# STUDY OF GENERALIZED FUNCTIONAL IDENTITIES IN PRIME AND SEMIPRIME RINGS

*Thesis submitted in partial fulfillment of the requirements for the degree of*

**Doctor of Philosophy**


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# CERTIFICATE FROM THE SUPERVISOR(S)

This is to certify that the thesis entitled "**STUDY OF GENERALIZED FUNCTIONAL IDENTITIES IN PRIME AND SEMIPRIME RINGS**" submitted by Ms. **Manami Bera**, who got her name registered on 22.03.2021 for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon her own work under the supervision of Prof. Sukhendu Kar, Dept. of Mathematics, Jadavpur University, and Dr. Basudeb Dhara, Dept. of Mathematics, Belda College, Paschim Medinipur, and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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*To My Parents*

# Contents

# Preface

Rings and algebras have long been studied for their own intrinsic properties, and an ongoing area of interest has been to explore the implications of various special identities and, conversely, identifying sufficient conditions for a given ring to satisfy a particular identity. Ring derivation is a branch of algebra that focuses on the study of the structure of additive maps and the structure of rings by analyzing functional identities related to these additive maps. These additive maps are derivation, skew derivation, generalized derivation, generalized skew derivation, $b$-generalized derivation, $X$-generalized skew derivation, generalized $(\alpha, \beta)$ derivation etc. It is well established that there is a significant relationship between functional identities involving derivations as well as generalized derivations and the structure of rings. The key purpose of this thesis is to investigate various functional identities and generalized functional identities within the context of prime and semiprime rings. Let $R$ be an associative ring. A basic example of a functional identity is the identity $[f(x), x] = 0$ for all $x \in R$, where $[x, y] = xy - yx$ and $f : R \to R$ is a mapping. In 1957, Posner [80] examined a specific type of functional identity in rings by considering the above function as a derivation. Later on, Brešar [10] demonstrated that if $f$ is an additive mapping satisfying the identity investigated by Posner [80] in a prime ring $R$, then $f$ must be of the form $f(x) = \lambda x + \xi(x)$, where $\lambda \in C$, $\xi : R \to C$ is an additive map and $C$ is the extended centroid of $R$. Subsequently, numerous researchers have explored various functional identities and achieved remarkable results. We have also investigated some problems within this area.

This thesis is composed of eight chapters. Chapter wise brief information is given bellow:

**Chapter 1** is basically devoted for introductory purpose. Some basic definitions, preliminaries and prerequisites which have been collected from other references and which are needed for the development of the subsequent chapters for this thesis.

In **Chapter 2**, we study some commutativity theorems involving generalized $(\alpha, \beta)$ derivations on left sided ideals in prime and semiprime rings. Some examples are given at the end of this chapter concluding that the hypothesis of semiprimeness or primeness in the results are not superfluous.

Recently, De Filippis [45] introduced the new map $X$-generalized skew derivation. This concept covers the concept of generalized skew derivation as well as $b$-generalized derivation.

In **Chapter 3**, we study an identity with annihilating and centralizing conditions involving $X$-generalized skew derivations in prime rings.

A number of authors have studied some functional identities on the evaluations of a non central valued multilinear polynomial.

In **Chapter 4**, we study an identity involving annihilating and centralizing conditions of generalized derivations acting on noncentral multilinear polynomials in prime ring.

In **Chapter 5**, we study an identity involving three generalized derivations acting on multilinear polynomial in prime rings.

In **Chapter 6**, we study a derivation which vanishes when applied over a identity involving two generalized derivations acting on noncentral multilinear polynomial. We obtain results for the inner case, and furthermore, we clarify all possible forms of the derivation and generalized derivations involved in the identity.

In **Chapter 7**, we study an identity involving two nonzero generalized skew-derivations acting on multilinear polynomials in prime rings.

An additive subgroup $L$ of $R$ is said to be a Lie ideal of $R$, if $[l, r] \in L$ for all $l \in L$ and $r \in R$. The $m$-th commutator of $a, b$ is defined as $[a, b]_m = [[a, b]_{m-1}, b]$, $m = 1, 2, \ldots$. It is easy to check that $[a, b]_m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} b^i a b^{m-i}$.

In **Chapter 8**, we study an $m$-th commutator identity involving three generalized derivations acting on elements in Lie Ideal $L$.

Last of all we give some references from where we get many valuable results which help us to develop the whole work.

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I am eager to thank all the authors in my field whose articles and books have greatly contributed to my understanding of the subject.

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# List of Publications

[1] B. Dhara, V. De Filippis, **M. Bera**: Vanishing Derivations on Some Subsets in Prime Rings Involving Generalized Derivations, *Bull. Malays. Math. Sci. Soc.*, 44 (2021), 2693-2714, DOI: 10.1007/s40840-021-01080-4.

[2] B. Dhara, S. Kar, **M. Bera**: $X$-generalized skew derivations with annihilating and centralizing conditions in prime rings, *Ann. Univ. Ferrara*, 68 (2022), 147-160, DOI: 10.1007/s11565-022-00393-x.

[3] **M. Bera**, B. Dhara, S. Kar: Some identities involving generalized $(\alpha, \beta)$-derivations in prime and semiprime rings, *Asian-Eur. J. Math.* , 16(04) (2023), Article No. 2350073, DOI: 10.1142/S1793557123500730.

[4] **M. Bera**, B. Dhara: Generalized derivations and multilinear polynomials in prime rings, *Rend. Circ. Mat. Palermo, II. Ser*, 73 (2024), 415-427, DOI: 10.1007/s12215-023-00915-2.

[5] **M. Bera**, B. Dhara: Generalized skew-derivations acting on multilinear polynomials in prime rings, *Advances in Ring Theory and Applications, WARA 2022, Springer Proceedings in Mathematics & Statistics*, 443 (2024), 279-300, DOI: 10.1007/978-3-031-50795-3-20.

[6] B. Dhara, S. Kar, **M. Bera**: Annihilating and Centralizing Condition of Generalized Derivation in Prime Ring, *Southeast Asian Bull. Math.* , 48(4) (2024), 467-476.

[7] **M. Bera**, B. Dhara, S. Kar: Generalized Derivation with Engel condition acting on Lie ideals in prime rings, *Communicated.*

# Chapter 1

# Outline, Foundations and Prerequisites

This chapter is focused on foundational concepts, notations and expressions that will be extensively utilized in the upcoming chapters.

Across this thesis, $R$ will refer to an associative ring and ideal of a ring will be understood as a two-sided ideal. Our focus has been on functional identities related to specific types of additive maps, particularly in the context of prime and semiprime rings. A functional identity on a ring $R$ is an identity that involves both functions and elements of $R$. The primary objective in studying functional identities is to identify the form of the mappings involved, or, when that is not feasible, to ascertain the structure of the ring. Functional identity theory has proven to be a valuable tool for addressing a range of problems across various fields. Primarily, this chapter covers special classes of rings, quotient rings, particular types of additive maps, and generalized polynomial identities. Since all basic notations are not possible to mention here, we refer to the books by Herstein [58], Jacobson [61, 62].

Before establishing the results, we give some fundamental concepts of rings.

## 1.1 Several Specific Categories of Rings

We restrict our attention specially to prime and semiprime rings. To grasp the theory of prime and semiprime rings, it is essential first to understand the concepts of prime and semiprime ideals.

Now, we revisit the definitions of prime and semiprime ideals in rings.

**Definition 1.1.1.** *Let $R$ be a ring. An ideal $P$ of $R$ is said to be a prime ideal if for any two ideals $A, B$ in $R$, $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.*

The following well-established theorem provide characterizations of prime ideal in a ring :

**Theorem 1.1.1.** *Let $P$ be an ideal of $R$. Then the following are equivalent :*

*(i) $P$ is a prime ideal of $R$;*

*(ii) If $a, b \in R$ such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$;*

*(iii) If $< a >, < b >$ are principal ideals in $R$ such that $< a >< b >\subseteq P$ then either $a \in P$ or $b \in P$;*

*(iv) If $V$ and $V'$ are right ideals in $R$, then $VV' \subseteq P$ implies either $V \subseteq P$ or $V' \subseteq P$;*

*(v) If $V$ and $V'$ are left ideals in $R$, then $VV' \subseteq P$ implies either $V \subseteq P$ or $V' \subseteq P$.*

**Definition 1.1.2.** *Let $R$ be a ring. An ideal $P$ of $R$ is said to be a semiprime ideal of $R$ if for any ideal $A$ in $R$, $A^2 \subseteq P$ implies $A \subseteq P$.*

For example, in the ring $\mathbb{Z}$ of integers, the semiprime ideals are the zero ideal, also those ideals of the form $n\mathbb{Z}$ where $n$ is a square-free integer (square-free integer is one divisible by no perfect square, except 1). Hence $15\mathbb{Z}$ is a semiprime ideal of $\mathbb{Z}$ but $18\mathbb{Z}$ is not.

- A prime ideal of a ring $R$ is a semiprime ideal of $R$.
- Intersection of any set of semiprime ideals of a ring $R$ is also a semiprime ideal of $R$.
- If $Q$ is a semiprime ideal of $R$ and $A$ is an ideal of $R$ such that $A^n \subseteq Q$ for some $n \in \mathbb{N}$, then $A \subseteq Q$.

The following well-established result detail the characteristics of semiprime ideals.

**Theorem 1.1.2.** *Let $Q$ be an ideal of $R$. Then the following are equivalent :*

*(i) $Q$ is a semiprime ideal of $R$;*

$(ii)$ *If* $a \in R$ *be such that* $aRa \subseteq Q$*, then* $a \in Q$*;*

$(iii)$ *If* $< a >$ *is a principal ideal in* $R$ *such that* $< a >^2 \subseteq Q$*, then* $a \in Q$*;*

$(iv)$ *If* $V$ *is a right ideal in* $R$*, then* $V^2 \subseteq Q$ *implies* $V \subseteq Q$*;*

$(v)$ *If* $V$ *is a left ideal in* $R$*, then* $V^2 \subseteq Q$ *implies* $V \subseteq Q$*.*

**Definition 1.1.3.** *A ring* $R$ *is said to be a prime ring if zero ideal is a prime ideal in* $R$*.*

**Example 1.1.1.** *Notable examples of prime ring are:*

*(i) Any domain is a prime ring.*

*(ii) Any matrix ring over an integral domain is a prime ring.*

*(iii) Every simple ring is a prime ring.*

**Theorem 1.1.3.** *The subsequent conditions are equivalent:*

$(i)$ $R$ *is a prime ring;*

$(ii)$ *If* $A, B$ *are two ideals of* $R$ *such that* $AB = (0)$*, then either* $A = (0)$ *or* $B = (0)$*;*

$(iii)$ *If* $a, b \in R$ *such that* $aRb = (0)$*, then either* $a = 0$ *or* $b = 0$*;*

$(iv)$ *There exist* $0 \neq r;\ 0 \neq t \in R$ *such that* $rst \neq 0$ *for some* $s \in R$*.*

**Definition 1.1.4.** *A ring* $R$ *is said to be a semiprime ring if zero ideal is a semiprime ideal in* $R$*.*

**Example 1.1.2.** *Notable examples of semiprime ring are:*

*(i) Every prime ring is semiprime ring.*

*(ii)* $R_1 \bigoplus R_2$ *is a semiprime ring if* $R_1$ *and* $R_2$ *are nonzero prime rings.*

**Theorem 1.1.4.** *The subsequent conditions are equivalent:*

$(i)$ $R$ *is a semiprime ring;*

$(ii)$ *If* $A$ *is an ideal of* $R$ *such that* $A^2 = (0)$*, then* $A = (0)$*;*

$(iii)$ *If* $a \in R$ *such that* $aRa = (0)$*, then* $a = 0$*;*

$(iv)$ *There exist* $0 \neq r;\ 0 \neq t \in R$ *such that* $rst \neq 0$ *for some* $s \in R$*.*

**Definition 1.1.5.** *An element* $a$ *of a ring* $R$ *is said to be nilpotent if* $a^n = 0$ *for some positive integer* $n$*.*

**Definition 1.1.6.** *For any nonempty subset* $A$ *of* $R$*,* $r(A) = \{x \in R : Ax = (0)\}$ *is called the right annihilator of* $A$ *in* $R$ *and* $l(A) = \{x \in R : xA = (0)\}$ *is called* *the left annihilator of A in R. If $r(A) = l(A)$, then $r(A)$ is called an annihilator ideal of R and is written as $ann_R(A)$. An element $a \in R$ is called annihilator of A if $a \in r(A)$ as well as $a \in l(A)$*

■ **Certain Characteristics of Prime and Semiprime Rings :**

In this section, we present several established facts about prime and semiprime rings. The more information about prime and semiprime rings can be found in the book of Herstein [59].

1. In general, if $R_1$ and $R_2$ are semiprime rings, then $R_1 \bigoplus R_2$ is semiprime but not prime.

2. Every prime ring is a semiprime ring but the converse is not true. For example, $\mathbb{Z} \bigoplus \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers, is a semiprime ring but not a prime ring.

3. Semiprime ring has no nonzero nilpotent ideal.

4. If $R$ is a semiprime ring and $I$ is an ideal of $R$, then $r(I) = l(I)$. Moreover, if $R$ is semiprime and $I$ is an ideal of $R$, then $I \cap ann_R(I) = (0)$.

5. Let $R$ be a semiprime ring and $\rho$ be a right ideal of $R$. Then $Z(\rho) \subseteq Z(R)$, where $Z(R)$ is the center of the ring $R$.

6. Let $R$ be a prime ring which contains a commutative one sided ideal. Then $R$ must be commutative.

7. If a prime ring $R$ contains a nonzero one sided central ideal, then $R$ must be a commutative ring.

8. Center of a prime ring contains no divisor of zero.

9. Let $R$ be a prime ring with center $Z(R)$. If $zr \in Z(R)$ for some $0 \neq z \in Z(R)$ and $r \in R$, then $r \in Z(R)$.

10. If $R$ is a prime ring with no nonzero nilpotent elements, then $R$ has no zero divisor.

11. $R$ is a prime ring if and only if $r(J) = (0)$, where $J$ is a nonzero right ideal of $R$.

**Definition 1.1.7.** *A ring $R$ is said to be $n-$torsion free, where $n$ is a positive integer, if whenever $nx = 0$, with $x \in R$, then $x = 0$.*

• For a prime ring $R$, $char(R) \neq n$ if and only if $R$ is $n-$torsion free. It is easy to note that whenever a ring $R$ is $n-$torsion free, then $char(R) \neq n$. But converse of the above result is not true for all rings. The primeness is required for the converse statement to be true.

## 1.2 Commutator Identities in Rings

For any $x, y \in R$, the symbol $[x, y]$ stands for the *commutator or Lie product* $xy - yx$ and the symbol $x \circ y$ stands for the *anti-commutator or Jordan product* $xy + yx$.

We recall some basic commutator identities in a ring $R$ as follows : For all $x, y, z \in R$,

$$[xy, z] = x[y, z] + [x, z]y; \quad [x, yz] = y[x, z] + [x, y]z;$$

$$(xy \circ z) = x(y \circ z) - [x, z]y = x[y, z] + (x \circ z)y;$$

$$(x \circ yz) = (x \circ y)z - y[x, z] = [x, y]z + y(x \circ z).$$

Moreover,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The last identity is called as Jacobi Identity.

For $x, y \in R$, set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$, and then an Engel type polynomial $[x, y]_k = [[x, y]_{k-1}, y]$, $k = 1, 2, \ldots$.

**Definition 1.2.1.** *An additive subgroup $L$ of $R$ is said to be a Lie ideal of $R$, if $[l, r] \in L$ for all $l \in L$ and $r \in R$.*

- ♦ Every ideal of a ring $R$ is a Lie ideal of $R$. It is to be noted that a Lie ideal of a ring $R$ may not be an ideal of $R$.

- ♦ A Lie ideal $L$ is said to be square closed if $u^2 \in L$ for all $u \in L$.

**Theorem 1.2.1.** *Let $S$ be a nonempty subset of a ring $R$. A map $f : R \to R$ is called commuting (resp. centralizing) on $S$ if $[f(x), x] = 0$ for all $x \in S$ (resp. $[f(x), x] \in Z(R)$ for all $x \in S$).*

**Theorem 1.2.2.** *Let $S$ be a nonempty subset of a ring $R$. Two maps $f : R \to R$ and $g : R \to R$ are called co-commuting (resp. co-centralizing) on $S$ if $f(x)x = xg(x)$ for all $x \in S$ (resp. $f(x)x - xg(x) \in Z(R)$ for all $x \in S$).*

## 1.3 Ring of Quotients

In the study of generalized identities in prime and semiprime rings we observe that rings of quotients play a crucial role. For us, the most important ring of quotients is the *maximal right ring of quotients* or *Utumi ring of quotients.* It was first constructed by Y. Utumi [89]. Another important ring of quotients is used here, *two-sided ring of quotients* or *Martindale ring of quotients.* This ring of quotients was introduced in [82] as a tool to study prime rings satisfying a generalized polynomial identity.

### 1.3.1 Utumi Ring of Quotients

Let $R$ be a prime ring and $\mathcal{D} = \{J\}$ be the collection of all dense right ideals of $R$, and consider $T$ to be the set of all $R$-homomorphisms $f : J_R \to R_R$, where $J$ ranges over $\mathcal{D}$, $J$ and $R$ are regarded as right $R$-modules. So $T = \{(f; J) |\ J \in \mathcal{D},\ f : J_R \to R_R\}$, where $(f; J)$ denotes $f$ acting on $J$.

We define $(f; J) \sim (g; K)$ if there exists $L \subseteq J \cap K$ such that $L \in \mathcal{D}$ and $f = g$ on $L$. One readily check that '$\sim$' is indeed an equivalence relation, let $[f; J]$ denote the equivalence class determined $(f; J) \in \mathcal{D}$ and we let $U$ denote the collection of all equivalence classes of $T$ with respect to '$\sim$'. We then define addition and multiplication of equivalence classes as follows:

$$[f; J] + [g; K] = [f + g; J \cap K] \text{ and } [f; J][g; K] = [fg; g^{-1}(J)].$$

One can easily check that the addition and multiplication is well-defined [7, pp. 55].

Under these operations it is readily seen that $U$ forms a ring with respect to above addition and multiplication. This ring $U$ is called *Utumi ring of quotients.*

Some important properties are given below :

**Proposition 1.3.1.** *[7, Proposition 2.1.7] For a semiprime ring $R$, the Utumi ring of quotients $U$ satisfies the following properties:*

*(1) $R$ is a subring of $U$;*

*(2) For all $q \in U$, there exists $J \in \mathcal{D}$ such that $qJ \subseteq R$;*

*(3) For all $q \in U$ and $J \in \mathcal{D}$, $qJ = 0$ if and only if $q = 0$;*

*(4) For all $J \in \mathcal{D}$ and $f : J_R \to R_R$, there exists $q \in U$ such that $f(x) = qx$ for all $x \in J$.*

*Furthermore, properties $(1) - (4)$ characterize the ring $U$ up to isomorphism.*

### 1.3.2 Martindale Ring of Quotients

For a prime ring $R$, a nonzero two-sided ideal is obviously a dense right ideal of $R$. In the above construction if we consider only nonzero two-sided ideals instead of dense right ideals, then we obtain the Martindale ring of quotient (see [82]). Here we denote this ring by $Q$.

**Proposition 1.3.2.** *[7, Proposition 2.2.1] Let $R$ be a semiprime ring. Then the Martindale ring of quotients $Q$ satisfies the following properties:*

*(1) $R$ is a subring of $Q$;*

*(2) For all $q \in Q$, there exists $J \in \mathcal{D}$ such that $qJ \subseteq R$;*

*(3) For all $q \in Q$ and $J \in \mathcal{D}$, $qJ = 0$ if and only if $q = 0$;*

*(4) For all $J \in \mathcal{D}$ and $f : J_R \to R_R$, there exists $q \in Q$ such that $f(x) = qx$ for all $x \in J$.*

*Furthermore, properties $(1) - (4)$ characterize ring $Q$ up to isomorphism.*

Some important facts are as follows:

• $Q$ can be naturally regarded as a subring of $U$ [7, Proposition 2.2.2] and can be characterized as follows: For $a \in U$, $a \in Q$ if and only if $aI \subseteq R$ for some nonzero two-sided ideal $I$ of $R$.

• Also for a prime ring $R$, the corresponding rings of quotients $Q$ and $U$ both are prime ring [40, p. 74].

• $Z(Q) = Z(U)$, where $Z(Q)$ and $Z(U)$ are centers of $Q$ and $U$ respectively [7, Remark 2.3.1].

**Definition 1.3.1.** *The center of the Martindale ring of quotients as well as the Utumi ring of quotients is called the extended centroid $C$ of $R$ and $S = RC$ is called the central closure of $R$.*

It is very well known that $C$ forms a field, when $R$ is prime ring [7, p. 70]. In fact, $S = RC$ is a prime ring containing $R$. Further $S$ is contained in $Q \subseteq U$. If $R$ has unity then $C = Z(S)$. If $R$ is a simple ring with unity then $Q = S = R$ i.e., $R$ is its own central closure. We refer to [59, 82] for more details.

## 1.4 Several Specific Additive Maps in Rings

Let $R$ be a ring. A map $f : R \to R$ is said to be additive if $f(x+y) = f(x) + f(y)$ holds for all $x, y \in R$ i.e., it preserves the additive structure of $R$.

**Definition 1.4.1.** *Let $R$ be a ring. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.*

**Example 1.4.1.** *Some examples of derivation are:*

*(i) The usual derivation $d$ on the polynomial ring $R = F[x]$ (where $F$ is a field) given by*

$$d(\sum_{i=0}^{t} a_i x^i) = \sum_{i=1}^{t} i a_i x^{i-1} = \sum_{i=0}^{t} i a_i x^{i-1}.$$

*(ii) The mapping $d_a : R \to R$ defined by $d_a(x) = [a, x]$ for all $x \in R$ and some fixed $a \in R$, we have $d_a(xy) = [a, xy] = x[a, y] + [a, x]y = xd_a(y) + d_a(x)y$. This kind of derivations are called as **inner derivations** of $R$. Inner derivation leads to a great*

*deal in the study of derivations. The derivation which is not inner, called* ***outer derivation****.*

*(iii) Let us consider the ring $R$ defined by,*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$

*where $\mathbb{Z}$ is the set of all integers. Let us define a map $d : R \to R$ by*

$$d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & nb \\ 0 & 0 \end{pmatrix}, a, b, c \in \mathbb{Z},$$

*and $n$ be any fixed integer. Then it is obvious that $d$ is a derivation on $R$.*

**Remark 1.4.1.** *Let $d$ be a derivation of a ring $R$. Then $d(Z(R)) \subseteq Z(R)$.*

The notion of derivation was extended by Brešar [9] in 1991. He first introduced the concept of generalized derivation which was further studied algebraically by Hvala [60] in 1998.

**Definition 1.4.2.** *Let $R$ be a ring. An additive mapping $F : R \to R$ is called a generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.*

Evidently, every derivation is a generalized derivation of $R$. When $d = 0$, then it is clear that $F$ turns to a left multiplier map of $R$. Thus generalized derivation covers the concept of derivation as well as the concept of left multiplier map. Some authors used the notion of left centralizer instead of left multiplier in a ring.

**Example 1.4.2.** *Let $R$ be a ring.*

*(i) For $a, b \in R$, the map $x \to ax + xb$ of $R$ is a generalized derivation.*

*(ii) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers. Let us define the mappings $d : R \to R$ and $F : R \to R$ by $d \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ and $F \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & a+b \\ 0 & 0 \end{pmatrix}$. Then $F$ is a generalized derivation of $R$ with the derivation $d$.*

*(iii) Addition of two generalized derivations of a ring $R$ is again a generalized derivation of $R$.*

In the literature, we can found many interesting results involving $(\alpha, \beta)$-derivations. These kinds of derivations defined as follows:

**Definition 1.4.3.** *An additive mapping $d : R \to R$ is called a $(\alpha, \beta)$-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$, where $\alpha, \beta$ are automorphisms of $R$.*

Of course every $(1, 1)$-derivation is a derivation of $R$, where 1 denotes the identity mapping of $R$.

Being inspired by the definition of $(\alpha, \beta)$-derivation, the notion of generalized $(\alpha, \beta)$-derivation was extended as follows:

**Definition 1.4.4.** *An additive mapping $F : R \to R$ is said to be a generalized $(\alpha, \beta)$-derivation of $R$, if there exists a $(\alpha, \beta)$-derivation $d : R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$, where $\alpha, \beta$ are automorphisms of $R$.*

Of course every generalized $(1, 1)$-derivation of $R$ is a generalized derivation of $R$, where 1 means an identity map of $R$. If $d = 0$, we have $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$, which is called a left $\alpha$-multiplier mapping of $R$. Thus generalized $(\alpha, \beta)$-derivation generalizes both the concepts, $(\alpha, \beta)$-derivation as well as left $\alpha$-multiplier mapping of $R$.

**Definition 1.4.5.** *Let $R$ be a Ring and $b \in R$. An additive mapping $F : R \to R$ is said to be $b$-generalized derivation of $R$, if there exists a derivation $d$ of $R$ such that*

$$F(xy) = F(x)y + bxd(y)$$

*holds for all $x, y \in R$.*

From the above definition we can easily see that generalized derivation is a 1-generalized derivation and we can define a map $F : R \to R$ by $F(x) = ax + bxc$ for all $x \in R$, where $a, b, c \in R$ which is called inner $b$-generalized derivation of $R$.

**Definition 1.4.6.** *Let $R$ be an associative ring and $\alpha$ be an automorphism of $R$. An additive mapping $d : R \to R$ is called a skew derivation of $R$ if*

$$d(xy) = d(x)y + \alpha(x)d(y)$$

*holds for all $x, y \in R$. Here $\alpha$ is called the associated automorphism of $d$.*

**Definition 1.4.7.** *An additive mapping $G : R \to R$ is said to be a generalized skew derivation of $R$, if there exists a skew derivation $d$ of $R$ with associated automorphism $\alpha$ such that*

$$G(xy) = G(x)y + \alpha(x)d(y)$$

*holds for all $x, y \in R$. Here $d$ is said to be an associated skew derivation of $G$ and $\alpha$ is called an associated automorphism of $G$.*

The concept of the map $X$-generalized skew derivation was introduced by De Filippis and Wei in [54]. The concept of $X$-generalized skew derivation generalizes the concept of generalized skew derivation as well as $b$-generalized derivation in $R$.

**Definition 1.4.8.** *Let $R$ be an associative ring, $b \in Q$, $d : R \to R$ an additive mapping and $\alpha$ be an automorphism of $R$. An additive mapping $F : R \to R$ is called an $X-$generalized skew derivation of $R$, with associated term $(b, \alpha, d)$ if $F(xy) = F(x)y + b\alpha(x)d(y)$ for all $x, y \in R$.*

It is very easy to check that $X-$generalized skew derivation generalizes the concept of generalized skew derivation as well as $b$-generalized derivation. The map $x \mapsto ax + b\alpha(x)c$ is an example of $X$-generalized skew derivation of $R$ with associated map $(b, \alpha, d)$, where $a, b, c \in R$ are fixed elements and $d(x) = \alpha(x)c - cx$ for all $x \in R$. Such $X$-generalized skew derivations of $R$ are called as inner $X$-generalized skew derivations of $R$.

## 1.5 Generalized Polynomial Identity (GPI)

Let $R$ be an associative ring and let $X = \{x_1, x_2, \ldots\}$ be an infinite set of non-commutative indeterminates. The classical approach to the theory of polynomial identities of a ring $R$ was to consider identical relations in $R$ of the form $p[x] = 0$, where $p[x] = \sum \alpha_{(i)} x_{i_1} x_{i_2} \cdots x_{i_n}$ is a polynomial in the $x_j$ with coefficients $\alpha_{(i)}$ which are integers or belong to a commutative field $F$ over which $R$ is an algebra. The main result in the theory of these identities is due to Kaplansky [61, p. 226] which states that a primitive ring satisfying a polynomial identity of degree $d$ is a finite-dimensional algebra over its center, and its dimension is $\leq [d/2]^2$.

The generalized polynomial identities to be dealt with are of the form:

$$P[x] = \sum \alpha_{i_1} \pi_{j_1} \alpha_{i_2} \pi_{j_2} \cdots \alpha_{i_k} \pi_{j_k} \alpha_{i_{k+1}} = 0,$$

where the $\pi_j$ are monomials in the indeterminates $x_j$ and the elements $a_{i_\lambda}$ appear both as coefficients and between the monomials $\pi_j$. More precisely, one considers a prime ring $R$ and $S = RC$, its central closure. Consider $S < x >= S *_C \{X\}$, the free product of $S$ and $\{X\}$ over $C$. The elements of $S < x >$ are called the **generalized polynomials**. By a nontrivial generalized polynomial, we mean a nonzero element of $S < x >$. An element $m \in S < x >$ of the form $m = q_0 y_1 q_1 y_2 q_2 \ldots y_n q_n$, where $\{q_0, q_1, \ldots, q_n\} \subseteq S$ and $\{y_1, y_2, \ldots, y_n\} \subseteq X$, is called a monomial ( some of the $q_i$ can be 1 also); $q_0, q_1, \ldots, q_n$ are called the coefficients of $m$. Each $f \in S < x >$ can be represented as a finite sum of monomials. Such a representation is certainly not unique.

Let $B$ be a set of $C$-independent vectors of $S$. By a $B$-monomial, we mean a monomial of the form $u_0 y_1 u_1 y_2 u_2 \ldots y_n u_n$, where $\{u_0, \ldots, u_n\} \subseteq B$ and $\{y_1, y_2, \ldots, y_n\}$ $\subseteq X$. Let $V = BC$, the $C$-subspace spanned by $B$. Then any $V$-generalized polynomial $f$ can be written in the form $\sum \alpha_i m_i$, where $\alpha_i \in C$ and $m_i$ are $B$-monomials, in the following manner: First fix a representation of $f$ with all of its coefficients in $V$ and express each coefficient of the given representation as a linear combination of elements of $B$. Then substitute these linear combinations into the representation of $f$ and expand the resulting expression using the distributive law. Finally, we collect similar terms to get our desired form.

It is also obvious that such representation of a given $f$ in terms of $B$-monomials is unique. If $B$ is chosen to be a basis of $S$ over $C$, the $B$-monomials span the whole $S < x >$.

The uniqueness of representation in terms of $B$-monomials gives a practical criterion to decide whether a given generalized polynomial $f$ is trivial or not: Pick a basis $B$ for the $C$-subspace spanned by the coefficients of a given representation of $f$. Express $f$ as a linear combination of $B$-monomials in the way explained above. Let us say $f = \sum \alpha_i m_i$, where $\alpha_i \in C$ and $m_i$ are $B$-monomials. Then $f$ is trivial if and only if $\alpha_i = 0$ for each $i$. This simple criterion will be used frequently in the subsequent chapters.

**Remark 1.5.1.** *As a consequence, if we consider $T = U *_C C\{X\}$, the free product of $U$ and the free algebra $C\{X\}$ over $C$. If $a_1, a_2 \in U$ are linearly independent over $C$ and $a_1 g_1(x_1, \ldots, x_n) + a_2 g_2(x_1, \ldots, x_n) = 0 \in T$, where*

$$g_1(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i h_i(x_1, \ldots, x_n)$$

*and*

$$g_2(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i k_i(x_1, \ldots, x_n)$$

*for $h_i(x_1, \ldots, x_n), k_i(x_1, \ldots, x_n) \in T$, then both $g_1(x_1, \ldots, x_n)$ and $g_2(x_1, \ldots, x_n)$ are zero element of $T$.*

**Definition 1.5.1.** *$S$ is said to satisfy generalized polynomial identity if there exists an $0 \neq f \in S < x >$ such that $f(s_1, s_2, \ldots, s_n) = 0$ for all $s_i \in S$.*

**Definition 1.5.2.** *The polynomial with $n$ variables*

$$s_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)},$$

*where $(-1)^{\sigma}$ is $+1$ or $-1$ according as $\sigma$ being an even or odd permutation in symmetric group $S_n$, is called the standard polynomial of degree $n$.*

**Theorem 1.5.1.** ***Amitsur-Levitzki Theorem:*** *Let $R$ be a commutative ring. Then $M_n(R)$ satisfies $s_{2n}$.*

**Theorem 1.5.2.** *[21, Theorem 2] Let $R$ be a prime ring with its Utumi ring of quotients $U$. For any dense submodule $M$ of $U$, the GPIs satisfied by $M$ are the same as the GPIs satisfied by $U$.*

**Theorem 1.5.3.** *[21, Theorem 3] Let $R$ be a prime ring with its Utumi ring of quotients $U$. Let $M$ and $N$ be two dense submodules of $U$. If $M$ satisfies a GPI, then $M$ satisfies a GPI of $N$.*

Let $R$ be a prime ring with its Utumi ring of quotients $U$. Let $Der(U)$ be the set of all derivations of $U$. By a derivation word we mean an additive map $\Delta$ of the form $\Delta = d_1 \dots d_n$ with each $d_i \in Der(U)$.

A differential polynomial is a generalized polynomial of the form $\Phi(\Delta_j(x_i))$ involving non-commutative indeterminates $x_i$ which are acted by derivation words $\Delta_j$ as uniary operation and with coefficients from $U$. $\Phi(\Delta_j(x_i))$ is said to be *differential identity* on $S \subseteq U$, if $\Phi(\Delta_j(x_i))$ assumes the constant value 0 for any assignment of values from $S$ to its indeterminates $x_i$.

**Theorem 1.5.4.** *[71, Theorem 3] Let $R$ be a semiprime ring, $U$ its Utumi ring of quotients and $I_R$ a dense $R$-submodule of $U_R$. Then $I$ and $U$ satisfy the same differential identities.*

## 1.6 Some Crucial Outcomes

**Theorem 1.6.1.** *[7, Proposition 2.5.1] Every derivation of a prime ring $R$ can be uniquely extended to a derivation of the Utumi ring of quotients $U$.*

**Theorem 1.6.2.** *[72, Theorem 3] Every generalized derivation $g$ on a dense right ideal of $R$ can be uniquely extended to $U$ and assumes the form $g(x) = ax + \delta(x)$ for some $a \in U$ and a derivation $\delta$ on $U$.*

**Theorem 1.6.3.** *[51, Lemma 1.5] Suppose that $A_1, \dots, A_k$ are non scalar matrices in $M_t(C)$, where $t \geq 2$ and $C$ is infinite field. Then there exists an invertible matrix $P \in M_m(C)$ such that any matrices $PA_1P^{-1}, \dots, PA_kP^{-1}$ have all nonzero entries.*

**Theorem 1.6.4.** *[64, Theorem 2]* ***Kharchenko's Theorem:***
*Let $R$ be a prime ring, $U$ be its Utumi ring of quotients and $I$ be an ideal of $R$. Let $\Phi(\Delta_j(x_i)) = 0$ be a reduced differential identity for $I$. Then $\Phi(z_{ij}) = 0$ is GPI for $U$, where $z_{ij}$ are distinct indeterminates.*

In particular, we have:
If $d$ is a nonzero outer derivation of $R$ and $\Phi(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)) = 0$ is a differential identity on $R$, then $U$ satisfies GPI $\Phi(x_1, \ldots, x_n, z_1, \ldots, z_n) = 0$, where $x_1, \ldots, x_n, z_1, \ldots, z_n$ are distinct indeterminates.

**Theorem 1.6.5.** *[61]* ***Jacobson Density Theorem:***
*Let $R$ be a (left) primitive ring with $R^V$ a faithful irreducible $R$-module and $D = End(R^V)$. Then for any natural number $n$, if $v_1, \ldots, v_n$ are $D$-independent in $V$ and $w_1, \ldots, w_n$ are arbitrary in $V$, then there exists $r \in R$ such that $rv_i = w_i, i = 1, \ldots, n$.*

**Theorem 1.6.6.** *[82, Theorem 3]* ***Martindale Theorem:***
*Let $R$ be a prime ring with its extended centroid $C$. Then $S = RC$ satisfies a GPI over $C$ if and only if $S$ contains a minimal right ideal $eS$ (hence $S$ is primitive) and $eSe$ is a finite dimensional division algebra over $C$, where $e$ is idempotent.*

**Theorem 1.6.7.** *[7, 61]* ***Litoff's Theorem:***
*Let $R$ be a primitive ring with nonzero socle $H = Soc(R)$ and $b_1, \ldots, b_m \in H$. Then there exists an idempotent $e \in H$ such that $b_1, \ldots, b_m \in eRe$ and the ring $eRe$ is isomorphic to $M_n(C)$.*

# Chapter 2

# Some Identities Involving Generalized $(\alpha, \beta)$-derivations in Prime and Semiprime Rings

## 2.1 Introduction

Throughout this chapter $R$ denotes an associative ring with its center $Z(R)$. There is ongoing interest to study the functional identities replacing generalized derivations with generalized $(\alpha, \beta)$-derivations. We have already mentioned in Chapter-1 that an additive map $d : R \to R$ is called a $(\alpha, \beta)$-derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$ where $\alpha, \beta$ are two automorphisms of $R$. For some fixed $a \in R$, the map $x \mapsto a\alpha(x) - \beta(x)a$ is an example of $(\alpha, \beta)$-derivation which is called inner $(\alpha, \beta)$-derivation.

Inspired by the definition of $(\alpha, \beta)$-derivation, the notion of generalized derivation was extended to generalized $(\alpha, \beta)$-derivation. An additive mapping $F : R \to R$ is said to be a generalized $(\alpha, \beta)$-derivation, if there exists a $(\alpha, \beta)$-derivation $d$ of $R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. Every $(\alpha, \beta)$-derivation is generalized $(\alpha, \beta)$-derivation. The map of the form $x \mapsto a\alpha(x) + \beta(x)b$ for some $a, b \in R$ is an example of generalized $(\alpha, \beta)$-derivation which is said to be an inner generalized $(\alpha, \beta)$-derivation.

In [6], Ashraf et al. have studied the following identities in prime ring $R$: (i) $F(xy) - xy \in Z(R)$ for all $x,\ y \in I$, (ii) $F(xy) + xy \in Z(R)$ for all $x,\ y \in I$, (iii)

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$F(xy) - yx \in Z(R)$ for all $x, y \in I$, (iv) $F(xy) + yx \in Z(R)$ for all $x, y \in I$, (v) $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, (vi) $F(x)F(y) + xy \in Z(R)$ for all $x, y \in I$, where $F$ is a generalized derivation of $R$ associated with a non-zero derivation $d$ and $I$ is a non-zero two-sided ideal of $R$ and obtained the commutativity of prime ring $R$.

In [33], Dhara studied the identities (i) $F(x)F(y) - yx \in Z(R)$ and (ii) $F(x)F(y) + yx \in Z(R)$ for all $x, y$ in some suitable subset of $R$. Recently, in [85], Tiwari et al. considered the identities as follows:

(i) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$;

(ii) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$;

(iii) $G(xy) \pm F(y)F(x) \pm xy \in Z(R)$ for all $x, y \in I$;

(iv) $G(xy) \pm F(y)F(x) \pm yx \in Z(R)$ for all $x, y \in I$;

(v) $G(xy) \pm F(y)F(x) \pm [x, y] \in Z(R)$ for all $x, y \in I$;

(vi) $G(xy) \pm F(x)F(y) \pm [\alpha(x), y] \in Z(R)$ for all $x, y \in I$,

where $I$ is a non-zero ideal in prime ring $R$ and $\alpha : R \to R$ is any mapping and obtained commutativity of prime rings.

Several authors studied the commutativity in prime and semiprime rings admitting $(\alpha, \beta)$-derivations and generalized $(\alpha, \beta)$-derivations which satisfy appropriate algebraic conditions on appropriate subsets of the rings (see [3], [34], [56], [57], [63], [79]).

Let $R$ be a prime ring and $F$, $G$ be two generalized $(\alpha, \beta)$-derivations with associated $(\alpha, \beta)$-derivations $d$ and $g$, respectively. Recently, in [56] Garg studied the following identities in prime rings

(i) $G(xy) + d(x)F(y) = 0$;

(ii) $G(xy) + d(x)F(y) + \alpha(yx) = 0$;

(iii) $G(xy) + d(y)F(x) = 0$;

(iv) $G(xy) + d(y)F(x) + \alpha(yx) = 0$;

(v) $G(xy) + F(x)F(y) = 0$;

(vi) $G(xy) + F(y)F(x) = 0$,

for all $x, y \in L$, where $L$ is a square closed Lie ideal of $R$.

In the present chapter our motivation is to study above identities involving generalized $(\alpha, \beta)$-derivations on left-sided ideals in prime and semiprime rings. More precisely, we study the following algebraic identities:

1. $G(xy) + d(x)F(y) + \alpha(xy) = 0$;
2. $G(xy) + d(x)F(y) = 0$;
3. $G(xy) + d(x)F(y) + \alpha(yx) = 0$;
4. $G(xy) + d(x)F(y) + \alpha(yx) + \alpha(xy) = 0$;
5. $G(xy) + d(x)F(y) + \alpha(yx) - \alpha(xy) = 0$;
6. $G(xy) + d(y)F(x) + \alpha(yx) = 0$;
7. $G(xy) + d(y)F(x) + \alpha(yx) + \alpha(xy) = 0$;
8. $G(xy) + d(y)F(x) + \alpha(yx) - \alpha(xy) = 0$;
9. $G(xy) + F(x)F(y) = 0$;
10. $G(xy) + F(y)F(x) = 0$;
11. $F(xy) + G(x)\alpha(y) + \alpha(yx) = 0$;
12. $F(x)F(y) + G(x)\alpha(y) + \alpha(yx) = 0$,

for all $x, y \in \lambda$, where $F$, $G$ are two generalized $(\alpha, \beta)$-derivations of $R$ associated with $(\alpha, \beta)$-derivations $d$ and $g$, respectively and $\lambda$ is a nonzero left-sided ideal of $R$.

## 2.2 Preliminaries

To prove our Theorems, we need the following Lemmas.

**Lemma 2.2.1.** *Let $R$ be a semiprime ring and $d$ be a nonzero $(\alpha, \beta)$-derivation of $R$. If $[d(x), \beta(x)] = 0$ for all $x \in R$, then $d$ maps $R$ to $Z(R)$.*

*Proof.* By hypothesis,

$$[d(x+y), \beta(x+y)] = 0 \tag{2.2.1}$$

for all $x, y \in R$, which implies

$$[d(x), \beta(y)] + [d(y), \beta(x)] = 0 \tag{2.2.2}$$

for all $x, y \in R$. Replacing $y$ with $xy$, we have

$$[d(x), \beta(x)\beta(y)] + [d(x)\alpha(y) + \beta(x)d(y), \beta(x)] = 0 \tag{2.2.3}$$

which gives by using $[d(x), \beta(x)] = 0$ for all $x \in R$ that

$$\beta(x)[d(x), \beta(y)] + [d(x)\alpha(y), \beta(x)] + \beta(x)[d(y), \beta(x)] = 0 \tag{2.2.4}$$

for all $x, y \in R$. By using (2.2.2), it reduces to

$$[d(x)\alpha(y), \beta(x)] = 0 \tag{2.2.5}$$

for all $x, y \in R$. Replacing $y$ by $yt$, for $t \in R$, we have

$$0 = [d(x)\alpha(y)\alpha(t), \beta(x)] = [d(x)\alpha(y), \beta(x)]\alpha(t) + d(x)\alpha(y)[\alpha(t), \beta(x)]$$

that gives

$$d(x)\alpha(y)[\alpha(t), \beta(x)] = 0$$

for all $x, y, t \in R$. This implies that $d(x)R[R, \beta(x)] = (0)$ for all $x \in R$. Since $R$ is semiprime, it must contain a family $\Omega = \{P_\alpha : \alpha \in \Lambda\}$ of prime ideals such that $\bigcap_{\alpha\in\Lambda} P_\alpha = (0)$. If $P$ is typical member of $\Omega$, then for each $x \in R$, we have either $d(x) \in P$ or $[R, \beta(x)] \subseteq P$. For fixed $P$, the sets $T_1 = \{x \in R : d(x) \in P\}$ and $T_2 = \{x \in R : [R, \beta(x)] \subseteq P\}$ form two additive subgroups of $R$ such that $T_1 \cup T_2 = R$. Therefore, either $T_1 = R$ or $T_2 = R$, that is, either $d(R) \subseteq P$ or $[R, \beta(R)] \subseteq P$. Both of these two conditions together imply that $[d(R), \beta(R)] \subseteq P$ for any $P \in \Omega$. Since $\bigcap_{\alpha\in\Lambda} P_\alpha = (0)$, we have $[d(R), \beta(R)] = (0)$ i.e., $[d(R), R] = (0)$. Hence $d(R) \subseteq Z(R)$ which implies that $d$ maps from $R$ to $Z(R)$. □

**Lemma 2.2.2.** *Let $R$ be a prime ring, $\lambda$ a nonzero left-sided ideal of $R$ and $d$ be a $(\alpha, \beta)$-derivation of $R$. If $\beta(\lambda)[d(x), \beta(x)] = (0)$ for all $x \in \lambda$, then $\beta(\lambda)d(\lambda) = (0)$ or $\alpha(\lambda)[\alpha(\lambda), \beta(\lambda)] = (0)$.*

*Proof.* By hypothesis,

$$\beta(\lambda)[d(x), \beta(x)] = (0) \tag{2.2.6}$$

for all $x \in \lambda$. Linearizing, we obtain

$$\beta(\lambda)[d(x), \beta(y)] + \beta(\lambda)[d(y), \beta(x)] = (0) \tag{2.2.7}$$

for all $x, y \in \lambda$. Replacing $y$ by $xy$, we get

$$\beta(\lambda)[d(x), \beta(x)\beta(y)] + \beta(\lambda)[d(x)\alpha(y) + \beta(x)d(y), \beta(x)] = (0) \tag{2.2.8}$$

that is

$$\begin{aligned}\beta(\lambda)\beta(x)[d(x), \beta(y)] + \beta(\lambda)[d(x), \beta(x)]\beta(y) + \beta(\lambda)d(x)[\alpha(y), \beta(x)]\\ +\beta(\lambda)[d(x), \beta(x)]\alpha(y) + \beta(\lambda)\beta(x)[d(y), \beta(x)] = (0)\end{aligned} \tag{2.2.9}$$

for all $x, y \in \lambda$. By using (2.2.6) and (2.2.7), above relation yields

$$\beta(\lambda)d(x)[\alpha(y), \beta(x)] = (0) \tag{2.2.10}$$

for all $x, y \in \lambda$. Replacing $y$ by $yt$, $t \in \lambda$, we get

$$\beta(\lambda)d(x)\alpha(y)[\alpha(t), \beta(x)] = (0) \tag{2.2.11}$$

for all $x, y, t \in \lambda$. We replace $y$ with $ry$, for $r \in R$, and then obtain

$$\beta(\lambda)d(x)R\alpha(y)[\alpha(t), \beta(x)] = (0) \tag{2.2.12}$$

for all $x, y, t \in \lambda$. Since $R$ is prime, for each $x \in \lambda$, either $\beta(\lambda)d(x) = (0)$ or $\alpha(\lambda)[\alpha(\lambda), \beta(x)] = (0)$. Let $T_1 = \{x \in \lambda | \beta(\lambda)d(x) = (0)\}$ and

$$T_2 = \{x \in \lambda | \alpha(\lambda)[\alpha(\lambda), \beta(x)] = (0)\}.$$

Then $T_1$ and $T_2$ are two additive subgroups of $\lambda$ such that $T_1 \cup T_2 = \lambda$. Since a group cannot be union of its two proper subgroups, either $T_1 = \lambda$ or $T_2 = \lambda$. Hence $\beta(\lambda)d(\lambda) = (0)$ or $\alpha(\lambda)[\alpha(\lambda), \beta(\lambda)] = (0)$. □

## 2.3 Main Results

**Theorem 2.3.1.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(x)F(y) + \alpha(xy) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)[d(x), \beta(x)] = (0)$ and $\beta(\lambda)[g(x), \beta(x)] = (0)$ for all $x \in \lambda$. If $\lambda = R$ then $d$ and $g$ map from $R$ to $Z(R)$.*

*Proof.* By the given condition

$$G(xy) + d(x)F(y) + \alpha(xy) = 0 \tag{2.3.1}$$

for all $x, y \in \lambda$. Substituting $y = yx$ we get

$$G(xyx) + d(x)F(yx) + \alpha(xyx) = 0 \tag{2.3.2}$$

that is

$$G(xy)\alpha(x) + \beta(xy)g(x) + d(x)(F(y)\alpha(x) + \beta(y)d(x)) + \alpha(xy)\alpha(x) = 0 \tag{2.3.3}$$

for all $x, y \in \lambda$. Using (2.3.1) we get

$$\beta(xy)g(x) + d(x)\beta(y)d(x) = 0 \tag{2.3.4}$$

for all $x, y \in \lambda$. Putting $y = xy$ in (2.3.4) we have

$$\beta(x)\beta(xy)g(x) + d(x)\beta(x)\beta(y)d(x) = 0 \tag{2.3.5}$$

which reduces by using (2.3.4) to $[d(x), \beta(x)]\beta(y)d(x) = 0$ for all $x, y \in \lambda$. This relation implies that $[d(x), \beta(x)]\beta(y)[d(x), \beta(x)] = 0$ and so,
$\beta(y)[d(x), \beta(x)]R\beta(y)[d(x), \beta(x)] = (0)$ for all $x, y \in \lambda$. Since $R$ is semiprime, $\beta(y)[d(x), \beta(x)] = 0$ for all $x, y \in \lambda$, as desired.

