A STUDY ON STRUCTURE SPACES OF SEMIRINGS, Γ-SEMIRINGS AND RELATED SEMIMODULES

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CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled **"A STUDY ON STRUCTURE SPACES OF SEMIRINGS, Γ-SEMIRINGS AND RELATED SEMIMODULES"** submitted by **Smt. Soumi Basu**, who got her name registered on 30.09.2019 **(Registration No.: SMATH1111619, Index No. 116/19/Maths./26)** for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon her own work under the supervision of Dr. Sujit Kumar Sardar, Professor, Department of Mathematics, Jadavpur University and co-supervision of Dr. Sarbani Mukherjee (Goswami), Assistant Professor, Department of Mathematics, Lady Brabourne College, Kolkata and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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Dedicated to
my parents
Sipra Basu and
Loknath Basu

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Abstract

The present work is a study on the structure spaces of semirings, semimodules, Γ-semirings and ΓS-semimodules showing a nice interplay between the algebraic structures and the topological structures. At first, different kinds of congruences on a Γ-semiring, viz. cancellative congruence, left and right regular congruence, maximal congruence and prime congruence have been defined and several results on those congruences have been studied. Then in order to study a Γ-semiring via its operator semirings, one-to-one correpondences between the sets of different types congruences on a Γ-semiring with those on its operator semirings have been established. Next, the structure space of semirings, consisting of all prime congruences on a semiring has been defined and its topological properties have been studied. Likewise the structure space of Γ-semirings, topologizing the set of all prime congruences on a commutative Γ-semiring with strong unities equipped with the Hull Kernel topology, has been defined and investigation of the topological properties of that space via the left operator semiring has been done establishing the topological isomorphism between the structure space of a Γ-semiring and that of its left operator semiring. Next, it has been acknowledged that the set $C_{-}(X)$ of all non-positive valued continuous functions over a topological space X falls into the algebraic structure viz. Γ-semiring with pointwise addition and multiplication taking $\Gamma = C_{-}(X)$ and not only that it is found that the left operator semiring of $C_{-}(X)$ is isomorphic to the semiring $C_{+}(X)$ of all non-negative valued continuous functions over a topological space X. Then many results have been proved on the congruences (such as maximal regular, prime congruences) and ideals (such as z-ideals, z^o-ideals) of the Γ-semiring $C_{-}(X)$ via the connection with those of the semiring $C_{+}(X)$. It has been established that the structure space and the real structure space of $C_{-}(X)$ are models of Stone-Čech compactification and Hewitt Realcompactification of a Tychonoff space X respectively. Also the Γ-semiring analogues of the 'Banach-Stone Theorem' and 'Hewitt Isomorphism Theorem' are obtained. In addition some results on z-ideals, z^o-ideals of $C_{-}(X)$, characterizing the topological space X, have been studied. After that the notion of prime ΓS-subsemimodules of an unitary ΓS-semimodule has been introduced and the structure space of a ΓS-semimodule is defined. Then a correspondence between the set of all prime ΓS-subsemimodules of a ΓS-semimodule and the set of all prime subsemimodules of the associated L-semimodule is obtained, where S is a Γ-semiring and L is its left operator semiring. After that several properties of the prime ΓS-subsemimodules over multiplication ΓS-semimodules, finitely generated ΓS-semimodules, k-finitely generated ΓS-semimodules, are studied via the associated L-semimodules. Thereafter the Hull Kernel topology is defined on the prime subsemimodules of a multiplication semimodule as well as on the set of all prime ΓS-subsemimodules of a multiplication ΓS-semimodule and those spaces have been named as the structure spaces of semimodules and ΓS-semimodules respectively. Then the topological properties of the structure space of semimodules have been examined and applying those the structure space of ΓS-semimodules has been studied via the homeomorphism of that space with the space of prime L-subsemimodules of the associated L-semimodule.

Contents

Introduction

The notion of semirings was introduced by Vandiver [81] in 1934. Semirings constitute a fairly natural generalization of both rings and distributive lattices. Since unlike in rings, additive inverses are not required in semirings, there are considerable differences between the theories of the two structures, viz. there is no bijection between the set of all ideals and congruences on semirings in general. As a result many theories like k-ideals have been developed to enrich this algebraic system and narrow the gap between rings and semirings. Also the structure spaces of semirings, formed by the class of prime ideals, prime k-ideals and maximal k-ideals etc. have been studied by many authors [7, 41, 74]. Moreover the properties, viz. separation axioms, compactness and connectedness of those spaces have been investigated as well. In [63] authors studied various separation axioms of the prime spectrum of a commutative semiring. In [50] Lescot proved that the set of all prime k-ideals in a characteristic one semiring with Zariski topology is a spectral space. He also defined the prime congruences on the characteristic one semirings which are basically additively idempotent commutative semiring with zero and unity, to accomplish his study to provide a proper algebraic geometry in characteristic one semirings. Recently in [32], it has been shown that the space of all prime k-congruences (following the notion of prime congruence as of Lescot) is homeomorphic to the space of all prime k-ideals equipped with Zariski topology which is known as the k-prime spectrum in any commutative semiring with zero and unity and it has been found that both the spaces are spectral spaces as well.

Also many authors have been interested in studying the semimodules and extended the results on ideals of a semiring to the setting of semimodules over semirings. For instance, in [14], [13], [85], [35], the authors extended some characterizations of prime ideals (k-ideals) and maximal ideals (k-ideals) of a semiring to prime subsemimodules (k-subsemimodules) and maximal subsemimodules (k-subsemimodules) of a semimodule. In [33], [34], [35] different properties of the prime subsemimodules and prime k-subsemimodules have been generalized over top semimodules/ multiplication semimodules/ finitely generated/ k-finitely generated semimodules. There are several works on the topology defined on the prime spectra of modules over commutative rings [53] as well as non-commutative rings [78]. For a semiring with identity, Golan [30] proved that its prime spectrum, endowed with the Zariski topology, is a quasicompact T_0 space. While for semimodules over semirings, Atani et al. [13] studied Zariski topology defined on the k-prime spectrum consisting of the prime k-subsemimodules of a semimodule, in [85], [14], [84] studies on the properties of prime subsemimodules, prime k-subsemimodules can be found. Later, Han et al. [34] defined top semimodule over a semiring, analogous to the notion of a top module (i.e., module whose spectrum of prime submodules attains a Zariski topology [53]) and studied some of its topological properties along with several other results regarding multiplication semimodules over commutative semirings (in [10], Ameri studied the prime spectrum for a multiplication module over a commutative ring with nonzero identity). Recently Han et al. in [35] studied the space of subtractive prime spectrum for a top semimodule over a semiring with zero and nonzero identity as well.

To study the representation of arbitrary rings as rings of continuous functions on topological spaces, Jacobson [43] defined a topology for the set of primitive ideals of a ring : the Hull Kernel topology. In the context of semirings, many authors have used a similar construction to topologize sets of ideals and congruences (see, for instance, [42], [75]). In [7] structure space of semirings consisting of prime k-ideals has been studied and further the structure space of the semiring Z_0^+ of all non-negative integers has also been studied. In the paper of Sen and Bandyopadhyay [75], they introduced the notion of Hull Kernel topology for the space of maximal regular congruences for the semialgebra and studied its topological properties. In 1995, Ray et al [67] constructed the hemiring structure space consisting of all maximal congruences, adopting the definition of congruence, regular congruence, maximal congruence etc. from the paper of Sen and Bandyopadhyay [75] on hemirings. They also introduced the concepts of residue class hemirings modulo regular congruences, prime congruences, maximal congruences [67]. In 1993, they introduced the notion of prime congruences on a semiring which is different from that of Lescot. In 2014, Joo and Mincheva [45, 54] followed the same approach of twisted products of pair of elements to define prime congruence on an additively idempotent semiring as this class of congruences exhibit some analogous properties to the prime ideals of commutative rings. In order to establish a good notion of radical congruences they showed that the intersection of all prime congruences on a semiring can be characterized by certain twisted power formulas. In 1993, the structure of $C_+(X)$ was first studied by S. K. Acharyya, K. C. Chattopadhyay and G. G. Ray [2] (the collection $C_+(X)$ of all non-negative real valued continuous functions over a topological space X forms a semiring with respect to pointwise addition and multiplication of functions). To establish the structure space of the semiring $C_+(X)$ of all non-negative real-valued continuous functions on a topological space X as another model of the Stone-$\check{C}$ech compactification, Acharyya et al. [67] considered the structure space of all maximal congruences which are prime on a semiring with zero equipped with the Hull Kernel topology. They proved that the space is T_1 and it is compact if the semiring is with unity and found a necessary and sufficient condition for T_2. They established that the structure space of maximal regular congruences on $C_+(X)$ is the Stone-$\check{C}$ech compactification of X and the structure space of maximal regular congruences that are real on $C_+(X)$ is the Hewitt realcompactification of X, where X is a Tychonoff space. They obtained the semiring analogue of the 'Banach-Stone Theorem' and 'Hewitt Isomorphism Theorem' as well. In 2019, E. M. Vechtomov *et. al.* in their paper [82], reviewed all the results intensively studied by them and many other authors on the theory of semirings of continuous functions, especially $C_+(X)$. In [22] authors have studied the z-ideals, z^o-ideals of $C_+(X)$ to characterize the topological space X. These abovementioned studies exhibit sustained research interest in this structure.

The notion of Γ was introduced in Algebra by Nobusawa in order to give a meaningful algebraic structure to $Hom(A, B)$, where A, B are additive abelian groups. Nobusawa introduced the notion of Γ–ring in 1964. Then Γ-structure was introduced in semigroup setting by M. K. Sen in 1981 in the form of Γ-semigroup and in semiring setting by M. M. K. Rao in 1995 in the form of Γ-semiring as a generalization of Γ-ring as well as of semiring [65]. After that the theory of Γ-semiring has been enriched by the introduction of operator semirings by T. K. Dutta and S. K. Sardar in 2002 [27]. Also they established the correspondence between the ideals of a Γ-semiring and those of its operator semirings which is an important and effective tool for studying Γ-semirings. The study of Γ-semiring has a sustained research interest which is evident from numerous recent papers on Γ-semirings. Γ-semirings are also studied from different perspective viz its connection with Morita context / equivalence of semirings obtained by Sardar, Gupta, Saha [71]. They showed that the left operator and the right operator semiring of a Nobusawa Γ-semiring with unities are Morita equivalent and also deduced that for any two Morita equivalent semirings there exists a Nobusawa Γ-semiring whose left and right operator semirings are isomorphic to those two semirings [71]. Pal, Chakraborty, Mukherjee, Sardar considered Γ-semirings as bi-linked semigroups (*cf.* (1) of Concluding remarks of [61]) and studied bi-linked group congruences on them. Sardar and Gupta established that the lattices of congruences of two Morita equivalent semirings are isomorphic [72]. Hedayati and Shum [37] studied the homomorphism of a Γ-semiring induced by the congruences, for establishing isomorphism theorem and some fundamental theorems of Γ-semirings. Many authors have studied the topological structure spaces of semirings, Γ-semirings and ternary semirings of prime ideals, prime k-ideals, maximal k-ideals etc. equipped with Hull Kernel topology which is evident from numerous papers on semirings [75], [2], Γ-semirings [38], [44], [55] and ternary semirings [47]. In [47] author has studied the structure space of prime k-ideals of Z_0^- of all non-positive integers, as a ternary semiring. The study of congruences in ternary semirings and Γ-semirings has been necessitated by the well known fact that there is no bijection between the set of all ideals and congruences on semirings, Γ-semirings and ternary semirings.

In order to introduce the concept of primitive Γ-semiring and Jacobson radicals, in 2004 Sardar and Dasgupta [69] introduced the notion of ΓS-semimodule in Γ-semiring theory. Also Dutta and Dasgupta [28] studied the properties of a ΓS-Semimodule via its associated semimodules. They constructed a semimodule over the left operator semiring L of the Γ-semiring S and called that the associated L-semimodule of the ΓS-semimodule in [28]. They established a lattice isomorphism between the lattices of all ΓS-subsemimodules of a ΓS-semimodule and of all subsemimodules of the associated L-semimodule and studied the properties of a ΓS-semimodule via its associated L-semimodule. In [23] and [77] properties of prime ΓS-modules and prime ΓS-submodules over a Γ-ring S has been studied respectively.

As a continuation of the above works, we study the properties of structure space (equipped with the Hull Kernel topology) of prime congruences on a semiring. We also study the notions of maximal, cancellative, regular, prime congruences on a Γ-semiring and establish one to one correspondences between the sets of those congruences on a Γ-semiring and those on its operator semirings. After that we study the structure spaces consisting of those above mentioned congruences on a Γ-semiring, especially on the Γ-semiring $C_-(X)$ of all non-positive valued continuous functions over a topological space X, in order to establish those spaces of $C_-(X)$ as models of the Stone-Čech compactification and the Hewitt realcompactification of X. Towards this we proved an algebraic as well as topological connection between the semiring $C_+(X)$ and the Γ-semiring $C_-(X)$ via the operator semiring approach. In the next segment we characterize the notions of prime ΓS-subsemimodules, multiplication ΓS-semimodules and study the properties of the prime ΓS-subsemimodules over an unitary ΓS-semimodule via the associated semimodule approach. For this we establish an one to one correspondence between the set of all prime ΓS-subsemimodules of a ΓS-semimodule and the set of all prime subsemimodules of its associated left operator semimodule. Next we define the structure space of all prime subsemimodules of a semimodule and study the topological properties of that space. Finally we define the structure space consisting of all prime ΓS-subsemimodules of a ΓS-semimodule and study the topological properties of that space via those of the structure space of its associated semimodule.

The thesis consists of six chapters. In the first chapter we recall the preliminaries. We give a brief description of each chapter below:

• **Chapter 1:** Here we mainly recall some preliminary notions and results of semirings, Γ-semirings, semimodules, ΓS-semimodules, lattices and topology.

• **Chapter 2:** Here we study different types of congruences such as cancellative congruence, left and right regular congruence, maximal congruence and prime congruence on a Γ-semiring. This study is accomplished via the operator semirings of a Γ-semiring by obtaining various lattice isomorphisms (bijection / inclusion preserving bijection) between the lattice of congruences / cancellative congruences / regular congruences (respectively, maximal congruences / prime congruences) on a Γ-semiring and those on the operator semirings.

• **Chapter 3:** Here we study some of the topological properties of the space of prime congruences on a semiring endowed with the Hull Kernel topology. The structure space, endowed with the Hull Kernel topology, of maximal regular congruences which are prime on a Γ-semiring as well as the structure space of prime congruences on a Γ-semiring have been studied via operator semirings.

• **Chapter 4:** Here we obtain some important results of the structure space of $C_-(X)$ of non-positive real valued continuous functions over a topological space X. It has been found that the structure spaces of the semiring $C_+(X)$ of non-negative real valued continuous functions and the Γ-semiring $C_-(X)$ are homeomorphic. Moreover it has been shown that the structure space of $C_-(X)$ is the Stone-$\check{C}$ech compactification of X, where X is a Tychonoff space. Furthermore, the Γ-semiring analogue of the 'Banach-Stone Theorem' has been obtained. In addition it is obtained that the real structure space of $C_{-}(X)$ is the Hewitt realcompactification of X and as a consequence the Γ-semiring analogue of the 'Hewitt isomorphism Theorem' is obtained. Several results on z-ideals, z^o-ideals of $C_{-}(X)$, characterizing the topological space X has been studied.

- **Chapter 5:** Suppose S is a Γ-semiring and L is its left operator semiring. We first obtain a correspondence between the set of all prime ΓS-subsemimodules of a ΓS-semimodule and the set of all prime subsemimodules of the associated L-semimodule. Also correspondence between the set of all finitely generated (k-finitely generated) ΓS-subsemimodules of a ΓS-semimodule and the set of all finitely generated (resp. k-finitely generated) subsemimodules of the associated L-semimodule has been established. It has been found that the associated L-semimodule is multiplication (finitely generated, k-finitely generated) semimodule if and only if the ΓS-semimodule is multiplication (resp. finitely generated, k-finitely generated) ΓS-semimodule. Using these correspondences, several properties of the prime ΓS-subsemimodules of ΓS-semimodules (especially of multiplication ΓS-semimodules, finitely generated ΓS-semimodules, k-finitely generated ΓS-semimodules) have been studied via those of the associated L-semimodule.

- **Chapter 6:** Suppose S is a Γ-semiring and L is its left operator semiring. We define the Hull Kernel topology on the set of all prime ΓS-subsemimodules of a multiplication ΓS-semimodule as well as on the prime subsemimodules of a multiplication semimodule and call those spaces the structure spaces of ΓS-semimodules and semimodules respectively. We then observe that the Zariski topology and the Hull Kernel topology on prime subsemimodules are the same. Using this observation we study the topological space of prime ΓS-subsemimodules via the topological space of prime L-subsemimodules of its associated L-semimodule.

CHAPTER 1

PRELIMINARIES

1 Preliminaries

In this chapter certain basic definitions and results are presented for their use in the sequel.

1.1 Lattice

Definition 1.1.1. [24] Let P be a set. An order (or partially order) on P is a binary relation $\leq$ on P such that for all $x, y, z \in P$,

(i) $x \leq x$ (known as reflexivity)

(ii) $x \leq y$ and $y \leq x$ imply $x = y$ (known as antisymmetry)

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (known as transitivity).

A set P equipped with an order relation $\leq$ is said to be an ordered set (or partially ordered set or poset).

Definition 1.1.2. [24] Let P and Q be two ordered sets. Then a map $\phi : P \to Q$ is said to be

(i) an *order preserving* if $x \leq y$ in P implies $\phi(x) \leq \phi(y)$ in Q,

(ii) an *order-embedding* if $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q,

(iii) an *order-isomorphism* if it is an order-embedding which maps P onto Q and P and Q are called order-isomorphic.

Definition 1.1.3. [24] Let L be an ordered set and $S \subseteq L$. An element $x \in L$ is an upper bound of S if $s \leq x$ for all $s \in S$. A lower bound is defined dually. x is called the least upper bound of S (supremum of S) and denoted by $supS$ if x is an upper bound of S and $x \leq y$ for all upper bounds y of S. Dually x is called the greatest lower bound of S (infimum of S) and denoted by $infS$ if x is a lower bound of S and $x \geq y$ for all lower bounds y of S.

Definition 1.1.4. [24] Let L be a nonempty ordered set. Then L is called a *lattice* if $sup\{x, y\}$ and $inf\{x, y\}$ exist for any two elements $x, y \in L$.
$sup\{x, y\}, inf\{x, y\}$ are denoted by $a \vee b, a \wedge b$ respectively.

Definition 1.1.5. [24] Let L be a lattice and $\emptyset \neq M \subseteq L$. Then M is called a *sublattice* of L if $a, b \in M$ implies $sup\{x, y\}, inf\{x, y\} \in M$.

Definition 1.1.6. [24] Let L and K be two lattices. A mapping $f : L \to K$ is said to be a *lattice homomorphism* if f is both join-preserving and meet-preserving i.e., for all $a, b \in L$, $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$. A bijective lattice homomorphism is called a *lattice isomorphism.*

Proposition 1.1.7. *[24] Let L and K be two lattices and $f : L \to K$ be a map.*
(i) Then the following are equivalent.

(a) f is order preserving.

(b) $f(a \vee b) \geq f(a) \vee f(b)$ for all $a, b \in L$.

(c) $f(a \wedge b) \leq f(a) \wedge f(b)$ for all $a, b \in L$.

In particular, if f is a homomorphism then f is order preserving.
(ii) f is a lattice isomorphism if and only if it is an order-isomorphism.

Definition 1.1.8. [24] A lattice L is said to be a *complete lattice* if the join (supremum), $\vee S$ and the meet (infimum), $\wedge S$ exist for every subset S of L.

1.2 Topology

Definition 1.2.1. [76] An operator that assigns to each subset A of a topological space X a subset $\overline{A}$ of X is called a *Kuratowski closure operator* if following four axioms called *Kuratowski closure axioms* hold:

(i) $\overline{\emptyset} = \emptyset$

(ii) $A \subseteq \overline{A}$

(iii) $\overline{\overline{A}} = \overline{A}$

(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all $A, B \subseteq X$.

From these axioms, we have, for $A, B \subseteq X$, $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$.
The topology induced by the Kuratowski closure operator is called the *Hull Kernel topology*.

Definition 1.2.2. [48] A topological space X is called *completely regular* if for each point x of X and each closed set A not containing x, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(A) = \{0\}$.
A T_1 completely regular topological space is called a *Tychonoff space*.

Definitions 1.2.3. [29] Let $C(X)$ be the ring of continuous functions over a topological space X.
An ideal I of $C(X)$ is called *fixed* if $\bigcap\{Z(f) : f \in I\} \neq \emptyset$.
A maximal ideal M in $C(X)$ is called *real* if $C(X)\diagup M$ is isomorphic to $\mathbb{R}$.
A space X is called *realcompact* if every real maximal ideal in $C(X)$ is fixed.

Definitions 1.2.4. [29] In a space X, a G_δ-subset of X is a countable intersection of open subsets of X.

Definition 1.2.5. [29] For any subset A of a topological space X, the set $rclA = \{x \in X$: each G_δ-set in X containing x intersects $A\}$ is called the *realclosure* of A. A is called *realclosed* if $rclA = A$.
Every closed set in X is realclosed.

Theorem 1.2.6. *[29] Every realclosed subset of a realcompact space is realcompact. It is to be noted that a compact space is always realcompact.*

Definition 1.2.7. [67] Let X be a topological space. A pair (α, Y) is a said to be an extension of X if Y is a space and α is a homeomorphism of X into Y such that $\alpha(X)$ is dense in Y.
If Y is compact T_2 then (α, Y) is called a *T_2-compactification* of X.
If Y is a realcompact space then (α, Y) is called a *realcompactification* of X.

Theorem 1.2.8. *[29] For any Tychonoff space X, $(\alpha, \beta X)$ is the Stone-Čech compactification of X if for every continuous function f from X into any compact T_2 space Y there exists a function F from βX to Y such that $F \circ \alpha = f$.*

Theorem 1.2.9. *[29] For any Tychonoff space X, $(\alpha, \nu X)$ is the Hewitt realcompactification of X if for every continuous function f from X into any realcompact space Y there exists a function F from νX to Y such that $F \circ \alpha = f$*

Definition 1.2.10. [39] A nonempty topological space X is said to be *irreducible* if every pair of nonempty open subsets in X have the nonempty intersection. A subset Y of a topological space X is said to be *irreducible* if the subspace Y is irreducible. A subset Y of a topological space X is *irreducible* if and only if for any closed subsets Y_1 and Y_2 in X, $Y \subseteq Y_1 \cup Y_2$ implies that $Y \subseteq Y_1$ or $Y \subseteq Y_2$.

Definition 1.2.11. [39] A point y of a topological space X is called a *generic point* of a closed subset Y of X if $\overline{\{y\}} = Y$.

Definition 1.2.12. [39] If a topological space X is *connected* then only subsets which are both open and closed (clopen sets) are X and the empty set. Otherwise the space is called *disconnected*.

Definitions 1.2.13. [63] A point x of a space X is *kerneled* if $\{x\}$ is the intersection of all open neighbourhoods of x. A space is called a $T_{1/4}$*-space* if each of its points is either kerneled or closed.

A point x of a space X is *isolated* if $\{x\}$ is open. A space is a $T_{1/2}$*-space* if each of its points is either isolated or closed.

Further a space is a $T_{3/4}$*-space* if each of its points is either closed or open-regular, where an *open-regular* set is an open set which is the interior of its closure.

In general $T_{3/4} \Leftarrow T_{1/2} \Leftarrow T_{1/4}$.

Note that every isolated point is kerneled.

Definitions 1.2.14. [29] A *P-space* is a topological space in which every G_δ-set is open.

An *almost-P-space* is a space in which G_δ-sets have dense interiors.

1.3 Semiring

Definition 1.3.1. [31] Let R be a nonempty set and '+' and '·' be two binary operations on R, called addition and multiplication respectively. Then $(R, +, \cdot)$ is called a *semiring* if (*i*) $(R, +)$ is a commutative semigroup,

(*ii*) $(R, \cdot)$ is a semigroup and

(*iii*) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$.

If there exists an element $0 \in R$ such that $a + 0 = a$ for all $a \in R$ then 0 is called additive neutral element or the zero of R and R is called a *semiring with zero*.
Moreover if $a \cdot 0 = 0 \cdot a = 0$ for all $a \in R$ then R is called a *semiring with absorbing zero*.
Again if there exists an element $1 \in R$ such that $a.1 = 1.a = a$ for all $a \in R$ then 1 is called multiplicative identity or simply identity element of R and R is called a *semiring with identity*.
Further if $a.b = b.a$ for all $a, b \in R$ then R is called a *commutative semiring*.
Note that this semiring is same as a semiring in [2] and [30] without the additive identity and multiplicative identity. Throughout our work we have considered semiring with absorbing zero.

Example 1.3.2. [30] Let $C_+(X)$ be the set of all non-negative valued continuous functions over a topological space X. This set with pointwise addition and multiplication forms a semiring.

Definitions 1.3.3. [30] An *ideal* I of a semiring R is a nonempty subset of R such that $a + b, as, sa \in I$ for all $a, b \in I$ and $s \in R$.
A proper ideal I of a semiring R is called *prime* if $ab \in I$ implies $a \in I$ or $b \in I$ for any elements $a, b \in R$.
A proper ideal I of a semiring R is called *maximal* if it is not contained in any other proper ideal of the semiring R.
An ideal I of a semiring R is called a k-*ideal* if for $x, y \in R$, $x + y \in I$ and $y \in I$ implies that $x \in I$.
For an ideal I of a semiring R, the k-*closure* of I, denoted by $\overline{I}$, is defined by
$\overline{I} = \{a \in R : a + b = c \text{ for some } b, c, \in I\}$.
Clearly I is a k-ideal of R if and only if $\overline{I} = I$. Also for an ideal I of R, $\overline{I}$ is a k-ideal of R.

