Classical and Quantum Cosmology From Various Gravity Theories: Noether Symmetry Approach

A thesis submitted in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy (Science) of Jadavpur University

by

Shriton Hembrom

Department of Mathematics

Jadavpur University

Kolkata

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Certificate From the Supervisors

This is certify that the thesis entitled **"Classical and Quantum Cosmology From Various Gravity Theories: Noether Symmetry Approach "** submitted by **Sri. Shriton Hembrom,** who got his name registered on January 31, 2023 (Index No.:1/23/Maths./28) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under our supervision and that neither this thesis nor any part of it has been submitted for either any degree/ diploma or any other academic award anywhere before.

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Declaration From the Author

I solemnly declare that this thesis is the product of my independent research, conducted at the Department of Mathematics, Jadavpur University, Kolkata - 700032, India. I further affirm that no part of this work has been submitted for the attainment of any degree, diploma, or academic qualification at any other institution.

The author has meticulously crafted all figures in this thesis using Maple and Mathematica software. The manuscript has undergone rigorous scrutiny to eliminate discrepancies and typographical errors. However, despite these diligent efforts, astute readers may identify inadvertent mistakes, and certain sections may appear unwarranted or imprecise. The author assumes full responsibility for any such errors arising from lapses in subject knowledge or inadvertent oversight.

Finally, I affirm that, to the best of my knowledge, all assistance utilized in the preparation of this thesis has been duly cited and acknowledged with the utmost integrity

Shriton Hembrom

This Thesis is dedicated to my Parents

Late Chhiman Hembrom , Fulmani Kisku

And my brother

Ajit Hembrom

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While a doctoral thesis is typically considered an individual effort, the extensive list that follows proves otherwise.

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Shriton Hembrom

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"Cultivation of mind should be the ultimate aim of human existence."

–*Dr. B R Ambedkar*

"Success is not final, failure is not fatal: it is the courage to continue that counts."

—*Winston Churchill*

Abstract

The thesis consists of eight chapters. Chapter one presents an overview of symmetry analysis as well as of Einstein gravity and other modified gravity theories related to the thesis. Chapters two- seven describe the research work presented in the thesis. Finally, a brief summary and future perspective has been presented in chapter eight.

In chapter two, a cosmological model having two scalar field (an extension of the Brans-Dicke scalar field model) has been studied for flat FLRW cosmological model. Due to Noether symmetry it is possible to have classical solutions which are analyzed graphically. In the context of quantum cosmology symmetry analysis helps to solve the Wheeler-DeWitt equation.

Modified fourth-order gravity theory $f(T, B)$ is considered in chapter three both classically and quantum mechanically using Noether symmetry analysis. The classical solutions have been examined from observational point of view while probability amplitude in quantum cosmology has been analyzed for classical analogy.

Chapter four analyzed in details Noether symmetry for the multi scalar-torsion gravity theory. The symmetry analysis identifies the conserved charge and energy of the model. Also the conformal symmetry of the physical metric has been studied to evaluate homothetic and Killing vector fields. In quantum description, both canonical approach as well as quantum trajectories has been investigated.

Chapter five deals with anisotropic Bianchi-I cosmological model. Here Noether symmetry analysis has been used for quantum description of the modified Hamiltonian-Jacobi equation considering quantum potential. Also conformal symmetry of the kinetic metric has been investigated to identify homothetic and Killing vector fields.

The modified teleparallel gravity theory has been studied in chapter six both for classical as well as quantum cosmological description. In quantum description the operator version of the conserved charge identifies the oscillatory part of the wave function and

as a consequence a complete solution is possible for the Wheeler-DeWitt equation. The corresponding probability amplitude has been examined whether classical singularity can be avoided or not by quantum description.

The non-minimally coupled scalar field cosmology in the background of flat FLRW space-time model has been analyzed in chapter seven using Noether symmetry. The symmetry analysis not only identifies the symmetry vector but also the associated conserved charge. The quantum version of this conserved charge identifies the oscillatory part of the wave function of the universe. Lastly, using Causal interpretation, quantum Bohmian trajectories has been evaluated and their classical limits has been examined.

List of Publications

The work of this thesis has been carried out at the Department of Mathematics, Jadavpur University, Kolkata-700032, India. This thesis is based on the following published papers.

- **Chapter 2** has been published as "Classical and quantum cosmology for two scalar field Brans–Dicke type theory: a Noether symmetry approach", **S. Hembrom**, R. Bhaumik, S. Dutta and S. Chakraborty, **Eur. Phys. Jour. C, 84**, 110 (2024), DOI:10.1140/epjc/s10052-024-12441-1
- **Chapter 3** has been published as "Noether symmetry analysis for fourth-order modified gravity theory: Cosmological solution and evolution ", **S. Hembrom**, R. Bhaumik and S. Chakraborty, **Int. Jour. of Mod. Phys. A, 39**, 13n14, id.2450058 (2024), DOI: 10.1142/S0217751X24500581
- **Chapter 4** has been published as "Symmetry analysis in multi scalar-torsion cosmological model with quantum description", **S. Hembrom**, R. Bhaumik, S. Dutta, S. Chakraborty, **Eur. Phys. Jour. C, 84**, 1054 (2024), DOI:10.1140/epjc/s10052-024-13356-7
- **Chapter 5** has been published as "Quantum Cosmology in Bianchi-I model: A study of symmetry analysis", D.Laya, **S. Hembrom**,S. Dutta, S. Chakraborty, **Journal of Subatomic Particles and Cosmology 1-2**, 100013 (2024), DOI:10.1016/j.jspc.2024.100013
- **Chapter 6** has been published as "A study of classical and quantum cosmology in modified teleparallel gravity and the role of Noether symmetry", D. Laya, **S. Hembrom**,S. Dutta, S. Chakarborty, **Eur.Phys. Jour.C,85**, 58 (2025), DOI:10.1140/epjc/s10052-025-13812-y
- **Chapter 7** has been published as "Symmetry of the Physical space and quantum cosmological description for non-minimally coupled scalar field cosmology", **S. Hembrom**, D. Laya, S. Dutta, S. Chakraborty, **Journal of Subatomic Particles and Cosmology 3**, 100048 (2025), DOI:10.1016/j.jspc.2025.100048

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Chapter 1

Introduction

1.1 Differential Equations in Symmetries

Symmetry can be described as a precise alignment in position or form relative to a specific point, line or plane. The term symmetry originates from the Latin word *"Symmetria."* Essentially, symmetry is an operation that preserves the invariance of the object. The concept of symmetry is closely linked to the work of mathematicians Felix Klein and Sophus Lie, who developed the essential mathematical frameworks to elucidate this notion.

Differential equations plays a pivotal role in addressing the various physical problems. However, we are often encounter couple non-linear partial differential equations that defy straight forward solutions. Symmetry analysis emerges as a powerful tool for resolving such challenges. In the late 19th century, the eminent mathematician Sophus Lie pioneered Lie symmetry analysis. His seminal work, *" Theory of Transformation Groups"*, published in 1888, paved the way for a revolutionary new direction in mathematics centered on the concept of symmetries. Lie theory is undeniably a vital mathematical instrument in modern science. Through this symmetry analysis, one identifies a canonical coordinate system for a given differential equation, transforming it into a new coordinate framework to simply its solution.

In the General Theory of Relativity (GTR), the gravitational fields equations are the non-linear second order partial differential equation which are quite difficult to finding the exact solutions for it. Symmetry analysis offers a method to simplify these equations by reducing the number of independent variables or equations, facilitating a more manageable approach to solving them. Lie symmetries are confined to the realm of independent variables, whereas continuous transformations may occur within the space of dependent variables. Consequently, field equations remain invariant.

Conversely, symmetry analysis has dominated the study of global continuous symmetries, internal symmetries of space-time, gauge symmetries, and permutation symmetries in Quantum Field Theory since the last century. In Noether symmetry analysis, conserved charges can pinpoint the true nature of similar physical processes. Here, Noether's theorem has been employed to simplify the couple non-linear differential equations. Furthermore, Noether symmetry has paved the way for new advancements in Quantum Cosmology.

This thesis primarily delves into Noether's theorem and its extensive applications. We can employ Noether symmetry analysis to derive classical cosmological solutions of different cosmological models, such as Brans-Dicke type two scalar field cosmological model, fourth order modified gravity $f(T,B)$ field model, Weyl Integrable Gravity (WIG) cosmological model, multi-scalar torsion field cosmological model and non-minimally coupled scalar field cosmological model. Additionally, we explore Quantum cosmology within the context of several dark energy models and also we discuss the Bohmian trajectories of a model. After that by solving the Wheeler-Dewitt equation, we are able to obtain the wave function of the Universe corresponding to specific cosmological models.

1.1.1 Symmetry Group

In abstract algebra, the symmetric group defined over any set is the group whose elements are all the bijections from the set to itself, and whose group operation is the composition of functions.

In classical mechanics, the symmetry group G_0 of a Hamiltonian system refers to the group of canonical transformation that remain invariant under the system's phase flow [1]. A notable instance of this is the rotation group, which examplifies the symmetries of systems governed by a central potential.

In transitioning from classical to quantum mechanics, the classical phase space is replaced by a quantum-mechanical Hilbert space (V), and the symmetry group (G_0) is represented through a (projective) representation of unitary $C-$ linear operators acting on (V). While the one-parameter continuous subgroups, underscored by Noether's theorem, are of paramount importance, the elements of G_0 not connected to the identity also play a crucial role. A prominent example is the space reflection operator, whose eigenspaces correspond to the subspaces of states with positive and negative parity. These eigenspaces decompose the matrix representation of any reflection-invariant Hamiltonian into two distinct blocks.

FIGURE 1.1: Represents the design of symmetry

1.1.2 One Parameter Point Transformation

In mathematics, a one- parameter group, or more precisely a one- parameter subgroup, typically refers to a continuous homomorphism of groups

$$\mathcal{F} : \mathbf{R} \longrightarrow \mathbf{G}$$

from the real line $\mathbf{R}$ (considered as an additive group) to another topological group $(\mathbf{G})$. If the homomorphism $\mathcal{F}$ is injective, then the image $\mathcal{F}(\mathbf{R})$ forms a subgroup of $\mathbf{G}$ that is isomorphic to $\mathbf{R}$ as an additive group.

One-parameter groups were first introduced by Sophus Lie in 1893 to formalize the concept of infinitesimal transformations. Lie defined an infinitesimal transformation as an infinitesimally small movement within the one-parameter group it generates [2]. These infinitesimal transformations are fundamental in constructing the corresponding Lie algebra, which serves as the framework for describing a Lie group of any dimension.

To streamline a differential equation, it is often advantageous to transform the dependent and independent variables into new ones. Let 'p' be the independent variable and 'q' the dependent variable in the given equation. We perform a point transformation from (p, q) to $(\tilde{p}, \tilde{q})$; where $\tilde{p} = \tilde{p}(p, q)$ and $\tilde{q} = \tilde{q}(p, q)$.

In the context of symmetry analysis, this point transformation must involve at least one arbitrary parameter, denoted as ϵ.

$$\begin{aligned} \tilde{p} &= \tilde{p}(p, q; \epsilon), \\ \tilde{q} &= \tilde{q}(p, q; \epsilon). \end{aligned} \tag{1.1}$$

Furthermore, this transformation must be invertible, and its repeated application should yield a transformation that remains within the same family.

$$\tilde{\tilde{p}} = \tilde{\tilde{p}}(\tilde{p}, \tilde{q}; \tilde{\epsilon}) = \tilde{\tilde{p}}(p, q; \tilde{\tilde{\epsilon}}), \tag{1.2}$$

where $\tilde{\tilde{\epsilon}} = \tilde{\tilde{\epsilon}}(\tilde{\epsilon}, \epsilon)$.

Based on the above properties, it can be concluded that transformation (1.1) constitutes a one-parameter group of transformations [3].

Example:

I: The rotation is given by

$$\tilde{p} = p\ cos\epsilon - q\ sin\epsilon,$$
$$\tilde{q} = p\ sin\epsilon + q\ cos\epsilon.$$

This is an example of one parameter group.

II: $\tilde{p} = p + l$ and $\tilde{q} = q + l$ is translation which is another example of one parameter group of transformation.

III: Rotation around a fixed axis: $(p, q, r) \to (p\ cos\epsilon - r\ sin\epsilon, q, psin\epsilon + r\ cos\epsilon)$.

The action of a one-parameter group on a set is referred to as a flow. A smooth vector field on a manifold generates a local flow at a given point, manifesting as a one-parameter group of local diffeomorphisms that maps points along the integral curves of the vector field. This local flow is fundamental in defining the Lie derivative of tensor fields in the direction of the vector field.

1.1.3 Invariance

An invariant is an observable of a physical system that remains unaltered under a given transformation [4]. The invariants of a system are intrinsically linked to the symmetries imposed by its environment, reflecting the fundamental nature of these symmetries.

A function $f : A \to R$ is deemed invariant if it remains unchanged under the action of a group transformation.

$$f(u, x) = f(x), \tag{1.3}$$

for all $u \in G$ and $x \in A$. Here G is symmetry group of the system.

The Lie symmetry of a differential equation examplifies a one-parameter point transformation. Through the Lie symmetry approach, the differential equation retains its invariance [5]. Lie symmetry proves particularly valuable in the study of nonlinear differential equations. By identifying invariant functions, Lie symmetry can solve systems of equations and derive analytic solutions, known as invariant solutions.

Example: Heat conduction equation $\frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial a^2} = 0$ is invariant under the following transformation:

$$
\begin{aligned}
\tilde{t} &= t + d, \;\; \tilde{a} = a, \;\; \tilde{g} = g. \\
\tilde{t} &= t, \;\; \tilde{a} = a + d, \;\; \tilde{g} = g. \\
\tilde{t} &= t, \;\; \tilde{a} = a, \;\; \tilde{b} = b + df(t, a);
\end{aligned}
$$

where $f(t, a)$ satisfies the condition $f_t - f_{aa} = 0$.

1.1.4 Infinitesimal Generator

An infinitesimal generator, commonly known as a generator, is a mathematical operator that produces a continuous transformation. It encapsulates an infinitesimal change or adjustment within a system. In the realm of group theory, an infinitesimal generator is associated with the tangent vector to a curve on the group manifold, which delineates the transformation. These generators are pivotal in elucidating symmetries and transformations across diverse mathematical and physical domains, including Lie groups and differential equations.

Let (p, q) be an arbitrary point. We can write

$$\tilde{p}(p,q,\epsilon) = a + \epsilon\xi(p,q) + ... = p + \epsilon Xp + ...,$$
$$\tilde{q}(p,q,\epsilon) = q + \epsilon\eta(p,q) + ... = q + \epsilon Xq + ..., \tag{1.4}$$

where

$$\xi(p,q) = \frac{\partial\tilde{p}}{\partial\epsilon}|_{\epsilon=0},$$
$$\eta(p,q) = \frac{\partial\tilde{q}}{\partial\epsilon}|_{\epsilon=0}. \tag{1.5}$$

and

$$X = \xi(p,q)\frac{\partial}{\partial p} + \eta(p,q)\frac{\partial}{\partial q}. \tag{1.6}$$

The operator X is recognized as the infinitesimal generator of the transformation, with ξ and η representing its two key components. The term "generator" signifies that by iterating the transformation, one can achieve a finite transformation; in other words, through integration, the complete transformation is obtained

$$\begin{aligned} \frac{\partial\tilde{p}}{\partial\epsilon} &= \xi(\tilde{p},\tilde{q}), \\ \textit{and} \quad \frac{\partial\tilde{q}}{\partial\epsilon} &= \eta(\tilde{p},\tilde{q}), \end{aligned} \tag{1.7}$$

with the initial values of p,q at $\epsilon = 0$; this yield the finite transformation.

Now we can rescale ϵ as $\epsilon = g(\hat{\epsilon})$, $g(0) = 0$ and $g'(0) \neq 0$.

Now from equation (1.5) we will get

$$\hat{\xi}\frac{\partial\tilde{p}}{\partial\hat{\epsilon}}|_{\epsilon=0} = \frac{\partial\tilde{p}}{\partial\epsilon}g'(\hat{\epsilon})|_{\epsilon=0} = g'(0)\xi.$$

Similarly, $\hat{\eta} = g'(0)\eta$.

1.1.5 Law of transformations

In general relativity, vectors can be represented in various coordinate systems, and it is often necessary to convert between these different representations. Unlike the linear Lorentz transformations, the transformations of coordinates in this context are not restricted to linear functions; instead, they can be any smooth, one-to-one mappings.

Equation (1.6) represents the infinitesimal generator. A pertinent question naturally arises: what will be the form of the infinitesimal generator when we substitute $u(p, q)$ and $v(p, q)$ in place of p and q. The generalized form of the infinitesimal generator, applicable to more than two variables, can be expressed as follows [4].

$$X = s^i(p^n)\frac{\partial}{\partial p^i}, \quad i = 1, 2, ...N \tag{1.8}$$

$$(\textit{summation over the dummy index } i).$$

Now the transformation is as

$$p^{i'} = p^{i'}(p^i), \qquad |\frac{\partial p^{i'}}{\partial p^i}| \neq 0 \tag{1.9}$$

From this transformation, we get,

$$\frac{\partial}{\partial p^i} = \frac{\partial p^{i'}}{\partial p^i}\frac{\partial}{\partial p^{i'}}. \tag{1.10}$$

Then the infinitesimal generator (1.8) can be written as

$$X = s^i(p^n)\frac{\partial p^{i'}}{\partial p^i}\frac{\partial}{\partial p^{i'}} = s^{i'}\frac{\partial}{\partial p^{i'}}, \tag{1.11}$$

where $s^{i'} = \frac{\partial p^{i'}}{\partial p^i}s^i$. Now,

$$Xp^n = s^i\frac{\partial}{\partial p^i}p^n = s^n. \tag{1.12}$$

(*using the above property*)

So, the infinitesimal generator can be written as

$$X = (Xp^i)\frac{\partial}{\partial p^i} = (Xp^{i'})\frac{\partial}{\partial p^{i'}}. \tag{1.13}$$

The above discussion establishes that, given the infinitesimal generator in a specific coordinate system p^i, it is straightforward to determine the corresponding infinitesimal generator in an alternative coordinate system $p^{i'}$. Consequently, we can infer that it is always possible to identify an appropriate coordinate system for any arbitrary number N of coordinates p^i in which the generator assumes a simplified form,

$$X = \frac{\partial}{\partial s}. \tag{1.14}$$

This is the normal form of the generator X.

Example: Consider the generator is given by

$$X = p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} \tag{1.15}$$

we can express the generator in terms of u and v, where $u = \frac{q}{p}$, $v = pq$. Then from (1.13) we get,

$$Xu = (p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q})u = 0,$$
$$Xv = 2ab = 2v. \tag{1.16}$$

Hence, the generator takes the form as

$$X = 2v\frac{\partial}{\partial v}. \tag{1.17}$$

1.1.6 Extensions of transformations

Assume a differential equation,

$$J(p, q', q'', q''', ..., q^{(n)}) = 0, \tag{1.18}$$

where $q' = \frac{dq}{dp}$ etc.

To apply a point transformation to a differential equation, it's crucial to first understand the transformation of the $q^{(n)}$ term. The extension of the point transformation to the derivatives can then be expressed as follows [4, 6]

$$\begin{aligned}\tilde{q'} = \frac{d\tilde{q}}{d\tilde{p}} &= \frac{d\tilde{q}(p,q;\epsilon)}{d\tilde{p}(p,q;\epsilon)} \\ &= \frac{q'\left(\frac{\partial\tilde{q}}{\partial q} + \frac{\partial\tilde{q}}{\partial p}\right)}{q'\left(\frac{\partial\tilde{p}}{\partial q} + \frac{\partial\tilde{p}}{\partial p}\right)} \\ &= \tilde{q'}(p,q,q';\epsilon). \end{aligned} \tag{1.19}$$

Similarly, $\tilde{q''} = \frac{d\tilde{q'}}{d\tilde{p}} = \tilde{q''}(p,q,q',q'';\epsilon)$ and so on.

Now, we can write the extension of the infinitesimal generator X.

$$\begin{aligned} \tilde{p} &= p + \epsilon\xi(p,q) + ... = p + \epsilon X p + ..., \\ \tilde{q} &= q + \epsilon\eta(p,q) + ... = q + \epsilon X q + ..., \\ ... &= ... ... \\ ... &= ... ... \\ \tilde{q}^{(n)} &= q^{(n)} + \epsilon\eta^{(n)}(p,q,q',...q^{(n)}) + ... = q^{(n)} + \epsilon X q^{(n)} + ... \; . \end{aligned} \tag{1.20}$$

Here $\eta, \eta', ..., \eta^{(n)}$ are of the form,

$$\begin{aligned} \eta' &= \frac{\partial\tilde{q'}}{\partial\epsilon}|_{\epsilon=0} \quad , \\ ... \quad ... & \quad ... ... \\ ... \quad ... & \quad ... ... \\ \eta^{(n)} &= \frac{\partial\tilde{q^{(n)}}}{\partial\epsilon}|_{\epsilon=0} \; . \end{aligned} \tag{1.21}$$

Using the equations (1.20) and (1.21) we get,

$$\begin{aligned}\tilde{q'} = q' + \epsilon\eta' + ... = \frac{dq'}{d\tilde{p}} &= \frac{aq + \epsilon d\eta + ...}{dp + \epsilon d\xi + ...} \\ &= \frac{q' + \epsilon\left(\frac{d\eta}{dp}\right) + ...}{1 + \epsilon\left(\frac{d\xi}{dp}\right) + ...} \\ &= q' + \epsilon\left(\frac{d\eta}{dp} - q'\frac{d\xi}{dp}\right) + ..., \end{aligned} \tag{1.22}$$

$$\begin{aligned}\tilde{q}^{(n)} &= q^{(n)} + \epsilon\eta^{(n)} + ... \\ &= \frac{d\tilde{q^{(n-1)}}}{d\tilde{q}} \\ &= q^{(n)} + \epsilon\left(\frac{d\eta^{(n-1)}}{dp} - q^{(n)}\frac{d\xi}{dp}\right) + ... \;. \end{aligned} \tag{1.23}$$

From which one get,

$$\begin{aligned}\eta' &= \frac{deta}{dp} - q'\frac{d\xi}{ap} \\ &= \frac{deta}{dp} + q'\left(\frac{d\eta}{dq} - \frac{d\xi}{sp}\right) - q''\frac{d\xi}{dq}, \\ \eta^{(n)} &= \frac{d\eta^{(n-1)}}{dp} - q^{(n)}\frac{d\xi}{dp}. \end{aligned} \tag{1.24}$$

Using the method of induction, one can show that,

$$\eta^{(n)} = \frac{d^n}{dp^n}(\eta - q'\xi) + q^{(n+1)}\xi. \tag{1.25}$$

Here $\eta^{(n)}$ is not the nth derivative of η.

To summarize the findings, we can assert that when the infinitesimal generator of a point transformation assumes the form

$$X = \xi(p,q)\frac{\partial}{\partial p} + \eta(p,q)\frac{\partial}{\partial q}, \tag{1.26}$$

its extension (prolongation) up to the nth derivative assumes the form,

$$X = \xi\frac{\partial}{\partial p} + \eta\frac{\partial}{\partial q} + \eta'\frac{\partial}{\partial q'} + ... + \eta^{(n)}\frac{\partial}{\partial q'^{(n)}} \tag{1.27}$$

where $\eta^{(n)}(p, q, q', ..., q^{(n)})$ are already defined in equation (1.24).

1.1.6.1 Multiple-parameter groups of transformations

We shall now undertake a transformation contingent upon multiple parameters, denoted as ϵ. We consider the transformation as

$$\tilde{p} = \tilde{p}(p, q; \epsilon_N), \qquad \tilde{q} = \tilde{q}(p, q; \epsilon_N), \tag{1.28}$$

Here $N = 1, 2, ..., r$.

We further assume that these ϵ_N parameters are mutually independent. This transformation constitutes an r-parameter group (G_r) if it encompasses the identity element and permits repeated application, potentially involving distinct ϵ_N values.

So, we can write the infinitesimal generator as

$$X_N = \xi_N \frac{\partial}{\partial p} + \eta_N \frac{\partial}{\partial q}, \tag{1.29}$$

where,

$$\begin{aligned} \xi_N(p,q) &= \frac{\partial \tilde{p}}{\partial \epsilon_N}\Big|_{\epsilon_M=0} \ , \\ \eta_N(p,q) &= \frac{\partial \tilde{q}}{\partial \epsilon_N}\Big|_{\epsilon_M=0}. \end{aligned} \tag{1.30}$$

It is important to observe that rescaling ϵ_N results in the corresponding rescaling of X_N by a proportional constant factor.

1.1.7 The definition of symmetry

A geometric shape or object is considered symmetric if it can be partitioned into two or more identical segments that are arranged in a systematic manner [7]. In essence, an object exhibits symmetry if a transformation can reposition its individual components without altering the overall form. The nature of symmetry is defined by the arrangement of these segments or by the specific type of transformation applied.

In physics, symmetry has been extended to encompass the concept of invariance—meaning an absence of change—under any transformation, such as arbitrary coordinate transformations [8]. This notion has emerged as one of the most formidable tools in theoretical physics, revealing that nearly all fundamental laws of nature stem from symmetrical principles. Nobel laureate P.W. Anderson, in his influential 1972 article "More is Different," aptly remarked that "it is only slightly overstating the case to say that physics is the study of symmetry." This idea is further illustrated by Noether's theorem, according to him for every continuous mathematical symmetry, there exists a corresponding conserved quantity—such as energy or momentum—known as a conserved current in Noether's terminology [9]. Additionally, Wigner's classification underscores that the symmetries governing the laws of physics dictate the intrinsic properties of the particles observed in nature [10].

A point transformation defined as $\tilde{p} = \tilde{p}(p, q)$ and $\tilde{q} = \tilde{q}(p, q)$, which may involve certain parameters, is termed a symmetry transformation of an ordinary differential equation if it maps one solution of the equation to another. In other words, under this transformation, the image of any solution remains a valid solution of the differential equation.

Consider a *nth* order differential equation

$$J\left(p, q, q', ..., q^{(n)}\right) = 0. \tag{1.31}$$

This ordinary differential equation remains invariant under the symmetry condition. Then one get

$$J\left(\tilde{q}, \tilde{q}, \tilde{q'}, ..., \widetilde{q^{(n)}}\right) = 0. \tag{1.32}$$

This definition suggests that the existence of a symmetry is unaffected by the selection of variables used to express the differential equation and its solutions. Consequently, it is reasonable to anticipate that simple differential equations possess multiple symmetries.

Symmetries that do not constitute a Lie group can be valuable in the analysis of differential equations, yet no practical method exists to identify them. Therefore, we can conclude that the symmetry transformation includes at least one parameter ϵ.

$$\begin{aligned} \tilde{q} &= \tilde{p}(p,q;\epsilon) \\ \tilde{q} &= \tilde{q}(p,q,\epsilon) \\ \tilde{q'} &= \tilde{q}'(p,q;\epsilon), \text{ and so on,} \end{aligned} \tag{1.33}$$

which is called a Lie point symmetry [4].

1.1.8 Canonical Co-ordinates

In mathematics and classical mechanics, canonical coordinates refer to specific sets of co-ordinates within the phase space that characterize a physical system at any given instant. These coordinates are integral to the Hamiltonian formulation of classical mechanics, providing a framework for describing the system's evolution. This concept also has a pertinent counterpart in quantum mechanics, reflecting its foundational significance across both classical and quantum realms.

As Hamiltonian mechanics are extended by symplectic geometry and canonical transformations are broadened to include contact transformations, the 19th-century notion of canonical coordinates in classical mechanics can be abstracted to a more sophisticated 20th-century concept. This modern interpretation involves coordinates on the cotangent bundle of a manifold, representing the advanced mathematical framework of phase space [11].

Let us consider $\vec{X}$, a vector field in the augmented space (p, q), given by

$$\vec{X} = \xi(p,q)\frac{\partial}{\partial p} + \eta(p,q)\frac{\partial}{\partial q}. \tag{1.34}$$

Now we will make a point transformation $(p, q) \longrightarrow (u, v)$.

Then the co-ordinate (u, v) will be called canonical co-ordinate of $\vec{X}$ if

$$\vec{X}_u = 0 \;\; and \;\; \vec{X}_v = 1, \tag{1.35}$$

holds.

Example: Let us assume the scaling group defined as follows:

$$\tilde{x} = e^t x \;,\; \tilde{y} = e^{4t} y$$

and the corresponding infinitesimal generator $\vec{X} = x\frac{\partial}{\partial x} + 4y\frac{\partial}{\partial y}$. Then, from equation (1.35), we get

$$\vec{X}_u = 0 \implies x\frac{\partial u}{\partial x} + 4y\frac{\partial u}{\partial y} = 0.$$

Then the characteristic differential equation becomes:

$$\frac{dx}{x} = \frac{dy}{4y} = \frac{du}{0}.$$

after simplifying, we get

$$u(x, y) = \frac{x}{y^4}.$$

In the same way, from another condition of Equation (1.35), we get

$$v(x, y) = logx.$$

So, the canonical co-ordinates are $(u, v) = (\frac{x}{y^4}), logx$.

1.1.9 Noether's Symmetry

Noether's theorem asserts that every continuous symmetry of the action of a physical system governed by conservative forces is associated with a corresponding conservation law. This landmark result, the first of two theorems established by mathematician Emmy Noether in 1915 and published in 1918, underscores a fundamental connection between symmetries and conservation principles [12]. The action of a physical system, defined as the time integral of the Lagrangian function, dictates the system's dynamics according to the principle of least action. This theorem specifically applies to continuous and smooth symmetries within physical space.

Noether's theorem plays a pivotal role in theoretical physics and the calculus of variations by elucidating the intrinsic link between the symmetries of a physical system and its conservation laws. It has profoundly influenced modern theoretical physicists, steering their focus towards the symmetries inherent in physical systems. Serving as a generalization of the concepts of constants of motion in Lagrangian and Hamiltonian mechanics—formulated in 1788 and 1833, respectively—Noether's theorem does not extend to systems that cannot be described by a Lagrangian alone, such as those incorporating a Rayleigh dissipation function. Specifically, dissipative systems exhibiting continuous symmetries may not necessarily conform to a corresponding conservation law.

- Noether's first theorem: The symmetry groups associated with a variational problem exhibit a one to one correspondence with the conservation laws in the corresponding Euler-Lagrange equation, revealing a profound and intrinsic relationship between these symmetries and the resulting conserved quantities.

  Noether's first theorem reveals a fundamental link between symmetries and conservation laws in physics. It asserts that every continuous symmetry observed in a physical system corresponds to a conserved quantity. Here, symmetry encompasses transformations that leave the system invariant, such as translations in time or space, rotations, or specific gauge transformations. These conserved quantities, derived from symmetries, remain invariant over time and are essential for analyzing the dynamics of physical systems. Notable examples include the conservation of energy, momentum, and angular momentum in classical mechanics. Noether's first theorem is profoundly significant across all of physics, providing a powerful framework for deriving and understanding conservation laws grounded in the intrinsic symmetries of physical systems.

- Noether's second theorem: A non-trivial differential relationship invariably exists between an infinite-dimensional variational symmetry group, dependent on an arbitrary function, and the corresponding non-trivial differential relation within its

Euler-Lagrange equations. This underscores a deep and inherent connection between the symmetry and the governing equations.

Noether's second theorem is pivotal as it broadens the profound relationship between symmetries and conservation laws first elucidated by her initial theorem to encompass systems with gauge symmetries. This extension is vital in contemporary theoretical physics, especially in the exploration of fundamental forces and elementary particles, where gauge theories are foundational. By pinpointing conserved currents linked to gauge symmetries, the second theorem offers deeper insights into the dynamics of these systems, thereby advancing our comprehension of fundamental interactions in nature.

Lat us consider a point-like Lagrangian for a physical system as

$$L = L\left(q^{\alpha}\ (x^{i})\ , \partial_j q^{\alpha}\ (x^{i})\right), \tag{1.36}$$

where $q^{\alpha}(x^{i})$ is the generalized co-ordinates.

By imposing symmetry constraints, one can either fully solve the evolution equations of a physical system or significantly simplify them, leading to a more streamlined and tractable form.

The Euler-Lagrange equation for the above Lagrangian can be expressed as

$$\partial_j \left(\frac{\partial L}{\partial \partial_j q^{\alpha}}\right) = \frac{\partial L}{\partial q^{\alpha}}, \quad \alpha = 1, 2, ..., N. \tag{1.37}$$

If we contract the above equation with some unknown function $\lambda^{\alpha}(q^{\beta})$,

$$\lambda^{\alpha}\left[\partial_j \left(\frac{\partial L}{\partial \partial_j q^{\alpha}}\right) - \frac{\partial L}{\partial q^{\alpha}}\right] = 0. \tag{1.38}$$

So the Lie derivative of the Lagrangian takes the form

$$\mathcal{L}_{\vec{X}} L = \vec{X} L = \lambda^{\alpha} \frac{\partial L}{\partial q^{\alpha}} + (\partial_j \lambda^{\alpha}) \frac{\partial L}{\partial(\partial_j q^{\alpha})} = \partial_j \left(\lambda^{\alpha} \frac{\partial L}{\partial(\partial_j q^{\alpha})}\right). \tag{1.39}$$

Here the vector field $\vec{X}$ express as the form [13, 14]

$$\vec{X} = \lambda^{\alpha} \frac{\partial}{\partial q^{\alpha}} + (\partial_j \lambda^{\alpha}) \frac{\partial}{\partial \partial_j q^{\alpha}}. \tag{1.40}$$

The vector field is recognized as the infinitesimal generator of the Noether symmetry. The existence of Noether's theorem requires that the Lie derivative of the Lagrangian must vanish i.e., $\mathcal{L}_{\vec{X}} L = 0$ i.e.,

$$\partial_j \left(\lambda^{\alpha} \frac{\partial L}{\partial(\partial_j q^{\alpha})} \right) = 0. \tag{1.41}$$

The system has a conserved current, known as the Noether current, which is fundamental to its structure [15, 16, 17].

$$Q^i = i_{\vec{X}} \Theta_L = \lambda^{\alpha} \frac{\partial L}{\partial \partial_i q^{\alpha}}, \tag{1.42}$$

satisfying the condition $\partial_i Q^i = 0$. Here $i_{\vec{X}}$ represents the inner product with the vector field $\vec{X}$, and the one-form Θ_L (known as Cartan one-form) is expressed as follows

$$\Theta_L = \frac{\partial L}{\partial q^{\alpha}} dq^{\alpha}. \tag{1.43}$$

Also the physical system has the energy function takes the form as

$$E = \frac{\partial L}{\partial \dot{q}^{\alpha}} \dot{q}^{\alpha} - L. \tag{1.44}$$

The energy function, referred to as the Hamiltonian of the system, remains a constant of motion in the absence of explicit time dependence in the Lagrangian [15, 16, 17]. The Noether symmetry approach stands as a robust framework, as the conserved quantities derived from it possess significant physical meaning.

Moreover, the aforementioned geometric inner product is ideally suited for identifying the cyclic variables within the augmented space. For a transformation $q^{\alpha} \longrightarrow s^{\alpha}$ in the augmented space, the symmetry vector express as,

$$\vec{X}_T = (i_{\vec{X}} ds^{\alpha}) \frac{\partial}{\partial s^{\alpha}} + \left(\frac{d}{dt} (i_{\vec{X}} ds^{\alpha}) \right) \frac{d}{ds^{\alpha}}. \tag{1.45}$$

The transformed symmetry vector $\vec{X}_T$ represents an elevated or augmented version of the original vector $\vec{X}$ in the expanded space. Without any loss of generality, if this point-like transformation is confined to a specific case,

$$\begin{aligned} i_{\vec{X}_T} ds^{\alpha} &= 1 \; for \; \alpha = m, \\ i_{\vec{X}_T} ds^{\alpha} &= 0 \; for \; \alpha \neq m. \end{aligned} \tag{1.46}$$

Then

$$\vec{X}_T = \frac{\partial}{\partial s^m} \; and \; \frac{\partial L_T}{\partial s^m} = 0, \tag{1.47}$$

Here, s^m functions as a cyclic variable within the augmented space. This geometric process proves instrumental in identifying a cyclic vector aligned with the direction of the symmetry vector $\vec{X}$.

Example: Let us consider a system of ordinary differential equations which is given by $\ddot{u} = f(t, u, \dot{u}, \phi, \dot{\phi}, \psi, \dot{\psi})$ and $\ddot{\phi} = g(t, u, \dot{u}, \phi, \dot{\phi}, \psi, \dot{\psi})$ with the point-like Lagrangian in four dimensional augmented space (t, u, ϕ, ψ) defined as $L = L(u, \dot{u}, \phi, \dot{\phi}, \psi, \dot{\psi})$.

Now, we consider the infinitesimal generator as

$$\vec{X} = \alpha \frac{\partial}{\partial u} + \beta \frac{\partial}{\partial \phi} + \gamma \frac{\partial}{\partial \psi} + \dot{\alpha} \frac{\partial}{\partial \dot{u}} + \dot{\beta} \frac{\partial}{\partial \dot{\phi}} + \dot{\gamma} \frac{\partial}{\partial \dot{\psi}}. \tag{1.48}$$

Here α, β, γ are coefficients of the infinitesimal generator and they are function of (u, ϕ, ψ). The term $\dot{\alpha}, \dot{\beta}, \; and \; \dot{\gamma}$ are expressed as

$$\begin{aligned} \dot{\alpha} &= \frac{d}{dt}(\alpha(u, \phi, \psi)) = \dot{u}\frac{\partial \alpha}{\partial u} + \dot{\phi}\frac{\partial \alpha}{\partial \phi} + \dot{\psi}\frac{\partial \alpha}{\partial \psi} \\ \dot{\beta} &= \frac{d}{dt}(\beta(u, \phi, \psi)) = \dot{u}\frac{\partial \beta}{\partial u} + \dot{\phi}\frac{\partial \beta}{\partial \phi} + \dot{\psi}\frac{\partial \beta}{\partial \psi} \\ \dot{\gamma} &= \frac{d}{dt}(\gamma(u, \phi, \psi)) = \dot{u}\frac{\partial \gamma}{\partial u} + \dot{\phi}\frac{\partial \gamma}{\partial \phi} + \dot{\psi}\frac{\partial \gamma}{\partial \psi}. \end{aligned} \tag{1.49}$$

The existence of Noether symmetry in the point-like Lagrangian necessitates that the Lie derivative of the Lagrangian with respect to the vector field, $\mathcal{L}_{\vec{X}}L$ must vanish, i.e., $\vec{X}L = 0$. This condition leads to a system of partial differential equations. By applying the method of separation of variables to these equations, we can systematically solve for the parameters $\alpha, \beta,$ *and* γ.

1.1.10 Symmetry and Laws of Conservation

Symmetry principles in physics, epitomized by Noether's theorem, reveal a deep and fundamental connection between symmetries and conservation laws. They assert that every continuous symmetry in a physical system gives rise to a corresponding conserved current. This intrinsic symmetry underpins the existence of conservation laws, establishing a powerful framework for understanding physical invariants.

- Symmetry under space translations $\implies$ Conservation of linear momentum
- Symmetry under rotations $\implies$ Conservation of angular momentum
- Symmetry under boosts (moving coordinates) $\implies$ Linear motion of the center of mass.
- Symmetry under time translations $\implies$ Conservation of energy.

1.1.11 Noether symmetries of ODEs

In the previous sections, we analyzed the invariance of functions under point transformations. Now, we turn our attention to investigating how ordinary differential equations (ODEs) preserve their structure when subjected to a one-parameter point transformation. Let us consider the 'n' dimensional system of ODEs

$$p^{(n)i} = y^i \left(t, p^k, \dot{p}^k, \ddot{p}^k, ..., (n-1)th\ derivative\ of\ p^k, \right) \tag{1.50}$$

where $\dot{p}^i = \frac{dp^i}{dt}$, $p^{(n)} = \frac{d^n p}{dt^n}$. The infinitesimal point transformation is

$$\tilde{p} = t + \epsilon\xi(t, p^k) \ , \tag{1.51}$$

$$\tilde{p^i} = p^i + \epsilon\eta^i(t, p^k) \ . \tag{1.52}$$

In Analytical Mechanics, the Lagrangian, expressed as $L = L(t, p^k, \dot{p}^k)$, defines the dynamics of a system. The equations governing the motion of the system are derived from the application of the Euler-Lagrange operator, denoted as E_i, on the Lagrangian L. In other terms,

$$E_i(L) = 0 \tag{1.53}$$

with

$$E_i = \frac{d}{dt}\frac{\partial}{\partial \dot{p}^k} - \frac{\partial}{\partial p^k}. \tag{1.54}$$

If the Lagrangian remains invariant under the transformation defined by equations (1.51)–(1.52), it follows that the Euler-Lagrange equations (1.53) likewise retain their form and are unaffected by this transformation. Emmy Noether proved that if a one-parameter point transformation preserves the Euler-Lagrange equations (1.53), a corresponding conserved quantity is inherently linked to that transformation.

Theorem-I: Let us Consider

$$X = \xi(t, p^k)\partial_t + \eta^i(t, p^k)\partial_i \ , \tag{1.55}$$

is the infinitesimal generator of transformation (1.51) – (1.52) and

$$L = L(t, p^k, \dot{p}^k) \ , \tag{1.56}$$

be a Lagrangian which describe the dynamical system (1.53). The transformation (1.51)–(1.52), when applied to (1.56), maintains the invariance of the Euler-Lagrange equations (1.53) if and only if there exists a function $f = f(t, x^k)$ that satisfies the following condition [18]:

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt} \ , \tag{1.57}$$

where $X^{[1]} = \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial p^i} + \eta^i_{[1]}\frac{\partial}{\partial \dot{p}^i}$, ($\eta^i_{[1]} = [\eta^i_{,t} + \dot{p}^i(\eta^i_{,p} - \eta^i_{,t}) - \dot{p}^{i2}\xi_{,p}]$, the first prolongation of η) is the first prolongation of (1.55)

If the generator (1.55) satisfies condition (1.57), it qualifies as a Noether symmetry of the dynamical system governed by the Lagrangian (1.56). These Noether symmetries collectively form a Lie algebra, referred to as the Noether algebra. In cases where the dynamical system (1.53) admits Lie symmetries that span a Lie algebra G_m of dimension $m \geq 1$, the Noether symmetries of (1.53) form a distinct Lie algebra, denoted as G_h, where $h \geq 0$. This algebra G_h is a subalgebra of G_m, specifically $G_h \subseteq G_m$.

Theorem-II: For each Noether point symmetry (1.55) associated with the Lagrangian (1.56), there exists a corresponding function $\Phi(t, p^k\dot{p}^k)$ intricately defined as:

$$\Phi = \xi\left(\dot{p}^i\frac{\partial L}{\partial \dot{p}^i} - L\right) - \eta^i\frac{\partial L}{\partial p^i} + f \ , \tag{1.58}$$

that is a first integral i.e., $\frac{d\Phi}{dt} = 0$ of the equations of motion, is known a Noether integral (first integral) of (1.53).

In further detail, a Noether symmetry, by conserving the differential equations (1.53), simultaneously qualifies as a Lie symmetry for equation (1.58). Consequenly, this relationship can be expressed as $[X^{[n]}, F] = \omega F$, where $X^{[n]}$ represents the n^{th} prolongation of the infinitesimal generator X, and ω is a function and $F = \frac{\partial}{\partial t} + \dot{p}_i\frac{\partial}{\partial p_i} + ... + y_i\left(t, p_k, \dot{p}_k, \ddot{p}_k, ..., p_i^{(n-1)}\right)\frac{\partial}{\partial p_i^{(n)}}$. We can affirm that equation (1.58) satisfies the condition $X(\Phi) = 0$, indicating that Noether integrals remain invariant as functions under the action of the Noether symmetry vector X.

The presence of Noether symmetries enables us to ascertain the defining characteristics of a dynamical system. When a dynamical system (1.53) with n degrees of freedom possesses (at least) n linearly independent first integrals Φ_J, for $J = 1\ to\ n$, that are in involution i.e., where $\{\}$ denotes the Poisson bracket, in that case, the solution to the dynamical system can be determined through the method of quadratures.

1.1.12 Noether symmetries of PDEs

Noether's theorem definitively establishes a profound link between symmetries and conservation laws for partial differential equations with a variational structure. However, the practical application of Noether's approach is critically constrained by two specific conditions, which significantly impede the construction of conservation laws.(i) The partial differential equations in question must be explicitly derived from a variational principle, meaning they are strictly Euler–Lagrange equations, and (ii) The symmetries employed must leave the variational integral invariant; thus, not every symmetry of the PDEs can produce a conservation law through Noether's theorem.

When tackling partial differential equations

$$H = H(p^i, q, q_i, q_{ij}), \tag{1.59}$$

derived from a rigorous variational principle, the following powerful theorem holds.

Theorem: The action of the transformation (1.20) on the Lagrangian

$$L_P = L_P(p^k, q, q_k), \tag{1.60}$$

remains (1.59) invariant if there exists a vector field $F^i = F^i(p^i, q)$ such that

$$X^{[1]}L_P + L_P\frac{d\xi^i}{dx^i} = \frac{dF^i}{dx^i}. \tag{1.61}$$

The point transformation generator identified in equation (1.20) is known as a Noether symmetry. The associated Noether flow is defined as

$$\Psi_i = \xi^k\left(q_k\frac{\partial L_P}{\partial q_i} - L_P\right) - \eta\frac{\partial L}{\partial q_i} + F^i, \tag{1.62}$$

by satisfying the condition $\frac{d\Psi}{dp^i} = 0$.

The method for applying Noether flows in PDEs significantly deviates from that used in ODEs. The conservation flow mentioned in equation (1.62) is utilized to lower the order of equation (1.59) by introducing a new dependent variable, $u(p^i)$. It has been rigorously demonstrated that the solution to this system is

$$u_{,i}(p^k) = \Psi_i(p^k, q, q_k), \tag{1.63}$$

is also a solution to equation (1.59). Furthermore, additional symmetries can arise from the system in equation (1.63) that are absent in equation (1.59). These newly uncovered symmetries are known as potential symmetries [19].

1.1.13 Generalized Symmetries

We have thus far addressed cases where the infinitesimal transformation ξ and η are solely functions of p and q. However, Noethers theorem [20] extends its reach to encompass dependencies on higher-order derivatives. Now, let us examine a functional of the form $X = X(p, q, \dot{q})$. In this context, ξ and η may incorporate terms involving q', q'', q''' etc., are inherently dependent. Consequently, Noether symmetries can be derived independently of prior knowledge of the Euler-Lagrange equation. This insight equally applies to higher-order Lagrangian.

In this case of point symmetries, separation by powers of q' is employed for a first-order Lagrangian. However, this approach becomes ineffective when both ξ and η exhibit functional dependence on q'. Instead, we must now separate terms based on the powers of q'', allowing us to account for this higher-order dependency.

