0 and 1, respectively.*

The following Lemmas 5.3.3 and 5.3.4 show the well-definedness of the functor $\mathcal{C}$.

Lemma 5.3.3. *For an object (S, R) in PSpa, $(HOM_{PSpa}((S, R), (\mathcal{L}, \leq)), \vee, \wedge, 0, 1)$ is an object in $ISP(\mathcal{L})$.*

Proof. This arises from the fact that $(HOM_{PSpa}((S, R), (\mathcal{L}, \leq)), \vee, \wedge, 0, 1)$ is a subalgebra of a direct power $\mathcal{L}^S$ of $\mathcal{L}$. Consequently, $(HOM_{PSpa}((S, R), (\mathcal{L}, \leq)), \vee, \wedge, 0, 1)$ is an object in $ISP(\mathcal{L})$. □

Lemma 5.3.4. *For an arrow $f : (S_1, R_1) \longrightarrow (S_2, R_2)$ in PSpa, $\mathcal{C}(f)$ is an arrow in $ISP(\mathcal{L})$.*

Proof. Well-definedness of the map $\mathcal{C}(f)$ is followed by the construction of $\mathcal{C}(f)$, and the fact that ϕ is an order-preserving continuous map. It is easy to follow that $\mathcal{C}(f)$ preserves all the defined operations. Thus, $\mathcal{C}(f)$ is a homomorphism between objects in $ISP(\mathcal{L})$. Consequently, $\mathcal{C}(f)$ is an arrow in $ISP(\mathcal{L})$. □

Theorem 5.3.1. *For an object A in $ISP(\mathcal{L})$, A is isomorphic to $\mathcal{C} \circ \mathcal{G}(A)$ in $ISP(\mathcal{L})$.*

Proof. Define a map $\sigma_A : A \longrightarrow \mathcal{C} \circ \mathcal{G}(A)$ by $\sigma_A(a)(v) = v(a)$, where $a \in A$, and $v \in \mathcal{G}(A)$. Now it is easily observed that, for each $a \in A$, $\sigma_A(a) \in \mathcal{C} \circ \mathcal{G}(A)$. For each $a \in A$, and $s \in \mathcal{L}$, $\sigma_A(a)^{-1}(\{s\}) = \{v \in HOM_{ISP(\mathcal{L})}(A, \mathcal{L}) : v(a) = s\} = \{v \in HOM_{ISP(\mathcal{L})}(A, \mathcal{L}) : T_s(v(a)) = 1\} = \langle T_s(a) \rangle$. Henceforth, $\{\sigma_A(a) : a \in A\} \subseteq \mathcal{C} \circ \mathcal{G}(A)$. So, σ_A is well-defined. Since the operations are defined point-wise on $\mathcal{C} \circ \mathcal{G}(A)$, σ_A preserves all the operations. Thus, σ_A is a homomorphism.
We now show that σ_A is one-one. For any members $a, b \in A$, if $a \neq b$, then we claim that $\sigma_A(a) \neq \sigma_A(b)$. As $A \in \mathbb{ISP}(\mathcal{L})$, by definition of $\mathbb{ISP}(\mathcal{L})$ we have A is isomorphic to a sub-algebra of direct power $\mathcal{L}^J$ of $\mathcal{L}$. Therefore, $a, b \in \mathcal{L}^J$, and thus $a(\lambda) \neq b(\lambda)$, for some $\lambda \in J$. Define a homomorphism $v : A \longrightarrow \mathcal{L}$ by $v(a) = a(\lambda)$. Then, v is well-defined, since the operations are defined pointwise on $\mathcal{L}^J$. Henceforth, $\sigma_A(a)(v) \neq \sigma_A(b)(v)$.
Now if $\psi \in \mathcal{C} \circ \mathcal{G}(A)$, we claim that $\psi = \sigma_A(a)$, for some $a \in A$. For $\ell \in \mathcal{L}$, $\psi^{-1}(\{\ell\})$ is a clopen subset of $\mathcal{G}(A)$, and hence $\psi^{-1}(\{\ell\})$ is a compact subset of $\mathcal{G}(A)$. So, $\psi^{-1}(\{\ell\})$ can be expressed as finite union of basis open sets in $HOM_{ISP(\mathcal{L})}(A, \mathcal{L})$.

Let $\psi^{-1}(\{\ell\}) = \langle a_\ell \rangle$. Now we claim that $\sigma_A(\bigvee_{\ell\in\mathcal{L}} \chi_\ell(a_\ell)) = \psi$. If $v \in \psi^{-1}(\{\ell\})$, then $\sigma_A(\bigvee_{\ell\in\mathcal{L}} \chi_\ell(a_\ell))(v) = \psi(v)$. Also, if $v \in \psi^{-1}(\{p\})$, then $v(\chi_\ell(a_\ell)) = \ell$, if $\ell = p$, and $v(\chi_\ell(a_\ell)) = 0$, if $\ell \neq p$. Consequently, $\sigma_A(\bigvee_{\ell\in\mathcal{L}} \chi_\ell(a_\ell))(v) = v(\bigvee_{\ell\in\mathcal{L}} \chi_\ell(a_\ell)) = \bigvee_{\ell\in\mathcal{L}} v(\chi_\ell(a_\ell)) = p = \psi(v)$. Therefore, $\psi = \sigma_A(a)$, where $a = \bigvee_{\ell\in\mathcal{L}} \chi_\ell(a_\ell)$. Hence, σ_A is surjective. Finally, we have σ_A is an isomorphism. This completes the proof. □

Theorem 5.3.2. *For an object (S, R) in PSpa, (S, R) is homeomorphic to $\mathcal{G} \circ \mathcal{C}(S, R)$.*

Proof. Define a map $\delta_S : (S, R) \longrightarrow \mathcal{G} \circ \mathcal{C}(S, R)$ by $\delta_S(s)(f) = f(s)$. For each $s \in S$, $\delta_S(s)$ is a homomorphism, as the operations are defined pointwise on $\mathcal{C}(S, R)$. Therefore, δ_S is well-defined. Now we observe that if $f \in \mathcal{C}(S, R)$, then $\delta_S^{-1}(\langle f \rangle) = \{s \in S : \delta_S(s)(f) = 1\} = f^{-1}(\{1\})$, is an open up-set in (S, R). δ_S is also an order preserving map, because if $s_1 R s_2$ then $f(s_1) \leq f(s_2)$. Therefore, $\delta_S(s_1) R' \delta_S(s_2)$, where R' is interpreted as a partial order relation on $\mathcal{G} \circ \mathcal{C}(S, R)$ in accordance with Definition 5.2.5.
Let $s \neq t$ in S. We claim that $\delta_S(s) \neq \delta_S(t)$. The claim is demonstrated by the fact that $\mathcal{G} \circ \mathcal{C}(S, R)$ is an object in $PSpa$, ensuring that it is zero-dimensional and Hausdorff. Thus, there exists $\phi \in \mathcal{C}(S, R)$ such that $\phi(s) \neq \phi(t)$. As a result, δ_S is one-one.
We now show that δ_S is surjective. We already observe that $\{\delta_S(s) : s \in S\} \subseteq \mathcal{G} \circ \mathcal{C}(S, R)$. As $\delta_S(S)$ is compact subset of $\mathcal{G} \circ \mathcal{C}(S, R)$, hence $\delta_S(S)$ is closed. If δ_S is not surjective, then there exists $v \in \mathcal{G} \circ \mathcal{C}(S, R)$ such that $v \notin \delta_S(S)$ i.e., $v \neq \delta_S(s)$, for any $s \in S$. Therefore, there exists a clopen up-set W in $\mathcal{G} \circ \mathcal{C}(S, R)$ containing v, but not $\delta_S(S)$. As W is compact, so W can be expressed as finite union of basis open sets. We may consider $W = \langle f \rangle \wedge \langle g \rangle^c$, for some $f, g \in \mathcal{C}(S, R)$. Now $\delta_S^{-1}(W) = \delta_S^{-1}(\langle f \rangle) \wedge \delta_S^{-1}(\langle g \rangle^c)$. Since $\delta_S^{-1}(W) = \emptyset$, hence $\delta_S^{-1}(\langle f \rangle) \subseteq \delta_S^{-1}(\langle g \rangle)$. Therefore, we have $f^{-1}(\{1\}) \subseteq g^{-1}(\{1\})$. Then, $T_1(f) \leq T_1(g)$. Consequently, we get $v(T_1(f)) = 1$ and $v(T_1(g)) = 1$. Since, $v(g) \neq 1$, this contradicts the fact that $v(T_1(g)) = 1$. Therefore, $\delta_S(S) = \mathcal{G} \circ \mathcal{C}(S, R)$.
It is easy to observed that δ_S is a closed map.
We now demonstrate that for any two members $s_1, s_2 \in S$, if $\delta_S(s_1) R' \delta_S(s_2)$ then $s_1 R s_2$. We demonstrate an equivalent statement, which reads as follows: if $s_1 \not R s_2$, then $\delta_S(s_1) \not R' \delta_S(s_2)$. Since $s_1 \not R s_2$, then there exists a clopen set $\mathcal{U} \subset S$ such that

$s_2 \in \mathcal{U}$ and $\mathcal{U} \cap R(s_1) = \emptyset$. Define $f : S \longrightarrow \mathcal{L}$ by

$$f(s) = \begin{cases} 0 & s \in \mathcal{U} \\ 1 & s \notin \mathcal{U} \end{cases}$$

Then, f is continuous, and $f(s_1) \nleq f(s_2)$. Hence, $\delta_S(s_1)(f) \not R' \delta_S(s_2)(f)$. Therefore, for any $s_1, s_2 \in S$, $s_1 R s_2$ iff $\delta_S(s_1) R' \delta_S(s_2)$. Now δ_S^{-1} is an order-preserving map, since δ_S is bijective, and the relation $s_1 R s_2 \iff \delta_S(s_1) R' \delta_S(s_2)$ holds. □

Theorem 5.3.3. *The category $ISP(\mathcal{L})$ is dually equivalent to the category $PSpa$.*

Proof. Let $ID_{ISP(\mathcal{L})}$, and ID_{PSpa} denote the identity functors on $ISP(\mathcal{L})$ and $PSpa$, respectively. We consider two natural transformations $\sigma : ID_{ISP(\mathcal{L})} \longrightarrow \mathcal{C} \circ \mathcal{G}$, and $\delta : ID_{PSpa} \longrightarrow \mathcal{G} \circ \mathcal{C}$. Now for each object A of $ISP(\mathcal{L})$, we consider $\sigma_A : A \longrightarrow \mathcal{C} \circ \mathcal{G}(A)$ by $\sigma_A(a)(v) = v(a)$, where $v \in \mathcal{G}(A)$. Moreover, for an object (S, R) of $PSpa$, consider $\delta_S(s)(f) = f(s)$, where $f \in \mathcal{C}(S, R)$. Then, it is easily shown that σ and δ are natural transformations. Also, σ and δ are natural isomorphisms by Theorem 5.3.1, and Theorem 5.3.2. Therefore, we can conclude that the category $ISP(\mathcal{L})$ is dually equivalent to the category $PSpa$. □

We now use Theorem 5.3.3 to develop a duality for $\mathbb{ISP_I}(\mathcal{L})$.

5.4 Duality for $\mathbb{ISP_I}(\mathcal{L})$

Category: $Hspa$

Definition 5.4.1. *We take a category $HSpa$ as follows.*

1. *Object: An object in $HSpa$ is defined by a triple (S, τ, R) such that (S, τ, R) is an object in $PSpa$ which additionally satisfies the following condition:*

   (i) if $\mathcal{C}$ is a clopen subset of S, then $R^{-1}(\mathcal{C})$ is a clopen down-set of S.

2. *Arrow: An arrow $\phi : (S_1, \tau_1, R_1) \longrightarrow (S_2, \tau_2, R_2)$ in $HSpa$ is an arrow in $PSpa$ which satisfies the following condition:*

   (i) for any members $s_1 \in S_1$, and $s_2 \in S_2$, if $\phi(s_1) R_2 s_2$ then there exists $s \in S_1$ such that $s_1 R_1 s$ and $\phi(s) = s_2$.

Functors: $\mathcal{G}_I$ and $\mathcal{C}_I$

Definition 5.4.2. *We define a functor $\mathcal{G}_I : ISP_I(\mathcal{L}) \longrightarrow HSpa$ as follows:*

- *$\mathcal{G}_I$ acts on an object $(A, \rightarrow)$ in $ISP_I(\mathcal{L})$ as $\mathcal{G}_I(A) = (HOM_{ISP_I(\mathcal{L})}(A, \mathcal{L}), R_A)$.*
- *$\mathcal{G}_I$ acts on an arrow $f : A \longrightarrow B$ in $ISP_I(\mathcal{L})$ as $\mathcal{G}_I(f) : \mathcal{G}_I(B) \longrightarrow \mathcal{G}_I(A)$ defined by $\mathcal{G}_I(f)(\phi) = \phi \circ f$, $\phi \in \mathcal{G}_I(B)$.*

Lemma 5.4.1. *For an object $(A, \rightarrow)$ in $ISP_I(\mathcal{L})$, $\mathcal{G}_I(A)$ is an object in $HSpa$.*

Proof. By Lemma 5.3.1, $\mathcal{G}_I(A)$ is an object in $PSpa$. We shall show that for each clopen subset $\mathcal{U}$ of S, $R_A^{-1}(\mathcal{U})$ is a clopen down-set. Since, $\{\langle a\rangle : a \in A\}$ is a clopen basis for the topology on $HOM_{ISP(\mathcal{L})}(A, \mathcal{L})$, and R_A^{-1} preserves union, therefore we show that $R_A^{-1}(\langle a\rangle)$ is a clopen down-set, for each $a \in A$. We now verify that $R_A^{-1}(\langle a\rangle) = \langle a \rightarrow 0\rangle^c$. If $v \in \langle a \rightarrow 0\rangle^c$, then $v(a \rightarrow 0) \neq 1$. By Definition 5.2.6, we have $v(a \rightarrow 0) = \bigwedge\{u(a \rightarrow 0) : vR_A u\} = \bigwedge\{u(a) \rightarrow 0 : vR_A u\}$. Since, $u(a) \rightarrow 0 = 0$ or 1, so that there exists $u \in HOM_{ISP(\mathcal{L})}(A, \mathcal{L})$ such that $vR_A u$ and $u(a) \rightarrow 0 = 0$. Thus, $u(a) = 1$. Henceforth $v \in R_A^{-1}(\langle a\rangle)$. Again, if $v \in R_A^{-1}(\langle a\rangle)$, then there exists $u \in \langle a\rangle$ such that $vR_A u$. So $v(a \rightarrow 0) = 0$. Hence, $v \in \langle a \rightarrow 0\rangle^c$. Finally, we have $R_A^{-1}(\langle a\rangle) = \langle a \rightarrow 0\rangle^c$, a clopen down-set. □

Lemma 5.4.2. *For an arrow $f : (A, \rightarrow) \longrightarrow (B, \rightarrow)$ in $ISP_I(\mathcal{L})$, $\mathcal{G}_I(f)$ is an arrow in $HSpa$.*

Proof. Here $\mathcal{G}_I(f) : \mathcal{G}_I(B) \longrightarrow \mathcal{G}_I(A)$ is defined as $\mathcal{G}_I(f)(\phi) = \phi \circ f$, where $\phi \in \mathcal{G}_I(B)$. Then by Lemma 5.3.2, $\mathcal{G}_I(f)$ is an arrow in $PSpa$. Next, we show the condition found in the arrow part of Definition 5.4.1. We demonstrate the equivalent condition that $\mathcal{G}_I(f)(R_B(v)) = R_A(\mathcal{G}_I(f)(v))$, $\forall v \in \mathcal{G}_I(B)$. We verify that if $\psi \notin \mathcal{G}_I(f)(R_B(v))$, then $\psi \notin R_A(\mathcal{G}_I(f)(v))$. Since, $\psi \notin \mathcal{G}_I(f)(R_B(v))$, then $\psi \neq \mathcal{G}_I(f)(w)$, for any $w \in HOM_{ISP(\mathcal{L})}(B, \mathcal{L})$ such that $vR_B w$. Then by the object part of Definition 5.3.4, we can take $\psi \in \langle a\rangle$ and $\mathcal{G}_I(f)(w) \in \langle a\rangle^c$. Suppose for contradiction, if $\psi \in R_A(\mathcal{G}_I(f)(v))$, then $(v \circ f)R_A\psi$. Therefore, by definition of R_A, we have $(v \circ f)(a) \leq \psi(a)$, $\forall a \in A$. Then $(v \circ f)(a \rightarrow 0) \leq \psi(a \rightarrow 0)$. But $\psi(a \rightarrow 0) = \psi(a) \rightarrow 0 = 0$, as $\psi(a) = 1$. Now $\mathcal{G}_I(f)(v)(a \rightarrow 0) = \bigwedge\{\mathcal{G}_I(f)(w)(a \rightarrow 0) : vR_B w\}$. Since $\mathcal{G}_I(f)(v)(a \rightarrow 0) = 0$, hence there exists $w \in HOM_{ISP(\mathcal{L})}(B, \mathcal{L})$ such that $vR_B w$ and $\mathcal{G}_I(f)(w)(a \rightarrow 0) = 0$. We see that $\mathcal{G}_I(f)(w)(a \rightarrow 0) = \mathcal{G}_I(f)(w)(a) \rightarrow 0 = 0 \Rightarrow \mathcal{G}_I(f)(w)(a) = 1$. This contradicts the assumption that $\mathcal{G}_I(f)(w) \in \langle a\rangle^c$. Hence $\psi \notin R_A(\mathcal{G}_I(f)(v))$. Thus equivalently we have, if $\psi \in R_A(\mathcal{G}_I(f)(v))$ then

$\psi \in \mathcal{G}_I(f)(R_B(v))$. Therefore, $R_A(\mathcal{G}_I(f)(v)) \subseteq \mathcal{G}_I(f)(R_B(v))$. It is easy to show that $\mathcal{G}_I(f)(R_B(v)) \subseteq R_A(\mathcal{G}_I(f)(v))$. $\square$

Therefore, the functor $\mathcal{G}_I$ is well-defined by Lemma 5.4.1, and Lemma 5.4.2.

Definition 5.4.3. *We define a functor $\mathcal{C}_I : HSpa \longrightarrow ISP_I(\mathcal{L})$ as follows:*

- *$\mathcal{C}_I$ acts on an object (S, R) in $HSpa$ as $\mathcal{C}_I(S, R) = (HOM_{PSpa}((S, R), (\mathcal{L}, \leq)), \rightarrow)$.*
- *$\mathcal{C}_I$ acts on an arrow $f : (S_1, R_1) \longrightarrow (S_2, R_2)$ in $HSpa$ as $\mathcal{C}_I(f) : \mathcal{C}_I(S_2, R_2) \longrightarrow \mathcal{C}_I(S_1, R_1)$ defined by $\mathcal{C}_I(f)(\phi) = \phi \circ f$, where $\phi \in \mathcal{C}_I(S_2, R_2)$.*

Lemma 5.4.3. *For an object (S, R) in $HSpa$, $\mathcal{C}_I(S, R)$ is an object in $ISP_I(\mathcal{L})$.*

Proof. As per Note 5.3.1, $HOM_{PSpa}((S, R), (\mathcal{L}, \leq))$ is an object in $ISP(\mathcal{L})$. To prove $\mathcal{C}_I(S, R)$ is an object in $ISP_I(\mathcal{L})$, we shall demonstrate that if $f, g \in \mathcal{C}_I(S, R)$, then $f \rightarrow g \in \mathcal{C}_I(S, R)$. Now $(f \rightarrow g)^{-1}(\{\ell\}) = \{s \in S : (f \rightarrow g)(s) = \ell\}$. By Definition 5.2.2, we observe that $(f \rightarrow g)^{-1}(\{\ell\}) = R^{-1}(g^{-1}(\{\ell\})) \cap (R^{-1}(f^{-1}(\{\ell\})))^c$. Then by Definition 5.4.1, $R^{-1}(g^{-1}(\{\ell\})) \cap (R^{-1}(f^{-1}(\{\ell\})))^c$ is a clopen set in S. Hence $f \rightarrow g \in \mathcal{C}_I(S, R)$. As a result, the intuitionistic implication operation($\rightarrow$) is well-defined. Therefore, $\mathcal{C}_I(S, R)$ is a sub-algebra of intuitionistic power $\mathcal{L}^S$ of $\mathcal{L}$. So, $\mathcal{C}_I(S, R)$ is an object in $ISP_I(\mathcal{L})$. $\square$

Lemma 5.4.4. *For an arrow $f : (S_1, R_1) \longrightarrow (S_2, R_2)$ in $HSpa$, $\mathcal{C}_I(f)$ is an arrow in $ISP_I(\mathcal{L})$.*

Proof. $\mathcal{C}_I(f)$ is an arrow in $ISP(\mathcal{L})$, according to Lemma 5.3.4. The only thing left to prove is that $\mathcal{C}_I(f)(g_1 \rightarrow g_2) = \mathcal{C}_I(f)(g_1) \rightarrow \mathcal{C}_I(f)(g_2)$. Now for $s_1 \in S_1$, $\mathcal{C}_I(f)(g_1 \rightarrow g_2)(s_1) = (g_1 \rightarrow g_2) \circ f(s_1) = \bigwedge\{g_1(y) \rightarrow g_2(y) : f(s_1)R_2y\}$, and $(\mathcal{C}_I(f)(g_1) \rightarrow \mathcal{C}_I(f)(g_2))(s_1) = (g_1 \circ f \rightarrow g_2 \circ f)(s_1) = \bigwedge\{(g_1 \circ f)(s_2) \rightarrow (g_2 \circ f)(s_2) : s_1R_1s_2\} = \bigwedge\{g_1(f(s_2)) \rightarrow g_2(f(s_2)) : s_1R_1s_2\}$. Because f is an order preserving map, we notice that $\bigwedge\{g_1(y) \rightarrow g_2(y) : f(s_1)R_2y\} \leq \bigwedge\{g_1(f(s_2)) \rightarrow g_2(f(s_2)) : s_1R_1s_2\}$. Furthermore, f meets the requirement stated in the arrow part of Definition 5.4.1, so we have $\bigwedge\{g_1(f(s_2)) \rightarrow g_2(f(s_2)) : s_1R_1s_2\} \leq \bigwedge\{g_1(y) \rightarrow g_2(y) : f(s_1)R_2y\}$. Thus $\mathcal{C}_I(f)(g_1 \rightarrow g_2) = \mathcal{C}_I(f)(g_1) \rightarrow \mathcal{C}_I(f)(g_2)$. $\square$

Therefore, the functor $\mathcal{C}_I$ is well-defined by Lemma 5.4.3, and Lemma 5.4.4.

Theorem 5.4.1. *For an object $(A, \rightarrow)$ in $ISP_I(\mathcal{L})$, A is isomorphic to $\mathcal{C}_I \circ \mathcal{G}_I(A)$ in $ISP_I(\mathcal{L})$.*

Proof. Define $\sigma_{(A,\to)} : A \longrightarrow \mathcal{C}_I \circ \mathcal{G}_I(A)$ by $\sigma_{(A,\to)}(a)(v) = v(a)$, where $a \in A$, and $v \in \mathcal{G}_I(A)$. Theorem 5.3.1 proves that $\sigma_{(A,\to)}$ is an isomorphism in $ISP(\mathcal{L})$. It is therefore necessary to demonstrate that $\sigma_{(A,\to)}(a \to b) = \sigma_{(A,\to)}(a) \to \sigma_{(A,\to)}(b)$. Now $[\sigma_{(A,\to)}(a) \to \sigma_{(A,\to)}(b)](v) = \bigwedge\{\sigma_{(A,\to)}(a)(w) \to \sigma_{(A,\to)}(b)(w) : vR_Aw\} = \bigwedge\{w(a) \to w(b) : vR_Aw\}$, where R_A is a partial order relation on $HOM_{ISP_I(\mathcal{L})}(A, \mathcal{L})$ defined in line with Definition 5.2.5. It is seen from Definition 5.2.6 that $\bigwedge\{w(a) \to w(b) : vR_Aw\} = v(a \to b) = \sigma_{(A,\to)}(a \to b)(v)$. Hence, $\sigma_{(A,\to)}(a \to b) = \sigma_{(A,\to)}(a) \to \sigma_{(A,\to)}(b)$. □

Theorem 5.4.2. *For an object (S, R) in HSpa, (S, R) is isomorphic to $\mathcal{G}_I \circ \mathcal{C}_I(S, R)$ in the category HSpa.*

Proof. Define $\delta_{(S,R)} : (S, R) \longrightarrow \mathcal{G}_I \circ \mathcal{C}_I(S, R)$ by $\delta_{(S,R)}(s)(f) = f(s)$, where $s \in S$, and $f \in \mathcal{C}_I(S, R)$. It is observed from Theorem 5.3.2 that $\delta_{(S,R)}$ is an isomorphism in the category $PSpa$. In Theorem 5.3.2, we see that the relation $s_1Rs_2 \iff \delta_S(s_1)R'\delta_S(s_2)$ holds for all $s_1, s_2 \in S$, where R' is the partial order relation on $\mathcal{G} \circ \mathcal{C}(S, R)$. So by definition of $\delta_{(S,R)}$, we can conclude that $\delta_{(S,R)}$ satisfies the relation $s_1Rs_2 \iff \delta_{(S,R)}(s_1)R'\delta_{(S,R)}(s_2)$. Because $\delta_{(S,R)}$ is bijective and fulfils the relation $s_1Rs_2 \iff \delta_{(S,R)}(s_1)R'\delta_{(S,R)}(s_2)$, it is readily proved that $\delta_{(S,R)}$ and $\delta^{-1}_{(S,R)}$ satisfy the requirement stated in the arrow part of Definition 5.4.1. Thus, $\delta_{(S,R)}$ is an isomorphism in $Hspa$. This wraps up the proof.