Now putting $y = yx$ in (2.3.4) we get

$$\beta(xy)\beta(x)g(x) + d(x)\beta(y)\beta(x)d(x) = 0. \tag{2.3.6}$$

Now right multiplying (2.3.4) by $\beta(x)$ and then substracting from (2.3.6) we get

$$\beta(xy)[g(x), \beta(x)] + d(x)\beta(y)[d(x), \beta(x)] = 0$$

for all $x, y \in \lambda$. Since $\beta(y)[d(x), \beta(x)] = 0$, we get $\beta(xy)[\beta(x), g(x)] = 0$ for all $x, y \in \lambda$. Since $\lambda$ is a left-sided ideal of $R$, $\beta(x)R\beta(y)[\beta(x), g(x)] = (0)$ for all $x, y \in \lambda$. This relation yields $[\beta(x), g(x)]R\beta(y)[\beta(x), g(x)] = (0)$ for all $x, y \in \lambda$. Due to semiprimeness of $R$, $\beta(\lambda)[g(x), \beta(x)] = (0)$ for all $x \in \lambda$.

When $\lambda = R$, then $[d(x), \beta(x)] = 0$ and $[g(x), \beta(x)] = 0$ for all $x \in R$. Then by Lemma 2.2.1, $d$ and $g$ map from $R$ to $Z(R)$. □

Since $G$ is a generalized $(\alpha, \beta)$-derivation on $R$ associated to $(\alpha, \beta)$ derivation $g$ of $R$, $G - \alpha$ is also a generalized $(\alpha, \beta)$-derivation on $R$ associated to $(\alpha, \beta)$ derivation $g$ of $R$. Thus replacing $G$ with $G - \alpha$ in Theorem 2.3.1, the following Theorem is straightforward.

**Theorem 2.3.2.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(x)F(y) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)[d(x), \beta(x)] = (0)$ and $\beta(\lambda)[g(x), \beta(x)] = (0)$ for all $x \in \lambda$. If $\lambda = R$ then $d$ and $g$ map from $R$ to $Z(R)$.*

**Corollary 2.3.3.** *Let $R$ be a prime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(x)F(y) + \alpha(xy) = 0$$

*for all $x, y \in \lambda$, then either $\beta(\lambda)d(\lambda) = (0)$ and $\beta(\lambda)g(\lambda) = (0)$ or $\alpha(\lambda)[\alpha(\lambda), \beta(\lambda)] = (0)$.*

*If $\lambda = R$ then $d = g = 0$ and $G = -\alpha$.*

*Proof.* By Theorem 2.3.1, we have $\beta(\lambda)[d(x), \beta(x)] = (0)$ and $\beta(\lambda)[g(x), \beta(x)] = (0)$ for all $x \in \lambda$. By applying Lemma 2.2.2, either $\alpha(\lambda)[\alpha(\lambda), \beta(\lambda)] = (0)$ or $\beta(\lambda)d(\lambda) = (0)$ and $\beta(\lambda)g(\lambda) = (0)$.

When $\lambda = R$, then the conclusions give either $R$ is commutative or $d = g = 0$. Thus we consider the following two cases:

**Case-i:** *When $d = g = 0$.*

Then our hypothesis $G(xy) + d(x)F(y) + \alpha(xy) = 0$ reduces to $(G(x) + \alpha(x))\alpha(y) = 0$ for all $x, y \in R$. This implies $G(x) + \alpha(x) = 0$ for all $x \in R$.

**Case-ii:** *When* $R$ *is commutative.*

Then substituting $y = yt$ in $G(xy) + d(x)F(y) + \alpha(xy) = 0$ we have

$$G(xy)\alpha(t) + \beta(xy)g(t) + d(x)(F(y)\alpha(t) + \beta(y)d(t)) + \alpha(xy)\alpha(t) = 0 \quad (2.3.7)$$

for all $x, y, t \in R$. By our hypothesis, it reduces to $\beta(y)(\beta(x)g(t) + d(x)d(t)) = 0$ for all $x, y, t \in R$. By primeness of $R$, $\beta(x)g(t) + d(x)d(t) = 0$ for all $x, t \in R$. We replace $x$ with $xs$ and then obtain $0 = \beta(s)(\beta(x)g(t) + d(x)d(t)) + d(s)d(t)\alpha(x) = d(s)d(t)\alpha(x)$ for all $x, s, t \in R$. This implies $d = 0$.

Then our hypothesis becomes $G(xy) + \alpha(xy) = 0$ for all $x, y \in R$. Replacing $y$ with $yz$, it yields $\beta(xy)g(z) = 0$ for all $x, y, z \in R$. This implies $g = 0$ and hence $(G(x) + \alpha(x))\alpha(y) = 0$ for all $x, y \in R$. This implies $G(x) + \alpha(x) = 0$ for all $x \in R$. □

**Theorem 2.3.4.** *Let* $R$ *be a semiprime ring,* $\lambda$ *be a left-sided ideal of* $R$*,* $F$ *and* $G$ *be two generalized* $(\alpha, \beta)$*-derivations on* $R$ *with associated* $(\alpha, \beta)$ *derivations* $d$ *and* $g$*, respectively. If*

$$G(xy) + d(x)F(y) + \alpha(yx) = 0$$

*for all* $x, y \in \lambda$*, then* $\beta(\lambda)[d(x), \beta(x)] = (0)$ *and* $\beta(\lambda)[g(x), \beta(x)] = (0)$ *for all* $x \in \lambda$*. If* $\lambda = R$ *then* $d$ *and* $g$ *map from* $R$ *to* $Z(R)$*.*

*Proof.* By the given condition

$$G(xy) + d(x)F(y) + \alpha(yx) = 0 \quad (2.3.8)$$

for all $x, y \in \lambda$. Substituting $y = yx$ we get

$$G(xyx) + d(x)F(yx) + \alpha(yx)\alpha(x) = 0 \quad (2.3.9)$$

i.e.,

$$G(xy)\alpha(x) + \beta(xy)g(x) + d(x)(F(y)\alpha(x) + \beta(y)d(x)) + \alpha(yx)\alpha(x) = 0$$

for all $x, y \in \lambda$. Using (2.3.8) we get

$$\beta(xy)g(x) + d(x)\beta(y)d(x) = 0 \quad (2.3.10)$$

for all $x, y \in \lambda$, which is same as relation (2.3.4) in Theorem 2.3.1. Hence by same argument, we have our conclusions. □

Since $G$ is a generalized $(\alpha, \beta)$-derivation on $R$ associated to $(\alpha, \beta)$ derivation $g$ of $R$, $G+\alpha$ and $G-\alpha$ are also generalized $(\alpha, \beta)$-derivations on $R$ associated to $(\alpha, \beta)$ derivation $g$ of $R$. Thus replacing $G$ with $G+\alpha$ and $G-\alpha$ in Theorem 2.3.4 respectively, the following Theorems are straightforward.

**Theorem 2.3.5.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(x)F(y) + \alpha(yx) + \alpha(xy) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)[d(x), \beta(x)] = (0)$ and $\beta(\lambda)[g(x), \beta(x)] = (0)$ for all $x \in \lambda$. If $\lambda = R$ then $d$ and $g$ map from $R$ to $Z(R)$.*

**Theorem 2.3.6.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(x)F(y) + \alpha(yx) - \alpha(xy) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)[d(x), \beta(x)] = (0)$ and $\beta(\lambda)[g(x), \beta(x)] = (0)$ for all $x \in \lambda$. If $\lambda = R$ then $d$ and $g$ map from $R$ to $Z(R)$.*

**Corollary 2.3.7.** *Let $R$ be a prime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(x)F(y) + \alpha(yx) = 0$$

*for all $x, y \in \lambda$, then either $\beta(\lambda)d(\lambda) = (0)$ and $\beta(\lambda)g(\lambda) = (0)$ or $\alpha(\lambda)[\alpha(\lambda), \beta(\lambda)] = (0)$.*

*If $\lambda = R$ then $R$ is commutative, $d = g = 0$ and $G = -\alpha$.*

*Proof.* By Theorem 2.3.4, we have $\beta(\lambda)[d(x), \beta(x)] = (0)$ and $\beta(\lambda)[g(x), \beta(x)] = (0)$ for all $x \in \lambda$. By applying Lemma 2.2.2, either $\alpha(\lambda)[\alpha(\lambda), \beta(\lambda)] = (0)$ or $\beta(\lambda)d(\lambda) = (0)$ and $\beta(\lambda)g(\lambda) = (0)$.

When $\lambda = R$, then the conclusions give either $R$ is commutative or $d = g = 0$. If $d = g = 0$, then our hypothesis $G(xy) + d(x)F(y) + \alpha(yx) = 0$ reduces to $G(x)\alpha(y) + \alpha(yx) = 0$ for all $x, y \in R$. Replacing $y$ by $yt$, we have $G(x)\alpha(y)\alpha(t) + \alpha(ytx) = 0$ for all $x, y, t \in R$. This relation implies by using $G(x)\alpha(y) + \alpha(yx) = 0$

that $\alpha(y[x,t]) = 0$ for all $x, y, t \in R$. Therefore, $R$ must be commutative. Hence by Corollary 2.3.3, conclusion follows. □

**Theorem 2.3.8.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(y)F(x) + \alpha(yx) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)[d(\lambda), \beta(\lambda)] = (0)$ and $\beta(\lambda)[\beta(y)g(y), \beta(\lambda)] = (0)$ for all $y \in \lambda$.*

*If $\lambda = R$ then $d$ maps from $R$ to $Z(R)$ and $\beta(x)g(x) \in Z(R)$ for all $x \in R$.*

*Proof.* By the given condition

$$G(xy) + d(y)F(x) + \alpha(yx) = 0 \tag{2.3.11}$$

for all $x, y \in \lambda$. Substituting $x = xy$ we get

$$G(xy)\alpha(y) + \beta(xy)g(y) + d(y)(F(x)\alpha(y) + \beta(x)d(y)) + \alpha(yx)\alpha(y) \tag{2.3.12}$$

which gives by using (2.3.11)

$$\beta(xy)g(y) + d(y)\beta(x)d(y) = 0 \tag{2.3.13}$$

for all $x, y \in \lambda$. Substituting $x = ux$ we get

$$\beta(uxy)g(y) + d(y)\beta(ux)d(y) = 0 \tag{2.3.14}$$

for all $x, y, u \in \lambda$. Left multiplying (2.3.13) by $\beta(u)$ and then subtracting from (2.3.14), we obtain $[d(y), \beta(u)]\beta(x)d(y) = 0$ for all $x, y, u \in \lambda$. Replacing $x$ with $rx$, where $r \in R$, we get $[d(y), \beta(u)]R\beta(x)d(y) = (0)$ for all $x, y, u \in \lambda$. This relation implies $[d(y), \beta(u)]R\beta(x)[d(y), \beta(u)] = (0)$ for all $x, y, u \in \lambda$. Since $R$ is semiprime, $\beta(x)[d(y), \beta(u)] = 0$ for all $x, y, u \in \lambda$.

Replacing $x$ with $xz$ in (2.3.13), we get

$$\beta(xzy)g(y) + d(y)\beta(xz)d(y) = 0 \tag{2.3.15}$$

for all $x, y, z \in \lambda$. Right multiplying (2.3.13) by $\beta(z)$ we get

$$\beta(xy)g(y)\beta(z) + d(y)\beta(x)d(y)\beta(z) = 0 \tag{2.3.16}$$

for all $x, y \in \lambda$. Subtracting (2.3.16) from (2.3.15) and then using the fact $\beta(x)[d(y), \beta(u)] = 0$ for all $x, y, u \in \lambda$, we get $\beta(x)[\beta(y)g(y), \beta(z)] = 0$ that is $\beta(\lambda)[\beta(y)g(y), \beta(\lambda)] = (0)$ for all $y \in \lambda$.

In particular, when $\lambda = R$, then $[d(R), \beta(R)] = (0)$ and $[\beta(x)g(x), \beta(R)] = (0)$ for all $x \in R$. The first case gives $d$ maps from $R$ to $Z(R)$. The second case gives $[\beta(x)g(x), R] = (0)$ for all $x \in R$, i.e., $\beta(x)g(x) \in Z(R)$ for all $x \in R$. □

Thus replacing $G$ with $G + \alpha$ and $G - \alpha$ in Theorem 2.3.8, respectively, the following Theorems are straightforward.

**Theorem 2.3.9.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(y)F(x) + \alpha(yx) + \alpha(xy) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)[d(\lambda), \beta(\lambda)] = (0)$ and $\beta(\lambda)[\beta(y)g(y), \beta(\lambda)] = (0)$ for all $y \in \lambda$.*

*If $\lambda = R$ then $d$ maps from $R$ to $Z(R)$ and $\beta(x)g(x) \in Z(R)$ for all $x \in R$.*

**Theorem 2.3.10.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(y)F(x) + \alpha(yx) - \alpha(xy) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)[d(\lambda), \beta(\lambda)] = (0)$ and $\beta(\lambda)[\beta(y)g(y), \beta(\lambda)] = (0)$ for all $y \in \lambda$.*

*If $\lambda = R$ then $d$ maps from $R$ to $Z(R)$ and $\beta(x)g(x) \in Z(R)$ for all $x \in R$.*

**Corollary 2.3.11.** *Let $R$ be a prime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + d(y)F(x) + \alpha(yx) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)[\beta(y)g(y), \beta(\lambda)] = (0)$ for all $y \in \lambda$ and either $\alpha(\lambda)[\alpha(\lambda), \beta(\lambda)] = (0)$ or $\beta(\lambda)d(\lambda) = (0)$.*

*If $\lambda = R$ then $R$ is commutative, $d = g = 0$ and $G = -\alpha$.*

*Proof.* By Theorem 2.3.8, we have $\beta(\lambda)[d(x),\beta(x)]=(0)$ and $\beta(\lambda)[\beta(x)g(x),\beta(\lambda)]=(0)$ for all $x\in\lambda$. By applying Lemma 2.2.2, either $\alpha(\lambda)[\alpha(\lambda),\beta(\lambda)]=(0)$ or $\beta(\lambda)d(\lambda)=(0)$.

When $\lambda=R$, then the conclusions give either $R$ is commutative or $d=0$. If $R$ is commutative, then by Corollary 2.3.3, $d=g=0$ and $G=-\alpha$.

If $d=0$, then our hypothesis $G(xy)+d(y)F(x)+\alpha(yx)=0$ reduces to $G(x)\alpha(y)+\beta(x)g(y)+\alpha(yx)=0$ for all $x,y\in R$. Replacing $y$ by $yt$, we have $G(x)\alpha(y)\alpha(t)+\beta(x)\{g(y)\alpha(t)+\beta(y)g(t)\}+\alpha(ytx)=0$ for all $x,y,t\in R$. This relation implies by using $G(x)\alpha(y)+\beta(x)g(y)+\alpha(yx)=0$ that $\beta(x)\beta(y)g(t)+\alpha(y[t,x])=0$ for all $x,y,t\in R$. Assuming $x=t$, we have $\beta(x)Rg(x)=(0)$ for all $x\in R$. Since $R$ is prime ring, for each $x\in R$, either $\beta(x)=0$ or $g(x)=0$. Since the set $\{x\in R:\beta(x)=0\}$ and the set $\{x\in R:g(x)=0\}$ both form two additive subgroups of $R$ whose union is $R$, thus we conclude as earlier argument that $\beta(R)=(0)$ or $g(R)=(0)$. Since for any automorphism $\beta$ of $R$, $\beta(R)\neq(0)$, we must have $g=0$. Thus our hypothesis $G(xy)+d(y)F(x)+\alpha(yx)=0$ reduces to $G(x)\alpha(y)+\alpha(yx)=0$ for all $x,y\in R$. Replacing $y$ by $yt$, we have $G(x)\alpha(y)\alpha(t)+\alpha(ytx)=0$ for all $x,y,t\in R$. This relation implies by using $G(x)\alpha(y)+\alpha(yx)=0$ that $\alpha(y[x,t])=0$ for all $x,y,t\in R$. Therefore, $R$ must be commutative. Hence by Corollary 2.3.3, conclusion follows. □

**Theorem 2.3.12.** *Let $R$ be a prime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha,\beta)$-derivations on $R$ with associated $(\alpha,\beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy)+F(x)F(y)=0$$

*for all $x,y\in\lambda$, then $F$ is $\beta$-commuting on $\lambda$ or $\beta(\lambda)d(\lambda)=(0)$ and $\beta(\lambda)g(\lambda)=(0)$.*

*Proof.* By the given condition

$$G(xy)+F(x)F(y)=0 \tag{2.3.17}$$

for all $x,y\in\lambda$. Now substituting $y=yz$ we get

$$G(xy)\alpha(z)+\beta(xy)g(z)+F(x)F(y)\alpha(z)+F(x)\beta(y)d(z)=0 \tag{2.3.18}$$

for all $x,y,z\in\lambda$. Using (2.3.17) we get

$$\beta(xy)g(z)+F(x)\beta(y)d(z)=0 \tag{2.3.19}$$

for all $x, y, z \in \lambda$. Putting $y = xy$ in (2.3.19), we get

$$\beta(x)\beta(xy)g(z) + F(x)\beta(xy)d(z) = 0 \tag{2.3.20}$$

for all $x, y, z \in \lambda$. Now pre-multiplying (2.3.19) by $\beta(x)$ and then subtracting from (2.3.20) we get

$$[F(x), \beta(x)]\beta(y)d(z) = 0 \tag{2.3.21}$$

for all $x, y, z \in \lambda$. Replacing $y$ with $ry$, for $r \in R$, we have

$$[F(x), \beta(x)]R\beta(y)d(z) = (0) \tag{2.3.22}$$

for all $x, y, z \in \lambda$. Since $R$ is prime, either $[F(x), \beta(x)] = 0$ for all $x \in \lambda$ that is $F$ is $\beta$-commuting on $\lambda$ or $\beta(\lambda)d(\lambda) = (0)$. When $\beta(\lambda)d(\lambda) = (0)$, by (2.3.20), $\beta(x)\beta(xy)g(z) = 0$ for all $x, y \in \lambda$. This implies that $\beta(\lambda)g(\lambda) = (0)$. □

**Theorem 2.3.13.** *Let $R$ be a prime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$G(xy) + F(y)F(x) = 0$$

*for all $x, y \in \lambda$, then one of the following holds: (i) $[F(\lambda), \beta(\lambda)] = (0)$; (ii) $\beta(\lambda)d(\lambda) = (0)$ and $\beta(\lambda)g(\lambda) = (0)$; (iii) $\beta(\lambda)d(\lambda) = (0)$ and $R$ is commutative.*

*Proof.* By the given condition

$$G(xy) + F(y)F(x) = 0 \tag{2.3.23}$$

for all $x, y \in \lambda$. Now substituting $x = xy$ we get

$$G(xy)\alpha(y) + \beta(xy)g(y) + F(y)F(x)\alpha(y) + F(y)\beta(x)d(y) = 0 \tag{2.3.24}$$

for all $x, y \in \lambda$. Using (2.3.23) we get

$$\beta(xy)g(y) + F(y)\beta(x)d(y) = 0 \tag{2.3.25}$$

for all $x, y \in \lambda$. Putting $x = zx$ in (2.3.25) we get

$$\beta(z)\beta(xy)g(y) + F(y)\beta(zy)d(y) = 0 \tag{2.3.26}$$

for all $x, y, z \in \lambda$. Now pre-multiplying (2.3.25) by $\beta(z)$ and then subtracting from (2.3.26) we get

$$[F(y), \beta(z)]\beta(x)d(y) = 0 \tag{2.3.27}$$

for all $x, y, z \in \lambda$. Replacing $x$ with $rx$, for $r \in R$, we have $[F(y), \beta(z)]R\beta(x)d(y) = (0)$ for all $x, y, z \in \lambda$. Primeness of $R$ implies that for each $y \in \lambda$ either $[F(y), \beta(\lambda)] = (0)$ or $\beta(\lambda)d(y) = (0)$. We consider two additive subgroups $T_1 = \{y \in \lambda | [F(y), \beta(\lambda)] = (0)\}$ and $T_2 = \{y \in \lambda | \beta(\lambda)d(y) = (0)\}$. Then $T_1 \cup T_2 = \lambda$. Since a group can not be union of its two proper subgroups, either $T_1 = \lambda$ or $T_2 = \lambda$. Thus either $[F(\lambda), \beta(\lambda)] = (0)$ or $\beta(\lambda)d(\lambda) = (0)$. If $\beta(\lambda)d(\lambda) = (0)$, then by (2.3.25), $\beta(y)g(y) = 0$ for all $y \in \lambda$. Linearizing it yields

$$\beta(x)g(y) + \beta(y)g(x) = 0 \tag{2.3.28}$$

for all $x, y \in \lambda$. Putting $x = tx$ we get

$$\beta(tx)g(y) + \beta(y)g(t)\alpha(x) + \beta(yt)g(x) = 0 \tag{2.3.29}$$

for all $x, y \in \lambda$. Left multiplying (2.3.28) by $\beta(t)$ and then subtracting from (2.3.29) we get

$$[\beta(y), \beta(t)]g(x) + \beta(y)g(t)\alpha(x) = 0 \tag{2.3.30}$$

for all $x, y, t \in \lambda$. Substituting $x$ with $xz$ in (2.3.30) and then using it, we obtain that $[\beta(y), \beta(t)]\beta(x)g(z) = 0$ which gives $[\beta(y), \beta(t)]R\beta(x)g(z) = (0)$ for all $x, y, t, z \in \lambda$. By primeness of $R$, either $[\lambda, \lambda] = (0)$ or $\beta(\lambda)g(\lambda) = (0)$. Note that $[\lambda, \lambda] = (0)$ implies $R$ must be commutative. Hence we have our conclusions. □

**Theorem 2.3.14.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$F(xy) + G(x)\alpha(y) + \alpha(yx) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)d(\lambda) = (0)$, $\alpha(\lambda)\beta(\lambda)g(\lambda) = (0)$ and $\lambda[\lambda, \lambda] = (0)$.*

*In particular, for $\lambda = R$, $R$ must be commutative and $d = g = 0$, $\alpha(y)(F(x) + G(x) + \alpha(x)) = 0$ for all $x, y \in \lambda$.*

*Proof.* By hypothesis

$$F(xy) + G(x)\alpha(y) + \alpha(yx) = 0 \tag{2.3.31}$$

for all $x, y \in \lambda$. Putting $y = yx$, we have

$$F(xy)\alpha(x) + \beta(xy)d(x) + G(x)\alpha(yx) + \alpha(yx^2) = 0 \tag{2.3.32}$$

for all $x, y \in \lambda$. Right multiplying (2.3.31) by $\alpha(x)$ and then subtracting from (2.3.32), we obtain

$$\beta(xy)d(x) = 0 \tag{2.3.33}$$

for all $x, y \in \lambda$. Replacing $y$ with $ry$, $r \in R$, we get $\beta(x)R\beta(y)d(x) = (0)$ for all $x, y \in \lambda$.

Let $\Omega = \{P_\alpha | \alpha \in I\}$ be a family of prime ideals of $R$ such that $\bigcap P_\alpha = (0)$.

If $P$ is typical member of $\Omega$, then for each $x \in \lambda$, either $\beta(x) \in P$ or $\beta(\lambda)d(x) \subseteq P$. For fixed $P$, the sets $T_1 = \{x \in \lambda : \beta(x) \in P\}$ and $T_2 = \{x \in \lambda : \beta(\lambda)d(x) \subseteq P\}$ form two additive subgroups of $\lambda$ such that $T_1 \cup T_2 = \lambda$. Therefore, either $T_1 = \lambda$ or $T_2 = \lambda$, that is, either $\beta(\lambda) \subseteq P$ or $\beta(\lambda)d(\lambda) \subseteq P$. Both of these two conditions together imply that $\beta(\lambda)d(\lambda) \subseteq P$ for any $P \in \Omega$. Since $\bigcap_{\alpha\in\Lambda} P_\alpha = (0)$, we have $\beta(\lambda)d(\lambda) = (0)$.

Thus replacing $y$ with $yz$ in (2.3.31), we get

$$F(xy)\alpha(z) + G(x)\alpha(yz) + \alpha(yzx) = 0 \tag{2.3.34}$$

for all $x, y \in \lambda$. Right multiplying (2.3.31) by $\alpha(z)$ and then subtracting from (2.3.34) we get $\alpha(y[z, x]) = 0$ for all $x, y, z \in \lambda$, that is, $\lambda[\lambda, \lambda] = (0)$.

Moreover, replacing $x = xz$ and $y = zy$ in (2.3.31) respectively and then subtracting one from another implies that $\beta(x)g(z)\alpha(y) + \alpha([y, z]x) = 0$ for all $x, y, z \in \lambda$. Then left multiplying by $\alpha(t)$ and using $\lambda[\lambda, \lambda] = (0)$, the relation gives $\alpha(t)\beta(x)g(z)\alpha(y) = 0$ for all $x, y, t, z \in \lambda$. Replace $y$ with $rt$, $r \in R$, yields $\alpha(t)\beta(x)g(z)R\alpha(t) = (0)$ and hence $\alpha(t)\beta(x)g(z)R\alpha(t)\beta(x)g(z) = (0)$ for all $x, t, z \in \lambda$. Since $R$ is semiprime, $\alpha(\lambda)\beta(\lambda)g(\lambda) = (0)$.

In particular, for $\lambda = R$, $R$ must be commutative and $d = g = 0$ and hence by (2.3.31), $(F(x) + G(x) + \alpha(x))\alpha(y) = 0$ for all $x, y \in \lambda$. Thus $\alpha(y)(F(x) + G(x) + \alpha(x))R\alpha(y)(F(x)+G(x)+\alpha(x)) = (0)$ for all $x, y \in \lambda$ and hence $\alpha(y)(F(x)+G(x)+\alpha(x)) = 0$ for all $x, y \in \lambda$. □

**Corollary 2.3.15.** *Let $R$ be a prime ring, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$F(xy) + G(x)\alpha(y) + \alpha(yx) = 0$$

*for all $x, y \in R$, then $R$ must be commutative and $d = g = 0$, $F + G = -\alpha$.*

**Theorem 2.3.16.** *Let $R$ be a semiprime ring, $\lambda$ be a left-sided ideal of $R$, $F$ and $G$ be two generalized $(\alpha, \beta)$-derivations on $R$ with associated $(\alpha, \beta)$ derivations $d$ and $g$, respectively. If*

$$F(x)F(y) + G(x)\alpha(y) + \alpha(yx) = 0$$

*for all $x, y \in \lambda$, then $\beta(\lambda)d(\lambda) = (0)$ and $\lambda[\lambda, \lambda] = (0)$.*

*In particular, for $\lambda = R$, $R$ must be commutative and $d = g = 0$.*

*Proof.* By hypothesis

$$F(x)F(y) + G(x)\alpha(y) + \alpha(yx) = 0 \tag{2.3.35}$$

for all $x, y \in \lambda$. Putting $y = yx$, we have

$$F(x)\{F(y)\alpha(x) + \beta(y)d(x)\} + G(x)\alpha(yx) + \alpha(yx^2) = 0 \tag{2.3.36}$$

for all $x, y \in \lambda$. Right multiplying (2.3.35) by $\alpha(x)$ and then subtracting from (2.3.36), we obtain

$$F(x)\beta(y)d(x) = 0 \tag{2.3.37}$$

for all $x, y \in \lambda$. Replacing $y$ with $ry$, $r \in R$, we get $F(x)R\beta(y)d(x) = (0)$ for all $x, y \in \lambda$.

Let $\Omega = \{P_\alpha | \alpha \in I\}$ be a family of prime ideals of $R$ such that $\bigcap P_\alpha = (0)$.

If $P$ is typical member of $\Omega$, then for each $x \in \lambda$, either $F(x) \in P$ or $\beta(\lambda)d(x) \subseteq P$. For fixed $P$, the sets $T_1 = \{x \in \lambda : F(x) \in P\}$ and $T_2 = \{x \in \lambda : \beta(\lambda)d(x) \subseteq P\}$ form two additive subgroups of $\lambda$ such that $T_1 \cup T_2 = \lambda$. Therefore, either $T_1 = \lambda$ or $T_2 = \lambda$, that is, either $F(\lambda) \subseteq P$ or $\beta(\lambda)d(\lambda) \subseteq P$. Now since for $x, y \in \lambda$, $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$, therefore $F(\lambda) \subseteq P$ implies $\beta(\lambda)d(\lambda) \subseteq P$. Thus we conclude that in any case $\beta(\lambda)d(\lambda) \subseteq P$ for any $P \in \Omega$. Since $\bigcap_{\alpha \in \Lambda} P_\alpha = (0)$, we have $\beta(\lambda)d(\lambda) = (0)$.

Thus replacing $y$ with $yz$ in (2.3.35), we get

$$(F(x)F(y)+G(x)\alpha(y)+\alpha(yx))\alpha(z)+\alpha(y[z,x])=0 \tag{2.3.38}$$

for all $x,y,z\in\lambda$. By (2.3.35), above relation gives $\lambda[\lambda,\lambda]=(0)$.

In particular for $\lambda=R$, $R$ must be commutative and $d=0$. Then putting $x=xz$ in (2.3.35) we have $(F(x)F(y)+G(x)\alpha(y)+\alpha(yx))\alpha(z)+\beta(x)g(z)\alpha(y)=0$ for all $x,y,z\in R$, implying $\beta(x)g(z)\alpha(y)=0$ for all $x,y,z\in R$. This again implies that $g=0$. □

## 2.4 Examples

We end this chapter with four examples, which show that hypothesis of primeness or semiprimeness in our theorems is not superfluous.

**Example 1:**

Consider the ring $R=\left\{\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}:a,b,c\in\mathbb{Z}\right\}$ where $\mathbb{Z}$ is the set of all integers. Since $\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}\begin{pmatrix}0&1&1\\0&0&0\\0&0&0\end{pmatrix}=0$, $R$ is not a prime or semiprime ring.

Define automorphisms $\alpha,\beta:R\to R$ such that $\alpha\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}$ and $\beta\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&-a&-b\\0&0&c\\0&0&0\end{pmatrix}$. Now define the maps $F$, $G$ and $d$, $g$ on $R$ by $F\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=G\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&-b\\0&0&0\\0&0&0\end{pmatrix}$ and $d\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=g\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&a&0\\0&0&c\\0&0&0\end{pmatrix}$. Then clearly $F$ and $G$ are generalized $(\alpha,\beta)$-derivations associated with $(\alpha, \beta)$-derivations $d$ and $g$ on $R$ respectively.

Now we can see that $G(xy)+d(x)F(y)+\alpha(xy)=0$ for all $x, y \in R$. Since $d, g$ do not map from $R$ to $Z(R)$ and $G \neq -\alpha$, the semiprimeness or primeness hypothesis in Theorem 2.3.1 and Corollary 2.3.3 is not superfluous.

**Example 2:**

Consider the ring $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ where $\mathbb{Z}$ is the set of all integers. Then as above $R$ is not semiprime ring.

Define automorphisms $\alpha, \beta : R \to R$ such that $\alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Now define the maps $F$, $G$ and $d$, $g$ on $R$ by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = G \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $g \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then clearly $F$, $G$ are two generalized $(\alpha, \beta)$-derivations associated with $(\alpha, \beta)$-derivations $d$, $g$ respectively on $R$. Now we can see that $G(xy) + d(x)F(y) = 0$ for all $x, y \in R$. Since $g$ does not map from $R$ to $Z(R)$, the semiprimeness hypothesis in Theorem 2.3.2 is not superfluous.

**Example 3:**

Consider the ring $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$ where $\mathbb{Z}$ is the set of all integers. Then as above $R$ is not a prime or semiprime ring.

Define automorphisms $\alpha, \beta : R \to R$ such that $\alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} =$

$\begin{pmatrix} 0 & -a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Now define the maps $F$, $G$ and $d$, $g$ on $R$ by $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} =$ $G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$ and $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = g\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then clearly $F$, $G$ are generalized $(\alpha,\beta)$-derivations associated with $(\alpha,\beta)$-derivations $d$, $g$, respectively on $R$. Now we can see that (i) $G(xy)+d(y)F(x)+\alpha(yx)=0$, (ii) $G(xy)+F(x)F(y)=0$ and (iii) $G(xy)+F(y)F(x)=0$ for all $x,y\in R$. Since $d$ and $g$ do not map from $R$ to $Z(R)$, $R$ is not commutative and $F$ is not $\beta$-commuting on $R$, the semiprimeness or primeness hypothesis in Theorem 2.3.8, Corollary 2.3.11, Theorem 2.3.12 and Theorem 2.3.13 can not be omitted.

**Example 4:**

Consider the ring $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a,b,c \in \mathbb{Z} \right\}$ where $\mathbb{Z}$ is the set of all integers. Then $R$ is not a semiprime ring.

Define automorphisms $\alpha,\beta : R \to R$ such that $\alpha\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $\beta\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$. Now define the maps $F$, $G$ and $d$, $g$ on $R$ by $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$, $G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} =$ $\begin{pmatrix} 0 & -a & -b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $g\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. Then clearly $F$, $G$ are generalized $(\alpha,\beta)$-derivations associated with $(\alpha,\beta)$-derivations $d$, $g$ respectively on $R$. Now we can see that $G(xy)+d(y)F(x)+\alpha(yx)+\alpha(xy)=0$ for all $x,y\in R$. Since $d$ does not map from $R$ to $Z(R)$, the semiprimeness hypothesis in Theorem 2.3.9 is essential.

# Chapter 3

# $X$-generalized Skew Derivations with Annihilating and Centralizing Conditions in Prime Rings

## 3.1 Introduction

Throughout this chapter $R$ denotes an associative prime ring with $char(R) \neq 2$, center $Z(R)$, extended centroid $C$ and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is noncentral-valued on $R$ and its right Martindale quotient ring $Q_r$. In [66], Kosan and Lee introduced the notion of $b$-generalized derivations. We already know that, if $F, d : R \to Q_r$ are additive maps and $b \in Q_r$ such that $F(xy) = F(x)y + bxd(y)$ for all $x, y \in R$, then $F$ is said to be $b$-generalized derivation of $R$ with associated map $d$. For some $a, b, c \in Q_r(R)$, the map $x \mapsto ax+bxc$ is an example of $b$-generalized derivation; which is called inner $b$-generalized derivation of $R$. Kosan and Lee [66] proved that if $R$ is a prime ring and $b \neq 0$, then the associated map $d$ must be a derivation of $R$. Recently, few papers studied $b$-generalized derivations (viz. [27], [73], [77]).

In [45], De Filippis introduced the new map $X-$generalized skew derivation. Let $b \in Q_r$, $d : R \to R$ an additive mapping and $\alpha$ be an automorphism of $R$. An additive mapping $F : R \to R$ is called an $X-$generalized skew derivation of $R$, with associated term $(b, \alpha, d)$ if $F(xy) = F(x)y + b\alpha(x)d(y)$ for all $x, y \in R$. In [55, Lemma 3.2], De Filippis and Wei proved that if $F : R \to R$ is an $X-$generalized

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skew derivation of a prime ring $R$ with associated term $(b, \alpha, d)$, then $d$ must be a skew derivation of $R$ with associated automorphism $\alpha$.

There are few papers which recently introduced and studied the $X$-generalized skew derivations (viz. [53], [54], [55]). In the present chapter our motivation is to study $X$-generalized skew derivation in prime rings.

Argaç and De Filippis [5, Theorem 2] proved a result for generalized derivation as follows:

*Let $K$ be a commutative ring with unity, $R$ be a noncommutative prime $K$-algebra with center $Z(R)$, $U$ be the Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$, $I$ a nonzero ideal of $R$. Suppose that $f(x_1, \ldots, x_n)$ is a noncentral multilinear polynomial over $K$, $F$ is a nonzero generalized derivation of $R$ such that $[F(f(x)), f(x)] = 0$ for all $x = (x_1, \ldots, x_n) \in I^n$. Then one of the following holds:*

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $a \in U$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on $R$;*
3. *$char(R) = 2$ and $R$ satisfies $s_4$.*

Recently, Dhara et al. [35, Corollary 2.7] proved the following:

*Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$, $C$ the extended centroid of $R$. Let $0 \neq a \in R$ and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is noncentral valued on $R$. Suppose that $F$ is a nonzero generalized derivation of $R$ such that $a[F(f(x)), f(x)] \in C$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then one of the following holds:*

1. *$f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exist $p \in U$ and $\lambda \in C$ such that $F(x) = px + xp + \lambda x$ for all $x \in R$;*
2. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
3. *$R$ satisfies $s_4$ and there exist $p \in U$ and $\lambda \in C$ such that $F(x) = px + xp + \lambda x$ for all $x \in R$.*

In [52], De Filippis and Wei proved a result for generalized skew derivations as follows:

*Let $R$ be a prime ring of $char(R) \neq 2$, $Q_r$ its right Martindale quotient ring, $C$ its extended centroid, $f(x_1, \ldots, x_n)$ a noncentral valued multilinear polynomial over $C$ and $F$ a nonzero generalized skew derivation of $R$. If there exists $0 \neq a \in R$ such that*

$$a[F(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] = 0$$

*for all $x_1, \ldots, x_n \in R$, then one of the following holds:*

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $q \in Q_r$ and $\lambda \in C$ such that $F(x) = (q + \lambda)x + xq$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$.*

In a recent paper [24], Das et al. considered the situation with central values, that is,

$$a[F(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in C$$

for all $x_1, \ldots, x_n \in R$, where $F$ is a generalized skew derivation of $R$, and then obtain same conclusions of [35, Corollary 2.7].

In the present chapter, we generalize all the above situations replacing the map $F$ with an $X$-generalized skew derivation of $R$. More precisely we prove the following Theorem:

**Theorem 3.1.1.** *Let $R$ be a prime ring of $char(R) \neq 2$, $Q_r$ its right Martindle quotient ring and $C$ its extended centroid, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ that is noncentral-valued on $R$ and $F$ an $X$-generalized skew derivation of $R$. If for some $0 \neq a \in R$,*

$$a[F(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in C$$

*for all $x_1, \ldots, x_n \in R$, then one of the following holds:*

*(1) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*

*(2) there exist $\lambda \in C$ and $a \in Q_r$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central-valued on $R$;*

*(3) $R$ satisfies $s_4$ and there exist $\lambda \in C$ and $a \in Q_r$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$.*

## 3.2 The matrix ring case and outcomes for inner generalized derivations

We need the following Lemma.

**Lemma 3.2.1.** *[35, Lemma 2.5] Let $R$ be a noncommutative prime ring of characteristic different from 2 with the extended centroid $C$ of $R$. Let $0 \neq a \in R$ and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$ which is not central-valued on $R$. Suppose that $b, c, p, q \in R$ such that $a(bf(x)^2 + f(x)(c-p)f(x) - f(x)^2 q) \in C$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then $c - p \in C$ and one of the following holds:*

1. *$f(x_1, \ldots, x_n)^2$ is central-valued on $R$ and $a(b + c - p - q) \in C$;*
2. *$q \in C$ and $a(b + c - p - q) = 0$;*
3. *$R$ satisfies $s_4$.*

**Proposition 3.2.2.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$ and $F(x) = kx + mxp$ for all $x \in R$ for some $k, m, p \in Q_r$. If $0 \neq a \in R$ such that $a[F(f(r)), f(r)] \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

*(1) There exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*

*(2) There exist $\lambda \in C$ and $a \in Q_r$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central-valued on $R$;*

*(3) $R$ satisfies $s_4$ and there exist $\lambda \in C$ and $a \in Q_r$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$.*

Since $F(x) = kx + mxp$,

$$a[F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in C$$

for all $r_1, \ldots, r_n \in R$ gives

$$akf(r)^2 + amf(r)pf(r) - af(r)kf(r) - af(r)mf(r)p \in C$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Thus $R$ satisfies the following GPI

$$\begin{aligned}&[akf(r_1, \ldots, r_n)^2 + amf(r_1, \ldots, r_n)pf(r_1, \ldots, r_n)\\ &-af(r_1, \ldots, r_n)kf(r_1, \ldots, r_n) - af(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p, r_{n+1}]. \quad (3.2.1)\end{aligned}$$

Now we will investigate this GPI in prime rings. We consider another GPI

$$\begin{aligned}&bf(r_1, \ldots, r_n)^2 + cf(r_1, \ldots, r_n)pf(r_1, \ldots, r_n)\\ &-af(r_1, \ldots, r_n)kf(r_1, \ldots, r_n) - af(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p \in C \quad (3.2.2)\end{aligned}$$

where $a, b, c, p, k, m \in Q_r$.

**Lemma 3.2.3.** *If $a \in C$, we have conclusions (1) and (2) of Proposition 3.2.2.*

*Proof.* In case $0 \neq a \in C$, then from (3.2.1) we get

$$\begin{aligned}&kf(r_1, \ldots, r_n)^2 + mf(r_1, \ldots, r_n)pf(r_1, \ldots, r_n)\\ &-f(r_1, \ldots, r_n)kf(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p \in C \quad (3.2.3)\end{aligned}$$

for all $r_1, \ldots, r_n \in R$. Then by [27, Corollary 1.2], we have conclusions (1) and (2). □

**Lemma 3.2.4.** *If $m \in C$, we have conclusions (1), (2) and (3) of Proposition 3.2.2.*

*Proof.* In case $m \in C$, we have from (3.2.1)

$$a(kf(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)(mp - k)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2 mp) \in C$$

for all $x_1, \ldots, x_n \in R$. Then by Lemma 3.2.1, $mp - k \in C$ and one of the following holds:

1. $f(x_1, \ldots, x_n)^2$ is central-valued on $R$. In this case, $F(x) = kx + mxp = kx + xmp = kx + xk + \lambda x$ for all $x \in R$, where $mp - k = \lambda \in C$. This is our conclusion (2).

2. $mp \in C$. As $mp - k \in C$, we have $k \in C$ and hence $F(x) = kx + mxp = kx + xmp = (k+mp)x = \mu x$ where $\mu = k + mp \in C$. This gives our conclusion (1).

3. $R$ satisfies $s_4$. In this case $F(x) = kx + mxp = kx + xmp = kx + xk + \lambda x$ for all $x \in R$, where $mp - k = \lambda \in C$. This is our conclusion (3).

