- In the last module we had discussed the introductory analysis of 2nd order systems. These systems are also known as 2d systems.
- The concepts of vector field and phase portrait were introduced for this purpose.
- The concepts of different types of stability were introduced for linear 2nd order systems.
- Discussed the concept of linearization of systems at equilibrium points using Jacobian matrix .
- In this lecture we are going to introduce Lyapunov's method of stability analysis with the formal definition of stability. This is an alternative to phase portrait method.

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Formal definitions of stability

- Let us consider the following system equation

$$\dot{x} = f\big(x(t)\big), \qquad \text{with initial value } x(0) = x_0$$

here $x(t)$ is a vector of n components and the values of the components of x are real including 0.

- In short hand we can write

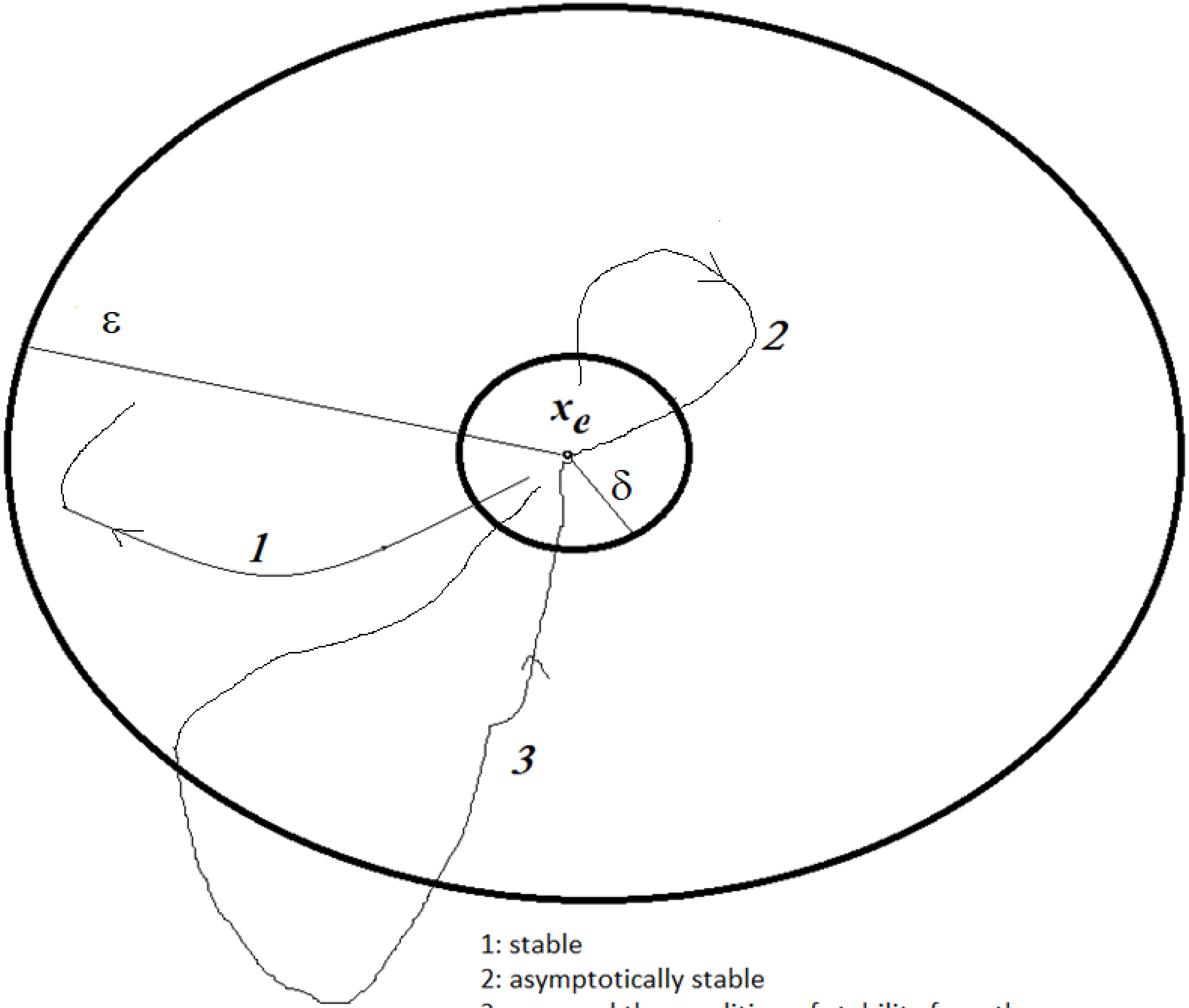
$$x(t) \in \mathfrak{D} \subseteq \Re^n$$

where $\mathfrak{D}$ is a set containing the origin and $f\colon \mathfrak{D} \to \Re^n$ is a continuous vector field on $\mathfrak{D}$.

