FREE VIBRATION ANALYSIS OF LAMINATED COMPOSITE CYLINDRICAL SHELLS

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Prepared By

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This is to certify that the thesis entitled "**Free Vibration Analysis of Laminated Composite Cylindrical shells**" is being submitted by **Ruman Biswas** (Class Roll No - 002110402008, Exam Roll No.-M4CIV23023, Registration No.- 160030 of 2021-2022) in partial fulfilment of the requirements for the award of the Master degree in Civil Engineering from Jadavpur University and it is delightfully declared that it is a record of bonafide research work carried out by her under my direct supervision in the year of 2022-2023.

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I, Ruman Biswas, a student of Master of Engineering in Civil Engineering (Structural Engineering), Jadavpur University, Faculty of Engineering & Technology, hereby declare that the work being presented in the thesis work entitled, **"Free Vibration Analysis of Laminated Composite Cylindrical Shells"**, is authentic record of work that has been carried out at the Department of Civil Engineering, Jadavpur University, under the guidance of Dr. Sreyashi Das (nee Pal), Associate Professor, Department of Civil Engineering, Jadavpur University.

The work contained in the thesis has not yet been submitted in part or full to any other university, institution, or professional body for award of any degree or diploma or any fellowship.

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MY PARENTS

AND MY TEACHER

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List of Symbols

Symbol	Description
X, Y, Z	Global coordinate system
x,y,z	local coordinate system
σ_{ij}	Stress components
ε_{ij}	Strain components
C_{ij}	Elastic constants
E_{ij}	Young modulus
G_{ij}	Shear modulus
ν_{ij}	Poisson's ratio
ρ	Density of Element
u , v, w	Displacement component
N_{ij}	Stress resultant
M_{ij}	Moment resultant
$[B]$	Strain-displacement matrix
$[D]$	constitutive matrix
$[M]_e$	Mass matrix
$[K]_e$	Stiffness matrix
J	Jacobian
N_i	Shape or interpolation function
$\emptyset_x$	Rotation about y-axis
$\emptyset_y$	Rotation about x-axis
ω	Frequency in rad/s
ϖ	Non-dimensional fundamental freq.

Abstract

Free vibration of cylindrical shells made up of laminated composite material is studied by using Finite Element approach in software-based method. Vibration theory of laminated composite shell using First-order Shear Deformation Theory (FSDT) is presented in this paper.

The equations of motions are based on First-order Shear Deformation Theory (FSDT) of shell. The effects of transverse shear deformation and rotatory inertia are taken in to account. An eight-noded serendipity elements with 5 degrees of freedom at each node is considered. Finite element approach is employed for finding the shape functions. The natural frequency of the laminated composite shell is computed using MATLAB programming.

Some of the results obtained in the analysis are compared with those present in the existing literature. Several numerical results are also presented for selected material parameters, shell geometry and support condition to study the free vibration characteristics of various laminated composite cylindrical shells. The natural frequency for different number of layers and thickness and other conditions found out. The obtained values are compared with the values available in the literature Reddy [20] & Liu [25].

Effects of material properties and geometric parameters on the free vibration of laminated composite shells are discussed and some related mode the constants generating by the integrating process are disposed by gauss quadrature integration, and thus the equations of motion of total system including the boundary condition are transformed into an algebraic program. Then natural frequencies of the laminated composite structures are directly obtained by solving these programs. Stability and accuracy of the present method are verified through convergence and validation studies. Some new results for laminated composite cylindrical shell with variable thickness and arbitrary boundary conditions are presented, which may serve as benchmark solution.

CHAPTER 1

Introduction

1.1 Background and Motivation

Composite materials are those in which not less than two materials are combined on a naturally visible scale to shape useful material. The individual materials are easily identifiable. The properties of composite material are far better when compared to the properties of its constituents, if designed properly, which is one of the major advantages. They have a major application in many modern fields of engineering. Laminated composites are being used more extensively as structural components in aerospace, automobile, civil, marine and other related weight sensitive engineering applications requiring high strength to weight and stiffness to weight ratios. The mechanical behaviour of laminated composites is strongly dependent on the degree of orthotropy of the individual layers, the ratio of transverse shear modulus to the in-plane modulus and the stacking sequence of laminate. By appropriate orientation of the fibres in each lamina, desired strength and stiffness parameters can be achieved.

The Classical Laminate Plate Theory (CLPT) is inadequate for the analysis of laminated composite plates as it ignores the effect of transverse shear deformation. Hence, FSDT comes into picture. The first-order theories which are based on Mindlin and Reissner assume that the displacements and in-plane stresses through the thickness of the laminate are linear. The simplifying assumptions, made in classical and first-order theories, are reflected by the high percentage error in the results of thick composite and sandwich shell with highly stiff facings. The effect of shell aspect ratio and transverse shear rigidities of stiff layers on fundamental frequencies are more pronounced in thicker shell than they are for thin shell. In contrast to the first-order shear deformation theories, higher-order shear deformation theories do not require a shear correction coefficient, owing to more realistic representation of the cross-sectional deformation. Because of these limitations, the need is obvious to use the refined theories, which include the consideration of realistic parabolic variation of transverse shear stress through the laminate thickness and warping of the transverse cross-section. Thus, the use of higher-order shear deformation theory is very important for the vibration analysis of laminated composite shell, especially for thick sandwich laminates.

Many analytical methods of analysis have been used to study the vibration of plates and shells. In closed form solutions, the analytical difficulties in solving the equations have until now been overcome only in some special cases, while the general case has not yet received a satisfactory treatment.

FEM is normally used for the analysis of composite structures. In this method, there are numerous sub-domains that form a physical domain. These sub-domains are called finite elements. The finite element approach has proved to be a powerful and widely applicable method for the vibration analysis of complex problems for which analytical solutions are nearly impossible to find. A variety of new elements have been proposed based on different structural theories, interpolation functions and formulation procedures in order to achieve a more accurate prediction of the free vibration of plates and shells.

The analysis of thin shells attracted attention of researchers from the first half of the nineteenth century. While many researchers were improving the theory of shell structures from time to time, another group of researchers started developing exotic materials with high strength and stiffness properties. This resulted in the use of laminated composite materials to fabricate shell forms. The researchers had realized that the configuration like folded plates, conoidal, saddle, spherical, elliptic and hyperbolic parabolic and hyper shell can offer a number of parallel advantages that suit to the requirements of the industry. In fact, in industrial applications a shell may have complicated boundary conditions and may be subjected to complex loading. The advent of high-speed computers in the second half of the twentieth century was a major development that paved the way of researchers to get involved in analysis and design of shells of arbitrary geometry and loading conditions using numerical techniques.

From the above review, it is evident that the free vibration behaviour of laminated composite cylindrical shells is currently an active area of research. It is also important to mention that now days the MATLAB is well accepted modelling tool by different industries. However, MATLAB is capable to analyse the different linear and or nonlinear responses of laminated structures with ease and the available literature.

1.2 Overview of Composite Materials

In its most basic form a composite material one, which is composed of at least two elements working together to produce material properties that are different to the properties of those elements on their own. In practice, most composites consist of a bulk material (the 'matrix'), and a reinforcement of some kind, added primarily to increase the strength and stiffness of the matrix. This reinforcement is usually in fibre form.

Composite: Necessity

The composite materials are used to enhance the desired properties, which are as follows-

- Strength
- Stiffness
- Toughness
- Corrosion resistance
- Wear resistance
- Reduced weight
- Fatigue life
- Thermal/Electrical insulation and conductivity
- Acoustic insulation
- Energy dissipation
- Attractiveness, cost
- Tailorable properties

Composite: Constituents:

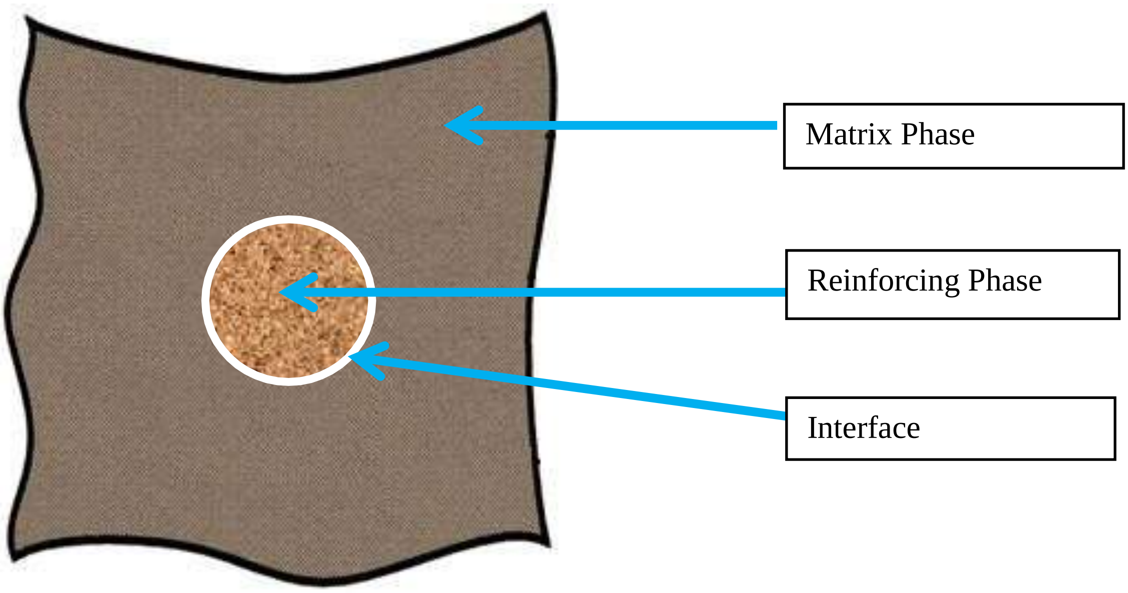


Fig 1.1: Internal structure of composite

Composite material

1. Reinforcement: A. Discontinuous
   B. Stronger
   C. Harder
2. Matrix: A. Continuous

Functions of reinforcement

1. Contribute desired properties
2. Load carrying capability
3. Transfer the strength to matrix

Functions of matrix

1. Holds the fibres together
2. Protects the fibres from environment
3. protects the fibres from abrasion (with each other)
4. Helps to maintain the distribution of fibres
5. Distributes the loads evenly between fibres
6. Enhances properties like transverse strength, impact resistance etc. of the resulting material.
7. Provides better finish to final product

Classification of Composites:

Based on the type of matrix material:

The most common manmade composites can be divided into four main groups:

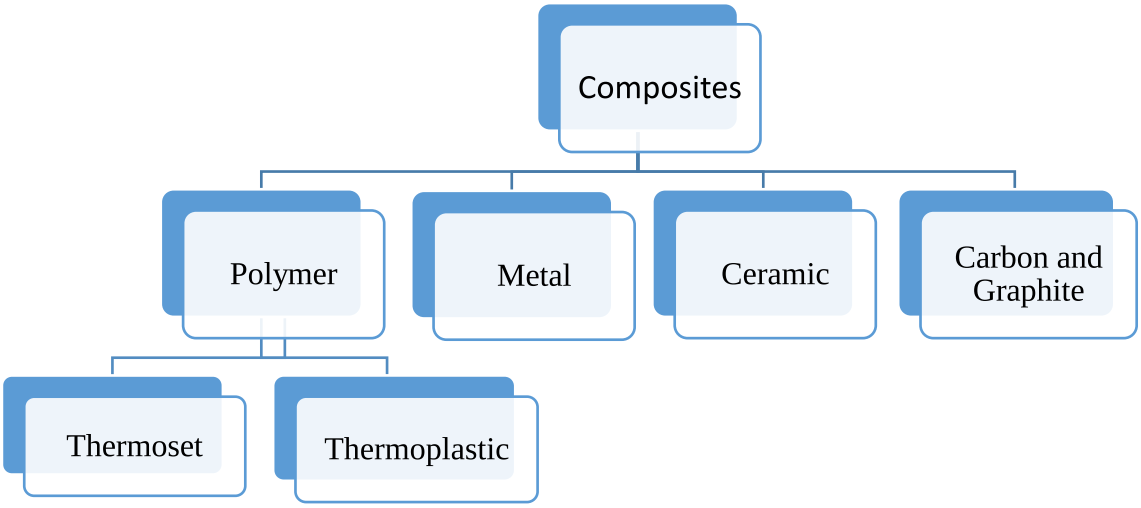


Fig 1.2: Classification of composite

1. **Polymer Matrix Composites (PMC's)-**These are the most common and will be discussed here. Also known as FRP - Fibre Reinforced Polymers (or Plastics) these materials use a polymer-based resin as the matrix, and a variety of fibres such as glass, carbon and aramid as the reinforcement.
2. **Metal Matrix Composites (MMC's)** -Increasingly found in the automotive industry, the materials use a metal such as aluminum as the matrix, and reinforce it with fibres, or particles, such as silicon carbide.
3. **Ceramic Matrix Composites (CMC's) -**Used in very high temperature environments, these materials use a ceramic as the matrix and reinforce it with short fibres, or whiskers such as those made from silicon carbide and boron nitride.
4. **Carbon and Graphite Composite (CGMC)-**Carbon (graphite) fibres are very stiff, strong and light filaments used in polymer (usually epoxy) matrix composites for aircraft structures and jet engine parts. Carbon fibres are up to five times stronger than mild steel for structural parts, yet are five times lighter.

1(a). Thermoset:

• Become cross-linked during fabrication & do not soften upon reheating**.**

Examples: Polypropylene, Polyvinyl chloride (PVC), Nylon, Polyurethane, Polyphenylene sulfide (PPS), Polysulpone.

• Higher toughness
• High volume
• Low-cost processing
• Temperature range ≥ 225ºC

Thermoplastics are increasingly used over thermosets because of the following Reasons:

• Processing is faster than thermoset composites since no curing reaction is Required Thermoplastic composites require only heating, shaping and cooling.
• Better properties:
- High toughness (delamination resistance) and damage tolerance,
- Low moisture absorption
- Chemical resistance
• They have low toxity.
• Cost is high!

1(b). Thermoplastic: Soften upon heating and can be reshaped with heat &pressure.

