Describing Function Analysis

- The frequency response method is a powerful tool for design and analysis of control systems involving linear components. (Bode plot, Nyquist plot, Laplace transform, Fourier transform etc.)
- The describing function method is an extension of the frequency response method of linear systems to nonlinear systems.
- This method can be used to approximately analyse and predict the behaviour of important classes of nonlinear systems, including systems with “hard” nonlinearities.
- Main applications of describing function method :-
    - prediction of limit cycles (self sustained oscillations) in nonlinear systems-
      Limit cycles may occur frequently in real life nonlinear systems.
      Sometimes, a limit cycle can be desirable - **electronic oscillators or signal generators**.
      In many cases, limit cycle is detrimental to a control system:
        - limit cycle tends to reduce control accuracy – onset of instability
        - limit cycle oscillations can cause increase in wear and tear or even failure of mechanical components in the control system
        - limit cycle can cause other detrimental effects like discomfort of passengers in aircrafts running under autopilot
    - analysis of response of nonlinear systems to sinusoidal inputs

Illustration of describing function

Let us consider the following system:

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0$$

Van der Pol oscillator:
We first assume the existence of a limit cycle with undetermined amplitude and frequency, and then determine whether the system equation can indeed support such a solution.

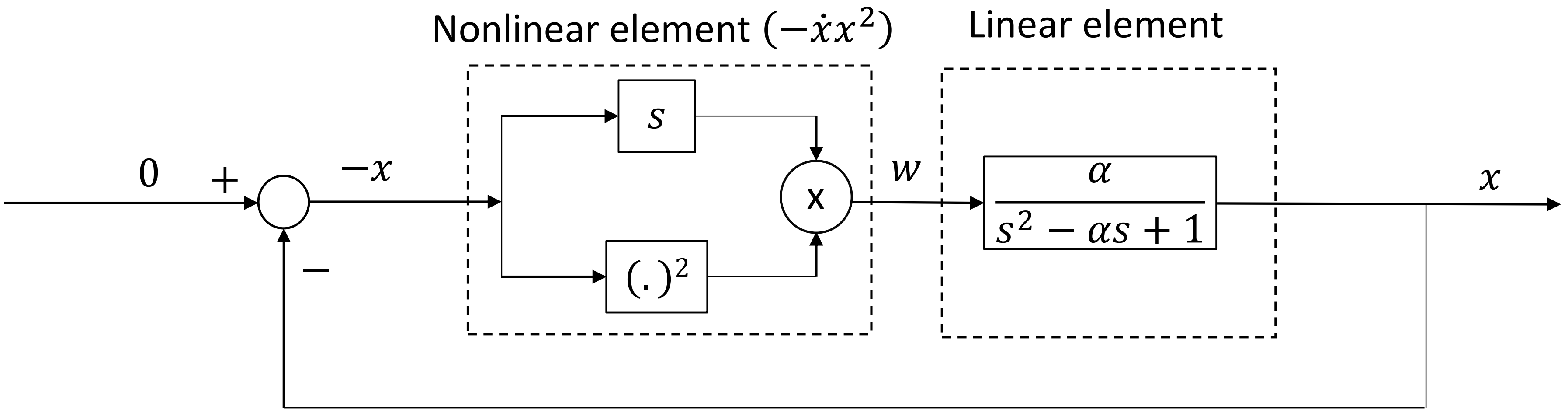
Taking Laplace transform we have:

$$s^2 x(s) + \alpha(x^2(s) - 1)sx(s) + x(s) = 0$$

This implies:

$$(s^2 - \alpha s + 1)x(s) = -\alpha s x^2(s)x(s) \Longrightarrow \frac{x(s)}{w(s)} = \frac{\alpha}{(s^2 - \alpha s + 1)},$$ where $w(s) = -sx^2(s)x(s)$

Hence we can represent the system as –ve feedback system with 0 input:



Let us assume that the output oscillation is of the following form:

$\cos(2\theta) = 1 - 2\sin^2(\theta)$

$$x(t) = A\sin(\omega t)$$

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

Therefore the output of the nonlinear element is given by

$$w = -x^2\dot{x} = -A^2\sin^2(\omega t)\, A\omega\cos(\omega t) = -\frac{A^3\omega}{2}[1 - \cos(2\omega t)]\cos(\omega t) = -\frac{A^3\omega}{4}[\cos(\omega t) - \cos(3\omega t)]$$

A 3rd harmonic term is present in w. The linear element has a low pass transfer function.

We may reasonably assume that this third harmonic term is sufficiently attenuated by the linear element.