□

Finally, we obtain the duality theorem for $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$.

Theorem 5.4.3. *The category $ISP_I(\mathcal{L})$ is dually equivalent to the category HSpa.*

Proof. Let $ID_{ISP_I(\mathcal{L})}$, and ID_{HSpa} denote the identity functors on $ISP_I(\mathcal{L})$ and $HSpa$, respectively. We consider two natural transformations $\sigma : ID_{ISP_I(\mathcal{L})} \longrightarrow \mathcal{C}_I \circ \mathcal{G}_I$, and $\delta : ID_{HSpa} \longrightarrow \mathcal{G}_I \circ \mathcal{C}_I$. Then for each object $(A, \to)$ of $ISP_I(\mathcal{L})$, we define $\sigma_{(A,\to)} : A \longrightarrow \mathcal{C}_I \circ \mathcal{G}_I(A)$ by $\sigma_{(A,\to)}(a)(v) = v(a)$, $v \in \mathcal{G}_I(A)$. Moreover, for an object (S, R) in $HSpa$, we define $\delta_{(S,R)} : (S, R) \longrightarrow \mathcal{G}_I \circ \mathcal{C}_I(S, R)$ by $\delta_{(S,R)}(s)(f) = f(s)$, $f \in \mathcal{C}_I(S, R)$. Then, it is simple to verify that σ and δ are, in fact, natural transformations. Theorems 5.4.1 and 5.4.2 demonstrate that σ and δ are natural isomorphisms. Thus, the categories $ISP_I(\mathcal{L})$ and $HSpa^{op}$ are equivalent. □

We have extended the duality: $ISP(\mathcal{L}) \equiv Pspa^{op}$ to the duality: $ISP_I(\mathcal{L}) \equiv Hspa^{op}$. It would be difficult to achieve an intuitionistic version of natural duality theory without the innovative concept of $\mathbb{ISP}_{\mathbb{I}}$.

5.5 Conclusion

To wrap up this chapter, we have introduced the novel notion of $\mathbb{ISP}_{\mathbb{I}}$ and developed an intuitionistic version of natural duality theory. As a result, this extended version of the natural duality theory incorporated the Esakia duality for the class of all Heyting algebras into the natural duality theory. $\mathbb{ISP}_{\mathbb{I}}$ thus serves as a natural foundation for the theory of intuitionistic natural dualities. Technically, we have began by developing duality theory for $\mathbb{ISP}(\mathcal{L})$. While switching our interest from $\mathbb{ISP}(\mathcal{L})$ to $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$, we indicated the intutionistic Kripke condition for $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$, where $\mathcal{L}$ is a finite algebra with a bounded lattice reduct. As a major finding, we obtained a duality for $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$.

Chapter 6

Bitopological duality for many-valued modal logic

6.1 Introduction

This chapter aims to demonstrate an intriguing duality theory for algebras of Fitting's many-valued modal logic in the context of bitopological languages. Thus, this has led to an extension of the natural duality theory for modal algebras in bitopological context. Algebraic axiomatization of a modified version of Fitting's Heyting-valued modal logic has already been addressed in [20]. In addition to algebraic axiomatizations with the completeness of Fitting's Heyting-valued modal logic, topological duality theorems have also been developed. Bitopological methods have already been employed to investigate duality theory for Fitting's Heyting-valued logic (see Chapter 4). However, bitopological approaches have not been used to develop duality for Fitting's many-valued modal logic. This chapter attempts to fill that gap.

Maruyama [21] proposed Jónsson-Tarski topological duality (see [9, 29, 54]) for $\mathcal{L}$-$\mathcal{ML}$-algebras (algebras of Fitting's Heyting-valued modal logic). Jónsson-Tarski duality for $\mathcal{L}$-$\mathcal{ML}$-algebras is essentially a $\mathcal{L}$-valued version of Jónsson-Tarski duality for modal algebras.

We aim to construct a bitopological duality for algebras of Fitting's Heyting- valued modal logic by setting up a notion of $PRBS_{\mathcal{L}}$ as a category of $\mathcal{L}$-valued pairwise

The results of this chapter appear in [61] **Das, Litan Kumar., Ray, Kumar sanakar., Mali, Prakash Chandra.: Duality for Fitting's Heyting-valued modal logic via Bitopology and Bi-Vietoris coalgebra. Theoretical Computer Science, Elsevier (Under Review). https://doi.org/10.48550/arXiv.2312.16276**

Boolean spaces with a relation. As a result, natural duality theory for modal algebras is extended in the context of bitopological languages. The main result is bitopological duality for $\mathcal{L}$-$\mathcal{ML}$-algebras, where $\mathcal{L}$ is a semi-primal algebra having a bounded lattice reduct. Our general theory extends the Jónsson-Tarski duality in the setting of bitopological language.

6.2 The notion of Bitopological spaces

We assume that the readers are familiar with the basic concepts of topology and category theory. We refer the reader to [5, 36] for information on universal algebra and lattice theory. For category theory, we refer to [2].
A bitopological space is defined as a triple (X, τ_1, τ_2) in which (X, τ_1) and (X, τ_2) are topological spaces. Consider δ_1 and δ_2 represent, respectively, the collections of τ_1-closed sets and τ_2-closed sets. We set $\beta_1 = \tau_1 \cap \delta_2$ and $\beta_2 = \tau_2 \cap \delta_1$.

Definition 6.2.1 ([39]). *(i) A bitopological space (X, τ_1, τ_2) is said to be pairwise Hausdorff space if for every pair (x, y) of distinct points $x, y \in X$ there exists disjoint open sets $U_x \in \tau_1$ and $U_y \in \tau_2$ containing x and y, respectively.*

(ii) A bitopological space (X, τ_1, τ_2) is said to be pairwise zero-dimensional if β_1 is a basis for τ_1 and β_2 is a basis for τ_2.

(iii) A bitopological space (X, τ_1, τ_2) is said to be pairwise compact if the topological space (X, τ), where $\tau = \tau_1 \vee \tau_2$, is compact.

According to Alexander's Lemma (a classical result in general topology), the idea of pairwise compactness described in Definition 7.1.3 is equivalent to the condition that every cover $\{U : U \in \tau_1 \cup \tau_2\}$ of X has a finite subcover. A pairwise Boolean space is a bitopological space that is pairwise Hausdorff, pairwise zero-dimensional, and pairwise compact. A map $f : (P, \tau_1, \tau_2) \to (P_1, \tau_1^1, \tau_2^1)$ is said to be pairwise continuous if the map $f : (P, \tau_i) \to (P_1, \tau_i^1)$ is continuous for $i \in \{1, 2\}$. Pairwise Boolean spaces and pairwise continuous maps form a category, denoted by PBS.

Proposition 6.2.1 ([49]). *If T_1 and T_2 are subbasis for the topologies τ_1 and τ_2, respectively, then $T_1 \cup T_2$ is a subbasis for the topology $\tau_1 \vee \tau_2$.*

Proposition 6.2.2 ([49]). *Let (X, τ_1, τ_2) be a pairwise compact bitopological space. Consider a finite collection $\{C_i : C_i \in \delta_1 \cup \delta_2, i = 1, 2, \cdots, n\}$ of subsets of X. Then $\bigcap_{i=1}^{n} C_i$ is pairwise compact.*

It is clear from the above proposition that any τ_1-closed or τ_2-closed subset of a pairwise compact space X is pairwise compact.

6.3 Fitting's Heyting-valued modal logic

Fitting [23] proposed $\mathcal{L}$-valued logics and $\mathcal{L}$-valued modal logics for a finite distributive lattice $\mathcal{L}$ (i.e., $\mathcal{L}$ is a Heyting algebra) in 1991. Maruyama [20] introduced algebraic axiomatization of Fitting's logics. In [20] the author studied Fitting's Heyting-valued logic and Heyting-valued modal logic without regard for fuzzy truth constants other than 0 and 1, and added a new operation $T_\ell(-)$, $\ell \in \mathcal{L}$. From a logical perspective, $T_\ell(p)$ infers that the truth value of a proposition p is ℓ. The operations of $\mathcal{L}$-valued logic, denoted by $\mathcal{L}$-$\mathcal{VL}$, are $\vee, \wedge, \to, 0, 1$ and $T_\ell(-)$, $\ell \in \mathcal{L}$, where $\vee, \wedge, \to$ are binary operations, 0 and 1 are nullary operations and T_ℓ is a unary operation. For $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \to \ell_2$ means the pseudo-complement of ℓ_1 relative to ℓ_2.

In universal algebra, the concept of semi-primal algebra holds great significance. The semi-primal algebra concept will now be defined as follows.

Definition 6.3.1. *Let A be an algebra. Then a function $f : A^n \to A$, $n \in \mathbb{N}$, is said to be conservative $\iff$ for any $a_1, a_2, \cdots, a_n \in A$, $f(a_1, a_2, \cdots a_n)$ is in the subalgebra of A generated by $\{a_1, a_2, \cdots, a_n\}$. A finite algebra A is said to be a semi-primal algebra if every conservative function $f : A^n \to A$, $n \in \mathbb{N}$, is a term function of A.*

The following lemmas describe some term functions.

Lemma 6.3.1. *Let $\mathcal{L}$ be a semi-primal algebra having bounded lattice reduct. Define a function $f : \mathcal{L}^4 \to \mathcal{L}$ by*

$$f(\ell_1, \ell_2, \ell_3, \ell_4) = \begin{cases} \ell_3 & (\ell_1 = \ell_2) \\ \ell_4 & (\ell_1 \neq \ell_2) \end{cases}$$

Then, f is a term function of $\mathcal{L}$.

Lemma 6.3.2. *Let $\mathcal{L}$ be a semi-primal algebra having bounded lattice reduct. For every $\ell \in \mathcal{L}$, define $T_\ell : \mathcal{L} \to \mathcal{L}$ by*

$$T_\ell(\ell') = \begin{cases} 1 & (\ell' = \ell) \\ 0 & (\ell' \neq \ell) \end{cases}$$

Then, T_ℓ is a term function of $\mathcal{L}$.

Lemma 6.3.3. *Let $\mathcal{L}$ be a semi-primal algebra having bounded lattice reduct. Let $\ell \in \mathcal{L}$. Then the function $U_\ell : \mathcal{L} \to \mathcal{L}$ defined by*

$$U_\ell(\ell') = \begin{cases} 1 & (\ell' \geq \ell) \\ 0 & (\ell' \ngeq \ell) \end{cases}$$

, is a term function of $\mathcal{L}$.

Observation 6.3.1. *The term function $U_\ell : \mathcal{L} \to \mathcal{L}$ can alternatively be defined using T_ℓ as follows:*

$$U_\ell(\ell') = \bigvee \{T_{\ell_1}(\ell') : \ell \leq \ell_1, \ell_1 \in \mathcal{L}\}$$

It is simple to demonstrate that U_ℓ commutes with $\wedge$, i.e., $U_\ell(a \wedge b) = U_\ell(a) \wedge U_\ell(b)$ for all $a \in A$, where A is a $\mathcal{L}$-$\mathcal{VL}$- algebra. Furthermore, we note that $U_1(a) = T_1(a)$.

We now recall the idea of $\mathcal{L}$-$\mathcal{VL}$-algebras, which provides sound and complete semantics of $\mathcal{L}$-valued logic $\mathcal{L}$-$\mathcal{VL}$.

Definition 6.3.2 ([20])**.** *An algebraic structure $(\mathcal{A}, \wedge, \vee, \to, T_\ell(\ell \in \mathcal{L}), 0, 1)$ is said to be a $\mathcal{L}$-$\mathcal{VL}$-algebra iff for any $\ell_1, \ell_2 \in \mathcal{L}$, and $a, b \in \mathcal{A}$, the following conditions hold :*

(i) $(\mathcal{A}, \wedge, \vee, \to, T_\ell(\ell \in \mathcal{L}), 0, 1)$ *is a Heyting algebra;*

(ii) $T_{\ell_1}(a) \wedge T_{\ell_2}(b) \leq T_{\ell_1 \to \ell_2}(a \to b) \wedge T_{\ell_1 \wedge \ell_2}(a \wedge b) \wedge T_{\ell_1 \vee \ell_2}(a \vee b)$;
$T_{\ell_2}(a) \leq T_{T_{\ell_1}(\ell_2)}(T_{\ell_1}(a))$;

(iii) $T_0(0) = 1$; $T_\ell(0) = 0$ $(\ell \neq 0)$; $T_1(1) = 1$; $T_\ell(1) = 0$, *if* $\ell \neq 1$;

(iv) $\bigvee\{T_\ell(a) : \ell \in \mathcal{L}\} = 1$; $T_{\ell_1}(a) \vee (T_{\ell_2}(a) \to 0) = 1$;
$T_{\ell_1}(a) \wedge T_{\ell_2}(a) = 0$, $(\ell_1 \neq \ell_2)$;

(v) $T_1(T_\ell(a)) = T_\ell(a)$, $T_0(T_\ell(a)) = T_\ell(a) \to 0$, $T_{\ell_2}(T_{\ell_1}(a)) = 0$, $(\ell_2 \neq 0, 1)$;

(vi) $T_1(a) \leq a$, $T_1(a \wedge b) = T_1(a) \wedge T_1(b)$;

(vii) $\bigwedge_{\ell \in \mathcal{L}} (T_\ell(a) \leftrightarrow T_\ell(b)) \leq (a \leftrightarrow b)$.

Note 6.3.1. *The class of all $\mathcal{L}$-$\mathcal{VL}$-algebras forms a variety (in the sense of universal algebra). If $\mathcal{L} = \{0, 1\}$, then $\mathcal{L}$-$\mathcal{VL}$-algebras becomes Boolean algebras.*

Definition 6.3.3. *A function between $\mathcal{L}$-$\mathcal{VL}$-algebras is said to be homomorphism if it preserves the operations $\vee, \wedge, \rightarrow, T_\ell(\ell \in \mathcal{L}), 0, 1$.*

Let $\mathcal{VA}_\mathcal{L}$ denote the category of $\mathcal{L}$-$\mathcal{VL}$-algebras and homomorphisms between them.
$\mathcal{L}$-valued modal logic denoted by $\mathcal{L}$-$\mathcal{ML}$, is defined by $\mathcal{L}$-valued Kripke semantics. The idea of $\mathcal{L}$-valued Kripke semantics can be found in [21]. The operations of $\mathcal{L}$-valued modal logic $\mathcal{L}$-$\mathcal{ML}$ are the operations of $\mathcal{L}$-$\mathcal{VL}$ and a unary operation $\Box$, called modal operation. We now recall the concept of $\mathcal{L}$-$\mathcal{ML}$-algebras, which define a sound and complete semantics for $\mathcal{L}$-$\mathcal{ML}$.

Definition 6.3.4 ([20])**.** *An algebraic structure $(\mathcal{A}, \wedge, \vee, \rightarrow, T_\ell(\ell \in \mathcal{L}), \Box, 0, 1)$ is said to be a $\mathcal{L}$-$\mathcal{ML}$-algebra iff it satisfies the following conditions:*

(i) $(\mathcal{A}, \wedge, \vee, \rightarrow, T_\ell(\ell \in \mathcal{L}), 0, 1)$ *is a $\mathcal{L}$-$\mathcal{VL}$-algebra;*

(ii) $\Box(a \wedge b) = \Box a \wedge \Box b$*;*

(iii) $\Box U_\ell(a) = U_\ell(\Box a)$, $\forall \ell \in \mathcal{L}$, *where the unary operation $U_\ell(\ell \in \mathcal{L})$ is defined by $U_\ell(a) = \bigvee\{T_{\ell'}(a) : \ell \leq \ell', \ell' \in \mathcal{L}\}$, $a \in \mathcal{A}$. Logically, it means that the truth value of a is greater than or equal to ℓ.*

A homomorphism of $\mathcal{L}$-$\mathcal{ML}$-algebras is a function that preserves all the operations of $\mathcal{L}$-$\mathcal{VL}$-algebras and the modal operation $\Box$. Let $\mathcal{MA}_\mathcal{L}$ denote the category of $\mathcal{L}$-$\mathcal{ML}$-algebras and homomorphisms of $\mathcal{L}$-$\mathcal{ML}$-algebras.
For a Kripke frame $(P, \mathcal{R})$, $\mathcal{R}[x] = \{y \in P : x\mathcal{R}y\}$, where $x \in P$, and $\mathcal{R}^{-1}[P'] = \{y \in P : \exists x \in P', y\mathcal{R}x\}$, where $P' \subseteq P$. We recall a modal operation $\Box_\mathcal{R}$ on $\mathcal{L}$-valued powerset algebra $\mathcal{L}^P$ of P.

Definition 6.3.5 ([20])**.** *Let $(P, \mathcal{R})$ be a Kripke frame and $f \in \mathcal{L}^P$. Then $\Box_\mathcal{R} f : P \rightarrow \mathcal{L}$ is defined by $(\Box_\mathcal{R} f)(x) = \bigwedge\{f(y) : y \in \mathcal{R}[x]\}$.*

Definition 6.3.6 ([21])**.** *Let $\mathcal{A}$ be an object in $\mathcal{MA}_\mathcal{L}$. A binary relation $\mathcal{R}_\Box$ on $HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}, \mathcal{L})$ is defined as follows:*
$\psi\mathcal{R}_\Box\phi \iff \forall \ell \in \mathcal{L}, \forall a \in \mathcal{A}, \psi(\Box a) \geq \ell \Rightarrow \phi(a) \geq \ell.$

A $\mathcal{L}$-valued map $\mathcal{D} : HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}, \mathcal{L}) \times \mathcal{A} \rightarrow \mathcal{L}$ is defined by $\mathcal{D}(\psi, a) = \psi(a), \psi \in HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}, \mathcal{L})$.

Lemma 6.3.4 ([21])**.** *The $\mathcal{L}$-valued canonical model $(HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}, \mathcal{L}), \mathcal{R}_\Box, \mathcal{D})$ of $\mathcal{A}$ is a $\mathcal{L}$-valued Kripke model. Then, $\mathcal{D}(\psi, \Box a) = \psi(\Box a) = \bigwedge\{\phi(a) : \phi \in \mathcal{R}_\Box[\psi]\}$.*

6.4 Bitopological duality for Fitting's Heyting-valued logic

We will introduce the key ideas and findings from the bitopological duality theory for Fitting's Heyting-valued logic. We refer to Chapter 4 for a more thorough explanation of the bitopological duality for Fitting's Heyting-valued logic. Let $\mathfrak{S}_{\mathcal{L}}$ denote the collection of subalgebras of $\mathcal{L}$. For a pairwise Boolean space $\mathcal{B}$, $\Lambda_{\mathcal{B}}$ denotes the collection of pairwise closed subspaces of $\mathcal{B}$. It is shown in [62] that a pairwise closed subset of a pairwise compact space is also pairwise compact. Hence, each member of $\Lambda_{\mathcal{B}}$ is a pairwise Boolean space. A finite distributive lattice $\mathcal{L}$ endowed with unary operation $T_\ell(\ell \in \mathcal{L})$ forms a semi-primal algebra. We have expanded the theory of natural duality [13] by creating a bitopological duality for $\mathcal{L}$-$\mathcal{VL}$-algebras [58].
We now recall the category $PBS_{\mathcal{L}}$ from [58].

6.4.1 Category

Definition 6.4.1 ([58]). *The category $PBS_{\mathcal{L}}$ is defined as follows:*

(1) Objects: An object in $PBS_{\mathcal{L}}$ is a tuple $(\mathcal{B}, \alpha_{\mathcal{B}})$ where $\mathcal{B}$ is a pairwise Boolean space and a mapping $\alpha_{\mathcal{B}} : \mathfrak{S}_{\mathcal{L}} \to \Lambda_{\mathcal{B}}$ satisfies the following conditions:

(i) $\alpha_{\mathcal{B}}(\mathcal{L}) = \mathcal{B}$;

(ii) if $\mathcal{L}_1 = \mathcal{L}_2 \wedge \mathcal{L}_3(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in \mathcal{L})$, then $\alpha_{\mathcal{B}}(\mathcal{L}_1) = \alpha_{\mathcal{B}}(\mathcal{L}_2) \cap \alpha_{\mathcal{B}}(\mathcal{L}_3)$.

(2) Arrows: An arrow $\psi : (\mathcal{B}_1, \alpha_{\mathcal{B}_1}) \to (\mathcal{B}_2, \alpha_{\mathcal{B}_2})$ in $PBS_{\mathcal{L}}$ is a pairwise continuous map $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ that satisfies the criterion that if $x \in \alpha_{\mathcal{B}_1}(\mathcal{L}_1)(\mathcal{L}_1 \in \mathfrak{S}_{\mathcal{L}})$, then $\psi(x) \in \alpha_{\mathcal{B}_2}(\mathcal{L}_1)$ i.e., ψ is a subspace preserving map.

Note 6.4.1. *(1) The bitopological space $(\mathcal{L}, \tau, \tau)$, where τ is the discrete topology on $\mathcal{L}$, is a pairwise Boolean space. Hence, $(\mathcal{L}, \tau, \tau, \alpha_{\mathcal{L}})$, where $\alpha_{\mathcal{L}}$ is a mapping from $\mathfrak{S}_{\mathcal{L}}$ to $\Lambda_{\mathcal{L}}$ that is defined by $\alpha_{\mathcal{L}}(\mathcal{L}') = \mathcal{L}'$, is an object in $PBS_{\mathcal{L}}$.*

(2) For an object $\mathcal{A}$ in $\mathcal{VA}_{\mathcal{L}}$, consider a bitopological space $(HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}), \tau_1, \tau_2)$, where the topologies τ_1 and τ_2 are generated by the bases $B^{\tau_1} = \{\langle a \rangle : a \in \mathcal{A}\}$, where $\langle a \rangle = \{h \in HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}) : h(a) = 1\}$, and $B^{\tau_2} = \{B^c : B \in B^{\tau_1}\}$, respectively. Here, B^c denotes the complement of B.

Fact 6.4.1 ([58]). *The bitopological space $(HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}), \tau_1, \tau_2)$ is a pairwise Boolean space.*

6.4.2 Functors

The duality between the categories $\mathcal{VA}_{\mathcal{L}}$ and $PBS_{\mathcal{L}}$ is obtained via the following functors.

Definition 6.4.2 ([58]). *A contravariant functor $\mathfrak{F} : PBS_{\mathcal{L}} \to \mathcal{VA}_{\mathcal{L}}$ is defned as follows:*

(i) For an object $(\mathcal{B}, \alpha_{\mathcal{B}})$ in $PBS_{\mathcal{L}}$, define $\mathfrak{F}(\mathcal{B}, \alpha_{\mathcal{B}}) = (HOM_{PBS_{\mathcal{L}}}((\mathcal{B}, \alpha_{\mathcal{B}}), (\mathcal{L}, \alpha_{\mathcal{L}})), \vee, \wedge, \rightarrow, T_\ell(\ell \in \mathcal{L}), 0, 1)$, where $\vee, \wedge, \rightarrow, T_\ell(\ell \in \mathcal{L}), 0, 1$ are pointwise operations on the set $HOM_{PBS_{\mathcal{L}}}((\mathcal{B}, \alpha_{\mathcal{B}}), (\mathcal{L}, \alpha_{\mathcal{L}}))$. The operations 0 and 1 are regarded as constant functions, with 0 and 1 being their respective values.

(ii) For an arrow $\phi : (\mathcal{B}, \alpha_{\mathcal{B}}) \to (\mathcal{B}', \alpha_{\mathcal{B}'})$ in $PBS_{\mathcal{L}}$, define $\mathfrak{F}(\phi) : \mathfrak{F}((\mathcal{B}', \alpha_{\mathcal{B}'})) \to \mathfrak{F}((\mathcal{B}, \alpha_{\mathcal{B}}))$ by $\mathfrak{F}(\phi)(\zeta) = \zeta \circ \phi$, where $\zeta \in HOM_{PBS_{\mathcal{L}}}((\mathcal{B}', \alpha_{\mathcal{B}'}), (\mathcal{L}, \alpha_{\mathcal{L}}))$.

Definition 6.4.3 ([58]). *A contravariant functor $\mathfrak{G} : \mathcal{VA}_{\mathcal{L}} \to PBS_{\mathcal{L}}$ is defined as follows:*

(i) $\mathfrak{G}$ acts on an object $\mathcal{A}$ in $\mathcal{VA}_{\mathcal{L}}$ as $\mathfrak{G}(\mathcal{A}) = (HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}), \tau_1, \tau_2, \alpha_{\mathcal{A}})$, where $\alpha_{\mathcal{A}}$ is a mapping from $\mathfrak{S}_{\mathcal{L}}$ to $\Lambda_{HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L})}$ which is defined by $\alpha_{\mathcal{A}}(\mathcal{L}^*) = HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}^*)$, $\mathcal{L}^* \in \mathfrak{S}_{\mathcal{L}}$.