Let us assume $\xi = \xi(p, q, q')$ and $\eta = \eta(p, q, q')$. Then Killing-type equation can be written as

$$\begin{aligned} \frac{\partial f}{\partial p} + q'\frac{\partial f}{\partial q} + q''\frac{\partial f}{\partial q'} &= \xi\frac{\partial X}{\partial p} + \eta\frac{\partial X}{\partial q} \\ + \left(\frac{\partial \eta}{\partial p} + q'\frac{\partial \eta}{\partial q} + q''\frac{\partial \eta}{\partial q'} - q'\frac{\partial \xi}{\partial p} - q'^2\frac{\partial \xi}{\partial q}q'q''\frac{\partial \xi}{\partial q'}\right)&\frac{\partial X}{\partial q'} \\ + \left(\frac{\partial \xi}{\partial p} + q'\frac{\partial \xi}{\partial q} + q''\frac{\partial \xi}{\partial q'}\right)& X. \end{aligned} \tag{1.64}$$

Here we comparing the coefficient of q'', so we obtained,

$$\frac{\partial f}{\partial q'} = \frac{\partial \eta}{\partial q'}\frac{\partial X}{\partial q'} - q'\frac{\partial \xi}{\partial q'}\frac{\partial X}{\partial q'} + X\frac{\partial \xi}{\partial q'}. \tag{1.65}$$

Hence the equation (1.64) simplifies to

$$\frac{\partial f}{\partial p} + q' \frac{\partial f}{\partial q} = \xi \frac{\partial X}{\partial p} + \eta \frac{\partial X}{\partial q} + \frac{\partial X}{\partial q'} \left(\frac{\partial \eta}{\partial p} + q' \frac{\partial \eta}{\partial q} \right)$$
$$-q' \frac{\partial X}{\partial q'} \left(\frac{\partial \xi}{\partial p} + q' \frac{\partial \xi}{\partial q} \right) + X \left(\frac{\partial \xi}{\partial p} + q' \frac{\partial \xi}{\partial q} \right) . \quad (1.66)$$

Here ξ and η are linear with q'.

1.1.14 Higher Dimensional Systems

In this discussion, we will delve into Noether's theorem as it applies to Lagrangian in systems with multiple degrees of freedom. The focus will be on the underlying symmetries,

$$G = \xi \frac{\partial}{\partial p} + \eta_i \frac{\partial}{\partial q_i'} \ , \quad (1.67)$$

it will be termed a Noether symmetry if it fulfills the following general equation

$$f' = \xi \frac{\partial X}{\partial p} + \eta_i \frac{\partial X}{\partial q_i} + (\eta' - q_i' \xi') \frac{\partial X}{\partial q_i'} . \quad (1.68)$$

Here ξ and η can be expressed as the form

$$\begin{aligned} \xi &= \xi(p, q_1, q_2, ..., q_n, q_1', q_2', ..., q_n') \ , \\ \eta_i &= \eta_i(p, q_1, q_2, ..., q_n, q_1', q_2', ..., q_n') \ . \end{aligned} \quad (1.69)$$

Thus the first integral can be expressed as

$$I = f - \left[\xi X + (\eta_i - q_i' \xi) \frac{\partial X}{\partial q_i'} \right] \ . \quad (1.70)$$

In this context, the repeated index i signifies a summation.

1.1.15 Movements within Riemannian spaces

Previously, we examined instances where a function remains invariant under point transformations. In the following sections, we will investigate cases where a function demonstrates invariance under these transformations. To support this exploration, we will focus on the geometric objects of interest, the metric tensor g_{ij} and the connection coefficients Γ^i_{jk} within a Riemannian space.

Definition: In an $n-$ dimensional space, denoted as $\mathcal{R}^n$, belonging to class C^p, an object is recognized as a geometric entity of class r (where $r \leq p$) if it possesses the following defining characteristics.

I) In every coordinate system x^i, there exists a well-defined set of components $\Psi^a(x^k)$.

II) During a coordinate transformation $x^{i'} = J^i(x^k)$, the transformed components $\Psi^{a'}$ in the new coordinate system $x^{i'}$ are rigorously defined as functions of class $r' = p - r$, which depend on the original components Ψ^a in the initial coordinates x^i, the transformation functions J^i, and their s^{th}- order derivatives $(1 \leq s \leq p)$. In essence, the new components $\Psi^{a'}$ are expressible through equations of the following form:

$$\Psi^{a'} = \Phi^a(\Psi^k, x^k, x^{k\prime}) \tag{1.71}$$

III) The functions Φ^a exhibit group properties, meaning they adhere to the following fundamental relations.

$$\Phi^a(\Phi(\Lambda, x^k, x^{k\prime}), x^k, x^{k'}) = \Phi^k(\Lambda, x^k, x^{k'}), \tag{1.72}$$

$$\Phi^a(\Phi(\Lambda, x^k, x^{k'}), x^k, x^{k'}) = \Lambda^a \ . \tag{1.73}$$

The rule of coordinate transformation $\Phi(\Lambda, x^k, x^{k'})$ characterizes the essence of the geometric entity. If the function Φ depends exclusively on Λ and the partial derivatives of J^K with respect to x^k, the geometric entity classified as a differential geometric object.

Additionally, a geometric object is deemed linear if the transformation law $\Phi(\Lambda, x^k, x^{k'})$ satisfies the condition that,

$$\Phi(\Lambda, x^k, x^{k'}) = J^a_b(x^k, x^{k'})\Lambda^b + C(x^k, x^{k'}). \tag{1.74}$$

Then the transformation law is

$$\Phi(\Lambda, x^k, x^{k'}) = J^a_b(x^k, x^{k'})\Lambda^b \tag{1.75}$$

The geometric object is defined as a linear homogeneous geometric entity. A significant category of such objects is the class of tensors.

Collineations: A linear differential geometric entity, $\Lambda(x^i)$, remains invariant under a one-parameter point transformation:

$$\tilde{x^i} = \tilde{x^i}(x^k, \epsilon) \ , \tag{1.76}$$

if and only if $\tilde{\Lambda}(\tilde{x^i})$=$\Lambda(x^i)$ at every point where the transformation is applied, the geometric entity Λ remains unaffected. Alternatively, the geometric object Λ remains unchanged under the influence of the infinitesimal generator of equation (1.76), denoted as $L_\xi \Lambda = 0$.

A direct consequence of the Lie derivative's definition and the transformation law governing linear differential geometric objects is that, if an object Λ remains invariant under the transformation (1.76), then a coordinate system exists in which the components of Λ are independent of one of the coordinates.

The concept of symmetry can be generalized by allowing the Lie derivative not to vanish, but instead to equate to another tensor. In this scenario, the Lie derivative of the linear differential geometric object Λ with respect to the infinitesimal generator ξ is expressed as $L_\xi \Lambda = \Psi$, where Ψ has the same number of components and index symmetries as Λ.

When this condition holds, the infinitesimal generator ξ is referred to as a collineation of Λ, and the specific nature of the collineation is defined by Ψ. Collineations are powerful tools for analyzing the geometric properties of Riemannian manifolds.

In Riemannian geometry, the geometric entity Λ may represent the metric tensor g_{ij} or any other derived geometric construct, such as the connection coefficients, that emerges from it.

Definition: Collineations involving geometric objects Λ derived from the metric g_{ij} of a Riemannian manifold are known as geometric collineations. In particular, the collineation associated with the metric, $L_{\xi}g_{ij}$, is termed the generic collineations can be formulated in relation to it. Furthermore, these geometric collineations can be decomposed into irreducible components, denoted by ω and H_{ij}, as follows:

$$L_{\xi}g_{ij} = 2\omega g_{ij} + 2H_{ij} \ . \tag{1.77}$$

Here, the function ω is referred to as the conformal factor, while H_{ij} is a symmetric, traceless tensor.

The importance of the quantities ω and H_{ij} lies in their effectiveness as tools for analyzing geometric collineations. To investigate this further, it is essential to express how any metric tensor transforms under the action of these symmetry variables and their derivatives, using the Lie derivative. In the following discussion, our attention will center on examining geometric collineations, particularly, those influencing the metric and connection coefficients within a Riemannian manifold.

1.1.16 Motions in Riemannian Spaces

Consider a space $\mathcal{R}^n$, an $n-$ dimensional manifold governed by the principles of Riemannian geometry. Within this space, distances are measured using an expression known as the line element, represented as,

$$ds^2 = g_{ij} dx^i dx^j, \tag{1.78}$$

here g_{ij} denotes the components of the metric tensor.

Definition: The transformation described in equation (1.76) is classified as a motion of the space R^n if, when applied, it preserves the line element. In other words, the metric tensor g_{ij} remains unchanged under the action of the transformation. This condition is mathematically expressed by stating that the Lie derivative of g_{ij} with respect to the infinitesimal generator ξ of the transformation is zero, i.e.,

$$L_\xi g_{ij} = 0. \tag{1.79}$$

The transformations defined by equation (1.76), which preserve the space $\mathcal{R}^n$, collectively form a group known as the group of motions. Since g_{ij} denotes a metric, condition (1.79) can be equivalently expressed as:

$$L_\xi g_{ij} = 2\ \xi_{(i;j)} = 0. \tag{1.80}$$

The equation is known as the Killing equation, and the vector field ξ is referred to as an isometry or Killing vector (KV). The collection of Killing vectors associated with a given metric forms a Lie algebra, termed the Killing algebra.

Motions hold profound significance in physics. For example, in Euclidean space, two fundamental motion groups exist: translations and rotations, represented as $T(3)\times SO(3)$. These motions correspond to the conservation of linear and angular momentum, respectively. Another notable example arises in cosmology, where the assumption that the universe is uniform in every direction and location gives rise to the Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime, which models a homogeneous and isotropic universe.

Theorem-I: In an $n-$dimensional Riemannian space $\mathcal{R}^n$, if a set of Killing vectors forms a Killing algebra G_{KV}, then the dimension of G_{KV} is constrained by $0 \leq dim(G_{KV}) \leq \frac{1}{2}n(n+1)$.

A Riemannian space with a Killing algebra of dimension $\frac{1}{2}n(n+1)$ is termed a maximally symmetric space. example of such spaces includes Euclidean space E_3 and Minkowski spacetime M^4.

Among Killing vectors, there exists a special class known as gradient Killing vectors. A Killing vector ξ is classified as a gradient Killing vector if its covariant derivative satisfies $\xi_{i;j} = 0$, which implies both $\xi_{(i;j)} = 0$ and $\xi[i;j] = 0$. For each gradient KV ξ, there exists a function T such that $T_{,k}g^{ik} = \xi^i$ and $T_{;ij} = 0$.

Theorem-II: If $\mathcal{R}^n$ has p gradient Killing vectors (where $p \leq n$), it is referred to as a p-decomposable space. In such instances, there exists a coordinate system in which the line element (1.78) can be expressed as :

$$ds^2 = M_{\alpha\beta}d\phi^\alpha d\phi^\beta + h_{\gamma\delta}(y^\gamma)dx_\gamma dx_\delta. \tag{1.81}$$

In this context, $M_{\alpha\beta}$ denotes the components associated with the p directions corresponding to the gradient Killing vectors. These components are given by $M_{\alpha\beta} = diag(d_1, d_2, d_3, ..., d_p)$, where each c_p is a constant. Conversely, $h_{\gamma\delta}$ represents the components related to the remaining $n-p$ directions. Here, α and β range from 1 to p, while γ and δ range from $p+1$ to n.

Example: Determine the Killing vectors (KVs) of the Euclidean sphere, provided its specific line element,

$$ds^2 = d\alpha^2 + sin^2\alpha \ d\beta^2. \tag{1.82}$$

To obtain the Killing vectors, we must solve the Killing equation (1.80), which leads to a system of equations

$$\begin{aligned} \xi^{\beta}_{,\beta} &= 0 \ , \\ 2\xi^{\alpha}_{,\alpha} + 2\xi^{\beta} \ sin\beta \ cos\beta &= 0 \ , \\ \xi^{\beta}_{,\alpha} + \xi^{\alpha}_{,\beta} - 2\xi^{\alpha} \ cot\beta &= 0. \end{aligned}$$

these solutions are the elements of the $SO(3)$ Lie algebra.

The two -dimensional Euclidean sphere (1.82) exhibits a three-dimensional Killing algebra, signifying its status as a maximally symmetric space. Moreover, any space with constant curvature is similarly classified as a maximally symmetric space.

Conformal Motion: The transformation described by equation (1.78) is referred to as a conformal motion when it preserves the angle between two directions at a given point. Formally, it is defined as follows:

The infinitesimal displacement ξ corresponding to the point transformation (1.78) is designated as a Conformal Killing Vector (CKV) when its influence on the metric g_{ij}, as determined by the Lie derivative, produces a scalar multiple of the original metric. This criterion indicates that,

$$L_{\xi} g_{ij} = 2\omega g_{ij}, \tag{1.83}$$

where $\omega = \frac{1}{n}\xi^k_{;k}$. When $\omega_{;ij} = 0$, ξ is classified as a special Conformal Killing Vector (sp. CKV). If ω remains constant, ξ becomes a Homothetic Killing Vector (HKV).

The Conformal Killing Vectors (CKVs) associated with a given metric form a Lie algebra known as the conformal algebra, denoted as G_{CV}. Let G_{HV} represent the algebra that contains the Homothetic Vectors (HVs), which, in turn, subsumes the algebra G_{KV} of Killing Vectors. Accordingly, the following theorem holds true.

- $G_{KV} \subseteq G_{HV} \subseteq G_{CV}$.
- $G_{H-K} = G_{HV} - (G_{HV} \bigcap G_{KV})$, for arbitrary n, then $0 \leq dim(G_{H-K}) \leq 1$; i.e., $\mathcal{R}^n$ admits at most one HV.
- $\mathcal{R}^2$ admits an infinite dimensional conformal algebra G_{CV}.
- $0 \leq dim(G_{CV}) \leq \frac{1}{2}(n+1)(n+2)$, for n¿2.

1.1.17 Symmetries of the connection

Consider ξ as the generator of an infinitesimal transformation corresponding to equation (1.78). In the context of Riemannian manifold governed by the metric tensor g_{ij}, a fundamental identity arises:

$$L_\xi \Gamma^i_{jk} = g^{ir}\left[(L_\xi g_{rk})_{;j} + (L_\xi g_{rj})_{;k} - (L_\xi g_{jk})_{;r}\right]. \tag{1.84}$$

If ξ is a hypersurface-orthogonal vector (HV) or a Killing vector (KV), then as stated in equation (1.84), the Lie derivative of the Christoffel symbols Γ^i_{jk} with respect to ξ vanishes. This indicates that the Christoffel symbols Γ^i_{jk} remain invariant under the transformation governed by equation (1.78).

Definition: The infinitesimal generator ξ, associated with the point transformation (1.78), maps one geodesic into another while preserving the affine parameter if and only if the Lie derivative of the connection coefficients, Γ^i_{jk}, with respect to ξ vanishes. This condition is satisfied when ,

$$L_\xi \Gamma^i_{jk} = 0. \tag{1.85}$$

The infinitesimal generator ξ is referred to as an Affine Killing vector or Affine collineation (AC).

Affine collineations (ACs) within $\mathcal{R}^n$ form a Lie algebra, known as the Affine algebra G_{AC}. Evidently, the homothetic algebra G_{HV} is a subset of G_{AC}, expressed as $G_{HV} \subseteq G_{AC}$. A spacetime admits proper ACs if and only if the dimension of G_{Hv} is strictly less than that of G_{AC}.

In flat space, when condition (1.85) reduces to $\xi_{i,jk} = 0$ the solution assumes the form $\xi_i = C_{ij}x^j + D_i$, where C_{ij} and D_i are constants defined by $n(n+1)$ parameters. As a result, flat space possesses an Affine algebra of dimension $n(n+1)$, incorporating both Killing vectors (KVs) and Hypersurface-Orthogonal Vectors (HVs). This conclusion leads to the inverse outcome.

Theorem: If an $n-$dimensional Riemannian space $\mathcal{R}^n$ possesses an Affine algebra G_{AC} with a dimension of $n(n+1)$, then $\mathcal{R}^n$ is necessarily a flat space.

Another significant form of affine symmetry is the Projective collineation.

Definition: The infinitesimal generator ξ of the point transformation (1.78) is designated as a Projective Collineation (PC) if there exists a function ψ such that,

$$L_\xi \Gamma^i_{jk} = \psi_{,j}\gamma^i_k + \psi_{,k}\gamma^i_j \ , \tag{1.86}$$

or similarly,

$$\xi_{(i;j);k} = 2g_{ij}\psi_{,k} + 2g_k(_i\psi_{,j}). \tag{1.87}$$

The function ψ, known as the projective function, is designated as such when it satisfies the condition $\psi_{;ij} = 0$, at which point we classify ξ as a special projective connection (sp. PC). Projective transformations modify the configuration of geodesics (autoparallel curves) within the manifold $\mathcal{R}^n$, preserving their shape yet failing to maintain the affine parameter.

The Projective Collineations (PCs) of the manifold $\mathcal{R}^n$ constitute a Lie algebra referred to as the Projective algebra, G_{PC}. In the context of flat space, condition (1.86) defines the fundamental framework governing projective collineations.

In flat space, condition (1.86) establishes the general form of the projective collineation.

$$\xi_i = C_{ij}x^j + (D_j x^j)x_i + E_i \ , \tag{1.88}$$

here C_{ij}, D_j, E_i are arbitrary constants.

Theorem: In a Riemannian space $\mathcal{R}^n$ of dimension n, the projective algebra G_{PC} is constrained to a miximum dimension of $n(n+2)$. If G_{CP} attains this upper limit, the space $\mathcal{R}^n$ is classified as a maximally symmetric space.

Propositions: Let $\mathcal{R}^n$ be an $n-$dimensional Riemannian space, then

(i)If a manifold $\mathcal{R}^n$ admits a Lie algebra of special Projective Collineations (sp. PC) with dimension p (where $p \leq n$), it also admits p gradient Killing vectors (KVs) and a gradient homothetic vector (HV). When $p = n$, the space is flat. This assertion is equally valid in reverse.

(ii) A maximally symmetric space that permits a proper Affine Collineation (AC) or a special Projective Collineation (sp. PC) is intrinsically flat.

A Riemannian space may admit a wider spectrum of collineations, including curvature collineations. The comprehensive classification of collineations in a Riemannian space, irrespective of whether the metric is definite or indefinite, is thoroughly examined in the reference [21]. A succinct summary of these definitions is presented in Table (1.1). In the following chapters, the claim will be rigorously substantiated and applied across various fields of physical metric and symmetry analysis.

Collineation $L_\xi A = B$	A	B
Killing Vector (KV)	g_{ij}	0
Homothetic Vector (HV)	g_{ij}	$2\psi g_{ij},\ \psi_{,i} = 0$
Conformal Vector (CKV)	g_{ij}	$2\psi g_{ij},\ \psi_{,i} \neq 0$
Affine Collinreation (AC)	Γ^i_{jk}	0
Proj. Collineations (PC)	Γ^i_{jk}	$2\phi_{(,j}\delta^i_{k)}, \phi_{,i} = 0$
Sp. Proj. Collineation (sp. PC)	Γ^i_{jk}	$2\phi_{(,j}\delta^i_{k)}, \phi_{,jk} = 0$

1.1.18 Formulation of Hamiltonian

In physics, Hamiltonian mechanics represents a profound reformulation of Lagrangian mechanics, originating in 1833 through the pioneering work of Sir William Rowan Hamilton. This framework supplants the use of generalized velocities $\dot{q}^i$ in Lagrangian mechanics with their conjugate variables, the generalized momenta. While both Hamiltonian and Lagrangian formalisms offer distinct mathematical treatments, they ultimately describe the same physical phenomena within classical mechanics.

Hamiltonian mechanics is deeply intertwined with advanced geometric concepts, particularly symplectic geometry and Poisson structures. Its formulation not only enhances the mathematical elegance of classical mechanics but also establishes a critical bridge to quantum mechanics, where the Hamiltonian function plays a central role in the transition from classical trajectories to quantum state evolution.

Now, applying the Legendre transformation,

$$H = p_i \dot{q}_i - L \ , \tag{1.89}$$

here q_i represent the generalized coordinate, with $L = L(q, \dot{q}, t)$ as the Lagrangian, and $p_i = \frac{\partial L}{\partial \dot{q}_i}$ denoting the conjugate momentum.

The 2nd-order Euler-Lagrange equation is expressed in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{1.90}$$

The 1st-order Hamiltonian equation of motion can be expressed as

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \, , \\ \dot{p} &= -\frac{\partial H}{\partial q} \, . \end{aligned} \tag{1.91}$$

Hence the Noetherian integral takes the form

$$I = f + \eta H - \tau p. \tag{1.92}$$

Where η and τ represent the coefficient functions of a generalized symmetry. Within the Hamiltonian formulation, time t serves as the independent variable, while p and q treated as dependent variables.

Let us consider an infinite transformation as bellow

$$\begin{aligned}\bar{t} &= t + \epsilon\eta \ , \\ \bar{q} &= q + \epsilon\tau \ , \\ \bar{p} &= p + \epsilon\xi \ . \end{aligned} \tag{1.93}$$

Produced by the differential operator,

$$G = \eta\frac{\partial}{\partial t} + \tau\frac{\partial}{\partial q} + \xi\frac{\partial}{\partial p}. \tag{1.94}$$

The action integral is expressed as

$$\mathcal{A} = \int_{t_0}^{t_1} \left[p\dot{q} - H(q,p,t)\right] dt, \tag{1.95}$$

here $\dot{q} = \dot{q}(q,p,t)$.

The transformed action, under the infinitesimal transformation, assumes the form:

$$\begin{aligned}\bar{\mathcal{A}} &= \int_{\bar{t_0}}^{\bar{t_1}} \{\bar{p}\dot{\bar{q}} - H(\bar{q},\bar{p},\bar{t})\}dt \\ &= \int_{t_0}^{t_1} \Big\{p\dot{q} - H + \epsilon\Big[p\psi + \dot{q}\xi - \tau\frac{\partial H}{\partial q} - \xi\frac{\partial H}{\partial p} - \eta\frac{\partial H}{\partial t} + \dot{\eta}(p\dot{q} - H)\Big]\Big\}dt \\ &\qquad\qquad +\epsilon\big\{(p_1\dot{q}_1 - H_1)\eta_1 - (p_0\dot{q}_0 - H_0)\eta_0\big\}. \end{aligned} \tag{1.96}$$

The subscripts 0 and 1 denote the values evaluated at t_0 and t_1, respectively. The term $\epsilon\psi$ represents the infinitesimal variation in $\dot{q}$ included by the infinitesimal transformation.

In this context, we have assumed

$$\psi = \eta \frac{\partial \dot{q}}{\partial t} + \tau \frac{\partial \dot{q}}{\partial q} + \xi \frac{\partial \dot{q}}{\partial p} \ . \tag{1.97}$$

The differential operator G (as shown in equation (1.94)) is identified as the Noether symmetry of the action integral if the transformed action integral is equivalent to the original action integral; that is $\bar{\mathcal{A}} = \mathcal{A}$, which yields the following result.

$$\dot{s} = p\psi + \dot{q}\xi - \tau \frac{\partial H}{\partial q} - \xi \frac{\partial H}{\partial p} - \eta \frac{\partial H}{\partial t} + \dot{\eta}(p\dot{q} - H) \ . \tag{1.98}$$

Now we can write,

$$(p_1\dot{q}_1 - H_1)\eta_1 - (p_0\dot{q}_0 - H_0)\eta_0 = -\int_{t_0}^{t_1} \dot{s} \ dt \ . \tag{1.99}$$

The transformation in $\dot{q}$ arises as a differential consequence of the transformations in q and η. Therefore, we have,

$$\psi = \dot{\tau} - \dot{q}\dot{\eta} \ . \tag{1.100}$$

Consequently, Hamilton's equation of motion for q assumes the following form:

$$\dot{s} = p\dot{\tau} - \tau \frac{\partial H}{\partial q} - \eta \frac{\partial H}{\partial t} - \dot{\eta} H \ . \tag{1.101}$$

Therefore, the 1st integral becomes

$$I = s + \eta H - \tau p \ . \tag{1.102}$$

The infinitesimal transformation in p is not independent of the transformations in t and $\dot{q}$. Therefore, we have

$$p = \frac{1}{\frac{\partial^2 H}{\partial p^2}} \left\{ \dot{\tau} - \dot{\eta} \frac{\partial H}{\partial p} - \eta \frac{\partial^2 H}{\partial p \partial t} - \tau \frac{\partial^2 H}{\partial p \partial q} \right\} . \tag{1.103}$$

This highlights the fact that beneath the Hamiltonian formalism in $(2n+1)$ variables lies a fundamental space of $(n+1)$ dimensions.

1.2 A Short Introduction of Cosmology

Cosmology, derived from the Ancient Greek kosmos meaning “universe” or “world,” and logia meaning “study” is a profound field within both physics and metaphysics that delves into the fundamental nature of the universe. The term was first introduced into English in 1656 through Thomas Blount’s Glossographia [22], and later formalized in 1731 by German philosopher Christian Wolff in his Latin work Cosmologia Generalis [23]. Religious or mythological cosmology encompasses a system of beliefs rooted in sacred texts, esoteric traditions, and creation myths, addressing both the origins and the ultimate fate of existence. In contrast, within the realm of scientific astronomy, cosmology focuses on unraveling the temporal evolution and structure of the universe.

Physical cosmology is the rigorous exploration of the observable universe’s origins, its vast structures, dynamics, and ultimate destiny, governed by the fundamental laws of science [24]. This field is pursued by a wide array of experts, including astronomers, physicists, and philosophers—such as metaphysicians and philosophers of physics, space, and time. Owing to its overlap with philosophical inquiry, theories in physical cosmology may incorporate both empirical and speculative elements, often relying on assumptions beyond the realm of direct testing. As a sub-discipline of astronomy, physical cosmology addresses the universe in its entirety. Modern cosmology is predominantly shaped by the Big Bang

Theory, which seeks to unify observational astronomy with particle physics [25, 26], particularly through the Λ-CDM model—a standard framework that incorporates dark matter and dark energy to explain the evolution of the universe.

The Big Bang is a foundational physical theory that articulates the universe's expansion from an initial state of extreme density and temperature [27]. The concept of an expanding universe was first rigorously formulated by physicist Alexander Friedmann in 1922 through his mathematical derivation of the Friedmann equations, which laid the groundwork for modern cosmological models [28, 29].

Independently of Friedmann's contributions, the Big Bang theory was first proposed in 1931 by Georges Lemaître, a Roman Catholic priest and physicist, who posited that the universe originated from a "primeval atom." Various cosmological models based on the Big Bang framework describe the evolution of the observable universe from its earliest stages to its vast large-scale structure [30]. These models provide a comprehensive account of numerous observed phenomena, such as the abundance of light elements, the cosmic microwave background (CMB) radiation, and the formation of large-scale cosmic structures. The uniformity of the universe, a challenge known as the flatness problem, is addressed by the theory of cosmic inflation, which suggests a period of rapid and exponential expansion of space in the universe's earliest moments.

The unequal abundances of matter and antimatter, a phenomenon known as baryon asymmetry, remains one of the great unresolved mysteries of cosmology. This asymmetry allowed the formation of primordial elements—predominantly hydrogen, with traces of helium and lithium—which later coalesced under the influence of gravity to form the first stars and galaxies. Astronomical observations reveal the presence of an enigmatic substance, dark matter, whose gravitational effects encircle galaxies. The majority of the universe's gravitational potential appears to arise from this dark matter, as both Big Bang models and empirical observations confirm that this gravitational force is not produced by baryonic matter, the normal atomic matter we interact with. Furthermore, redshift measurements from distant supernovae reveal that the universe's expansion is accelerating, a phenomenon attributed to an equally mysterious force known as dark energy [31].

Cosmology is a specialized branch of astronomy that explores the origins and evolution of the universe, spanning from the Big Bang to the present and projecting into the future. NASA defines cosmology as “the scientific study of the large scale properties of the universe as a whole.” emphasizing its focus on understanding the universe’s fundamental structure and dynamics over vast cosmic timescales.

Since the late 20th century, mounting observational evidence [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42] from the cosmic microwave background (CMB), baryon acoustic oscillations (BAO) [43, 44], and Type Ia supernovae [32, 34, 45] has consistently demonstrated that the Universe is undergoing an accelerated expansion, contrary to the previously held belief of a decelerating cosmos. To account for this accelerated phase, cosmologists have diverged into two main schools of thought. One faction postulates the existence of an exotic form of matter, termed dark energy (DE), within the framework of Einstein’s general relativity. This substance violates the strong energy condition, driving the observed acceleration. The alternative approach advocates for modifications to the theory of gravity itself, introducing additional terms into the Einstein-Hilbert action to explain the phenomenon.

The cosmological constant [31, 46] (Λ) and the Λ cold dark matter (ΛCDM) model are frequently employed as candidates for dark energy (DE) [47, 48, 49]. However, due to two significant limitations, the cosmological constant is not widely accepted as a robust model for DE. Instead, dynamical dark energy models have gained prominence in the field. These two major issues are the (i) fine-tuning problem [50, 51] and (ii) the coincidence problem [52, 53] . To address these challenges, cosmologists have developed alternative dark energy models to better explain the universe’s recent phase of accelerated expansion.

As a result, a wide variety of dark energy models have emerged, including the Multiscalar field dark energy model [54, 55, 56, 57, 58, 59, 60, 61], Brans-Dicke type scalar field dark energy model [62, 63], Wyel Integrable gravity field dark energy model [64], Quitessence model [65, 66], Phantom model [67], Quintom model [68, 69] and Teleparallel gravity model [70, 71], among others. In this thesis, we delve into several dark energy models, deriving their classical solutions. Additionally, we explore the quantum cosmology of these models by formulating the Wheeler-DeWitt (WD) equation.

1.2.1 Homogeneity and Isotropy

In modern physical cosmology, the cosmological principle posits that the Universe is homogeneous and isotropic on a grand scale. “Homogeneous” signifies that the Universe lacks any preferred or distinguished location, meaning it appears uniform regardless of where one observes. “Isotropic” indicates that no particular direction is favoured, implying that the Universe looks the same in every direction. The phrase “large scales” refers to vast ditances of approximately 100 megapersecs (MPc) or greater, where these principles hold true [72].

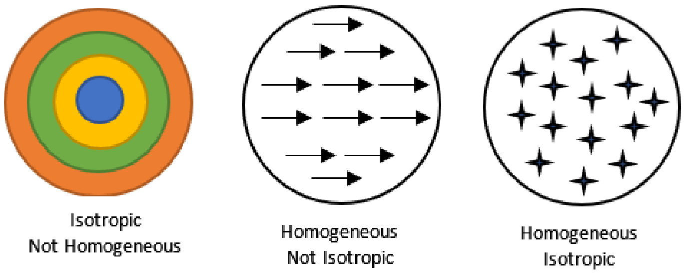

FIGURE 1.2: Represents the example of Homogeneity and Isotropy

On small scales, the Universe is neither uniform nor symmetric in all directions. Moreover, it’s crucial to recognize that homogeneity does not necessarily entail isotropy. In figure(1.2), we see some stripes. On the left side stripe is isotropic around the center but not homogeneous. The middle stripe is homogeneous but not isotropy and the right side stripe is both homogeneous and isotropic.

1.2.2 FLRW Universe

The evolution of the Universe is governed by the Einstein field equations, which establish a profound connection between the geometry of space-time and the distribution of matter within the cosmos. The Einstein equations, in their explicit form, serve as a crucial framework, linking the curvature of space-time to the energy and matter content of the Universe. The Einstein equation takes the form

$$G_{\mu\nu} = \kappa T_{\mu\nu} \ . \tag{1.104}$$

Here, $\kappa = 8\pi G$, represents the gravitational coupling constant. The term $T_{\mu\nu}$ denotes the energy-momentum tensor encapsulating the distribution and flow of energy and matter. $G_{\mu\nu}$, referred to as the Einstein tensor, is defined by its relationship to the curvature of space-time, describing how gravity arises from the matter and energy content of the Universe.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,. \tag{1.105}$$

Here. R represents the Ricci scalar, a fundamental quantity that encapsulates the curvature of space-time, defined through the contraction of the Ricci tensor.

$$R = R_{\mu\nu} g^{\mu\nu} \tag{1.106}$$

1.2.3 Friedmann Equation

In 1922, Alexander Friedmann first derived the set of equations that explain the expansion of the Universe within a homogeneous and isotropic framework, directly from Einstein's field equations. These equations, now known as the Friedmann equations, provide a foundational understanding of how space itself evolves on a large scale. The overall geometry of the Universe, characterized by its homogeneity and isotropy, is captured by a specific metric that defines its structure and behaviour.

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + sin\theta \; d\phi^2) \right] . \tag{1.107}$$

The metric, known as the Friedmann- Lemaitre-Robertson-Walker (FLRW) metric, serves as the foundation for describing the large-scale structure of the Universe [73, 74, 75, 76]. In this framework, $a(t)$ represents the scale factor, which dictates how distances within the Universe evolve over time, while κ symbolizes the scalar curvature. Specially, $\kappa = 1$ corresponds to a closed Universe, $\kappa = -1$ characterizes an open Universe, and $\kappa = 0$ signifies a flat Universe, each model reflecting distinct geometric and cosmological properties.

The Einstein field equations for a non-flat model ($\kappa \neq 0$) can be expressed as follows:

$$3H^2 + \frac{3\kappa}{a^2} = 8\pi G\rho \ , \tag{1.108}$$

$$2\dot{H} + 3H^2 + \frac{\kappa}{a^2} = -8\pi G p. \tag{1.109}$$

In the given context, the overdot denotes differentiation with respect to cosmic time, t. Here, ρ represents the energy density, while p denotes the thermodynamic pressure associated with the scalar field. The Hubble parameter, H, is conventionally expressed as $\frac{\dot{a}}{a}$, quantifying the rate at which the Universe expands. The two equations referenced, equations (1.108) and (1.109), are recognized as the first and second Friedmann equations, respectively, which form the foundation of modern cosmological models.

The conservation of the energy-momentum tensor gives rise to the continuity equation

$$\nabla_\mu \, T^\mu_\nu = 0 \ . \tag{1.110}$$

From above equation, we derive

$$\dot{\rho} + 3H(p + \rho) = 0 \ . \tag{1.111}$$

By eliminating $\frac{\kappa}{a^2}$ from equations (1.108) and (1.109), we obtain the following result.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho) \ . \tag{1.112}$$

The term $\frac{\ddot{a}}{a}$ quantifies the acceleration of the expansion.

If $(3p+\rho) > 0$, then $\frac{\ddot{a}}{a} < 0$, indicating that the Universe is decelerating . Conversely, if $(3p+\rho) < 0$, then $\frac{\ddot{a}}{a} > 0$, signifying that the Universe is accelerating.

Matter for which $(3p+\rho) > 0$, satisfying the strong energy condition, is classified as normal matter. Conversely, matter that violates the strong energy condition is referred to as exotic matter.

The first Friedmann equation can be expressed as follows

$$\Omega(t) = 1 + \frac{\kappa}{a^2 H^2} \ , \tag{1.113}$$

Here $\Omega(t) \equiv \frac{\rho}{\rho_c}$ is defined as the density parameter, where $\rho_c = \frac{3H^2}{8\pi G}$ represents the critical density.

From the equation (1.113), we can derive the following conclusions:

$$\begin{aligned} \Omega(t) > 1 &= \rho > \rho_c \Longrightarrow \kappa = +1 \ , \\ \Omega(t) < 1 &= \rho < \rho_c \Longrightarrow \kappa = -1 \ , \\ \Omega(t) = 1 &= \rho = \rho_c \Longrightarrow \kappa = 0 \ . \end{aligned} \tag{1.114}$$

Thus, we can categorize the geometry of the Universe based on the distribution of matter.

$$q = -\left(1 + \frac{\dot{H}}{H^2}\right) \ . \tag{1.115}$$

The deceleration parameter, denoted as q, is a dimensionless quantity. A negative value of q ($q < 0$) signifies that the Universe is accelerating, while a positive value ($q > 0$) indicates that the Universe is decelerating.

1.2.4 Dynamics of the Universe Filled with a Perfect Fluid

In this section, we examine the evolution of the Universe filled with a perfect fluid that adheres to the barotropic equation of state as

$$\omega = \frac{p}{\rho} \, , \tag{1.116}$$

Here, p represents the thermodynamic pressure, and ρ denotes the energy density.

By substituting the equation of state from equation (1.116) into the FLRW equation, we can derive the following results.

$$H = \frac{2}{3(1+\omega)(t-t_0)}, \text{ when } \kappa = 0 \tag{1.117}$$

and

$$\rho \propto a^{-3(1+\omega)} \, . \tag{1.118}$$

As a result, the scale factor assumes the following form.

$$a \propto t^{\frac{2}{3(1+\omega)}} \, , \tag{1.119}$$

Here, t represents the cosmic time, and t_0 is the integration constant. This solution, however, is not applicable when $\omega = -1$.

For a universe dominated by radiation, where the equation of state parameter $\omega = \frac{1}{3}$, the dynamics and evolution of the cosmos are fundamentally governed by radiation pressure, significantly influencing the expansion rate during this phase.

$$\begin{aligned} \rho &\propto a^{-4} \,, \\ a &\propto (t - t_0)^{\frac{1}{2}} . \end{aligned} \tag{1.120}$$

For a universe dominated by dust, where the equation of state parameter $\omega = 0$, the evolution is primarily driven by matter, with gravitational attraction playing a dominant role in shaping the cosmic expansion during this phase, as pressure becomes negligible.

$$\begin{aligned} \rho &\propto a^{-3}, \\ a &\propto (t - t_0)^{\frac{2}{3}} . \end{aligned} \tag{1.121}$$

In the stiff fluid era, characterized by a state equation with $\omega = 1$, the dynamics are dominated by an extremely rigid equation of state, where the pressure equals the energy density, resulting in the most intense and non-relativistic behaviour possible for any cosmological fluid.

$$\begin{aligned} \rho &\propto a^{-6}, \\ a &\propto (t - t_0)^{\frac{1}{3}} . \end{aligned} \tag{1.122}$$

1.2.5 Dark Energy

The cosmological constant, first introduced by Albert Einstein in 1917 into his field equations of general relativity, was designed to support the concept of a static universe [31, 46]. However, following Hubble's groundbreaking discovery in 1929, which revealed the universe's expansion, Einstein discarded the idea, considering it unnecessary. Consequently, the cosmological constant was largely disregarded and presumed to have no impact on the gravitational field. from 1929 until the early 1990s, the prevailing view among cosmologists was that the cosmological constant held no significance and was effectively set to zero.

Incorporating the cosmological constant, Einstein's equation can be expressed in an enhanced form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{1.123}$$

here Λ represent the cosmological constant.

The modified Friedmann equations now assume the following form

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(3p + \rho) + \frac{\Lambda}{3}. \end{aligned} \tag{1.124}$$

These equations allow for a static solution characterized by positive spatial curvature, a configuration referred to as the "Einstein Static Universe."

The energy density and pressure can be expressed in terms of the cosmological constant as follows

$$\begin{aligned} \rho_\Lambda &= \frac{\Lambda}{8\pi G}, \\ p_\lambda &= -\frac{\Lambda}{8\pi G}. \end{aligned} \tag{1.125}$$

Hence one can obtain, $p_\Lambda + \rho_\lambda = 0$.

So the state parameter equation can be expressed as

$$\omega_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -1 \tag{1.126}$$

The strong energy condition is violated in this context, suggesting that the cosmological constant could be a viable candidate for dark energy. However, the cosmological constant faces two significant challenges: the Coincidence Problem and the Fine-Tuning Problem.

Coincidence Problem: Recent observational evidence suggests that the energy density of the cosmological constant and that of matter are of comparable magnitude. Specifically, if the energy density of the cosmological constant is denoted as ρ_Λ and that of matter as ρ_m, we find that $\rho_\Lambda \propto \rho_m$. This implies we are in a unique epoch of the Universe's evolution, where the energy densities of both matter and the cosmological constant are approximately equal. This phenomenon is referred to as the **Cosmic Coincidence Problem**, underscoring the intriguing question of why these densities coincide at this particular moment in cosmic history.

Fine Tuning Problem: The recent cosmological observations have confirmed that the cosmological constant is a non-zero quantity, with its predicted value being

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = 10^{-47} GeV^4. \tag{1.127}$$

According to quantum field theory, the predicted value of the cosmological constant is extraordinarily large, approximately $10^{74} GeV^4$. However, this theoretical estimate starkly contrasts with the significantly lower observed value, creating a profound discrepancy. This mismatch represents one of the most critical and unresolved issues concerning the cosmological constant in modern physics.

To address these two significant shortcomings of the cosmological constant, cosmologists have introduced dynamic dark energy models.

1.2.6 Cosmographic Parameter

In standard cosmology, we employ a set of critical parameters known as cosmographic parameters. These are invaluable tools for analyzing the different evolutionary phases of the Universe [77]. Before defining these parameters, we assume the Universe is homogeneous and isotropic on large scales. The FLRW line element describes the geometric structure of the Universe. In this framework, the scale factor, $a(t)$, characterizes the Universe's expansion. To analyze its behavior, we expand the scale factor $a(t)$ around the present cosmic time, t_0, using a Taylor series expansion, resulting in the following expression:

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t-t_0) + \frac{1}{2!} q_0 H_0^2 (t-t_0)^2 + \frac{1}{3!} J_0 H_0^3 (t-t_0)^3 + \frac{1}{4!} s_0 H_0^4 (t-t_0)^4 + \frac{1}{5!} l_0 H_0^5 (t-t_0)^5 + \frac{1}{6!} m_0 H_0^6 (t-t_0)^6 + \ldots \tag{1.128}$$

Here, t_0 denotes the present time, with the subscript zero indicating the current values of the respective parameters. The coefficients corresponding to the various powers of $(t - t_0)$

in the expansion are defined as follows:

$$\begin{aligned} H &= \frac{\dot{a}}{a} = \text{Hubble Parameter.} \\ q &= -\frac{\ddot{a}}{aH^2} = \text{Deceleration Parameter.} \\ J &= \frac{\dddot{a}}{aH^3} = \text{Jerk Parameter.} \\ s &= \frac{\ddddot{a}}{aH^4} = \text{Snap Parameter.} \\ l &= \frac{\overset{\cdot\cdot\cdot\cdot\cdot}{a}}{aH^5} = \text{Lerk Parameter.} \\ m &= \frac{\overset{\cdot\cdot\cdot\cdot\cdot\cdot}{a}}{aH^6} = \text{m - Parameter.} \end{aligned}$$

The parameters serve as crucial tools for determining the distance-redshift relation and various cosmic distances within the Universe [78, 79, 80]. The sign of the Hubble parameter reveals whether the Universe is expanding ($H > 0$) or contracting ($H < 0$). Likewise, the sign of the deceleration parameter q indicates if the Universe is accelerating ($q < 0$) or decelerating ($q > 0$). Moreover, the deceleration parameter is defined as:

$$q = -(1 + \frac{\dot{H}}{H^2}) = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{1.129}$$

Similarly, a shift in the sign of the Jerk parameter in an expanding Universe signifies whether the rate of cosmic acceleration is increasing or decreasing. Moreover, the deceleration parameter q can be expressed through a Taylor series expansion in terms of the redshift parameter, providing further insight into the dynamics of the Universe.

$$q(z) = q_0 + (-q_0 - 2q_0^2 + J_0)z + \frac{1}{2}(2q_0 + 8q_0^2 - 7q_0 J_0 + 8q_0^3 - 4J_0 - s_0)z^2 + O(z^3), \tag{1.130}$$

The fundamental relation $\frac{1}{a} = 1 + z$, which connects the canonical redshift with the scale factor in Big-Bang cosmology, is pivotal in the study of the Universe's expansion. The redshift parameter plays a crucial role in cosmology, as the expansion of the Universe has been definitely characterized by the observed redshifts of distance galaxies. Furthermore, the Hubble parameter, when expressed in terms of the redshift parameter, can also be expanded using a Taylor series, providing a more detailed understanding of cosmic evolution.

$$H(z) = H_0 + \frac{\dot{H}_0}{1!}z + \frac{\ddot{H}_0}{2!}z^2 + \frac{\dddot{H}_0}{3!}z^3 + \frac{\ddddot{H}_0}{4!}z^4 + ... \tag{1.131}$$

Here, the dot denotes differentiation with respect to cosmic time, and the redshift H_0 represents the value at $z = 0$. Additionally, the current values of the Hubble and deceleration parameters are defined as follows [81, 34, 36]:

$$\begin{aligned} H_0 &= 73.04 + / + -1.04 km/s/Mpc \\ q_0 &= -0.55615 \end{aligned}$$

1.2.7 Modified Gravity Theory

General Relativity (GR) remains the foundational theory of gravity, providing a robust framework for understanding the geometric nature of spacetime. In a universe that exhibits uniformity and isotropy, the Einstein field equations lead to the Friedmann equations, which describe the evolution of the cosmos. The standard Big-Bang cosmology, characterized by radiation and matter dominated eras, is coherently explained within the GR framework. In this section, we will offer a succinct overview of the action principle, field equations, and key aspects of modified gravity theories. This sets the stage for an in-depth exploration of $f(R)$ and $f(T)$ gravity theories, a pivotal extension of gravitational theory [82, 83, 84].

f(R) Gravity: $f(R)$ gravity represents an extension of Einstein's general Relativity, broadening its theoretical framework. It encompasses a variety of models, each defined by a unique solution, f, of the Ricci scalar R. The simplest pathway to recover General Relativity within $f(R)$ gravity is to set the function equal to the Ricci scalar itself. This variable function introduces the possibility of accounting for the accelerated expansion and structure formation of the universe without invoking elusive entities like dark energy or dark matter [85, 86]. Certain functions in this framework may be inspired by modifications arising from quantum gravity theories. First introduced by Hans Adolph Buchdahl in 1970 [87], $f(R)$ gravity has advanced considerably over time, with significant progress driven by Starobinsky's influential work on cosmic inflation.

The Einstein-Hilbert action forms the foundational framework og general relativity, serving as the core from which the Einstein field equations are derived through the principle of least action. Employing the metric signature $(-,+,+,+)$ to represent spacetime, the gravitational term of this action is expressed as:

$$S_{EH} = \frac{1}{2\kappa}\int R\sqrt{-g}d^4x + \int \sqrt{-g}d^4x\mathcal{L}_m(g_{\mu\nu},\phi). \tag{1.132}$$

Here, g represents the determinant of the metric tensor $g_{\mu\nu}$, where R,the Ricci scalar, is defined as $R_{\mu\nu}g_{\mu\nu}$. The term $\mathcal{L}_m$ denotes the matter Lagrangian, which depends on the metric tensor $g_{\mu\nu}$ and the matter field ϕ. The constant $\kappa = \frac{8\pi G}{c^4}$ is the Einstein gravitational constant, where G represents the universal gravitational constant and c signifies the speed of light in a vacuum. If the integral converges, it is evaluated over the entire spacetime. In cases where the integral does not converge, S becomes ill-defined. However, by adopting a modified approach integrating over arbitrarily large but compact domains the Einstein field equations still emerge as the Euler-Lagrange equations of the Einstein-Hilbert action.

One of the most direct modifications to General Relativity is the $f(R)$ gravity theory, widely regarded as a well-established framework among modified gravity theories used to explain the current phase of cosmic acceleration. In this theory, f denotes an arbitrary function of the Ricci scalar R, effectively replacing R with $f(R)$.

$$S = \frac{1}{2\kappa}\int f(R)\sqrt{-g}d^4x + \int \sqrt{-g}d^4x\mathcal{L}_m(g_{\mu\nu}, \psi). \tag{1.133}$$

All other quantities remain unchanged in this framework. The field equations are derived by varying the action with respect to both the metric tensor $g_{\mu\nu}$ and the Ricci scalar R.

f(T) Gravity : In the framework of $f(T)$ gravity, as with all torsional formulations, we employ the vierbein fields e^A_μ, which establish an orthonormal basis in the tangent space at each point on the manifold x^μ [88]. The metric is then expressed as $g_{\mu\nu} = \eta_{AB}e^A_\mu e^B_\nu$, where Greek indices refer to the coordinate space and Latin indices correspond to the tangent space. In contrast to the torsion-free Levi-Civita connection, we utilize the curvature-free Weitzenböck connection, defined as $\Gamma^\lambda_{\nu\mu} = e^\lambda_A \partial_\mu e^A_\nu$ [98]. Consequently, the gravitational field is characterized by the torsion tensor.

$$T^\rho_{\mu\nu} \equiv e^\rho_A(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu). \tag{1.134}$$

The Lagrangian for the teleparallel equivalent of general relativity, represented by the torsion scalar T, is rigorously defined through the contractions of the torsion tensor, as established in [89].

$$T \equiv \frac{1}{4}T^{\rho\mu\nu}T_{\rho\mu\nu} + \frac{1}{2}T^{\rho\mu\nu}T_{\nu\mu\rho} - T_{\rho\mu}{}^{\rho}T^{\nu\mu}_{\nu}. \tag{1.135}$$

Building on the $f(R)$ extensions of general relativity, the torsion scalar T can be extended to a function $f(T)$, thereby formulating the action for $f(T)$ gravity as outlined in [90]

$$S = \frac{1}{16\pi G}\int d^4\, xe\,[f(T)], \tag{1.136}$$

here, $e = det(e^A_\mu) = \sqrt{-g}$ represents the determinant of the tetrad field, and G is the gravitational constant, with units chosen such that the speed of light is set to 1. it is important to note that the teleparallel equivalent of general relativity (TEGR), and thus general relativity itself, is recovered when $f(T) = T$. additionally, general relativity with a cosmological constant is regained when $f(T) = T + \Lambda$.