Examples:

Polyesters
Epoxies
Polyimides
Other resins

Polyesters:

Advantages:

• Low cost

• Good mechanical strength
• Low viscosity and versatility
• Good electrical properties
• Good heat resistance
• Cold and hot molding
• Curing temperature is 120°C

Epoxy: Epoxy resins are widely used for most advanced composites.
Advantages:
• Low shrinkage during curing
• High strength and flexibility
• Adjustable curing range
• Better adhesion between fibre and matrix
• Better electrical properties
• Resistance to chemicals and solvents
Disadvantages:
• Somewhat toxic in nature
• Limited temperature application range up to 175°C
• Moisture absorption affecting dimensional properties
• High thermal coefficient of expansion
• slow curing

Polyimides:
Advantages:
• Excellent mechanical strength
• Excellent strength retention for long term in 260-315°C (500-600°F) range in addition, short term in 370°C (700°F) range.
• Excellent electrical properties
• Good fire resistance and low smoke emission
• Hot molding under pressure and
• Curing temperature is 175°C (350°F) and 315°C

2. Metals:
Examples:
Aluminum
Titanium
Copper
Advantages:
• Higher use temperature range
Aluminum matrix composite – use temperature range above 300ºC
And titanium at 800 ºC
• Higher transfer strength, toughness (in contrast with brittle behavior of Polymers and ceramics)
• The absence of moisture & high thermal conductivity (copper)

Disadvantages:
• Heavier
• More susceptible to interface degradation at the fiber/matrix interface and to Corrosion

3. Ceramics:

A. Carbon,
B. Silicon carbide and
C. Silicon nitride

Advantages:

• Ceramic have use very high temperature range > 2000 ºC
• High elastic modulus
• Low density

Disadvantages:

• Brittleness
• Susceptible to flows

4. Carbon and Graphite:

Carbon fibres in carbon matrix – carbon/carbon composites
Used under extreme mechanical and thermal loads (space applications)

Advantages:

• Low specific weight
• High heat absorption capacity
• Resistance to thermal shock
• High resistance to damage
• Exceptional frictional properties at high energy levels
• Resistance to high temperatures
• Chemical inertness
• Low coefficient of thermal expansion (excellent dimensional stability)

Disadvantages:

• Low resistance to oxidation above 500°C
• High cost of materials and manufacturing

Types of fibre:

a) Glass
b) Carbon
c) Organic
d) Ceramic

Fibrous Composites:

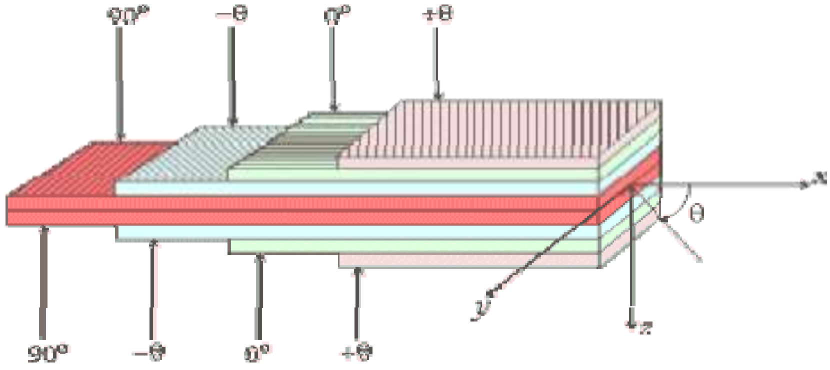


Fig1.3: Component of fibre composite

1.3. Use of composite:

•Aerospace/Military:
• Civil:
• Electronic:
• Energy:
• Automobile/Transportation:
• Sports:
• Medical:
• Marine:

Aerospace: Stiffer and lightweight graphite composites are used in the construction of space antennae, mirrors and optical instruments. The lesser values of thermal and hygric expansion coefficients of these materials make them suitable for such uses in severe hygrothermal environment where extreme dimensional stability is required for accurate functioning of these equipment's. Carbon fibres are used in extreme temperatures as the heat shielding material of rocket nozzles, re-entry structures and also in jet engines. The lightweight and high rigidity of these materials makes them suitable to be used in rocket motor casings and rocket launchers. Reduction in weight nearly up to 25% can be achieved by using fibre reinforced plastics instead of conventional materials in an aircraft. Graphite-epoxy and boron-epoxy fibres are widely used in aircraft structures. Nearly 50% of the Boeing 787 including the fuselage, fairings, floor, beams, wing trailing edge surfaces and empennage are made up of carbon-epoxy and graphite-titanium composites. Advanced composites do not corrode like metals. The combination of corrosion and fatigue cracking is a significant problem for aluminium commercial fuselage structure which is eliminated by the use of composites. The lightweight, damage tolerant and stealth characteristics of composites make them suitable for many military aircrafts. Carbon and Kevlar fibre composites are used in making many unmanned vehicles by NASA. Airbus A380 uses a substantial amount of a hybrid glass/epoxy/aluminium laminate which combines the advantages and mitigates the disadvantages of metals.

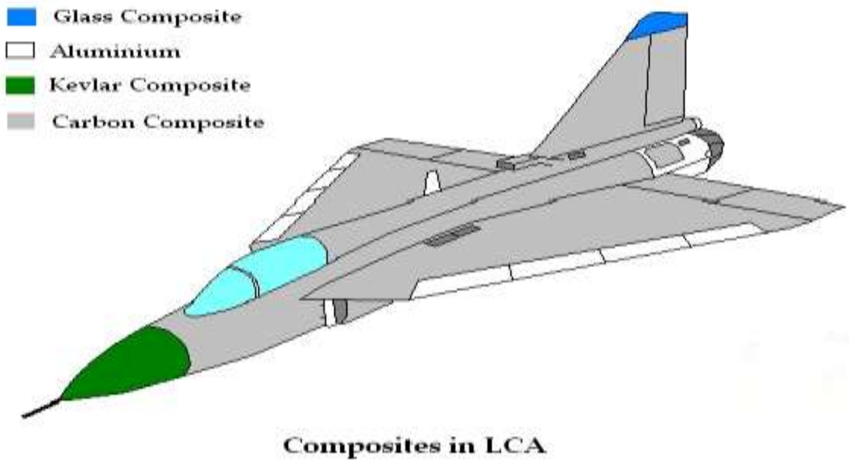


Fig 1.4: use of composite in aircraft

Transportation: Composites are widely used for manufacturing automotive parts and automobiles, truck and railway coaches. With increasing number of passengers and stringent rules of safety being laid for comfortable journey of passengers, trains are getting heavy day by day. Thus to reduce the weight of trains use of composites is becoming necessary. Glass reinforced polymers and sandwich materials are some of the materials used in modern railway coaches. The stiffness and cost effectiveness offered, apart from easy availability of raw materials, make composite materials the obvious choice for applications in surface transportation. In heavy transport vehicles, composites are used with cost effectiveness and for reducing weight. A combination of polyester resin with a variety of reinforcements offers low cost, easily designable production of functional parts of road vehicles.

Fig 1.5: use of composite in transportation

Civil/Construction: Applications of composites range from non-structural gratings and claddings to full structural systems for industrial supports, doors and windows, panelling, furniture, buildings, long span roof structures, tanks, bridge components complete bridge systems and other interiors. Acrylic resin with quartz sand composite is used for manufacturing kitchen sinks. Usage of composites for damage repairing, seismic retrofitting and upgrading of concrete bridges finds increased adoption as a way to extend the service life of existing structures, they are also being considered as an economic solution for new bridge structures. High-performance fibres such as glass, carbon, aramid and hybrids impregnated with resin systems ranging from vinyl esters and other thermosetting resin systems to thermoplastics are used as grid-type reinforcement for concrete structures. Decks for both pedestrian and vehicle bridges across waterways, railways and roadways are now built entirely from composites. The composite deck has six to seven times the load capacity of a reinforced concrete deck with only 20 percent of the weight. Among a wide array of composite products, pultruded profiles such as gratings, ladders, cable trays, solid rods and other sections are used in many structural application with Class I flame retardancy. The Fiber-line Bridge, Kolding, Denmark was designed by the Danish engineering Company, Ramboll using the pultruded profiles.

Nowadays, composites are used in peripheral structures of aerodromes. The potential and application of high-performance composites has revolutionized space structural technology. Composites are an attractive proposition considering the embedded energy (energy required to manufacture) especially against steel, aluminium and other metals. Composites are considered an ideal choice for building materials because of their high impact resistance, corrosion resistance, thermal and acoustic insulation. Natural fibre composites are gaining wider acceptance from the point of view of wood substitution.

Fig 1.6: use of composite in construction

Power: High voltage electrical transmission towers are now being constructed from pultruded composite sections using a "snap and build" assembly procedure, which eliminates the use of fasteners and adhesives. Composite power and lighting poles are finding increased application for both performance and environmental reasons. Composite modular acoustic enclosures for DG (Diesel Generator) sets are used to control of Noise Pollution. Other potential applications of composites in this sector are third rail covers for underground railway, structures for overhead transmission lines for railway, fibre optic tensile members, switchgear frames, and aerial lift-truck booms. Composites made from refractory metals combined with metals are used as electrical contacts and also to interrupt very high currents and sustain mechanical action.

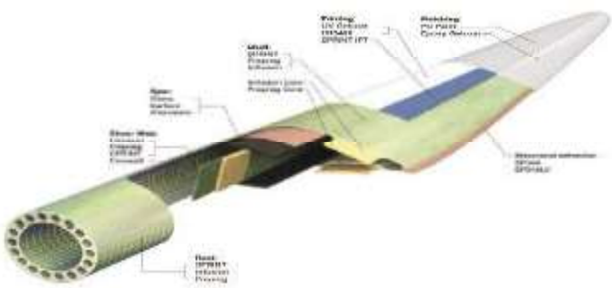


Fig 1.7: use of composite at power station

Marine Application: Composites are used in high pressure and aggressive environmental situations for applications in oil gas, piping system, topside applications, down-hole tubing in sub-sea, and others. The tailorability of composites to suit specific applications has been one of its greater advantages such as imparting low thermal conductivity and low coefficient of thermal expansion, high axial strength and stiffness etc. Due to corrosion resistance, mouldability and maintenance free service of composites they are used in making houseboats. Sandwich composite with PUF foam is used in making hull. Deck portion of the houseboats is composed of moulded resin infused composite gratings supported vertically along the centre line of the hull.

Water lubricate propeller shaft bearings:

Fig 1.8: use of composite in marine structure

Bio-medical: Bio-medical prosthetic devices are artificial replacements that are used in the human body to function as original parts. Composite material has been identified as the new class of synthetic bio-materials. Lightweight carbon-fibre reinforced polymer-matrix containing polysulfone or poly-ether-ketone is used for composite limb. The Mahaveer Vikalang Sahayata Samithi, Jaipur developed an artificial leg made of high-density polyethylene (HDPE) that permits squatting and walking on uneven ground. It is waterproof, simple, durable and lighter in weight and looks like a natural foot. Prosthesis is used to replace not only lost arms and legs, but also bone, artery, heart valve replacements, artificial eyes, teeth, optical lenses and hearing aids.

Fig 1.9: use of composite in medicals

Sports: For manufacturing sports goods consideration of characteristics like strength, ductility, density, fatigue resistance, toughness modulus, damping coefficient, cost, etc. are required to be considered. To meet the requirements of sports equipment composites is the primary material of choice. Composite materials are used in manufacturing Canoes and Kayaks, Vaulting Pole, Golf and Polo rods, tennis rackets, skis, Archery equipment, Javelin, Hand gliders, Wind surfer boards, Protective sportswear. Carbon composite bike frame is a complex structure with performance characteristics that include lightness, rigidity, durability, shock absorption etc. Hybrid fibre (carbon and aramid), carbon/kevlar epoxy materials are ideal composite materials for bicycle components. The composites are finding application in bicycle components such as Forks, Handle bars and Connecting bar ends, Seat posts, etc. Radius Engineering- Salt Lake City, Utah developed Swix carbon fibre ski poles which have been used by Gold medal Olympic skiers since 1990s. Radius developed the Trek carbon fibre bicycle frame which is much lighter than the corresponding steel frame.

Fig 1.10: use of composite in sports industry

Offshore Engineering: Composites meet diverse design requirements with significant weight savings and exhibit high strength-to-weight ratio compared to conventional materials. Composites have found extensive applications in the oil and gas industry since last two decades. In the offshore oil and gas industry, the cost of manufacturing and erecting oil rigs has reduced significantly by replacing heavy metal pipelines with lighter ones made of composites. Composite pipes are also used for fire water piping, sea water cooling, draining systems and sewerage. The high cost to replace steel piping in retrofit applications and increased longevity in new construction are driving the use of composites, which withstand severe conditions as experienced in offshore environment. Use of composite pipes has reduced problems, like corrosion and blockage of fire lines, reduction in structural support sizes and material handling during construction. Glass Reinforced Epoxy (GRE) piping system is successfully used in offshore environments against highly corrosive fluids at various pressures, temperatures, adverse soil and weather conditions (especially in sea water cooling lines, air vent systems, drilling fluids, firefighting, ballasts and drinking water lines in offshore application, oil exploration, desalination, chemical plants, fire mains, dredging, portable water etc.). Conventionally, grids/gratings are made of mild steel/cast iron. Due to the limitations on corrosion resistance, weight, durability, lifecycle costs etc. for the metallic gratings, composite grids/gratings perform much better due to their superior properties under aggressive environments as in chemical process industry.

Automotive: Carbon fibre reinforced epoxies have been used in racing cars and recently for the safety of cars. Thermoplastics are also used in manufacturing vehicle parts. In manufacturing of automobile parts, glass and sisal fibres usually find the maximum use. A reinforced-plastic composite costs more than sheet steel, when considered on the basis of cost and performance but other qualities justify the high expenditure. Mechanical properties of the parts, which affect the thickness and weight offer enough savings to render them more effective than steel. Some complicated parts of light commercial vehicles, which need casting, may be compression moulded from composites of the sheet or bulk variety. State-of-art technologies of moulding, tooling and fabricating have thrown open possibilities of increased manufacturing of vehicles that use reinforced polyesters. Materials used in automotive body parts show high tensile strength and flexural moduli. The material is not ductile and hence will not yield and the failure is accounted only in terms of fracture. These properties and thickness, determine the maximum bending moment which is several times higher than the point of fracture for steel sheets. Composite panels are used as the complete outer skin of the body to give a unique look. Good stability against corrosion or impact makes the composites widely used in vulnerable valance panels below the front and rear bumpers. Signal lamps, indicator lamps of vehicles are fabricated from glass-reinforced composites. For tractors the most crucial parameter is weight reduction as it directly affects efficiency, payload and the economy. Durability is the chief factor as these vehicles are normally realizations of capital investments. Time required, cost and frequency of maintenance add substantially to the total costs. Therefore, it is natural to try and reduce these factors to a minimum.

1.4. Free Vibration

A structure is said to be undergoing free vibration when it is disturbed from its static equilibrium position and then allowed to vibrate without any external dynamic excitation. Vibration of a system is affected by its inertial properties like mass and stiffness and damping properties like damping ratio. In practical situations damping may be due to a number of mechanical forces but mathematically they are idealized by equivalent viscous damping forces.

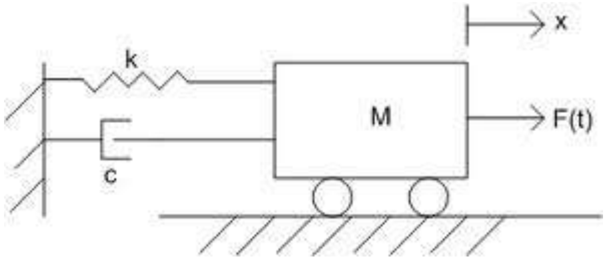


Fig. 1.11: Diagrammatic representation of SDOF system

The motion of a linear SDOF system can be idealized as a mass spring-damper system subjected to external force F(t) can be expressed by the following differential equation,

$$[\boldsymbol{M}][\ddot{\boldsymbol{x}}] + [\boldsymbol{c}][\dot{\boldsymbol{x}}] + [\boldsymbol{k}][\boldsymbol{x}] = \boldsymbol{F}(\boldsymbol{t})$$

For free vibration setting this external force equal to zero we get the differential equation for free vibration as,

$$[\boldsymbol{M}][\ddot{\boldsymbol{x}}] + [\boldsymbol{c}][\dot{\boldsymbol{x}}] + [\boldsymbol{k}][\boldsymbol{x}] = \boldsymbol{0}$$

For systems without damping, i.e. c = 0 this equation reduces to,

$$[\boldsymbol{M}][\ddot{\boldsymbol{x}}] + [\boldsymbol{k}][\boldsymbol{x}] = \boldsymbol{0}$$

Free vibration initiated by disturbing the system from its static equilibrium position by imparting the mass some displacement, x (0) and velocity, $\dot{x}(0)$ at time zero, defined at the instant the motion is initiated:

$$x = x\,(0) \qquad \dot{x} = \dot{x}\,(0)$$

Subject to these initial conditions, the solution to the homogeneous differential equation is obtained as,

$$\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{0})\,\boldsymbol{cos}\,\boldsymbol{\omega_n t} + \frac{\dot{\boldsymbol{x}}(\boldsymbol{0})}{\boldsymbol{\omega_n}}\boldsymbol{sin}\,\boldsymbol{\omega_n t} \qquad ; \text{ Where, } \boldsymbol{\omega_n} = \sqrt{\frac{\boldsymbol{k}}{\boldsymbol{M}}}$$

1.5. Objective of present study:

To perform free vibration analysis of cylindrical shells made up of laminated composite material using Finite Element approach. Different parametric studies will be performed to understand the effect of curvature, thickness, and boundary conditions on natural frequencies of the shell.