(ii) $\mathfrak{G}$ acts on an arrow $\psi : \mathcal{A} \to \mathcal{A}^*$ in $\mathcal{VA}_{\mathcal{L}}$ as follows: $\mathfrak{G}(\psi) : \mathfrak{G}(\mathcal{A}^*) \to \mathfrak{G}(\mathcal{A})$ is defined by $\mathfrak{G}(\psi)(\phi) = \phi \circ \psi$, $\phi \in \mathfrak{G}(\mathcal{A}^*)$.

In [58], the following duality result is proved for $\mathcal{L}$-$\mathcal{VL}$-algebras:

Theorem 6.4.1. *The categories $\mathcal{VA}_{\mathcal{L}}$ and $PBS_{\mathcal{L}}$ are dually equivalent.*

6.5 Bitopological duality for Fitting's modal logic

In this section, we use bitopological approaches to demonstrate a duality for Fitting's many-valued modal logic. This extends Jónsson-Tarski topological duality for modal algebras from the standpoint of universal algebra.
Let $\mathcal{R}$ be a relation on P and $C \subseteq P$. We define $[\mathcal{R}]C = \{p \in P : \mathcal{R}[p] \subseteq C\}$ and $\langle\mathcal{R}\rangle C = \{p \in P : \mathcal{R}[p] \cap C \neq \emptyset\}$.

6.5.1 Category $PRBS_{\mathcal{L}}$

Definition 6.5.1. *We define a category $PRBS_{\mathcal{L}}$ as follows:*

(1) Objects: An object in $PRBS_{\mathcal{L}}$ is a triple $(P, \alpha_P, \mathcal{R})$ such that (P, α_P) is an object in $PBS_{\mathcal{L}}$ and $\mathcal{R}$ is a binary relation on P that satisfies the following conditions:

(i) for each p in P, $\mathcal{R}[p]$ is a pairwise compact subset of P;

(ii) $\forall \mathcal{C} \in \beta_1$, $[\mathcal{R}]\mathcal{C}, \langle\mathcal{R}\rangle\mathcal{C} \in \beta_1$;

(iii) for any $\mathcal{L}' \in \mathfrak{S}_{\mathcal{L}}$, if $m \in \alpha_P(\mathcal{L}')$ then $\mathcal{R}[m] \subseteq \alpha_P(\mathcal{L}')$.

(2) Arrows: An arrow $f : (P, \alpha_P, \mathcal{R}) \to (P', \alpha_{P'}, \mathcal{R}')$ in $PRBS_{\mathcal{L}}$ is an arrow in $PBS_{\mathcal{L}}$ which additionally satisfies the following conditions:

(i) if $p_1\mathcal{R}p_2$ then $f(p_1)\mathcal{R}'f(p_2)$;

(ii) if $f(p)\mathcal{R}'p'$ then $\exists p^* \in P$ such that $p\mathcal{R}p^*$ and $f(p^*) = p'$.

Note 6.5.1. *We see that $[\mathcal{R}]U^c = (\langle\mathcal{R}\rangle U)^c$, and $\langle\mathcal{R}\rangle U^c = ([\mathcal{R}]U)^c$. Since $\beta_2 = \{U^c : U \in \beta_1\}$, hence if the relation $\mathcal{R}$ satisfies condition (ii) that is given in the object part of Definition 6.5.1, then $[\mathcal{R}]Q, \langle\mathcal{R}\rangle Q \in \beta_2$, $\forall Q \in \beta_2$.*

6.5.2 Functors

In this subsection, we introduce functors $\mathcal{F}$ and $\mathcal{G}$ to establish the dual equivalence between the categories $\mathcal{MA}_{\mathcal{L}}$ and $PRBS_{\mathcal{L}}$.

Definition 6.5.2. *We define a functor $\mathcal{G} : \mathcal{MA}_{\mathcal{L}} \to PRBS_{\mathcal{L}}$.*

(i) $\mathcal{G}$ acts on an object $(\mathcal{A}, \Box)$ in $\mathcal{MA}_{\mathcal{L}}$ as $\mathcal{G}(\mathcal{A}) = (HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}), \tau_1, \tau_2, \alpha_{\mathcal{A}}, \mathcal{R}_{\Box})$, where $\alpha_{\mathcal{A}}$ is a mapping from $\mathfrak{S}_{\mathcal{L}}$ to $\Lambda_{HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A},\mathcal{L})}$ defined by $\alpha_{\mathcal{A}}(\mathcal{L}_1) = HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L}_1)$, and $\mathcal{R}_{\Box}$ is a binary relation on $HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L})$ that is described in Definition 6.3.6.

*(ii) $\mathcal{G}$ acts on an arrow $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ in $\mathcal{MA}_{\mathcal{L}}$ as follows:
Define $\mathcal{G}(\psi) : \mathcal{G}(\mathcal{A}_2) \to \mathcal{G}(\mathcal{A}_1)$ by $\mathcal{G}(\psi)(\phi) = \phi \circ \psi$, where $\phi \in HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}_2, \mathcal{L})$.*

Lemma 6.5.1 and Lemma 6.5.2 demonstrate the well-definedness of $\mathcal{G}$.

Lemma 6.5.1. *For an object $(\mathcal{A}, \Box)$ in $\mathcal{MA}_{\mathcal{L}}$, $\mathcal{G}(\mathcal{A})$ is an object in $PRBS_{\mathcal{L}}$.*

Proof. Definition 6.4.3 shows that $(HOM_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \tau_1, \tau_2, \alpha_{\mathcal{A}})$ is an object in $PBS_{\mathcal{L}}$. So it is enough to show that $\mathcal{R}_\Box$ meets the conditions specified in the object part of Definition 6.5.1. We first show that for $\mathcal{W} \in HOM_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L})$, $\mathcal{R}_\Box[\mathcal{W}] \in \delta_1 \cup \delta_2$. Let $\mathcal{U} \notin \mathcal{R}_\Box[\mathcal{W}]$. Then by Definition 6.3.6, there is an element $a \in \mathcal{A}$ such that there is $L_1 \in \mathcal{L}$, for which $\mathcal{W}(\Box a) \geq L_1$ but $\mathcal{U}(a) \not\geq L_1$. It follows that $\mathcal{U} \in \langle \neg U_{L_1}(a) \rangle \in \tau_2$ and $\mathcal{R}_\Box[\mathcal{W}] \cap \langle \neg U_{L_1}(a) \rangle = \emptyset$ i.e., $\langle \neg U_{L_1}(a) \rangle \subseteq (\mathcal{R}_\Box[\mathcal{W}])^c$. Hence, $\mathcal{U} \notin \overline{\mathcal{R}_\Box[\mathcal{W}]}^{\tau_2}$, where $\overline{\mathcal{R}_\Box[\mathcal{W}]}^{\tau_2}$ denotes the closure of $\mathcal{R}_\Box[\mathcal{W}]$ in $(HOM_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \tau_2)$. Equivalently, we have $\overline{\mathcal{R}_\Box[\mathcal{W}]}^{\tau_2} \subset \mathcal{R}_\Box[\mathcal{W}]$. Therefore, $\mathcal{R}_\Box[\mathcal{W}]$ is τ_2-closed. Since $(HOM_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \tau_1, \tau_2)$ is pairwise compact, by Proposition 6.2.2, we have $\mathcal{R}_\Box[\mathcal{W}]$ is pairwise compact.

Now we verify the condition (ii) in the object part of Definition 6.5.1. Since $\{\langle a \rangle : a \in \mathcal{A}\} \in \beta_1$ and $\{\langle T_1(a) \to 0 \rangle : a \in \mathcal{A}\} \in \beta_2$ are the basis for the topologies τ_1 and τ_2, respectively, so we show that for each $a \in \mathcal{A}$, $\langle \mathcal{R}_\Box \rangle(\langle a \rangle) \in \beta_1$ and $[\mathcal{R}_\Box]\langle a \rangle \in \beta_1$. We see that

$$\begin{aligned} \langle \mathcal{R}_\Box \rangle \langle a \rangle &= \{\mathcal{W} \in HOM_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}) : \mathcal{R}_\Box[\mathcal{W}] \cap \langle a \rangle \neq \emptyset\} \\ &= ([\mathcal{R}_\Box]\langle T_1(a) \to 0 \rangle)^c \\ &= \{\mathcal{W} \in HOM_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}) : \mathcal{R}_\Box[\mathcal{W}] \not\subset \langle T_1(a) \to 0 \rangle\} \end{aligned}$$

We show that $([\mathcal{R}_\Box]\langle T_1(a) \to 0 \rangle)^c$ is τ_1-open and τ_2-closed. Let $\mathcal{U} \in ([\mathcal{R}_\Box]\langle T_1(a) \to 0 \rangle)^c$. Then $\mathcal{R}_\Box[\mathcal{U}] \not\subset \langle T_1(a) \to 0 \rangle$. It is easy to see that $\exists$ τ_1-open set $\langle \Box T_1(a) \rangle$ such that $\mathcal{U} \in \langle \Box T_1(a) \rangle$. Let $\mathcal{E} \in \langle \Box T_1(a) \rangle$. Then $\mathcal{E}(\Box T_1(a)) = 1$. Using the Kripke condition we have $1 = \mathcal{E}(\Box T_1(a)) = \bigwedge\{\mathcal{U}(T_1(a)) : \mathcal{E}\mathcal{R}_\Box\mathcal{U}\}$. According to Lemma 6.3.2, $\mathcal{U}(T_1(a))$ is either 0 or 1. Henceforth, for all $\mathcal{U} \in HOM_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L})$ with $\mathcal{E}\mathcal{R}_\Box\mathcal{U}$ we have $\mathcal{U}(T_1(a)) = 1$. As a result, $\mathcal{R}_\Box[\mathcal{E}] \not\subset \langle T_1(a) \to 0 \rangle$ i.e., $\mathcal{E} \in ([\mathcal{R}_\Box]\langle T_1(a) \to 0 \rangle)^c$. Henceforth, $\mathcal{U} \in \langle \Box T_1(a) \rangle \subset ([\mathcal{R}_\Box]\langle T_1(a) \to 0 \rangle)^c$. Therefore, $[\mathcal{R}_\Box]\langle T_1(a) \to 0 \rangle)^c$ is τ_1-open i.e., $\langle \mathcal{R}_\Box \rangle \langle a \rangle$ is τ_1-open.

Let $\mathcal{W} \in (\langle \mathcal{R}_\Box \rangle \langle a \rangle)^c$. Then $\mathcal{R}_\Box[\mathcal{W}] \cap \langle a \rangle = \emptyset$. It is easy to see that there is τ_1-open set $\langle \Box(T_1(a) \to 0) \rangle$ such that $\mathcal{W} \in \langle \Box(T_1(a) \to 0) \rangle$. Also, by applying the Kripke condition, we have $\langle \Box(T_1(a) \to 0) \rangle \subset (\langle \mathcal{R}_\Box \rangle \langle a \rangle)^c$. Therefore, $\mathcal{W} \in \langle \Box(T_1(a) \to 0) \rangle \subset (\langle \mathcal{R}_\Box \rangle \langle a \rangle)^c$. It shows that $(\langle \mathcal{R}_\Box \rangle \langle a \rangle)^c$ is τ_1-open i.e., $\langle \mathcal{R}_\Box \rangle \langle a \rangle$ is τ_1-closed. It follows from Proposition 6.2.2 that $\langle \mathcal{R}_\Box \rangle \langle a \rangle$ is pairwise compact. Since the topological space $(HOM_{\mathcal{VA_L}}(\mathcal{A}, \mathcal{L}), \tau_2)$ with basis $\{\langle T_1(a) \to 0 \rangle : a \in \mathcal{A}\}$ is a Hausdorff space, so $\langle \mathcal{R}_\Box \rangle \langle a \rangle$ is τ_2-closed. Hence, $\langle \mathcal{R}_\Box \rangle \langle a \rangle \in \beta_1$.

Next, we show that $[\mathcal{R}_\Box]\langle a\rangle \in \beta_1$. We see that

$$\begin{aligned}[\mathcal{R}_\Box]\langle a\rangle &= \{\mathcal{W} \in HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A},\mathcal{L}) : \mathcal{R}_\Box[\mathcal{W}] \subseteq \langle a\rangle\} \\ &= (\langle\mathcal{R}_\Box\rangle\langle T_1(a) \to 0\rangle)^c\end{aligned}$$

We claim that $\langle\mathcal{R}_\Box\rangle\langle T_1(a) \to 0\rangle = \langle\Box T_1(a) \to 0\rangle$. Let $\mathcal{W} \in \langle\Box T_1(a) \to 0\rangle$. Then $\mathcal{W}(\Box T_1(a) \to 0) = 1$. Hence, $\mathcal{W}(\Box T_1(a)) = 0$. Using the Kripke condition, we have, $0 = \mathcal{W}(\Box T_1(a)) = \bigwedge\{\mathcal{U}(T_1(a)) : \mathcal{W}\mathcal{R}_\Box\mathcal{U}\}$. Since $\mathcal{U}(T_1(a)) = 0$ or 1, hence $\exists$ $\mathcal{U} \in HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A},\mathcal{L})$ with $\mathcal{W}\mathcal{R}_\Box\mathcal{U}$ such that $\mathcal{U}(T_1(a)) = 0$. Then $\mathcal{U} \in \langle T_1(a) \to 0\rangle$. Therefore, $\mathcal{R}_\Box[\mathcal{W}] \cap \langle T_1(a) \to 0\rangle \neq \emptyset$. Thus, $\mathcal{W} \in \langle\mathcal{R}_\Box\rangle\langle T_1(a) \to 0\rangle$. Similarly, by employing the Kripke condition, we can show that if $\mathcal{W} \in \langle\mathcal{R}_\Box\rangle\langle T_1(a) \to 0\rangle$ then $\mathcal{W} \in \langle\Box T_1(a) \to 0\rangle$. Since $\langle\Box T_1(a) \to 0\rangle \in \beta_2$, we have $\langle\mathcal{R}_\Box\rangle\langle T_1(a) \to 0\rangle \in \beta_2$. As a result, $[\mathcal{R}_\Box]\langle a\rangle \in \beta_1$.
Finally, we demonstrate that $\mathcal{G}(\mathcal{A})$ meets condition (*iii*) in the object part of Definition 6.5.1. Let $u \in \alpha_\mathcal{A}(\mathcal{L}') = HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A},\mathcal{L}')$. Suppose $\mathcal{R}_\Box[u] \not\subset \alpha_\mathcal{A}(\mathcal{L}')$. Then $\exists v \in \mathcal{R}_\Box[u]$ such that $v \notin \alpha_\mathcal{A}(\mathcal{L}')$. Hence, $\exists a^* \in \mathcal{A}$ such that $v(a^*) \notin \mathcal{L}'$. Let $v(a^*) = \ell^*$. Now for any element $\psi \in \alpha_\mathcal{A}(\mathcal{L}')$,

$$\psi(T_{\ell^*}(a^*) \to a^*) = \begin{cases} \ell^* & \text{if } \psi(a^*) = \ell^* \\ 1 & \text{if } \psi(a^*) \neq \ell^* \end{cases}$$

Using Kripke condition, we have $u(\Box(T_{\ell^*}(a^*) \to a^*)) = \bigwedge\{\psi(T_{\ell^*}(a^*) \to a^*) : \psi \in \mathcal{R}_\Box[u]\}$. This shows that $u(\Box(T_{\ell^*}(a^*) \to a^*)) = \ell^* \notin \mathcal{L}'$. But this contradicts the fact that $u \in \alpha_\mathcal{A}(\mathcal{L}')$. As a result, $\mathcal{G}(\mathcal{A})$ satisfies condition (*iii*). □

Lemma 6.5.2. *Let* $(\mathcal{A}_1, \Box_1)$, $(\mathcal{A}_2, \Box_2)$ *be the objects in* $\mathcal{MA}_\mathcal{L}$ *and* $\psi : \mathcal{A}_1 \to \mathcal{A}_2$ *an arrow in* $\mathcal{MA}_\mathcal{L}$. *Then,* $\mathcal{G}(\psi)$ *is an arrow in* $PRBS_\mathcal{L}$.

Proof. Here $\mathcal{G}(\psi) : \mathcal{G}(\mathcal{A}_2) \to \mathcal{G}(\mathcal{A}_1)$ is defined by $\mathcal{G}(\psi)(\phi) = \phi \circ \psi$, $\phi \in HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}_2,\mathcal{L})$. It follows from Definition 6.4.3 that $\mathcal{G}(\psi)$ is an arrow in $PBS_\mathcal{L}$. Therefore, it is still necessary to demonstrate that $\mathcal{G}(\psi)$ satisfies conditions (*i*) and (*ii*) listed in the arrow portion of Definition 6.5.1. We first check condition (*i*). Let $v_1\mathcal{R}_{\Box_2}v_2$, where $v_1, v_2 \in \mathcal{G}(\mathcal{A}_2)$. We are to show that $\mathcal{G}(\psi)(v_1)\mathcal{R}_{\Box_1}\mathcal{G}(\psi)(v_2)$. Now, if $v_1 \circ \psi(\Box_1 a_1) \geq \ell$ for $a_1 \in \mathcal{A}_1$ and $\ell \in \mathcal{L}$, then we have $v_1(\Box_2\psi(a_1)) \geq \ell$. As $v_1\mathcal{R}_{\Box_2}v_2$, so we get $v_2(\psi(a_1)) \geq \ell$. Hence, $\mathcal{G}(\psi)(v_1)\mathcal{R}_{\Box_1}\mathcal{G}(\psi)(v_2)$. We then check condition (*ii*), which is mentioned in the arrow part of Definition 6.5.1. This is equivalent to verifying $\mathcal{R}_{\Box_1}[\mathcal{G}(\psi)(v_1)] = \mathcal{G}(\psi)(\mathcal{R}_{\Box_2}[v_1])$. Let $\mathcal{W} \in \mathcal{R}_{\Box_1}[v_1 \circ \psi]$, where $\mathcal{W} \in HOM_{\mathcal{VA}_\mathcal{L}}(\mathcal{A}_1,\mathcal{L})$. Then $(v_1 \circ \psi)\mathcal{R}_{\Box_1}\mathcal{W}$. Suppose

$\mathcal{W} \notin \mathcal{G}(\psi)(\mathcal{R}_{\Box_2}[v_1])$. Then $\mathcal{W} \neq \mathcal{G}(\psi)(v^*)$, $\forall v^* \in HOM_{\mathcal{VA_L}}(\mathcal{A}_2, \mathcal{L})$ such that $v_1\mathcal{R}_{\Box_2}v^*$. As $(HOM_{\mathcal{VA_L}}(\mathcal{A}_1, \mathcal{L}), \tau_1, \tau_2)$ is a pairwise Hausdorff space, so we can consider $\mathcal{W} \in \langle a_1 \rangle$ and $\mathcal{G}(\psi)(v^*) = v^* \circ \psi \in \langle T_1(a_1) \to 0 \rangle$. Since $\mathcal{W} \in \mathcal{R}_{\Box_1}[\mathcal{G}(\psi)(v_1)]$ and $\mathcal{W}(a_1) = 1$, we have $\mathcal{G}(\psi)(v_1)(\Box_1 a_1) = 1$ i.e., $(v_1 \circ \psi)(\Box_1 a_1) = 1$. Since $v_1\mathcal{R}_{\Box_2}v^*$, we have $\mathcal{G}(\psi)(v_1)\mathcal{R}_{\Box_1}\mathcal{G}(\psi)(v^*)$ using the condition (i) specified in the arrow part of Definition 6.5.1. As $\mathcal{G}(\psi)(v_1)(\Box_1 a_1) = 1$, Lemma 6.3.4 shows that $\mathcal{G}(\psi)(v^*)(a_1) = 1$, i.e., $v^* \circ \psi \in \langle a_1 \rangle$. This contradicts the fact that $\mathcal{G}(\psi)(v^*) \in \langle T_1(a_1) \to 0 \rangle$. Therefore, $\mathcal{R}_{\Box_1}[\mathcal{G}(\psi)(v_1)] \subseteq \mathcal{G}(\psi)(\mathcal{R}_{\Box_2}[v_1])$. Similarly, we can show the reverse direction. □

Definition 6.5.3. *We define a functor $\mathcal{F} : PRBS_{\mathcal{L}} \to \mathcal{MA_L}$.*

(i) Define $\mathcal{F}(P, \alpha_P, \mathcal{R}) = (HOM_{PBS_{\mathcal{L}}}((P, \alpha_P), (\mathcal{L}, \alpha_{\mathcal{L}})), \wedge, \vee, \to, T_\ell(\ell \in \mathcal{L}), 0, 1, \Box_{\mathcal{R}})$ for an object $(P, \alpha_P, \mathcal{R})$ in $PRBS_{\mathcal{L}}$. Definition 6.3.5 describes the modal operation $\Box_{\mathcal{R}}$. Here $\wedge, \vee, \to, T_\ell$ are pointwise operations defined on the set $HOM_{PBS_{\mathcal{L}}}((P, \alpha_P), (\mathcal{L}, \alpha_{\mathcal{L}}))$.

(ii) Let $\psi : (P_1, \alpha_{P_1}, \mathcal{R}_1) \to (P_2, \alpha_{P_2}, \mathcal{R}_2)$ be an arrow in $PRBS_{\mathcal{L}}$. Define $\mathcal{F}(\psi) : \mathcal{F}(P_2, \alpha_{P_2}, \mathcal{R}_2) \to \mathcal{F}(P_1, \alpha_{P_1}, \mathcal{R}_1)$ by $\mathcal{F}(\psi)(\phi) = \phi \circ \psi$ for $\phi \in \mathcal{F}(P_2, \alpha_{P_2}, \mathcal{R}_2)$.

Note 6.5.2. *If $\psi, \phi : (P, \tau_1^P, \tau_2^P, \alpha_P) \to (\mathcal{L}, \tau, \tau, \alpha_{\mathcal{L}})$ are pairwise continuous maps then $\psi \wedge \phi, \psi \vee \phi, \psi \to \phi, T_\ell(\psi)$ are also pairwise continuous maps. Thus, $(HOM_{PBS_{\mathcal{L}}}((P, \alpha_P), (\mathcal{L}, \alpha_{\mathcal{L}})), \wedge, \vee, \to, T_\ell(\ell \in \mathcal{L}), 0, 1)$ is a $\mathcal{L}$-$\mathcal{VL}$-algebra.*

The following lemmas (Lemma 6.5.3 and Lemma 6.5.4) show that the functor $\mathcal{F}$ is well-defined.

Lemma 6.5.3. *Let $(P, \alpha_P, \mathcal{R})$ be an object in $PRBS_{\mathcal{L}}$. Then, $\mathcal{F}(P, \alpha_P, \mathcal{R})$ is an object in $\mathcal{MA_L}$.*

Proof. It is clear from Definition 6.4.2 that $\mathcal{F}(P, \alpha_P)$ is an object in $\mathcal{VA_L}$. We need to show that the modal operation $\Box_{\mathcal{R}}$ on $\mathcal{F}(P, \alpha_P, \mathcal{R})$ is well-defined. Let $\eta \in \mathcal{F}(P, \alpha_P, \mathcal{R})$. We then verify $\Box_{\mathcal{R}}\eta \in \mathcal{F}(P, \alpha_P, \mathcal{R})$. For any $\ell \in \mathcal{L}$,

$$\begin{aligned}(\Box_{\mathcal{R}}\eta)^{-1}(\{\ell\}) &= \{p \in P : \bigwedge\{\eta(p') : p' \in \mathcal{R}[p] = \ell\} \\ &= \langle \mathcal{R} \rangle((T_\ell(\eta))^{-1}(\{1\})) \cap (\langle \mathcal{R} \rangle((U_\ell(\eta))^{-1}(\{0\})))^c\end{aligned}$$

As both $T_\ell(\eta)$ and $U_\ell(\eta)$ are pairwise continuous maps, henceforth $(T_\ell(\eta))^{-1}(\{1\}) \in \beta_1^P \cap \beta_2^P$ and $(U_\ell(\eta))^{-1}(\{0\}) \in \beta_1^P \cap \beta_2^P$, where $\beta_1^P = \tau_1^P \cap \delta_2^P$ and $\beta_2^P = \tau_2^P \cap \delta_1^P$. Therefore, $(\Box_{\mathcal{R}}\eta)^{-1}(\{\ell\}) \in \tau_1^P$. Also, $(\Box_{\mathcal{R}}\eta)^{-1}(\{\ell\}) \in \tau_2^P$. As a result, $\Box_{\mathcal{R}}\eta$ is a pairwise continuous map from P to $\mathcal{L}$. Furthermore, by applying condition (iii)

that is stated in the object part of Definition 6.5.1, we see that for any subalgebra $\mathcal{M} \in \mathfrak{S}_{\mathcal{L}}$, and if $m \in \alpha_{\mathcal{L}}(\mathcal{M})$ then $(\Box_{\mathcal{R}}\eta)(m) = \bigwedge\{\eta(m') : m' \in \mathcal{R}[m]\} \in \alpha_{\mathcal{L}}(\mathcal{M})$. Thus $\Box_{\mathcal{R}}\eta$ is a subspace preserving map. Hence, $\Box_{\mathcal{R}}\eta \in \mathcal{F}(P, \alpha_P, \mathcal{R})$. □

Lemma 6.5.4. *Let $\psi : (P_1, \alpha_{P_1}, \mathcal{R}_1) \to (P_2, \alpha_{P_2}, \mathcal{R}_2)$ be an arrow in $PRBS_{\mathcal{L}}$. Then, $\mathcal{F}(\psi)$ is an arrow in $\mathcal{MA}_{\mathcal{L}}$.*

Proof. According to Definition 6.4.2, $\mathcal{F}(\psi)$ is an arrow in $\mathcal{VA}_{\mathcal{L}}$. Therefore, it is sufficient to demonstrate that $\mathcal{F}(\psi)(\Box_{\mathcal{R}_2}\phi_2) = \Box_{\mathcal{R}_1}(\mathcal{F}(\psi)\phi_2)$, where $\phi_2 \in HOM_{PBS_{\mathcal{L}}}((P_2, \alpha_{P_2}), (\mathcal{L}, \alpha_{\mathcal{L}}))$. For any $p_1 \in P_1$, we have $\mathcal{F}(\psi)(\Box_{\mathcal{R}_2}\phi_2)(p_1) = \Box_{\mathcal{R}_2}\phi_2 \circ \psi(p_1) = \bigwedge\{\phi_2(p_2) : p_2 \in \mathcal{R}_2[\psi(p_1)]\}$, and $\Box_{\mathcal{R}_1}(\mathcal{F}(\psi)\phi_2)(p_1) = \Box_{\mathcal{R}_1}(\phi_2 \circ \psi)(p_1) = \bigwedge\{\phi_2 \circ \psi(p) : p \in \mathcal{R}_1[p_1]\}$. As ψ satisfies conditions (i) and (ii) listed in item 2 of Definition 6.5.1, it is easy to show that $(\mathcal{F}(\psi)(\Box_{\mathcal{R}_2}\phi_2))(p_1) \leq \Box_{\mathcal{R}_1}(\mathcal{F}(\psi)\phi_2)(p_1)$ and $\Box_{\mathcal{R}_1}(\mathcal{F}(\psi)\phi_2)(p_1) \leq (\mathcal{F}(\psi)(\Box_{\mathcal{R}_2}\phi_2))(p_1)$. As a result, $\mathcal{F}(\psi)(\Box_{\mathcal{R}_2}\phi_2) = \Box_{\mathcal{R}_1}(\mathcal{F}(\psi)\phi_2)$. □

6.5.3 Bitopological Duality for Fitting's Heyting-valued modal logic

In this subsection, we develop bitopological duality for algebras of Fitting's Heyting-valued modal logic.