Then we can approximate w as follows:

$$w \approx -\frac{A^3\omega}{4}\cos(\omega t) = \frac{A^2}{4}\frac{d}{dt}[-A\sin(\omega t)]$$

Crux of the describing function approach

In Laplace transform domain we can write as:

$$w(s) = N(A,s)[-x(s)], \text{ where } N(A,s) = \frac{A^2}{4}s$$

If we analyse the close loop response of the system, we can write:

$$1 + N(A,s)G(s) = 0 \Rightarrow 1 + \frac{A^2 s}{4}\frac{\alpha}{s^2 - \alpha s + 1} = 0, \quad \text{where } G(s) = \text{transfer function of the linear element}$$

Setting $s = j\omega$ and equating real parts we get $A = 2$ and equating imaginary parts we get $\omega = 1$

Solving the closed loop characteristic equation

$$1 + \frac{A^2 s}{4} \frac{\alpha}{s^2 - \alpha s + 1} = 0$$

for s we get,

$$s_{1,2} = -\frac{1}{8}\alpha(A^2 - 4) \pm \sqrt{\frac{1}{64}\alpha^2(A^2 - 4)^2 - 1}.$$

Now setting $A = 2$, we get,

$$s_{1,2} = \pm j$$

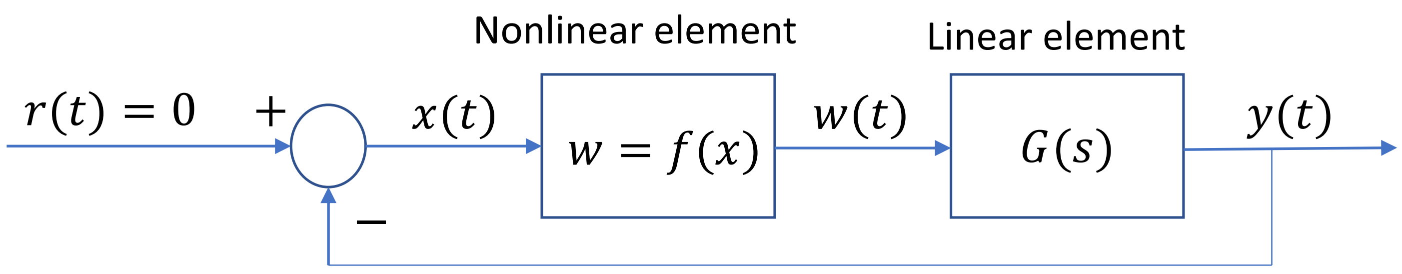
and this corresponds to an oscillation of amplitude $A = 2$ and angular frequency $\omega = 1$.
So are justified in the assumption that there is a self-sustained oscillation present in the system.
The function $\boldsymbol{N(A, s)}$ is called the **describing function** of the nonlinear element $\Longrightarrow$ nothing but the **approximate frequency response** of the nonlinear element where the **higher order harmonic terms** are neglected.

Basic assumptions or conditions regrading applicability of describing functions:

1. there is only a single nonlinear element in the system under consideration
2. the nonlinear element is time-invariant
3. corresponding to a sinusoidal input $x = \sin(\omega t)$, only the fundamental component in the output $w(t)$ of the nonlinear element has to be considered
4. the nonlinearity is of odd type.

Discussion on the assumptions or conditions:

1. In case there are two or more nonlinear elements, one has to combine them together in a single nonlinearity (two nonlinearities in parallel) or neglect the secondary nonlinearities.
2. We shall consider autonomous systems only.
3. Fundamental assumption. For this assumption to be valid, it is imperative for the linear element following the nonlinearity to have a low pass transfer function i.e. $|G(j\omega)| \gg |G(jn\omega)|,\ n = 2,3,\ldots$ …. often referred as the "***filtering hypothesis***".
4. Input is of the type $A\sin(\omega t)$ (odd) not $A\cos(\omega t)$ (even).



Definition of describing function of a nonlinear element (NE) :

- Input to a NE is $x(t) = A\sin(\omega t)$
- Output ($w(t)$)from the NE is generally periodic, not necessarily sinusoidal. Using Fourier series, the periodic function $w(t)$ can be expanded as

$$w(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)]$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)d(\omega t)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(n\omega t)\,d(\omega t)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(n\omega t)\,d(\omega t)$$

- Condition 4 => $a_0 = 0$
- Condition 3 =>

$$w(t) \approx w_1(t) = a_1\cos(\omega t) + b_1\sin(\omega t) = M\sin(\omega t + \phi)$$

where

$$M = \sqrt{a_1^2 + b_1^2} \qquad \phi = \tan^{-1}\left(\frac{a_1}{b_1}\right)$$

- In complex representation this sinusoid is

$$w_1(t) = Me^{j(\omega t+\phi)} = (b_1 + ja_1)e^{j\omega t}$$

- Definition of describing function:

$$N(A,s) \triangleq \frac{Me^{j(\omega t+\phi)}}{Ae^{j\omega t}} = \frac{M}{A}e^{j\phi} = \frac{1}{A}(b_1 + ja_1)$$

Complex ratio of the fundamental component of the NE o/p to sinusoid i/p

Ways to calculate describing function:

- Analytical calculation - When the input-output relationship of nonlinear characteristics $w = f(x)$ has an explicit analytical (functional) expression. In absence of any closed form functional relationship, the method of analytical calculation fails.