Proposition 1.3.4. *[82] Prime (maximal) ideals of the semiring $C_+(X)$ are precisely the ideals $P \cap C_+(X)$ for the prime (resp. maximal) ideals P of the ring $C(X)$.*

Proposition 1.3.5. *[82] Any prime ideal of a semiring $C_+(X)$ is contained in a unique maximal ideal.*

Theorem 1.3.6. *[82] For any topological spaces X and Y, the following are equivalent.*

(1) *$C(X)$ is isomorphic to $C(Y)$.*

(2) *$C_+(X)$ is isomorphic to $C_+(X)$.*

Definition 1.3.7. [30] If R_1 and R_2 are semirings then a function $\phi : R_1 \to R_2$ is a *morphism of semirings* if and only if:

(1) $\phi(0_{R_1}) = 0_{R_2}$,

(2) $\phi(1_{R_1}) = 1_{R_2}$ and

(3) $\phi(r + r') = \phi(r) + \phi(r')$ and $\phi(rr') = \phi(r)\phi(r')$ for all $r, r' \in R_1$.

A morphism of semirings which is both injective and surjective is called an *isomorphism*.

Definition 1.3.8. [30] A semiring R is called *additively cancellative* if for all $x, y, z \in R$, $x + z = y + z$ implies $x = y$.

Definition 1.3.9. [30] A semiring R is said to be *zero-sum free* if for $a + b = 0$, where $a, b \in R$, $a = 0$ and $b = 0$.

Definition 1.3.10. [30] A semiring R is said to be *zero-divisor free (ZDF)* if for $a, b \in R$, $ab = 0$ implies that $a = 0$ or $b = 0$.

Definition 1.3.11. [30] A semiring R is a *division semiring* if for every nonzero element $r \in R$ there exists an element $x \in R$ such that $xr = 1 = rx$. A commutative division semiring is a *semifield*.

Proposition 1.3.12. *[30] A division semiring is either zerosumfree or is a division ring.*

Definition 1.3.13. [30] An equivalence relation ρ on a semiring R is called a *congruence* on R if for any $a, b, c \in R$,

$$(a, b) \in \rho \text{ implies } (a + c, b + c) \in \rho,\ (ac, bc) \in \rho \text{ and } (ca, cb) \in \rho.$$

Remark 1.3.14. [30] The set of all congruences on a semiring R, partially ordered with the set inclusion, forms a complete lattice with meet and join defined as follows: for two congruences ρ_1, ρ_2, $\rho_1 \wedge \rho_2 = \rho_1 \cap \rho_2$ and $\rho_1 \vee \rho_2$ is defined as $(a, b) \in \rho_1 \vee \rho_2$ if and only if for some natural number n, there exists elements x_1, x_2,..., x_{n-1} in S and $a = x_0$, $x_n = b$ such that $(x_{k-1}, x_k) \in \rho_1$ or $(x_{k-1}, x_k) \in \rho_2$ for $1 \leq k \leq n$.

Proposition 1.3.15. *[30] If R is a commutative semiring with more than two elements having no nontrivial proper congruence relations then R is a field.*

Definition 1.3.16. [67] A congruence ρ on a semiring R is said to be a *cancellative congruence* if $(a + c, b + c) \in \rho$ implies $(a, b) \in \rho$, where $a, b, c \in R$.

Definition 1.3.17. [67] A cancellative congruence ρ on a semiring R is called *left regular congruence* if there exist elements $e_1, e_2 \in R$ such that for all $a \in R$, $(a + e_1a, e_2a) \in \rho$. The pair (e_1, e_2) is called *left unity pair* of ρ.
A cancellative congruence ρ is called *right regular congruence* if there exist elements $f_1, f_2 \in R$ such that for all $a \in R$, $(a + af_1, af_2) \in \rho$. The pair (f_1, f_2) is called *right unity pair* of ρ.
ρ is called *regular congruence* if it is both left and right regular congruence.

Note 1.3.18. It can be shown that if ρ is a proper left regular congruence on R and (e_1, e_2) is a left unity pair of ρ then $(e_1, e_2) \notin \rho$. Therefore from this fact it follows that e_1, e_2 are distinct elements of R.

Definition 1.3.19. [67] A proper congruence is called maximal congruence if it is not contained in any proper congruence on semiring R.
A proper left regular congruence is called maximal left regular congruence if it is not properly contained in any other proper left regular congruence on semiring R.

Definition 1.3.20. [67] A proper congruence ρ on a semiring R is called a *prime congruence* if it satisfies the following condition:

$(ad + bc, ac + bd) \in \rho$ implies either $(a, b) \in \rho$ or $(c, d) \in \rho$, where $a, b, c, d \in R$.

In the following we give two examples of prime congruences.

Examples 1.3.21. [67]

(i) Let us consider the semiring $C_+(X)$ of all non-negative real-valued continuous functions on a topological space X. Then for any element $x \in X$, the relation ρ_x defined by,

$$(f, g) \in \rho_x \text{ if and only if } f(x) = g(x), \text{ where } f, g \in C_+(X)$$

is a prime congruence on $C_+(X)$.

(ii) For any prime number p, the relation ρ_p on the semiring Z_0^+ of all non-negative integers defined by,

$$\rho_p = \{(m, n) \in Z_0^+ \times Z_0^+ : m - n \text{ is divisible by } p\}$$

is a prime congruence.

Proposition 1.3.22. *[45] If a congruence is prime then it can not be obtained as the intersection of two strictly larger congruences.*

Definition 1.3.23. [67] A pair (e, f) of elements of a semiring R is called a *left identity pair* in R if for all $a \in R$, $a + ea = fa$ and a pair (e', f') is called a *right identity pair* in R if for all $a \in R$, $a + ae' = af'$. A pair is called an *identity pair* in R if it is both a left and a right identity pair.

Note 1.3.24. It can be shown that if ρ is a proper cancellative congruence on R and (e, f) is an identity pair of ρ then $(e, f) \notin \rho$. Therefore from this fact it follows that e, f are distinct elements of R.

Remark 1.3.25. [67] If a semiring contains a left (right) identity pair then every cancellative congruence on it is left (resp. right) regular. Also if a semiring contains a multiplicative identity 1 then (0, 1) is an identity pair of the semiring and hence every cancellative congruence on it is a regular congruence.

Theorem 1.3.26. *[67] Every left regular congruence on a semiring is contained in a maximal left regular congruence on it.*

Theorem 1.3.27. *[67] Every maximal left regular congruence on a semiring is maximal cancellative.*

Theorem 1.3.28. *[67] Let R be a semiring and ρ be a congruence on R. Then the quotient set $R/\rho = \{\rho(x) : x \in R\}$, with the compositions '+' and '.' defined by,*

$$\rho(x) + \rho(y) = \rho(x + y) \text{ and } \rho(x).\rho(y) = \rho(xy)$$

for all $x, y \in R$, is a semiring if and only if ρ is a congruence on R.

Lemma 1.3.29. *[31] If ρ is a congruence on a semiring R and R/ρ has a zero $[0]_\rho$ then $[0]_\rho$ is a k-ideal of R.*

Example 1.3.30. [30] Any k-ideal of Z_0^+ is the form aZ_0^+ for $a \in Z_0^+$.

Theorem 1.3.31. *[67] Let R be a semiring and ρ be a congruence on R. R/ρ is additively cancellative semiring if and only if ρ is cancellative congruence on R.*

Theorem 1.3.32. *[67] Let R be a semiring and ρ be a congruence on R. R/ρ contains a left identity pair if and only if ρ is left regular congruence on R.*

Theorem 1.3.33. *[67] Let R be a semiring and ρ_0 be a congruence on R. There is an order preserving bijection between set of all congruences (cancellative, left regular, right regular, regular congruences) on R containing ρ_0 and the set of all congruences (respectively cancellative, left regular, right regular, regular congruences) on R/ρ_0.*

Theorem 1.3.34. *[67] Let R be a semiring and ρ_0 be a congruence on R. Then ρ_0 is a maximal congruence on R if and only if the semiring R/ρ_0 contains no congruence other than diagonal.*

Theorem 1.3.35. *[67] Let R be a semiring and ρ be a prime congruence on R. Then R/ρ has no divisor of zero.*

Theorem 1.3.36. *[67] The space of all maximal regular congruences which are prime on a semiring with unity equipped with Hull Kernel topology is a T_1, compact space.*

Corollary 1.3.37. *Singletons are closed if and only if every prime congruence is maximal.*

Theorem 1.3.38. *[67] There is an order preserving bijection between the sets of all congruences (regular, maximal regular congruences) of two isomorphic semirings.*

Theorem 1.3.39. *[67] If two semirings are isomorphic then their structure spaces of maximal regular congruences are homeomorphic.*

Theorem 1.3.40. *[67] Every maximal regular congruence on the semiring $C_+(X)$ is a prime congruence on $C_+(X)$.*

Theorem 1.3.41. *[67] For $x \in X$, $\rho_x = \{(f, g) : f, g \in C_+(X)$ and $f(x) = g(x)\}$ is a maximal regular congruence on $C_+(X)$.*

Theorem 1.3.42. *[67] The structure space of all maximal regular congruences on $C_+(X)$ with the Hull Kernel topology is a compact T_2 space, where X is a Tychonoff space.*

Corollary 1.3.43. *[2] If X is a Tychonoff space then $(\eta_X, \mathcal{B}_{C_+(X)})$ is the Stone-Čech compactification of X, where $\eta_X : X \to \mathcal{B}_{C_+(X)}$ is defined as $\eta_X(x) := \rho_x$ and $(f,g) \in \rho_x$ if and only if $f(x) = g(x)$ for $f, g \in C_+(X)$ and $\mathcal{B}_{C_+(X)}$ = the structure space of all maximal regular congruences on the semiring $C_+(X)$.*

Note that in [2] and [67], the structure space of all maximal regular congruences on the semiring $C_+(X)$ is denoted by $W(C_+(X))$ by the authors.

Definition 1.3.44. [67] A maximal regular congruence ρ on $C_+(X)$ is called *real* if $C_+(X)/\rho$ is isomorphic to R_0^+.

Lemma 1.3.45. *[67] The real structure space $\mathcal{B}^R_{C_+(X)}$ consisting of all maximal regular congruences which are real on $C_+(X)$ is realcompact.*

Note that in [67], the space $\mathcal{B}^R_{C_+(X)}$ of all maximal regular real congruences on the semiring $C_+(X)$ is denoted by $W_R(C_+(X))$ by the authors.

Corollary 1.3.46. *[67] If X is a Tychonoff space then $(\eta_X, \mathcal{B}^R_{C_+(X)})$ is the Hewitt realcompactification of X, where $\eta_X : X \to \mathcal{B}^R_{C_+(X)}$ is defined as $\eta_X(x) := \rho_x$ and $(f,g) \in \rho_x$ if and only if $f(x) = g(x)$ for $f, g \in C_+(X)$ and $\mathcal{B}^R_{C_+(X)}$ = the structure space of all maximal regular real congruences on the semiring $C_+(X)$.*

Theorem 1.3.47. *[2] If X and Y are compact T_2 spaces then the two semirings $C_+(X)$ and $C_+(Y)$ are isomorphic if and only if X and Y are homeomorphic.*

Theorem 1.3.48. *[3] If X and Y are realcompact spaces then the two semirings $C_+(X)$ and $C_+(Y)$ are isomorphic if and only if X and Y are homeomorphic.*

Definition 1.3.49. [30] A proper ideal I is called *strong* if and only if $a+b \in I$ implies $a, b \in I$.

Theorem 1.3.50. *[30] Let R and S be two isomorphic semirings and $Id(R)$, $Id(S)$ be the sets of all ideals of R and S (respectively). Then there exists an order preserving bijective correspondence between $Id(R)$ and $Id(S)$.*

Theorem 1.3.51. *[30] If R and S are two isomorphic semirings then the sets of all k-ideals, prime ideals, prime k-ideals (maximal, maximal k-ideals) of R and S are in order preserving bijective correspondences (resp. bijective correpondences).*

Lemma 1.3.52. [22] *Prime ideals of $C_+(X)$ are strong ideals.*

Theorem 1.3.53. [22] *For every ring ideal I in $C(X)$, the corresponding Γ-semiring ideal $I \cap C_+(X)$ is a k-ideal of $C_+(X)$.*

Definition 1.3.54. [22] Let R be a commutative semiring with unity. Then an ideal I of R is said to be a *z-ideal* if $f \in I$ implies $M_f^+ \subseteq I$, where M_f^+ is the intersection of all maximal ideals of $C_+(X)$ containing f.
Equivalently, a proper ideal I in $C_+(X)$ is called a *z-ideal* if $Z(f) \subseteq Z(g)$ and $f \in I$ implies $g \in I$.

Proposition 1.3.55. [22] *For $a \in C_+(X)$, $M_a^+ = \{g \in C_+(X) : Z(a) \subseteq Z(g)\}$.*

Theorem 1.3.56. [22] *Any z-ideal of $C_+(X)$ is of the form $I \cap C_+(X)$ for various z-ideals I of $C(X)$.*

Proposition 1.3.57. [22] *z-ideals of $C_+(X)$ are strong.*

Theorem 1.3.58. [22] *The minimal prime ideals of $C_+(X)$ are z-ideals.*

Theorem 1.3.59. [22] *For any z-ideal I in $C_+(X)$, the following are equivalent.*

(*i*) *I is a prime ideal.*

(*ii*) *I contains a prime ideal.*

(*iii*) *For all $k, l \in C_+(X)$, if $kl = \mathbf{0}$ then $k \in I$ or $l \in I$.*

Definition 1.3.60. [22] A commutative semiring R is called a *regular semiring* if for any $a \in R$ there exist $r \in R$ such that $a = a^2 r$.

Theorem 1.3.61. [22] *The following are equivalent.*

(*i*) *X is a P-space.*

(*ii*) *Each ideal of $C(X)$ is a z-ideal.*

(*iii*) *Each ideal of $C_+(X)$ is a z-ideal.*

(*iv*) *Each strong ideal is a z-ideal in $C_+(X)$.*

(*v*) *Each prime ideal is a z-ideal in $C_+(X)$.*

(*vi*) *$C_+(X)$ is a regular semiring.*

Definition 1.3.62. [22] An ideal of a semiring is called *essential* if it intersects every nonzero ideal nontrivially.

Theorem 1.3.63. *[22] The following are equivalent for any essential ideal E of the semiring $C_+(X)$.*

(i) *E intersects every nonzero z-ideals nontrivially.*

(ii) *E intersects every ideal nontrivially.*

(iii) $Ann^+(E) = (\mathbf{0})$.

(iv) $\bigcap Z[E]$ *is nowhere dense.*

Theorem 1.3.64. *[22] Any essential z-ideal in $C_+(X)$ is of the form $I \cap C_+(X)$, where I is an essential z-ideal of $C(X)$.*

Theorem 1.3.65. *[22] X is a P-space if and only if every essential ideal is a z-ideal in $C_+(X)$.*

Definition 1.3.66. [22] An ideal I in $C_+(X)$ is called *z^o-ideal* if $a \in I$ implies $P_a^+ \subseteq I$, where P_a^+ is the intersection of all minimal prime ideals of $C_+(X)$ containing a.

Proposition 1.3.67. *[22] z^o-ideals in $C_+(X)$ are strong.*

Theorem 1.3.68. *[22] Any z^o-ideal in $C_+(X)$ is of the form $J \cap C_+(X)$ for various z^o-ideals J in $C(X)$.*

Definition 1.3.69. [30] A nonzero element a in a semiring R is said to be a *zero-divisor* if there exists nonzero $b \in R$ such that $ab = 0$.

Theorem 1.3.70. *[22] The following are equivalent.*

(i) *X is an almost P-space.*

(ii) *Every z-ideal in $C(X)$ is a z^o-ideal.*

(iii) *Every maximal ideal in $C(X)$ is a z^o-ideal.*

(iv) *Every maximal ideal in $C(X)$ consists entirely of zero-divisors.*

(v) *For each nonunit element $f \in C(X)$, there exists $0 \neq g \in C(X)$ with $P_f \subseteq Ann(g)$.*

(vi) *Every z-ideal in $C_+(X)$ is a z^o-ideal.*

(vii) *Every maximal ideal in $C_+(X)$ is a z^o-ideal.*

(viii) *Every maximal ideal in $C_+(X)$ consists entirely of zero-divisors.*

(ix) *For each nonunit element $f \in C_+(X)$, there exists $0 \neq g \in C_+(X)$ with $P_f^+ \subseteq Ann^+(g)$.*

Definition 1.3.71. [22] A semiring R is called *almost regular* if for any nonunit $a \in S$ there exists $1 \neq b \in S$ such that $ab = a$.

Theorem 1.3.72. *[22] X is an almost P-space if and only if $C_+(X)$ is almost regular.*

Definition 1.3.73. [30] A semiring R is said to be *Noetherian (k-Noetherian)* if for every ascending chain $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ of ideals (respectively k-ideals) in R there exists a positive integer n such that $A_i = A_n$ for all $i \leq n$.

Definition 1.3.74. [30] A semiring is said to be *Artinian (k-Artinian)* if for every descending chain $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$ of ideals (respectively k-ideals) in R there exists a positive integer n such that $A_i = A_n$ for all $i \leq n$.

Theorem 1.3.75. *[22] The following are equivalent.*

(i) *X is finite.*

(ii) *$C_+(X)$ is Noetherian.*

(iii) *$C_+(X)$ is k-Noetherian.*

(iv) *$C_+(X)$ is Artinian.*

(v) *$C_+(X)$ is k-Artinian.*

1.4 Γ-semiring

Definition 1.4.1. [65] Let S and Γ be two additive commutative monoids. S is called a Γ-*semiring* if there exist mapping from $S \times \Gamma \times S$ to S (images to be denoted by $a\gamma b$, where $a, b \in S$ and $\gamma \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$

(iii) $a(\alpha+\beta)c = a\alpha c + a\beta c$

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$

(v) $a\alpha 0_S = 0_S\alpha a = 0_S$ and $a0_\Gamma b = 0_S$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
S is said to be a *commutative* Γ*-semiring* if $a\alpha b = b\alpha a$ for all $a, b \in S$, $\alpha \in \Gamma$.

Example 1.4.2. [65] Let A and B be two additive commutative semigroups. Let $Hom(A, B)$ be the additive commutative semigroup of all homomorphishms from A to B and Γ be additive commutative semigroup of all homomorphisms from B to A. Then $Hom(A, B)$ is a Γ-semiring, where $a\alpha b$ denotes the usual composition of homomorphisms, where $a, b \in Hom(A, B)$ and $\alpha \in \Gamma$.

Example 1.4.3. Let $C_-(X)$ be the set of all non-positive valued continuous functions on a topological space X. Then with pointwise addition and multiplication of functions $C_-(X)$ forms a Γ-semiring, where $\Gamma = C_-(X)$.

Definition 1.4.4. [27] Let S be a Γ-semiring and F be the free additive commutative semigroup generated by $S \times \Gamma$. Then the relation ρ on F defined by,

$$\sum_{i=1}^{m}(x_i, \alpha_i)\rho\sum_{j=1}^{n}(y_j, \beta_j) \text{ if and only if } \sum_{i=1}^{m} x_i\alpha_i a = \sum_{j=1}^{n} y_j\beta_j a,$$

for all $a \in S$ $(m, n \in Z^+)$, is a congruence on F.
We denote the congruence class containing $\sum_{i=1}^{m}(x_i, \alpha_i)$ by $\sum_{i=1}^{m}[x_i, \alpha_i]$.
Then F/ρ is an additive commutative semigroup.
Now F/ρ forms a semiring with the multiplication defined by,

$$(\sum_{i=1}^{m}[x_i, \alpha_i])(\sum_{j=1}^{n}[y_j, \beta_j]) = \sum_{i=1}^{m}[x_i\alpha_i y_j, \beta_j].$$

We denote this semiring by L and call it the *left operator semiring* of the Γ-semiring S.
Dually we define the *right operator semiring* R of the Γ-semiring S.

Theorem 1.4.5. *[27] If a Γ-semiring S is commutative then its left (right) operator semiring L (resp. R) is commutative.*

Theorem 1.4.6. *[27] If S is a commutative Γ-semiring then L and R are isomorphic.*

Definitions 1.4.7. [27] Let S be a Γ-semiring and L be the left operator semiring and R be the right one. If there exists an element $\sum_{i=1}^{m}[e_i, \delta_i] \in L$ (respectively, $\sum_{j=1}^{n}[\gamma_j, f_i] \in R$) such that $\sum_{i=1}^{m} e_i\delta_i a = a$ (respectively, $\sum_{j=1}^{n} a\gamma_j f_j = a$), for all $a \in S$, then S is said to have the *left unity* $\sum_{i=1}^{m}[e_i, \delta_i]$ (respectively, the *right unity* $\sum_{j=1}^{n}[\gamma_j, f_i]$).
Also if there exists an element $[e, \delta] \in L$ (respectively, $[\gamma, f] \in R$) such that $e\delta a = a$ (respectively, $a\gamma f = a$) for all $a \in S$, then S is said to have the *strong left unity* $[e, \delta]$ (respectively, *strong right unity* $[\gamma, f]$).

Definition 1.4.8. [69] A Γ-semiring S is called *additively cancellative* if
for all $x, y, z \in S$, $x + z = y + z$ implies $x = y$.

Lemma 1.4.9. *[26] Let S be a Γ-semiring and L be its left operator semiring. Then for any $\alpha \in \Gamma$, $[0, \alpha]$ is the zero of L.*

Proposition 1.4.10. *[26] Let S be a Γ-semiring and L and R be its left and right operator semirings. If $\sum_{i=1}^{m}[e_i, \delta_i]$ ($\sum_{j=1}^{n}[\gamma_j, f_i]$) is the left (resp. right) unity of S then it is the identity of L (resp. R).*

Theorem 1.4.11. *[70] Let S be a Γ-semiring and L be its left operator semiring. Then S is additively cancellative if and only if so is L.*

Definition 1.4.12. [68] A Γ-semiring S is called a *regular* Γ*-semiring* if for any $a \in S$, $a \in a\Gamma S\Gamma a$.

Definition 1.4.13. [68] A nonzero element a in a Γ-semiring S is said to be a *zero-divisor* if there exist nonzero $\gamma \in \Gamma$ and $b \in R$ such that $a\gamma b = 0$.

Definition 1.4.14. [27] A Γ-semiring S is said to be *zero-divisor free (ZDF)* if for $a, b \in S$ and $\alpha \in \Gamma$, $a\alpha b = 0$ implies that $a = 0$ or $\alpha = 0$ or $b = 0$.

Definition 1.4.15. [26] A Γ-semiring S is said to be *weak zero-divisor free (WZDF)* if for $a, b \in S$, $a\Gamma' b = \{0\}$ implies that either $a = 0$ or $b = 0$, where $\Gamma' = \Gamma \setminus \{0\}$.
A commutative Γ-semiring S with unities is said to be a Γ*-semi-integral domain (*Γ*-SID)* if it is WZDF.

Definitions 1.4.16. [26] Let S be a Γ-semiring. An additive subsemigroup I of S is called an *left (right) ideal* of S if $S\Gamma I \subseteq I$ (resp. $I\Gamma S \subseteq I$). If I is both a left and a right ideal then I is called an *ideal* of S.
An ideal I of a Γ-semiring S is called a *k-ideal* if for $x, y \in S$, $x + y \in I$ and $y \in I$ implies that $x \in I$.

Definitions 1.4.17. [68] For an ideal I of a Γ-semiring S, the *k-closure* of I, denoted by $\overline{I}$, is defined by $\overline{I} = \{a \in S : a + b = c \text{ for some } b, c, \in I\}$.
Clearly I is a k-ideal of S if and only if $\overline{I} = I$. Also $\overline{I}$ is a k-ideal of S.

Definition 1.4.18. [26] Let S be a Γ-semiring. A proper ideal P of S is said to be *prime* if for any two ideals I and J of S, $I\Gamma J \subseteq P$ implies that either $I \subseteq P$ or $J \subseteq P$, where $I\Gamma J$ denotes the set of all finite sums of the form $\sum_i a_i\alpha_i b_i$ with $a_i \in I$, $b_i \in J$ and $\alpha_i \in \Gamma$.
A prime ideal which is also a k-ideal is called a *prime k-ideal* of S.

Theorem 1.4.19. *[27] The set of all ideals (k-ideals, prime ideals, prime k-ideals) of a Γ-semiring S and its left operator semiring L are in order preserving bijective correspondence via the mapping $I \mapsto I^{+'}$, where I is an ideal (resp. k-ideal, prime ideal, prime k-ideal) of S and $I^{+'} = \{\sum_{i=1}^{m}[x_i, \alpha_i] \in L : (\sum_{i=1}^{m}[x_i, \alpha_i])S \subseteq I\}$ and the inverse mapping $J \mapsto J^{+}$, where J is an ideal (resp. k-ideal, prime ideal, prime k-ideal) of L and $J^{+} = \{a \in S : [a, \Gamma] \subseteq J\}$.*

Definition 1.4.20. [27] A Γ-semiring S is said to be *Noetherian (k-Noetherian)* if for every ascending chain $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ of ideals (respectively k-ideals) in S there exists a positive integer n such that $A_i = A_n$ for all $i \leq n$.

Definition 1.4.21. [27] A Γ-semiring is said to be *Artinian (k-Artinian)* if for every descending chain $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$ of ideals (respectively k-ideals) in S there exists a positive integer n such that $A_i = A_n$ for all $i \leq n$.

