Sliding (mode) control

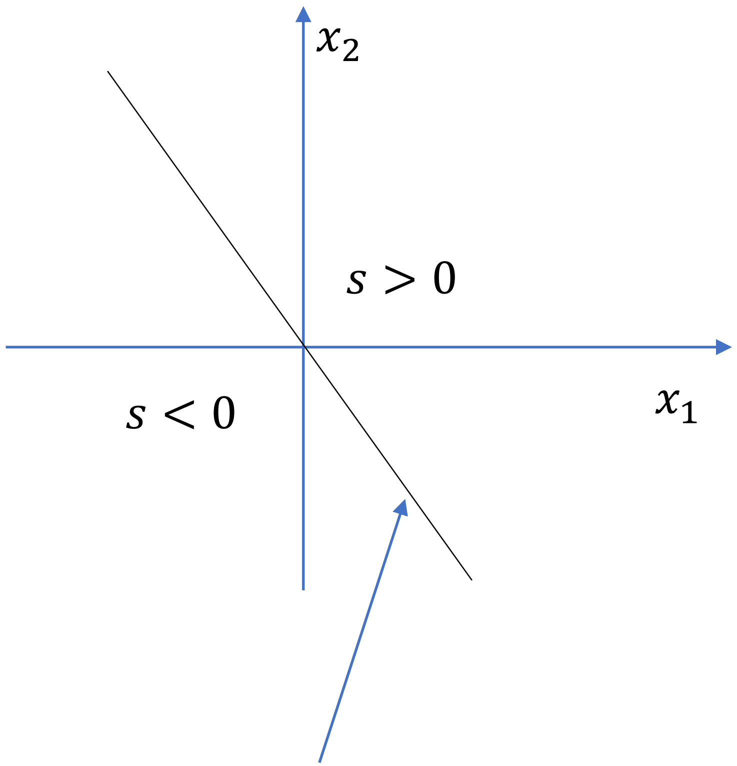
- We consider again the control of nonlinear systems.
- We now allow the models to be imprecise or uncertain.
- Probable cause of model uncertainty:
  - actual uncertainty about the plant e.g. unknown plant parameter.
  - purposeful choice of a simplified representation of the system's dynamics *e.g.,* modeling friction as linear.
- From a control point of view, modeling inaccuracies can be classified into two major kinds:
  - structured (or parametric) uncertainties : corresponds to inaccuracies on the terms actually included in the model
  - unstructured uncertainties (or unmodeled dynamics): corresponds to inaccuracies on *(i.e.,* underestimation of) the system order.
- Inaccuracies in modeling can have strong and adverse effects on nonlinear control systems.
- Any practical design must address them explicitly.
- Two major and complementary approaches to dealing with model uncertainties:
  - Robust control – sliding control
  - Adaptive control

Illustration of the concepts of sliding mode control:

Consider the following 2d ($n = 2$) single input system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = h(x) + g(x)u,\ \ g(x) \geq g_0 > 0 \ \text{ for all } x,$$

where the functions $h(x)$ and $g(x)$ are unknown.
Let us now introduce a variable s given by



$$s = ax_1 + x_2, \quad a > 0$$

Now if we impose the condition $s = 0$, we have

$$ax_1 + x_2 = 0,$$

which is nothing but the equation of a st. line passing through the origin and lying in quadrants II and IV in the (x_1, x_2) plane. In general the equation $s = 0$ is called a **manifold**.

With this condition imposed we can write

$$x_2 = -ax_1 \Rightarrow \dot{x}_1 = -ax_1 \Rightarrow x_1(t) \to 0 \text{ when } t \to \infty.$$

Since x_2 is related to x_1 through a scaler multiplier, it also $\to 0$ when $t \to \infty$. Hence the equilibrium point (0,0) is asymptotically stable. So even **without the knowledge of the system**, we can achieve an automatic asymptotic stability of the equilibrium point if the state trajectory lies on the sliding manifold (surface) $s = 0$.

Now the question is how to drive the system to the sliding surface and maintain it there?