- Numerical integration – In this case the relationship $w = f(x)$ is available in the form of graphs or numerical tables. It is convenient to use numerical integration using computers.

- Experimental evaluation – Particularly suitable for complex nonlinearities. Comes handy when a system nonlinearity can be isolated and excited with sinusoidal input with varying frequencies and amplitudes. The fundamental component of the output can be determined using a harmonic analyzer. (similar to experimental determination of frequency response of linear systems – with the difference that in this case the amplitude is also varied). The results of the experiments are a set of curves on complex planes representing the describing function, instead of analytical expressions. Specialized instruments are available for this purpose.

Another example of analytical method:

Describing function of a type of a certain type of spring.

The nonlinear block

$$w = x + x^3/2$$

Where x = input, and w = output

Now for $x(t) = A\sin(\omega t)$, we have $w(t) = A\sin(\omega t) + \frac{A^3}{2}\sin^3(\omega t)$

The fundamental component of the output is $w_1(t) = a_1\cos(\omega t) + b_1\sin(\omega t)$

Since $x(t)$ is an odd function, we have $a_0 = 0$, $a_1 = 0$ and

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[A\sin(\omega t) + \frac{A^3}{2}\sin^3(\omega t)\right]\sin(\omega t)d(\omega t) = A + \frac{3}{8}A^3$$

From the definition of describing function we can write $N(A,s) = N(A) = \frac{b_1}{A} = 1 + \frac{3}{8}A^2$

Describing function of saturation nonlinearity:

- a: range, k: slope of the linear portion of the nonlinear element, input: $x(t) = A\sin(\omega t)$
- <u>Case I</u>: when $A \le a$, output $w(t) = kA\sin(\omega t)$; describing function $= k$.
- <u>Case II</u>: when $A > a$,
  - output (in the 1st quadrant)

$$w(t) = \begin{cases} kA\sin(\omega t), & 0 \le \omega t \le \gamma \\ ka, & \gamma < \omega t \le \dfrac{\pi}{2} \end{cases}$$

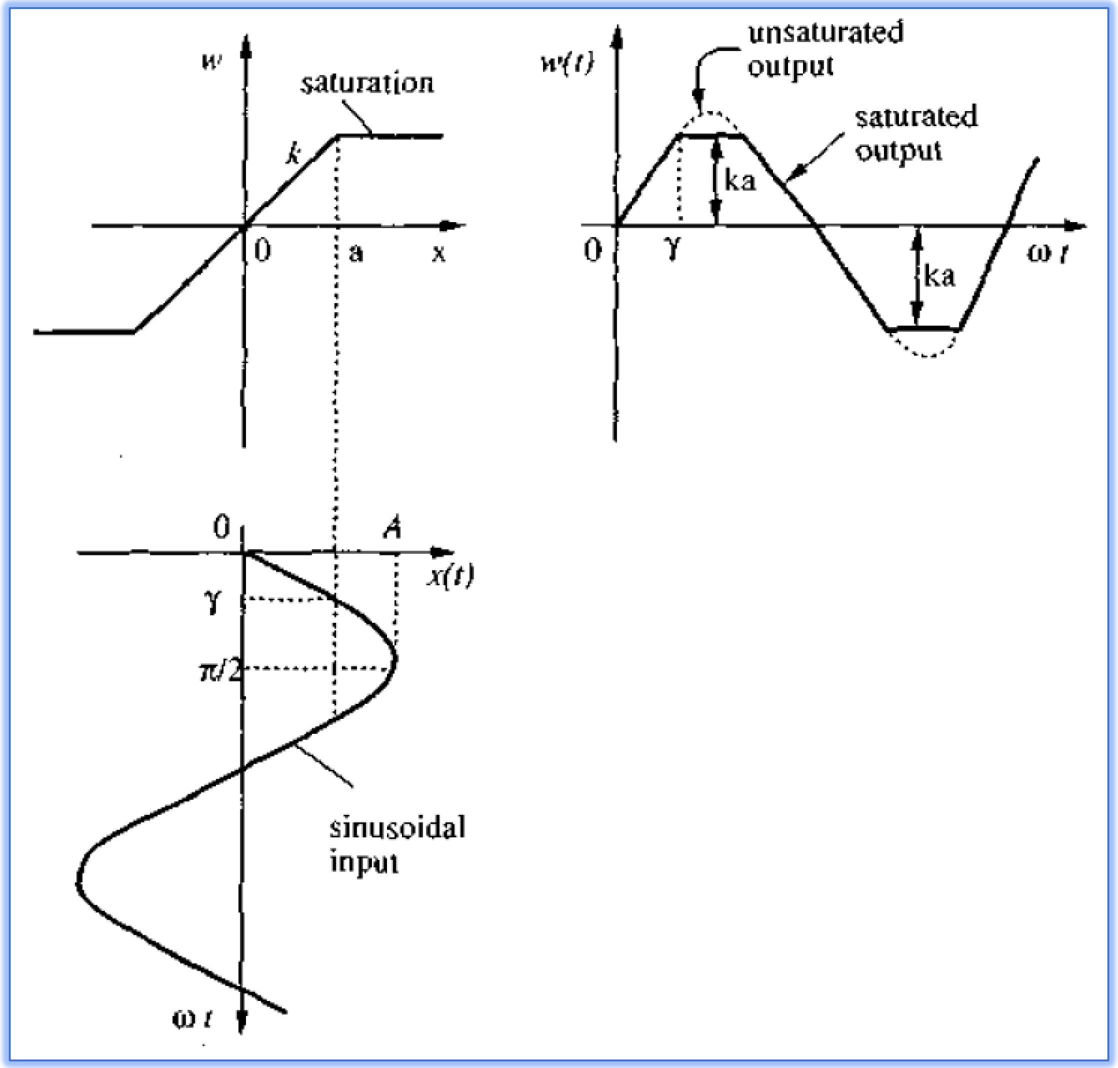
where $\gamma = \sin^{-1}\left(\frac{a}{A}\right)$