Theorem 1.4.22. *[27] A Γ-semiring S is Noetherian (k-Noetherian) if and only if its right operator semiring R or left operator semiring L is Noetherian (resp. k-Noetherian).*

Theorem 1.4.23. *[27] A Γ-semiring S is Artinian (k-Artinian) if and only if its right operator semiring R or left operator semiring L is Artinian (resp. k-Artinian).*

Definition 1.4.24. [27] A commutative Γ-semiring S with unities is a Γ-*semifield if for any $a(\neq 0) \in S$ and for any $\alpha(\neq 0) \in \Gamma$ there exists $b \in S$, $\beta \in \Gamma$ such that $a\alpha b\beta d = d$ for all $d \in S$.*

Theorem 1.4.25. *[27] A commutative Γ-semiring S with unities is a Γ-semifield if and only if it is ZDF and has no nonzero proper ideals.*

Theorem 1.4.26. *[27] If a commutative* Γ*-semiring* S *is a* Γ*-semifield then the left operator semiring* L *of* S *is a semifield.*

Definition 1.4.27. [37] Let S_1 be a Γ_1-semiring and S_2 a Γ_2-semiring.
Then $(\phi, g) : (S_1, \Gamma_1) \to (S_2, \Gamma_2)$ is called a *homomorphism* if $\phi : S_1 \to S_2$ and $g : \Gamma_1 \to \Gamma_2$ are homomorphisms of semigroups such that $\phi(x\gamma y) = \phi(x)g(x)\phi(y)$ for all $x, y \in S_1$ and $\gamma \in \Gamma$. The mapping (ϕ, g) is called an *epimorphism* if (ϕ, g) is a homomorphism and ϕ and g are epimorphisms of semigroups. Similarly we can define a *monomorphism.* A homomorphism (ϕ, g) is an *isomorphism* if (ϕ, g) is an epimorphism and a monomorphism.

Definition 1.4.28. [65] A *congruence* ρ on a Γ-semiring S is an equivalence relation which satisfies the following axioms:

(*i*) $(a, b) \in \rho$ implies $(a + c, b + c) \in \rho$,

(*ii*) $(a, b) \in \rho$ implies $(a\gamma c, b\gamma c) \in \rho$ and $(c\gamma a, c\gamma b) \in \rho$, for all $a, b, c \in S$ and for all $\gamma \in \Gamma$.

Equivalently, ρ is a congruence on a Γ-semiring S if for $a, b, c, d \in S$, $(a, b) \in \rho$ and $(c, d) \in \rho$ implies $(a + c, b + d) \in \rho$ and $(a\gamma c, b\gamma d), (c\gamma a, d\gamma b) \in \rho$ for all $\gamma \in \Gamma$.

Theorem 1.4.29. *[37] Let* ρ *be an equivalence relation on a* Γ*-semiring* S*. If* ρ *is a congruence on* S *then the quotient set* $S/\rho = \{\rho(x) : x \in S\}$ *with the compositions '+' and '·' defined by,*

$$\rho(x) + \rho(y) = \rho(x + y) \text{ and } \rho(x) \cdot \gamma \cdot \rho(y) = \rho(x\gamma y)$$

for all $x, y \in S$ *and for all* $\gamma \in \Gamma$*, is a* Γ*-semiring.*

We have the following theorem due to the 'Isomorphism Theorem' proved by Hedayati and Shum, 2011.

Theorem 1.4.30. *[37] Let* S_1 *and* S_2 *be* Γ_1*-semiring and* Γ_2*-semiring respectively. If* $(\phi, g) : (S_1, \Gamma_1) \to (S_2, \Gamma_2)$ *is an epimorphism then there exists an isomorphism* $(\psi, g) : (S_1/\theta_{(\phi,g)}, \Gamma_1) \to (S_2, \Gamma_2)$*, where* $\theta_{(\phi,g)} = \{(x, y) : \phi(x) = \phi(y)\}$ *is a congruence on the* Γ_1*-semiring* S_1*.*

1.5 Semimodule

Definition 1.5.1. [34] A left *semimodule M over a semiring R or a left R-semimodule* is a commutative semigroup $(M, +)$ together with a scalar multiplication $R \times M \to M$ defined by $(r, x) \mapsto rx$ such that

(i) $(r + s)x = rx + sx$,

(ii) $r(x + y) = rx + ry$,

(iii) $r(sx) = (rs)x$,

for all $r, s \in R$ and $x, y \in M$.
If a left R-semimodule M has an additively neutral element 0_M satisfying $r0_M = 0_M$ for all $r \in R$ then 0_M is called a *zero of M*.
A left R-semimodule M is said to be *unitary* if the semiring R has a zero 0 and an identity 1, M has a zero 0_M and $0x = 0_M$ and $1x = x$ hold for all $x \in M$.
Analogously a right R-semimodule is defined.

Throughout this chapter we consider R-semimodule M as an unitary left R-semimodule.

Definition 1.5.2. [34] A nonempty subset N of an R-semimodule M is called a *subsemimodule* of M if $x + y \in N$ and $rx \in N$ for all $x, y \in N$ and all $r \in R$. A subsemimodule N of an R-semimodule M is called a *subtractive subsemimodule or a k-subsemimodule* of M if $x + y \in N$ and $y \in N$ imply $x \in N$.

Definition 1.5.3. [34] For a subsemimodule N of an R-semimodule M, put $(N : M) = \{r \in R : rM \subseteq N\}$. Then $(N : M)$ is an ideal of R, called the *associated ideal* of N. If N is a k-subsemimodule of M then $(N : M)$ is a k-ideal of R.

Definition 1.5.4. [34] A subsemimodule N of an R-semimodule M is said to be proper if $N \neq M$.
A proper subsemimodule P of an R-semimodule M is said to be *maximal* in M if for each subsemimodule N of M, $P \subseteq N \subseteq M$ implies that $N = P$ or $N = M$.
A proper k-subsemimodule P of an R-semimodule M is said to be *k-maximal* in M if for each k-subsemimodule N of M, $P \subseteq N \subseteq M$ implies that $N = P$ or $N = M$.

Definition 1.5.5. [35] A proper subsemimodule P of an R-semimodule M is said to be *prime* in M if $rRm \subseteq P$ with $r \in R$ and $m \in M$ implies that $m \in P$ or $r \in (P : M)$.
A proper k-subsemimodule which is also prime is called a *prime k-subsemimodule*.

Definition 1.5.6. [34] An R-semimodule M is said to be *multiplication semimodule* if for every subsemimodule N of M there exists an ideal I of R such that $N = IM$. Also $N = (N : M)M$.

Remark 1.5.7. [34] If P is a prime subsemimodule of M then $(P : M)$ is a prime ideal of R.

Remark 1.5.8. [35] If P is a prime k-subsemimodule of M then $(P : M)$ is a prime k-ideal of R.

Lemma 1.5.9. *[34] Every maximal subsemimodule K of an R-semimodule M is prime.*

Lemma 1.5.10. *[35] Every k-maximal subsemimodule K of an R-semimodule M is prime.*

Lemma 1.5.11. *[34] Let M be a finitely generated R-semimodule. If N is a proper subsemimodule of M then there exists a maximal subsemimodule of M containing N. Therefore M has a maximal subsemimodule.*

Definition 1.5.12. [33] An R-semimodule M is said to be *subtractively finitely generated or k-finitely generated* if there exists a nonempty finite subset F of M such that $M = \overline{\langle F \rangle}$.

Lemma 1.5.13. *[33] Let M be a k-finitely generated R-semimodule. If N is a proper k-subsemimodule of M then there exists a k-maximal subsemimodule of M containing N. Therefore M has a k-maximal subsemimodule.*

Note that in [85] and [33], the associated ideal is denoted as $A_N(M)$ instead of $(N : M)$.

Theorem 1.5.14. *[85] Let M be a multiplication R-semimodule and N be a subsemimodule of M. Then N is a prime subsemimodule of M if and only if $(N : M)$ is a prime ideal of R.*

Proposition 1.5.15. *[85] An R-semimodule M is a multiplication semimodule if and only if there exists an ideal I of R such that $mR = MI$ for each $m \in M$*

Proposition 1.5.16. *[85] Let P be a prime subsemimodule and $N_1, N_2, ..., N_t$ are subsemimodules (k-subsemimodules) of the multiplication semimodule M over the semiring R. Then $\bigcap_{i=1}^{t} N_i \subseteq P$ if and only if $N_j \subseteq P$ for some j with $1 \leq j \leq t$.*

Definition 1.5.17. [84] Let M be an R-semimodule over a semiring R. We call M a *k-multiplication semimodule* if for all subsemimodules N of M there exists a k-ideal I of R such that $N = IM$.

Theorem 1.5.18. *[84] Let M be a k-multiplication semimodule of the semiring R. Then a k-subsemimodule N is prime if and only if $(N : M)$ is a prime k-ideal.*

Definition 1.5.19. [35] For a subsemimodule N of an R-semimodule M, the *radical* $rad(N)$ of N is defined to be the intersection of all prime subsemimodules of M containing N.
$rad_k(N)$, the *k-radical of* N, is defined to be the intersection of all prime k-subsemimodules of M containing N.

Note that in [34] and [35], the intersection of all prime subsemimodules (k-subsemimodules) of a R-semimodule M containing a subsemimodule N of M is denoted by $\sqrt{N}$ (resp. $\sqrt{N}^{(k)}$) instead of $rad(N)$ (resp. $rad_k(N)$).

Proposition 1.5.20. *[34] If N is a subsemimodule of a multiplication R-semimodule M then $rad(N) = Mrad((N : M))$.*

Theorem 1.5.21. *[35] If N is a k-subsemimodule of a multiplication R-semimodule M then $rad(N) = rad_k(N)$.*

Corollary 1.5.22. *[35] If N is a k-subsemimodule of a multiplication R-semimodule M then $rad_k(N) = Mrad((N : M))$.*

Lemma 1.5.23. *[34] Let M be a multiplication R-semimodule. If N is a proper subsemimodule of M then there exists a prime subsemimodule of M containing N. Therefore $Spec(M) \neq \emptyset$, where $Spec(M)$ is the set of all prime subsemimodules of M.*

Theorem 1.5.24. *[35] Let M be a multiplication R-semimodule. If N is a proper k-subsemimodule of M then there is a prime k-subsemimodule of M containing N. Therefore $Spec_k(M) \neq \emptyset$, where $Spec_k(M)$ is the set of all prime k-subsemimodules of M.*

For a nonempty subset S of M, put $V(S) = \{P \in Spec(M) : S \subseteq P\}$, where $Spec(M)$ is the set of all prime subsemimodules of an R-semimodule M.
An R-semimodule M is called a *top semimodule* if for any subsemimodules N and L of M, there exists a subsemimodule T of M such that $V(N) \cup V(L) = V(T)$.

For a top R-semimodule M, the collection $\{V(S) : \emptyset \neq S \subseteq M\}$ of subsets of $Spec(M)$ satisfies the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology* on $Spec(M)$ and then $Spec(M)$ is called the *prime spectrum* of M. The open set $Spec(M) \setminus V(S)$ of $Spec(M)$ is denoted by $D(S)$. For every element $m \in M$, put $V(m) = \{P \in Spec(M) : m \in P\}$ and $D(m) = Spec(M) \setminus V(m)$. $D(m)$ is called a basic open set of $Spec(M)$.
By Corollary 4.1 of [34], a multiplication R-semimodule M is a top semimodule. Therefore the topological properties of $Spec(M)$ over a top semimodule hold for that over a multiplication semimodule.

Lemma 1.5.25. *[34] If M is a multiplication R-semimodule then the collection $\{D(m)|m \in M\}$ is a base for the Zariski topology on $Spec(M)$.*

Lemma 1.5.26. *[34] If N is a subsemimodule of M then $V(N) = V(rad(N))$.*

Lemma 1.5.27. *[34] If N and K are subsemimodules of M with $V(N) \subseteq V(K)$, then $K \subseteq rad(N)$.*

Theorem 1.5.28. *[34] If M is a multiplication R-semimodule, then $Spec(M)$ is a T_0-space.*

Theorem 1.5.29. *[34] Let M be a multiplication R-semimodule. Then $Spec(M)$ is a T_1-space if and only if every prime subsemimodule is not contained in the other prime subsemimodule in M.*

Theorem 1.5.30. *[34] Let M be a multiplication R-semimodule and $\emptyset \neq Y \subseteq Spec(M)$. Then Y is an irreducible subset of $Spec(M)$ if and only if the intersection of all prime subsemimodules belonging to Y is a prime subsemimodule of M.*

Theorem 1.5.31. *[34] If N is a subsemimodule of a multiplication R-semimodule M then the following are equivalent to one another.*

(1) $V(N)$is an irreducible subset of $Spec(M)$.

(2) $rad(N)$ is a prime subsemimodule of M.

(3) $rad(N)$ is a generic point of $V(N)$ in $Spec(M)$.

Theorem 1.5.32. *[34] A multiplication R-semimodule M is finitely generated if and only if $Spec(M)$ is a compact space.*

Theorem 1.5.33. *[34] If M is a multiplication R-semimodule then every basic open set of $Spec(M)$ is compact.*

Corollary 1.5.34. *[34] Let M be a multiplication R-semimodule. An open set of $Spec(M)$ is compact if and only if it is a union of a finite number of basic open sets.*

Corollary 1.5.35. *[34] If N is a finitely generated subsemimodule of a multiplication R-semimodule M then $D(N)$ is compact in $Spec(M)$.*

Theorem 1.5.36. *[34] If M is a multiplication R-semimodule then the intersection of finitely many basic open sets is compact in $Spec(M)$.*

Let us consider the space of all prime k-subsemimodules of an R-semimodule M, denoted by $Spec_k(M)$, the *k-prime spectrum* of M, along with the subspace topology as the subspace of $Spec(M)$. Then the closed sets are $V_k(S) = \{P \in Spec_k(M) : S \subseteq P\}$, where S is a nonempty subset of M. $V_k(S) = V(S) \cap Spec_k(M)$.
The open sets are of the form $D_k(S) = D(S) \cap Spec_k(M)$. For every element $m \in M$, put $V_k(m) = \{P \in Spec_k(M) : m \in P\}$ and $D_k(m) = Spec_k(M) \setminus V_k(m)$. $D_k(m)$ is a basic open set of $Spec_k(M)$.
Again since a multiplication R-semimodule M is a top semimodule, the topological properties of $Spec_k(M)$ over a top semimodule hold for that over a multiplication semimodule.

Lemma 1.5.37. *[35] If M is a multiplication R-semimodule then $\{D_k(m) : m \in M\}$ is a base for the topology on $Spec_k(M)$.*

Theorem 1.5.38. *[35] If M is a multiplication R-semimodule then $Spec_k(M)$ is a T_0 space.*

Theorem 1.5.39. *[35] Let M be a multiplication R-semimodule. Then $Spec_k(M)$ is a T_1 space if and only if every prime k-subsemimodule is not contained in the other prime k-subsemimodule in M.*

Theorem 1.5.40. *[35] If M is a multiplication R-semimodule and $\emptyset \neq Spec_k(M)$ then the following are equivalent to one another.*

(1) *Y is an irreducible subset in $Spec_k(M)$.*

(2) *Y is an irreducible subset in $Spec(M)$.*

(3) *The intersection of all prime subsemimodules belonging to Y is a prime subsemimodule of M.*

(4) *The intersection of all prime subsemimodules belonging to Y is a prime k-subsemimodule of M.*

Theorem 1.5.41. *[35] If N is a subsemimodule of a multiplication R-semimodule M then the following are equivalent to one another.*

(1) $V_k(N)$ is an irreducible subset of $Spec_k(M)$.

(2) $rad_k(N)$ is a prime subsemimodule of M.

(3) $rad_k(N)$ is a generic point of $V_k(N)$ in $Spec_k(M)$.

Theorem 1.5.42. *[35] A multiplication R-semimodule M is k-finitely generated if and only if $Spec_k(M)$ is a compact space.*

Theorem 1.5.43. *[35] If M is a multiplication R-semimodule then every basic open set of $Spec_k(M)$ is compact.*

Corollary 1.5.44. *[35] Let M be a multiplication R-semimodule. An open set of $Spec_k(M)$ is compact if and only if it is a union of a finite number of basic open sets.*

Corollary 1.5.45. *[35] If N is a k-finitely generated subsemimodule of a multiplication R-semimodule M then $D_k(N)$ is compact in $Spec_k(M)$.*

Theorem 1.5.46. *[35] If M is a multiplication R-semimodule then the intersection of finitely many basic open sets is compact in $Spec_k(M)$.*

1.6 ΓS-semimodule

Definition 1.6.1. [69] Let S be a Γ-semiring and L and R be the left and right operator semirings of S respectively. An additive commutative monoid M is said to be a *right* Γ*-semiring* S*-semimodule* or simply a ΓS*-semimodule* if there exists a mapping $M \times \Gamma \times S \to M$ (images to be denoted by $a\gamma s$, where $a \in M, \gamma \in \Gamma$ and $s \in S$) satisfying the following conditions:

(i) $a\alpha(s+t) = a\alpha s + a\alpha t$

(ii) $(a+b)\alpha s = a\alpha s + b\alpha s$

(iii) $a(\alpha+\beta)s = a\alpha s + a\beta s$

(iv) $a\alpha(s\beta t) = (a\alpha s)\beta t$

(v) $a\alpha 0_S = 0_M\alpha s = 0_M$

for all $a, b \in M$, for all $s, t \in S$ and for all $\alpha, \beta \in \Gamma$.
Left Γ-semimodule of S can be defined in a similar manner and it is called an $S\Gamma$-*semimodule*.
A ΓS-semimodule M is called *unitary* if S has a right unity $\sum_{j=1}^{n}[\gamma_j, f_i] \in R$ and $\sum_{j=1}^{n} a\gamma_j f_j = a$ for all $a \in M$.
Throughout the chapter, we take M as an unitary ΓS-semimodule.

Example 1.6.2. [68] Let us consider the Γ-semiring of all non-positive integers Z_0^-, where $\Gamma = Z_0^-$ and let $M = Z_0^+$ be the additive commutative semigroup of non-negative integers. Then M is a ΓS-semimodule, where $S = \Gamma = Z_0^-$.

Definition 1.6.3. [69] A nonempty subset N of a ΓS-semimodule M is said to be a ΓS-*subsemimodule* of M if $a + b \in N$ and $a\alpha s \in N$ for all $a, b \in N$, for all $s \in S$ and for all $\alpha \in \Gamma$.
N contains the zero of M.

Example 1.6.4. [68] Let N be the additive commutative semigroup $\{0\} \cup \{n : n \geq 6$ is an even integer$\}$. Then N is a ΓS-subsemimodule of the ΓS-semimodule Z_0^+, where $S = \Gamma = Z_0^-$.

Definition 1.6.5. [69] A ΓS-subsemimodule N of a ΓS-semimodule M is said to be a $k\Gamma S$-*subsemimodule* of M if $a + b, b \in N$, $a \in M$ imply that $a \in N$.

Example 1.6.6. [68] Let K be the additive commutative semigroup $\{n : n$ is an even non-negative integer$\}$. Then K is a $k\Gamma S$-subsemimodule of the ΓS-semimodule Z_0^+, where $S = \Gamma = Z_0^-$.

Remark 1.6.7. [28] Intersection of any family of ΓS-subsemimodules ($k\Gamma S$-subsemimodules) of a ΓS-semimodule is a ΓS-subsemimodule (resp. $k\Gamma S$-subsemimodule).

Definition 1.6.8. [69] Let N be a ΓS-subsemimodule of a ΓS-semimodule M. Then *k-closure* of N, denoted by $\overline{N}^k$, is defined by $\overline{N}^k = \{a \in M : a + b = c$ for some $b, c, \in N\}$.

Clearly N is a $k\Gamma S$-subsemimodule of M if and only if $\overline{N}^k = N$. Also if N is a ΓS-subsemimodule of M then $\overline{N}^k$ is a $k\Gamma S$-subsemimodule of M.

Example 1.6.9. [68] The set $K = \{n : n$ is an even non-negative integer$\}$ is the k-closure of the ΓS-subsemimodule $N = \{0\} \cup \{n : n \geq 6$ is an even integer$\}$ of Z_0^+, where $S = \Gamma = Z_0^-$ i.e., K is the smallest $k\Gamma S$-subsemimodule of Z_0^+ containing N. In fact, if there exists a $k\Gamma S$-subsemimodule N_1 of Z_0^+ such that $N \subset N_1 \subset K$ then either $N_1 = N \cup \{2\}$ or $N_1 = N \cup \{4\}$. For the first case, $2+4, 2 \in N_1$ though $4 \notin N_1$ and for the second case, $2+4, 4 \in N_1$ though $2 \notin N_1$. This leads to a contradiction to our assumption that N_1 is a $k\Gamma S$-subsemimodule of Z_0^+. Therefore K is the k-closure of the ΓS-subsemimodule N of Z_0^+.

Definition 1.6.10. [28] Let M be a right ΓS-semimodule over a Γ-semiring S with unities, L be the left operator semiring of S and G be the free additive commutative semigroup generated by $M \times \Gamma$. Then the relation ρ on G defined by,

$$\sum_{i=1}^{m}(m_i, \alpha_i)\rho\sum_{j=1}^{n}(n_j, \beta_j) \text{ if and only if } \sum_{i=1}^{m} m_i\alpha_i x = \sum_{j=1}^{n} n_j\beta_j x,$$

for all $x \in S$ $(m, n \in Z^+)$, is a congruence on G.
We denote the congruence class containing $\sum_{i=1}^{m}(m_i, \alpha_i)$ by $\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle$.
Then G/ρ is an additive commutative semigroup.
A multiplication from right of the elements of G/ρ by the elements of L is defined as follows: for any $\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle \in G/\rho$ and $\sum_{j=1}^{n}[x_j, \beta_j] \in L$,

$$(\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle)(\sum_{j=1}^{n}[x_j, \beta_j]) = \sum_{i,j}\langle m_i\alpha_i x_j, \beta_j\rangle.$$

With respect to this multiplication, the additive commutative semiring G/ρ is a unitary right L-semimodule. We call it the *associated L-semimodule* of the ΓS-semimodule M and denote it by $M^{\#}$.
If N and Δ are nonempty subsets of M and Γ respectively then we denote by $\langle N, \Delta\rangle$ the set $\{\sum_{i=1}^{n}\langle n_i, \alpha_i\rangle : n_i \in N, \alpha \in \Delta, n \in Z^+\}$. Also $\langle M, \Gamma\rangle = M^{\#}$.

Theorem 1.6.11. *[28] The lattices of all ΓS-subsemimodules ($k\Gamma S$-subsemimodules) of a ΓS-semimodule M and its associated L-semimodule $M^{\#}$ are isomorphic via the mapping $N \mapsto N^{+'}$, where $N^{+'} = \{\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle : (\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle)S \subseteq N\}$, for a ΓS-subsemimodule (resp. $k\Gamma S$-subsemimodule) N of M and the inverse mapping $K \mapsto K^+$, where $K^+ = \{a \in M : \langle a, \Gamma\rangle \subseteq K\}$, for a subsemimodule (resp. k-subsemimodule) K of L.*

CHAPTER 2

A STUDY ON CONGRUENCES ON A Γ-SEMIRING VIA ITS OPERATOR SEMIRINGS

2

A study on congruences on a $\Gamma-$semiring via its operator semirings

The studies on the congruences corroborates the well known fact that the structure theory of semigroups and semirings / hemirings and their generalizations such as Γ-semirings are heavily dependent on the study of congruences. So we extend the study of congruences on Γ-semirings by generalizing the notions and results of semirings studied by Acharyya et al. Among various ways of studying Γ-semiring, one approach is to study it via its operator semirings [27] and this technique is used by many authors while studying this Γ-algebraic structure (for instance [27], [69], [70], [71]). We have also applied the same technique in order to accomplish our present study. We obtain a lattice isomorphism between the set of all congruences on a Γ-semiring and those on its operator semirings. We have also obtained such lattice isomorphisms for cancellative congruences and for regular congruences. For maximal congruences and prime congruences we have obtained bijection and inclusion preserving bijection respectively. These correspondences have been applied to obtain some results on quotient Γ-semirings induced by the congruences.

In **section 1**, we introduce the notions of different types of congruences on a Γ-semiring and discuss some examples of those. We also prove the congruence version of the correspondence theorem of Γ-semirings in Theorem 2.1.27. Besides, we give some remarks

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This chapter is mainly based on the work of the following paper:
Sarbani Mukherjee (Goswami), Soumi Basu and Sujit Kumar Sardar, ***A study on congruences on a Γ-semiring via its operator semirings,*** **Communicated.**

and observations on those congruences.
In **section 2**, we obtain a lattice isomorphism (*cf.* Theorem 2.2.4) between the set of all congruences on a Γ-semiring and those on its operator semirings. We have also obtained such lattice isomorphisms for cancellative congruences (*cf.* Theorem 2.2.8) and for regular congruences (*cf.* Theorem 2.2.12). For maximal congruences and prime congruences we have obtained bijection and inclusion preserving bijection respectively (*cf.* Theorems 2.2.14, 2.2.20). We prove that the lattices of congruences of two isomorphic Γ-semirings are isomorphic (*cf.* Corollary 2.2.23).
In **section 3**, we obtain a connection between the quotient (by congruences) left operator semiring and the left operator semiring of a quotient Γ-semiring (by congruences) in Theorem 2.3.1. Those correspondences established in section 2 have been applied to obtain Theorem 2.3.4, Theorem 2.3.5, Theorem 2.3.8 which are analogous to the results on prime ideals and maximal ideals and quotient Γ-semiring (by ideals).

2.1 Congruences on a $\Gamma-$semiring

Let us start this section with some examples of congruences on Γ-semirings which are also prevalent in our whole study.

Example 2.1.1. Let us consider the Γ-semiring $C_-(X)$ over a topological space X (*cf.* Example 1.4.3). Let us choose an element $x \in X$. Then the relation ρ_x defined by, $(f,g) \in \rho_x$ if and only if $f(x) = g(x)$, where $f,g \in C_-(X)$ is a congruence on the Γ-semiring $C_-(X)$ of all non-positive valued continuous functions on a topological space X, where $\Gamma = C_-(X)$. Clearly it is an equivalence relation. Indeed for all $f,g,h \in C_-(X)$, $(f,g) \in \rho_x$ implies $f(x) = g(x)$, for all $x \in X$ which implies $f(x) + h(x) = g(x) + h(x)$ for all $x \in X$. Then $(f+h, g+h) \in \rho_x$.
Also for all $h, \gamma \in C_-(X)$, $(f.\gamma.h, g.\gamma.h), (h.\gamma.f, h.\gamma.g) \in \rho_x$.
Therefore ρ_x is a congruence.