First we observe that, s satisfies the following equation:

$$\dot{s} = a\dot{x}_1 + \dot{x}_2 = ax_2 + h(x) + g(x)u$$

Let us now impose the following condition on $h(x)$ and $g(x)$

$$\left|\frac{ax_2+h(x)}{g(x)}\right| \leq \mathfrak{r}(x), \text{ for all } x \in \Re^2, \mathfrak{r}(x) \text{ is a function of } x.$$

$$\Rightarrow \; |ax_2 + h(x)| \leq \mathfrak{r}(x)|g(x)|$$

Let us now introduce a Lyapunov candidate function

$$V = \frac{1}{2}s^2.$$

Then we have

$$\dot{V} = s\dot{s} = s[ax_2 + h(x) + g(x)u] = s[ax_2 + h(x)] + sg(x)u$$

Now

$$|s[ax_2 + h(x)]| = |s||ax_2 + h(x)| \leq |s|\mathfrak{r}(x)|g(x)| = |s|\mathfrak{r}(x)g(x)$$

Since we know that for any function $f(x) \leq |f(x)|$, we can write

$$\dot{V} \leq |s|\mathfrak{r}(x)g(x) + sg(x)u$$

Let us choose u as

$$u = -\beta(x)\,\mathrm{sgn}(s)$$

where

$$\beta(x) \geq \mathfrak{r}(x) + \beta_0\,, \qquad \beta_0 > 0 \quad \text{and}$$

Signum function

$$\mathrm{sgn}(x) \stackrel{\text{def}}{=} \begin{cases} +1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$$

Now we have

$$\dot{V} \leq |s|\mathfrak{r}(x)g(x) - g(x)s\,\mathrm{sgn}(s)\beta(x) = |s|\mathfrak{r}(x)g(x) - g(x)|s|\beta(x) = |s|g(x)[\mathfrak{r}(x) - \beta(x)]$$

Now from $\beta(x) \geq \mathfrak{r}(x) + \beta_0$ we can derive

$$\beta(x) - \mathfrak{r}(x) \geq \beta_0 \Rightarrow \mathfrak{r}(x) - \beta(x) \leq -\beta_0$$

Hence

$$\dot{V} \leq -|s|g(x)\beta_0 \qquad \Longleftarrow \text{upper bound of } \dot{V}$$

We have (from the Lyapunov candidate function)

$$s = \pm\sqrt{2V} \Rightarrow |s| = \sqrt{2V}$$

So we can further refine the upper bound of $\dot{V}$ as

$$\dot{V} \leq -g_0\beta_0|s| = -g_0\beta_0\sqrt{2V}$$

Integrating both sides from 0 to t we have

Valid until the system reaches the sliding surface.

$$\sqrt{V} \leq \sqrt{V\big(s(0)\big)} - \frac{1}{\sqrt{2}}g_0\beta_0 t$$

$$\text{or, } |s(t)| \leq |s(0)| - g_0\beta_0 t$$

$|s(t)|$ is a +ve quantity. So as t increases it reaches a minimum value of 0.

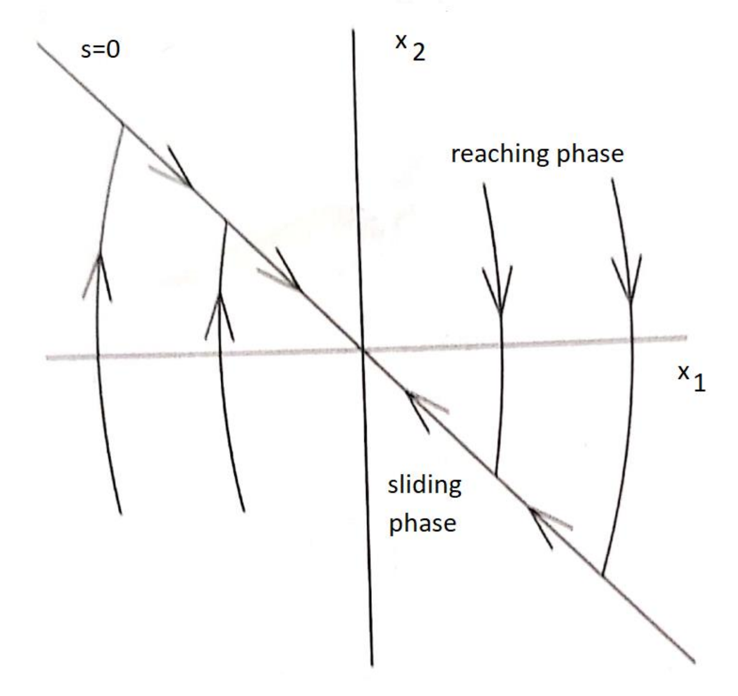
Therefore, the system reaches the sliding surface $s = 0$ in finite time and once on the sliding surface it cannot leave. When it is on the sliding surface, we have