Theorem 6.5.1. *Let $\mathcal{A}$ be a $\mathcal{L}$-$\mathcal{ML}$ algebra. Then $\mathcal{A}$ is isomorphic to $\mathcal{F} \circ \mathcal{G}(\mathcal{A})$ in $\mathcal{MA}_{\mathcal{L}}$.*

Proof. We define $\gamma^{\mathcal{A}} : \mathcal{A} \to \mathcal{F} \circ \mathcal{G}(\mathcal{A})$ by $\gamma^{\mathcal{A}}(a)(g) = g(a)$, where $a \in \mathcal{A}$ and $g \in HOM_{\mathcal{VA}_{\mathcal{L}}}(\mathcal{A}, \mathcal{L})$. It is known from Theorem 6.4.1 that $\gamma^{\mathcal{A}}$ is an isomorphism in the category $\mathcal{VA}_{\mathcal{L}}$. The only thing left to prove is that $\gamma^{\mathcal{A}}$ preserves the modal operation $\Box$, i.e., $\gamma^{\mathcal{A}}(\Box a) = \Box_{\mathcal{R}_\Box}\gamma^{\mathcal{A}}(a)$, $a \in \mathcal{A}$. Let $g \in \mathcal{G}(\mathcal{A})$. Then

$$\begin{aligned}(\Box_{\mathcal{R}_\Box}\gamma^{\mathcal{A}}(a))(g) &= \bigwedge\{\gamma^{\mathcal{A}}(g^*) : g^* \in \mathcal{R}_\Box[g]\} \\ &= \bigwedge\{g^*(a) : g^* \in \mathcal{R}_\Box[g]\} \\ &= g(\Box a) \text{ (by Lemma 6.3.4)} \\ &= \gamma^{\mathcal{A}}(\Box a)(g)\end{aligned}$$

Hence the result follows. □

Theorem 6.5.2. *Consider an object $(P, \alpha_P, \mathcal{R})$ in $PRBS_{\mathcal{L}}$. Then, $(P, \alpha_P, \mathcal{R})$ is isomorphic to $\mathcal{G} \circ \mathcal{F}(P, \alpha_P, \mathcal{R})$ in the category $PRBS_{\mathcal{L}}$.*

Proof. Define $\zeta_{(P,\alpha_P,\mathcal{R})} : (P,\alpha_P,\mathcal{R}) \to \mathcal{G}\circ\mathcal{F}(P,\alpha_P,\mathcal{R})$ by $\zeta_{(P,\alpha_P,\mathcal{R})}(p)(\psi) = \psi(p)$, where $p \in P$ and $\psi \in HOM_{PBS_{\mathcal{L}}}((P,\alpha_P),(\mathcal{L},\alpha_{\mathcal{L}}))$. Theorem 6.4.1 shows that $\zeta_{(P,\alpha_P,\mathcal{R})}$ is a bi-homeomorphism in the category $PBS_{\mathcal{L}}$. We show that $\zeta_{(P,\alpha_P,\mathcal{R})}$ and $\zeta^{-1}_{(P,\alpha_P,\mathcal{R})}$ satisfy the conditions given in item 2 of Definition 6.5.1. We claim that for any $p,p' \in P$, $p' \in \mathcal{R}[p] \iff \zeta_{(P,\alpha_P,\mathcal{R})}(p') \in \mathcal{R}_{\Box_{\mathcal{R}}}[\zeta_{(P,\alpha_P,\mathcal{R})}(p)]$. Let $p' \in \mathcal{R}[p]$. Suppose $\zeta_{(P,\alpha_P,\mathcal{R})}(p)(\Box_{\mathcal{R}}\psi) \geq \ell$, where $\ell \in \mathcal{L}$ and $\psi \in HOM_{PBS_{\mathcal{L}}}((P,\alpha_P),(\mathcal{L},\alpha_{\mathcal{L}}))$. Then $\zeta_{(P,\alpha_P,\mathcal{R})}(p)(\Box_{\mathcal{R}}\psi) = (\Box_{\mathcal{R}}\psi)(p) = \bigwedge\{\psi(p^*) : p* \in \mathcal{R}[p]\}$. Since $p' \in \mathcal{R}[p]$ and $\zeta_{(P,\alpha_P,\mathcal{R})}(p)(\Box_{\mathcal{R}}\psi) \geq \ell$, we have $\zeta_{(P,\alpha_P,\mathcal{R})}(p')(\psi) \geq \ell$. Hence, $\zeta_{(P,\alpha_P,\mathcal{R})}(p)\mathcal{R}_{\Box_{\mathcal{R}}}\zeta_{(P,\alpha_P,\mathcal{R})}(p')$, i.e., $\zeta_{(P,\alpha_P,\mathcal{R})}(p') \in \mathcal{R}_{\Box_{\mathcal{R}}}[\zeta_{(P,\alpha_P,\mathcal{R})}(p)]$. Now we verify if $\zeta_{(P,\alpha_P,\mathcal{R})}(p') \in \mathcal{R}_{\Box_{\mathcal{R}}}[\zeta_{(P,\alpha_P,\mathcal{R})}(p)]$ then $p' \in \mathcal{R}[p]$. We verify its contrapositive statement. Suppose $p' \notin \mathcal{R}[p]$. By Definition 6.5.1, $\mathcal{R}[p]$ is a pairwise compact subset of pairwise Boolean space P. Then it is easy to show that $\mathcal{R}[p]$ is pairwise closed. Therefore we can get a τ_1^P-basis open set $\mathcal{O} \in \beta_1^P$ such that $p' \in \mathcal{O}$ and $\mathcal{O} \subseteq P - \mathcal{R}[p]$, i.e., $\mathcal{O} \cap \mathcal{R}[p] = \emptyset$. Define a mapping $f : P \to \mathcal{L}$ by

$$f(p) = \begin{cases} 0 & \text{if } p \in \mathcal{O} \\ 1 & \text{if } p \in \mathcal{O}^c \end{cases}$$

Then f is a pairwise continuous map from (P,τ_1^P,τ_2^P) to $(\mathcal{L},\tau,\tau)$. As a result, it can be shown that $f \in HOM_{PBS_{\mathcal{L}}}((P,\alpha_P),(\mathcal{L},\alpha_{\mathcal{L}}))$. Now, $\Box_{\mathcal{R}}f(p) = \bigwedge\{f(z) : z \in \mathcal{R}[p]\} = 1$ and $f(p') = 0$. Hence, $\zeta_{(P,\alpha_P,\mathcal{R})}(p)(\Box_{\mathcal{R}}f) = 1$ but $\zeta_{(P,\alpha_P,\mathcal{R})}(p')(f) \neq 1$. Therefore, $\zeta_{(P,\alpha_P,\mathcal{R})}(p') \notin \mathcal{R}_{\Box_{\mathcal{R}}}[\zeta_{(P,\alpha_P,\mathcal{R})}(p)]$. Hence, we have for any $p,p' \in P$, $p' \in \mathcal{R}[p] \iff \zeta_{(P,\alpha_P,\mathcal{R})}(p') \in \mathcal{R}_{\Box_{\mathcal{R}}}[\zeta_{(P,\alpha_P,\mathcal{R})}(p)]$. As a result, $\zeta_{(P,\alpha_P,\mathcal{R})}$ and $\zeta^{-1}_{(P,\alpha_P,\mathcal{R})}$ satisfy conditions (i) and (ii) mentioned in item 2 of Definition 6.5.1. Thus, $\zeta_{(P,\alpha_P,\mathcal{R})}$ is a homeomorphism. This finishes the proof. □

Finally, we obtain the bitopological duality for Fitting's Heyting-valued modal logic.

Theorem 6.5.3. *The categories $\mathcal{MA}_{\mathcal{L}}$ and $PRBS_{\mathcal{L}}$ are dually equivalent.*

Proof. Let ID_1 and ID_2 be the identity functors on $\mathcal{MA}_{\mathcal{L}}$ and $PRBS_{\mathcal{L}}$, respectively. This theorem will be proved by defining two natural isomorphisms, $\gamma : ID_1 \to \mathcal{F}\circ\mathcal{G}$ and $\zeta : ID_2 \to \mathcal{G}\circ\mathcal{F}$. For an object $\mathcal{A}$ in $\mathcal{MA}_{\mathcal{L}}$ define $\gamma^{\mathcal{A}} : \mathcal{A} \to \mathcal{F}\circ\mathcal{G}(\mathcal{A})$ by $\gamma^{\mathcal{A}}(a)(g) = g(a)$, where $a \in \mathcal{A}$ and $g \in \mathcal{G}(\mathcal{A})$. For an object $(P,\alpha_P,\mathcal{R})$ in $PRBS_{\mathcal{L}}$ define $\zeta_{(P,\alpha_P,\mathcal{R})} : (P,\alpha_P,\mathcal{R}) \to \mathcal{G}\circ\mathcal{F}(P,\alpha_P,\mathcal{R})$ by $\zeta_{(P,\alpha_P,\mathcal{R})}(p)(\psi) = \psi(p)$, where $p \in P$ and $\psi \in HOM_{PBS_{\mathcal{L}}}((P,\alpha_P),(\mathcal{L},\alpha_{\mathcal{L}}))$. Then it can be shown that γ and ζ are natural transformations. According to Theorems 6.5.1 and 6.5.2, γ and ζ are natural isomorphisms. □

Thus we have extended bitopological duality for Fitting's many-valued logic:

$$\mathcal{VA}_{\mathcal{L}} \equiv PBS^{op}_{\mathcal{L}}$$

to the duality for Fitting's many-valued modal logic:

$$\mathcal{MA}_{\mathcal{L}} \equiv PRBS^{op}_{\mathcal{L}}$$

We have obtained a duality for the class of all algebras of a version of Fitting's Heyting-valued modal logic in bitopological language via the novel notion of $PRBS_{\mathcal{L}}$, without which it would be challenging to achieve such a modalized version of the bitopological duality for many-valued logic. This has led to an extension of the natural duality theory for modal algebras.
In the next chapter, we shall demonstrate how to characterize the category $PRBS_{\mathcal{L}}$ using the coalgebra theory, thereby obtaining a coalgebraic interpretation of the duality $\mathcal{MA}_{\mathcal{L}} \equiv PRBS^{op}_{\mathcal{L}}$.

6.6 Conclusion

We have defined the category $PRBS_{\mathcal{L}}$ and connected it to the category $\mathcal{VA}_{\mathcal{L}}$ using the appropriate functors. Consequently, we have found a duality for the class of all algebras of a version of Fitting's Heyting-valued modal logic in a bitopological setting. This has led to an extension of the natural duality theory for modal algebras. It has been noted that the methodology laid out in this chapter extends the Jonssion-Tarski duality for algebras of Fitting's Heyting-valued modal logic (e.g., see [21]) in a bitopological context.

Chapter 7

Coalgebraic Duality for many-valued modal logic

This chapter is primarily concerned with establishing a coalgebraic duality for Fitting's many-valued modal logic. In chapter 6, we have established a bitopological duality for algebras of Fitting's Heyting-valued modal logic by building up a notion of $PRBS_{\mathcal{L}}$ as a category of $\mathcal{L}$-valued pairwise Boolean spaces with a relation. This chapter will show how the category $PRBS_{\mathcal{L}}$ can be characterized using the theory of coalgebras, leading to a coalgebraic description of the bitopological duality for Fitting's Heyting-valued modal logic.

We aim to construct a bi-Vietoris functor on the category $PBS_{\mathcal{L}}$ of $\mathcal{L}$-valued ($\mathcal{L}$ is a Heyting algebra) pairwise Boolean spaces. Finally, we obtain a dual equivalence between categories of biVietoris coalgebras and algebras of Fitting's Heyting-valued modal logic. Thus, we conclude that Fitting's many-valued modal logic is sound and complete with respect to the coalgebras of a biVietoris functor. The key conclusion is coalgebraic duality for algebras of Fitting's Heyting-valued modal logic represented by $\mathcal{L}$-$\mathcal{ML}$-algebras, where $\mathcal{L}$ is a semi-primal algebra having a bounded lattice reduct. Our general theory extends the Abramsky-Kupke-Kurz-Venema coalgebraic duality [1, 55] in the setting of bitopological language. Furthermore, it introduces a novel coalgebraic duality for $\mathcal{L}$-$\mathcal{ML}$-algebras.
An exemplary story in coalgebraic logic can be found in [63]. The Stone duality [51] between Boolean algebras and sets represents the syntax and semantics of a propo-

The outcomes of this chapter appear in [61] **Das, Litan Kumar., Ray, Kumar sanakar., Mali, Prakash Chandra.: Duality for Fitting's Heyting-valued modal logic via Bitopology and Bi-Vietoris coalgebra. Theoretical Computer Science, Elsevier (Under Review). https://doi.org/10.48550/arXiv.2312.16276** sitional logic. The algebras and coalgebras of the endofunctors define the syntax and semantics of the modal propositional logic. As an illustration, the modal logic K and Kripke semantics derive from the Stone duality by taking an endofunctor on sets. So, in acceptable circumstances, we can achieve duality between the relevant algebras and coalgebras. In addition to demonstrating the fact that the widely recognized Stone duality could be articulated in coalgebraic terms, Abramsky [1] also showed that a coalgebraic formulation could be provided for the Jónsson-Tarski duality between descriptive general Kripke frames and modal algebras (see also [55] for further information). In particular, the category of descriptive general Kripke frames is isomorphic to the category of Boolean spaces. Esakia [64] also noticed this connection. Therefore, coalgebras for the Vietoris functor on the category of Boolean spaces can represent sound and complete semantics for modal logic. In [65], the author showed that coalgebras of a Vietoris functor on the category of Priestley spaces, i.e., compact, totally ordered disconnected spaces, provide sound and complete semantics for positive modal logic. The objective of this chapter is to combine the idea that the semantics of Fitting's many-valued modal logic can be understood as coalgebras for the bi-Vietoris functor on the category $PBS_{\mathcal{L}}$ of $\mathcal{L}$-valued pairwise Boolean spaces and pairwise continuous maps.

We first define an endofunctor $V^{bi}_{\mathcal{L}} : PBS_{\mathcal{L}} \to PBS_{\mathcal{L}}$, called $\mathcal{L}$-biVietoris functor. Then we demonstrate that the category $COALG(V^{bi}_{\mathcal{L}})$ of coalgebras for the endofunctor $V^{bi}_{\mathcal{L}}$ is isomorphic to the category $PRBS_{\mathcal{L}}$.

7.1 The notions of Coalgebra and Bitopological spaces

The notion of Coalgebra

Let's review the definitions of coalgebra and coalgebra morphisms. We refer the reader to [6] for an overview of coalgebras.

Definition 7.1.1. *A coalgebra for an endofunctor $\mathfrak{T} : \mathbf{C} \to \mathbf{C}$ on a category $\mathbf{C}$, called $\mathfrak{T}$-coalgebra, is defined by a tuple (C, T), where C is an object in $\mathbf{C}$ and $T : C \to \mathfrak{T}(C)$ is an arrow in $\mathbf{C}$.*

Definition 7.1.2. *Let (C_1, T_1) and (C_2, T_2) be any two $\mathfrak{T}$-coalgebras. Then* f :

$(C_1, T_1) \to (C_2, T_2)$ *is said to be a* $\mathfrak{T}$*-coalgebra morphism if* $f : C_1 \to C_2$ *is an arrow in* $\mathbf{C}$ *which satisfies* $T_2 \circ f = \mathfrak{T}(f) \circ T_1$*, i.e., the following diagram commutes:*

$$\begin{array}{ccc} C_1 & \xrightarrow{\;f\;} & C_2 \\ {\scriptstyle T_1}\big\downarrow & & \big\downarrow{\scriptstyle T_2} \\ f(C_1) & \xrightarrow[\mathfrak{T}(f)]{} & f(C_2) \end{array}$$

$\mathfrak{T}$-coalgebras and $\mathfrak{T}$-coalgebra morphisms form a category, denoted by $COALG(\mathfrak{T})$.

Basic concept of Bitopological spaces

A bitopological space is defined by a triple (X, τ_1, τ_2) in which (X, τ_1) and (X, τ_2) are topological spaces. Consider δ_1 and δ_2 represent, respectively, the collections of τ_1-closed sets and τ_2-closed sets. We set $\beta_1 = \tau_1 \cap \delta_2$ and $\beta_2 = \tau_2 \cap \delta_1$.

Definition 7.1.3 ([39])**.** *(i) A bitopological space* (X, τ_1, τ_2) *is said to be pairwise Hausdorff space if for every pair* (x, y) *of distinct points* $x, y \in X$ *there exists disjoint open sets* $U_x \in \tau_1$ *and* $U_y \in \tau_2$ *containing* x *and* y*, respectively.*

(ii) A bitopological space (X, τ_1, τ_2) *is said to be pairwise zero-dimensional if* β_1 *is a basis for* τ_1 *and* β_2 *is a basis for* τ_2*.*

(iii) A bitopological space (X, τ_1, τ_2) *is said to be pairwise compact if the topological space* (X, τ)*, where* $\tau = \tau_1 \vee \tau_2$*, is compact.*

7.2 The structure of the endofunctor $V_{\mathcal{L}}^{bi}$

In this section, we introduce the concept of pairwise Vietoris spaces and construct an endofunctor $V_{\mathcal{L}}^{bi}$ on the category $PBS_{\mathcal{L}}$.

We define the pairwise Vietoris space as follows:

Definition 7.2.1. *Let* (S, τ_1^S, τ_2^S) *be a pairwise topological space and* $\mathcal{K}(S)$ *the set of all pairwise closed subsets of* S*. We define* $\Box U = \{C \in \mathcal{K}(S) : C \subseteq U\}$ *and* $\Diamond U = \{C \in \mathcal{K}(S) : C \cap U \neq \emptyset\}$*,* $U \subseteq S$*. Let* β_1^S *and* β_2^S *be the basis for the* *topologies* τ_1^S *and* τ_2^S*, respectively. The pairwise Vietoris space* $V_P(S)$ *of the pairwise topological space* (S, τ_1^S, τ_2^S) *is defined as a pairwise topological space* $(\mathcal{K}(S), \tau_1^V, \tau_2^V)$*, where* τ_1^V *is the topology on* $\mathcal{K}(S)$ *generated by subbasis* $\{\Box U, \Diamond U : U \in \beta_1^S\}$ *and the topology* τ_2^V *on* $\mathcal{K}(S)$ *is generated by subbasis* $\{\Box U, \Diamond U : U \in \beta_2^S\}$*.*

We then show that $V_P(S)$ is a pairwise Boolean space whenever S is a pairwise Boolean space.

Lemma 7.2.1. *If* (S, τ_1^S, τ_2^S) *is a pairwise Boolean space then* $V_P(S) = (\mathcal{K}(S), \tau_1^V, \tau_2^V)$ *is pairwise zero-dimensional.*

Proof. We shall show that $\beta_1^V = \tau_1^V \cap \delta_2^V$ is a basis for $\tau_!^V$, where δ_2^V is the set of τ_2^V-closed sets. Let $\mathcal{O} \in \tau_1^V$. Then $\mathcal{O}$ can be expressed as $\mathcal{O} = \bigcup_{\lambda \in \Lambda} (\bigcap_{j=1}^{n_\lambda} \Box U_j \cap \bigcap_{k=1}^{m_\lambda} \Diamond U_k)$, $U_j, U_k \in \beta_1^S = \tau_1^S \cap \delta_2^S$. In order to show that β_1^V is a basis for τ_1^V, it is necessary to show that $\bigcap_{j=1}^{n_\lambda} \Box U_j \cap \bigcap_{k=1}^{m_\lambda} \Diamond U_k \in \beta_1^V$. Because the finite intersection of the members of β_1^V is again in β_1^V, it is sufficient to establish that for $U \in \beta_1^S$, $\Box U, \Diamond U \in \beta_1^V$. As τ_1^V is the topology generated by the subbasis $\{\Box U, \Diamond U : U \in \beta_1^S\}$, hence $\Box U, \Diamond U \in \tau_1^V$. Now we see that $(\Box U)^c = \Diamond U^c$ and $(\Diamond U)^c = \Box U^c$. Since $U \in \beta_1^S$, so $U^c \in \beta_2^S$. As a result, $\Box U, \Diamond U \in \delta_2^V$. Henceforth, $\Box U, \Diamond U \in \beta_1^V$. Similarly, it can be shown that $\beta_2^V = \tau_2^V \cap \delta_1^V$, δ_1^V is the set of τ_1^V-closed sets, is a basis for τ_2^V. □

Lemma 7.2.2. *If* (S, τ_1^S, τ_2^S) *is a pairwise Boolean space then* $V_P(S) = (\mathcal{K}(S), \tau_1^V, \tau_2^V)$ *is pairwise Hausdorff.*

Proof. Let $C, C' \in \mathcal{K}(S)$ and $C \neq C'$. Let $z \in C$ such that $z \neq z'$, $\forall z' \in C'$. For each point $z' \in C'$, we choose disjoint open sets $U_{z'}^c \in \beta_2^S$ and $U_{z'} \in \beta_1^S$ (using the condition that (S, τ_1^S, τ_2^S) is pairwise Hausdorff space.) containing points z' and z, respectively. So the collection $\{U_{z'}^c : z' \in C'\}$ is τ_2^S-open covering of C'. As C' is pairwise compact, so there is a finite collection $\{U_{z_i'}^c : i = 1, 2, \cdots, n\}$ such that $C' \subseteq \bigcup_{i=1}^n U_{z_i'}^c$. Let $V' = \bigcup_{i=1}^n U_{z_i'}^c$ and $U = \bigcap_{i=1}^n U_{z_i'}$. As $z \in C \cap U$, hence $C \cap U \neq \emptyset$. Also, $C' \cap U = \emptyset$ because $C' \subseteq U^c$. It follows that $C \in \Diamond U \in \tau_1^V$ and $C' \notin \Diamond U$ i.e., $C' \in (\Diamond U)^c = \Box U^c \in \tau_2^V$. So we have two disjoint open sets $\Diamond U \in \tau_1^V$ and $\Box U^c \in \tau_2^V$ containing C and C', respectively. □

Lemma 7.2.3. *If* (S, τ_1^S, τ_2^S) *is a pairwise Boolean space then* $V_P(S) = (\mathcal{K}(S), \tau_1^V, \tau_2^V)$ *is pairwise compact.*

Proof. It is known from Proposition 6.2.1 that $\{\Box U, \Diamond U : U \in \beta_1^S \cup \beta_2^S\}$ is a subbasis for the topology $\tau_1^S \vee \tau_2^S$. We shall show that every cover of $\mathcal{K}(S)$ by subbasis-open sets has a finite subcover. Let $\mathcal{K}(S) = \bigcup_{\lambda\in\Lambda} \Box U_\lambda \cup \bigcup_{i\in I} \Diamond V_i$. Consider $S_1 = S - \bigcup_{i\in I} V_i$. Then S_1 is a pairwise closed subset of S. Hence, $S_1 \in \mathcal{K}(S)$. Since, $S_1 \notin \Diamond V_i$ for each $i \in I$, so that $S_1 \in \bigcup_{\lambda\in\Lambda} \Box U_\lambda$. Then for some $\lambda' \in \Lambda$, $S_1 \in \Box U_{\lambda'}$. As a result, $S_1 \subseteq U_{\lambda'}$ and hence $S - U_{\lambda'} \subseteq S - S_1 = \bigcup\limits_{i\in I} V_i$. Then, $S = U_{\lambda'} \cup \bigcup\limits_{i\in I} V_i$. As S is pairwise compact, we have $S = U_{\lambda'} \cup \bigcup_{i=1}^{n} V_i$. Let A be an arbitrary element of $\mathcal{K}(S)$. If $A \subseteq U_{\lambda'}$ then $A \in \Box U_{\lambda'}$ otherwise $A \subseteq \bigcup_{i\in I} V_i$ i.e., $A \cap V_i \neq \emptyset$ for some $i \in \{1, 2, \cdots, n\}$. As a result, $A \in \Box U_{\lambda'} \cup \bigcup_{i\in I} \Diamond V_i$. Therefore, $V_P(S) = (\mathcal{K}(S), \tau_1^V, \tau_2^V)$ is pairwise compact. □

Lemmas 7.2.1, 7.2.2 and 7.2.3 establish the following result:

Theorem 7.2.1. *If (S, τ_1^S, τ_2^S) is a pairwise Boolean space then $V_P(S) = (\mathcal{K}(S), \tau_1^V, \tau_2^V)$ is also a pairwise Boolean space.*

We now construct the $\mathcal{L}$-biVietoris functor $V_{\mathcal{L}}^{bi}$.