  - Observed: $a_0 = 0$, and $a_1 = 0$.

$$b_1 = \frac{4}{\pi}\int_0^{\pi/2} w(t)\sin(\omega t)d(\omega t)$$

$$= \frac{4}{\pi}\int_0^{\gamma} kA\, sin^2(\omega t)d(\omega t) + \frac{4}{\pi}\int_{\gamma}^{\pi/2} ka\sin(\omega t)d(\omega t)$$

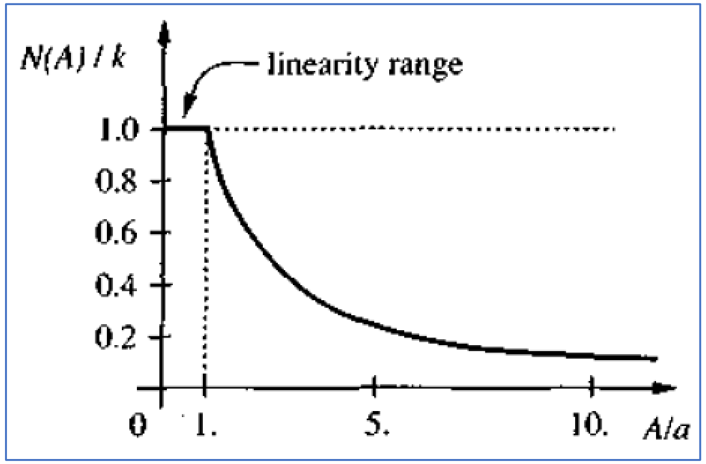
$$= \frac{2kA}{\pi}\left[\gamma + \frac{a}{A}\sqrt{1 - \frac{a^2}{A^2}}\right]$$



- The describing function:

$$N(A) = \frac{b_1}{A} = \frac{2k}{\pi}\left[\sin^{-1}\left(\frac{a}{A}\right) + \frac{a}{A}\sqrt{1 - \frac{a^2}{A^2}}\right]$$

- Three salient features of this describing function:
  - $N(A) = k$ if the input amplitude is within the linear range
  - $N(A)$ decreases as input amplitude increases
  - No phase shift ⇒ no delay between input and output



$N(A)/k$ plotted against A/a

Describing function of relay type (on-off) nonlinearity:

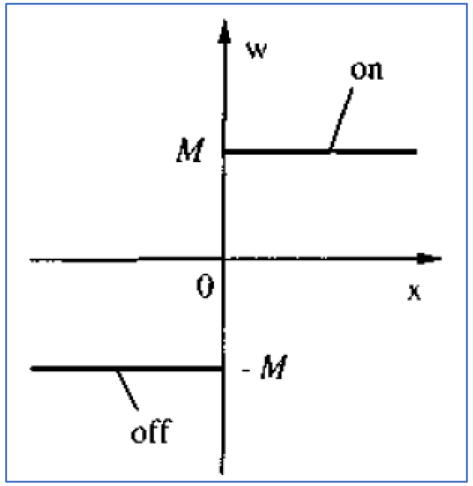
- Special limiting case of saturation type nonlinearity when $a \to 0$ and $k \to \infty$
- In this case we have