Scalar tensor Gravity Theory: The scalar-tensor theory, pioneered by John Moffat, stands as a prominent framework within modified gravitational theories [91, 92]. Grounded in the action principle, this theory introduces a vector field while transforming its three constants into scalar fields. Recognized for its explanatory power, the theory provides insights into the universe's accelerated expansion, encompassing both metric and scalar fields within gravitational dynamics. The Brans-Dicke theory emerges as a specific case within this framework, distinguished by its constant coupling parameter. Originally influential in cosmological studies [62], the scalar-tensor theory has recently gained renewed focus from researchers examining the current acceleration era [93]. Notably, this theory offers distinct advantages in addressing both the fine-tuning and coincidence problems in cosmology.

The action integral governing this gravitational theory is expressed as

$$S = \frac{1}{2\kappa}\int [f(\phi, R) - \xi(\phi)(\Delta\phi)^2]\sqrt{-g}d^4x + \int \sqrt{-g}d^4x\mathcal{L}_m(g_{\mu\nu}, \Psi), \tag{1.137}$$

Here, f denotes a versatile function dependent on both the scalar field ϕ and the Ricci scalar R. $\mathcal{L}_m$ represents the matter Lagrangian, while ξ is a function explicitly tied to ϕ. Furthermore, κ is defined as $\frac{8\pi G}{c^4}$.

The cation integral reduces to $f(R)$ gravity when $f(\phi, R) = f(R)$ and $\xi(\phi) = 0$. Likewise, it adopts the form of the Brans-Dicke (BD) theory by setting $f(\phi, R) = \phi R$ and $\xi(\phi) = \frac{\omega_{BD}}{\phi}$, where ω_{BD} represents the Brans-Dicke parameter. Subsequently, two field equations can be derived by varying the action with respect to ϕ and the metric tensor $g_{\mu\nu}$. The BD theory will not be elaborated here, as it has been thoroughly discussed in the following section.

Within this framework, it is crucial to recognize that a relationship can be drawn between the Brans-Dicke (BD) theory and $f(R)$ theory through both metric and Palatini formalisms. To demonstrate this connection, we consider the following correspondence:

$$\begin{aligned} U(\phi) &= \frac{1}{2}[R(\phi)F - f(R(\phi))], \\ \phi &= F(R). \end{aligned} \tag{1.138}$$

In this context, $R = R(T)$ in the Palatini formalism and $R = R(g)$ in the metric formalism. At this stage, we can juxtapose the field equations of this theory with those previously discussed.

1.3 Quantum Cosmology:

In cosmology, the Hamiltonian formulation offers a profound framework for understanding the Universe's dynamics. Originating from physicist William Rowan Hamilton's work, this approach defines the evolution of systems though Hamilton's equations, which govern the temporal progression of variables via a Hamiltonian function. In cosmological studies, this formulation is especially valuable for examining the behavior of gravitational fields and matter distribution on cosmic scales. By translating Einstein's equations of general relativity into Hamiltonian form, scientists can rigorously investigate phenomena such as cosmic expansion, structure formation, and other fundamental aspects shaping the Universe. This approach unveils profound insights into the intrinsic properties of spacetime, the mechanism driving cosmic inflation, and the genesis of galaxies and large-scale structures. Furthermore, the Hamiltonian formulation is indispensable in quantum cosmology, offering a rigorous framework to explore the quantum nature of the Universe from its nascent moments. In essence, the Hamiltonian formulation is pivotal in deepening our comprehension of the cosmos and its evolutionary journey.

Quantum cosmology is a fascinating realm of theoretical physics that applies quantum mechanics to the universe on a grand scale. It embarks on an ambitious quest to uncover the profound mysteries of the cosmos its origin, evolution, and ultimate fate guided by the principles of quantum mechanics. This field strives to reveal the fundamental quantum nature underlying the vast expanse of spacetime.

At its essence, quantum cosmology confronts humanity's most profound questions: What ignited the universe's inception? How has it transformed across billions of years? What forces dictate its path? Such questions drive physicists to probe the deepest reaches of the infinitesimally small and the primordial universe, unraveling the intricate fabric of spacetime and examining the dynamics of matter and energy on a cosmic scale [94, 95, 96, 97].

A pivotal approach in quantum cosmology is the minisuperspace approximation, a method that reduces the universe's complexity by isolating a finite set of fundamental degrees of freedom. These include variables like the scale factor, which encapsulates cosmic expansion, and scalar fields representing various forms of matter and energy. By distilling the problem into a simplified model, resembling a system of interacting particles, researchers aim to probe the profound quantum foundations underlying the structure and evolution of the cosmos [98].

Quantum cosmology, however, faces significant challenges. Shifting from classical to quantum descriptions of the universe introduces deep theoretical and conceptual difficulties, defining time, and interpreting quantum states, present formidable obstacles that require groundbreaking approaches and solutions.

Quantum cosmology is a field of theoretical physics focused on applying quantum mechanics to understand the universe at cosmological scales. This branch of study aims to answer fundamental questions about the cosmos's origin, evolution, and eventual fate within a quantum framework. By examining the universe's behavior at the smallest scales and earliest moments, quantum cosmology explores phenomena such as the initial singularity, the nature of space-time, and the development of cosmic structures. Through mathematical modeling and theoretical methods, quantum cosmologists work to uncover the quantum foundations of the universe and their implications for our understanding of cosmic evolution.

The connection between quantum cosmology and Noether symmetry analysis lies in their mutual pursuit of unveiling the fundamental principles that govern the universe. Noether's theorem offers a profound framework for identifying symmetries and corresponding conservation laws, which are crucial in characterizing the dynamics of quantum cosmological models. By leveraging Noether symmetry analysis within quantum cosmology, researchers strive to uncover latent symmetries that underpin the universe's evolution, thereby attaining a more profound understanding of its foundational nature.

A quantum theory of gravity seeks to resolve singularities and eliminate the divergences inherent in classical cosmology. This drives the application of quantum gravity to the homogeneous, isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime with minor perturbations, as it precisely models the observable universe at large scales. Comparable simplifications are expected within the quantum framework, making an exhaustive understanding of quantum gravity in the nonlinear regime unnecessary for deriving observationally significant conclusions.

The study of quantum cosmology initiates with the "minisuperspace approximation" and the formulation of a quantum theory for FLRW universes. This process entails reducing field theory to a finite-dimensional dynamical system, akin to a system of interacting particles. The universe's degrees of freedom in this context include the scale factor a, the matter scalar field ϕ, and the spatial curvature parameter κ. A cosmological constant λ may also be included, integrated into the scalar field's potential energy $V(\phi)$. In the Hamiltonian framework, the requirement for a vanishing total Hamiltonian introduces complex technical and interpretational challenges. These encompass ordering ambiguities, the absence of a unique inner product to define Hilbert space, lack of time evolution within the system, an overabundance of solutions to dynamical equations, and the absence of an observer within the universe. Such challenges would be addressed by a more comprehensive quantum gravity theory, positioning quantum cosmology as a simplified yet insightful domain for tackling these foundational issues.

1.3.1 Canonical Quantization:

In physics, canonical quantization is a formal procedure for translating a classical theory into a quantum framework, with a primary aim of preserving the original structure and symmetries inherent in the classical theory as closely as possible. Although this approach was not precisely the path Werner Heisenberg took to develop quantum mechanics, Paul Dirac pioneered it in his 1926 doctoral thesis, referring to it as the "method of classical analogy" for quantization [99]. He later elaborated on this concept in his foundational text, Principles of Quantum Mechanics [100]. The term "canonical" stems from the Hamiltonian formulation of classical mechanics, where a system's evolution is governed by canonical Poisson brackets- a structural element that, though modified, remains partially intact within the framework of classical quantization.

In the Hamiltonian formulation of classical mechanics, the Poisson bracket serves as a fundamental concept. A " canonical coordinate system" comprises position and momentum variables that adhere to specific canonical Poisson-bracket relation, which are essential in defining the system's dynamics and structure.

$$\{q_i, p_j\} = \delta_{ij}. \tag{1.139}$$

In which the Poisson bracket is defined by

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right). \tag{1.140}$$

For any arbitrary functions $f(q_i, p_j)$ and $g(q_i, p_j)$ defined over phase space, Hamilton's equations can be elegantly reformulated using Poisson brackets, yielding a more concise and powerful expression.

$$\begin{aligned} \dot{q}_i &= \{q_i, H\}, \\ \dot{p}_i &= \{p_i, H\}. \end{aligned} \tag{1.141}$$

These equations delineate a "flow" or trajectory within phase space, governed by the Hamiltonian H. For any arbitrary phase space function $F(q, p)$, the formalism provides that:

$$\frac{d}{dt} F(q_i, p_i) = \{F, H\}.$$

In the framework of canonical quantization, phase space variables are elevated to quantum operators acting within a Hilbert space, while the Poisson brackets between these variables are supplanted by the canonical commutation relations.

$$[\hat{q}, \hat{p}] = i\hbar \ . \tag{1.142}$$

Within the position representation, this commutation relation is explicitly realized through the choice:

$$\hat{q}\psi(q) = q\psi(q)$$

and

$$\hat{p}\psi(q) = -i\hbar\frac{d}{dq}\psi(q)$$

The system's dynamics are governed by the Schr$\ddot{o}$dinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi. \tag{1.143}$$

Here $\hat{H}$ represents the operator constructed from the Hamiltonian $H(p, q)$, with the substitutions $q \longrightarrow q$ and $p \longrightarrow -i\hbar\frac{d}{dq}$.

1.3.2 Formation of the Wheeler-DeWitt Equations:

The Wheeler-DeWitt equation stands as a foundational element in quantum cosmology, endeavoring to deliver a quantum description of the entire Universe. It represents a pivotal equation in the canonical quantization of general relativity, embedding quantum mechanical principles within the cosmological framework. The Wheeler-DeWitt equation encapsulates the dynamics of the Universe's wave function, enabling researchers to investigate its behavior across distinct cosmological epochs. The solutions to this equation offer profound insights into the essence of space, time, and matter, illuminating the origins and evolution of the cosmos. Ultimately, the Wheeler-DeWitt equation is instrumental in addressing essential question about the Universe's quantum nature and the potential for a singular beginning.

Within quantum cosmology, the Hamiltonian formulation assumes greater importance than the Lagrangian approach. It is particularly notable that the canonically conjugate momenta linked to the conserved current-arising from Noether symmetry-remains invariant, highlighting the constancy of these momenta in alignment with the symmetry-conserved currents dictated by Noether's theorem.

$$\Pi_\xi = \frac{\partial L}{\partial \dot{q}^\xi} = \Sigma_\eta, \tag{1.144}$$

here ξ varies from 0 to r, and Σ_η is a constant. In the quantization framework, the operator form of the previously mentioned conserved momentum is expressed as follows:

$$-i\partial_{q^l}|\psi>= \Sigma_l|\psi> . \tag{1.145}$$

Thus, for a real Σ_l, the differential equation described above admits an oscillatory solution, expressed as:

$$|\psi>= \sum_{l=1}^{r} e^{ip_{0\rho}q^\rho}|\psi(\phi^k)>, \;\; k < n. \tag{1.146}$$

Here k denotes directions lacking symmetry, while n specifies the dimension of the minisuperspace. Following Hurtle's perspective [101], oscillations within the Universe's wave function signal the presence of Noether symmetry. This symmetry implies the conservation of conjugate momenta along the symmetry vector, and, conversely, such conservation signifies the existence of Noether symmetry. In other words, the symmetry vector aids in identifying the oscillatory character of the physical system, helping reveal the periodic properties within solutions to the Wheeler-DeWitt equation.

1.3.3 Causal Interpretation and Bohmian Trajectory:

The de Broglie–Bohm theory is a compelling interpretation of quantum mechanics that asserts the existence of an actual configuration of particles alongside the wave function, even when these particles are unobserved. The temporal evolution of this configuration is governed by a guiding equation, while the wave function itself evolves according to the Schrödinger equation. This theory, named after the influential physicists Louis de Broglie (1892–1987) and David Bohm (1917–1992), offers a deterministic framework that challenges conventional views of quantum phenomena.

The theory is fundamentally deterministic and explicitly nonlocal [102]: the velocity of any individual particle is influenced by the guiding equation, which in turn depends on the configuration of all particles within the system.

In the de Broglie–Bohm theory, measurements represent a specific instance of quantum processes, yielding predictions that align with those of other interpretations of quantum mechanics. Notably, the theory circumvents the "measurement problem" by asserting that particles possess a definite configuration at all times. The Born rule is not merely a postulate in this framework; instead, it emerges as a theorem derived from an additional postulate known as the "quantum equilibrium hypothesis," which complements the fundamental principles governing the wave function. The theory is also characterized by several mathematically equivalent formulations.

Bohmian Mechanics: In theoretical physics, the pilot-wave theory, also known as Bohmian mechanics, stands as the earliest known example of a hidden-variable theory, first articulated by Louis de Broglie in 1927. Its contemporary formulation, the de Broglie–Bohm theory, interprets quantum mechanics as a deterministic framework, effectively circumventing challenges such as wave–particle duality, instantaneous wave function collapse, and the paradox of Schrödinger's cat through its inherently nonlocal nature.

The de Broglie–Bohm pilot-wave theory is among several interpretations of non-relativistic quantum mechanics.

Bohmian mechanics represents the same theoretical framework but emphasizes the concept of current flow, which is grounded in the quantum equilibrium hypothesis that dictates probability adheres to the Born rule. The term "Bohmian mechanics" often encompasses various extensions beyond the spin-less version initially proposed by Bohm. While the de Broglie–Bohm theory focuses primarily on Lagrangians and Hamilton-Jacobi equations, featuring the quantum potential as its central icon, Bohmian mechanics prioritizes the continuity equation, with the guiding equation as its hallmark. Despite these differences in emphasis, both formulations are mathematically equivalent in contexts where the Hamilton-Jacobi framework applies, specifically concerning spin-less particles.

The entirety of non-relativistic quantum mechanics is comprehensively encapsulated within this theoretical framework [103] . Recent advancements have leveraged this formalism to simulate the evolution of many-body quantum systems, achieving a substantial acceleration in computation compared to traditional quantum methodologies.

Causal Interpretation: Bohm initially introduced his ideas under the term Causal Interpretation. However, he later recognized that causal carried connotations of determinism, which led him to favor the term Ontological Interpretation to better capture the essence of his theory. The primary reference for this work is The Undivided Universe (Bohm and Hiley, 1993).

This phase of research encompasses Bohm's work, including his collaborations with Jean-Pierre Vigier and Basil Hiley. Bohm is explicit in asserting that this theory is fundamentally non-deterministic, with his work alongside Hiley incorporating a stochastic approach. Consequently, this theory does not constitute a strict formulation of de Broglie–Bohm theory. Nevertheless, it warrants discussion here, as the term Bohm Interpretation remains ambiguous, often conflating this theory with de Broglie–Bohm interpretations.

In 1996, philosopher of science Arthur Fine provided a comprehensive analysis of potential interpretations of Bohm's 1952 model [104].

William Simpson has proposed a hylomorphic interpretation of Bohmian mechanics, envisioning the cosmos as an Aristotelian substance constituted by material particles and a substantial form. Within this framework, the wave function assumes a dispositional role,[105] orchestrating the trajectories of the particles.

Chapter 2

Classical and Quantum cosmology for two scalar field Brans-Dicke type theory: A Noether Symmetry approach

2.1 Prelude

In cosmology, to describe the evolution of the Universe particularly at the early inflationary era and at the late era of accelerated expansion, the scalar field has a significant role [106, 107]. The scalar field can be considered in the gravitational action integral either with a minimal coupling between the scalar filed and the gravity or with a non–minimal coupling between them due to Mach's principle. The simplest model where gravity and scalar filed are minimally coupled [108, 109, 110] is the quintessence model while Brans-Dicke (BD) theory [111] is the common example of the other. The constant w_{BD}, known as the BD parameter, characterizes the theory in the sense that for w_{BD} implies the significant role of the scalar field while for large w_{BD} the major contribution to the dynamics comes from tensor part. Interestingly, BD theory and Einstein theory are two distinct theory in the sense that $w_{BD} \to \infty$ does not recover the usual Einstein gravity. Other examples of scalar field theory are O'Hanlon theory (a particular case of BD theory) and Galileon theory [112, 113, 114]. It is to be noted that these theories belong to a family of scalar-tensor theory [115] termed as Horndeski theory [116].

In the context of higher–order alternative theories of gravity, the scalar field identifies the degrees of freedoms. In [117], a higher order gravity theory was considered where action function contains Ricci scalar and its first and second order derivatives and they have found that the theory is equivalent to a two scalar field theory of which one is the usual BD scalar field while the other scalar field is minimally coupled to gravity [118, 119, 120, 121, 122, 123]. In this chapter we shall consider such two scalar field cosmology in Jordan frame where the choice of the interaction between the two scalar fields are (i) in the potential part and (ii) in the kinetic part of the action integral. It is to be noted that in an Einstein frame by considering a conformal transformation the above two models may be identified as a quintom model and chiral cosmological model (i.e., σ model) respectively. However, for cosmological prediction, exact analytical solutions of the field equations are essential and it is very hard to find them due to highly coupled and non–linear nature of the differential equations. The theory of symmetries of differential equations and associated conserved quantities are very helpful to simplify the field equations or even solve them in some cases.

The geometric symmetries of the space time have a crucial role for investigating any physical problem. Precisely, the Noether point symmetry has an extra advantage over other geometric symmetries due to the presence of a conserved quantity termed as Noether charge. Also for distinguishing different similar physical processes this conserved charge can be considered as a selection criterion. Further, the Noether integral can be applied either to have the integrability of the system or at-least to simplify the system to a great extend. Moreover, it is possible to have self–consistency of any phenomenological physical model or constraining physical parameters involved by the Noether symmetry analysis. The plan of this chapter is as follows:

We have discussed the basic concept of coupled BD theory in Section II, Section III gives the general idea about the Noether symmetry analysis. The application of this chapter using Noether symmetry analysis has been discussed in Section IV and the solution of the model is presented in Section V. Section VI deals with the solution of wave function using Wheeler DeWitt equation. Finally, we draw our brief review of this model in Section VII.

2.2 Basic Equations for coupled Brans-Dicke scalar field cosmology

The action for usual Brans-Dicke scalar field is given by [111, 115, 124]

$$A_{BD} = \int d^4x\sqrt{-g}\left[\frac{1}{2}\phi R - \frac{1}{2}\frac{w_{BD}}{\phi}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + L_m\right] \tag{2.1}$$

where as usual ϕ is the BD scalar field, w_{BD} is the BD coupling parameter and L_m is the matter source Lagrangian. As a further generalization let us consider another scalar field $\psi(x^k)$ which is either minimally coupled to BD scalar field or coupled to the BD field in a non–minimal way. So the above BD action integral modifies to

$$A_{MBD} = \int d^4x\sqrt{-g}\left[\frac{1}{2}\phi R - \frac{1}{2}\frac{w_{BD}}{\phi}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - \frac{\epsilon}{2}g^{\mu\nu}\psi_{;\mu}\psi_{;\nu} - V(\phi,\psi)\right] \tag{2.2}$$

for the minimally coupled scalar field ψ while the action takes the form

$$A_{NMBD} = \int d^4x\sqrt{-g}\left[\frac{1}{2}\phi R - \frac{1}{2}\frac{w_{BD}}{\phi}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - \frac{\epsilon}{2}\phi\ g^{\mu\nu}\psi_{;\mu}\psi_{;\nu} - V(\phi,\psi)\right] \tag{2.3}$$

for non-minimal coupling between the two scalar fields. Here $\epsilon = \pm 1$ identifies the scalar field ψ to be quintessence or phantom in nature.

The above two action integrals in Jordan frame are quite distinct and it is intersecting to show the equivalent conformal theories in the Einstein frame. Now, under the conformal transformation $\bar{g}_{ij} = \phi g_{ij}$, the action (2.1) for the BD theory becomes

$$\bar{A}_{BD} = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu} + \bar{L}_m\right] \tag{2.4}$$

Here $\Phi = \Phi(\phi(x^k))$ can be considered as a minimally coupled scalar field and $\bar{L}_m$ is the equivalent Lagrangian due to conformal transformation. Similarly, the actions (2.2) and (2.3) for the BD theory with minimally or non-minimally coupled second scalar field with the above conformal transformation take the form as

$$\bar{A}_{MBD} = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu} - \frac{\epsilon}{2}e^{k\Phi}g^{\mu\nu}\psi_{;\mu}\psi_{;\nu} - \bar{V}(\Phi,\psi)\right] \tag{2.5}$$

and

$$\bar{A}_{NMBD} = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu} - \frac{\epsilon}{2}\ g^{\mu\nu}\psi_{;\mu}\psi_{;\nu} - \bar{V}(\Phi,\psi)\right] \tag{2.6}$$

with $\bar{V}$, the form of V after the conformal transformation and $k = k(w_{BD})$. In equations (2.5) and (2.6) we have the actions for two scalar fields which are coupled in the kinetic part for equation (2.5) while there is no interaction among the scalar fields for the action (2.6). In cosmology the action (2.5) is known as chiral cosmology and is related to the description of the inflationary era in the form of hyperinflation, α–attractors etc. while the action (2.6) corresponds to quintom model for the description of dark energy in late phase and may cross the cosmological constant boundary. One may note that even in simplest FLRW space–time model the field equations for both the above models are non–linear and coupled in nature. So to obtain exact cosmological solution Noether symmetry analysis will be very much appropriate.

By varying the action (2.1) with respect to the metric tensor, one can obtain the modified Einstein field equations as

$$\phi G_{\mu\nu} = \frac{w_{BD}}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}g^{k\lambda}\phi_{;k}\phi_{;\lambda}\right) - \frac{1}{\phi}\left(g_{\mu\nu}g^{k\lambda}\phi_{;k\lambda} - \phi_{;\mu}\phi_{;\nu}\right) - g_{\mu\nu}\frac{V(\phi)}{\phi} + \frac{1}{\phi}T_{\mu\nu} \quad (2.7)$$

Similarly, one can obtain the second order differential equation for the action (2.2) by varying the scalar field $\phi(x^k)$ and $\psi(x^k)$ as

$$g^{r\lambda}\phi_{;r\lambda} - \frac{1}{2\phi}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \frac{\phi}{2w_{BD}}(R - 2V_{,\phi}) = 0 \quad (2.8)$$

and

$$g^{r\lambda}\psi_{;r\lambda} + V_{,\psi} = 0 \quad (2.9)$$

and for action (**??**) as

$$g^{r\lambda}\phi_{;r\lambda} - \frac{1}{2\phi}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + \frac{\epsilon\phi}{w_{BD}}g^{\mu\nu}\psi_{;\mu}\psi_{;\nu} + \frac{\phi}{2w_{BD}}(R - V_{,\phi}) = 0 \quad (2.10)$$

and

$$g^{r\lambda}\psi_{;r\lambda} + \frac{1}{\phi}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + V_{,\psi} = 0 \quad (2.11)$$

2.3 Brief Idea About Noether Symmetry

According to the Mathematician Emmy Noether, there are some conserved quantity associated to any physical system provided the Lagrangian of the system is invariant with respect to the Lie derivative along an appropriate vector filed. This is known as Noethers first theorem.

If $L(q^\alpha(x^i), \dot{q}^\alpha(x^i))$ be a point like Lagrangian of the physical system then the Euler–Lagrange equation can be written as [125, 126, 127, 127, 128, 129, 130]

$$\partial_j \left(\frac{\partial L}{\partial \partial_j q^\alpha} \right) = \frac{\partial L}{\partial q^\alpha} \tag{2.12}$$

So, if we contract this equation (2.12) with some unknown function $\lambda^\alpha(q^\beta)$ then it simplifies to

$$\begin{aligned} \lambda^\alpha \left[\partial_j \left(\frac{\partial L}{\partial \partial_j q^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} \right] &= 0 \\ \text{i.e.,} \quad \lambda^\alpha \frac{\partial L}{\partial q^\alpha} + (\partial_j \lambda^\alpha) \left(\frac{\partial L}{\partial \partial_j q^\alpha} \right) &= \partial_j \left(\lambda^\alpha \frac{\partial L}{\partial \partial_j q^\alpha} \right) \end{aligned} \tag{2.13}$$

and the Lie derivative of the Lagrangian can be defined as

$$\mathcal{L}_{\overrightarrow{X}} L = \lambda^\alpha \frac{\partial L}{\partial q^\alpha} + (\partial_j \lambda^\alpha) \frac{\partial L}{\partial (\partial_j q^\alpha)} = \partial_j \left(\lambda^\alpha \frac{\partial L}{\partial \partial_j q^\alpha} \right) \tag{2.14}$$

Here the vector $\overrightarrow{X}$ represents the infinitesimal generator of the Noether symmetry and is defined as

$$\overrightarrow{X} = \lambda^\alpha \frac{\partial}{\partial q^\alpha} + (\partial_j \lambda^\alpha) \frac{\partial}{\partial (\partial_j q^\alpha)} \tag{2.15}$$

Now, according to Noethers theorem if the Lie derivative of the Lagrangian along the infinitesimal generator (2.15) vanishes then we can say that there exist Noether symmetry for the physical system, i.e.,

$$\begin{aligned} \mathcal{L}_{\overrightarrow{X}} L &= 0 \\ \partial_j \left(\lambda^\alpha \frac{\partial L}{\partial \partial_j q^\alpha} \right) &= 0 \end{aligned} \tag{2.16}$$

So, from (2.16), one can say that associated to this symmetry criteria, there is a constant of motion which can be written as

$$Q^i = \lambda^\alpha \frac{\partial L}{\partial (\partial_i q^\alpha)} \tag{2.17}$$

This is known as Noether current or conserved current, which satisfies the condition $\partial_i Q^i = 0$. But if in the Lagrangian there is no explicit time dependence, then energy function is also a constant of motion. Then the energy function of the system is

$$E = \dot{q}^{\alpha}\frac{\partial L}{\partial \dot{q}^{\alpha}} - L \tag{2.18}$$

This energy function is also known as Hamiltonian. In the context of quantum cosmology the Noether symmetry condition can be written as

$$\mathcal{L}_{\overrightarrow{X}_H} H = 0$$

where H is the Hamiltonian of the given system. Here the vector field $\overrightarrow{X}_H$ takes the form as

$$\overrightarrow{X}_H = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}$$

and for the presence of Noether symmetry the canonically conjugate momenta corresponding to the conserved current is constant, that means

$$\pi_l = \frac{\partial L}{\partial \dot{q}^l} = \Sigma_l \ , \quad \text{a constant} \tag{2.19}$$

where l goes to 1 to r(number of symmetries). Operator version of (2.19) can be written as

$$-i\partial_{q^l}|\psi >= \Sigma_l|\psi > \tag{2.20}$$

where $|\psi >$ represents the wave function of the Universe. Now, by solving equation (2.20) one can determine the oscillatory part of the wave function. The solution of equation (2.20) takes the form

$$|\psi >= \sum_{l=1}^{r} e^{i\Sigma_l q^l}|\phi(q^{\sigma}) >, \quad \sigma < n \tag{2.21}$$

'σ' represents the directions along which the symmetry does not exist. Thus the oscillatory part of the wave function assures the existence of Noether symmetry.

2.4 Application of Noether Symmetry in Brans–Dicke cosmological Model

Cosmological principle states that in the large scale structure the Universe is assumed to be homogeneous and isotropic and is described by the metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{2.22}$$

Here $N(t)$ is the lapse function and $a(t)$ is the scale factor of the Universe and the usual Ricci scalar from (2.22) can be written as

$$R = \frac{6}{N^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - \frac{\dot{a}\dot{N}}{aN}\right] \tag{2.23}$$

where the over dots indicates the derivative with respect to the cosmic time t.

Now from the action (2.2), using equation (2.23) one can obtain the point like Lagrangian of the given system

$$\mathcal{L} = \frac{1}{2N}\left(-6a\phi\dot{\phi}^2 - 6a^2\dot{a}\dot{\phi} - \frac{w_{BD}}{\phi}a^3\dot{\phi}^2 - \epsilon a^3\dot{\psi}^2\right) + a^3NV(\phi,\psi) \tag{2.24}$$

By varying the above Lagrangian with respect to the variables (a, ϕ, ψ) and choosing $N = 1$ one can obtain the following field equations of the Brans–Dicke cosmological model as

$$3\phi H^2 + 2\phi\dot{H} + 2H\dot{\phi} - \frac{w_{BD}}{2}\dot{\phi}^2 - \frac{\epsilon}{2}\dot{\psi}^2 + \ddot{\phi} + V(\phi,\psi) = 0 \tag{2.25}$$

$$3\dot{H} + 6H^2 + 3w_{BD}H\frac{\dot{\phi}}{\phi} - \frac{\epsilon}{2}\dot{\psi}^2 - \frac{w_{BD}}{2}\left(\frac{\dot{\phi}^2}{\phi^2} - 2\frac{\ddot{\phi}}{\phi}\right) + V_{,\phi}(\phi,\psi) = 0 \tag{2.26}$$

$$\epsilon\ddot{\psi} + 3\epsilon H\dot{\psi} + V_{,\psi}(\phi,\psi) = 0 \tag{2.27}$$

Now by the variation with respect to the lapse function provides the constraint equation

$$6\phi H^2 + 6H\dot{\phi} + \frac{w_{BD}}{\phi}\dot{\phi}^2 + \frac{\epsilon}{2}\dot{\psi}^2 + 2V(\phi,\psi) = 0 \tag{2.28}$$

Here H is the Hubble parameter defined by $\frac{\dot{a}}{a}$. For $N = N(a,\phi,\psi)$ the above field equations (2.25–2.28) describe a point particle's element takes the form as

$$ds_1^2 = \frac{1}{N}\left(-6a\phi d\phi^2 - 6a^2 da d\phi - \frac{w_{BD}}{\phi}a^3 d\phi^2 - \epsilon\phi a^3 d\psi^2\right) \tag{2.29}$$

with effective potential $V_{eff} = a^3NV(\phi,\psi)$.

In this section our main focus to apply Noether theorem in the given Lagrangian (2.24). So for the present three dimensional point like Lagrangian the infinitesimal generator takes the form as

$$\overrightarrow{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial\phi} + \gamma\frac{\partial}{\partial\psi} + \delta\frac{\partial}{\partial N} + \dot{\alpha}\frac{\partial}{\partial\dot{a}} + \dot{\beta}\frac{\partial}{\partial\dot{\phi}} + \dot{\gamma}\frac{\partial}{\partial\dot{\psi}} \tag{2.30}$$

Now by using the condition (2.16) we can see that the co-efficients of the infinitesimal generator $(\alpha, \beta, \gamma, \delta)$ have to satisfy the following set of partial differential equations

$$-\frac{3\alpha\phi}{N} - \frac{3a\beta}{N} + \frac{3a\phi}{N^2}\delta - \frac{6a\phi}{N}\frac{\partial\alpha}{\partial a} - \frac{3a^2}{N}\frac{\partial\beta}{\partial a} = 0 \tag{2.31}$$

$$\frac{w_{BD}a^3}{2\phi^2 N}\beta + \frac{w_{BD}}{2N^2\phi}a^3\delta - \frac{3a^2}{N}\frac{\partial\alpha}{\partial\phi} - \frac{w_{BD}}{N\phi}a^3\frac{\partial\beta}{\partial\phi} - \frac{3w_{BD}}{2\phi N}a^2\alpha = 0 \tag{2.32}$$

$$-\frac{3\epsilon}{2N}a^2\alpha + \frac{\epsilon}{2N^2}a^3\delta - \frac{\epsilon}{N}a^3\frac{\partial\gamma}{\partial\psi} = 0 \tag{2.33}$$

$$\frac{-6}{N}a\alpha + \frac{3}{N^2}a^2\delta - \frac{3a^2}{N}\frac{\partial\alpha}{\partial a} - \frac{6a\phi}{N}\frac{\partial\alpha}{\partial\phi} - \frac{3a^2}{N}\frac{\partial\beta}{\partial\phi} - \frac{w_{BD}}{N\phi}a^3\frac{\partial\beta}{\partial a} = 0 \tag{2.34}$$

$$-\frac{3a^2}{N}\frac{\partial\alpha}{\partial\psi} - \frac{w_{BD}}{N\phi}a^3\frac{\partial\beta}{\partial\psi} - \frac{\epsilon}{N}a^3\frac{\partial V}{\partial\phi} = 0 \tag{2.35}$$

$$3\alpha a^2 NV(\phi,\psi) + a^3\beta N\frac{\partial V(\phi,\psi)}{\partial\phi} + \gamma a^3 N\frac{\partial V(\phi,\psi)}{\partial\psi} + \delta a^3 V(\phi,\psi) = 0 \tag{2.36}$$

Hence by solving the over determined set of equations (2.31-2.36) by the method of separation of variable we can find that the symmetry vector (2.30) admits the following set of Noether symmetries

Case:I $X_1 = \alpha_0 a\frac{\partial}{\partial a} + \beta_0\phi\frac{\partial}{\partial\phi} + \gamma_0\psi\frac{\partial}{\partial\psi} + \delta_0 N\frac{\partial}{\partial N}$, with the unknown potential $V(\phi,\psi) = v_0\phi^{\frac{k}{\beta_0}}\psi^{\frac{k_1}{\gamma_0}}$ where $\alpha_0, \beta_0, \gamma_0, \delta_0, k, k_1$ are the arbitrary constants related by the relations $3\alpha_0 + \beta_0 - \delta_0 = 3\alpha_0 + 2\gamma_0 - \delta_0 = 0,\ k_1 = -3\alpha_0 - \delta_0 - k$.

Case:II $X_2 = \alpha_0' a\frac{\partial}{\partial a} + \beta_0'\phi\frac{\partial}{\partial\phi} + \gamma_0'\psi\frac{\partial}{\partial\psi}$, here we have the potential function as $V(\phi,\psi) = \psi^2$, with $\alpha_0', \beta_0', \gamma_0'$ are the arbitrary constants related to $3\alpha_0' + \beta_0' = 3\alpha_0' + 2\gamma_0' = 0$.

Case:III $X_2 = \alpha_0'' a\frac{\partial}{\partial a} - \frac{6\alpha_0''}{\lambda}\frac{\partial}{\partial\psi} + 3\alpha_0'' N\frac{\partial}{\partial N}$, and in this case the potential function $V(\phi,\psi) = e^{\lambda\psi}$, with α_0'', λ as the arbitrary constants.

Case:IV $X_4 = \alpha_0''' a\frac{\partial}{\partial a} + \beta_0'''\phi\frac{\partial}{\partial\phi} + \gamma_0'''\psi\frac{\partial}{\partial\psi} + \delta_0' N\frac{\partial}{\partial N}$, with the unknown potential $V(\phi,\psi) = \phi^2$ where $\alpha_0''', \beta_0''', \gamma_0''', \delta_0'$ are the arbitrary constants related by the relations $2\alpha_0''' = -\beta_0''' = 2\delta_0' = -2\gamma_0'''$.

2.5 Solution of the model

In this section we are trying to find the exact solution of the present model. For this purpose we choose a suitable co-ordinate transformation $(a, \phi, \psi, N) \rightarrow (u, v, w, w_1)$ in such a way that the transformed Lagrangian i.e., the Lagrangian in terms of new variables contains at least one cyclic variable. So, now for each case we have restricted this transformation as

$$i_{\vec{X}}du = 1,\ i_{\vec{X}}dv = 0,\ i_{\vec{X}}dw = 0,\ i_{\vec{X}}dw_1 = 0 \tag{2.37}$$

here the transformed symmetry vector is along the direction of u and perpendicular to the directions of v, w, w_1. $i_{\vec{X}}$ in equation (2.37) stands for the inner-product. Hence by solving equation (2.37) one can gets the transformed Lagrangian for the following cases:

Case:I: The relation between the old and new variables can be expressed as

$$\left.\begin{array}{rcl} a & = & e^{-u} \\ a^2\phi & = & e^{v} \\ a\psi & = & e^{w} \\ \frac{a^3\phi}{N} & = & e^{w_1} \end{array}\right\} \tag{2.38}$$

Hence by using (2.38) the transformed Lagrangian takes the form

$$L_T = e^{w_1}\left[(3-2w_{BD})\dot{u}^2+(3-2w_{BD})\dot{u}\dot{v}-\frac{w_{BD}}{2}\dot{v}^2-\frac{\epsilon}{2}e^{2w-v}(\dot{w}^2+2\dot{w}\dot{u}+\dot{u}^2)+v_0e^{-2w_1}e^{(1+\frac{k}{2})v}e^{(4-k)w}\right] \tag{2.39}$$

where the new variable u acts as a cyclic coordinate. So corresponding to this transformed Lagrangian the field equations are much simpler than the previous one. Hence the Euler Lagrange equations can be written as

$$2\dot{u}(3-2w_{BD}) + \dot{v}(3-2w_{BD}) - \epsilon e^{2w-v}(\dot{w}+\dot{u}) = A\ (constant). \tag{2.40}$$

$$(3-2w_{BD})\ddot{u} - w_{BD}\ddot{v} + \frac{\epsilon}{2}(\dot{w}^2+2\dot{w}\dot{u}+\dot{u}^2)e^{2w-v} - v_0e^{-2w_1}(1+\frac{k}{2})e^{(1+\frac{k}{2})v}e^{(4-k)w} = 0 \tag{2.41}$$

$$\frac{d}{dt}\left\{e^{2w-v}(\dot{w}+\dot{u})\right\} - \epsilon(\dot{w}+\dot{u})^2e^{2w-v} - v_0e^{-2w_1}e^{(1+\frac{k}{2})v}(4-k)e^{(4-k)w} = 0 \tag{2.42}$$

$$(3-2w_{BD})\dot{u}^2+(3-2w_{BD})\dot{u}\dot{v}-\frac{w_{BD}}{2}\dot{v}^2-\frac{\epsilon}{2}(\dot{w}^2+2\dot{w}\dot{u}+\dot{u}^2)e^{2w-v}-v_0e^{-2w_1}e^{(1+\frac{k}{2})v}e^{(4-k)w} = 0 \tag{2.43}$$

Solving the above set of equations(2.40–2.43), one can obtain the explicit solution for the following cases:

Subcase I: $w(t) = w_0$, a constant

$$\left.\begin{aligned} u(t) &= 2c_1 p \tan\left(\frac{t+c_2}{2c_1}\right) + c_3 \\ v(t) &= \ln\left\{\frac{\sec^2\left(\frac{t+c_2}{2c_1}\right)}{2Bc_1^2}\right\} \\ \text{and} \quad w_{BD} &= \frac{3}{2} \end{aligned}\right\} \tag{2.44}$$

Where $p = \frac{-A}{2\epsilon B c_1^2} e^{-2w_0}$, $B = \frac{2A^2 e^{-2w_0}}{\epsilon(4-k)}$, A, B, c_1, c_2, c_3 are the arbitrary constants.

Subcase:II $w(t) = lv(t), c > 0$, (l, c are constants)

$$\left.\begin{aligned} u(t) &= \frac{-2l}{2l-1} \ln\left[\sqrt{\frac{2m}{(1-2l)c}} \sinh\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right] - \frac{A\sqrt{c}}{m\epsilon} \coth\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_4)\right) + c_5 \\ v(t) &= \frac{2}{2l-1} \ln\left[\sqrt{\frac{2m}{(1-2l)c}} \sinh\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right] \\ w(t) &= \frac{2l}{2l-1} \ln\left[\sqrt{\frac{2m}{(1-2l)c}} \sinh\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right] \\ \text{and} \quad w_{BD} &= \frac{3}{2} \end{aligned}\right\} \tag{2.45}$$

here c_4, c_5, m are arbitrary constants.

Subcase:III $w(t) = lv(t), c < 0$

$$\left.\begin{aligned} u(t) &= \frac{-2l}{2l-1} \ln\left[\sqrt{\frac{2m}{(1-2l)c}} \sin\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)\right] - \frac{A\sqrt{c}}{m\epsilon} \cot\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_4)\right) + c_7 \\ v(t) &= \frac{2}{2l-1} \ln\left[\sqrt{\frac{2m}{(1-2l)c}} \sin\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)\right] \\ w(t) &= \frac{2l}{2l-1} \ln\left[\sqrt{\frac{2m}{(1-2l)c}} \sin\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)\right] \\ \text{and} \quad w_{BD} &= \frac{3}{2} \end{aligned}\right\} \tag{2.46}$$

Hence the classical cosmological solutions in the old variables for the above three cases:

Subcase:I

$$\left.\begin{aligned} a(t) &= e^{-2c_1 p \tan\left(\frac{t+c_2}{2c_1}\right) - c_3} \\ \phi(t) &= \frac{1}{2Bc_1^2}\left(\sec^2\left(\frac{t+c_2}{2c_1}\right)\right) e^{4c_1 p \tan\left(\frac{t+c_2}{2c_1}\right) + 2c_3} \\ \psi(t) &= e^{w_0} e^{2c_1 p \tan\left(\frac{t+c_2}{2c_1}\right) + c_3} \end{aligned}\right\} \tag{2.47}$$

Subcase:II

$$
\left.\begin{aligned}
a(t) &= \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sinh\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right)^{\frac{-2l}{1-2l}} e^{\frac{A\sqrt{c}}{m\epsilon}\coth\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)+c_5} \\
\phi(t) &= \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sinh\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right)^{\frac{4l}{2l-1}} e^{-\frac{2A\sqrt{c}}{m\epsilon}\coth\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)-2c_5} \\
&\quad \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sinh\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right)^{\frac{2}{2l-1}} \\
\psi(t) &= \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sinh\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right)^{\frac{4l}{1-2l}} e^{-\frac{A\sqrt{c}}{s\epsilon}\coth\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)+c_5} \\
&\quad \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sinh\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right)^{\frac{2l}{2l-1}}
\end{aligned}\right\} \tag{2.48}
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Subcase:III

$$
\left.\begin{aligned}
a(t) &= \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sin\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)\right)^{\frac{-2l}{1-2l}} e^{\frac{A\sqrt{c}}{m\epsilon}\cot\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)+c_7} \\
\phi(t) &= \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sin\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)\right)^{\frac{4l}{2l-1}} e^{-\frac{2A\sqrt{c}}{m\epsilon}\cot\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)-2c_7} \\
&\quad \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sin\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right)^{\frac{2}{2l-1}} \\
\psi(t) &= \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sin\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)\right)^{\frac{4l}{1-2l}} e^{-\frac{A\sqrt{c}}{s\epsilon}\cot\left(\frac{(2l-1)\sqrt{c}}{2}(t+c_6)\right)+c_7} \\
&\quad \left(\sqrt{\tfrac{2m}{(1-2l)c}}\sin\left(\tfrac{(2l-1)\sqrt{c}}{2}(t+c_4)\right)\right)^{\frac{2l}{2l-1}}
\end{aligned}\right\} \tag{2.49}
$$

Case: II: For the symmetry vector X_2 one can get the interrelation between the old and the new variables using equation(2.37) as

$$a = e^{-2u}, \; a^3\phi = e^v, \; a^3\psi^2 = e^w$$

As a consequence, the transformed Lagrangian in new variables takes the form

$$L_T = e^v\Big[-12\dot{u}^2 + 6\dot{u}(\dot{v}+6\dot{u}) - \frac{w_{BD}}{2}(\dot{v}+6\dot{u})^2\Big] - \frac{\epsilon}{8}e^w(\dot{w}+2\dot{u})^2 + e^w$$

Though the above transformed Lagrangian as well as the transformed field equations are much simpler than the old variable, but still it is not possible to have an analytic solution in this case.

Case: III: Due to symmetry vector X_3, it is possible to have a transformation of the variables in the augmented space so that the system gets simplified using equation(2.37). The interrelation between the old and the new variables are given by

$$u = \ln a,\ v = \phi,\ w = 6\ln a + \lambda\psi,\ D = \ln(\frac{a^3}{N})$$

So the transformed Lagrangian has the expression

$$L_T = e^D\left[-3v\dot{v}^2 - 3\dot{u}\dot{v} - \frac{w_{BD}\dot{v}^2}{2v}\right] - \frac{\epsilon}{2\lambda^2}e^D(\dot{w} - 6\dot{u})^2 + e^{w-D}$$

similar to case II, no analytic solution of the field equations corresponding to the above Lagrangian is possible.

