- Control theory is divided into two branches:
  - Linear control theory -
    - Applies to systems made of devices which are linear – What is a linear system or device?
    - Governed by linear differential equations
    - A major subclass: systems with parameters that do not change with time, called *linear time invariant* (LTI) systems.
    - Can be analyzed by powerful frequency domain mathematical techniques, such as the Laplace transform, Fourier transform, Z transform, Bode plot, root locus, and Nyquist stability criterion.

  - Nonlinear control theory -
    - Covers a wider class of systems that do not obey the fundamental principles of linear systems/devices.
    - This type of systems play a vital role in the control systems from an engineering point of view - in practice all systems/devices are nonlinear in nature.
    - These systems are often governed by nonlinear differential equations.
    - Some of the mathematical techniques include - limit cycles and describing functions.

Types of nonlinearity effects:

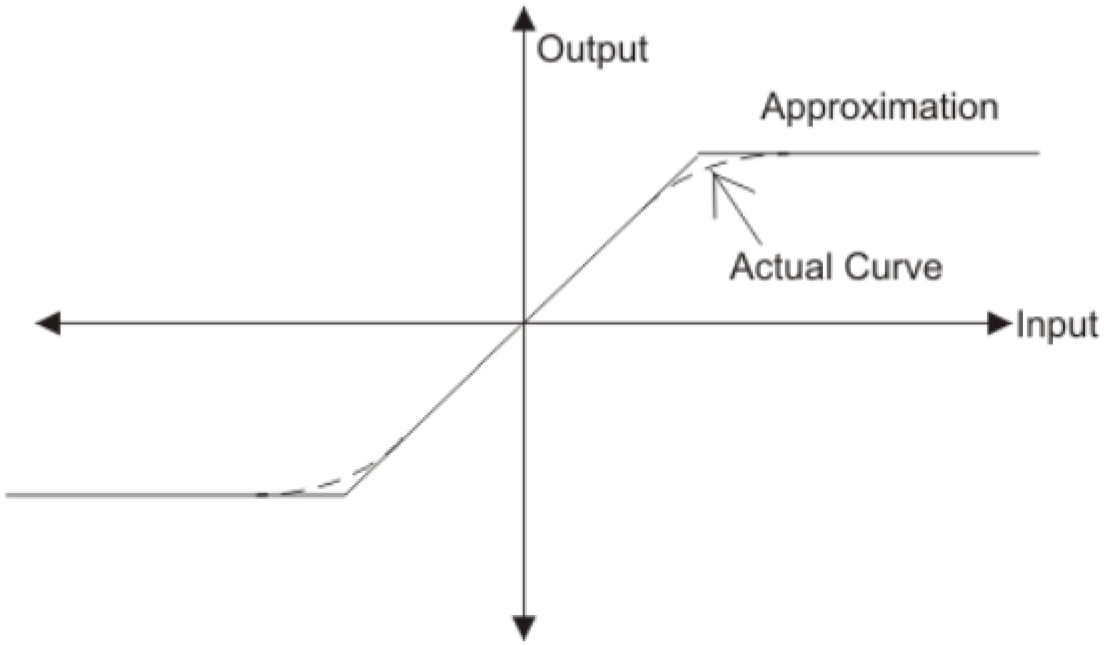
- On the basis of magnitude
  - Saturation
  - Friction
  - Dead zone
  - Relay
  - Backlash

- On the basis of frequency
  - Limit cycle
  - Harmonics
  - Chaos

Saturation:

Saturation nonlinearity is a common type of nonlinearity.

From the curve we can see that the output showing linear behavior over a certain range of the input but after that there is a saturation in the curve which is a kind of non linearity in the system.



Saturation non-linearity

Example: ??

