- Let us consider linear systems with the following
- form:

$$\dot{x}_1 = f_1(x_1, x_2) = a x_1 + b x_2 \ldots\ldots(1)$$
$$\dot{x}_2 = f_2(x_1, x_2) = c x_1 + d x_2 \ldots\ldots(2)$$

a, *b*, *c*, and *d* are constants

- Such systems are called 2nd order systems.
- The x_1, x_2 plane is called the **phase plane** or **state plane.**
- When $b \neq 0$, from (1) we get

$$x_2 = \frac{1}{b}(\dot{x}_1 - a x_1) \ldots\ldots(3)$$

- Taking time derivative of (3) we get

$$\dot{x}_2 = \frac{1}{b}(\ddot{x}_1 - a \dot{x}_1) \ldots\ldots(4)$$

- Using (2) and (4) we get

$$\ddot{x}_1 - (a+d)\dot{x}_1 + (ad - bc)x_1 = 0 \ldots\ldots(5)$$

- (5) can be solved in the usual way of solving 2nd order differential equations.
- Once we get x_1, we can use (3) to find x_2

- <u>**Recapitulation of solution of 2nd order differential equations:**</u>
- The general form is $\ddot{x}_1 + a\dot{x}_1 + b x_1 = 0$ and the trial solution is $x_1 = e^{\lambda t}$
- So we get $\lambda^2 e^{\lambda t} + a\lambda e^{\lambda t} + b e^{\lambda t} = 0$ and the characteristic equation is $\lambda^2 + a\lambda + b = 0$
- The solutions for λ are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

- <u>Case I</u>: real and distinct roots of λ. $(a^2 - 4b) > 0$
  General solution:

$$x_1 = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

- <u>Case II</u>: equal real roots of λ. $(a^2 - 4b) = 0$ and $\lambda_{1,2} = -a/2$
  General solution:

$$x_1 = (c_1 + c_2 t)e^{-at/2}$$

- <u>Case III</u>: complex roots of λ. $(a^2 - 4b) < 0$.

$$\lambda_{1,2} = -\frac{1}{2}a \pm j\omega \quad \text{where, } \omega = b - \frac{a^2}{4}$$

General solution:

$$x_1 = e^{-at/2}(A\cos(\omega t) + B\sin(\omega t))$$

- Here c_1, c_2, A and B are arbitrary constants to be determined from the initial conditions

Vector field and phase portrait in phase plane:

- We shall consider the linear 2nd order system:

$$\dot{x} = Ax\,, \text{where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Here we can observe that $x = [0\ 0]^T$ is a solution

- the equilibrium solution - the origin of the phase plane (x_1, x_2).

- Example:

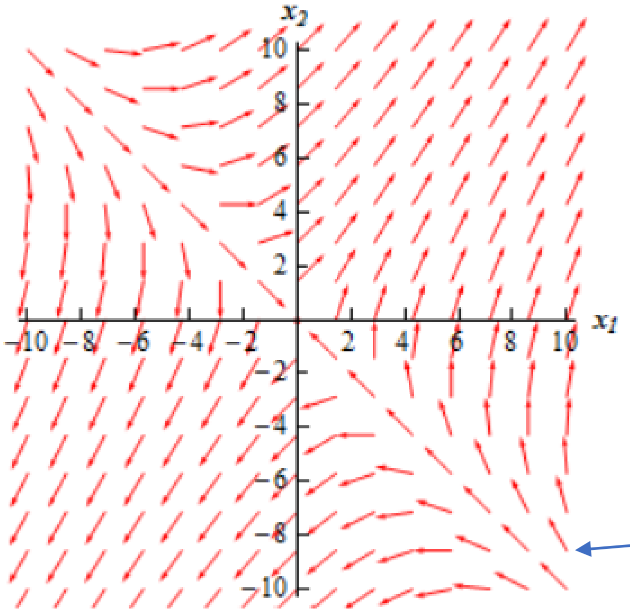
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