- Let x_e be an equilibrium point for f so that $f(x_e) = 0$, then
  1. The equilibrium point is said to be **stable** (Lyapunov stable), if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, if $\|x(0) - x_e\| < \delta$, then for every $t > 0$ we have $\|x(t) - x_e\| < \epsilon$.
  2. The equilibrium point is said to be **asymptotically stable** if it is Lyapunov stable and there exists a $\delta > 0$ such that, if $\|x(0) - x_e\| < \delta$, then $\lim_{t\to\infty} \|x(t) - x_e\| = 0$.
- If the solutions that start out near an equilibrium point x_e stay near x_e forever (i.e. as *t* increases), then the equilibrium point x_e is **Lyapunov stable**. (interpretation of condition 1)
- More strongly, if x_e is Lyapunov stable and all solutions that start out near x_e converge to x_e, then x_e is **asymptotically stable**. (interpretation of condition 2)

Intuitive concept of equilibrium point: If you start from this point, then you remain at this point

Visualization of stability



1: stable
2: asymptotically stable
3: removed the condition of stability from the definition of asymptotic stability

Lyapunov's Linearization theorem:

- We have discussed in the last lecture how we can linearize a nonlinear system using the 2d Taylor expansion and neglecting the H.O.T.
- In essence we are considering that the trajectories of the nonlinear system in a small neighborhood of an equilibrium point are close to the trajectories of its linearized version about that point.

- The essence of Lyapunov's Linearization theorem may be stated as follows:

- If the origin (equilibrium point) of the linearized state equation is
  - an asymptotically stable (unstable) node, or an asymptotically stable (unstable) improper node or an asymptotically stable (unstable) spiral or a saddle point,
- then in a small neighborhood of the equilibrium point, the trajectory of the nonlinear system will behave like
  - an asymptotically stable (unstable) node, or an asymptotically stable (unstable) improper node or an asymptotically stable (unstable) spiral or a saddle point.

- One major limitation of this linearization scheme -
  - The size of the neighborhood around an equilibrium point is unknown. We do not know for what range of values of δ and ε the approximation of linearization holds good around an equilibrium point.
- This leads to the concept of Lyapunov's direct method.

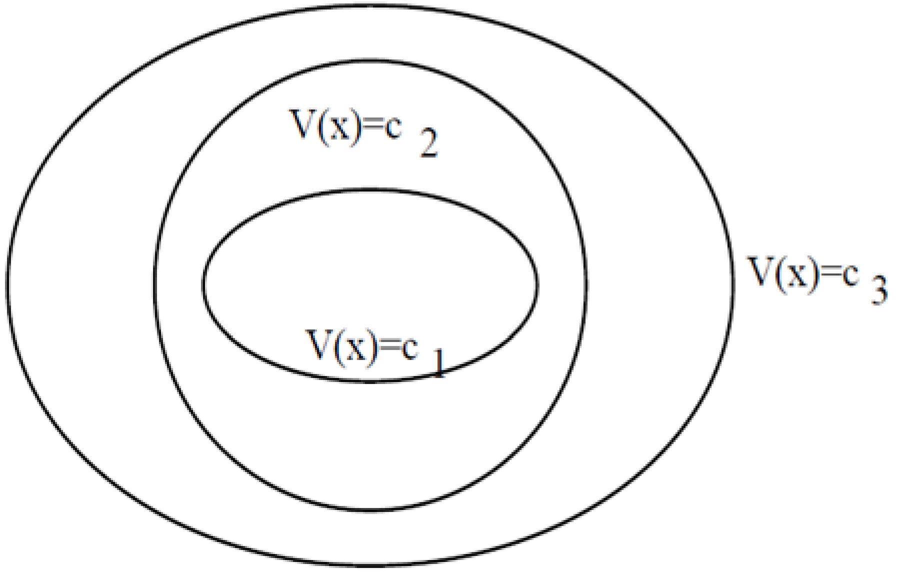
Lyapunov's direct method of determining stability of dynamical systems