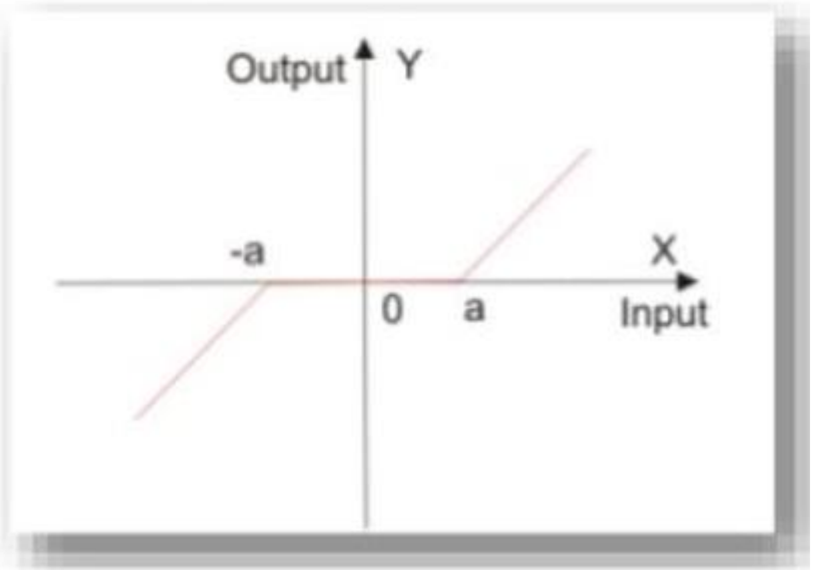
Friction:

Friction is the force resisting the relative motion of solid surfaces, fluid layers, and material elements sliding against each other. It is a kind of nonlinearity present in the system.

A common example in an electric motor in which we find friction drag due to the rubbing contact between the brushes and the commutator.

Dead zone:

Nonlinearity effect where the system does not respond to the given input until the input reaches within a particular range of values.
The output remains zero when the value of input falls within a certain range.



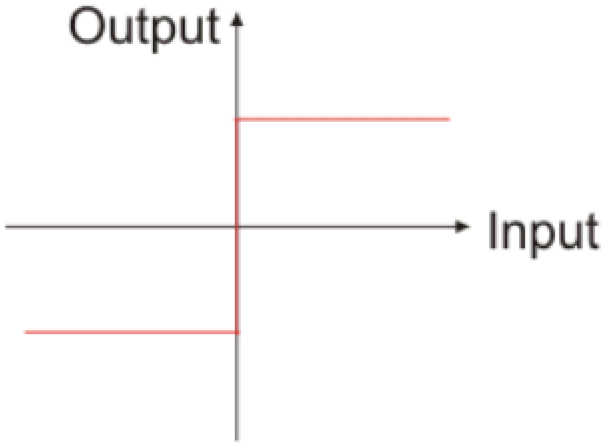
Effects of dead zone:

- System performance degradation
- Reduced accuracy
- May destabilize the system.

Example: ??

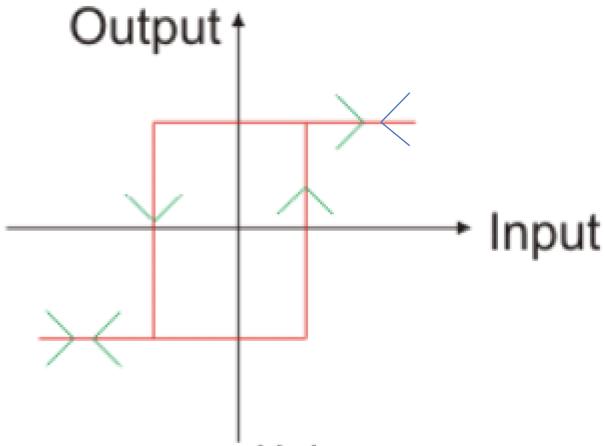
Relay:

Electromechanical relays are frequently used in control systems where the control strategy requires a control signal with only two or three states. This is also called as ON/OFF controller or two state controller.