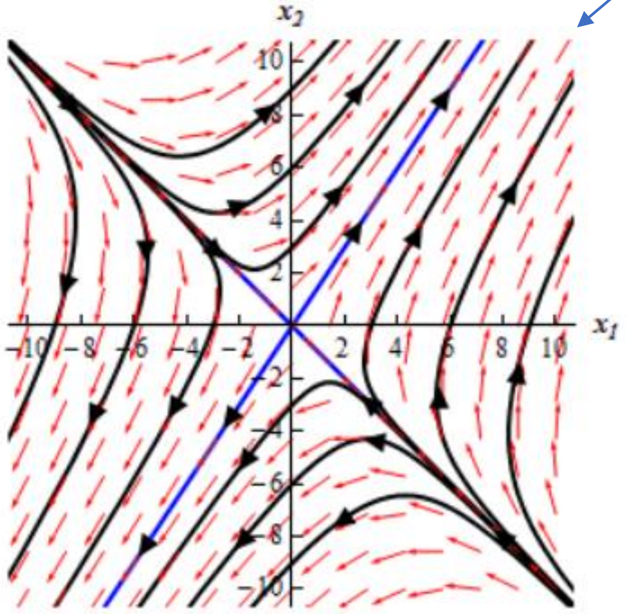
This indicates that the point (-1,1) is a vector pointing towards (1,-1) – (draw a line of fixed length from (-1,1) in the dir. of (1,-1))

- Similarly we can find that
  - (1,-1) → (-1,1)
  - (2,0) → (2,6)
  - (-3,-2) → $(-7, -13)$
- Repeat the step for a large number of points in the phase plane – vector field.
- Trajectories – vectors in the sketch are tangents to the trajectories – the directions of the vectors give the directions of the trajectories as t increases. (phase portrait)

Phase portrait: family or group of trajectories in phase plane



vector field



phase portrait (contains only trajectories)

Qualitative analysis

The locus of the solution x(t) in x_1-x_2 plane, starting from an initial point - trajectory

How to construct a trajectory:
- Start from an initial point (x_{10}, x_{20})
- Calculate (x1dot, x2dot)
- Draw an element line from (x_{10}, x_{20}) to (x1dot,x2dot)
- Repeat the last two steps from the end of the element line

- Seems to be several groups of trajectories that have slightly different behaviors.
- Some groups of trajectories start at (or near at least) the equilibrium solution and then move straight away from it.
- Some other groups of trajectories start away from the equilibrium solution and then move straight in towards the equilibrium solution.
- We call the equilibrium point a **saddle point** - we call the equilibrium point in this case **unstable** since some of trajectories are moving away from it as t increases.

- Let us revisit our example:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

- Using (1) and (2) we have

a=1, b=2, c=3 and d=2

- Using (5) we can write:

$$\ddot{x}_1-3\dot{x}_1-4x_1=0$$

- The characteristic equation is:

$$\lambda^2-3\lambda-4=0$$

and the roots are:

$$\lambda_{1,2}=-1,4$$

- Since the roots are real and distinct (Case I), the general solution for x_1 is:

$$x_1(t)=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}=c_1e^{-t}+c_2e^{4t}$$

- Since we now know x_1 ,from (3) we get:

$$x_2(t)=-c_1e^{-t}+\frac{3}{2}c_2e^{4t}$$

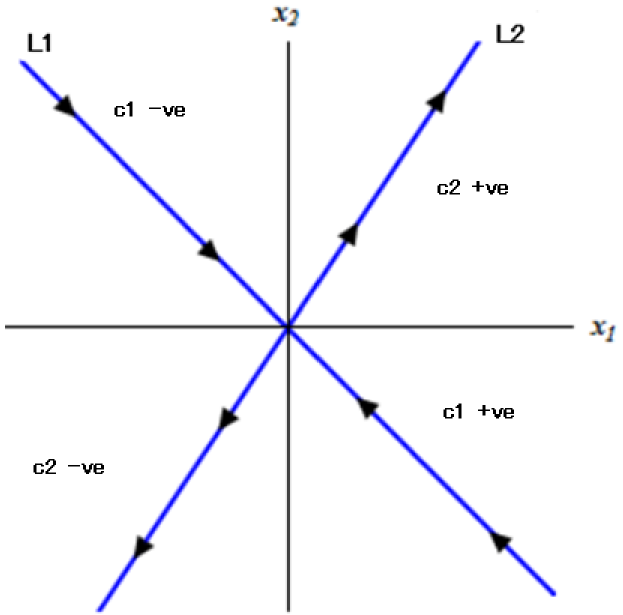
- In vector notation we can write:

$$x(t)=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=c_1e^{-t}\begin{bmatrix}1\\ -1\end{bmatrix}+c_2e^{4t}\begin{bmatrix}1\\ 3/2\end{bmatrix}$$