$$x_2 = -ax_1 \Rightarrow \dot{x}_1 = -ax_1 \Rightarrow x_1(t) \rightarrow 0 \text{ when } t \rightarrow \infty. \text{ (from page 2)}$$

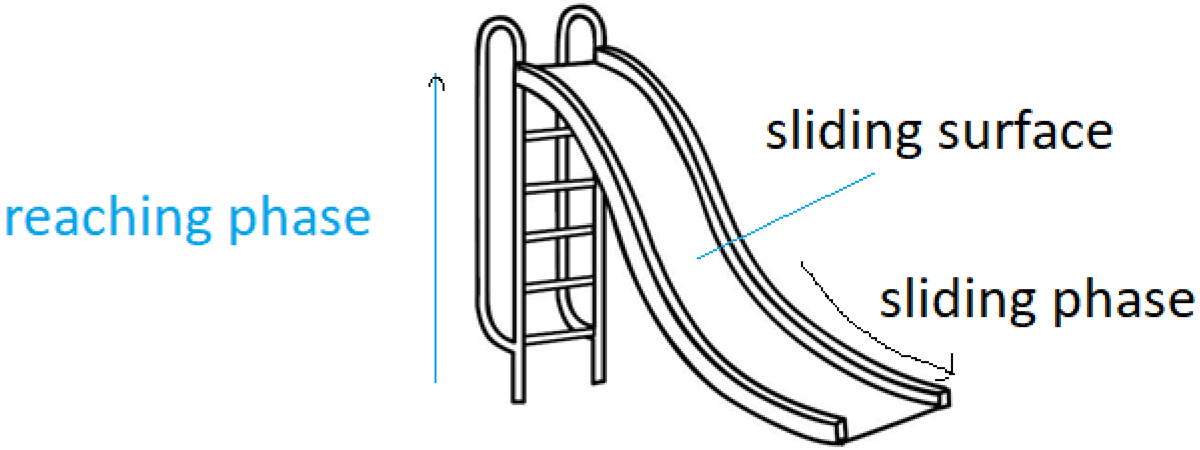
Since x_2 is related to x_1 through a scaler multiplier, it also $\rightarrow 0$ when $t \rightarrow \infty$. Hence the equilibrium point (0,0) is asymptotically stable. (from page 2)

Sum-up discussion:

- State trajectories confined to a properly selected sliding manifold ($s = 0$) leads to asymptotic stability of the equilibrium point of the system.
- The motion of the state variables in time on the sliding manifold is independent of the system model parameters h and g.
- By designing a proper input $u = -\beta(x)\,\mathrm{sgn}(s)$, where β depends only on the upper bound $\mathfrak{r}(x)$, we can move the system from a region (in the state space) outside the sliding manifold onto it. This movement is called the ***reaching phase***.
- The input $u = -\beta(x)\,\mathrm{sgn}(s)$ is called the ***sliding (mode) control***. The noteworthy feature of sliding control is its **robustness** against with respect to uncertainties in h and g.
- Once the system reaches the sliding manifold, its movement remains confined to the sliding manifold. This is called the ***sliding phase***.