- Lyapunov stability is named after Aleksandr Mikhailovich Lyapunov, a Russian mathematician who defended the thesis *The General Problem of Stability of Motion* at Kharkov University in 1892.
- A. M. Lyapunov was a pioneer in successful endeavoring to develop the global approach to the analysis of the stability of dynamical systems.
- Let us now consider (without the **loss of generality**) that the **origin is an equilibrium point** of the system under consideration i.e. $x_e = 0$.
- The idea behind Lyapunov's method is to establish properties of the equilibrium point (or, more generally, of the nonlinear system) by studying how certain ***carefully selected scalar functions of the state*** evolve as the system state evolves.
- The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation: if the total *energy* of a mechanical (or electrical) system is continuously dissipated, then the system, *whether linear or nonlinear,* must eventually settle down to an equilibrium point. Thus, we may conclude the stability of a system by examining the variation of a single *scalar* function.
- Consider, a continuous scalar function $V(x)$ (Lyapunov function) that is 0 at the origin and positive elsewhere in some ball (region) enclosing the origin, i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ within this ball. Such a $V(x)$ may be thought of as an "energy" function. Let $\dot{V}(x)$ denote the time derivative of $V(x)$ along any trajectory of the system, i.e. its rate of change as $x(t)$ varies according to the system equation.
- If this derivative is negative throughout the region (except at the origin), then this implies that the energy is strictly decreasing over time. In this case, because the energy is lower bounded by 0, the energy must go to 0, which implies that all trajectories converge to the zero state.

Some formal definitions:

- Let V be a continuous map from $\Re^n$ to $\Re$. We call $V(x)$ a locally **positive definite** (lpd) function around $x = 0$ if
  1. $V(0) = 0$.
  2. $V(x) > 0, 0 < \|x\| < r$ for some r.
- Similarly, the function is called locally **positive semidefinite** (lpsd) if the strict inequality on the function in the second condition is replaced by $V(x) \geq 0$. The function $V(x)$ is locally **negative definite** (lnd) if $-V(x)$ is lpd, and locally **negative semidefinite** (lnsd) if $-V(x)$ is lpsd.

- What may be useful in forming a mental picture of an lpd function $V(x)$ is to think of it as having “contours" of constant V that form (at least in a small region around the origin) a nested set of closed surfaces surrounding the origin. The situation for $n = 2$ is illustrated in the figure below.



Level lines for a Lyapunov function, where $c_1 < c_2 < c_3$.

We shall denote the derivative of such a V with respect to time along a trajectory of the system under consideration by $\dot{V}(x)$ and it is given by (using the concept of total derivative)

$$\dot{V}(x) = \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_i \frac{\partial V}{\partial x_i} f_i(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \left[\frac{\partial V}{\partial x}\right]^T f(x)$$

- **Definition of Lyapunov function**:

Let V be an lpd function (a "candidate Lyapunov function"), and let $\dot{V}$ be its time derivative along the trajectories of the system under consideration . If $\dot{V}$ is lnsd, then V is called a Lyapunov function of the system under consideration.

- **Lyapunov Theorem for Local Stability**:

- If there exists a Lyapunov function for the system under consideration, then $x = 0$ is a stable equilibrium point in the sense of Lyapunov.
- If in addition $\dot{V}(x) < 0,\ 0 < \|x\| < r_1$ for some r_1 , i.e. if $\dot{V}$ is lnd, then $x = 0$ is an asymptotically stable equilibrium point.

- The region in the state space for which our earlier results hold is determined by the region over which $V(x)$ serves as a Lyapunov function.
- It is of special interest to determine the “**basin of attraction**" of an asymptotically stable equilibrium point, i.e. the set of initial conditions whose subsequent trajectories end up at this equilibrium point.
- An equilibrium point is **globally asymptotically stable** (or asymptotically stable “in the large") if its basin of attraction is the entire state space.
- In other words, if we set $\epsilon = +\infty$ in our earlier definition of asymptotic stability, we can say that the equilibrium point is globally asymptotically stable.

- **Lyapunov Theorem for Global Asymptotic Stability**:

If a function $V(x)$ is positive definite on the entire state space, and has the additional property that $V(x) \to \infty$ as $\|x\| \to \infty$, and if its derivative $\dot{V}$ is negative definite on the entire state space, then the equilibrium point at the origin is globally asymptotically stable.