Case: IV: For the symmetry vector X_4 we have proceed as for the earlier three cases using equation(2.37) and the interrelation between the old variable and the new variable are given by

$$a = e^{-u},\ \phi = e^{2u+v}\ , \psi = e^{u+w}\ , N = e^{v-u-D}$$

The corresponding form of the Lagrangian in the new variable is given by

$$L_T = e^D\left[-3\dot{u}^2 + 3\dot{u}(\dot{v} + 2\dot{u}) - \frac{w_{BD}}{2}(\dot{v} + 2\dot{u})^2 - \frac{\epsilon}{2}e^{2w-v}(\dot{u} + \dot{w})^2\right] + e^{3v-D}$$

Thus the solutions of the corresponding Euler-Lagrange equations for different choices for w are expressed as

Subcase I: $w(t) = m$, a constant

$$\left.\begin{aligned} u(t) &= \frac{-Be^{-2m}}{\epsilon\sqrt{M}}\ln\left|\sec\left(\frac{t+c_8}{c_9}\right) + \tan\left(\frac{t+c_8}{c_9}\right)\right| + c_{10} \\ v(t) &= \frac{1}{2}\ln\left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right) \\ \text{and}\ \ w_{BD} &= \frac{3}{2} \end{aligned}\right\} \tag{2.50}$$

Subcase II: $w(t) = -\frac{1}{2}v(t)$

$$\left.\begin{aligned} u(t) &= \ln\left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right)^{\frac{1}{4}} - \frac{B}{Mc_9}\tan\left(\frac{t+c_8}{c_9}\right) + c_{10} \\ v(t) &= \ln\left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right)^{\frac{1}{2}} \\ w(t) &= \ln\left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right)^{\frac{-1}{4}} \\ \text{and} \;\; w_{BD} &= \frac{3}{2} \end{aligned}\right\} \tag{2.51}$$

Subcase III: $w(t) = \frac{1}{2}v(t)$

$$\left.\begin{aligned} u(t) &= \ln\left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right)^{-\frac{1}{4}} - Bt + c_{10} \\ v(t) &= \ln\left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right)^{\frac{1}{2}} \\ w(t) &= \ln\left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right)^{\frac{1}{4}} \\ \text{and} \;\; w_{BD} &= \frac{3}{2} \end{aligned}\right\} \tag{2.52}$$

with $B, m, M = 2e^{-2D}, c_8, c_9, c_{10}$ are arbitrary constants in the above solutions. Thus the classical solutions in old variables takes the form

Subcase I:

$$\left.\begin{aligned} a(t) &= \left(\sec\left(\frac{t+c_8}{c_9}\right) + \tan\left(\frac{t+c_8}{c_9}\right)\right)^{\frac{Be^{-2m}}{\epsilon\sqrt{M}}} e^{c_{10}} \\ \phi(t) &= \frac{1}{\sqrt{M}c_9}\sec\left(\frac{t+c_8}{c_9}\right)\left(\sec\left(\frac{t+c_8}{c_9}\right) + \tan\left(\frac{t+c_8}{c_9}\right)\right)^{\frac{-2Be^{-2m}}{\epsilon\sqrt{M}}} e^{-2c_{10}} \\ \psi(t) &= e^m\left(\sec\left(\frac{t+c_8}{c_9}\right) + \tan\left(\frac{t+c_8}{c_9}\right)\right)^{\frac{-Be^{-2m}}{\epsilon\sqrt{M}}} e^{-c_{10}} \end{aligned}\right\} \tag{2.53}$$

Subcase II:

$$\left.\begin{aligned} a(t) &= \left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right)^{\frac{-1}{4}} e^{\frac{B}{Mc_9}\tan\left(\frac{t+c_8}{c_9}\right) - c_{10}} \\ \phi(t) &= \frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2} e^{\frac{-2B}{Mc_9}\tan\left(\frac{t+c_8}{c_9}\right) + 2c_{10}} \\ \psi(t) &= e^{\frac{-B}{Mc_9}\tan\left(\frac{t+c_8}{c_9}\right) + c_{10}} \end{aligned}\right\} \tag{2.54}$$

Subcase III:

$$\left.\begin{array}{rcl} a(t) &=& \left(\frac{\sec^2\left(\frac{t+c_8}{c_9}\right)}{Mc_9^2}\right)^{\frac{1}{4}} e^{Bt-c_{10}} \\ \phi(t) &=& e^{-2Bt+2c_{10}} \\ \psi(t) &=& e^{-Bt+c_{10}} \end{array}\right\} \tag{2.55}$$

To examine whether the above six set of solutions for the two different Noether symmetry vectors (and different choices for the parameters) are compatible to classical cosmology, the three leading cosmological parameters namely the scale factor, Hubble parameter and the acceleration parameter have been plotted in Fig (2.1) for case I, considering various choices for the parameters involved. Note that we have drawn the figures only for subcase I as the figures for the other two subcases are almost similar. For Case IV, the graphical representation for the scale factor and the Hubble parameter are of same nature in all the subcases and also they are similar to those in Fig (2.1). So we have not present them again. However, the variation of the acceleration parameter has some distinct character in the three subcases and are presented in Fig (2.2). From the graph it is clear that the present cosmological model is an expanding model of the Universe with the rate of expansion gradually decreases. For case I the Universe evolves from an accelerated phase to decelerated phase then again in the accelerated era as noted in observation data. For Case IV, the model describes the evolution from decelerated era to late time accelerated epoch for subcase I and III while the subcase II corresponds to the entire evolution of the Universe starting from the earlier accelerated era of evolution.

2.6 Solution of the wave function using Wheeler-DeWitt equation

In the above transformed Lagrangian (2.39) the variable u acts as a cyclic coordinate, so in this case the canonically conjugate momenta can be written as [17, 131]

$$p_u = \frac{\partial L}{\partial \dot{u}} = \epsilon \dot{u} e^{2w+w_1-v} \quad (\text{choosing} \quad w_{BD} = \frac{3}{2}) \tag{2.56}$$

$$p_v = \frac{\partial L}{\partial \dot{v}} = -\frac{3}{2} e^{w_1} \dot{v} \tag{2.57}$$

and consequently the Hamiltonian of the system takes the form

$$H = \frac{1}{2\epsilon} e^{v-2w-w_1} p_u^2 - \frac{1}{3} e^{-w_1} p_v^2 - v_0 e^{(1+\frac{k}{2})v - w_1} e^{(4-k)w} \tag{2.58}$$

Hence the corresponding Wheeler-DeWitt equation can be written as

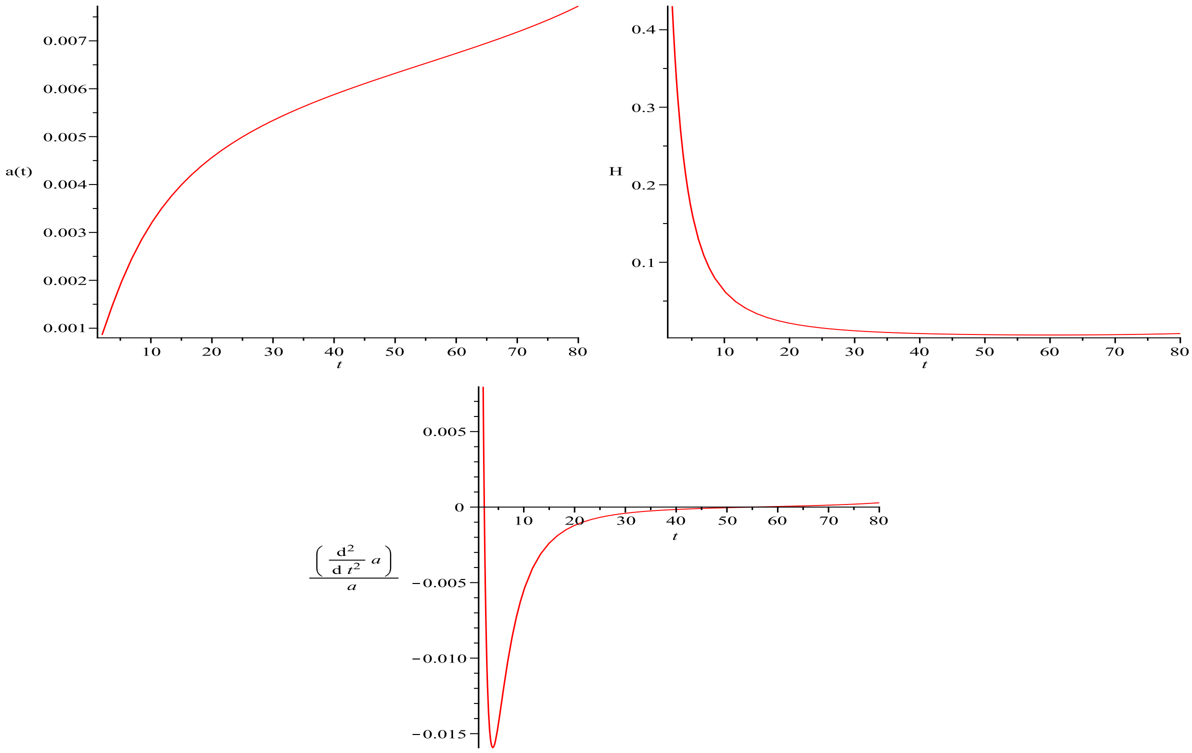

FIGURE 2.1: shows the graphical representation of scale factor $a(t)$ (top left), Hubble parameter $H(t)$ (top right) and $\frac{\ddot{a}}{a}$ (bottom) with respect to cosmic time t for Case I.

$$\left[-\frac{1}{2\epsilon} e^{v-2w-w_1}\frac{\partial^2}{\partial u^2} + \frac{1}{3}e^{-w_1}p_v^2\frac{\partial^2}{\partial v^2} - v_0 e^{(1+\frac{k}{2})v-w_1} e^{(4-k)w}\right]\Psi(u,v) = 0 \tag{2.59}$$

where Ψ is the wave function of the Universe for this model. Now, using the method of separation of variables $\Psi(u,v) = \Psi_1(u)\Psi_2(v)$ one can solve the above WD equation. Since u is a cyclic coordinate, so the corresponding conjugate momenta (p_u) will be conserved in nature i.e.,

$$\epsilon\dot{u}e^{2w+w_1-v} = \text{conserved} = \Sigma_0 \ \text{(say)} \tag{2.60}$$

The solution of the operator version of the equation (2.60) gives the oscillatory part of the wave function as

$$\Psi_1(u) = e^{i\Sigma_0 u} \tag{2.61}$$

using (2.61) in the WD equation (2.59), one gets the complete expression of the wave function as

$$\Psi(u,v) = e^{i\Sigma_0 u}\left\{c_1\ J\left(2\sqrt{3v_0}e^{3w}, \frac{\sqrt{6}\Sigma_0 e^{-w+\frac{v}{2}}}{\sqrt{\epsilon}}\right) + c_2\ Y\left(2\sqrt{3v_0}e^{3w}, \frac{\sqrt{6}\Sigma_0 e^{-w+\frac{v}{2}}}{\sqrt{\epsilon}}\right)\right\} \tag{2.62}$$

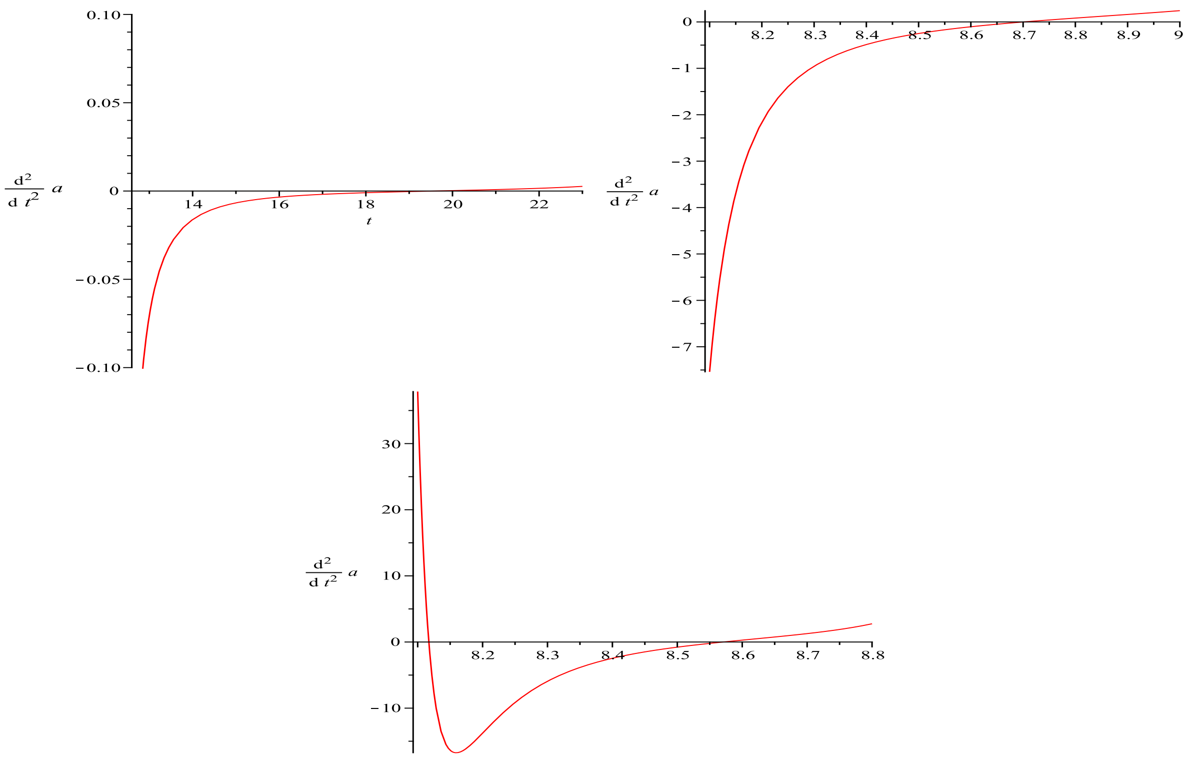

FIGURE 2.2: Represents $\frac{\ddot{a}}{a}$ (top left) for $w = 0$, $\frac{\ddot{a}}{a}$ (top right) when $w(t) = \frac{1}{2}v(t)$ and $\frac{\ddot{a}}{a}$ (bottom) for $w(t) = -\frac{1}{2}v(t)$ with respect to cosmic time t for Case IV.

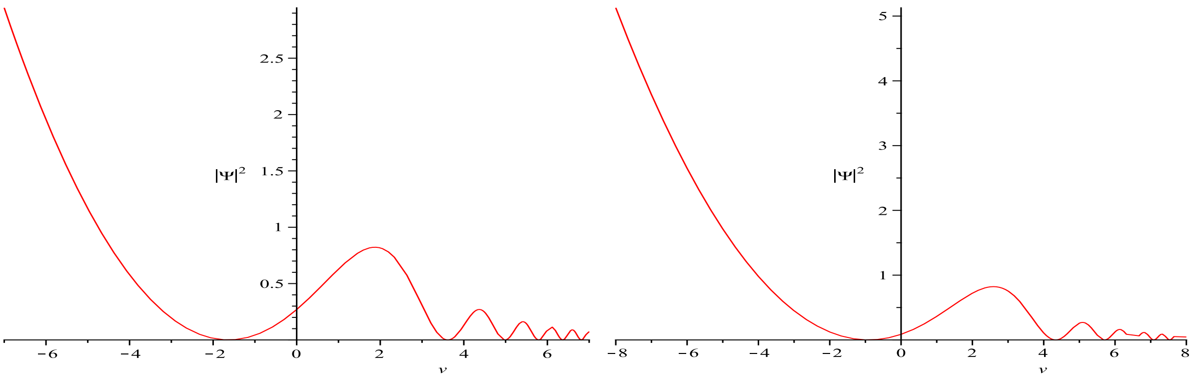

FIGURE 2.3: shows the graphical representation of wave function $|\psi|^2$ (top left) when $k = 0$, and (top right) when $k = -2$ against v for Case I.

where c_2, c_2 are the integration constants and J, Y are the usual Bessel and Neumann functions. Note that the quantum cosmological description for case IV is very similar to the above description for case I. Hence we have only given the graphical description. The graphical representation of $|\Psi|^2$ against v has been shown in Fig.(2.3) and Fig.(2.4) for case I and case IV respectively. It is to be noted that in the subcases for each case the nature of the graph for $|\Psi|^2$ almost same and hence only two graphs are presented for the

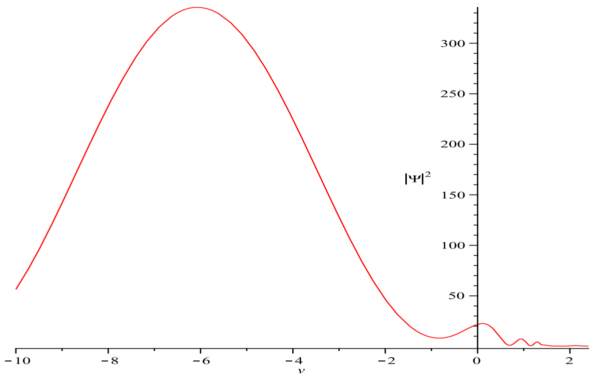

FIGURE 2.4: Represents the wave function $|\psi|^2$ for case IV

case I (with $k = 0, k = -2$ respectively). For case I, $|\Psi|^2$ shows a damping oscillation at large u while $|\Psi|^2$ increases as v decreases to $-\infty$. Hence at zero volume (i.e., $v \to -\infty$) the probability measure has finite nonzero value and hence classical singularity can not be eliminated by quantum description. Similar is the situation in case IV. The oscillatory part of the probability measure gradually dies out as v increases.

2.7 Brief Summary and Conclusion

From cosmological point of view this chapter deals with a modified gravity theory where the Brans-Dicke theory has been extended for two scalar fields which are minimally coupled among themselves. This model in Jordan frame can be viewed as the interacting two scalar field model with interaction in the potential term. However, Einstein frame, one may interpret this two scalar field model as quintom model and the second scalar field is of quintessence or phantom in nature depending on the choice of the parameter ϵ to be 1 or -1. To judge the merit of the present cosmological model in FLRW space time, exact solutions to the modified Einstein field equations are essential. The challenge to solve these higher non-linear coupled set of differential equations Noether symmetry analysis has been used in this work. Though four distinct symmetry vectors are obtained but exact solution is possible only in two cases (for I and IV). In the other two cases cyclic coordinate is identified and Lagrangian is simplified to some extend but that is not sufficient to have the exact solutions. By analyzing the possible cosmological solutions it is found that the present model either describes the evolution starting from the earlier accelerated era to the present dark energy dominated expansion through the matter dominated era or the model describes the evolution from the decelerated matter dominated phase to the present late time accelerated epoch. to address the issue of initial big-bang singularity quantum cosmology has been formulated in Hamiltonian approach by constructing the WD equation. The wave function i.e., The solution of the WD equation has been used to have the probability measure and it has been plotted in Fig.(2.3) and Fig.(2.4). From the graphs it is evident that probability measure does not vanish at zero volume. Hence one may conclude that the classical singularity can not be eliminated by quantum description.

Chapter 3

Noether Symmetry analysis for fourth order modified gravity theory: Cosmological Solution and Evolution

3.1 Prelude

In standard Cosmology, the action integral contains two terms- one is the usual Einstein-Hilbert action and the other one corresponds to matter Lagrangian mainly for baryonic matter. Before 1998 all observational evidences are well in accord with the theoretical predictions of standard cosmology. But difficulty started with the observational prediction of Type (Ia) supernova data in 1998 [132, 133, 134, 135, 134, 136, 137, 138]. The prediction of accelerated expansion from observational data can not be supported by the prediction of standard cosmology. Since then the Cosmologists have been speculating either to modify the matter Lagrangian part (inclusion of dark energy) or to modify the Einstein-Hilbert action (i.e. modification of gravity theory). Still now cosmologists are not able to have a definite conclusion regarding the modification of the theory to match the observational fact, rather the ΛCDM model is the best choice so far. In recent years, modification of gravity theory has drawn more attention than the choice of unknown exotic matter (i.e. DE) due to lack of any evidence for it. The popular modified gravity theory is the well known $f(R)$ gravity model [139, 140]. Also the teleparallel equivalence of general relativity i.e. Teleparallel gravity and its generalization $f(T)$ gravity theory [141, 142, 143] (T is related to the antisymmetric connection) is also widely used in the literature. The basic difference between these two modified gravity theories are (i) $f(R)$ gravity theory is in general a fourth-order gravity theory while the gravitational field equations in $f(T)$ gravity theory are of second order.(ii) $f(R)$-gravity theory can be shown to be equivalent with Brans-Dicke (BD) scalar field [111, 106, 144, 111, 145, 146, 147] with zero BD parameter while there is no scalar-field or scalar tensor description for $f(T)$-gravity theory. (iii) Like a usual general relativity in $f(R)$ gravity the metric is considered as the gravitational potential which in turn determines the curvature of the space -time. However, in the teleparallel framework torsion tensor characterizes the gravitational field without any role of the curvature.(iv) $f(R)$ gravity model is invariant under local Lorentz transformation while $f(T)$ does not have Lorentz invariance, but it can be made so suitably (for details see appendix A). However, it is possible to have a teleparallel equivalence of $f(R)$ gravity as a particular subset of models depending on the torsion scalar with a boundary term. There are other modified gravity theories due to the existence of other invariants in the Action integral, namely $f(R, G)$-gravity theory (G is the Gauss-Bonnet invariant scalar), $f(R, T^{(m)})$-gravity theory ($T^{(m)}$ is the trace of the energy-momentum tensor) and so on. In this chapter, we shall consider another modified gravity theory namely $f(T, B)$ theory [148] where as usual T is the invariant of the teleparallel gravity and $B = 2e_{\mu}^{-1}\partial_{\mu}(eT_{\rho}^{\rho\mu})$, the boundary term relating R and T ($T^{\alpha}_{\rho\ \beta}$ is the curvature less Weitzenböck connection and T is the invariant of Weitzenböck connection). The motivation for considering $f(T, B)$ gravity theory is that this general framework contains both $f(R)$ as well as $f(T)$- gravity as special subcases. Though the Ricci scalar is invariant under local Lorentz transformation but not the torsion scalar(T). However, the particular combination $T - B$ is invariant though none separately is invariant under Lorentzian transformation. Further, from $f(T, B)$ gravity theory one may recover the standard $f(R)$ gravity theory as well as the teleparallel equivalent of $f(R)$ gravity (for details see appendix B). Clearly, the field equations of the present modified gravity theory will be highly non-linear and coupled in nature and it is very hard to find any analytic solution from them.

Noether symmetry analysis is a very powerful tool to simplify differential equations to a great extend and some times it is possible even to solve them by identifying a cyclic variable along the symmetry direction. The motivation of the chapter is to examine whether Cosmological investigation has been possible for such complicated modified gravity theory using Noether symmetry analysis. The plan of the chapter is as follows: Section (II) presents a brief overview of $f(T, B)$ gravity theory while a general formulation of Noether symmetry has been described in section (III). Section (IV) deals with present $f(T, B)$ gravity theory with Noether symmetry for classical cosmological solution in $FLRW$ model.

Quantum cosmology has been studied for the present model in section (V). The paper ends with a brief summary in section (VI).

3.2 A brief review of $f(T,B)$-gravity theory:

The gravitational field equations for the present modified gravity theory [148, 149] can be written as

$$16\pi G e \mathcal{T}_a^{\lambda} = 2eh_a^{\lambda}(f_{,B})^{;\mu\nu}g_{\mu\nu} - 2eh_a^{\sigma}(f_{,B})_{;\sigma}{}^{;\lambda} + eBh_a^{\lambda}f_{,B} + 4(eS_a{}^{\mu\lambda})_{,\mu}f_T$$
$$+4e\Big[(f_{,B})_{,\mu} + (f_{,T})_{,\mu}\Big]S_a{}^{\mu\lambda} - 4ef_{,T}T^{\sigma}{}_{\mu a}S_{\sigma}{}^{\lambda\mu} - efh_a^{\lambda}, \tag{3.1}$$

Here $\mathcal{T}_\mu{}^{\nu}$ is the energy-momentum tensor of the matter source under consideration, a comma before an index indicates partial differentiation while a semicolon stands for covariant differentiation and $e_i = h_i^{\mu}(x)\partial_i$ is the vierbein field, defined the Weitzenböck connection as $\hat{\Gamma}^{\delta}_{\mu\nu} = h^{\delta}_{\alpha}\partial_{\mu}h^{\alpha}_{\nu}$. The antisymmetric part of this connection is defined as

$$T^{\rho}_{\alpha\beta} = \hat{\Gamma}^{\rho}_{\beta\alpha} - \hat{\Gamma}^{\rho}_{\alpha\beta} = h^{\rho}_{i}\big(\partial_{\alpha}h^{i}_{\beta} - \partial_{\beta}h^{i}_{\alpha}\big) \tag{3.2}$$

Further, the torsion tensor is defined as

$$S_{\rho}{}^{\alpha\beta} = \frac{1}{2}\big(K^{\alpha\beta}{}_{\rho} + \delta^{\alpha}_{\rho}T^{\delta\beta}{}_{\delta} - \delta^{\beta}_{\rho}T^{\delta\alpha}{}_{\delta}\big) \tag{3.3}$$

with $K^{\alpha\beta}{}_{\rho} = -\frac{1}{2}\big(T^{\alpha\beta}{}_{\rho} - T^{\beta\alpha}{}_{\rho} - T_{\rho}{}^{\alpha\beta}\big)$, the contorsion tensor. One may identify the contorsion tensor as the difference between the Levi-Civita connections in the holonomic and in the non holonomic frame [150] and $e = det(e^{i}_{\alpha}) = \sqrt{-g}$.

The above field equations are of fourth-order in nature and they reduce to gravitational field equations for $f(T)$ teleparallel gravity if f is linear in B i.e.$f_{,BB} = 0$ while $f(R)$ will be recovered if $f(T,B) = \phi(B-T)$

In the context of cosmology the space-time geometry should be homogeneous and isotropic FLRW model having line-element

$$ds^2 = -N^2(t)dt^2 + a^2(t)\big(dx^2 + dy^2 + dz^2\big) \tag{3.4}$$

so that the vierbein fields are given by

$$h^i_\mu(t) = diag\big(N(t), a(t), a(t), a(t)\big) \tag{3.5}$$

with $N(t)$ being the Lapse function and $a(t)$ the scale factor. Then the scalars T and B have the explicit expressions

$$T = -\frac{6}{N^2}H^2, B = -\frac{6}{N^2}\big(\dot{H} + 3H^2 - \frac{\dot{N}}{N}H\big) \tag{3.6}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Now for the comoving observer having $u^\mu = \frac{1}{N}\delta^t_0$ with $u_\mu u^\mu = -1$, the explicit form of the field equations are

$$\frac{f}{2} - \frac{3\dot{a}\dot{f}_{,B}}{aN^2} + \frac{6f_T\dot{a}^2}{a^2N^2} + \frac{3f_B}{N^2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{N}}{aN} + 2\big(\frac{\dot{a}}{a}\big)^2\right) = \rho \tag{3.7}$$

$$\frac{1}{f} + \frac{2\dot{a}\dot{f}_{,B}}{aN^2} + \frac{(3f_{,B} + 2f_{,T})}{N^2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}\dot{N}}{aN} + 2\big(\frac{\dot{a}}{a}\big)^2\right) - \frac{\ddot{f}_{,B}}{N^2} + \frac{\dot{f}_{,B}\dot{N}}{N^3} = -p \tag{3.8}$$

Here ρ and p are the energy density and thermodynamic pressure of the matter field. As the present modified gravity theory is a fourth-order theory similar to $f(R)$ gravity, so it is desirable to introduce Lagrange multipliers so that the order of the field equations reduce at the cost of an increase of the degrees of freedom. Using the definition of T and B for the present $FLRW$ model from equation (3.6) one may write the gravitational action integral using Lagrange multipliers as [148, 149]

$$I = \int dt\left[Nfa^3 - \mu_1\left(T + 6\big(\frac{\dot{a}}{Na}\big)\right) - \mu_2\left(B + \frac{6}{N^2}\big(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{N}}{aN}\big)\right)\right] \tag{3.9}$$

Here μ_1 and μ_2 are the Lagrangian multipliers and for simplicity matter part of the action is not considered. Now from the variation of the action integral with respect to the variables T and B give the estimate of μ_1 and μ_2 as

$$\mu_1 = Na^3\frac{\partial f}{\partial T} \quad and \quad \mu_2 = Na^3\frac{\partial f}{\partial B} \tag{3.10}$$

Thus using (3.10) in equation (3.9) the above action takes the form

$$I = \int dt\left[Nfa^3 - Na^3f_{,T}\big(T + 6\frac{H^2}{N^2}\big) - Na^3f_{,B}\left(B + \frac{6}{N^2}\big(\dot{H} + 3H^2 - H\frac{\dot{N}}{N}\big)\right)\right] \tag{3.11}$$

Now eliminating the second derivative term (i.e. $\dot{H}$) in the action integral by integration by parts the Lagrangian takes the form as

$$L_{TB} = -\frac{6}{N^2} a\dot{a}^2 f_{,T} + \frac{6}{N} a^2 \dot{a} \dot{f_{,B}} + Na^3 \big(f - Tf_{,T} - Bf_{,B}\big) \tag{3.12}$$

So the Euler-Lagrange equations corresponding to the variables a,T and B give the required field equations for $f(T, B)$-gravity while the Euler-Lagrange equation corresponding to the Lapse function gives the constraint equation. Now without loss of generality the present model can be simplified further by choosing $N(t) = 1$ and denoting $\phi = \frac{\partial f}{\partial B}$. So the simple form of the Lagrangian becomes

$$L_{TB} = -6a\dot{a}^2 f_{,T} + 6a^2 \dot{a}\dot{\phi} - a^3 V\big(\phi, T\big) \tag{3.13}$$

with $V\big(\phi, T\big) = Tf_{,T} + Bf_{,B} - f\big(T, B\big)$.

Further, if $f(T, B)$ is assumed to be a homogeneous function of its arguments of degree n then (5.13) further simplifies to

$$L_{TB} = -6a\dot{a}^2 f_{,T} + 6a^2 \dot{a}\dot{\phi} - (n-1)a^3 f(T, B) \tag{3.14}$$

If the explicit form of $f(T, B)$ is chosen as $f(T, B) = C_1 T^n + C_2 B^n$ then $\phi = \frac{\partial f}{\partial B} = C_2 n B^{n-1}$, and the Lagrangian (5.14) further simplifies to

$$L(T, \phi) = -6a\dot{a}^2 C_1 n T^{n-1} + 6a^2 \dot{a}\dot{\phi} - (n-1)a^3 \big[C_1 T^n + C_3 \phi^{\frac{n}{n-1}}\big] \tag{3.15}$$

with C_1, C_3 as arbitrary constants.

3.3 The general formulation of Noether Symmetry approach

The Noether symmetry approach is a powerful tool to simplify differential equations and in some cases it helps us to solve them. The basic idea behind this geometric approach is to identify a symmetry vector in the augmented space. Then by appropriate transformation of the variables in the augmented space. One may identify a cyclic variable so that the differential equations become much simplier in form. Also it may be possible to have an analytic solution in some cases.

According to Noether, the determination of the symmetry vector is characterized by the invariance of the Lagrangian with respect to the Lie derivative [151, 152, 153, 154] along the vector field along the symmetry vector(Noether's 1st theorem) i.e.

$$\mathcal{L}_{\overrightarrow{X}} L = \overrightarrow{X} L = 0 \tag{3.16}$$

where $\overrightarrow{X}$ is the vector field along the symmetry direction. If $L = L\big[q^{\mu}(x^i), \dot{q}^{\mu}(x^i)\big]$ is the point like canonical Lagrangian of the system then contracting the Euler-Lagrange equations with some unknown functions $A^{\alpha}(q^{\mu})$ one obtains (after some algebra) [155, 127]

$$\mathcal{L}_{\overrightarrow{X}} L = \overrightarrow{X} L = A^{\alpha} \frac{\partial L}{\partial q^{\alpha}} + \big(\partial_j A^{\alpha}\big)\big(\frac{\partial L}{\partial \partial_j q^{\alpha}}\big) = \partial_j \big(A^{\alpha} \frac{\partial L}{\partial \partial_j q^{\alpha}}\big) \tag{3.17}$$

where the symmetry vector in the augmented space has the expression

$$\overrightarrow{X} = A^{\alpha} \frac{\partial}{\partial q^{\alpha}} + \big(\partial_j A^{\alpha}\big)\big(\frac{\partial L}{\partial \partial_j q^{\alpha}}\big) \tag{3.18}$$

It is also termed as infinitesimal generators of the symmetry. Thus due to Noether's theorem as $\mathcal{L}_{\overrightarrow{X}} L = 0$ identifies the symmetry vector and consequently there is a conserved current (known as Noether current) [156]

$$I^j = \lambda^{\alpha} \frac{\partial L}{\partial \partial_j q^{\alpha}} \quad , \quad \partial_j I^j = 0 \tag{3.19}$$

Thus Noether symmetry approach not only determine the symmetry vector along which the Lagrangian is invariant but also it identifies the conserved quantities of a physical system. More specifically, the symmetry condition is associated with a constant of motion of the Lagrangian with conserved phase flux along the symmetry direction. Further, integrating the time component of the conserved current over the spatial volume one obtain the conserved charge associate to the symmetry. Also, if the physical system has no explicit time dependence then the energy function (also known as Hamiltonian of the system)

$$E = \dot{q}^{\alpha} \frac{\partial L}{\partial \dot{q}^{\alpha}} - L \tag{3.20}$$

is another constant of motion. It is to be noted that though there is no well defined notion of energy for gravitational field in a given space -time still there is an associated conserved notion of energy provided there is a time-like Killing vector field [157, 158, 159]. Using Cartan one-form it is possible to express the conserved charge as the inner product of the infinitesimal generator with the one-form [160] as

$$Q = i_{\overrightarrow{X}} \theta_L \tag{3.21}$$

where $\theta_L = \frac{\partial L}{\partial \dot{q}^\alpha}\tilde{d}q^\alpha$, is the cartan one form and $i_{\overrightarrow{X}}$ is the notation for the inner product with the vector field $\overrightarrow{X}$. Moreover, to simplify the cosmic evolution equations Noether symmetry vector has an important role. Suppose there is a point transformation : $q^\alpha \longrightarrow Q^\alpha$ in the augmented space so that one of the new variable is along the direction of the symmetry, then one can restrict the transformation as

$$i_{\overrightarrow{X}}dQ^\rho = 1 \quad and \quad i_{\overrightarrow{X}}dQ^\mu = 0\ ,\ \mu \neq \rho \tag{3.22}$$

i.e. the co-ordinate Q^ρ becomes cyclic. Also due to the above transformation the form of the infinitesimal generators becomes

$$\overrightarrow{X}_T = \left(i_{\overrightarrow{X}}dQ^\alpha\right)\frac{\partial}{\partial Q^\alpha} + \left[\frac{d}{dt}\left(i_{\overrightarrow{X}}dQ^\alpha\right)\right]\frac{\partial}{\partial \dot{Q}^\alpha} \tag{3.23}$$

which due to the above restriction (3.22) simplifies to

$$\overrightarrow{X}_T = \frac{\partial}{\partial Q^\rho} \tag{3.24}$$

Thus in the new transformed co-ordinates the Lagrangian as well as the evolution equations simplify to a great extend and sometimes they may be solvable after the transformation.

Finally, the Noether symmetry analysis has a great role in formulation of quantum cosmology and possible solution of the Wheeler-DeWitt (WD) equation.The symmetries in superspace are usually identified by the metric and matter field. However, for simplicity one deals with minisuperspaces by imposing restrictions on geometrodynamics of the superspace. The simplest minisuperspace model is usually homogeneous and isotropic in nature and as a result the Lapse function is a function of 't' alone while the shift vector vanishes identically. Thus the line element can be written in (3+1)-decomposition as

$$ds^2 = -N^2(t)dt^2 + h_{ab}(x,t)dx^a dx^b \tag{3.25}$$

and the Einstein-Hilbert action can be written in the form[161, 162]

$$\mathcal{I}(h_{ab}, N) = \int dt\ d^3x\ N\sqrt{h}\big[K_{ab}K^{ab} - K^2 + \ ^{(3)}R - 2\Lambda\big] \tag{3.26}$$

where K_{ab} denotes the extrinsic curvature with $K = K_{ab}q^{ab}$, the trace of it, $^{(3)}R$ is the usual 3-space curvature and Λ stands for the cosmological constant. Further, the homogeneity of the three space demands the three space metric should correspond to a finite number of functions $u^\mu(t)$,$\mu = 0, 1, ..., (n-1)$ and equation (3.26) becomes

$$\mathcal{I}\big[u^\mu(t), N(t)\big] = \int_0^t Ndt\Big[\frac{1}{2N^2}\ \rho_{\mu\nu}\ \dot{u}^\mu(t)\dot{u}^\nu - V(u)\Big] \tag{3.27}$$

where the metric on the minisuperspace is denoted by $\rho_{\mu\nu}$.One may interpret this action integral for a relativistic point particle moving in nD space (having metric $\rho_{\mu\nu}$) with self interacting potential $V(u)$.Thus the second order differential equation representing the equation of motion of the particle are given by

$$\frac{1}{N}\frac{d}{dt}(\frac{\dot{u}^\mu}{N}) + \frac{1}{N^2}\ \Gamma^\mu_{\rho\sigma}\ \dot{u}^\rho\ \dot{u}^\sigma + \rho^{\mu\nu}\ \frac{\partial V}{\partial q^\nu} = 0 \tag{3.28}$$

It is to be noted that the above equation of motion is restricted by the following constraint equation (obtained by variation of the action w.r.t the lapse function)

$$\frac{1}{2N^2}\ \rho_{\mu\nu}\dot{u}^\mu\dot{u}^\nu + V(u) = 0 \tag{3.29}$$

For quantum cosmology, it is desirable to formulate the Hamiltonian dynamics for which the canonical momenta corresponding to the dynamical variables are

$$p_\mu = \frac{\partial L}{\partial \dot{u}^\mu} = \rho_{\mu\nu}\ \frac{\dot{u}^\nu}{N} \tag{3.30}$$

As a result, the Hamiltonian takes the form

$$\mathcal{H} = p_\mu\dot{u}^\mu - L = N\Big[\frac{1}{2}\ \rho^{\mu\nu}p_\mu p_\nu + V(q)\Big] \equiv N\mathcal{H} \tag{3.31}$$

It is interesting to note that the above form of $\mathcal{H}$ is nothing but the constraint equation (3.29) in the momentum formulation. Now for quantization scheme one has to make the operator conversion $p_\mu \longrightarrow -i\frac{\partial}{\partial q^\mu}$ and as a result the above Hamiltonian becomes a second order hyperbolic partial differential equation

$$\mathcal{H}(u^\mu - i\frac{\partial}{\partial u^\mu})\psi(u^\mu) = 0 \tag{3.32}$$

In quantum cosmology, it is termed as Wheeler-DeWitt (WD) equation with $\psi(u^\mu)$ is called the wave function of the Universe. However, in formulation of the WD equation one has to encounter an ambiguity associated with factor ordering which may be eliminated demanding the quantization scheme to be covariant in nature i.e, invariant under the transformation $u^\mu \longrightarrow u^{-\mu}(u^\mu)$. Now due to hyperbolic nature of the WD equation, one may have probability measure having conserved probability current as

$$\vec{J} = \frac{i}{2}\left(\psi^*\Delta\psi - \psi\Delta\psi^*\right) \tag{3.33}$$

with $\vec{\Delta}.\vec{J} = 0$, Hence one may define the probability measure as

$$dp = |\psi(u^\mu)|^2 \; dV \tag{3.34}$$

with dV, a volume element in minisuperspace.

As Hamiltonian is very useful in formation of Wheeler-DeWitt (WD) equation so in minisuperspace models of quantum cosmology symmetry analysis may appropriately interpret the wave function of the Universe. Due to the cyclic nature of the co-ordinate Q^ρ in the transformed variables, the conserved canonically conjugate momentum [162] has the expression

$$p_\rho = \frac{\partial L}{\partial \dot{Q}^\rho} = \Sigma_\rho \;, \quad \text{a constant} \tag{3.35}$$

So in quantization program the operator version of this conserved momentum is a first order differential equation as

$$-i\partial_Q^\rho|\psi >= \sigma_\rho|\psi > \;, \tag{3.36}$$

which has the oscillatory solution of the form $e^{(i\Sigma_\rho Q^\rho)}$ and hence the solution of the WD equation i.e. the wave function of the Universe can be written as

$$|\psi >= e^{i\Sigma_\rho Q^\rho}|\Phi(Q^\delta) > \tag{3.37}$$

where $\delta(\neq \rho)$ runs over 1 to $n-1$ (n is the dimension of the minisuperspace). Also δ represents the directions along which there is no symmetry. So the oscilatory part of the wave function is associated with some conserved momentum due to Noether symmetry. Hence the oscilatory part of the wave function may indicate the existence of Noether symmetry. Lastly, according to Hartle [163] the Noether symmetries are associated with the classical trajectories (in minisuperspace) which are nothing but the solutions to the classical field equations. Therefore, Noether symmetry can be imagine to be a bridge connecting the classical observable universe to quantum cosmology.

3.4 Noether Symmetry in $f(T, B)$ Cosmology: Classical solution

The present particular $f(T, B)$ model has $3D$ configuration space (a, ϕ, T) and the Lagrangian is given by equation (3.15). So the infinitesimal generator of the Noether symmetry has the expression

$$\overrightarrow{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \gamma\frac{\partial}{\partial T} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}} + \dot{\gamma}\frac{\partial}{\partial \dot{T}} \tag{3.38}$$

with $\alpha = \alpha(a, \phi, T)$, $\beta = \beta(a, \phi, T)$, $\gamma = \gamma(a, \phi, T)$, the coefficients of the generator and $\dot{\alpha} = \frac{\partial \alpha}{\partial a}\dot{a} + \frac{\partial \alpha}{\partial \phi}\dot{\phi} + \frac{\partial \alpha}{\partial T}\dot{T}$, and so on.

Due to Noether's first theorem

$$\mathcal{L}_{\overrightarrow{X}} L = \overrightarrow{X} L = 0 \tag{3.39}$$

Now equating the coefficients of $\dot{a}^2, \dot{\phi}^2, \dot{T}^2, \dot{a}\dot{\phi}, \dot{a}\dot{T}$ *and* $\dot{\phi}\dot{T}$ in the above symmetry condition one gets the following system of 1st order partial differentials as

$$-6\alpha C_1 n T^{n-1} - 6\gamma a C_1 n(n-1)T^{n-2} - 12aC_1 nT^{n-1}\frac{\partial \alpha}{\partial a} + 6a^2\frac{\partial \beta}{\partial a} = 0 \tag{3.40}$$

$$6a^2\frac{\partial \alpha}{\partial \phi} = 0 \tag{3.41}$$

$$12\alpha a - 12aC_1 nT^{n-1}\frac{\partial \alpha}{\partial \phi} + 6a^2\frac{\partial \alpha}{\partial a} + 6a^2\frac{\partial \beta}{\partial \phi} = 0 \tag{3.42}$$

$$-12aC_1 nT^{n-1}\frac{\partial \alpha}{\partial T} + 6a^2\frac{\partial \beta}{\partial T} = 0 \tag{3.43}$$

$$6a^2\frac{\partial \alpha}{\partial T} = 0 \tag{3.44}$$

$$-3a^2\alpha(n-1)[C_1 T^n + C_3\phi^{\frac{n}{n-1}}] - C_3 a^3\beta n\phi^{\frac{1}{n-1}} - C_1 a^3\gamma n(n-1)T^{n-1} = 0 \tag{3.45}$$

Using the technique of separation of variables of the form $\alpha = \alpha_0\alpha_1(a)\alpha_2(\phi)\alpha_3(T)$, $\beta = \beta_0\beta_1(a)\beta_2(\phi)\beta_3(T)$ *ans* $\gamma = \gamma_0\gamma_1(a)\gamma_2(\phi)\gamma_3(T)$ the above set of partial differential equations has the explicit solution

$$\alpha = \alpha_0 a^{-2} \ , \ \beta = \beta_0 \ , \ \gamma = -6\alpha_0 a^{-3} T \tag{3.46}$$

where α_0 , is an arbitrary integration constant and for consistency of the above partial differential equation one has to constraint $n = \frac{1}{2}$ and $C_3 = 0$. Usually, for a field theory in curved background there is no definite notion of energy for gravitational field, however, there is an associated conserved energy if the space-time admits time-like Killing vector field.Though for $FLRW$ model there is no time-like Killing vector field yet one may have an associated energy function due to explicit independence of time variable in the Lagrangian.

The explicit form of the conserved charge and conserved energy for the present model has the explicit form

$$Q = 6\alpha_0[-C_1\frac{\dot{a}}{a}T^{-\frac{1}{2}} + \dot{\phi}] + 6\beta_0 a^2\dot{a} \tag{3.47}$$

$$E = -3C_1 a\dot{a}^2 T^{-\frac{1}{2}} + 6a^2\dot{a}^2 - \frac{C_1}{2}a^3 T^{-\frac{1}{2}} \tag{3.48}$$

In order to simplify the Lagrangian as well as the evolution equations a point transformation $(a, \phi, T) \longrightarrow (u, v, w)$ in the configuration space is done using the condition (3.22) so that one of the new variables (*say* u) becomes cyclic. The 1st order partial differential equations due to equation (3.22) can be solved using Lagrange's method to give,

$$a^3 = 3\alpha_0 u \quad , \quad \phi = \beta_0(u - v) \quad and \quad T = \frac{e^w u^{-2}}{9\alpha_0^2} \tag{3.49}$$

Thus the Lagrangian in equation (3.15) ($n = \frac{1}{2}$ *and* $C_3 = 0$) in terms of the new variables takes the form

$$L = -3C_1{\alpha_0}^2 e^{-\frac{w}{2}}\dot{u}^2 + 6\alpha_0\beta_0(\dot{u}^2 - \dot{u}\dot{v}) + \frac{C_1}{2}e^{\frac{w}{2}} \tag{3.50}$$

Now solving the Euler-Lagrange equations the explicit solution both in new and in old variables has the form

$$u = \frac{B}{6\alpha_0\beta_0}t + C \quad , \quad v = Nt + D \quad , \quad w = ln(M) \tag{3.51}$$

and

$$a = \left(\frac{B}{2\beta_0}t + 3\alpha_0 C\right)^{\frac{1}{3}} \quad , \quad \phi = \beta_0[(\frac{B}{6\alpha_0\beta_0} - N)t + C - D] \quad , \quad T = \frac{M}{9{\alpha_0}^2}(\frac{B}{6\alpha_0\beta_0}t + C)^{-2} \tag{3.52}$$

where $M = -\frac{B^2}{6\alpha_0{\beta_0}^2}$, $N = \frac{1}{6\alpha_0\beta_0}\left(C_1\sqrt{6}{\alpha_0}^{\frac{3}{2}}{\beta_0}^{-\frac{1}{2}} + 2B - A\right)$ and A , B , C , D are arbitrary constant.

The cosmological parameters namely the Hubble parameter and acceleration and the scale factor a have been plotted in FIG.3.2, FIG.3.3 and FIG.3.1 respectively.