Example 2.1.2. Let us consider the set Z_0^-, the set of all non-positive integers. Then with pointwise addition and multiplication of integers Z_0^- forms a Γ-semiring, where $\Gamma = Z_0^-$. Let us choose a positive integer k. Then the relation ρ_k on the Γ-semiring Z_0^-, defined by $\rho_k = \{(m,n) \in Z_0^- \times Z_0^- : m-n \text{ is divisible by } k\}$ is a congruence. Clearly it is an equivalence relation. Indeed for all $m,n,p \in Z_0^-$, $(m,n) \in \rho_k$ implies $(m-n)$ is divisible by k which implies $(m+p)-(n+p)$ is divisible by k. Then $(m+p, n+p) \in \rho_k$.
Also for all $m,n,p,\gamma \in Z_0^-$, $(m\gamma p, n\gamma p) \in \rho_k$ and $(p\gamma m, p\gamma m) \in \rho_k$.

Therefore ρ_k is a congruence.

Remark 2.1.3. The set of all congruences on a Γ-semiring S, partially ordered with the set inclusion, forms a complete lattice with meet and join defined as follows:
for two congruences ρ_1, ρ_2, $\rho_1 \wedge \rho_2 = \rho_1 \cap \rho_2$ and $\rho_1 \vee \rho_2$ is defined as $(a, b) \in \rho_1 \vee \rho_2$ if and only if for some natural number n, there exists elements x_1, x_2,..., x_{n-1} in S with $a = x_0$, $x_n = b$ such that $(x_{k-1}, x_k) \in \rho_1$ or $(x_{k-1}, x_k) \in \rho_2$ for $1 \leq k \leq n$.

Let us now introduce the notions of different types of congruences on a Γ-semiring. First we will study cancellative congruence on a Γ-semiring.

Definition 2.1.4. A congruence ρ on a Γ-semiring S is said to be a *cancellative congruence* if

$$(a + c, b + c) \in \rho \text{ implies } (a, b) \in \rho, \text{ where } a, b, c \in S.$$

Example 2.1.5. Consider the congruence ρ_x in Example 2.1.1. $(f + h, g + h) \in \rho_x$ implies $f(x) + h(x) = g(x) + h(x)$ which implies $f(x) = g(x)$. So $(f, g) \in \rho_x$. Thus ρ_x is a cancellative congruence on $C_-(X)$.

Example 2.1.6. Consider the congruence ρ_k in Example 2.1.2. Let $(m+p, n+p) \in \rho_k$ where $m, n, p \in Z_0^-$. Then $m - n = (m + p) - (n + p)$ is divisible by k which implies $(m, n) \in \rho_k$. Hence ρ_k is a cancellative congruence on Z_0^-.

Theorem 2.1.7. *If S is a Γ-semiring and ρ is a congruence on S then S/ρ is an additively cancellative Γ-semiring if and only if ρ is a cancellative congruence.*

Proof. Let S/ρ be an additively cancellative Γ-semiring. Let $(a + c, b + c) \in \rho$, where $a, b, c \in S$. Then $\rho(a + c) = \rho(b + c)$, i.e., $\rho(a) + \rho(c) = \rho(b) + \rho(c)$ implies $\rho(a) = \rho(b)$ which means $(a, b) \in \rho$. Therefore ρ is a cancellative congruence on S.
Conversely, let ρ be a cancellative congruence.
Then $\rho(a) + \rho(c) = \rho(b) + \rho(c)$ implies $(a + c, b + c) \in \rho$. $(a, b) \in \rho$, i.e., $\rho(a) = \rho(b)$. This completes the proof. □

Let us introduce the notion of regular congruence on a Γ-semiring.

Definition 2.1.8. A cancellative congruence ρ on a Γ-semiring S is called a *left regular congruence* if there exist $e_1, e_2 \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ such that

$$(a + e_1\gamma_1 a, e_2\gamma_2 a) \in \rho \text{ for all } a \in S.$$

The pair (e_1, e_2) is called a *left regular unity pair* of ρ. A cancellative congruence ρ on a Γ-semiring S is called a *right regular congruence* if there exist $f_1, f_2 \in S$ and $\delta_1, \delta_2 \in \Gamma$ such that

$$(a + a\delta_1 f_1, a\delta_2 f_2) \in \rho \text{ for all } a \in S.$$

The pair (f_1, f_2) is called a *right regular unity pair* of ρ.
ρ is called a *regular congruence* if it is both a left and a right regular congruence on S.

Note that if a Γ-semiring is commutative then a left regular congruence is also a right regular congruence and hence a regular congruence on it.

Example 2.1.9. In Example 2.1.1, there exist $\mathbf{0}, -\mathbf{1} \in C_-(X)$, where $\mathbf{0}(x) = 0$ for all $x \in X$ and $-\mathbf{1}(x) = -1$ for all $x \in X$. Then for all $f \in C_-(X)$, $(f + \mathbf{0}(-\mathbf{1})f, (-\mathbf{1})(-\mathbf{1})f) \in \rho_x$. Also ρ_x is a cancellative congruence on $C_-(X)$. Hence ρ_x is a left regular congruence. Since $C_-(X)$ is a commutative Γ-semiring, ρ_x is also a right regular congruence on the Γ-semiring $C_-(X)$, where $\Gamma = C_-(X)$. Thus ρ_x is a regular congruence on $C_-(X)$.

Example 2.1.10. In Example 2.1.2, for all non-positive integer m, $(m+0(-1)m, (-1)(-1)m) \in \rho_k$. Also ρ_k is a cancellative congruence on the Γ-semiring Z_0^-, where $\Gamma = Z_0^-$. Thus ρ_k is a left regular congruence on Z_0^-. Since Z_0^- is a commutative Γ-semiring, ρ_k is also a right regular congruence on Z_0^-. Hence ρ_k is a regular congruence on Z_0^-.

The following examples exhibit that a right regular congruence may not be a left regular congruence and vice versa.

Examples 2.1.11. (1) Let S be the additive commutative semigroup of all 5×2 matrices over the set of all non-negative rationals and Γ be the additive commutative semigroup of all 2×5 matrices over the same set. Then S is a Γ-semiring if AXB denotes the usual matrix product, where $A, B \in S$ and $X \in \Gamma$. This Γ-semiring has the strong right unity $[E, F]$,

where $E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then the identity relation on S is a right regular congruence on S, since $A+AEO = AEF$, for all $A \in S$, where O denotes the zero matrix of order 5×2. But it is not a left regular congruence on S. For this, let us assume that there exist $X_1, X_2 \in S$ and $Y_1, Y_2 \in \Gamma$ such that $A + X_1Y_1A = X_2Y_2A$, for all $A \in S$. Then $A = (X_2Y_2 - X_1Y_1)A$, for all $A \in S$ which is not possible, since the rank of the matrix $X_2Y_2 - X_1Y_1$ can be atmost 4 and it cannot be equal to the identity matrix of order 5.

(2) If we take the Γ-semiring S, where S is the set of all 2×5 matrices over the set of all non-negative rationals and Γ be the set of all 5×2 matrices over the same set then in a similar fashion as above we can show that the identity relation is a left regular congruence but not a right regular congruence on S.

Remark 2.1.12. It is easy to observe that every cancellative congruence containing a left (right) regular congruence is always a left (resp. right) regular congruence.

Analogous to the concept of identity pair in semirings, here we define the identity pair of Γ-semirings.

Definition 2.1.13. In a Γ-semiring S, a pair $(e, f) \in S \times S$ is called a *left identity pair* in S if there exist $\gamma, \delta \in \Gamma$ such that $a + e\gamma a = f\delta a$ for all $a \in S$ and a pair $(e', f') \in S \times S$ is called a *right identity pair* in S if there exist $\gamma', \delta' \in \Gamma$ such that $a + a\gamma' e' = a\delta' f'$ for all $a \in S$. A pair is called an *identity pair* if it is both a left and a right identity pair on S.
Clearly if the Γ-semiring is commutative then a left identity pair is also a right identity pair and hence becomes an *identity pair* in it.

Theorem 2.1.14. *Let S be a Γ-semiring and ρ be a congruence on S. Then S/ρ is an additively cancellative Γ-semiring containing a left (right) identity pair if and only if ρ is a left (resp. right) regular congruence on the Γ-semiring S.*

Proof. Let ρ be a left regular congruence on a Γ-semiring S. Then in view of Theorem 2.1.7, S/ρ is an additively cancellative Γ-semiring. Now let (e_1, e_2) be a left regular unity pair. Therefore there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $(a + e_1\gamma_1 a, e_2\gamma_2 a) \in \rho$ for all $a \in S$. Then

$$\rho(a) + \rho(e_1) \cdot \gamma_1 \cdot \rho(a) = \rho(e_2) \cdot \gamma_2 \cdot \rho(a) \text{ for all } a \in S.$$

Therefore S/ρ contains a left identity pair $(\rho(e_1), \rho(e_2))$.
Conversely, let S/ρ be an additively cancellative Γ-semiring containing a left identity pair $(\rho(f_1), \rho(f_2))$. Therefore ρ is a cancellative congruence on S, by Theorem 2.1.7. Also there exist $\delta_1, \delta_2 \in \Gamma$ such that

$$\rho(a) + \rho(f_1) \cdot \delta_1 \cdot \rho(a) = \rho(f_2) \cdot \delta_2 \cdot \rho(a) \text{ for all } \rho(a) \in S/\rho.$$

This implies $(a + f_1\delta_1 a, f_2\delta_2 a) \in \rho$ for all $a \in S$. Therefore ρ is a left regular congruence on S. □

In the following we define the maximal congruence and maximal regular congruence on a Γ-semiring.

Definition 2.1.15. A proper congruence ρ on a Γ-semiring S is called a *maximal congruence* if there is no proper congruence on S properly containing ρ.
A proper left (right) regular congruence on a Γ-semiring S is called a *maximal left (right) regular congruence* if it is not properly contained in any proper left (resp. right) regular congruence on S.

Example 2.1.16. In the Example 2.1.2, for a prime number p, if we take the congruence ρ_p on the Γ-semiring Z_0^-, where $\Gamma = Z_0^-$ then ρ_p is a maximal congruence on Z_0^-.

Example 2.1.17. From Examples 2.1.16 and 2.1.10 we get that Example 2.1.16 is also an example of a maximal regular congruence

Theorem 2.1.18. *Let S be a Γ-semiring. Every maximal left (right) regular congruence on S is a maximal cancellative congruence on S.*

Proof. Proof follows from Remark 2.1.12. □

The converse of the above theorem is not true. It is evident from the following example.

Example 2.1.19. Let us consider the set of all non-positive integers Z_0^- and the set of all non-positive even integers $2Z_0^-$. Then under usual addition and multiplication of integers, if we take $S = Z_0^-$ and $\Gamma = 2Z_0^-$ then S is a Γ-semiring. Therefore the relation $\rho = \{(m, n) \in Z_0^- \times Z_0^- : m - n \text{ is divisible by } 2\}$ on the Γ-semiring Z_0^-, is a maximal cancellative congruence. But ρ is neither a left regular congruence nor a right regular congruence on Z_0^-, since for $e_1, e_2 \in Z_0^-$ and $\gamma_1, \gamma_2 \in 2Z_0^-$, $(a + e_1\gamma_1 a, e_2\gamma_2 a) \notin \rho$ for all odd non-positive integer a.

Recall that G. G. Ray et. al defined the concept of prime congruences on a semiring as follows: a proper congruence ρ on a semiring R is called a *prime congruence* if $(ad + bc, ac + bd) \in \rho$ implies either $(a, b) \in \rho$ or $(c, d) \in \rho$, where $a, b, c, d \in R$.
Let us now introduce the notion of prime congruences on a Γ-semiring.

Definition 2.1.20. A proper congruence ρ on a Γ-semiring S is called a *prime congruence* if it satisfies the following condition:

$$(a\gamma d + b\gamma c, a\gamma c + b\gamma d) \in \rho \text{ for all } \gamma \in \Gamma \text{ implies either } (a, b) \in \rho \text{ or } (c, d) \in \rho,$$

where $a, b, c, d \in S$.

below we give two examples of prime congruences on Γ-semiring.

Example 2.1.21. In the Example 2.1.1, let us consider $(f\gamma i + g\gamma h, f\gamma h + g\gamma i) \in \rho_x$ for all $\gamma \in C_-(X)$, where $f, g, h, i \in C_-(X)$. This implies

$$f(x)\gamma(x)i(x) + g(x)\gamma(x)h(x) = f(x)\gamma(x)h(x) + g(x)\gamma(x)i(x) \text{ for all } \gamma \in C_-(X).$$

Then $(f(x) - g(x))\gamma(x)(h(x) - i(x)) = 0$ for all $\gamma \in C_-(X)$. Therefore in particular, for any nonzero $\gamma(x)$, either $f(x) = g(x)$ or $h(x) = i(x)$ whence either $(f, g) \in \rho_x$ or $(h, i) \in \rho_x$. Hence ρ_x is a prime congruence on $C_-(X)$.

Example 2.1.22. Let us consider Example 2.1.2 and take ρ_p for any prime number p. Now let $(a\gamma d + b\gamma c, a\gamma c + b\gamma d) \in \rho_p$ for all $\gamma \in Z_0^-$, where $a, b, c, d \in Z_0^-$. This implies $(a\gamma d + b\gamma c) - (a\gamma c + b\gamma d)$ is divisible by p for all $\gamma \in Z_0^-$, i.e., $(a - b)\gamma(c - d)$ is divisible by p for all $\gamma \in Z_0^-$. Since p is prime, p divides $a - b$ or $c - d$. Hence it follows that either $(a, b) \in \rho_p$ or $(c, d) \in \rho_p$. Therefore ρ_p is a prime congruence on Z_0^-.

The following example shows that the intersection of two prime congruences may not be a prime congruence.

Example 2.1.23. Let us consider Example 2.1.2 and take two congruences ρ_2 and ρ_3 for $k = 2, 3$. Clearly ρ_2 and ρ_3 are prime congruences but $\rho_2 \bigcap \rho_3 = \rho_6$ is not a prime congruence on the Γ-semiring Z_0^-, where $\Gamma = Z_0^-$.

Remark 2.1.24. In view of the above example we conclude that the set of all prime congruences on a Γ-semiring does not form a sublattice of the lattice of congruences on a Γ-semiring.

The following two examples show that none of the maximal congruence and the prime congruence implies the other.

Example 2.1.25. In example 2.1.19, the congruence ρ is an example of a maximal congruence on the Γ-semiring Z_0^-, where $\Gamma = 2Z_0^-$. But it is not a prime congruence on Z_0^-, as, for $a = -1$, $b = -4$, $c = -5$, $d = -6$ and for any $\gamma \in 2Z_0^-$,
$(a\gamma d + b\gamma c, a\gamma c + b\gamma d) \in \rho$ does not imply $(a, b) \in \rho$ or $(c, d) \in \rho$.

Example 2.1.26. Let us consider the Γ-semiring $Z_0^+[x]$, of all polynomials over x with non-negative coefficients, where $\Gamma = Z_0^+$ with usual addition and multiplication of polynomials. Consider the relation

$$\rho = \{(p(x), q(x)) \in Z_0^+[x] \times Z_0^+[x] : p(0) = q(0)\} \text{ on } Z_0^+[x].$$

It is a prime congruence on $Z_0^+[x]$. But it is not a maximal congruence on $Z_0^+[x]$, since it is contained in the congruence

$$\rho' = \{(p(x), q(x)) \in Z_0^+[x] \times Z_0^+[x] : p(0) - q(0) \text{ is divisible by } 2\} \text{ on } Z_0^+[x].$$

The following theorem is the correspondence theorem for Γ-semirings induced by congruences.

Theorem 2.1.27. *Let S be a Γ-semiring and ρ_0 be a congruence on S. There is an order preserving bijection between the set of all congruences (cancellative, left regular, right regular, prime congruences) on S containing ρ_0 and the set of all congruences (respectively, cancellative, left regular, right regular, prime congruences) on S/ρ_0.*

Proof. Let ρ be a congruence on a Γ-semiring S containing ρ_0. Let us consider the relation

$$\rho_\sigma = \{(\rho_0(x), \rho_0(y)) \in S/\rho_0 \times S/\rho_0 : (x, y) \in \rho\}$$

on the Γ-semiring S/ρ_0. Routine verification shows that ρ_σ is an equivalence relation on S/ρ_0. Let $(\rho_0(x), \rho_0(y)) \in \rho_\sigma$. Then for all $x, y, z \in S$ and $\gamma \in \Gamma$, $(x, y) \in \rho$ implies $(x + z, y + z), (x\gamma z, y\gamma z), (z\gamma x, z\gamma y) \in \rho$. This shows that

$$(\rho_0(x + z), \rho_0(y + z)), (\rho_0(x\gamma z), \rho_0(y\gamma z)) \in \rho_\sigma$$

whence $(\rho_0(x) + \rho_0(z), \rho_0(y) + \rho_0(z)), (\rho_0(x).\gamma.\rho_0(z), \rho_0(y).\gamma.\rho_0(z))$ and $(\rho_0(z).\gamma.\rho_0(x), \rho_0(z).\gamma.\rho_0(y)) \in \rho_\sigma$. Therefore ρ_σ is a congruence on S/ρ_0.
Again let π be a congruence on the Γ-semiring S/ρ_0 and consider the relation on S

$$\pi_\delta = \{(x, y) \in S \times S : (\rho_0(x), \rho_0(y)) \in \pi\}.$$

Clearly π_δ is a congruence on S containing ρ_0. Using definition of ρ_σ and π_δ, we can deduce that for a congruence ρ on S containing ρ_0, $\rho = (\rho_{\sigma)_\delta}$ and for a congruence π on S/ρ_0, $\pi = (\pi_{\delta)_\sigma}$. These two equalities show that there is a bijective correspondence between the set of all congruences on S containing ρ_0 and the set of all congruences on S/ρ_0. Now let ρ_1, ρ_2 be two congruences on S containing ρ_0. Then $\rho_1 \subseteq \rho_2$ if and only

if $(\rho_1)_\sigma \subseteq (\rho_2)_\sigma$. Therefore the correspondence is order preserving.
Also if ρ is a cancellative (left regular, right regular, prime) congruence on S containing ρ_0 then ρ_σ is a cancellative (respectively, left regular, right regular, prime) congruence on S/ρ_0. Similarly for a cancellative (left regular, right regular, prime) congruence π on S/ρ_0, π_δ is a cancellative (respectively, left regular, right regular, prime) congruence on S containing ρ_0.

□

The following theorem gives a necessary and sufficient condition for maximal congruences.

Theorem 2.1.28. *Let ρ_0 be a congruence on a Γ-semiring S. Then ρ_0 is a maximal congruence on S if and only if the Γ-semiring S/ρ_0 contains no congruence other than diagonal.*

Proof. The proof follows from Theorem 2.1.27. □

2.2 Bijections between the sets of congruences on a $\Gamma-$semiring and those on its operator semirings

Throughout this chapter, we denote S as a Γ-semiring with zero unless or otherwise mentioned and L and R to be the *left operator semiring* and *right operator semiring* of the Γ-semiring S.
In this section we are going to establish the bijection between the set of all congruences on a Γ-semiring and the sets of all congruences on its operator semirings.

Definitions 2.2.1. Let ρ be a relation on the left operator semiring L (right operator semiring R) of a Γ-semiring S. We define

$$\rho^+ = \{(a,b) \in S \times S : ([a,\alpha],[b,\alpha]) \in \rho \text{ for all } \alpha \in \Gamma\}$$

(respectively, $\rho^* = \{(a,b) \in S \times S : ([\alpha,a],[\alpha,b]) \in \rho \text{ for all } \alpha \in \Gamma\}$).
Let σ be a relation on a Γ-semiring S. We define

$$\sigma^{+'} = \{(\textstyle\sum_{i=1}^m [x_i,\alpha_i], \sum_{j=1}^n [y_j,\beta_j]) \in L \times L : (\sum_{i=1}^m x_i\alpha_i s, \sum_{j=1}^n y_j\beta_j s) \in \sigma \text{ for all } s \in S\}$$

(respectively, $\sigma^{*'} = \{(\sum_{i=1}^m [\alpha_i,x_i], \sum_{j=1}^n [\beta_j,y_j]) \in R \times R : (\sum_{i=1}^m s\alpha_i x_i, \sum_{j=1}^n s\beta_j y_j) \in \sigma$ for all $s \in S\}$).

Remark 2.2.2. Let σ_1, σ_2 be two relations on S. Then $\sigma_1 \subseteq \sigma_2$ implies $\sigma_1^{+'} \subseteq \sigma_2^{+'}$ (and $\sigma_1^{*'} \subseteq \sigma_2^{*'}$).

Proposition 2.2.3. *(i) If ρ is a congruence on L (on R) then ρ^+ (resp. ρ^*) is a congruence on S.*

(ii) If σ is a congruence on S then $\sigma^{+'}$ ($\sigma^{*'}$) is a congruence on L (resp. on R).

Proof. (i) Clearly ρ^+ is an equivalence relation on S as ρ is an equivalence relation on L. Let $(a,b) \in \rho^+$. Then $([a,\alpha],[b,\alpha]) \in \rho$ for all $\alpha \in \Gamma$. Since ρ is a congruence on L, $([a,\alpha]+[c,\alpha],[b,\alpha]+[c,\alpha]) \in \rho$ for all $c \in S$ and for all $\alpha \in \Gamma$. Therefore $([a+c,\alpha],[b+c,\alpha]) \in \rho$ for all $c \in S$ and for all $\alpha \in \Gamma$. Hence $(a+c,b+c) \in \rho^+$ for all $c \in S$.
Also $([a,\alpha][c,\beta],[b,\alpha][c,\beta]) \in \rho$ for all $c \in S$ and for all $\alpha,\beta \in \Gamma$. Therefore $([a\alpha c,\beta],[b\alpha c,\beta]) \in \rho$ for all $c \in S$ and for all $\alpha,\beta \in \Gamma$. Hence $(a\alpha c,b\alpha c) \in \rho^+$ for all $c \in S$ and for all $\alpha \in \Gamma$. $([c,\alpha].[a,\alpha],[c,\alpha].[b,\alpha]) \in \rho$ for all $c \in S$ and $\alpha \in \Gamma$. Therefore $([c\alpha a,\alpha],[c\alpha b,\alpha]) \in \rho$ for all $c \in S$ and $\alpha \in \Gamma$.
Similarly we can show that $(c\alpha a,c\alpha b) \in \rho^+$ for all $c \in S$ and for all $\alpha \in \Gamma$. Therefore ρ^+ is a congruence on the Γ-semiring S.

(ii) Clearly $\sigma^{+'}$ is an equivalence relation on S as σ is an equivalence relation on S. Let $(\sum_{i=1}^{m}[x_i,\alpha_i], \sum_{j=1}^{n}[y_j,\beta_j]) \in \sigma^{+'}$ and $\sum_{k=1}^{p}[z_k,\gamma_k] \in L$. Since σ is a congruence on S, then $(\sum_{i=1}^{m} x_i\alpha_i s + \sum_{k=1}^{p} z_k\gamma_k s, \sum_{j=1}^{n} y_j\beta_j s + \sum_{k=1}^{p} z_k\gamma_k s) \in \sigma$ for all $s \in S$. This implies that $(\sum_{i=1}^{m}[x_i,\alpha_i] + \sum_{k=1}^{p}[z_k,\gamma_k], \sum_{j=1}^{n}[y_j,\beta_j] + \sum_{k=1}^{p}[z_k,\gamma_k]) \in \sigma^{+'}$.
Also $(\sum_{i,k} x_i\alpha_i z_k\gamma_k s', \sum_{j,k} y_j\beta_j z_k\gamma_k s') \in \sigma$ for all $s' \in S$.
This shows that $(\sum_{i,k}[x_i\alpha_i z_k,\gamma_k], \sum_{j,k}[y_j\beta_j z_k,\gamma_k]) \in \sigma^{+'}$.
Therefore $(\sum_{i=1}^{m}[x_i,\alpha_i]\sum_{k=1}^{p}[z_k,\gamma_k], \sum_{j=1}^{n}[y_j,\beta_j]\sum_{k=1}^{p}[z_k,\gamma_k]) \in \sigma^{+'}$.
Similarly we can show that $(\sum_{k=1}^{p}[z_k,\gamma_k]\sum_{i=1}^{m}[x_i,\alpha_i], \sum_{k=1}^{p}[z_k,\gamma_k]\sum_{j=1}^{n}[y_j,\beta_j]) \in \sigma^{+'}$.
Therefore $\sigma^{+'}$ is a congruence on L. □

Theorem 2.2.4. *Let S be a Γ-semiring with left and right unities. The lattices of all congruences (partially ordered with the set inclusion) on S and its left operator semiring L are isomorphic via the mapping $\sigma \mapsto \sigma^{+'}$, where σ is a congruence on S.*

Proof. Let σ be a congruence on a Γ-semiring S with unities.
Then $(\sigma^{+'})^+ = \{(x,y) \in S \times S : ([x,\alpha],[y,\alpha]) \in \sigma^{+'} \text{ for all } \alpha \in \Gamma\}$
$= \{(x,y) \in S \times S : (x\alpha s, y\alpha s) \in \sigma \text{ for all } s \in S,\ \alpha \in \Gamma\}$. Let $(x,y) \in (\sigma^{+'})^+$. Since σ is a congruence, taking in particular $\alpha = \gamma_i$ and $s = f_i$, for $i = 1,2,...,m$, we

get, $(\sum_{i=1}^{m} x\gamma_i f_i, \sum_{i=1}^{m} y\gamma_i f_i) \in \sigma$, where $\sum_{i=1}^{m}[\gamma_i, f_i]$ is the right unity of S. Therefore $(x, y) \in \sigma$ whence $(\sigma^{+'})^{+} \subseteq \sigma$. From the definition of $(\sigma^{+'})^{+}$, we see that $(a, b) \in \sigma$ implies $(a\alpha s, b\alpha s) \in \sigma$ for all $s \in S$, $\alpha \in \Gamma$ which further implies $([a, \alpha], [b, \alpha]) \in \sigma^{+'}$ for all $\alpha \in \Gamma$ and it means $(a, b) \in (\sigma^{+'})^{+}$. So $\sigma \subseteq (\sigma^{+'})^{+}$. Therefore we get $(\sigma^{+'})^{+} = \sigma$. Hence the mapping is injective.

