- So far we have studied some aspects of how to analyze the behavior of a nonlinear control system, assuming that the control system had been designed.
- We shall now shift our attention to the problem of designing nonlinear control systems.
- In general the objective of controller design can be stated as:
    - *given a physical system to be controlled and the specifications of its desired behavior, construct a feedback control law to make the closed-loop system display the desired behaviour.*

- Generally, the tasks of control systems can be divided into two categories: stabilization (or regulation) and tracking (or servo).
- In stabilization problems, a control system, called a ***stabilizer*** (or a regulator), is to be designed so that the state of the closed-loop system will be stabilized around an equilibrium point. Examples of stabilization tasks are temperature control of refrigerators, ACs, Geysers etc., altitude control of aircraft and position control of robot arms.
- In tracking control problems, the design objective is to construct a controller, called a ***tracker,*** so that the system output tracks a given time-varying trajectory. Problems such as making an aircraft fly along a specified path (autopilot) or making a robot hand draw straight lines or circles are typical tracking control tasks. Mangalyaan is another example.

Formal definition of stabilization problem:

Given a nonlinear dynamic system described by

$$\dot{x} = f(x, u, t)$$

find a control law $u = u(t)$ *such that, starting from any region* Ω *in the state space, the state* x *tends to* 0 *as* $t \rightarrow \infty$.

Points to pay attention:

1. In the above definition, we allow the size of the region Ω to be large; otherwise, the stabilization problem may be adequately solved using the theories of linear control.
2. If the objective of the control task is to drive the state to some non-zero set-point x_d, one can simply transform the problem into a zero-point regulation by taking $x - x_{\mathrm{d}}$ as the state.

Formal definition of tracking problem:

Given a nonlinear dynamic system described by

$$\dot{x} = f(x, u, t)$$
$$y = h(x)$$

and a desired output trajectory y_d, find a control law for the input u such that, starting from any initial state in a region Ω, the tracking errors $y(t) - y_d(t)$ go to zero, while the whole state x remains bounded.

Points to pay attention:

1. It is assumed that the desired trajectory y_d and its derivatives up to a sufficiently high order are continuous and bounded.
2. When the closed-loop system is such that proper initial states imply zero tracking error for all the time, i.e. $y(t) \equiv y_d(t)$, for all $t \geq 0$, then the control system is said to be capable of *perfect tracking.*
3. Similarly we can define *asymptotic tracking.*

Standard procedure for controller design in nonlinear system:

Given a physical system to be controlled, one typically goes through the following standard procedure, possibly with a few iterations:

1. specify the desired behavior, and select actuators and sensors;
2. model the physical plant by a set of differential equations;
3. design a control law for the system;
4. analyze and simulate the resulting control system;
5. implement the control system in hardware.

Specifying the desired behaviour in nonlinear systems:

- In linear control, the desired behavior of a control system can be *systematically* specified, either in the time-domain (rise time, overshoot, settling time etc.) or in frequency domain (3db frequencies, slopes etc.). In linear control design such quantitative specs. are first laid down and then the controller is designed to meet these specs.
- Systematic specification for nonlinear systems is much less obvious because the response of a nonlinear can be indeterministic from one command to another. Furthermore an exact frequency response analysis not available. Hence one can look for some qualitative specs. like **stability**, **accuracy** and **speed of response**, **robustness** (sensitivity to effects which are not considered in the design, such as disturbances, measurement noise, unmodeled dynamics, etc.*)*, cost etc.

Modelling Nonlinear Systems:

- Modelling: process of deriving a math. description (set of diff. eqn.) of the physical system to be controlled.
- Good understanding of the system: more accurate model is not always better, since it leads to unnecessarily complex design and time consuming computations.
- The key is to keep the “essential” effects and discard the insignificant effects in the system dynamics; good judgement comes from **knowledge** and **experience** of the system.