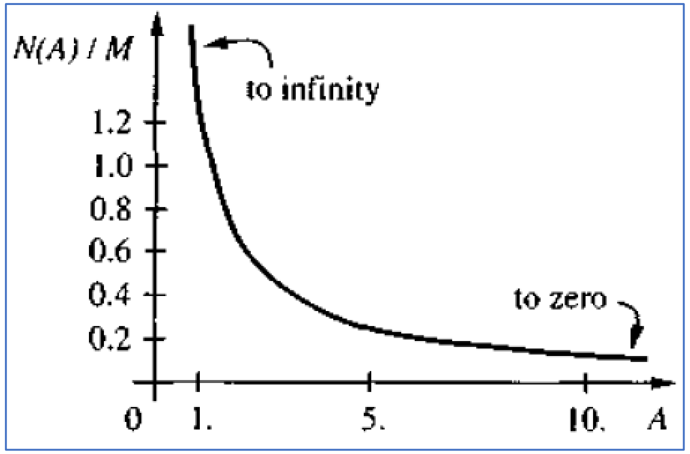
$$w(t) = \begin{cases} -M, & -\pi \le \omega t < 0 \\ +M, & 0 \le \omega t < \pi \end{cases}$$

- One can verify

$$a_0 = 0, a_1 = 0 \text{ and } b_1 = \frac{4M}{\pi}$$

- So the describing function: $N(A) = \frac{4M}{\pi A}$





$N(A)/M$ plotted against A

Describing function of dead-zone nonlinearity:

Dead-zone width $= 2\delta$, slope $= k$.

Input: $x(t) = A\sin(\omega t)$

In one quarter of a period we have

$$w(t) = \begin{cases} 0, & 0 \le \omega t \le \gamma \\ k(A\sin(\omega t) - \delta), & \gamma \le \omega t \le \dfrac{\pi}{2} \end{cases}$$

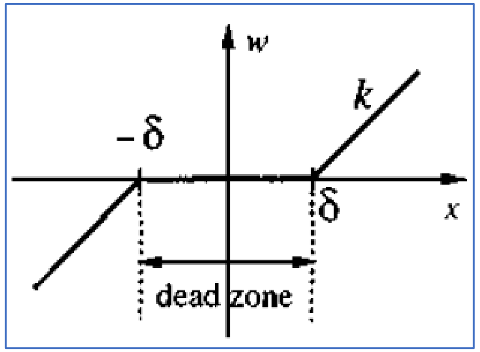
where $\gamma = \sin^{-1}\left(\frac{\delta}{A}\right)$.

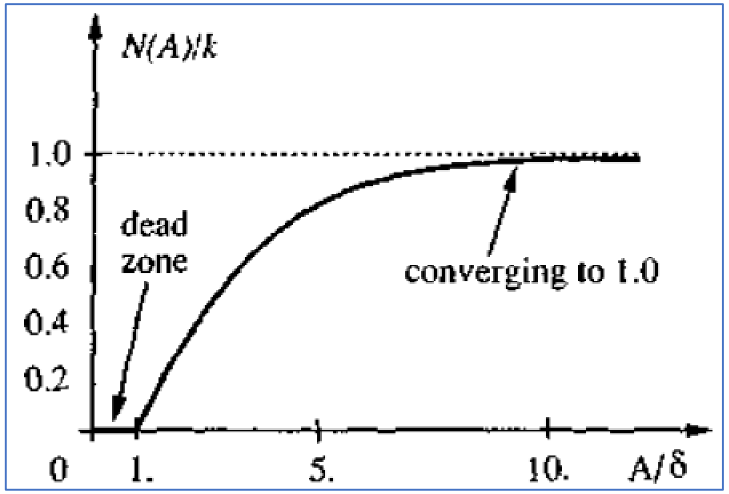
Here also we have $a_0 = 0$ and $a_1 = 0$.