- It can be shown that the roots of the characteristic equation are nothing but the eigenvalues of the system matrix

$$A=\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix}$$

- The eigenvector (L1) corresponding to the eigenvalue λ_1=-1 can be determined as $[1\ \ -1]^T$
- The eigenvector (L2) corresponding to the eigenvalue λ_2=4 can be determined as $[1\ \ 3/2]^T$

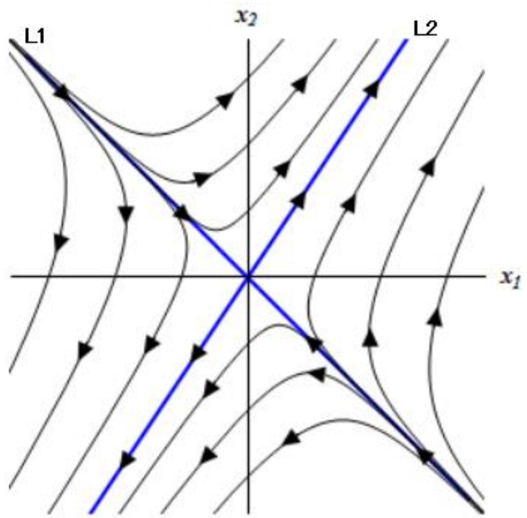


Sketch of lines that follow the directions of the two eigenvectors. Since λ_1 is –ve, an increase in t along L1 will cause the trajectory to move towards the origin.
The opposite happens for L2 since λ_2 is +ve.

- Consider c_2= 0
  - the solution is an exponential times a vector
  - the exponential term affects the magnitude of the vector
  - the constant c_1 affects both the sign and the magnitude of the vector
  - If $c_1 > 0$ the trajectory will be in Quadrant IV
  - if $c_1 < 0$ the trajectory will be in Quadrant II.

For c_1 =0, Complimentary things will happen

- If both constants are nonzero the trajectories will have a combination of these behaviors.
- For large negative t's the trajectories will be dominated by the portion that has the negative eigenvalue since in these cases the exponent will be large and positive.
- Trajectories for large positive t's will be dominated by the portion with the positive eigenvalue.
- **<u>In general if we know the eigenvalues and the corresponding eigenvectors, we can predict the qualitative behavior of the trajectories and draw conclusions about the stability of the equilibrium point or solution</u>**.
- In this case (<u>real eigenvalues with real and opposite signs</u>) we call the equilibrium solution a **saddle point** where few trajectories move towards it and all the other trajectories move away from it as t increases.

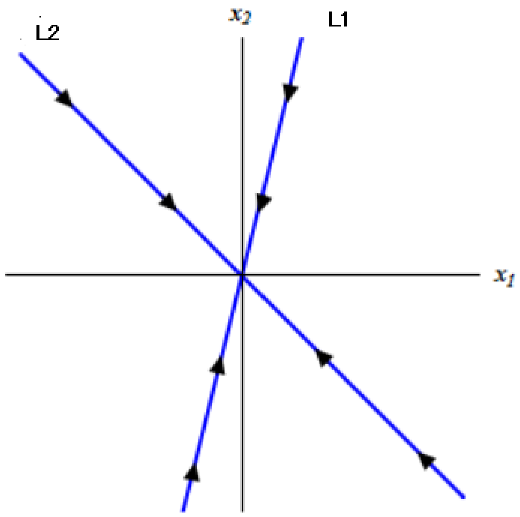


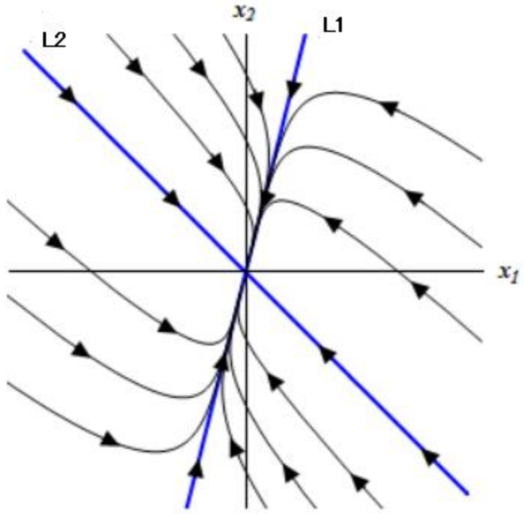
- Example (<u>real eigenvalues with –ve signs</u>):

$$\dot{x}=\begin{bmatrix}-5 & 1\\ 4 & -2\end{bmatrix}x$$

- The eigenvalues and the corresponding eigenvectors are:

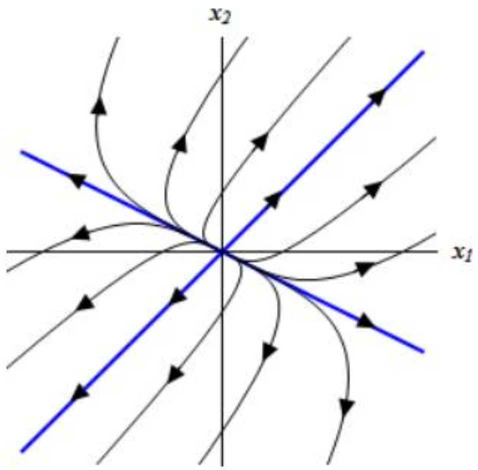
$$\lambda_1=-1,\ \ \lambda_2=-6$$
$$L1=\begin{bmatrix}1\\4\end{bmatrix},\ \ L2=\begin{bmatrix}-1\\4\end{bmatrix}$$





In this case the equilibrium point is called a **proper node** which is **asymptotically stable**. Equilibrium solutions are asymptotically stable if all the trajectories move in towards it as t increases.

- If both the eigenvalues are +ve and, we will call the equilibrium point as a **proper node** which is **asymptotically unstable** where all the trajectories move away from it as t increases.



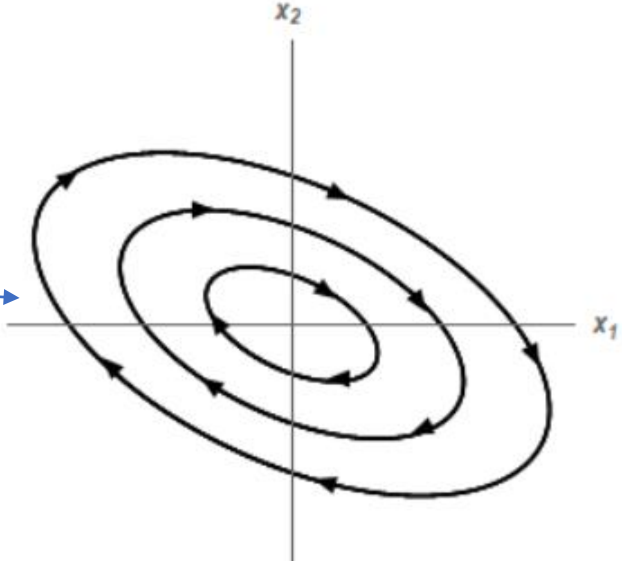
- Example: (purely complex eigenvalues)

$$\dot{x}=\begin{bmatrix} 3 & 9 \\ -4 & -3 \end{bmatrix}x$$

In this case the eigenvalues are:

$$\lambda_1=3\sqrt{3}i, \qquad \lambda_1=-3\sqrt{3}i$$

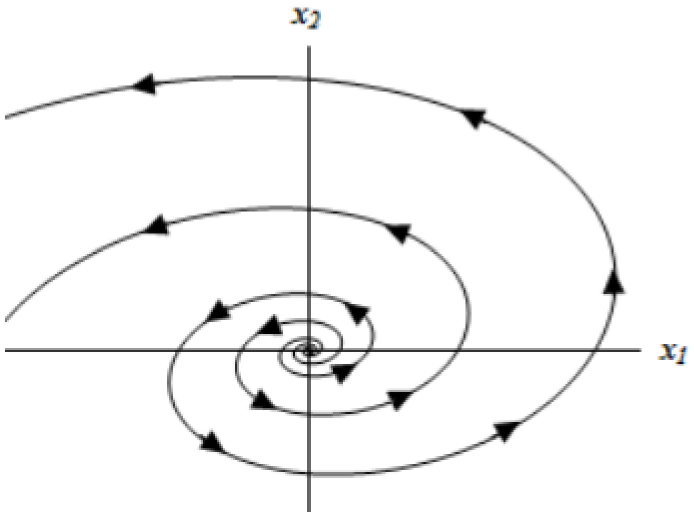
- The trajectories are circles or ellipses in this that are centered at the origin.