(a)

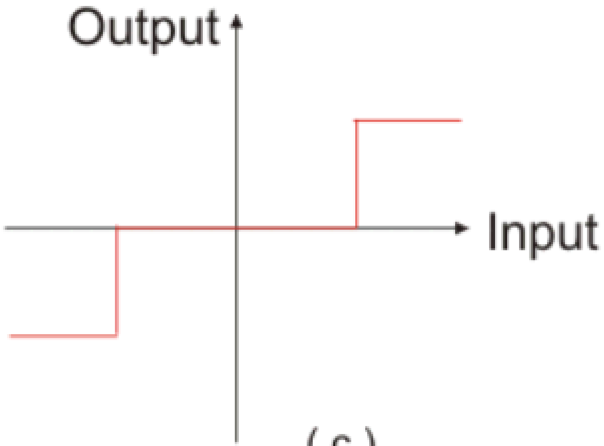
figure (a): ideal characteristics of a bidirectional relay



(b)

figure (b): relay with hysteresis

What is the beneficial effect of hysteresis in this case?

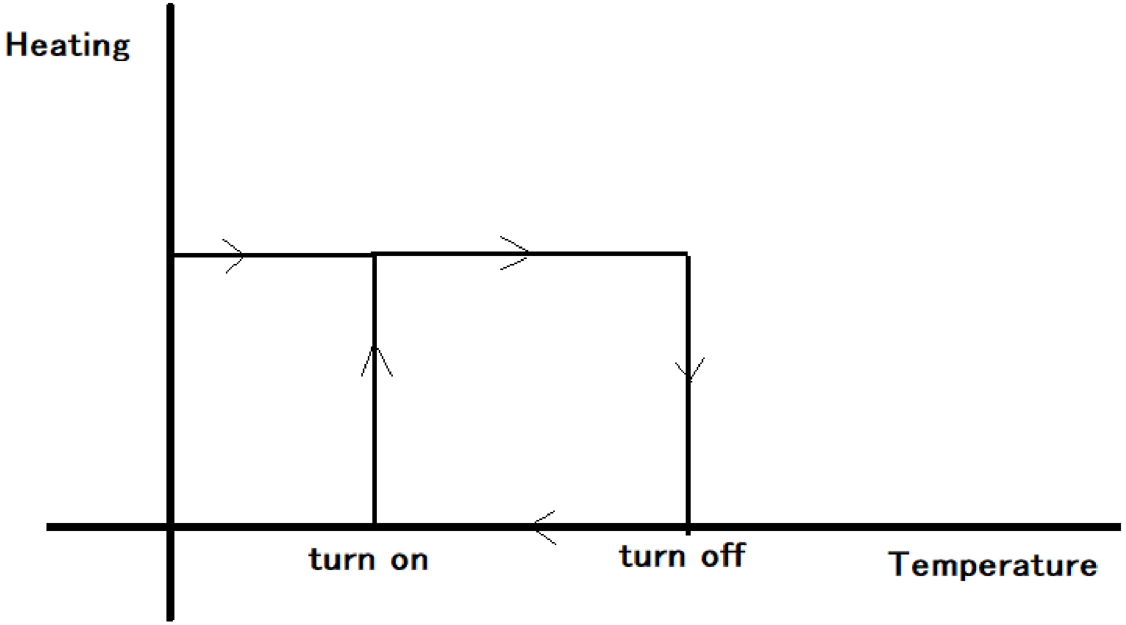


(c)

figure (c): relay with hysteresis and dead zone

Limit cycle:

An example of a nonlinear control system is a thermostat-controlled heating system. A building heating system such as a furnace has a nonlinear response to changes in temperature; it is either "on" or "off", it does not have the fine control in response to temperature differences that a proportional (linear) device would have. Therefore, the furnace is off until the temperature falls below the "turn on" setpoint of the thermostat, when it turns on. Due to the heat added by the furnace, the temperature increases until it reaches the "turn off" setpoint of the thermostat, which turns the furnace off, and the cycle repeats. This cycling of the temperature about the desired temperature is called a ***limit cycle***, and is one of the characteristics of nonlinear control systems.