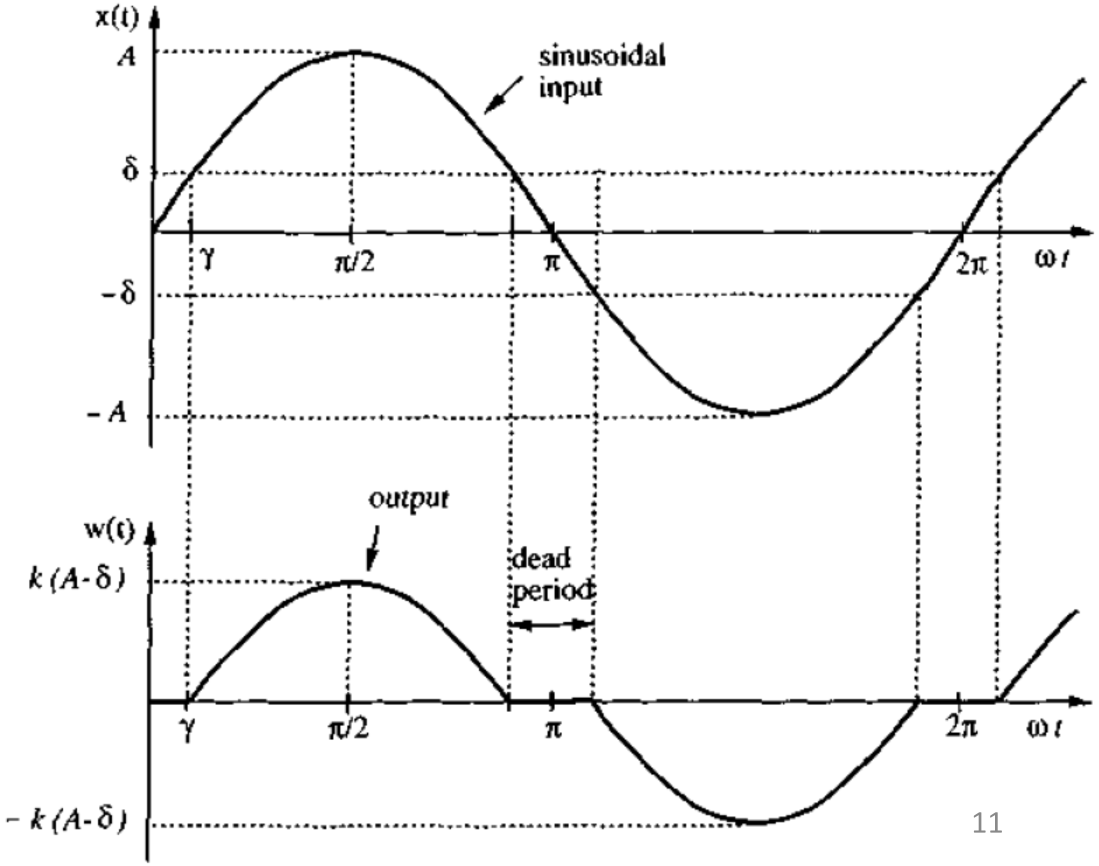
$$\begin{aligned} b_1 &= \frac{4}{\pi}\int_0^{\pi/2} w(t)\sin(\omega t)d(\omega t) \\ &= \frac{4}{\pi}\int_\gamma^{\pi/2} k(A\sin(\omega t) - \delta)\sin(\omega t)d(\omega t) \\ &= \frac{2kA}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{\delta}{A}\right) - \frac{\delta}{A}\sqrt{1 - \frac{\delta^2}{A^2}}\right) \end{aligned}$$

Hence the describing function is

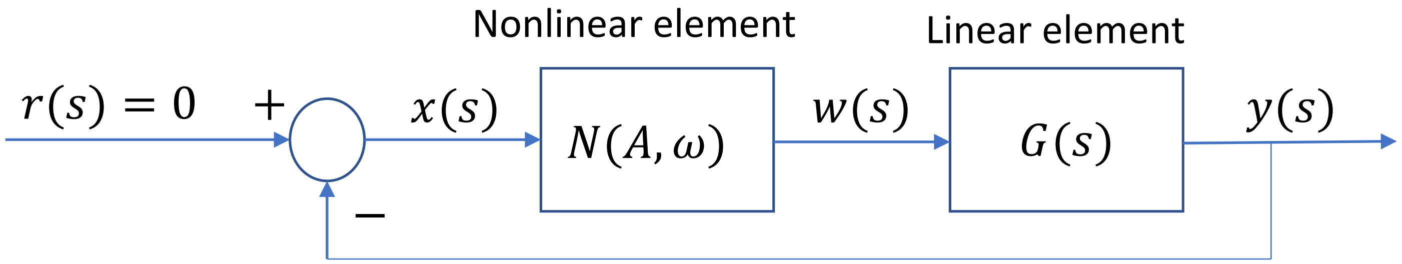
$$N(A) = \frac{b_1}{A} = \frac{2k}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{\delta}{A}\right) - \frac{\delta}{A}\sqrt{1 - \frac{\delta^2}{A^2}}\right)$$







Existence of Limit Cycles:



We have

$$w(s) = N(A, \omega)x(s), \text{ and}$$

$$y(s) = w(s)G(s) = N(A, \omega)x(s)G(s) = -N(A, \omega)y(s)G(s)$$

$$y(s)[1 + N(A, \omega)G(s)] = 0$$

We shall consider that there is a sustained oscillation at the output and then derive the condition of its existence.