□

**Lemma 3.2.5.** *If $p \in C$, we have conclusions (1) of Proposition 3.2.2.*

*Proof.* In case $p \in C$, we have from (3.2.1)

$$a((k+mp)f(x_1,\dots,x_n)^2 - f(x_1,\dots,x_n)(k+mp)f(x_1,\dots,x_n)) \in C$$

for all $x_1,\dots,x_n \in R$. Then by Lemma 3.2.1, $k+mp \in C$. Then $F(x) = kx+mxp = kx+mpx = (k+mp)x = \lambda x$ where $\lambda = (k+mp) \in C$ which gives conclusion (1). □

**Lemma 3.2.6.** *If $R = M_t(C), t \geq 2$ be the ring of all $t \times t$ matrices over the infinite field $C$ and for some $a, b, c, k, m, p \in R$. If $R$ satisfies (3.2.2), then either $a \in C.I_t$ or $m \in C.I_t$ or $p \in C.I_t$.*

*Proof.* On contrary, we assume that $a \notin C.I_t, m \notin C.I_t$ and $p \notin C.I_t$. By Theorem 1.6.3, there exists an invertible matrix $Q$ such that the matrices $\phi(a) = QaQ^{-1}$, $\phi(m) = QmQ^{-1}$ and $\phi(p) = QpQ^{-1}$ have all non-zero entries. By assumption, $R$ satisfies

$$bf(r_1,\dots,r_n)^2 + cf(r_1,\dots,r_n)pf(r_1,\dots,r_n) - af(r_1,\dots,r_n)kf(r_1,\dots,r_n) \\ -af(r_1,\dots,r_n)mf(r_1,\dots,r_n)p \in C.I_t. \tag{3.2.4}$$

Evidently, $\phi$ is an inner automorphism and so $R$ satisfies the condition

$$\phi(b)f(r_1,\dots,r_n)^2 + \phi(c)f(r_1,\dots,r_n)\phi(p)f(r_1,\dots,r_n) - \\ \phi(a)f(r_1,\dots,r_n)\phi(k)f(r_1,\dots,r_n) \\ -\phi(a)f(r_1,\dots,r_n)\phi(m)f(r_1,\dots,r_n)\phi(p) \in C.I_t. \tag{3.2.5}$$

Let $e_{ij}$ denotes the usual matrix unit with 1 in $(i,j)$-entry and zero elsewhere. Since $f(x_1,\dots,x_n)$ is not central, by [71] (see also [76]), there exist a sequence of matrices

$r_1, \ldots, r_n \in M_t(C)$ and $0 \neq \gamma \in C$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$ with $i \neq j$. Thus replacing $f(r_1, \ldots, r_n)$ by $\gamma e_{ij}$ in (3.2.5), we get

$$\phi(b)e_{ij}^2 + \phi(c)e_{ij}\phi(p)e_{ij} - \phi(a)e_{ij}\phi(k)e_{ij} - \phi(a)e_{ij}\phi(m)e_{ij}\phi(p) \in C.I_t.$$

that is

$$\phi(c)e_{ij}\phi(p)e_{ij} - \phi(a)e_{ij}\phi(k)e_{ij} - \phi(a)e_{ij}\phi(m)e_{ij}\phi(p)$$

is a scalar matrix whose $(j, i)$ entry is zero. Thus we get

$$\phi(a)_{ji}\phi(m)_{ji}\phi(p)_{ji} = 0$$

which is a contradiction, since $\phi(a)$, $\phi(p)$ and $\phi(m)$ have all non-zero entries. Thus we conclude that either $a \in C.I_t$ or $p \in C.I_t$ or $m \in C.I_t$. □

**Lemma 3.2.7.** *Let $R = M_t(C), t \geq 2$ be the ring of all $t \times t$ matrices over the field $C$ with $char(C) \neq 2$ and for some $a, b, c, p, k, m \in R$, $R$ satisfies (3.2.2), then either $a \in C.I_t$ or $m \in C.I_t$ or $p \in C.I_t$.*

*Proof.* If $C$ is an infinite field, then the conclusions follow by Lemma 3.2.6.

Thus we assume that $C$ is a finite field. Let $K$ be an infinite field which is an extension of the field $C$. Let $\bar{R} = M_t(K) \cong R \bigotimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on $R$ if and only if it is central-valued on $\bar{R}$. Let

$$\chi(r_1, \ldots, r_n, r_{n+1}) = [bf(r_1, \ldots, r_n)^2 + cf(r_1, \ldots, r_n)pf(r_1, \ldots, r_n)$$

$$-af(r_1, \ldots, r_n)kf(r_1, \ldots, r_n) - af(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p, r_{n+1}].$$

It is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$. Hence the complete linearization of $\chi(r_1, \ldots, r_n, r_{n+1})$ yields a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, s_1, \ldots, s_n, r_{n+1})$ in $2n + 1$ indeterminates such that

$$\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n, r_{n+1}) = 2^n\chi(r_1, \ldots, r_n, r_{n+1}).$$

Clearly the multilinear polynomial $\Theta(r_1, \ldots, r_n, s_1, \ldots, s_n, r_{n+1})$ is a generalized polynomial identity for $R$ and $\bar{R}$ too. Since $char(C) \neq 2$, we obtain

$$\chi(r_1, \ldots, r_n, r_{n+1}) = 0$$

for all $r_1, \ldots, r_n, r_{n+1} \in \bar{R}$ and thus the conclusion follows by Lemma 3.2.6. □

**Corollary 3.2.8.** *If $R = M_t(C), t \geq 2$ be the ring of all $t \times t$ matrices over the field $C$ with $char(C) \neq 2$ and for some $a, b, c, p, k, m \in Q_r$, $R$ satisfies*

$$br^2 + crpr - arkr - armrp \in C$$

*then either $a \in C.I_t$ or $m \in C.I_t$ or $p \in C.I_t$.*

**Lemma 3.2.9.** *Let $R$ be a prime ring of $char(R) \neq 2$. If for some $a, b, c, p, k, m \in R$, $R$ satisfies*

$$bf(r_1, \ldots, r_n)^2 + cf(r_1, \ldots, r_n)pf(r_1, \ldots, r_n) - af(r_1, \ldots, r_n)kf(r_1, \ldots, r_n)$$

$$-af(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p = 0,$$

*then either $a \in C$ or $m \in C$ or $p \in C$.*

*Proof.* Since $R$ and $Q_r$ satisfy same generalized polynomial identity (GPI) (see [22]), $Q_r$ satisfies

$$bf(r_1, \ldots, r_n)^2 + cf(r_1, \ldots, r_n)pf(r_1, \ldots, r_n) - af(r_1, \ldots, r_n)kf(r_1, \ldots, r_n)$$

$$-af(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p = 0.$$

First we assume that this is a trivial GPI for $Q_r$. If $p \in C$, we have our conclusion. If $p \notin C$, then the term $-af(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p = 0$ can not be cancelled and so $-af(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p = 0$ implying $a = 0$ or $m = 0$. Thus we have reached to our conclusion.

Next assume that the above GPI is a non-trivial GPI for $Q_r$. Then by [21], $Q_r$ also satisfies the same GPI $\Psi(x_1, \ldots, x_n) = 0$. Using Martindale's Theorem (see Theorem 1.6.6), $Q_r$ is a primitive ring having nonzero socle soc$(Q_r)$ with $C$ as its associated division ring. Hence, by Jacobson's theorem (see Theorem 1.6.5), $Q_r$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$.

If $V$ is finite dimensional over $C$, then $dim_C V = t$ and also $R \cong M_t(C)$. Moreover, since $f(r_1, \ldots, r_n)$ is not central valued on $R$, $R$ must be noncommutative. Hence $t \geq 2$. In this case by Lemma 3.2.7, either $a$ or $m$ or $p$ is in $C$; and so we are done.

Now if $V$ is infinite dimensional over $C$, then by [90, Lemma 2], the set $f(Q_r)$ is dense on $Q_r$ and hence $Q_r$ satisfies

$$br^2 + crpr - arkr - armrp = 0.$$

For any $e^2 = e \in soc(Q_r)$, we have $eQ_re \cong M_t(C)$ with $t = dim_C Ve$. We want to show that in this case also either $a$ or $m$ or $p$ is in $C$. To prove this, let none of $a, m, p$ be in $C$. Then $a, m, p$ do not centralize the nonzero ideal $soc(Q_r)$. Hence there exist $h_1, h_2, h_3 \in soc(R)$ such that $[a, h_1] \neq 0$, $[m, h_2] \neq 0$, $[p, h_3] \neq 0$. By Litoff's Theorem (see Theorem 1.6.7), there exists an idempotent $e \in soc(Q_r)$ such that $ah_1, h_1a, mh_2, h_2m, ph_3, h_3p, h_1, h_2, h_3 \in eQ_re$. We have $eQ_re \cong M_t(C)$ with $t = dim_C Ve$. Since

$$eber^2 + ecereper - eaereker - eaeremerepe = 0$$

for all $r \in eQ_re$, by Corollary 3.2.8, either $eae$ or $eme$ or $epe$ is in $Ce$. Thus

$$ah_1 = eah_1e = (eae)h_1e = eh_1eae = eh_1ae = h_1a$$

or

$$mh_2 = emh_2e = (eme)h_2e = eh_2eme = eh_2me = h_2m$$

or

$$ph_3 = eph_3e = (epe)h_3e = eh_3epe = eh_3pe = h_3p,$$

a contradiction.

Thus we have proved that either $a$ or $m$ or $p$ is in $C$. □

**Proof of Proposition 3.2.2**

By hypothesis,

$$[akf(r_1, \ldots, r_n)^2 + amf(r_1, \ldots, r_n)pf(r_1, \ldots, r_n)$$
$$-af(r_1, \ldots, r_n)kf(r_1, \ldots, r_n) - af(r_1, \ldots, r_n)mf(r_1, \ldots, r_n)p, r_{n+1}] = 0 \quad (3.2.6)$$

for all $r_1, \ldots, r_n, r_{n+1} \in R$. If $0 = a[F(x), x]$ for all $x \in f(R)$, then by Lemma 3.2.9 either $a \in C$ or $m \in C$ or $p \in C$.

Next we assume that there exists $x_0 \in f(R)$ such that $0 \neq a[F(x_0), x_0] \in C$. Thus $R$ satisfies a central generalized polynomial identity $a[F(x), x] \in C$ for all $x \in f(R)$. By [18, Theorem 1], $RC$ is a finite dimensional central simple $C$-algebra, so that $Soc(Q_r) = RC = Q_r$, in particular, $R$ is a PI-ring. Denote by $K$ the algebraic closure of $C$, if $C$ is infinite, otherwise let $K = C$. Then $RC \bigotimes_C K \cong M_l(K)$

for some $l \geq 2$. Moreover, $RC \bigotimes_C K$ satisfies the same generalized polynomial identities of $RC = Q_r$, in particular, $a[F(x), x]$ is central in $RC \bigotimes_C K$ for any $x \in f(RC \bigotimes_C K)$. Therefore, by Lemma 3.2.7, we have that either $[a, RC \bigotimes_C K] = (0)$ or $[m, RC \bigotimes_C K] = (0)$ or $[p, RC \bigotimes_C K] = (0)$. Since $RC \bigotimes_C K$ satisfies the same generalized polynomial identities of $RC = Q_r$, it follows that either $[a, Q_r] = (0)$ or $[m, Q_r] = (0)$ or $[p, Q_r] = (0)$. In other words, we have proved that either $a \in C$ or $m \in C$ or $p \in C$.

Thus in any case, we have obtained that either $a \in C$ or $m \in C$ or $p \in C$. In any case conclusion follows by Lemma 3.2.3, Lemma 3.2.4 and Lemma 3.2.5.

**Proposition 3.2.10.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ is a noncentral multilinear polynomial over $C$, $F(x) = kx + m\alpha(x)p$, where $k, m, p \in Q_r$ and $\alpha$ is an automorphism of $R$. Let $0 \neq a \in R$ be such that*

$$a[F(f(r)), f(r)] \in C$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

*(1) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*

*(2) there exist $\lambda \in C$ and $a \in Q_r$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$ and $f(x_1, \ldots, x_n)^2$ is central-valued on $R$;*

*(3) $R$ satisfies $s_4$ and there exist $\lambda \in C$ and $a \in Q_r$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$.*

*Proof.* If $\alpha$ is an inner automorphism, then the result follows by Proposition 3.2.2. So we assume that $\alpha$ is an outer automorphism of $R$. By hypothesis, we have

$$a[kf(x_1, \ldots, x_n) + m\alpha(f(x_1, \ldots, x_n))p, f(x_1, \ldots, x_n)] \in C \tag{3.2.7}$$

for all $x_1, \ldots, x_n \in R$.

Let $f^\alpha(x_1, \ldots, x_n)$ be the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficients $\gamma_\sigma$ with $\alpha(\gamma_\sigma)$. Since the degree of any $(x_i)^\alpha$-word in (3.2.7) is equal to one, by [22, Theorem 3], $Q_r$ satisfies the generalized polynomial identity

$$[a[kf(x_1, \ldots, x_n) + mf^\alpha(y_1, \ldots, y_n)p, f(x_1, \ldots, x_n)], x_{n+1}] = 0.$$

In particular, $Q_r$ satisfies the blended component

$$[a[mf^{\alpha}(y_1,\ldots,y_n)p, f(x_1,\ldots,x_n)], x_{n+1}] = 0.$$

Then

$$a[z, f(x_1,\ldots,x_n)] \in C,$$

where $z = mf^{\alpha}(y_1,\ldots,y_n)p$. Hence, $z = mf^{\alpha}(y_1,\ldots,y_n)p \in C$ for all $y_1,\ldots,y_n \in Q_r$. Then $[mf^{\alpha}(y_1,\ldots,y_n)p, f^{\alpha}(y_1,\ldots,y_n)] = 0$. Hence for $q \notin C$, $Q_r$ satisfies

$$[q, [mf^{\alpha}(y_1,\ldots,y_n)p, f^{\alpha}(y_1,\ldots,y_n)]] = 0.$$

Then by [27, Lemma 2.9], one of the following holds:

1. $m \in C, mp \in C$;
2. $p \in C, mp \in C$.

Since $z = mf^{\alpha}(y_1,\ldots,y_n)p \in C$ for all $y_1,\ldots,y_n \in Q_r$, then for both the above cases, we have $mpf^{\alpha}(y_1,\ldots,y_n) \in C$ for all $y_1,\ldots,y_n \in Q_r$. If $mp \neq 0$, then $f^{\alpha}(y_1,\ldots,y_n) \in C$ which is a contradiction. Therefore $mp = 0$ with either $m \in C$ or $p \in C$. Hence as $mp = 0$, we have either $m = 0$ or $p = 0$. Therefore, $Q_r$ satisfies the following relation $a[kf(x_1,\ldots,x_n), f(x_1,\ldots,x_n)] \in C$, which is

$$a(kf(x_1,\ldots,x_n)^2 - f(x_1,\ldots,x_n)kf(x_1,\ldots,x_n)) \in C.$$

By Lemma 3.2.1, $k \in C$. Since either $m$ or $p$ is zero, therefore $F(x) = kx$, which is our conclusion (1). □

In particular, we have the following corollary.

**Corollary 3.2.11.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 and $C$ be its extended centroid. Suppose that $f(x_1,\ldots,x_n)$ is a noncentral multilinear polynomial over $C$, $F(x) = kx - k\alpha(x)$, where $k \in Q_r$ and $\alpha$ is an automorphism of $R$. Let $0 \neq a \in R$ be such that*

$$a[F(f(r)), f(r)] \in C$$

*for all $r = (r_1,\ldots,r_n) \in R^n$, then $F = 0$.*

## 3.3 Proof of Main Theorem

The $X$-generalized skew derivation $F$ has its form $F(x) = bx + cd(x)$ for all $x \in R$, where $b, c \in Q_r$ and $d$ is a skew-derivation of $R$. Since any skew derivations of $R$ can be uniquely extended in $Q_r$ and by [23, Theorem 2], $R$ and $Q_r$ satisfy the same generalized polynomial identities with a single skew derivation, $Q_r$ satisfies

$$a[bf(x_1,\ldots,x_n) + cd(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] \in C.$$

If $d$ is inner skew derivation of $Q_r$, i.e., $d(x) = px - \alpha(x)p$, then $F(x) = (b+cp)x - c\alpha(x)p$ for all $x \in Q_r$. Then by Proposition 3.2.10, we have our conclusions.

Thus we assume that $d$ is the outer skew derivation of $Q_r$. We denote

$$f(x_1,\ldots,x_n) = \sum_{\sigma\in S_n} \gamma_\sigma x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$$

where $\gamma_\sigma \in C$. Let $f^d(x_1,\ldots,x_n)$ be the polynomial obtained from $f(x_1,\ldots,x_n)$ by replacing each coefficients $\gamma_\sigma$ with $d(\gamma_\sigma)$. Hence

$$d(f(x_1,\ldots,x_n)) = f^d(x_1,\ldots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)}.$$

Thus $Q_r$ satisfies

$$a[bf(x_1,\ldots,x_n) + cf^d(x_1,\ldots,x_n) + c\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots\ x_{\sigma(j)})$$
$$d(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)}, f(x_1,\ldots,x_n)] \in C. \tag{3.3.1}$$

Then by [23, Theorem 1], $Q_r$ satisfies

$$a[bf(x_1,\ldots,x_n) + cf^d(x_1,\ldots,x_n)$$
$$+c\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}, f(x_1,\ldots,x_n)] \in C.$$

In particular, $Q_r$ satisfies the blended component

$$a[c\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)},$$
$$f(x_1,\ldots,x_n)] \in C. \tag{3.3.2}$$

Then two cases arise:

- Case i: Let $\alpha$ be an inner automorphism of $Q_r$. Then there exists $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$ for all $x \in Q_r$. Moreover, $\alpha(\gamma_\sigma) = \gamma_\sigma$ for all coefficients involved in $f(x_1, \ldots, x_n)$. Then replacing $y_i$ by $x_i - \alpha(x_i)$ in above relation, we have that $Q_r$ satisfies

$$a[c(f(x_1, \ldots, x_n) - \alpha(f(x_1, \ldots, x_n))), f(x_1, \ldots, x_n)] \in C$$

i.e.,

$$a[cf(x_1, \ldots, x_n) - c\alpha(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] \in C$$

Then by Corollary 3.2.11, $c(x - \alpha(x)) = 0$ for all $x \in Q_r$. This gives $c(x - qxq^{-1}) = 0$ i.e., $c[q, x] = 0$ for all $x \in Q_r$. This implies that either $c = 0$ or $q \in C$. In any case $F$ becomes a generalized derivation and hence conclusion follows by [35, Corollary 2.7].

- Case ii: Let $\alpha$ be an outer automorphism of $Q_r$. Then from (3.2.7), using [22, Theorem 3], $Q_r$ satisfies generalized polynomial identity

$$a[c \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} z_{\sigma(1)} z_{\sigma(2)} \ldots z_{\sigma(j)} y_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}, f(x_1, \ldots, x_n)] \in C.$$

In particular, $Q_r$ satisfies the blended component

$$a[c \textstyle\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) z_{\sigma(1)} z_{\sigma(2)} \ldots z_{\sigma(n-1)} y_{\sigma(n)}, f(x_1, \ldots, x_n)] \in C. \quad (3.3.3)$$

Replacing $z_i$ by $\alpha(z_i)$ and $y_i$ by $\alpha(z_i)$ for $i = 1, 2, \ldots, n$, we get that $Q_r$ satisfies

$$a[c\alpha(f(z_1, \ldots, z_n)), f(x_1, \ldots, x_n)] \in C$$

i.e.,

$$a[c', f(x_1, \ldots, x_n)] \in C$$

where $c' = c\alpha(f(z_1, \ldots, z_n))$. Since $a$ and $f(x_1, \ldots, x_n)$ are noncentral then $c' = c\alpha(f(z_1, \ldots, z_n)) \in C$ for all $z_1, \ldots, z_n \in Q_r$. Commuting both sides with $\alpha(f(z_1, \ldots, z_n))$ we get,
$0 = [c\alpha(f(z_1, \ldots, z_n)), \alpha(f(z_1, \ldots, z_n))] = [c, \alpha(f(z_1, \ldots, z_n))]\alpha(f(z_1, \ldots, z_n))$

for all $z_1, \ldots, z_n \in Q_r$ which implies $c \in C$. Therefore, $c\alpha(f(z_1, \ldots, z_n)) \in C$ for all $z_1, \ldots, z_n \in Q_r$ implies $c = 0$. Then from (3.3.1) we get

$$a[bf(x_1, \ldots, x_n), f(x_1, \ldots, x_n)] \in C$$

i.e., $a(bf(x_1, \ldots, x_n)^2 - f(x_1, \ldots, x_n)bf(x_1, \ldots, x_n)) \in C$. Then by Lemma 3.2.1, $b \in C$ and hence $F(x) = bx$ for all $x \in R$, with $b \in C$. This is our conclusion (1).

Thus the Proof of Theorem 3.1.1 is completed. $\square$

# Chapter 4

# Annihilating and Centralizing Condition of Generalized Derivation in Prime Ring

## 4.1 Introduction

Throughout this chapter, we denote $R$ as a prime ring with $char(R) \neq 2$, center $Z(R)$, extended centroid $C$. Here $f(x_1, \ldots, x_n)$ denotes a multilinear polynomial over $C$ that is noncentral-valued on $R$, $I$ a nonzero ideal of $R$. By $d$ and $F$, we mean a derivation and a generalized derivation of $R$. Posner initiated the study of commuting and centralizing maps in 1957. Posner [80] proved that a prime ring must be commutative if it possesses a nonzero centralizing derivation i.e., $[d(x), x] \in Z(R)$ for all $x \in R$, where $d$ is a nonzero derivation of $R$. Since then many authors investigated these centralizing and commuting maps in different way (see [12], [25], [41], [43], [48]). For example Lee and Lee [69] showed that if $[d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in Z(R)$ for all $r_1, \ldots, r_n$ in some nonzero ideal of $R$, then $f(r_1, \ldots, r_n)$ is central valued on $R$, except when $char(R) \neq 2$ and $R$ satisfies $s_4$. In [38], Dhara and Sharma studied the situations $[d^2(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$ and $[d^2(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in Z(R)$ for all $r_1, \ldots, r_n$ in some nonzero right sided ideal of $R$.

Argac and De Filippis [5] studied the commuting generalized derivation maps. In [5], Argac and De Filippis obtained the following:

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Let $K$ be a commutative ring with unity, $R$ be a noncommutative prime $K$-algebra with center $Z(R)$, $U$ be the Utumi quotient ring of $R$, $C = Z(U)$ the extended centroid of $R$ and $I$ a nonzero ideal of $R$. Suppose that $f(x_1, \ldots, x_n)$ is a noncentral multilinear polynomial over $K$, $F$ is a nonzero generalized derivation of $R$ such that

$$[F(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)] = 0$$

for all $x_1, \ldots, x_n \in I$. Then one of the following holds:

1. there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
2. $f(x_1, \ldots, x_n)^2$ is central-valued on $R$ and there exist $a \in U$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$;
3. $char(R) = 2$ and $R$ satisfies $s_4$.

In [42], De Filippis obtained the following:

Let $R$ be a prime ring of characteristic different from 2, with extended centroid $C$, $U$ its Utumi quotient ring, $F \neq 0$ a non-zero generalized derivation of $R$, $f(r_1, \ldots, r_n)$ a noncentral multilinear polynomial over $C$ in $n$ non-commuting variables, $a \in R$ such that

$$a[F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$$

for any $r_1, \ldots, r_n \in R$. Then one of the following holds:

1. $a = 0$;
2. there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;
3. there exist $q \in U$ and $\lambda \in C$ such that $F(x) = (q + \lambda)x + xq$ for all $x \in R$ and $f(r_1, \ldots, r_n)^2$ is central valued on $R$.

In [35, Corollary 2.7], Dhara et al. obtained the following:

Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ the Utumi quotient ring of $R$ and $C$ the extended centroid of $R$. Let $0 \neq a \in R$ and

$f(x_1,\ldots,x_n)$ be a multilinear polynomial over $C$ which is noncentral-valued on $R$. Suppose that $F$ is a nonzero generalized derivation of $R$ such that

$$a[F(f(x_1,\ldots,x_n)), f(x_1,\ldots,x_n)] \in C$$

for all $x_1,\ldots,x_n \in R$. Then one of the following holds:

1. $f(x_1,\ldots,x_n)^2$ is central-valued on $R$ and there exist $p \in U$ and $\lambda \in C$ such that $F(x) = px + xp + \lambda x$ for all $x \in R$;

2. there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;

3. $R$ satisfies $s_4$ and there exist $p \in U$ and $\lambda \in C$ such that $F(x) = px + xp + \lambda x$ for all $x \in R$.

In [16], Carini et al. already obtained the following:

Let $R$ be a non-commutative prime ring of characteristic different from 2, $U$ its right Utumi quotient ring, $C$ its extended centroid, $F$ a generalized derivation on $R$, and $f(r_1,\ldots,r_n)$ a noncentral multilinear polynomial over $C$. If there exists $a \in R$ such that, for all $r_1,\ldots,r_n \in R$,

$$a[F^2(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)] = 0$$

then one of the following statements hold:

(i) $a = 0$;

(ii) there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$;

(iii) there exists $c \in U$ such that $F(x) = cx$, for all $x \in R$, with $c^2 \in C$;

(iv) there exists $c \in U$ such that $F(x) = xc$, for all $x \in R$, with $c^2 \in C$.

In this present chapter, we will study the following situation

$$a[F^2(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)] \in C$$

for all $r_1,\ldots,r_n \in R$. More precisely, we will prove the following Theorem.

**Theorem 4.1.1.** *Let $R$ be a prime ring of characteristic different from 2, $U$ its Utumi ring of quotients and $C$ its extended centroid, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ that is noncentral-valued on $R$, $I$ a nonzero ideal of $R$ and $F$ a nonzero generalized derivation of $R$. If for some $0 \neq a \in R$,*

$$a[F^2(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in C$$

*for all $r_1, \ldots, r_n \in I$, then one of the following holds:*

*(i) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*

*(ii) there exists $p \in U$ such that $F(x) = px$ with $p^2 \in C$;*

*(iii) there exists $p \in U$ such that $F(x) = xp$ with $p^2 \in C$.*

**Corollary 4.1.2.** *Let $R$ be a prime ring of characteristic different from 2, $C$ its extended centroid, $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ that is noncentral-valued on $R$, $I$ a nonzero ideal of $R$ and $d$ a derivation of $R$. If for some $a \in R$,*

$$a[d^2(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] \in C$$

*for all $r_1, \ldots, r_n \in I$, then either $a = 0$ or $d = 0$.*

**Corollary 4.1.3.** *Let $R$ be a prime ring of characteristic different from 2, $C$ its extended centroid, $I$ a nonzero ideal of $R$ and $d$ a nonzero derivation of $R$. If for some $a \in R$,*

$$a[d^2(x), x] \in C$$

*for all $x \in I$, then either $a = 0$ or $R$ is commutative.*

## 4.2 The case of inner generalized derivations

**Lemma 4.2.1.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ its Utumi ring of quotients and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$ and $F(x) = bx$ for all $x \in R$ for some $b \in U$. If $0 \neq a \in R$ such that $a[F^2(f(r)), f(r)] \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$, then $b^2 \in C$.*

*Proof.* By hypothesis, $R$ satisfies

$$a[b^2 f(r), f(r)] \in C \tag{4.2.1}$$

that is

$$a[b^2, f(r)]f(r) \in C \tag{4.2.2}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by [35, Corollary 2.8], $b^2 \in C$. □

**Lemma 4.2.2.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ its Utumi ring of quotients and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$ and $F(x) = xc$ for all $x \in R$ for some $c \in U$. If $0 \neq a \in R$ such that $a[F^2(f(r)), f(r)] \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$, then $c^2 \in C$.*

*Proof.* By hypothesis, $R$ satisfies

$$a[f(r)c^2, f(r)] \in C \tag{4.2.3}$$

that is

$$af(r)[c^2, f(r)] \in C \tag{4.2.4}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by [35, Corollary 2.8], $c^2 \in C$. □

**Lemma 4.2.3.** *Let $R = M_m(C), m \geq 2$ be the ring of all $m \times m$ matrices over the field $C$ with $char(R) \neq 2$, $f(r_1, \ldots, r_n)$ a non-central multilinear polynomial over $C$ and $a(\neq 0), b, c \in R$. If*

$$a[b^2 f(r) + 2bf(r)c + f(r)c^2, f(r)] \in C.I_m$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then either $b \in C.I_m$ or $c \in C.I_m$.*

*Proof.* Assume first that $C$ is an infinite field. By assumption, $R$ satisfies

$$\begin{aligned} &\phi(r_1, \ldots, r_n, r_{n+1}) \\ &= [ab^2 f(r_1, \ldots, r_n)^2 + 2abf(r_1, \ldots, r_n)cf(r_1, \ldots, r_n) + af(r_1, \ldots, r_n)c^2 f(r_1, \ldots, r_n) \\ &\quad - af(r_1, \ldots, r_n)b^2 f(r_1, \ldots, r_n) - 2af(r_1, \ldots, r_n)bf(r_1, \ldots, r_n)c \\ &\quad - af(r_1, \ldots, r_n)^2 c^2, r_{n+1}] = 0. \end{aligned} \tag{4.2.5}$$

We first assume that $a \notin C.I_m$, $b \notin C.I_m$ and $c \notin C.I_m$. Then under this assumption, by Theorem 1.6.3, there exists an invertible matrix $Q$ such that $\phi(a) = QaQ^{-1}$, $\phi(b) = QbQ^{-1}$ and $\phi(c) = QcQ^{-1}$ have all non-zero entries. Evidently, $R$ satisfies the condition

$$\begin{aligned}
&[\phi(ab^2)f(r_1,\ldots,r_n)^2 + 2\phi(ab)f(r_1,\ldots,r_n)\phi(c)f(r_1,\ldots,r_n)\\
&+\phi(a)f(r_1,\ldots,r_n)\phi(c)^2f(r_1,\ldots,r_n) - \phi(a)f(r_1,\ldots,r_n)\phi(b)^2f(r_1,\ldots,r_n)\\
&-2\phi(a)f(r_1,\ldots,r_n)\phi(b)f(r_1,\ldots,r_n)\phi(c)\\
&-\phi(a)f(r_1,\ldots,r_n)^2\phi(c)^2, r_{n+1}] = 0. \qquad (4.2.6)
\end{aligned}$$

Since $f(x_1,\ldots,x_n)$ is not central valued, by [71] (see also [76]), there exist $r_1,\ldots,r_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(r_1,\ldots,r_n) = \gamma e_{ij}$ with $i \neq j$, where $e_{ij}$ denotes the usual matrix unit with 1 in $(i,j)$-entry and zero elsewhere. Thus by (4.2.6),

$$\begin{aligned}
&[\phi(ab^2)e_{ij}^2 + 2\phi(ab)e_{ij}\phi(c)e_{ij} + \phi(a)e_{ij}\phi(c)^2e_{ij} - \phi(a)e_{ij}\phi(b)^2e_{ij}\\
&-2\phi(a)e_{ij}\phi(b)e_{ij}\phi(c) - \phi(a)e_{ij}^2\phi(c)^2, e_{ij}] = 0.
\end{aligned}$$

Left multiplying by $e_{ij}$ and then using $char(R) \neq 2$, we have

$$e_{ij}\phi(a)e_{ij}\phi(b)e_{ij}\phi(c)e_{ij} = 0$$

which is a contradiction, since $\phi(a)$, $\phi(b)$ and $\phi(c)$ have all non-zero entries. Thus we conclude that either $a \in C.I_m$ or $b \in C.I_m$ or $c \in C.I_m$.

If $0 \neq a \in C.I_m$, by assumption, we have

$$b^2f(r)^2 + 2bf(r)cf(r) + f(r)c^2f(r) - f(r)b^2f(r) - 2f(r)bf(r)c - f(r)^2c^2 \in C.I_m$$

for all $r = (r_1,\ldots,r_n) \in R^n$. Then by similar process replacing $f(r_1,\ldots,r_n)$ with $e_{ij}$ and then commuting by $e_{ij}$ yields $2\phi(b)_{ji}\phi(c)_{ji} = 0$, which is a contradiction, since $\phi(b)$ and $\phi(c)$ have all non-zero entries. Thus finally we conclude that either $b \in C.I_m$ or $c \in C.I_m$.

Next we assume that $C$ is a finite fields. Let $K$ be an infinite field which is an extension of the field $C$. Let $\bar{R} = M_m(K) \cong R \bigotimes_C K$. Notice that the multilinear polynomial $f(r_1,\ldots,r_n)$ is central-valued on $R$ if and only if it is central-valued on $\bar{R}$. $R$ satisfies

$$\begin{aligned}
\phi(r_1,\ldots,r_n,r_{n+1}) = &[ab^2f(r)^2 + 2abf(r)cf(r) + af(r)c^2f(r)\\
&-af(r)b^2f(r) - 2af(r)bf(r)c - af(r)^2c^2, r_{n+1}] \qquad (4.2.7)
\end{aligned}$$

Moreover, it is a multi-homogeneous of multi-degree (2,..., 2) in the indeterminates $r_1, \ldots, r_n$. Hence the complete linearization of $\phi(r_1, \ldots, r_n, r_{n+1})$ yields a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, s_1, \ldots, s_n, r_{n+1})$ in $2n+1$ indeterminates such that $\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n, r_{n+1}) = 2^n \phi(r_1, \ldots, r_n, r_{n+1})$.

Clearly the multilinear polynomial $\Theta(r_1, \ldots, r_n, s_1, \ldots, s_n, r_{n+1})$ is a generalized polynomial identity for $R$ and $\bar{R}$ too. Since $char(C) \neq 2$, we obtain

$$\phi(r_1, \ldots, r_n, r_{n+1}) = 0$$

for all $r_1, \ldots, r_n, r_{n+1} \in \bar{R}$ and hence the conclusion follows from above argument when associated field was infinite field. □

To prove our next Lemma, we need the following:

**Lemma 4.2.4.** *[35, Lemma 2.5] Let $R$ be a noncommutative prime ring of characteristic different from 2 with the extended centroid $C$ of $R$. Let $0 \neq a \in R$ and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$ which is not central-valued on $R$. Suppose that $b, c, p, q \in R$ such that $a(bf(x)^2 + f(x)(c-p)f(x) - f(x)^2 q) \in C$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then $c - p \in C$ and one of the following holds:*

*(i) $f(x_1, \ldots, x_n)^2$ is central-valued on $R$ and $a(b+c-p-q) \in C$;*

*(ii) $q \in C$ and $a(b+c-p-q) = 0$;*

*(iii) $R$ satisfies $s_4$.*

**Lemma 4.2.5.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $U$ be its Utumi ring of quotients and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$ and $F(x) = bx + xc$ for all $x \in R$, for some $b, c \in U$. If $0 \neq a \in R$ such that $a[F^2(f(r)), f(r)] \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

*(1) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*

*(2) there exists $p \in U$ such that $F(x) = px$ for all $x \in R$ with $p^2 \in C$;*

*(3) there exists $p \in U$ such that $F(x) = xp$ for all $x \in R$ with $p^2 \in C$.*

*Proof.* By hypothesis,

$$a[b^2f(r) + 2bf(r)c + f(r)c^2, f(r)] \in C \tag{4.2.8}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. If $a[b^2f(r) + 2bf(r)c + f(r)c^2, f(r)] = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then by [16], we have our conclusions (1)-(3).

Thus we assume that there exists $r = (r_1, \ldots, r_n) \in R^n$ such that $a[b^2f(r) + 2bf(r)c + f(r)c^2, f(r)] \neq 0$. Therefore, $a[b^2f(r) + 2bf(r)c + f(r)c^2, f(r)] \in C$ is a nonzero central generalized identity for $R$. In this case by [18, Theorem 1], $R$ is a PI- ring and hence $RC = U$ is a nontrivial GPI-ring simple with 1. By [67, Lemma 2] and [81, Theorem 2.3.29], there exists a field $K$ such that $U \subseteq M_k(K)$, $k \geq 2$; moreover $U$ and $M_k(K)$ satisfy the same generalized polynomial identities. Hence $a[b^2f(r)+2bf(r)c+f(r)c^2, f(r)] \in Z(M_k(K))$ for all $r = (r_1, \ldots, r_n) \in M_k(K)$. By Lemma 4.2.3, either $b \in C$ or $c \in C$.

If $b \in C$, then by Lemma 4.2.2, $F(x) = x(b + c)$ for all $x \in R$ with $(b + c)^2 \in C$, as desired in conclusion (3).

If $c \in C$, then by Lemma 4.2.1, $F(x) = (b + c)x$ for all $x \in R$ with $(b + c)^2 \in C$, as desired in conclusion (2). □

Now we are ready to prove our Theorem 4.1.1.

## 4.3 Proof of Main Theorem in the general case

To prove the Main Theorem 4.1.1, we need the following Remark:

**Remark 4.3.1.** *Suppose that $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$, then*

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$$

*where $\gamma_\sigma \in C$ and $S_n$ be the symmetric group of $n$ symbols. If $d$ be a derivation on $R$ and $f^d(x_1, \ldots, x_n)$, $f^{d^2}(x_1, \ldots, x_n)$ be the polynomials obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficients $\gamma_\sigma$ with $d(\gamma_\sigma)$ and $d^2(\gamma_\sigma)$, respectively, then*

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$$

*and*

$$d^2(f(r_1, \ldots, r_n)) = f^{d^2}(r_1, \ldots, r_n) + 2\sum_i f^d(r_1, \ldots, d(r_i), \ldots, r_n)$$

$$+\sum_i f(r_1,\ldots,d^2(r_i),\ldots,r_n) + 2\sum_{i\neq j} f(r_1,\ldots,d(r_i),\ldots,d(r_j)\ldots,r_n).$$

By [72, Theorem 3], there exist $b \in U$ and derivation $d$ on $U$ such that $F(x) = bx + d(x)$ for all $x \in U$. Then $F^2(x) = b(bx+d(x)) + d(bx+d(x)) = F(b)x + 2bd(x) + d^2(x)$. By [21] and [71], both $I$, $R$ and $U$ satisfy the same generalized polynomial identities and same differential identities. So we have

$$a[F(b)f(r_1,\ldots,r_n) + 2bd(f(r_1,\ldots,r_n)) + d^2(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)] \in C$$

for all $r_1,\ldots,r_n \in U$. If $d$ is inner, that is, $F$ is inner, then conclusions follow by Lemma 4.2.5. Thus we assume that $d$ is not inner. Using Remark 4.3.1, above relation can be written as

$$a[F(b)f(r_1,\ldots,r_n) + 2bf^d(r_1,\ldots,r_n) + 2b\textstyle\sum_i f(r_1,\ldots,d(r_i),\ldots,r_n)$$
$$+f^{d^2}(r_1,\ldots,r_n) + 2\textstyle\sum_i f^d(r_1,\ldots,d(r_i),\ldots,r_n) + \sum_i f(r_1,\ldots,d^2(r_i),\ldots,r_n)$$
$$+2\textstyle\sum_{i\neq j} f(r_1,\ldots,d(r_i),\ldots,d(r_j)\ldots,r_n), f(r_1,\ldots,r_n)] \in C$$

for all $r_1,\ldots,r_n \in U$. Then by Kharchenko's result (from Theorem 1.6.4), $U$ satisfies

$$a[F(b)f(r_1,\ldots,r_n) + 2bf^d(r_1,\ldots,r_n) + 2b\textstyle\sum_i f(r_1,\ldots,t_i,\ldots,r_n)$$
$$+f^{d^2}(r_1,\ldots,r_n) + 2\textstyle\sum_i f^d(r_1,\ldots,t_i,\ldots,r_n) + \sum_i f(r_1,\ldots,s_i,\ldots,r_n)$$
$$+2\textstyle\sum_{i\neq j} f(r_1,\ldots,t_i,\ldots,t_j\ldots,r_n), f(r_1,\ldots,r_n)] \in C.$$

In particular, $U$ satisfies the blended component

$$a[\sum_i f(r_1,\ldots,s_i,\ldots,r_n), f(r_1,\ldots,r_n)] \in C.$$

Replacing $s_i$ with $[q, r_i]$ for some $q \notin C$, $U$ satisfies

$$a[[q, f(r_1,\ldots,r_n)], f(r_1,\ldots,r_n)] \in C$$

which gives

$$a(qf(r_1,\ldots,r_n)^2 - 2f(r_1,\ldots,r_n)qf(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)^2q) \in C.$$

Then by Lemma 4.2.4, $q \in C$ which leads to a contradiction. Thus the proof of the Theorem 4.1.1 is completed.

# Chapter 5

# Generalized Derivations and Multilinear Polynomials in Prime Rings

## 5.1 Introduction

In this Chapter, $R$ always denotes a prime ring which is associative and the center is $Z(R)$. $U$ denotes the Utumi quotient ring of the prime ring $R$. It is noted that $R$ is a subring of the Utumi quotient ring $U$. The center of $U$ is called the extended centroid of $R$ and it is denoted by $C$. Let $f(r_1, \ldots, r_n)$ be a noncentral multilinear polynomial over $C$ in $n$ noncommuting variables which is written as $f(r_1, \ldots, r_n) = r_1 r_2 \ldots r_n + \sum\limits_{I \neq \sigma \in S_n} \alpha_\sigma r_{\sigma(1)} \ldots r_{\sigma(n)}$ for some $\alpha_\sigma \in C$.

In [50], De Filippis and Di Vincenzo studied the situation

$$\delta([d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]) = 0$$

for all $r_1, \ldots, r_n \in R$ where $R$ is a prime $K$-algebra of characteristic different from 2, $K$ is a non-commutative ring with unity, $d$ and $\delta$ are two non-zero derivations of $R$ and $f(r_1, \ldots, r_n)$ is a multilinear polynomial over $K$.

In [51], De Filippis and Di Vincenzo investigated the above result by replacing the derivation $d$ with a generalized derivation $F$. They studied the situation

$$d([F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]) = 0$$

for all $r_1, \ldots, r_n \in R$.

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In [26], Dhara extended the above result by replacing derivation $d$ with a generalized derivation $G$ i.e.,

$$G([F(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)])=0$$

for all $r_1,\ldots,r_n \in R$.

In this line of investigation very recent, Tiwari [87] has studied the case

$$G(F(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n))=H(f(r_1,\ldots,r_n)^2)$$

for all $r_1,\ldots,r_n \in R$, where $F,G$ and $H$ are three generalized derivations in prime ring $R$ and then described the forms of the maps.

In this present chapter, our aim is to study

$$G([F(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)])=H(f(r_1,\ldots,r_n)^2)$$

for all $r_1,\ldots,r_n \in R$. More precisely, we prove the following:

**Theorem 5.1.1.** *Suppose that $R$ is a prime ring with Utumi quotient ring $U$, extended centroid $C$ and $f(r_1,\ldots,r_n)$ a noncentral multilinear polynomial over $C$. Let $I$ be a nonzero ideal of $R$ and $char(R)\neq 2$. If $F$, $G$ and $H$ are three generalized derivations of $R$ such that*

$$F\Big([G(f(r)),f(r)]\Big)=H(f(r)^2)$$

*for all $r=(r_1,\ldots,r_n)\in I^n$, then one of the following holds:*

*(1) there exist $\lambda,\mu\in C$ and $a,b\in U$ such that $F(x)=\mu x$, $G(x)=ax+xa+\lambda x$, $H(x)=[b,x]$ for all $x\in R$ with $b-\mu a\in C$;*

*(2) there exist $\lambda,\mu\in C$ and $a,b,q\in U$ such that $F(x)=qx+xq+\mu x$, $G(x)=ax+xa+\lambda x$, $H(x)=[b,x]$ for all $x\in R$ with $q+\alpha a\in C$ and $q^2+\mu q+\alpha b\in C$ for some $0\neq\alpha\in C$;*

*(3) $f(r_1,\ldots,r_n)^2$ is central valued on $R$ and there exist $\lambda\in C$, $a,b,p\in U$ and a derivation $d$ in $U$ such that $F(x)=px+d(x)$, $G(x)=ax+xa+\lambda x$ and $H(x)=[b,x]$ for all $x\in R$;*

*(4) $R$ satisfies $s_4$.*

## 5.2 The case of inner generalized derivations

**Lemma 5.2.1.** *[5] Suppose that $R$ is a noncommutative prime ring with Utumi quotient ring $U$, extended centroid $C$ and $f(r_1,\dots,r_n)$ a noncentral multilinear polynomial over $C$. If there exist $a,b,c \in U$ such that*

$$f(r)af(r) + f(r)^2 b - cf(r)^2 = 0$$

*for all $r = (r_1,\dots,r_n) \in R^n$, then one of the following holds:*

*(1) $a,b,c \in C$ and $a+b-c=0$;*

*(2) $a \in C$, $f(r_1,\dots,r_n)^2$ is central valued on $R$ and $a+b-c=0$;*

*(3) $char(R)=2$ and $R$ satisfies $s_4$.*

**Lemma 5.2.2.** *[5, Lemma 1] Let $R$ be a noncommutative prime ring and $\eta(x_1,\dots,x_n)$ be any polynomial over $C$, which is not an identity for $R$. If $a,b \in U$ such that*

$$a\eta(r) - \eta(r)b = 0$$

*for all $r = (r_1,\dots,r_n) \in R^n$, then one of the following holds:*

*(1) $a = b \in C$;*

*(2) $a = b$ and $\eta(x_1,\dots,x_n)$ is central valued on $R$;*

*(3) $char(R)=2$ and $R$ satisfies $s_4$.*

**Lemma 5.2.3.** *[26, Proposition 2.7] Let $R$ be a prime ring of $char(R) \neq 2$, $C$ the extended centroid of $R$ and $f(r_1,\dots,r_n)$ a non-central multilinear polynomial over $C$. If $a,b,p,q,s,p',b' \in R$ such that $p'f(r)^2 + pf(r)sf(r) - pf(r)^2b + af(r)^2q + f(r)sf(r)q - f(r)^2b' = 0$ for all $r=(r_1,\dots,r_n) \in R^n$, then either $p,q \in C$ or $s \in C$.*

**Lemma 5.2.4.** *Let $R$ be a noncommutative prime ring of $char(R) \neq 2$ and $C$ be the extended centroid of $R$. Let $b,p,q,m,u \in U$ and $\eta(x_1,\dots,x_n)$ be any polynomial over $C$, which is not central valued on $R$. If*

$$p[b,\eta(r)] + [b,\eta(r)]q = m\eta(r) + \eta(r)u$$

*for all $r = (r_1,\dots,r_n) \in R^n$, then one of the following holds:*

1. *$b \in C$;*

2. *$p,q \in C$;*

3. $q, b, u \in C$;

4. $p, b, m \in C$;

5. *there exists* $0 \neq \alpha \in C$ *such that* $p + \alpha b \in C$ *and* $q + \alpha b \in C$.