1.6. Scope of present study:

1. Eight-noded Serendipity elements with first order shear deformation theory has been used in the study.
2. A MATLAB program will be used.
3. Rotary inertia is taken in to consideration.
4. A 3×3 integration of Gauss Quadrature is used in the evaluation of bending stiffness whereas a 2×2 reduced integration is employed for shear stiffness terms.
5. Parametric studies have been performed.

CHAPTER 2

Literature Review

2.0 Introduction

In general, most of laminated composite shell are frequently used in aircraft and automobiles. Also, most of the structures are subjected to dynamic load which is become a very important parametric study how to overcome this dynamic load, where these shells are subjected to dynamic loads due to its purpose of use in dynamic place. The modal analysis is a very efficient technique for the assessment of stiffness of structure. Therefore, laminated composite shell are expected to have adequate stiffness to resist failure due to vibration. The frequency of vibration of the shell should be within a certain limit so that it does not affect the function of nearby parts in the structure and does not produce any discomfort. Any deviation if found in the experiment and numerical frequency indicate some crack, failure plane or any sort of weakness in the composite shell. Therefore, it is important to predict the natural frequencies of composite shell to study the behavior of the structure and to avoid resonance of large structures under dynamic loading. Many investigators have analyzed the free vibration characteristics of laminated composite shell and a number of theoretical and experimental methods have been proposed to predict their natural mode of vibration. The related literature was critically reviewed to provide background information on the problem to be considered in the project work and to emphasize the relevance of the present study. The historical background of published literatures can be broadly classified into the following categories:

1. **Review on**
    - **C. Finite Element Analysis of Laminated Composite plate and shell**
    - **D. First order shear deformation theories of the plate and shells**

2. **Review on**
    - **C. Free Vibration Analysis of Laminated Composite plates**
    - **D. Free Vibration Analysis of Laminated Composites cylindrical Shells**

2.1. (A) Review on Finite Element Method:

A great progress has been made over the few decade for better understanding of the free vibration characteristics of laminated composite plate and shells. Due to insufficient of availability of analytical solution for practical applications, the numerical approximate method have become the most effective tools. The finite element method (FEM) is considered a very effective and versatile approach for these problems. There is a vast amount of literature on free vibration analysis of laminated plate/shall which is too large to list here.

Bert [1] and Mohamad [2] have conducted surveys and provided details on the development of the finite element methods for modelling and model analysis of laminated plates/shells.

Yang et al. [3]. Zhang et al [4] present a review of the recent development of the finite element analysis for laminated composite plates from 1990. The literature review is devoted to the recently developed finite elements based on the various laminated plate and shell theories for the free vibration and dynamics, buckling and post-buckling analysis, geometric nonlinearity and large deformation analysis, and failure and damage analysis of composite laminated plates. The material nonlinearity effects and thermal effects on the buckling and post-buckling analysis, the first-ply failure analysis and the failure and damage analysis were emphasized specially.

Moon and Choi [5] have formulated the Transfer Stiffness Coefficient method for Vibration Analysis of Frame Structures. They developed the concept based on the transfer of the dynamic stiffness coefficient, which is related to the force and displacement vector at each node from the left end to the right end of the structure.

Myung [6] has developed the Finite Element-Transfer Stiffness Coefficient Method for free vibration analysis of plate structures. His approach is based on the combination of the modelling techniques in FEM and the transfer technique of the stiffness coefficient in the transfer stiffness coefficient method.

Lal et al. [7] have dealt with nonlinear free vibration of laminated composite plates on elastic foundation with random system properties. The basic formulation of the problem is based on higher-order shear displacement theory. A direct iterative method in conjunction with first-order Taylor series based perturbation technique procedure is developed to solve random nonlinear generalized eigenvalue problem. The developed probabilistic procedure is successfully used for the nonlinear free vibration problem with a reasonable accuracy.

Gajbir et al. [8] have studied Nonlinear vibration analysis of composite laminated and sandwich plates with random material properties Nonlinear vibration analysis is performed using a C^0 assumed strain interpolated finite element plate model based on Reddy's third order theory. An earlier model is modified to include the effect of transverse shear variation along the plate thickness and Von-Karman strain terms.

2.1 (B) Review on First order shear deformation theory:

Beams and plates usually are modelled as either Kirchhoff or Mindlin theory. For the Kirchhoff plate, the ratio of the thickness to the lengths is relatively small, and thus the transverse shear and rotary inertia are neglected, unlike in the Mindlin model where this ratio is no longer considered small.

Mindlin [9] developed a theory of the transverse vibration of thick plates including the effect of transverse shear and rotary inertia, in which few analytical solutions were obtained for plates with free edges.

Lanhe et al. [10] presented vibration analysis of non-symmetric angle-ply laminated composite thick plate based on first-order shear deformation theory using moving least square differential quadrature method.

Xiang et al. [11] proposed a nth-order shear deformation theory for solving vibration analysis of general composite laminated plate. This theory can satisfies the zero transverse shear stress boundary conditions on the top and bottom surface of the plate and Reddy's plate theory can be considered as a special case of this theory. Natural frequencies are computed by a mesh less radial point collocation method based on the thin plate spline radial basis function.

Liew and Liu [12] presented free vibration analysis of moderately thick annular sector isotropic plates for various boundary conditions, sector angle and thickness ratio based on the Mindlin first-order shear deformation theory using differential quadrature method.

Liew et al. [13] solved the free vibration analysis of circular Mindlin plates using the same method. **Viswanathan and Kim [14]** applied point collocation and spline function method for solving non-symmetric laminated thick plate.

Tai and Kim [15] proposed a refined plate theory, which accounts for parabolic distribution of the transverse shear strains through the plate thickness and satisfies the zero traction boundary conditions on the surfaces of the plate without using shear correction factors. Equations were derived using Hamilton's principle and naiver solution was applied for obtaining closed-form solution. Results are compared with three-dimensional elasticity, first- and third-order shear deformation theory.

Gurses et al. [16] presented free vibration analysis of symmetric laminated trapezoidal using first-order shear deformation theory of plate by discrete singular convolution method.

Civalek [17] was developed discrete singular convolution for vibration analysis of moderately thick symmetrically laminated composite plates based on the first-order shear deformation.

Sharma et al. [18] studied free vibration analysis of non-symmetric composite laminated sector annular plate using analytical method and Chebyshev polynomials based on first-order shear deformation theory.

2.2 A. Free vibration Analysis of laminated composite plate:

Fiber reinforced composite laminates in aerospace, structural components, automotive, marine and other engineering applications are being used on large scale. Some properties of the composites like, lightweight, high specific strength, high specific stiffness, excellent fatigue and corrosion resistance have made the analysis essential for the researchers.

Noor AK [19] (1973), presented complete list of FSDTs and HSDTs for the static, free vibration and buckling analysis of laminate composites. He presented exact three-dimensional elasticity solutions for the free vibration of isotropic, orthotropic and anisotropic composite laminate plates, which serve as benchmark solutions for comparison by many researchers.

J.N. Reddy, M. ASCE [20] (1984), Preaented Exact solution of moderately thick laminated shells.

E. Carrera [21] (1998), presented the dynamic analysis of multi-layered plates using layer-wise mixed theories with respect to existing two-dimensional theories. They have employed Reissner's mixed variational equation to derive the differential equations, in terms of the introduced stress and displacement variables, that govern the dynamic equilibrium and compatibility of each layer.

Ganapathi and Makhecha [22] (2001), proposed an improved ZIGT for free vibration analysis of laminated composite plates in which the C0 interpolation functions are only required during their finite element implementation.

Matsunaga [23] (2002) developed a higher-order theory based on a complete power series expansion of the displacement field in the thickness coordinate. He presented closed-form solutions for the vibration of simply supported cross-ply laminated plates and demonstrated that, for expansion orders higher than three, a noticeable improvement is obtained in comparison with TSDT.

A.R. Setoodeh, G. Karami [24] (2003), proposed a generalized layer-wise laminated plate theory based on a three-dimensional approach for static, vibration, and buckling analysis of fibre reinforced laminated composite plates.

Liu ML, To CWS [25] (2003), developed the computational models for the free vibration and damping analysis based on the FSDT and HSDT, relatively few models were developed based on the Layer wise theories. The computational model developed based on the layer wise theories include the 18-node, three-dimensional higher-order mixed model free vibration analysis of multi-layered thick composite plates, in which the continuity of the transverse stress and the displacement fields were enforced through the thickness of laminated composite plate.

Lathes wary et.al, [26] (2004), investigated the static and free vibration analysis of moderately thick laminated composite plates using a 4-node finite element formulation based on higher order shear deformation theory, and the transient analysis of layered anisotropic plates using a shear deformable 9-noded Lagrangian element-based on first-order shear deformation theory.

Akhras G, Li W [27] (2005), studied free linear vibration behaviour of laminated composite rectangular plates is by using moving least squares differential quadrature procedure, based on the first order shear deformation. He developed a spline finite strip method for static and free linear vibration analysis of composite square plates using Reddy's higher-order shear deformation theory.

Wu Z, Chen WZ [28] (2006), extended the global–local higher-order theory to predict natural frequencies of laminated composite and sandwich plates. These theories can predict more accurate natural frequencies of laminated composite and sandwich plate, and the number of unknowns involved in these models is independent of the number of layers.

Ferreira et.al, [29] (2008), used layer wise theory based on Mindlin's first-order shear deformation theory in each layer to analyse the vibration and static responses of composite and sandwich plates. Flexural analysis based on multiquadric radial basis function is also done.

Marjanovic et.al, [30] (2013), presented the structural analysis of laminated composite and sandwich plates and observed the different forms of damage. They observed that Delamination is the most common type of damage for laminated composite plates.

2.2 (B). Review on Free Vibration Analysis of Laminated composite shells:

In the last decade the demand for composite materials have gone up drastically and has resulted in extensive research in the field of composite and their vibrational characteristics.

Mochida et al. [31] presented vibration analysis of isotropic thin double curve shallow shell of rectangular platform using the superposition-Galerkin method of seven seta of boundary conditions.

Monterubbio [32] presented the Rayleigh-Ritz and penalty function method for solving free vibration of isotropic thin shallow shells of rectangular platform with spherical, cylindrical and hyperbolic paraboloid geometries.

Liew and Lim [33] considered the vibration analysis of thin doubly curved shallow shells of rectangular platform in many boundary condition and with many Gaussian curvatures. The pb-Ritz energy based approach along with in-plane and transverse deflections assumed in the form of a product of mathematically complete two-dimensional orthogonal polynomials and a basic function, was employed to model of vibration.

Ruotolo [34] compared donnell, loves, sanders and flugge's thin shells theories in the evaluation of natural frequency of cylindrical shells based on Kirchhoff Hypothesis. He used energy function for soling the equation of motion and levy solution for satisfaction of boundaries.

Civalek [35, 36] presented free vibration analysis of laminated conical and cylindrical shell using discrete singular convolution approach. This study carried out using love's first approximation thin shell theory. The effect of circumferential wave number and number of layer on natural frequencies were considered.

Lim et al. [37] presented the vibration analysis of composite shallow conical shells including the effects of pretwist using energy method for symmetric, non-symmetric and various number of layer.

Kabir [38, 39, and 40] presented an analytical solution to the boundary value problem for free vibration analysis of antisymmetric angle-ply laminated cylindrical panels. The solution is based on a boundary-continuous double Fourier series. Equations including rotary and in-plane inertias. The characteristics equation of the panel are defined by five highly coupled second and third order partial differential equations in five unknowns.

Nosier and Reddy [41] presented donnell shear deformation type theory and donnell's classical theory for vibration and stability of cross ply laminated circular cylindrical shells using analytical solution based on levy-type.

Soldatos [42] reported vibration analysis of composite laminated thin cylindrical shallow shell panels based on thin hypothesis theory of shell and compared four type of theory of shell including Flugge, Sander, Love and donnell.

Soldatos [43] considered vibration analysis of laminated shell either circular or non-circular using refined shear deformation theory. This theory accounts for parabolic variation of transverse shear strains and it is capable pf satisfying zero shear traction boundary condition at the external shell surface, and make no use of transverse shear correction factor.

Shu [44, 45, and 46] applied generalized differential quadrature method for vibration analysis of isotropic and composite laminated conical shells based on Love's first approximation thin shell theory. The displacement field were expressed as product of un-known function along the axial direction and Fourier function along the circumferential direction. The same author considered vibration analysis of laminated composite cylindrical shells using the same method and based on the same theory.

Ganapati et al. [47, 48] characterized free vibration of thick laminated composite non-circular shells using higher-order theory. The formulation accounts for the variation of the in plane and transverse displacement through the thickness, abrupt discontinuity in slope of the in-plane displacement at the interfaces, and includes in plane, rotary inertia terms, and the inertia contribution due to the coupling between the different order displacement terms. The energy method and finite element procedure used for solving the governing equation.

Leissa et al. [49] considered vibration analysis of circular cantilevered thin shallow shells of rectangular platform using Ritz method.

Leissa [50] presented a closed from solution for free vibration analysis of shallow shells and studied the effects of shallowness on the natural frequencies.

Soldatos and hadjigeorgiou [51] studied three-dimensional vibrational analysis of isotropic cylindrical thick shells and panels according to levy solution.

Asadi and Quta [52,53] presented the vibrational analysis of generally moderately thick deep composite laminated cylindrical shells with various lay-up using differential quadrature method based on the first order shear deformation theory.

Hosseini-Hashmi [54] and fadaee [55] presented an exact closed-form procedure using new auxiliary and potential functions free vibration analysis of moderately thick spherical shell panels based on the first order shear deformation theory. The strain displacement relation of both Donnell and Sander theories were used and compared with 3D finite element analysis. Shell has two opposite simply supported (Levy-type). The effect of various stretching-bending coupling on the frequency parameters were discussed.

Nanda and Bandyopadhyay [56] investigated the nonlinear free vibration of laminated composite cylindrical shell panels in the presence of cut-outs using finite element model.

Chakraborty et al. [57] presented finite element analysis for the free vibration behaviour of point supported laminated composite cylindrical shells.

Lam and Qian [58] established Analytical solutions for the free vibrations of thick symmetric angle-ply laminated composite cylindrical shells using the first order shear deformation theory.

Zhang [59] analysed the natural frequencies of cross-ply laminated composite cylindrical shells by the wave propagation approach for the influences of different boundary conditions on circumferential modes.

Narita et al. [60] presented a finite element solution for the free vibration problem of cross ply laminated, closed cylindrical shells using classical lamination theory Based on the energy expressions.

The present study deals with the free vibration analysis of laminated composite cylindrical shell structure. Using this program, effect of varying lay-up angle, Side by thickness ratios, orthotropy ratio of the composite shells on the variation of natural frequency were studied subjected to different boundary conditions (All side simply supported and all side clamped). The frequencies are obtained by computer programing using MATLAB, which is compared to the numerical results obtained from literature.