Now let ρ be a congruence on L. Then
$(\rho^{+})^{+'} = \{(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) : (\sum_{i=1}^{m} x_i\alpha_i s, \sum_{j=1}^{n} y_j\beta_j s) \in \rho^{+}$ for all $s \in S\}$
$= \{(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) : ([\sum_{i=1}^{m} x_i\alpha_i s, \gamma], [\sum_{j=1}^{n} y_j\beta_j s, \gamma]) \in \rho$ for all $\gamma \in \Gamma$, $s \in S\}$
$= \{(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) : (\sum_{i=1}^{m}[x_i, \alpha_i][s, \gamma], \sum_{j=1}^{n}[y_j, \beta_j][s, \gamma]) \in \rho$ for all $\gamma \in \Gamma$, $s \in S\}$.

Let $(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) \in (\rho^{+})^{+'}$. Therefore
$(\sum_{i=1}^{m}[x_i, \alpha_i][e_k, \delta_k], \sum_{j=1}^{n}[y_j, \beta_j][e_k, \delta_k]) \in \rho$, for $k = 1, 2, ..., p$ and whence
$(\sum_{i=1}^{m}[x_i, \alpha_i]\sum_{k=1}^{p}[e_k, \delta_k], \sum_{j=1}^{n}[y_j, \beta_j]\sum_{k=1}^{p}[e_k, \delta_k]) \in \rho$, where $\sum_{k=1}^{p}[e_k, \delta_k]$ is the left unity of S. Thus $(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) \in \rho$. Therefore $(\rho^{+})^{+'} \subseteq \rho$. From the definition of $(\rho^{+})^{+'}$, we can deduce that $\rho \subseteq (\rho^{+})^{+'}$. Therefore $(\rho^{+})^{+'} = \rho$. Hence the mapping is bijective.

Let σ_1, σ_2 be two relations on S. Then $\sigma_1 \subseteq \sigma_2$ implies $\sigma_1^{+'} \subseteq \sigma_2^{+'}$ (and $\sigma_1^{*'} \subseteq \sigma_2^{*'}$). To prove this, let $(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) \in \sigma_1^{+'}$. Then $(\sum_{i=1}^{m} x_i\alpha_i s, \sum_{j=1}^{n} y_j\beta_j s) \in \sigma_1 \subseteq \sigma_2$ for all $s \in S$ implies $(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) \in \sigma_2^{+'}$. Therefore $\sigma_1^{+'} \subseteq \sigma_2^{+'}$.
Similarly it can be proved that for any two relations ρ_1, ρ_2 on L (on R), $\rho_1 \subseteq \rho_2$ implies $\rho_1^{+} \subseteq \rho_2^{+}$ (respectively, $\rho_1^{*} \subseteq \rho_2^{*}$). So the mapping preserves inclusion. Since the set of all congruences on S and its left operator semiring L form lattices (*cf.* Remark 2.1.3 and Remark 1.3.14), then by Proposition 1.1.7, the mapping is a lattice isomorphism. □

Dually we can prove the following result which is the right analogue of Theorem 2.2.4.

Theorem 2.2.5. *Let S be a Γ-semiring with left and right unities. The lattices of all congruences (partially ordered with set inclusion) on S and its right operator semiring R are isomorphic via the mapping $\sigma \mapsto \sigma^{*'}$, where σ is a congruence on S.*

Combining Theorems 2.2.4 and 2.2.5 we have the following.

Theorem 2.2.6. *Let S be a Γ-semiring with unities. Then the following lattices (under the set inclusion) are isomorphic:*

(i) $\mathcal{C}(S)$*, the lattice of all congruences on the* Γ*-semiring* S*,*

(ii) $\mathcal{C}(L)$*, the lattice of all congruences on* L*,*

(iii) $\mathcal{C}(R)$*, the lattice of all congruences on* R*.*

In the rest of this section, we obtain correspondences between the set of all cancellative (regular, maximal, prime) congruences on S and those on L.

Proposition 2.2.7. *(i) If* ρ *is a cancellative congruence on* L *(on* R*) then* ρ^+ *(resp.* ρ^**) is a cancellative congruence on* S*.*

(ii) If σ *is a cancellative congruence on* S *then* $\sigma^{+'}$ *(*$\sigma^{*'}$*) is a cancellative congruence on* L *(resp. on* R*).*

Proof. (i) Let $(a+c, b+c) \in \rho^+$, where $a, b, c \in S$. Then $([a+c, \alpha], [b+c, \alpha]) \in \rho$ for all $\alpha \in \Gamma$ which implies $([a, \alpha] + [c, \alpha], [b, \alpha] + [c, \alpha]) \in \rho$ for all $\alpha \in \Gamma$. Therefore $([a, \alpha], [b, \alpha]) \in \rho$ for all $\alpha \in \Gamma$, since ρ is cancellative. So $(a, b) \in \rho^+$. Therefore ρ^+ is a cancellative congruence on the Γ-semiring S.

(ii) Let $(\sum_{i=1}^{m}[x_i, \alpha_i] + \sum_{k=1}^{p}[z_k, \gamma_k], \sum_{j=1}^{n}[y_j, \beta_j] + \sum_{k=1}^{p}[z_k, \gamma_k]) \in \sigma^{+'}$,
where $\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j], \sum_{k=1}^{p}[z_k, \gamma_k] \in L$.
Then $(\sum_{i=1}^{m} x_i\alpha_i s + \sum_{k=1}^{p} z_k\gamma_k s, \sum_{j=1}^{n} y_j\beta_j s + \sum_{k=1}^{p} z_k\gamma_k s) \in \sigma$ for all $s \in S$. Since σ is cancellative, $(\sum_{i=1}^{m} x_i\alpha_i s, \sum_{j=1}^{n} y_j\beta_j s) \in \sigma$ for all $s \in S$.
Therefore $(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) \in \sigma^{+'}$. Hence $\sigma^{+'}$ is a cancellative congruence on L. □

Theorem 2.2.8. *Let* S *be a* Γ*-semiring with unities. Then the following lattices (under the set inclusion) are isomorphic:*

(i) the lattice of all cancellative congruences on the Γ*-semiring* S*,*

(ii) the lattice of all cancellative congruences on L*,*

(iii) the lattice of all cancellative congruences on R*.*

Proof. In view of Theorems 2.2.4, 2.2.5 and Proposition 2.2.7, we obtain an inclusion preserving bijection between the set of all cancellative congruences on S and that on L (on R). Again the sets of all cancellative congruences on a semiring and on a Γ-semiring form sublattices of the lattices of all congruences on that semiring and on that Γ-semiring respectively. Hence the theorem. □

Remark 2.2.9. Let S be a Γ-semiring with left unity. Therefore left operator semiring L of the Γ-semiring S is with identity and $[0,\alpha]$ is zero of L, for any $\alpha \in \Gamma$. Then by Remark 1.3.25, every cancellative congruence on L is a left regular congruence on L. Similarly if S is a Γ-semiring with right unity then every cancellative congruence on R is a right regular congruence on R.

Proposition 2.2.10. *Let S be a Γ-semiring with strong unities. Then every congruence on S is a cancellative congruence if and only if it is a regular congruence on S.*

Proof. Let ρ be a cancellative congruence on a Γ-semiring S with strong unities. Then for any $\beta \in \Gamma$, $(a + 0\beta a, e\delta a) \in \rho$ for all $a \in S$, where $[e, \delta]$ is the strong left unity of S. Therefore ρ is a left regular congruence on S. Since S is a Γ-semiring with strong right unity, we can similarly deduce that ρ is a right regular congruence on S.
The converse part follows from the definitions of left and right regular congruences (*cf.* Definition 2.1.8). □

Proposition 2.2.11. *(i) If σ is a left regular congruence on S then $\sigma^{+'}$ is a left regular congruence on L.*

(ii) If σ is a right regular congruence on S then $\sigma^{*'}$ is a right regular congruence on R.

Proof. (i) Let σ be a left regular congruence on a Γ-semiring S. Then σ is a cancellative congruence on S, whence $\sigma^{+'}$ is cancellative congruence on L by Proposition 2.2.7. Let $\sum_{i=1}^{m}[x_i, \alpha_i] \in L$. Since σ is a left regular congruence on S, let (e_1, e_2) be a left regular unity pair of σ. Therefore there exist $\delta_1, \delta_2 \in \Gamma$ such that
$(\sum_{i=1}^{m} x_i\alpha_i s + e_1\delta_1 \sum_{i=1}^{m} x_i\alpha_i s, e_2\delta_2 \sum_{i=1}^{m} x_i\alpha_i s) \in \sigma$ for all $s \in S$.
Therefore $(\sum_{i=1}^{m}[x_i, \alpha_i] + [e_1, \delta_1] \sum_{i=1}^{m}[x_i, \alpha_i], [e_2, \delta_2] \sum_{i=1}^{m}[x_i, \alpha_i]) \in \sigma^{+'}$. Hence $\sigma^{+'}$ is a left regular congruence on L, where $([e_1, \delta_1], [e_2, \delta_2])$ is a left unity pair of $\sigma^{+'}$.
(ii) The proof can be derived similarly. □

Theorem 2.2.12. *Let S be a Γ-semiring with strong unities. Then the following lattices (under the set inclusion) are isomorphic:*

(i) the lattice of all regular congruences on S,

(ii) the lattice of all left regular congruences on L,

(iii) the lattice of all right regular congruences on R.

Proof. In view of Theorems 2.2.4, 2.2.5, Remark 2.2.9 and Propositions 2.2.10, 2.2.11, there is an inclusion preserving bijection between the set of all regular congruences on S and left regular (right regular) congruences on L (resp. on R). The rest of the proof follows from the fact that in a Γ-semiring S with strong unities, the set of all regular congruences on S and the set of all left (right) regular congruences on L (resp. on R) form lattices. □

Proposition 2.2.13. *Let S be a Γ-semiring with unities.*

(*i*) *If ρ is a maximal congruence on L (on R) then ρ^+ (resp. ρ^*) is a maximal congruence on S.*

(*ii*) *If σ is a maximal congruence on S then $\sigma^{+'}$ ($\sigma^{*'}$) is a maximal congruence on L (resp. on R).*

Proof. (*i*) Let ρ be a maximal congruence on L. Since ρ is a proper congruence on L, then ρ^+ is a proper congruence on S (*cf.* Theorem 2.2.3). Let σ be a proper congruence on S containing ρ^+, i.e., $\rho^+ \subseteq \sigma$. If $\rho^+ \neq \sigma$, then in view of the Theorem 2.2.4, $\rho = (\rho^+)^{+'} \subset \sigma^{+'}$ and this contradicts the maximality of ρ. Therefore $\rho^+ = \sigma$ which means ρ^+ is a maximal congruence on the Γ-semiring S.

(*ii*) Let σ be maximal congruence on S. Since σ is a proper congruence on S, then $\sigma^{+'}$ is a proper congruence on L. Let ρ be a proper congruence on L containing $\sigma^{+'}$, i.e., $\sigma^{+'} \subseteq \rho$. If $\sigma^{+'} \neq \rho$, then in view of the Theorem 2.2.4, $\sigma = (\sigma^{+'})^+ \subset \rho^+$. This contradicts the maximality of σ. Therefore $\sigma^{+'} = \rho$ which means $\sigma^{+'}$ is maximal congruence on L. □

In view of Theorems 2.2.4, 2.2.5 and Proposition 2.2.13, we have the following theorem.

Theorem 2.2.14. *Let S be a Γ-semiring with unities. Then the following sets are in bijective correspondence:*

(*i*) *the set of all maximal congruences on the Γ-semiring S,*

(*ii*) *the set of all maximal congruences on L,*

(*iii*) *the set of all maximal congruences on R.*

Remark 2.2.15. The set of all maximal congruences on a Γ-semiring does not form a lattice, as the intersection of two maximal congruences is not a maximal congruence on a Γ-semiring. Hence Theorem 2.2.14 can be considered to be the maximal congruence analogue of Theorem 2.2.8 and Theorem 2.2.12.

From Theorems 2.2.12, 2.2.14 we obtain the following theorem.

Theorem 2.2.16. *Let S be a Γ-semiring with strong unities. Then the following sets are in bijective correspondences:*

(i) the set of all maximal regular congruences on S,

(ii) the set of all maximal left regular congruences on L,

(iii) the set of all maximal right regular congruences on R.

Theorem 2.2.17. *Let S be a Γ-semiring with strong unities. Every left regular congruence on S is contained in a maximal left regular congruence on S.*

Proof. Let σ be a left regular congruence on S. Then $\sigma^{+'}$ is left regular congruence on L by Theorem 2.2.12. So by Theorem 1.3.26, $\sigma^{+'}$ is contained in a maximal left regular congruence ρ on L(say). Then $\sigma = (\sigma^{+'})^{+} \subset \rho^{+}$ and ρ^{+} is a maximal left regular congruence on S by Theorem 2.2.16. This completes the proof. □

Proposition 2.2.18. *(i) If S has strong right unity then ρ is a prime congruence on L implies ρ^{+} is a prime congruence on S.*

(ii) If S has strong left unity then ρ is a prime congruence on R implies ρ^{*} is a prime congruence on S.

Proof. (i) Let $[\gamma, f]$ be the strong right unity of a Γ-semiring S.
Let $(a\gamma' d + b\gamma' c, a\gamma' c + b\gamma' d) \in \rho^{+}$ for all $\gamma' \in \Gamma$, where $a, b, c, d \in S$.
Then $([a\gamma' d + b\gamma' c, \alpha], [a\gamma' c + b\gamma' d, \alpha]) \in \rho$ for all $\alpha, \gamma' \in \Gamma$
whence $([a, \gamma'][d, \alpha] + [b, \gamma'][c, \alpha], [a, \gamma'][c, \alpha] + [b, \gamma'][d, \alpha]) \in \rho$ for all $\alpha, \gamma' \in \Gamma$.
Therefore $([a, \gamma][d, \gamma] + [b, \gamma][c, \gamma], [a, \gamma][c, \gamma] + [b, \gamma][d, \gamma]) \in \rho$.
Then either $([a, \gamma], [b, \gamma]) \in \rho$ or $([c, \gamma], [d, \gamma]) \in \rho$, since ρ is a prime congruence.
This shows that

$$\text{either } ([a, \gamma][f, \delta], [b, \gamma][f, \delta]) \in \rho \text{ or } ([c, \gamma][f, \delta], [d, \gamma][f, \delta]) \in \rho \text{ for all } \delta \in \Gamma.$$

Therefore either $([a,\delta],[b,\delta]) \in \rho$ or $([c,\delta],[d,\delta]) \in \rho$ for all $\delta \in \Gamma$. Hence either $(a,b) \in \rho^+$ or $(c,d) \in \rho^+$. Therefore ρ^+ is a prime congruence on the Γ-semiring S.

(ii) The proof follows similarly. □

Proposition 2.2.19. *Suppose S be a commutative Γ-semiring. Let σ be a prime congruence on S.*

(i) *If S has strong left unity then $\sigma^{+'}$ is a prime congruence on L.*

(ii) *If S has strong right unity then $\sigma^{*'}$ is a prime congruence on R.*

Proof. (i) Let $[e,\delta]$ be the strong left unity of a commutative Γ-semiring S. Let $(\sum_{i=1}^m [x_i,\alpha_i]\sum_{l=1}^q [w_l,\delta_l] + \sum_{j=1}^n [y_j,\beta_j]\sum_{k=1}^p [z_k,\gamma_k], \sum_{i=1}^m [x_i,\alpha_i]\sum_{k=1}^p [z_k,\gamma_k] + \sum_{j=1}^n [y_j,\beta_j]\sum_{l=1}^q [w_l,\delta_l]) \in \sigma^{+'}$. Therefore $(\sum_{i=1}^m [x_i,\alpha_i]\sum_{l=1}^q [w_l,\delta_l][e,\gamma] + \sum_{j=1}^n [y_j,\beta_j]\sum_{k=1}^p [z_k,\gamma_k][e,\gamma], \sum_{i=1}^m [x_i,\alpha_i]\sum_{k=1}^p [z_k,\gamma_k][e,\gamma] + \sum_{j=1}^n [y_j,\beta_j]\sum_{l=1}^q [w_l,\delta_l][e,\gamma]) \in \sigma^{+'}$ for all $\gamma \in \Gamma$.
Since S is commutative then L is also commutative (*cf.* Theorem 1.4.5). Therefore $(\sum_{i=1}^m [x_i,\alpha_i][e,\gamma]\sum_{l=1}^q [w_l,\delta_l] + \sum_{j=1}^n [y_j,\beta_j][e,\gamma]\sum_{k=1}^p [z_k,\gamma_k],$ $\sum_{i=1}^m [x_i,\alpha_i][e,\gamma]\sum_{k=1}^p [z_k,\gamma_k] + \sum_{j=1}^n [y_j,\beta_j][e,\gamma]\sum_{l=1}^q [w_l,\delta_l]) \in \sigma^{+'}$ for all $\gamma \in \Gamma$. Then $(\sum_{i,l}[x_i\alpha_i e\gamma w_l,\delta_l] + \sum_{j,k}[y_j\beta_j e\gamma z_k,\gamma_k], \sum_{i,k}[x_i\alpha_i e\gamma z_k,\gamma_k] + \sum_{j,l}[y_j\beta_j e\gamma w_l,\delta_l] \in \sigma^{+'}$ for all $\gamma \in \Gamma$. This shows that $(\sum_{i,l} x_i\alpha_i e\gamma w_l\delta_l s + \sum_{j,k} y_j\beta_j e\gamma z_k\gamma_k s, \sum_{i,k} x_i\alpha_i e\gamma z_k\gamma_k s + \sum_{j,l} y_j\beta_j e\gamma w_l\delta_l s) \in \sigma$ for all $s \in S$, for all $\gamma \in \Gamma$. Putting $s = e$, in particular, we get, $(\sum_{i,l}(x_i\alpha_i e)\gamma(w_l\delta_l e) + \sum_{j,k}(y_j\beta_j e)\gamma(z_k\gamma_k e), \sum_{i,k}(x_i\alpha_i e)\gamma(z_k\gamma_k e) + \sum_{j,l}(y_j\beta_j e)\gamma(w_l\delta_l e)) \in \sigma$ for all $\gamma \in \Gamma$. Since σ is a prime congruence on S then

$$\text{either } (\textstyle\sum_{i=1}^m x_i\alpha_i e, \sum_{j=1}^n y_j\beta_j e) \in \sigma \text{ or } (\sum_{k=1}^p z_k\gamma_k e, \sum_{l=1}^q w_l\delta_l e) \in \sigma.$$

This implies that

$$(\textstyle\sum_{i=1}^m x_i\alpha_i e\delta s', \sum_{j=1}^n y_j\beta_j e\delta s') \in \sigma \text{ or } (\sum_{k=1}^p z_k\gamma_k e\delta s', \sum_{l=1}^q w_l\delta_l e\delta s') \in \sigma \text{ for all } s' \in S.$$

Then $(\sum_{i=1}^m x_i\alpha_i s', \sum_{j=1}^n y_j\beta_j s') \in \sigma$ or $(\sum_{k=1}^p z_k\gamma_k s', \sum_{l=1}^q w_l\delta_l s') \in \sigma$ for all $s' \in S$. Therefore $(\sum_{i=1}^m [x_i,\alpha_i], \sum_{j=1}^n [y_j,\beta_j]) \in \sigma^{+'}$ or $(\sum_{k=1}^p [z_k,\gamma_k], \sum_{l=1}^q [w_l,\delta_l]) \in \sigma^{+'}$. Hence $\sigma^{+'}$ is a prime congruence on L.

(ii) The proof follows similarly. □

In view of Theorems 2.2.4, 2.2.5 and Propositions 2.2.18 and 2.2.19, we have the following theorem for prime congruences which is the counterpart of Theorem 2.2.14.

Theorem 2.2.20. *Let S be a commutative Γ-semiring with strong unities. Then the following sets are in inclusion preserving bijective correspondence:*

(i) the set of all prime congruences on the Γ-semiring S,

(ii) the set of all prime congruences on L,

(iii) the set of all prime congruences on R.

In the following, we observe that the left operator semirings of two isomorphic Γ-semirings are isomorphic and dually we can obtain the same result for right operator semiring.

Theorem 2.2.21. *If two Γ-semirings are isomorphic then their corresponding left operator semirings are isomorphic.*

Proof. Let S_1 be a Γ_1-semiring and S_2 be a Γ_2-semiring which are isomorphic (*cf.* Definition 2.2.21) and L_1 and L_2 be the corresponding left operator semirings respectively. Let (θ, ϕ) be a Γ-semiring isomorphism of S_1 onto S_2. Let $\psi : L_1 \to L_2$ be defined by $\psi(\sum_{i=1}^{m}[x_i, \alpha_i]) := \sum_{i=1}^{m}[\theta(x_i), \phi(\alpha_i)]$. Then $\sum_{i=1}^{m}[x_i, \alpha_i] = \sum_{j=1}^{n}[y_j, \beta_j]$ implies $\sum_{i=1}^{m} x_i\alpha_i s = \sum_{j=1}^{n} y_j\beta_j s$ for all $s \in S_1$ whence it follows that $\theta(\sum_{i=1}^{m} x_i\alpha_i s) = \theta(\sum_{j=1}^{n} y_j\beta_j s)$ for all $s \in S_1$. Therefore $\sum_{i=1}^{m} \theta(x_i\alpha_i s) = \sum_{j=1}^{n} \theta(y_j\beta_j s)$ for all $s \in S_1$ from which it follows that $\sum_{i=1}^{m} \theta(x_i) \cdot \phi(\alpha_i) \cdot \theta(s) = \sum_{j=1}^{n} \theta(y_j) \cdot \phi(\beta_j) \cdot \theta(s)$ for all $s \in S_1$. This implies for all $s' \in S_2$,

$$\sum_{i=1}^{m} \theta(x_i) \cdot \phi(\alpha_i) \cdot s' = \sum_{j=1}^{n} \theta(y_j) \cdot \phi(\beta_j) \cdot s',$$

since θ in onto and $\theta(s) = s'$. So $\sum_{i=1}^{m}[\theta(x_i), \phi(\alpha_i)] = \sum_{j=1}^{n}[\theta(y_j), \phi(\beta_j)]$ which implies that $\psi(\sum_{i=1}^{m}[x_i, \alpha_i]) = \psi(\sum_{j=1}^{n}[y_j, \beta_j])$. Therefore ψ is well defined and one-one and since θ and ϕ are onto, by the definition of mapping ψ is onto.
Now $\psi(\sum_{i=1}^{m}[x_i, \alpha_i]) + \psi(\sum_{j=1}^{n}[y_j, \beta_j]) = \psi(\sum_{i=1}^{m}[x_i, \alpha_i] + \sum_{j=1}^{n}[y_j, \beta_j])$.
Also $\psi(\sum_{i=1}^{m}[x_i, \alpha_i] \cdot \sum_{j=1}^{n}[y_j, \beta_j]) = \psi(\sum_{i,j}[x_i\alpha_i y_j, \beta_j]) = \sum_{i,j}[\theta(x_i\alpha_i y_j), \phi(\beta_j)]$
$= \sum_{i,j}[\theta(x_i) \cdot \phi(\alpha_i) \cdot \theta(y_j), \phi(\beta_j)] = \sum_{i=1}^{m}[\theta(x_i), \phi(\alpha_i)] \cdot \sum_{j=1}^{n}[\theta(y_j), \phi(\beta_j)]$
$= \psi(\sum_{i=1}^{m}[x_i, \alpha_i]) \cdot \psi(\sum_{j=1}^{n}[y_j, \beta_j])$. Therefore ψ is a semiring isomorphism. □

By virtue of the above theorem we have the following.

Corollary 2.2.22. *Let S_1 and S_2 be Γ_1-semiring and Γ_2-semiring respectively and let the two Γ-semirings be isomorphic. If S_1 is proper nontrivial congruence free Γ_1-semiring then so is Γ_2-semiring S_2.*

Proof. Let L_1 and L_2 be the left operator semirings of (S_1, Γ_1) and (S_2, Γ_2) respectively. Let S_1 be a proper nontrivial congruence free Γ_1-semiring. Then its left operator semiring L_1 contains no proper nontrivial congruence by Theorem 2.2.4. Since (S_1, Γ_1) and (S_2, Γ_2) are isomorphic, L_1 and L_2 are isomorphic semirings by Theorem 2.2.21. Therefore by Theorem 1.3.38, L_2 contains no proper nontrivial congruence whence it follows that S_2 is a proper nontrivial congruence free Γ_2-semiring by Theorem 2.2.4. □

Corollary 2.2.23. *Let S_1 and S_2 be Γ_1-semiring and Γ_2-semiring respectively and let the two Γ-semirings be isomorphic. Then their lattices of congruences are isomorphic.*

Proof. Let L_1 and L_2 be the left operator semirings of (S_1, Γ_1) and (S_2, Γ_2) respectively. Then by Theorem 2.2.21, L_1 and L_2 are isomorphic. So by Theorem 1.3.38, lattices of congruences on L_1 and L_2 are isomorphic. Again from Theorem 2.2.4 it follows that there exist lattice isomorphisms between the lattices of congruences on S_1, L_1 and S_2, L_2. Therefore we obtain that lattices of congruences on S_1 and S_2 are isomorphic. □

2.3 Some results on quotient $\Gamma-$semiring

In this section we establish a connection between the quotient left operator semiring and the left operator semiring of a quotient Γ-semiring.

Theorem 2.3.1. *Let ρ be a congruence on S. Then $L_{S/\rho}$ and $L/\rho^{+'}$ are isomorphic via the mapping*

$$\sum_{i=1}^{m}[\rho(x_i), \alpha_i] \mapsto \rho^{+'}(\sum_{i=1}^{m}[x_i, \alpha_i]),$$

where $L_{S/\rho}$ is the left operator semiring of the Γ-semiring S/ρ.