Backlash

An important nonlinearity effect that commonly occurs in mechanical transmissions such as gear trains and linkages.

It is a form of mechanical hysteresis.

Backlash is the play between the teeth of the drive gear and those of the driven gear.

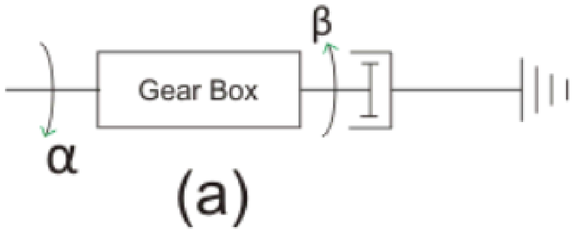


figure (a): gearbox

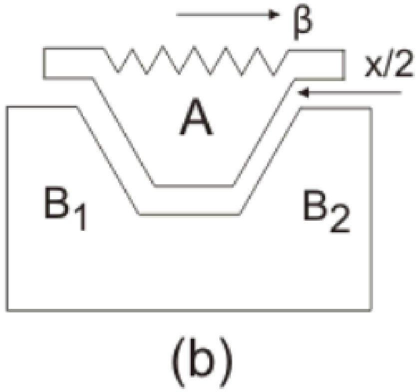


figure (b): backlash

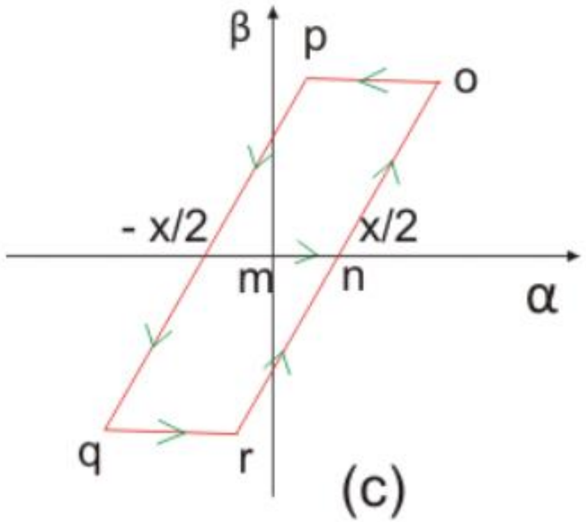