6. $R$ *satisfies* $s_4$.

*Proof.* $G$ denotes the additive subgroup of $R$ generated by the set

$$S = \{\eta(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}.$$

Since $\eta(x_1, \ldots, x_n)$ is noncentral valued on $R$, $S \neq \{0\}$. By assumption, $p[b, x] + [b, x]q = mx + xu$ for all $x \in G$. Since $char(R) \neq 2$, by [20] either $G \subseteq Z(R)$ or $G$ contains a noncentral Lie ideal $L$ of $R$. Since $\eta(x_1, \ldots, x_n) \in G$ and $\eta(x_1, \ldots, x_n)$ is a polynomial which is not central valued on $R$, therefore $G \not\subseteq Z(R)$. Thus we conclude that $G$ contains a noncentral Lie ideal $L$ of $R$. Then by [8, Lemma 1], there exists a noncentral two sided ideal $I$ of $R$ such that $[I, R] \subseteq L$. In particular,

$$p[b, [x_1, x_2]] + [b, [x_1, x_2]]q = m[x_1, x_2] + [x_1, x_2]u$$

for all $x_1, x_2 \in I$. By [21],

$$p[b, [x_1, x_2]] + [b, [x_1, x_2]]q = m[x_1, x_2] + [x_1, x_2]u$$

is a generalized polynomial identity for $R$ and for $U$. Then by [44, Proposition 2.5], conclusions follow. □

**Lemma 5.2.5.** *Suppose that $R$ is a noncommutative prime ring with Utumi quotient ring $U$, extended centroid $C$ and $f(r_1, \ldots, r_n)$ a noncentral multilinear polynomial over $C$. Let $char(R) \neq 2$. If $F$, $G$ and $H$ are three nonzero inner generalized derivations of $R$ such that*

$$F\Big([G(f(r)), f(r)]\Big) = H(f(r)^2)$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

(1) *there exist $\lambda, \mu \in C$ and $a, b \in U$ such that $F(x) = \mu x$, $G(x) = ax + xa + \lambda x$, $H(x) = [b, x]$ for all $x \in R$ with $b - \mu a \in C$;*

(2) *there exist* $\lambda, \mu \in C$ *and* $a, b, q \in U$ *such that* $F(x) = qx + xq + \mu x$, $G(x) = ax + xa + \lambda x$, $H(x) = [b, x]$ *for all* $x \in R$ *with* $q + \alpha a \in C$ *and* $q^2 + \mu q + \alpha b \in C$ *for some* $0 \neq \alpha \in C$*;*

(3) $f(r_1, \ldots, r_n)^2$ *is central valued on* $R$ *and there exist* $\lambda \in C$ *and* $a, b, p, q \in U$ *such that* $F(x) = px + xq$, $G(x) = ax + xa + \lambda x$ *and* $H(x) = [b, x]$ *for all* $x \in R$*;*

(4) $R$ *satisfies* $s_4$.

*Proof.* Assume that $F(x) = px + xq$, $G(x) = ax + xb$ and $H(x) = mx + xu$ for all $x \in R$ and for some $a, b, p, q, m, u \in U$. Then by hypothesis

$$F\Big([G(f(r)), f(r)]\Big) = H(f(r)^2)$$

gives

$$(pa - m)f(r)^2 + pf(r)(b - a)f(r) - pf(r)^2 b + af(r)^2 q \\ +f(r)(b - a)f(r)q - f(r)^2(bq + u) = 0 \tag{5.2.1}$$

for all $r = (r_1, \ldots, r_n) \in R^n$.

By Lemma 5.2.3, either $p, q \in C$ or $b - a \in C$.

If $p, q \in C$, then $F(x) = (p + q)x$ and hence (5.2.1) reduces to

$$\big((p + q)a - m\big)f(r)^2 + f(r)(b - a)(p + q)f(r) - f(r)^2\big((p + q)b + u\big) = 0.$$

Then by Lemma 5.2.1, we get $(b - a)(p + q) \in C$. If $p + q = 0$, then $F = 0$, a contradiction. Thus $0 \neq p + q \in C$. Hence $b - a \in C$.

Therefore, in any case $b - a \in C$. Let $b = a + \lambda$ for some $\lambda \in C$. Then

$$F\Big([G(f(r)), f(r)]\Big) = H(f(r)^2)$$

implies $F([af(r) + f(r)a + \lambda f(r), f(r)]) = H(f(r)^2)$ i.e., $F([a, f(r)^2]) = H(f(r)^2)$ for all $r = (r_1, \ldots, r_n) \in R^n$. Thus we have

$$p[a, f(r)^2] + [a, f(r)^2]q = mf(r)^2 + f(r)^2 u \tag{5.2.2}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Lemma 5.2.4, one of the following holds:

1. $f(r_1, \ldots, r_n)^2$ is central valued on $R$. Then (5.2.2) reduces to $(m+u)f(r)^2 = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$. This yields $m + u = 0$. Thus we have $F(x) = px+xq$, $G(x) = ax+xa+\lambda x$ and $H(x) = [m, x]$ for all $x \in R$. Thus conclusion (3) is obtained.

2. $a \in C$. Then (5.2.2) implies $mf(r)^2 + f(r)^2u = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$. By Lemma 5.2.2, one of the following holds:

   (i) $m = -u \in C$. Then $H(x) = 0$ which is a contradiction, since $H \neq 0$.

   (ii) $m = -u$ and $f(r_1, \ldots, r_n)^2$ is central valued on $R$. Thus $F(x) = px + xq$, $G(x) = x(a+b) = \mu x$ and $H(x) = [m, x]$ for all $x \in R$, where $\mu = a+b = 2a + \lambda \in C$. This is a particular case of conclusion (3).

3. $p, q \in C$. Let $\mu = p + q$. By (5.2.2), for all $r = (r_1, \ldots, r_n) \in R^n$,

$$\mu[a, f(r)^2] = mf(r)^2 + f(r)^2u$$

   i.e.,

$$(\mu a - m)f(r)^2 - f(r)^2(\mu a + u) = 0.$$

   By Lemma 5.2.2, one of the following holds:

   (i) $\mu a - m = \mu a + u \in C$, i.e., $m + u = 0$ and $m = \mu a + \eta$ for some $\eta \in C$. Then $F(x) = \mu x$, $G(x) = ax + xa + \lambda x$ and $H(x) = [m, x]$ for all $x \in R$ which is conclusion (1).

   (ii) $\mu a - m = \mu a + u$ and $f(r_1, \ldots, r_n)^2$ is central valued on $R$. Then $F(x) = \mu x$, $G(x) = ax + xa + \lambda x$ and $H(x) = [m, x]$ for all $x \in R$, which is a particular case of conclusion (3).

4. $q, a, u \in C$. By (5.2.2), $(m+u)f(r)^2 = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$ implying $m + u = 0$, i.e., $H(x) = 0$ which leads to a contradiction.

5. $p, a, m \in C$. By (5.2.2), $f(r)^2(m+u) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$ implying $m + u = 0$, i.e., $H(x) = 0$, a contradiction.

6. there exists $0 \neq \alpha \in C$ such that $p + \alpha a \in C$ and $q + \alpha a \in C$. Then $p - q \in C$, i.e., $p = q + \mu$ for some $\mu \in C$. Since $q + \alpha a \in C$, $0 = [q + \alpha a, x] = [q, x] + \alpha[a, x]$

and so $[a,x] = \beta[q,x]$, where $\beta = -\alpha^{-1} \in C$. Now, by (5.2.2),

$$\beta\Big((q+\mu)[q, f(r)^2] + [q, f(r)^2]q\Big) = mf(r)^2 + f(r)^2u$$

that is

$$[\beta(q^2+\mu q), f(r)^2] = mf(r)^2 + f(r)^2u.$$

We re-writing it as

$$(\beta(q^2+\mu q) - m)f(r)^2 - f(r)^2(\beta(q^2+\mu q)+u) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Lemma 5.2.2, one of the following holds:

(i) $\beta(q^2+\mu q) - m = \beta(q^2+\mu q) + u \in C$. It gives $m + u = 0$. So $F(x) = qx + xq + \mu x$, $G(x) = ax + xa + \lambda x$ and $H(x) = [m,x]$ for all $x \in R$ with $\beta(q^2+\mu q) - m \in C$ i.e., $q^2 + \mu q + \alpha m \in C$ (as $\beta = -\alpha^{-1}$) which is conclusion (2).

(ii) $\beta(q^2+\mu q) - m = \beta(q^2+\mu q) + u$ and $f(r_1, \ldots, r_n)^2$ is central valued on $R$. Thus $m + u = 0$. So $F(x) = qx + xq + \mu x$, $G(x) = ax + xa + \lambda x$ and $H(x) = [m,x]$ for all $x \in R$, which is conclusion (3).

7. $R$ satisfies $s_4$.

□

## 5.3 Proof of Main Theorem for general case

In all that follows, throughout this section, $R$ always be a prime ring with extended centroid $C$ and $U$ its Utumi ring of quotients. We always denote by $f(r_1, \ldots, r_n)$ a noncentral multilinear polynomial over $C$ and $f(r_1, \ldots, r_n) = \sum_{\sigma \in S_n} \alpha_\sigma r_{\sigma(1)} \ldots r_{\sigma(n)}$ for some $\alpha_\sigma \in C$.

The following facts are frequently used to prove our Theorem 5.1.1.

**Fact 5.3.1:** Let $d$ and $\delta$ be two derivations of $R$. By $f^d(r_1, \ldots, r_n)$ and $f^{\delta d}(r_1, \ldots, r_n)$, we denote the polynomial obtained from $f(r_1, \ldots, r_n)$ replacing each coefficients $\alpha_\sigma$ with $d(\alpha_\sigma)$ and $\delta d(\alpha_\sigma)$ respectively. Then we have

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$$

and

$$\delta d(f(r_1,\dots,r_n)) = f^{\delta d}(r_1,\dots,r_n) + \sum_i f^{\delta}(r_1,\dots,d(r_i),\dots,r_n)$$
$$+\sum_i f^{d}(r_1,\dots,\delta(r_i),\dots,r_n) + \sum_i f(r_1,\dots,\delta d(r_i),\dots,r_n)$$
$$+\sum_{i\neq j} f(r_1,\dots,\delta(r_i),\dots,d(r_j),\dots,r_n).$$

**Fact 5.3.2:** [64, Theorem 2] (see also [71, Theorem 1]) Let $Der(U)$ be the set of all derivations on $U$ and $D_{int}$ be the $C$-subspace of $Der(U)$ consisting of all inner derivations on $U$.

By a derivation word we mean an additive map $\Delta$ of the form $\Delta = d_1^{s_1} d_2^{s_2} \cdots d_m^{s_m}$, with each $d_i \in Der(U)$ and $s_i \geq 1$.

For prime ring $R$, let $\Phi(x_i^{\Delta_j})$ be a differential identity on $R$, involving $n$ derivation words $\Delta_1,\dots,\Delta_n$. Assume next that each $\Delta_j$ is a derivations word in the following form

$$\Delta_j = d_1^{s_{1,j}} d_2^{s_{2,j}} \cdots d_m^{s_{m,j}} \quad j = 1,\dots,n$$

and

$$s = max\{s_{i,j},\ i = 1,\dots,m\ j = 1,\dots,n\}.$$

Let $char(R) = p \neq 0$. If $d_1,\dots,d_m$ are linearly $C-$ independent modulo $D_{int}$ and $s < p$, then $\Phi(y_{ji})$ is a generalized polynomial identity on $R$, where $y_{ji}$ are distinct indeterminates.

**Fact 5.3.3:** [38, Lemma 1.2] Let $R$ be a prime ring of characteristic different from 2 and $f(r_1,\dots,r_n)$ a multilinear polynomial over $C$. If for any $i = 1,\dots,n$,

$$[f(r_1,\dots,z_i,\dots,r_n), f(r_1,\dots,r_n)] = 0$$

for all $r_1,\dots,r_n,z_i \in R$, then the polynomial $f(r_1,\dots,r_n)$ is central-valued on $R$.

In view of [72, Theorem 3], there exist $a,p,m \in U$ and derivations $d,g,h$ of $U$ such that $F(x) = px + d(x)$, $G(x) = ax + g(x)$ and $H(x) = mx + h(x)$. We know that $I$, $R$ and $U$ satisfy the same generalized polynomial identities (GPIs) (see [21]) and also the same differential identities (see [71]) and hence by hypothesis

$$p\Big[af(r) + g(f(r)), f(r)\Big] + d\Big(\Big[af(r) + g(f(r)), f(r)\Big]\Big)$$
$$= mf(r)^2 + h(f(r)^2) \qquad (5.3.1)$$

for all $r = (r_1, \dots, r_n) \in U^n$.

If $H = 0$, then by [26] and if $F = 0$ or $G = 0$, then by [5], we get our conclusions. Thus we assume that $F$, $G$ and $H$ all are nonzero maps.

If $F$, $G$ and $H$ are all inner generalized derivations, then by Lemma 5.2.5, we have our all conclusions of Theorem 5.1.1. Thus in all that follows we may assume that all of $F$, $G$ and $H$ are not inner. Therefore, we have the following cases:

- $d$, $g$ are inner and $h$ is outer.
- $d$, $h$ are inner and $g$ is outer.
- $g$, $h$ are inner and $d$ is outer.
- $d$ is inner, $g$ and $h$ are outer.
- $g$ is inner and $d$, $h$ are outer.
- $h$ is inner, $d$ and $g$ are outer.
- $d$, $g$ and $h$ all are outer.

We consider these cases into the following Lemmas.

**Lemma 5.3.1.** *If $d$, $g$ are inner and $h$ is outer, then no conclusion of Theorem 5.1.1 holds.*

*Proof.* Let $d(x) = [k, x]$ and $g(x) = [q, x]$ for all $x \in R$ and fixed $k, q \in U$. Then (5.3.1) reduces to

$$p\Big[af(r) + [q, f(r)], f(r)\Big] + \Big[k, \Big[af(r) + [q, f(r)], f(r)\Big]\Big] \\ = mf(r)^2 + h(f(r)^2) \tag{5.3.2}$$

for all $r = (r_1, \dots, r_n) \in U^n$. Since $h$ is not inner derivation on $U$, by Fact 5.3.2, we can replace each $h(r_i)$ with $z_i$ in (5.3.2) and then $U$ satisfies blended component

$$0 = \sum_i f(r_1, \dots, z_i, \dots, r_n) f(r_1, \dots, r_n) \\ + f(r_1, \dots, r_n) \sum_i f(r_1, \dots, z_i, \dots, r_n). \tag{5.3.3}$$

In particular, for $z_1 = r_1$ and $z_i = 0$ for all $i \geq 2$, $U$ satisfies

$$2f(r_1, \ldots, r_n)^2 = 0. \tag{5.3.4}$$

Since $char(R) \neq 2$, $f(r_1, \ldots, r_n)^2 = 0$, implying $f(r_1, \ldots, r_n) = 0$, a contradiction. $\square$

**Lemma 5.3.2.** *If $d$, $h$ are inner and $g$ is outer, then no conclusion of Theorem 5.1.1 holds.*

*Proof.* Let $d(x) = [k, x]$ and $h(x) = [k', x]$ for all $x \in R$ and for fixed $k, k' \in U$. Then (5.3.1) reduces to

$$\begin{aligned} p\Big[af(r) + g(f(r)), f(r)\Big] + \Big[k, \Big[af(r) + g(f(r)), f(r)\Big]\Big] \\ = mf(r)^2 + [k', f(r)^2] \end{aligned} \tag{5.3.5}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. In this case as $g$ is not inner derivation on $U$, by Fact 5.3.2, we replace $g(r_i)$ with $y_i$ in equation (5.3.5) and then $U$ satisfies the blended component

$$\begin{aligned} p\Big[\sum_i f(r_1, \ldots, y_i, \ldots, r_n), f(r_1, \ldots, r_n)\Big] \\ +\Big[k, \Big[\sum_i f(r_1, \ldots, y_i, \ldots, r_n), f(r_1, \ldots, r_n)\Big]\Big] = 0. \end{aligned} \tag{5.3.6}$$

Then replacing $y_i$ by $[A', r_i]$ for some $A' \notin C$ we get

$$\begin{aligned} p\Big[[A', f(r_1, \ldots, r_n)], f(r_1, \ldots, r_n)\Big] \\ +\Big[k, \Big[[A', f(r_1, \ldots, r_n)], f(r_1, \ldots, r_n)\Big]\Big] = 0. \end{aligned} \tag{5.3.7}$$

This is the situation

$$F([\delta(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in U^n$, where $\delta(x) = [A', x]$. Then by [26], $\delta = 0$, i.e., $A' \in C$, a contradiction. $\square$

**Lemma 5.3.3.** *If $g$, $h$ are inner and $d$ is outer, then the conclusion (3) of Theorem 5.1.1 holds.*

*Proof.* Let $g(x) = [q, x]$ and $h(x) = [k', x]$ for all $x \in R$ and for some fixed $q, k' \in U$. Then (5.3.1) reduces to

$$p\Big[af(r) + [q, f(r)], f(r)\Big] + d\Big(\Big[af(r) + [q, f(r)], f(r)\Big]\Big) \\ = mf(r)^2 + [k', f(r)^2)] \quad (5.3.8)$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

Since $d$ is not inner derivation on $U$, in this case by Fact 5.3.2, we can replace $d(r_i)$ with $x_i$ in the above relation and then $U$ satisfies the blended component

$$\Big(\Big[a\sum_i f(r_1, \ldots, x_i, \ldots, r_n) + [q, \sum_i f(r_1, \ldots, x_i, \ldots, r_n)], f(r_1, \ldots, r_n)\Big]\Big) \\ +\Big(\Big[af(r_1, \ldots, r_n) + [q, f(r_1, \ldots, r_n)], \sum_i f(r_1, \ldots, x_i, \ldots, r_n)\Big]\Big) = 0. \quad (5.3.9)$$

In particular, for $x_1 = r_1$ and $x_i = 0$ for all $i \geq 2$ and using $char(R) \neq 2$, we have

$$\Big(\Big[af(r) + [q, f(r)], f(r)\Big]\Big) = 0 \quad (5.3.10)$$

that is $[G(f(r)), f(r)] = 0$ for all $r = (r_1, \ldots, r_n) \in U^n$.
Since $[G(f(r)), f(r)] = 0$, by hypothesis, we have $H(f(r)^2) = 0$ for all $r = (r_1, \ldots, r_n) \in U^n$. By [5, Theorem 1], there exists $b' \in U$ such that $H(x) = [b', x]$ for all $x \in R$ with $f(r_1, \ldots, r_n)^2$ is central valued in $R$.

Again, since $[G(f(r)), f(r)] = 0$ for all $r = (r_1, \ldots, r_n) \in U^n$, then by [5, Theorem 2], one of the following holds:

(i) there exists $\lambda \in C$ such that $G(x) = \lambda x$ for all $x \in R$ which is a particular case of conclusion (3).

(ii) there exist $\lambda \in C$ and $b \in U$ such that $G(x) = bx + xb + \lambda x$ for all $x \in R$ which is conclusion (3). □

**Lemma 5.3.4.** *If $d$ is inner and $g$, $h$ are outer, then no conclusion of Theorem 5.1.1 holds.*

*Proof.* Let $d(x) = [k, x]$ for all $x \in R$ and for fixed $k \in U$. Then (5.3.1) reduces to

$$p\Big[af(r) + g(f(r)), f(r)\Big] + \Big[k, \Big[af(r) + g(f(r)), f(r)\Big]\Big] \\ = mf(r)^2 + h(f(r)^2) \quad (5.3.11)$$

for all $r = (r_1, \ldots, r_n) \in U^n$. Since $g$ and $h$ are outer, we consider the following cases:

**Case-1.** *Let $g$ and $h$ be linearly $C$-dependent.*

Then $h(x) = \alpha_1' g(x) + [k', x]$ for all $x \in R$ and for some $\alpha_1' \in C$ and $k' \in U$. Then (5.3.11) becomes

$$p\Big[af(r) + g(f(r)), f(r)\Big] + \Big[k, \Big[af(r) + g(f(r)), f(r)\Big]\Big] = mf(r)^2 + \alpha_1' g(f(r)^2) + [k', f(r)^2] \quad (5.3.12)$$

for all $r = (r_1, \dots, r_n) \in U^n$. By using Fact 5.3.2, we can replace $g(r_i)$ with $y_i$ and then $U$ satisfies the blended component

$$p\Big[\sum_i f(r_1, \dots, y_i, \dots, r_n), f(r_1, \dots, r_n)\Big] + \Big[k, \Big[\sum_i f(r_1, \dots, y_i, \dots, r_n), f(r_1, \dots, r_n)\Big]\Big]$$
$$= \alpha_1' \Big(\sum_i f(r_1, \dots, y_i, \dots, r_n) f(r_1, \dots, r_n) + f(r_1, \dots, r_n) \sum_i f(r_1, \dots, y_i, \dots, r_n)\Big)$$

In particular, for $y_1 = r_1$ and $y_i = 0$ for all $i \geq 2$, we get

$$0 = 2\alpha_1' f(r_1, \dots, r_n)^2$$

which implies $\alpha_1' = 0$ or $f(r_1, \dots, r_n) = 0$. In any cases, we have contradiction.

**Case-2.** *Let $g$ and $h$ be linearly $C$-independent.*

By Fact 5.3.2, we can replace $g(r_i)$ with $y_i$ and $h(r_i)$ with $z_i$ respectively in (5.3.11) and then $U$ satisfies the blended component

$$\sum_i f(r_1, \dots, z_i, \dots, r_n) f(r_1, \dots, r_n) + f(r_1, \dots, r_n) \sum_i f(r_1, \dots, z_i, \dots, r_n) = 0. \quad (5.3.13)$$

In particular, for $z_1 = r_1$ and $z_2 = \cdots = z_n = 0$, $U$ satisfies $2f(r_1, \dots, r_n)^2 = 0$. Since $char(R) \neq 2$, this implies that $f(r_1, \dots, r_n) = 0$, a contradiction. □

**Lemma 5.3.5.** *If $g$ is inner and $d$, $h$ are outer, then no conclusion of Theorem 5.1.1 holds.*

*Proof.* Let $g(x) = [q, x]$ for all $x \in R$ and for fixed $q \in U$. Then (5.3.1) reduces to

$$p\Big[af(r) + [q, f(r)], f(r)\Big] + d\Big(\Big[af(r) + [q, f(r)], f(r)\Big]\Big) = mf(r)^2 + h(f(r)^2) \quad (5.3.14)$$

for all $r = (r_1, \dots, r_n) \in U^n$. In this case $h$ and $d$ are outer.

**Case-1.** *Let $h$ and $d$ be linearly $C$-dependent.*

Then for some $\alpha_1$, $\alpha_2 \in C$ and $k \in U$, $\alpha_1 h(x) + \alpha_2 d(x) = [k, x]$. Since $d$ is outer, there exist $\alpha_1' \in C$ and $k' \in U$ such that $h(x) = \alpha_1' d(x) + [k', x]$ for all $x \in R$. By (5.3.14),

$$\begin{aligned} p\Big[af(r) + [q, f(r)], f(r)\Big] &+ d\Big(\Big[af(r) + [q, f(r)], f(r)\Big]\Big) \\ &= mf(r)^2 + \alpha_1' d(f(r)^2) + [k', f(r)^2] \end{aligned} \tag{5.3.15}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By Fact 5.3.2, we can replace $d(r_i)$ with $x_i$ and then $U$ satisfies the blended component

$$\begin{aligned} \Big[a\sum_i f(r_1, \ldots, x_i, \ldots, r_n) &+ [q, \sum_i f(r_1, \ldots, x_i, \ldots, r_n)], f(r_1, \ldots, r_n)\Big] \\ +\Big[af(r_1, \ldots, r_n) &+ [q, f(r_1, \ldots, r_n)], \sum_i f(r_1, \ldots, x_i, \ldots, r_n)\Big] \\ &= \alpha_1'\Big(\sum_i f(r_1, \ldots, x_i, \ldots, r_n) f(r_1, \ldots, r_n) \\ &\quad + f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, x_i, \ldots, r_n)\Big). \end{aligned} \tag{5.3.16}$$

In particular, for $x_1 = r_1$ and $x_i = 0$ for all $i \geq 2$, we have by using $char(R) \neq 2$,

$$\Big[af(r) + [q, f(r)], f(r)\Big] = \alpha_1' f(r)^2 \tag{5.3.17}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. This can be re-written as

$$(a + q - \alpha_1')f(r)^2 - f(r)(2q + a)f(r) + f(r)^2 q = 0 \tag{5.3.18}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By Lemma 5.2.1, one of the following holds:

1. $q, a + q - \alpha_1', 2q + a \in C$ and $a + q - \alpha_1' - 2q - a + q = 0$. This gives $\alpha_1' = 0$, leads to a contradiction.

2. $f(r_1, \ldots, r_n)^2$ is central valued on $R$ and $a + q - \alpha_1' - 2q - a + q = 0$ implying $\alpha_1' = 0$, a contradiction.

**Case-2.** *Let $d$ and $h$ be linearly $C$-independent.*

By applying Fact 5.3.2 to (5.3.14), we can replace $d(r_i)$ with $x_i$ and $h(r_i)$ with $z_i$ respectively, and then $U$ satisfies the blended component

$$\begin{aligned} &\sum_i f(r_1, \ldots, z_i, \ldots, r_n) f(r_1, \ldots, r_n) \\ &+ f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, z_i, \ldots, r_n) = 0. \end{aligned} \tag{5.3.19}$$

In particular, $U$ satisfies $2f(r_1, \ldots, r_n)^2 = 0$. Since $char(R) \neq 2$, this implies that $f(r_1, \ldots, r_n) = 0$, a contradiction. □

**Lemma 5.3.6.** *If $h$ is inner, $d$ and $g$ are outer, then no conclusion of Theorem 5.1.1 holds.*

*Proof.* Let $h(x) = [k, x]$ for all $x \in R$ and for some $k \in U$. Then (5.3.1) reduces to

$$p\Big[af(r) + g(f(r)), f(r)\Big] + d\Big(\Big[af(r) + g(f(r)), f(r)\Big]\Big)$$
$$= mf(r)^2 + [k, f(r)^2] \tag{5.3.20}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. Now we have the following two cases:

**Case-1.** *Let $d$ and $g$ be linearly $C$-dependent.*

Then $d(x) = \alpha_1' g(x) + [k', x]$ for all $x \in R$, where $\alpha_1' \in C$ and $k' \in U$. Then by (5.3.20)

$$p\Big[af(r) + g(f(r)), f(r)\Big] + \alpha_1' g\Big(\Big[af(r) + g(f(r)), f(r)\Big]\Big)$$
$$+\Big[k', \Big[af(r) + g(f(r)), f(r)\Big]\Big] = mf(r)^2 + [k, f(r)^2]. \tag{5.3.21}$$

By Fact 5.3.2, we can replace $g(r_i)$ with $y_i$ and $g^2(r_i)$ with $c_i$ respectively in (5.3.21) and then $U$ satisfies the blended component

$$\alpha_1'\Big[\sum_i f(r_1, \ldots, c_i, \ldots, r_n), f(r_1, \ldots, r_n)\Big] = 0. \tag{5.3.22}$$

Since $\alpha_1' \neq 0$,

$$\Big[\sum_i f(r_1, \ldots, c_i, \ldots, r_n), f(r_1, \ldots, r_n)\Big] = 0. \tag{5.3.23}$$

By Fact 5.3.3, it leads to a contradiction.

**Case-2.** *Let $d$ and $g$ be linearly $C$-independent.*

By Fact 5.3.2, we can replace $d(r_i)$ with $x_i$, $g(r_i)$ with $y_i$ and $dg(r_i)$ with $z_i$ respectively in (5.3.20) and then $U$ satisfies the blended component

$$[\sum_i f(r_1, \ldots, z_i, \ldots, r_n), f(r_1, \ldots, r_n)] = 0. \tag{5.3.24}$$

By Fact 5.3.3, this leads to a contradiction. □

**Lemma 5.3.7.** *If all of $d$, $g$ and $h$ are outer, then no conclusion of Theorem 5.1.1 holds.*

*Proof.* We have the following cases.

**Case-1.** *$d$, $g$ and $h$ are linearly $C$-dependent.*

Then there exist $\alpha_1, \alpha_2, \alpha_3 \in C$, $q \in U$ such that $\alpha_1 d(x)+\alpha_2 g(x)+\alpha_3 h(x) = [q, x]$ for all $x \in U$. Since $d$ is not inner, $(\alpha_2, \alpha_3) \neq (0, 0)$.

Without loss of generality, we may assume $\alpha_3 \neq 0$. Then we can write $h(x) = \alpha'_1 d(x) + \alpha'_2 g(x) + [q', x]$ for all $x \in U$, where $\alpha'_1 = -\alpha_1\alpha_3^{-1}$, $\alpha'_2 = -\alpha_2\alpha_3^{-1}$ and $q' = \alpha_3^{-1} q$. Then (5.3.1) gives

$$\begin{aligned} p\Big[af(r) + g(f(r)), f(r)\Big] + d\Big(\Big[af(r) + g(f(r)), f(r)\Big]\Big) \\ = mf(r)^2 + \alpha'_1 d(f(r)^2) + \alpha'_2 g(f(r)^2) + [q', f(r)^2] \end{aligned} \qquad (5.3.25)$$

for all $r = (r_1, \ldots, r_n) \in U^n$. Now we have the following two sub-cases.

**Sub-case-i.** Let $d$ and $g$ be $C$-dependent modulo inner derivations of $U$. Then since $d$ and $g$ are outer, $g(x) = \beta'_1 d(x) + [t', x]$, where $\beta'_1 \in C$ and $t' \in C$. By (5.3.25),

$$\begin{aligned} p\Big[af(r) + \beta'_1 d(f(r)) + [t', f(r)], f(r)\Big] \\ +d\Big(\Big[af(r) + \beta'_1 d(f(r)) + [t', f(r)], f(r)\Big]\Big) \\ = mf(r)^2 + (\alpha'_1 + \beta'_1\alpha'_2)d(f(r)^2) + [\alpha'_2 t' + q', f(r)^2] \end{aligned} \qquad (5.3.26)$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

By Fact 5.3.2, we can replace $d(r_i)$ with $x_i$, $d^2(r_i)$ with $w_i$ in above relation and then $U$ satisfies the blended component

$$\beta'_1[\sum_i f(r_1, \ldots, w_i, \ldots, r_n), f(r_1, \ldots, r_n)] = 0,$$

where $w_i = d^2(r_i)$. Since $\beta'_1 \neq 0$, by Fact 5.3.3, this gives to a contradiction.

**Sub-case-ii.** Let $d$ and $g$ be $C$-independent modulo inner derivations of $U$.

Again, by Fact 5.3.2, we replace $d(r_i)$ with $x_i$, $g(r_i)$ with $y_i$ and $dg(r_i)$ with $z_i$ in (5.3.25) and then $U$ satisfies the blended component

$$\Big[\sum_i f(r_1, \ldots, z_i, \ldots, r_n), f(r_1, \ldots, r_n)\Big] = 0,$$

where $z_i = dg(r_i)$. Again, by Fact 5.3.3, above relation gives to a contradiction.

**Case-2.** *$d$, $g$ and $h$ are linearly $C$-independent.*

By Fact 5.3.2, substituting $d(r_i)$ with $x_i$, $g(r_i)$ with $y_i$, $h(r_i)$ with $t_i$ and $dg(r_i)$ with $z_i$ in (5.3.1) and then using Fact 5.3.3 to (5.3.1), $U$ satisfies the blended component

$$\sum_i f(r_1, \ldots, t_i, \ldots, r_n) f(r_1, \ldots, r_n) + f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, t_i, \ldots, r_n) = 0.$$

This is same as (5.3.13) and hence by same argument, it leads to a contradiction. □

# Chapter 6

# Vanishing Derivations on Some Subsets in Prime Rings Involving Generalized Derivations

## 6.1 Introduction

Let $R$ be a noncommutative prime ring with center $Z(R)$, extended centroid $C$ and Utumi quotient ring $U$. A well-known result of Posner [80] states that if $d$ is a derivation of $R$ such that $d(x)x - xd(x) \in Z(R)$ for all $x \in R$, then either $d = 0$ or $R$ is commutative. Numerous articles appear in the literature which aim to generalize Posner's result. For instance, in [74], Lee and Shiue study the case when $d(x)x - x\delta(x) \in C$ for all $x \in \{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in R\}$, where $d$ and $\delta$ are derivations of $R$ and $f(x_1, \ldots, x_n)$ is any polynomial over $C$ and obtained that either $d = \delta = 0$, or $\delta = -d$ and $f(x_1, \ldots, x_n)^2$ is central valued on $RC$, except when $char(R) = 2$ and $dim_C RC = 4$.

Recently, in [5] Argac and De Filippis, determine a complete description of two generalized derivations $F$ and $G$ of $R$ satisfying the condition $F(x)x - xG(x) = 0$ for all $x \in \{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in I\}$, where $f(x_1, \ldots, x_n)$ is a multilinear polynomial over $C$ and $I$ is an ideal of $R$.

More recently, in [86], Tiwari considers the situation when $F^2(u)u - G(u^2) = 0$ for all $u \in f(R)$ and then obtained all possible forms of the maps. Starting from the result in [86], Tiwari and Singh study its central-valued version (see [88]), that

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is $F^2(u)u - G(u^2) \in C$ for all $u \in \{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in R\}$.

The present chapter is then motivated by wanting to generalize the previous last two cited results. To do this, we will investigate the situation when $d(F^2(u)u - G(u^2)) = 0$ for all $u \in \{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in I\}$, where $f(x_1, \ldots, x_n)$ is a multilinear polynomial over $C$, $I$ is an ideal of $R$, $d$ is a nonzero derivation of $R$ and $F, G$ are two generalized derivations of $R$. More precisely, we prove the following:

**Theorem 6.1.1.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 and 3, $U$ be the Utumi quotient ring of $R$ and $C = Z(U)$ be the extended centroid of $R$. Suppose that $f(x_1, \ldots, x_n)$ is a noncentral multilinear polynomial over $C$ and $F$, $G$ are two generalized derivations of $R$ and $d$ is a nonzero derivation of $R$. Let $I$ be a nonzero ideal of $R$ and $f(I) = \{f(x_1, \ldots, x_n)|x_1, \ldots, x_n \in I\}$. If*

$$d(F^2(u)u - G(u^2)) = 0$$

*for all $u \in f(I)$, then one of the following holds:*

*(1) there exist $a, b, p \in U$ and $\delta$ a derivation of $R$ such that $d(x) = [a, x]$, $F(x) = xb$, $G(x) = px + \delta(x)$ for all $x \in R$ with $b^2 \in C$, $[a, p] = 0$ and $f(r_1, \ldots, r_n)^2$ is central valued on $R$;*

*(2) there exist $a, b, p, q \in U$ and $\alpha, \lambda, \mu \in C$ such that $d(x) = [a, x]$, $F(x) = xb$, $G(x) = px + xq$ for all $x \in R$ with $b^2 \in C$, $p + \alpha a = \lambda$, $q + \alpha a = \mu$ and $\alpha a^2 + (b^2 - \lambda - \mu)a \in C$;*

*(3) there exist $a, b, p, q \in U$ and $\alpha, \alpha', \beta \in C$ such that $d(x) = [a, x]$, $F(x) = bx$, $G(x) = px + xq$ for all $x \in R$ with $b^2 - p + q = \alpha'$, $q + \alpha a = \beta$ and $\alpha a^2 + (\alpha' - 2\beta)a \in C$;*

*(4) there exist $a, b, p \in U$ and $\delta$ a derivation of $R$ such that $d(x) = [a, x]$, $F(x) = bx$, $G(x) = px + \delta(x)$ for all $x \in R$ with $[a, b^2 - p] = 0$ and $f(r_1, \ldots, r_n)^2$ is central valued on $R$;*

*(5) there exists $b \in U$ such that $F(x) = xb$, $G(x) = b^2x$ for all $x \in R$ with $b^2 \in C$;*

*(6) there exists $b \in U$ such that $F(x) = bx$, $G(x) = b^2x$ for all $x \in R$;*

(7) *there exist* $b, p' \in U$ *such that* $F(x) = bx$, $G(x) = xb^2 + [p', x]$ *for all* $x \in R$ *and* $f(r_1, \dots, r_n)^2$ *is central valued on R;*

(8) *there exist* $b, p' \in U$ *such that* $b^2 \in C$, $F(x) = xb$, $G(x) = b^2x + [p', x]$ *for all* $x \in R$ *and* $f(r_1, \dots, r_n)^2$ *is central valued on R;*

(9) $R \subseteq M_2(C)$*, the* $2 \times 2$ *matrix ring over* $C$*, and there exist* $a, b, p, q \in U$ *such that* $d(x) = [a, x]$, $G(x) = px + xq$ *and either* $F(x) = xb$*, for all* $x \in R$*, or* $F(x) = bx$ *for all* $x \in R$*. Moreover* $b^2 \in C$*, and* $b^2 - p = \gamma I_2 - q$*, where* $I_2 \in M_2(C)$ *is the identity matrix and* $\gamma \in C$ *is the trace of the matrix* $q$*.*

## 6.2 Results for Inner Cases

In this section, we always assume that $R$ is a noncommutative prime ring of characteristic different from 2, with $U$ the Utumi ring of quotients of $R$, $C = Z(U)$ the extended centroid of $R$, $f(X_1, \dots, X_n)$ a multilinear polynomial over $C$ which is not central-valued on $R$, $f(R)$ the set of all evaluations of the multilinear polynomial $f(X_1, \dots, X_n)$ in $R$. Suppose that $char(R) \neq 2$.

Moreover, all the results contained in the present section Here we study a special generalized identity which will be useful for the proof of our theorem, in case the involved generalized derivations are inner. More precisely we analyze the case when there exist $a, b, c, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \in U$ such that

$$a_1u^2 + a_2ua_3u + a_4ua_5u + au^2a_6 + a_7u^2a + bucua + ua_8ua + u^2a_9 = 0 \qquad (6.2.1)$$

for all $u \in f(R)$.

We need the following:

**Lemma 6.2.1.** *Let* $R = M_m(C)$*,* $m \geq 2$*, be the ring of all* $m \times m$ *matrices over the field* $C$*. If* $R$ *satisfies (6.2.1), then either* $a \in Z(R)$ *or* $b \in Z(R)$ *or* $c \in Z(R)$*.*

*Proof.* We assume first that $C$ is an infinite field.

To prove this Lemma, we assume that $a$, $b$ and $c$ are noncentral. Then by Theorem 1.6.3, there exists an invertible matrix $P$ such that $PaP^{-1}$, $PbP^{-1}$ and $PcP^{-1}$ have all non-zero entries. Let $\varphi(x) = PxP^{-1}$ for all $x \in R$ be an automorphism of $R$. Since $f(R)$ is invariant under the action of all inner automorphisms of $R$, by (6.2.1),

$$\begin{aligned}&\varphi(a_1)u^2 + \varphi(a_2)u\varphi(a_3)u + \varphi(a_4)u\varphi(a_5)u+\\&+\varphi(a)u^2\varphi(a_6) + \varphi(a_7)u^2\varphi(a) + \varphi(b)u\varphi(c)u\varphi(a)\\&+u\varphi(a_8)u\varphi(a) + u^2\varphi(a_9) = 0\end{aligned} \tag{6.2.2}$$

for all $u \in f(R)$.

Here $e_{ij}$ denotes the usual matrix unit with 1 in $(i, j)$-entry and zero elsewhere. Since $f(r_1, \ldots, r_n)$ is noncentral multilinear polynomial in $M_m(C)$, by [71] (see also [76]), there exist $r_1, \ldots, r_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij} \in f(R)$, with $i \neq j$. Hence in particular, we may replace $u$ with $\gamma e_{ij}$ in (6.2.2). Then, since $u^2 = 0$, both left and right multiplying by $e_{ij}$, we get

$$\gamma^2 e_{ij}\varphi(b)e_{ij}\varphi(c)e_{ij}\varphi(a)e_{ij} = 0 \tag{6.2.3}$$

which is a contradiction, since $\gamma \neq 0$ and $\varphi(a)$, $\varphi(b)$ and $\varphi(c)$ have all non-zero entries. Thus we conclude that either $a$ or $b$ or $c$ is central matrix, when $C$ is an infinite field.

Next, we assume that $C$ is a finite field. Let $K$ be an infinite field which is an extension of the field $C$ and let $\overline{R} = M_m(K) \cong R \otimes_C K$. Note that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on $R$ if and only if it is central-valued on $\overline{R}$. The generalized polynomial

$$\begin{aligned}\chi(r_1, \ldots, r_n) &= a_1 f(r_1, \ldots, r_n)^2 + a_2 f(r_1, \ldots, r_n)a_3 f(r_1, \ldots, r_n)\\&+ a_4 f(r_1, \ldots, r_n)a_5' f(r_1, \ldots, r_n) + a f(r_1, \ldots, r_n)^2 a_6\\&+ a_7 f(r_1, \ldots, r_n)^2 a + b f(r_1, \ldots, r_n)c f(r_1, \ldots, r_n)a\\&+ f(r_1, \ldots, r_n)a_8 f(r_1, \ldots, r_n)a + f(r_1, \ldots, r_n)^2 a_9\end{aligned}$$

is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$, which is satisfied by $R$, that is, $\chi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$.

Complete linearization of $\chi(r_1, \ldots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, s_1, \ldots, s_n)$ in $2n$ indeterminates; moreover,

$$\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n) = 2^n \chi(r_1, \ldots, r_n).$$

Clearly the multilinear polynomial $\Theta(r_1, \ldots, r_n, s_1, \ldots, s_n)$ is a generalized polynomial identity for $R$ and $\overline{R}$ too. Since $char(C) \neq 2$, we obtain $\chi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and then, conclusion follows from the first argument. □

**Lemma 6.2.2.** *Let $R$ be a noncommutative prime ring and assume that (6.2.1) is a trivial generalized polynomial identity for $R$. Then either $a \in C$ or $b \in C$ or $c \in C$.*

*Proof.* In this proof, for sake of clearness, we denote $X = f(x_1, \ldots, x_n)$. Moreover we assume that $a, b$ and $c$ are noncentral elements. Since $R$ and $U$ satisfy same generalized polynomial identities (see [21]), $U$ satisfies (6.2.1). Moreover, by hypothesis, (6.2.1) is a trivial generalized polynomial identity for $R$. Hence the generalized polynomial identity

$$a_1X^2 + a_2Xa_3X + a_4Xa_5X + aX^2a_6$$
$$+ a_7X^2a + bXcXa + Xa_8Xa + X^2a_9$$

is the zero element in the free product $T = U *_C C\{x_1, \ldots, x_n\}$, that is

$$\begin{aligned} &a_1X^2 + a_2Xa_3X + a_4Xa_5X + aX^2a_6 \\ &+ a_7X^2a + bXcXa + Xa_8Xa + X^2a_9 = 0 \in T. \end{aligned} \tag{6.2.4}$$

Suppose $a, a_6, a_9, 1$ are linearly independent over $C$. We have $aX^2a_6 = 0 \in T$, forcing either $a = 0$ or $a_6 = 0$, which is a contradiction.

Thus, we assume that there exist $\alpha, \beta, \gamma, \eta \in C$ such that

$$\alpha a + \beta a_9 + \gamma a_6 + \eta = 0.$$

Moreover, $a \notin C$ implies $(\beta, \gamma) \neq (0, 0)$. We divide the rest of the proof into two cases.

**Assume** $\beta \neq 0$

In this case, we may write $a_9 = \alpha' a + \gamma' a_6 + \eta'$, for suitable $\alpha', \gamma', \eta' \in C$. Then, by (6.2.4)

$$\begin{aligned} &a_1X^2 + a_2Xa_3X + a_4Xa_5X + aX^2a_6 \\ &+ a_7X^2a + bXcXa + Xa_8Xa + X^2(\alpha' a + \gamma' a_6 + \eta') = 0 \in T. \end{aligned} \tag{6.2.5}$$

Notice that, if we suppose that $a, a_6, 1$ are linearly $C$-independent, by relation (6.2.5) and since $U$ is not GPI ring, it follows $(a + \gamma')X^2a_6 = 0 \in T$, which is again a contradiction. Hence there are $\lambda, \mu, \nu \in C$ such that

$$\lambda a + \mu a_6 + \nu = 0 \quad \mu \neq 0.$$

So we may write $a_6 = \lambda' a + \nu'$, for suitable $\lambda', \nu' \in C$. In this case (6.2.5) reduces to

$$\begin{aligned} &a_1X^2 + a_2Xa_3X + a_4Xa_5X + aX^2(\lambda' a + \nu') \\ &+ a_7X^2a + bXcXa + Xa_8Xa + X^2(\alpha' a + \gamma'\lambda' a + \gamma'\nu' + \eta') = 0 \in T. \end{aligned} \tag{6.2.6}$$

Once again, since $U$ is not GPI ring and $a \notin C$, by (6.2.6) it follows that

$$(\lambda' a + a_7 + \alpha' + \lambda'\gamma')X + bXc + Xa_8 = 0 \in T. \tag{6.2.7}$$

In case $c, a_8, 1$ are linearly $C$-independent, then the contradiction $bXc = 0 \in T$ follows. On the other hand, if $c, a_8, 1$ are linearly $C$-dependent, since $c \notin C$, we may assume there exist $\sigma, \tau \in C$ such that $a_8 = \sigma c + \tau$. Therefore, by (6.2.7) we get

$$(\lambda' a + a_7 + \alpha' + \lambda'\gamma')X + bXc + X(\sigma c + \tau) = 0 \in T$$

which implies $(b+\sigma)Xc = 0 \in T$. This contradicts the assumption $b \notin C$ and $c \notin C$.