CHAPTER 3

Governing Equations

3.1 Basic Constitutive Equations

From the law of conservation of linear momentum, the equations of motion are given by,

$$\sigma_{ij,i} + b_j = \rho \ddot{u}_j \qquad (1)$$

From the law of conservation of angular momentum, one may obtain,

$$\sigma_{ij} = \sigma_{ji} \qquad (2)$$

Constitutive relation of general linear anisotropic material is given by,

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad (3)$$

Combining above equations, and neglecting body force terms, equation of dynamic equilibrium may be represented as,

$$\rho \ddot{u}_I - (C_{ijkl}\varepsilon_{kl})_j = 0 \qquad (4)$$

Since σ_{ij} is symmetrical, C_{ijkl} must be symmetrical in i and j i.e.

$$C_{ijkl} = C_{jikl} \qquad (5)$$

Again for linear elastic problems, the strain tensor, given by $\varepsilon_{kl} = 0.5(u_{k,l} + u_{l,k})$, is also symmetrical in l and k, therefore,

$$C_{ijkl} = C_{ijlk} \qquad (6)$$

Using strain energy density function, W, defined as the strain energy contained in a unit volume of the structure undergoing deformation from an arbitrary datum, and given by,

$$W = \frac{1}{2} C_{ijkl}\varepsilon_{ij}\,\varepsilon_{kl} \qquad (7)$$

It can be obtained from mathematical identity,

$$\frac{\partial^2 W}{\partial \varepsilon_{ij}\varepsilon_{kl}} = \frac{\partial^2 W}{\partial \varepsilon_{ijkl}\varepsilon_{ij}} \tag{8}$$

That, $C_{ijkl} = C_{klij}$. Finally, it can be concluded that,

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk} \tag{9}$$

Now it can be shown that in general anisotropic element among the 81-elastic constant, 21 are independent elastic constants.

Now the constitutive relationship can be written with 21 material constants in pseudo-tonsorial form as,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \ or \ \tau_{23} \\ \sigma_5 or \ \tau_{13} \\ \sigma_6 \ or \ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \ or \ \gamma_{23} \\ \varepsilon_5 \ or \ \gamma_{13} \\ \varepsilon_6 \ or \ \gamma_{12} \end{Bmatrix} \tag{10}$$

3.2 Elastic Stiffness Matrix

Generally, the constitutive relationship matrix c_{ij} depend on the orientation of the co-ordinate system. An isotropic material is characterized by infinite number of planes of material symmetry through a point. An isotropic material is fully characterized by only two independent constants, c_{11}, c_{12}.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (c_{11} - c_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

In case of an orthotropic material, which has three mutually perpendicular plane of symmetry, number of independent elastic constant reduced to nine. For orthotropic material independent coefficients are c_{11}, c_{12}, c_{13}, c_{22}, c_{23}, c_{33}, c_{44}, c_{55}, c_{66}

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \quad (11)$$

For an orthotropic material following reciprocal law holds:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \quad \text{(i, j=1, 2, 3)}$$

The relation between the stiffness constant and the engineering constant for orthotropic material is given by:

$$c_{11} = E_1 \frac{1 - \nu_{23}\nu_{32}}{\Delta}, \; c_{12} = E_1 \frac{\nu_{21} - \nu_{31}\nu_{23}}{\Delta} = E_2 \frac{\nu_{12} - \nu_{32}\nu_{13}}{\Delta}$$

$$c_{12} = E_1 \frac{\nu_{31} - \nu_{21}\nu_{32}}{\Delta} = E_3 \frac{\nu_{13} - \nu_{12}\nu_{23}}{\Delta}, \quad c_{22} = E_2 \frac{1 - \nu_{13}\nu_{31}}{\Delta} \quad (12)$$

$$c_{23} = E_2 \frac{\nu_{32} - \nu_{12}\nu_{31}}{\Delta} = E_3 \frac{\nu_{23} - \nu_{21}\nu_{13}}{\Delta}, \quad c_{33} = E_3 \frac{1 - \nu_{12}\nu_{21}}{\Delta}$$

$$c_{44} = G_{23}, \quad c_{55} = G_{31}, \quad c_{66} = G_{12}, \quad \Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}$$

Here, directions 1, 2 and 3 are directions of orthotropy.

E_1, E_2, E_3 = Young moduli in 1, 2 and 3 directions, respectively

ν_{ij} = Poisson's ratio of transverse strain in the j^{th} direction to the axial strain in the i^{th} direction, when stressed in the i^{th} direction.

= $-\varepsilon_j / \varepsilon_i$

G_{23}, G_{13}, G_{12} = shear moduli in the 2-3, 1-3 and 1-2 planes, respectively.

A thin, unidirectional lamina is essentially an orthotropic material with presence of fibre in direction 1. Hence simplified elastic relations are as follows:

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \nu_{12} Q_{22} = \nu_{21} Q_{11}, \quad (13)$$

$$Q_{44} = G_{23}, \quad Q_{55} = G_{31}, \quad Q_{66} = G_{12}$$

Though the transverse shear stresses σ_4 (or τ_{23}), σ_5 (or τ_{31}), have been incorporated for use of Mindlin's plate theory instead of conventional Kirchhoff's theory.

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & 0 & 0 & 0\\ Q_{12} & Q_{22} & 0 & 0 & 0\\ 0 & 0 & Q_{44} & 0 & 0\\ 0 & 0 & 0 & Q_{55} & 0\\ 0 & 0 & 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{Bmatrix} \quad (14)$$

The above equation is based on the assumption that the fibres are oriented along the direction 1, while the lamina lies on the 1-2 plane, directions 1 and 2 being mutually perpendicular. These relations are termed as the on-axis relations. The off-axis relations are obtained by operating suitable transformations on the on-axis relations. The off-axis relations are transformations needed to get the stress-strain relations along the local axes defined on the shell:

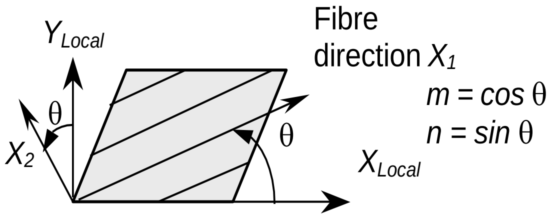


Fig.3.1. Arbitrary lamina, Orthotropy directions and local axis.

The elastic coefficients transformed from the orthotropic axes to the user-defined local co-ordinates are written with primes to identify them properly. These transformation rules in explicit form may be written as furnished below:

Where,

$Q'_{11} = m^4Q_{11} + 2m^2n^2 (Q_{12}+2Q_{66}) + n^4Q_{22}$

$Q'_{22} = n^4Q_{11} + 2m^2n^2 (Q_{12}+2Q_{66}) + m^4Q_{22}$

$Q'_{12} = m^2n^2 (Q_{11}+ Q_{12} - 4Q_{66}) + (m^4 + n^4) Q_{12}$

$Q'_{16} = m^3n (Q_{11}- Q_{12} - 2Q_{66}) - mn^3 (Q_{22} - Q_{12} - 2Q_{66})$ (15)

$Q'_{26} = mn^3 (Q_{11} - Q_{12} - 2Q_{66}) - m^3n (Q_{22}- Q_{12} - 2Q_{66})$

$Q'_{66} = m^2n^2 (Q_{11}+ Q_{22} - 2Q_{12}) + (m^2 - n^2) Q_{66}$

$Q'_{44} = m^2Q_{44} + n^2Q_{55}$

$Q'_{45} = mn (Q_{55} - Q_{44})$

$Q'_{55} = m^2Q_{55} + n^2Q_{44}$

3.3 First-Order Assumptions

The general construction of a laminated composite of thickness h consisting of unidirectional laminae bonded together to act as an integral continuum, is pictorially explained in Fig.3.2. The bonds are infinitesimally thin and are not shear deformable.

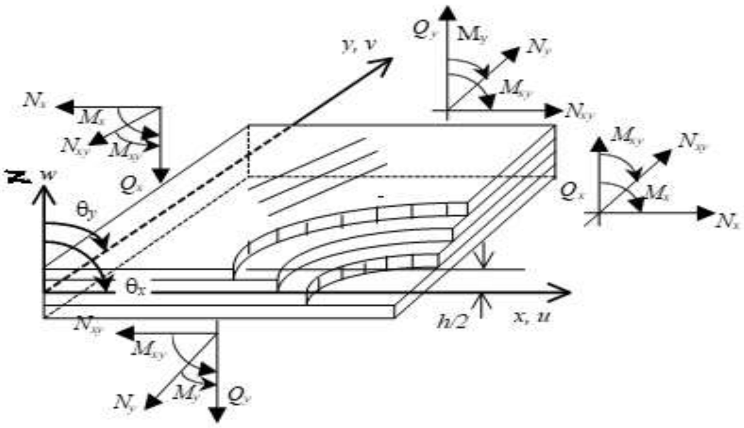


Fig.3.2 The laminated composite with positive displacements, rotations andstress resultants.

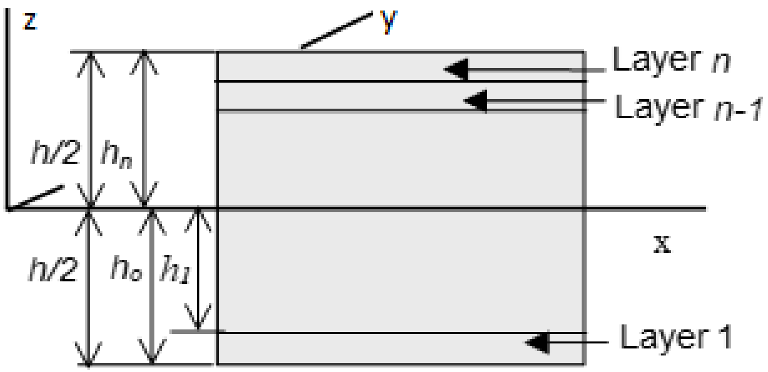


Fig.3.3 Composite nomenclature.

In the present analysis the first order transverse shear deformation is assumed. Thus a vertical plane cross section before deformation is assumed to remain plane after deformation but it may not remain normal to the middle surface as it happens with the Kirchhoff's plates. The following are the assumptions made in the Yang-Norris-Stavsky (YNS) theory, which is a generalization of the Mindlin theory to accommodate composite laminates

The following assumption is made according to the YNS theory, which is the generalization of the Mindlin theory to laminated composite plates:

1. The material behavior is linear and elastic.
2. The thickness, t, of the laminate is small compared to the other two dimensions.
3. Displacements u, v, and w are small compared to the laminate thickness, h.
4. Normal to the mid-plane before deformation remains straight but not necessarily normal to the mid-plane after deformation.
5. Stresses normal to the mid-plane are neglected.

3.4 Displacement Model

The deformed geometry of the laminated plate is shown in Fig. 3.4 and Fig. 3.5. It may happen that the mid plane has in-plane translation u_0 and v_0, out of plane displacement w_0 and two rotations θ_x and θ_y.

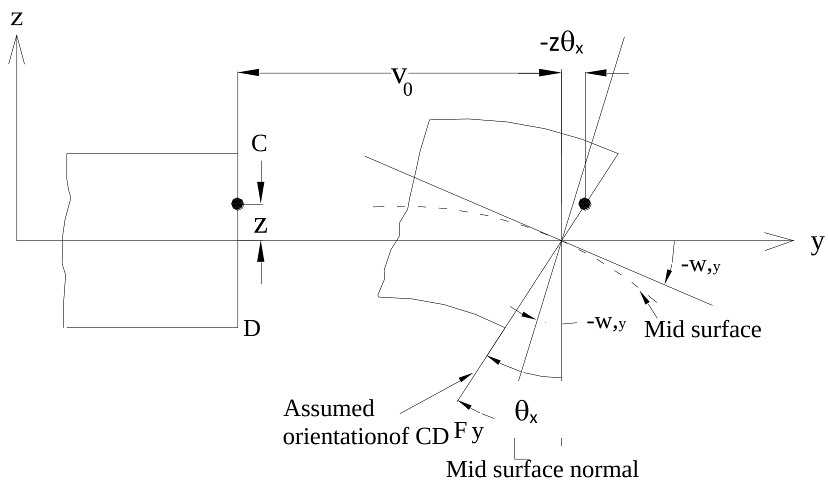


Fig. 3.4: *A 2-dimensional view of deformation of the plate along a section parallel to the x-z plane*

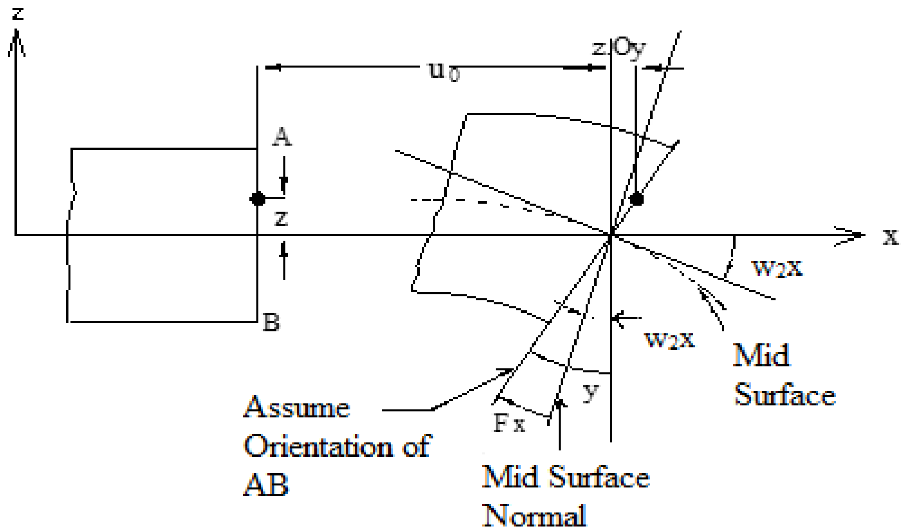


Fig. 3.5: *A 2-dimensional view of deformation of the plate along a section parallel to the y-z plane*

The in-plane displacements u and v of any point at and distance z from the mid-plane are given by:

$$u = u_0 + z\theta_y \, , \; v = v_0 - z\theta_x \quad (16)$$

The shear rotation of the plate can be expressed as,

$\phi_x = \theta_y + w_{,x}$, $\phi_y = -\theta_x + w_{,y}$

3.5 Strain-Displacement Relations

The linear in-plane strain of laminate at a distance z from the mid-surface are given by,

$$\varepsilon_x = u_{,x} = u_{0,x} + z\,\theta_{y,x} = \varepsilon^0{}_x + z\kappa_x$$
$$\varepsilon_y = v_{,y} = v_{0,y} - z\,\theta_{x,y} = \varepsilon^0{}_y + z\kappa_y \qquad (17)$$
$$\varepsilon_{xy} = u_{,y} + v_{,x} = u_{0,y} + v_{0,x} + z(\theta_{y,y} - \theta_{x,x}) = \varepsilon^0{}_{xy} + z\kappa_{xy}$$

Since, the transverse shear deformation is assumed same across the thickness of the laminate are given by,

$$\varepsilon_{xz} = \phi_x\ ,\ \varepsilon_{yz} = \phi_y, \qquad (18)$$

3.5.1 Constitutive Relations for a Laminated Composite Plate

The internal force and moment resultants of the laminate are obtained by integrating the lamina stresses over the entire plate thickness. Thus, stress terms are replaced by the stress-resultant terms, while the conventional strain terms are replaced by the mid-plane strain terms. Thus, assuming that the laminate comprises of n laminae, for the in-plane force,

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k \mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Q'^{k}_{ij}\, [e]_{x,y}\, \mathrm{dz} - \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Q'^{k}_{ij}\, [e]_k\, \mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Q'^{k}_{ij} \left([\varepsilon^0]_{x,y} + [z\kappa]_{x,y}\right) \mathrm{dz} - \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Q'^{k}_{ij}\, [e]_k\, \mathrm{dz}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k zdz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_k z\mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} z\, Q'^{k}_{ij}\, [\varepsilon]_{x,y} \mathrm{dz} - \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} z\, Q'^{k}_{ij}\, [e]_k \mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Q'^{k}_{ij} \left([z\varepsilon^0]_{x,y} + [z^2\kappa]_{x,y}\right) d\mathrm{z} - \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} z\, Q'^{k}_{ij}\, [e]_k \mathrm{dz}$$

(19)

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}_k dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}_k \mathrm{dz}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \alpha \left[{Q'}^{k}_{ij} \right]_k \begin{Bmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix}_k \mathrm{dz}$$

$$= \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix}_k$$

where, z_k , z_{k-1} are the top and bottom distances of a lamina, k from the mid-surface.