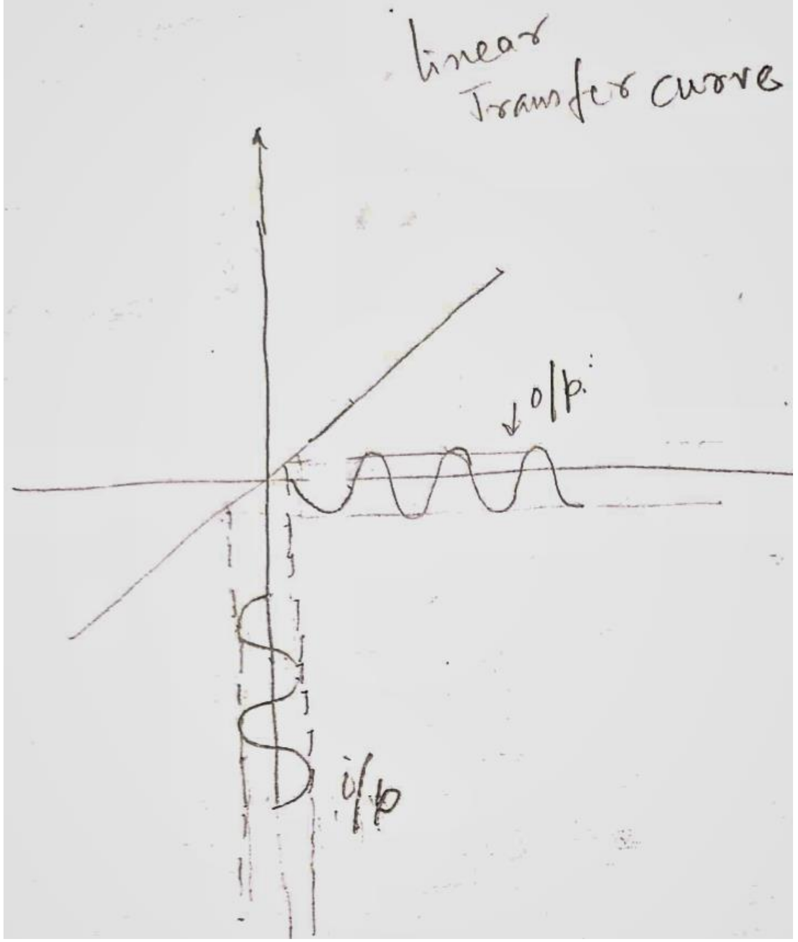
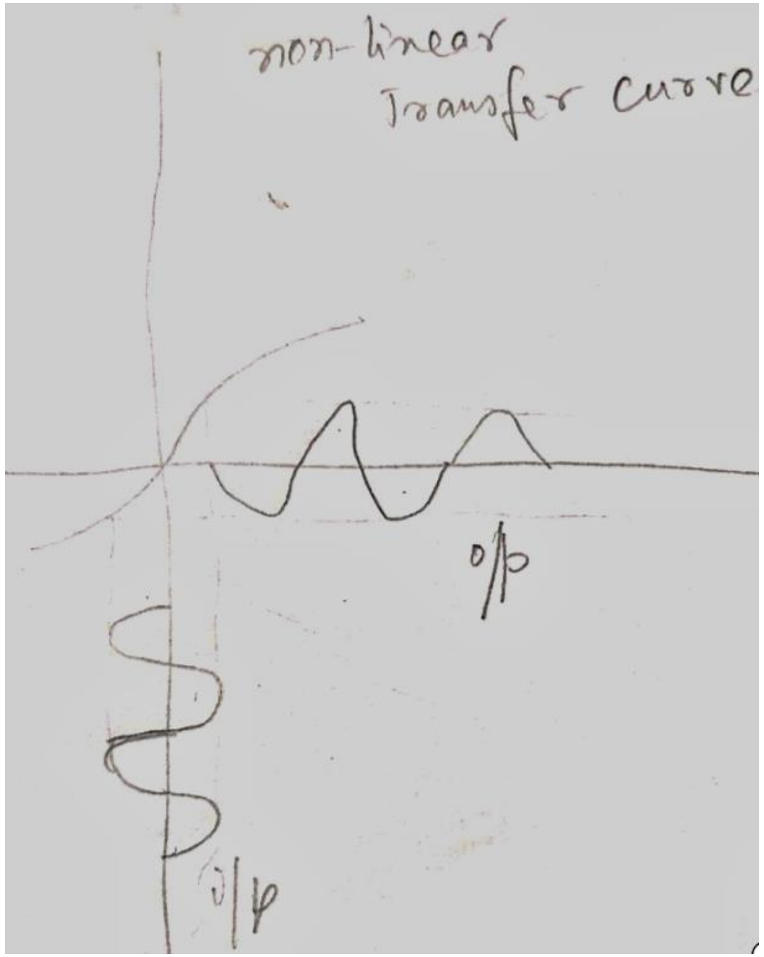


figure (c): relationship between input and output motions

Harmonics:

Linear system: If a pure sinusoid is applied at the input, output waveform will have same frequency. The amplitude and phase may vary.
Nonlinear system: If a pure sinusoid is applied at the input, the output waveform will not be a pure sinusoid. It may contain other frequency components called the harmonics.