**Assume now** $\gamma \neq 0$

Here, we write $a_6 = \alpha'' a + \beta'' a_9 + \eta''$, for suitable $\alpha'', \beta'', \eta'' \in C$. Thus (6.2.4) reduces to

$$\begin{aligned} &a_1X^2 + a_2Xa_3X + a_4Xa_5X + aX^2(\alpha'' a + \beta'' a_9 + \eta'') \\ &+ a_7X^2a + bXcXa + Xa_8Xa + X^2a_9 = 0 \in T. \end{aligned} \tag{6.2.8}$$

If $a, a_9, 1$ are linearly $C$-independent, (6.2.8) implies that $(1 + \beta'' a)X^2a_9 = 0 \in T$, a contradiction. Hence, $a, a_9, 1$ are linearly $C$-dependent, and since $a \notin C$, there are suitable $\lambda'', \mu'' \in C$ such that $a_9 = \lambda'' a + \mu''$. Substitution of $a_9$ in (6.2.8) leads to

$$\begin{aligned} &a_1X^2 + a_2Xa_3X + a_4Xa_5X + aX^2\big(\alpha'' a + \beta''(\lambda'' a + \mu'') + \eta''\big) \\ &+ a_7X^2a + bXcXa + Xa_8Xa + X^2(\lambda'' a + \mu'') = 0 \in T. \end{aligned} \tag{6.2.9}$$

In particular, since $a \notin C$,

$$(\alpha'' a + \lambda''\beta'' a + a_7 + \lambda'')X^2a + (bXc + Xa_8)Xa = 0 \in T$$

that is

$$(\alpha'' a + \lambda''\beta'' a + a_7 + \lambda'')X + (bXc + Xa_8) = 0 \in T.$$

In case $c, a_8, 1$ are linearly $C$-independent, then the contradiction $bXc = 0 \in T$ follows. Thus we assume $c, a_8, 1$ are linearly $C$-dependent, and since $c \notin C$, there exist $\sigma', \tau' \in C$ such that $a_8 = \sigma' c + \tau'$. Therefore

$$(\alpha'' a + \lambda''\beta'' a + a_7 + \lambda'')X + bXc + \sigma' Xc + \tau' X = 0 \in T.$$

Since $c \notin C$ and $U$ is not GPI ring, it follows $(b+\sigma')Xc = 0 \in T$, a contradiction again. □

**Lemma 6.2.3.** *Let $R$ be a noncommutative prime ring. If $R$ satisfies (6.2.1), then either $a \in Z(R)$ or $b \in Z(R)$ or $c \in Z(R)$.*

*Proof.* By hypothesis, since $R$ and $U$ satisfy same generalized polynomial identities (see [21]), $U$ satisfies (6.2.1). We denote

$$\begin{aligned}\chi(r) &= a_1 f(r)^2 + a_2 f(r) a_3 f(r) + a_4 f(r) a_5 f(r) + a f(r)^2 a_6 \\ &+ a_7 f(r)^2 a + b f(r) c f(r) a + f(r) a_8 f(r) a + f(r)^2 a_9\end{aligned} \tag{6.2.10}$$

where $r = (r_1, \ldots, r_n) \in U^n$. If (6.2.10) is trivial generalized polynomial identity for $U$, then we conclude from Lemma 6.2.2.

Thus we may assume that (6.2.10) is not trivial generalized polynomial identity for $U$; moreover, by contradiction we suppose simultaneously that $a \notin Z(R)$, $b \notin Z(R)$ and $c \notin Z(R)$.

If $C$ is infinite, $\chi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of $C$. Since both $U$ and $U \otimes_C \overline{C}$ are prime and centrally closed [39, Theorems 2.5 and 3.5], we may replace $R$ by $U$ or $U \otimes_C \overline{C}$ according to $C$ finite or infinite. Then $R$ is centrally closed over $C$ and $\chi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$. By Martindale's result (see Theorem 1.6.6), $R$ is then a primitive ring with nonzero socle $soc(R)$ and with $C$ as its associated division ring. Then, by Jacobson's theorem (see Theorem 1.6.5), $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$.

If $\dim_C V = m$, we have $R \cong M_m(C)$ and both $R$ and $M_m(C)$ satisfy (6.2.10). As $R$ is noncommutative, $m \geq 2$. By Lemma 6.2.1, we get that $a$ or $b$ or $c$ is in $C$, a contradiction.

If $\dim_C V = \infty$, then for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since $a, b$ and $c$ are not in $C$, they cannot commute any nonzero ideal of $R$. Thus $[a, soc(R)] \neq (0)$, $[b, soc(R)] \neq (0)$ and $[c, soc(R)] \neq (0)$. Hence there exist $h_1, h_2, h_3 \in soc(R)$ such that either $[a, h_1] \neq 0$, $[b, h_2] \neq 0$ and $[c, h_3] \neq 0$. By Litoff's theorem (see Theorem 1.6.7), there exists idempotent $e \in soc(R)$ such that $ah_1, h_1a, bh_2, h_2b, ch_3, h_3c, h_1, h_2, h_3 \in eRe$. We have $eRe \cong M_k(C)$ with

$k = \dim_C Ve$. By (6.2.10)

$$a_1 f(r)^2 + a_2 f(r) a_3 f(r) + a_4 f(r) a_5 f(r) + a f(r)^2 a_6$$
$$+ a_7 f(r)^2 a + b f(r) c f(r) a + f(r) a_8 f(r) a + f(r)^2 a_9 = 0$$

for all $r = (er_1e, \ldots, er_ne) \in R^n$. Thus, both right and left multiplying by $e$ in above relation, we have that

$$ea_1ef(r)^2 + ea_2ef(r)ea_3ef(r) + ea_4ef(r)ea_5ef(r) + eaef(r)^2ea_6e$$
$$+ ea_7ef(r)^2eae + ebef(r)ecef(r)eae + f(r)ea_8ef(r)eae + f(r)^2ea_9e = 0$$

for all $r = (r_1, \ldots, r_n) \in (eRe)^n$. As $eRe \cong M_k(C)$, by above arguments, we conclude that either $eae$ or $ebe$ or $ece$ are central elements of $eRe$. Thus $ah_1 = (eae)h_1 = h_1eae = h_1a$ or $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$ or $ch_3 = (ece)h_3 = h_3(ece) = h_3c$, a contradiction. □

In order to apply the previous result to our work, we assume that there exist $a, b, c, p, q \in U$ such that $d(x) = [a, x]$, $F(x) = bx + xc$ and $G(x) = px + xq$ for all $x \in R$. Thus $d, F, G$ are an inner derivation and two inner generalized derivations of $R$, respectively. As $d$ is nonzero, $a \notin C$. Then $d(F^2(u)u - G(u^2)) = 0$ for all $u \in f(R)$ yields

$$[a, b^2u^2 + 2bucu + uc^2u - pu^2 - u^2q] = 0$$

for all $u \in f(R)$. Expanding the identity, we obtain

$$a(b^2 - p)u^2 + 2abucu + auc^2u - au^2q - (b^2 - p)u^2a - 2bucua - uc^2ua + u^2qa = 0$$

for all $u \in f(R)$. We re-write it as

$$a_1u^2 + a_2ua_3u + a_4ua_5u - au^2a_6 + a_7u^2a + b'ucua + ua_8ua + u^2a_9 = 0 \quad (6.2.11)$$

for all $u \in f(R)$, where

$$a_1 = a(b^2 - p), a_2 = 2ab, a_3 = c, a_4 = a, a_5 = c^2,$$

$$a_6 = q, a_7 = -(b^2 - p), b' = -2b, a_8 = -c^2, a_9 = qa.$$

Moreover, we need the following results:

**Lemma 6.2.4.** *[11] Let $R$ be a noncommutative prime ring of char$(R) \neq 2$, $C$ the extended centroid of $R$ and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$. If $d$ is a nonzero derivation of $R$, $G$ a nonzero generalized derivation of $R$ such that $d(G(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n)) = 0$ for all $x_1, \ldots, x_n \in R$, then $f(x_1, \ldots, x_n)^2$ is central-valued on $R$ and there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$, and $d$ is an inner derivation of $R$ such that $d(a) = 0$.*

**Lemma 6.2.5.** *[30, Proposition 2.6] Let $R$ be a prime ring of char$(R) \neq 2$, $C$ the extended centroid of $R$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over $C$. If $a, b, c, w, p, q \in R$ such that*

$$bf(r)^2 + cf(r)wf(r) - cf(r)^2q - af(r)^2c - f(r)wf(r)c + f(r)^2p = 0$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then either $c$ or $w$ is central.*

**Lemma 6.2.6.** *Let $R$ be a noncommutative prime ring with char$(R) \neq 2$, $a, c, q \in U$, $p(x_1, \ldots, x_n)$ be any polynomial over $C$, which is not an identity for $R$. If $[c, ap(r) + p(r)q] = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

1. *$c \in C$;*
2. *$[c, a + q] = 0$ and $p(x_1, \ldots, x_n)$ is central valued on $R$;*
3. *$a = -q \in C$;*
4. *$a - q \in C$ and $q + \alpha c \in C$ for some $\alpha \in C$;*
5. *$R \subseteq M_2(C)$, the $2 \times 2$ matrix ring over $C$, and $a = q - \gamma I_2$, where $I_2 \in M_2(C)$ is the identity matrix and $\gamma \in C$ is precisely the trace of the matrix $q$.*

*Proof.* Under the assumption of the present Lemma and by [30, Lemma 2.12], we have that one of the following conclusion occurs:

(i) $c \in C$;

(ii) $[c, a + q] = 0$ and $p(x_1, \ldots, x_n)$ is central valued on $R$;

(iii) $a = -q \in C$;

(iv) $a - q \in C$ and $q + \alpha c \in C$ for some $\alpha \in C$;

(v) $R$ satisfies the standard identity $s_4$ and there exists $\gamma \in Z(R)$ such that $a = q - \gamma$ .

Hence we have to analyze the only case (v). To do this, of course we may suppose that $p(x_1, \ldots, x_n)$ is not central valued on $R$, otherwise $[c, a+q] = 0$ easily follows, and we would have finished. Analogously, we suppose in all that follows $c \notin C$ and $q \notin C$ (if not we easily obtain the conclusion $a = -q \in C$).

Since $R$ satisfies the standard identity $s_4$, without loss of generality we assume that $U = M_2(C)$, the $2\times 2$ matrix ring over $C$. We also recall that, since $char(R) \neq 2$ and $p(x_1, \ldots, x_n)$ is not central valued on $R$, the additive subgroup generated by $p(x_1, \ldots, x_n)$ contains a non-central Lie ideal $L$ of $U$ (see [20]). Moreover, it is well known that, in case of characteristic different from 2, since $U = M_2(C)$ is a simple ring, $[U, U] \subseteq L$. Therefore

$$\big[c, q[r_1, r_2] + [r_1, r_2]q - \gamma[r_1, r_2]\big] = 0 \quad \forall r_1, r_2 \in U. \qquad (6.2.12)$$

In particular, for $r_1 = q$ in (6.2.12) we get

$$\big[c, [q^2 - \gamma q, r_2]\big] = 0 \quad \forall r_2 \in U \qquad (6.2.13)$$

which implies

$$q^2 - \gamma q \in C = Z(U). \qquad (6.2.14)$$

We denote $c = \sum c_{ij}e_{ij}$, $q = \sum q_{ij}e_{ij}$, $cq = \sum a'_{ij}e_{ij}$, $qc = \sum b'_{ij}e_{ij}$, where any $c_{ij}, q_{ij}, a'_{ij}, b'_{ij}$ are elements of $C$. By relation (6.2.14) we obtain the following conditions:

$$\begin{aligned} &(q_{11} - q_{22})(q_{11} + q_{22} - \gamma) = 0 \\ &q_{12}(q_{11} + q_{22} - \gamma) = 0 \\ &q_{21}(q_{11} + q_{22} - \gamma) = 0 \end{aligned} \qquad (6.2.15)$$

If we suppose $q_{11}+q_{22}-\gamma \neq 0$, relations (6.2.15) say that $q_{12} = q_{21} = 0$ and $q_{11} = q_{22}$, then $q$ is a central matrix, a contradiction.

We may then conclude that $\gamma$ is precisely the trace of the matrix $q$. □

Now we are ready to prove our theorem for inner cases as follows:

**Lemma 6.2.7.** *Let $R$ be a prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, $I$ an ideal of $R$ and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$ which is not central valued on $R$. Assume that there exist*

*$a,b,c,p,q \in U$ such that $d(x)=[a,x]$, $F(x)=bx+xc$ and $G(x)=px+xq$ for all $x \in R$. If $d \neq 0$ and*

$$d(F^2(u)u - G(u^2)) = 0$$

*for all $u \in f(I)$, then one of the following holds:*

*(1) there exist $b', p' \in U$ such that $F(x)=xb'$, $G(x)=p'x+[q,x]$ for all $x \in R$ with $b'^2 \in C$, $[a,p']=0$ and $f(r_1,\dots,r_n)^2$ is central valued on $R$;*

*(2) there exists $b' \in U$ such that $F(x)=xb'$, $G(x)=b'^2x$ for all $x \in R$, with $b''^2 \in C$;*

*(3) there exist $b' \in U$ and $\alpha,\lambda,\mu \in C$ such that $F(x)=xb'$, for all $x \in R$, with $b'^2 \in C$, $p+\alpha a=\lambda$, $q+\alpha a=\mu$ and $(b'^2-\lambda-\mu)a+\alpha a^2 \in C$;*

*(4) there exist $b', p' \in U$ such that $F(x)=b'x$, $G(x)=p'x+[q,x]$ for all $x \in R$, with $[a,b'^2-p']=0$ and $f(r_1,\dots,r_n)^2$ is central valued on $R$;*

*(5) there exists $b' \in U$ such that $F(x)=b'x$, $G(x)=b'^2x$ for all $x \in R$;*

*(6) there exist $b' \in U$ and $\alpha,\alpha',\beta \in C$ such that $F(x)=b'x$, for all $x \in R$, with $b'^2-p+q=\alpha'$, $q+\alpha a=\beta$ and $\alpha a^2+(\alpha'-2\beta)a \in C$;*

*(7) $R \subseteq M_2(C)$ and there exists $b' \in U$ such that either $F(x)=xb'$, for all $x \in R$, or $F(x)=b'x$ for all $x \in R$. Moreover, $b'^2-p=\gamma I_2-q$, where $I_2 \in M_2(C)$ is the identity matrix and $\gamma \in C$ is the trace of $q$.*

*Proof.* As $d$ is nonzero, $a \notin C$. Then $d(F^2(u)u-G(u^2))=0$ for all $u \in f(I)$ yields

$$[a, b^2u^2+2bucu+uc^2u-pu^2-u^2q]=0$$

for all $u \in f(I)$. Expanding the identity, we obtain

$$a(b^2-p)u^2+2abucu+auc^2u-au^2q-(b^2-p)u^2a-2bucua-uc^2ua+u^2qa=0$$

for all $u \in f(I)$. Since $I$, $R$ and $U$ satisfy the same generalized polynomial identities (see [21]),

$$a(b^2-p)u^2+2abucu+auc^2u-au^2q-(b^2-p)u^2a-2bucua-uc^2ua+u^2qa=0 \quad (6.2.16)$$

for all $u \in f(U)$. By Lemma 6.2.3, either $b \in C$ or $c \in C$. Thus we consider the following two cases:

**Case I:** $b \in C$

In this case, $F(x) = xb'$ and so $F^2(x) = xb'^2$ for all $x \in R$, where $b' = b + c$. By hypothesis,

$$[a, ub'^2u - pu^2 - u^2q] = 0 \tag{6.2.17}$$

that is

$$[a, (ub'^2 - pu)u - u(uq)] = 0 \tag{6.2.18}$$

for all $u \in f(U)$. By Lemma 6.2.5, $b'^2 \in C$. From above relation,

$$[a, (b'^2 - p)u^2 - u^2q)] = 0 \tag{6.2.19}$$

for all $u \in f(U)$. Again, by Lemma 6.2.6, one of the following holds:

(i) $[a, b'^2 - p - q] = 0$ and $f(r_1, \ldots, r_n)^2$ is central valued. In this case, $G(x) = p'x - [q, x]$ for all $x \in R$, where $p' = p + q$. This is conclusion (1).

(ii) $b'^2 - p = q \in C$; in this case $G(x) = px + xq = (p + q)x$ for all $x \in R$. This is conclusion (2).

(iii) $b'^2 - p + q \in C$ with $q + \alpha a \in C$ for some nonzero $\alpha \in C$. Thus, $p - q \in C$, $p + \alpha a \in C$, so that there exist $\lambda, \mu \in C$ such that $p + \alpha a = \lambda$ and $q + \alpha a = \mu$. Then, both $[a, p] = 0$ and $[a, q] = 0$. Then, by (6.2.19) $[a, (b'^2 - p)u^2 - u^2q)] = 0$ that is $[(b'^2 - \lambda - \mu)a + \alpha a^2, u^2] = 0$, for all $u \in f(U)$. By Lemma 1 in [5], either $(b'^2 - \lambda - \mu)a + \alpha a^2 \in C$ or $f(r_1, \ldots, r_n)^2$ is central valued. Hence we get conclusion (3) or conclusion (1), respectively.

(iv) $R \subseteq M_2(C)$ and $b'^2 - p = \gamma I_2 - q$, where $I_2 \in M_2(C)$ is the identity matrix and $\gamma \in C$ is the trace of $q$. This is conclusion (7).

**Case II:** $c \in C$

In such case, $F(x) = b'x$ and so $F^2(x) = b'^2x$ for all $x \in R$, where $b' = b + c$. By hypothesis,

$$[a, (b'^2 - p)u^2 - u^2q] = 0 \tag{6.2.20}$$

for all $u \in f(U)$. By Lemma 6.2.6, one of the following holds:

(i) $[a, b'^2 - p - q] = 0$ and $f(r_1, \ldots, r_n)^2$ is central valued. In this case, $G(x) = px + xq = (p + q)x - [q, x]$ for all $x \in R$. This is conclusion (4).

(ii) $b'^2 - p = q \in C$; in this case, $G(x) = px + xq = (p+q)x$ for all $x \in R$. This is conclusion (5).

(iii) $b'^2 - p + q = \alpha' \in C$ with $q + \alpha a = \beta \in C$ for some nonzero $\alpha \in C$. Then, $[a, (b'^2 - p)u^2 - u^2 q] = 0$ implies $[\alpha a^2 + (\alpha' - 2\beta)a, u^2] = 0$ for all $u \in f(U)$. By Lemma 1 in [5], either $\alpha a^2 + (\alpha' - 2\beta)a \in C$ or $f(r_1, \ldots, r_n)^2$ is central valued. Thus, we get conclusion (6) or conclusion (4), respectively.

(iv) $R \subseteq M_2(C)$ and $b'^2 - p = \gamma I_2 - q$, where $I_2 \in M_2(C)$ is the identity matrix and $\gamma \in C$ is the trace of $q$. This is conclusion (7). □

## 6.3 Proof of Main Theorem

In light of the results contained in the previous section, Theorem 6.1.1 is proved if $d$, $F$ and $G$ are simultaneously inner. In this section, we then consider the case when at least one of them is not inner. We recall that, in view of [72, Theorem 3], there exist $a, b \in U$ and derivations $d', \delta$ of $U$ such that $F(x) = ax + d'(x)$ and $G(x) = bx + \delta(x)$. We know the fact that $I$, $R$ and $U$ satisfy the same generalized polynomial identities (GPIs) (see [21]) and also the same differential identities (see [71]) and hence by hypothesis

$$\begin{aligned} d\Big((a^2 - b)f(r_1, \ldots, r_n)^2 + ad'\big(f(r_1, \ldots, r_n)\big)f(r_1, \ldots, r_n) + d'(a)f(r_1, \ldots, r_n)^2 \\ + af(r_1, \ldots, r_n)d'\big(f(r_1, \ldots, r_n)\big) + d'^2\big(f(r_1, \ldots, r_n)\big)f(r_1, \ldots, r_n) \\ - \delta\big(f(r_1, \ldots, r_n)\big)f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)\delta\big(f(r_1, \ldots, r_n)\big)\Big) = 0 \end{aligned} \tag{6.3.1}$$

for all $r_1, \ldots, r_n \in U$.

Now, we denote by $Der(U)$ the set of all derivations on $U$. By a derivation word, we mean an additive map $\Delta$ of the form $\Delta = d_1^{s_1} d_2^{s_2} \cdots d_m^{s_m}$, with each $d_i \in Der(U)$ and $s_i \geq 1$. Then, a differential polynomial is a generalized polynomial, with coefficients in $U$, of the form $\Phi(x_i^{\Delta_j})$ involving noncommutative indeterminates $x_i$ on which the derivations words $\Delta_j$ act as unary operations. The differential polynomial $\Phi(x_i^{\Delta_j})$ is said to be a differential identity on a subset $T$ of $U$ if it vanishes for any assignment of values from $T$ to its indeterminates $x_i$.

Let $D_{int}$ be the $C$-subspace of $Der(U)$ consisting of all inner derivations on $U$. By [64, Theorem 2], we have the following result (see also [71, Theorem 1]):

**Fact 6.3.1** Let $R$ be a prime ring, $d_1, \ldots, d_m \in Der(U)$, $\Phi(x_i^{\Delta_j})$ is a differential identity on $R$, involving $n$ derivation words $\Delta_1, \ldots, \Delta_n$. Assume that each $\Delta_j$ is a derivations word of the following form

$$\Delta_j = d_1^{s_{1,j}} d_2^{s_{2,j}} \cdots d_m^{s_{m,j}} \quad j = 1, \ldots, n$$

and let

$$s = max\{s_{i,j},\ i = 1, \ldots, m\ j = 1, \ldots, n\}.$$

If $d_1, \ldots, d_m$ are $C-$linearly independent modulo $D_{int}$ and $s < p$, if $char(R) = p \neq 0$, then $\Phi(y_{ji})$ is a generalized polynomial identity on $R$, where $y_{ji}$ are distinct indeterminates.

Since $f(x_1, \ldots, x_n)$ a multilinear polynomial, we can write

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

where $S_n$ is the permutation group over $n$ elements and any $\alpha_\sigma \in C$.

In all that follows and for any derivations $d$ and $\delta$ of $R$, we denote by $f^d(x_1, \ldots, x_n)$, $f^{d^2}(x_1, \ldots, x_n)$, $f^{d^3}(x_1, \ldots, x_n)$, $f^{\delta d}(x_1, \ldots, x_n)$, $f^{\delta^2 d}(x_1, \ldots, x_n)$ the polynomials obtained from $f(x_1, \ldots, x_n)$ replacing each coefficient $\alpha_\sigma$ with $d(\alpha_\sigma)$, $d^2(\alpha_\sigma)$, $d^3(\alpha_\sigma)$, $d(\delta(\alpha_\sigma))$ and $d(\delta^2(\alpha_\sigma))$, respectively. In this way, we have

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n),$$

$$d^2(f(r_1, \ldots, r_n)) = f^{d^2}(r_1, \ldots, r_n) + 2\sum_i f^d(r_1, \ldots, d(r_i), \ldots, r_n)$$
$$+ \sum_i f(r_1, \ldots, d^2(r_i), \ldots, r_n) + 2\sum_{i \neq j} f(r_1, \ldots, d(r_i), \ldots, d(r_j), \ldots, r_n),$$

$$
\begin{aligned}
&d^3(f(r_1,\ldots,r_n)) = \\
&f^{d^3}(r_1,\ldots,r_n) + 3\sum_i f^{d^2}(r_1,\ldots,d(r_i),\ldots,r_n) \\
&+ 3\sum_i f^{d}(r_1,\ldots,d^2(r_i),\ldots,r_n) + 6\sum_{i\neq j} f^{d}(r_1,\ldots,d(r_i),\ldots,d(r_j),\ldots,r_n) \\
&+ \sum_i f(r_1,\ldots,d^3(r_i),\ldots,r_n) + 3\sum_{i\neq j} f(r_1,\ldots,d(r_i),\ldots,d^2(r_j),\ldots,r_n) \\
&+ 6\sum_{i\neq j\neq k\neq i} f(r_1,\ldots,d(r_i),\ldots,d(r_j),\ldots,d(r_k),\ldots,r_n)
\end{aligned}
$$

and

$$
\begin{aligned}
&d\delta(f(r_1,\ldots,r_n)) = f^{\delta d}(r_1,\ldots,r_n) \\
&+ \sum_i f^{\delta}(r_1,\ldots,d(r_i),\ldots,r_n) + \sum_i f^{d}(r_1,\ldots,\delta(r_i),\ldots,r_n) \\
&+ \sum_i f(r_1,\ldots,d\delta(r_i),\ldots,r_n) + \sum_i f(r_1,\ldots,d(r_i),\ldots,\delta(r_j),\ldots,r_n).
\end{aligned}
$$

We begin with the following:

**Proposition 6.3.1.** *Assume char$(R) \neq 2$. Let $d(x) = [c,x]$ for all $x \in R$ and fixed $c \in U \setminus C$. Then, one of the following cases occurs:*

*(1) there exist $b, p \in U$ and $\delta$ a derivation of $R$ such that $F(x) = xb$, $G(x) = px + \delta(x)$ for all $x \in R$ with $b^2 \in C$, $[c,p] = 0$ and $f(r_1,\ldots,r_n)^2$ is central valued on $R$;*

*(2) there exist $b, p \in U$ and $\delta$ a derivation of $R$ such that $F(x) = bx$, $G(x) = px + \delta(x)$ for all $x \in R$ with $[c, b^2 - p] = 0$ and $f(r_1,\ldots,r_n)^2$ is central valued on $R$.*

*Proof.* For $d(x) = [c,x]$, we write (6.3.1) as

$$
\begin{aligned}
&\Big[c, (a^2-b)f(r_1,\ldots,r_n)^2 + ad'\big(f(r_1,\ldots,r_n)\big)f(r_1,\ldots,r_n) + d'(a)f(r_1,\ldots,r_n)^2 \\
&+ af(r_1,\ldots,r_n)d'\big(f(r_1,\ldots,r_n)\big) + d'^2\big(f(r_1,\ldots,r_n)\big)f(r_1,\ldots,r_n) \\
&- \delta\big(f(r_1,\ldots,r_n)\big)f(r_1,\ldots,r_n) - f(r_1,\ldots,r_n)\delta\big(f(r_1,\ldots,r_n)\big)\Big] = 0
\end{aligned}
\tag{6.3.2}
$$

for all $r_1,\ldots,r_n \in U$.

Assume firstly that $d', \delta$ are linearly $C$-independent modulo inner derivations. In this case, by Fact 6.3.1, we replace each $d'^2(r_i)$ with $y_i$ in (6.3.2), and then, $U$ satisfies blended component

$$[c, \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n)]. \tag{6.3.3}$$

Then, replacing $y_i$ with $[q, r_i]$ in (6.3.3), where $q \in U \setminus C$, $U$ satisfies

$$[c, \sum_i f(r_1, \ldots, [q, r_i], \ldots, r_n) f(r_1, \ldots, r_n)] = 0 \tag{6.3.4}$$

for all $r_i \in U$, that is

$$\big[c, [q, f(r_1, \ldots, r_n)] f(r_1, \ldots, r_n)\big] = 0$$

for all $r_i \in U$. By [11, Theorem], it follows either $c \in C$ or $q \in C$, in any case a contradiction.

Let now $\lambda, \mu \in C$ and $q \in U$ be such that $\lambda d'(x) + \mu\delta(x) = [q, x]$, for any $x \in R$.

In case $\mu = 0$ and $\lambda \neq 0$, we have $d'(x) = [\lambda^{-1}q, x]$ and $\delta$ is not an inner derivation of $R$. Hence, for $\lambda^{-1}q = c'$, $U$ satisfies

$$\begin{aligned}\Big[&c, (a^2 - b) f(r_1, \ldots, r_n)^2 + 2a[c', f(r_1, \ldots, r_n)] f(r_1, \ldots, r_n) \\ &+ d'(a) f(r_1, \ldots, r_n)^2 + \big[c', [c', f(r_1, \ldots, r_n)]\big] f(r_1, \ldots, r_n) \\ &- \delta(f(r_1, \ldots, r_n)) f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)\delta(f(r_1, \ldots, r_n))\Big] = 0\end{aligned} \tag{6.3.5}$$

for all $r_1, \ldots, r_n \in U$.

Since $\delta$ is not inner and by Fact 6.3.1, we replace $\delta(r_i)$ with $y_i$ in relation (6.3.5), and then, $U$ satisfies blended component

$$[c, \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n) + f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, y_i, \ldots, r_n)] = 0 \tag{6.3.6}$$

for all $r_i, y_i \in U$.

In particular, $U$ satisfies $[c, 2f(r_1, \ldots, r_n)^2] = 0$ implying $f(r_1, \ldots, r_n)^2$ is central valued (see Lemma 1 in [5]). Then, by (6.3.5),

$$\begin{aligned}\Big[&c, (a^2 - b) f(r_1, \ldots, r_n)^2 + 2a[c', f(r_1, \ldots, r_n)] f(r_1, \ldots, r_n) \\ &+ d'(a) f(r_1, \ldots, r_n)^2 + \big[c', [c', f(r_1, \ldots, r_n)]\big] f(r_1, \ldots, r_n)\Big] = 0\end{aligned} \tag{6.3.7}$$

for all $r_1, \dots, r_n \in U$.

Then by Lemma 6.2.3, either $a + c' \in C$ or $c' \in C$. If $a + c' \in C$, then $F(x) = (a + c')x - xc' = xa$ for all $x \in R$ and so

$$[c, f(r_1, \dots, r_n)a^2 f(r_1, \dots, r_n) - bf(r_1, \dots, r_n)^2] = 0 \tag{6.3.8}$$

which gives

$$[c, \big(f(r_1, \dots, r_n)a^2 - bf(r_1, \dots, r_n)\big)f(r_1, \dots, r_n)] = 0 \tag{6.3.9}$$

for all $r_1, \dots, r_n \in U$. By Lemma 6.2.4, $a^2 \in C$ and $[c, a^2 - b] = 0$. Thus, in this case $F(x) = xa$ and $G(x) = bx + \delta(x)$ for all $x \in R$, with $a^2 \in C$, $[c, b] = 0$ and $f(r_1, \dots, r_n)^2$ is central valued, as desired in conclusion (1).

On the other hand, if $c' \in C$ with $f(r_1, \dots, r_n)^2$ is central valued, then $d' = 0$ and hence $F(x) = ax$ for all $x \in R$. By (6.3.5),

$$[c, a^2 - b]f(r_1, \dots, r_n)^2 = 0 \tag{6.3.10}$$

for all $r_1, \dots, r_n \in U$. This implies either $f(r_1, \dots, r_n)^2 = 0$ or $[c, a^2 - b] = 0$. In the first case, $f(r_1, \dots, r_n)^2 = 0$ implies $f(r_1, \dots, r_n) = 0$, a contradiction. Thus, the conclusion (2) is obtained.

Let now $\lambda = 0$ and $\mu \neq 0$, so that $\delta(x) = [\mu^{-1}q, x]$ and $d'$ is not an inner derivation of $R$. Hence, for $\mu^{-1}q = c''$ and by (6.3.2),

$$\begin{aligned}\Big[&c, (a^2 - b)f(r_1, \dots, r_n)^2 + ad'(f(r_1, \dots, r_n))f(r_1, \dots, r_n)\\ &+ d'(a)f(r_1, \dots, r_n)^2 + ad'(f(r_1, \dots, r_n))f(r_1, \dots, r_n)\\ &+ d'^2(f(r_1, \dots, r_n))f(r_1, \dots, r_n) - [c'', f(r_1, \dots, r_n)]f(r_1, \dots, r_n)\\ &- f(r_1, \dots, r_n)[c'', f(r_1, \dots, r_n)]\Big] = 0\end{aligned} \tag{6.3.11}$$

for all $r_1, \dots, r_n \in U$.

Since $d'$ is not inner and by Fact 6.3.1, we replace $d'^2(r_i)$ with $y_i$ in relation (6.3.11), and then $U$ satisfies blended component (6.3.3) and a contradiction follows.

Finally consider the case both $\lambda \neq 0$ and $\mu \neq 0$. Hence, $\delta(x) = \alpha d'(x) + [a', x]$, where $\alpha = -\mu^{-1}\lambda$ and $a' = \mu^{-1}q$. Moreover, $d'$ is not inner, if not both $d'$ and $\delta$

must be inner. By (6.3.2)

$$\Big[c, (a^2-b)f(r_1,\ldots,r_n)^2 + ad'(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)$$
$$+ d'(a)f(r_1,\ldots,r_n)^2 + ad'(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n)$$
$$+ d'^2(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) \tag{6.3.12}$$
$$- \alpha d'(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) - f(r_1,\ldots,r_n)\alpha d'(f(r_1,\ldots,r_n))$$
$$- [a', f(r_1,\ldots,r_n)^2]\Big] = 0$$

for all $r_1,\ldots,r_n \in U$. In particular, by Fact 6.3.1, we replace $d'^2(r_i)$ with $y_i$ in above relation, and then, again $U$ satisfies the blended component (6.3.3). As above, we have a contradiction. □

We now consider another special case of our Theorem 6.1.1:

**Proposition 6.3.2.** *Assume char$(R) \neq 2$. Let $d'(x) = [q,x]$ for all $x \in R$ and fixed $q \in U \setminus C$. Then one of the following cases occurs:*

*(1) there exist $c,b,p \in U$ and $\delta$ a derivation of $R$ such that $d(x) = [c,x]$, $F(x) = xb$, $G(x) = px + \delta(x)$ for all $x \in R$ with $b^2 \in C$, $[c,p] = 0$ and $f(r_1,\ldots,r_n)^2$ is central valued on $R$;*

*(2) there exist $c,b,p \in U$ and $\delta$ a derivation of $R$ such that $d(x) = [c,x]$, $F(x) = bx$, $G(x) = px + \delta(x)$ for all $x \in R$ with $[c,b^2-p] = 0$ and $f(r_1,\ldots,r_n)^2$ is central valued on $R$;*

*(3) there exists $a \in U$ such that $a^2 \in C$, $F(x) = xa$, $G(x) = a^2x$ for all $x \in R$;*

*(4) there exist $a,p' \in U$ such that $a^2 \in C$, $F(x) = xa$, $G(x) = a^2x + [p',x]$ for all $x \in R$ and $f(r_1,\ldots,r_n)^2$ is central valued on $R$;*

*(5) there exist $a \in U$ such that $F(x) = ax$, $G(x) = a^2x$ for all $x \in R$;*

*(6) there exist $a,p' \in U$ such that $F(x) = ax$, $G(x) = xa^2 + [p',x]$ for all $x \in R$ and $f(r_1,\ldots,r_n)^2$ is central valued.*

*Proof.* Under the assumption of the present proposition, we have that

$$\begin{aligned} d\Big( &(a^2-b)f(r_1,\ldots,r_n)^2 + 2a[q,f(r_1,\ldots,r_n)]f(r_1,\ldots,r_n) \\ &+ [q,a]f(r_1,\ldots,r_n)^2 + \big[q,[q,f(r_1,\ldots,r_n)]\big]f(r_1,\ldots,r_n) \\ &- \delta(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) - f(r_1,\ldots,r_n)\delta(f(r_1,\ldots,r_n))\Big) = 0 \end{aligned} \tag{6.3.13}$$

for all $r_1,\ldots,r_n \in U$.

In case $d,\delta$ are linearly $C$-independent modulo inner derivations and by Fact 6.3.1, we may replace in (6.3.13) each $d\delta(r_i)$ with arbitrary $y_i$. It follows that $U$ satisfies the component

$$\sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n)$$

and in particular

$$2f(r_1,\ldots,r_n)^2 = 0 \quad \forall r_1,\ldots,r_n \in U.$$

Thus, $f(x_1,\ldots,x_n)$ should be an identity for $U$, a contradiction.

Hence, we must assume that there exist $\lambda,\mu \in C$ and $p \in U$ such that $\lambda d(x) + \mu\delta(x) = [p,x]$, for any $x \in R$. In the case, $\mu = 0$ and $\lambda \neq 0$, $d$ is an inner derivation of $R$ and the conclusion follows from Proposition 6.3.1. Therefore, we may suppose $\mu \neq 0$, so that $\delta(x) = \alpha d(x) + [p',x]$, for any $x \in R$, where $\alpha = -\mu^{-1}\lambda$ and $p' = \mu^{-1}p$. Ofcourse, we also assume that $d$ is not an inner derivation of $R$; otherwise, by Proposition 6.3.1, we obtain conclusions (1) and (2) of the present proposition.

By (6.3.13) we have that

$$\begin{aligned} d\Big( &(a^2-b)f(r_1,\ldots,r_n)^2 + 2a[q,f(r_1,\ldots,r_n)]f(r_1,\ldots,r_n) \\ &+ [q,a]f(r_1,\ldots,r_n)^2 + \big[q,[q,f(r_1,\ldots,r_n)]\big]f(r_1,\ldots,r_n) \\ &- \alpha d(f(r_1,\ldots,r_n))f(r_1,\ldots,r_n) - [p',f(r_1,\ldots,r_n)]f(r_1,\ldots,r_n) \\ &- f(r_1,\ldots,r_n)\alpha d(f(r_1,\ldots,r_n)) - f(r_1,\ldots,r_n)[p',f(r_1,\ldots,r_n)]\Big) = 0 \end{aligned} \tag{6.3.14}$$

for all $r_1,\ldots,r_n \in U$. The derivation words that appear in (6.3.14) have the form $d(r_i)$ and $d^2(r_i)$. Since $d$ is not inner, by Fact 6.3.1, we may replace each $d(r_i)$ and $d^2(r_i)$ with arbitrary $t_i$ and $y_i$, respectively. Thus, $U$ satisfies the component

$$\alpha\{\sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\}$$

and, as above, a contradiction follows, unless $\alpha = 0$, that is $\lambda = 0$. Therefore, for $\lambda = 0$, it follows that $\delta(x) = [p', x]$ and, by (6.3.14),

$$\begin{aligned} d\Big( &(a^2 - b)f(r_1, \dots, r_n)^2 + 2a[q, f(r_1, \dots, r_n)]f(r_1, \dots, r_n) \\ &+ [q, a]f(r_1, \dots, r_n)^2 + \big[q, [q, f(r_1, \dots, r_n)]\big]f(r_1, \dots, r_n) \\ &- [p', f(r_1, \dots, r_n)]f(r_1, \dots, r_n) - f(r_1, \dots, r_n)[p', f(r_1, \dots, r_n)]\Big) = 0 \end{aligned} \tag{6.3.15}$$

for all $r_1, \dots, r_n \in U$. Hence, the derivation words that appear in (6.3.15) have only the form $d(r_i)$ and, replacing each $d(r_i)$ with arbitrary $t_i$, $U$ satisfies the blended component

$$\begin{aligned} &(a^2 - b)\sum_i f(x_1, \dots, t_i, \dots, x_n)f(r_1, \dots, r_n) \\ &+(a^2 - b)f(r_1, \dots, r_n)\sum_i f(x_1, \dots, t_i, \dots, x_n) \\ &+2a[q, \sum_i f(x_1, \dots, t_i, \dots, x_n)]f(r_1, \dots, r_n) \\ &+2a[q, f(r_1, \dots, r_n)]\sum_i f(x_1, \dots, t_i, \dots, x_n) \\ &+[q, a]\sum_i f(x_1, \dots, t_i, \dots, x_n)f(r_1, \dots, r_n) \\ &+[q, a]f(r_1, \dots, r_n)\sum_i f(x_1, \dots, t_i, \dots, x_n) \\ &+\big[q, [q, \sum_i f(x_1, \dots, t_i, \dots, x_n)]\big]f(r_1, \dots, r_n) \\ &+\big[q, [q, f(r_1, \dots, r_n)]\big]\sum_i f(x_1, \dots, t_i, \dots, x_n) \\ &-[p', \sum_i f(x_1, \dots, t_i, \dots, x_n)]f(r_1, \dots, r_n) \\ &-[p', f(r_1, \dots, r_n)]\sum_i f(x_1, \dots, t_i, \dots, x_n) \\ &-\sum_i f(x_1, \dots, t_i, \dots, x_n)[p', f(r_1, \dots, r_n)] \\ &-f(r_1, \dots, r_n)[p', \sum_i f(x_1, \dots, t_i, \dots, x_n)] = 0. \end{aligned} \tag{6.3.16}$$

In particular, replacing $t_1$ with $x_1$ and $t_2 = \cdots = t_n = 0$, and then using $char(R) \neq 2$, we obtain the identity

$$F^2(f(r_1, \dots, r_n))f(r_1, \dots, r_n) - G(f(r_1, \dots, r_n)^2) = 0$$

for all $r_1, \ldots, r_n \in U$, where $F(x) = ax + [q, x]$ and $G(x) = bx + [p', x]$ for all $x \in R$.

Then, by [86], we have our conclusions (3)-(6). □

Now we have the following remark:

**Remark 6.3.3.** We would like to point out that, in each of the previous proved results, the only restriction regarding the characteristic was $char(R) \neq 2$.

In all that follows, we assume in addition $char(R) \neq 2, 3$.

**Proof of Theorem 6.1.1** Thanks to the results obtained in Propositions 6.3.1 and 6.3.2, in all that follows we may assume both $d$ and $d'$ are not inner derivations of $R$. Under this assumption, here we prove that a number of contradictions follows.

Let $d$, $d'$ and $\delta$ be $C$-independent modulo inner derivations of $U$.

Applying Fact 6.3.1 and replacing in (6.3.1) each $dd'^2(r_i)$ with $y_i$, we note that $U$ satisfies the blended component

$$\sum_i f(x_1, \ldots, y_i, \ldots, x_n) f(x_1, \ldots, x_n)$$

and in particular $f(r_1, \ldots, r_n)^2 = 0$, for any $r_1, \ldots, r_n \in U$. This implies the contradiction $f(r_1, \ldots, r_n) = 0$, for any $r_1, \ldots, r_n \in R$.

In all that follows, we then assume that $d$, $d'$ and $\delta$ are $C$-dependent modulo inner derivations of $U$, i.e., there exist $\lambda, \mu, \nu \in C$ and $q \in U$ such that

$$\lambda d(x) + \mu d'(x) + \nu \delta(x) = [q, x] \quad \forall x \in R. \tag{6.3.17}$$

Notice that, since $d$ is not inner, then $(\mu, \nu) \neq (0, 0)$.

**Case 1**

In case $\mu \neq 0$, by (6.3.17) we have that

$$d'(x) = \lambda' d(x) + \nu' \delta(x) + [q', x] \quad \forall x \in R \tag{6.3.18}$$

where $\lambda' = -\mu^{-1}\lambda$, $\nu' = -\mu^{-1}\nu$ and $q' = \mu^{-1}q$. Substitution of $d'$ in (6.3.1) leads to a differential identity in which the derivation words that appear are of the type

$$d, \delta, d\delta, \delta d, d^2, \delta^2, d^2\delta, d\delta d, d^3, d\delta^2.$$

We recall that

$$d\delta^2(f(r_1,\dots,r_n)) = f^{\delta^2 d}(r_1,\dots,r_n) + \sum_i f^{\delta^2}(r_1,\dots,d(r_i),\dots,r_n)$$
$$+2\sum_i f^{\delta d}(r_1,\dots,\delta(r_i),\dots,r_n) + 2\sum_i f^{\delta}(r_1,\dots,d\delta(r_i),\dots,r_n)$$
$$+2\sum_{i\neq j} f^{\delta}(r_1,\dots,d(r_i),\dots,\delta(r_j),\dots,r_n)$$
$$+\sum_i f^{d}(r_1,\dots,\delta^2(r_i),\dots,r_n) + \sum_i f(r_1,\dots,d\delta^2(r_i),\dots,r_n)$$
$$+\sum_{i\neq j} f(r_1,\dots,d(r_i),\dots,\delta^2(r_j),\dots,r_n)$$
$$+2\sum_{i\neq j} f^{d}(r_1,\dots,\delta(r_i),\dots,d(r_j),\dots,r_n)$$
$$+2\sum_{i\neq j} f(r_1,\dots,d\delta(r_i),\dots,d(r_j),\dots,r_n)$$
$$+2\sum_{i\neq j\neq k\neq i} f(r_1,\dots,\delta(r_i),\dots,d(r_j),\dots,\delta(r_k),\dots,r_n).$$

If we suppose that $d,\delta$ are linearly $C$-independent modulo inner derivations, using again Fact 6.3.1, we may replace each $d^3(r_i)$ and $d\delta^2(r_i)$ with $y_i$ and $t_i$, respectively. By computations, $U$ satisfies both the blended components

$$\lambda'^2\sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)$$

and

$$\nu'^2\sum_i f(x_1,\dots,t_i,\dots,x_n)f(x_1,\dots,x_n).$$

As above, we arrive at

$$\lambda'^2 f(r_1,\dots,r_n)^2 = 0 \quad \forall r_1,\dots,r_n\in U$$

and

$$\nu'^2 f(r_1,\dots,r_n)^2 = 0 \quad \forall r_1,\dots,r_n\in U.$$

The previous relations imply $\lambda'=0$ and $\nu'=0$, that is both $\lambda=0$ and $\nu=0$. On the other hand, this last conclusion forces $d'$ to be inner, which is a contradiction.

Consider now the case when there are suitable $\eta,\vartheta\in C$ and $p\in U$ such that

$$\eta d(x) + \vartheta\delta(x) = [p,x] \quad \forall x\in R$$

where $\vartheta \neq 0$ since $d$ is assumed to be not inner. Then, we may write

$$\delta(x) = \eta' d(x) + [p', x] \quad \forall x \in R \tag{6.3.19}$$

where $\eta' = -\vartheta^{-1}\eta$ and $p' = \vartheta^{-1}p$. Hence, (6.3.18) and (6.3.19) leads to

$$d'(x) = (\lambda' + \nu'\eta')d(x) + [q' + \nu' p', x] \quad \forall x \in R. \tag{6.3.20}$$

Once again, we substitute $d'$ in (6.3.1). In this last case, we arrive at a differential identity in which the derivation words that appear are of the type

$$d, d^2, d^3.$$

Since $d$ is not inner, we may replace each $d(r_i), d^2(r_i), d^3(r_i)$ with $y_i, t_i, z_i$, respectively. In particular, $U$ satisfies the blended component

$$(\lambda' + \nu'\eta')^2 \sum_i f(x_1, \ldots, z_i, \ldots, x_n) f(x_1, \ldots, x_n)$$

that is

$$(\lambda' + \nu'\eta')^2 f(r_1, \ldots, r_n)^2 = 0 \quad \forall r_1, \ldots, r_n \in U.$$

Thus, $\lambda' + \nu'\eta' = 0$ and $d'$ is an inner derivation, which is again a contradiction.