- The equilibrium point in the case is called a **center** and is stable – but not asymptotically stable.

- Example: (<u>complex eigenvalues with a real part</u>)

$$\dot{x}=\begin{bmatrix}3 & -13\\ 5 & 1\end{bmatrix}x$$



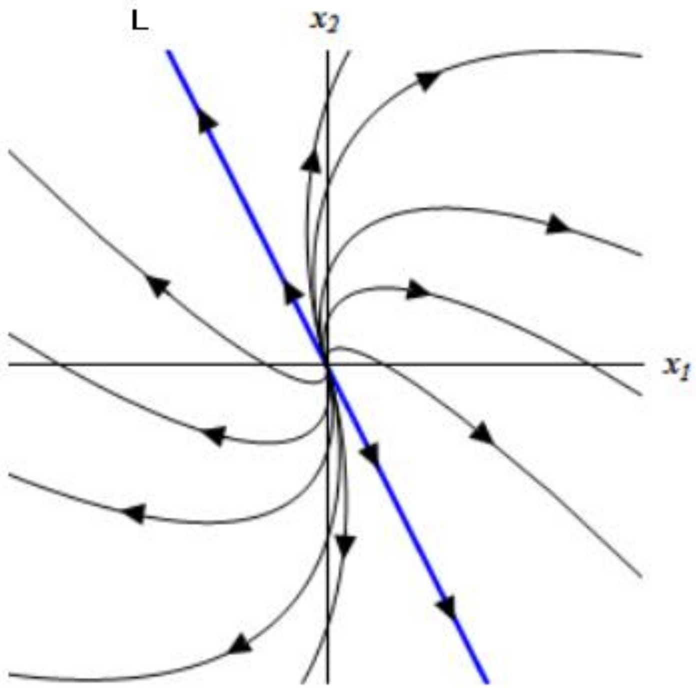
In this case the eigenvalues are:

$$\lambda_1=2+8i,\ \ \lambda_2=2-8i$$

- When the eigenvalues of a system are complex with a real part the trajectories will spiral into or out of the origin.
- We can determine which one it will be by looking at the real portion.
- Since the real portion will end up being the exponent of an exponential function (as we saw earlier), if the real part is positive the trajectories will grow very large as t increases.
- If the real part is negative the solution will die out as t increases.
- Hence if the <u>**real part is positive**</u> the trajectories will spiral out from the origin
- Conversely if the <u>**real part is negative**</u> they will spiral into the origin.

- Example: (repeated real +ve eigenvalues)

$$\dot{x}=\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}x$$



In this case the eigenvalues are:

$$\lambda_{1,2}=5$$

The eigenvector is:

$$L=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

- Trajectories in this case always emerge from the origin in a direction that is parallel to the eigenvector.
- The equilibrium point is called a improper **node** and is unstable in this case.

- Example: (repeated real -ve eigenvalues)

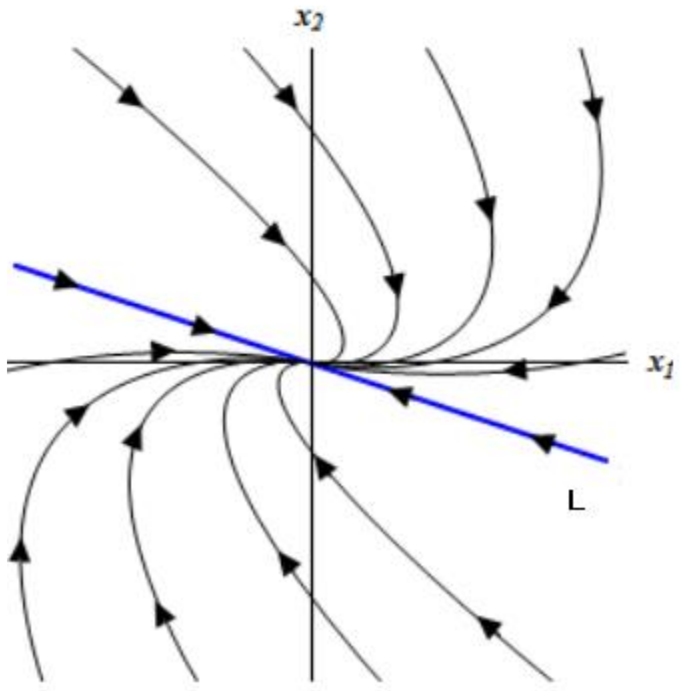
$$\dot{x}=\begin{bmatrix}-1 & \frac{3}{2}\\ -\frac{1}{6} & -2\end{bmatrix}x$$