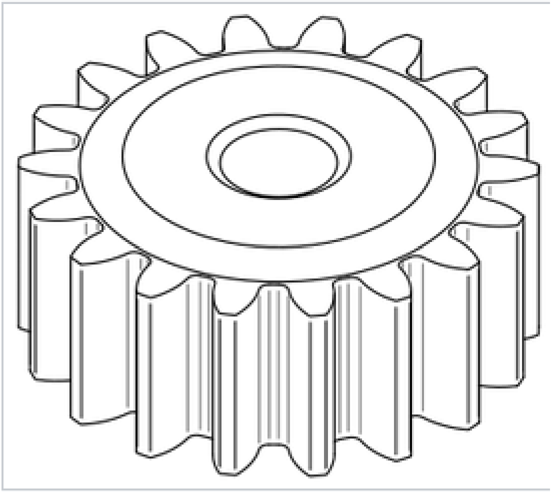




Chaos

Sometimes a nonlinear system may exhibit complex and irregular motion that are extremely sensitive to initial conditions and hence unpredictable.

Example: Spur gear system with backlash - the resulting vibration has chaotic behaviour.
Spur gear: the simplest type of gear that is straight cut



State space representation of a nonlinear system:

- In general the behaviour of a dynamic system can be represented by a finite number of coupled first-order differential equations:

$$\dot{x}_1 = f_1\left(t, x_1, \dots, x_n, u_1, \dots, u_p\right)$$
$$\dot{x}_2 = f_2\left(t, x_1, \dots, x_n, u_1, \dots, u_p\right)$$
$$\vdots \qquad \vdots$$
$$\dot{x}_n = f_n\left(t, x_1, \dots, x_n, u_1, \dots, u_p\right)$$
$$x_1, \dots, x_n \triangleq \text{state variables} = \text{memory of the system about its past}$$
$$u_1, \dots, u_p \triangleq \text{input variables}$$

- In vector notation we have,

$$x = [x_1, x_2, \dots, x_n]^T$$
$$u = \left[u_1, u_2, \dots, u_p\right]^T$$
$$f(t, x, u) = [f_1(t, x, u), f_2(t, x, u), \dots, f_n(t, x, u)]^T$$

- Reformatting the coupled first-order differential equations into a single n-dimensional vector equation we have, the state vector equation as,

$$\dot{x} = f(t, x, u)$$

Similarly the q-dimensional output equation as,

$$y = h(t, x, u)$$

$y = [y_1, y_2, \dots, y_q] \triangleq$ output variables = variables of particular interest in the analysis of the dynamical system.

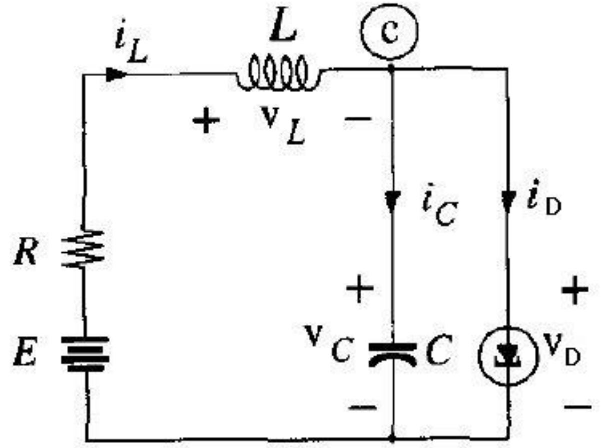
- Example: variables that can be physically measured or variables that are required or expected to behave in a specified manner.
- Together the state vector equation and the output vector equation is called the **state-space model** or simply the **state model.**
- For linear systems the state model takes the following familiar form:

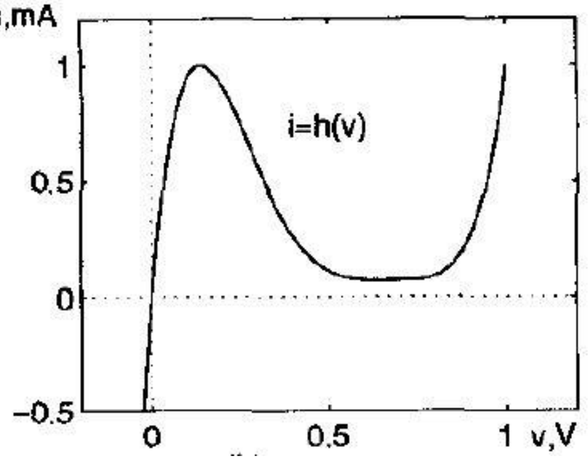
$$\dot{x} = A(t)x + B(t)y$$
$$y = C(t)x + D(t)y$$

- Since there is a plethora of powerful analytical tools available for linear systems as mentioned earlier, an obvious step for the analysis of nonlinear systems will be to linearize the system about a nominal operating point and analyse the resulting linear system – a useful and common practice in engineering.
- Problems with the approach of linearization:
  - Linearization is an approximation in the neighbourhood of a quiescent point
    - predicts the behaviour in the vicinity of that point
    - cannot predict “nonlocal” or “global” behaviour
  - There are several essentially nonlinear phenomena that can take place in a nonlinear system only. Linearization cannot predict these bahaviours.

Example of state space representation of a nonlinear system -

- Tunnel diode circuit:





States: $x_1 \triangleq v_C$ $x_2 \triangleq i_L$ input: $u = E$ For the capacitor : $i_C = C\frac{dv_C}{dt}$ For the inductor : $v_L = L\frac{di_L}{dt}$

In node C: $i_C + i_D - i_L = 0 \quad \Longrightarrow i_C = i_L - i_D = x_2 - h(v_D) = x_2 - h(x_1)$

In the left loop: $i_L R + v_L + v_c - E = 0 \quad \Longrightarrow v_L = E - v_C - i_L R = u - x_1 - x_2 R$

We have

$$\frac{dv_C}{dt} = \dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2]$$

$$\frac{di_L}{dt} = \dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

The state equations of the system.

Equilibrium points:

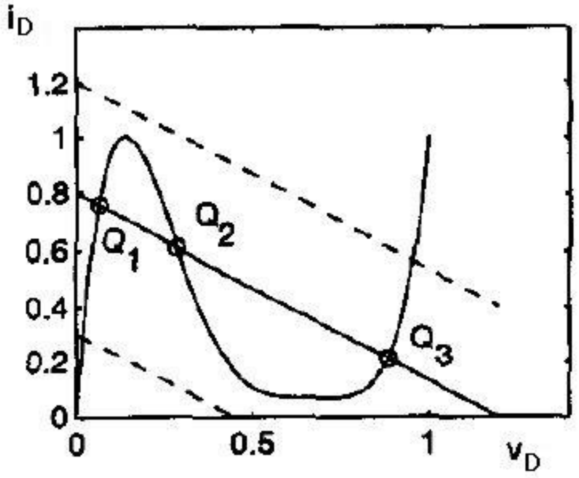
Set $\dot{x}_1 = \dot{x}_2 = 0$. Then we have :

1) $0 = \frac{1}{C}[-h(x_1) + x_2] \implies h(x_1) = x_2$

2) $0 = \frac{1}{L}[-x_1 - Rx_2 + u] \implies x_2 = \frac{E}{R} - \frac{1}{R}x_1$

From the last two equations we have the equilibrium points as the roots of the following equation:

$$h(x_1) = \frac{E}{R} - \frac{1}{R}x_1$$



The diagonal lines are called load lines. Consider the last equation. When $x_1 = 0, h(x_1) = \frac{E}{R}$
When $x_1 = E, h(x_1) = 0$.

Case 1 (middle, solid line): 3 equilibrium point Q1, Q2 and Q3
Case 2 (top, dotted line): $E \uparrow, R \leftrightarrow$, only one equilibrium point
Case 3 (bottom, dotted line): $E \downarrow, R \leftrightarrow$, only one equilibrium point