**Case 2**

Start again by (6.3.17) and consider now the case $\nu \neq 0$. Thus,

$$\delta(x) = \lambda'' d(x) + \mu'' d'(x) + [q'', x] \quad \forall x \in R \tag{6.3.21}$$

where $\lambda'' = -\nu^{-1}\lambda$, $\mu'' = -\nu^{-1}\mu$ and $q'' = \nu^{-1}q$. In light of (6.3.21), the derivation words that appear in (6.3.1) are

$$d, d', dd', d^2, d'^2, dd'^2.$$

If $d, d'$ are linearly $C$-independent modulo inner derivations, we may replace each $dd'^2(r_i)$ with $y_i$. As a consequence, $U$ satisfies the blended component

$$\sum_i f(x_1, \ldots, y_i, \ldots, x_n) f(x_1, \ldots, x_n)$$

implying the contradiction $f(r_1, \ldots, r_n)^2 = 0$, for any $r_1, \ldots, r_n \in U$.

Hence, we finally consider $\sigma d(x) + \tau d'(x) = [v, x]$ for any $x \in R$, for suitable elements $\sigma, \tau \in C$ and $v \in U$. Moreover, since both $d$ and $\delta$ cannot be inner derivations, it follows that $\sigma \neq 0$ and $\tau \neq 0$. Thus, we write

$$d'(x) = \sigma' d(x) + [v', x] \quad \forall x \in R$$

where $\sigma' = -\tau^{-1}\sigma \neq 0$ and $v' = \tau^{-1}v$. Hence, by (6.3.21),

$$\delta(x) = (\lambda'' + \mu''\sigma')d(x) + [\mu''v' + q'', x] \quad \forall x \in R. \qquad (6.3.22)$$

If we substitute $\delta$ in (6.3.1), we obtain a differential identity containing the derivation words $d, d^2, d^3$. Since $d$ is not inner, we replace each $d(r_i), d^2(r_i), d^3(r_i)$ with $y_i, t_i, z_i$, respectively. Thus, $U$ satisfies the blended component

$$\sigma'^2 \sum_i f(x_1, \ldots, z_i, \ldots, x_n) f(x_1, \ldots, x_n).$$

In particular,

$$\sigma'^2 f(r_1, \ldots, r_n)^2 = 0 \quad \forall r_1, \ldots, r_n \in U$$

which is again a contradiction, since $\sigma' \neq 0$ and $f(x_1, \ldots, x_n)$ is not an identity for $R$.

This completes the proof of the Theorem.

# Chapter 7

# Generalized Skew Derivations Acting on Multilinear Polynomials in Prime Rings

## 7.1 Introduction

Throughout this chapter, unless specifically stated, $R$ always denotes an associative prime ring with center $Z(R)$ and with extended centroid $C$. Always $Q_r$ denotes the right Martindale quotient ring of $R$. Let $S \subseteq R$ and $\xi : R \to R$ be an additive map. The map $\xi$ is said to act as a homomorphism on $S$ if $\xi(xy) = \xi(x)\xi(y)$ for all $x, y \in S$. The map $\xi$ is said to be a Jordan homomorphism acting on $S$, if $\xi(x^2) = \xi(x)^2$ for all $x \in S$. So every homomorphism map is Jordan homomorphism, but the converse is not true in general. Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $C$ in $n$ non-commuting indeterminates and $f(R) = \{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in R\}$.

In [13], Carini et al. studied the case when $F(u)G(u) = 0$ for all $u \in f(R)$, where $F$ and $G$ are generalized derivations of $R$ and then describe all possible forms of $F$ and $G$. Recently Tiwari [84] studied $T_2(u)T_1(u) = T_1(u)u - uT_3(u)$ for all $u \in f(R)$, when $T_1, T_2, T_3$ are all generalized derivations of $R$.

In [47], De Filippis and Dhara studied the situation $F(u)F(u) = G(u^2)$ for all $u \in f(R)$, where $F$ and $G$ are generalized skew derivations of $R$ associated to the same automorphism and then obtain all possible forms of the maps.

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In a recent paper, Dhara et al. [32] generalize the concept of Jordan homomorphisms by considering two generalized skew derivations $F$ and $G$ such that $F(x^2) = F(x)G(x)$ for all $x \in R$. In this paper authors considered the situation $F(u^2) = F(u)G(u)$ for all $u \in f(R)$ and then described all possible forms of the maps.

In [14], Carini et al. studied the case $F(u)G(u) = 0$ for all $u \in f(R)$, where $F$ and $G$ are generalized skew derivations of $R$ and then describe all possible forms of $F$ and $G$. In another article [15], Carini et al. considered the situation $F(u)u - uG(u) \in C$ for all $u \in f(R)$, where $F$ and $G$ are generalized skew derivations of $R$. It is natural to consider a case combining both the situations, that is, $F(u)G(u) = F(u)u - uG(u)$ for all $u \in f(R)$. In this present chapter our motivation is to investigate this situation. More precisely, we prove the following Theorem.

**Theorem 7.1.1.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $Q_r$ be its right Martindale quotient ring, $C$ be its extended centroid and $f(r_1, \ldots, r_n)$ be a noncentral multilinear polynomial over $C$. Suppose that $T_1$, $T_2$ are nonzero generalized skew derivations on $R$. If $R$ satisfies the identity*

$$T_1(u)T_2(u) = T_1(u)u - uT_2(u)$$

*where $u \in f(R)$, then one of the following holds:*

1. *there exist $a, c \in Q_r$ such that $T_1(x) = xa$, $T_2(x) = cx$ for all $x \in R$ with $ac + c - a = 0$;*
2. *$f(x_1, \ldots, x_n)^2$ is central valued on $R$ and one of the following holds:*
   - *(a) there exists $\mu \in C$ such that $T_1(x) = -x + \mu pxp^{-1}$ and $T_2(x) = x - \mu^{-1}pxp^{-1}$ for all $x \in R$;*
   - *(b) $T_1(x) = ax$ and $T_2(x) = xc$ for all $x \in R$ with $ac + c - a = 0$.*

## 7.2 Some reductions in inner case

The following Lemma is a particular case of Lemma 3 of [5].

**Lemma 7.2.1.** *Let $R$ be a prime ring and $f(r_1,\dots,r_n)$ be a noncentral multilinear polynomial over $C$. If $a,b,c\in R$ such that*

$$f(r)af(r)+f(r)^2b-cf(r)^2=0$$

*for all $r=(r_1,\dots,r_n)\in R^n$, then one of the following holds:*

*(1)* $a,b,c\in C$ *and* $a+b-c=0$*;*

*(2)* $f(x_1,\dots,x_n)^2$ *is central valued on* $R$*,* $a\in C$ *and* $a+b-c=0$*;*

*(3)* $char(R)=2$ *and* $R$ *satisfies* $s_4$*.*

**Lemma 7.2.2.** *[28, Lemma 2.9] Let $R$ be a prime ring of $char(R)\neq 2$ and $p(x_1,\dots,x_n)$ be any polynomial over $C$ which is nonzero valued on $R$. If $a,b,c,c'\in R$ such that*

$$ap(r)+p(r)b+cp(r)c'=0$$

*for all $r=(r_1,\dots,r_n)\in R^n$, then one of the following holds:*

*(1)* $b,c'\in C$ *and* $a+b+cc'=0$*;*

*(2)* $a,c\in C$ *and* $a+b+cc'=0$*;*

*(3)* $a+b+cc'=0$ *and* $p(x_1,\dots,x_n)$ *is central valued on* $R$*.*

In all that follows we assume that $T_1(x)=ax+pxp^{-1}b$ and $T_2(x)=cx+pxp^{-1}q$ for all $x\in R$, where $a,b,c,p,q\in Q_r$. Suppose that $R$ satisfies

$$T_1(f(r))T_2(f(r))=T_1(f(r))f(r)-f(r)T_2(f(r))$$

for all $r=(r_1,\dots,r_n)\in R^n$. This gives after pre-multiplying by $p^{-1}$

$$\begin{aligned}p^{-1}(a+1)f(r)cf(r)+f(r)p^{-1}(bc-b)f(r)+p^{-1}(a+1)f(r)pf(r)p^{-1}q\\+f(r)p^{-1}bpf(r)p^{-1}q-p^{-1}af(r)^2=0.\end{aligned}\qquad(7.2.1)$$

Now we prove the following Lemmas.

**Lemma 7.2.3.** *Let $R$ be a prime ring of characteristic different from 2, $Q_r$ be its right Martindale quotient ring, $C$ be its extended centroid and $f(r_1,\dots,r_n)$ be a noncentral multilinear polynomial over $C$. Let $T_1(x)=ax+pxp^{-1}b$ and $T_2(x)=cx+pxp^{-1}q$ for all $x\in R$, for some $a,b,c,q,p\in Q_r$. Suppose that $T_1(u)T_2(u)=T_1(u)u-uT_2(u)$ for all $u\in f(R)$. If $p^{-1}(a+1)\in C$, then one of the following holds:*

(1) $T_1(x) = x(a+b)$, $T_2(x) = (c+q)x$ *for all* $x \in R$ *with* $(a+b)(c+q)+(c+q)-(a+b) = 0$;

(2) $f(x_1,\ldots,x_n)^2$ *is central valued on* $R$ *and there exists* $\mu \in C$ *such that* $T_1(x) = -x+\mu pxp^{-1}$ *and* $T_2(x) = x-\mu^{-1}pxp^{-1}$ *for all* $x \in R$.

*Proof.* Since $p^{-1}(a+1) \in C$, we get from (7.2.1)

$$f(r)p^{-1}(ac+c+bc-b)f(r)+f(r)p^{-1}(ap+p+bp)f(r)p^{-1}q-p^{-1}af(r)^2 = 0 \quad (7.2.2)$$

Then by Lemma 3.7 in [17], either $p^{-1}(ap+p+bp) \in C$ or $p^{-1}q \in C$. Now we consider the following two cases:

**(i)** *When* $p^{-1}(ap+p+bp) \in C$.

This implies $a+b \in C$. Let $a+b+1 = \mu \in C$. Thus (7.2.2) reduces to

$$f(r)p^{-1}(ac+c+bc-b)f(r)+f(r)^2p^{-1}(aq+bq+q)-p^{-1}af(r)^2 = 0 \qquad (7.2.3)$$

Then by Lemma 7.2.1, any one of the following holds:

• $p^{-1}(ac+c+bc-b) \in C$, $p^{-1}a \in C$, $p^{-1}(aq+bq+q) \in C$ and $p^{-1}(ac+c+bc-b+aq+bq+q-a) = 0$ i.e., $(a+b)(c+q)+(c+q)-(a+b) = 0$, i.e., $\mu(c+q)-\mu+1 = 0$. Now Since $p^{-1}(a+1) \in C$ and $p^{-1}a \in C$ together yields $a, p \in C$. Again, since $a+b \in C$, we have $b \in C$. Now $p^{-1}(aq+bq+q) \in C$ implies $a+b+1 = \mu = 0$ or $q \in C$. Now $\mu = 0$ contradicts with the fact $\mu(c+q)-\mu+1 = 0$. Hence $q \in C$. Now $\mu(c+q)-\mu+1 = 0$ gives $c \in C$. Therefore, $T_1(x) = ax+pxp^{-1}b = (a+b)x$ and $T_2(x) = cx+pxp^{-1}q = (c+q)x$ for all $x \in R$. Then conclusion (1) holds.

• $f(x_1,\ldots,x_n)^2$ is central valued on $R$, $p^{-1}(ac+c+bc-b) \in C$ and $p^{-1}(ac+c+bc-b+aq+bq+q-a) = 0$. The last relation yields $(a+b+1)(c+q)-(a+b) = 0$, that is, $\mu(c+q-1)+1 = 0$. Now $p^{-1}(ac+c+bc-b) \in C$ implies that $p^{-1}(\mu c-b) \in C$. Since $p^{-1}(a+1) \in C$, we can write $p^{-1}(\mu c-a-b-1) \in C$ and hence $p^{-1}(\mu c-\mu) \in C$, that is, $\mu p^{-1}(c-1) \in C$. This implies either $\mu = 0$ or $p^{-1}(c-1) \in C$. But if $\mu = 0$, then $\mu(c+q-1)+1 = 0$ gives contradiction. Thus we conclude that $p^{-1}(c-1) \in C$. Let $p^{-1}(c-1) = \lambda \in C$. Now $\mu(c+q-1)+1 = 0$ yields $p^{-1}(c+q-1)+p^{-1}\mu^{-1} = 0$. This gives $\lambda+p^{-1}q+p^{-1}\mu^{-1} = 0$, i.e., $p^{-1}q = -p^{-1}\mu^{-1}-\lambda$.

Let $p^{-1}(a+1) = \nu \in C$. Then $a+b+1 = \mu \in C$ implies $p^{-1}b = \mu p^{-1}-\nu$. Thus $T_1(x) = ax+pxp^{-1}b = ax+px(\mu p^{-1}-\nu) = -x+\mu pxp^{-1}$, since $a-\nu p = -1$ and

$T_2(x) = cx + pxp^{-1}q = cx + px(-p^{-1}\mu^{-1} - \lambda) = x - \mu^{-1}pxp^{-1}$, since $c - \lambda p = 1$. This is our conclusion (2).

**(ii)** *When $p^{-1}q \in C$.*

By (7.2.2), we get

$$f(r)p^{-1}(ac + c + bc - b + aq + bq + q)f(r) - p^{-1}af(r)^2 = 0. \tag{7.2.4}$$

By Lemma 7.2.1, $p^{-1}(ac + c + bc - b + aq + bq + q) \in C$, $p^{-1}a \in C$ and $p^{-1}(ac + c + bc - b + aq + bq + q - a) = 0$ i.e., $(a + b)(c + q) + (c + q) - (a + b) = 0$. Since $p^{-1}(a + 1) \in C$, we have $p \in C$ and hence $a, q \in C$. Thus $T_1(x) = x(a + b)$ and $T_2(x) = (c + q)x$ for all $x \in R$. Then conclusion (1) holds. □

**Lemma 7.2.4.** *Let $R$ be a prime ring of characteristic different from 2, $Q_r$ be its right Martindale quotient ring, $C$ be its extended centroid and $f(r_1, \ldots, r_n)$ be a noncentral multilinear polynomial over $C$. Let $T_1(x) = ax + xb$ and $T_2(x) = cx + xq$ for all $x \in R$, for some $a, b, c, q \in Q_r$. Suppose that $T_1(u)T_2(u) = T_1(u)u - uT_2(u)$ for all $u \in f(R)$. Then one of the following holds:*

*(1) $T_1(x) = x(a + b)$, $T_2(x) = (c + q)x$ for all $x \in R$ with $(a + b)(c + q) + (c + q) - (a + b) = 0$;*

*(2) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and one of the following holds:*

- *(a) there exists $\mu \in C$ such that $T_1(x) = -x + \mu x$ and $T_2(x) = x - \mu^{-1}x$ for all $x \in R$;*
- *(b) $T_1(x) = (a + b)x$ and $T_2(x) = x(c + q)$ for all $x \in R$ with $(a + b)(c + q) + (c + q) - (a + b) = 0$.*

*Proof.* By (7.2.1), we get

$$af(r)cf(r) + f(r)(bc - b + c)f(r) + af(r)^2q + f(r)bf(r)q - af(r)^2 + f(r)^2q = 0. \tag{7.2.5}$$

By Lemma 3.7 in [17], either $a \in C$ or $c \in C$ and either $b \in C$ or $q \in C$. If $a \in C$, then conclusion follows by Lemma 7.2.3. Thus we consider the following two cases:

**(i)** *When $c, b \in C$.*

Then from (7.2.5) we get

$$(ac + bc - a - b + c)f(r)^2 + (a + b)f(r)^2q + f(r)^2q = 0. \tag{7.2.6}$$

Then using Lemma 7.2.2, we get any one of the following:

1. $q \in C$ and $(a+b)(c+q)+(c+q)=(a+b)$. In this case, $c+q \in C$ and hence, $a+b=(c+q)/(1-c-q)$. From the relation $(a+b)(c+q)+(c+q)=(a+b)$, we observe that $c+q \neq 1$. Hence, $a+b \in C$. Then $T_1(x)=(a+b)x$ and $T_2(x)=(c+q)x$ for all $x \in R$, which gives particular case of conclusion (1).

2. $a+b \in C$ and $(a+b)(c+q)+(c+q)=(a+b)$. This implies $a+b \neq -1$. Thus $c+q=(a+b)/(1+a+b) \in C$. Then $T_1(x)=(a+b)x$ and $T_2(x)=x(c+q)$ for all $x \in R$, which gives particular case conclusion (1).

3. $f(x_1,\ldots,x_n)^2$ is central valued on $R$ and $(a+b)(c+q)+(c+q)=(a+b)$. In this case $T_1(x)=(a+b)x$ and $T_2(x)=x(c+q)$ which gives conclusion (2)-(b).

**(ii)** *When $c, q \in C$.*
Then from (7.2.5) we get

$$(ac+aq-a+q)f(r)^2+f(r)(bc+bq-b+c)f(r)=0. \tag{7.2.7}$$

By Lemma 7.2.1, $bc+bq-b+c \in C$ and $(a+b)(c+q)+(c+q)=(a+b)$. Now $bc+bq-b+c \in C$ implies $b(c+q-1) \in C$. This gives either $c+q=1$ or $b \in C$. Now $c+q=1$ contradicts with the fact $(a+b)(c+q)+(c+q)=(a+b)$. Thus $b \in C$ and $a+b=(c+q)/(1-c-q) \in C$. Hence $T_1(x)=(a+b)x$, $T_2(x)=(c+q)x$ for all $x \in R$. This is a particular case of conclusion (1). □

**Lemma 7.2.5.** *Let $R$ be a prime ring of characteristic different from 2, $Q_r$ be its right Martindale quotient ring, $C$ be its extended centroid and $f(r_1,\ldots,r_n)$ be a noncentral multilinear polynomial over $C$. Let $T_1(x)=ax+pxp^{-1}b$ and $T_2(x)=cx+pxp^{-1}q$ for all $x \in R$, for some $a,b,c,q,p \in Q_r$. Suppose that $T_1(u)T_2(u)=T_1(u)u-uT_2(u)$ for all $u \in f(R)$. If $p^{-1}q \in C$, then one of the following holds:*

*(1)* $T_1(x)=x(a+b)$, $T_2(x)=(c+q)x$ *for all* $x \in R$ *with* $(a+b)(c+q)+(c+q)-(a+b)=0$;

*(2)* $f(x_1,\ldots,x_n)^2$ *is central valued on* $R$ *and there exists* $\mu \in C$ *such that* $T_1(x)=-x+\mu pxp^{-1}$ *and* $T_2(x)=x-\mu^{-1}pxp^{-1}$ *for all* $x \in R$.

*Proof.* Since $p^{-1}q \in C$, $T_1(x)=ax+pxp^{-1}b$ and $T_2(x)=(c+q)x$. Thus by (7.2.1),

$$p^{-1}(a+1)f(r)(c+q)f(r)+f(r)p^{-1}(bc-b+bq)f(r)-p^{-1}af(r)^2=0. \tag{7.2.8}$$

By Lemma 3.7 in [17], either $p^{-1}(a+1) \in C$ or $(c+q) \in C$. If $p^{-1}(a+1) \in C$, we get conclusions by Lemma 7.2.3. Now, if $(c+q) \in C$, then from above

$$p^{-1}(ac+c+aq+q-a)f(r)^2 + f(r)p^{-1}(bc-b+bq)f(r) = 0. \tag{7.2.9}$$

By Lemma 7.2.1, $p^{-1}(ac+c+aq+q-a) \in C$, $p^{-1}(bc-b+bq) \in C$ and $p^{-1}(ac+c+bc-b+aq+bq+q-a) = 0$. Now $p^{-1}b(c-1+q) \in C$ implies either $c+q=1$ or $p^{-1}b \in C$. Now $c+q=1$ contradicts with the fact that $(a+b)(c+q)+(c+q)-(a+b)=0$. Thus, $a+b=(c+q)/(1-c-q)$ and $p^{-1}b \in C$. Then $T_1(x)=(a+b)x$ for all $x \in R$, which gives particular case of conclusion (1). □

**Proposition 7.2.6.** *Suppose that $R = M_k(C)$ is the ring of all $k \times k$ matrices over the field $C$ with $k \geq 2$ and $p_1, p_2, p_3, p_4, p_5, p_6, p_7 \in R$. If*

$$p_1 f(r) p_2 f(r) + f(r) p_3 f(r) + p_1 f(r) p_4 f(r) p_5 + f(r) p_6 f(r) p_5 - p_7 f(r)^2 = 0 \tag{7.2.10}$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then either $p_1 \in C \cdot I_k$ or $p_4 \in C \cdot I_k$ or $p_5 \in C \cdot I_k$.*

*Proof.* We consider the following two cases:
**Case-I:** *Suppose that $C$ is infinite field.*
Let $p_1 \notin Z(R)$, $p_4 \notin Z(R)$ and $p_5 \notin Z(R)$, that is, $p_1$, $p_4$ and $p_5$ are not scalar matrices.
By Theorem 1.6.3, there exists an invertible matrix $\eta \in R$ such that $\phi(x) = \eta x \eta^{-1}$ an inner automorphism of $R$ and $\phi(p_1)$, $\phi(p_4)$, $\phi(p_5)$ have all non-zero entries. Clearly $R$ satisfies

$$\begin{aligned} \phi(p_1) f(r) \phi(p_2) f(r) + f(r) \phi(p_3) f(r) + \phi(p_1) f(r) \phi(p_4) f(r) \phi(p_5) \\ + f(r) \phi(p_6) f(r) \phi(p_5) - \phi(p_7) f(r)^2 = 0 \end{aligned} \tag{7.2.11}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Let $e_{ij}$ be the matrix whose $(i,j)$-entry is 1 and rest entries are zeros. Since $f(x_1, \ldots, x_n)$ is not central, by [71] (see also [76]), there exist $r_1, \ldots, r_n \in M_k(C)$ and $\gamma \in C - \{0\}$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$, with $i \neq j$. For the value of $f(r_1, \ldots, r_n) = \gamma e_{ij}$, (7.2.11) implies

$$\begin{aligned} \phi(p_1) e_{ij} \phi(p_2) e_{ij} + e_{ij} \phi(p_3) e_{ij} + \phi(p_1) e_{ij} \phi(p_4) e_{ij} \phi(p_5) \\ + e_{ij} \phi(p_6) e_{ij} \phi(p_5) - \phi(p_7) e_{ij}^2 = 0. \end{aligned} \tag{7.2.12}$$

Right multiplying above relation by $e_{ij}$, we obtain

$$\phi(p_1) e_{ij} \phi(p_4) e_{ij} \phi(p_5) e_{ij} + e_{ij} \phi(p_6) e_{ij} \phi(p_5) e_{ij} = 0. \tag{7.2.13}$$

Again left multiplying above relation by $e_{ij}$, we obtain

$$e_{ij}\phi(p_1)e_{ij}\phi(p_4)e_{ij}\phi(p_5)e_{ij} = 0 \tag{7.2.14}$$

which gives

$$\phi(p_1)_{ji}\phi(p_4)_{ji}\phi(p_5)_{ji} = 0,$$

a contradiction. Therefore, either $p_1$ or $p_4$ or $p_5$ is scalar matrix.

**Case-II:** *Suppose that $C$ is finite filed.*

Let $K'$ be an infinite field which is an extension of the field $C$. Let $\overline{R} = M_k(K') \cong R \otimes_C K'$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on $R$ if and only if it is central valued on $\overline{R}$. Let

$$\begin{aligned} P(r_1, \ldots, r_n) = p_1 f(r_1, \ldots, r_n)p_2 f(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)p_3 f(r_1, \ldots, r_n) \\ +p_1 f(r_1, \ldots, r_n)p_4 f(r_1, \ldots, r_n)p_5 + f(r_1, \ldots, r_n)p_6 f(r_1, \ldots, r_n)p_5 \\ -p_7 f(r_1, \ldots, r_n)^2. \end{aligned} \tag{7.2.15}$$

Since the generalized polynomial $P(r_1, \ldots, r_n)$ on $R$ is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$, the complete linearization of $P(r_1, \ldots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n)$ in $2n$ indeterminates. Moreover,

$$\Theta(r_1, \ldots, r_n, r_1, \ldots, r_n) = 2^n P(r_1, \ldots, r_n).$$

It is clear that the multilinear polynomial $\Theta(r_1, \ldots, r_n, x_1, \ldots, x_n)$ is a generalized polynomial identity for both $R$ and $\overline{R}$. Since $char(R) \neq 2$, we obtain $P(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and then we get conclusions as desired by Case-I. □

**Corollary 7.2.7.** *Let $R = M_m(C)$, $m \geq 2$ be the ring of all matrices over the field $C$ with $char(R) \neq 2$ and $p_1, p_2, p_3, p_4, p_5, p_6, p_7 \in R$. If*

$$p_1 r p_2 r + r p_3 r + p_1 r p_4 r p_5 + r p_6 r p_5 - p_7 r^2 = 0$$

*for all $r \in R$, then either $p_1 \in C \cdot I_k$ or $p_4 \in C \cdot I_k$ or $p_5 \in C \cdot I_k$.*

**Lemma 7.2.8.** *Let $R$ be a noncommutative prime ring and $p_1, p_2, p_3, p_4, p_5, p_6, p_7 \in R$. If*

$$\begin{aligned} p_1 f(x_1, \ldots, x_n)p_2 f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)p_3 f(x_1, \ldots, x_n) \\ +p_1 f(x_1, \ldots, x_n)p_4 f(x_1, \ldots, x_n)p_5 \\ +f(x_1, \ldots, x_n)p_6 f(x_1, \ldots, x_n)p_5 - p_7 f(x_1, \ldots, x_n)^2 = 0 \end{aligned} \tag{7.2.16}$$

*is a trivial GPI for $R$, then either $p_1$ or $p_4$ or $p_5$ is central.*

*Proof.* Let none of $p_1$, $p_4$ and $p_5$ be central. Since $R$ and $Q_r$ satisfy same generalized polynomial identity (GPI) (see [21]), $Q_r$ satisfies (7.2.16). Also (7.2.16) is a trivial GPI for $R$. Therefore,

$$\begin{aligned} p_1f(x_1,\ldots,x_n)p_2f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)p_3f(x_1,\ldots,x_n) \\ +p_1f(x_1,\ldots,x_n)p_4f(x_1,\ldots,x_n)p_5 \\ +f(x_1,\ldots,x_n)p_6f(x_1,\ldots,x_n)p_5 - p_7f(x_1,\ldots,x_n)^2 \end{aligned} \tag{7.2.17}$$

is the zero element in the free product $T = Q_r *_C C\{x_1,\ldots,x_n\}$. Since $p_5$ and 1 are linearly independent over $C$, from above

$$\Big\{p_1f(x_1,\ldots,x_n)p_4 + f(x_1,\ldots,x_n)p_6\Big\}f(x_1,\ldots,x_n)p_5 = 0 \in T.$$

Again, since $p_1$ and 1 are linearly independent over $C$,

$$p_1f(x_1,\ldots,x_n)p_4f(x_1,\ldots,x_n)p_5 = 0 \in T.$$

This implies either $p_1 = 0$ or $p_4 = 0$ or $p_5 = 0$, contradiction. □

**Lemma 7.2.9.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $Q_r$ be its right Martindale quotient ring, $C$ be its extended centroid and $p_1, p_2, p_3, p_4, p_5, p_6, p_7 \in Q_r$. Suppose that $f(r_1,\ldots,r_n)$ be a noncentral multilinear polynomial over $C$. If $R$ satisfies*

$$p_1f(r)p_2f(r) + f(r)p_3f(r) + p_1f(r)p_4f(r)p_5 + f(r)p_6f(r)p_5 - p_7f(r)^2 = 0,$$

*for all $r = (r_1,\ldots,r_n) \in R^n$, then either $p_1$ or $p_4$ or $p_5$ is central.*

*Proof.* Let

$$\begin{aligned} P(r_1,\ldots,r_n) = p_1f(r_1,\ldots,r_n)p_2f(r_1,\ldots,r_n) + f(r_1,\ldots,r_n)p_3f(r_1,\ldots,r_n) \\ +p_1f(r_1,\ldots,r_n)p_4f(r_1,\ldots,r_n)p_5 + f(r_1,\ldots,r_n)p_6f(r_1,\ldots,r_n)p_5 \\ -p_7f(r_1,\ldots,r_n)^2. \end{aligned} \tag{7.2.18}$$

If $P(r_1,\ldots,r_n) = 0$ is the trivial GPI for $R$, then by Lemma 7.2.8 we get our conclusion. Thus we assume that $P(r_1,\ldots,r_n) = 0$ is the non-trivial GPI for $R$ and so for $Q_r$ (see [21]). In case $C$ is infinite, we have $P(x_1,\ldots,x_n) = 0$ for all

$x_1, \dots, x_n \in Q_r \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of $C$. Since both $Q_r$ and $Q_r \otimes_C \overline{C}$ are prime and centrally closed [39, Theorems 2.5 and 3.5], we may replace $R$ by $Q_r$ or $Q_r \otimes_C \overline{C}$ according to $C$ finite or infinite. Then $R$ is centrally closed over $C$ and $P(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in R$. By Martindale's theorem (see Theorem 1.6.6), $R$ is then a primitive ring with nonzero socle $soc(R)$ and with $C$ as its associated division ring. Then, by Jacobson's theorem (see Theorem 1.6.5), $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$. Now we have the following cases.

**Case-I:** Suppose that $V$ is finite dimensional over $C$, that is, $\dim_C V = t$.
Then by density of $R$, we have $R \cong M_t(C)$. Since $f(r_1, \dots, r_n)$ is not central valued on $R$, $R$ must be noncommutative and so $t \geq 2$. In this case, by Proposition 7.2.6, we get either $p_1$ or $p_4$ or $p_5$ is central.

**Case-II:** Suppose that $V$ is infinite dimensional over $C$.
Then by [90, Lemma 2], the set $f(R)$ is dense on $R$. Then by hypothesis, $R$ satisfies

$$p_1 r p_2 r + r p_3 r + p_1 r p_4 r p_5 + r p_6 r p_5 - p_7 r^2 = 0. \tag{7.2.19}$$

If any one of $p_1$ or $p_4$ or $p_5$ is central, then we get our conclusions. Therefore on contrary, we assume that none of them be central elements. Then by the property of primeness, none of them can commute with nonzero ideal $soc(R)$, that is, $[p_1, soc(R)] \neq (0)$, $[p_4, soc(R)] \neq (0)$, $[p_5, soc(R)] \neq (0)$. Then there exist $h_1, h_2, h_3 \in soc(R)$ such that

$$[p_1, h_1] \neq 0, [p_4, h_2] \neq 0 \text{ and } [p_5, h_3] \neq 0.$$

By Martindale's theorem (see Theorem 1.6.6), for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. By Litoff's Theorem (see Theorem 1.6.7), there exists idempotent $e \in soc(R)$ such that $h_1, h_2, h_3, , p_1h_1, h_1p_1, p_4h_2, h_2p_4, p_5h_3, h_3p_5 \in eRe$. Since $R$ satisfies generalized identity (7.2.19), the subring $eRe$ satisfies

$$(ep_1e)r(ep_2e)r + r(ep_3e)r + (ep_1e)r(ep_4e)r(ep_5e) + r(ep_6e)r(ep_5e) - (ep_7e)r^2 = 0. \tag{7.2.20}$$

Then by Corollary 7.2.7, any one of the following holds:

1. $ep_1e \in eC$ which contradicts with existence of $h_1$;

2. $ep_4e \in eC$ which contradicts with existence of $h_2$;

3. $ep_5e \in eC$ which contradicts with existence of $h_3$.

Hence the Lemma is proved. □

**Lemma 7.2.10.** *Let $R$ be a prime ring of characteristic different from 2, $Q_r$ be its right Martindale quotient ring, $C$ be its extended centroid and $f(r_1, \ldots, r_n)$ be a noncentral multilinear polynomial over $C$. Let $T_1(x) = ax + pxp^{-1}b$ and $T_2(x) = cx + pxp^{-1}q$ for all $x \in R$, for some $a, b, c, q, p \in Q_r$. If $T_1(u)T_2(u) = T_1(u)u - uT_2(u)$ for all $u \in f(R)$, then one of the following holds:*

1. *$T_1(x) = x(a + b)$, $T_2(x) = (c + q)x$ for all $x \in R$ with $(a + b)(c + q) + (c + q) - (a + b) = 0$;*
2. *$f(x_1, \ldots, x_n)^2$ is central valued on $R$ and one of the following holds:*
   (a) *there exists $\mu \in C$ such that $T_1(x) = -x + \mu pxp^{-1}$ and $T_2(x) = x - \mu^{-1}pxp^{-1}$ for all $x \in R$;*
   (b) *$T_1(x) = (a + b)x$ and $T_2(x) = x(c + q)$ for all $x \in R$ with $(a + b)(c + q) + (c + q) - (a + b) = 0$.*

*Proof.* By hypothesis

$$\Big(af(r) + pf(r)p^{-1}b\Big)\Big(cf(r) + pf(r)p^{-1}q\Big) = \Big(af(r) + pf(r)p^{-1}b\Big)f(r) - f(r)\Big(cf(r) + pf(r)p^{-1}q\Big)$$

that is

$$\begin{aligned} af(r)cf(r) + pf(r)p^{-1}bcf(r) + af(r)pf(r)p^{-1}q + pf(r)p^{-1}bpf(r)p^{-1}q \\ = \Big(af(r) + pf(r)p^{-1}b\Big)f(r) - f(r)\Big(cf(r) + pf(r)p^{-1}q\Big) \end{aligned} \tag{7.2.21}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Pre-multiply by $p^{-1}$, we get,

$$\begin{aligned} p^{-1}af(r)cf(r) + f(r)p^{-1}bcf(r) + p^{-1}af(r)pf(r)p^{-1}q + f(r)p^{-1}bpf(r)p^{-1}q \\ = \Big(p^{-1}af(r) + f(r)p^{-1}b\Big)f(r) - p^{-1}f(r)\Big(cf(r) + pf(r)p^{-1}q\Big) \end{aligned} \tag{7.2.22}$$

which gives

$$\begin{aligned} p^{-1}(a + 1)f(r)cf(r) + f(r)p^{-1}(bc - b)f(r) + p^{-1}(a + 1)f(r)pf(r)p^{-1}q \\ +f(r)p^{-1}bpf(r)p^{-1}q - p^{-1}af(r)^2 = 0 \end{aligned} \tag{7.2.23}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. By Lemma 7.2.9, either $p^{-1}(a+1) \in C$ or $p \in C$ or $p^{-1}q \in C$. In any case by Lemma 7.2.3, Lemma 7.2.4, Lemma 7.2.5, we obtain our conclusions. □

**Lemma 7.2.11.** *Let $R$ be a noncommutative prime ring of characteristic different from 2, $Q_r$ be its right Martindale quotient ring and $C$ be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C$, $T_1(x) = ax + \alpha(x)b$ and $T_2(x) = cx + \alpha(x)q$ for all $x \in R$, where $a, b, c, q \in Q_r$, $\alpha \in Aut(R)$. If $R$ satisfies*

$$T_1(f(r))T_2(f(r)) = T_1(f(r))f(r) - f(r)T_2(f(r))$$

*for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

1. *$T_1(x) = x(a+b)$, $T_2(x) = (c+q)x$ for all $x \in R$ with $(a+b)(c+q) + (c+q) - (a+b) = 0$;*

2. *$f(x_1, \ldots, x_n)^2$ is central valued on $R$ and one of the following holds:*

   (a) *there exists $\mu \in C$ such that $T_1(x) = -x + \mu p x p^{-1}$ and $T_2(x) = x - \mu^{-1} p x p^{-1}$ for all $x \in R$;*

   (b) *$T_1(x) = (a+b)x$ and $T_2(x) = x(c+q)$ for all $x \in R$ with $(a+b)(c+q) + (c+q) - (a+b) = 0$.*

*Proof.* If $\alpha$ is an inner automorphism, then by Lemma 7.2.10, we have our conclusions. Next, we assume that $\alpha$ is an outer automorphism. Any $(x_i)^\alpha$-word degree in $\Phi(x_1, \ldots, x_n)$ is equal to 2 and $char(R) = 0$ or $char(R) = p \geq 2$ and hence by [22, Theorem 3], $R$ satisfies the generalized polynomial identity

$$\begin{aligned}\Big(af(x_1, \ldots, x_n) + f^\alpha(y_1, \ldots, y_n)b\Big)\Big(cf(x_1, \ldots, x_n) + f^\alpha(y_1, \ldots, y_n)q\Big)\\ -af(x_1, \ldots, x_n)^2 - f^\alpha(y_1, \ldots, y_n)bf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)cf(x_1, \ldots, x_n)\\ +f(x_1, \ldots, x_n)f^\alpha(y_1, \ldots, y_n)q \end{aligned} \tag{7.2.24}$$

where we denote by $f^\alpha(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient $\gamma_\sigma$ with $\alpha(\gamma_\sigma)$. By (7.2.24), $R$ satisfies blended component

$$af(x_1, \ldots, x_n)cf(x_1, \ldots, x_n) - af(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)cf(x_1, \ldots, x_n) = 0. \tag{7.2.25}$$

By Lemma 3.7 in [17], either $a \in C$ or $c \in C$. If $a \in C$, then $R$ satisfies

$$f(x_1, \ldots, x_n)(ac + c)f(x_1, \ldots, x_n) - af(x_1, \ldots, x_n)^2 = 0. \tag{7.2.26}$$

By Lemma 7.2.1, $ac+c \in C$ with $ac+c-a = 0$. Thus $(a+1)c \in C$ implies $a+1 = 0$ or $c \in C$. If $a + 1 = 0$, then $ac + c - a = 0$ implies $a = 0$ contradicting with the fact $a + 1 = 0$.

Next we assume that $c \in C$. Then (7.2.25) reduces to

$$(ac - a + c)f(x_1, \ldots, x_n)^2 = 0 \tag{7.2.27}$$

which implies $ac-a+c = 0$. we see that $a(c-1) = -c \in C$. This implies either $c = 1$ or $a \in C$. As before, $a \in C$ leads to a contradiction and hence, $c = 1$ contradicting with the fact $ac - a + c = 0$. This completes the proof of the Lemma. □

## 7.3 Proof of Main Theorem

Let $f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ be any multilinear polynomials over $C$. We know the following facts.

**Fact 7.3.1:** The action of any skew derivation $d$ on any monomial of $f(x_1, \cdots, x_n)$ can be described as follows:

$$d(\gamma_\sigma . x_{\sigma(1)} \ldots x_{\sigma(n)}) = d(\gamma_\sigma) x_{\sigma(1)} \ldots x_{\sigma(n)}$$
$$+\alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \ldots x_{\sigma(n)}.$$

Hence, we have

$$d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n)$$
$$+ \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \ldots x_{\sigma(n)}.$$

**Fact 7.3.2:** Let $R$ be a prime ring, $D$ be an $X$-outer skew derivation of $R$ and $\alpha$ be an $X$-outer automorphism of $R$. If $\Phi(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for $R$, then $R$ also satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where $x_i$, $y_i$ and $z_i$ are distinct indeterminates ([23, Theorem 1]).

Now we are ready to prove our Theorem 7.1.1.

By [19], there exist $a, c \in Q_r$ and $d, \delta$ skew derivations of $R$ such that $T_1(x) = ax + d(x)$ and $T_2(x) = cx + \delta(x)$ for all $x \in R$. By the results contained in the previous section, the Theorem is proved if one of the following cases occurs:

- $d = 0$, $\delta = 0$;
- both $d, \delta$ are inner skew derivations of $R$, that is, both $F$ and $G$ are inner generalized skew derivations of $R$.

Thus in all that follows, we assume that

- either $d \neq 0$ or $\delta \neq 0$;
- $d$ and $\delta$ are not simultaneously inner skew derivations of $R$.

By [23, Theorem 2] and our hypothesis, $Q_r$ satisfies

$$\begin{aligned}\Big(af(x_1,\dots,x_n) + d(f(x_1,\dots,x_n))\Big)\Big(cf(x_1,\dots,x_n) + \delta(f(x_1,\dots,x_n)\Big)\\ = \Big(af(x_1,\dots,x_n) + d(f(x_1,\dots,x_n))\Big)f(x_1,\dots,x_n)\\ -f(x_1,\dots,x_n)\Big(cf(x_1,\dots,x_n) - \delta(f(x_1,\dots,x_n))\Big).\end{aligned} \tag{7.3.1}$$

Thus from above, $Q_r$ satisfies

$$\begin{aligned}&\Big(af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n)\\ &+ \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big).\\ &\Big(cf(x_1,\dots,x_n) + f^\delta(x_1,\dots,x_n)\\ &+ \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big)\\ &= \Big(af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n)\\ &+ \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big)f(x_1,\dots,x_n)\\ &-f(x_1,\dots,x_n)\Big(cf(x_1,\dots,x_n) + f^\delta(x_1,\dots,x_n)\\ &+ \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big).\end{aligned} \tag{7.3.2}$$

Thus we have to consider the following cases:

**Case-1:** *$d$ is inner and $\delta$ is outer.*

Assume that $d(x) = ux - \alpha(x)u$, for some $u \in Q_r$. Then, by [23, Theorem 2], $Q_r$ also satisfies

$$\begin{aligned}\Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big).\\ \Big(cf(x_1,\dots,x_n) + f^{\delta}(x_1,\dots,x_n)\\ + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big)\\ = \Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big)f(x_1,\dots,x_n)\\ -f(x_1,\dots,x_n)\Big(cf(x_1,\dots,x_n) + f^{\delta}(x_1,\dots,x_n)\\ + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big).\end{aligned} \tag{7.3.3}$$

Since $\delta$ is not inner, $Q_r$ satisfies

$$\begin{aligned}\Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big).\\ \Big(cf(x_1,\dots,x_n) + f^{\delta}(x_1,\dots,x_n)\\ + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big)\\ = \Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big)f(x_1,\dots,x_n)\\ -f(x_1,\dots,x_n)\Big(cf(x_1,\dots,x_n) + f^{\delta}(x_1,\dots,x_n)\\ + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big).\end{aligned} \tag{7.3.4}$$

In particular, $Q_r$ satisfies,

$$\begin{aligned}\Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u + f(x_1,\dots,x_n)\Big)\\ .\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big) = 0.\end{aligned} \tag{7.3.5}$$

Let $\alpha$ be inner automorphism, that is, there is an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$. Then from (7.3.5)

$$\begin{aligned}\Big((a+u+1)f(x_1,\dots,x_n) - qf(x_1,\dots,x_n)q^{-1}u\Big).\\ \Big(\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}qx_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)}q^{-1}y_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big) = 0.\end{aligned} \tag{7.3.6}$$

Replacing any $y_{\sigma(j+1)}$ by $qy_{\sigma(j+1)}$, it follows that $Q_r$ satisfies

$$\begin{aligned}&\Big((a+u+1)f(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}u\Big)\cdot\\ q\Big(&\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)}y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)=0.\end{aligned} \tag{7.3.7}$$

that is

$$\Big((a+u+1)f(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}u\Big)qf(x_1,\ldots,x_n)=0. \tag{7.3.8}$$

Left multiplying by $q^{-1}$, $Q_r$ satisfies

$$\Big(q^{-1}(a+u+1)f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)q^{-1}u\Big)qf(x_1,\ldots,x_n)=0. \tag{7.3.9}$$

By Lemma 3.7 in [17], either $q\in C$ or $q^{-1}(a+u+1)\in C$.

- If $q\in C$, then $\alpha=Id$, identity map. Then from above $Q_r$ satisfies

$$\Big((a+u+1)f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)u\Big)f(x_1,\ldots,x_n)=0. \tag{7.3.10}$$

By Lemma 7.2.1, $u\in C$ and $a+1=0$. Thus $d=0$ and $T_1(x)=-x$ for all $x\in R$.

- If $q^{-1}(a+u+1)\in C$, then from above $Q_r$ satisfies

$$f(x_1,\ldots,x_n)q^{-1}(a+1)qf(x_1,\ldots,x_n)=0 \tag{7.3.11}$$

which implies $q^{-1}(a+1)q=0$, that is, $a=-1$. Thus $q^{-1}(a+u+1)\in C$ implies $q^{-1}u\in C$ and hence $d=0$ and $T_1(x)=-x$ for all $x\in R$.

Therefore, in any cases we obtain that $T_1(x)=-x$ for all $x\in R$. Then situation $T_1(x)T_2(x)=T_1(x)x-xT_2(x)$ for all $x\in f(R)$ implies $f(x_1,\ldots,x_n)^2=0$ which gives $f(x_1,\ldots,x_n)=0$ for all $x_1,\ldots,x_n\in R$, a contradiction.