A typical phase portrait under sliding mode control



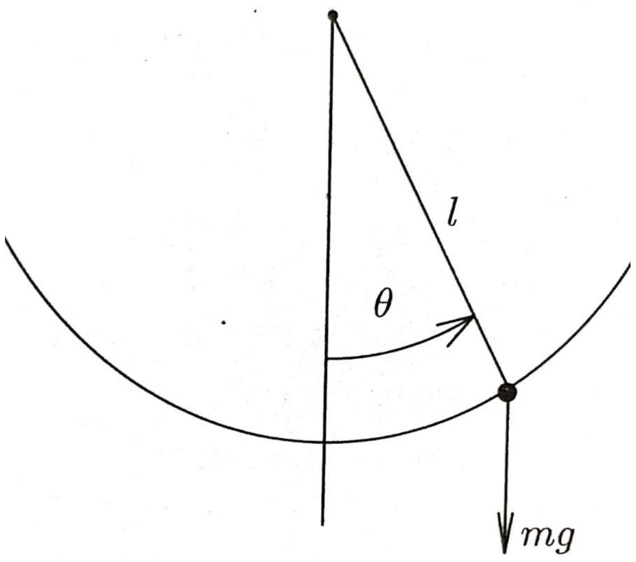
Example 1:

Let us consider the pendulum equation:

$$\ddot{\theta} + \sin(\theta) + b\dot{\theta} = cu$$

where the system parameters b and c are dependent on the physical parameters -

b: function of the coefficient of friction, mass of the bob and frequency of oscillation
c: function of the mass of the bob
u: function of the input (applied) torque and length of the massless connecting rod.



Let us introduce the modelling uncertainties through the inequalities

$$0 \leq b \leq 0.2$$

$$0.5 \leq c \leq 2\ .$$

Let us assume that we want to stabilize the pendulum at $\theta = \pi/2$.

Taking the states as $x_1 = \theta - \pi/2$ and $x_2 = \dot{\theta}$, we have the state model as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\cos(x_1) - bx_2 + cu \end{aligned}$$

By analogy we have

$$h(x) = -\cos(x_1) - bx_2 \text{ and } g(x) = c$$

Let us consider the sliding surface

$$s = x_1 + x_2 \Rightarrow a = 1 \text{ by analogy.}$$

The control input

$$u = -\beta(x)\,\mathrm{sgn}(s)$$

where $\beta(x) \geq \mathfrak{r}(x) + \beta_0; \beta_0 \geq 0.$

Now we have

$$\left|\frac{ax_2 + h(x)}{g(x)}\right| \leq \mathfrak{r}(x)$$

A

$$or, \boxed{\left|\frac{(1-b)x_2 - \cos(x_1)}{c}\right|} \leq \mathfrak{r}(x)$$

Since $0.5 \leq c \leq 2$, we can write,

$$0.5|(1-b)x_2 - \cos(x_1)| \leq A \leq 2|(1-b)x_2 - \cos(x_1)|.$$

Hence the upper bound of A is given by

$$A \leq 2|(1-b)x_2 - \cos(x_1)|.$$

Since $0 \leq b \leq 0.2$, we can write

$$2|(1-b)x_2 - \cos(x_1)| \leq 2|x_2 - \cos(x_1)|.$$

Using the triangular inequality, we arrive at

$$2|x_2 - \cos(x_1)| \leq 2(|x_2| + |\cos(x_1)|)$$
$$\leq 2(|x_2| + 1).$$

So we have $\mathfrak{r}(x) = 2(|x_2| + 1)$. Hence

$$\beta(x) \geq 2(|x_2| + 1) + \beta_0.$$

Since β_0 is an arbitrary +ve constant, we can assign any +ve value to it. If we assume $\beta_0 = 0.5$, we can write