- Example 1: Consider the following dynamical system

$$\begin{aligned}\dot{x}_1 &= -x_1 + 4x_2\\ \dot{x}_2 &= -x_1 - x_2^3.\end{aligned}$$

One can verify that the only equilibrium point for this system is the origin $x = (x_1, x_2) = (0,0)$.
Let us investigate the stability of the origin using a suitable Lyapunov function.
Let us propose the following quadratic Lyapunov function:

$$V = x_1^2 + ax_2^2$$

where a is a positive constant to be determined.
It is clear that V is positive definite on the entire state space $\Re^2$.
The time derivative of V along the trajectories of the system is given by

$$\begin{aligned}\dot{V} &= \begin{bmatrix} 2x_1 & 2ax_2 \end{bmatrix}\begin{bmatrix} -x_1 + 4x_2 \\ -x_1 - x_2^3 \end{bmatrix}\\ &= -2x_1^2 + (8 - 2a)x_1x_2 - 2ax_2^4.\end{aligned}$$

If we choose $a = 4$ then we can eliminate the cross term x_1x_2, and the derivative of V becomes

$$\dot{V} = -2x_1^2 - 8x_2^4$$

This is clearly a negative definite function on the entire state space.
Therefore we can conclude that $x = (x_1, x_2) = (0,0)$ is a globally asymptotically stable equilibrium point.

- Example 2: A highly studied example in the area of dynamical systems and chaos is the famous Lorenz system, which is a nonlinear system that evolves in $\mathfrak{N}^3$ whose equations are given by:

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz, \end{aligned}$$

where σ , r and b are positive constants.
This system of equations provides an approximate model of a horizontal fluid layer that is heated from below. The warmer fluid from the bottom rises and thus causes convection currents. This approximates what happens in the atmosphere. Under intense heating this model exhibits complex dynamical behavior.
In this example we would like to analyze the stability of the origin under the condition $0 < r < 1$.
Let us define the Lyapunov function as

$$V = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2$$

where α_1, α_2 and α_3 are positive constants to be determined.
It is clear that V is positive definite on $\Re^3$.
The derivative of V along the trajectories of the system is given by

$$\dot{V} = \begin{bmatrix} 2\alpha_1 x & 2\alpha_2 y & 2\alpha_3 z \end{bmatrix} \begin{bmatrix} \sigma(y-x) \\ rx - y - xz \\ xy - bz \end{bmatrix} = \begin{array}{l} -2\alpha_1\sigma x^2 - 2\alpha_2 y^2 - 2\alpha_3 b z^2 \\ +xy(2\alpha_1\sigma + 2r\alpha_2) + (2\alpha_3 - 2\alpha_2)xyz. \end{array}$$

If we choose $\alpha_2 = \alpha_3 = 1$ and $\alpha_1 = \frac{1}{\sigma}$, then $\dot{V}$ becomes

$$\begin{aligned} \dot{V} &= -2\left(x^2 + y^2 + 2bz^2 - (1+r)xy\right) \\ &= -2\left[\left(x - \frac{1}{2}(1+r)y\right)^2 + \left(1 - (\frac{1+r}{2})^2\right)y^2 + bz^2\right] \end{aligned}$$

Since we have assumed $0 < r < 1$, it follows that $0 < \frac{1+r}{2} < 1$.
Therefore $\dot{V}$ is negative definite on the entire state space $\Re^3$.
This implies that the origin is globally asymptotically stable.

Example 3: Consider the following system

$$\dot{x}_1 = -x_1 + x_1 x_2$$

$$\dot{x}_2 = -x_2$$

We shall use a quadratic Lyapunov function candidate to determine the stability of the equilibrium point.
Let the Lyapunov function be $V = \frac{1}{2}(x_1^2 + x_2^2)$
We have

$$\dot{V} = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{bmatrix} = -x_1^2(1 - x_2) - x_2^2$$

$\dot{V} < 0,$ irrespective of the value of x_1 if $x_2 < 1$

Hence the equilibrium point (0,0) is asymptotically stable.

Discussion on Lyapunov's theorem:

- Lyapunov's theorem can be applied without solving the differential equation $\dot{x} = f(x)$.
- But there is no systematic method for finding Lyapunov functions.
- In some cases there are natural Lyapunov function candidates like energy functions in electrical or mechanical systems.
- In other cases, it is basically a matter of heuristics i.e. by trial and error. One can use experience, intuition, and physical insights to search for an appropriate Lyapunov function.
- Failure of a particular Lyapunov candidate function to satisfy the conditions for asymptotic stability or globally asymptotic stability does not mean that the equilibrium point is not asymptotically stable or globally asymptotically stable. It only means that such stability properties cannot be established by using this particular Lyapunov candidate function. We have to search for a suitable Lyapunov function.
- On the other hand if can establish the stability even for a single Lyapunov function, it is enough.