Let $\alpha$ be outer. Then from (7.3.5), $Q_r$ satisfies (see [22])

$$\begin{aligned}&\Big((a+u)f(x_1,\ldots,x_n)-f^\alpha(z_1,\ldots,z_n)u+f(x_1,\ldots,x_n)\Big)\cdot\\ &\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}z_{\sigma(1)}z_{\sigma(2)}\ldots z_{\sigma(j)}y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)=0.\end{aligned} \tag{7.3.12}$$

In particular, for $x_1=\cdots=x_n=0$, we have from (7.3.12) that $Q_r$ satisfies

$$f^\alpha(z_1,\ldots,z_n)u\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)z_{\sigma(1)}z_{\sigma(2)}\ldots z_{\sigma(j)}\ldots y_{\sigma(n)}\Big)=0. \tag{7.3.13}$$

Replacing $y_i$ with $z_{\sigma(i)}$ in above relation, $Q_r$ satisfies

$$f^{\alpha}(z_1,\ldots,z_n)uf^{\alpha}(z_1,\ldots,z_n)=0. \qquad (7.3.14)$$

This gives $u=0$ and hence $d=0$. Thus (7.3.12) reduces to

$$\Big(af(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)\Big).\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}z_{\sigma(1)}z_{\sigma(2)}\ldots z_{\sigma(j)}y_{\sigma(j+1)} x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)=0. \qquad (7.3.15)$$

In particular, $Q_r$ satisfies the blended component

$$\Big(af(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)\Big).\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)z_{\sigma(1)}z_{\sigma(2)}\ldots z_{\sigma(n-1)}y_{\sigma(n)}\Big)=0. \qquad (7.3.16)$$

Replacing $z_i$ and $y_i$ by $\alpha(z_i)$ for $i=1,\ldots,n$, we have from above relation that $Q_r$ satisfies

$$(a+1)f(x_1,\ldots,x_n)\alpha(f(z_1,\ldots,z_n))=0. \qquad (7.3.17)$$

Since $f(x_1,\ldots,x_n)$ is non-central valued, $\alpha(f(z_1,\ldots,z_n))$ is also non-central valued in $Q_r$ and hence by Lemma 7.2.2, $a+1=0$ i.e., $a=-1$. Thus $T_1(x)=-x$ for all $x\in R$ which leads to a contradiction as before.

**Case-2:** *$\delta$ is inner and $d$ is outer.*

Assume $\delta(x)=vx-\alpha(x)v$, for some $v\in Q_r$, then as above $Q_r$ satisfies

$$\begin{aligned}&\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).\\&\Big((c+v)f(x_1,\ldots,x_n)-\alpha(f(x_1,\ldots,x_n))v\Big)\\&=\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big((c+v)f(x_1,\ldots,x_n)-\alpha(f(x_1,\ldots,x_n))v\Big).\end{aligned} \qquad (7.3.18)$$

Since $d$ is not inner, $Q_r$ satisfies

$$\begin{aligned}&\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).\\&\Big((c+v)f(x_1,\ldots,x_n)-\alpha(f(x_1,\ldots,x_n))v\Big)\\&=\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big((c+v)f(x_1,\ldots,x_n)-\alpha(f(x_1,\ldots,x_n))v\Big).\end{aligned}\tag{7.3.19}$$

In particular, $Q_r$ satisfies blended component

$$\begin{aligned}&\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).\\&\Big((c+v-1)f(x_1,\ldots,x_n)-\alpha(f(x_1,\ldots,x_n))v\Big)=0.\end{aligned}\tag{7.3.20}$$

Let $\alpha$ be inner, that is, there exists $q\in Q_r$ such that $\alpha(x)=qxq^{-1}$ for all $x\in R$. By (7.3.20)

$$\begin{aligned}&\Big(\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}qx_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)}q^{-1}y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).\\&\Big((c+v-1)f(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}v\Big)=0.\end{aligned}\tag{7.3.21}$$

Replacing any $y_{\sigma(j+1)}$ by $qy_{\sigma(j+1)}$, it follows that $Q_r$ satisfies

$$\begin{aligned}&q\Big(\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)}y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).\\&\Big((c+v-1)f(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}v\Big)=0.\end{aligned}\tag{7.3.22}$$

In particular, $Q_r$ satisfies

$$qf(x_1,\ldots,x_n)\Big((c+v-1)f(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}v\Big)=0.\tag{7.3.23}$$

Left multiplying by $q^{-1}$, $Q_r$ satisfies

$$f(x_1,\ldots,x_n)\{(c+v-1)f(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}v\}=0.\tag{7.3.24}$$

Then by Lemma 3.7 in [17], either $q \in C$ or $q^{-1}v \in C$.

• If $q \in C$, then $\alpha = Id$, identity map. Then from above $Q_r$ satisfies

$$f(x_1, \ldots, x_n)\{(c + v - 1)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)v\} = 0. \tag{7.3.25}$$

By Lemma 7.2.1, $v, c \in C$ with $c - 1 = 0$. Thus $\delta = 0$ and $T_2(x) = x$ for all $x \in R$.

• If $q^{-1}v \in C$, then $\delta = 0$ and (7.3.24) reduces to $f(x_1, \ldots, x_n)(c-1)f(x_1, \ldots, x_n) = 0$ which implies $c - 1 = 0$. Therefore, $T_2(x) = x$ for all $x \in R$. Thus in any cases we have $T_2(x) = x$ for all $x \in R$ and hence our hypothesis $T_1(x)T_2(x) = T_1(x)x - xT_2(x)$ for all $x \in f(R)$ implies $f(x_1, \ldots, x_n)^2 = 0$ for all $x_1, \ldots, x_n \in R$, a contradiction.

Next, let $\alpha$ be outer. Then from (7.3.20), $Q_r$ satisfies (see [22])

$$\begin{aligned}\Big(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} z_{\sigma(1)} z_{\sigma(2)} \ldots z_{\sigma(j)} y_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}\Big)\cdot \\ \Big((c + v - 1)f(x_1, \ldots, x_n) - f^\alpha(z_1, \ldots, z_n))v\Big) = 0.\end{aligned} \tag{7.3.26}$$

For $x_1 = \cdots = x_n = 0$, one has that $Q_r$ satisfies

$$\Big(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) z_{\sigma(1)} z_{\sigma(2)} \ldots z_{\sigma(n-1)} y_{\sigma(n)}\Big) f^\alpha(z_1, \ldots, z_n)v = 0. \tag{7.3.27}$$

In particular, replacing $y_i$ with $z_i$ we have from above that $f^\alpha(z_1, \ldots, z_n)^2 v = 0$ is an identity for $Q_r$. This gives $v = 0$ and hence $\delta = 0$. By (7.3.26), $Q_r$ satisfies

$$\begin{aligned}\Big(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} z_{\sigma(1)} z_{\sigma(2)} \ldots z_{\sigma(j)} y_{\sigma(j+1)} x_{\sigma(j+2)} \ldots x_{\sigma(n)}\Big)\cdot \\ (c - 1)f(x_1, \ldots, x_n) = 0.\end{aligned} \tag{7.3.28}$$

In particular, $Q_r$ satisfies the blended component

$$\Big(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) z_{\sigma(1)} z_{\sigma(2)} \ldots z_{\sigma(n-1)} y_{\sigma(n)}\Big)(c - 1)f(x_1, \ldots, x_n) = 0. \tag{7.3.29}$$

Replacing $y_i$ and $z_i$ by $\alpha(z_i)$ for $i = 1, \ldots, n$, $Q_r$ satisfies

$$\alpha(f(z_1, \ldots, z_n))(c - 1)f(x_1, \ldots, x_n) = 0 \tag{7.3.30}$$

i.e., $c'f(x_1, \ldots, x_n) = 0$ where $c' = \alpha(f(z_1, \ldots, z_n))(c - 1)$. This implies $c' = 0$ i.e., $\alpha(f(z_1, \ldots, z_n))(c - 1) = 0$. Again this implies $c = 1$. Thus, $T_2(x) = x$ for all $x \in R$

and hence by same argument as above it leads to a contradiction.

**Case** $3:\{d,\delta\}$ **is linearly** $C$**-independent modulo inner skew derivations of** $R$**.**

In this case, since $d$ and $\delta$ are associated with the same automorphism, by relation (7.3.2) and [65, Theorem 6.5.9] it follows that $Q_r$ satisfies the generalized identity

$$\begin{aligned}&\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).\\&\Big(cf(x_1,\ldots,x_n)+f^\delta(x_1,\ldots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)\\&=af(x_1,\ldots,x_n)^2+\Big(f^d(x_1,\ldots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)cf(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)\Big(f^\delta(x_1,\ldots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).\end{aligned}\tag{7.3.31}$$

Thus $Q_r$ satisfies the blended component

$$\begin{aligned}&\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).\\&\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)=0.\end{aligned}\tag{7.3.32}$$

In above relation we replace $y_{\sigma(j+1)}$ with $z_{\sigma(j+1)}$ for any $j=0,\ldots,n-1$ and then $Q_r$ satisfies

$$\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)^2=0.\tag{7.3.33}$$

Now if $\alpha$ is inner, then $\alpha(x)=pxp^{-1}$ for any $x\in R$ and for some $p\in Q_r$. From above $Q_r$ satisfies

$$\Big(\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}px_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)}p^{-1}z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)^2=0.\tag{7.3.34}$$

Replacing $z_{\sigma(j+1)}$ with $px_{\sigma(j+1)}$ for any $j = 0, \dots, n-1$, $Q_r$ satisfies

$$\Big(pf(x_1, \dots, x_n)\Big)^2 = 0. \tag{7.3.35}$$

Left multiplying by $p^{-1}$ it follows that

$$f(x_1, \dots, x_n)pf(x_1, \dots, x_n) = 0. \tag{7.3.36}$$

By Lemma 7.2.1, it follows that $p = 0$, a contradiction. On the other hand, if $\alpha$ is not inner, by [22]

$$\Big(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} t_{\sigma(1)} t_{\sigma(2)} \dots t_{\sigma(j)} z_{\sigma(j+1)} x_{\sigma(j+2)} \dots x_{\sigma(n)}\Big)^2 = 0. \tag{7.3.37}$$

is an identity for $Q_r$ and hence for $x_1 = \dots = x_n = 0$,

$$\Big(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) t_{\sigma(1)} \dots t_{\sigma(n-1)} z_{\sigma(n)}\Big)^2 = 0. \tag{7.3.38}$$

Replacing $z_i$ with $t_i$ for all $i = 1, \dots, n$, we have $f^\alpha(t_1, \dots, t_n)^2 = 0$ is an identity for $Q_r$ implying $f^\alpha(t_1, \dots, t_n) = 0$, a contradiction.

**Case** $4 : \{d, \delta\}$ **is linearly** $C$**-dependent modulo inner skew derivations of** $R$**.**

Under our last assumption, there exist $\lambda, \mu \in C$, $q \in Q_r$ and $\gamma \in Aut(R)$ such that $\lambda d(x) + \mu\delta(x) = qx - \gamma(x)q$, for any $x \in R$. Recall that $d$ and $\delta$ are not inner skew derivations, so that both $\lambda \neq 0$ and $\mu \neq 0$. Here we write $\delta(x) = px - \gamma(x)p + \eta d(x)$, where $p = \mu^{-1}q$ and $\eta = -\lambda\mu^{-1} \neq 0$. Hence $Q_r$ satisfies

$$\begin{gathered}
\Big(af(x_1, \dots, x_n) + f^d(x_1, \dots, x_n) \\
+ \textstyle\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \dots x_{\sigma(n)}\Big) \cdot \\
\Big((c+p)f(x_1, \dots, x_n) - \gamma(f(x_1, \dots, x_n))p + \eta f^d(x_1, \dots, x_n) \\
+ \eta \textstyle\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \dots x_{\sigma(n)}\Big) \\
= \Big(af(x_1, \dots, x_n)^2 + (f^d(x_1, \dots, x_n) \\
+ \textstyle\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \dots x_{\sigma(n)}) f(x_1, \dots, x_n)\Big) \\
- f(x_1, \dots, x_n)(c+p)f(x_1, \dots, x_n) \\
+ f(x_1, \dots, x_n)\Big(\gamma(f(x_1, \dots, x_n))p - \eta f^d(x_1, \dots, x_n) \\
- \eta \textstyle\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \dots x_{\sigma(n)}\Big).
\end{gathered}$$

Since $d$ is outer, by [23, Theorem 1], it follows that $Q_r$ satisfies

$$\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).$$
$$\Big((c+p)f(x_1,\ldots,x_n)-\gamma(f(x_1,\ldots,x_n))p+\eta f^d(x_1,\ldots,x_n)$$
$$+\eta\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)$$
$$=\Big(af(x_1,\ldots,x_n)^2+(f^d(x_1,\ldots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)})f(x_1,\ldots,x_n)\Big)$$
$$-f(x_1,\ldots,x_n)(c+p)f(x_1,\ldots,x_n)$$
$$+f(x_1,\ldots,x_n)\Big(\gamma(f(x_1,\ldots,x_n))p-\eta f^d(x_1,\ldots,x_n)$$
$$-\eta\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).$$

In particular, $Q_r$ satisfies blended components (taking only $y_i$ terms)

$$\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).$$
$$\Big(\eta\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)$$
$$+\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).$$
$$\Big((c+p)f(x_1,\ldots,x_n)-\gamma(f(x_1,\ldots,x_n))p+\eta f^d(x_1,\ldots,x_n)$$
$$+\eta\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)$$
$$=\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)\Big(\eta\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).$$

The above equations contains some terms of $y_i^2$ and some other terms of $y_i$. If we replace $y_i$ with $-y_i$ in above equation and then adding both the equations, we will have the only terms of $y_i^2$. Then by using $char(R)\neq 2$, we get that $Q_r$ satisfies

$$\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big)^2=0 \qquad (7.3.39)$$

which is same as (7.3.33). Thus by same argument, we arrive at a contradiction. Thus the proof of the theorem 7.1.1 is completed.

# Chapter 8

# Generalized Derivation with Engel Condition Acting on Lie Ideals in Prime Rings

## 8.1 Introduction

Throughout this chapter, $R$ always stands for an associative prime ring, characteristic of $R$ is different from 2, the center of $R$ is $Z(R)$ and $U$ stands for the Utumi quotient ring of $R$. It is noted that $R$ is a subring of $U$. $C = Z(U)$, the center of $U$ is called the extended centroid of $R$.

The $m$-th commutator of $a, b$ is defined as $[a, b]_m = [[a, b]_{m-1}, b]$, $m = 1, 2, \ldots$. It is easy to check that $[a, b]_m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} b^i a b^{m-i}$. Also denote $[a_1, a_2, \ldots, a_n] = [[a_1, a_2, \ldots, a_{n-1}], a_n]$ for all $a_1, \ldots, a_n \in R$ and for every positive integer $n \geq 2$.

Now we shall provide the background of our study. Suppose that $d$ and $\delta$ stands for derivations of $R$ and $L$ stands for noncentral Lie ideal of $R$. A well-known theorem of Posner [80] asserts that $R$ is commutative, if $[d(a), a] \in Z(R)$ for all $a \in R$. In [67], Lanski extended the above result by substituting $a \in R$ with $a \in L$, incorporating the $m$-Engel condition and subsequently established that if $[d(a), a]_m = 0$ for all $a \in L$, $m \geq 1$ is a fixed integer, then $char(R) = 2$ and $R$ satisfies $s_4$. Bresar [10] commenced the investigation of co-centralizing derivations and proved that if $d(a)a - a\delta(a) \in Z(R)$ for all $a \in R$, then $d = \delta = 0$ or $R$ is commutative. Further, Lee and Wong [70] generalized this outcome to Lie ideals

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and demonstrated that if $d(a)a - a\delta(a) \in Z(R)$ for all $a \in L$, then $d = \delta = 0$ or $R \subseteq M_2(K)$.

Later, by swapping $d$ with a generalized derivation $F$, the aforementioned problem was studied by Argac et al. in [4]. More precisely, authors proved that if $[F(a), a]_m = 0$ for all $a \in L$, $m \geq 1$ a fixed integer, then either $R$ satisfies $s_4$ or $F(a) = \alpha a$ for all $a \in R$ and for some $\alpha \in C$. Carini et al. [12] explored a result under the left annihilator condition and replaced derivations with generalized derivations. Specifically, the authors demonstrated the structure of generalized derivations $F$ and $G$, if for some $0 \neq b \in R$, $b(F(a)a - aG(a)) = 0$ for all $a \in L$.

Let $f(r_1, \ldots, r_n)$ be a noncentral multilinear polynomial over $C$ and denote the set $f(S) = f(r_1, \ldots, r_n)$, where $r_1, \ldots, r_n \in S$ for some $S \subseteq R$. In [5], the authors examined the co-commuting condition $F(a)a - aG(a) = 0$ for all $a \in f(I)$ where $I$ is an ideal of $R$. In [46], the authors investigated the co-centralizing condition, $F(a)a - aG(a) \in C$ for all $a \in f(\lambda)$, where $\lambda$ is a nonzero right ideal of $R$.

Let $I$ be a left sided ideal of $R$, $L$ be a Lie ideal of $R$ and $F$, $G$, $H$ be three generalized derivations of $R$. Identity with Engel condition was studied by Lanski in [68]. Lanski studied in semiprime ring that $[d(a^{t_0}), a^{t_1}, \ldots, a^{t_n}] = 0$ for all $a \in I$, where $t_0, \ldots, t_n \geq 1$ are fixed integers. Lee and Shiue [75] studied $[d(a^m)a^n - a^p\delta(a^q), a^r]_k = 0$ for all $a \in I$, where $m, n, p, q, r, k$ are fixed positive integers.

In [4] Argac et al. studied $[F(a), a]_k = 0$ for all $a \in L$, where $k \geq 1$ a fixed integer.

In the same flavour, Albas et al. [2] studied $[F(a^k), a^k]_n = 0$ for all $a \in I$.

Above result is extended by Dhara [29] et al. to the case $[F(a^{n_1}), a^{n_2}, \ldots, a^{n_k}] = 0$ for all $a \in L$ and $[F(a^{n_1}), a^{n_2}, \ldots, a^{n_k}] = 0$ for all $a \in [I, I]$, where $n_1, \ldots, n_k$ are all fixed positive integers.

Recently, in [1], Alahmadi et al. studied $[F(a^m)a^n + a^n d(a^m), a^r]_k = 0$ for all $a \in R$, where derivation $d$ is independent of $F$. Further this result was studied by Dhara and De Filippis [31] when generalized derivation $F$ and the derivation $d$ act on left ideals and Lie ideals of $R$.

In [83], Tiwari examined the scenario $F(a)G(a) - aH(a) = 0$ for all $a \in f(R)$

and obtain all possible forms of the maps $F$ and $G$. In [37], Dhara et al. and in [36], Dhara et al. examined the identities $[F(a)G(a) - aH(a), a] = 0$ for all $a \in L$ and the $n$-Engel condition $[F(a)G(a), a]_n = 0$ for all $a \in L$, respectively and then authors described the structure of the maps in all the cases. So it is quite apparent to determine the structure of the maps whenever

$$[F(a)G(a) - aH(a), a^{t_1}, a^{t_2}, \ldots, a^{t_n}] = 0$$

for all $a \in L$, where $t_1, \ldots, t_n$ are some fixed positive integers.

More precisely we prove the following theorem.

**Theorem 8.1.1.** *Let $R$ be a prime ring with $char(R) \neq 2$, $U$ Utumi quotient ring, $C = Z(U)$ extended centroid of $R$ and $F$, $G$, $H$ three generalized derivations of $R$. If*

$$[F(X)G(X) - XH(X), X^{t_1}, X^{t_2}, \ldots, X^{t_n}] = 0$$

*for all $X \in L$ and for some fixed positive integers $t_1, \ldots, t_n$, where $L$ is a non-central Lie ideal of $R$, then either $R \subseteq M_2(K)$, the $2 \times 2$ matrix ring over a field $K$ or one of the following holds:*

*(1) there exist $a, b, m, u \in U$, $\lambda \in C$ such that $F(x) = \lambda x$, $G(x) = ax + xb$, $H(x) = mx + xu$ for all $x \in R$, with $\lambda a - m, \lambda b - u \in C$;*

*(2) there exist $a, p, m \in U$ such that $F(x) = xp$, $G(x) = ax$, $H(x) = mx$ for all $x \in R$, with $pa - m \in C$;*

*(3) there exist $a \in U$, $\mu \in C$ and a derivation $\delta$ on $R$ such that $F = 0$, $G(x) = ax + \delta(x)$, $H(x) = \mu x$ for all $x \in R$;*

*(4) there exist $p \in U$, $\mu \in C$ and a derivation $d$ on $R$ such that $F(x) = px + d(x)$, $G = 0$, $H(x) = \mu x$ for all $x \in R$;*

*(5) there exist $a, m \in U$, $0 \neq \lambda, \alpha \in C$ and a derivation $\eta$ on $R$ such that $F(x) = \lambda x$, $G(x) = ax + \alpha\eta(x)$, $H(x) = mx + \eta(x)$ for all $x \in R$, with $\lambda a - m \in C$, $\alpha\lambda = 1$;*

*(6) $t = lcm\{t_1, t_2, \ldots, t_n\}$ is even, $R \cong M_l(C)$, the $l \times l$ matrix ring over a finite field $C$ for some integer $l \geq 3$ and one of the following holds:*

(i) *there exist* $a, b, m, u \in U$, $\lambda \in C$ *such that* $F(x) = \lambda x$, $G(x) = ax + xb$, $H(x) = mx + xu$ *for all* $x \in R$, *with* $\lambda b - u \in C$;

(ii) *there exist* $a, p, m \in U$ *such that* $F(x) = xp$, $G(x) = ax$, $H(x) = mx$ *for all* $x \in R$;

(iii) *there exist* $a, m \in U$ *and a derivation* $\delta$ *on* $R$ *such that* $F = 0$, $G(x) = ax + \delta(x)$, $H(x) = mx$ *for all* $x \in R$;

(iv) *there exist* $p, m \in U$ *and a derivation* $d$ *on* $R$ *such that* $F(x) = px + d(x)$, $G = 0$, $H(x) = mx$ *for all* $x \in R$;

(v) *there exist* $a, m \in U$, $0 \neq \lambda, \alpha \in C$ *and a derivation* $\eta$ *on* $R$ *such that* $F(x) = \lambda x$, $G(x) = ax + \alpha\eta(x)$, $H(x) = mx + \eta(x)$ *for all* $x \in R$, *with* $\alpha\lambda = 1$.

**Corollary 8.1.2.** *Let* $R$ *be a prime ring with* $char(R) \neq 2$ *and* $d$, $\delta$, $h$ *three nonzero derivations of* $R$. *If*

$$[d(X)\delta(X) - Xh(X), X^{t_1}, X^{t_2}, \ldots, X^{t_n}] = 0$$

*for all* $X \in L$ *and for some fixed positive integers* $t_1, \ldots, t_n$, *where* $L$ *is a non-central Lie ideal of* $R$, *then* $R \subseteq M_2(K)$, *the* $2 \times 2$ *matrix ring over a field* $K$.

Let $R = M_2(K)$ be a matrix ring over a field $K$. Let us consider a noncentral Lie ideal $L = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ of $R$. If we choose $d(x) = \delta(x) = [e_{12}, x]$ for all $x \in R$ and $h = Id$, identity map, we see that

$$[d(X)\delta(X) - Xh(X), X] = 0$$

holds for all $X \in L$. Thus the conclusion $R \subseteq M_2(K)$ in the above Corollary can not be omitted. Hence this conclusion is also essential in our Theorem 8.1.1.

## 8.2 The Case: maps are inner

Throughout this section, we always assume that $R$ be a noncommutative prime ring with $char(R) \neq 2$ and $U$ be its Utumi ring of quotients. For a field $K$, $M_l(K)$ denotes the $l \times l$ matrix ring over the field $K$. Assume that $a, b, p, q, m, u \in U$ such that $F(x) = px + xq$, $G(x) = ax + xb$ and $H(x) = mx + xu$ for all $x \in R$. Let $t = lcm\{t_1, t_2, \ldots, t_n\}$. First we examine the identity $[F(X)G(X) - XH(X), X^t]_n = 0$ for all $X \in [R, R]$. Thus identity becomes

$$\Big[p[x_1, x_2]a[x_1, x_2] + p[x_1, x_2]^2 b + [x_1, x_2]c[x_1, x_2] + [x_1, x_2]q[x_1, x_2]b - [x_1, x_2]^2 u, [x_1, x_2]^t\Big]_n = 0 \quad (8.2.1)$$

for all $x_1, x_2 \in R$, where $c = qa - m$.

**Lemma 8.2.1.** *Let $C$ be a finite field and $R \cong M_l(C)$, for some integer $l \geq 3$. If $char(R) \neq 2$ and $p \in R$ such that*

$$\Big[[y_1, y_2]p[y_1, y_2], [y_1, y_2]^t\Big]_n = 0$$

*for all $y_1, y_2 \in R$, where $n, t$ are some fixed positive integers and $t$ is odd, then $p \in Z(R)$.*

*Proof.* Let us assume that $y_1 = e_{ij}$, $y_2 = e_{ji}$ for different indices $i, j$. Since $t$ is odd, we have $[y_1, y_2]^t = [y_1, y_2] = X$ (say). Hence, $\Big[[y_1, y_2]p[y_1, y_2], [y_1, y_2]^t\Big]_n = [XpX, X]_n$.

For $n = 1$, $[XpX, X] = XpX^2 - X^2pX$.

For $n = 2$, $[XpX, X]_2 = [[XpX, X], X] = XpX^3 - 2X^2pX^2 + X^3pX = 2XpX - 2X^2pX^2 = 2X(p - XpX)X$.

For, $n = 3$, $[XpX, X]_3 = [[XpX, X]_2, X] = 2^2XpX^2 - 2^2X^2pX = 2^2[XpX, X]$.

Therefore, by same manner we have for $n$ odd integer that

$$[XpX, X]_n = 2^{n-1}[XpX, X]$$

and for $n$ even integer

$$[XpX, X]_n = 2^{n-1}X(p - XpX)X.$$

So for $n$ odd integer, replacing $X = e_{ii} - e_{jj}$, $[XpX, X] = 0$ implies

$$(e_{ii} - e_{jj})p(e_{ii} + e_{jj}) - (e_{ii} + e_{jj})p(e_{ii} - e_{jj}) = 0.$$

Left and right multiplying by $e_{ii}$ and $e_{jj}$ respectively, we obtain $2e_{ii}pe_{jj} = 0$ and hence $p_{ij} = 0$. So, $p$ is diagonal.

Again, for $n$ odd integer, replacing $X = e_{ii} - e_{jj}$, $X(p - XpX)X = 0$ implies

$$2\{(e_{ii} - e_{jj})p(e_{ii} - e_{jj}) - (e_{ii} + e_{jj})p(e_{ii} + e_{jj})\} = 0.$$

Left and right multiplying by $e_{ii}$ and $e_{jj}$ respectively, we obtain $2e_{ii}pe_{jj} = 0$ and hence $p_{ij} = 0$. So, $p$ is diagonal.

Thus, in any cases, we have $p$ is diagonal and hence by standard argument $p \in Z(R)$. □

**Lemma 8.2.2.** *[78, Proposition 2.5] If*

$$\Big[p_1[y_1, y_2]^{m+n} + [y_1, y_2]^m p_2[y_1, y_2]^n + [y_1, y_2]^s p_3[y_1, y_2]^t + [y_1, y_2]^{s+t} p_4, [y_1, y_2]^r\Big]_k = 0$$

*for all $y_1, y_2 \in R$, where $p_1, p_2, p_3, p_4 \in U$ and $m, n, s, t, r, k$ are fixed positive integers, then either $R \subseteq M_2(K)$ for some field $K$ or $p_1, p_4 \in C$ and one of the following holds:*

*(1) $p_2, p_3 \in C$;*

*(2) $m = s$, $n = t$ and $p_2 + p_3 \in C$;*

*(3) $C$ is a finite field and $R \cong M_l(C)$ for some integer $l \geq 3$.*

**Lemma 8.2.3.** *Let $K$ be a field with char$(K) \neq 2$, $R = M_l(K)$ and $a, b, p, q, c, u \in R$. If $l \geq 3$ and $R$ satisfies (8.2.1), then one of the following holds:*

*(1) $p, q \in K.I_l$;*

*(2) $p, b \in K.I_l$;*

*(3) $a, b \in K.I_l$.*

*Proof.* Let us assume $a = \sum_{hk} a_{hk}e_{hk}$, $b = \sum_{hk} b_{hk}e_{hk}$, $p = \sum_{hk} p_{hk}e_{hk}$, $q = \sum_{hk} q_{hk}e_{hk}$, $c = \sum_{hk} c_{hk}e_{hk}$ and $u = \sum_{hk} u_{hk}e_{hk}$ for $0 \neq a_{hk}, b_{hk}, p_{hk}, q_{hk}, c_{hk}, u_{hk} \in K$, where $e_{hk}$ is the usual matrix with 1 in the $(h, k)$-th entry and zero elsewhere. Let $i, j, h$ be three different indices. Choose $[x_1, x_2] = [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$ in (8.2.1). Then $[x_1, x_2]^t = [e_{ij}, e_{ji}]^t = e_{ii} + (-1)^t e_{jj}$ for $t \geq 1$ and so (8.2.1) gives

$$\begin{aligned}&\big[p(e_{ii} - e_{jj})a(e_{ii} - e_{jj}) + p(e_{ii} + e_{jj})b + (e_{ii} - e_{jj})c(e_{ii} - e_{jj})\\ &+(e_{ii} - e_{jj})q(e_{ii} - e_{jj})b - (e_{ii} + e_{jj})u, e_{ii} + (-1)^t e_{jj}\big]_n = 0. \qquad (8.2.2)\end{aligned}$$

Right multiplying by $e_{ii}$ and left multiplying by $e_{hh}$, we obtain

$$p_{hi}(a_{ii}+b_{ii})+p_{hj}(b_{ji}-a_{ji})=0. \tag{8.2.3}$$

For any automorphism $\varphi$ of $M_l(K)$, we have

$$\begin{aligned}\big[\varphi(p)[x_1,x_2]\varphi(a)[x_1,x_2]+\varphi(p)[x_1,x_2]^2\varphi(b)+[x_1,x_2]\varphi(c)[x_1,x_2]\\ +[x_1,x_2]\varphi(q)[x_1,x_2]\varphi(b)-[x_1,x_2]^2\varphi(u),[x_1,x_2]^t\big]_n=0\end{aligned} \tag{8.2.4}$$

is a GPI for $R$. In particular, let $\varphi(x)=(1+e_{ih})x(1-e_{ih})$ for all $x\in R$. If we denote $\varphi(a)=\sum_{hk}a'_{hk}e_{hk}$, $\varphi(b)=\sum_{hk}b'_{hk}e_{hk}$, $\varphi(p)=\sum_{hk}p'_{hk}e_{hk}$ and $\varphi(q)=\sum_{hk}q'_{hk}e_{hk}$ for $a'_{hk}$, $b'_{hk}$, $p'_{hk}$, $q'_{hk}\in K$, using relation (8.2.3), it follows that

$$p'_{hi}(a'_{ii}+b'_{ii})+p'_{hj}(b'_{ji}-a'_{ji})=0$$

i.e.,

$$p_{hi}(a_{ii}+a_{hi}+b_{ii}+b_{hi})+p_{hj}(b_{ji}-a_{ji})=0.$$

By using (8.2.3), it yields

$$p_{hi}(a+b)_{hi}=0.$$

Hence by [49, Proposition 1], either $p\in K.I_l$ or $a+b\in K.I_l$.

Again, if we choose the automorphism $\eta(x)=(1+e_{jh})x(1-e_{jh})$ for all $x\in R$ and if we denote $\eta(a)=\sum_{hk}a''_{hk}e_{hk}$, $\eta(b)=\sum_{hk}b''_{hk}e_{hk}$, $\eta(p)=\sum_{hk}p''_{hk}e_{hk}$ and $\eta(q)=\sum_{hk}q''_{hk}e_{hk}$ for $a''_{hk}$,$b''_{hk}$, $p''_{hk}$, $q''_{hk}\in K$, then using above relation (8.2.3), we have

$$p''_{hi}(a''_{ii}+b''_{ii})+p''_{hj}(b''_{ji}-a''_{ji})=0$$

i.e.,

$$p_{hi}(a_{ii}+b_{ii})+p_{hj}(b_{ji}+b_{hi}-a_{ji}-a_{hi})=0.$$

By using (8.2.3), it yields

$$p_{hj}(a-b)_{hi}=0. \tag{8.2.5}$$

Now if we choose the automorphism $\chi(x)=(1+e_{ji})x(1-e_{ji})$ for all $x\in R$ and if we denote $\chi(a)=\sum_{hk}a'''_{hk}e_{hk}$, $\chi(b)=\sum_{hk}b'''_{hk}e_{hk}$, $\chi(p)=\sum_{hk}p'''_{hk}e_{hk}$ and $\chi(q)=\sum_{hk}q'''_{hk}e_{hk}$ for $a'''_{hk}$,$b'''_{hk}$, $p'''_{hk}$, $q'''_{hk}\in K$, then by the above relation, it follows that

$$p'''_{hj}(a'''_{hi}-b'''_{hi})=0 \tag{8.2.6}$$

i.e.,

$$p_{hj}(a_{hi} - a_{hj} - b_{hi} + b_{hj}) = 0.$$

By using (8.2.5), it leads to

$$p_{hj}(a - b)_{hj} = 0. \tag{8.2.7}$$

Then by [49, Proposition 1], either $p \in K.I_l$ or $a - b \in K.I_l$.

So, if $p \notin K.I_l$, then $a - b$, $a + b \in K.I_l$ i.e., $a, b \in K.I_l$. This indicates that either $p \in K.I_l$ or $a, b \in K.I_l$.

Now, let us consider $p \in K.I_l$. Then we have $R$ satisfies

$$\begin{aligned}\Big[[x_1, x_2](pa + c)[x_1, x_2] + [x_1, x_2]^2(pb - u)\\ +[x_1, x_2]q[x_1, x_2]b, [x_1, x_2]^t\Big]_n = 0.\end{aligned} \tag{8.2.8}$$

Substituting $[x_1, x_2]^t = e_{ii} + (-1)^t e_{jj}$ for $t \geq 1$, we obtain

$$\begin{aligned}\Big[(e_{ii} - e_{jj})(pa + c)(e_{ii} - e_{jj}) + (e_{ii} + e_{jj})(pb - u)\\ +(e_{ii} - e_{jj})q(e_{ii} - e_{jj})b, (e_{ii} + (-1)^t e_{jj})\Big]_n = 0.\end{aligned} \tag{8.2.9}$$

Right multiplying by $e_{l'l'}$ and left multiplying by $e_{jj}$ (for $l' \neq i, j$) in this relation, we get

$$(-1)^n e_{jj}\Big\{(e_{ii} + e_{jj})(pb - u) + (e_{ii} - e_{jj})q(e_{ii} - e_{jj})b\Big\}e_{l'l'} = 0 \tag{8.2.10}$$

which gives

$$(pb)_{jl'} - u_{jl'} - q_{ji}b_{il'} + q_{jj}b_{jl'} = 0. \tag{8.2.11}$$

Since, we considered $p \in K.I_l$, we obtain

$$(p + q)_{jj}b_{jl'} - u_{jl'} - q_{ji}b_{il'} = 0. \tag{8.2.12}$$

Now we choose the automorphism $\Psi(x) = (1 + e_{ij})x(1 - e_{ij})$ for all $x \in R$ and we denote $\Psi(a) = \sum_{hk} a''''_{hk}e_{hk}$, $\Psi(b) = \sum_{hk} b''''_{hk}e_{hk}$, $\Psi(p) = \sum_{hk} p''''_{hk}e_{hk}$ and $\Psi(q) = \sum_{hk} q''''_{hk}e_{hk}$ for $a''''_{hk}$,$b''''_{hk}$, $p''''_{hk}$, $q''''_{hk} \in K$. By above relation, it follows that

$$(p + q)''''_{jj}b''''_{jl'} - u''''_{jl'} - q''''_{ji}b''''_{il'} = 0 \tag{8.2.13}$$

i.e.,

$$\big((p+q)_{jj}-(p+q)_{ji}\big)b_{jl'}-u_{jl'}-q_{ji}b_{il'}-q_{ji}b_{jl'}=0.$$

This gives

$$\{(p+q)_{jj}-q_{ji}\}b_{jl'}-u_{jl'}-q_{ji}b_{il'}-q_{ji}b_{jl'}=0. \tag{8.2.14}$$

Using (8.2.12), we obtain $2q_{ji}b_{jl'}=0$ and since $char(R)\neq 2$, we obtain $q_{ji}b_{jl'}=0$.

Again by similar method, considering the automorphism $\zeta(x)=(1+e_{il'})x(1-e_{il'})$ for all $x\in R$ and using the above relation, it follows that

$$q_{ji}\{b_{jl'}-b_{ji}\}=0. \tag{8.2.15}$$

Since $q_{ji}b_{jl'}=0$, we obtain $q_{ji}b_{ji}=0$. Then by [49, Proposition 1], either $q\in K.I_l$ or $b\in K.I_l$. □

**Lemma 8.2.4.** *If (8.2.1) is a trivial generalized polynomial identity (GPI) for $R$, then one of the following holds:*

*(1)* $a=-b\in C$, $u, c+bq\in C$;

*(2)* $p, b, u, ap+bq+c\in C$;

*(3)* $p, q, ap+c, b(p+q)-u\in C$.

*Proof.* Assume that $T=U_{*_C}C\{x_1,x_2\}$, the free product of $U$ and $C\{x_1,x_2\}$, the free $C$-algebra in noncommuting indeterminates $x_1,x_2$.

By [21], (8.2.1) is a trivial GPI for $U$, therefore

$$\begin{aligned}\big[p[x_1,x_2]a[x_1,x_2]+p[x_1,x_2]^2b+[x_1,x_2]c[x_1,x_2]+[x_1,x_2]q[x_1,x_2]b\\ -[x_1,x_2]^2u,[x_1,x_2]^t\big]_n\end{aligned} \tag{8.2.16}$$

is a zero element in the free product $T$, i.e.,

$$\begin{aligned}\textstyle\sum(-1)^i\binom{n}{i}[x_1,x_2]^{ti}\Big(p[x_1,x_2]a[x_1,x_2]+p[x_1,x_2]^2b+[x_1,x_2]c[x_1,x_2]\\ +[x_1,x_2]q[x_1,x_2]b-[x_1,x_2]^2u\Big)[x_1,x_2]^{t(n-i)}=0\in T.\end{aligned} \tag{8.2.17}$$

If $p\notin C$, then from above

$$p[x_1,x_2](a[x_1,x_2]+[x_1,x_2]b)[x_1,x_2]^{tn}=0\in T.$$

This implies $a = -b \in C$. Using this in (8.2.16), we have

$$\big[[x_1, x_2](c + bq)[x_1, x_2] - [x_1, x_2]^2 u, [x_1, x_2]^t\big]_n = 0 \in T \tag{8.2.18}$$

i.e.,

$$\sum(-1)^i \binom{n}{i} [x_1, x_2]^{ti+1}\big((c + bq)[x_1, x_2] - [x_1, x_2]u\big)[x_1, x_2]^{t(n-i)} = 0 \in T.$$

Then clearly $u = c + bq \in C$ which indicates conclusion (1).

Again, if $p \in C$, we obtain from (8.2.16),

$$\begin{aligned}\big[[x_1, x_2](ap + c)[x_1, x_2] + [x_1, x_2]^2(pb - u) + [x_1, x_2]q[x_1, x_2]b,\\ [x_1, x_2]^t\big]_n = 0 \in T\end{aligned} \tag{8.2.19}$$

i.e.,

$$\begin{aligned}\sum(-1)^i \tbinom{n}{i} [x_1, x_2]^{ti+1}\Big((ap + c)[x_1, x_2] + [x_1, x_2](pb - u)\\ +q[x_1, x_2]b\Big)[x_1, x_2]^{t(n-i)} = 0 \in T.\end{aligned} \tag{8.2.20}$$

This implies $\{b, pb - u, 1\}$ is linearly $C$-dependent. Let $\gamma_1 b + \gamma_2(pb - u) + \gamma_3 1 = 0$. If $\gamma_2 = 0$, then $b \in C$ which gives

$$\begin{aligned}\sum(-1)^i \tbinom{n}{i} [x_1, x_2]^{ti+1}\Big((ap + c + qb)[x_1, x_2] + [x_1, x_2](pb - u)\Big)\\ .[x_1, x_2]^{t(n-i)} = 0 \in T.\end{aligned} \tag{8.2.21}$$

Then $ap + c + qb = u - pb \in C$. This gives conclusion (2).

If $\gamma_2 \neq 0$, then we can write $pb - u = \lambda_1 b + \lambda_2$, where $\lambda_1, \lambda_2 \in C$. Then (8.2.20) indicates

$$\begin{aligned}\sum(-1)^i \tbinom{n}{i} [x_1, x_2]^{ti+1}\Big((ap + c + \lambda_2)[x_1, x_2] + (q + \lambda_1)[x_1, x_2]b\Big)\\ .[x_1, x_2]^{t(n-i)} = 0 \in T.\end{aligned} \tag{8.2.22}$$

From above equation, either $b \in C$ or

$$[x_1, x_2]^{tn+1}(ap + c + \lambda_2)[x_1, x_2] = 0 \in T$$

and

$$[x_1, x_2]^{tn+1}(q + \lambda_1)[x_1, x_2]b = 0 \in T.$$

Thus if $b \notin C$, first equation gives $(ap + c + \lambda_2) = 0$, i.e., $ap + c \in C$ and the second equation gives $q \in C$. Since $c = qa - m$, we have $ap + qa - m \in C$. Now by (8.2.20), using $q \in C$, we can obtain $ap + c = u - pb - qb \in C$ which indicates conclusion (3).

On the other hand, if $b \in C$, then from (8.2.22),

$$\sum(-1)^i \binom{n}{i}[x_1, x_2]^{ti+1}(ap + c + bq + \lambda_1 b + \lambda_2)[x_1, x_2]^{t(n-i)+1} = 0 \in T.$$

Since $\lambda_1 b + \lambda_2 = pb - u$, then we now have

$$\big[[x_1, x_2](ap + c + pb - u + qb)[x_1, x_2], [x_1, x_2]^t\big]_n = 0 \in T \qquad (8.2.23)$$

i.e.,

$$[x_1, x_2]\Big[(ap + c + pb - u + qb), [x_1, x_2]^t\Big]_n [x_1, x_2] = 0 \in T. \qquad (8.2.24)$$

Hence $(ap+c+pb-u+qb) \in C$. Moreover, since $b \in C$, we have $pb-u = \lambda_1 b+\lambda_2 \in C$ and hence $ap + qb + c \in C$ which indicates conclusion (2). □

**Lemma 8.2.5.** *If $R$ satisfies (8.2.1) and $p, q \in C$, then one of the following holds:*

*(1)* $ap + c, b(p + q) - u \in C$;

*(2)* *$t$ is even, $b(p + q) - u \in C$ and $R = U \cong M_l(C)$, $l \geq 3$, $C$ is finite;*

*(3)* *$R \subseteq M_2(K)$, over a field $K$.*

*Proof.* By the hypothesis, $R$ satisfies

$$\big[[x_1, x_2](ap + c)[x_1, x_2] + [x_1, x_2]^2(bp + bq - u), [x_1, x_2]^t\big]_n = 0. \qquad (8.2.25)$$

By Lemma 8.2.2, we obtain one of the following results:

(1) $ap + c, b(p + q) - u \in C$;

(2) $b(p + q) - u \in C$ and $R = U \cong M_l(C)$, $l \geq 3$, $C$ is finite field. If $t$ is odd, by Lemma 8.2.1, we obtain $ap+c \in C$ which is conclusion (1) otherwise $t$ is even;

(3) $R \subseteq M_2(K)$, over a field $K$.

□

**Lemma 8.2.6.** *If $R$ satisfies (8.2.1) and $p, b \in C$, then one of the following holds:*

*(1)* $ap + bq + c, u \in C$;

*(2)* *$t$ is even, $u \in C$ and $R = U \cong M_l(C)$, $l \geq 3$, $C$ is finite field;*

*(3)* *$R \subseteq M_2(K)$, over a field $K$.*

*Proof.* By the hypothesis, $R$ satisfies

$$[x_1, x_2]\big[(ap + bq + c)[x_1, x_2] - [x_1, x_2]u, [x_1, x_2]^t\big]_n = 0.$$

By Lemma 8.2.2, we obtain one of the following results:

(1) $ap + bq + c, u \in C$;

(2) $u \in C$ and $R = U \cong M_l(C)$, $l \geq 3$, $C$ is finite field. If $t$ is odd then using Lemma 8.2.1, we obtain $ap + bq + c \in C$ which is conclusion (1) otherwise $t$ is even;

(3) $R \subseteq M_2(K)$, over a field $K$.

□

**Lemma 8.2.7.** *If $R$ satisfies (8.2.1) and $a, b \in C$, then following one holds:*

*(1)* *$u, c + bq \in C$ with $a + b = 0$;*

*(2)* $p, u, c + bq \in C$;

*(3)* *$t$ is even, $u \in C$, $R = U \cong M_l(C)$, $l \geq 3$, $C$ is finite field and either $a + b = 0$ or $p \in C$;*

*(4)* *$R \subseteq M_2(K)$, over a field $K$.*

*Proof.* By the hypothesis, $R$ satisfies

$$\big[(a + b)p[x_1, x_2]^2 + [x_1, x_2](c + bq)[x_1, x_2] - [x_1, x_2]^2u, [x_1, x_2]^t\big]_n = 0.$$

Then using Lemma 8.2.2, we obtain one of the following results:

(1) $(a + b)p, u \in C$, $(c + bq) \in C$; Since $a, b \in C$ and $(a + b)p \in C$ then either $a + b = 0$ or $p \in C$;

(2) $(a+b)p, u \in C$ and $R = U \cong M_l(C)$, $l \geq 3$, $C$ is finite field. If $t$ is odd, then by Lemma 8.2.1, we obtain $c+bq \in C$ which is same as (1) otherwise $t$ is even. Also $a, b \in C$ and $(a+b)p \in C$ together imply either $a+b=0$ or $p \in C$;

(3) $R \subseteq M_2(K)$, over a field $K$.

□

**Lemma 8.2.8.** *Let $R$ be a primitive ring with $char(R) \neq 2$ having nonzero socle $Soc(R)$, where $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$ along with $dim_C V = \infty$. If $R$ satisfies (8.2.1) for some $a, b, p, q, c, u \in R$, then either $p, q \in C$ or $p, b \in C$ or $a, b \in C$.*

*Proof.* Since $\dim_C V = \infty$, $R$ can not satisfy PI $s_4$.