$$\beta(x) \geq 2|x_2| + 2.5$$

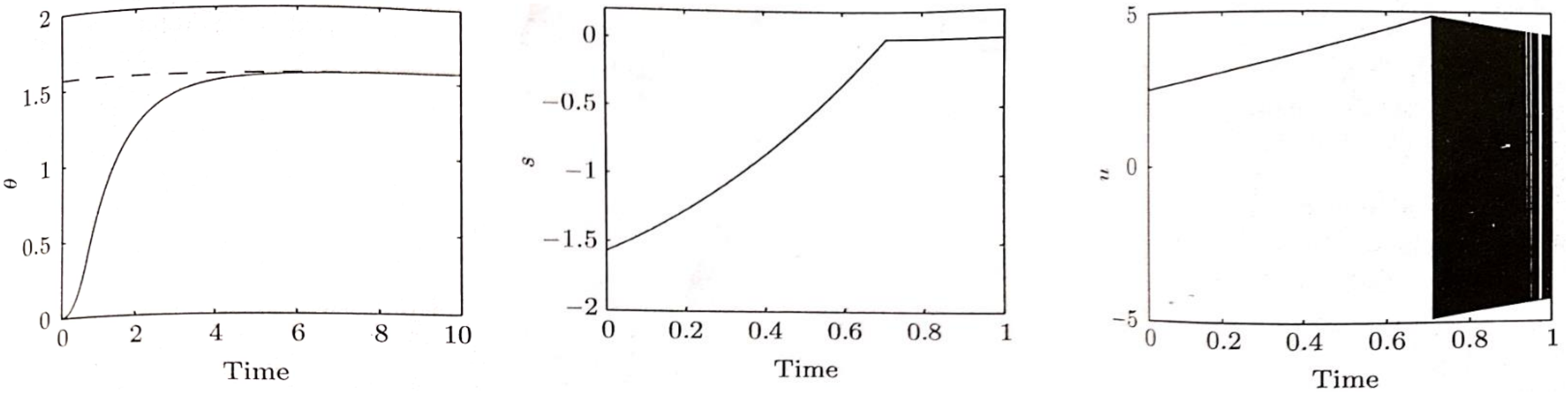
and consequently

$$u = -(2|x_2| + 2.5)\,\mathrm{sgn}(s).$$

Triangular inequality:
$|a \pm b| \leq |a| + |b|$

$\mathfrak{r}(x)$ is the upper bound of A

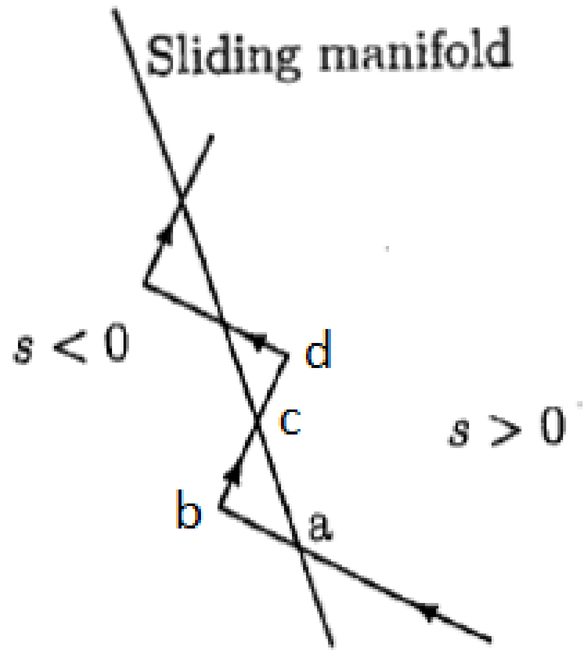
If we simulate the system with $b = 0.01$, $c = 0.5$ and $\theta(0) = \dot{\theta}(0) = 0$, we get the following plots



We can observe that the duration of the reaching phase is about 0.7 second and from that point onwards the control input oscillates at a very high frequency. Ideally to maintain the state trajectory on the sliding manifold $s = 0$, the <u>control input needs to oscillate at infinite frequency</u>.

Let us try to understand this statement:

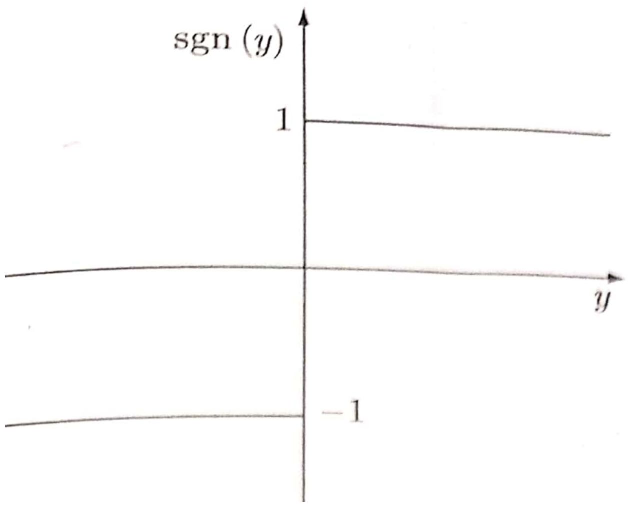
- Let us assume the signum function got a built-in delay. This will be the actual case where e.g. there will be switching delays in a real life system.
- Let us assume without the loss of generality that the state trajectory is approaching the sliding manifold $s = 0$ from the region $s > 0$.
- Once it hits $s = 0$ for the first time at point a, s will change sign but due to delay in switching of the control let us assume it reaches the point b $(s < 0)$.
- Now again the control act to bring it back to point c and again due to delay in switching in control it will reach point d. It is clear that if the switching delay is lesser, the control will switch at smaller intervals and the magnitude of trajectory deviations around $s = 0$ will be lesser. So when the frequency of control switching tends to infinity, the trajectory will tend to follow the sliding manifold with no deviation at all.
- This chattering or “zig-zag” motion of the trajectory causes low control accuracy, high switching heat loss in electrical circuits and high wear and tear of mechanical moving parts in a mechanical system.

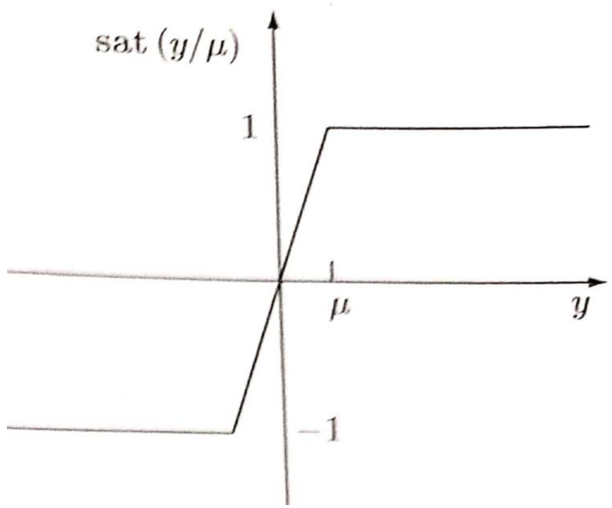


How to avoid infinite frequency oscillation?

Replace the signum function with high-slope saturation function. Hence the control takes the form

$$u = -\beta(x)\,\mathrm{sat}\left(\frac{s}{\mu}\right)$$





where the saturation function is defined as

$$\mathrm{sat}\,(y) = \begin{cases} y, & \text{when } |y| \le 1 \\ \mathrm{sgn}(y), & \text{when } |y| > 1 \end{cases}$$

and μ is a +ve constant.

We can observe that the slope of the linear portion of $\mathrm{sat}(s/\mu)$ is $1/\mu$ and as $\mu \to 0,\ \mathrm{sat}(s/\mu) \to \mathrm{sgn}(s)$.

An analysis of the behaviour of the performance of the “continuous” sliding mode control:

The reaching phase:

In this phase when $|s| > \mu$ we have the control as $u = -\beta(x)\,\mathrm{sgn}(s)$.
So here we can use our earlier analysis with $V = \left(\frac{1}{2}\right) s^2$.
We had earlier established the upper bound of $\dot{V}$ as