The internal force and moment resultants can be expressed as,

$$\{F\} = [D]\ \{\varepsilon\} - \{F^N\} \tag{20}$$

where, $\{F\} = \{N_x\ N_y\ N_{xy}\ M_x\ M_y\ M_{xy}\ Q_x\ Q_y\}^T$

$\{\varepsilon\} = \{\varepsilon_x\ \varepsilon_y\ \varepsilon_{xy}\ \kappa_x\ \kappa_y\ \kappa_{xy}\ \phi_x\ \phi_y\}^T = \{\varepsilon_x\ \varepsilon_y\ \varepsilon_{xy}\ \kappa_x\ \kappa_y\ \kappa_{xy}\ \varepsilon_{xz}\ \varepsilon_{yz}\}^T$

$\{F^N\} = \{N^N_x\ N^N_y\ N^N_{xy}\ M^N_x\ M^N_y\ M^N_{xy}\ 0\ 0\}^T$

$$[D] = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45} & A_{55} \end{bmatrix}$$

The stiffness of the laminate are,

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} {Q'}^{k}_{ij}\ [1, z, z^2\ ]_k \mathrm{dz} \qquad \text{(i, j = 1, 2 and 6)}$$

$$(A_{ij}) = \alpha \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} {Q'}^{k}_{ij} \mathrm{dz} \qquad \text{(i, j = 4, 5)}$$

3.6 Finite Element Formulation

The element stiffness, geometric stiffness and mass matrices are derived using the Principle of Total Potential Energy. An eight noded serendipity isoparametric element have been used. Both the geometry and displacement field are expressed in terms of same shape functions. The parent element in local natural co-ordinate system can be mapped to an arbitrary shape in the Cartesian co-ordinate system.

3.6.1 Isoparametric Element

The isoparametric means "same parameters" and is applied here because same interpolation functions are used to interpolate the magnitude of co-ordinates as well as the degree of freedom. Eight noded serendipity element (Fig. 3.6) with five degree of freedom at each node, i.e., u_0, v_0, w, θ_x, θ_y is considered for finite element formulation.

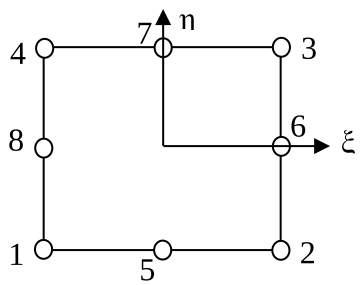


Fig. 3.6: *8 Noded Serendipity elements*

The co-ordinates and the elastic parameters can be interpolated using shape function (interpolation function) N_i as follows,

$$x = \sum_{i=1}^{8} N_i(\xi,\eta)x_i \;,\; y = \sum_{i=1}^{8} N_i(\xi,\eta)y_i \tag{21}$$

where, x_i and y_i are the global co-ordinate at a node i.

$$u_0 = \sum_{i=1}^{8} N_i(\xi,\eta)u_{o_i} \quad v_0 = \sum_{i=1}^{8} N_i(\xi,\eta)v_{o_i} \quad w = \sum_{i=1}^{8} N_i(\xi,\eta)w_i$$
$$\theta_x = \sum_{i=1}^{8} N_i(\xi,\eta)\theta_{x_i} \quad \theta_y = \sum_{i=1}^{8} N_i(\xi,\eta)\theta_{y_i} \tag{22}$$

in which u_{oi}, u_{oi}, v_{oi}, w_i, θ_{xi}, θ_{yi} are the displacement at a node i.

The shape functions N_i in equations are defined as,

$$N_i = (1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i + \eta\eta_i - 1)/4 \qquad i = 1 \text{to } 4$$
$$N_i = (1-\xi^2)(1+\eta\eta_i)/2 \qquad i = 5,7 \qquad (23)$$
$$N_i = (1-\eta^2)(1+\xi\xi_i)/2 \qquad i = 6,8$$

Where, ξ and η are the local natural co-ordinates of the element and ξ_i and η_i are the value of them at node i.

The derivatives of the shape functions N_i with respect to x and y are expressed in terms of their derivatives with respect to ξ and η by the following relationship,

$$\begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} = [J]^{-1} \begin{bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{bmatrix} \qquad (24)$$

Where, $[J] = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix}$ = Jacobian matrix

THEORETICAL FORMULATION OF LAMINATED COMPOSITE CYLINDRICAL SHELL

Considering a laminated composite cylindrical shell of uniform thickness h and consisting of a number of thin laminae, each of which may be arbitrarily oriented at an angle θ with reference to the x-axis of the co-ordinate system.

The constitutive equations for the cylindrical shells are given by-

$\{\sigma\} = [D]\ \{\varepsilon\}$, where

$$\{\sigma\} = \{N_X\ \ N_Y\ \ N_{XY}\ \ M_X\ \ M_Y\ \ M_{XY}\ \ Q_X\ \ Q_Y\}\ ;\ \text{and}$$

$$\{\epsilon\} = \{\epsilon_x^0\ , \epsilon_y^0\ , \gamma_{zy}^0, \kappa_x, \kappa_y, \kappa_{xy}, \gamma_{xz}^0, \gamma_{yz}^0\}$$

$$[D] = \begin{bmatrix} A11 & A12 & A16 & B11 & B12 & B16 & 0 & 0 \\ A12 & A22 & 26 & B12 & B22 & B26 & 0 & 0 \\ A16 & A26 & B12 & B22 & B26 & B66 & 0 & 0 \\ B11 & B12 & B16 & D11 & D12 & D16 & 0 & 0 \\ B12 & B22 & B26 & D16 & D26 & D66 & 0 & 0 \\ B16 & B26 & B66 & D16 & D26 & D66 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & S11 & S12 \\ 0 & 0 & 0 & 0 & 0 & 0 & S12 & S22 \end{bmatrix}$$

The strain–displacements relations are established as-

$$\begin{Bmatrix} \epsilon_X \\ \epsilon_Y \\ \gamma_{XY} \\ \gamma_{XZ} \\ \gamma_{YZ} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_X^0 \\ \varepsilon_Y^0 \\ \gamma_{XY}^0 \\ \gamma_{XZ}^0 \\ \gamma_{YZ}^0 \end{Bmatrix} + \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \\ k_{xz} \\ k_{yz} \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial U}{\partial x} \\ \frac{\partial V}{\partial y} - \frac{w}{R} \\ \frac{\partial U}{\partial y} + \frac{\partial v}{\partial x} \\ \alpha + \frac{\partial w}{\partial x} \\ \beta + \frac{\partial w}{\partial y} \end{Bmatrix}; \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \\ k_{xz} \\ k_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \alpha}{\partial x} \\ \frac{\partial \beta}{\partial y} \\ \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \\ 0 \\ 0 \end{Bmatrix}$$

Stiffness matrix for an element

The strain–displacement relation is given by,

$$\{\epsilon\} = [B]\,\{\delta\}$$

$$\{\delta\}=\{u_1,v_1,w_1,\alpha_1,\beta_1,\ldots..,u_8,v_8,w_8,\alpha_8,\beta_8\}^T$$

$$[B] = \sum_{i=1}^{8} \begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 \\ 0 & N_{i,y} & -N_i/R & 0 & 0 \\ N_{i,y} & N_{i,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & N_{i,x} & 0 \\ 0 & 0 & 0 & 0 & N_{i,y} \\ 0 & 0 & 0 & N_{i,y} & N_{i,x} \\ 0 & 0 & N_{i,x} & N_i & 0 \\ 0 & 0 & N_{i,y} & 0 & N_i \end{bmatrix}$$

The element stiffness matrix is given by-

$$[K_e] = \iint [B]^T[D][B]\,dx\,dy.$$

Mass matrix for an element-

The mass matrix of the element is given by,

$$[M_e] = \iint [N]^T[P][N]\,dx\,dy,$$

$$[N] = \sum_{i=1}^{8} \begin{bmatrix} Ni & 0 & 0 & 0 & 0 \\ 0 & Ni & 0 & 0 & 0 \\ 0 & 0 & Ni & 0 & 0 \\ 0 & 0 & 0 & Ni & 0 \\ 0 & 0 & 0 & 0 & Ni \end{bmatrix}, \quad [P] = \begin{bmatrix} P & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 \\ 0 & 0 & P & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

In which, $P=\sum_{K=1}^{N} \int_{Z_{K-1}}^{Z_K} \rho\,dz.$ and $I= \sum_{K=1}^{N} \int_{Z_{K-1}}^{Z_K} Z^2 \rho\,dz$

Fundamental natural frequency (rad/s-

The element stiffness and mass matrices are evaluated first by expressing the integrals in the local natural co-ordinates ξ and η of the element and then performing numerical integration by using Gauss quadrature. Then the element matrices are assembled after performing appropriate transformations due to the curved shell surface to obtain the respective global matrices.

The condition for free vibration analysis,

$$[[K] \quad - \omega_n^2 \ [\mathrm{M}]] = 0$$

CHAPTER 4

Numerical Study and Results

In this section, a number of numerical examples are presented to demonstrate the behaviour of the laminated composite shell for the free vibration analysis. Particular shell structure with various boundary conditions, span to thickness ratio and modular ratio (the degree of orthotropy) has been analyzed. A MATLAB program is developed based on first order shear deformation theory (FSDT) using eight noded serendipity isoparametric shell element.

In all examples, the composite material properties taken are same for all the layers. The ply angle of each layer is measured about the global Y-axis (Xe) as shown in the figure 4.1. All layers have the same thickness and mass density.

Initially, a mesh convergence study is performed to obtain the minimum number of meshes required for analysis. Validation studies are then performed to validate the results of the program against those published in established literature. A series of parametric studies are conducted for different orthotropy ratios, boundary conditions, fibre angles and thicknesses.

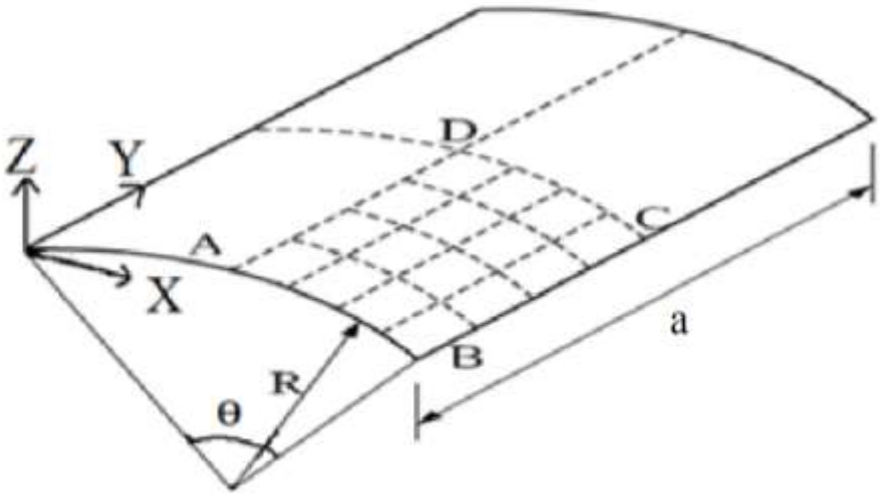


Fig.4.1: Global co-ordinate axis for shell structure.

5.1 Mesh Convergence Study:

A mesh convergence study has been carried out to obtain minimum number of elements required to achieve the non-dimensional natural frequencies as accurately as possible optimizing the computation time. For this purpose, a square laminated composite shell is taken and analyzed under all side simply supported (SSSS) and all side clamped (CCCC) boundary conditions. The side-to-thickness ratio (=a/h) is taken as 100). Mesh sizes containing 4x4, 6x6, 8x8, 10x10, 12x12 and 16x16 elements as shown in fig. 4.2, are considered for the convergence study. The material properties of the shell are as given below.

- **Material property:**
  - E_1=2 x10^{11} GPa;
  - E_2=0.08x10^{11} GPa;(where, E1/E2=25)
  - g_{12}=0.04x10^{11} GPa;
  - g_{13}=0.04 x10^{11} GPa;
  - g_{23}=0.016 x10^{11} GPa;
  - ν_{12}=0.25;
  - ν_{21}=0.25;
  - Rho (ρ)=1000 Kg/m^3;
  - θ_1(layer-1) =0;
  - θ_2(layer-2) =90;
- **Dimension of element(a*b) =**20mx20m
- **Radius of curvature of shell(R) =**100m
- **Initial thickness of shell element(t) =**0.2m
- **No. of layers(n):**2
- **Lay-up sequence:**(0°/90°)
- **The non-dimensional frequency is expressed as,**

  $f = \frac{wa^2}{h} * \sqrt{\rho/E2}$

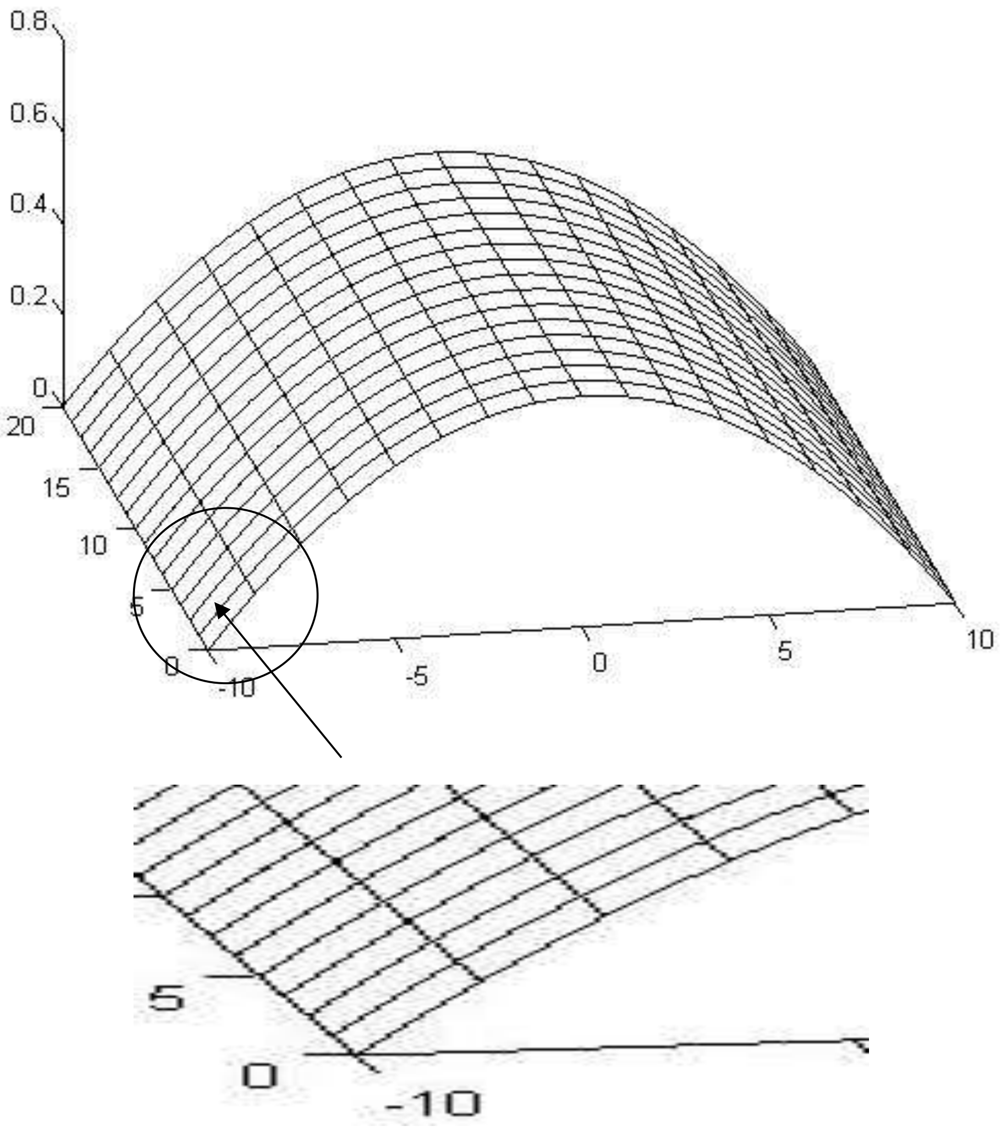


Fig.4.2: Details of a typical mesh having 256 elements.