Some of the available methods of nonlinear controller design

Trial-and-error

- The idea is to use the analysis tools to guide the search for a controller which can then be justified by analysis and simulations. The **phase plane** method, the **describing function** method, and **Lyapunov analysis** can all be used for this purpose. Experience and intuition are critical in this process. For complex systems trial-and-error often fails.

Feedback linearization

- A vital step in designing a control system for a given physical plant (system) is to derive a meaningful (nominal) *model* of the plant, *i.e.,* a model that captures the key dynamics of the plant in the operational range of interest. Some forms of models, however, lend themselves more easily to controller design. Feedback linearization deals with techniques for *transforming original (nonlinear) system models into equivalent models of a simpler (linear) form.*

Robust nonlinear control

- In robust nonlinear control (e.g. sliding control) the controller is designed based on the consideration of both the nominal model *and* some characterization of the model uncertainties.

Adaptive control

- Adaptive control is an approach to dealing with uncertain or time-varying systems. The adaptive control designs apply mainly to systems with a known dynamic structure but unknown constant or slowly-varying parameters. Adaptive controllers, whether developed for linear systems or for nonlinear systems, are inherently nonlinear.

Gain-scheduling

- This was originally developed for the trajectory control of aircraft. The idea: to select a number of operating points which cover the range of the system operation. At each of these points a linear time-invariant approximation to the plant dynamics is made and a linear controller is designed for each linearized plant. Between operating points, the parameters of the controllers are then interpolated, or *scheduled,* thus resulting in a global controller.

Feedback Linearization:

- The central idea: algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied.
- Differs entirely from conventional linearization *(i.e.,* Jacobian linearization) in that linearization is achieved by exact state transformations and feedback, rather than by linear approximations of the dynamics.
- Feedback linearization techniques can be viewed as ways of *transforming original system models into equivalent models of a simpler form.*
- Some of the successful applications to practical control problems: helicopters, high performance aircraft, industrial robots, and biomedical devices.

Example 1: Fluid level control in a tank

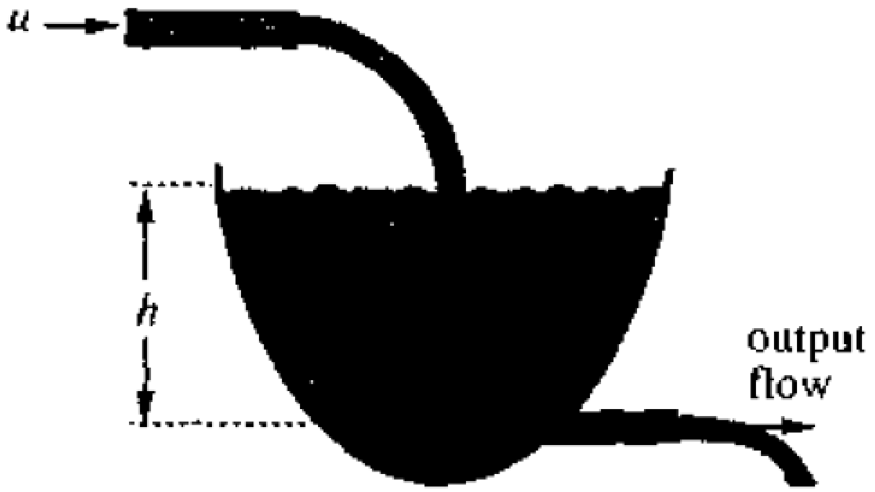
Control of level h of a fluid in a tank to a desired level h_d.
h_0 : initial level; u : input vol. flow rate; $A(h)$: cross section of the tank as a function of h; a : cross section of the outlet pipe.
The dynamic system model:

$$A(h)\dot{h} = u - a\sqrt{2gh}$$

If h_d is quite different from h_0, the control problem becomes nonlinear.
Let us choose:

$$u(t) = a\sqrt{2gh} + A(h)v$$

where v is the "equivalent" input to be determined.