Proof. We define a mapping, $\phi : L_{S/\rho} \to L/\rho^{+'}$ by $\phi(\sum_{i=1}^{m}[\rho(x_i), \alpha_i]) := \rho^{+'}(\sum_{i=1}^{m}[x_i, \alpha_i])$. Let $\sum_{i=1}^{m}[\rho(x_i), \alpha_i] = \sum_{j=1}^{n}[\rho(y_j), \beta_j]$ in $L_{S/\rho}$. Then $\sum_{i=1}^{m}\rho(x_i)\cdot\alpha_i\cdot\rho(s) = \sum_{j=1}^{n}\rho(y_j)\cdot\beta_j\cdot\rho(s)$ for all $\rho(s) \in S/\rho$. Therefore $\sum_{i=1}^{m}\rho(x_i\alpha_i s) = \sum_{j=1}^{n}\rho(y_j\beta_j s)$ for all $s \in S$. Then $(\sum_{i=1}^{m}x_i\alpha_i s, \sum_{j=1}^{n}y_j\beta_j s) \in \rho$ for all $s \in S$
whence $(\sum_{i=1}^{m}[x_i, \alpha_i], \sum_{j=1}^{n}[y_j, \beta_j]) \in \rho^{+'}$. Therefore $\rho^{+'}(\sum_{i=1}^{m}[x_i, \alpha_i]) = \rho^{+'}(\sum_{j=1}^{n}[y_j, \beta_j])$. Hence $\phi(\sum_{i=1}^{m}[\rho(x_i), \alpha_i]) = \phi(\sum_{j=1}^{n}[\rho(y_j), \beta_j])$ in $L/\rho^{+'}$. Therefore ϕ is well defined. By the reverse implication it follows that ϕ is injective. Also by the definition of the mapping it is surjective. It remains to show that ϕ is a semiring homomorphism.
Let $\sum_{i=1}^{m}[\rho(x_i), \alpha_i] + \sum_{j=1}^{n}[\rho(y_j), \beta_j] = \sum_{k=1}^{p}[\rho(z_k), \gamma_k]$.
Then $\sum_{i=1}^{m}\rho(x_i).\alpha_i.\rho(s) + \sum_{j=1}^{n}\rho(y_j).\beta_j.\rho(s) = \sum_{k=1}^{p}\rho(z_k).\gamma_k.\rho(s)$ for all $s \in S$
which implies $\sum_{i=1}^{m}\rho(x_i\alpha_i s) + \sum_{j=1}^{n}\rho(y_j\beta_j s) = \sum_{k=1}^{p}\rho(z_k\gamma_k s)$ for all $s \in S$. Then

$$\rho(\sum_{i=1}^{m} x_i\alpha_i s + \sum_{j=1}^{n} y_j\beta_j s) = \rho(\sum_{k=1}^{p} z_k\gamma_k s) \text{ for all } s \in S$$

which implies $(\sum_{i=1}^{m} x_i\alpha_i s + \sum_{j=1}^{n} y_j\beta_j s, \sum_{k=1}^{p} z_k\gamma_k s) \in \rho$ for all $s \in S$ whence it follows that $(\sum_{i=1}^{m}[x_i, \alpha_i] + \sum_{j=1}^{n}[y_j, \beta_j]), \sum_{k=1}^{p}[z_k, \gamma_k]) \in \rho^{+'}$.
Therefore $\rho^{+'}(\sum_{i=1}^{m}[x_i, \alpha_i] + \sum_{j=1}^{n}[y_j, \beta_j]) = \rho^{+'}(\sum_{k=1}^{p}[z_k, \gamma_k])$ which means $\rho^{+'}(\sum_{i=1}^{m}[x_i, \alpha_i]) + \rho^{+'}(\sum_{j=1}^{n}[y_j, \beta_j]) = \rho^{+'}(\sum_{k=1}^{p}[z_k, \gamma_k])$.
Hence $\phi(\sum_{i=1}^{m}[\rho(x_i), \alpha_i]) + \phi(\sum_{j=1}^{n}[\rho(y_j), \beta_j]) = \phi(\sum_{i=1}^{m}[\rho(x_i), \alpha_i] + \sum_{j=1}^{n}[\rho(y_j), \beta_j])$.
Also $\phi(\sum_{i=1}^{m}[\rho(x_i), \alpha_i]. \sum_{j=1}^{n}[\rho(y_j), \beta_j]) = \phi(\sum_{i,j}[\rho(x_i)\alpha_i\rho(y_j), \beta_j])$
$= \rho^{+'}(\sum_{i,j}[x_i\alpha_i y_j, \beta_j]) = \rho^{+'}(\sum_{i=1}^{m}[x_i, \alpha_i]).\rho^{+'}(\sum_{j=1}^{n}[y_j, \beta_j])$
$= \phi(\sum_{i=1}^{m}[\rho(x_i), \alpha_i]).\phi(\sum_{j=1}^{n}[\rho(y_j), \beta_j])$.
Clearly ϕ is a semiring homomorphism. Therefore ϕ is a semiring isomorphism, i.e., $L_{S/\rho}$ and $L/\rho^{+'}$ are isomorphic. □

Dually we obtain the following result which is the right analogue of Theorem 2.3.1.

Theorem 2.3.2. *Let ρ be a congruence on S. Then $R_{S/\rho}$ and $R/\rho^{*'}$ are isomorphic via the mapping*

$$\sum_{i=1}^{m}[\alpha_i, \rho(x_i)] \mapsto \rho^{*'}(\sum_{i=1}^{m}[\alpha_i, x_i]),$$

where $R_{S/\rho}$ is the right operator semiring of the Γ-semiring S/ρ.

Below we prove a criteria for a Γ-semiring to be a Γ-semi-integral domain. For this, we first prove the following lemma.

Lemma 2.3.3. *Let S be a Γ-semiring with strong right unity. Then S is weak zero divisor free if its left operator semiring L contains no divisor of zero.*

Proof. Let $a\Gamma' b = \{0\}$, where $a, b \in S$, $\Gamma' = \Gamma \setminus \{0\}$. Then $a\alpha b = 0$ for any nonzero $\alpha \in \Gamma$. Therefore $[a\gamma b, \gamma] = [0, \gamma]$ in L, where $[\gamma, f]$ is the strong right unity of S. Since L contains no divisor of zero, then either $[a, \gamma] = [0, \gamma]$ or $[b, \gamma] = [0, \gamma]$. Therefore either $a = 0$ or $b = 0$. Hence in view of Definition 1.4.15, S is weak zero divisor free. □

Using the above lemma we now prove the following.

Theorem 2.3.4. *Let S be a commutative Γ-semiring with strong unities. If ρ is a prime congruence on S then S/ρ is a Γ-semi-integral domain.*

Proof. Let ρ be a prime congruence on a Γ-semiring S with strong unities. Then $\rho^{+'}$ is a prime congruence on the left operator semiring L by Proposition 2.2.19. It implies $L/\rho^{+'}$ has no divisor of zero (*cf.* Theorem 1.3.35). Therefore Theorem 2.3.1 together with Lemma 2.3.3 implies that S/ρ is weak zero divisor free. Hence, in view of Definition 1.4.15, S/ρ is a Γ-semi-integral domain. □

As a maximal congruence analogue of the above result we obtain the following. Before going to the proof we recall that a commutative Γ-semiring S with unities is a Γ-semifield if and only if it is ZDF and has no nonzero proper ideals.

Theorem 2.3.5. *Let S be a commutative Γ-semiring with unities where S contains more than two elements and ρ be a maximal congruence such that S/ρ is a ZDF Γ-semiring. Then S/ρ is a Γ-semifield.*

Proof. S/ρ contains no nontrivial proper congruences, since ρ is maximal congruence on S (*cf.* Theorem 2.1.28). Then by Theorem 2.2.4, $L_{S/\rho}$ contains no nontrivial proper congruences. This implies that $L_{S/\rho}$ is a field (*cf.* Proposition 1.3.15). Hence $L_{S/\rho}$ has no nontrivial proper ideals. Therefore using 1.4.19 it follows that S/ρ has no nontrivial proper ideals. Hence it follows that S/ρ is a Γ-semifield (*cf.* Theorem 1.4.25). □

Though we could not obtain the converse of Theorem 2.3.4, we could obtain a sort of converse of Theorem 2.3.5 viz., Theorem 2.3.8 for which we need the following lemma. First we note the following definition.

Definition 2.3.6. A Γ-semiring S is said to be *zero-sum free* if for $a + b = 0$, where $a, b \in S$, $a = 0$ and $b = 0$.

Lemma 2.3.7. *If a Γ-semiring S with unities is not zero-sum free then its left operator semiring L is not zero-sum free.*

Proof. Let L be zero-sum free and $a+b = 0$, where $a, b \in S$. Then $[a, \alpha]+[b, \alpha] = [0, \alpha]$ for all $\alpha \in \Gamma$ which implies $[a, \alpha] = [0, \alpha]$ and $[b, \alpha] = [0, \alpha]$ for all $\alpha \in \Gamma$. Therefore we obtain that $a = 0$ and $b = 0$. Hence Γ-semiring S is zero-sum free and this completes the proof. □

Theorem 2.3.8. *Let S be a commutative Γ-semiring with unities and ρ be a congruence such that S/ρ is a Γ-semifield which is not zero-sum free. Then ρ is a maximal congruence on S.*

Proof. By Theorem 1.4.26, if a commutative Γ-semiring S is a Γ-semifield then the left operator semiring L of S is a semifield. So $L_{S/\rho}$ is a semifield, since S/ρ is a Γ-semifield.

Now by Lemma 2.3.7, if S/ρ is not zero-sum free then so is also $L_{S/\rho}$. Therefore it follows that $L_{S/\rho}$ is a field, by Proposition 1.3.12. So $L_{S/\rho}$ contains no nontrivial proper congruence whence it follows that S/ρ contains no nontrivial proper congruence (by 2.2.4). So applying Theorem 2.1.28 we get that ρ is a maximal congruence on S. □

CHAPTER 3

STRUCTURE SPACES OF SEMIRINGS AND Γ-SEMIRINGS

3

Structure spaces of semirings and $\Gamma-$semirings

In this chapter we have defined the structure space of prime congruences on any semiring without any restrictions of assumptions like unity or commutativity. We study the topological properties of that space which is certainly a larger space than the space considered by Acharyya et al. and found a base and characterized the closed sets of the space. Then we investigate for the topological properties viz. separation axioms, compactness, connectedness, irreducibility etc. of that space and have found some necessary and / or sufficient conditions for T_1, T_2, compactness, connectedness etc. Also we identify the structure space of the semiring Z_0^+ of all non-negative integers and found that it is T_1, compact, connected but neither T_2 nor regular.

In previous chapter we introduced the notions of various types of congruences viz. cancellative congruence, regular congruence, maximal congruence, prime congruence on a Γ-semiring and established the bijections between the set of all congruences, cancellative congruences, regular congruences, maximal congruences, prime congruences on a Γ-semiring and those on its operator semirings respectively. Motivated by the studies on structure spaces based on various types of congruences, we have constructed the structure space of a commutative Γ-semiring with strong unities considering the

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This chapter is based on the work published in the following papers:

(i) **Soumi Basu, Sarbani Mukherjee (Goswami) and Kamalika Chakraborty,** ***On the structure space of prime congruences on semirings, Discussiones Mathematicae - General Algebra and Applications,*** **43 (2) (2023) 389-401.**

(ii) **Soumi Basu, Sarbani Mukherjee (Goswami) and Sujit Kumar Sardar,** ***The structure space of*** $C_{-}(X)$ ***via that of*** Γ***-semirings, Asian-European Journal of Mathematics,*** **16 (8) (2023), 2350142 (20 pages).**

space as the set of all prime congruences endowed with Hull Kernel topology. In order to accomplish our study we follow the approach of studying a Γ-semiring via its operator semirings as was done in [55] while studying the topological structure space of all prime k-ideals of a Γ-semiring. Here also using the results established in the setting of semiring, we have studied several principal topological axioms and properties (such as seperation axioms, compactness, connectedness etc.) of the structure space of all prime congruences on a Γ-semiring via its operator semiring and also studied those on the space of all maximal regular congruences those are prime, as a subspace of that space. We have also studied, via operator semirings, the topological properties of the structure space of the Γ-semiring Z_0^- of all non-positive integers (where $\Gamma = Z_0^-$).

In **section 1**, we study the topological properties of the space of prime congruences on a semiring endowed with the Hull Kernel topology. Various properties viz. separation axioms, compactness and connectedness etc. of the space of prime congruences on a semiring are studied (*cf.* Theorems 3.1.10, 3.1.18, 3.1.20, 3.1.22). Also the topological properties of the structure space of the semiring Z_0^+ of all non-negative integers are studied (*cf.* Theorem 3.1.25, Example 3.1.26). Additionally it is shown that the structure spaces of two isomorphic semirings are homeomorphic (*cf.* Theorem 3.1.24).

In **section 2**, the structure space, endowed with the Hull Kernel topology, of prime congruences on a Γ-semiring is studied via its operator semiring (for instance, *cf.* Theorems 3.2.20, 3.2.21, 3.2.22, 3.2.24, 3.2.27, 3.2.28). For this, it is established that the structure space of a Γ-semiring and its left operator semiring are homeomorphic (*cf.* Theorem 3.2.8). Also the structure space of the Γ-semiring Z_0^- of all non-positive integers is characterized (*cf.* Corollary 3.2.36). In this connection, the space of maximal regular congruences which are prime on a Γ-semiring is studied as a subspace of the above mentioned space. In addition it is proved that if two Γ-semirings are isomorphic then their corresponding structure spaces are homeomorphic (*cf.* Theorem 3.2.47).

3.1 Structure space of prime congruences on a semiring

In this section we define the structure space of prime congruences on a semiring.

Definition 3.1.1. Let R be a semiring and $\mathcal{A}_R$ be the collection of all prime congruences on R. For any subset A of $\mathcal{A}_R$, we define

$$\overline{A} = \{\rho \in \mathcal{A}_R : \bigcap_{\rho_i \in A} \rho_i \subseteq \rho\}.$$

Evidently, $\overline{\emptyset} = \emptyset$.

The following Lemma 3.1.2 and Theorem 3.1.3 can be considered as counterparts of Lemma 4.1 and Theorem 4.1 of [75], in the setting of prime congruences on a semiring R.

Lemma 3.1.2. *Let ρ be a prime congruence on R and ρ_1, ρ_2 be two congruences on R. Then*

$$\rho_1 \cap \rho_2 \subseteq \rho \text{ implies either } \rho_1 \subseteq \rho \text{ or } \rho_2 \subseteq \rho.$$

Proof. Let ρ be a prime congruence on R and $\rho_1 \cap \rho_2 \subseteq \rho$ and $\rho_2 \nsubseteq \rho$. Then there exists $(c, d) \in R \times R$ such that $(c, d) \in \rho_2 \setminus \rho$. Now let $(a, b) \in \rho_1$. Then $(ad + bc, ac + bd) \in \rho_1$. Also $(ad + bc, ac + bd) \in \rho_2$. Therefore

$$(ad + bc, ac + bd) \in \rho_1 \cap \rho_2 \subseteq \rho \text{ implies } (a, b) \in \rho$$

Hence $\rho_1 \subseteq \rho$. □

Theorem 3.1.3. *Let R be a semiring and the set of all prime congruences on R be $\mathcal{A}_R$. Then the mapping from $A \mapsto \overline{A}$ is a Kuratowski closure operator on $\mathcal{A}_R$.*

Proof. Let $A, B \subseteq \mathcal{A}_R$.
(i) $\bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho_\alpha$ for each α and hence $A \subseteq \overline{A}$.

(ii) From (i) it is clear that $\overline{A} \subseteq \overline{\overline{A}}$. Let $\rho_\beta \in \overline{\overline{A}}$. Then $\bigcap_{\rho_\alpha \in \overline{A}} \rho_\alpha \subseteq \rho_\beta$. Again $\bigcap_{\rho_\gamma \in A} \rho_\gamma \subseteq \rho_\alpha$ for all α. Then

$$\bigcap_{\rho_\gamma \in A} \rho_\gamma \subseteq \bigcap_{\rho_\alpha \in \overline{A}} \rho_\alpha \subseteq \rho_\beta \text{ implies } \rho_\beta \in \overline{A}.$$

Thus $\overline{\overline{A}} \subseteq \overline{A}$. Therefore $\overline{\overline{A}} = \overline{A}$.

(iii) Let us suppose that $A \subseteq B$. Let $\rho_\beta \in \overline{A}$. Then $\bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho_\beta$. Since $A \subseteq B$, it follows that $\bigcap_{\rho_\alpha \in B} \rho_\alpha \subseteq \bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho_\beta$. This implies that $\rho_\beta \in \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$.

(iv) Clearly $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Now let $\rho_\beta \in \overline{A \cup B}$. Then $\bigcap_{\rho_\alpha \in A \cup B} \rho_\alpha \subseteq \rho_\beta$. It can be easily seen that

$$\bigcap_{\rho_\alpha \in A \cup B} \rho_\alpha = (\bigcap_{\rho_\alpha \in A} \rho_\alpha) \bigcap (\bigcap_{\rho_\alpha \in B} \rho_\alpha).$$

Since $(\bigcap_{\rho_\alpha \in A} \rho_\alpha)$ and $(\bigcap_{\rho_\alpha \in B} \rho_\alpha)$ are congruences on the semiring R and ρ_β is a prime congruence on R then by Lemma 3.1.2,

$$(\bigcap_{\rho_\alpha \in A} \rho_\alpha) \bigcap (\bigcap_{\rho_\alpha \in B} \rho_\alpha) \subseteq \rho_\beta \text{ implies either } \bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho_\beta \text{ or } \bigcap_{\rho_\alpha \in B} \rho_\alpha \subseteq \rho_\beta.$$

Hence $\rho_\beta \in \overline{A} \cup \overline{B}$. Consequently, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Therefore $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

□

Definition 3.1.4. The topology τ_R induced by the Kuratowski closure operator on $\mathcal{A}_R$ is known as Hull Kernel topology. We consider this topological space to be the *structure space* of the semiring R.

Throughout this chapter R is a semiring and the space of all prime congruences on R with the Hull Kernel topology is denoted as $\mathcal{A}_R$.

Notations: Let ρ be a congruence on a semiring R and $(x, y) \in R \times R$. We define,

$$\Delta(x, y) = \{\rho \in \mathcal{A}_R : (x, y) \in \rho\};\ C\Delta(x, y) = \{\rho \in \mathcal{A}_R : (x, y) \notin \rho\}$$

$$\Delta(\rho) = \{\rho' \in \mathcal{A}_R : \rho \subseteq \rho'\};\ C\Delta(\rho) = \{\rho' \in \mathcal{A}_R : \rho \nsubseteq \rho'\}.$$

Below we obtain forms of the closed sets, a closed base of the space.

Proposition 3.1.5. *Any closed set in $\mathcal{A}_R$ is of the form $\Delta(\rho)$, where ρ is a congruence on R.*

Proof. Let $\overline{A}$ be any closed set in $\mathcal{A}_R$, where $A \subseteq \mathcal{A}_R$. Let $A = \{\rho_\alpha : \alpha \in \Lambda\}$ and $\rho = \bigcap_{\rho_\alpha \in A} \rho_\alpha$. Then ρ is a congruence on R. Let $\rho' \in \overline{A}$. Then $\bigcap_{\rho_\alpha \in A} \rho_\alpha \subseteq \rho'$, i.e., $\rho \subseteq \rho'$. Consequently, $\rho' \in \Delta(\rho)$. So $\overline{A} \subseteq \Delta(\rho)$. By the reverse implication, we obtain that $\Delta(\rho) \subseteq \overline{A}$. Thus $\overline{A} = \Delta(\rho)$. □

Corollary 3.1.6. $\Delta(\rho) = \bigcap\{\Delta(x, y) : (x, y) \in \rho\}$, *where ρ is a congruence on R.*

Proof. Let $\rho' \in \Delta(\rho)$. Then $\rho \subseteq \rho'$ implies that $\rho' \in \Delta(x, y)$ for all $(x, y) \in \rho$. Therefore $\Delta(\rho) \subseteq \bigcap_{(x,y)\in\rho} \Delta(x, y)$. Conversely, let $\rho_1 \in \bigcap\{\Delta(x, y) : (x, y) \in \rho\}$. Then $(x, y) \in \rho_1$ for all $(x, y) \in \rho$ which implies $\rho \subseteq \rho_1$, i.e., $\rho_1 \in \Delta(\rho)$. Therefore $\Delta(\rho) = \bigcap\{\Delta(x, y) : (x, y) \in \rho\}$. □

Proposition 3.1.7. $\{C\Delta(a, b) : (a, b) \in R \times R\}$ *is an open base for $\mathcal{A}_R$.*

Proof. Let U be an open set in $\mathcal{A}_R$. Then $A = \mathcal{A}_R \setminus U$ is a closed set in $\mathcal{A}_R$. By the Proposition 3.1.5, $A = \Delta(\rho)$ for some congruence ρ on R. Then $\sigma \in U$ implies $\sigma \notin A$, i.e., $\rho \nsubseteq \sigma$. Then there exists $(a, b) \in \rho$ such that $(a, b) \notin \sigma$. Hence $\sigma \in C\Delta(a, b)$. Now let $\sigma' \in C\Delta(a, b)$. Then $(a, b) \notin \sigma'$. This implies that $\rho \nsubseteq \sigma'$ whence it follows that $\sigma' \notin \Delta(\rho) = A$ which implies $\sigma' \in \mathcal{A}_R \setminus A = U$. Hence $C\Delta(a, b) \subseteq U$. Consequently, $\sigma \in C\Delta(a, b) \subseteq U$. Thus $\{C\Delta(a, b) : (a, b) \in R \times R\}$ is an open base for $\mathcal{A}_R$. □

Now we will study the separation properties of the space $\mathcal{A}_R$. Before going to the main result, we prove the following lemma.

Lemma 3.1.8. *For $a, b, c, d \in R$,*

$$\Delta(a, b) \cup \Delta(c, d) = \Delta(ac + bd, ad + bc).$$

Proof. Let $\rho \in \Delta(a, b) \cup \Delta(c, d)$. Then either $\rho \in \Delta(a, b)$ or $\rho \in \Delta(c, d)$. If $\rho \in \Delta(a, b)$ then $(a, b) \in \rho$. Hence $(ac + bd, ad + bc) \in \rho$ which implies $\rho \in \Delta(ac + bd, ad + bc)$. Therefore

$$\Delta(a, b) \cup \Delta(c, d) \subseteq \Delta(ac + bd, ad + bc).$$

Again let $\sigma \in \Delta(ac + bd, ad + bc)$. Then

$$(ac + bd, ad + bc) \in \sigma \text{ implies that either } (a, b) \in \sigma \text{ or } (c, d) \in \sigma$$

as σ is a prime congruence. Therefore $\sigma \in \Delta(a, b) \cup \Delta(c, d)$ from which it follows that

$$\Delta(ac + bd, ad + bc) \subseteq \Delta(a, b) \cup \Delta(c, d).$$

Hence for $a, b, c, d \in R$,

$$\Delta(a, b) \cup \Delta(c, d) = \Delta(ac + bd, ad + bc).$$

□

From above result we have the following.

Corollary 3.1.9. *For $a, b, c, d \in R$,*

$$C\Delta(a, b) \cap C\Delta(c, d) = C\Delta(ac + bd, ad + bc).$$

In the following theorem we proof the separation properties.

Theorem 3.1.10. *(i) The space $\mathcal{A}_R$ is T_0.*

(ii) The space $\mathcal{A}_R$ is T_1 if and only if no element of $\mathcal{A}_R$ is contained in any other element of $\mathcal{A}_R$.

(iii) $\mathcal{A}_R$ is T_2 if and only if for any two distinct elements ρ_1, ρ_2 of $\mathcal{A}_R$, there exists two pairs $(a, b), (c, d)$ of elements of $R \times R$ such that

$$(a, b) \notin \rho_1,\ (c, d) \notin \rho_2 \text{ and } (ac + bd, ad + bc) \in \rho \text{ for all } \rho \in \mathcal{A}_R.$$

(*iv*) *The space* $\mathcal{A}_R$ *is a regular space if and only if for any* $\rho \in \mathcal{A}_R$ *and* $(a, b) \notin \rho$, *there exists a congruence* σ *on* R *and* $(c, d) \in R \times R$ *such that*

$$\rho \in C\Delta(c, d) \subseteq \Delta(\sigma) \subseteq C\Delta(a, b).$$

Proof. (*i*) Let ρ_1 and ρ_2 be two distinct elements of $\mathcal{A}_R$. Then either $\rho_1 \setminus \rho_2 \neq \emptyset$ or $\rho_2 \setminus \rho_1 \neq \emptyset$. Let us suppose that $\rho_1 \setminus \rho_2 \neq \emptyset$ and $(a, b) \in \rho_1 \setminus \rho_2$. Then $C\Delta(a, b)$ is a neighbourhood of ρ_2 not containing ρ_1. Hence the space $\mathcal{A}_R$ is T_0.

(*ii*) Let the space $\mathcal{A}_R$ be T_1. Let us suppose that ρ_1 and ρ_2 be any two distinct elements of $\mathcal{A}_R$. Then each of ρ_1 and ρ_2 has a neighbourhood not containing the other. Since ρ_1 and ρ_2 are arbitrary elements of $\mathcal{A}_R$, it follows that no element of $\mathcal{A}_R$ is contained in any other element of $\mathcal{A}_R$.
Conversely, let us suppose that no element of $\mathcal{A}_R$ is contained in any other element of $\mathcal{A}_R$. Let ρ_1 and ρ_2 be any two distinct elements of $\mathcal{A}_R$. Then by hypothesis, $\rho_1 \nsubseteq \rho_2$ and $\rho_2 \nsubseteq \rho_1$. This implies that there exist $(a, b), (c, d) \in R \times R$ such that $(a, b) \in \rho_1$ but $(a, b) \notin \rho_2$ and $(c, d) \in \rho_2$ but $(c, d) \notin \rho_1$. Consequently, we have $\rho_1 \in C\Delta(c, d)$ but $\rho_1 \notin C\Delta(a, b)$ and $\rho_2 \in C\Delta(a, b)$ but $\rho_2 \notin C\Delta(c, d)$, i.e., each of ρ_1 and ρ_2 has a neighbourhood not containing the other. Hence the space $\mathcal{A}_R$ is T_1.