Hence $y(s) \neq 0$ and consequently we have

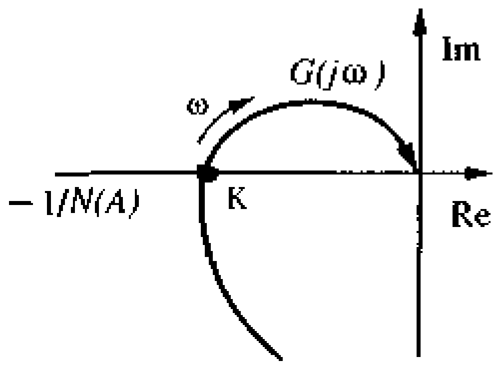
$$G(s) = -\frac{1}{N(A, \omega)} \cdots\cdots (I)$$

So the limit cycle to exist the amplitude A and frequency ω must satisfy (I). If there is no solution to *(I),* the limit cycle for the system does not exist.

It is generally difficult to solve (I) through analytical methods. Therefore graphical methods are used.

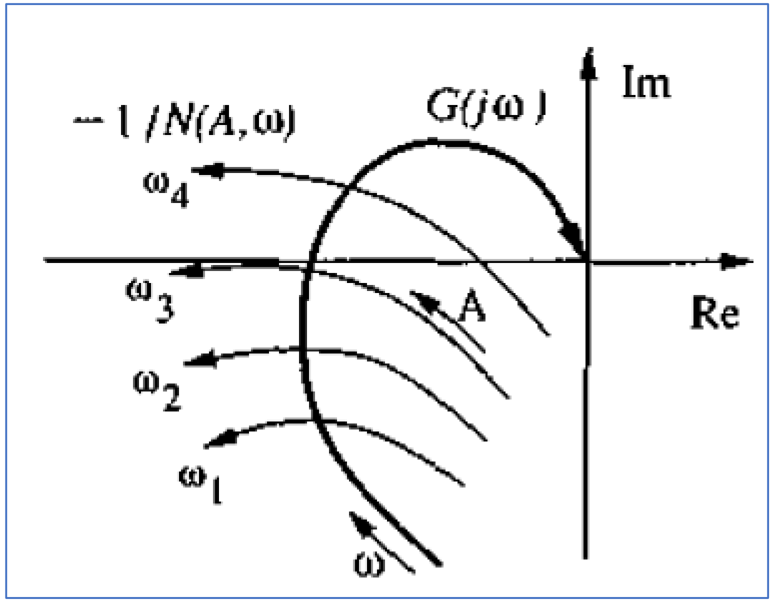
Graphical solution approach for frequency independent describing function