With this selection of $u(t)$, the resulting system dynamics becomes:

$$\dot{h} = v \Rightarrow \text{linear system}$$

Still we need to determine v. Now let

$$v = -\alpha\tilde{h}, \qquad \text{where } \tilde{h} \triangleq h(t) - h_d = \text{level error and } \alpha > 0, \text{a constant.}$$

Then we have the system dynamics as

$$\dot{h} = -\alpha\tilde{h} \Rightarrow \dot{\tilde{h}} = -\alpha\tilde{h}, \text{since } \dot{\tilde{h}} = \dot{h}.$$

We can observe that $\tilde{h}(t) \to 0$ as $t \to \infty$, since $\tilde{h}(t) = \tilde{h}(0)e^{-\alpha t} \Rightarrow$ **regulation (stabilization) problem**.

The input flow is determined by the control law

$$u(t) = a\sqrt{2gh} - A(h)\alpha\tilde{h}.$$

Now suppose the desired level varies with time i.e. $h_d = h_d(t)$. Then we can set v as

$$v(t) = \dot{h}_d - \alpha\tilde{h}; \text{ where } \tilde{h} = h(t) - h_d(t) \Rightarrow \dot{\tilde{h}} = \dot{h} - \dot{h}_d.$$

From the system dynamics we can now write

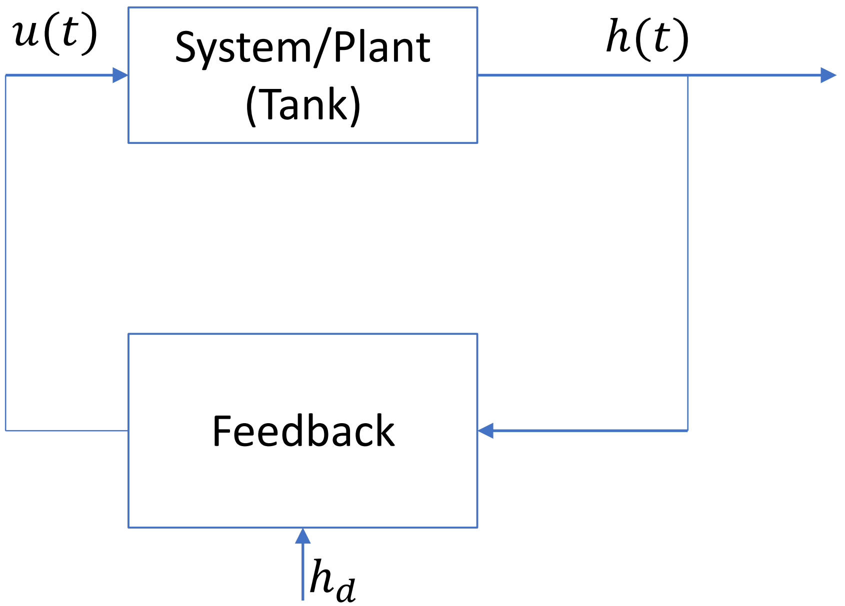
$$\dot{h} = v = \dot{h}_d - \alpha\tilde{h}$$

$$or, \dot{h} - \dot{h}_d = \dot{h}_d - \alpha\tilde{h} - \dot{h}_d \Rightarrow \dot{\tilde{h}} = -\alpha\tilde{h} \Rightarrow \tilde{h}(t) = \tilde{h}(0)e^{-\alpha t}.$$

In this case also we find that $\tilde{h}(t) \to 0$ as $t \to \infty \Rightarrow$ **tracking problem**.

Points to be noted:

1. The state equation has become linear due to proper selection of input function $u(t)$.
2. The input function itself is not linear.
3. The input function can be thought as a feedback function of the state.


$u(t)$
System/Plant
(Tank)
$h(t)$
Feedback
h_d

Example 2:

Let us consider the following system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a[\sin(x_1 + \delta) - \sin(\delta)] - bx_2 + cu$$

where a, b, c and δ are system constants.