Definition 7.2.2. *We define a $\mathcal{L}$-biVietoris functor $V_{\mathcal{L}}^{bi} : PBS_{\mathcal{L}} \to PBS_{\mathcal{L}}$ as follows:*

(i) For an object $(S, \tau_1^S, \tau_2^S, \alpha_S)$ in $PBS_{\mathcal{L}}$, we define $V_{\mathcal{L}}^{bi}(S, \tau_1^S, \tau_2^S, \alpha_S) = (V_P(S), V_P \circ \alpha_S)$ where α_S is a mapping from $\mathfrak{S}_{\mathcal{L}}$ to Λ_S, then for any $\mathcal{L}_1 \in \mathfrak{S}_{\mathcal{L}}$, $V_P \circ \alpha_S(\mathcal{L}_1)$ is the pairwise Vietoris space of a pairwise closed subspace (i.e., pairwise Boolean subspace) $\alpha_S(\mathcal{L}_1)$ of S;

(ii) For an arrow $f : (S_1, \tau_1^{S_1}, \tau_2^{S_1}, \alpha_{S_1}) \to (S_2, \tau_1^{S_2}, \tau_2^{S_2}, \alpha_{S_2})$ in $PBS_{\mathcal{L}}$, $V_{\mathcal{L}}^{bi}(f) : (V_P(S_1), V_P \circ \alpha_{S_1}) \to (V_P(S_2), V_P \circ \alpha_{S_2})$ is defined by $V_{\mathcal{L}}^{bi}(f)(K) = f[K]$, where $K \in V_P(S_1)$.

We verify the well-definedness of the functor $V_{\mathcal{L}}^{bi}$.

Lemma 7.2.4. *Let $(S, \tau_1^S, \tau_2^S, \alpha_S)$ be an object in $PBS_{\mathcal{L}}$. Then $V_{\mathcal{L}}^{bi}(S, \tau_1^S, \tau_2^S, \alpha_S)$ is an object in $PBS_{\mathcal{L}}$.*

Proof. Theorem 7.2.1 shows that $V_P(S)$ is a pairwise Boolean space. Now we shall show that $V_P \circ \alpha_S$ is a pairwise closed subspace of $V_P(S)$. For $\mathcal{L}_1 \in \mathfrak{S}_{\mathcal{L}}$, an element of $V_P(S) \circ \alpha_S(\mathcal{L}_1)$ is a pairwise compact subset of $\alpha_S(\mathcal{L}_1)$. As $\alpha_S(\mathcal{L}_1)$ is also pairwise compact subspace of S, so that an element of $V_P \circ \alpha_S(\mathcal{L}_1)$ is a pairwise compact subset of S. As a result, $V_P \circ \alpha_S(\mathcal{L}_1)$ is a subset of $V_P(S)$. For $U \in \beta_1^S$,

we get $\Box U \cap V_P \circ \alpha_S(\mathcal{L}_1) = \{C \in V_P \circ \alpha_S(\mathcal{L}_1) : C \subset U\} = \Box(U \cap \alpha_S(\mathcal{L}_1))$ and $\Diamond U \cap V_P \circ \alpha_S(\mathcal{L}_1) = \{C \in V_P \circ \alpha_S(\mathcal{L}_1) : C \cap U \neq \emptyset\} = \Diamond(U \cap \alpha_S(\mathcal{L}_1))$. Similarly for $U \in \beta_2^S$. Hence, $V_P \circ \alpha_S(\mathcal{L}_1)$ is a pairwise subspace of $V_P(S)$. Since $\alpha_S(\mathcal{L}_1)$ is a pairwise Boolean subspace of S, by Theorem 7.2.1 we have $V_P \circ \alpha_S(\mathcal{L}_1)$ is a pairwise Boolean space. Henceforth, $V_P \circ \alpha_S(\mathcal{L}_1)$ is a pairwise closed subspace of $V_P(S)$.
Now we show that $V_P \circ \alpha_S$ satisfies the conditions given in the object part of Definition 6.4.1. If $\alpha_S(\mathcal{L}) = S$ then $V_P \circ \alpha_S(\mathcal{L}) = V_P(S)$.
Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in \mathfrak{S}_{\mathcal{L}}$. If $\mathcal{L}_1 = \mathcal{L}_2 \cap \mathcal{L}_3$ then we show that $V_P(\alpha_S(\mathcal{L}_1)) = V_P(\alpha_S(\mathcal{L}_2)) \cap V_P(\alpha_S(\mathcal{L}_3))$. Now $V_P(\alpha_S(\mathcal{L}_1)) = V_P(\alpha_S(\mathcal{L}_2 \cap \mathcal{L}_3)) = V_P(\alpha_S(\mathcal{L}_2) \cap \alpha_S(\mathcal{L}_3))$. The element structure of $V_P(\alpha_S(\mathcal{L}_2) \cap \alpha_S(\mathcal{L}_3))$ is of the form $P \cap (\alpha_S(\mathcal{L}_2) \cap \alpha_S(\mathcal{L}_3))$ and $Q \cap (\alpha_S(\mathcal{L}_2) \cap \alpha_S(\mathcal{L}_3))$, where P and Q are τ_1^S-closed set and τ_2^S-closed set, respectively. The elements of $V_P(\alpha_S(\mathcal{L}_2)) \cap V_P(\alpha_S(\mathcal{L}_3))$ are of the form $(P_1 \cap \alpha_S(\mathcal{L}_2)) \cap (P_2 \cap \alpha_S(\mathcal{L}_3))$ and $(Q_1 \cap \alpha_S(\mathcal{L}_2)) \cap (Q_2 \cap \alpha_S(\mathcal{L}_3))$, where P_1, P_2 are τ_1^S-closed and Q_1, Q_2 are τ_2^S-closed. Then it is straightforward to demonstrate that $V_P(\alpha_S(\mathcal{L}_2) \cap \alpha_S(\mathcal{L}_3)) \subseteq V_P(\alpha_S(\mathcal{L}_2)) \cap V_P(\alpha_S(\mathcal{L}_3))$ and $V_P(\alpha_S(\mathcal{L}_2)) \cap V_P(\alpha_S(\mathcal{L}_3)) \subseteq V_P(\alpha_S(\mathcal{L}_2) \cap \alpha_S(\mathcal{L}_3))$. □

Lemma 7.2.5. *Let* $f : (S_1, \tau_1^{S_1}, \tau_2^{S_1}, \alpha_{S_1}) \to (S_2, \tau_1^{S_2}, \tau_2^{S_2}, \alpha_{S_2})$ *be an arrow in* $PBS_{\mathcal{L}}$. *Then* $V_{\mathcal{L}}^{bi}(f)$ *is an arrow in* $PBS_{\mathcal{L}}$.

Proof. Given that f is a pairwise continuous map from a pairwise Boolean space S_1 to a pairwise Boolean space S_2. Let $K \in V_P(S_1)$. Then K is a pairwise closed subset of S_1 and hence K is pairwise compact. Now $V_{\mathcal{L}}^{bi}(f)(K) = f[K]$ is a pairwise compact subset of S_2. Since S_2 is a pairwise Boolean space, $f[K]$ is a pairwise closed subset of S_2. As a result, $V_{\mathcal{L}}^{bi}(f)(K) \in V_P(S_2)$. To show that $V_{\mathcal{L}}^{bi}(f)$ is pairwise continuous, let $U \in \beta_1^{S_2}$ and $V \in \beta_2^{S_2}$. Then

$$\begin{aligned} V_{\mathcal{L}}^{bi}(f)^{-1}(\Box U) &= \{K \in V_P(S_1) : V_{\mathcal{L}}^{bi}(f)(K) \in \Box U\} \\ &= \{K \in \mathcal{K}(S_1) : f[K] \subseteq U\} \\ &= \{K \in \mathcal{K}(S_1) : K \subseteq f^{-1}(U)\} \\ &= \Box f^{-1}(U) \end{aligned}$$

and

$$\begin{aligned} V^{bi}_{\mathcal{L}}(f)^{-1}(\Diamond U) &= \{K \in V_P(S_1) : V^{bi}_{\mathcal{L}}(f)(K) \in \Diamond U\} \\ &= \{K \in \mathcal{K}(S_1) : f[K] \cap U \neq \emptyset\} \\ &= \{K \in \mathcal{K}(S_1) : K \cap f^{-1}(U) \neq \emptyset\} \\ &= \Diamond f^{-1}(U) \end{aligned}$$

Similarly, $V^{bi}_{\mathcal{L}}(f)^{-1}(\Box V) = \Box f^{-1}(V)$ and $V^{bi}_{\mathcal{L}}(f)^{-1}(\Diamond V) = \Diamond f^{-1}(V)$. Therefore, $V^{bi}_{\mathcal{L}}(f)$ is pairwise continuous. It is still necessary to demonstrate that $V^{bi}_{\mathcal{L}}(f)$ is subspace preserving. Let $M \in V_P \circ \alpha_{S_1}(\mathcal{L}_1)$, $\mathcal{L}_1 \in \mathfrak{S}_{\mathcal{L}}$. Then $M \subseteq \alpha_{S_1}(\mathcal{L}_1)$. As f is an arrow in $PBS_{\mathcal{L}}$, hence f is a subspace preserving map. Thus, $f(M) \subseteq \alpha_{S_2}(\mathcal{L}_1)$. It shows that $V^{bi}_{\mathcal{L}}(f)(M) \subseteq \alpha_{S_2}(\mathcal{L}_1)$. Thus we have $V^{bi}_{\mathcal{L}}(f)(M) \in V_P \circ \alpha_{S_2}(\mathcal{L}_1)$. $\Box$

7.3 Coalgebraic duality for Fitting's many-valued modal logic

We first introduce two functors $\mathfrak{B}$ and $\mathfrak{C}$ between the categories $PRBS_{\mathcal{L}}$ and $COALG(V^{bi}_{\mathcal{L}})$ to show that these two categories are isomorphic.

Functors: $\mathfrak{B}$ and $\mathfrak{C}$

Definition 7.3.1. *We define a functor $\mathfrak{B} : PRBS_{\mathcal{L}} \to COALG(V^{bi}_{\mathcal{L}})$ as follows:*

(i) For an object $(S, \alpha_S, \mathcal{R})$ in $PRBS_{\mathcal{L}}$, define $\mathfrak{B}(S, \alpha_S, \mathcal{R}) = (S, \alpha_S, \mathcal{R}[-])$, where $\mathcal{R}[-] : (S, \alpha_S) \to V^{bi}_{\mathcal{L}}(S, \alpha_S)$ is an arrow in $PBS_{\mathcal{L}}$ defined by $\mathcal{R}[s] = \{p \in S : s\mathcal{R}p\}$, $s \in S$;

(ii) For an arrow $f : (S_1, \alpha_{S_1}, \mathcal{R}_1) \to (S_2, \alpha_{S_2}, \mathcal{R}_2)$ in $PRBS_{\mathcal{L}}$, define $\mathfrak{B}(f) : (S_1, \alpha_{S_1}, \mathcal{R}_1[-]) \to (S_2, \alpha_{S_2}, \mathcal{R}_2[-])$ by $\mathfrak{B}(f) = f$.

The well-definedness of the functor $\mathfrak{B}$ is shown by the following two lemmas:

Lemma 7.3.1. *Let $(S, \alpha_S, \mathcal{R})$ be an object in $PRBS_{\mathcal{L}}$. Then $\mathfrak{B}(S, \alpha_S, \mathcal{R})$ is an object in $COALG(V^{bi}_{\mathcal{L}})$.*

Proof. We shall show that $\mathcal{R}[-] : (S, \alpha_S) \to V^{bi}_{\mathcal{L}}(S, \alpha_S)$ is an arrow in $PBS_{\mathcal{L}}$. By the conditions given in the object part of Definition 6.5.1, we know that for each

$s \in S$, $\mathcal{R}[s]$ is pairwise compact subset of S. As S is pairwise Boolean space, hence $\mathcal{R}[s]$ is a pairwise closed subset of S. Thus $\mathcal{R}[s] \in V_P(S)$. Let $U \in \beta_1^S$. Then

$$\begin{aligned}\mathcal{R}[-]^{-1}(\Box U) &= \{s \in S : \mathcal{R}[s] \in \Box U\} \\ &= \{s \in S : \mathcal{R}[s] \subseteq U\} \\ &= [\mathcal{R}]U \in \beta_1^S \text{ [by Definition 6.5.1]}\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}[-]^{-1}(\Diamond U) &= \{s \in S : \mathcal{R}[s] \in \Diamond U\} \\ &= \{s \in S : \mathcal{R}[s] \cap U \neq \emptyset\} \\ &= \langle \mathcal{R} \rangle U \in \beta_1^S \text{ [by Definition 6.5.1]}\end{aligned}$$

Similarly, for $U \in \beta_2^S$, $\mathcal{R}[-]^{-1}(\Box U) = [\mathcal{R}]U \in \beta_2^S$ and $\mathcal{R}[-]^{-1}(\Diamond U) = \langle \mathcal{R} \rangle U \in \beta_2^S$. Henceforth, $\mathcal{R}[-]$ is a pairwise continuous map. Now we show that $\mathcal{R}[-]$ is subspace preserving. Let $s \in \alpha_S(\mathcal{L}')$, $\mathcal{L}' \in \mathfrak{S}_{\mathcal{L}}$. It is known from Definition 6.5.1 that $\mathcal{R}[s]$ is a pairwise compact subset of $\alpha_S(\mathcal{L}')$. Since $\alpha_S(\mathcal{L}')$ is itself a pairwise Boolean space, thus we have $\mathcal{R}[s] \in V_P \circ \alpha_S(\mathcal{L}')$. Therefore, $\mathfrak{B}(S, \alpha_S, \mathcal{R})$ is a $V_{\mathcal{L}}^{bi}$-coalgebra. □

Lemma 7.3.2. *Let $f : (S_1, \alpha_{S_1}, \mathcal{R}_1) \to (S_2, \alpha_{S_2}, \mathcal{R}_2)$ be an arrow in $PRBS_{\mathcal{L}}$. Then $\mathfrak{B}(f)$ is an arrow in $COALG(V_{\mathcal{L}}^{bi})$.*

Proof. As f is an arrow in $PRBS_{\mathcal{L}}$, so $\mathfrak{B}(f) = f : (S_1, \alpha_{S_1}, \mathcal{R}_1[-]) \to (S_2, \alpha_{S_2}, \mathcal{R}_2[-])$ is a pairwise continuous map. Now using the conditions mentioned in the arrow part of Definition 6.5.1, it is straightforward to verify that $\mathcal{R}_2[-] \circ f = V_{\mathcal{L}}^{bi} \circ \mathcal{R}_1[-]$. Thus $\mathfrak{B}(f)$ is an arrow in $COALG(V_{\mathcal{L}}^{bi})$. □

Definition 7.3.2. *We define a functor $\mathfrak{C} : COALG(V_{\mathcal{L}}^{bi}) \to PRBS_{\mathcal{L}}$ as follows:*

(i) For an object $((C, \alpha_C), \xi)$ in $COALG(V_{\mathcal{L}}^{bi})$, define $\mathfrak{C}((C, \alpha_C), \xi) = (C, \alpha_C, \mathcal{R}_\xi)$, where $\mathcal{R}_\xi$ is a binary relation on C defined by $d \in \mathcal{R}_\xi[c] \iff d \in \xi(c)$, $c, d \in C$;

(ii) For an arrow $f : ((C_1, \alpha_{C_1}), \xi_1) \to ((C_2, \alpha_{C_2}), \xi_2)$ in $COALG(V_{\mathcal{L}}^{bi})$, define $\mathfrak{C}(f) : (C_1, \alpha_{C_1}, \mathcal{R}_{\xi_1}) \to (C_2, \alpha_{C_2}, \mathcal{R}_{\xi_2})$ by $\mathfrak{C}(f) = f$.

The well-definedness of the functor $\mathfrak{C}$ is shown by Lemma 7.3.3 and Lemma 7.3.4.

Lemma 7.3.3. *For an object $((C, \alpha_C), \xi)$ in $COALG(V_{\mathcal{L}}^{bi})$, $\mathfrak{C}((C, \alpha_C), \xi) = (C, \alpha_C, \mathcal{R}_\xi)$ is an object in $PRBS_{\mathcal{L}}$.*

Proof. In order to show that $\mathfrak{C}((C,\alpha_C),\xi)$ is an object in $PRBS_{\mathcal{L}}$, we must verify that $\mathfrak{C}((C,\alpha_C),\xi)$ satisfies the conditions given in the object part of Definition 6.5.1. For each $c \in C$, $\mathcal{R}_\xi[c] = \xi(c) \in V_P(C)$. Hence, $\mathcal{R}_\xi[c]$ is a pairwise closed subset of C. Thus $\mathcal{R}_\xi[c]$ is pairwise compact. Let $U \in \beta_1^C$. Then

$$\begin{aligned}[\mathcal{R}_\xi](U) &= \{c \in C : \mathcal{R}_\xi[c] \subseteq U\} \\ &= \{c \in C : \xi(c) \subseteq U\} \\ &= \{c \in C : \xi(c) \in \Box U\} \\ &= \xi^{-1}(\Box U) \in \beta_1^C\end{aligned}$$

and

$$\begin{aligned}\langle \mathcal{R}_\xi \rangle U &= \{c \in C : \mathcal{R}_\xi[c] \cap U \neq \emptyset\} \\ &= \{c \in C : \xi(c) \cap U \neq \emptyset\} \\ &= \{c \in C : \xi(c) \in \Diamond U\} \\ &= \xi^{-1}(\Diamond U) \in \beta_1^C\end{aligned}$$

Finally, let $m \in \alpha_C(\mathcal{L}')$ for $\mathcal{L}' \in \mathfrak{S}_{\mathcal{L}}$. As ξ is a subspace preserving map from (C,α_C) to $V_{\mathcal{L}}^{bi}(C,\alpha_C)$, we have $\mathcal{R}_\xi[m] = \xi(m) \in V_P \circ \alpha_C(\mathcal{L}')$. Henceforth, $\mathcal{R}_\xi[m] \subset \alpha_C(\mathcal{L}')$. □

Lemma 7.3.4. *For an arrow $f : ((C_1,\alpha_{C_1}),\xi_1) \to ((C_2,\alpha_{C_2}),\xi_2)$ in $COALG(V_{\mathcal{L}}^{bi})$, $\mathfrak{C}(f) : (C_1,\alpha_{C_1},\mathcal{R}_{\xi_1}) \to (C_2,\alpha_{C_2},\mathcal{R}_{\xi_2})$ is an arrow in $PRBS_{\mathcal{L}}$.*

Proof. It is straightforward to prove that $\mathfrak{C}$ is an arrow in $PRBS_{\mathcal{L}}$. □

Now we obtain the following result:

Theorem 7.3.1. *The categories $PRBS_{\mathcal{L}}$ and $COALG(V_{\mathcal{L}}^{bi})$ are isomorphic.*

Proof. We shall show that the categories $PRBS_{\mathcal{L}}$ and $COALG(V_{\mathcal{L}}^{bi})$ are isomorphic via the functors $\mathfrak{B}$ and $\mathfrak{C}$. Let $(S,\alpha_S,\mathcal{R})$ be an object in $PRBS_{\mathcal{L}}$. Then $\mathfrak{C} \circ \mathfrak{B}(S,\alpha_S,\mathcal{R}) = \mathfrak{C}(S,\alpha_S,\mathcal{R}[-]) = (S,\alpha_S,\mathcal{R}_{\mathcal{R}[-]})$. Now $t \in \mathcal{R}_{\mathcal{R}[-]}(s) \iff t \in \mathcal{R}[s]$. Thus, $(S,\alpha_S,\mathcal{R}) = \mathfrak{C} \circ \mathfrak{B}(S,\alpha_S,\mathcal{R})$. Let $((C,\alpha_C),\xi)$ be an object in $COALG(V_{\mathcal{L}}^{bi})$. Then $\mathfrak{B} \circ \mathfrak{C}((C,\alpha_C),\xi) = \mathfrak{B}(C,\alpha_C,\mathcal{R}_\xi) = ((C,\alpha_C),\mathcal{R}_\xi[-])$. We have $c_2 \in \mathcal{R}_\xi[c_1] \iff c_2 \in \xi(c_1)$. As a result, $((C,\alpha_C),\xi) = \mathfrak{B} \circ \mathfrak{C}((C,\alpha_C),\xi)$. It is clear that for an arrow f in $COALG(V_{\mathcal{L}}^{bi})$, $\mathfrak{B} \circ \mathfrak{C}(f) = f$ and for an arrow f in $PRBS_{\mathcal{L}}$, $\mathfrak{C} \circ \mathfrak{B}(f) = f$. □

Coalgebraic Duality

Using Theorems 6.5.3 and 7.3.1, we arrive at the following duality theorem:

Theorem 7.3.2. *The categories $\mathcal{MA}_{\mathcal{L}}$ and $COALG(V^{bi}_{\mathcal{L}})$ are dually equivalent.*

Thus the modal semi-primal duality for algebras of Fitting's Heyting-valued modal logic (for more information, see [21]) can potentially be represented in terms of the coalgebras of $\mathcal{L}$-biVietoris functor $V^{bi}_{\mathcal{L}}$.
Finally, based on the preceding theorems, we can conclude:

Theorem 7.3.3. *Fitting's Heyting-valued modal logic is sound and complete with respect to coalgebras of the biVietoris functor $V^{bi}_{\mathcal{L}}$.*

7.4 Conclusion

We have demonstrated how the theory of coalgebras can be used to characterise the category $PRBS_{\mathcal{L}}$ and thus obtained a coalgebraic description of the bitopological duality for Fitting's Heyting-valued modal logic. In this chapter, we have explicitly constructed the Vietoris functor on the category $PBS_{\mathcal{L}}$ of $\mathcal{L}$-pairwise Boolean spaces and we have finally concluded that coalgebras for this functor provide sound and complete semantics for Fitting's Heyting-valued modal logic.
As an application of this coalgebraic duality, we may establish the existence of a final coalgebra and cofree coalgebras in the category $COALG(V^{bi}_{\mathcal{L}})$, and we can also develop the coalgebraic duality theorem for many-valued modal logics in a bitopological scenario.

Chapter 8

Coalgebraic Fuzzy geometric logic

The goal of this chapter is to develop coalgebraic fuzzy geometric logic by incorporating modalities into the language of fuzzy geometric logic. A generalized form of modal logic can be created within the context of coalgebraic logic. Coalgebraic geometric logic was recently developed by adding modalities to the language of propositional geometric logic using the coalgebra approach. However, as far as we are aware, no studies have been done specifically on modal fuzzy geometric logic. This chapter study the modal fuzzy geometric logic using coalgebra theory. This new logic might potentially be used to model and reason about transition systems that involve uncertainty in behaviour. We propose a theoretical framework based on coalgebra theory to add modalities into the language of fuzzy geometric logic. Coalgebras for an endofunctor on a category of fuzzy topological spaces and fuzzy continuous maps serve as the foundation for models of this logic. Our key finding is the existence of a final model in the category of models for endofunctors defined on sober fuzzy topological spaces. Furthermore, we present a comparative analysis of the notions of behavioural equivalence, bisimulation, and modal equivalence on the resulting class of models.