Non-dimensional natural frequencies obtained for different meshes under different boundary conditions are tabulated in Table 4.1 and 4.2. It can be observed that the fundamental frequency converges for a mesh containing 16x16 elements. Hence, for subsequent studies, a mesh containing 16x16 elements has been chosen.

Table 4.1: First Three Non-Dimensional Frequency f= $\frac{wa^2}{h} * \sqrt{\rho/E2}$ under SSSS condition

MODE NO.	SSSS					
	4*4	6*6	8*8	10*10	12*12	16*16
1	15.875	15.984	16.032	16.071	16.102	16.134
2	31.938	30.618	30.295	30.227	30.205	30.196
3	32.971	30.707	30.404	30.318	30.291	30.266

Table 4.2: First Three Non-Dimensional Frequency f= $\frac{wa^2}{h} * \sqrt{\rho/E2}$ under CCCC condition

MODE NO.	CCCC					
	4*4	6*6	8*8	10*10	12*12	16*16
1	23.045	20.924	20.655	20.592	20.572	20.555
2	48.345	42.442	41.494	41.289	41.219	41.172
3	48.482	42.519	41.596	41.393	41.330	41.282

5.2 Validation Study:

Validation studies are made to test the accuracy of the computer program and the results are compared with those available in the literature (Reddy (20). The cases are as follows:

- Validation for laminated composite shell with JN-Reddy, Material property:
  - E_1=2 x10^{11} GPa;
  - E_2=0.08 x10^{11} GPa;(where, E1/E2=25)
  - g_{12}=0.04 x10^{11} GPa;
  - g_{13}=0.04 x10^{11} GPa;
  - g_{23}=0.016 x10^{11} GPa;
  - ν_{12}=0.25;
  - ν_{21}=0.25;
  - Rho (ρ)=1000 Kg/m^3;
  - θ_1(layer-1) =0;
  - θ_2(layer-2) =90;
  - Thickness(t) =0.2 m
  - Side(a,b)=20mx20m
  - Radius(R) =100 m

Table 4.3: Validation of Non-Dimensional Frequency f= $\frac{wa^2}{h} * \sqrt{\rho/E2}$ for Mesh Sizes 16 x16

Mode	R/a	layer	Non-Dim Frequency Reddy	Non-Dim Frequency Present
1	**5**	**0/90**	**16.67**	**16.14**
2			**-**	**30.20**
3			**-**	**30.27**

First three non-dimensional frequencies are obtained and the results are compared with those of Reddy (20) in Table 4.3. From the table it can be seen that the first mode result for this shell are in good agreement.

4.3. Case Studies

A series of parametric case studies has been conducted on the two layered laminated composite shell. Both external and internal parameters are varied to get a complete idea about the effect of material characteristics on the dynamic behavior of laminated composite Shells. The following case studies are provided in this report:

1. Variation of Lay-up angle (θ)
2. Variation of E_1/E_2 Ratio
3. Variation of Side by Thickness Ratio (a/h)
4. Variation of Boundary condition

4.3.1. **Case Study 1: Study of Dynamic Behavior of Laminated Composite shell structure for Varying Lay-up angle (θ) of SSSS and CCCC condition**.

Various lay-up angles are considered to examine the effect on the natural frequency (rad/s) of the laminated shell of lay-up sequence ($0/\theta$). The angle θ varies from 45° to 90°, which are given in Table 4.4. The material properties of the lamina are as follows:

- **Material property:**
  $E_1 = 2x10^{11}$ GPa, $E_2 = 0.08x10^{11}$ GPa, $g_{13}=g_{12} = 0.04x10^{11}$ GPa, $g_{23} = 0.016x10^{11}$ GPa,
  $\nu_{12} = 0.25$, $\nu_{21} = 0.25$
- **No. of layers:** 2
- **Lay-up sequence:** $0/\theta$
- **Density(ρ):** 1000 kg/m^3
- **Side-to-thickness ratio (a/h):**100
- **Boundary Conditions:** All side simply supported (SSSS) and All side clamped (CCCC)
- **Mesh Size:** 16 x 16

Table 4.4: First five Natural Frequency in (rad/s) under SSSS and CCCC condition for varying Lay-up Angle

		LAY-UP ANGLE (θ)			
Boundary Condition	**Mode No.**	**45**	**60**	**75**	**90**
SSSS	1	**21.56**	**21.70**	**22.41**	**22.82**
	2	36.38	38.36	40.74	42.71
	3	48.98	47.51	44.74	42.81
	4	51.80	53.05	55.38	56.35
	5	61.85	75.16	80.97	81.47
CCCC	1	**31.23**	**29.69**	**29.12**	**29.07**
	2	50.12	51.69	55.52	58.23
	3	66.06	62.01	59.82	58.38
	4	68.59	72.34	73.00	73.51
	5	81.08	92.83	103.04	106.13

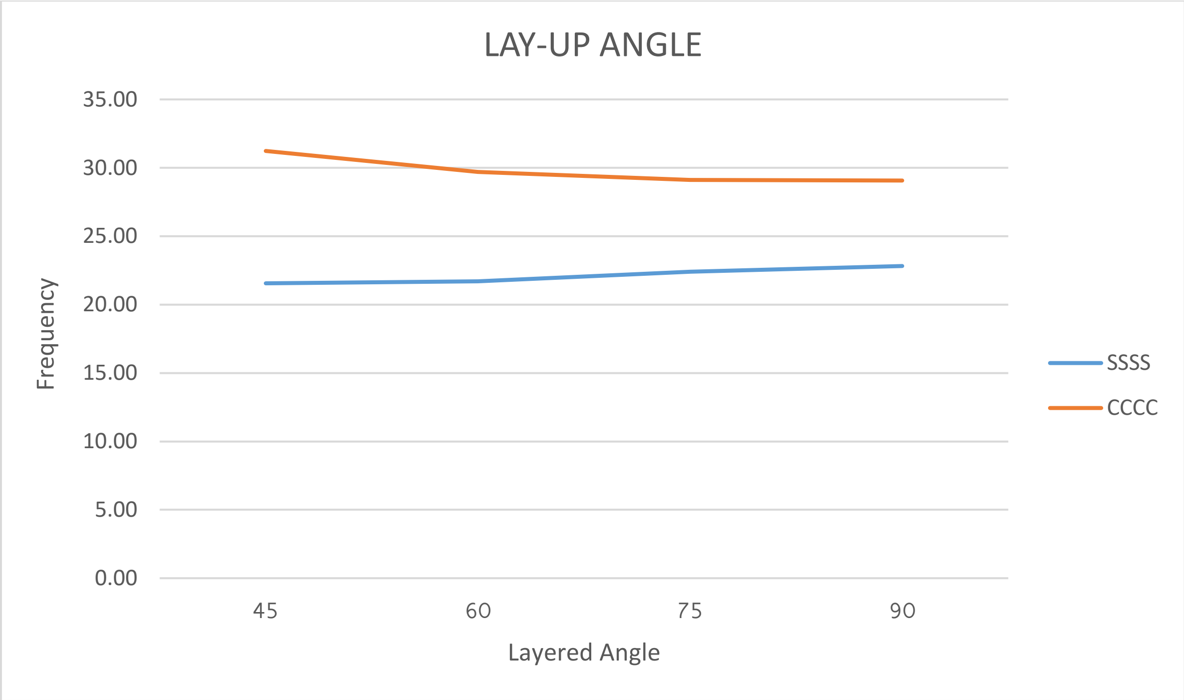


Fig. 4.3: Natural Frequency (rad/s) Vs Lay-up Angle under SSSS and CCCC condition

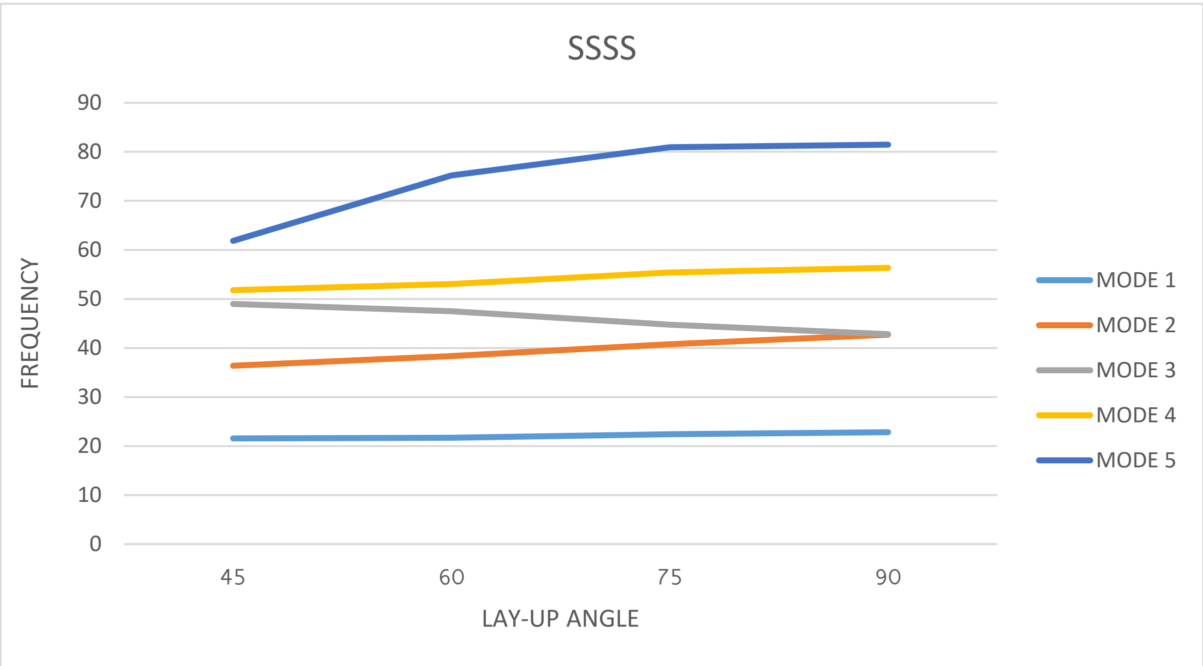


Fig. 4.4: For 5-mode variation of natural frequency (rad/s) in SSSS condition for case 1

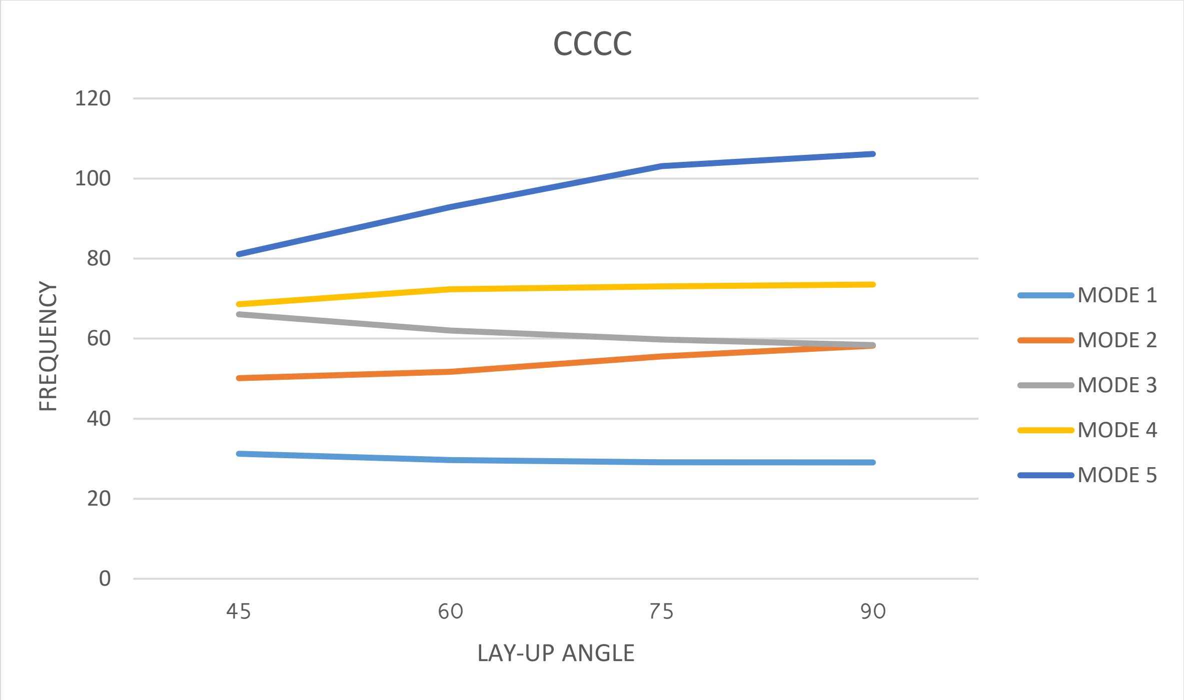


Fig. 4.5: For 5-mode variation of natural frequency (rad/s) in CCCC condition for case 1

From Table 4.4 and Fig. 4.3, 4.4 and 4.5 following observations can be made.

1. For all side simply supported edge condition, increasing lay-up angle from 45^o to 90^o, the fundamental natural frequency increases. This indicates that stiffness of the shell element increases with increase in lay-up angle and become maximum at θ =90^o.

2. For all side clamped condition, there is a sharp decrease in fundamental frequency from 45^o to 90^o and for 60^o onward the variation in natural frequency is marginal.

3. For all side simply supported case, frequency is maximum at θ =90^o, as better cross fibre makes the shell element stiffer and stronger. We know the natural frequency is directly proportional to the stiffness of that structure (w=$\sqrt{k/m}$; where 'k'=stiffness of the structure and 'm' is the mass of the structure).

Fig. 4.3 shows first three mode shapes for θ = 60^o. with SSSS and CCCC boundary conditions. It is seen that mode 1 forms a half sine curve and mode 2 forms a full sine curve.