Through inspection one find that if u is chosen as

$$u = \frac{a}{c}[\sin(x_1 + \delta) - \sin(\delta)] + \frac{v}{c} \ ,$$

then the nonlinear term can be eliminated and the resulting linear system is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -bx_2 + v$$

Now if we set $v = -k_1 x_1 - k_2 x_2$, then the eigen values of the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -k_1 x_1 - (k_2 + b)x_2$$

will be located on the open left-half plane.

The control law becomes

$$u = \frac{a}{c}[\sin(x_1 + \delta) - \sin(\delta)] - \frac{1}{c}(k_1 x_1 + k_2 x_2)$$

As in Example1, here also the input u is derived from the state variables $\Rightarrow$ **state feedback**

Discussion:

How general is the idea of nonlinearity cancellation?

- Not expected to cancel each and every nonlinear system.
- Certain forms or structures of the system dynamics are responsible for this type on nonlinearity elimination
- We can observe –
  1. Cancellation of a nonlinearity term $\alpha(x)$ by subtraction $\Rightarrow u + \alpha(x)$ must appear in the system eqn.
  2. Cancellation of a nonlinearity term $\gamma(x)$ by division $\Rightarrow \gamma(x)u$ must appear. If $|\gamma(x)| \neq 0$ in the domain of x, then this cancellation is possible by $u = \beta(x)$ where $\beta(x) = \gamma^{-1}(x)$.
- We can conclude that to use state feedback as a means to convert a nonlinear state equation into a linear one by cancelling the nonlinearity requires the nonlinear state equation to have the form (structure):

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)] \ldots\ldots\ldots (1)$$

where

$$A = [n \times n]$$

$$B = [n \times p]$$

$$\gamma(x): \Re^n \to \Re^{p \times p}, |\gamma(x)| \neq 0 \text{ in every } x \in \mathcal{D}$$

$$\alpha: \Re^n \to \Re^p$$

A matrix A is called Hurwitz if $\text{Re}(\lambda_i) < 0$

If the state equation is of the form of (1), then it can be linearized by

$$u = \alpha(x) + \beta(x)v$$

where $\beta(x) = \gamma^{-1}(x)$ and we obtain the linear state equation

$$\dot{x} = Ax + Bv.$$

To determine v we can set $v = -Kx$, such that $A - BK$ is Hurwitz and we get $u = \alpha(x) - \beta(x)Kx$.

Example 3:

Consider the following system:

$$\dot{x}_1 = a\sin(x_2)$$
$$\dot{x}_2 = -x_1^2 + u$$

Here we cannot choose u to eliminate the nonlinear term.
Since the choice of state variable is not unique, we can always change the state variables by transformation

$$\left.\begin{aligned} z_1 &= x_1 \\ z_2 &= a\sin(x_2) = \dot{x}_1 \end{aligned}\right\} \text{ , in short we can write } z = T(x)$$

The new state variables z_1 and z_2 now satisfy:

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a\cos(x_2)(-x_1^2 + u)$$

and the nonlinearity term can be eliminated by selecting

$$u = x_1^2 + \frac{1}{a\cos(x_2)}v$$

which is well defined for $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$.

The inverse transformation from $z \rightarrow x$ is given by

$$\left.\begin{aligned} x_1 &= z_1 \\ x_2 &= \sin^{-1}\left(\frac{z_2}{a}\right), -a < z_2 < a \end{aligned}\right\}, \quad x = T^{-1}(z)$$

The state equation with the transformed state variables $z = [z_1\ z_2]^T$ becomes

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= a(-z_1^2 + u)\cos\left[\sin^{-1}\left(\frac{z_2}{a}\right)\right]. \end{aligned}$$

This implies that when the transformation $z = T(x)$ exists, the inverse transformation (map) $x = T^{-1}(z)$ must exist.

A continuously differentiable transformation is called a **diffeomorphism** when its continuously differentiable inverse transformation exists.