In this case the eigenvalues are:

$$\lambda_{1,2}=-\frac{3}{2}$$

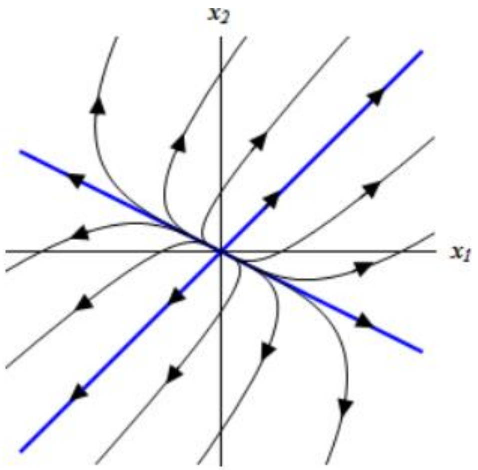
The eigenvector is:

$$L=\begin{bmatrix}-3\\ 1\end{bmatrix}$$

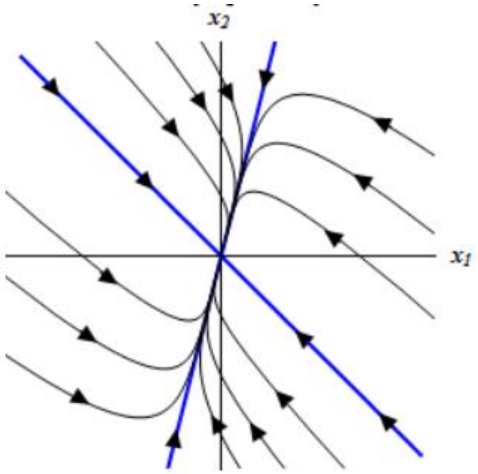


- Trajectories in this case always converge to the origin.
- The equilibrium point is called a improper **node** and is asymptotically stable in this case.

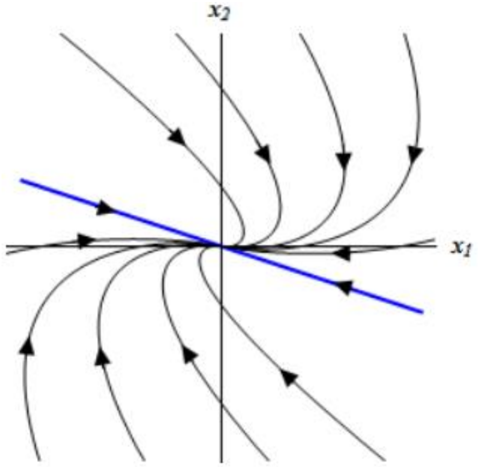
Some examples of phase portraits:



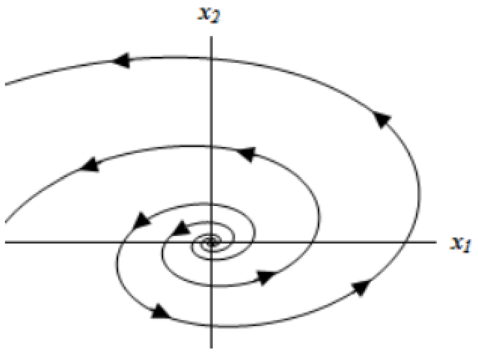
Node - Unstable



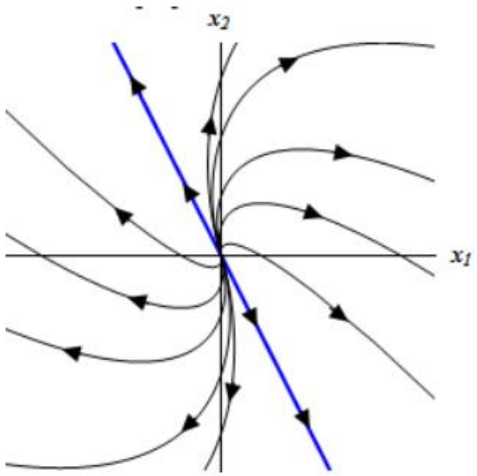
Node - Asymptotically stable



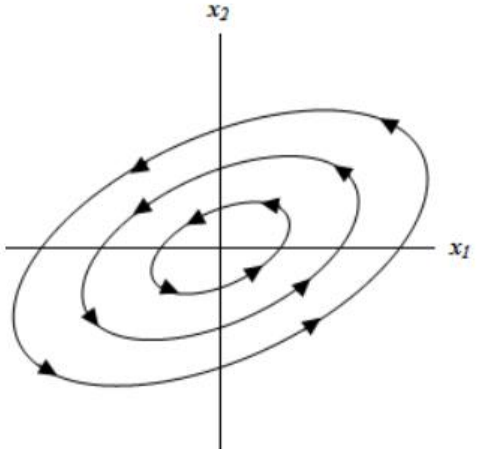
Improper node - Asymptotically stable



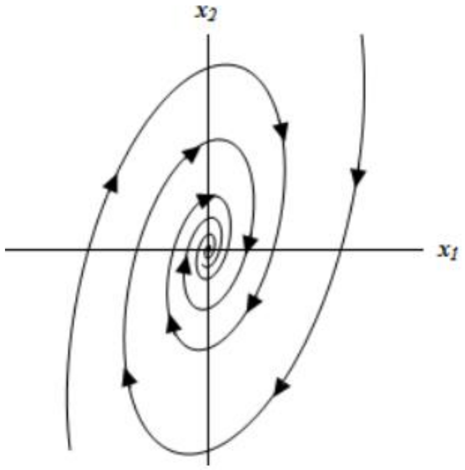
Spiral - Unstable



Improper node - Unstable



Center - Stable



Spiral – Asymptotically stable

Difference between **center stable** and **asymptotically stable** equilibrium points:

- In an asymptotically stable node or spiral all the trajectories will move in towards the equilibrium point as *t* increases.
- A center (which is always stable) trajectory will just move around the equilibrium point but never actually move in towards it.

Moving from linear to nonlinear 2nd order systems

- Qualitative behavior of nonlinear systems can be obtained locally by linearization around the equilibrium points
- Linear systems can have
  - an isolated equilibrium point

  or
  - a continuum of equilibrium points (When $det|A| = 0$)
- nonlinear systems can have multiple isolated equilibrium points.
- Qualitative behavior of second-order nonlinear system can be investigated by
  - linearizing the system around equilibria and then by studying the system behavior near them
- Let (x_{10}, x_{20}) be one of the equilibrium points of the nonlinear system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

- f_1, f_2 are continuously differentiable about (x_{1O}, x_{2O})
- Since we are interested in trajectories near (x_{1O}, x_{2O}), we define

$$x_1 = x_{1O} + y_1, \; x_2 = x_{2O} + y_2$$

- y_1, y_2 are small perturbations around the equilibrium point.

- The qualitative behaviour of the nonlinear system under consideration can be derived through 2d Taylor expansion around the equilibrium points:

$$\dot{x}_1 = \dot{x}_{1O} + \dot{y}_1 = f_1(x_{1O} + y_1, x_{2O} + y_2) = f_1(x_{1O}, x_{2O}) + \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_{1O}, x_{2O})} y_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_{1O}, x_{2O})} y_2 + \text{H.O.T.}$$

$$\dot{x}_2 = \dot{x}_{2O} + \dot{y}_2 = f_2(x_{1O} + y_1, x_{2O} + y_2) = f_2(x_{1O}, x_{2O}) + \left.\frac{\partial f_2}{\partial x_1}\right|_{(x_{1O}, x_{2O})} y_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_{1O}, x_{2O})} y_2 + \text{H.O.T.}$$

where H.O.T. = higher order terms.