In [77], fuzzy geometric logic is introduced as a natural extension of propositional geometric logic [98]. Vickers in [98] developed geometric logic based on point-free topology, propositional logic, and the logic of finite observations [4]. Several studies have mentioned it (e.g., [74, 11, 12, 99, 100]). The language of geometric logic

The outcomes of this chapter can be found in [60] **Das, Litan Kumar., Ray, Kumar Sankar., Mali, Prakash Chandra.: Coalgebraic Fuzzy geometric logic. International Journal of Information Technology, Springer (accepted).** and [59] **Das, Litan Kumar., Ray, Kumar Sankar., Mali, Prakash Chandra.: Bisimulations for Fuzzy Geometric Models. International Conference on Recent Trends in Artificial Intelligence and IoT, 1822, 152-163, CCIS, Springer, 2023.**

is created on a collection of propositional variables by applying propositional connectives: finite conjunction ($\wedge$) and arbitrary disjunction ($\bigvee$). These connectives preserve the property of finite observability. Vickers [98] investigated the connection between topological spaces, topological systems, and geometric logic. A topological system is defined by a triple $(X, \models, A)$ in which X occurs as a non-empty set of objects, A defines a frame and $\models$ is a satisfaction relation from X to A.
The authors in [77] have generalized geometric logic to the many-valued context by extending the notion of satisfiability relation. They noticed that if the satisfaction relation is fuzzy, there are two possible outcomes for the related consequence relation: crisp or fuzzy. They consequently introduced general fuzzy geometric logic as well as fuzzy geometric logic with graded consequences. In addition, their work demonstrated the link between fuzzy geometric logic, fuzzy topology, and fuzzy topological systems. The concept of fuzzy topological spaces has introduced in [78] and has been the focus of numerous studies (e.g. [79, 80, 81, 82, 83]). A comprehensive explanation of graded consequences and associated issues can be found in [84].
A thorough literature review on several aspects of coalgebraic logic has been accomplished, and its findings have been compiled in Table 8.1. The literature survey makes clear that the coalgebraization of fuzzy geometric logic has not been studied. Consequently, the goal of the current study is to investigate modal fuzzy geometric logic using the coalgebra process and to establish a criterion for the existence of final fuzzy geometric models.

We extend the predicate lifting approach [87] and apply it to build modal operators for fuzzy geometric logic, which can be interpreted in coalgebra-based models with a fuzzy topological space as the state space. The structures, known as fuzzy geometric models, provide the semantics of our coalgebraic logic. Final models are important in "state-based systems" because they create what is commonly referred to as minimal representations: they are canonical interpretations that include every possible behaviours that a system could exhibit. The duality between sober fuzzy topological spaces and spatial frames (cf. Theorem 8.1.1) enables us to approach challenges from several angles. As an instance, the duality between sober fuzzy spaces and spatial frames leads to the concrete creation of a final model (see Section 8.3). The concept of bisimulation [91] is widely used in computer science and mathematics. Bisimulation between coalgebras is a fundamental idea in "state-based systems" that associates states with the same behaviour. In [95], the author established various conceptions of coalgebraic bisimulation and investigated their relationship. In the current chapter, we study the notions of bisimulation for fuzzy

Table 8.1: Literature review (tabular form).

Sr. No.	**Investigator**	**logic**	**Findings of the study**
1	[98]	Propositional Geometric logic: logic of finite observations	Interrelation between systems and spaces
2	[77]	Fuzzy geometric logic	Interrelation among fuzzy geometric logic, fuzzy systems, and fuzzy spaces
3	[63]	Coalgebraic logic	Coalgebra is excellent for reasoning concepts relating to behaviour and observable indistinguishability.
4	[66, 67]	Coalgebraic modal logic	Coalgebraic logic on the category of sets is constructed, and modal operators are defined using the methods of predicate lifting.
5	[69]	Stone-based coalgebraic logic	Clopen-predicate liftings are used to define modal operators and explore various concepts of bisimulation.
6	[88, 89]	Coalgebraic geometric logic	Investigate open-predicate lifting and create a criterion to demonstrate the existence of final geometric models. .

geometric models. Our aim here is to show that the concepts of fuzzy geometric modal equivalence, bisimulation, and behavioural equivalence coincide on the categories of fuzzy topological spaces and the category of sober fuzzy topological spaces, provided the set of fuzzy predicate liftings and the endofunctors satisfy certain requirements.

Our research is comparable to that found in [88, 89]. From a mathematical perspective, the findings in this chapter might be referred to as generalizations of relevant classical concepts to the many-valued setting.

8.1 The preliminary findings from Fuzzy Set Theory, Fuzzy topological spaces and Coalgebra theory

Fuzzy set theory

Zadeh [107] explored fuzzy set theory. We review some essential concepts in fuzzy set theory.

Definition 8.1.1 ([107]). *A fuzzy set $\tilde{f}$ on a set S is defined by the membership function $\tilde{f} : S \to [0, 1]$.*

Let $\tilde{f}^c$ denote the complement of $\tilde{f}$. Define $\tilde{f}^c : S \to [0, 1]$ by $\tilde{f}^c(s) = 1 - \tilde{f}(s)$, $\forall s \in S$. $\tilde{f}^c$ is a fuzzy set on S.

Note 8.1.1. *If $\tilde{f}_1$ and $\tilde{f}_2$ are fuzzy sets on S, then $\tilde{f}_1 \vee \tilde{f}_2$ and $\tilde{f}_1 \wedge \tilde{f}_2$ are fuzzy sets on S, where the fuzzy sets $\tilde{f}_1 \vee \tilde{f}_2$ and $\tilde{f}_1 \wedge \tilde{f}_2$ are defined by $(\tilde{f}_1 \vee \tilde{f}_2)(s) = \tilde{f}_1(s) \vee \tilde{f}_2(s)$ and $(\tilde{f}_1 \wedge \tilde{f}_2)(s) = \tilde{f}_1(s) \wedge \tilde{f}_2(s)$, respectively.*

Remark 8.1.1. *For each $s \in S$, the grade of membership of s in the fuzzy set $\tilde{f}$ is given by the value $\tilde{f}(s)$. It is represented by the symbol $gr(s \in \tilde{f})$.*

Definition 8.1.2 ([107]). *Let S_1 and S_2 be two sets and $f : S_1 \to S_2$ be a given function. For a fuzzy set $\tilde{s_1}$ on S_1, the direct image $f(\tilde{s_1}) : S_2 \to [0, 1]$ of the fuzzy set $\tilde{s_1}$ under the function f is defined by $f(\tilde{s_1})(s) = \bigvee\{\tilde{s_1}(t) : t \in f^{-1}(\{s\})\}$, where $s \in S_2$.*

Definition 8.1.3 ([107]). *Let S_1 and S_2 be two sets and $f : S_1 \to S_2$ be a given function. For a fuzzy set $\tilde{s_2}$ on S_2, the inverse image $f^{-1}(\tilde{s_2}) : S_1 \to [0, 1]$ of the fuzzy set $\tilde{s_2}$ under the function f is defined by $f^{-1}(\tilde{s_2}) = \tilde{s_2} \circ f$.*

Definition 8.1.4 ([107]). *Let μ and η be fuzzy sets on S. Then, μ is a fuzzy subset of η, denoted by $\mu \leq \eta$, $\iff$ $\mu(s) \leq \eta(s)$, $\forall s \in S$.*

Definition 8.1.5 ([107]). *Consider a mapping $f : S \to T$ and a collection $\{\mu_i : i \in \mathcal{I}, \mathcal{I}$ is an index set $\}$ of fuzzy sets on T. Then*

(i) $f^{-1}(\bigvee_{i \in \mathcal{I}} \mu_i) = \bigvee_{i \in \mathcal{I}} f^{-1}(\mu_i)$;

(ii) $f^{-1}(\bigwedge_{i \in \mathcal{I}} \mu_i) = \bigwedge_{i \in \mathcal{I}} f^{-1}(\mu_i)$.

Goguen first considered the category of fuzzy sets in [102]. Several authors studied on the category of fuzzy sets (e.g.,[104, 105, 106]). Let **FS** denote the category of fuzzy sets.

Definition 8.1.6 ([104]). *The category **FS** is defined as follows:*

*(i) An object in **FS** is a pair $(S, \tilde{g})$, where S is a set and $\tilde{g} : S \to [0, 1]$ is a membership function;*

*(ii) A morphism $f : (S, \tilde{g}) \to (T, \tilde{h})$ in **FS** is a function $f : S \to T$ such that $\tilde{g}(s) \leq f^{-1}(\tilde{h})(s)$.*

Fuzzy topological spaces

We recall the definition of fuzzy topological spaces from [78].

Definition 8.1.7 ([78]). *Let S be a set. A collection τ_S of fuzzy sets on S is said to be fuzzy topology on S if the following conditions hold:*

(i) $\tilde{\emptyset}, \tilde{S} \in \tau_S$, where $\tilde{\emptyset}(s) = 0$, $\forall s \in S$ and $\tilde{S}(s) = 1$, $\forall s \in S$;

(ii) if $\tilde{g}_1, \tilde{g}_2 \in \tau_s$ then $\tilde{g}_1 \wedge \tilde{g}_2 \in \tau_S$, where $(\tilde{g}_1 \wedge \tilde{g}_2)(s) = \tilde{g}_1(s) \wedge \tilde{g}_2(s)$;

(iii) if $\tilde{g}_j \in \tau_S$ for $j \in \Lambda$, Λ is an index set, then $\bigvee_{j \in \Lambda} \tilde{g}_j \in \tau_S$, where $\bigvee_{j \in \Lambda} \tilde{g}_j(s) = \sup_{j \in \Lambda}\{\tilde{g}_j(s)\}$.

Then, the pair (S, τ_S) is referred to as a fuzzy topological space and members of τ_S are said to be fuzzy open sets on (S, τ_S).

Note 8.1.2. *Let (S, τ_S) be a fuzzy topological space. Then, the fuzzy topology τ_S on S can be considered as a frame.*

Definition 8.1.8 ([103]). *Let (S, τ_S) be a fuzzy topological space. Then a subset $\mathfrak{B}$ of τ_S is called a basis for (S, τ_S) if it satisfies the following conditions:*

(i) if $\tilde{b}_1, \tilde{b}_2 \in \mathfrak{B}$ then $\tilde{b}_1 \wedge \tilde{b}_2 \in \mathfrak{B}$;

(ii) for each member $\tilde{t} \in \tau_S$, there exists a subcollection $\mathcal{C} = \{\tilde{t}_j \in \mathfrak{B} : j \in \Lambda\}$ such that $\tilde{t} = \bigvee_{j \in \Lambda} \tilde{t}_j$.

Definition 8.1.9 ([103])**.** *Let τ_S be a fuzzy topology on S and $\mathfrak{S} \subset \tau_S$. Then, $\mathfrak{S}$ is a subbasis for a fuzzy space $(S, \tau_S) \iff$ the collection of all finite meets of members of S is a basis for (S, τ_S).*

Definition 8.1.10. *A fuzzy topological space (S, τ_S) is said to be Kolmogorov space or T_0-space if for any pair (x, y) of distinct points in S, there is a fuzzy open set $\tilde{g}$ on S such that $\tilde{g}(x) \neq \tilde{g}(y)$.*

Definition 8.1.11. *Let (F, τ_F) and (G, τ_G) be fuzzy topological spaces. A mapping $f : F \to G$ is fuzzy continuous if and only if, for every fuzzy open set $\tilde{g}$ on (G, τ_G), $f^{-1}(\tilde{g})$ is a fuzzy open set on (F, τ_F).*

Let **Fuzzy-Top** denote the category of fuzzy topological spaces.

Definition 8.1.12. *The category **Fuzzy-Top** is defined as follows:*

*(i) Objects in **Fuzzy-Top** are fuzzy topological spaces (S, τ_S);*

*(ii) Morphisms $f : (S, \tau_S) \to (T, \tau_T)$ in **Fuzzy-Top** are fuzzy continuous mappings.*

Definition 8.1.13. *A functor $\mathcal{Q}$ from the category **Fuzzy-Top** to the category **FS** of fuzzy sets can be defined as follows:*

*(i) For an object (S, τ_S) in **Fuzzy-Top**, define $\mathcal{Q}(S)$ = set of fuzzy open sets on (S, τ_S);*

*(ii) For a morphism $\phi : (S, \tau_S) \to (T, \tau_T)$ in **Fuzzy-Top**, define $\mathcal{Q}(\phi) = \phi^{-1} : \mathcal{Q}(T) \to \mathcal{Q}(S)$ by $\phi^{-1}(\mu) = \mu \circ \phi$, $\mu \in \mathcal{Q}(T)$.*

Definition 8.1.14. *We define a functor $\mathcal{P}$: **Fuzzy-Top** $\to$ **FRM** as follows:*

*(i) For an object S in **Fuzzy-top**, define $\mathcal{P}(S) = \tau_S$;*

*(ii) For an arrow $\eta : S_1 \to S_2$ in **Fuzzy-top**, define $\mathcal{P}(\eta) : \mathcal{P}(S_2) \to \mathcal{P}(S_1)$ by $\mathcal{P}(\eta)(\xi) = \xi \circ \eta$, where $\xi \in \mathcal{P}(S_2)$.*

Note 8.1.3. *Let F be a frame and (S, τ_S) be a fuzzy topological space. Then, the frame F is spatial if there is an isomorphism from F to $\mathcal{P}(S)$.*

Consider that **S-FRM** is the category of spatial frames and homomorphisms between frames. Let F denote a frame and PTF denote the collection of frame homomorphisms h from F to $[0, 1]$. Then the collection $\{\Psi(a) : a \in F\}$ is a fuzzy topology on PTF, where for each $a \in F$, $\Psi(a)$ is a membership function from PTF to $[0, 1]$ which is defined by $\Psi(a)(h) = h(a)$.

Definition 8.1.15 ([97]). *Let τ be a fuzzy topology on S. Assume that $\tilde{T} \in \tau$. A membership function $\Psi(\tilde{T}) : PT\tau \to [0,1]$ can be defined as $\Psi(\tilde{T})(h) = h(\tilde{T})$. Then, $\Psi(\tilde{T})$ is a fuzzy set on $PT\tau$.*

The collection $\{\Psi(\tilde{T}) : \tilde{T} \in \tau\}$ is a fuzzy topology on $PT\tau$.

Corollary 8.1.1 ([97]). *Consider a fuzzy topological space (S, τ). A mapping $f : S \to PT\tau$ is defined by $f(s)(\tilde{\phi}) = \tilde{\phi}(s)$, where $\tilde{\phi} \in \tau$. Then, (S, τ) becomes a sober space $\iff$ f is bijective.*

The category of sober fuzzy topological spaces and fuzzy continuous maps is denoted by **SFuzzy-Top**.

Definition 8.1.16. *We define a functor PT : **FRM** $\to$ **Fuzzy-Top** as follows:*

*(i) For an object F in **FRM**, define $PT(F) = PTF$;*

*(ii) For an arrow $f : F \to F'$ in **FRM**, define $PT(f) : PTF' \to PTF$ by $PT(f)(h) = h \circ f$, where $h \in PTF'$.*

Theorem 8.1.1. *The category **S-FRM** is dually equivalent to the category **SFuzzy-Top**.*

Proof. Let id_1 and id_2 be the identity functors on **S-FRM** and **SFuzzy-Top**, respectively. We define two natural transformations $\zeta : id_1 \to \mathcal{P} \circ PT$ and $\eta : id_2 \to PT \circ \mathcal{P}$. For a spatial frame F, we define $\zeta_F : F \to \mathcal{P} \circ PT(F)$ by $\zeta_F(u)(h) = h(u)$, where $h \in PTF$. Since F is a spatial frame, we have ζ_F is an isomorphism. It becomes easy to observe that ζ is a natural transformation. As a result, ζ is a natural isomorphism.
For an object S in **SFuzzy-Top**, define $\eta_S : S \to PT \circ \mathcal{P}S$ by $\eta_S(s)(\tilde{g}) = \tilde{g}(s), \forall s \in S$ and $\tilde{g} \in \mathcal{P}(S)$. As S is sober, so η_S is bijective. We observe that, for $\tilde{g} \in \mathcal{P}(S)$, $\eta_S^{-1}(\Psi(\tilde{g}))(s) = \Psi(\tilde{g})(\eta_S(s)) = \eta_S(s)(\tilde{g}) = \tilde{g}(s)$. Therefore, $\eta_S^{-1}(\Psi(\tilde{g})) = \tilde{g}$. Moreover, η_S is an open map because $\eta_S(\tilde{g})(h) = \bigvee\{\tilde{g}(s) : s \in \eta_S^{-1}(h)\} = h(\tilde{g}) = \Psi(\tilde{g})(h)$. Therefore, $\eta_S(\tilde{g}) = \Psi(\tilde{g})$. Consequently, η_S is a fuzzy homeomorphism. It can be shown that η is a natural transformation. Hence, η is a natural isomorphism. □

Coalgebra

Coalgebras are categorical structures that are dual or opposite (in the sense of category theory) to the notion of algebras. The coalgebraic approach is abundantly applied in computer science and artificial intelligence (e.g., knowledge representation, concurrency, logical reasoning, automata theory, etc.).

Definition 8.1.17 ([6]). *Assume that T is an endofunctor on a category $\mathcal{S}$. A T-coalgebra is a pair (A, ζ), where A is an object in $\mathcal{S}$ and $\zeta : A \to T(A)$ is a morphism in $\mathcal{S}$.*

Definition 8.1.18 ([6]). *A morphism between T-coalgebras (A, δ) and (B, β) is defined by a morphism $\psi : A \to B$ in $\mathcal{S}$ satisfying the equation $T(\psi) \circ \delta = \beta \circ \psi$, i.e., the Fig. 8.1 commutes.*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta} \\ T(A) & \xrightarrow[T(\psi)]{} & T(B) \end{array}$$

Figure 8.1: Illustration of coalgebra morphism

T-coalgebras and morphisms between T-coalgebras form a category, denoted by $\mathbf{COALG}(T)$.
A final coalgebra is a final or terminal object in $\mathbf{COALG}(T)$. It has a significant impact on computer science. The final coalgebra is crucial as it makes sense of behaviourally equivalent states in coalgebras.

Definition 8.1.19 ([6]). *A final coalgebra in $\boldsymbol{COALG}(T)$ is a T-coalgebra (A, δ) which satisfies that for each T-coalgebra (B, β), a unique morphism exists from (B, β) to (A, δ).*

Definition 8.1.20 ([96]). *Let (A, δ) and (B, β) be objects in $\boldsymbol{COALG}(T)$. We say that any two states $a \in A$ and $b \in B$ are behaviourally equivalent if there exists an object (C, α) in $\boldsymbol{COALG}(T)$ and T-coalgebra morphisms $g : (A, \delta) \to (C, \alpha)$ and $h : (B, \beta) \to (C, \alpha)$ such that $g(a) = h(b)$.*

Definition 8.1.21 ([96]). *Let (A, δ) and (B, β) be two T-coalgebras. Then a relation $\mathcal{R} \subseteq A \times B$ is said to be a bisimulation between (A, δ) and (B, β) if there exists a T-coalgebra $(\mathcal{R}, \gamma)$ such that the projection maps $\pi_1 : \mathcal{R} \to A$ and $\pi_2 : \mathcal{R} \to B$ are coalgebra morphisms and satisfy the relations $\delta \circ \pi_1 = T(\pi_1) \circ \gamma$, $\beta \circ \pi_2 = T(\pi_2) \circ \gamma$. So, the diagram shown in Fig. 8.2 is commutative.*

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & \mathcal{R} & \xrightarrow{\pi_2} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ T(A) & \xleftarrow[T(\pi_1)]{} & T(\mathcal{R}) & \xrightarrow[T(\pi_2)]{} & T(B) \end{array}$$

Figure 8.2: Illustration of coalgebraic bisimulation

8.2 Coalgebraic logic

It is assumed for this section that T is an arbitrary endofunctor on the category $\mathcal{C} =$ **Fuzzy-Top**. We define coalgebraic logic for **Fuzzy-Top**-coalgebras. First, we introduce a notion of a predicate lifting for the endofunctor T, called fuzzy-open predicate lifting.

Definition 8.2.1. *A natural transformation $\lambda : \mathcal{Q}^n \to \mathcal{Q} \circ T$ is called a fuzyy-open predicate lifting. So, by the naturality law the following diagram commutes:*

$$\begin{array}{ccccc} Y & \quad & Q^n X & \xrightarrow{\lambda_X} & Q \circ T(X) \\ {\scriptstyle\phi}\uparrow & & {\scriptstyle Q^n\phi}\uparrow & & \uparrow{\scriptstyle\phi^{-1}} \\ X & & Q^n Y & \xrightarrow[\lambda_Y]{} & Q \circ T(Y) \end{array}$$

Let $\breve{\lambda}$ is the dual of λ. Define $\breve{\lambda}$ as $\breve{\lambda}(\mu_1, \mu_2, \cdots, \mu_n) = 1 - \lambda(1-\mu_1, 1-\mu_2, \cdots, 1-\mu_n)$, where $\mu_i \in \mathcal{Q}(S)$, $i = 1, 2, \cdots, n$ and S is an object in ***Fuzzy-Top****.*

Definition 8.2.2. *The fuzzy-open predicate lifting λ is*

(i) monotone if for every object S in ***Fuzzy-Top*** *and $\mu_i, \eta_i \in \mathcal{Q}(S)$, $i = 1, 2, \cdots, n$ such that $\mu_1 \leq \eta_1, \cdots, \mu_n \leq \eta_n \Rightarrow \lambda_S(\mu_1, \cdots, \mu_n) \leq \lambda_S(\eta_1, \cdots, \eta_n)$.*

Let Σ be a collection of fuzzy-open predicate liftings for T. Then, the collection Σ is said to be a fuzzy geometric modal signature for T. Σ is referred to be monotone whenever every member of Σ is monotone.

Definition 8.2.3. *The collection Σ for an endofunctor T on* ***Fuzzy-Top*** *is considered to be characteristic for T if for each object S in* ***Fuzzy-Top****, the collection $\{\lambda_S(\mu_1, \cdots, \mu_n) : \lambda \in \Sigma, \mu_i \in \mathcal{Q}(S)\}$ meets the subbasis criteria for the fuzzy topology on TS.*

Let $\mathcal{L}(\Sigma)$ denote the modal language generated by Σ.

Definition 8.2.4. *The modal language $\mathcal{L}(\Sigma)$ is the collection $\boldsymbol{FGML}(\Sigma)$ of formulas defined as follows:*
$\beta ::= \top | p | \beta_1 \wedge \beta_2 | \bigvee_{j \in J} \beta_j | \heartsuit^{\lambda}(\beta_1, \beta_2, \cdots, \beta_n)$*, where $\lambda \in \Sigma$, Φ represents the set of propositional variables p and J represents an index set.*

Definition 8.2.5. *A fuzzy geometric model for the functor T is a mathematical structure $\mathcal{S} = (S, \sigma, \mathcal{V})$ consisting of a T-coalgebra (S, σ), and valuation mapping $\mathcal{V} : \Phi \to \mathcal{Q}(S) \subseteq [0,1]^S$.*

We now define a category $FMOD(T)$ as follows.

Definition 8.2.6. *The following describes a category $FMOD(T)$:*

1. *Objects: An object in $FMOD(T)$ is a fuzzy geometric model for T;*
2. *Arrows: An arrow $f : (S, \sigma_1, \mathcal{V}_S) \to (S', \sigma_2, \mathcal{V}'_{S'})$ in $FMOD(T)$ is a coalgebra morphism $f : (S, \sigma_1) \to (S', \sigma_2)$ which satisfies the condition: $f^{-1} \circ \mathcal{V}'_{S'} = \mathcal{V}_S$.*

Definition 8.2.7. *Consider a formula α in $\boldsymbol{FGML}(\Sigma)$. The semantics of α in terms of fuzzy geometric model $\mathcal{S} = (S, \sigma, \mathcal{V})$ is defined as shown below:*

(i) $[[\top]]_{\mathcal{S}}(s) = 1$;

(ii) $[[p]]_{\mathcal{S}}(s) = \mathcal{V}(p)(s)$;

(iii) $[[\alpha_1 \wedge \alpha_2]]_{\mathcal{S}}(s) = [[\alpha_1]]_{\mathcal{S}}(s) \wedge [[\alpha_2]]_{\mathcal{S}}(s)$;

(iv) $[[\bigvee_{i \in J} \alpha_i]]_{\mathcal{S}}(s) = Sup\{[[\alpha_i]]_{\mathcal{S}}(s)\}$;

(v) $[[\heartsuit^{\lambda}(\alpha_1, \alpha_2, \cdots, \alpha_n)]]_{\mathcal{S}}(s) = \lambda_S([[\alpha_1]]_{\mathcal{S}}, [[\alpha_2]]_{\mathcal{S}}, \cdots, [[\alpha_n]]_{\mathcal{S}}) \circ \sigma(s)$.

Grade of a formula α satisfied by a state or world s in S is denoted by $gr(s \models \alpha)$ and defined by $gr(s \models \alpha) = [[\alpha]]_{\mathcal{S}}(s)$. Two states s and t in S are modally equivalent if $gr(s \models \alpha) = gr(t \models \alpha)$, $\forall \alpha$ in **FGML**(Σ). We express it by the notation $s \equiv_{\Sigma} t$.