Proof of Lyapunov's theorem of local stability:

First, we prove stability in the sense of Lyapunov

Suppose $\varepsilon > 0$ is given. We need to find $\delta > 0$ such that for all $\|x(0)\| < \delta$, it follows that $\|x(t)\| < \varepsilon$ for all $t > 0$. The figure illustrates the constructions of the proof for the case $n = 2$. Let $\varepsilon_1 = \min(\varepsilon, r)$. Let us define $m = \min_{\|x\|=\varepsilon_1} V(x)$. Now since $V(x)$ is continuous m is well defined and positive. Choose δ satisfying $0 < \delta < \varepsilon_1$ such that for all $\|x\| < \delta$, $V(x) < m$. Such a choice is always possible, again because of the continuity of $V(x)$. Now, consider any $x(0)$ such that $\|x(0)\| < \delta$, $V(x(0)) < m$, and let $x(t)$ be the resulting trajectory. $V(x(t))$ is non-increasing (since $\dot{V}\big(x(t)\big) \leq 0$) which results in $V(x(t)) < m$. We will show that this implies that $\|x(t)\| < \varepsilon_1$. Suppose there exists t_1 such that $\|x(t_1)\| > \varepsilon_1$, then by continuity we must have that at an earlier time t2, $\|x(t_2)\| = \varepsilon_1$, and $\min_{\|x\|=\varepsilon_1} |V(x)| = m > V\big(x(t_2)\big)$, which is a contradiction. Thus stability in the sense of Lyapunov holds true.

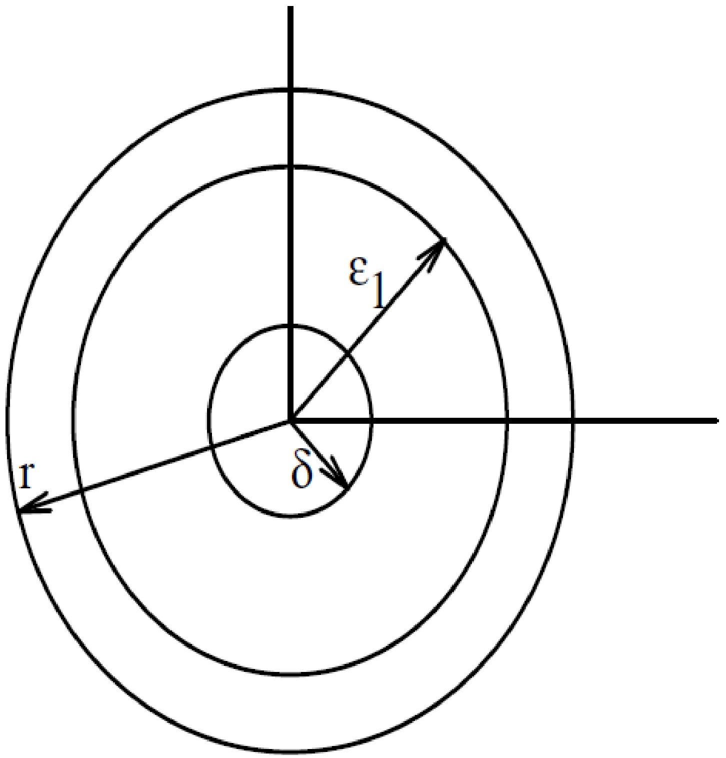


Illustration of the neighborhoods used in the proof

To prove asymptotic stability when $\dot{V}$ is lnd, we need to show that as $t \to \infty$, $V(x(t)) \to 0$; then, by continuity of V, $\|x(t)\| \to 0$. Since $V(x(t))$ is strictly decreasing, and $V(x(t)) \geq 0$ we know that $V(x(t)) \to c$, with $c \geq 0$. We want to show that c is in fact zero. We can argue by contradiction and suppose that $c > 0$. Let the set S be defined as

$$S = \{x \in \mathbb{R}^n | V(x) \leq c\},$$

and let B_α be a ball inside S of radius α,

$$B_\alpha = \{x \in S | \|x\| < \alpha\}.$$

Suppose $x(t)$ is a trajectory of the system that starts at $x(0)$, we know that $V(x(t))$ is decreasing monotonically to c and $V(x(t)) > c$ for all t. Therefore, $x(t) \notin B_\alpha$; recall that $B_\alpha \subset S$ which is defined as all the elements in $\mathbb{R}^n$ for which $V(x) \leq c$. In the first part of the proof, we have established that if $\|x(0)\| < \delta$ then $\|x(t)\| < \epsilon$. We can define the largest derivative of $V(x)$ as

$$-\gamma = \max_{\alpha \leq \|x\| \leq \epsilon} \dot{V}(x).$$

Clearly $-\gamma < 0$ since $\dot{V}(x)$ is lnd. Observe that,

$$\begin{aligned} V(x(t) &= V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \\ &\leq V(x(0)) - \gamma t, \end{aligned}$$

which implies that $V(x(t))$ will be negative which will result in a contradiction establishing the fact that c must be zero.

Proof of asymptotic stability