$$\dot{V} \leq -g_0\beta_0|s|$$

and the inequality

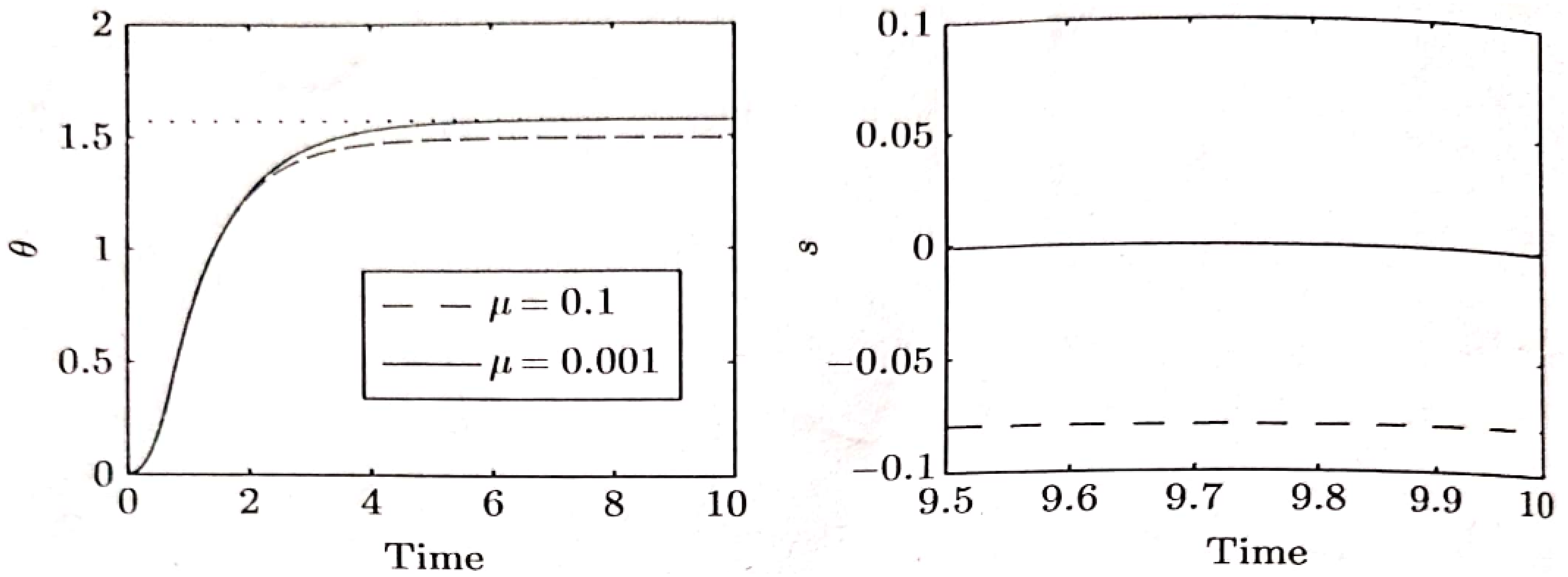
$$|s(t)| \leq |s(0)| - g_0\beta_0 t.$$

Hence following our earlier logic we can say that whenever $|s(0)| > \mu$, $|s(t)|$ will be strictly decreasing until it reaches the boundary layer (characterized by the set $\{|s| < \mu\}$) in finite time.

The boundary layer:

Here we have $\dot{x}_1 = -ax_1 + s$ and $|s| < \mu$.
Here we shall consider the Lyapunov candidate function as $V_0 = \frac{1}{2}x_1^2$. We have $\dot{V}_0 = x_1\dot{x}_1$. Now $\dot{V}_0$ satisfies the following
$\dot{V}_0 = -ax_1^2 + x_1 s \leq -ax_1^2 + |x_1 s| = -ax_1^2 + |x_1||s| \leq -ax_1^2 + |x_1|\mu$.
To further refine the upper bound of –ve semi definiteness of $\dot{V}_0$ we can observe that if we set $|x_1| \geq \frac{\mu}{a\theta_1}$ i.e. $\mu \leq a\theta_1|x_1|$ where $0 < \theta_1 < 1$, then $\dot{V}_0 \leq -(1-\theta_1)ax_1^2$.
So we can see that starting from the outside of the boundary layer we are reaching the region defined by the set $\Omega_\mu = \left\{|x_1| \leq \frac{\mu}{a\theta_1}, |s| \leq \mu\right\}$ in finite time.

The performance of the pendulum system in Example 1 under the “continuous” sliding mode control using the control $u = -(2|x_2| + 2.5)\text{sat}(s/\mu)$.



Limitations of sliding mode control

1. The system has to conform to the generic form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = h(x) + g(x)u.$$

   If not, we can try using the state transformation technique to achieve this form. If we fail to achieve this generic form then sliding mode control cannot be used.

2. The sliding mode control can handle the uncertainties present in the $\dot{x}_2$ term. But it cannot cope with the uncertainties if present in the $\dot{x}_1$ term.