Now let us gather all the information in the form of the following formal definition:

A nonlinear system

$$\dot{x} = f(x) + G(x)u$$

where $f: \mathcal{D} \to \Re^n$ and $G: \mathcal{D} \to \Re^{n \times p}$ are sufficiently smooth on a domain $\mathcal{D} \subset \Re^n$, is said to be feedback linearizable (or input-state linearizable) if there exists a diffeomorphism $T: \mathcal{D} \to \Re^n$ such that $\mathcal{D}_z = T(\mathcal{D})$ contains the origin and the change of variable $z = T(x)$ transforms the system under consideration into the form

$$\dot{x} = Az + B\gamma(x)[u - \alpha(x)]$$

with (A,B) controllable and $\gamma(x)$ nonsingular for all $x \in \mathcal{D}$.

Sufficiently smooth ⇒ all the partial derivatives that may appear later on are defined and continuous.

Example 4:

When certain output variables are of importance as in the case of **tracking** problem, the system dynamics is described by state and output equations.

Linearizing the state equations may not necessarily linearize the output equation.

Let us consider the system of Example 3 with an output equation as $y = x_2$.

Then the modification of state variables and state feedback: $z_1 = x_1, z_2 = a\sin(x_2)$ and $u = x_1^2 + \frac{1}{a\cos(x_2)}v$

yield the following system

$$\left.\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= v\end{aligned}\right\}\text{, linear state equation}$$

$$y = \sin^{-1}\left(\frac{z_2}{a}\right)\text{, nonlinear output equation}$$

We can easily remedy this situation by setting $u = x_1^2 + v$. Then we get

$$\dot{x}_1 = a\sin(x_2)$$

$$\dot{x}_2 = v$$

$$y = x_2 = \int v dt$$

<u>We can observe</u>: x_1 is not connected to output $y \Rightarrow$ output linearization has made a part of the system dynamics (described by one state variable x_1) unobservable from y. This part of the dynamics is called the ***internal dynamics,*** because it cannot be seen from the external i/p-o/p relationship. We must have the internal dynamics to be stable for this scheme to be effective.

Systematic approach towards input-output linearization:

Let us consider the following SISO system

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

where the mappings f, g and h are sufficiently smooth in the domain $\mathcal{D} \subset \Re^n$. $f: \mathcal{D} \to \Re^n$ and $g: \mathcal{D} \to \Re^n$.

We can write using the chain rule to write

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = \frac{\partial h}{\partial x}f(x) + \frac{\partial h}{\partial x}g(x)\, u \stackrel{\text{def}}{=} L_f h(x) + L_g h(x)u\ .$$

$L_f h(x) = \frac{\partial h}{\partial x} f(x)$ is defined as the ***lie derivative*** of h w.r.t. f or along f.

Named after Sophus Lie

Using this concept we can write the following notations:

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x)$$

$$L_f^2 h(x) = L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x), \qquad L_f^3 h(x) = \frac{\partial (L_f^2 h)}{\partial x} f(x), \ldots, L_f^k h(x) = \frac{\partial \left(L_f^{(k-1)} h\right)}{\partial x} f(x)$$

$$L_f^0 h(x) = h(x)$$

We have $\dot{y} = L_f h(x) + L_g h(x) u$. If $L_g h(x) u = 0$, then $\dot{y} = L_f h(x)$ i.e. independent of u.

We then calculate the 2nd derivative of y as

$$y^{(2)} = \frac{d(L_f h)}{dt} = \frac{\partial(L_f h)}{\partial x}\dot{x} = \frac{\partial(L_f h)}{\partial x}[f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x) u$$

Once again if $L_g L_f h(x) = 0$, then $y^{(2)} = L_f^2 h(x)$ is independent of u.

Repeating this process, we can see that if $h(x)$ satisfies

$$L_g L_f^{i-1} h(x) = 0 \text{ for } i = 1,2,\ldots,\rho-1 \text{ and } L_g L_f^{\rho-1} h(x) \neq 0,$$

then u does not appear in the expression of $y, \dot{y}, y^{(2)}, \ldots, y^{(\rho-1)}$ but it appears in the expression of $y^{(\rho)}$ with a nonzero value

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u.$$

It clearly shows that the system is input-output linearizable, since

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)}\left[v - L_f^{\rho} h(x)\right]$$

reduces the input-output map to $y^{(\rho)} = v$ which is a linear relationship between y and v. The integer ρ is called the ***relative degree*** of the system.