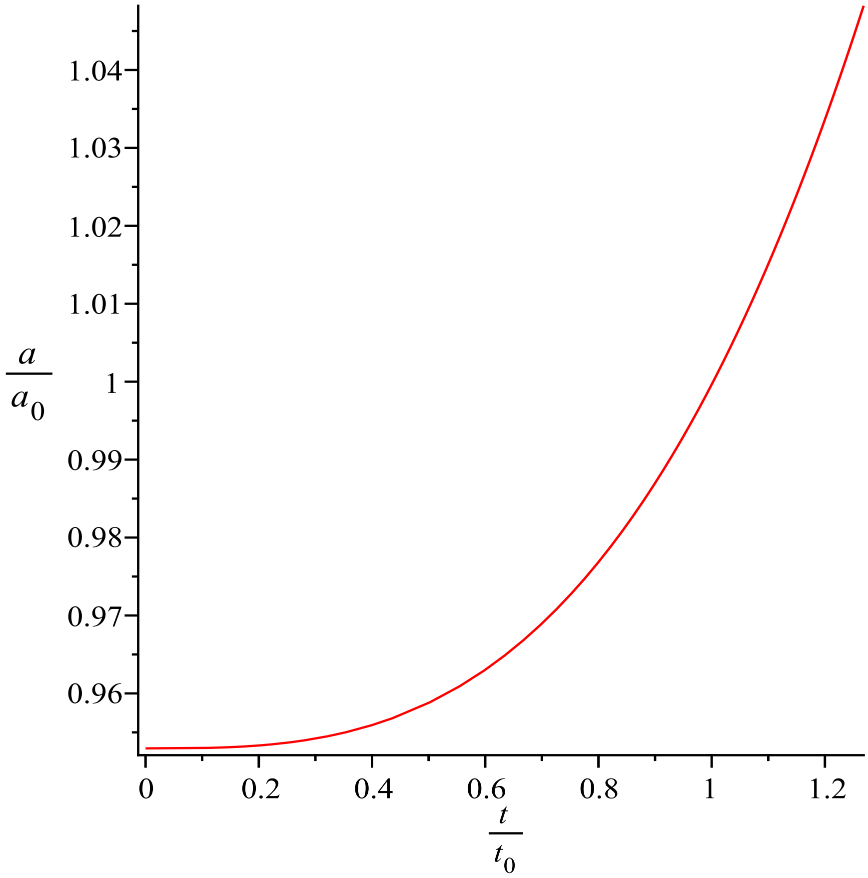

FIGURE 3.1: Graphical representation of the scale factor with respect to dimensionless cosmic time $\frac{t}{t_0}$ with $t_0 = 0.104938$ ($B = 6$, $\alpha_0 = 1$, $\beta_0 = 1$, $C = -0.0333$)

3.5 wheeler Dewitt Equation and Wave Function of the Universe : Aspects of Quantum Cosmology

In the 3D configuration space with new variables (u, v, w) the Lagrangian is given by equation (3.50). So the canonically conjugate momenta associated the model are given by

$$p_u = (-6C_1{\alpha_0}^2 e^{-\frac{w}{2}} + 12\alpha_0\beta_0)\dot{u} - 6\alpha_0\beta_0\dot{v} \tag{3.53}$$

$$p_v = -6\alpha_0\beta_0\dot{u} \tag{3.54}$$

$$p_w = 0 \tag{3.55}$$

and consequently the Hamiltonian of the system is given by

$$\mathcal{H} = Rp_u^2 - Sp_up_v + N_0 \tag{3.56}$$

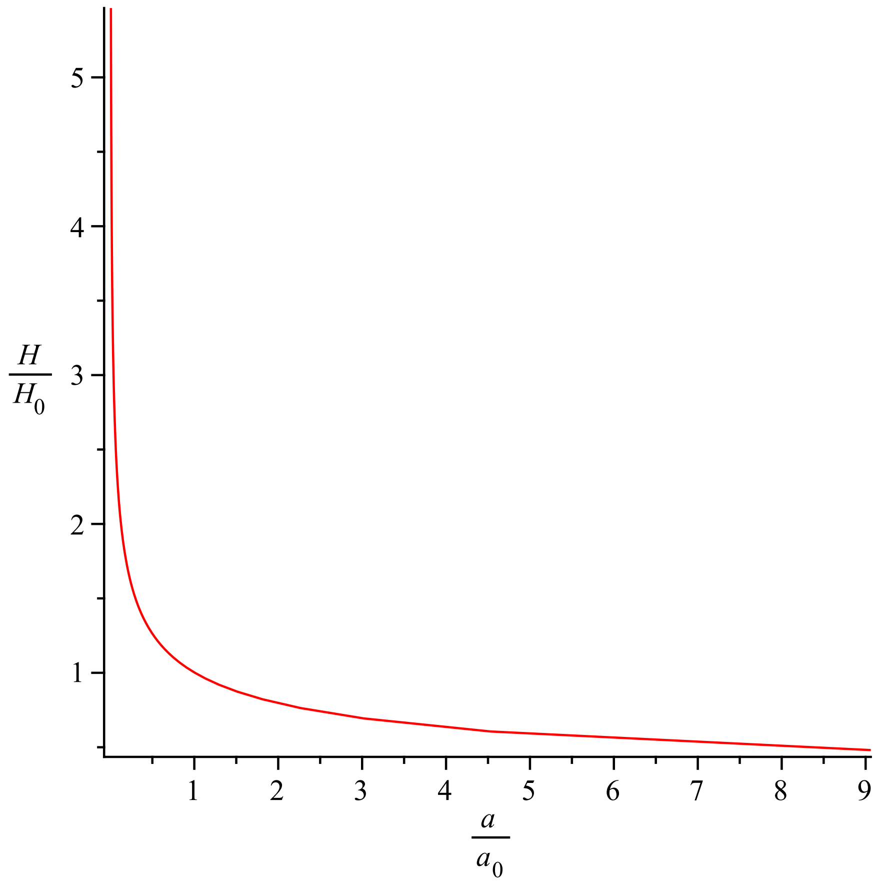

FIGURE 3.2: Graphical representation of the dimensionless Hubble parameter $\frac{H}{H_0}$ with respect to the dimensionless scale factor $\frac{a}{a_0}$ where $H_0 = 67.5037$

with $R = \frac{(3C_1\alpha_0^2 e^{-\frac{w}{2}} - 6\alpha_0\beta_0)}{(6\alpha_0\beta_0)^2}$, $S = \frac{1}{6\alpha_0\beta_0}$ and $N_0 = \frac{\alpha_0 C_1 e^{\frac{w}{2}}}{2}$

In quantum cosmology the basic equation is the Wheeler-DeWitt (WD) equation which is a second order hyperbolic partial differential equation and its solution is termed as the wave function of the Universe. The WD equation is nothing but the operator version of the the above Hamiltonian i.e.,$\hat{H}|\psi(u, v, w) >= 0$, with $\hat{H}$ the operator version and $|\psi >$ the wave function. It is to be noted that one has to take care of the operator ordering issue during conversion to the operators for the WD equation. The rule of conversion for the present model is $p_u \longrightarrow -i\frac{\partial}{\partial u}$, $p_v \longrightarrow -i\frac{\partial}{\partial v}$ and $p_w \longrightarrow -i\frac{\partial}{\partial w}$. So the explicit form of the WD equation is

$$\hat{H}|\psi >\equiv \left[-R\frac{\partial^2}{\partial v^2} + S\frac{\partial^2}{\partial u \partial v} + N_0\right] |\psi >= 0 \tag{3.57}$$

Note that here due to absence of u or v in the coefficients R and S no factor ordering

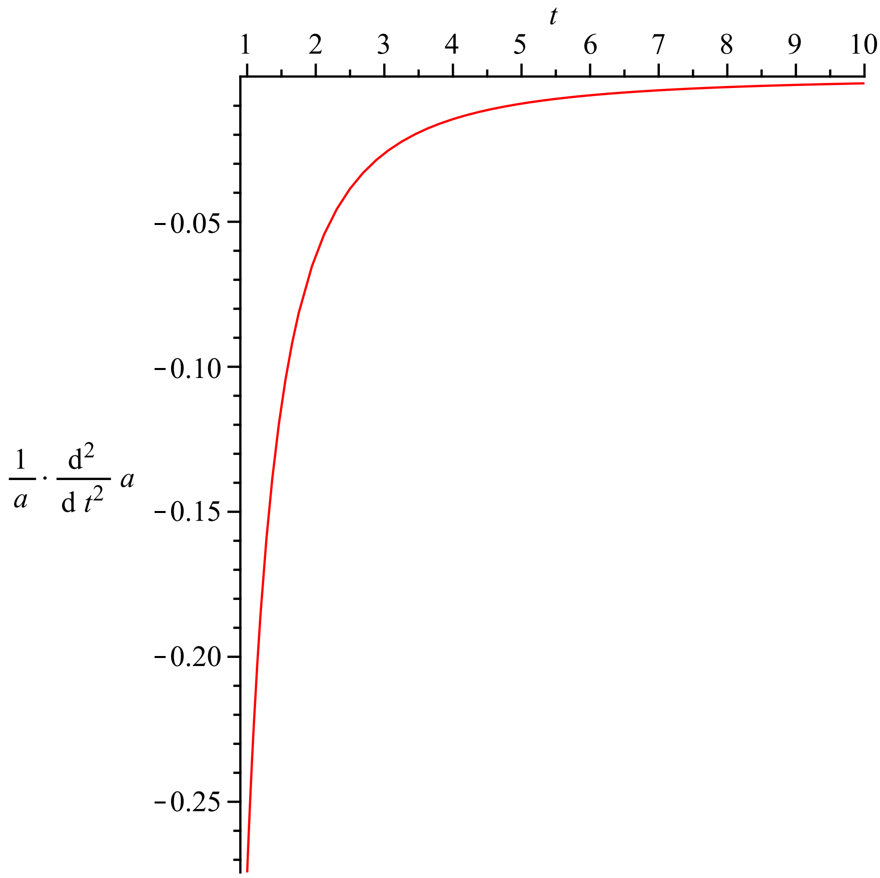

FIGURE 3.3: Graphical representation of $\frac{\ddot{a}}{a}$ with respect to the cosmic time t.

problem has arised. Now due to cyclic nature of the variable u, the associated momentum will be conserved i.e., $p_u = \Sigma_u$,a constant. As a result, the operator version of this conserved momentum equation takes the first order linear differential equation as

$$-i\frac{\partial}{\partial u}|\psi>= \Sigma_0|\psi> \tag{3.58}$$

i.e.,

$$|\psi>= exp(i\Sigma_0 u)|\phi(v)> \tag{3.59}$$

with $|\phi(v)>$ satisfying the second order ordinary differential equation as

$$R\frac{d^2|\phi(v)>}{dv^2} - i\Sigma_0 S\frac{d|\phi(v)>}{dv} - T|\phi(v)>= 0 \tag{3.60}$$

Hence the solution of equation (3.60) is

$$|\phi(v)>= e^{\frac{i\Sigma_0 S}{2R}v}\left[c_1 e^{\frac{\sqrt{4N_0 R-\Sigma_0^2 S^2}}{2R}v} + c_2 e^{-\frac{\sqrt{4N_0 R-\Sigma_0^2 S^2}}{2R}v}\right] \tag{3.61}$$

where c_1 and c_2 are arbitrary constants.

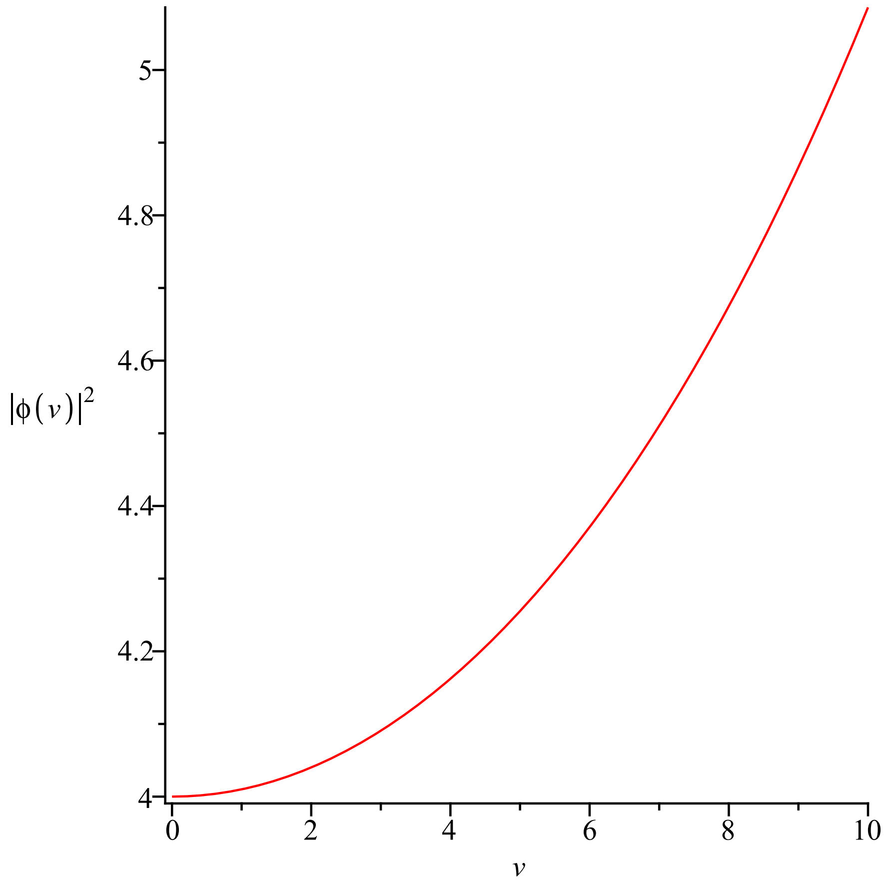

FIGURE 3.4: The graph represents square of the absolute value of the the wave function (non-oscillation part) when $c_1^2 = c_2^2$ ($\Sigma_o = 2$, $R = S = 1$, $N_0 = 1.0025$ *and* $c_1 = c_2 = 1$).

In figures 4 and 5 the probability measure has been plotted against the variable v. From the figures it is evident that there is a finite probability for $v = 0$. However, from equation (3.49) (i.e., the inter relation between the old and new variables) it is clear that $v = 0$ does not imply volume to be zero (i.e., $u = o$). Also the independence of $||\psi>|^2$ on u implies that there is a finite non-zero probability at zero volume. Hence quantum process does not avoid the classical singularity.

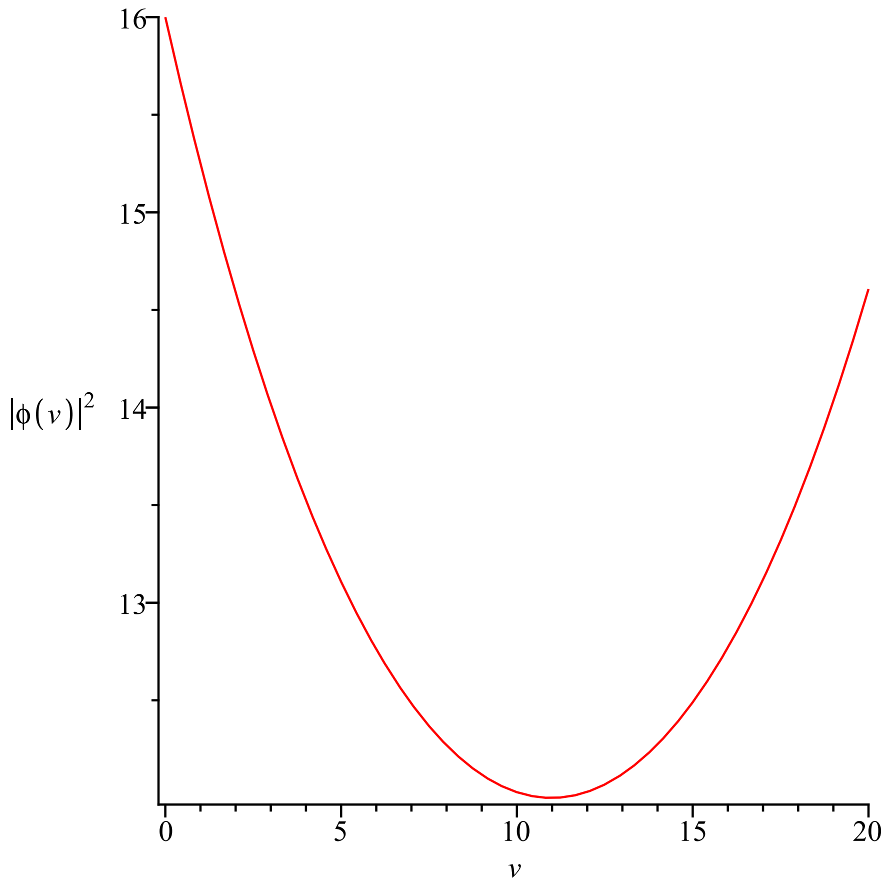

FIGURE 3.5: The graph represents the the variation of $||\psi>|^2$ when $c_1 \neq c_2$ ($\Sigma_o = 2$, $R = S = 1$, $N_0 = 1.0025$, $c_1 = 1$ *and* $c_2 = 3$).

3.6 A brief Summary

The present work deals with a very complicated modified gravity theory where the field equations are of fourth order. However, they are reduced to second order differential equations by using the technique of Lagrange's multiplier, still these second order differential equations are so coupled and non-linear in nature that one can not solve them with usual techniques. Noether symmetry analysis has been used not only to simplify the evolution equations but also it is possible to have the classical cosmological solution. The relevant cosmological parameters corresponding to this solution has been presented graphically in figures (3.1 − 3.3). From the behaviour of these parameters one may conclude that the present cosmological solution describes an expanding model of the universe but only in the matter dominated phase and it is analogous to the stiff matter solution in general relativity. As the parameter 'n' automatically has the value $n = \frac{1}{2}$ due to Noether symmetry analysis so any other choice of 'n' will not be consistent from the view point of symmetry analysis. Hence it is not possible to implement a realistic model covering the different phases of the cosmic history. Further to address the inherent singularity of the model, quantum cosmology has been formulated. The conserved momentum corresponding to Noether symmetry is found to have a significant role not only in solving the WD equation but also to identify the periodic part of the wave function. Finally, the quantum formulation of the present classical cosmological model does not avoid the inherent singularity. lastly, one may conclude that though the present cosmological model may not be of much interest from observational point of view but the present work shows a path how to deal with complicated physical models using the tool of Noether symmetry analysis.

Appendix

A. Lorentz invariance of the present gravity theory

The general field equations (3.1) can be rewritten in a covariant form using the Einstein tensor and the metric.For this we write the Ricci tensor in terms of contorsion tensor as

$$R_{\mu\nu} = \nabla_\nu K_\lambda{}^\lambda{}_\mu - \nabla_\lambda K_\nu{}^\lambda{}_\mu + K_\lambda{}^\rho{}_\mu K_\nu{}^\lambda{}_\rho - K_\lambda{}^\lambda{}_\rho K_\nu{}^\rho{}_\mu \tag{3.62}$$

i.e.

$$R^\lambda_\nu = \frac{1}{e}\Big(\partial_\sigma\big(eK_\nu{}^{\lambda\sigma}\big) + \partial_\nu\big(eT^\lambda\big)\Big) - 2S_\sigma{}^{\lambda\mu}W_\nu{}^\sigma{}_\mu \tag{3.63}$$

Using (3.63) the field equations (3.1) takes the form

$$\begin{aligned} 16\pi G e T^\lambda_\nu \;=\; & 2e\delta^\lambda_\nu \Box f_B - 2e\nabla^\lambda\nabla_\nu f_B + eBf_B\delta^\lambda_\nu + he\Big[\big(f_{BB}+f_{BT}\big)\big(\partial_\mu B\big) + \\ & \big(f_{TT}Tf_{BT}\big)\big(\partial_\mu T\big)\Big]S_\nu{}^{\mu\lambda} + h\, e^a_\nu\, \partial\mu\big(e\, S_a{}^{\mu\lambda}\big)f_T - 4ef_T T^\sigma{}_{\mu\nu}\, S_\sigma{}^{\lambda\mu} - ef\delta^\lambda_\nu \end{aligned} \tag{3.64}$$

This field equation can be expressed in terms of curvature scalar and Einstein tensor as

$$\begin{aligned} \Pi_{\mu\nu} \;\equiv\; & -f_T\, G_{\mu\nu} + g_{\mu\nu}\, \Box f_B - \nabla_\mu\nabla_\nu f_B + \frac{1}{2}\big(Bf_B + Tf_T - f\big)\, g_{\mu\nu} \\ & +2\Big[\big(f_{BB}+f_{BT}\big)\big(\nabla_\lambda B\big) + \big(f_{TT}+f_{BT}\big)\big(\nabla_\lambda T\big)\Big]S_\nu{}^\lambda{}_\mu \\ & = 8\pi G T_{\mu\nu} \end{aligned} \tag{3.65}$$

With $R^\mu_\nu \equiv G^\mu_\nu + \frac{1}{2}(B-T)\delta^\mu{}_\nu$ and $R = -T + B = -T + 2\partial_\mu T^\mu$.

Now if f is chosen to be independent of the boundary term i.e. $f = f(T)$ then the above field equations reduce to the usual $f(T)$-gravity. Though the reduced field equations for $f(T)$ gravity are not in general invariant under local Lorentz transformation yet they are manifestly covariant. In principle, torsion is an essential ingredient of the $f(T)$ theories and it may be Lorentz invariant. In fact, in its classical pure tetrad formulation (without spin connection) $f(T)$ violates the local Lorentz symmetry still by using an appropriate spin connection it can be brought to a formally Lorentz invariant shape. In particular, a kind of stuckelberg trick can be used to restore the invariance. The equation for the spin connection is simply the antisymmetric part of the equation for the tetrad and the spin connection itself can always be made equal to zero by a gauge choice (for details see the book "Teleparallel gravity: an introduction", Springer, 2013 by R. Aldrovandi and J.G. Pereira). As a result the coefficient of $S_{\nu}{}^{\lambda}{}_{\mu}$ should be identically Zero. i.e

$$f_{BB} + f_{BT} = 0 \quad and \quad f_{TT} + f_{BT} = 0 \tag{3.66}$$

The general solution can be written as

$$f(T, B) = g(R) + C_1 B \tag{3.67}$$

with C_1 as arbitrary constant. Due to invariance of the field equations with a linear term in 'B' (a total derivative term) one may choose $C_1 = 0$ without any loss of generality.Thus essentially the present model reduces to $f(R)$-gravity which is manifestly Lorentz invariant.Therefore, one may conclude that the teleparallel equivalent of $f(R)$ gravity is the only possible Lorentz-invariant gravity theory but the field equations contain higher-order derivative terms.

B. $f(T)$ and $f(R)$ gravity theory from the present $f(T, B)$-gravity model:

$f(T)$ gravity: If the function f is such that $f_B = 0$ i.e. $f(T, B) = f(T)$ then the field equations (3.1) simplifies to

$$16\Pi G e T_{\nu}^{\lambda} = 4e\Big[f_{TT}\big(\partial_{\mu}T\big)\Big]S_{\nu}{}^{\mu\lambda} + 4e_{\nu}^{a}\partial_{\mu}\big(eS_{\mu}{}^{\mu\lambda}\big)f_T - 4ef_T T^{\sigma}_{\mu\nu}S_{\sigma}{}^{\lambda\mu} - ef\delta_{\nu}^{\lambda} \, , \tag{3.68}$$

which is nothing but the field equations for $f(T)$-gravity. It is to be noted that the above field equations are second order field equations and they remain same if f_B is a non-zero constant i.e. f depends on B linearly.

$f(R)$ **gravity:** To reduce the $f(T,B)$-gravity model into $f(R)$ gravity theory, it is useful to consider f as

$$f(T,B) = f(-T+B) = f(R) \tag{3.69}$$

so that

$$F(R) = f'(-T+B) = -f_T + f_B \tag{3.70}$$

This choice of the function 'f' reduces the general field equations (3.1) in the simple form as

$$\begin{aligned} 16\Pi K e\, T_\nu^\lambda &= 2e\, \delta_\nu^\lambda \Box\, F - 2e\nabla^\lambda \nabla_\nu F + eBF\, \delta_\nu^\lambda - 4e^a{}_\nu\, \partial_\mu \big(e\, S_a{}^{\mu\lambda}\big) F \\ &\quad + 4eF\, T^\sigma{}_{\mu\nu} S_\sigma{}^{\lambda\mu} - ef \delta^\lambda{}_\nu \end{aligned} \tag{3.71}$$

These field equations are nothing but the field equations for teleparallel equivalent of $f(R)$-gravity.However, using the relation $\nabla_\mu V^\mu = \frac{1}{e}\, \partial_\mu \big(eV^\mu\big)$ in the teleparallel frame work (∇_μ stands for the covariant derivative of the Levi-Civita connection) and using the relations (3.62) and (3.63) for Ricci-tensor and Ricci scalar the field equations (3.71) becomes the standard field equations for $f(R)$ as

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + g_{\mu\nu}\, \Box F(R) - \nabla_\mu \nabla_\nu F = 8\pi G T_{\mu\nu} \tag{3.72}$$

Therefore, by suitable choice of the $f(T,B)$ function it is possible to have $f(T)$-gravity or teleparallel equivalent of $f(R)$-gravity.

Chapter 4

Symmetry analysis in multi scalar-torsion cosmological model with quantum description

4.1 Prelude

The history of cosmic evolution starting from big bang scenario shows that so far the Universe undergoes two phase transitions: early inflationary era $\longrightarrow$ matter dominated epoch $\longrightarrow$ present late time accelerated expansion phase. It is interesting to note that both the accelerated expansion phases have the common feature that matter component responsible for these eras have negative pressure components. So it is reasonable to think that both these scenarios may have the same underlying physical features, controlling the evolution at two distinct phases. Hence it is interesting to have a unified theory which may describe these two phases, having a huge difference in energy scales.

Usually, in inflationary era the common choice for the matter field is the scalar field having non-zero mass. There are various choices for the potential of the scalar field [164, 165, 166, 167, 168, 169, 170, 171], (for a detailed description see the review [172]) which in the slow-roll regime describes the accelerated expansion. Though cosmological constant is best fitted with observational data as a candidate for dark energy (DE) yet it is not so popular due to its conceptual discrepancies. The quintessence model [172, 167] is a very popular candidate for dynamical DE. According to this model, an adjusting or tracker scalar field [48], during rolling down the potential field ,is able to accelerate the Universe

[31, 173, 174, 175, 176, 177, 178, 179]. Further, another interesting feature for this DE model is to describe a unified model for the various DE components of the Universe [180, 181], for example it can provide the well known DE candidate namely, the Chaplygin gas fluids [182, 170].

Subsequently, cosmologists opted for other scalar field models of which a common one is the phantom scalar field [183, 184, 185] having negative energy density and it is possible to have the equation of state parameter $w \leq -1$. As a result, the rate of accelerated expansion is much larger than the exponential growth due to de Sitter inflation and the Universe encounters another (end) singularity termed as big-rip singularity [184]. Subsequently, cosmologists have opted for interacting scalar field models namely (a) scalar field interacts with other components of matter in the Universe (known as chameleon theory) [186], (b) nonminimally coupled scalar field models (i.e., scalar fields interacts with gravity through the gravitational action) [187, 188, 189, 190, 191] and also multi scalar field models [192, 193, 194, 57] (due to extra degrees of freedom these models may describe more than one era of evolution).

On the otherhand, another group of cosmologists prefer modified gravity theories as an alternative to DE to explain this late time accelerated expansion. The extra geometric invariants in the Einstein-Hilbert action provide additional geometric terms in the field equations and as a result , it is possible to have the required accelerated expansion. However, from the point of view of a general (metric independent) formulation of gravity theory, a general connection (note that the Ricci scalar R in the usual Einstein gravity is constructed using the Levi-civita connection) may be associated with a non-metricity scalars Q and the torsion scalar T. In particular, for the antisymmetric Weitzenböck connection as the fundamental connection of physical space [195, 196], the torsion scalar T is considered as the gravitational Lagrangian density and one gets the teleparallel equivalent of general relativity. Subsequently, a multi scalar field cosmology has been developed in the teleparallelism framework [197, 198] in which one of the scalar field is coupled to the torsion scalar (T) with a diatonic coupling function and another scalar field is minimally coupled to gravity, with non-vanishing interaction between the two fields.The present work deals with Noether symmetry analysis of the multi scalar field torsion gravity in the background of flat FLRW space-time as well as symmetry of the kinematic metric. Also an explicit study of quantum cosmology i.e., solution of the Wheeler DeWitt equation as well as quantum Bohiman trajectories has been analyzed. The plan of the chapter is as follows: Section II deals with a brief study of the present physical model in FLRW background. An overall Noether symmetry analysis has been given in section III. In section IV , an explicit derivation of the Noether symmetry vector for the present study has been shown. A detailed conformal symmetry of the physical metric has been described in section V. Canonical quantization scheme for quantum cosmology has been formulated in section VI. Section VII shows casual interpretation and Bohiman trajectory for the present model. The chapter ends with a brief summary in section VIII.

4.2 A brief study of multi scalar-torsion gravity and FLRW space time

In the framework of teleparallel theory of gravity, the action integral for multi scalar field gravitaional model can be written as [70, 199]

$$\mathcal{A}=\frac{1}{16\pi G}\int d^4x\; e\left(\mu(\phi)\Big[T+\frac{\delta}{2}\phi_{;\mu}\phi^{;\mu}+V(\phi)+\frac{\lambda}{2}\psi_{;\mu}\psi^{;\mu}\Big]+U(\phi,\psi)\right) \tag{4.1}$$

Here the scalar field ϕ is non-minimally coupled to the torsion scalar T through the coupling function $\mu(\phi)$, δ and λ stand for the coupling parameters between gravity and the scalar fields ϕ and ψ respectively and $V(\phi)$ is the potential function of the scalar field ϕ. The other potential function $U(\phi,\psi)$ stands for interaction between the two scalar fields. The sign of the coupling parameters δ and λ characterize the nature of the scalar fields.

In teleparallel gravity theory the torsion scalar T is related to the antisymmetric Weitzenb$\ddot{o}$ck connection $\Gamma^{a}{}_{bc}$ as

$$T=S_a{}^{bc}\;T^{a}_{\;bc} \tag{4.2}$$

where the torsion tensor $T^{a}_{\;bc}$ is defined as

$$T^{a}_{\;bc}=\Gamma^{a}{}_{cb}-\Gamma^{a}{}_{bc} \tag{4.3}$$

with

$$S_a{}^{bc}=\frac{1}{2}\left(K^{bc}_{\;\;a}-\delta_a{}^{b}T_d{}^{dc}+\delta^c_aT_d{}^{db}\right) \tag{4.4}$$

The contortion tensor is defined as

$$K^{a}_{\;bc}=\frac{1}{2}\left(T_b{}^a{}_c-T_c{}^a{}_b-T^{a}_{\;bc}\right) \tag{4.5}$$

The vierbein fields ${h^i}_a$ and metric tensor g_{ab} are related to the Weitzenb$\ddot{o}$ck connection ${\Gamma^a}_{bc}$ by the relations [200]

$$e_i = {h^a}_i\ \partial_a\ ,\ g_{ab} = \eta_{ij} {h^i}_a {h^j}_b \tag{4.6}$$

with $\Gamma^a_{bc} = {h_i}^a \partial_b\ h^i_c + {w^i}_{cd}\ h^d_b\ h^a_i$ and ${w^i}_{ab}$ is the spin connection.

The $4D$ space-time geometry is chosen as homogeneous and isotropic flat FLRW model with line element

$$ds^2 = -dt^2 + a^2(t)\big(dx^2 + dy^2 + dz^2\big) \tag{4.7}$$

where $a(t)$ is the scale factor. As a result, the vierbein fields take the diagonal form [197]

$${h^i}_a(t) = diag\big[1, a(t), a(t), a(t)\big] \tag{4.8}$$

while spin connections vanish identically for the above cartesian co-ordinates. So the torsion scalar simplifies to $T = 6H^2$ with $H = \frac{\dot{a}}{a}$, the Hubble parameter. Thus the Lagrangian for this simplified space-time geometry corresponding to the action integral (4.1) can be expressed in point-like form as

$$\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}, \psi, \dot{\psi}) = \mu(\phi)\Big[6a\dot{a}^2 + a^3\Big(\frac{\delta}{2}\dot{\phi}^2 + \frac{\lambda}{2}\dot{\psi}^2\Big) - a^3 F(\phi, \psi)\Big] \tag{4.9}$$

So the modified Einstein field equations are the Euler-Lagrange equations (i.e. $\frac{d}{dt}\frac{\partial L}{\partial \dot{\sigma}} - \frac{\partial L}{\partial \sigma} = 0,\ \sigma = a, \phi, \psi$) of the above Lagrangian, having explicit form

$$\left.\begin{array}{lcl}
6H^2 + \frac{\delta}{2}\dot{\phi}^2 + \frac{\lambda}{2}\dot{\psi}^2 + F(\phi, \psi) & = & 0 \\
\dot{H} + \frac{3}{2}H^2 - \frac{1}{4}\Big(\frac{\delta}{2}\dot{\phi}^2 + \frac{\lambda}{2}\dot{\psi}^2 + F(\phi, \psi)\Big) + H\dot{\phi}\ \frac{\mu'(\phi)}{\mu(\phi)} & = & 0 \\
\ddot{\phi} + 3H\dot{\phi} + \frac{1}{\delta}\ \frac{\mu'(\phi)}{\mu(\phi)}\Big(\frac{\delta}{2}\dot{\phi}^2 - \frac{\lambda}{2}\dot{\psi}^2 - 6H^2 - F(\phi, \psi)\Big) - \frac{1}{\delta}\frac{\partial F}{\partial \phi} & = & 0 \\
\ddot{\psi} + 3H\dot{\psi} + \frac{\mu'(\phi)}{\mu(\phi)}\dot{\phi}\dot{\psi} - \frac{1}{\lambda}\frac{\partial F}{\partial \psi} & = & 0
\end{array}\right\} \tag{4.10}$$

with $F(\phi, \psi) = -U(\phi, \psi)\mu(\phi) - V(\phi)\mu(\phi)$

Note that the first equation in the set of equations (4.10) is the constraint equation while the other three equations are the evolution equations for the scale factor 'a' and the scalar fields ϕ ans ψ respectively. Further, the first two equations in (4.10) are the modified Friedmann equations as

$$3H^2 = \rho_{eff}\ and\ 2\dot{H} + 3H^2 = -p_{eff} \tag{4.11}$$

with

$$\rho_{eff} = \frac{1}{2}\Big[F(\phi,\psi) - \frac{\delta}{2}\dot{\phi}^2 - \frac{\lambda}{2}\dot{\psi}^2\Big] \ \textit{and} \ p_{eff} = 2H\dot{\phi}\,\frac{\mu'(\phi)}{\mu(\phi)} - \frac{1}{2}\Big(\frac{\delta}{2}\dot{\phi}^2 + \frac{\lambda}{2}\dot{\psi}^2 + F(\phi,\psi)\Big) \quad (4.12)$$

So the effective equation of state parameter has the complicated expression

$$w_{eff} = -1 + \frac{2\Big[2H\dot{\phi}\,\frac{\mu'(\phi)}{\mu(\phi)} - \frac{\delta}{2}\dot{\phi}^2 - \frac{\lambda}{2}\dot{\psi}^2\Big]}{\Big[F(\phi,\psi) - \frac{\delta}{2}\dot{\phi}^2 - \frac{\lambda}{2}\dot{\psi}^2\Big]} \quad (4.13)$$

Now in the limit $\dot{\phi} \longrightarrow 0$ (i.e $\phi \longrightarrow$ a constant), $F(\phi,\psi) \longrightarrow F_0(\psi)$

$$w_{eff} = -\frac{\big[F_0(\psi) + \frac{\lambda}{2}\dot{\psi}^2\big]}{\big[F_0(\psi) - \frac{\lambda}{2}\dot{\psi}^2\big]} \quad (4.14)$$

which corresponds to a real scalar field for $\lambda < 0$ while it represents a phantom scalar for $\lambda > 0$. Further, for both $\dot{\phi} \longrightarrow 0$ and $\dot{\psi} \longrightarrow 0$ the effective equation of state corresponds to a cosmological constant.

4.3 A ground work on Noether Symmetry Analysis:

The first theorem due to Noether [200, 159, 201] states that if a physical system is provided with a Lagrangian then the system should be associated with some conserved quantities provided the Lagrangian is invariant with respect to some vector field i.e.

$$\mathcal{L}_{\overrightarrow{X}} L = \overrightarrow{X} L = 0 \quad (4.15)$$

In fact, for a point like canonical Lagrangian $L = L\big[u^\alpha(x^i), \partial u^\alpha(x^i)\big]$ with $u^\alpha(x^i)$ being the generalized coordinates, the Euler-Lagrange equations are

$$\partial_j\Big(\frac{\partial L}{\partial(\partial_j u^\alpha)}\Big) - \frac{\partial L}{\partial u^\alpha} = 0 \quad (4.16)$$

Now contracting with some function $\rho^\alpha(u^\beta)$ in the augmented space one gets (after some simplification)

$$\mathcal{L}_{\overrightarrow{X}} L = \rho^\alpha \frac{\partial L}{\partial u^\alpha} + \big(\partial_j \rho^\alpha\big)\frac{\partial L}{\partial(\partial_j u^\alpha)} = \partial_j\Big(\rho^\alpha \frac{\partial L}{\partial(\partial_j u^\alpha)}\Big) \quad (4.17)$$

Now if we define

$$\mathcal{I}^j = \rho^\alpha \frac{\partial L}{\partial(\partial_j u^\alpha)} \quad (4.18)$$

and

$$\vec{X} = \rho^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \left(\partial_j \rho^{\alpha}\right) \frac{\partial}{\partial(\partial_j u^{\alpha})} \tag{4.19}$$

then we have both the possibilities:

(a) By Noether's theorem if $\vec{X}$ is the symmetry vector which keeps the Lagrangian to be invariant (i.e. $\mathcal{L}_{\vec{X}}$) then $\mathcal{I}^j$ is the associated conserved current (i.e. $\partial_j \mathcal{I}^j = 0$).

(b) Conversely, if $\mathcal{I}^j$ is a conserved vector field (i.e. current) then Lagrangian is always associated with some vector field (symmetry vector) along which it will be invariant.

The energy function associated with the physical system can be expressed in terms of the Lagrangian as

$$E = \dot{u}^{\alpha} \frac{\partial L}{\partial \dot{u}^{\alpha}} - L \tag{4.20}$$

Further, if the time component of the conserved current is integrated over the spatial volume the scalar quantity so obtained is termed as conserved Noether charge (Q). As the space-time geometry under consideration is homogeneous in nature so all variables depend only on time and the Noether charge coincides with the conserved current. Moreover, from geometric point of view, Noether charge can be interpreted as the inner product of the infinitesimal generator (i.e. symmetry vector) with cartan one form [202] i.e.

$$Q = i_{\vec{X}} \Theta_L \tag{4.21}$$

where

$$\Theta_L = \frac{\partial L}{\partial u^{\alpha}} du^{\alpha} \tag{4.22}$$

is the cartan one form and $i_{\vec{X}}$ stands for the inner product with the vector field $\vec{X}$.

Moreover, this geometric notion of inner product representation can be employed to identify cyclic variables in the augmented space. In particular, if $u^{\alpha} \longrightarrow U^{\alpha}$ is a transformation in the augmented space, then the symmetry vector takes the form

$$\overrightarrow{X_U} = \left(i_{\vec{X}} dU^{\alpha}\right) \frac{\partial}{\partial U^{\alpha}} + \frac{d}{dt}\Big(\left(i_{\vec{X}} dU^{\alpha}\right)\Big) \frac{\partial}{\partial \dot{U}^{\alpha}} \tag{4.23}$$

So this transformed symmetry vector can be interpreted as a lift of the vector field $\vec{X}$ in the augmented space. Now, due to simplicity if the above transformation in the augmented space is restricted by

$$i_{\vec{X}} dU^{\alpha} = \delta^{\alpha}_{\theta} \tag{4.24}$$

then $\overrightarrow{X}_U = \frac{\partial}{\partial U^\theta}$ with $\frac{\partial L_T}{\partial U^\theta} = 0$. Thus the augmented variable U^θ is a cyclic variable and the transformed symmetry vector (i.e. infinitesimal generator) is directed along the coordinate line of the cyclic variable.

On the otherhand, due to homogeneous nature of the space-time geometry, the energy function (defined in equation (4.20)) E is not only conserved but also coincides with the Hamiltonian of the system. Further as the Hamiltonian (a constant of motion) is very useful for description of quantum cosmology so Noether symmetry condition can be rewritten as

$$\mathcal{L}_{\overrightarrow{X}_H} H = 0 \tag{4.25}$$

with $\overrightarrow{X}_H = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \ddot{q}^\alpha \frac{\partial}{\partial \dot{q}^\alpha}$, the symmetry vector. Now, due to Noether symmetry the conserved conjugate momenta can be written as

$$\Pi_\alpha = \frac{\partial L}{\partial u^\alpha} = \Sigma_\alpha \text{ (a constant)}, \alpha = 1, 2, ...m \tag{4.26}$$

where 'm' denotes the number of symmetries. These conserved momenta reflect an interesting feature in quantum cosmology as follows:

The operator version of the above conserved momenta results a first order partial differential equations as

$$-i\frac{\partial}{\partial u^\alpha}|\psi >= \Sigma_\alpha|\psi > \tag{4.27}$$

with $|\psi >$, the wave function of the universe. For real Σ_α, the solution of the above equation shows that the wave function has the oscillatory part as

$$|\psi >= \sum_{r=1}^{m} e^{i\Sigma_r u^r}|\phi(u^\mu) >, \ \mu < n \tag{4.28}$$

where μ stands for the directions along which there is no symmetry with n the dimension of the minisuperspace [163]. Thus the wave function has oscillatory part along those directions along which there are conserved momenta which in turn implies the existence of Noether symmetry [201]. Thus, Noether symmetry helps us to consider the entire class of speculative Lagrangians (with the given symmetry) for description of a physical system.

4.4 Noether Symmetry Analysis: An explicit derivation

The configuration space for the present model is a $3D$ space (a, ϕ, ψ) with Lagrangian and the field equations in equations (4.9) and (4.10) respectively. So the explicit form of the infinitesimal generator is

$$\overrightarrow{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \gamma\frac{\partial}{\partial \psi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}} + \dot{\gamma}\frac{\partial}{\partial \dot{\psi}} \tag{4.29}$$

and the coefficients α ,β and γ are functions of the configuration space variables i.e. $\alpha = \alpha(a, \phi, \psi)$,$\beta = \beta(a, \phi, \psi)$ and $\gamma = \gamma(a, \phi, \psi)$.

Also one has

$$\dot{\alpha} = \frac{\partial \alpha}{\partial a}\dot{a} + \frac{\partial \alpha}{\partial \phi}\dot{\phi} + \frac{\partial \alpha}{\partial \psi}\dot{\psi} \tag{4.30}$$

and similarly $\dot{\beta}$ and $\dot{\gamma}$. By imposing the Noether's 1st theorem on the Lagrangian (4.9) one gets

$$\mathcal{L}_{\overrightarrow{X}} L = 0 \tag{4.31}$$

The explicit form of the equation (4.31) gives a system of partial differential equations as the following

$$\left.\begin{array}{lcl}
\mu(\phi)\big(6\alpha + 12a\frac{\partial \alpha}{\partial a}\big) + \mu'(\phi)6a\beta & = & 0 \\
\mu(\phi)\big(\frac{3}{2}\delta\alpha a^2 + \delta a^3\frac{\partial \beta}{\partial \phi}\big) + \mu'(\phi)\frac{\delta}{2}a^3\beta & = & 0 \\
\mu(\phi)\big(\frac{3}{2}\lambda\alpha a^2 + \lambda a^3\frac{\partial \gamma}{\partial \psi}\big) + \mu'(\phi)\frac{\lambda}{2}a^3\beta & = & 0 \\
\mu(\phi)\big(12a\frac{\partial \alpha}{\partial \phi} + \delta a^3\frac{\partial \beta}{\partial a}\big) & = & 0 \\
\mu(\phi)\big(12a\frac{\partial \alpha}{\partial \psi} + \lambda a^3\frac{\partial \gamma}{\partial a}\big) & = & 0 \\
\mu(\phi)\big(\delta a^3\frac{\partial \beta}{\partial \psi} + \lambda a^3\frac{\partial \gamma}{\partial \phi}\big) & = & 0 \\
\mu(\phi)\big[-3a^2\alpha F(\phi,\psi) - \beta a^3 F_1'(\phi)F_2(\psi) - \gamma a^3 F_1(\phi)F_2'(\psi)\big] - \mu'(\phi)a^3\beta F(\phi,\psi) & = & 0
\end{array}\right\} \tag{4.32}$$

with $F(\phi, \psi) = F_1(\phi)F_2(\psi)$.

To solve these set of partial differential equations it is desirable to use the method of separation of variables so that α,β and γ can be written in separable product form as

$$\alpha = \alpha_1(a)\alpha_2(\phi)\alpha_3(\psi)\ ,\ \ \beta = \beta_1(a)\beta_2(\phi)\beta_3(\psi)\ ,\ \ \gamma = \gamma_1(a)\gamma_2(\phi)\gamma_3(\psi) \tag{4.33}$$

Using this technique, the above p.d.e determine the coefficients α, β and γ, the potential function, coupling function and the parameters, rather than choosing them phenomenologically (in normal practice). The explicit solutions give:

Case-I: $\delta = \frac{4}{3}$, $\lambda > 0$, $\omega = \frac{\sqrt{3\lambda}}{2}$

$$\left.\begin{array}{rrl} & \alpha & = \frac{e^{\phi}}{a^2}\{c_1 \sin(\omega\psi) - c_2 \cos(\omega\psi)\} \\ & \beta & = -3\frac{e^{\phi}}{a^3}\{c_1 \sin(\omega\psi) - c_2 \cos(\omega\psi)\} \\ & \gamma & = \frac{2e^{\phi}}{\omega a^3}\{c_1 \cos(\omega\psi) + c_2 \sin(\omega\psi)\} \\ \text{and} & F(\phi,\psi) & = \left[c_1' \cos(\omega\psi) + c_2' \sin(\omega\psi)\right]^M e^{(2-M)\phi} \end{array}\right\} \tag{4.34}$$

with $\mu(\phi) = \mu_0 e^{-\phi}$. Here c_1, c_2,c_1',c_2' μ_0 and M are arbitrary constants.

Case-II: $\delta = -\frac{4}{3}$, $\lambda > 0$

$$\left.\begin{array}{rrl} & \alpha & = \frac{e^{\phi}}{a^2}\{c_1 e^{\omega\psi} - c_2 e^{-\omega\psi}\} \\ & \beta & = -3\frac{e^{\phi}}{a^3}\{c_1 e^{\omega\psi} - c_2 e^{-\omega\psi}\} \\ & \gamma & = -\frac{2e^{\phi}}{\omega a^3}\{c_1 e^{\omega\psi} + c_2 e^{-\omega\psi}\} \\ \text{and} & F(\phi,\psi) & = \left[c_1' e^{\omega\psi} + c_2' e^{-\omega\psi}\right]^N e^{(2-N)\phi} \end{array}\right\} \tag{4.35}$$

with $\mu(\phi) = \mu_0 e^{-\phi}$. Here N is the arbitrary constant.

Case-III: $\delta = \frac{4}{3}$, $\lambda < 0$, $\omega_0 = \frac{\sqrt{3|\lambda|}}{2}$

$$\left.\begin{array}{rrl} & \alpha & = \frac{e^{\phi}}{a^2}\{c_1 \sinh(\omega_0\psi) + c_2 \cosh(\omega_0\psi)\} \\ & \beta & = -3\frac{e^{\phi}}{a^3}\{c_1 \sinh(\omega_0\psi) + c_2 \cosh(\omega_0\psi)\} \\ & \gamma & = -\frac{2e^{\phi}}{\omega_0 a^3}\{c_1 \cosh(\omega_0\psi) + c_2 \sinh(\omega_0\psi)\} \\ \text{and} & F(\phi,\psi) & = \left[c_{10}' \cosh(\omega_0\psi) + c_{20}' \sinh(\omega_0\psi)\right]^M e^{(2-M)\phi} \end{array}\right\} \tag{4.36}$$

Here c_{10}' and c_{20}' are arbitrary constants.