(*iii*) Let the space $\mathcal{A}_R$ be T_2. Then for any two distinct congruences ρ_1, ρ_2 of $\mathcal{A}_R$, there exist two open sets $C\Delta(a, b)$ and $C\Delta(c, d)$ such that $\rho_1 \in C\Delta(a, b)$ and $\rho_2 \in C\Delta(c, d)$ and $C\Delta(a, b) \cap C\Delta(c, d) = \emptyset$.
Therefore $(a, b) \notin \rho_1$ and $(c, d) \notin \rho_2$. Let if possible there exists ρ in $\mathcal{A}_R$ such that $(ac + bd, ad + bc) \notin \rho$. That means, by Corollary 3.1.9,

$$\rho \in C\Delta(ac + bd, ad + bc) = C\Delta(a, b) \cap C\Delta(c, d) = \emptyset,$$

which is a contradiction. Hence the given condition holds in $\mathcal{A}_R$. Conversely, let the condition hold. Let ρ_1, ρ_2 be two distinct elements of $\mathcal{A}_R$. Then there exists two pairs $(a, b), (c, d)$ of elements of $R \times R$ such that

$$(a, b) \notin \rho_1 \text{ and } (c, d) \notin \rho_2 \text{ and } (ac + bd, ad + bc) \in \rho \text{ for all } \rho \in \mathcal{A}_R.$$

Therefore

$$\rho_1 \in C\Delta(a, b) \text{ and } \rho_2 \in C\Delta(c, d) \text{ and } \rho \in \Delta(ac + bd, ad + bc) \text{ for all } \rho \in \mathcal{A}_R,$$

i.e., $C\Delta(ac + bd, ad + bc) = \emptyset$. Thus there exist two open sets $C\Delta(a, b)$ and $C\Delta(c, d)$ containing ρ_1 and ρ_2 respectively such that

$$C\Delta(a, b) \cap C\Delta(c, d) = C\Delta(ac + bd, ad + bc) = \emptyset.$$

Therefore the space is T_2.

(iv) Let the space $\mathcal{A}_R$ be regular. Let $\rho \in \mathcal{A}_R$ and $(a,b) \notin \rho$. Then $\rho \in C\Delta(a,b)$ and $\mathcal{A}_R \setminus C\Delta(a,b)$ is a closed set not containing ρ. Since $\mathcal{A}_R$ is a regular space, there exists two disjoint open sets U and V such that $\rho \in U$ and $\mathcal{A}_R \setminus C\Delta(a,b) \subseteq V$, i.e., $\mathcal{A}_R \setminus V \subseteq C\Delta(a,b)$. $\mathcal{A}_R \setminus V$ is a closed set which means $\mathcal{A}_R \setminus V = \Delta(\sigma) \subseteq C\Delta(a,b)$ for some congruence σ on R (*cf.* Proposition 3.1.5). ..... (1)
Since $U \cap V = \emptyset$, $V \subseteq \mathcal{A}_R \setminus U$ and $\mathcal{A}_R \setminus U$ being a closed set, is of the form $\mathcal{A}_R \setminus U = \Delta(\sigma')$ for some congruence σ' on R. Since $\rho \in U$ then $\rho \notin \mathcal{A}_R \setminus U = \Delta(\sigma')$ which implies $\sigma' \nsubseteq \rho$. Therefore there exists $(c,d) \in \sigma'$ such that $(c,d) \notin \rho$ whence it follows that $\rho \in C\Delta(c,d)$. ..... (2)
Now we are to show that $V \subseteq \Delta(c,d)$. Let $\rho_1 \in V$. Then $V \subseteq \Delta(\sigma')$ implies $\sigma' \subseteq \rho_1$. Since $(c,d) \in \sigma'$, $(c,d) \in \rho_1$ and hence $\rho_1 \in \Delta(c,d)$. Thus $V \subseteq \Delta(c,d)$.
Consequently, $C\Delta(c,d) \subseteq \mathcal{A}_R \setminus V = \Delta(\sigma)$. ..... (3)
Thus combining (1), (2), (3), we find that

$$\rho \in C\Delta(c,d) \subseteq \Delta(\sigma) \subseteq C\Delta(a,b).$$

Conversely, let the given condition hold and let $\rho \in \mathcal{A}_R$ and A be a closed set not containing ρ. Then $A = \Delta(\sigma')$ for some congruence σ' on R. Since $\rho \notin \Delta(\sigma')$, we have $\sigma' \nsubseteq \rho$. This implies that there exists $(a,b) \in \sigma'$ such that $(a,b) \notin \rho$. Now by the given condition, there exists a congruence σ on R and $(c,d) \in R \times R$ such that

$$\rho \in C\Delta(c,d) \subseteq \Delta(\sigma) \subseteq C\Delta(a,b).$$

Since $(a,b) \in \sigma'$, $C\Delta(a,b) \cap \Delta(\sigma') = \emptyset$. Indeed, if $C\Delta(a,b) \cap \Delta(\sigma') \neq \emptyset$ then $\rho' \in C\Delta(a,b) \cap \Delta(\sigma')$ would imply that $(a,b) \notin \rho'$ and $\sigma' \subseteq \rho'$ which is a contradiction to the fact that $(a,b) \in \sigma'$. Hence

$$\Delta(\sigma') \subseteq \mathcal{A}_R \setminus C\Delta(a,b) \subseteq \mathcal{A}_R \setminus \Delta(\sigma).$$

Therefore $\mathcal{A}_R \setminus \Delta(\sigma)$ is an open set containing $\Delta(\sigma')$. It is clear that $C\Delta(c,d) \cap (\mathcal{A}_R \setminus \Delta(\sigma)) = \emptyset$. So we find that $C\Delta(c,d)$ and $\mathcal{A}_R \setminus \Delta(\sigma)$ are two disjoint open sets containing ρ and $\Delta(\sigma')$ respectively. Hence the space $\mathcal{A}_R$ is a regular space. □

Theorem 3.1.11. *$\mathcal{A}_R$ is compact if and only if for any collection of pairs $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ of elements in R, there exists a finite subcollection $\{(a_i,b_i) : i = 1,2,...,n\}$ in $R \times R$ such that for any $\rho \in \mathcal{A}_R$, there exists some (a_i,b_i) from the subcollection such that $(a_i,b_i) \notin \rho$.*

Proof. Let $\mathcal{A}_R$ be compact. Then the open cover $\{C\Delta(a_\alpha, b_\alpha) : (a_\alpha, b_\alpha) \in R \times R\}$ of $\mathcal{A}_R$ has a finite subcover $\{C\Delta(a_i, b_i) : i = 1, 2, ..., n\}$. Then for any $\rho \in \mathcal{A}_R$, $\rho \in C\Delta(a_i, b_i)$ for some $(a_i, b_i) \in R \times R$. This implies that $(a_i, b_i) \notin \rho$. Hence $\{(a_i, b_i) : i = 1, 2, ..., n\}$ is the required finite subcollection of elements of $R \times R$ such that for any $\rho \in \mathcal{A}_R$, there exists some (a_i, b_i) for $i = 1, 2, ..., n$ such that $(a_i, b_i) \notin \rho$.
Conversely, let us suppose that the given condition holds.
Let $\{C\Delta(a_i, b_i) : (a_i, b_i) \in R \times R\}$ be an open cover of $\mathcal{A}_R$. Suppose to the contrary that no finite subcollection of $\{C\Delta(a_i, b_i) : (a_i, b_i) \in R \times R\}$ covers $\mathcal{A}_R$. This means that for any finite set $\{(a_i, b_i) : i = 1, 2, ..., n\}$ of elements of $R \times R$, $\bigcup_{i=1}^n C\Delta(a_i, b_i) \neq \mathcal{A}_R$ whence $\bigcap_{i=1}^n \Delta(a_i, b_i) \neq \emptyset$. Then there exists $\rho \in \mathcal{A}_R$ such that $\rho \in \bigcap_{i=1}^n \Delta(a_i, b_i)$ which implies $(a_i, b_i) \in \rho$ for $i = 1, 2, ..., n$ and this leads to a contradiction. So the open cover $\{C\Delta(a_i, b_i) : (a_i, b_i) \in R \times R\}$ has a finite subcover and hence $\mathcal{A}_R$ is compact. □

Theorem 3.1.12. *$\mathcal{A}_R$ is disconnected if and only if there exists a congruence ρ on R and a collection of pairs $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ of elements in R not belonging to ρ such that if $\rho' \in \mathcal{A}_R$ and $(a_\alpha, b_\alpha) \in \rho'$ for all $\alpha \in \Lambda$ then $\rho \setminus \rho' \neq \emptyset$.*

Proof. Let $\mathcal{A}_R$ be disconnected. Then there exists a nontrivial clopen subset of $\mathcal{A}_R$. Let ρ be a congruence on R for which $\Delta(\rho)$ is closed as well as open.
Then $\Delta(\rho) = \bigcup_{\alpha \in \Lambda} C\Delta(a_\alpha, b_\alpha)$, where $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ is a collection of pairs of elements in R. Now if $\rho' \in \mathcal{A}_R$ and $(a_\alpha, b_\alpha) \in \rho'$ for all $\alpha \in \Lambda$ then we have $\rho' \in \Delta(a_\alpha, b_\alpha)$ for all $\alpha \in \Lambda$. Therefore $\rho' \notin C\Delta(a_\alpha, b_\alpha)$ for all $\alpha \in \Lambda$ which implies $\rho' \notin \Delta(\rho)$. So $\rho \nsubseteq \rho'$, i.e., $\rho \setminus \rho' \neq \emptyset$.
Conversely, let the given condition hold. Clearly $\bigcup_{\alpha \in \Lambda} C\Delta(a_\alpha, b_\alpha) \subseteq \Delta(\rho)$. Now let $\sigma \in \bigcap_{\alpha \in \Lambda} \Delta(a_\alpha, b_\alpha)$. Then $(a_\alpha, b_\alpha) \in \sigma$ for all $\alpha \in \Lambda$. So by the given condition $\rho \setminus \sigma \neq \emptyset$ which implies that σ does not contain ρ, i.e., $\sigma \in C\Delta(\rho)$. So $\bigcap_{\alpha \in \Lambda} \Delta(a_\alpha, b_\alpha) \subseteq C\Delta(\rho)$. Therefore $\Delta(\rho) \subseteq \bigcup_{\alpha \in \Lambda} C\Delta(a_\alpha, b_\alpha)$. Hence $\Delta(\rho) = \bigcup_{\alpha \in \Lambda} C\Delta(a_\alpha, b_\alpha)$ which is a clopen subset of $\mathcal{A}_R$. Therefore $\mathcal{A}_R$ is disconnected. □

In general intersection of two prime congruences is not a prime congruence on a semiring. In the following we study that under certain condition this result is true.

Theorem 3.1.13. *Let $\{\rho_i : i \in \Lambda\}$ be the collection of prime congruences on R such that $\{\rho_i : i \in \Lambda\}$ forms a chain of congruences. Then $\bigcap_{i \in \Lambda} \rho_i$ is a prime congruence on R.*

Proof. Let $(a, b) \notin \bigcap_{i \in \Lambda} \rho_i$ and $(c, d) \notin \bigcap_{i \in \Lambda} \rho_i$, where $a, b, c, d \in R$. That means there exists $\alpha, \beta \in \Lambda$ such that $(a,b) \notin \rho_\alpha$ and $(c,d) \notin \rho_\beta$. As $\{\rho_i : i \in \Lambda\}$ forms a chain of prime congruences, let us suppose that $\rho_\beta \subseteq \rho_\alpha$. Then

$$(a,b) \notin \rho_\beta \text{ implies } (ac+bd, ad+bc) \notin \rho_\beta.$$

Therefore $(ac+bd, ad+bc) \notin \bigcap_{i\in\Lambda} \rho_i$. So $\bigcap_{i\in\Lambda} \rho_i$ is a prime congruence on R. □

Theorem 3.1.14. *Let A be a nonempty closed subset of $\mathcal{A}_R$. Then A is irreducible if and only if $\bigcap_{\rho_i\in A} \rho_i$ is a prime congruence on R.*

Proof. Let A be a closed subset of $\mathcal{A}_R$ which is irreducible.
Let $(ac+bd, ad+bc) \in \bigcap_{\rho_i\in A} \rho_i$. Since ρ_i is a prime congruence on R for each i then either $(a,b) \in \rho_i$ or $(c,d) \in \rho_i$ for each i. Hence either $\rho_i \in \Delta(a,b)$ or $\rho_i \in \Delta(c,d)$ for each i. Thus $A \subseteq \Delta(a,b) \cup \Delta(c,d)$. Since A is irreducible, it implies that $A \subseteq \Delta(a,b)$ or $A \subseteq \Delta(c,d)$. Therefore it follows that

$$(a,b) \in \textstyle\bigcap_{\rho_i\in A} \rho_i \text{ or } (c,d) \in \bigcap_{\rho_i\in A} \rho_i.$$

Conversely, let us suppose that $\bigcap_{\rho_i\in A} \rho_i$ is a prime congruence on R. Let $A \subseteq B \cup C$, where B, C are closed subsets of A. So

$$\textstyle\bigcap_{\rho_i\in A} \rho_i \subseteq \bigcap_{\rho_i\in B} \rho_i \text{ and } \bigcap_{\rho_i\in A} \rho_i \subseteq \bigcap_{\rho_i\in C} \rho_i.$$

So $\bigcap_{\rho_i\in A} \rho_i \subseteq (\bigcap_{\rho_i\in B} \rho_i) \bigcap (\bigcap_{\rho_i\in C} \rho_i)$. Therefore by the fact that a prime congruence can not be obtained as the intersection of two strictly larger congruences (*cf.* Proposition 1.3.22), it follows that

$$\text{either } \textstyle\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in B} \rho_i \text{ or } \bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in C} \rho_i.$$

If we assume that $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in B} \rho_i$ then for any $\rho_k \in A$, $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in B} \rho_i \subseteq \rho_k$. It follows that $\rho_k \in \overline{B} = B$ as B is a closed subset of A. Hence $A \subseteq B$. Similarly if we assume that $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in C} \rho_i$ then this implies that $A \subseteq C$. Hence A is irreducible. □

Theorem 3.1.15. *Let A be a subset of $\mathcal{A}_R$. A is dense in $\mathcal{A}_R$ if and only if $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in \mathcal{A}_R} \rho_i$.*

Proof. Let A be a subset of $\mathcal{A}_R$ which is dense in $\mathcal{A}_R$. Obviously, $\bigcap_{\rho_i\in \mathcal{A}_R} \rho_i \subseteq \bigcap_{\rho_i\in A} \rho_i$. Since A is dense in $\mathcal{A}_R$ then by the definition of $\overline{A}$, $\bigcap_{\rho_i\in A} \rho_i \subseteq \bigcap_{\rho_i\in \mathcal{A}_R} \rho_i$. Therefore $\bigcap_{\rho_i\in A} \rho_i = \bigcap_{\rho_i\in \mathcal{A}_R} \rho_i$. To prove the converse, let us assume that $\mathcal{A}_R \setminus \overline{A} \neq \emptyset$. Then there exists a prime congruence ρ on R such that $\rho \in \mathcal{A}_R \setminus \overline{A}$. Therefore there exists an open neighbourhood U of ρ in $\mathcal{A}_R$ such that $U \cap A = \emptyset$, i.e., $C\Delta(x,y) \cap A = \emptyset$ for

some $(x,y) \in R \times R$ with $U = C\Delta(x,y)$. Then $A \subseteq \Delta(x,y)$ implies $(x,y) \in \bigcap_{\rho_i \in A} \rho_i$.
Now if possible let $(x,y) \in \bigcap_{\rho_i \in \mathcal{A}_R} \rho_i$.
Then $\rho_i \in \Delta(x,y)$ for each $\rho_i \in \mathcal{A}_R$. It implies that $\mathcal{A}_R = \Delta(x,y)$, i.e., $C\Delta(x,y) = \emptyset$ which is a contradiction to the fact that it is an open neighbourhood of ρ. That means $(x,y) \notin \bigcap_{\rho_i \in \mathcal{A}_R} \rho_i$. Therefore $\bigcap_{\rho_i \in \mathcal{A}_R} \rho_i \subsetneqq \bigcap_{\rho_i \in A} \rho_i$.
Hence $\bigcap_{\rho_i \in A} \rho_i = \bigcap_{\rho_i \in \mathcal{A}_R} \rho_i$ implies that A is dense in $\mathcal{A}_R$. □

Let us now introduce the notions of absolutely prime-irreducible congruences on a semiring.

Definitions 3.1.16. Let $\mathcal{F}$ be a family of congruences on R. A congruence ρ is said to be *absolutely $\mathcal{F}$-irreducible* if for every subset $\{\rho_i\}_{i \in \Lambda}$ of $\mathcal{F}$ such that $\bigcap_{i \in \Lambda} \rho_i \subseteq \rho$, there exists j in Λ such that $\rho_j \subseteq \rho$.
If in particular, $\mathcal{F} =$ the set of all prime congruences on R then ρ is said to be *absolutely prime-irreducible*.

Lemma 3.1.17. *Let ρ be a prime congruence on R. Then ρ is a kerneled point of $\mathcal{A}_R$ if and only if ρ is a minimal prime congruence.*

Proof. Let ρ be a kerneled point of $\mathcal{A}_R$. Then $\{\rho\} = \bigcap\{C\Delta(\sigma) : \rho \in C\Delta(\sigma)\}$. Let us suppose, if possible that ρ is not a minimal prime. Then there exists a prime congruence ρ' properly contained in ρ, i.e., $\rho' \subset \rho$ which implies that every neighbourhood of ρ is also a neighbourhood of ρ'. So $\rho' \in \bigcap\{C\Delta(\sigma) : \rho \in C\Delta(\sigma)\}$. This leads to a contradiction to our assumption. Therefore ρ is a minimal prime.
Conversely, let ρ be a minimal prime congruence on R. Let us suppose that ρ' is a prime congruence such that $\rho' \in \bigcap\{C\Delta(\sigma) : \rho \in C\Delta(\sigma)\}$. Let us consider the family $\{C\Delta(a,b) : \rho \in C\Delta(a,b)\}$ of neighbourhoods of ρ.
Then $\rho' \in \bigcap\{C\Delta(a,b) : \rho \in C\Delta(a,b)\}$. We observe that if $(a,b) \in \rho'$ then $(a,b) \in \rho$, since otherwise, if $(a,b) \notin \rho$ then $\rho \in C\Delta(a,b)$ would imply that $\rho' \in C\Delta(a,b)$ which is a contradiction. Therefore $\rho \cap \rho' \neq \emptyset$. If possible let $\rho' \nsubseteq \rho$.
Since $\rho' \in \bigcap\{C\Delta(\sigma) : \rho \in C\Delta(\sigma)\}$, $\rho' \nsubseteq \rho$ implies that $\rho \in C\Delta(\rho')$ whence it follows that $\rho' \in C\Delta(\rho')$ which is absurd. Hence $\rho' \subseteq \rho$. By minimality of ρ, it is only possible if $\rho = \rho'$. Therefore ρ is a kerneled point of $\mathcal{A}_R$.

□

Theorem 3.1.18. *$\mathcal{A}_R$ is a $T_{1/4}$-space if and only if every prime congruence on R is either maximal or minimal prime.*

Proof. From Lemma 3.1.17 and Corollary 1.3.37 we get that every prime congruence on R is either maximal or minimal prime if and only if each point in $\mathcal{A}_R$ is either closed or kerneled. This completes the proof. □

Lemma 3.1.19. *Let ρ be a prime congruence on R. Then ρ is a isolated point of $\mathcal{A}_R$ if and only if ρ is an absolutely prime-irreducible minimal prime congruence.*

Proof. If $\mathcal{A}_R = \{\rho\}$ then the proof is immediate.
So let us suppose that $\mathcal{A}_R \neq \{\rho\}$ and ρ is an isolated point of $\mathcal{A}_R$. Therefore $\{\rho\} = C\Delta(\sigma)$ for some congruence σ on R. Since every isolated point is kerneled, ρ is a kerneled point of $\mathcal{A}_R$. Therefore by Lemma 3.1.17, ρ is a minimal prime congruence. Let $\{\rho_i\}_{i\in\Lambda}$ be a subset of $\mathcal{A}_R$ such that $\bigcap_{i\in\Lambda}\rho_i \subseteq \rho$. If for every j, $\rho_j \nsubseteq \rho$ then $\rho_j \in \mathcal{A}_R \setminus \{\rho\} = \Delta(\sigma)$ implies $\sigma \subseteq \rho_j$ for all j. This implies $\sigma \subseteq \rho$ which is a contradiction. Hence there exists j such that $\rho_j \subseteq \rho$. So ρ is an absolutely prime-irreducible minimal prime congruence.
Conversely, suppose that ρ is an absolutely prime-irreducible minimal prime congruence. If we take the collection of all prime congruences which are different from ρ and take the intersection over the collection, we obtain a congruence σ (say) on R which is not contained in ρ. Therefore $\{\rho\} = C\Delta(\sigma)$, i.e., ρ is an isolated point of $\mathcal{A}_R$.

□

Theorem 3.1.20. *$\mathcal{A}_R$ is a $T_{1/2}$-space if and only if every prime congruence on R is either maximal or absolutely prime-irreducible minimal prime.*

Proof. From Lemma 3.1.19 and Corollary 1.3.37 we get that every prime congruence on R is either maximal or minimal prime if and only if each point in $\mathcal{A}_R$ is either closed or kerneled. This completes the proof. □

Lemma 3.1.21. *Let ρ be a prime congruence on R. Then the following are equivalent.*

(i) *ρ is an open-regular point of $\mathcal{A}_R$.*

(ii) *ρ is the unique interior point of $\Delta(\rho)$.*

(iii) $\mathcal{A}_R \setminus \{\rho\} = \overline{C\Delta(\rho)}$.

(iv) *ρ is an isolated point of $\mathcal{A}_R$ such that $\bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\} = \bigcap C\Delta(\rho)$.*

Proof. (i) ⇔ (ii) : ρ is an open-regular point of $\mathcal{A}_R$ if and only if ρ is the unique interior point of $\overline{\{\rho\}}$ if and only if ρ is the unique interior point of $\Delta(\rho)$.

$(ii) \Leftrightarrow (iii)$: Let us Consider (ii) holds. Then $\mathcal{A}_R \setminus \{\rho\}$ is a closed set containing $C\Delta(\rho)$. So $\overline{C\Delta(\rho)} \subseteq \mathcal{A}_R \setminus \{\rho\}$. Now let $\rho' \in \mathcal{A}_R \setminus \{\rho\}$. This implies $\rho' \in \overline{C\Delta(\rho)}$, i.e., $\mathcal{A}_R \setminus \{\rho\} \subseteq \overline{C\Delta(\rho)}$. Therefore $\mathcal{A}_R \setminus \{\rho\} = \overline{C\Delta(\rho)}$.
Conversely, let us consider (iii) holds. Therefore $\{\rho\}$ is an open set in $\mathcal{A}_R$. Then ρ is an interior point of $\Delta(\rho)$. If ρ' is an interior point of $\Delta(\rho)$ then there exists $(a, b) \in R \times R$ such that $\rho' \in C\Delta(a, b) \subseteq \Delta(\rho)$. Then ρ' must be equal to ρ, since, otherwise, $\rho' \in \mathcal{A}_R \setminus \{\rho\} = \overline{C\Delta(\rho)}$ and this implies that $C\Delta(a, b) \cap C\Delta(\rho) \neq \emptyset$ which is a contradiction. Hence ρ is the unique interior point of $\Delta(\rho)$.

$(iii) \Leftrightarrow (iv)$: Let $\mathcal{A}_R \setminus \{\rho\} = \overline{C\Delta(\rho)}$. Then clearly ρ is an isolated point. Let us consider the two collections $\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\}$ and $\{\sigma \in \mathcal{A}_R : \rho \nsubseteq \sigma\}$.
So $\{\sigma \in \mathcal{A}_R : \rho \nsubseteq \sigma\} \subseteq \{\sigma \in \mathcal{A}_R : \rho \neq \sigma\}$
whence it follows that $\bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\} \subseteq \bigcap C\Delta(\rho)$.
Now let $(a, b) \in \bigcap C\Delta(\rho)$ and $\sigma \neq \rho$. Then $\sigma \in \mathcal{A}_R \setminus \{\rho\} = \overline{C\Delta(\rho)}$.
Now if $(a, b) \notin \sigma$ then $C\Delta(a, b)$ and $C\Delta(\rho)$ intersect which is a contradiction. In fact if $\rho' \in C\Delta(a, b) \cap C\Delta(\rho)$ then $\rho' \in C\Delta(a, b)$ and $\rho' \in C\Delta(\rho)$. Now $(a, b) \in \bigcap C\Delta(\rho)$ implies $(a, b) \in \rho_1$ for all ρ_1 such that $\rho \nsubseteq \rho_1$. Therefore $\rho' \in C\Delta(\rho)$ implies that $(a, b) \in \rho'$, i.e., $\rho' \in \Delta(a, b)$. Hence $(a, b) \in \sigma$. Therefore $(a, b) \in \bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\}$ whence it follows that $\bigcap C\Delta(\rho) \subseteq \bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\}$.
Hence $\bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\} = \bigcap C\Delta(\rho)$.
Conversely, let ρ be an isolated point of $\mathcal{A}_R$ and $\bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\} = \bigcap C\Delta(\rho)$.
Then $\{\rho\}$ is open, i.e., $\mathcal{A}_R \setminus \{\rho\}$ is closed which contains $C\Delta(\rho)$.
Thus $\overline{C\Delta(\rho)} \subseteq \mathcal{A}_R \setminus \{\rho\}$. Again by hypothesis,

$$\overline{C\Delta(\rho)} = \Delta(\bigcap C\Delta(\rho)) = \Delta(\bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\}).$$

Therefore $\mathcal{A}_R \setminus \{\rho\} \subseteq \overline{C\Delta(\rho)}$. Hence $\mathcal{A}_R \setminus \{\rho\} = \overline{C\Delta(\rho)}$.

□

From Lemmas 3.1.21 and 3.1.19 we get the following theorem.