- For a sufficiently small neighbourhood around the equilibrium point we can neglect the H.O.T.
- Since (x_{1O}, x_{2O}) is one of the equilibrium points of the nonlinear system, we can write $f_1(x_{1O}, x_{2O})=0$ and $f_2(x_{1O}, x_{2O})=0$.
- Consequently we land up into the following locally linearized system

$$\begin{cases} \dot{y}_1 = a_{11}y_1 + a_{12}y_2 \\ \dot{y}_2 = a_{21}y_1 + a_{22}y_2 \end{cases}, \quad a_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{(x_{10}, x_{20})}, \qquad i, j = 1,2$$

- The equilibrium point of the derived linear system is $(y_1, y_2) = (0,0)$ and we can write

$$\dot{y} = Ay, \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \left.\frac{\partial f}{\partial x}\right|_{(x_{10}, x_{20})}$$

- The Matrix $\frac{\partial f}{\partial x}$ is called Jacobian Matrix.
- The trajectories of the nonlinear system in a small neighborhood of an equilibrium point are close to the trajectories of its linearization about that point.
- If the origin (equilibrium point) of the linearized state equation is
    - an asymptotically stable (unstable) node, or an asymptotically stable (unstable) improper node or an asymptotically stable (unstable) spiral or a saddle point,
- then in a small neighborhood of the equilibrium point, the trajectory of the nonlinear system will behave like
    - an asymptotically stable (unstable) node, or an asymptotically stable (unstable) improper node or an asymptotically stable (unstable) spiral or a saddle point.
- This linearization scheme is not valid for a "center" equilibrium point.

- Example: we have the following nonlinear system -

$$\dot{x}_1 = -x_1 + 2x_1^3 + x_2$$
$$\dot{x}_2 = -x_1 - x_2$$

We want to find all the equilibrium points and determine the nature of each equilibrium point.

- First let us determine the equilibrium points:

$$0 = -x_1 + 2x_1^3 + x_2, \quad 0 = -x_1 - x_2 \Rightarrow x_1 = -x_2$$
$$x_1 = -x_2 \Rightarrow 2x_1(x_1^2 - 1) = 0 \Rightarrow x_1 = 0, 1, -1$$

The equilibrium points are at (0,0), (1,-1) and (-1,1).

- The Jacobian matrix is given by,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + 6x_1^2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm j \Rightarrow (0,0) \text{ is an asymptotically stable spiral}$$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2 \pm \sqrt{8} \Rightarrow (1,-1) \text{ is a saddle}$$

Similarly it can be shown that (-1,1) is also a saddle

Review : Eigenvalues & Eigenvectors

- If we multiply an n x n matrix (A) by an n x 1 vector (L) we will get a new n x 1 vector (y) back. In other words:

$$AL = y$$

- What we want to know is if it is possible for the following to happen –
  - Instead of just getting a brand new vector out of the multiplication is it possible instead to get the following,

$$AL = \lambda L \quad ....(1)$$

- In other words, is it possible, at least for certain λ and L , to have matrix multiplication be the same as just multiplying the vector by a constant?
- If we do happen to have a λ and L for which this works (and they will always come in pairs) then we call λ an **eigenvalue** of A and L an **eigenvector** of A.
- How to find the eigenvalues and eigenvectors for a matrix ?
  - first we notice that if $L = 0$ then (1) is going to be true for any value of λ. So we impose that $L \neq 0$.
  - We can now rewrite (1) as

$$(A - \lambda I_n) L = 0$$

  - Since $L \neq 0$, we can solve

$$\det(A - \lambda I_n) = 0$$

    to find the values λ.
  - Once we have the eigenvalues we can then go back and determine the eigenvector for each eigenvalue.

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- First we determine the eigenvalues:

$$\det(A - \lambda I_n) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 3\lambda - 4$$

$$= (\lambda+1)(\lambda-4) \quad \Rightarrow \quad \lambda_1 = -1,\ \lambda_2 = 4$$

- Now let's find the eigenvectors for each of these
- eigenvalues:

$\lambda_1 = -1$:

We have to solve

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} [L1] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{where} \quad [L1] = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

This implies

$$2l_1 + 2l_2 = 0 \quad \Rightarrow \quad l_1 = -l_2$$

This determines an eigenvector corresponding to λ_1 = -1 up to a scalar multiple. If we choose l_1 =1, we obtain the eigenvector

$$L1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_2 = 4$:

We have to solve

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} [L2] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This implies

$$-3l_1 + 2l_2 = 0 \quad \Rightarrow \quad l_1 = \frac{2}{3} l_2$$

If we choose l_1 =1, we obtain the eigenvector

$$L2 = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$$