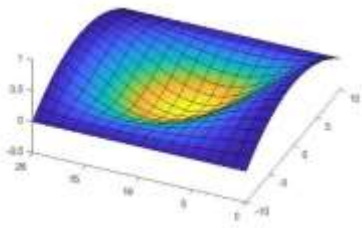
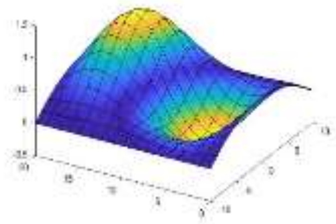
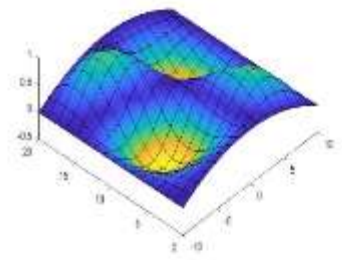
L.A SSSS	MODE 1	MODE 2	MODE 3
60			

Fig 4.6: First Three Mode Shape for SSSS condition for case 1

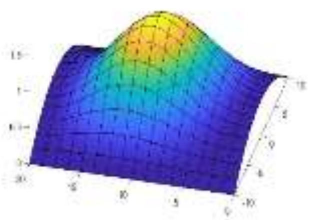
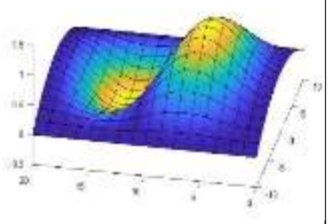
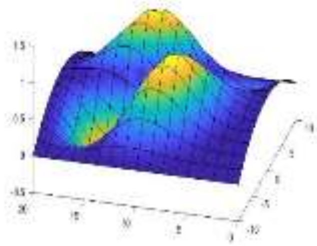
L.A CCCC	MODE 1	MODE 2	MODE 3
60			

Fig 4.7: First Three Mode Shape for CCCC condition for case 1

4.3.2. **Case Study 2: Study of Dynamic Behavior of Laminated Composite shell Structure for Varying E_1/E_2 ratio of SSSS and CCCC condition.**

Various E_1/E_2 ratio are considered to examine the effect on the natural frequency (rad/s) of the laminate. This E_1/E_2 ratio varied from 10 to 30 as shown in table 4.5. The material properties of the lamina are as follows:

- **Material property:** E_1/E_2 ratio=10, 15, 20, 25 & 30.
  As E_1/E_2=25, E_1 = 2*10^{11} GPa, E_2 =Variable, GPa, G_{13}=G_{12} =0.04*10^{11} GPa,
  G_{23} =0.016*10^{11} GPa,
  ν_{12} = 0.25, ν_{21} = 0.25
- **No. of layers:** 2
- **Lay-up sequence:** 0/90°
- **Density(ρ):** 1000 kg/m^3
- **Side-to-thickness ratio (a/h):**100
- **Boundary Conditions:** All side simply supported (SSSS) and All side clamped (CCCC)
- **Mesh Size:** 16 x 16

Table 4.5: First five Natural Frequency in rad/s under SSSS and CCCC condition for varying E_1/E_2 Ratio

		E1/E2 RATIO				
Boundary Condition	mode	10	15	20	25	30
SSSS	1	**25.87**	**24.43**	**23.52**	**22.82**	**22.21**
	2	54.86	49.08	45.43	42.71	40.43
	3	54.95	49.17	45.52	42.81	40.54
	4	81.56	69.54	61.93	56.35	51.81
	5	107.13	95.03	87.25	81.47	76.60
CCCC	1	**35.44**	**33.26**	**30.76**	**29.07**	**27.77**
	2	71.73	67.14	61.87	58.23	55.39
	3	71.92	67.31	62.03	58.38	55.54
	4	94.17	90.08	80.43	73.51	67.95
	5	131.55	122.37	112.88	106.13	100.63

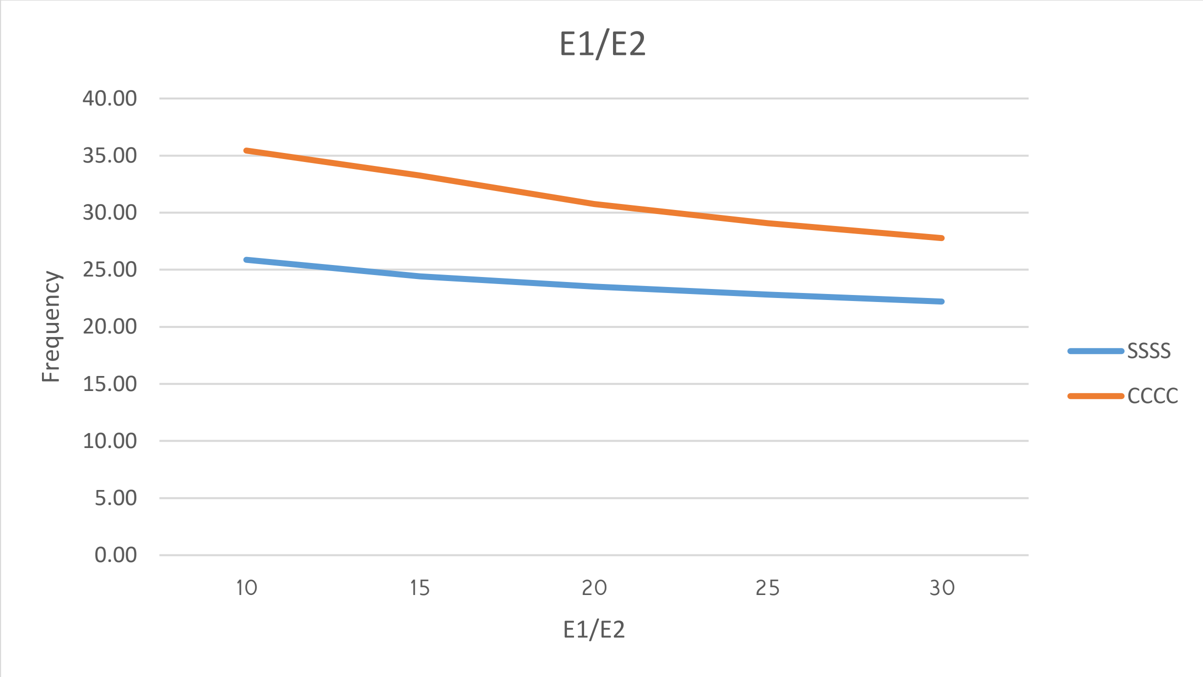


Fig 4.8: For Natural Frequency (rad/s) Vs E_1/E_2 ratio of shell Element under SSSS and CCCC condition

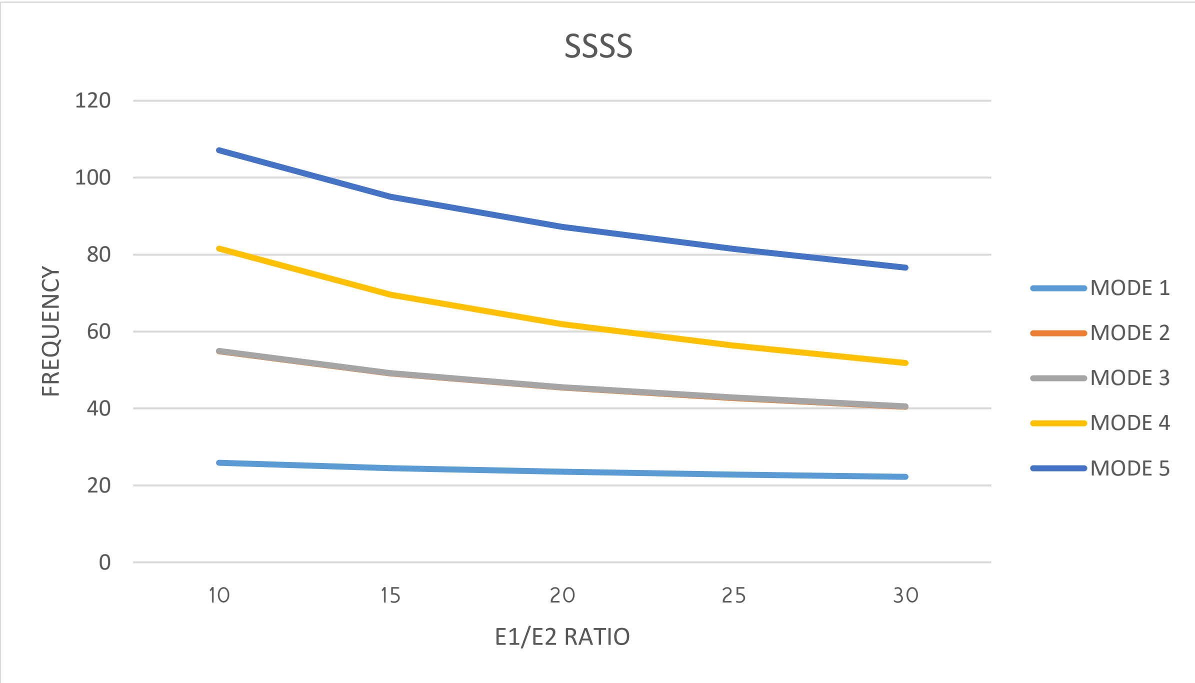


Fig 4.9: For 5-mode variation of natural frequency (rad/s) in SSSS condition for case 2

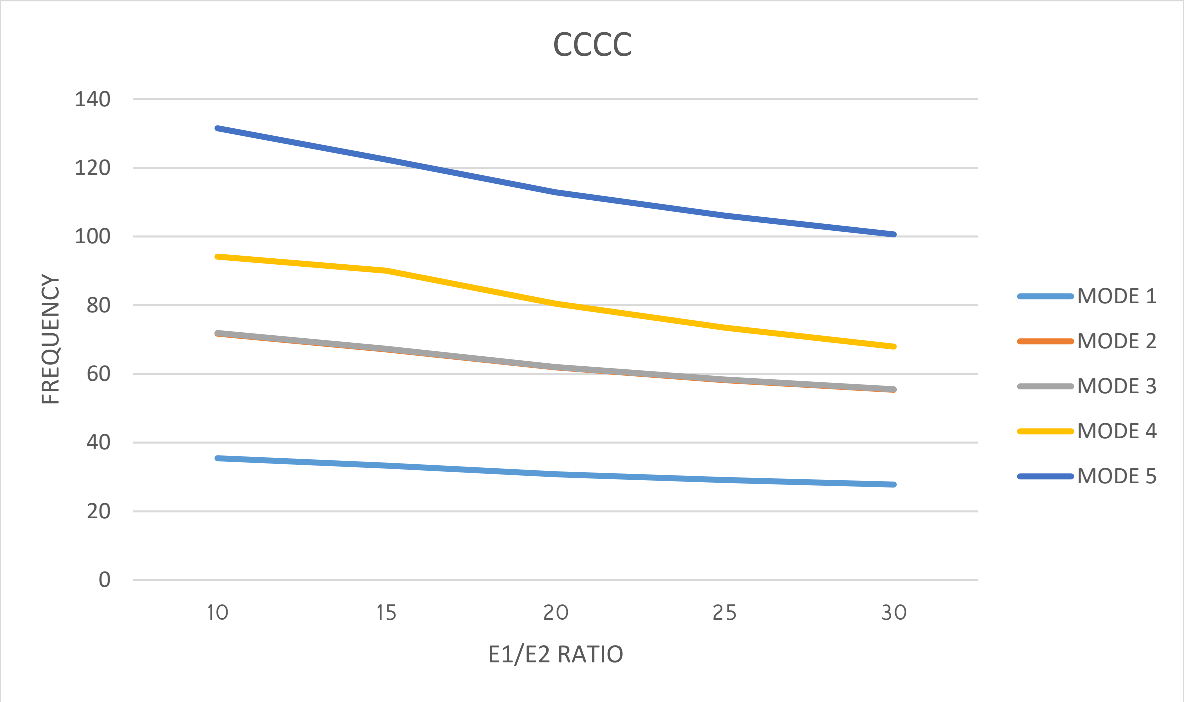


Fig 4.10: For 5-mode variation of natural frequency (rad/s) in CCCC condition for case 2

From above table and figures the following observations can be made.

1. Both for all side SSSS and CCCC conditions, with increase in orthotropy ratio there is decrease of natural frequencies. The rate of decrease of natural frequency is more for higher modes.
2. It is obvious because in this case, increase in orthotropy ratio denotes decrease in value of E_2 keeping E_1 value same which means the structure are becoming weak or flexible in nature, hence there is decrease in the natural frequency as well as stiffness of the structure.

4.3.3. **Case Study 3: Study of Dynamic Behavior of Laminated Composite shell Structure for varying side by thickness ratio (a/h) of shell element for SSSS and CCCC condition.**

Various thicknesses are considered to examine the effect of change in thickness on the natural frequency (rad/s) of the structure. These thicknesses varied as 0.1333, 0.2, 0.4, 1& 2 m (i.e. side to thicknesses ratio means a/h ratio varied as 20, 50, 100, and 150 as shown in table 4.6. The material properties of the lamina are as follows:

- **Material property:**
  $E_1 = 2x10^{11}$ GPa, $E_2 = 0.08x10^{11}$ GPa, G_{13}=G_{12} =$0.04x10^{11}$ GPa, G_{23} =$0.016x10^{11}$ GPa, ν_{12} = 0.25, ν_{21} = 0.25
- **No. of layers:** 2
- **Lay-up sequence:** 0/90°
- **Density(ρ):** 1000 kg/m^3
- **Side-to-thickness ratio (a/h):** varies 150, 100, 50 & 20.
- **Boundary Conditions** All side simply supported (SSSS) and all side clamped (CCCC)
- **Mesh Size:** 16 x 16

Table 4.6: First five Natural Frequency in rad/s under SSSS and CCCC condition for varying a/h Ratio

		a/h RATIO			
Boundary Condition	Mode, a/h	20	50	100	150
SSSS	1	**108.05**	**45.35**	**22.82**	**15.21**
	2	195.14	84.34	42.70	28.53
	3	195.60	84.52	42.80	28.61
	4	252.51	110.97	56.34	37.64
	5	352.38	159.48	81.43	54.49
CCCC	1	**133.29**	**57.44**	**29.07**	**19.42**
	2	249.97	113.85	58.2256	38.98
	3	250.72	114.18	58.38	39.09
	4	310.86	143.27	73.51	49.25
	5	420.77	204.56	106.13	71.26

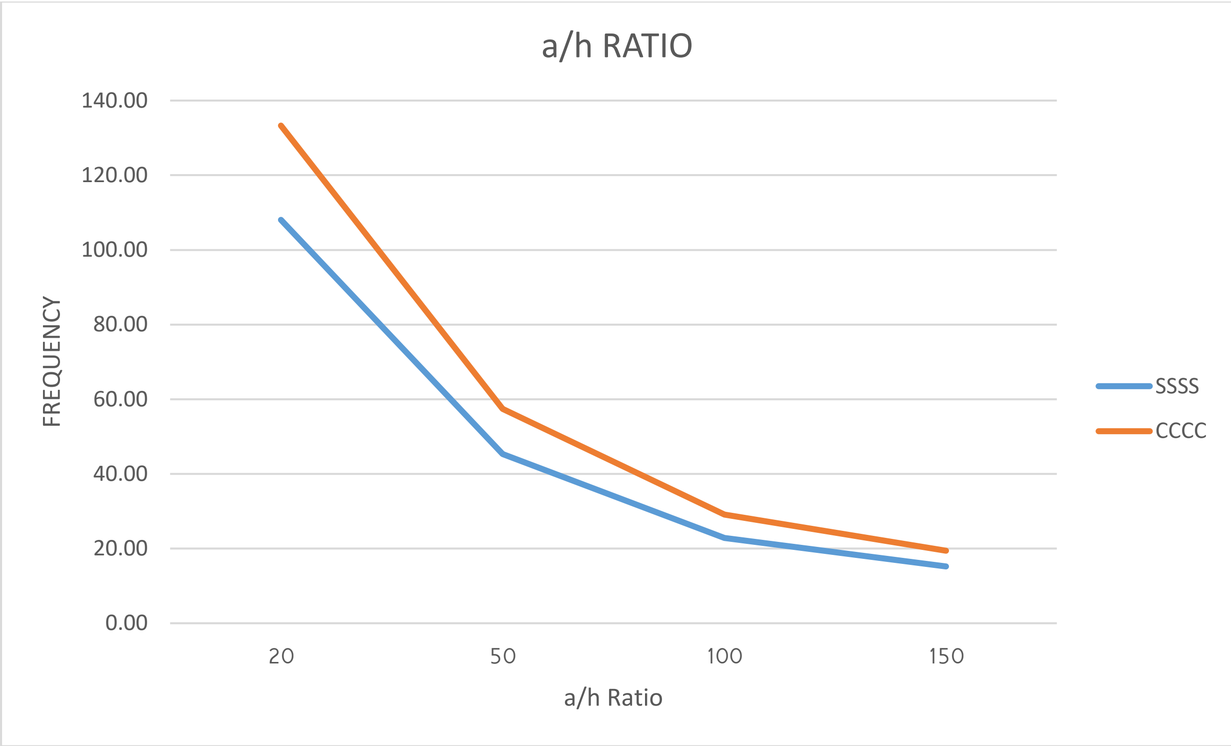


Fig: 4.11. For Natural Frequency (rad/s) Vs a/h Ratio of shell Element under SSSS and CCCC condition