Case-IV: $\delta = -\frac{4}{3}$, $\lambda < 0$

$$\left.\begin{array}{rrl} & \alpha & = \frac{e^{\phi}}{a^2}\{c_1 \cos(\omega_0\psi) - c_2 \sin(\omega_0\psi)\} \\ & \beta & = -3\frac{e^{\phi}}{a^3}\{c_1 \cos(\omega_0\psi) - c_2 \sin(\omega_0\psi)\} \\ & \gamma & = -\frac{2e^{\phi}}{\omega_0 a^3}\{c_1 \sin(\omega_0\psi) + c_2 \cos(\omega_0\psi)\} \\ \text{and} & F(\phi,\psi) & = \left[c_{10}' \sin(\omega_0\psi) + c_{20}' \cos(\omega_0\psi)\right]^N e^{(2-N)\phi} \end{array}\right\} \tag{4.37}$$

Another important aspect of Noether symmetry is that there are some conserved quantities associated with it. Though for a field theory in a curved space there is no well defined notion of energy, yet if there is no time-like Killing vector of the space, an associated conserved energy will exist. For the present FLRW model there is no time like Killing vector and further, the Lagrangian density has no explicit time dependence. Thus the present symmetry analysis corresponds to two conserved quantities namely the conserved energy and the conserved Noether charge. The explicit form of these two conserved quantities are

Case-I: $\delta = \frac{4}{3}$, $\lambda > 0$, $\omega = \frac{\sqrt{3\lambda}}{2}$

$$\left.\begin{aligned} Q &= \Big[c_1\sin(\omega\psi) - c_2\cos(\omega\psi)\Big]\Big(12\mu_0\tfrac{\dot{a}}{a} - 4\mu_0\dot{\phi}\Big) + \tfrac{2\mu_0\lambda}{\omega}\dot{\psi}\Big[c_1\cos(\omega\psi) + c_2\sin(\omega\psi)\Big] \\ E &= \mu_0 e^{-\phi}\Big[6a\dot{a}^2 + \tfrac{2}{3}a^3\dot{\phi}^2 + \tfrac{\lambda}{2}a^3\dot{\psi}^2 + a^3\big(c_1'\cos(\omega\psi) + c_2'\sin(\omega\psi)\big)^M e^{(2-M)\phi}\Big] \end{aligned}\right\} \tag{4.38}$$

Case-II: $\delta = -\frac{4}{3}$, $\lambda > 0$

$$\left.\begin{aligned} Q &= \Big[c_1 e^{\omega\psi} - c_2 e^{-\omega\psi}\Big]\Big(12\mu_0\tfrac{\dot{a}}{a} + 4\mu_0\dot{\phi}\Big) - \tfrac{2\mu_0\lambda}{\omega}\dot{\psi}\Big[c_1 e^{\omega\psi} + c_2 e^{-\omega\psi}\Big] \\ E &= \mu_0 e^{-\phi}\Big[6a\dot{a}^2 - \tfrac{2}{3}a^3\dot{\phi}^2 + \tfrac{\lambda}{2}a^3\dot{\psi}^2 + a^3\big(c_1' e^{\omega\psi} + c_2' e^{-\omega\psi}\big)^N e^{(2-N)\phi}\Big] \end{aligned}\right\} \tag{4.39}$$

Case-III: $\delta = \frac{4}{3}$, $\lambda < 0$, $\omega_0 = \frac{\sqrt{3|\lambda|}}{2}$

$$\left.\begin{aligned} Q &= \Big[c_1\sinh(\omega_0\psi) + c_2\cosh(\omega_0\psi)\Big]\Big(12\mu_0\tfrac{\dot{a}}{a} + 4\mu_0\dot{\phi}\Big) - \tfrac{2\mu_0\lambda}{\omega}\dot{\psi}\Big[c_1\cosh(\omega_0\psi) + c_2\sinh(\omega_0\psi)\Big] \\ E &= \mu_0 e^{-\phi}\Big[6a\dot{a}^2 - \tfrac{2}{3}a^3\dot{\phi}^2 + \tfrac{\lambda}{2}a^3\dot{\psi}^2 + a^3\big(c_{10}'\cosh(\omega_0\psi) + c_{20}'\sinh(\omega_0\psi)\big)^M e^{(2-M)\phi}\Big] \end{aligned}\right\} \tag{4.40}$$

Case-IV: $\delta = -\frac{4}{3}$, $\lambda < 0$

$$\left.\begin{aligned} Q &= \Big[c_1\cos(\omega_0\psi) - c_2\sin(\omega_0\psi)\Big]\Big(12\mu_0\tfrac{\dot{a}}{a} + 4\mu_0\dot{\phi}\Big) - \tfrac{2\mu_0\lambda}{\omega}\dot{\psi}\Big[c_1\sin(\omega_0\psi) + c_2\cos(\omega_0\psi)\Big] \\ E &= \mu_0 e^{-\phi}\Big[6a\dot{a}^2 - \tfrac{2}{3}a^3\dot{\phi}^2 + \tfrac{\lambda}{2}a^3\dot{\psi}^2 + a^3\big(c_{10}'\sin(\omega_0\psi) + c_{20}'\cos(\omega_0\psi)\big)^N e^{(2-N)\phi}\Big] \end{aligned}\right\} \tag{4.41}$$

We shall now (in the next section) investigate the symmetry of the physical state by examining the kinetic metric.

4.5 An overview of conformal symmetry and the present physical model

Conformal invariance in differential geometry shows a very rich geometrical structure and is related to the metric of the manifold. A vector field μ^a is said to be a conformal Killing vector (CKV) of the metric provided

$$\mathcal{L}_{\overrightarrow{\mu}} g_{ij} = \Lambda(x^\alpha) g_{ij} \tag{4.42}$$

with Λ an arbitrary function of the geometric space. Now, depending on the choice of this arbitrary function one may classify the above symmetry vector as follows:

(i) if $\Lambda(x^\alpha) = 0$ then $\overrightarrow{\mu}$ is termed as Killing vector field.

(ii) if $\Lambda(x^\alpha) = \Lambda_0 (\neq 0)$, a constant then $\overrightarrow{\mu}$ is termed as homothetic vector field.

Note that in a particular geometric space these three types of invariant vector fields individually forms an algebra and they are inter-related as follows:

$$KA \subseteq HA \subseteq CA \tag{4.43}$$

where KA is the Killing algebra, formed by all the above invariant Killing vector fields, similarly, HA and CA are respectively homothetic algebra and conformal algebra. For a $nD(n > 2)$ geometrical space the dimension of these algebra are $\frac{n(n+1)}{2}$ (for Killing vector fields), $\frac{n(n+1)}{2} + 1$ (for homothetic vector fields) and $\frac{(n+1)(n+2)}{2}$ (for conformal vector fields) respectively. Note that Noether symmetries of the physical system are related to the homothetic algebra of the metric.

Further, in a given space two metrics g and q are said to be conformally related if $\exists$ a function $\Theta(x^\alpha)$ such that

$$g_{ij} = \Theta(x^\alpha) q_{ij} \tag{4.44}$$

It is easy to show that two metrics, conformally related have the same conformal algebra but the two subalgebras (namely the Killing algebra and the homothetic algebra) may not be identical for the two metrices. In particular, for the common conformal algebra if $\mu(x^\alpha)$ and $\mu'(x^\alpha)$ are the conformal functions for the two metrics for the conformal Killing vector field $\overrightarrow{\rho_{C_K}}$ then one get

$$\mu'(x^\alpha) = \mu(x^\alpha) + \mathcal{L}_{\overrightarrow{\rho}_{C_K}} (\ln \Theta) \tag{4.45}$$

and hence two conformally related physical systems are not identical. In particular, the equations of motions (i.e. Euler-Lagrange equations) corresponding to two conformal Lagrangians transform covariantly under conformal transformation provided the Hamiltonian (i.e. total energy) is zero. As a consequence, for quantum cosmology, the systems are conformally invariant.

Now, if the Lagrangian of a physical system can be written in the form of a point particle as $L = T - V$ then the Noether point symmetries are identified by the elements of the homothetic group of the kinetic metric. As the present Lagrangian (in equation (4.9)) can be expressed as that for a point particle so it is interesting to identify the homothetic group of the kinetic metric so that one can identify the Noether point symmetries. However, even if there is no Noether point symmetries of the field equations i.e., they are not Noether integrable still one may have additional Noether point symmetries due to the above homothetic algebra. The kinetic part of the Lagrangian in equation (4.9) gives the kinetic metric as

$$ds_k^2 = a^3\mu(\phi)\big[\frac{6}{a^2}da^2 + \frac{\delta}{2}d\phi^2 + \frac{\lambda}{2}d\psi^2\big] \tag{4.46}$$

having effective potential

$$F_{eff}(a, \phi, \psi) = a^3 F(\phi, \psi)\mu(\phi) \tag{4.47}$$

Now choosing $\mu(\phi) = \mu_0 e^{-\phi}$ and for the sign choices of the parameters δ and λ one may write the above kinetic metric as

$$\left.\begin{aligned} ds_k^2 &= \mu_0 e^{3u-\phi}\Big[6du^2 + \tfrac{2}{3}d\phi^2 + \tfrac{\lambda}{2}d\psi^2\Big], \text{ case-I} \\ &= \mu_0 e^{3u-\phi}\Big[6du^2 - \tfrac{2}{3}d\phi^2 + \tfrac{\lambda}{2}d\psi^2\Big], \text{ case-II} \\ &= \mu_0 e^{3u-\phi}\Big[6du^2 + \tfrac{2}{3}d\phi^2 - \tfrac{|\lambda|}{2}d\psi^2\Big], \text{ case-III} \\ &= \mu_0 e^{3u-\phi}\Big[6du^2 - \tfrac{2}{3}d\phi^2 - \tfrac{|\lambda|}{2}d\psi^2\Big], \text{ case-IV} \end{aligned}\right\} \tag{4.48}$$

with $u = \ln a$. Note that among the above four cases the first one is not of much interest due to its Euclidean nature. The other three cases are conformal to the $3D$ Minkowskian metric as

$$ds_k^2 = -d\alpha^2 + d\beta^2 + d\gamma^2 \tag{4.49}$$

where

$$\left.\begin{aligned} \alpha &= \sqrt{\tfrac{2}{3}}\phi\ ,\ \beta = \sqrt{6}u\ ,\ \gamma = \sqrt{\tfrac{\lambda}{2}}\psi\ ,\ \text{case-II} \\ \alpha &= \sqrt{\tfrac{|\lambda|}{2}}\psi\ ,\ \beta = \sqrt{6}u\ ,\ \gamma = \sqrt{\tfrac{2}{3}}\phi\ ,\ \text{case-III} \\ \alpha &= \sqrt{6}u\ ,\ \beta = \sqrt{\tfrac{2}{3}}\phi\ ,\ \gamma = \sqrt{\tfrac{|\lambda|}{2}}\psi\ ,\ \text{case-IV} \end{aligned}\right\} \tag{4.50}$$

Further,with the above transformation the Lagrangian of the physical system may be considered in FLRW model as

(a) minimally coupled two ghost scalar fields (case-I)

(b) minimally coupled one normal scalar field with a ghost scalar field (quintom model) (case-II and III)

(c) minimally coupled two real scalar fields (case-IV)

The kinetic metric (4.46) has in general a gradient homothetic vector (HV) [203, 204, 205] $\overrightarrow{v}_H = \frac{2}{3}a\ln a$, with $\psi\overrightarrow{v}_H = 1$. However, this gradient HV does not generate a Noether point symmetry of the present Lagrangian.

On the otherhand, to obtain further Noether-point symmetries one has to consider the $2D$ metric in the (ϕ,ψ) plane so that the $3D$ kinetic metric has larger homothetic algebra. Thus, in addition to the Hamiltonian the present physical system should have two more first integrals i.e. if $H_{\mu\nu}$ be the homothetic algebra then dim $H_{\mu\nu} \geq 3$. Depending on the nature of the $2D$ metric one has the following two possibilities:

(i) if the $2D$ metric is flat (i.e. curvature $\kappa = 0$) then it admits three Killing vectors, spanning the $E(2)$ group.

(ii) if the $2D$ metric is a space of constant (non-zero) curvature (i.e. $\kappa \neq 0$, a constant) then the three possible Killing vectors span the $So(3)$ group.

Now with suitable transformation of variables the kinetic metric can be written as conformal to the metric

$$ds_k^2 = -dl^2 + l^2\left(\epsilon_1 d\Phi^2 + \epsilon_2 d\Psi^2\right) \tag{4.51}$$

with $l = \sqrt{6}a$, $\Phi = \sqrt{\frac{|\delta|}{12}}\phi$, $\Psi = \sqrt{\frac{|\lambda|}{12}}\psi$, $\epsilon_{1,2} = \pm 1$.

As we have mentioned earlier case II and III correspond to quintom model [206, 207, 201] which has been explicitly studied in references [208] so we shall discuss only the fourth case for which $\epsilon_1 = 1 = \epsilon_2$. Further, due to conformally flat nature of the above kinetic metric there are 7 ($\frac{n(n+1)}{2} + 1$ for $n = 3$) homothetic Lie algebra of which the gradient homothetic vector has been described above and the others can be classified as follows:

(a) Three gradient Killing vectors K_i^α , $\alpha, i = 1, 2, 3$ which are translational in nature, has the explicit expression:

$$\left.\begin{array}{ll} K_1^\alpha & = -\frac{1}{2}(2+\Psi^2)\partial_l + \frac{1}{2l}\Psi^2\partial_\Phi + \frac{1}{l}\Psi\partial_\Psi \\ K_2^\alpha & = \frac{1}{2}\Psi^2\partial_l + \frac{1}{2l}(2-\Psi^2)\partial_\Phi + \frac{1}{2}\Psi\partial_\Psi \\ K_3^\alpha & = -\Psi\partial_l + \frac{1}{l}\Psi\partial_\Phi + \frac{1}{l}\partial_\Psi \end{array}\right\} \tag{4.52}$$

having gradient Killing functions:

$\Theta_1 = \frac{l}{2}\big[2+\Psi^2\big],\ \Theta_2 = \frac{l}{2}\big[-\Psi^2\big],\ \Theta_3 = l\Psi$

(b) Three non-gradient Killing vectors (representing rotation) generate the $So(3)$ algebra :

$\vec{r}_{12} = \partial_\Psi,\ \vec{r}_{23} = \partial_\Phi + \Psi\partial_\Psi$ and $\vec{r}_{31} = \Psi\partial_\Psi - \frac{1}{2}(1-\Psi^2)\partial_\Psi$

The above homothetic Lie algebra reflects that Noether point symmetries are very much associated with the differential geometric structure of the physical metric.

We shall now discuss the quantum description of the present cosmological model in the following two sections. Section VI describes the canonical quantization scheme, while section VII deals with the causal interpretation.

4.6 A detailed analysis of quantum cosmology: The canonical approach

In quantum cosmology with canonical quantization program there is a significant role of Noether symmetry analysis. Usually superspace symmetries are characterized by the metric and matter field. Due to simplification, superspace geometrodynamics are restricted to minisuperspaces which are usually chosen as homogeneous and isotropic in nature. As a result, the shift vector vanishes identically and the lapse function is only time dependent. So the line element in $(3+1)$- decomposition takes the form

$$ds^2 = -N^2(t)dt^2 + h_{\alpha\beta}(x,t)dx^\alpha dx^\beta \tag{4.53}$$

As a result, the explicit form of the Einstein-Hilbert action becomes

$$\mathcal{A}\big[e_{\alpha\beta}, N\big] = \int dt\ d^3x\ N\sqrt{e}\Big[K_{\alpha\beta}K^{\alpha\beta} - K^2 + {}^{(3)}R - 2\Lambda\Big] \tag{4.54}$$

Here ${}^{(3)}R$ and Λ are the usual 3-space curvature and the cosmological constant with $K = K_{\alpha\beta}h^{\alpha\beta}$, the trace of the extrinsic curvature $K_{\alpha\beta}$. Further, due to homogeneity of the 3-space the metric on it is characterized by a finite number of function $u^{\alpha}(t)$, $\alpha = 0, 1, ...(n-1)$ so that the above action becomes

$$\mathcal{A}\big[u^{\alpha}(t), N(t)\big] = \int_0^t N\ dt\Big[\frac{1}{2N^2}e_{\alpha\beta}\ \dot{u}^{\alpha}(t)\dot{u}^{\beta}(t) - V(u)\Big] \tag{4.55}$$

The above action corresponds to a relativistic particle moving in nD space having self interacting potential $V(u)$. It is interesting to note that the evolution equations of the particle i.e.

$$\frac{1}{N}\frac{d}{dt}\Big(\frac{\dot{u}^{\alpha}}{N}\Big) + \frac{1}{N^2}\Gamma^{\alpha}_{\beta\delta}\ \dot{u}^{\beta}\dot{u}^{\delta} + e^{\alpha\beta}\frac{\partial V}{\partial u^{\beta}} = 0 \tag{4.56}$$

are restricted by the constraint equation

$$\frac{1}{2N^2}\ e_{\alpha\beta}\ \dot{u}^{\alpha}\dot{u}^{\beta} + V(u) = 0 \tag{4.57}$$

It is interesting to note that due to quantum description the Hamiltonian formulation is useful and for the present minisuperspace description the Hamiltonian is nothing but the constraint equation with phase space form

$$\mathcal{H} = N\big[\frac{1}{2}\ e^{\alpha\beta}\Pi_{\alpha}\Pi_{\beta} + V(u)\big] \equiv 0 \tag{4.58}$$

where $\Pi_{\alpha} = e_{\alpha\beta}\ \frac{\dot{u^{\beta}}}{N}$, is the momentum conjugate to u^{α}. Now due to operator conversion in the quantization scheme the equation (4.58) becomes a second order hyperbolic partial differential equation

$$\mathcal{H}\Big[u^{\alpha}, -i\frac{\partial}{\partial u^{\alpha}}\Big]\Psi(u^{\alpha}) = 0 \tag{4.59}$$

In quantum cosmology it is termed as Wheeler DeWitt (WD) equation with $\Psi(u^{\alpha})$, the wave function of the Universe. The ambiguity in operator ordering for the formulation of the WD equation can be resolved by imposing the restriction that the quantization scheme should be covariant in nature i.e. invariant under the transformation $u^{\alpha} \longrightarrow \bar{u}^{\alpha} = \bar{u}^{\alpha}(u^{\beta})$. The conserved probability current due to hyperbolic nature of the WD equation takes the form [209, 210]

$$\overrightarrow{J} = \frac{i}{2}\big(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\big) \tag{4.60}$$

with $\overrightarrow{\nabla}.\overrightarrow{J} = 0$.

So the probability measure can be defined as

$$dp = |\Psi(u^{\alpha})|^2 dV \tag{4.61}$$

with minisuperspace volume element dV.

Now the explicit form of the Hamiltonian for the present physical scenario is

$$\mathcal{H} = \frac{1}{24a\mu(\phi)}\, p_a^2 + \frac{\delta}{2a^3\mu(\phi)}\, p_\phi^2 + \frac{\lambda}{2a^3\mu(\phi)}\, p_\psi^2 + a^3\mu(\phi)F(\phi,\psi) \equiv 0 \tag{4.62}$$

This is also known as scalar constraint equation due to the appearance of the lapse function as Lagrange multiplier. From the point of view of Dirac quantization program the quantum states should be annihilated by the operator version of the above scalar constraint equation [163]i.e. $\hat{H}\Psi(a,\phi,\psi) = 0$ i.e.

$$\Big[-\frac{1}{24a\mu(\phi)}\frac{\partial^2}{\partial a^2} - \frac{\delta}{2a^3\mu(\phi)}\frac{\partial^2}{\partial \phi^2} - \frac{\lambda}{2a^3\mu(\phi)}\frac{\partial^2}{\partial \psi^2} + a^3\mu(\phi)F(\phi,\psi)\Big]\Psi(a,\phi,\psi) = 0 \tag{4.63}$$

To solve the above 2nd order hyperbolic partial differential it is desirable to write the wave function Ψ in a separable form as

$$\Psi(a,\phi,\psi) = \sum_{m,n} A_m(a)B_{mn}(\phi)C_n(\psi) \tag{4.64}$$

with the ordinary differential equations for these separable functions as :

$$\left.\begin{array}{ll} a^2\frac{d^2A_m}{da^2} - 24\mu_0^2a^6A_m - 12\epsilon_1 m^2A_m & = 0 \\ \delta\frac{d^2B_{mn}}{d\phi^2} + \left(\epsilon_1 m^2 - \epsilon_2 n^2\right)B_{mn}(\phi) & = 0 \\ \text{and} \qquad \lambda\frac{d^2C_n(\psi)}{d\psi^2} + \epsilon_2 n^2 C_n(\psi) & = 0 \end{array}\right\} \tag{4.65}$$

Here $\mu(\phi) = \mu_0 e^{-\phi}$ and $F(\phi,\psi) = e^{2\phi}$ by choosing $M = 0$ in equation (4.34) and $\epsilon_{1,2} = \pm 1$.

Thus the explicit solution can be written as

Case-I: $\epsilon_1 = 1 = \epsilon_2$

$$\left.\begin{array}{lll} A_m(a) & = & c_1\sqrt{a}J\left(\frac{1}{6}\sqrt{1+48m^2}, \frac{2}{3}\sqrt{-6\mu_0^2}a^3\right) + c_2\sqrt{a}Y\left(\frac{1}{6}\sqrt{1+48m^2}, \frac{2}{3}\sqrt{-6\mu_0^2}a^3\right) \\ B_{mn}(\phi) & = & c_3\cos(\omega_{01}\phi) + c_4\sin(\omega_{01}\phi)\ ,\ m>n,\ \omega_{01}^2 = \frac{m^2-n^2}{\delta} \\ & = & c_3\phi + c_4\ ,\ m=n \\ & = & c_3\cosh(|\omega_{01}|\phi) + c_4\sinh(|\omega_{01}|\phi)\ ,\ m<n \\ C_n(\psi) & = & c_5\cos(p_0\psi) + c_6\sin(p_0\psi)\ ,\ p_0^2 = \frac{n^2}{\lambda} \end{array}\right\} \tag{4.66}$$

where $c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6$ are the arbitrary constants and J and Y stands for the usual Bessel functions of first and second kind respectively.

Case-II: $\epsilon_1 = +1\ ,\ \epsilon_2 = -1$

$$\left.\begin{array}{rcl} A_m(a) &=& c_1\sqrt{a}J\big(\frac{1}{6}\sqrt{1+48m^2}, \frac{2}{3}\sqrt{-6\mu_0^2 a^3}\big) + c_2\sqrt{a}Y\big(\frac{1}{6}\sqrt{1+48m^2}, \frac{2}{3}\sqrt{-6\mu_0^2 a^3}\big) \\ B_{mn}(\phi) &=& c_3\cos(\omega_{02}\phi) + c_4\sin(\omega_{02}\phi)\ ,\ \omega_{02}^2 = \frac{m^2+n^2}{\delta} \\ C_n(\psi) &=& c_5\cosh(p_0\psi) + c_6\sinh(p_0\psi) \end{array}\right\} \tag{4.67}$$

Case-III: $\epsilon_1 = -1\ ,\ \epsilon_2 = +1$

$$\left.\begin{array}{rcl} A_m(a) &=& c_1\sqrt{a}J\big(\frac{1}{6}\sqrt{1-48m^2}, \frac{2}{3}\sqrt{-6\mu_0^2 a^3}\big) + c_2\sqrt{a}Y\big(\frac{1}{6}\sqrt{1-48m^2}, \frac{2}{3}\sqrt{-6\mu_0^2 a^3}\big) \\ B_{mn}(\phi) &=& c_3\cosh(\omega_{02}\phi) + c_4\sinh(\omega_{02}\phi) \\ C_n(\psi) &=& c_5\cos(p_0\psi) + c_6\sin(p_0\psi) \end{array}\right\} \tag{4.68}$$

Case-IV: $\epsilon_1 = -1\ ,\ \epsilon_2 = -1$

$$\left.\begin{array}{rcl} A_m(a) &=& c_1\sqrt{a}J\big(\frac{1}{6}\sqrt{1-48m^2}, \frac{2}{3}\sqrt{-6\mu_0^2 a^3}\big) + c_2\sqrt{a}Y\big(\frac{1}{6}\sqrt{1-48m^2}, \frac{2}{3}\sqrt{-6\mu_0^2 a^3}\big) \\ B_{mn}(\phi) &=& c_3\cos(\omega_{02}\phi) + c_4\sin(\omega_{02}\phi)\ ,\ n>m\ ,\ \omega_{02}^2 = \frac{n^2-m^2}{\delta} \\ &=& c_3\phi + c_4\ ,\ n=m \\ &=& c_3\cosh(\omega_{02}\phi) + c_4\sinh(\omega_{02}\phi)\ ,\ n<m \\ C_n(\psi) &=& c_5\cosh(p_0\psi) + c_6\sinh(p_0\psi). \end{array}\right\} \tag{4.69}$$

As a result, the general solution can be expressed in superposition form as

$$\Psi(a,\phi,\psi) = \int W(a,\phi,\psi)\ A_m(a)\ B_{mn}(\phi)\ C_n(\psi) da d\phi d\psi \tag{4.70}$$

with W denoting some weight factor.

Now for WKB approximation to have semi classical limit one may write

$$\Psi = exp(\frac{i}{\hbar}S) \tag{4.71}$$

where the classical Hamiltonian-Jacobi(HJ) function S can be expanded in powers of $\hbar$ as

$$S = S_0 + \hbar\ S_1 + \hbar^2 S_2 + ... \tag{4.72}$$

As a result, one gets the wave packet

$$\Psi = \int S(\overrightarrow{k}) exp(\frac{1}{\hbar}S_0) d\overrightarrow{k} \tag{4.73}$$

which describes the classical solution and $\overrightarrow{k} = (k_1, k_2, k_3)$ stands for arbitrary separation constants. Now using (4.71) as the form for Ψ the WD equation (4.63) gives in the zeroth order in $\hbar$ the first order non-linear differential equation for S_0 (in explicit form):

$$-\frac{1}{24a\mu(\phi)}(\frac{\partial S_0}{\partial a})^2 - \frac{\delta}{2a^3\mu(\phi)}(\frac{\partial S_0}{\partial \phi})^2 - \frac{\lambda}{2a^3\mu(\phi)}(\frac{\partial S_0}{\partial \psi})^2 + a^3\mu(\phi)F(\phi,\psi) = 0 \tag{4.74}$$

Now choosing S_0 in the additive separable form as

$$S_0(a,\phi,\psi) = S_{01}(a) + S_{02}(\phi) + S_{03}(\psi) \tag{4.75}$$

One gets

$$\left.\begin{array}{rcl} S_{01}(a) & = & a\sqrt{k_1 + 12\mu_0 a^6}\ {}_2F_1\big[-\frac{1}{2},\frac{1}{6},\frac{7}{6},-\frac{12\mu_0 a^6}{k_1}\big] + a_0,\ k_1 > 0 \\ & = & a\sqrt{-k_1 + 12\mu_0 a^6}\ {}_2F_1\big[-\frac{1}{2},\frac{1}{6},\frac{7}{6},\frac{12\mu_0 a^6}{k_1}\big] + a_0,\ k_1 < 0 \\ & = & \frac{a^4\sqrt{\mu_0}}{4} + a_0\ ,\ k_1 = 0 \\ S_{02}(\phi) & = & \sqrt{-\frac{(k_1+k_2)}{\delta}}\phi + \phi_0 \\ S_{03}(\psi) & = & \sqrt{\frac{k_2}{\lambda}}\psi + \psi_0. \end{array}\right\} \tag{4.76}$$

wih a_0, ϕ_0 and ψ_0, the constants of integration.

Hence the wave packet (4.73) has an explicit form as

$$\Psi(a,\phi,\psi) = \iint \gamma_0(k_1,k_2)exp\big[\frac{i}{\hbar}S_{01}(a,k_1)\ S_{02}(\phi,k_1,k_2)\ S_{03}(k_2,\psi)\big]dk_1dk_2 \tag{4.77}$$

with the weight function γ_0 having Gaussian distribution.

Using this wave function the probability amplitude ($|\psi|^2$) has been plotted in Fig.(4.1) for the above cases. From the figure it is seen that probability amplitude approaches a finite non zero constant as $a \to 0$ in the left figure (i.e., case-I and Case-II), while in the right figure (i.e., case-III and Case-IV) the probability amplitude becomes zero as $a \to 0$. Hence one may conclude that the classical singularity may be avoided in the second case, while it is unavoidable in the first case through the quantum description .

4.7 Bohmian Trajectories and causal Interpretation

This section deals with an alternative interpretation of quantum mechanics to cosmology. The notion of quantum potential is responsible for the quantum effects in this ontological interpretation. As a result, in minisuperspace the schrӧdinger equation is replaced by

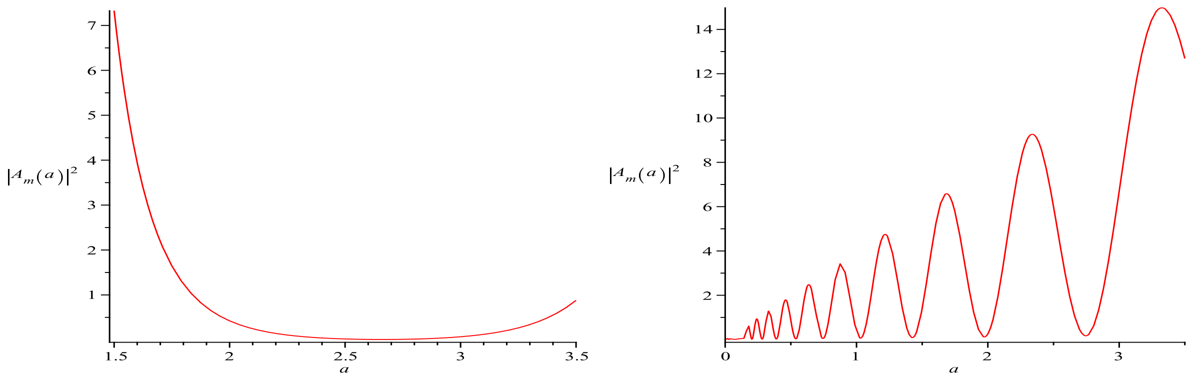

FIGURE 4.1: shows the graphical representation of wave function $|A_m(a)|^2$ (left) for case -I and Case -II, $(c_1 = 0.5,\ c_2 = 5.05,\ m = 1.05,\ \mu_0 = 0.03)$ and (right) for case -III and case -IV $(c_1 = 3,\ c_2 = 2.5,\ m = 2.8,\ \mu_0 = 0.03)$ against a.

the wheeler-Dewitt equation. In this formulation, the Bohmian trajectories (known as quantum trajectories) are the time evolution of the metric and field variables which follow the quantum Hamilton-Jacobi equation. Note that for small values of the scale factor Bohmian trajectories are dominated by quantum effects while for large scale factor these trajectories identify classical features.

In standard Einstein gravity with metric formulation there are four constraint equations which can be classified in two groups namely the vector constraints (or super momentum constraints) and the scalar constraint (Hamiltonian constraint). However, from cosmological point of view as the space-time is homogeneous and isotropic so the vector constraints vanish identically while the operator version of the Hamiltonian constraint is the standard WD equation i.e.

$$\mathcal{H}\big[u^\alpha(t), \hat{p}_\alpha(t)\big]\,\Psi(u^\alpha) = 0 \tag{4.78}$$

where $u^\alpha(t)$ and $p_\alpha(t)$ represent the homogeneous degrees of freedom due to the three metric and its conjugate momenta. In analogy with WKB ansatz the wave function Ψ can be written as

$$\Psi(u^\alpha) = A(u^\alpha)\ exp\big[\frac{i}{\hbar}B(u^\alpha)\big] \tag{4.79}$$

which as an ansatz in equation (4.78) gives the usual Hamilton-Jacobi(HJ) equation

$$\frac{1}{2}e^{\alpha\beta}(u^\alpha)\ \frac{\partial B}{\partial u^\alpha}\frac{\partial B}{\partial u^\beta} + D(u^\alpha) + E(u^\alpha) = 0 \tag{4.80}$$

with $E(u^\alpha) = -\frac{1}{A}e^{\alpha\beta}\frac{\partial^2 B}{\partial q^\alpha \partial q^\beta}$, the quantum potential. Also $e_{\alpha\beta}$ and $D(u^\alpha)$ are respectively the reduced super metric to the given minisuperspace [209, 210, 211] and a particularization of the scalar curvature density (i.e.$-e^{\frac{1}{2}}R$) of the space-like hypersurfaces. From the point of view of causal interpretation, the quantum cosmological trajectories $u^\alpha(t)$ should be real and observer independent. So they are characterized by the HJ equation through equating the expressions for the momentum as:

The momentum from the HJ equation (4.80) can be written as

$$p_\alpha = \frac{\partial B}{\partial u^\alpha} \tag{4.81}$$

which comparing with the usual momentum-velocity relation (i.e. $p_\alpha = e_{\alpha\beta}\frac{\partial u^\beta}{\partial t}$) gives the quantum trajectories as

$$e_{\alpha\beta}\frac{\partial u^\beta}{\partial t} = \frac{\partial B}{\partial u^\alpha} \tag{4.82}$$

(throughout the work the lapse function N is chosen as a constant). These first order differential equations are the Bohmian trajectories [212, 213]. Note that these trajectories are invariant under time re-parameterization and as a consequence, casual interpretation of minisuperspace quantization program has no time problem.

In the present problem the Hamilton-Jacobi equation after quantum correction takes the form:

$$\frac{a^2}{12}(\frac{\partial B}{\partial a})^2 + \delta(\frac{\partial B}{\partial \phi})^2 + \lambda(\frac{\partial B}{\partial \psi})^2 - E - 2a^6\mu^2(\phi)F(\phi,\psi) = 0 \tag{4.83}$$

with $E = \frac{1}{A}\left[\frac{a^2}{12}\frac{\partial^2 A}{\partial a^2} + \delta\frac{\partial^2 A}{\partial \phi^2} + \lambda\frac{\partial^2 A}{\partial \psi^2}\right]$, the quantum potential. As a result, the Bohmian trajectories are identified by the following (1st order) differential equations.

$$\frac{\partial B}{\partial a} = \frac{6\dot{a}}{a^2}\ ,\ \frac{\partial B}{\partial \phi} = \frac{\dot{\phi}}{2\delta}\ ,\ \frac{\partial B}{\partial \psi} = \frac{\dot{\psi}}{2\lambda}. \tag{4.84}$$

Now choosing for simplicity the separable form of the H-J function as

$$B(a,\phi,\psi) = B_1(a) + B_2(\phi) + B_3(\psi) \tag{4.85}$$

with $A(a,\phi,\psi) = A_1(a)A_2(\phi)A_3(\psi)$, the quantum potential takes the separable form as

$$E(a,\phi,\psi) = E_1(a) + E_2(\phi) + E_3(\psi) \tag{4.86}$$

Thus trajectory differential equations now become ordinary differential equations as

$$6\frac{\dot{a}}{a^2} = \frac{dB_1}{da}\ ,\ \frac{d\phi}{dt} = 2\delta\frac{dB_2}{d\phi}\ ,\ \frac{d\psi}{dt} = 2\lambda\frac{dB_3}{d\psi} \tag{4.87}$$

Now, if one chooses

$$B_1(a) = B_{10}a^{n_1} \ , \ B_2(\phi) = B_{20}\phi^{n_2} \ and \ B_3(\psi) = B_{30}\psi^{n_3} \tag{4.88}$$

with B_{i0} and n_i (i=1,2,3) are constants, then one has

$$a = a_0 t^{\frac{2}{1-n_1}} \ , \ \phi = \phi_0 t^{\frac{1}{2-n_2}} \ , \ \psi = \psi_0 t^{\frac{1}{2-n_3}} \tag{4.89}$$

Thus it is easy to see that the above quantum trajectory passes through classical singularity ($a = 0$) provided $n_1 < 1$. For $n_1 > 1$ the Universe may have a future singularity. The behaviour of ϕ and ψ are similar to the scale factor.

4.8 Brief summary

The present chapter is an example to show how symmetry analysis helps us to explore a complicated physical system. In cosmological context the (modified) Einstein field equations are highly nonlinear coupled 2nd order differential equations and in most of the situations they can not solved by the usual mathematical techniques in the literature. Noether symmetry analysis has a leading role to resolve this issue. The advantage of using this symmetry technique is three fold:(i) simplify the field equations to a great extend so that in most of the cases they become solvable, (ii) identification of conserved currents (charge), (iii) determination of parameters or arbitrary functions (without phenomenologically choice). In the present context it is possible to have the conserved charge and also the arbitrary (potential) function has been determined by using Noether symmetry approach but the Noether symmetry approach can not identify any cyclic variable so that the field equations can be so simplified that they becomes solvable. This puts a general question namely " It is always possible to a cyclic variable along the Noether symmetry vector"? Subsequently, symmetry of the physical space i.e. kinetic metric has been examined for conformal symmetry. Also Homothetic as well as Killing vector fields of the physical symmetry has been evaluated. As a consequence rich geometrical structure as well as some physical inference is possible by this symmetry approach. Hence present work is an example how the conformal geometric of the physical metric identifies the symmetry of the system.

Subsequently quantum cosmological description of the physical model has been done extensively. At first, in canonical quantization scheme the Hamiltonian constraint has been translated into the operator form, giving rise to the Wheeler DeWitt equation and it is possible to have the wave function of the Universe by solving this second order hyperbolic partial differential equation. By plotting the probability amplitude it has been inferred whether classical singularity can be avoided by quantum description or not. Also Hamilton-Jacobi(HJ) formulation in WKB approximation has been presented. Lastly, from casual interpretation it is possible to have quantum paths known as Bohmian trajectories. Here quantum potential in the modified HJ equation has a significant role to identify the Bohmian trajectories. Also one can examine whether these quantum trajectories passes through the classical singularity or not. In the present context by choosing power-law form for the quantum potential components, it is found that the classical singularity may be avoided with proper choice of the power index. Finally, for future work it will be interesting to address the above question namely how existence of cyclic variable which associated with the Noether symmetry vector.

Chapter 5

Quantum Cosmology in Bianchi-I model: A study of symmetry analysis

5.1 Prelude

The inherent geometric singularity of Einstein gravity leads to space-time singularity in standard cosmology and and is commonly known as initial big-bang singularity or late-time big-rip singularity [214, 215, 216, 217]. The indefinite extension of geodesic for any value of the affine parameter is termed as geodesic completeness otherwise one has the notion of incomplete geodesic [218, 214, 219]. Geometrically, the existence of singularity is associated with the existence of incomplete geodesics. In the context of cosmology, the space-time singularity is usually occur either at the initial instant or at infinite (or finite) future time instant. In this connection, it is worthy to mention that there are cosmological solutions which are singularity free usually known as bouncing solution or emergent scenario. However, from the observational view point particularly the cosmic microwave background radiation (CMBR) observation [220, 221] is the strongest support in favour of initial big-bang singularity [222, 223, 224]. From cosmological principle as well as for simplicity most of the studies in cosmology are in the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) space-time model [225, 226]. As CMBR observations show small anisotropy locally so it is interesting to study anisotropic cosmological models. The present work in Bianchi-I cosmological model is a consequence of such motivation [153, 227]. The main problem of these anisotropic cosmological models is the coupled non-linear field equations. So usual mathematical techniques are not sufficient to solve them analytically. However, if Noether symmetry analysis has been imposed then the evolution equations get simplified and even they may be solvable in some cases. Also quantum formulation [228] particularly the canonical quantization scheme and causal interpretation can be developed. The plan of this chapter is as follows: at first the basic equation of this model are discussed in Section-II, then the application of this model using Noether symmetry analysis are presented in Section-III. Section IV represents the Noether symmetry analysis and quantum formulation of this model. Also the causal interpretation and Bohmian trajectory are discussed in Section V. Finally the manuscripts ends with a brief discussion.

5.2 Bianchi-I cosmological model: Basic Equations

The action for a non-minimally coupled scalar field having self interacting potential can be written as [229, 230, 231, 232]

$$\mathcal{A} = \int d^4x\sqrt{-g}\left[A(\phi)R - \frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + V(\phi)\right] \tag{5.1}$$

Here the coupling function $A(\phi)$ and the potential $V(\phi)$ are c'-functions of the scalar field ϕ. One may introduce a conformal transformation:

$$g_{\alpha\beta} = \frac{\alpha_0}{A(\phi)}q_{\alpha\beta}, \quad \alpha_0, \text{a constant} \tag{5.2}$$

so that the above action corresponds to a minimally coupled scalar field as

$$\mathcal{A} = \int d^4x\sqrt{-g}\left[\alpha_0 R(q) - \frac{1}{2}\tilde{q}^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta} + V_\sigma(\sigma)\right] \tag{5.3}$$

where $\sigma = \int \frac{\sqrt{\alpha_0(A+3A'^2)^{\frac{1}{2}}}}{A}d\phi$ is the newly defined scalar field and $V_\sigma(\sigma) = \frac{\alpha_0^2 V(\phi(\sigma))}{A^2(\phi(\sigma))}$ is the modified potential function.

Essentially the above transformation of the metric tensor and the scalar field is nothing but the transformation from Jordan frame to the Einstein frame. Now for the present Bianchi-I space-time model the line-element can be written as

$$ds^2 = -\tilde{N}^2dt^2 + \tilde{a}^2(t)\left[e^{2\beta_1(t)}dx_1^2 + e^{2\beta_2(t)}dx_2^2 + e^{2\beta_3(t)}dx_3^2\right] \tag{5.4}$$

where as usual $\tilde{N}$ is the lapse function, $\tilde{a}(t)$ is the scale factor and $\beta_1, \beta_2, \beta_3$ are anisotropic parameters along the three spatial directions with the restrictions: $\beta_1 + \beta_2 + \beta_3 = 0$. However, writing these anisotropy components in terms of two dependent parameters as

$$\beta_1 = \frac{1}{\sqrt{6}}\alpha_1 + \frac{1}{\sqrt{2}}\alpha_2,\ \beta_2 = \frac{1}{\sqrt{6}}\alpha_1 - \frac{1}{\sqrt{2}}\alpha_2,\ \beta_3 = -\sqrt{\frac{2}{3}}\alpha_1, \tag{5.5}$$

$\bar{\mu}$ the Ricci scalar has the explicit expression

$$R = \frac{1}{N^2}\left[6\frac{\dot{N}}{N}\frac{\dot{a}}{a} - 6\frac{\ddot{a}}{a} - 6\frac{\dot{a}^2}{a^2} - \dot{\alpha_1}^2 - \dot{\alpha_2}^2\right]. \tag{5.6}$$

and the explicit expression for the above action (5.3) takes the form: [232]

$$\mathcal{A} = \int dtd^3x\left[\frac{6\alpha_0 a^3}{N}\left(\frac{\dot{N}}{N}\frac{\dot{a}}{a} - \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - \frac{\alpha_0 a^3}{N}(\dot{\alpha_1}^2 + \dot{\alpha_2}^2) - \frac{a^3\dot{\sigma}^2}{2N} + a^3 N V_\sigma(\sigma)\right] \tag{5.7}$$

with $N = \sqrt{\frac{A}{\alpha_0}}\tilde{N}$ and $a = \sqrt{\frac{A}{\alpha_0}}\tilde{a}$.

Now integrating by parts the second derivative term one can write the Lagrangian as (with $N = \text{constant} = 1$)

$$L = 6\alpha_0 a\dot{a}^2 - \alpha_0 a^3(\dot{\alpha_1}^2 + \dot{\alpha_2}^2) - \frac{a^3\dot{\sigma}^2}{2} + a^3 V_\sigma(\sigma). \tag{5.8}$$

Note that for the Lagrangian (7.8) one may define $r = \alpha_1 + i\alpha_2$ with $r.r^* = |r|^2 = \alpha_1^2 + \alpha_2^2$ so that $\dot{\alpha_1}^2 + \dot{\alpha_2}^2 = |\dot{r}|^2 = \dot{\alpha}^2$(say). Thus the physical system is equivalent to a single complex scalar field γ and a self interacting scalar field σ.

5.3 Noether symmetry Analysis

According to Noether's theorem [131, 233, 234, 235, 236], the invariance of the Lagrangian along an infinitesimal generator (i.e., symmetry vector) in the augmented space can be written as

$$\mathcal{L}_{\overrightarrow{X}}L = \lambda^\alpha\frac{\partial L}{\partial q^\alpha} + (\partial_j\lambda^\alpha)\frac{\partial L}{\partial(\partial_j q^\alpha)} = 0, \tag{5.9}$$

with

$$\overrightarrow{X} = \lambda^\alpha\frac{\partial}{\partial q^\alpha} + (\partial_j\lambda^\alpha)\frac{\partial}{\partial(\partial_j q^\alpha)}, \tag{5.10}$$

the symmetry vector. Further associated with this symmetry vector there is a conserved current of the physical system namely

$$I^i = \lambda^\alpha \frac{\partial L}{\partial\big(\partial_i q^\alpha\big)}. \tag{5.11}$$

However, if the Lagrangian has no explicit time dependence then the essential system has two conserved quantities namely the energy function and the associate Noether charge (the integral of the time component of the Noether current).

For the present model one has $4D$ configuration space $\{a, \alpha_1, \alpha_2, \sigma\}$ and the symmetry vector has the explicit form:

$$\overrightarrow{X} = l\frac{\partial}{\partial a} + m\frac{\partial}{\partial \sigma} + \dot{l}\frac{\partial}{\partial \dot{a}} + \dot{m}\frac{\partial}{\partial \dot{\sigma}} + \dot{s}\frac{\partial}{\partial \dot{\alpha_1}} + \dot{u}\frac{\partial}{\partial \dot{\alpha_2}}, \tag{5.12}$$

with l, m, s and u are functions of the configuration variables. Now imposing the symmetry condition (7.9) one gets a set of first order partial differential equations as

$$\left.\begin{array}{lcl} l + 2a\frac{\partial l}{\partial a} & = & 0, \\ \frac{3l}{2} + a\frac{\partial m}{\partial \sigma} & = & 0, \\ \frac{3l}{2} + a\frac{\partial s}{\partial \alpha_1} & = & 0, \\ \frac{3l}{2} + a\frac{\partial u}{\partial \alpha_2} & = & 0, \\ 12u_1\frac{\partial l}{\partial \sigma} - a^2\frac{\partial m}{\partial a} & = & 0, \\ \frac{\partial l}{\partial \alpha_1} = \frac{\partial l}{\partial \alpha_2} & = & 0, \\ \frac{\partial m}{\partial \alpha_1} = \frac{\partial m}{\partial \alpha_2} & = & 0, \\ 3V_\sigma(\sigma)l + amV'_\sigma(\sigma) & = & 0. \end{array}\right\} \tag{5.13}$$

Now using separation of variables the above set of partial differential equations has an explicit solution

$$\begin{aligned} l &= \frac{l_0\cosh(p\sigma)}{\sqrt{a}},\ m = \frac{-3l_0\sinh(p\sigma)}{2a^{\frac{3}{2}}p}, \\ s &= \frac{-3l_0\alpha_1^*\cosh(p\sigma)}{2a^{\frac{3}{2}}},\ u = \frac{-3l_0\alpha_2^*\cosh(p\sigma)}{2a^{\frac{3}{2}}}, \end{aligned} \tag{5.14}$$

with $p = \sqrt{\frac{3}{16\alpha_0}}$. Also the potential function has an explicit solution by solving the last equation of (5.13) as

$$V_\sigma(\sigma) = V_0\sinh^2(p\sigma). \tag{5.15}$$

The explicit form of the conserved quantities are given by:

$$Q = 12\alpha_0 l_0 \sqrt{a}\dot{a}\cosh(p\sigma) + \frac{3l_0\dot{\sigma}a^{\frac{3}{2}}}{2p}\sinh(p\sigma) + 3l_0\alpha_0 a^{\frac{3}{2}}(\alpha_1^*\dot{\alpha_1} + \alpha_2^*\dot{\alpha_2})\cosh(p\sigma), \quad (5.16)$$

$$E = 6\alpha_0 a\dot{a}^2 - \frac{a^3\dot{\sigma}^2}{2} - \alpha_0 a^3(\dot{\alpha_1}^2 + \dot{\alpha_2}^2) - V_0 a^3\sinh^2(p\sigma). \quad (5.17)$$

To study the physical space singularity we start with the kinetic metric associated with present Lagrangian (7.8)

$$dS_4^{(k)^2} = \alpha_0 a^3\left[-\frac{6}{a^2}da^2 + d\alpha^2 + \frac{1}{2\alpha_0}d\phi^2\right], \quad (5.18)$$

with $d\alpha^2 = d\alpha_1^2 + d\alpha_2^2$. There is associated effective potential $V_{eff} = a^3V_\sigma(\sigma)$. As the Lagrangian can be written for a point-like of the form $L = T - V$ so the Noether point symmetries correspond to the elements of the homothetic group of the kinetic metric $dS_4^{(k)^2}$ [237]. Further, due to conformal decomposable nature of the above kinetic metric as

$$dS_4^{(k)^2} = -\frac{6}{a^2}da^2 + d\alpha^2 + \frac{1}{2\alpha_0}d\phi^2, \quad (5.19)$$

there exists a gradient homothetic vector (HV) $G_V = \frac{2}{3}a\partial_a$ with $\psi G_V = 1$ and it is not possible to generate a Noether point symmetry for the given Lagrangian corresponding to this gradient HV. Moreover, the $2D$ space in $\{\alpha, \phi\}$ is a space of constant curvature which admits 3 Killing vectors spanning the $So(3)$ group. Further, the Lagrangian (7.8) can be written as

$$L = -\frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2(\dot{\delta_1}^2 + \dot{\delta_2}^2) - u^2V_\delta(\delta_3), \quad (5.20)$$

by choosing

$$\begin{aligned} u &= 4\sqrt{\frac{\alpha_0}{3}}a^{\frac{3}{2}}, \ \delta_1 = \frac{1}{2}\sqrt{\frac{3}{2}}\alpha, \\ \delta_2 &= \frac{1}{4}\sqrt{\frac{3}{2\alpha_0}}\phi \text{ and } V_\delta(\delta_3) = \frac{3}{16\alpha_0}V_\sigma, \end{aligned} \quad (5.21)$$

and consequently the kinetic metric takes the simple form

$$dS_4^{(k)^2} = -du^2 + u^2(d\delta_1^2 + d\delta_2^2). \quad (5.22)$$

Due to conformal flat nature of the above kinetic metric there are $7D$ ($\frac{n(n+1)}{2}+1$ for $n = 3$) homothetic Lie algebra formed by the following vectors:

(a) Three translational gradient killing vectors $\overrightarrow{\mu}^{i},\ i=1,2,3$

$$\overrightarrow{\mu}^{(1)} = -\frac{1}{2}\Big[(1+\delta_2^2)e^{\delta_1}+e^{-\delta_1}\Big]\partial_u+\frac{1}{2u}\Big[(1+\delta_2^2)-e^{-\delta_1}\Big]\partial_{\delta_1}+\frac{1}{u}\delta_2 e^{-\delta_1}\partial_{\delta_2},$$
$$\overrightarrow{\mu}^{(2)} = -\frac{1}{2}\Big[(1-\delta_2^2)e^{\delta_1}-e^{-\delta_1}\Big]\partial_u+\frac{1}{2u}\Big[(1-\delta_2^2)+e^{-\delta_1}\Big]\partial_{\delta_1}-\frac{1}{u}\delta_2 e^{-\delta_1}\partial_{\delta_2},$$
$$\overrightarrow{\mu}^{(3)} = -e^{\delta_1}\partial_u+\frac{1}{u}e^{\delta_1}\partial_{\delta_1}+\frac{1}{u}e^{-\delta_1}\partial_{\delta_2}.$$

The associated gradient Killing function are $K^{(1)} = \frac{1}{2}u\big[(1+\delta_2^2)e^{\delta_1}+e^{-\delta_1}\big]$, $K^{(2)} = \frac{1}{2}u\big[(1-\delta_2^2)e^{\delta_1}-e^{-\delta_1}\big]$ and $K^{(3)} = u\delta_3 e^{\delta_2}$.