The formal definition of relative degree:

The nonlinear system under consideration is said to have a relative degree ρ, where $1 \leq \rho \leq n$ in a region $\mathcal{D}_0 \subset \mathcal{D}$ if

$$L_g L_f^{i-1} h(x) = 0,\ i = 1,2,3, \dots, \rho - 1 \text{ and } L_g L_f^{\rho-1} h(x) \neq 0$$

for all $x \in \mathcal{D}_0$.

It is apparent that our purpose will be served if we can find a simple (linear) and direct relationship between the i/p and o/p.

Example 5:

Let us consider the following 3rd order system (i.e. $n = 3$)

$$\dot{x}_1 = \sin(x_2) + (x_2 + 1)x_3$$
$$\dot{x}_2 = x_1^5 + x_3$$
$$\dot{x}_3 = x_1^2 + u$$
$$y = x_1$$

An apparent difficulty with this model is that the output y is only indirectly related to the input u through the state vector x and the nonlinear state equation.

Let us follow the recipe of the systematic approach of repeated derivatives.

We have $\dot{y} = \dot{x}_1 = \sin(x_2) + (x_2 + 1)x_3 \Rightarrow \dot{y}$ is not directly related to u. Let us do the derivative once more.

$\ddot{y} = f_1(x) + (x_2 + 1)u$, where $f_1(x) = \left(x_1^5 + x_3\right)(\cos(x_2) + x_3) + (x_2 + 1)x_1^2$.

If we set (the control law)

$$u = \frac{[v - f_1(x)]}{(x_2+1)},$$

we get $\ddot{y} = v$, a direct and linear relationship between y and v.

If our aim is to design a tracking controller to minimize the tracking error $e(t) = y(t) - y_d(t)$, for this we set

$v = \ddot{y}_d - k_1 e - k_2 \dot{e}$, where k_1, k_2 are positive constants.

or, $\ddot{y} = \ddot{y}_d - k_1 e - k_2 \dot{e}$

or, $\ddot{e} = -k_1 e - k_2 \dot{e}$

or, $\ddot{e} + k_2 \dot{e} + k_1 e = 0$.

Since $k_1, k_2 > 0$, we can see that $e(t) \to 0$ exponentially (if $k_2^2 \geq 4k_1$) as $t \to \infty$.

Points to pay attention:

1. The control law is defined everywhere, except at the singularity points such $x_2 = -1$.
2. The system has a **relative degree** = 2. It can be shown formally that for any controllable system of order n, it will take *at most* n differentiations of any output for the control input to appear, i.e., $r \leq n$.
3. *The effectiveness of this control scheme depends on the stability of the internal dynamics (introduced in Example 4)*
4. The dynamics $\ddot{e} + k_2 \dot{e} + k_1 e = 0$ is of order 2 but the whole system has a dynamics of order 3. A part of the system dynamics has been rendered "unobservable" by the i/p-o/p linearization scheme. This is the internal dynamics that is not visible from the external i/p-o/p relation $\ddot{y} = f_1(x) + (x_2 + 1)u$.

Points to remember:

5. In this case it can be shown that we can represent the internal dynamics by the state equation

$$\dot{x}_3 = x_1^2 + u = x_1^2 + \frac{\ddot{y}_d - k_1 e - k_2 \dot{e} - f_1}{(x_2 + 1)}$$

If the states as dictated by this equation remain bounded during tracking, then we can consider our control problem to be solved. Otherwise the above tracking controller is practically meaningless, because the instability of the internal dynamics may imply undesirable phenomena such as the burning-up of fuses or the violent (uncontrolled) vibration of mechanical members.