Definition 8.2.8. *Let $\mathcal{B} = (B, \sigma_1, \mathcal{V}_B)$ and $\mathcal{B}' = (B', \sigma_2, \mathcal{V}_{B'})$ be fuzzy geometric models for T. States $b \in B$ and $b' \in B'$ are said to be behaviourally equivalent in $FMOD(T)$ if there exists an object $\mathcal{C} = (C, \gamma, \mathcal{V}_C)$ in $FMOD(T)$ and morphisms $g : \mathcal{B} \to \mathcal{C}$ and $h : \mathcal{B}' \to \mathcal{C}$ in $FMOD(T)$ such that $g(b) = h(b')$.*

In Proposition 8.2.1, we shall demonstrate that fuzzy geometric model morphisms preserve truth degrees.

Proposition 8.2.1. *Assume that $f : \mathcal{S} = (S, \sigma_1, \mathcal{V}_S) \to \mathcal{K} = (K, \sigma_2, \mathcal{V}_K)$ is a morphism in $FMOD(T)$. Then, we have $gr(s \models \alpha) = gr(f(s) \models \alpha)$, $\forall \alpha \in \boldsymbol{FGML}(\Sigma)$ and $s \in S$.*

Proof. We are to show that for all formulas α, $[[\alpha]]_{\mathcal{S}}(s) = [[\alpha]]_{\mathcal{K}}(f(s))$. If p is a propositional variable then by using Definition 8.2.6, we can show that $[[p]]_{\mathcal{S}}(s) = [[p]]_{\mathcal{K}}(f(s))$ i.e., $gr(s \models p) = gr(f(s) \models p)$. It is straightforward to demonstrate that $gr(s \models \bigvee_{j \in J} \alpha_j) = gr(f(s) \models \bigvee_{j \in J} \alpha_j)$ and $gr(s \models \alpha_1 \wedge \alpha_2) = gr(f(s) \models \alpha_1 \wedge \alpha_2)$. The only part we have to show is that $gr(s \models \heartsuit^\lambda(\alpha_1, \alpha_2, \cdots, \alpha_n) = gr(f(s) \models \heartsuit^\lambda(\alpha_1, \alpha_2, \cdots, \alpha_n))$. Since f is the coalgebra morphism, henceforth $Tf \circ \sigma_1 = \sigma_2 \circ f$. So, the following diagram (Fig.8.3) commutes.

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & K \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ TS & \xrightarrow[\quad Tf \quad]{} & TK \end{array}$$

Figure 8.3: Coalgebra morphism

Applying the functor $\mathcal{Q}$ to the previous diagram (Fig. 8.3) yields the following diagram, which commutes as well.

$$\begin{array}{ccc} \mathcal{Q}S & \xleftarrow{\quad \mathcal{Q}f = f^{-1} \quad} & \mathcal{Q}K \\ {\scriptstyle \mathcal{Q}\sigma_1 = \sigma_1^{-1}}\uparrow & & \uparrow{\scriptstyle \mathcal{Q}\sigma_2 = \sigma_2^{-1}} \\ \mathcal{Q}(TS) & \xleftarrow[\mathcal{Q}(Tf) = (Tf)^{-1}]{} & \mathcal{Q}(TK) \end{array}$$

Now,

$$\begin{aligned} & [[\heartsuit^\lambda(\alpha_1, \alpha_2, \cdots, \alpha_n)]]_{\mathcal{S}}(s) \\ & = \lambda_S([[\alpha_1]]_{\mathcal{S}}, \cdots, [[\alpha_n]]_{\mathcal{S}}) \circ \sigma_1(s) \\ & = \lambda_S([[\alpha_1]]_{\mathcal{K}} \circ f, \cdots, [[\alpha_n]]_{\mathcal{K}} \circ f) \circ \sigma_1(s) \text{ [as } f^{-1}([[\alpha]]_{\mathcal{K}}) = [[\alpha]]_{\mathcal{S}} \text{]} \\ & = \lambda_K([[\alpha_1]]_{\mathcal{K}}, \cdots, [[\alpha_n]]_{\mathcal{K}}) \circ Tf \circ \sigma_1(s) \text{ [by naturality of } \lambda \text{]} \\ & = \lambda_K([[\alpha_1]]_{\mathcal{K}}, \cdots, [[\alpha_n]]_{\mathcal{K}}) \circ \sigma_2 \circ f(s) \\ & = [[\heartsuit^\lambda(\alpha_1, \cdots, \alpha_n)]]_{\mathcal{K}}(f(s)) \end{aligned}$$

Therefore, $gr(s \models \heartsuit^\lambda(\alpha_1, \alpha_2, \cdots, \alpha_n)) = gr(f(s) \models \heartsuit^\lambda(\alpha_1, \alpha_2, \cdots, \alpha_n))$. □

We now arrive at the following outcome by utilizing Proposition 8.2.1.

Proposition 8.2.2. *Behaviourally equivalent states are modally equivalent.*

Proof. Let $\mathcal{B} = (B, \sigma_1, \mathcal{V}_B)$ and $\mathcal{B}' = (B', \sigma_2, \mathcal{V}_{B'})$ be fuzzy geometric models for T. Consider $b \in B$ and $b' \in B'$ are two states. Suppose, the states b and b' are behaviourally equivalent. We shall show that they are modally equivalent. Since b and b' are behaviourally equivalent in $FMOD(T)$, so there exists an object $\mathcal{C} = (C, \gamma, \mathcal{V}_C)$ in $FMOD(T)$ and morphisms $g : \mathcal{B} \to \mathcal{C}$ and $h : \mathcal{B}' \to \mathcal{C}$ in $FMOD(T)$ such that $g(b) = h(b')$. Now, by Proposition 8.2.1, we have $gr(b \models \alpha) = gr(g(b) \models \alpha)$ and $gr(b' \models \alpha) = gr(h(b') \models \alpha)$, $\forall \alpha \in \mathbf{FGML}(\Sigma)$. As $g(b) = h(b')$, hence $gr(b \models \alpha) = gr(b' \models \alpha)$, $\forall \alpha \in \mathbf{FGML}(\Sigma)$. Therefore, the states b and b' are modally equivalent. □

8.3 Final model

In this section, we assume that T is an endofunctor on **SFuzzy-Top**, the category of sober fuzzy topological spaces, and consider a characteristic fuzzy geometric modal signature Σ for the endofunctor T. We shall create a final model in $FMOD(T)$ for the endofunctor T. Let $\mathcal{B} = (B, \gamma, \mathcal{V}_B)$ be a fuzzy geometric model for T.

Definition 8.3.1. *Any two formulas α and β are equivalent in FMOD(T) iff $gr(b \models \alpha) = gr(b \models \beta)$, $\forall b \in B$. Let $\alpha \equiv \beta$ denote the formulas α and β are equivalent.*

Let $[\alpha]$ denote the equivalence class of a formula $\alpha \in \mathbf{FGML}(\Sigma)$. Let $\mathcal{E}$ be the collection of equivalence classes of formulas in $\mathbf{FGML}(\Sigma)$. We define $gr(b \models [\alpha]) = gr(b \models \alpha)$, for any $b \in B$.

We shall now show that $\mathcal{E}$ is a frame.

Proposition 8.3.1. *$\mathcal{E}$ is a frame.*

Proof. The order relation on $\mathcal{E}$ is defined as: $[\alpha] \leq [\beta] \iff gr(b \models \alpha) \leq gr(b \models \beta)$, $\forall b \in B$. As $gr(b \models \alpha) = gr(b \models \alpha)$, the order relation $\leq$ is reflexive. It is easy to see that if $[\alpha_1] \leq [\beta_1]$ and $[\beta_1] \leq [\beta_3]$ then $[\alpha_1] \leq [\beta_3]$. Thus the order relation $\leq$ is transitive. Now, if $[\alpha] \leq [\beta]$ and $[\beta] \leq [\alpha]$ then by the defined order relation we have $gr(b \models \alpha) = gr(b \models \beta)$, $\forall b \in B$. Hence, $\alpha \equiv \beta$. As a result, $[\alpha] = [\beta]$. So the order relation $\leq$ is antisymmetric. Therefore, $\mathcal{E}$ is a poset with this order relation. As $gr(b \models \alpha \wedge \beta) = [[\alpha \wedge \beta]]_{\mathcal{B}}(b) = [[\alpha]]_{\mathcal{B}}(b) \wedge [[\beta]]_{\mathcal{B}}(b) = gr(b \models \alpha) \wedge gr(b \models \beta)$, hence $[\alpha \wedge \beta] \in \mathcal{E}$. As a result, $[\alpha] \wedge [\beta] \in \mathcal{E}$. Similarly, arbitrary join exists in

$\mathcal{E}$. We observe that, $[\alpha] \wedge \bigvee_{j\in J}[\beta_j] = [\alpha] \wedge [\bigvee_{j\in J} \beta_j] = [\alpha \wedge \bigvee_{j\in J} \beta_j]$. Now, we have $gr(b \models \alpha \wedge \bigvee_{j\in J} \beta_j) = gr(b \models \alpha) \wedge gr(b \models \bigvee_{j\in J} \beta_j) = [[\alpha]]_{\mathcal{B}}(b) \wedge [[\bigvee_{j\in J} \beta_j]]_{\mathcal{B}}(b) = [[\alpha]]_{\mathcal{B}}(b) \wedge Sup_{j\in J}\{[[\beta_j]]_{\mathcal{B}}(b)\} = Sup_{j\in J}\{[[\alpha]]_{\mathcal{B}}(b) \wedge [[\beta_j]]_{\mathcal{B}}(b)\} = [[\bigvee_{j\in J}(\alpha \wedge \beta_j)]](b) = gr(b \models \bigvee_{j\in J}(\alpha \wedge \beta_j))$, $\forall b \in B$. Consequently, $[\alpha \wedge \bigvee_{j\in J} \beta_j] = [\bigvee_{j\in J}(\alpha \wedge \beta_j)]$. Henceforth, $[\alpha] \wedge \bigvee_{j\in J}[\beta_j] = [\bigvee_{j\in J}(\alpha \wedge \beta_j)] = \bigvee_{j\in J}[\alpha \wedge \beta_j] = \bigvee_{j\in J}([\alpha] \wedge [\beta_j])$. Therefore, $\mathcal{E}$ is a frame. $\square$

Definition 8.3.2. *Let $\mathcal{F} = PT(\mathcal{E})$. A map $\tilde{f} : B \to \mathcal{F}$ is defined by $\tilde{f}(b) = h_b$, where h_b is a frame homomorphism from $\mathcal{E}$ to $[0,1]$ defined by $h_b([\alpha]) = gr(b \models \alpha)$.*

Note 8.3.1. *The mapping $\tilde{f} : B \to \mathcal{F}$ is fuzzy continuous. Let $[\alpha] \in \mathcal{E}$. Then we show that $\tilde{f}^{-1}(\Psi([\alpha])) = [[\alpha]]_{\mathcal{B}}$ by the following:*

$$\tilde{f}^{-1}(\Psi([\alpha]))(b) = \Psi([\alpha])\tilde{f}(b) = \tilde{f}(b)([\alpha]) = gr(b \models \alpha) = [[\alpha]]_{\mathcal{B}}(b).$$

*Hence, $\forall \alpha \in$ **FGML**(Σ), $\tilde{f}^{-1}(\Psi([\alpha]))$ is a fuzzy open set on B. Therefore, $\tilde{f}$ is a fuzzy continuous map.*

Let $\mathcal{G} = \mathcal{P} \circ T \circ PT$. Then $\mathcal{G} :$ **FRM** $\to$ **FRM** is a functor. Since the category **S-FRM** of spatial frames is equivalent to the opposite category of **SFuzzy-Top**, the endofunctor defined on the category **S-FRM** is a restriction of $\mathcal{G}$. As Σ is characteristic, so the collection $\{\lambda_B(\widehat{[\alpha_1]}, \cdots, \widehat{[\alpha_n]}) : \lambda \in \Sigma, \alpha_i \in \mathbf{FGML}(\Sigma), \widehat{[\alpha_i]} \in \mathcal{Q}(PT\mathcal{E}), i = 1, \cdots, n\}$ generates the frame $\mathcal{G}(\mathcal{E})$. So, an assignment can be defined on the generators of $\mathcal{G}(\mathcal{E})$, and by Remark 1.0.1, it can be extended to a frame homomorphism from $\mathcal{G}(\mathcal{E})$ to $\mathcal{E}$.

Definition 8.3.3. *Define a morphism $\xi : \mathcal{G}(\mathcal{E}) \to \mathcal{E}$ in $\mathcal{FRM}$ by $\xi(\lambda_{\mathcal{F}}(\widehat{[\alpha_1]}, \widehat{[\alpha_2]}, \cdots, \widehat{[\alpha_n]})) = [\heartsuit^{\lambda}(\alpha_1, \alpha_2, \cdots, \alpha_n)]$.*

The well-definedness of the morphism ξ is shown by Lemma 8.3.1.

Lemma 8.3.1. *Suppose $\bigvee_{i\in\Lambda}(\bigwedge_{j\in K_i} \lambda_{\mathcal{F}}^{i,j}(\widehat{[\alpha_1]}^{i,j}, \widehat{[\alpha_2]}^{i,j}, \cdots, \widehat{[\alpha_{n_{i,j}}]}^{i,j})) = \bigvee_{r\in\mathcal{I}}(\bigwedge_{s\in\mathcal{J}_r} \lambda_{\mathcal{F}}^{r,s}(\widehat{[\alpha'_1]}^{r,s}, \widehat{[\alpha'_2]}^{r,s}, \cdots, \widehat{[\alpha'_{n_{r,s}}]}^{r,s}))$, where Λ and $\mathcal{I}$ are the arbitrary index sets, and K_i, $\mathcal{J}_r$ are the finite index sets. Then, formulas $\bigvee_{i\in\Lambda}(\bigwedge_{j\in K_i} \heartsuit^{\lambda^{i,j}}(\alpha_1^{i,j}, \alpha_2^{i,j}, \cdots, \alpha_{n_{i,j}}^{i,j}))$ and $\bigvee_{r\in\mathcal{I}}(\bigwedge_{s\in\mathcal{J}_r} \heartsuit^{\lambda^{r,s}}(\alpha_1'^{r,s}, \alpha_2'^{r,s}, \cdots, \alpha_{n_{r,s}}'^{r,s}))$ are equivalent in $FMOD(T)$.*

Proof. We shall show that, for an object $\mathcal{B} = (B, \gamma, \mathcal{V}_B)$ in $FMOD(T)$, $gr(b \models \bigvee_{i\in\Lambda}(\bigwedge_{j\in K_i} \heartsuit^{\lambda^{i,j}}(\alpha_1^{i,j}, \alpha_2^{i,j}, \cdots, \alpha_{n_{i,j}}^{i,j}))) = gr(b \models \bigvee_{r\in\mathcal{I}}(\bigwedge_{s\in\mathcal{J}_r} \heartsuit^{\lambda^{r,s}}(\alpha_1^{\prime r,s}, \alpha_2^{\prime r,s}, \cdots, \alpha_{n_{r,s}}^{\prime r,s})))$, $\forall b \in B$.

Now we observe that,

$$\bigvee_{i\in\Lambda}(\bigwedge_{j\in K_i} \lambda_B^{i,j}([[\alpha_1^{i,j}]]_{\mathcal{B}}, [[\alpha_2^{i,j}]]_{\mathcal{B}}, \cdots, [[\alpha_{n_{i,j}}^{i,j}]]_{\mathcal{B}}))$$
$$= \bigvee_{i\in\Lambda}(\bigwedge_{j\in K_i} \lambda_B^{i,j}(\tilde{f}^{-1}(\Psi([\alpha_1^{i,j}])), \tilde{f}^{-1}(\Psi([\alpha_2^{i,j}])), \cdots, \tilde{f}^{-1}(\Psi([\alpha_{n_{i,j}}^{i,j}])))) \text{ [By Note 8.3.1]}$$
$$= \bigvee_{i\in\Lambda}(\bigwedge_{j\in K_i} (T\tilde{f})^{-1}(\lambda_{\mathcal{F}}^{i,j}(\Psi([\alpha_1^{i,j}]), \Psi([\alpha_2^{i,j}]), \cdots, \Psi([\alpha_{n_{i,j}}^{i,j}])))) \text{ [By naturality of } \lambda \text{]}$$
$$= (T\tilde{f})^{-1}(\bigvee_{i\in\Lambda}(\bigwedge_{j\in K_i} \lambda_{\mathcal{F}}^{i,j}(\Psi([\alpha_1^{i,j}]), \Psi([\alpha_2^{i,j}]), \cdots, \Psi([\alpha_{n_{i,j}}^{i,j}])))) \text{ [By Definition 8.1.5]}$$
$$= (T\tilde{f})^{-1}(\bigvee_{r\in\mathcal{I}}(\bigwedge_{s\in\mathcal{J}_r} \lambda_{\mathcal{F}}^{r,s}(\Psi([\alpha_1^{\prime r,s}]), \Psi([\alpha_2^{\prime r,s}]), \cdots, \Psi([\alpha_{n_{r,s}}^{\prime r,s}])))) \text{ [By the given hypothesis]}$$
$$= \bigvee_{r\in\mathcal{I}}(\bigwedge_{s\in\mathcal{J}_r} (T\tilde{f})^{-1}(\lambda_{\mathcal{F}}^{r,s}(\Psi([\alpha_1^{\prime r,s}]), \Psi([\alpha_2^{\prime r,s}]), \cdots, \Psi([\alpha_{n_{r,s}}^{\prime r,s}])))) \text{ [By Definition 8.1.5]}$$
$$= \bigvee_{r\in\mathcal{I}}(\bigwedge_{s\in\mathcal{J}_r} \lambda_B^{r,s}(\tilde{f}^{-1}(\Psi([\alpha_1^{\prime r,s}])), \tilde{f}^{-1}(\Psi([\alpha_2^{\prime r,s}])), \cdots, \tilde{f}^{-1}(\Psi([\alpha_{n_{r,s}}^{\prime r,s}])))) \text{ [By naturality of } \lambda \text{]}$$
$$= \bigvee_{r\in\mathcal{I}}(\bigwedge_{s\in\mathcal{J}_r} \lambda_B^{r,s}([[\alpha_1^{\prime r,s}]]_{\mathcal{B}}, [[\alpha_2^{\prime r,s}]]_{\mathcal{B}}, \cdots, [[\alpha_{n_{r,s}}^{\prime r,s}]]_{\mathcal{B}}))$$

Therefore, for a fuzzy geometric model $\mathcal{B}$, we have $gr(b \models \bigvee_{i\in\Lambda}(\bigwedge_{j\in K_i} \heartsuit^{\lambda^{i,j}}(\alpha_1^{i,j}, \alpha_2^{i,j}, \cdots, \alpha_{n_{i,j}}^{i,j}))) = gr(b \models \bigvee_{r\in\mathcal{I}}(\bigwedge_{s\in\mathcal{J}_r} \heartsuit^{\lambda^{r,s}}(\alpha_1^{\prime r,s}, \alpha_2^{\prime r,s}, \cdots, \alpha_{n_{r,s}}^{\prime r,s})))$, $\forall b \in B$. □

So, $(\mathcal{E}, \xi)$ is a $\mathcal{G}$-algebra. Now we construct a T-coalgebra structure on $\mathcal{F} = PT\mathcal{E}$.

Definition 8.3.4. *Consider a morphism* $\phi = \eta_{T\mathcal{F}}^{-1} \circ PT(\xi) : \mathcal{F} \to T\mathcal{F}$, *where the morphism* $PT(\xi) : PT(\mathcal{E}) \to PT(\mathcal{G}(\mathcal{E}))$ *is defined by* $PT(\xi)(h)(\lambda_{\mathcal{F}}(\widehat{[\alpha_1]}, \widehat{[\alpha_2]}, \cdots, \widehat{[\alpha_n]})) = gr([\heartsuit^{\lambda}(\alpha_1, \alpha_2, \cdots, \alpha_n)] \in h)$, *where* $h \in PT(\mathcal{E}) = PT\mathcal{E}$, *and the morphism* $\eta_{T\mathcal{F}} : T\mathcal{F} \to PT(\mathcal{G}(\mathcal{E}))$ *is defined by* $\eta_{T\mathcal{F}}(h^*)(\lambda_{\mathcal{F}}(\widehat{[\alpha_1]}, \widehat{[\alpha_2]}, \cdots, \widehat{[\alpha_n]})) = \lambda_{\mathcal{F}}(\widehat{[\alpha_1]}, \widehat{[\alpha_2]}, \cdots, \widehat{[\alpha_n]})(h^*)$, *where* $h^* \in T\mathcal{F}$.

Note 8.3.2. *Since* $T\mathcal{F}$ *is a sober fuzzy topological space, so by Theorem 8.1.1,* $\eta_{T\mathcal{F}}$ *is an isomorphism. Consequently, the morphism* ϕ *is well-defined.*

Definition 8.3.5. *The triple* $(\mathcal{F}, \phi, \mathcal{V}_{\mathcal{F}})$ *is an object in* $FMOD(T)$*, where* $(\mathcal{F}, \phi)$ *is a* T*-coalgebra and the valuation function* $\mathcal{V}_{\mathcal{F}} : \Phi \to \mathcal{Q}(\mathcal{F})$ *is defined by* $\mathcal{V}_{\mathcal{F}}(p)(\tilde{g}) = gr(p \in \tilde{g}) = \tilde{g}(p)$*, where* $p \in \Phi$ *and* $\tilde{g} \in \mathcal{Q}(\mathcal{F})$*.*

Proposition 8.3.2. *The mapping* $\tilde{f} : \mathcal{B} \to (\mathcal{F}, \phi, \mathcal{V}_{\mathcal{F}})$ *is a morphism in* $FMOD(T)$*.*

Proof. We are to show that $\tilde{f}$ is a coalgebra morphism from $\mathcal{B}$ to $\mathcal{F}$, and $\tilde{f}^{-1} \circ \mathcal{V}_{\mathcal{F}} = \mathcal{V}_B$. It is observed that for every propositional variable p,

$$\begin{aligned}
\tilde{f}^{-1} \circ \mathcal{V}_{\mathcal{F}}(p)(b) &= \tilde{f}^{-1}(\mathcal{V}_{\mathcal{F}}(p))(b) \\
&= \mathcal{V}_{\mathcal{F}}(p)(\tilde{f}(b)) \\
&= gr(p \in \tilde{f}(b)) \\
&= \tilde{f}(b)(p) \\
&= gr(b \models p) \text{ [By Definition 8.3.2]} \\
&= \mathcal{V}_B(p)(b).
\end{aligned}$$

Henceforth, $\tilde{f}^{-1} \circ \mathcal{V}_{\mathcal{F}} = \mathcal{V}_B$. To prove $\tilde{f}$ is a T-coalgebra morphism, we show that $T\tilde{f} \circ \gamma = \phi \circ \tilde{f}$ i.e., the diagram shown in Fig.8.4 commutes.

$$\begin{array}{ccc}
B & \xrightarrow{\tilde{f}} & \mathcal{F} \\
{\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\phi} \\
TB & \xrightarrow[T\tilde{f}]{} & T\mathcal{F}
\end{array}$$

Figure 8.4: Illustration of T-coalgebra morphism

Now we observe that,

$$\begin{aligned}
& gr(T\tilde{f} \circ \gamma(b) \in \lambda_{\mathcal{F}}(\Psi([\alpha_1]), \Psi([\alpha_2]), \cdots, \Psi([\alpha_n]))) \\
&= gr(\gamma(b) \in (T\tilde{f})^{-1} \circ \lambda_{\mathcal{F}}(\Psi([\alpha_1]), \Psi([\alpha_2]), \cdots, \Psi([\alpha_n]))) \\
&= gr(\gamma(b) \in \lambda_B([[\alpha_1]]_{\mathcal{B}}, [[\alpha_2]]_{\mathcal{B}}, \cdots, [[\alpha_n]]_{\mathcal{B}})) \text{ [Since } \lambda \text{ is the natural transformation]} \\
&= \lambda_B([[\alpha_1]]_{\mathcal{B}}, [[\alpha_2]]_{\mathcal{B}}, \cdots, [[\alpha_n]]_{\mathcal{B}}) \circ \gamma(b) \\
&= [[\heartsuit^{\lambda}(\alpha_1, \alpha_2, \cdots, \alpha_n)]]_{\mathcal{B}}(b) \\
&= gr(b \models \heartsuit^{\lambda}(\alpha_1, \alpha_2, \cdots, \alpha_n)) \\
&= \tilde{f}(b)([\heartsuit^{\lambda}([\alpha_1], [\alpha_2], \cdots, [\alpha_n])]) \text{ [By Definition 8.3.2]} \\
&= gr(\phi \circ \tilde{f}(b) \in \lambda_{\mathcal{F}}(\Psi([\alpha_1]), \Psi([\alpha_2]), \cdots, \Psi([\alpha_n])) \text{ [By Definition 8.3.4]}
\end{aligned}$$

As $T\mathcal{F}$ is a sober fuzzy topological space, hence it is a T_0-space (Kolmogorov space). Therefore, we have $T\tilde{f} \circ \gamma = \phi \circ \tilde{f}$. □

Theorem 8.3.1. *The fuzzy geometric model* $\mathfrak{F} = (\mathcal{F}, \phi, \mathcal{V}_{\mathcal{F}})$ *in* $FMOD(T)$ *is a final model in* $FMOD(T)$.