- In this case we have $N(A,\omega) = N(A)$

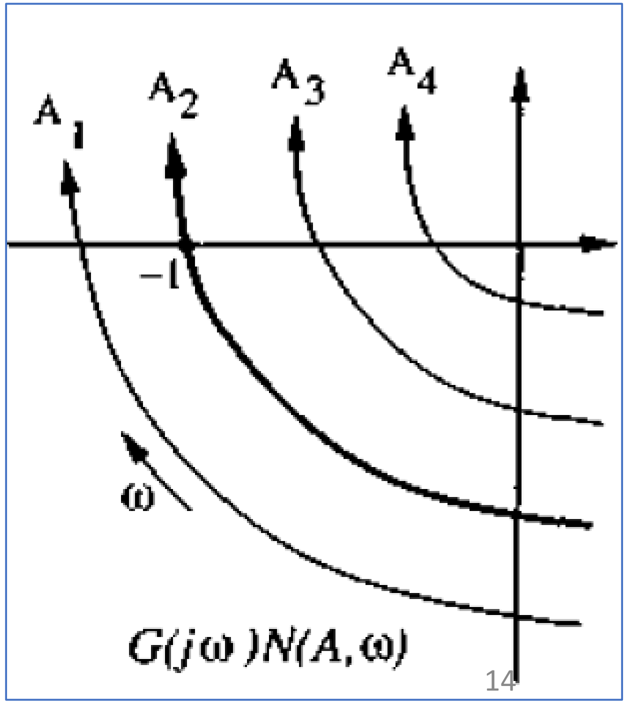


- We now plot $G(s)$ i.e. $G(j\omega)$ with varying ω and $-1/N(A)$ with varying A in the complex plane. Since $-1/N(A)$ is purely real, its plot will be a straight line along the real axis.
- If the two curves intersect, then there exist limit cycles, and the values of A and ω corresponding to the intersection point are the solutions of (I).
- It is shown in the adjacent figure that the two curves intersect at point K.
- The amplitude of the limit cycle is A_K, the value of A corresponding to the point K on the $-1/N(A)$ curve.
- The frequency of the limit cycle is ω_K, the value of ω corresponding to the point K on the $G(j\omega)$ curve.
- It is also useful to point out that the above procedure only gives a ***prediction*** of the existence of limit cycles. The validity and accuracy of this prediction should be confirmed by computer simulations.

Graphical solution approach for frequency dependent describing function



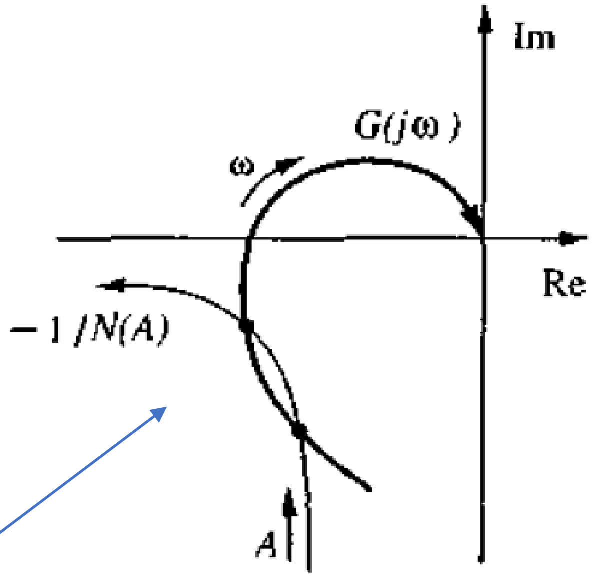
In this case we have $N = N(A, \omega)$
We have a complicated situation here : a family of $-1/N(A, \omega)$ curves.



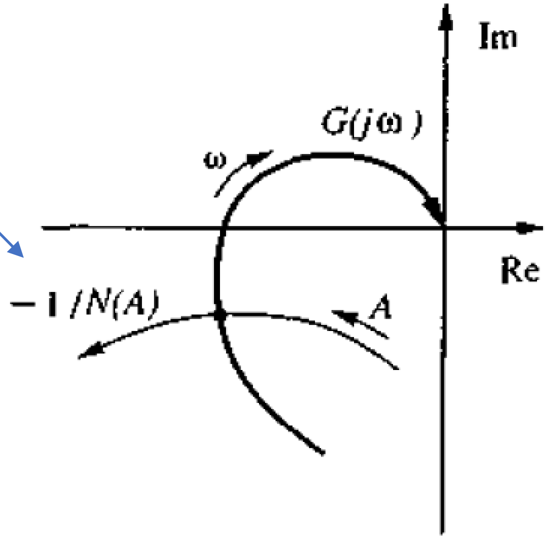
One way to avoid this complication : plot $N(A, \omega)G(j\omega)$ curves instead.
A curve passing *through the point* $(-1{,}0)$ in the complex plane indicates the existence of a limit cycle, with the value of A for the curve being the amplitude of the limit cycle, and the value of ω at the point $(-1{,}0)$ being the frequency of the limit cycle.

Reliability of Describing Function Analysis

- Empirical evidence over the last three decades, and later theoretical justification, indicate that the describing function method can effectively solve a large number of practical control problems involving limit cycles.
- Due to the approximate nature of the technique, three kinds of inaccuracies are possible:
  1. The amplitude and frequency of the predicted limit cycle are not accurate
  2. A predicted limit cycle does not actually exist
  3. An existing limit cycle is not predicted
- If the validity of filtering hypothesis is doubtful, it is observed that the $G(j\omega)$ locus is tangent or almost tangent to the $-1/N(A)$ locus. Then the conclusions from a describing function analysis might be erroneous. As a result, the second or third types of errors listed above may occur.
- Conversely, if the $-1/N(A)$ locus intersects the $G(j\omega)$ locus almost perpendicularly, then the results of the describing function are usually good.
- The first kind of inaccuracy is quite common. Generally, the predicted amplitude and frequency of a limit cycle always deviate somewhat from the true values. How much the predicted values differ from the true values depends on how well the nonlinear system satisfies the assumptions of the describing function method. In order to obtain accurate values of the predicted limit cycles, simulation of the nonlinear system is necessary in this case.



(a)



(b)