(b) Three nongradient Killing vectors which are rotational in nature and span the $So(3)$ algebra:

$$\overrightarrow{\mu}_{12}=\partial_{\delta_3},\ \overrightarrow{\mu}_{23}=\partial_{\delta_1}+\delta_2\partial_{\delta_2},\ \overrightarrow{\mu}_{32}=\delta_2\partial_{\delta_1}+\frac{1}{2}(\delta_2^2-1)\partial_{\delta_2}.$$

(c) The gradient HV which we have already defined.

5.4 Noethehr symmetry analysis and quantum formulation

The present section describes the formulation of canonical quantization of the Bianchi-I cosmological model using the Noether symmetry analysis. The conserved charge (or Noether charge) corresponding to the present Noether symmetry are nothing but the conserved momenta associated with the cyclic variables and as a result they identify the oscillatory part of the wave function of the Universe. Subsequently, the Wheeler DeWitt equation simplifies and may even solvable.

In cosmology, instead of superspaces (due to its complicated nature) usually the simplest and widely used minisuperspaces [236] are widely used in the literature. The homogeneous and isotropic nature of the minisuperspaces are characterized by the homogeneous Lapse function (i.e., $N = N(t)$) and vanishing shift vector. Thus the 4D space time metric can have the explicit form as

$$ds^2 = -N^2(t)\ dt^2 + h_{ab}(x,t)dx^a\ dx^b,\ (a,b)=(1,2,3). \tag{5.23}$$

Due to this (3+1)–decomposition, the Einstein-Hilbert action has the explicit expression

$$\mathcal{A}(h_{ab}, N) = \int dt\ d^3x\ N\sqrt{h}\Big[k_{ab}\ k^{ab} - k^2 + (3)_R\Big], \tag{5.24}$$

where k_{ab} represents the extrinsic curvature of the space, $k = k_{ab}h^{ab}$ and $(3)_R$ is the curvature scalar of the space. Further due to homogeneous nature, h_{ab} can be described by a finite number of time functions $l^\alpha(t)$, $\alpha = 0, 1, 2...., n-1$ and consequently the above action takes the form [238, 239]

$$\mathcal{A}(l^\alpha(t), N) = \int_0^1 Ndt\Big[\frac{1}{2N^2}\mu_{\alpha\beta}(l)\ \dot{l}^\alpha(t)\dot{l}^\beta(t) - U(l)\Big]. \tag{5.25}$$

This action can be interpreted as that of a relativistic point particle with $U(l)$ as the self interacting potential in a nD curved space time and the equation of motion of this particle has the explicit form

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{l}^\alpha}{N}\right) + \frac{1}{N^2}\Gamma^\alpha_{\beta\gamma}\ \dot{l}^\beta\dot{l}^\gamma + \mu^{\alpha\beta}\frac{\partial v}{\partial l^\beta} = 0. \tag{5.26}$$

The variation of the action (7.23) with respect to the Lapse function gives the scalar constant equation (also known as Hamiltonian constraint)

$$\frac{1}{2N^2}\mu_{\alpha\beta}(l)\ \dot{l}^\alpha(t)\dot{l}^\beta(t) + U(l) = 0. \tag{5.27}$$

(Note that in general there are vector constraint equations which vanish for the present minisuperspace model).

For the Hamiltonian formulation, the momenta conjugate to l^α is defined as

$$\Pi_\alpha = \frac{\partial L}{\partial \dot{l}^\alpha} = \mu_{\alpha\beta}\frac{\dot{l}^\beta}{N}. \tag{5.28}$$

So the Hamiltonian of the system takes the form

$$H = \Pi_\alpha\dot{l}^\alpha - L = N\left[\frac{1}{2}\mu^{\alpha\beta}\Pi_\alpha\Pi_\beta + U(q)\right] \equiv N\mathcal{H}. \tag{5.29}$$

It is interesting to note that the Hamiltonian constraint (7.25) when expressed in terms of momenta from equation (7.26) then the above Hamiltonian coincides with the scalar constraint equation (i.e., $H \equiv 0$).

Now using the operator conversion of the momenta variables for quantization programme the above Hamiltonian (7.27) (i.e., Hamiltonian constraint) becomes the well known Wheeler DeWitt (WD) equation in quantum cosmology as

$$\mathcal{H}\left(l^{\alpha}, -i\hbar\frac{\partial}{\partial l^{\alpha}}\right)\psi(l^{\alpha}) = 0. \tag{5.30}$$

This second order hyperbolic differential equation has issue with ambiguity in operator ordering and a possible way of resolution is to impose the covariant nature of the minisuperspace quantization.

Moreover, in this minisuperspace quantization scheme, the conserved current associated with the probability measure is given by

$$\vec{J} = \frac{1}{2}\left(\psi^{*} \bigtriangledown \psi - \psi \bigtriangledown \psi^{*}\right), \tag{5.31}$$

with $\vec{\bigtriangledown}.\vec{J} = 0$. Also the probability measure can be written as

$$dp = |\psi(l^{\alpha})|^{2}\; dV, \tag{5.32}$$

with dV, a volume element on minisuperspace.

In the present Bianchi-I cosmological model, the minisuperspace is a $3D$ space $\{a, \alpha, \phi\}$ with the corresponding conjugate momenta as

$$\begin{aligned} p_a &= \frac{\partial L}{\partial \dot{a}} = 12U_1 a\dot{a}, \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = -a^3\phi, \\ p_\alpha &= \frac{\partial L}{\partial \dot{\alpha}} = -2U_1 a^3\dot{\alpha} = \sigma(\text{say}). \end{aligned} \tag{5.33}$$

So the Hamiltonian (or Hamiltonian constraint) of the system takes the form

$$\mathcal{H} = \frac{p_a^2}{24U_1 a} - \frac{1}{4U_1 a^3}p_\alpha^2 + \frac{p_\phi^2}{2a^3} - V_0 a^3 \sinh^3(p\phi) \equiv 0. \tag{5.34}$$

The operator conversion i.e., $p_a \to -i\frac{\partial}{\partial a}, p_\phi \to -i\frac{\partial}{\partial \phi}, p_\alpha \to -i\frac{\partial}{\partial \alpha}$ of the above Hamiltonian constraint gives the WD equation in explicit form as

$$\left[-\frac{1}{24U_1 a}\frac{\partial^2}{\partial a^2} + \frac{1}{4U_1 a^3}\frac{\partial^2}{\partial \alpha^2} - \frac{1}{2a^3}\frac{\partial^2}{\partial \phi^2} - V_0 a^3 \sinh^3(p\phi)\right]\psi(a,\phi,\alpha) = 0. \tag{5.35}$$

Now due to cyclic nature of the variable α, the momentum p_α is conserved quantity, so in the operator version one has the first order differential as

$$-i\frac{\partial \psi}{\partial \alpha} = \sigma_1 \psi. \tag{5.36}$$

Now, if ψ has the following product form

$$\psi(a,\alpha,\phi) = \psi_1(a,\phi)\psi_2(\alpha), \tag{5.37}$$

then equation (7.34) gives ψ_2 as a periodic function as

$$\psi_2(\alpha) = \alpha_0 e^{i\sigma_1 \alpha}. \tag{5.38}$$

Using equation (7.35) and (7.36) in equation (7.33) one has the $2D$ partial differential equation as

$$\frac{\partial^2 \psi_1}{\partial a^2} + 12U_1 \frac{\partial^2 \psi_1}{\partial \phi^2} + \{6a^3\sigma_1^2 + 24U_1 a^6 \sinh^3(p\phi)\}\psi_1 = 0. \tag{5.39}$$

The above differential equation (7.37) can not be solved analytically. So ψ_1 has been evaluated numerically and $|\psi|^2$ is presented graphically in figures (6.3) and (5.2) for different choices of parameters involved. The graphical presentation shows that as $a \to 0$, $|\psi|^2$ approaches a non zero finite constant i.e., classical big-bang singularity has finite non zero probability in the quantum description while in Fig 5.2. the probability amplitude approaches to zero values as scale factor approaches to classical singularity.

5.5 Causal Interpretation: Bohmian Trajectory

In metric formulation of Einstein gravity four constraints appear and are classified as scalar constraint (or Hamiltonian constraint) and vector (or super momentum) constraint. However, for homogeneous minisuperspace model the vector constraint vanishes identically and the quantum version of the remaining scalar constraint gives the WD equation (in equation (7.28)) [240, 241, 242, 243, 244]. Now in analogy with WKB approach one may

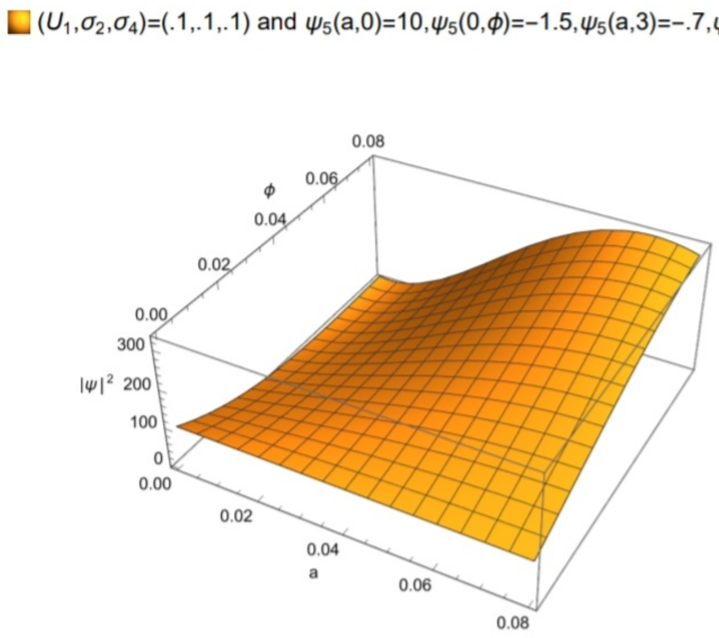

FIGURE 5.1: Represents nonzero probability amplitude when $a \to 0$

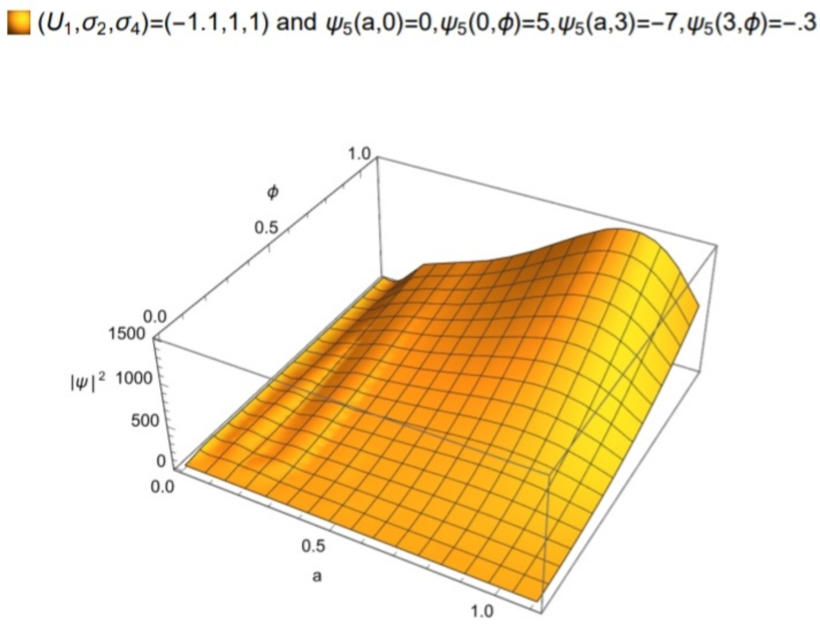

FIGURE 5.2: Represents zero probability amplitude when $a \to 0$

write the wave function ψ as

$$\psi = M(l)\exp\left[\frac{i}{\hbar}T(l)\right]. \tag{5.40}$$

Now substituting (7.38) into the WD equation (7.28) one has the quantum version of the Hamilton-Jacobi (HJ) as

$$\frac{1}{2}s_{\alpha\beta}(l_\mu)\frac{\partial T}{\partial l_\alpha}\frac{\partial T}{\partial l_\beta} + U_0(l_\mu) + \Sigma(l_\mu) = 0. \tag{5.41}$$

Here $s_{\alpha\beta}$ is the reduced supermetric due to the given minisuperspace and $U_0(l_\mu)$ represents the scalar curvature density particularization of the space like hyper-surfaces and the extra term $\Sigma(l_\mu)$ having expression

$$\Sigma(l_\mu) = -\frac{1}{M}s_{\alpha\beta}\frac{\partial^2 M}{\partial l_\alpha \partial l_\beta}, \tag{5.42}$$

is termed as quantum potential.

Now in quantum cosmology, by virtue of causal interpretation, the trajectories $l_\alpha(t)$ should be real observer independent are classified by the above HJ equation (7.39). Now if one compares the momenta

$$\Pi^\alpha = \frac{\partial T}{\partial l_\alpha}$$

with the usual momentum–velocity relation namely

$$\Pi^\alpha = s^{\alpha\beta}\frac{1}{N}\frac{\partial l_\beta}{\partial t}, \tag{5.43}$$

then the above equivalence identifies the quantum trajectories (or the Bohmian trajectories) by the first order differential equation as [245]

$$\frac{\partial S(l_\alpha)}{\partial l_\alpha} = s^{\alpha\beta}\frac{1}{N}\frac{\partial l_\beta}{\partial t}. \tag{5.44}$$

Thus due to time re-parametrization invariance, there should not be any problem of time in the causal interpretation of minisuperspace quantum cosmology and hence one may choose the gauge freedom $\dot{N} = 0$. In the present context the quantum potential has the expression

$$\Sigma(a, \phi, \alpha) = -\frac{1}{M}\left[\frac{a^2}{\sigma}\frac{\partial^2 M}{\partial a^2} - 2U_1\frac{\partial^2 M}{\partial \phi^2} - \frac{\partial^2 M}{\partial \alpha^2}\right], \tag{5.45}$$

with quantum modified HJ equation has the explicit form

$$\frac{1}{\hbar^2}\left[\frac{a^2}{\sigma}\left(\frac{\partial T}{\partial a}\right)^2-2U_1\left(\frac{\partial T}{\partial \phi}\right)^2-\left(\frac{\partial T}{\partial \alpha}\right)^2\right]+\Sigma+4U_1V_0a^6\sinh^2(p\phi)=0. \tag{5.46}$$

Thus using (7.42), the explicit form of the Bohmian trajectories are given by

$$\frac{\partial T}{\partial a}=12U_1a\dot{a},\quad \frac{\partial T}{\partial \phi}=-a^3\dot{\phi},\quad \frac{\partial T}{\partial \alpha}=-2U_1a^3\dot{\alpha}. \tag{5.47}$$

Now for simplicity, if the wave function is in the product separable form as

$$\psi=A_{r_1}(a)B_{r_1r_2}(\phi)C_{r_2}(\alpha), \tag{5.48}$$

then the WKB ansatz takes the form as

$$\psi=M_1(a)M_2(\phi)M_3(\alpha)\exp\left[i\{s_1(x)+s_2(y)+s_3\}\right]. \tag{5.49}$$

Note that due to this spliting of the above WKB ansatz the Bohmian trajectories (7.45) are independent of $(\phi,\alpha),(\alpha,a)$ and (a,ϕ) respectively. Also the quantum potential (7.43) has the separable form as

$$\Sigma(a,\phi,\alpha)=\Sigma_1(a)+\Sigma_2(\phi)+\Sigma_3(\alpha). \tag{5.50}$$

Now to examine the behaviour of classical singularity using the above quantum trajectory one should have scale factor 'a' to be infinitesimally small. Using (7.45) and (7.48) into the quantum modified HJ equation (7.44) one has (choosing $\hbar=1$)

$$24a^4U_1^2\left(\frac{da}{dt}\right)^2-2U_1a^6\left(\frac{d\phi}{dt}\right)^2-4U_1^2a^6\left(\frac{d\alpha}{dt}\right)^2+\Sigma_1(a)+\Sigma_2(\phi)+\Sigma_3(\alpha)+4U_1V_0a^6\sinh^2(p\phi)=0. \tag{5.51}$$

We shall now solve the above HJ equation for the following particular choice of the quantum potential. For simplicity we choose $\Sigma_1(a)=\Sigma_0$, a constant, $\Sigma_2=0=\Sigma_3$. Then the above differential equation simplifies to the following separable form as

$$24\frac{U_1^2}{a^2}\left(\frac{da}{dt}\right)^2+\frac{\Sigma_0}{a^6}=2U_1\left(\frac{d\phi}{dt}\right)^2-4U_1V_0\sinh^2(p\phi)=4U_1\left(\frac{d\alpha}{dt}\right)^2=\lambda^2(\text{say}), \tag{5.52}$$

with λ, the separable constant. Thus we have the following ordinary differential equations to determine the configuration space variables (a, ϕ, α) :

$$24\frac{U_1^2}{a^2}\left(\frac{da}{dt}\right)^2 + \frac{\Sigma_0}{a^6} = \lambda^2, \tag{5.53}$$

$$2U_1\left(\frac{d\phi}{dt}\right)^2 - 4U_1V_0\sinh^2(p\phi) = \lambda^2, \tag{5.54}$$

$$4U_1\left(\frac{d\alpha}{dt}\right)^2 = \lambda^2. \tag{5.55}$$

The solution takes the form

$$\begin{aligned} a^3 &= \left(a_0e^{\lambda t} + a_1e^{-\lambda t}\right), \\ \phi(t) &= \frac{1}{p}\sinh^{-1}\left[\tan\{p(\sqrt{2V_0}t + \phi_0)\}\right], \\ \alpha(t) &= \frac{\lambda}{2U_1}t + \alpha_0. \end{aligned}$$

The above solution shows that 'a' can not approach to singularity (i.e., $a \longrightarrow 0$) for any finite 't', provided a_0 and a are of same sign. Thus classical singularity may be avoided by the above quantum description with causal interpretation.

5.6 Brief Summary

The present work is an example how symmetry analysis can describe complex gravity theories. Here modified teleparallel gravity theory has been examined both classically and in quantum formulation. Though due to its complexity the Noether symmetry analysis can not simplify the Lagrangian to that extend that classical solution can be evaluated. However, the physical kinetic metric has been analyzed from the point of view of the conformal symmetry and it is possible to identify the homothetic and killing vector fields. Subsequently, quantum description has been presented in details. At first, in canonical quantization programme the WD equation has been formulated and its periodic part has been identified by using the conserved Noether charge(in operator version). Then the WD equation has been solved numerically and the probability amplitude (i.e., $|\psi|^2$) has been presented graphically in FIG. 6.3 and 5.2 for two different choices of the parameters involved. FIG 6.3. shows that classical singularity still persists in quantum description while in FIG 5.2. shows that quantum prescription may avoided the classical singularity. Finally, causal interpretation of quantum mechanics as Bohmian trajectory has been formulated. The classical trajectory has been modified through quantum potential and a particular Bohmian trajectory has been presented which avoids classical singularity. Finally, one may conclude that symmetry analysis may provide physical consequences for complicated cosmological models.

Chapter 6

A study of classical and quantum cosmology in modified teleparallel gravity and the role of Noether symmetry

6.1 Prelude

The cosmologists have been trying their best to identify the modification in standard cosmology so that it is in accord with the recent observational evidences [246, 137, 247, 248, 249]. So far they have opted for two possible modification namely (i) introduction of some new kind of exotic matter having large negative pressure (known as dark energy (DE)) (ii) modification of Einstein gravity so that the extra geometric (or physical) terms may act as repulsive matter component. For modified gravity approach it is desirable that at a certain limit of the parameter(s) the theory should corresponds to Einstein gravity. It is well known that an alternate consistent description of gravitational interaction is the torsion of space-time, eliminating the effect of curvature. This approach is known as the Teleparallel gravity theory [250, 143] and is demonstrably equivalent to Einstein gravity. Here, the Teleparallel action has been modified further by considering a scalar field having coupling between the scalar field potential and electromagnetic field. The modified Einstein field equations of the present modified gravity model are coupled and non-linear in nature. So it is very hard (impossible) to find solutions by the usual known mathematical techniques. As a result, it is almost impossible to infer any cosmological prediction. In this context, Noether symmetry approach (NSA) [251, 252, 253, 254] is an appropriate mathematical tool to describe such situations [255, 256, 257, 258, 259, 260, 261, 262]. The advantage of NSA is of three fold namely: (i) the coupling/potential (arbitrary) functions and parameters in the theory may be determined in course of symmetry analysis is rather than choosing them phenomenologically, (ii) by identification of the cyclic variable along the symmetry vector the field equations get simplified or even be solved, (iii) the associated conserved currents (charges) put one step forward for quantum cosmological description. The following sections are planned as follows: at first the basic equation of this modified gravity model has been discussed in Section II and then a brief overview of Noether symmetry is discussed in Section III. After that the derivation of the Noether symmetry vector and the classical solution is presented in Section IV. Section V deals with the Symmetry analysis in quantum cosmology and the paper ends with a short discussion.

6.2 The Modified gravity Model: Basic equations

A single scalar field in Einstein gravity is well known in cosmology. This scalar field in inflationary era is termed as inflation field. It is generally speculated that gauge fields has the dominant role in the early exponential expansion. So it is a natural question whether vector fields and non-linear electromegnetic fields may produce such negative pressure. In this context, a well known gravitational action used in the literature [263, 264, 265, 266] is

$$\mathcal{A}_0 = \int d^4x\sqrt{-g}\left[\frac{M_{PL}^2}{2}R + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) - \frac{1}{4}f(\phi)^2 F_{\mu\nu}F^{\mu\nu}\right], \tag{6.1}$$

and also it has been examined whether the present late-time accelerated phase may be described by such action integral. As a further extension it is reasonable have a unified model describing all the cosmological scenarios with a single scalar field model, teleparallel equivalence of gravity theory is considered and the action integrals becomes

$$\mathcal{A} = \int d^4xe\left[\frac{M_{PL}^2}{2}T + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) - \frac{1}{4}\{f(\phi)\}^2 F_{\mu\nu}F^{\mu\nu}\right]. \tag{6.2}$$

Here $e = \det e^i_\mu = \sqrt{-g}$, e^i_μ are the vierbein (tetrad) basis, T is the usual torsion scalar, notationally $\phi_{,\mu}$ denotes the components of the gradient of ϕ, $V(\phi)$ is the standard scalar field potential, $f(\phi)$ is the coupling function (with $f^2(\phi)$, the gauge kinetic function), $F_{\mu\nu}$ is the electromagnetic field tensor and M_{PL} is the reduced Planck mass. For the electromagnetic four vector A_μ, the field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv A_{\nu,\mu} - A_{\mu,\nu}. \tag{6.3}$$

In the background of homogeneous and isotropic flat FLAT model i.e.,

$$dS^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{6.4}$$

the torsion scalar simplifies to $T = -6H^2$ and the electromagnetic four vector can be written as

$$A_\mu = (A_0, A_0, A_2, A_3) = \left(A_0(t), \frac{A(t)}{\sqrt{3}}, \frac{A(t)}{\sqrt{3}}, \frac{A(t)}{\sqrt{3}}\right), \tag{6.5}$$

so that $F_{\mu\nu}F^{\mu\nu} = -\frac{2\dot{A}^2(t)}{a^2(t)}$. Thus without any loss of generality one may choose the gauge $A_0 = 0$, using the gauge invariance in ref.[267, 268].

$$L(T, \dot{T}) = -3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 - a^3V(\phi) + \frac{1}{2}af^2(\phi)\dot{A}^2. \tag{6.6}$$

So here the 3D configuration space is $T = (a, \phi, A)$ having 6D tangent space $\{a, \phi, A, \dot{a}, \dot{\phi}, \dot{A}\}$.

6.3 NOETHER SYMMETRY ANALYSIS: A brief OVERVIEW

Noether in her 1st theorem [161, 131, 233, 235, 269, 270] identified a symmetry vector and an associated conserved current corresponding to a physical system provided

- there must be a Lagrangian of the physical system
- the Lagrangian is invariant with respect to a vector field i.e., $\mathcal{L}_{\vec{\mathcal{X}}}L = \vec{\mathcal{X}}L = 0$

where $\vec{\mathcal{X}}$ lies in the tangent space and is termed as symmetry vector. Further, for a point-like canonical $L = L[\dot{\phi}^\alpha(x^i), \phi^\alpha(x^i)]$ with $\phi^\alpha(x^i)$being the generalized co-ordinates then a simple algebra with Euler-Lagrange equations gives up the symmetry vector as

$$\vec{\mathcal{X}} = \sigma^\alpha \frac{\partial}{\partial \phi^\alpha} + (\partial_j \sigma^\alpha)\frac{\partial}{\partial(\partial_j \phi^\alpha)} \tag{6.7}$$

$$\mathcal{L}_{\vec{\mathcal{X}}}L = 0$$

and a conserved current vector as (known as Noether current vector)

$$I^j = \sigma^\alpha \frac{\partial L}{\partial(\partial_j \phi^\alpha)} \tag{6.8}$$

$$\partial_j I^j = 0.$$

Now from the Lagrangian dynamics the associated energy function is given by

$$E = \dot{\phi}^\alpha \frac{\partial L}{\partial \dot{\phi}^\alpha} - L. \tag{6.9}$$

Also there is an associated conserved charge (known as Noether charge) as

$$Q = \int I^t d^3x. \tag{6.10}$$

However, as in the context of the cosmology the background geometry is considered as homogeneous and isotropic so all the variables depend only on time and as a result Noether current coincides with Noether current and also the energy function is conserved.

From geometrical point view, one express the Noether charge as the inner product of the symmetry vector (i.e., infinitesimal generator) with the Cartan one form [160] i.e.,

$$Q = i_{\overrightarrow{\mathcal{X}}} \Theta_L \tag{6.11}$$

where

$$\Theta_L = \frac{\partial L}{\partial \phi^\alpha} d\phi^\alpha,$$

is the Cartan one form and $i_{\overrightarrow{\mathcal{X}}}$ indicates the inner product with the symmetry vector field $\overrightarrow{\mathcal{X}}$.

If a transformation: $\phi^\alpha \rightarrow \psi^\alpha$ is considered in the augmented space then the symmetry vector becomes

$$\overrightarrow{\mathcal{X}}_T = (i_{\overrightarrow{\mathcal{X}}} d\psi^\alpha) \frac{\partial}{\partial \psi^\alpha} + \frac{d}{dt}(i_{\overrightarrow{\mathcal{X}}} d\psi^\alpha) \frac{\partial}{\partial \dot{\psi}^\alpha}. \tag{6.12}$$

Symmetrically, this transformed symmetry vector can be interpreted as a lift of the vector field $\overrightarrow{\mathcal{X}}$ in the augmented space. Now by restricting the above transformation so that the inner product along a typical direction vanishes i.e.,

$$i_{\overrightarrow{\mathcal{X}}} d\psi^\alpha = \delta^\alpha_\theta. \tag{6.13}$$

As a result, the transformed symmetry vector simplifies to $\overrightarrow{\mathcal{X}}_T = \frac{\partial}{\partial \psi^\theta}$ and one has $\frac{\partial L}{\partial \psi^\theta} = 0$. This shows that the transformed augmented variable ψ^θ is a cyclic variables and the transformed infinitesimal generator is directed along the coordinate line of the cyclic variable. Further, due to cyclic nature of the transformed augmented variable ψ^θ, from Lagrangian mechanics the momentum associated with it namely,

$$p_\theta = \frac{\partial L_T}{\partial \dot{\psi^\theta}} = \Pi_0 \tag{6.14}$$

is also conserved momentum equation (6.14) can be written as a first order partial differential equation replacing the momentum variable by the corresponding differential operator (operating on the wave function as (choosing $\hbar = 1$))

$$-i\frac{\partial}{\partial \psi^\theta} |\psi\rangle = \Pi_0 |\psi\rangle \,, \tag{6.15}$$

which has a simple oscillatory solution as

$$|\psi\rangle = e^{i\Pi_0\psi_0} |\psi\rangle \,. \tag{6.16}$$

Here Π_0 is assumed to be real and $|\psi\rangle$ is the wave function corresponding to the other augmented variables. It is to be noted that if we have more than one conserved momenta then the oscillatory parts of the wave function correspond to those directions. Thus it is reasonable to speculate that oscillatory part of the wave function indicates the existence of Noether symmetry [271] of the physical system.

6.4 Derivation of Symmetry Vector and Classical Solution

As the present physical problem has $3D$ configuration space so the symmetry vector in explicit form has the expression

$$\vec{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}} + \dot{\gamma}\frac{\partial}{\partial \dot{A}} \,. \tag{6.17}$$

The unknown coefficients are function of the configuration space variables i.e.,

$$\alpha = \alpha(a, \phi, A), \beta = \beta(a, \phi, A), \gamma = \gamma(a, \phi, A)$$

with $\dot{\alpha} = \frac{\partial \alpha}{\partial a}\dot{a} + \frac{\partial \alpha}{\partial \phi}\dot{\phi} + \frac{\partial \alpha}{\partial A}\dot{A}$ and so on.

The coefficients are determines from the Noether smmetry condition (i.e., $\mathcal{L}_{\overrightarrow{\mathcal{X}}} L = \overrightarrow{\mathcal{X}} L = 0$) by equating the coefficients of different powers of $\dot{a}, \dot{\phi}, \dot{A}$ and their products. As a result we have following set of first order partial differential equation

$$\alpha + 2a\frac{\partial \alpha}{\partial a} = 0, \tag{6.18}$$

$$3\alpha + 2a\frac{\partial \beta}{\partial \phi} = 0, \tag{6.19}$$

$$6\frac{\partial \alpha}{\partial \phi} - a^2\frac{\partial \beta}{\partial a} = 0, \tag{6.20}$$

$$\alpha f(\phi) + 2af'(\phi)\beta + 2af(\phi)\frac{\partial \gamma}{\partial A} = 0, \tag{6.21}$$

$$3aV(\phi) + a\beta V'(\phi) = 0. \tag{6.22}$$

The explicit solution for the coefficients and the unknown functions $f(\phi)$ and $V(\phi)$ are given by

$$\alpha = \frac{\alpha_0}{\sqrt{a}}\sinh(m\phi),\ \beta = \frac{-3\alpha_0}{2m}a^{-\frac{3}{2}}\cosh(m\phi),\ \gamma = \gamma_0,\ f(\phi) = f_0\cosh^{\frac{1}{3}}(m\phi),\ V(\phi) = V_0\cosh^2(m\phi), \tag{6.23}$$

where α_0, γ_0, f_0 and V_0 are arbitrary integration constants and $m = \pm\sqrt{\frac{3}{8}}$.

It is to be noted that the above symmetry analysis not only identifies the symmetry vector but also determines the unknown coupling function $f(\phi)$ and the potential function $V(\phi)$. Moreover, due to this Noether symmetry the associated conserbved quantity namely the Noether charge and conserved eenergy (due to homogeneity and isotropy of the space time) have the explicit expression:

$$Q = -6\alpha_0\sqrt{a}\dot{a}\sinh(m\phi) - \sqrt{6}\alpha_0 a^{\frac{3}{2}}\dot{\phi}\cosh(m\phi) + \gamma_0 f_0^2\cosh^{\frac{2}{3}}(m\phi)\dot{A}$$

and

$$E = -3a\dot{a}^2 + a^3\frac{\dot{\phi}^2}{2} + \frac{f_0^2}{2}a\cosh^{\frac{2}{3}}(m\phi)\dot{A}^2 + V_0a^3\cosh^2(m\phi). \tag{6.24}$$

Now in order to obtain the classical cosmological solutions one has to transform the configuration variables so that the Lagrangian and consequently the field equations become in much simpler form. Here we shall use the property of Noether symmetry vector to identify a cyclic variable in the configuration space by imposing the condition (6.13).

In the present $3D$ space (a, ϕ, A) we choose the transformation as $(a, \phi, A) \to (u, v, A)$ so that the symmetry vector $\vec{X}$ satisfies the following inner products:

$$i_{\vec{X}}du = 1, \quad \text{and} \quad i_{\vec{X}}dv = 0. \tag{6.25}$$

Now solving the above two first order partial differential equation by Lagranges method the explicit form of u and v are

$$u = -\frac{2}{3\alpha_0} a^{\frac{3}{2}} \sinh(m\phi), \quad \text{and} \quad v = a^{\frac{3}{2}} \cosh(m\phi). \tag{6.26}$$

So the transformed Lagrangian takes the form

$$L_T = -\frac{4}{3}\dot{v}^2 + 6\alpha_0 \dot{u}^2 + \frac{f_0 A^2}{2} v^{\frac{2}{3}} - V_0 v^2. \tag{6.27}$$

The Euler-Lagrange equation corresponding to the above simplified Lagrangian (i.e., (7.23)) can be solved easily, which is given by

$$\begin{aligned}
u(t) &= \frac{c_1 t}{12\alpha_0} + c_2, \; v(t) = c_3 \cosh\left(\sqrt{\frac{3V_0}{4}} t + c_4\right), \; A(t) = c_5, \; \text{for } V_0 > 0, \\
u(t) &= \frac{c_1 t}{12\alpha_0} + c_2, \; v(t) = c_3 \cos\left(\sqrt{\frac{3V_0}{4}} t + c_4\right), \; A(t) = c_5, \; \text{for } V_0 < 0,
\end{aligned} \tag{6.28}$$

with c_1, c_2, c_3, c_4 and c_5 are arbitrary integration constants.

So the classical solution for the scale factor 'a', the scalar field 'ϕ' and the electromagnetic potential 'A' have the explicit expression:

Case: $V_0 > 0$

$$\begin{aligned}
a(t) &= \left\{ c_3^2 \cosh^2\left(\sqrt{\frac{3V_0}{4}} t + c_4\right) - \left(\frac{c_1 t}{8} + \frac{c_2 \alpha_0}{2}\right)^2 \right\}^{\frac{1}{3}}, \\
\phi(t) &= \frac{1}{m} \tanh^{-1}\left[-\frac{\frac{c_1 t}{8} + \frac{3 c_2 \alpha_0}{2}}{c_3 \cosh\left(\sqrt{\frac{3V_0}{4}} t + c_4\right)} \right], \\
A(t) &= c_5,
\end{aligned} \tag{6.29}$$

Case: $V_0 < 0$

$$\begin{aligned}
a(t) &= \left\{c_3^2\cos^2\left(\sqrt{\frac{3V_0}{4}}t+c_4\right)-\left(\frac{c_1 t}{8}+\frac{c_2\alpha_0}{2}\right)^2\right\}^{\frac{1}{3}}, \\
\phi(t) &= \frac{1}{m}\tan^{-1}\left[-\frac{\frac{c_1 t}{8}+\frac{3c_2\alpha_0}{2}}{c_3\cosh\left(\sqrt{\frac{3V_0}{4}}t+c_4\right)}\right], \\
A(t) &= c_5. \qquad (6.30)
\end{aligned}$$

Now to analyze the above cosmological solutions, we have plotted the important cosmological parameters namely the scale factor 'a', the Hubble parameter $H = \frac{\dot{a}}{a}$, and the deceleration parameter $q = -\frac{a\ddot{a}}{\dot{a}^2}$ against the cosmic time in figures (6.1, 6.2), for different values of the parameters involved. From the figures (i.e., FIG.6.1 and FIG.6.2) we see that the dynamics of the cosmic evolution is consistent with the observation at least qualitatively. The increasing nature of the scale factor indicates that the Universe is expanding throughout its evolution since the initial big bang singularity. The Hubble parameter gradually decreases since the beginning as predicted by the observational evidences. The variation of deceleration parameter is intersecting for $V_0 > 0$. The graph of 'q' in (FIG.6.1) shows that the Universe was in an accelerating phase in the early eras, subsequently it entered into an decelerating phase (where $q > 0$) and there after again the Universe goes to the present should acceleration expansion era. The graph also shows that in future the Universe should enter into an era of evolution where $q \to 0$ (asymptotically). For the other solution with $V_0 < 0$ the qualitative nature of the scale factor and the Hubble parameter remain same but the deceleration parameter in (FIG.6.2) has only one transition from the initial decelerated phase to present accelerated era of expansion–it can not predict the initial accelerated era of evolution.

6.5 Symmetry analysis in quantum cosmology: Canonical quantization

In the present section quantum cosmology has been developed with canonical quantization programme. Also it has been shown that Noether symmetry analysis, in particular the conserved charge and conserved energy (the Hamiltonian in phase space) corresponding to the symmetry has crucial role in this description. A general formulation of quantum cosmology is the following:

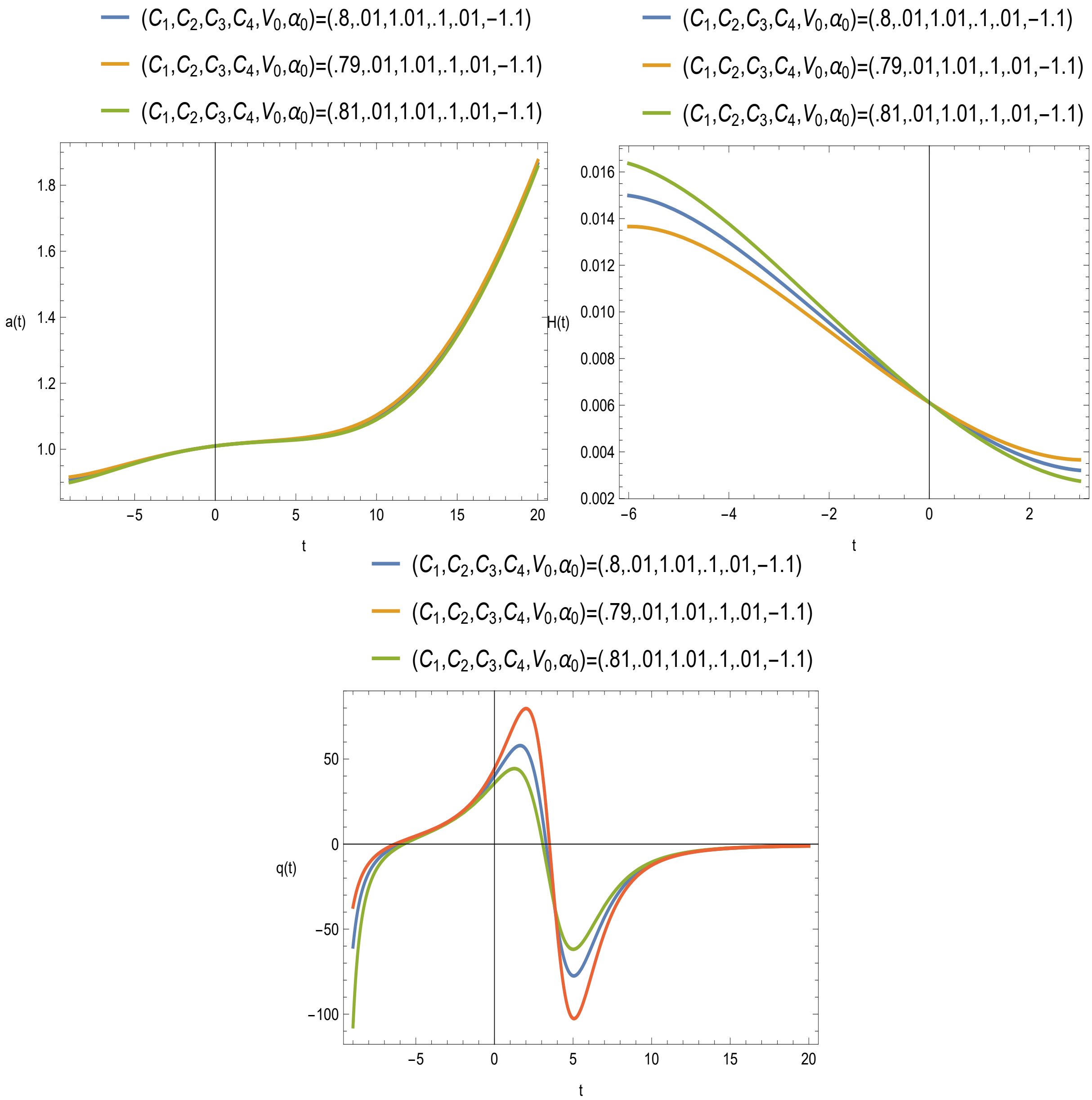

FIGURE 6.1: The graphical representation of scale factor $a(t)$ (top left), Hubble parameter $H(t)$ (top right) and deceleration parameter q (bottom) with respect to cosmic time t whenever $V_0 > 0$.

Quantum cosmology is formulated on superspace which is characterized by the metric and the matter field. Due to infinite dimensional nature of the superspace the geometrodynamics is usually restricted to minisuperspaces which are considered for simplicity as homogeneous and isotropic in nature, so that shift vector vanishes identically and lapse function has only time dependence. Thus using (3+1) decomposition the line element takes the form

$$ds^2 = -N^2(t)dt^2 + \mu_{\alpha\beta}(x,t)dx^\alpha dx^\beta, \tag{6.31}$$

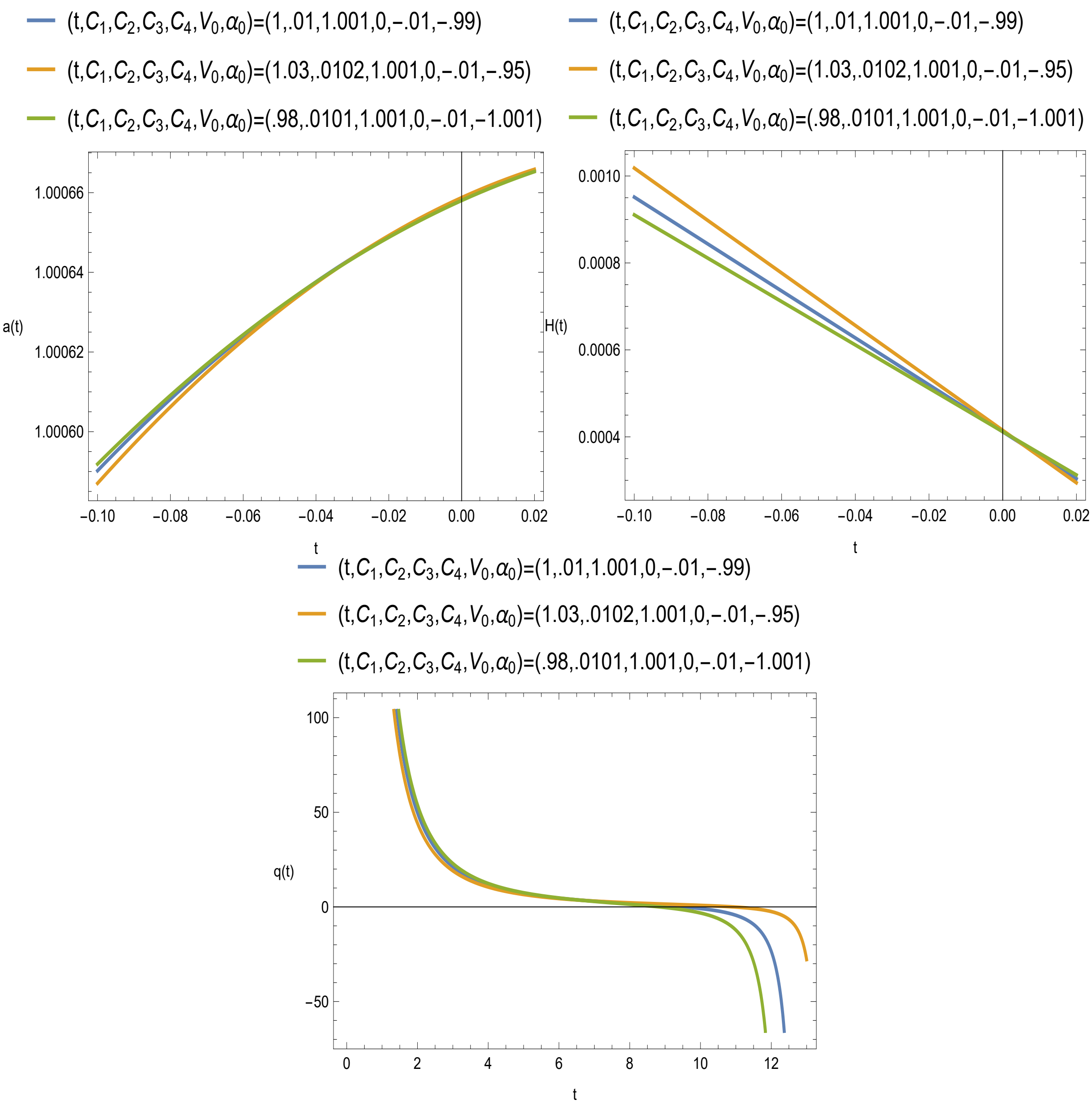

FIGURE 6.2: The graphical representation of scale factor $a(t)$ (top left), Hubble parameter $H(t)$ (top right) and deceleration parameter q (bottom) with respect to cosmic time t whenever $V_0 < 0$.

the Einstein-Hilbert action has the explicit form as

$$\mathcal{A}\big[\mu_{\alpha,\beta}, N\big] = \int dt d^3x N\sqrt{\mu}\Big[K_{\alpha\beta}K^{\alpha\beta} - K^2 + (3)_R\Big], \tag{6.32}$$

where the 3–space geometry is the characterized by the three space curvature $(3)_R$ and the extrinsic curvature $K_{\alpha\beta}$ of the 3–space hypersurface ($K = k_{\alpha\beta}h^{\alpha\beta}$ is the trace of the extrincic curvature). But due to simplified geometry (i.e., homogenity and isotropy) of the three space there is a finite number of degrees of freedom $q^{\alpha}(t)$, $(\alpha = 0, 1, 2, ..., n-1)$ so that equation-(7.28) takes the form

$$\mathcal{A}\big[q^{\alpha}(t), N(t)\big] = \int N dt \left[\frac{1}{2N^2}\mu_{\alpha\beta}\dot{q}^{\alpha}(t)\dot{q}^{\beta}(t) - V(u)\right]. \tag{6.33}$$

This action can be identified to a relativistic particle in nD having self interacting potential $V(u)$, having equation of motion

$$\frac{1}{N}\frac{d}{dt}\left(\frac{q^{\alpha}(t)}{N}\right) + \frac{1}{N^2}\Gamma^{\alpha}_{\beta\gamma}\dot{q}^{\beta}(t)\dot{q}^{\gamma}(t) + \mu^{\alpha\beta}\frac{\partial V}{\partial q^{\beta}} = 0 \tag{6.34}$$

together with a constraint equation

$$\frac{1}{2N^2}\mu_{\alpha\beta}\dot{q}^{\beta}\dot{q}^{\gamma} + V(q) = 0. \tag{6.35}$$

Further, it is interesting to note that the above constraint equation in the phase space is nothing but the Hamiltonian of the system i.e.,

$$\mathcal{H} = N\left[\frac{1}{2}\mu^{\alpha\beta}p_{\alpha}p_{\beta} + V(q)\right] \equiv 0 \tag{6.36}$$

with $p_{\alpha} = \mu^{\alpha\beta}\frac{\dot{q}^{\beta}}{N}$, the momentum conjugate to q^{α}.