Proof. We prove it by showing that for an object $\mathcal{B} = (B, \gamma, \mathcal{V}_B)$ in $FMOD(T)$, a unique T-coalgebra morphism exists from $\mathcal{B}$ to $\mathfrak{F}$. Following the Proposition 8.3.2, a T-coalgebra morphism $\tilde{f} : \mathcal{B} \to \mathfrak{F}$ exists. The only part that remains to be proven here is that $\tilde{f}$ is unique. Consider a morphism $f^* : \mathcal{B} \to \mathfrak{F}$ in $FMOD(T)$. By Proposition 8.2.1, we have $gr(b \models \alpha) = gr(f^*(b) \models [\alpha])$. Now $gr(\tilde{f}(b) \models [\alpha]) = \tilde{f}(b)[\alpha] = gr(b \models \alpha) = gr(f^*(b) \models [\alpha])$. Consequently, $\tilde{f}(b) = f^*(b)$. Therefore, $\mathfrak{F}$ is final in $FMOD(T)$. □

By Theorem 8.3.1, we derive the following result.

Theorem 8.3.2. *Modal equivalence implies behavioural equivalence.*

Proof. Let $\mathcal{B} = (B, \gamma, \mathcal{V}_B)$ and $\mathcal{B}_1 = (B_1, \gamma_1, \mathcal{V}_{B_1})$ be fuzzy geometric models for T. Let $b \in B$ and $b_1 \in B_1$ be states. If b and b_1 are modally equivalent then we have $gr(b \models \alpha) = gr(b_1 \models \alpha)$, for all formulas α. By Proposition 8.3.2, there exist morphisms $\tilde{f} : \mathcal{B} \to \mathfrak{F}$ and $\tilde{f}_1 : \mathcal{B}_1 \to \mathfrak{F}$ in $FMOD(T)$. Using Proposition 8.2.1, we have $gr(\tilde{f}(b) \models [\alpha]) = gr(b \models \alpha) = gr(b_1 \models \alpha) = gr(\tilde{f}_1(b_1) \models [\alpha])$, for all formulas α. Therefore, $\tilde{f}(b) = \tilde{f}_1(b_1)$. Hence, b and b_1 are behaviourally equivalent. □

Remark 8.3.1. *The converse of the statement mentioned in Theorem 8.3.2 is true by Proposition 8.2.2. Thus, modal equivalence and behavioural equivalence coincide when the endofunctor T is specified on **SFuzzy-Top**.*

8.4 Bisimulations

The aim of this section is to develop bisimulations for fuzzy geometric models for an endofunctor T, where T is defined on **Fuzzy-Top**.

Definition 8.4.1 ([102]). *Consider that F and F' are any two sets, and R is a relation between F and F'. Then, for a subset E of F, $R[E] = \{d' \in F' : \exists\ e \in E, eRd'\}$ and for a subset E' of F' $R^{-1}[E'] = \{d \in F : \exists\ e' \in E', dRe'\}$.*

Let μ be a fuzzy set on F. Then a fuzzy set $R[\mu]$ on F' can be defined by $R[\mu](d') = \bigvee_{d \in F} \{\mu(d) : dRd'\}$. For a fuzzy set η on F', we define an inverse image of η under the relation R by $R^{-1}[\eta](d) = \bigvee_{d' \in F'} \{\eta(d') : dRd'\}$. It is clear that $R^{-1}[\eta]$ is a fuzzy set on F.

We define the Aczel-Mendler bisimulation between fuzzy geometric models for T.

Definition 8.4.2. *Let $\mathcal{B}_1 = (B_1, \gamma_1, \mathcal{V}_{B_1})$ and $\mathcal{B}_2 = (B_2, \gamma_2, \mathcal{V}_{B_2})$ be two fuzzy geometric models for T. Then, a relation $\mathcal{R} \subseteq B_1 \times B_2$ is said to be an Aczel-Mendler bisimulation between $\mathcal{B}_1$ and $\mathcal{B}_2$ if for each $(b_1, b_2) \in \mathcal{R}$ and $p \in \Phi$, $\mathcal{V}_{B_1}(p)(b_1) = \mathcal{V}_{B_2}(p)(b_2)$, i.e., $gr(b_1 \models p) = gr(b_2 \models p)$ and there exists a coalgebra morphism $\gamma^* : \mathcal{R} \to T\mathcal{R}$ for which the projection maps $\pi_1 : \mathcal{R} \to B_1$ and $\pi_2 : \mathcal{R} \to B_2$ are coalgebra morphisms and satisfy the relations $\gamma_1 \circ \pi_1 = T(\pi_1) \circ \gamma^*$, $\gamma_2 \circ \pi_2 = T(\pi_2) \circ \gamma^*$, i.e. the diagram shown in Fig. 8.5 commutes:*

$$\begin{array}{ccccc} B_1 & \xleftarrow{\pi_1} & \mathcal{R} & \xrightarrow{\pi_2} & B_2 \\ \gamma_1 \downarrow & & \downarrow \gamma^* & & \downarrow \gamma_2 \\ TB_1 & \xleftarrow[T\pi_1]{} & T\mathcal{R} & \xrightarrow[T\pi_2]{} & TB_2 \end{array}$$

Figure 8.5: Illustration of Aczel-Mendler bisimulation between fuzzy geometric models

We now introduce a notion of Σ-bisimulation between fuzzy geometric models for T, adapting the "Λ-bisimulation" concepts discussed in [75, 76].
First, we introduce the notion of coherent pairs.

Definition 8.4.3. *Assume that $\mathcal{R}$ is a relation between B and B'. Let $\pi_1 : \mathcal{R} \to B$ and $\pi_2 : \mathcal{R} \to B'$ be projection maps. Then, a pair $(\tilde{r_1}, \tilde{r_2})$, where $\tilde{r_1}$ and $\tilde{r_2}$ are respectively the fuzzy sets on B and B', is called $\mathcal{R}$-coherent if $\mathcal{R}[\tilde{r_1}] \leq \tilde{r_2}$ and $\mathcal{R}^{-1}[\tilde{r_2}] \leq \tilde{r_1}$.*

Definition 8.4.4. *Let $\mathcal{B}_1 = (B_1, \gamma_1, \mathcal{V}_{B_1})$ and $\mathcal{B}_2 = (B_2, \gamma_2, \mathcal{V}_{B_2})$ be two fuzzy geometric models for T. A relation $\mathcal{R} \subseteq B_1 \times B_2$ is said to be a Σ-bisimulation between $\mathcal{B}_1$ and $\mathcal{B}_2$ if for all $(b_1, b_2) \in \mathcal{R}$, $p \in \Phi$ and each pair of fuzzy open sets $(\mu_i, \eta_i) \in \mathcal{Q}(B_1) \times \mathcal{Q}(B_2)$ such that $\mathcal{R}[\mu_i] \leq \eta_i$ and $\mathcal{R}^{-1}[\eta_i] \leq \mu_i$, we have :*

(i) $gr(b_1 \models p) = gr(b_2 \models p)$, *and*

(ii) $gr(\gamma_1(b_1) \in \lambda_{B_1}(\mu_1, \mu_2, \cdots, \mu_n)) = gr(\gamma_2(b_2) \in \lambda_{B_2}(\eta_1, \eta_2, \cdots, \eta_n))$.

Two states $b_1 \in B_1$ and $b_2 \in B_2$ are said to be Σ-bisimilar if there exists a Σ-bisimulation $\mathcal{R}$ such that $(b_1, b_2) \in \mathcal{R}$.
We now require the following observation:

Lemma 8.4.1. *Consider $\mathcal{R} \subseteq B_1 \times B_2$ is a relation that is equipped with the fuzzy subspace topology. Let $\pi_1 : \mathcal{R} \to B_1$ and $\pi_2 : \mathcal{R} \to B_2$ be projection maps. Then, a pair of fuzzy open sets $(\mu, \eta) \in \mathcal{Q}(B_1) \times \mathcal{Q}(B_2)$ is $\mathcal{R}$-coherent $\iff \pi_1^{-1}(\mu) = \pi_2^{-1}(\eta)$.*

Proof. Suppose the pair of fuzzy open sets (μ, η) is $\mathcal{R}$-coherent. We shall show that $\pi_1^{-1}(\mu) = \pi_2^{-1}(\eta)$. First, we show that $\pi_1^{-1}(\mu)$ is a fuzzy subset of $\pi_2^{-1}(\eta)$, i.e. $\pi_1^{-1}(\mu) \leq \pi_2^{-1}(\eta)$. We notice that $\pi_2(\pi_1^{-1}(\mu))$ and $\mathcal{R}[\mu]$ are both fuzzy sets on B_2. It is simple to demonstrate that $\pi_2(\pi_1^{-1}(\mu)) = \mathcal{R}[\mu]$.
Now,

$$\begin{aligned}\pi^{-1}(\mu) &\leq \pi_2^{-1}(\pi_2(\pi_1^{-1}(\mu))) \\ &= \pi_2^{-1}(\mathcal{R}[\mu]) \text{ [As } \pi_2(\pi_1^{-1}(\mu)) = \mathcal{R}[\mu] \text{]} \\ &\leq \pi_2^{-1}(\eta) \text{ [As } \mathcal{R}[\mu] \leq \eta \text{]}\end{aligned}$$

Similarly, we can show that $\pi_2^{-1}(\eta) \leq \pi_1^{-1}(\mu)$. It is straightforward to verify that if $\pi_1^{-1}(\mu) = \pi_2^{-1}(\eta)$ then the pair (μ, η) is $\mathcal{R}$-coherent. □

Now, we shall show that Σ-bisimilar states are modally equivalent.

Corollary 8.4.1. *Assume that T is an endofunctor on* ***Fuzzy-Top****. Then Σ-bisimilarity implies modal equivalence.*

Proof. Let $\mathcal{R}$ be a Σ-bisimulation between fuzzy geometric models $\mathcal{B}_1 = (B_1, \gamma_1, \mathcal{V}_{B_1})$ and $\mathcal{B}_2 = (B_2, \gamma_2, \mathcal{V}_{B_2})$. Let $b_1 \in B_1$ and $b_2 \in B_2$ be two states. Suppose $b_1 \mathcal{R} b_2$. We shall show that $gr(b_1 \models \alpha) = gr(b_2 \models \alpha)$, $\forall \alpha \in \mathbf{FGML}(\Sigma)$. If p is a propositional variable, then it follows from the definition of Σ-bisimulation that $gr(b_1 \models p) = gr(b_2 \models p)$. It can be easily shown that $gr(b_1 \models \alpha_1 \wedge \alpha_2) = gr(b_2 \models \alpha_1 \wedge \alpha_2)$ and $gr(b_1 \models \bigvee_{j \in J} \alpha_j) = gr(b_2 \models \bigvee_{j \in J} \alpha_j)$, J is an index set. Now, $gr(b_1 \models \heartsuit^{\lambda}(\alpha_1, \alpha_2, \cdots, \alpha_n)) = gr(\gamma_1(b_1) \in \lambda_{B_1}([[\alpha_1]]_{\mathcal{B}_1}, [[\alpha_2]]_{\mathcal{B}_1}, \cdots, [[\alpha_n]]_{\mathcal{B}_1}))$. By induction principle, we can show that, for each $i = 1, 2, \cdots, n$, $\mathcal{R}[[[\alpha_i]]_{\mathcal{B}_1}] \leq [[\alpha_i]]_{\mathcal{B}_2}$ and $\mathcal{R}^{-1}[[[\alpha_i]]_{\mathcal{B}_2}] \leq [[\alpha_i]]_{\mathcal{B}_1}$. As $\mathcal{R}$ is a Σ-bisimulation, we have $gr(\gamma_1(b_1) \in$

$\lambda_{B_1}([[\alpha_1]]_{\mathcal{B}_1}, [[\alpha_2]]_{\mathcal{B}_1}, \cdots, [[\alpha_n]]_{\mathcal{B}_1})) = gr(\gamma_2(b_2) \in \lambda_{B_2}([[\alpha_1]]_{\mathcal{B}_2}, [[\alpha_2]]_{\mathcal{B}_2}, \cdots, [[\alpha_n]]_{\mathcal{B}_2})$. Consequently, $gr(b_1 \models \heartsuit^{\lambda}(\alpha_1, \alpha_2, \cdots, \alpha_n)) = gr(b_2 \models \heartsuit^{\lambda}(\alpha_1, \alpha_2, \cdots, \alpha_n))$. Therefore, b_1 and b_2 are modally equivalent. □

Combining the results from Remark 8.3.1 and Corollary 8.4.1 yields the following result:

Corollary 8.4.2. *For an endofunctor T on **SFuzzy-Top**, Σ-bisimilarity implies behavioural equivalence.*

Corollary 8.4.3. *Let T be an endofunctor on **Fuzzy-Top**; let Σ be a monotone fuzzy geometric modal signature for T. Then Aczel-Mendler bisimulation is a Σ-bisimulation.*

Proof. Consider $\mathcal{B}_1 = (B_1, \gamma_1, \mathcal{V}_{B_1})$ and $\mathcal{B}_2 = (B_2, \gamma_2, \mathcal{V}_{B_2})$ are fuzzy geometric models for T. Let $\mathcal{R}$ be an Aczel-Mendler bisimulation between $\mathcal{B}_1$ and $\mathcal{B}_2$. Then the diagram shown in Fig.8.6 commutes.

$$\begin{array}{ccccc} B_1 & \xleftarrow{\pi_1} & \mathcal{R} & \xrightarrow{\pi_2} & B_2 \\ \gamma_1 \downarrow & & \downarrow \gamma^* & & \downarrow \gamma_2 \\ TB_1 & \xleftarrow[T\pi_1]{} & T\mathcal{R} & \xrightarrow[T\pi_2]{} & TB_2 \end{array}$$

Figure 8.6: Illustration of Aczel-Mendler bisimulation between fuzzy geometric models

We are to show that, $\mathcal{R}$ is a Σ-bisimulation. Consider $b_1 \mathcal{R} b_2$, where $b_1 \in B_1$ and $b_2 \in B_2$ are the states. Given that $\mathcal{R}$ is an Aczel-Mendler bisimulation, we have for any propositional variable $p \in \Phi$, $gr(b_1 \models p) = gr(b_2 \models p)$. Assume that for each pair of fuzzy open sets $(\xi_i, \zeta_i) \in \mathcal{Q}(B_1) \times \mathcal{Q}(B_2)$, $\mathcal{R}[\xi_i] \leq \zeta_i$ and $\mathcal{R}^{-1}[\zeta_i] \leq \xi_i$.

Now,

$gr(\gamma_1(b_1) \in \lambda_{B_1}(\xi_1, \xi_2, \cdots, \xi_n))$
$= \lambda_{B_1}(\xi_1, \xi_2, \cdots, \xi_n)(\gamma_1(b_1))$
$= \lambda_{B_1}(\xi_1, \xi_2, \cdots, \xi_n)(T\pi_1)(\gamma^*(b_1, b_2))$ [As $\gamma_1 \circ \pi_1 = T\pi_1 \circ \gamma^*$]
$= gr(\gamma^*(b_1, b_2) \in (T\pi_1)^{-1}(\lambda_{B_1}(\xi_1, \xi_2, \cdots, \xi_n)))$
$= gr(\gamma^*(b_1, b_2) \in \lambda_{\mathcal{R}}(\xi_1 \circ \pi_1, \xi_2 \circ \pi_1, \cdots, \xi_n \circ \pi_1))$ [As λ is a natural transformation]
$\leq gr(\gamma^*(b_1, b_2) \in \lambda_{\mathcal{R}}(\pi_2^{-1}(\pi_2(\pi_1^{-1}(\xi_1)))), \cdots, \pi_2^{-1}(\pi_2(\pi_1^{-1}(\xi_n)))$ [As λ is monotone]
$= gr(\gamma^*(b_1, b_2) \in \lambda_{\mathcal{R}}(\pi_2^{-1}(\mathcal{R}[\xi_1]), \pi_2^{-1}(\mathcal{R}[\xi_2]), \cdots, \pi_2^{-1}(\mathcal{R}[\xi_n])))$
$\leq gr(\gamma^*(b_1, b_2) \in \lambda_{\mathcal{R}}(\pi_2^{-1}(\zeta_1), \pi_2^{-1}(\zeta_2), \cdots, \pi_2^{-1}(\zeta_n)))$[As λ is monotone]
$= gr(\gamma^*(b_1, b_2) \in \lambda_{\mathcal{R}}(\zeta_1 \circ \pi_2, \cdots, \zeta_n \circ \pi_2))$
$= gr(\gamma^*(b_1, b_2) \in (T\pi_2)^{-1}(\lambda_{B_2}(\zeta_1, \zeta_2, \cdots, \zeta_n)))$ [As λ is natural]
$= gr(\gamma_2(b_2) \in \lambda_{B_2}(\zeta_1, \zeta_2, \cdots, \zeta_n))$

Therefore, $gr(\gamma_1(b_1) \in \lambda_{B_1}(\xi_1, \xi_2, \cdots, \xi_n)) \leq gr(\gamma_2(b_2) \in \lambda_{B_2}(\zeta_1, \zeta_2, \cdots, \zeta_n))$. Similarly, it can be shown that $gr(\gamma_2(b_2) \in \lambda_{B_2}(\zeta_1, \zeta_2, \cdots, \zeta_n)) \leq gr(\gamma_1(b_1) \in \lambda_{B_1}(\xi_1, \xi_2, \cdots, \xi_n))$. Finally, we have $gr(\gamma_1(b_1) \in \lambda_{B_1}(\xi_1, \xi_2, \cdots, \xi_n)) = gr(\gamma_2(b_2) \in \lambda_{B_2}(\zeta_1, \zeta_2, \cdots, \zeta_n))$. Hence, the result follows. □

We now have all of the necessary components to define a bisimilarity concept for fuzzy geometric models.

Theorem 8.4.1. *Let $\mathcal{B} = (B, \gamma, \mathcal{V}_B)$ be a fuzyy geometric model for T. Then $\mathcal{R}' = \bigcup\{\mathcal{R} : \mathcal{R}$ is a Σ-bisimulation from $\mathcal{B}$ to $\mathcal{B}$ $\}$ is a Σ-bisimulation from $\mathcal{B}$ to itself.*

Proof. Consider for each pair of fuzzy open sets (μ_i, μ_i') in B, $\mathcal{R}'[\mu_i] \leq \mu_i'$ and $\mathcal{R}'^{-1}[\mu_i'] \leq \mu_i$. Let $(b_1, b_2) \in \mathcal{R}'$. Then there exists $\mathcal{R} \in \mathcal{R}'$ such that $(b_1, b_2) \in \mathcal{R}$. Since $\mathcal{R}$ is Σ-bisimulation, we have $\mathcal{V}_B(p)(b_1) = \mathcal{V}_B(p)(b_2)$ i.e., $gr(b_1 \models p) = gr(b_2 \models p)$. We observe that $\mathcal{R}[\mu_i] \leq \mathcal{R}'[\mu_i] \leq \mu_i'$ and $\mathcal{R}^{-1}[\mu_i'] \leq \mathcal{R}'^{-1}[\mu_i'] \leq \mu_i$. Now, we get $gr(\gamma(b_1) \in \lambda_B(\mu_1, \mu_2, \cdots, \mu_n)) = gr(\gamma(b_2) \in \lambda_B(\mu_1', \mu_2', \cdots, \mu_n'))$. It follows that $\mathcal{R}'$ is Σ-bisimulation from $\mathcal{B}$ to itself. □

We emphasize that the above result is important for further progress of this work. The above finding paves the way for future theoretical advancements, notably the co-inductive proof principle [90].

8.5 Applications

The following is a hypothetical application of fuzzy geometric modal logic.

Example 8.5.1. *Assume that a serious virus has infected a particular location. Scientists believe that three medications m_1, m_2, and m_3 may be given in treating this condition. Furthermore, the scientists evaluate the patient's state after administering these medications and determine whether their health conditions: "no-change", "partially change", "change", which are denoted by e_1, e_2, e_3, respectively. So, the above scenario may lead a "State-based system" with truth degrees. For example, $\models (e_2, m_2) = \frac{1}{2}$ means that the patient's health condition is changed partially after applying the medicine m_2 with possibility $\frac{1}{2}$. Let $S =$ set of states $= \{e_1, e_2, e_3\}$. By defining an endofunctor T on a category S' of fuzzy topological spaces which contains S as a sub-category, we can construct a fuzzy geometric model $(S, \sigma, \mathcal{V})$, where $\mathcal{V}$ is a valuation mapping. When a patient's health changes, we naturally consider whether that shift is favourable or negative. Thus, an observer is required in this case. We may utilize our coalgebraic bisimulation theory to determine whether an observer exists between states.*

Another possible applications are listed below:

- We can also build fuzzy geometric models and develop bisimulation theorem to deal with erroneous and unpredictable featured values in multi-document summarising techniques [101].
- We can theoretically apply our coalgebraic logic to a fuzzy search query (e.g., [85, 93, 94, 92]) and use the coalgebraic bisimulation concept to assess whether the meaning of two keywords or strings is identical.

8.6 Conclusion

We have discussed coalgebraic logic for fuzzy-topological coalgebras. The structures referred to as the fuzzy geometric models for T, provide the semantics for our coalgebraic logics. We have shown that a final model exists in the category $FMOD(T)$ of fuzzy geometric models, where T is an endofunctor on **SFuzzy-Top**. Finally, we have studied bisimulations for fuzzy geometric models. In addition, we have demonstrated that for an endofunctor T on **SFuzzy-Top**, Σ-bisimilarity implies behavioural equivalence.

Chapter 9

Concluding Remarks and Future Research Directions

This chapter addresses several noteworthy queries and possible avenues for future research.

1. The dualities discussed in Chapter 2 and Chapter 3 are fundamentally distinct from those developed in [21]. The dualities delineated in [21] are grounded on the idea of natural dualities, whereas the dualities addressed in Chapters 2, 3 draw upon Vickers' concept [98].
   Several parallel studies (e.g., [31, 83, 26, 27]) were conducted during the advancement of the works reported in Chapters 2, 3. Some of these studies introduced more generalized concepts such as variety-based topology and topological systems rather than lattice-valued topology and topological systems (see [31]) to present a categorical connection between systems and spaces, whereas others shed light on fuzzy environments (e.g., see [83, 26, 27]) to accomplish the same purpose. In light of this, it is worth mentioning that similar research can be conducted by establishing a categorical link between the categories of variety-based topological spaces, variety-based topological systems, and algebras of many-valued-modal logics.
   It might be possible to develop modal geometric logic by adhering to our procedures covered in Chapter 3.

2. Concerning the findings presented in Chapter 4, we hope to have shown how the methods of universal algebra and bitopology provide an intriguing viewpoint on Fitting's many-valued logic.
   The findings in Chapter 4 and Chapter 5 can be extended in various ways.

(a) By establishing an appropriate Vietoris functor, Maruyama in [41] created a coalgebraic duality for the category $ISP_M(\mathcal{L})$. In this regard, we believe that modalizing the concept of $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$ will produce a tenable outcome. A coalgebraic duality may be developed for the modalized notion of $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$ by constructing an appropriate bi-Vietoris functor.

(b) The NU duality theorem was established in [42]. It seems that the intuitionistic version of natural duality theory allows for the generalization of the NU duality theorem to $\mathbb{ISP}_{\mathbb{I}}$. It could be possible to accomplish this using the methods of bitopology.

(c) It might be conceivable to connect categorically with $ISP_I(\mathcal{L})$ by creating an intuitionistic topological system (within the context of Vicker's work [98]). Therefore, there is a another method for developing a duality for $\mathbb{ISP}_{\mathbb{I}}(\mathcal{L})$.

3. In light of our work discussed in Chapters 6, 7, we can suggest some future research directions.

(a) As an application of this coalgebraic duality, we may establish the existence of a final coalgebra and cofree coalgebras in the category $COALG(V_{\mathcal{L}}^{bi})$, and we can also develop the coalgebraic duality theorem for many-valued modal logics in a bitopological scenario.

(b) Another interesting line of research would be to show that coalgebras of an endofunctor V on the category BES of bi-topological Esakia spaces (the idea of bitopological Esakia spaces can be found in [40]) can characterise lattice-valued intuitionistic modal logic. However, it is unclear to us how to characterise the relation $\mathcal{R}$ on bitopological Esakia spaces in terms of coalgebras of the functor V, and this appears to be an open problem at the moment.

4. In Chapter 8, we have started to lay the groundwork for coalgebraic fuzzy geometric logic. However, there are still many interesting, unresolved questions.

(a) We have not addressed the completeness of modal fuzzy geometric logic. However, it would be interesting to know under which conditions the completeness result will be attained.

(b) We have not examined the possibility that the notion of behavioural equivalence implies the Σ-bisimilarity idea. We may define a stronger fuzzy-open predicate lifting idea for endofunctors on **Fuzzy-Top** by emulating the topological predicate lifting notion for an endofunctor on the category of Stone spaces as stated in [69]. It may therefore be demonstrated that behavioural equivalence indicates Σ-bisimilarity.

(c) Considering an endofunctor T on the category of compact fuzzy Hausdorff spaces, we can show the bi-implication between modal equivalence and Σ-bisimilarity. In this case, it will be fruitful to adopt a stronger notion of fuzzy geometric modal signature Σ.

(d) We have already observed that if T is an endofunctor on **SFuzzy-Top** then behavioural equivalence coincides with modal equivalence. We will attempt to circumvent this limitation by determining whether behavioral equivalence and modal equivalence coincide when T is an endofunctor on the category of compact fuzzy Hausdorff spaces.

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