We know that if $[s, Soc(R)] = (0)$ for any element $s \in R$, then $s \in C$. Thus on contrary, we may assume that there exist $l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8, l_9, l_{10} \in Soc(R)$ such that

1. either $[p, l_1] \neq 0$ or $[q, l_2] \neq 0$;

2. either $[p, l_3] \neq 0$ or $[b, l_4] \neq 0$;

3. either $[a, l_5] \neq 0$ or $[b, l_6] \neq 0$;

4. $s_4(l_7, l_8, l_9, l_{10}) \neq 0$.

Now we will prove that a number of contradictions arise. By Litoff's theorem (see Theorem 1.6.7), there exists $e^2 = e \in Soc(R)$ such that

- $l_i \in eRe$ for all $i = 1, \dots, 10$;
- $pl_i, l_ip, ql_i, l_iq, bl_i, l_ib, al_i, l_ia \in eRe$ for all $i = 1, \dots, 10$

where $eRe \cong M_k(C)$, the matrix ring over $C$. By the hypothesis,

$$\Big[(epe)[x_1,x_2](eae)[x_1,x_2] + (epe)[x_1,x_2]^2(ebe) + [x_1,x_2](ece)[x_1,x_2] \\ +[x_1,x_2](eqe)[x_1,x_2](ebe) - [x_1,x_2]^2(eue), [x_1,x_2]^t\Big]_n = 0$$

is a GPI for $eRe$. Then by Lemma 8.2.3, one of the following holds:

1. $epe, eqe \in Ce$;

2. $epe, ebe \in Ce$;

3. $eae, ebe \in Ce$;

4. $eRe \subseteq M_2(K)$, over a field $K$, that is, $eRe$ satisfies $s_4$. In any case we have contradiction with the choice of $l_i \in Soc(R)$, $i = 1, \ldots, 10$.

□

**Proposition 8.2.9.** *If (8.2.1) is a GPI for R, then following one holds:*

*(1)* $p, q, ap + c, b(p+q) - u \in C$;

*(2)* $p, b, ap + bq + c, u \in C$;

*(3)* $p, a, b, u, c + bq \in C$;

*(4)* $a, b, u, c + bq \in C$ *with* $a + b = 0$;

*(5)* $R \subseteq M_2(K)$, *over a field* $K$;

*(6)* $t$ *is even,* $R = U \cong M_l(C)$, $l \geq 3$, $C$ *is finite field and one of the following holds:*

- $p, q, b(p+q) - u \in C$;
- $p, b, u \in C$;
- $a, b, u, p \in C$;
- $a, b, u \in C$ *with* $a + b = 0$.

*Proof.* If one of the following holds

(1) $p, q \in C$;

(2) $p, b \in C$;

(3) $a, b \in C$;

then conclusion follows by Lemma 8.2.5, Lemma 8.2.6 and Lemma 8.2.7. Thus on contrary, we assume

(1) either $p \notin C$ or $q \notin C$;

(2) either $p \notin C$ or $b \notin C$;

(3) either $a \notin C$ or $b \notin C$.

Then we prove a number of contradictions.
By Lemma 8.2.4, (8.2.1) is nontrivial GPI for $R$. By [21], (8.2.1) is a nontrivial GPI for $U$.

Assume that if $C$ is infinite, then $E$ be the algebraic closure of $C$ and if $C$ is finite, set $E = C$. So we may assume that $U$ is a subring $U \otimes_C E$. By [70], (8.2.1) is a nontrivial GPI for $U \otimes_C E$. In view of [39, Theorem 3.5], $U \otimes_C E$ is prime ring with extended centroid $E$. Let $\widetilde{U} = U \otimes_C E$. By Martindale's theorem (see Theorem 1.6.6), $\widetilde{U}$ is isomorphic to a dense subring of $\mathrm{End}(V_D)$, where $V$ is a vector space over a division ring $D$ and $D$ is a finite dimensional central division algebra over $E$. Note that $E$ is either finite or algebraically closed. Since $D$ is a finite dimensional over $E$, we must have $D = E$. If $\dim_E V = k$, $R \subseteq \widetilde{U} \cong M_k(E)$. If $k = 2$, then conclusion follows, otherwise by Lemma 8.2.3, either $p, q \in C$ or $p, b \in C$ or $a, b \in C$. In any case, contradiction arises.

Hence we assume that $\dim_E V = \infty$. Then using Lemma 8.2.8, again we arrive at a contradiction. □

**Lemma 8.2.10.** *Suppose that $F$, $G$, $H$ are three inner generalized derivations on $R$ such that*

$$[F([x,y])G([x,y]) - [x,y]H([x,y]), [x,y]^t]_n = 0$$

*for all $x, y \in U$, where $t \geq 1$ fixed integer, then either $R \subseteq M_2(K)$, over a field $K$ or one of the following holds:*

*(1) there exist $a, b, m, u \in U$, $\lambda \in C$ such that $F(x) = \lambda x$, $G(x) = ax + xb$, $H(x) = mx + xu$ for all $x \in R$ with $\lambda a - m, \lambda b - u \in C$ for all $x \in R$;*

*(2) there exist $a, p, m \in U$ such that $F(x) = xp$, $G(x) = ax$, $H(x) = mx$ for all $x \in R$ with $pa - m \in C$ for all $x \in R$;*

*(3) there exist $p, q \in U$, $\lambda \in C$ such that $F(x) = px + xq$, $G = 0$, $H(x) = \lambda x$ for all $x \in R$;*

*(4) $t$ is even, $R \cong M_l(C)$, $l \geq 3$, $C$ is finite field and one of the following holds:*

*(i) there exist $a, b, m, u \in U$, $\lambda \in C$ such that $F(x) = \lambda x$, $G(x) = ax + xb$, $H(x) = mx + xu$ for all $x \in R$ with $\lambda b - u \in C$ for all $x \in R$;*

*(ii) there exist $a, p, m \in U$ such that $F(x) = xp$, $G(x) = ax$, $H(x) = mx$ for all $x \in R$;*

*(iii) there exist $p, q, m \in U$ such that $F(x) = px + xq$, $G(x) = 0$, $H(x) = mx$ for all $x \in R$.*

*Proof.* Let us assume that there exist $a, b, p, q, m, u \in U$ such that $F(x) = px + xq$, $G(x) = ax + xb$ and $H(x) = mx + xu$ for all $x \in R$. By hypothesis, $R$ satisfies

$$\big[p[x,y]a[x,y] + p[x,y]^2 b + [x,y](qa - m)[x,y] \\ +[x,y]q[x,y]b - [x,y]^2 u, [x,y]^t\big]_n = 0.$$

Then using Proposition 8.2.9, we have either $R \subseteq M_2(K)$, over a field $K$ or one of the following holds:

1. $p, q, ap + qa - m, bp + bq - u \in C$; Then $F(x) = (p+q)x$, $G(x) = ax + xb$, $H(x) = mx + xu$ for all $x \in R$ with $(p+q)a - m \in C$, $(p+q)b - u \in C$. Thus we obtain conclusion (1).

2. $p, b, ap + bq + qa - m, u \in C$; Then $F(x) = x(p+q)$, $G(x) = (a+b)x$, $H(x) = (m+u)x$ for all $x \in R$ with $(p+q)(a+b) - (m+u) = pa + qa + pb + qb - m - u \in C$. Thus we obtain conclusion (2).

3. $p, a, b, u, qa - m + bq \in C$; Then $F(x) = x(p+q)$, $G(x) = (a+b)x$, $H(x) = (m+u)x$ for all $x \in R$ with $(a+b)(p+q) - (m+u) = ap + qa + pb + bq - m - u \in C$. This is a particular case of conclusion (2).

4. $a, b, u, qa - m + bq \in C$ with $a + b = 0$; Then $F(x) = px + xq$, $G = 0$, $H(x) = \lambda x$ for all $x \in R$, for $\lambda = m + u \in C$ which is conclusion (3).

5. $t$ is even, $R \cong M_l(C)$, $l \geq 3$, $C$ is finite field and one of the following holds:

   (i) $p, q, b(p+q) - u \in C$; Then $F(x) = (p+q)x$, $G(x) = ax + xb$, $H(x) = mx + xu$ for all $x \in R$ with $(p+q)b - u \in C$ which is conclusion (4)-(i).

   (ii) $p, b, u \in C$; Then $F(x) = x(p+q)$, $G(x) = (a+b)x$, $H(x) = (m+u)x$ for all $x \in R$ which is conclusion (4)-(ii).

   (iii) $a, b, u, p \in C$; Then $F(x) = x(p+q)$, $G(x) = (a+b)x$, $H(x) = (m+u)x$ for all $x \in R$ which is a particular case of conclusion (4)-(ii).

(iv) $a, b, u \in C$ with $a + b = 0$; Then $F(x) = px + xq$, $G = 0$, $H(x) = mx$ for all $x \in R$ which is conclusion (4)-(iii).

□

## 8.3 Proof of Theorem

Let $t = lcm\{t_1, t_2, \ldots, t_n\}$. Then we can write $t = pt_1$ for some positive integer $p$. Hence

$$\begin{aligned} & [F(X)G(X) - XH(X), X^t, X^{t_2}, \ldots, X^{t_n}] \\ = & [F(X)G(X) - XH(X), X^{pt_1}, X^{t_2}, \ldots, X^{t_n}] \\ = & \sum_{i=0}^{p-1} X^{it_1}[F(X)G(X) - XH(X), X^{t_1}, X^{t_2}, \ldots, X^{t_n}]X^{(p-1-i)t_1} \\ = & 0. \end{aligned}$$

By same argument, we have

$$[F(X)G(X) - XH(X), X^t, X^t, \ldots, X^t] = 0,$$

that is,

$$[F(X)G(X) - XH(X), X^t]_n = 0$$

for all $X \in L$.

Since $L$ is a noncentral, by [8, Lemma 1], $0 \neq [I, I] \subseteq L$ for some nonzero ideal $I$ of $R$. Thus $I$ satisfies

$$\Big[F([x, y])G([x, y]) - [x, y]H([x, y]), [x, y]^t\Big]_n = 0.$$

By [72, Theorem 3], there exist $a, p, m \in U$ and derivations $d, \delta, \eta$ of $U$ such that $F(x) = px + d(x)$, $G(x) = ax + \delta(x)$ and $H(x) = mx + \eta(x)$. By [21, 71], we can write

$$\begin{aligned} \Big[&(p[x, y] + d([x, y]))(a[x, y] + \delta([x, y])) \\ &-[x, y](m[x, y] + \eta([x, y])), [x, y]^t\Big]_n = 0 \end{aligned} \tag{8.3.1}$$

for all $x, y \in U$.

If $F$, $G$ and $H$ are all inner maps, then by Lemma 8.2.10, we have conclusions (1), (2) and (6)-(i), (6)-(ii) and particular cases of (4), (6)-(iv) of Theorem 8.1.1 Thus, it is enough to consider the cases where all of $F$, $G$ and $H$ are not inner maps. All these cases are considered in the following Lemmas:

**Lemma 8.3.1.** *If $d$, $\eta$ are inner and $\delta$ is outer, then one of the conclusions (3) and (6)-(iii) holds.*

*Proof.* Let $d(x) = [l, x]$, $\eta(x) = [k, x]$ for all $x \in R$ and for some $l, k \in U$. The (8.3.1) transferred to

$$\Big[\big(p[x,y] + [l,[x,y]]\big)\big(a[x,y] + \delta([x,y])\big) - [x,y](m[x,y] + [k,[x,y]]), \\ [x,y]^t\Big]_n = 0. \tag{8.3.2}$$

By [64, Theorem 2], $U$ satisfies

$$\Big[(p[x,y] + [l,[x,y]])(a[x,y] + [v_1,y] + [x,v_2]) - [x,y](m[x,y] + [k,[x,y]]), [x,y]^t\Big]_n = 0.$$

In particular, $U$ satisfies the blended component

$$\Big[\big(p[x,y] + [l,[x,y]]\big)\big([v_1,y] + [x,v_2]\big), [x,y]^t\Big]_n = 0.$$

Choose $b \in U - C$ and then replacing $v_1$ with $[b,x]$ and $v_2$ with $[b,y]$ in above relation, and then obtain

$$\Big[\big((p+l)[x,y] - [x,y]l\big)\big[b,[x,y]\big], [x,y]^t\Big]_n = 0.$$

Since $b \notin C$, using Proposition 8.2.9, either $R \subseteq M_2(K)$ or $p, l, bp \in C$. When $p, l, bp \in C$, we obtain $d = 0$, $p = 0$, $F = 0$. Then equation (8.3.2) reduces to

$$[x,y]\Big[(m+k)[x,y] - [x,y]k, [x,y]^t\Big]_n = 0.$$

Hence again using Proposition 8.2.9, we obtain either $m, k \in C$ which gives conclusion (3) or $t$ is even, $k \in C$ with $R \cong M_l(C)$, $l \geq 3$ and $C$ is a finite field, which gives conclusion (6)-(iii). □

**Lemma 8.3.2.** *If $d$, $\delta$ are inner and $\eta$ is outer, then $R \subseteq M_2(K)$ over a field $K$.*

*Proof.* Let $d(x) = [l, x]$, $\delta(x) = [k, x]$ for all $x \in R$ and for some $l, k \in U$. Then (8.3.1) transferred to

$$\Big[\big(p[x,y] + [l,[x,y]]\big)\big(a[x,y] + [k,[x,y]]\big) - [x,y](m[x,y] + \eta([x,y])), [x,y]^t\Big]_n = 0.$$

By [64, Theorem 2], $U$ satisfies

$$\Big[(p[x,y] + [l,[x,y]])(c[x,y] + [k,[x,y]]) \\ -[x,y](m[x,y] + [v_1,y] + [x,v_2]), [x,y]^t\Big]_n = 0. \tag{8.3.3}$$

In particular, $U$ satisfies the blended component

$$[x,y]\big[[v_1,y] + [x,v_2], [x,y]^t\big]_n = 0 \tag{8.3.4}$$

which is a polynomial identity (PI). By [67, Lemma 2] there exists a field $K$ such that $U \subseteq M_s(K)$, $s > 1$ and the matrix ring $M_s(K)$ satisfies

$$[x,y]\big[[v_1,y] + [x,v_2], [x,y]^t\big]_n = 0.$$

If $s = 2$, then $R \subseteq M_2(K)$, as desired. So, let us assume $s \geq 3$. But for $x = e_{12}$, $y = e_{21}$, $v_1 = 0$ and $v_2 = e_{23}$, we have

$$0 = [x,y]\big[([v_1,y] + [x,v_2]), [x,y]^t\big]_n = e_{13},$$

a contradiction. □

**Lemma 8.3.3.** *If $\delta$, $\eta$ are inner and $d$ is outer, then either $R \subseteq M_2(K)$ over a field $K$ or one of the conclusions (4) and (6)-(iv) holds.*

*Proof.* Let $\delta(x) = [l, x]$, $\eta(x) = [k, x]$ for all $x \in R$ and for some $l, k \in U$. Then (8.3.1) reduces to

$$\Big[\big(p[x,y] + d([x,y])\big)\big(a[x,y] + [l,[x,y]]\big) - [x,y](m[x,y] + [k,[x,y]]), \\ [x,y]^t\Big]_n = 0. \tag{8.3.5}$$

By [64, Theorem 2], $U$ satisfies

$$\Big[(p[x,y] + [v_1,y] + [x,v_2])(a[x,y] + [l,[x,y]]) - [x,y](m[x,y] + [k,[x,y]]), \\ [x,y]^t\Big]_n = 0. \tag{8.3.6}$$

In particular, $U$ satisfies the blended component

$$\Big[\big([v_1,y]+[x,v_2]\big)\big(a[x,y]+[l,[x,y]]\big),[x,y]^t\Big]_n=0. \tag{8.3.7}$$

We choose $b\in U-C$ and replacing $v_1$ with $[b,x]$ and $v_2$ with $[b,y]$ and then obtain

$$\Big[[b,[x,y]]\big((a+l)[x,y]-[x,y]l\big),[x,y]^t\Big]_n=0. \tag{8.3.8}$$

Since $b\notin C$, by Proposition 8.2.9, either $R\subseteq M_2(K)$ over a field $K$ or $l\in C$, $a=0$. For $l\in C$, $a=0$ we obtain $\delta=0$, $G=0$. Then equation (8.3.5) reduces to

$$[x,y]\Big[(m+k)[x,y]-[x,y]k,[x,y]^t\Big]_n=0. \tag{8.3.9}$$

Again using Proposition 8.2.9, we obtain either $m,k\in C$ which gives conclusion (4) or $t$ is even, $k\in C$ with $R=U\cong M_l(C)$, $l\geq 3$ and $C$ is finite field which gives conclusion (6)-(iv). □

**Lemma 8.3.4.** *If $d$ is inner and $\delta$, $\eta$ are outer, then either $R\subseteq M_2(K)$ over a field $K$ or one of the conclusions (5) and (6)-(v) holds.*

*Proof.* Let $d(x)=[l,x]$ for all $x\in R$, for some $l\in U$. Then (8.3.1) reduces to

$$\Big[(p[x,y]+[l,[x,y]])(a[x,y]+\delta([x,y])) -[x,y](m[x,y]+\eta([x,y])),[x,y]^t\Big]_n=0. \tag{8.3.10}$$

- *Let $\delta$ and $\eta$ be linearly $C$-dependent.*

Then $\delta(x)=\alpha\eta(x)+[k,x]$ for all $x\in R$, $k\in U$ where $0\neq\alpha\in C$ then (8.3.10) reduces to

$$\Big[(p[x,y]+[l,[x,y]])(a[x,y]+\alpha\eta([x,y])+[k,[x,y]]) -[x,y](m[x,y]+\eta([x,y])),[x,y]^t\Big]_n=0. \tag{8.3.11}$$

By [64, Theorem 2], $U$ satisfies

$$\big[(p[x,y]+[l,[x,y]])(a[x,y]+\alpha([v_1,y]+[x,v_2])+[k,[x,y]]) -[x,y](m[x,y]+[v_1,y]+[x,v_2]),[x,y]^t\big]_n=0. \tag{8.3.12}$$

$U$ satisfies the blended component

$$\big[\alpha(p[x,y]+[l,[x,y]])([v_1,y]+[x,v_2])-[x,y]([v_1,y]+[x,v_2]),[x,y]^t\big]_n=0.$$

Choose $b \in U - C$ and replace $v_1$ with $[b, x]$ and $v_2$ with $[b, y]$ in above equation, and then get

$$\Big[\big(\alpha(p+l)[x,y] - [x,y]\alpha l\big)\big[b,[x,y]\big] - [x,y][b,[x,y]], [x,y]^t\Big]_n = 0.$$

Since $b \notin C$, $\alpha \neq 0$, by Proposition 8.2.9, either $R \subseteq M_2(K)$ over a field $K$ or $p, l, b(\alpha p - 1) \in C$. When $p, l, b(\alpha p - 1) \in C$, we obtain $\alpha p = 1$, $d = 0$. Then by (8.3.11),

$$\big[[x,y](pa + pk - m)[x,y] - [x,y]^2 pk, [x,y]^t\big]_n = 0. \tag{8.3.13}$$

Again applying Proposition 8.2.9, we obtain either $pa - m, pk \in C$ or $t$ is even, $pk \in C$ with $R = U \cong M_l(C)$, $l \geq 3$, $C$ is a finite field. Now $p, pk \in C$ imply either $k \in C$ or $p = 0$. But $p = 0$ contradicts $\alpha p = 1$. Consequently, $k \in C$, $\delta(x) = \alpha\eta(x)$. Hence we obtain either conclusion (5) or (6)-(v).

- *Let $\delta$ and $\eta$ be linearly $C$-independent.*

By [64, Theorem 2], from (8.3.10), $U$ satisfies

$$\Big[(p[x,y] + [l,[x,y]])(a[x,y] + [v_1,y] + [x,v_2]) \\ -[x,y](m[x,y] + [u_1,y] + [x,u_2]), [x,y]^t\Big]_n = 0. \tag{8.3.14}$$

In particular, $U$ satisfies blended component $[x,y]\big[[u_1,y] + [x,u_2], [x,y]^t\big]_n = 0$ which is same as (8.3.4). Hence by similar manner, the conclusion follows. □

**Lemma 8.3.5.** *If $\delta$ is inner and $d$, $\eta$ are outer, then we obtain $R \subseteq M_2(K)$ over a field $K$.*

*Proof.* Let $\delta(x) = [l, x]$ for all $x \in R$, for some $l \in U$. Then (8.3.1) reduces to

$$\Big[(p[x,y] + d([x,y]))(a[x,y] + [l,[x,y]]) \\ -[x,y](m[x,y] + \eta([x,y])), [x,y]^t\Big]_n = 0. \tag{8.3.15}$$

- *Let $d$ and $\eta$ be linearly $C$-dependent.*

Then $d(x) = \alpha\eta(x) + [k, x]$ for all $x \in R$ where $0 \neq \alpha \in C$, by (8.3.15),

$$\Big[(p[x,y] + \alpha\eta([x,y]) + [k,[x,y]])(a[x,y] + [l,[x,y]]) \\ -[x,y](m[x,y] + \eta([x,y])), [x,y]^t\Big]_n = 0. \tag{8.3.16}$$

By [64, Theorem 2], $U$ satisfies

$$\Big[(p[x,y]+\alpha([v_1,y]+[x,v_2])+[k,[x,y]])(a[x,y]+[l,[x,y]] \\ -[x,y](m[x,y]+[v_1,y]+[x,v_2]),[x,y]^t\Big]_n=0. \tag{8.3.17}$$

In particular, $U$ satisfies the blended component

$$\big[\alpha([v_1,y]+[x,v_2])(a[x,y]+[l,[x,y]])-[x,y]([v_1,y]+[x,v_2]),[x,y]^t\big]_n=0.$$

Choose $b\in U-C$ and replacing $v_1$ with $[b,x]$ and $v_2$ with $[b,y]$. We get from above

$$\Big[\big[b,[x,y]\big]\big(\alpha(a+l)[x,y]-[x,y]\alpha l\big)-[x,y]\big[b,[x,y]\big],[x,y]^t\Big]_n=0.$$

Since $b\notin C$, $\alpha\neq 0$, using Proposition 8.2.9, $R\subseteq M_2(K)$ over a field $K$.

- *Let $d$ and $\eta$ be linearly $C$-independent.*

By [64, Theorem 2], from (8.3.15), $U$ satisfies

$$\Big[(p[x,y]+[v_1,y]+[x,v_2])(a[x,y]+[l,[x,y]]) \\ -[x,y](m[x,y]+[u_1,y]+[x,u_2]),[x,y]^t\Big]_n=0. \tag{8.3.18}$$

In particular, $U$ satisfies blended component $[x,y]\big[[u_1,y]+[x,u_2],[x,y]^t\big]_n=0$ which is same as (8.3.4) and hence the conclusion follows. □

**Lemma 8.3.6.** *If $\eta$ is inner and $d$, $\delta$ are outer, then we obtain $R\subseteq M_2(K)$ over a field $K$.*

*Proof.* Let $\eta(x)=[l,x]$ for all $x\in R$, for some $l\in U$. Then (8.3.1) reduces to

$$\Big[(p[x,y]+d([x,y]))(a[x,y]+\delta([x,y])) \\ -[x,y](m[x,y]+[l,[x,y]]),[x,y]^t\Big]_n=0. \tag{8.3.19}$$

- *Let $d$ and $\delta$ be linearly $C$-dependent.*

Then $d(x)=\alpha\delta(x)+[k,x]$ for all $x\in R$ where $0\neq\alpha\in C$. Thus (8.3.19) reduces to

$$\Big[(p[x,y]+\alpha\delta([x,y])+[k,[x,y]])(a[x,y]+\delta([x,y])) \\ -[x,y](m[x,y]+[l,[x,y]]),[x,y]^t\Big]_n=0. \tag{8.3.20}$$

By [64, Theorem 2], $U$ satisfies

$$\big[(p[x,y]+\alpha([v_1,y]+[x,v_2])+[k,[x,y]])(a[x,y]+[v_1,y]+[x,v_2]) \\ -[x,y](m[x,y]+[l,[x,y]]),[x,y]^t\big]_n=0. \qquad (8.3.21)$$

In particular, $U$ satisfies the blended component (for $v_1=0$)

$$\Big[\alpha[v_1,y]\Big(a[x,y]+[v_1,y]+[x,v_2]\Big)+\Big(p[x,y]+\alpha([v_1,y]+[x,v_2]) \\ +[k,[x,y]]\Big)[v_1,y],[x,y]^t\Big]_n=0. \qquad (8.3.22)$$

Since $\alpha\neq 0$, from above $U$ satisfies the blended component (for $v_2=0$),

$$\big[[v_1,y][x,v_2]+[x,v_2][v_1,y],[x,y]^t\big]_n=0. \qquad (8.3.23)$$

This is a polynomial identity (PI) and hence by [67, Lemma 2], $U\subseteq M_s(K)$, $s>1$ for some field $K$ and $M_s(K)$ satisfies

$$\big[[v_1,y][x,v_2]+[x,v_2][v_1,y],[x,y]\big]_n=0. \qquad (8.3.24)$$

If $s=2$, then $R\subseteq M_2(K)$ over a field $K$. Let us assume $s\geq 3$. Then for $v_1=e_{31}$, $y=e_{12}$, $x=e_{21}$, $v_2=e_{11}$, we have

$$\big[[v_1,y][x,v_2]+[x,v_2][v_1,y],[x,y]^t\big]_n=(-1)^{tn}e_{31}=0,$$

a contradiction.

- *Let $\delta$ and $d$ be linearly $C$-independent.*

By [64, Theorem 2], from (8.3.19), $U$ satisfies

$$\big[(p[x,y]+[v_1,y]+[x,v_2])(a[x,y]+[u_1,y]+[x,u_2]) \\ -[x,y](m[x,y]+[l,[x,y]]),[x,y]^t\big]_n=0. \qquad (8.3.25)$$

In particular, $U$ satisfies blended components (for $v_1=0$),

$$\big[[v_1,y](a[x,y]+[u_1,y]+[x,u_2]),[x,y]^t\big]_n=0. \qquad (8.3.26)$$

Assuming $v_1=x$, $U$ satisfies

$$[x,y]\big[([u_1,y]+[x,u_2]),[x,y]^t\big]_n=0$$

which is same as (8.3.4). Thus by similar method we obtain the conclusion. □

**Lemma 8.3.7.** *If all of $d$, $\delta$ and $\eta$ are outer, then we obtain $R \subseteq M_2(K)$ over a field $K$.*

*Proof.* We have the following cases.

**Case-i.** *$d$, $\delta$ and $\eta$ are linearly $C$-dependent.* Then for some $\alpha_1, \alpha_2, \alpha_3 \in C$, $l \in U$ such that $\alpha_1 d(x) + \alpha_2\delta(x) + \alpha_3\eta(x) = [l, x]$ for all $x \in U$. Since $\eta$ is not inner, $(\alpha_1, \alpha_2) \neq (0, 0)$.

Without loss of generality, we may assume $\alpha_1 \neq 0$. Then $d(x) = \alpha'_1\delta(x) + \alpha'_2\eta(x) + [l', x]$ for all $x \in U$, where $\alpha'_1 = -\alpha_2\alpha_1^{-1}, \alpha'_2 = -\alpha_3\alpha_1^{-1}$ and $l' = \alpha_1^{-1}l$. Then (8.3.1) gives,

$$\Big[(p[x,y] + \alpha'_1\delta([x,y]) + \alpha'_2\eta([x,y]) + [l',[x,y]])(a[x,y] + \delta([x,y])) \\ -[x,y](m[x,y] + \eta([x,y])), [x,y]^t\Big]_n = 0. \tag{8.3.27}$$

Then we have the following cases.

**Sub-case-i.** Let $\delta$ and $\eta$ be $C$-dependent modulo inner derivations of $U$. Then $\delta(x) = \beta'_2\eta(x) + [k', x]$, where $0 \neq \beta'_2 \in C$. Then (8.3.27) reduces to

$$\Big[\Big(p[x,y] + \beta''_2\eta([x,y]) + [k'',[x,y]]\Big)\Big(a[x,y] + \beta'_2\eta([x,y]) + [k',[x,y]]\Big) \\ -[x,y](m[x,y] + \eta([x,y])), [x,y]^t\Big]_n = 0 \tag{8.3.28}$$

where $\beta''_2 = \alpha'_1\beta'_2 + \alpha'_2$, $k'' = \alpha'_1k' + l'$, $d(x) = \beta''_2\eta(x) + [k'', x]$.

By [64, Theorem 2], $U$ satisfies

$$\Big[\Big(p[x,y] + \beta''_2([v_1,y] + [x,v_2]) + [k'',[x,y]]\Big)\Big(a[x,y] + \beta'_2([v_1,y] + [x,v_2]) \\ +[k',[x,y]]\Big) - [x,y](m[x,y] + [v_1,y] + [x,v_2]), [x,y]^t\Big]_n = 0. \tag{8.3.29}$$

Particularly, $U$ satisfies the blended component (for $v_1 = 0$)

$$\Big[\beta''_2[v_1,y].\Big(a[x,y] + \beta'_2([v_1,y] + [x,v_2]) + [k',[x,y]]\Big) + \Big(p[x,y] \\ +\beta''_2([v_1,y] + [x,v_2]) + [k'',[x,y]]\Big)\beta'_2[v_1,y] - [x,y][v_1,y], [x,y]^t\Big]_n = 0.$$

Again from above $U$ satisfies the blended component (for $v_2 = 0$),

$$\beta''_2\beta'_2\big[[v_1,y][x,v_2] + [x,v_2][v_1,y], [x,y]^t\big]_n = 0.$$

Using the same argument applied for (8.3.23), we obtain $\beta''_2 = 0$ (as $\beta'_2 \neq 0$) unless $R \subseteq M_2(K)$ holds. If $\beta''_2 = 0$, then $d$ is inner, a contradiction.

**Sub-case-ii.** Let $\delta$ and $\eta$ be $C$-independent modulo inner derivations of $U$. By [64, Theorem 2], (8.3.27) gives that $U$ satisfies

$$\Big[\Big(p[x,y]+\alpha_1'([v_1,y]+[x,v_2])+\alpha_2'([u_1,y]+[x,u_2])+[l',[x,y]]\Big).$$
$$\Big(a[x,y]+[v_1,y]+[x,v_2]\Big)-[x,y](m[x,y]+[u_1,y]+[x,u_2]),[x,y]^t\Big]_n=0.$$

Hence $U$ satisfies the blended component

$$\Big[\alpha_2'([u_1,y]+[x,u_2])(a[x,y]+[v_1,y]+[x,v_2])$$
$$-[x,y]([u_1,y]+[x,u_2]),[x,y]^t\Big]_n=0 \tag{8.3.30}$$

which implies $\Big[\alpha_2'([u_1,y]+[x,u_2])([v_1,y]+[x,v_2]),[x,y]^t\Big]_n=0$. Now for $u_1=x$, $u_2=0$, $U$ satisfies $\alpha_2'[x,y]\big[([v_1,y]+[x,v_2]),[x,y]\big]_n=0$. Then by similar argument applied to (8.3.4), either $R\subseteq M_2(K)$ over a field $K$, as desired or $\alpha_2'=0$. When $\alpha_2'=0$, (8.3.30) reduces to

$$[x,y]\big[([u_1,y]+[x,u_2]),[x,y]\big]_n=0.$$

This is same as (8.3.4) and again by similar method conclusion follows.

**Case-ii.** *d, $\delta$ and $\eta$ are linearly $C$-independent.*

By [64, Theorem 2], as before we obtain from (8.3.1) that

$$\Big[\big(p[x,y]+[u_1,y]+[x,u_2]\big)\big(a[x,y]+[v_1,y]+[x,v_2]\big)$$
$$-[x,y]\big(m[x,y]+[w_1,y]+[x,w_2]\big),[x,y]^t\Big]_n=0. \tag{8.3.31}$$

Hence $U$ satisfies the blended component (for $u_1=u_2=0$)

$$[x,y]\big[[w_1,y]+[x,w_2],[x,y]^t\big]_n=0.$$

This is same as (8.3.4). Hence by similar method conclusion follows. Thus the proof of the Theorem 8.1.1 is completed. □

# Bibliography


[1] A. Alahmadi, S. Ali, A. N. Khan, and M. S. Khan, *A characterization of generalized derivations on prime rings*, Commun. Algebra **44 (8)** (2016), 3201–3210.

[2] E. Albas, N. Argac, and V. De Filippis, *Generalized derivations with Engel conditions on one sided ideals*, Commun. Algebra **36** (2008), 2063–2071.

[3] F. Ali and M. A. Chaudhry, *On generalized* $(\alpha, \beta)$*-derivations of semiprime rings*, Turk. J. Math. **34** (2010), 1–6.

[4] N. Argac, L. Carini, and V. De Filippis, *An Engel condition with generalized derivations on Lie ideals*, Taiwanese J. Math. **12 (2)** (2008), 419–433.

[5] N. Argaç and V. De Filippis, *Actions of generalized derivations on multilinear polynomials in prime rings*, Algebra Colloq. **18 (spec 01)** (2011), 955–964.

[6] M. Ashraf, A. Ali, and S. Ali, *Some commutativity theorem for prime rings with generalized derivations*, Southeast Asian Bull. Math. **31** (2007), 415–421.

[7] K. I. Beidar, W. S. Martindale 3rd, and A. V. Mikhalev, *Rings with generalized identities*, Pure and Applied Math., 196, Marcel Dekker, New York, 1996.

[8] J. Bergen, I. N. Herstein, and J. W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra. **71 (1)** (1981), 259–267.

[9] M. Brešar, *On the distance of the composition of two derivations to the generalized derivations*, Glasgow Math. J. **33** (1991), 89–93.

[10] ———, *Centralizing mappings and derivations in prime rings*, J. Algebra **156** (1993), 385–394.

[11] L. Carini and V. De Filippis, *Centralizers of generalized derivations on multilinear polynomials in prime rings*, Siberian Math. J. **53 (6)** (2012), 1051–1060.

[12] L. Carini, V. De Filippis, and B. Dhara, *Annihilators on co-commutators with generalized derivations on Lie ideals*, Publ. Math. Debrecen **76 (4)** (2010), 395–409.

[13] L. Carini, V. De Filippis, and G. Scudo, *Identities with product of generalized derivations in prime rings*, Algebra Colloq. **20 (4)** (2013), 711–720.

[14] ______, *Identities with product of generalized skew derivations on multilinear polynomials*, Commun. Algebra **44 (7)** (2016), 3122–3138.

[15] L. Carini, V. De Filippis, and F. Wei, *Generalized skew derivations cocentralizing multilinear polynomials*, Mediterr. J. Math. **13** (2016), 2397–2424.

[16] L. Carini and G. Scudo, *An annihilator condition with generalized derivations on multilinear polynomials*, Commun. Algebra **42** (2014), 4084–4093.

[17] A. Chandrasekhar and S. K. Tiwari, *A note on generalized derivations as a jordan homomorphism*, Bull. Korean Math. Soc. **57 (3)** (2020), 709–737.

[18] C. M. Chang and T. K. Lee, *Annihilators of power values of derivations in prime rings*, Commun. Algebra **26 (7)** (1998), 2091–2113.

[19] J. C. Chang, *On the identity* $h(x) = af(x) + g(x)b$, Taiwanese. J. Math. **7 (1)** (2003), 103–113.

[20] C. L. Chuang, *The additive subgroup generated by a polynomial*, Israel J. Math. **59 (1)** (1987), 98–106.

[21] ______, *GPIs having coefficients in utumi quotient rings*, Proc. Amer. Math. Soc. **103 (3)** (1988), 723–728.

[22] ______, *Differential identities with automorphisms and antiautomorphisms ii*, J. Algebra **160 (1)** (1993), 130–171.

[23] C. L. Chuang and T. K. Lee, *Identities with a single skew derivations*, J. Algebra **288 (1)** (2005), 59–77.

[24] P. Das, B. Dhara, and S. Kar, *Generalized skew-derivations annihilating and centralizing on multilinear polynomials in prime rings*, Proc. Indian Acad. Sci. (Math. Sci.) **129 (2)** (2019), Article ID 0018.

[25] B. Dhara, *Derivations with Engel conditions on multilinear polynomials in prime rings*, Demonstr. Math. **42 (3)** (2009), 467–478.

[26] ———, *Vanishing and commuting generalized derivations on multiliear polynomials in prime rings*, Southeast Asian Bull. Math **41** (2017), 397–411.

[27] ———, *b-generalized derivation on multilinear polynomials in prime rings*, Bull. Korean Math. Soc. **55 (2)** (2018), 573–586.

[28] ———, *Generalized derivations acting on multilinear polynomials in prime rings*, Czechoslovak Math. J. **68 (1)** (2018), 95–119.

[29] B. Dhara, A. Ali, and D. Das, *Engel conditions of generalized derivations on Lie ideals and left sided ideals in prime rings and banach algebras*, Afr. Mat. **27 (7)** (2016), 1391–1401.

[30] B. Dhara, N. Argac, and E. Albas, *Vanishing derivations and co-centralizing generalized derivations on multilinear polynomials in prime rings*, Commun. Algebra **44 (5)** (2016), 1905–1923.

[31] B. Dhara and V. De Filippis, *Engel conditions of generalized derivations on left ideals and Lie ideals in prime rings*, Commun. Algebra **48 (1)** (2020), 154–167.

[32] B. Dhara, V. De Filippis, and G. S. Sandhu, *Generalized skew-derivations as generalizations of jordan homomorphisms in prime rings*, Commun. Algebra **50 (11)** (2022), 5016–5032.

[33] B. Dhara, S. Kar, and S. Mondal, *A result on generalized derivations on Lie ideals in prime rings*, Beitr. Algebra Geom. **54 (2)** (2013), 677–682.

[34] ———, *Commutativity theorems on prime and semiprime rings with generalized $(\sigma, \tau)$-derivations*, Bol. Soc. Paran. Mat. **32 (1)** (2014), 109–122.

[35] B. Dhara, S. Kar, and K. G. Pradhan, *Co-centralizing generalized derivations acting on multilinear polynomials in prime rings*, Bull. Iranian Math. Soc. **42 (6)** (2016), 1331–1342.

[36] B. Dhara, S. Kar, and K. Singh, *Product of generalized derivations with Engel condition on Lie ideals in prime rings*, Asian-European J. Math. **16 (12)** (2023), Art. No. 2350233.

[37] B. Dhara, S. Kuila, and S. Kar, *Generalized derivations commuting on Lie ideals in prime rings*, Ann. Univ. Ferrara **69** (2023), 159–181.

[38] B. Dhara and R. K. Sharma, *Right sided ideals and multilinear polynomials with derivations on prime rings*, Rend. Sem. Mat. Univ. Padova **121** (2009), 243–257.

[39] T. S. Erickson, W. S. Martindale 3rd, and J. M. Osborn, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975), 49–63.

[40] C. Faith, *Lectures on injective modules and quotient rings*, Lecture notes in Math. 49, Springer Verlag, New York and Berlin, 1967.

[41] V. De Filippis, *An Engel condition with generalized derivations on multilinear polynomials*, Israel J. Math. **162** (2007), 93–108.

[42] ______, *Posner's second theorem and an annihilator condition with generalized derivations*, Turk. J. Math. **32** (2008), 197–211.

[43] ______, *Generalized derivations with Engel condition on multilinear polynomials*, Israel J. Math. **171** (2009), 325–348.

[44] ______, *A product of two generalized derivations on polynomials in prime rings*, Collect. Math. **61 (3)** (2010), 303–322.

[45] ______, *Annihilators and power values of generalized skew derivations on Lie ideals*, Canad. Math. Bull. **59 (2)** (2016), 258–270.

[46] V. De Filippis and B. Dhara, *Cocentralizing generalized derivations on multilinear polynomial on right ideals of prime rings*, Demonstr. Math. **47 (1)** (2014), 22–36.

[47] ______, *Generalized skew-derivations and generalizations of homomorphism maps in prime rings*, Commun. Algebra **47 (8)** (2019), 3154–3169.

[48] V. De Filippis and F. Rania, *Multilinear polynomials and cocentralizing conditions in prime rings*, Siberian Math. J. **52 (5)** (2011), 813–823.

[49] V. De Filippis and G. Scudo, *Strong commutativity and Engel condition preserving maps in prime and semiprime rings*, Linear Multilinear Algebra **61 (7)** (2013), 917–938.

[50] V. De Filippis and O. M. Di Vincenzo, *Posner's second theorem, multilinear polynomials and vanishing derivations.*, J. Aust. Math. Soc. **76** (2004), 357–368.

[51] ———, *Vanishing derivations and centralizers of generalized derivations on multilinear polynomials*, Commun. Algebra **40 (6)** (2012), 1918–1932.

[52] V. De Filippis and F. Wei, *Posner's second theorems and annihilator conditions with generalized skew derivations*, Colloq. Math. **129 (1)** (2012), 61–74.

[53] ———, *b-generalized skew derivations on Lie ideals*, Mediterr. J. Math. **15 (2)** (2018), Article number: 65.

[54] ———, *Centralizers of X-generalized skew derivations on multilinear polynomials in prime rings*, Commun. Math. Stat. **6** (2018), 49–71.

[55] ———, *An Engel condition with X-generalized skew derivations on Lie ideals*, Commun. Algebra **46 (12)** (2018), 5433–5446.

[56] C. Garg and R. K. Sharma, *On generalized* $(\alpha, \beta)$*-derivations in prime rings*, Rend. Circ. Mat. Palermo **65** (2016), 175–184.

[57] O. Golbasi and E. Koc, *Some commutativity theorems of prime rings with generalized* $(\sigma, \tau)$*-derivation*, Commun. Korean Math. Soc. **26 (3)** (2011), 445–454.

[58] I. N. Herstein, *Topics in ring theory*, The University of Chicago Press, Chicago, 1969.

[59] ———, *Rings with involution*, Univ. of Chicago Press, Chicago, 1976.

[60] B. Hvala, *Generalized derivations in rings*, Commun. Algebra **26 (4)** (1998), 1147–1166.

[61] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Pub. 37, 1964.

[62] ———, *PI-algebras, an introduction*, Lecture notes in Math. 441, Springer Verlag, New York, 1975.

[63] Y. S. Jung and K. H. Park, *On generalized* $(\alpha, \beta)$*-derivations and commutativity in prime rings*, Bull. Korean Math. Soc. **43 (1)** (2006), 101–106.

[64] V. K. Kharchenko, *Differential identity of prime rings*, Algebra Log. **17** (1978), 155–168.

[65] ———, *Automorphisms and derivations of associative rings, mathematics and its applications*, Kluver Academic Publisher 69, Dordrecht, 1991.

[66] M. T. Košan and T. K. Lee, *b-generalized derivations having nilpotent values*, J. Aust. Math. Soc. **96** (2014), 326–337.

[67] C. Lanski, *An Engel condition with derivation*, Proc. Amer. Math. Soc. **118 (3)** (1993), 731–734.

[68] ———, *An Engel condition with derivation for left ideals*, Proc. Amer. Math. Soc. **125 (2)** (1997), 339–345.

[69] P. H. Lee and T. K. Lee, *Derivations with Engel conditions on multilinear polynomials*, Proc. Amer. Math. Soc. **124 (9)** (1996), 2625–2629.

[70] P.H. Lee and T. L. Wong, *Derivations cocentralizing Lie ideals*, Bull. Inst. Math. Acad. Sinica **23** (1995), 1–5.

[71] T. K. Lee, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sinica **20 (1)** (1992), 27–38.

[72] ———, *Generalized derivations of left faithful rings*, Commun. Algebra **27 (8)** (1999), 4057–4073.

[73] ———, *Additive maps having a generalized derivation expansion*, J. Algebra Appl. **14 (4)** (2015), Art. No. 1550048.

[74] T. K. Lee and W. K. Shiue, *Derivations cocentralizing polynomials*, Taiwanese J. Math. **2 (4)** (1998), 457–467.

[75] ———, *A result on derivations with Engel condition in prime rings*, Southeast Asian. Bull. Math. **23** (1999), 437–446.

[76] U. Leron, *Nil and power central polynomials in rings*, Trans. Amer. Math. Soc. **202** (1975), 97–103.

[77] C. K. Liu, *An Engel condition with b-generalized derivations*, Linear Multilinear Algebra **65 (2)** (2017), 300–312.

[78] ———, *An Engel condition with two generalized derivations on Lie ideals of prime rings*, J. Algebra Appl. **22 (5)** (2023), Art. No. 2350115.

[79] M. Marubayashi, M. Ashraf, N. Ur Rehman, and S. Ali, *On generalized* $(\alpha, \beta)$-*derivations in prime rings*, Algebra Colloq. **17 (Spec 1)** (2010), 865–874.

[80] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[81] L. H. Rowen, *Polynomial identities in ring theorys*, Pure and Applied Mathematics 84, Academic Press, New York, 1980.

[82] W. S. Martindale 3rd, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12 (4)** (1969), 576–584.

[83] S. K. Tiwari, *Generalized derivations with multilinear polynomials in prime rings*, Commun. Algebra **46 (2)** (2018), 5356–5372.

[84] ———, *Product and commuting generalized derivations in prime rings*, Rend. Circ. Mat. Palermo, II. Ser **72** (2023), 1377–1397.

[85] S. K. Tiwari, R. K. Sharma, and B. Dhara, *Identities related to generalized derivations on ideal in prime rings*, Beitr. Algebra Geom. **57 (4)** (2016), 809–821.

[86] S.K. Tiwari, *Identities with generalized derivations and multilinear polynomials*, Bull. Iran. Math. Soc. **46 (2)** (2020), 425–440.

[87] ———, *Identities with generalized derivations in prime rings*, Rend. Circ. Mat. Palermo, II. Ser. **71** (2021), 207–223.

[88] S.K. Tiwari and S. K. Singh, *A note on generalized derivations with multilinear polynomials in prime rings*, Commun. Algebra **48 (4)** (2020), 1770–1788.

[89] Y. Utumi, *On quotient rings*, Osaka J. Math. **8 (1)** (1956), 1–18.

[90] T. L. Wong, *Derivations with power central values on multilinear polynomials*, Algebra Colloq. **3 (4)** (1996), 369–378.