In canonical quantization scheme, the operator version of the Hamiltonian operates on the wave function (of the Universe) and it results a second order hyperbolic partial differential equation as

$$\hat{\mathcal{H}}\left[q^{\alpha}, -i\frac{\partial}{\partial q^{\alpha}}\right]\psi(q^{\alpha}) = 0. \tag{6.37}$$

In quantum cosmological context this is termed as Wheeler-DeWitt equation-the basic equation in the context of quantum description of the Universe. The common problem in this quantization programme is the operator ordering problem which may be addressed by restricting the quantization process to be covariant in nature. In the present context it is useful to calculate the probability measure as

$$dp = |\psi(q^{\alpha})|^2 dV \tag{6.38}$$

with dV, the minisuperspace volume element.

In the present problem due to the transformed Lagrangian (7.23) the Hamiltonian takes the form as

$$\mathcal{H} = \frac{p_u^2}{24\alpha_0} - \frac{3}{16}p_v^2 + \frac{p_A^2}{2f_0}v^{-\frac{2}{3}} + V_0 v^2 \equiv 0 \tag{6.39}$$

Where,

$$p_{_u} = 12\alpha_0 \dot{u} = \sigma, \; p_{_v}^2 = -\frac{8}{3}\dot{v} \text{ and } p_{_A}^2 = f_0 \dot{A} v^{\frac{2}{3}} = \Sigma \tag{6.40}$$

are the momenta corresponding to the configuration variables u, v and w and due to cyclic nature of the variables u and A, σ and Σ are constants of motion. Further, it is to be noted that the equation (7.35) for the Hamiltonian is also known as scalar constraint equation due to the lapse function. Now, in course of quantization converting the momenta variables by the corresponding operators, one has the resulting WD equation as

$$-\frac{1}{24\alpha_0}\frac{\partial^2\psi}{\partial u^2} + \frac{3}{16}\frac{\partial^2\psi}{\partial v^2} - \frac{v^{-\frac{2}{3}}}{2f_0}\frac{\partial^2\psi}{\partial A^2} + V_0 v^2\psi = 0. \tag{6.41}$$

Now to solve this second order hyperbolic coupled p.d.e., the symmetry analysis or more specifically the conserved momenta play a crucial role. The operator version of the above conserved momenta (i.e., equation (7.36)) takes the form

$$-i\frac{\partial\psi}{\partial u} = \sigma\psi \text{ and } -i\frac{\partial\psi}{\partial A} = \Sigma\psi. \tag{6.42}$$

So if we write $\psi(u, v, A)$ in the in the separable from as

$$\psi = L(u)M(v)N(A) \tag{6.43}$$

then solving the first order differential equations in (7.38) one may have

$$\psi = M(v)\exp[i\sigma u + i\Sigma A] \tag{6.44}$$

using this form of ψ in the above WD equation (7.37), the differential equation for $M(v)$ becomes

$$\frac{d^2M}{dv^2} + \left(\frac{2\sigma^2}{9\sigma_0} + \frac{8\Sigma^2}{3f_0}v^{-\frac{2}{3}} + \frac{16v_0}{3}v^2\right)M = 0 \tag{6.45}$$

The second order differential equation for 'M' can not be solvable for 'M' analytically. However, it has been evaluate numerically and then the probability amplitude $|\psi|^2$ has been plotted for both $V_0 > 0$ and $V_0 < 0$ in FIG. (6.3) and (6.4). From the graphical representation it is clear that the probability amplitude approaches a non zero value or zero value as the scale factor $a \to 0$ for $V_0 > 0$ or $V_0 < 0$ respectively. This indicates that there is finite nonzero probability or zero probability for the initial big bang singularity corresponding to $V_0 > 0$ or $V_0 < 0$. In other words, big bang singularity can not be avoided for $V_0 > 0$ and it can be eliminated for $V_0 < 0$ by the above quantum description of the classical cosmological model.

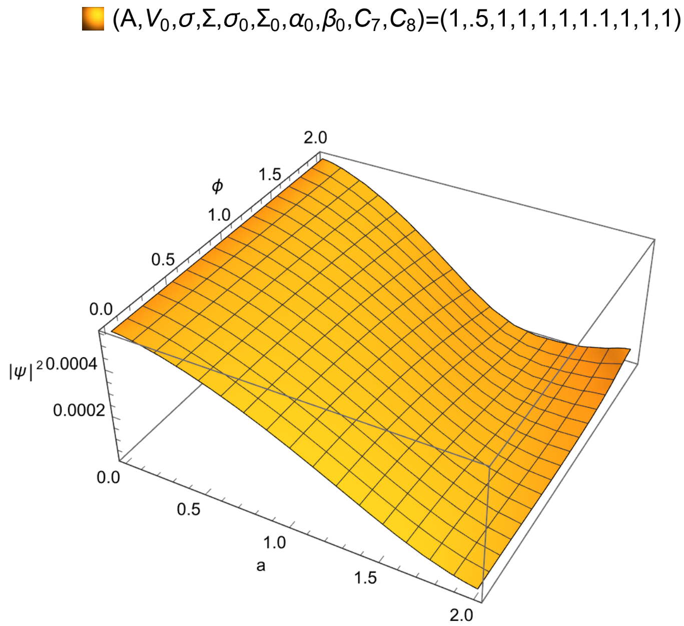

FIGURE 6.3: Representation of the probability amplitude for $V_0 > 0$

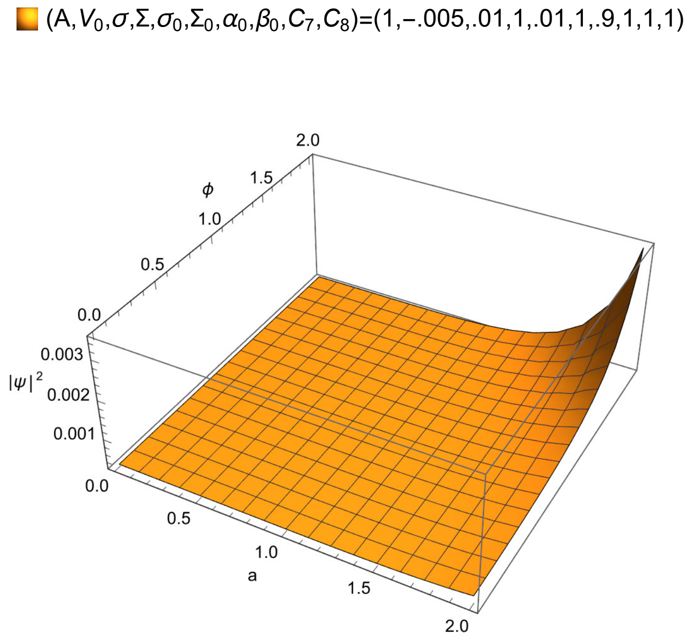

FIGURE 6.4: Representation of the probability amplitude for $V_0 < 0$

6.6 A brief overview of the work

The present chapter is a complicated classical cosmological model in the framework of Teleparallel gravity theory, an equivalent description of Einstein gravity. The present gravity theory has been modified further by introducing a self interacting scalar field with a coupling between the potential function and electromagnetic field. As a result the modified Einstein field equations become coupled and highly nonlinear in nature and they can not be solvable by our known mathematical tools. As an application of the Noether symmetry analysis to differential equations the classical solution has been obtained in two steps, namely (i) identification of the symmetry vector in the augmented space and subsequently (ii) determination of a cyclic variable by imposing a transformation using inner product with cartan one form. Due to presence of the cyclic variable the physical system has simplified field equations which are then solvable analytically. The graphical presentation of the cosmological parameters namely the scale factor, Hubble parameter and the deceleration parameter in FIG.(6.1 and 6.2) shows that the model is qualitatively consistent with the with the observed nature of the cosmic history. Finally, quantum cosmology has been formulated using the WD equation, the operator version of the classical scalar constraint equation. Among the two possible classical solutions it has been found that the quantum description may eliminate the big bang singularity while for other case it is not possible. Lastly, it is noted that due to Noether symmetry, gtere are conserved momenta of the physical system and the quantum description shows an oscillatory behaviour of the wave function along the direction and they have no role in probability measure.

Chapter 7

Symmetry of the Physical space and quantum cosmological description for non-minimally coupled scalar field cosmology

7.1 Prelude

In cosmology, we are at present in an interesting phase where the observational predictions (of accelerated rate of expansion) challenge the theory. The series of observational data since 1998 (namely Type Ia supernova [272, 273], cosmic microwave background (CMB) radiation [274, 221, 275, 276, 277, 278, 279], large scale structure [280], baryon acoustic oscillations (BAO) [281] and weak lensing [136]) challenge modern cosmological predictions due Einstein gravity namely the accelerated phase of of expansion. The theoreticians are divided into two groups- one group has opted for modified theory of gravity (for cosmological description) while the other group is inclined to introduce some exotic matter in the theory. Though still now ΛCDM model [282] (i.e., cosmological constant with cold dark matter) is the best observationally supported model yet the model itself has severe problems namely the cosmological constant problem [283] and the coincidence problem [284]. As a result, the second group favours several dynamical dark energy models [47, 285, 286, 287] which are completely illusive and unresolved issues.

The modification of gravity theory has been done either from the geometric aspect or from the matter sector [288, 289, 290, 291, 292]. The simplest modification of Einstein gravity is the well known $f(R)$ gravity theory where an arbitrary function of Ricci scalar is chosen as Lagrangian density [293, 294, 295, 296]. The Born-infeld inspired modified gravity theory has been extensively discussed in [297]. A comprehensive analysis of cosmological implications of these modified gravity theories has been done in [120] for inflation, bounce and late time accelerated phase of evolution. Another class of modified gravity theories deal with non-minimal coupling with gravity [298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309] and are important both from theoretical as well as from phenomenological perspective due to their potential applications in inflation and related issues [310, 311, 312, 313, 314, 315, 316, 317]. As there is no evidence of dark energy from observational view point so cosmologists are more inclined to use modified gravity theories (for related literature see ref [318, 319, 320, 321, 322, 323]) to match observational evidences. The present work is an example of such a non-minimally coupled scalar field cosmology and symmetry analysis has been employed for cosmological description. In this context it is relevant to mention that a detailed analysis of symmetry of the physical system has been done in [324].

The symmetry analysis has a great role in analyzing a physical theory. The analysis not only determine the symmetry vector but also evaluated the unknown functions or parameters in the system (rather than choosing them phenomenologically) as well as conserved current. Further, the Noether symmetry analysis has a significant role [325, 262, 326, 252, 327] to describe quantum cosmology. The plan of the paper is as follows: non-minimally coupled scalar field cosmology has been briefly presented in section-**II**. A general description of Noether symmetry and symmetry of the physical metric has been given in section-**III**. An explicit derivation of Noether point symmetry and symmetry of the physical metric has been done for the present gravity model in section-**IV**. In section-**V**, a detailed quantum cosmological analysis has been shown with special reference to Bohmian trajectories. Finally, the paper ends with a brief summary and concluding remarks in section-**VI**.

7.2 A brief overview of non-minimally coupled scalar field cosmology

The basic difference between minimal coupling and non-minimal coupling of a scalar field with gravity theory is that the effect of the scalar field is confined to the background geometry of the space-time while in case of non-minimal coupling the scalar field has an active role to modify the gravitational field itself. The interaction between the scalar field and gravity (for non-minimal coupling) has been carried out through a parameter (known as coupling parameter) which measures the strength of the coupling between the scalar field and the space-time curvature (R). In the present work, such a non-minimally coupled scalar field model has been studied having action

$$\mathcal{A}(g,\phi) = \int d^4x \sqrt{-g}\left[\frac{1}{2}m_p^2 R + \frac{1}{2}\xi R\,\phi^2 - \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)\right], \tag{7.1}$$

where ϕ is the non-minimally coupled scalar field, ξ is the coupling parameter and m_p is the usual Planck mass. Now, in the background of flat FLRW space-time having line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{7.2}$$

the Lagrangian density corresponding to the above action takes the form

$$\mathcal{L} = -6a\dot{a}^2(1-\xi\phi^2) + a^3\dot{\phi}^2 - 2a^3V(\phi). \tag{7.3}$$

Here the configuration space is a $2D$ space $\{a,\phi\}$ so that the Hessian matrix is a 2×2 matrix of the form

$$H_{ij} = \begin{bmatrix} \dfrac{\partial^2\mathcal{L}}{\partial\dot{a}^2} & \dfrac{\partial^2\mathcal{L}}{\partial\dot{a}\partial\dot{\phi}} \\ \dfrac{\partial^2\mathcal{L}}{\partial\dot{a}\partial\dot{\phi}} & \dfrac{\partial^2\mathcal{L}}{\partial\dot{\phi}^2} \end{bmatrix} = \begin{bmatrix} -12a(1-\xi\phi^2) & 0 \\ 0 & 2a^3 \end{bmatrix}, \tag{7.4}$$

having non-zero determinant (note that $\xi = 0$ gives the minimally coupled scalar field model). So one must have non-trivial dynamics related to gravitational interactions. Also, from mathematical point of view, non-zero Hessian determinant corresponds to non-degenerate or regular Lagrangian. Now, the Euler-Lagrange equations corresponding to the above Lagrangian (7.3) are

$$-(4a\ddot{a} + 2\dot{a}^2)(1-\xi\phi^2) - a^2\dot{\phi}^2 + 8\xi a\phi\dot{a}\dot{\phi} + 2a^2V(\phi) = 0, \tag{7.5}$$

$$a^3\ddot{\phi} + 3a^2\dot{a}\dot{\phi} - 6\xi a\phi\dot{a}^2 + a^3V'(\phi) = 0. \tag{7.6}$$

Note that the above two evolution equations are not independent, rather there is a scalar constraint equation

$$6a\dot{a}^2(1-\xi\phi^2) - a^3\dot{\phi}^2 - 2a^3V(\phi) = 0. \tag{7.7}$$

We shall employ Noether symmetry analysis for the present physical system in the subsequent sections.

7.3 A general description of Noether symmetry analysis and symmetry of the physical metric

The pioneering work in symmetry analysis of a physical system is due to the mathematician Emmy Noether. According to her, a physical system is always associated to some conserved quantity (called Noether current) provided the Lagrangian of the physical system is invariant with respect to the Lie derivative along a appropriate vector field. So mathematically, if $\mathcal{L}(q^{\mu}(x^i), \dot{q}^{\mu}(x^i))$ be the point-like Lagrangian of a physical system then $\mathcal{L}_{\vec{X}} L = 0$ always implies a conserved current (i.e., $\partial_i Q^i = 0$)

$$Q^i = \lambda^{\alpha} \frac{\partial L}{\partial(\partial_i q^{\alpha})}. \tag{7.8}$$

Here $X = \lambda^{\alpha} \frac{\partial}{\partial q^{\alpha}} + (\partial_j \lambda^{\alpha}) \frac{\partial}{\partial(\partial_j q^{\alpha})}$, the symmetry vector, is the infinitesimal generator of the Noether symmetry. Further, if the physical system has no explicit time dependence then the energy function (also known as the Hamiltonian of the system)

$$E = \dot{q}^{\alpha} \frac{\partial L}{\partial dotq^{\alpha}} - L, \tag{7.9}$$

is also a constant of motion.

From the geometrical point of view, both the Lagrangian as well as the symmetry vector is defined over the tangent space of the configuration $TQ(q^{\alpha}, \dot{q}^{\alpha})$. Further, the Noether point symmetries of Lagrange equations having first order Lagrangian are generated by the elements of the homothetic group of the kinetic metric. If the field equations do not have Noether point symmetries in general then they are not Noether integrable, but still there may have extra Noether point symmetries due to the above homothetic algebra [237].

The physical space has rich geometric structure if there are conformal invariances i.e.,

$$\mathcal{L}_{\vec{X}} g_{\mu\nu} = \Lambda(x^i) g_{\mu\nu}, \tag{7.10}$$

with $\vec{X}$, the conformal vector field. As a particular cases if $\Lambda(x^i) = \Lambda_0(\neq 0)$ then $\vec{X}$ is termed as homothetic vector field while $\vec{X}$ will be a killing vector field provide $\Lambda(x^i) = 0$. In a $n(> 2)$ dimensional space the collection of conformal/ homothetic/killing algebra i.e.,

$CA/HA/KA$ and are related as

$$\text{KA} \leq \text{HA} \leq \text{CA}.$$

Infact the dimension of such algebras are $\frac{n(n+1)}{2}$, $\frac{n(n+1)}{2}+1$ and $\frac{(n+1)(n+2)}{2}$ respectively. In general, though two conformally related metrics have the some CA but the above subalgebras are not identical. Physically, for a system with the Lagrangian, it has been shown [269] that the field equations for two conformally related Lagrangian are covariantly transformed under conformal transformation provided the total energy of the system is zero. Conversely, physical systems with zero total energy are conformally related and their equations of motion are covariantly invariant. However, from the point of view of Noether symmetry two conformally related physical systems are not identical as their homothetic algebras are distinct and Noether point symmetry depends on homothetic algebra of the metric.

7.4 Noether point symmetry and other symmetries of the physical space: An explicit derivation

The present non-minimally coupled scalar field model has 2D configuration space $\{a, \phi\}$ (i.e., tangent space over which the Lagrangian as well as the symmetry vector is defined as a 4D space $\{a, \phi, \dot{a}, \dot{\phi}\}$) for which the symmetry vector has the explicit form (for details see appendix I)

$$\vec{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}}, \tag{7.11}$$

with

$$\alpha = \alpha_0 a^{-2}\xi\phi(1-\xi\phi^2)^{-\frac{3}{2}},\ \beta = -\frac{3}{2}\alpha_0 a^{-3}(1-\xi\phi^2)^{-\frac{1}{2}}, \tag{7.12}$$

The present physical model has the kinetic metric having explicit form

$$ds^2_{(k)} = -6a^2(1-\xi\phi^2)da^2 + a^3 d\phi^2, \tag{7.13}$$

with effective potential $V_{eff} = -2a^3 V(\phi)$. Now with suitable transformation of variables the above kinetic metric can be written as

$$ds^2_{(k)} = \begin{cases} \exp(\sqrt{\frac{3}{2}}u)\cos^2(\sqrt{\xi}v)[-du^2 + dv^2], \ \xi > 0 \\ \exp(\sqrt{\frac{3}{2}}u)\cosh^2(\sqrt{|\xi|}v)[-du^2 + dv^2], \ \xi < 0 \end{cases} \tag{7.14}$$

with $u = \sqrt{6}\ln a$ and $v = \frac{1}{\sqrt{\xi}}\sin^{-1}(\sqrt{\xi}\phi)$ when $\xi > 0$ and $v = \frac{1}{\sqrt{|\xi|}}\sinh^{-1}(\sqrt{|\xi|}\phi)$ when $\xi < 0$. Thus the above kinetic metric is conformal to the flat $2D$ metric of the Minkowskian geometry and has $4D$ $\left(\frac{n(n+1)}{2} + 1 \text{ for } n = 2\right)$ homothetic Lie algebra characterized by a gradient homothetic vectorfield: $H_v = \frac{2}{3}u\partial u$ with $\psi_{H_v} = 1$ and three Killing vectors which span the $E(2)$ group. Now two gradient Killing vectors (translational in nature) are

$$\begin{cases} \overrightarrow{K}_{(1)} = -\frac{1}{2}(e^{\beta} + e^{-\beta})\frac{\partial}{\partial\alpha} + \frac{1}{2\alpha}(e^{\beta} - e^{-\beta})\frac{\partial}{\partial B}, \\ \overrightarrow{K}_{(2)} = -\frac{1}{2}(e^{\beta} - e^{-\beta})\frac{\partial}{\partial\alpha} + \frac{1}{2\alpha}(e^{\beta} + e^{-\beta})\frac{\partial}{\partial B}, \end{cases} \tag{7.15}$$

and the non-gradient Killing vector (rotational in nature) is

$$\overrightarrow{K}_{(12)} = \frac{\partial}{\partial\beta}, \tag{7.16}$$

with $u = \alpha\cos\beta$, $v = \alpha\sin\beta$. Note that the above homothetic vector field does not guarantee a Noether point symmetry for the Lagrangian. It is possible to have additional Noether symmetries due to the above Killing vector fields. Due to the above transformation the transformed Lagrangian has the simplified form :

$$L = -\frac{1}{2}\dot{u}^2 + \frac{1}{2}\dot{v}^2 + V_{eff}, \tag{7.17}$$

and it corresponds to a real scalar field.

7.5 A description of quantum cosmology and causal interpretation

The complete quantum description of cosmology is not possible due to complicated form of superspace. So the common practice is to use minisuperspace models in homogeneous and isotropic space-time. As a result, the lapse function in space-time metric is homogeneous (i.e., $N = N(t)$) and the shift vector becomes identically zero. Thus the space-time line element can be written in (3+1)-decomposition as

$$ds^2 = -N^2(t)dt^2 + e_{ij}(x,t)dx^i dx^j, \quad i,j = 1,2,3. \tag{7.18}$$

Due to this decomposition the Einstein-Hilbert action can be written in terms of 3-space geometry as

$$\mathcal{A}(e_{ij}, N) = \int dt\ d^3xN\sqrt{e}\big[K_{ij}K^{ij} - K^2 + {}^{(3)}R\big], \tag{7.19}$$

where K_{ij} is the extrinsic curvature tensor and ${}^{(3)}R$ is the three space curvature scalar. The above action can be simplified further buy choosing e_{ij} as a finite number of time functions $\gamma^a(t)$, $a = 0, 1, 2, ...n-1$ (due to homogeneity of the three space). As a result, the above action takes the form of a relativistic point particle having self-interacting potential in a nD curved background as

$$\mathcal{A}(g^a, N) = \int_0^1 Ndt\Big[\frac{1}{2N^2}l_{ab}\dot{r}^a\dot{r}^b - W(r)\Big], \tag{7.20}$$

and the path of the relativistic particle (by variation with $r^a(t)$) is characterized by

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{r}^a}{N}\right) + \frac{1}{N^2}\Gamma^a_{bc}\dot{r}^b\dot{r}^c + l^{ab}\frac{\partial W}{\partial r^b} = 0. \tag{7.21}$$

Note that the path of the relativistic particle is not independently described by the above equation (7.21), rather there is a constraint equation (by variation with respect to the lapse function N)

$$\frac{1}{N^2}l_{ab}\dot{r}^a\dot{r}^b + W(r) = 0, \tag{7.22}$$

which is known as Hamiltonian constraint or scalar constraint. Using the momenta Π_a conjugate to r^a, i.e., $\Pi_a = \frac{\partial L}{\partial \dot{r}^a} = l_{ab}\frac{\dot{r}^b}{N}$ the above constraint equation takes the form

$$\mathcal{H}(\Pi_a, r^a) \equiv \frac{1}{2}\ l^{ab}p_a p_b + W(r) = 0. \tag{7.23}$$

Here $\mathcal{H}$ is nothing but the Hamiltonian of the system. Now for quantization programme it is customary to write the operator version $p_\alpha \longrightarrow -i\hbar\frac{\partial}{\partial q^\alpha}$ so that we have the partial differential equation

$$\mathcal{H}\left(r^a, -i\hbar\frac{\partial}{\partial r^a}\right)\psi(r^a) = 0. \tag{7.24}$$

This second order hyperbolic type pde is known as the Wheeler-DeWitt (WD) equation. The common problem in this quantization scheme is the operator ordering problem. However, in this minisuperspace quantization approach one may resolve this restricting to covariant nature.

Additionally, associated with this quantization approach one can associate a conserved probability current as

$$\vec{J} = \frac{1}{2}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right), \tag{7.25}$$

with $\vec{\nabla}.\vec{J} = 0$.

Here the wave function ψ is a solution of the WD equation and it gives the probability measures as

$$dp = |\psi(r^a)|^2 dV, \tag{7.26}$$

with dV, a volume element in minisuperspace.

In the context of present cosmological model the minisuperspace is $2D$ $\{a, \phi\}$ in nature and the associated conjugate momenta to the variables are

$$\Pi_a = -12a\dot{a}(1-\xi\phi^2),\ \Pi_\phi = 2a^3\dot{\phi}. \tag{7.27}$$

So the Hamiltonian of the system has the expression

$$\mathcal{H} = -\frac{a^2}{24}\Pi_a^2 + (1-\xi\phi^2)\Pi_\phi^2 + 2V_0 a^6 = 0, \tag{7.28}$$

and consequently the WD equation takes the form

$$\frac{a^2}{24}\frac{\partial^2\psi(a,\phi)}{\partial a^2} - (1-\xi\phi^2)\frac{\partial^2\psi(a,\phi)}{\partial\phi^2} + 2a^6 V_0\psi(a,\phi) = 0. \tag{7.29}$$

Now using the separation of variables i.e., $\psi(a,\phi) = U(a)V(\phi)$ one has the ordinary differential equations for U and V as

$$\frac{a^2}{24}\frac{d^2U}{da^2} + 2V_0 a^6 U = \lambda U(a), \tag{7.30}$$

and

$$(1-\xi\phi^2)\frac{d^2V}{d\phi^2} = \lambda V(\phi), \tag{7.31}$$

having solution

$$\psi(a,\phi) = \left[C_1\sqrt{a}\ J\left(\frac{1}{6}\sqrt{1+96\lambda}, \frac{4}{3}\sqrt{3}\sqrt{V_0}a^3\right) + C_2\sqrt{a}\ Y\left(\frac{1}{6}\sqrt{1+96\lambda}, \frac{4}{3}\sqrt{3}\sqrt{V_0}a^3\right)\right] \tag{7.32}$$
$$\left[C_3(-1+\xi\phi^2)\text{hypergeom}\left(\left[\frac{1}{4}\frac{3\sqrt{\xi}+I\sqrt{4\lambda-\xi}}{\sqrt{\xi}}, -\frac{1}{4}\frac{-3\sqrt{\xi}+I\sqrt{4\lambda-\xi}}{\sqrt{\xi}}\right], \left[\frac{1}{2}\right], \xi\phi^2\right) + \right.$$
$$\left. C_4(-1+\xi\phi^2)\text{hypergeom}\left(\left[\frac{1}{4}\frac{5\sqrt{\xi}+I\sqrt{4\lambda-\xi}}{\sqrt{\xi}}, -\frac{1}{4}\frac{-5\sqrt{\xi}+I\sqrt{4\lambda-\xi}}{\sqrt{\xi}}\right], \left[\frac{3}{2}\right], \xi\phi^2\right)\phi\right],$$

here C_1, C_2, C_3 and C_4 are integration constant and J and Y stands for the usual Bessel functions for first and second kind respectively. Thus the general solution of the WD equation can be obtained as the superposition of eigen functions of the above WD operator in the form

$$\Psi(a,\phi) = \int \mu(s)\psi(a,\phi,s)ds, \tag{7.33}$$

with $\mu(s)$ the weight function associated to conserved charge 's' moreover, it is desirable to have wave function in quantum cosmology, consistent with classical result. So there should be a coherent wave packet with good asymptotic behaviour in the minisuperspace and have a maximum around classical trajectory. The probability amplitude (proportionally to $|\psi|^2$) has been plotted against the minisuperspace variable at ϕ in fig-7.1 and fig-7.2 for the choices $\xi = \frac{3}{4}$ & $-\frac{3}{8}$ respectively.

On the otherhand, for semi-classical description, it is desirable to consider the WKB approximation as

$$\psi = \exp\left(\frac{i}{\hbar}S\right). \tag{7.34}$$

Here the classical HJ function S can be expanded in order of $\hbar$ as

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + ... \tag{7.35}$$

so that the wave packet

$$\Psi = \int S(\vec{K})\ \exp(\frac{i}{\hbar}S_0)\ d(\vec{K}), \tag{7.36}$$

identifies the classical solution with $\vec{K} = (k_1, k_2)$ indicating some arbitrary separation parameters. Now, using (7.34) into the WD equation (7.29), the differential equation for S_0 (in zeroth order in $\hbar$) takes the form

$$\frac{a^2}{24}\left(\frac{\partial S_0}{\partial a}\right)^2 - (1-\xi\phi^2)\left(\frac{\partial S_0}{\partial \phi}\right)^2 + 2a^6 V_0 = 0. \tag{7.37}$$

Now, writing $S_0(a,\phi) = S_{0a}(a) + S_{0\phi}(\phi)$, one has the explicit form as

$$\begin{cases} S_{0a}(a) = 2\sqrt{\delta}\int\sqrt{\frac{\delta - 2V_0 a^6}{a^2}}\ da = \frac{\sqrt{\delta-2a^6V_0}}{3} - \frac{\sqrt{\delta}}{3}\tanh^{-1}\left(\frac{\sqrt{\delta-2a^6V_0}}{\sqrt{\delta}}\right) + \xi_0 \\ S_{0\phi}(\phi) = \sqrt{\delta}\int\frac{d\phi}{\sqrt{1-\xi\phi^2}} = \sqrt{\frac{\delta}{\xi}}\sin^{-1}(\sqrt{\xi}\phi) + \xi_1 \end{cases}, \tag{7.38}$$

with $\delta > 0$ be the separation constant and ξ_0, ξ_1 are integration constants. So the explicit form of the wave packet becomes (from eqn (7.37))

$$\psi(a,\phi) = \int \xi(\delta)\ \exp\left[\frac{i}{\hbar} S_{0a}(a) S_{0\phi}(\phi)\right]\ d\delta. \tag{7.39}$$

To introduce causal interpretation, we introduce similar to WKB approximation

$$\psi = A(r) \exp\left[\frac{i}{\hbar} B(r)\right]. \tag{7.40}$$

Then from the WD equation (7.24) we have the Hamilton-Jacobi (H-J) equation with quantum correction as

$$\frac{1}{2}\, l^{ab}\, \frac{\partial B}{\partial r^a}\frac{\partial B}{\partial r^b} + q(r^a) + X(r^a) = 0, \tag{7.41}$$

with

$$Q(r^a) = -\frac{1}{A}\, l^{ab}\, \frac{\partial^2 B}{\partial r^a \partial r^b}, \tag{7.42}$$

identifies as quantum potential. Note that $X(r^a)$ can be considered as the particularization of the scalar curvature density (i.e., $-r^{\frac{1}{2}}\ {}^{(3)}R$) of the spacelike hypersurfaces.

Now due to causal interpretation, even in quantum cosmology it is possible to have real trajectories which are observer independent and are classified by the above quantum modified H-J equation (7.41). Now identifying the definition of the momentum from the HJ function i.e., $\Pi_a = \frac{\partial B}{\partial r^a}$ with the usual momentum-velocity relation i.e., $\Pi_a = l_{ab}\frac{1}{N}\frac{\partial r^b}{\partial t}$, the quantum trajectories (also known as Bohmian trajectories) has the explicit form

$$N\frac{\partial B}{\partial r^a} = l_{ab}\frac{\partial r^b}{\partial t}. \tag{7.43}$$

These first order partial differential equations, describing the Bohmian trajectories are invariant under time reparametrization [131, 245] and hence one may choose the gauge $\dot{N} = 0$ without any loss of generality.

For the present cosmological model the quantum corrected H-J equation has the explicit form

$$\frac{a^2}{24}\left(\frac{\partial B}{\partial a}\right)^2 - (1-\xi\phi^2)\left(\frac{\partial B}{\partial \phi}\right)^2 + Q(r^a) + 2a^6 V_0 = 0, \tag{7.44}$$

with

$$Q = -\frac{1}{A}\left[\frac{a^2}{24}\left(\frac{\partial^2 A}{\partial a^2}\right) - (1-\xi\phi^2)\frac{\partial^2 A}{\partial \phi^2}\right], \tag{7.45}$$

the quantum potential. Thus the Bohmian trajectories are explicitly expressed as 1st order differential equations as

$$\frac{\partial B}{\partial a} = \frac{12}{a^2}\dot{a}\ ,\ \frac{\partial B}{\partial \phi} = -\frac{\dot{\phi}}{2(1-\xi\phi^2)}. \tag{7.46}$$

Now, using the standard technique of separation of variables we write equation (7.40) as

$$\psi = A_1(a)A_2(\phi)\ \exp\left[\frac{i}{\hbar}\Big(B_1(a)+B_2(\phi)\Big)\right], \tag{7.47}$$

so that the quantum potential (7.45) takes the form

$$Q = \frac{a^2}{24A_1}\frac{d^2A_1}{da^2} - \frac{(1-\xi\phi^2)}{A_2}\frac{d^2A_2}{d\phi^2}. \tag{7.48}$$

Thus the quantum trajectories now become first order ordinary differential equations as

$$\frac{dB_1}{da} = \frac{12}{a^2}\dot{a}\ ,\ \frac{dB_2}{d\phi} = -\frac{\dot{\phi}}{2(1-\xi\phi^2)}. \tag{7.49}$$

To have explicit form of the trajectories we consider the following phenomenological choices for the quantum potential

Case-I: $B_1 = B_0 a^{-n}\ ,\ B_2 = 0$. Then from equations (7.49) we have

$$a = \left[a_0 - \frac{n^2B_0}{12}(t-t_0)\right]^{\frac{1}{n}}\ ,\ \phi = \phi_0, \tag{7.50}$$

so clearly $a \longrightarrow 0$ as $t \longrightarrow t_0 + \frac{12a_0}{n^2B_0}$ i.e., the quantum trajectory passes through the classical big-bang singularity at finite time. Note that existence of classical big-bang singularity through quantum trajectory is not affected for a change of sign of B_0 and a_0.

Case-II: $B_1 = B_0 \log a$, $B_2 = 0$. Then the quantum trajectory is given by

$$a = a_0\ e^{\left(\frac{B_0}{12}\right)t},\ \phi = \phi_0. \tag{7.51}$$

Hence the trajectory can be identified as the de-Sitter expansion in cosmology. Due to non-singular nature of the de-Sitter solution, the initial big-bang singularity is avoided.

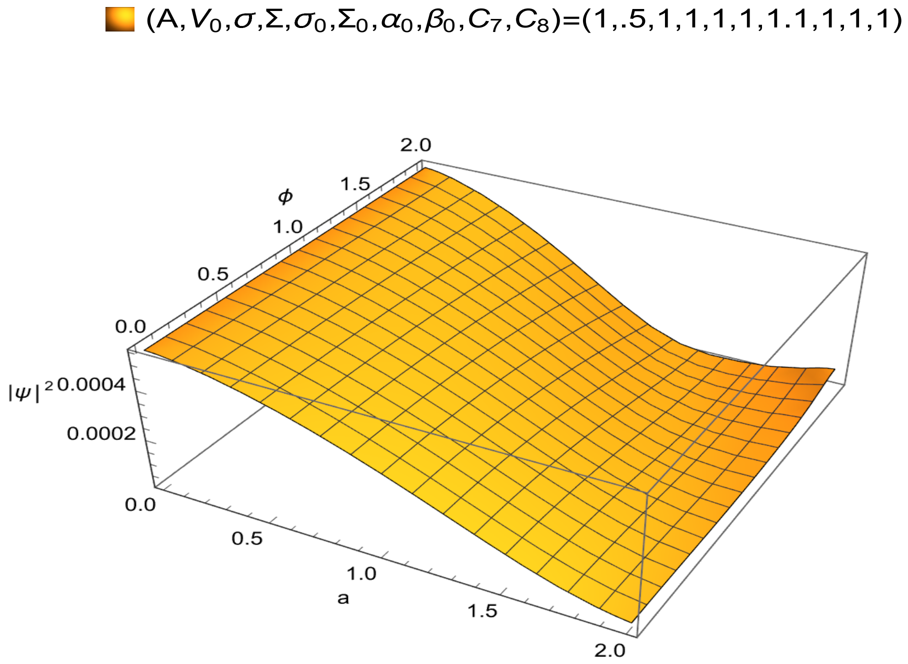

FIGURE 7.1: Graphical representation of wave function whenever $\xi = \frac{3}{4}$.

7.6 Brief summary and concluding remarks:

The present work is an example where symmetry of the physical system as well as symmetry of the physical space has been applied to a coupled gravity model. The Noether symmetry analysis is the popular and widely used symmetry process for physical systems. The only limitation for Noether symmetry analysis is that the physical system must be obtained from a well defined Lagrangian. The interesting feature of this symmetry analysis is that it can determine the unknown parameter or unknown function in the physical system without choosing them phenomenologically. In the present physical system, in course of evaluating the symmetry vector, the unknown coupling parameter ξ and the potential function of the scalar field has been determined. Also the conserved charge and conserved energy has been determined in association with the Noether symmetry vector. The homothetic vector fields as well as the Killing vector fields has been evaluated for the corresponding physical space. In quantum description, the wave function of the Universe has been evaluated as a solution of the WD equation by using the separation of variables approach. The graphical representation of the probability amplitude identifies whether the classical singularity is avoided or not by the quantum description. In fact fig-7.1 and fig-7.2 shows the graphically representation of the probability amplitude for $\xi = \frac{3}{4}$ & $-\frac{3}{8}$ respectively and it is found that, the quantum description can not avoided the classical

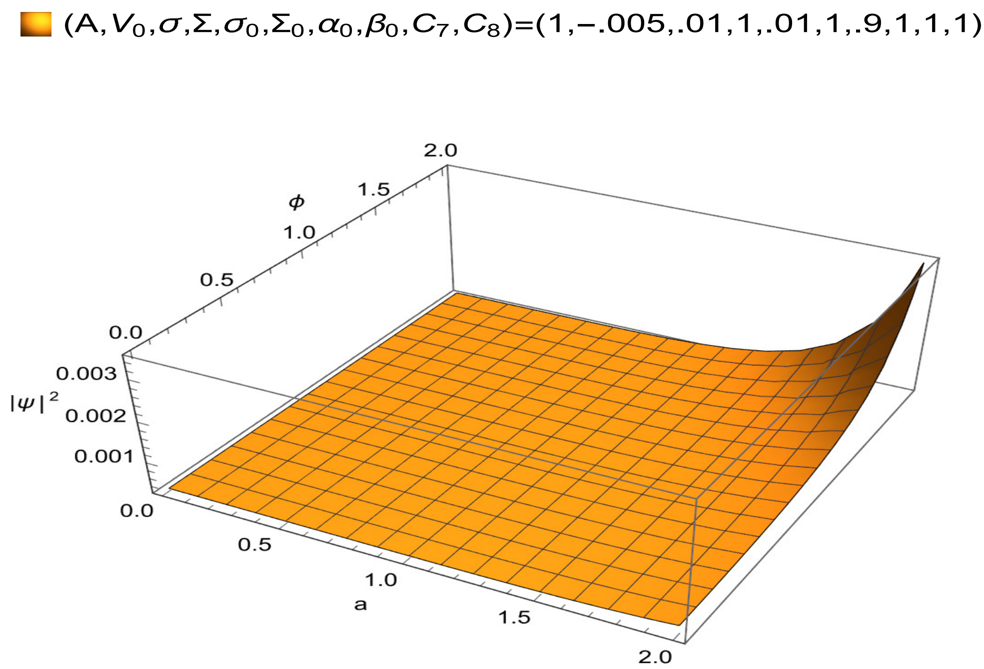

FIGURE 7.2: Graphical representation of wave function whenever $\xi = -\frac{3}{8}$.

singularity for $\xi = \frac{3}{4}$, while it is possible to avoided the classical singularity by quantum description for $\xi = -\frac{3}{8}$ (as shown in fig-7.2). So quantum description with canonical formulation may eliminate the classical gravitational singularity by proper choices of the parameters involved. Further, with causal interpretation it is possible to have quantum trajectories, some may avoid the classical singularity. In this context it is worthy to mention that any quantum formulation namely the path integral formulation determines the wave function of the Universe consider sum over histories and due to its complicated structure we have not considered it here. Finally, one may conclude that symmetry analysis is not only useful for classical cosmological description but also has an important role in quantum description and also in identifying symmetries of the physical space.

7.7 Appendix-I

Due to $2D$ configuration space the symmetry vector is defined over the $4D$ augmented space $\{a, \phi, \dot{a}, \dot{\phi}\}$. i.e.

$$\vec{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \phi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\phi}}, \tag{7.52}$$

where $\alpha = \alpha(a, \phi), \beta = \beta(a, \phi)$ and $\dot{\alpha} = \frac{\partial\alpha}{\partial a}\dot{a} + \frac{\partial\alpha}{\partial\phi}\dot{\phi}$, $\beta = \frac{\partial\beta}{\partial a}\dot{a} + \frac{\partial\beta}{\partial\beta}\dot{\phi}$.

The Noether symmetry namely $\mathcal{L}_{\vec{X}}L = 0$ gives rise to a set of first order partial differential equations

$$\left.\begin{array}{lcl} -6\alpha(1-\xi\phi^2) + 12\beta a\xi\phi - 12a(1-\xi\phi^2)\frac{\partial\alpha}{\partial a} & = & 0, \\ 3\alpha a^2 + 2a^3\frac{\partial\beta}{\partial\phi} & = & 0, \\ -12a(1-\xi\phi^2)\frac{\partial\alpha}{\partial\phi} + 2a^3\frac{\partial\beta}{\partial a} & = & 0, \\ -6a^2\alpha V(\phi) - 2a^3\beta V'(\phi) & = & 0. \end{array}\right\}$$

The above system of partial differential equations can be solved using the technique of separation of variables to give

$$\alpha = \alpha_0 a^{-2}\xi\phi(1-\xi\phi^2)^{-\frac{3}{2}}, \ \beta = -\frac{3}{2}\alpha_0 a^{-3}(1-\xi\phi^2)^{-\frac{1}{2}}, \tag{7.53}$$

with α_0, an integration constant. Further, in course of solving the above set of partial differential equations it is possible to have the unknown parameters ξ as well as the functional form of the potential function $V(\phi)$ (instead of phenomenological choice) as

$$\xi = \frac{3}{4} \ \ \textit{or} \ \ -\frac{3}{8} \ \ \textit{and} \ \ V(\phi) = \left(1-\xi\phi^2\right)^{-1}, \tag{7.54}$$

with V_0, the constant of integration. As the system has no explicit time dependence so the above symmetry analysis is associated with conserved charge (Noether charge) and conserved energy as

$$Q = -3\alpha_0(1-\xi\phi^2)^{-\frac{1}{2}}\left[4\xi\phi\frac{\dot{a}}{a} + \dot{\phi}\right], \tag{7.55}$$

and

$$E = -6a(1-\xi\phi^2)\dot{a}^2 + a^3\dot{\phi}^2 + 2a^3V_0(1-\xi\phi^2)^{-1}. \tag{7.56}$$

Chapter 8

Brief Summary and Future Prospect :

The thesis consists of an extensive study of Noether symmetry approach in various modified gravity models. Due to coupled and non-linear form of the evolution equations it is not possible to have an analytic solution of the cosmological models. But the thesis shows how the Noether symmetry analysis help to solve analytically the complicated cosmological models as well as to have some nice geometrical interpretation of the physical space involved.

Chapter two is related to a two scalar field cosmological model as an extension of the brans-Dicke theory. By imposing Noether symmetry, the Lagrangian as well as the field equations get much simplified and it is possible to have the classical solution. The relevant cosmological parameter has been plotted graphically and it is found that the present model either describes the evolutions starting from the earlier accelerated era to the present dark energy dominated expansion through the matter dominated phase or it start from decelerated era to the present accelerated expansion. The initial big-bang singularity has been studied through quantum description by analyzing the solution of the Wheeler-DeWitt equation. From the graph it is found that the quantum description can not eliminate the big-bang singularity.

In chapter three, a modified fourth order gravity theory has been considered in flat FLRW model both for classical and quantum description. The Noether symmetry simplifies the field equation so that it is possible to have the classical cosmological solution. The graph of the cosmological parameters show that here the matter is in the form of stiff fluid and it is not possible to describe all the phases of cosmic evolution. The quantum cosmological description finds the wave function of universe and here also the classical singularity can not be avoided by quantum description.

A multi-scalar torsion gravity theory for the flat FLRW model has been investigated in chapter four. The Noether symmetry analysis simplified the Lagrangian but it is not possible to have the solution due to inability of determining the cyclic variables. The conserved quantity associated with the physical system has been determined. Further the conformal symmetry of the physical metric has been studied. Both the homothetic as well as the Killing vector field of the physical symmetry has been evaluated. Here also quantum description of the present model has been done by analyzing the wheeler DeWitt equation as well as quantum part has been determined (Bohmian Trajectory) with Hamiltonian Jacobi formulation in WKB approximation.

Chapter five deals with Noether symmetry analysis for anisotropic Bianchi-I space time with conformal symmetry of the physical space. Here modified teleparallel gravity theory has been studied. The chapter identifies both the homothetic and killing vector field of the physical kinetic metric. In quantum description both the Wheeler DeWitt equation as well as the causal interpretation has been investigated in details. It is found that the classical trajectory has been modified through quantum potential and it is found that quantum trajectory avoids classical singularity.

Modified teleparallel gravity theory has been studied both classically and quantumically in chapter six, using Noether symmetry description. The geometry of the space time is chosen to be homogeneous and isotropic flat FLRW model. Using cartan one form for calculating the inner product it is possible to have the cyclic variables. As a result, the field equations are much simplified and become solvable. It is found that the solutions are qualitatively in agreement with the observational data. In quantum description it is found that the conserved charge helps to solve the Wheeler DeWitt equation as well as to identify the oscillatory behaviours of the wave function. Here the quantum description shows that big-bang singularity may be avoided by the quantum description.

Chapter seven deals with non minimally coupled scalar field cosmology in flat FLRW model. The Noether symmetry analysis not only identifies the symmetry vector but also the conserved charge associated with the symmetry. In quantum formulation, canonical quantization scheme as well as quantum bohemian trajectories has been formulated using causal interpretation and their classical limit has been examined.

Regarding the future prospect of the present work, Noether symmetry analysis has a wide scope of application in different physical context. An important area where Noether symmetry analysis play an important role is to study the geodesic equations in various space time model. Also the geodesic of a particle near a black hole or wormhole may be effectively studied using Noether symmetry analysis.

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