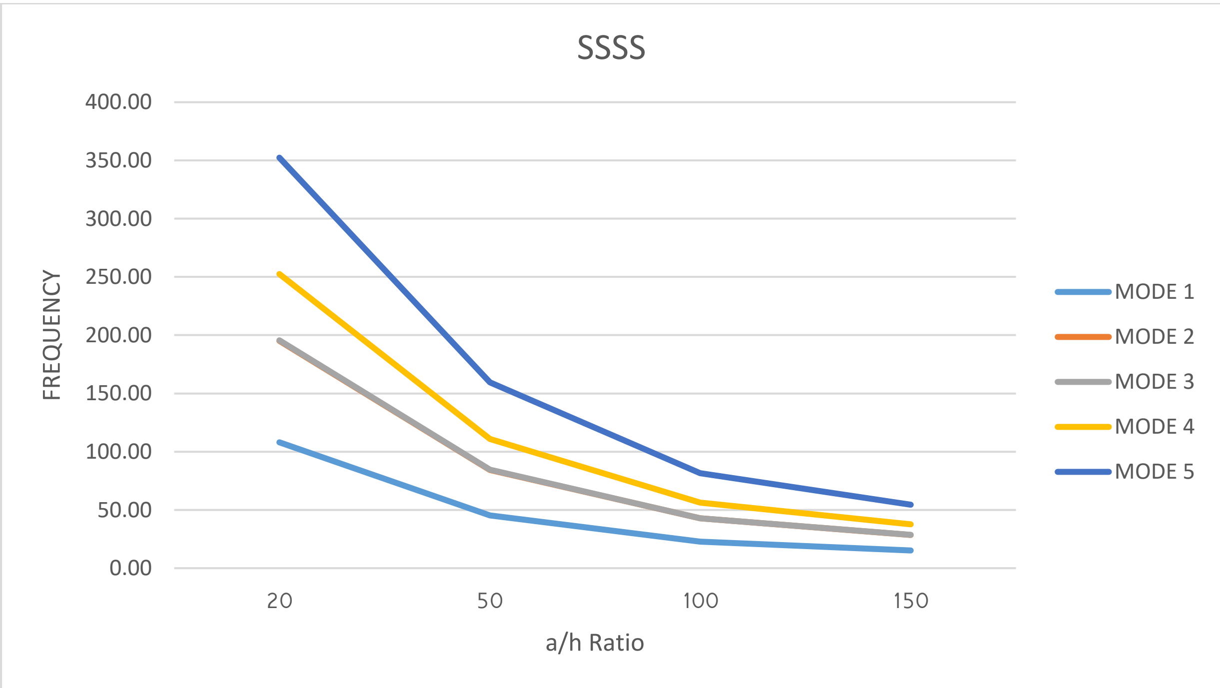


Fig: 4.12: For 5-mode variation of natural frequency (rad/s) for SSSS condition for case 3

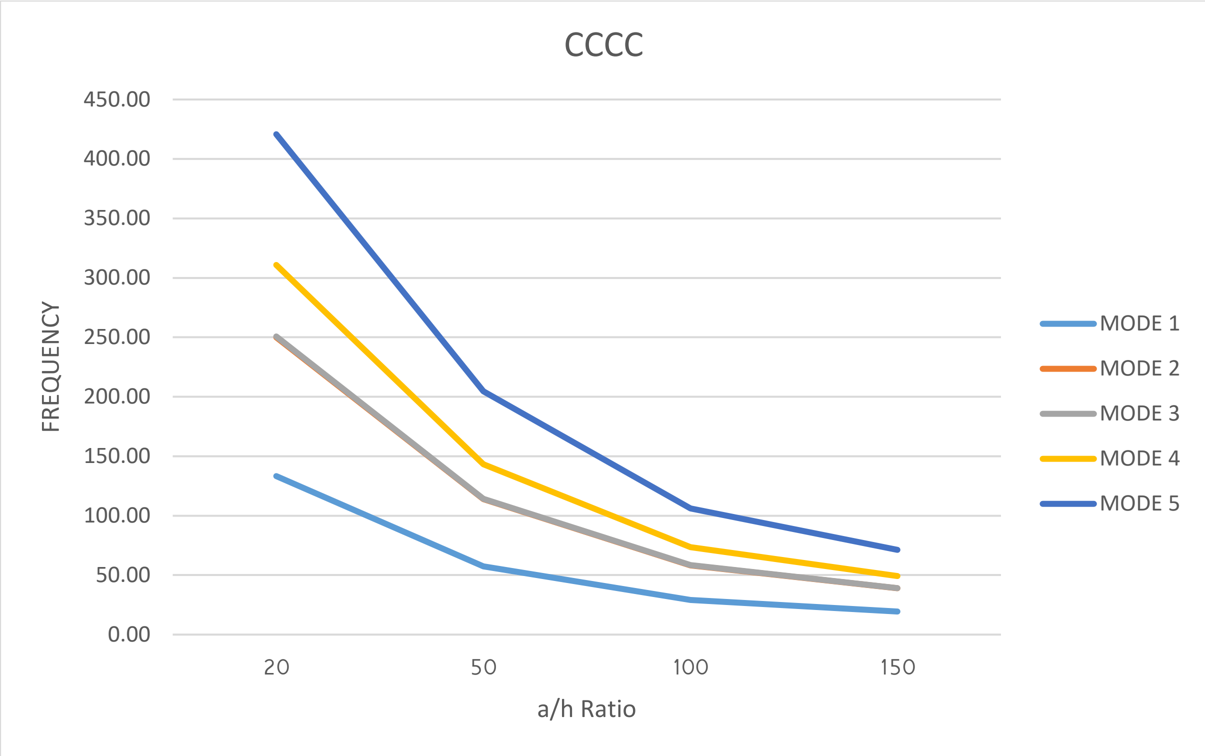


Fig: 4.13: For 5-mode variation of natural frequency (rad/s) of CCCC condition for case 3

From above table and figures the following observations can be made.

1. For both simply supported and clamped boundary conditions, the fundamental natural frequency is found to increasing up slightly for thickness up to 0.4m, then frequently increasing up the natural frequency for varying thickness from 0.4m to 2.0m.

2. For both boundary condition, it is observed from Table 4.6 that second and third modes frequency is approximately same.

3. Due to the curvature present in shell element, it can be concluded that clamped boundary condition imparts higher stiffness to the structure as compared to simply supported boundary conditions.

4. It is observed that decrease in natural frequency with increase in a/h ratio is closely linear.

5. From Table 4.6 it is observed that clamped boundary condition impart more restrains than SSSS case.

4.3.4. **Case Study 4: Study of Dynamic Behavior of Laminated Composite shell Structure for Varying Layered angle and a/h Ratio together of SSSS and CCCC conditions**.

Various layered angle and thickness are considered to examine the effect on the natural frequency (rad/s) of the laminate. The Layered angle varies from 45° to 90° and a/h Ratio 20, 50, 100 and 150 which are given in table 4.7. The material properties of the lamina are as follows:

- **Material property:**

  E_1 = 2x10^{11} GPa, E_2 = 0.08x10^{11} GPa, G_{13}=G_{12} =0.04x10^{11} GPa, G_{23} =0.016x10^{11} GPa,

  ν_{12} = 0.25, ν_{21} = 0.25
- **No. of layers:** 2
- **Lay-up sequence:** 0/θ
- **Thickness:**0.1333 to 1 (a/h as 20, 50, 100, 150)
- **Density(ρ):** 1000 kg/m^3
- **Boundary Conditions:** All side Simply Supported (SSSS) and all sides clamped (CCCC)
- **Mesh Size:** 16 x 16

Table 4.7: First five Natural Frequency in rad/s under SSSS and CCCC condition for varying Layered Angle and a/h Ratio

			LAY-UP ANGLE			
Boundary Condition	**MODE**	a/h, θ	45	60	75	90
SSSS	1	20	100.805	102.340	105.995	108.046
		50	42.465	43.047	44.519	45.354
		100	21.556	21.699	22.406	22.816
		150	14.392	14.487	14.940	15.208
CCCC	1	20	137.759	133.502	132.837	133.294
		50	61.179	58.440	57.482	57.436
		100	31.231	29.695	29.122	29.069
		150	20.876	19.867	19.464	19.424

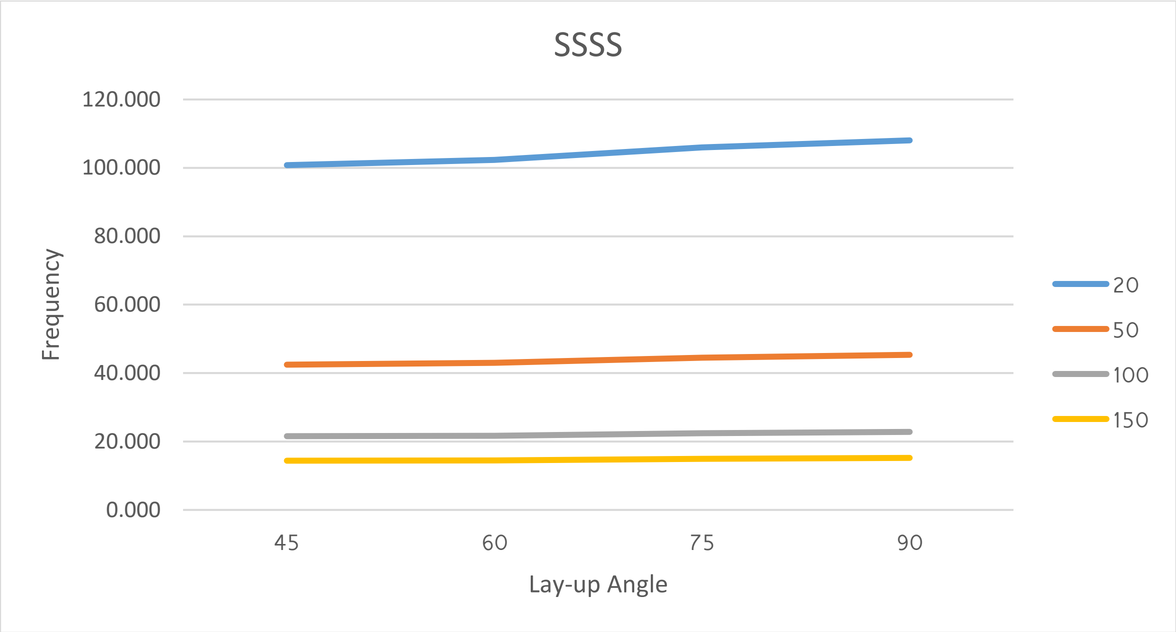


Fig: 4.14: For Natural Frequency (rad/s) Vs Layered Angle with varying a/h ratio of shell Element under SSSS condition

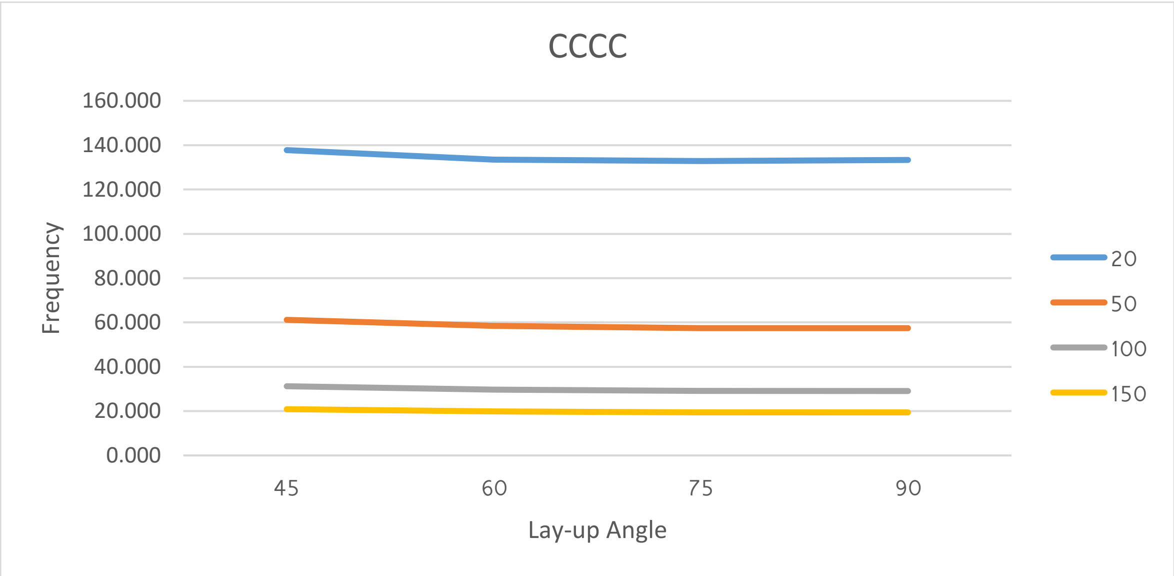


Fig: 4.15: For Natural Frequency (rad/s) Vs Layered Angle with varying a/h ratio of shell Element under CCCC condition

From above Table 4.7 and Figures 4.14 and 4.15 the following observations can be made.

1. For both simply supported (SSSS) and clamped (CCCC) conditions, decrease in frequencies observed for increase in a/h ratios. When a/h is 20, the variation becomes non-linear.

2. It is observed for SSSS case, maximum stiffness occurs at 0/90 lay-up sequence and for CCCC case maximum stiffness occurs at 0/45.

Hence it can be said lay-up angle along with a/h ratio, play a very important role for dynamic analysis of natural frequency.

CHAPTER 5

Conclusions

This present work is concerned with the free vibration analysis of laminated composite shells structures. For the numerical simulation of the problem finite element technique has been used. Various, shells thicknesses, orthotropy ratio, lay up angle and boundary configurations have been considered in the present study. From the numerical analyses the inferences obtained can be summarized as follows:

1. In case study 1, it is observed that due to increase in lay-up angle natural frequency increases with angle and becomes maximum at (0/90) in simply supported condition. In all side clamped condition, the stiffness becomes maximum at (0/45) giving maximum frequency.
2. In case study 2, it is observed that When orthotropy ratio near equal to 1, means same materials provided for both direction makes more stiff structure. The increase in orthotropy ratio denotes decrease in value of E_2 keeping E_1 value same which means the structure are becoming weak or flexible in nature, hence there is decrease in the natural frequency as well as stiffness of the structure.

3. In case study 3, it is seen that due to increase in shell thickness the structure becomes stiffer and hence the natural frequency increases.

4. For both simply supported (SSSS) and clamped (CCCC) conditions, decrease in frequencies observed for increase in a/h ratios. When a/h is 20, the variation becomes non-linear.

In general. It can be said that the stiffness of a shell structure depends on various parameter such as boundary condition, a/h ratios, layup angle and orthotropy ratio. Hence one can tailor-made the structure for various parameters to get the suitable frequency parameters. Though before applying these solutions, physical experimentation is necessary.

CHAPTER 6

Future Scope of Study

The finite element technique was adopted in the present study to investigate on the free vibration of multilayered laminated composites shells.

- These parametric studies can be conducted with other materials viz., E-glass/epoxy, Graphite/ epoxy etc.

- The present formulation is based on first order shear deformation theory. It can be modified using higher order shear deformation theory into account.

- This study may be extended to other practical complicated structural elements.

- Forced vibration, buckling and stability analysis can be other scopes of the present study.

CHAPTER 7

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