Theorem 3.1.22. *The following are equivalent.*

(i) *$\mathcal{A}_R$ is a $T_{3/4}$-space.*

(ii) *Every point ρ is either closed or satisfies $\mathcal{A}_R \setminus \{\rho\} = \overline{C\Delta(\rho)}$.*

(iii) *Every point ρ is either closed or an isolated point such that $\bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\} = \bigcap C\Delta(\rho)$.*

(*iv*) *Every prime congruence* ρ *on* R *is either maximal or an absolutely prime-irreducible congruence such that* $\bigcap\{\sigma \in \mathcal{A}_R : \rho \neq \sigma\} = \bigcap C\Delta(\rho)$.

Theorem 3.1.23. *There is an order preserving bijection between the sets of all prime congruences of two isomorphic semirings.*

Proof. Let $\phi : R_1 \to R_2$ be an isomorphism of the two semirings R_1 and R_2. Then there is an order preserving bijection between the sets of all congruences on two isomorphic semirings via the map $\rho \mapsto \rho_\phi$, where ρ is a congruence on R_1 and $\rho_\phi = \{(\phi(a), \phi(b)) : (a, b) \in \rho\}$ is a congruence on R_2. We need to show that ρ is a prime congruence on R_1 if and only if ρ_ϕ is a prime congruence on R_2. Let us first assume that ρ is a prime congruence on R_1. Let us consider that for $a, b, c, d \in R_1$,

$$(\phi(a)\phi(d) + \phi(b)\phi(c), \phi(a)\phi(c) + \phi(b)\phi(d)) \in \rho_\phi.$$

Since ϕ is an isomorphism of semirings, $(\phi(ad + bc), \phi(ac + bd)) \in \rho_\phi$ which means $(ad + bc, ac + bd) \in \rho$. Now ρ is a prime congruence whence it follows that either $(a, b) \in \rho$ or $(c, d) \in \rho$. That means

$$\text{either } (\phi(a), \phi(b)) \in \rho_\phi \text{ or } (\phi(c), \phi(d)) \in \rho_\phi.$$

Therefore ρ_ϕ is a prime congruence on R_2. With a similar argument we obtain the other part. This completes the proof. □

Theorem 3.1.24. *If two semirings* R_1 *and* R_2 *are isomorphic then their structure spaces of prime congruences (maximal regular congruences) respectively* $\mathcal{A}_{R_1}$ *and* $\mathcal{A}_{R_2}$ *are homeomorphic.*

Proof. Let $\phi : R_1 \to R_2$ be an isomorphism between two semirings R_1 and R_2. Let us define a mapping $\phi^* : \mathcal{A}_{R_1} \to \mathcal{A}_{R_2}$ by $\phi^*(\sigma) := \{(\phi(a), \phi(b)) : (a, b) \in \sigma\}$, where σ is a prime congruence on R_1 (Theorem 3.1.23 assures that $\phi^*(\sigma)$ is a prime congruence on R_2). We will show that ϕ^* is a homeomorphism. From the definition of mapping it is clear that ϕ^* is bijective. Now $\rho \in \phi^{*-1}(\Delta(a, b))$ if and only if $\phi^*(\rho) \in \Delta(a, b)$ if and only if $(a, b) \in \phi^*(\rho)$ if and only if $(\phi^{-1}(a), \phi^{-1}(b)) \in \rho$ if and only if $\rho \in \Delta(\phi^{-1}(a), \phi^{-1}(b))$. Therefore ϕ^* is continuous. With similar argument as above we can also prove that ϕ^{*-1} is continuous. Hence ϕ^* is a homeomorphism.
The homeomorphism of maximal regular part is already proved in [67].

□

In the following result we characterize the prime congruences on the semiring Z_0^+ of all non-negative integers.

Theorem 3.1.25. *The prime congruences on the semiring Z_0^+ of all non-negative integers are precisely of the form*

$$\rho_p = \{(m,n) \in Z_0^+ \times Z_0^+ : m-n \text{ is divisible by } p\} \text{ for some prime number } p.$$

Proof. For any prime number p,

$$\rho_p = \{(m,n) \in Z_0^+ \times Z_0^+ : m-n \text{ is divisible by } p\}$$

is a prime congruence (*cf.* Examples 1.3.21 (ii)) on Z_0^+. Let σ be any prime congruence on Z_0^+. We are to prove that σ is of the form ρ_p for some prime number p. The zero-class of σ denoted by $\sigma(0)$ is a k-ideal of Z_0^+, by Lemma 1.3.29. Since any k-ideal is the form aZ_0^+ for $a \in Z_0^+$ (*cf.* Example 1.3.30), $\sigma(0) = kZ_0^+$ for some $k \in Z_0^+$. Therefore it can be easily shown that the congruence

$$\rho_k = \{(m,n) \in Z_0^+ \times Z_0^+ : m-n \text{ is divisible by } k\}$$

is contained in σ, i.e., $\rho_k \subseteq \sigma$. Also k must be a prime number p (say). In fact, if k is not a prime then $k = n_1 n_2$ (say). Since σ is a prime congruence on Z_0^+, $(k,0) \in \rho_k \subseteq \sigma$ implies either $n_1 \in kZ_0^+$ or $n_2 \in kZ_0^+$ which is absurd. Again ρ_p is a maximal congruence on Z_0^+ which implies that $\rho_p = \sigma$. Therefore the prime congruences on Z_0^+ are precisely of the form ρ_p for any prime number p. □

To end this section we give one example of a semiring R whose structure space $\mathcal{A}_R$ is T_0, T_1, compact, connected but neither T_2 nor regular.

Example 3.1.26. Let us consider the semiring $R = Z_0^+$ of all non-negative integers and let $\mathcal{A}_{Z_0^+} = \{\rho_p : p \text{ is a prime}\}$ be the space of all prime congruences on Z_0^+ with the Hull Kernel topology defined in it (we have already proved in Theorem 3.1.25 that the prime congruences on Z_0^+ are ρ_p for any prime number p). Then we have the following properties of the structure space $\mathcal{A}_{Z_0^+}$ of the semiring R :

(*i*) $\mathcal{A}_{Z_0^+}$ is a T_0-space by (*i*) of Theorem 3.1.10.

(*ii*) $\mathcal{A}_{Z_0^+}$ is a T_1-space by (ii) of Theorem 3.1.10.

(*iii*) $\mathcal{A}_{Z_0^+}$ is a compact space by Theorem 3.1.11.
Now let ρ_{p_1} and ρ_{p_2} be two distinct elements of $\mathcal{A}_{Z_0^+}$ and let $(a,b),(c,d)$ be two

pairs of elements of Z_0^+ such that $(a,b) \notin \rho_{p_1}$ and $(c,d) \notin \rho_{p_2}$ which means p_1 does not divide $(a-b)$ and p_2 does not divide $(c-d)$. Then there always exists a prime number p such that p does not divide $(a-b)$ and p does not divide $(c-d)$, i.e., p does not divide $(a-b)(c-d) = (ac+bd) - (ad+bc)$. This implies that $(ad+bc, ac+bd) \notin \rho_p$. Again $C\Delta(a,b)$ for $(a,b) \in Z_0^+ \times Z_0^+$ is infinite and its complement $\Delta(a,b)$ is finite which is also a closed set. Hence it follows that any two nontrivial open sets intersect. Therefore we have the following:

(iv) $\mathcal{A}_{Z_0^+}$ is not a T_2-space by (iii) of Theorem 3.1.10.

(v) $\mathcal{A}_{Z_0^+}$ is not a regular space.

(vi) $\mathcal{A}_{Z_0^+}$ is a connected space.

It is to be noted that ρ_p is a maximal congruence on Z_0^+. Therefore in view of Theorem 3.1.25, every prime congruence is maximal. So by Corollary 1.3.37, we deduce that every point of $\mathcal{A}_{Z_0^+}$ is closed. Now using Theorems 3.1.18, 3.1.20, 3.1.22 we have the following.

Theorem 3.1.27. *$\mathcal{A}_{Z_0^+}$ is a $T_{1/4}$, $T_{1/2}$, $T_{3/4}$ space.*

3.2 Structure space of prime congruences on a Γ−semiring

In this section we define the structure space of prime congruences on a Γ-semiring and study the topological properties of the space via those of its left operator semiring.

Definition 3.2.1. Let $\mathcal{A}_S$ be the collection of all prime congruences on a commutative Γ-semiring S with strong left and right unities. For any subset A of $\mathcal{A}_S$, we define $\overline{A} = \{\rho \in \mathcal{A}_S : \bigcap_{\rho_i \in A} \rho_i \subseteq \rho\}$.

Throughout the section unless otherwise mentioned $\mathcal{A}_S$ stands for the space of all prime congruences on a commutative Γ-semiring S with strong unities and $\mathcal{A}_L$ denotes that of its left operator semiring L and $\mathcal{A}_R$ denotes the space of all prime congruences on a semiring R, all equipped with Hull Kernel topology.
Note that since left and right operator semirings of a commutative Γ-semiring are isomorphic, the structure space of left operator semiring is homeomorphic to the space of right operator semiring (see Theorem 3.1.24).

To prove the kuratowski closure axioms in the context of the Γ-semirings, we first prove few results.

Notations: For $A \subseteq \mathcal{A}_S$, $A^{+'} = \{\sigma^{+'} \in \mathcal{A}_L : \sigma \in A\}$.
For $B \subseteq \mathcal{A}_L$, $B^{+} = \{\rho^{+} \in \mathcal{A}_S : \rho \in B\}$.

Lemma 3.2.2. (i) *For any subset* A *of* $\mathcal{A}_S$, $\overline{A^{+'}} = (\overline{A})^{+'}$.
(ii) *For any subset* B *of* $\mathcal{A}_L$, $\overline{B^{+}} = (\overline{B})^{+}$.

Proof. (i) Let $\rho \in \overline{A^{+'}}$. Then $\rho \in \mathcal{A}_L$ and $\bigcap_{\sigma_i \in A} \sigma_i^{+'} \subseteq \rho$.
Clearly $\bigcap_{\sigma_i \in A} \sigma_i^{+'} = (\bigcap_{\sigma_i \in A} \sigma_i)^{+'}$. It implies that

$$\bigcap_{\sigma_i \in A} \sigma_i = \bigcap_{\sigma_i \in A} (\sigma_i^{+'})^{+} = ((\bigcap_{\sigma_i \in A} \sigma_i)^{+'})^{+} \subseteq \rho^{+}$$

whence it follows that $\rho^{+} \in \overline{A}$. Hence $\rho \in (\overline{A})^{+'}$. So $\overline{A^{+'}} \subseteq (\overline{A})^{+'}$. The reverse inclusion follows similarly. Consequently, $\overline{A^{+'}} = (\overline{A})^{+'}$.

(ii) The proof can be done similarly. □

Theorem 3.2.3. *There exists an inclusion preserving bijection between* $\wp(A_S)$ *and* $\wp(A_L)$ *via the mapping* $A \mapsto A^{+'}$*, where* $A \in \wp(A_S)$ *and* $\wp(A_S)$ *is the power set of* A_S*.*

Proof. Let $A \in \wp(A_L)$. Then $A^{+} \in \wp(A_S)$. We shall now prove that $(A^{+})^{+'} = A$.
Let $\rho^{+'} \in (A^{+})^{+'}$. Then $\rho \in A^{+}$. So there exists $\sigma \in A$ such that $\rho = \sigma^{+}$. Therefore $\rho^{+'} = (\sigma^{+})^{+'} = \sigma \in A$ implies that $(A^{+})^{+'} \subseteq A$. Again let $\sigma_1 \in A$.
Then $\sigma_1 = (\sigma_1^{+})^{+'} \in (A^{+})^{+'}$. Therefore $A \subseteq (A^{+})^{+'}$. Hence $(A^{+})^{+'} = A$. Similarly it can be proved that $B = (B^{+'})^{+}$ for all $B \in \wp(A_S)$. Therefore the mapping $A \mapsto A^{+'}$ is bijective. Now let $A, B \in \wp(A_S)$ such that $A \subseteq B$. Then $\sigma^{+'} \in A^{+'}$ implies $\sigma \in A \subseteq B$. So $\sigma^{+'} \in B^{+'}$ and hence $A^{+'} \subseteq B^{+'}$. This completes the proof. □

Lemma 3.2.4. (i) *For* $A, B \subseteq \mathcal{A}_S$, $A^{+'} \cup B^{+'} = (A \cup B)^{+'}$.
(ii) *For* $A, B \subseteq \mathcal{A}_L$, $A^{+} \cup B^{+} = (A \cup B)^{+}$.

Proof. (i) $A, B \subseteq A \cup B$. Then $A^{+'}, B^{+'} \subseteq (A \cup B)^{+'}$ implies $A^{+'} \cup B^{+'} \subseteq (A \cup B)^{+'}$.
Let $\rho^{+'} \in (A \cup B)^{+'}$. Then either $\rho \in A$ or $\rho \in B$. Therefore $\rho^{+'} \in A^{+'} \cup B^{+'}$ which implies $(A \cup B)^{+'} \subseteq A^{+'} \cup B^{+'}$. Hence $A^{+'} \cup B^{+'} = (A \cup B)^{+'}$.

(ii) The proof is similar as that of (i). □

In view of Theorems 3.1.3, 3.2.3 and Lemmas 3.2.2, 3.2.4, we prove the following Theorem.

Theorem 3.2.5. *For* $A, B \subseteq \mathcal{A}_S$, (*i*) $A \subseteq \overline{A}$

(*ii*) $\overline{\overline{A}} = \overline{A}$

(*iii*) $A \subseteq B$ *implies* $\overline{A} \subseteq \overline{B}$

(*iv*) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (*i*) $A^{+'} \subseteq \overline{A^{+'}} = (\overline{A})^{+'}$. This implies $(A^{+'})^{+} \subseteq ((\overline{A})^{+'})^{+}$. Therefore $A \subseteq \overline{A}$.

(*ii*) $\overline{\overline{A}} = ((\overline{\overline{A}})^{+'})^{+} = (\overline{(\overline{A})^{+'}})^{+} = (\overline{(\overline{A^{+'}})})^{+} = (\overline{A^{+'}})^{+} = ((\overline{A})^{+'})^{+} = \overline{A}$

(*iii*) $A \subseteq B$ implies $A^{+'} \subseteq B^{+'}$. Hence $\overline{A^{+'}} \subseteq \overline{B^{+'}}$ from which it follows that $(\overline{A})^{+'} \subseteq (\overline{B})^{+'}$. It implies that $((\overline{A})^{+'})^{+} \subseteq ((\overline{B})^{+'})^{+}$. Therefore $\overline{A} \subseteq \overline{B}$.

(*iv*) $\overline{A \cup B} = ((\overline{A \cup B})^{+'})^{+} = (\overline{(A \cup B)^{+'}})^{+} = (\overline{A^{+'} \cup B^{+'}})^{+} = (\overline{A^{+'}} \cup \overline{B^{+'}})^{+}$
$= (\overline{A^{+'}})^{+} \cup (\overline{B^{+'}})^{+} = ((\overline{A})^{+'})^{+} \cup ((\overline{B})^{+'})^{+} = \overline{A} \cup \overline{B}$. □

The following result follows from Theorem 3.2.5.

Theorem 3.2.6. *The mapping from* $A \mapsto \overline{A}$ *is a Kuratowski closure operator on* $\mathcal{A}_S$*, where* $A \subseteq \mathcal{A}_S$.

Definition 3.2.7. The topology $\tau_{\mathcal{A}_S}$ induced by the Kuratowski closure operator on $\mathcal{A}_S$ is known as Hull Kernel topology. The topological space is called the *structure space* of the Γ-semiring S.
Note that the Hull Kernel topology on $\mathcal{A}_L$ is denoted by $\tau_{\mathcal{A}_L}$.

Using Lemma 3.2.2 we have the following theorem.

Theorem 3.2.8. $(\mathcal{A}_S, \tau_{\mathcal{A}_S})$ *and* $(\mathcal{A}_L, \tau_{\mathcal{A}_L})$ *are homeomorphic.*

Proof. Let us define a mapping $f : \mathcal{A}_S \to \mathcal{A}_L$ by $f(\rho) = \rho^{+'}$, where $\rho \in \mathcal{A}_S$. f is a bijective map by Theorem 2.2.20. For any subset A of $\mathcal{A}_S$, $f(A) = A^{+'}$. Let A be any closed set in $\mathcal{A}_S$, then $f(A) = f(\overline{A}) = (\overline{A})^{+'} = \overline{A^{+'}}$ which is a closed set in $\mathcal{A}_L$. Again let B be any closed set in $\mathcal{A}_L$, then $f^{-1}(B) = f^{-1}(\overline{B}) = (\overline{B})^{+} = \overline{B^{+}}$ which is a closed set in $\mathcal{A}_S$. Therefore f is a homeomorphism. □

Notations: Let $x, y \in S$. We define,

$$\Delta(x,y) = \{\rho \in \mathcal{A}_S : (x,y) \in \rho\}; C\Delta(x,y) = \{\rho \in \mathcal{A}_S : (x,y) \notin \rho\}.$$

Lemma 3.2.9. (*i*) $(\Delta(a,b))^{+'} = \Delta([a,\gamma],[b,\gamma])$ *and*
(*ii*) $(\Delta([a,\gamma],[b,\gamma]))^{+} = \Delta(a,b)$*, where* $[\gamma, f]$ *is the strong right unity of the* Γ*-semiring* S *and* $a, b \in S$.

Proof. Let $a, b \in S$.
(i) $\rho \in \Delta([a,\gamma],[b,\gamma])$. Then $([a,\gamma],[b,\gamma]) \in \rho$ implies $([a,\gamma]\cdot[f,\alpha],[b,\gamma]\cdot[f,\alpha]) \in \rho$ for all $\alpha \in \Gamma$. Therefore $([a,\alpha],[b,\alpha]) \in \rho$, for all $\alpha \in \Gamma$ which implies $\rho^+ \in \Delta(a,b)$. Hence $\rho = (\rho^+)^{+'} \in (\Delta(a,b))^{+'}$ implies $\Delta([a,\gamma],[b,\gamma]) \subseteq (\Delta(a,b))^{+'}$.
Again let $\sigma^{+'} \in (\Delta(a,b))^{+'}$. Then $\sigma \in \Delta(a,b)$ implies $(a,b) \in \sigma = (\sigma^{+'})^+$. Therefore $([a,\alpha],[b,\alpha]) \in \sigma^{+'}$, for all $\alpha \in \Gamma$ implies $\sigma^{+'} \in \Delta([a,\gamma],[b,\gamma])$.
Hence $(\Delta(a,b))^{+'} \subseteq \Delta([a,\gamma],[b,\gamma])$. So $(\Delta(a,b))^{+'} = \Delta([a,\gamma],[b,\gamma])$.
(ii) $\Delta(a,b) = ((\Delta(a,b))^{+'})^+ = (\Delta([a,\gamma],[b,\gamma]))^+$, by (i). □

From the above lemma the following is easily derived.

Lemma 3.2.10. *(i) $(C\Delta(a,b))^{+'} = C\Delta([a,\gamma],[b,\gamma])$ and (ii) $(C\Delta([a,\gamma],[b,\gamma]))^+ = C\Delta(a,b)$, where $[\gamma,f]$ is the strong right unity of the Γ-semiring S and $a,b \in S$.*

Remark 3.2.11. $\Delta([a,\gamma],[b,\gamma]) \subsetneq \Delta([a,\alpha],[b,\alpha])$ for any $\alpha \in \Gamma$, where $[\gamma,f]$ is the strong right unity of the Γ-semiring S and $a,b \in S$.

The following result is the Γ-semiring analogue of Lemma 3.1.8.

Theorem 3.2.12. *$\Delta(a,b) \cup \Delta(c,d) = \Delta(a\gamma c + b\gamma d, a\gamma d + b\gamma c)$, where $[\gamma,f]$ is the strong right unity of the Γ-semiring S and $a,b,c,d \in S$.*

Proof. By Lemma 3.1.8, for $a,b,c,d \in S$,

$$\Delta([a,\gamma],[b,\gamma]) \cup \Delta([c,\gamma],[d,\gamma]) =$$
$$\Delta([a,\gamma]\cdot[c,\gamma]+[b,\gamma]\cdot[d,\gamma],[a,\gamma]\cdot[d,\gamma]+[b,\gamma]\cdot[c,\gamma]) = \Delta([a\gamma c+b\gamma d,\gamma],[a\gamma d+b\gamma c,\gamma]).$$

By Lemma 3.2.4,

$$((\Delta[a,\gamma],[b,\gamma]) \cup \Delta([c,\gamma],[d,\gamma]))^+ = (\Delta([a,\gamma],[b,\gamma]))^+ \cup (\Delta([c,\gamma],[d,\gamma]))^+.$$

This implies $(\Delta([a\gamma c + b\gamma d,\gamma],[a\gamma d + b\gamma c,\gamma]))^+ = \Delta(a,b) \cup \Delta(c,d)$ which implies $\Delta(a,b) \cup \Delta(c,d) = \Delta(a\gamma c + b\gamma d, a\gamma d + b\gamma c)$ (by Lemma 3.2.9). □

From the above theorem we have the following.

Theorem 3.2.13. *$C\Delta(a,b) \cap C\Delta(c,d) = C\Delta(a\gamma c + b\gamma d, a\gamma d + b\gamma c)$, where $[\gamma,f]$ is the strong right unity of the Γ-semiring S and $a,b,c,d \in S$.*

Notations: Let ρ be a congruence on S. We define,

$$\Delta(\rho) = \{\rho' \in \mathcal{A}_S : \rho \subseteq \rho'\}; C\Delta(\rho) = \{\rho' \in \mathcal{A}_S : \rho \nsubseteq \rho'\}.$$

Proposition 3.2.14. *Let ρ be a congruence on the Γ-semiring S. Then* $(\Delta(\rho))^{+'} = \Delta(\rho^{+'})$.

Proof. Let $\sigma^{+'} \in (\Delta(\rho))^{+'} \subseteq \mathcal{A}_L$.
Then $\sigma \in \Delta(\rho) \subseteq \mathcal{A}_S$, i.e., $\sigma \in \mathcal{A}_S$ and $\rho \subseteq \sigma$. Thus $\rho^{+'} \subseteq \sigma^{+'} \in \mathcal{A}_L$. So $\sigma^{+'} \in \Delta(\rho^{+'})$.
Therefore $(\Delta(\rho))^{+'} \subseteq \Delta(\rho^{+'})$.
The reverse inclusion follows similarly. Hence $\Delta(\rho)^{+'} = \Delta(\rho^{+'})$. □

We can derive the following proposition in a similar manner.

Proposition 3.2.15. *Let σ be a congruence on L. Then $(\Delta(\sigma))^{+} = \Delta(\sigma^{+})$.*

Proof. $\Delta(\sigma) = \Delta((\sigma^{+})^{+'}) = (\Delta(\sigma^{+}))^{+'}$, by Proposition 3.2.14.
Therefore $(\Delta(\sigma))^{+} = \Delta(\sigma^{+})$. □

Proposition 3.2.16. *Any closed set in $\mathcal{A}_S$ is of the form $\Delta(\rho)$, where ρ is a congruence on S.*

Proof. Let A be any closed set in $\mathcal{A}_S$. Then $A^{+'}$ is a closed set in $\mathcal{A}_L$ by Theorem 3.2.8. So $A^{+'} = \Delta(\sigma)$ for some congruence σ on L by Theorem 3.1.5 which implies $(A^{+'})^{+} = (\Delta(\sigma))^{+}$. Thus by Proposition 3.2.15, $A = \Delta(\rho)$, where $\rho = \sigma^{+}$ is a congruence on S. □

Now the following proposition is immediate.

Proposition 3.2.17. *Any open set in $\mathcal{A}_S$ is of the form $C\Delta(\rho)$, where ρ is a congruence on S.*

Proposition 3.2.18. *$\{C\Delta(a,b) : (a,b) \in S \times S\}$ is an open base for $\mathcal{A}_S$.*

Proof. Let U be an open set in $\mathcal{A}_S$. Then $A = \mathcal{A}_S \setminus U$ is a closed set in $\mathcal{A}_S$. By the Proposition 3.2.16, $A = \Delta(\rho)$ for some congruence ρ on S. Then $\sigma \in U$ implies $\sigma \notin A$, i.e., $\rho \nsubseteq \sigma$. Then there exists $(a,b) \in \rho$ such that $(a,b) \notin \sigma$. Hence $\sigma \in C\Delta(a,b)$. Now let $\sigma' \in C\Delta(a,b)$. Then $(a,b) \notin \sigma'$. This implies that $\rho \nsubseteq \sigma'$ whence it follows that $\sigma' \in U$. Hence $C\Delta(a,b) \subseteq U$. Consequently, $\sigma \in C\Delta(a,b) \subseteq U$.
Thus $\{C\Delta(a,b) : (a,b) \in S \times S\}$ is an open base for $\mathcal{A}_S$. □

Corollary 3.2.19. *$\{C\Delta([a,\gamma],[b,\gamma]) : (a,b) \in S \times S\}$ is an open base for $\mathcal{A}_L$, where $[\gamma, f]$ is the strong right unity of the Γ-semiring S.*

Proof. Let V be any open set in $\mathcal{A}_L$ and $\rho \in V$. So $\rho^+ \in V^+$. Then $V^+ \in \tau_{\mathcal{A}_S}$ implies that there exists some $(a,b) \in S \times S$ such that $\rho^+ \in C\Delta(a,b) \subseteq V^+$, by Theorem 3.2.18. Therefore $\rho = (\rho^+)^{+'} \in (C\Delta(a,b))^{+'} \subseteq (V^+)^{+'} = V$. This implies that $\rho \in C\Delta([a,\gamma],[b,\gamma]) \subseteq V$. □

Now we study the separation properties via operator.

Theorem 3.2.20. *The space $\mathcal{A}_S$ is T_0.*

Proof. Proof follows from Theorems 3.1.10 (i) and 3.2.8. □

Theorem 3.2.21. *The space $\mathcal{A}_S$ is T_1 if and only if no element of $\mathcal{A}_S$ is contained in any other element of $\mathcal{A}_S$.*

Proof. Let $\mathcal{A}_S$ is a T_1-space. Then $\mathcal{A}_L$ is a T_1-space by Theorem 3.2.8. Therefore by Theorem 3.1.10 (ii), it follows that no element of $\mathcal{A}_L$ is contained in any other element of $\mathcal{A}_L$ which further implies that no element of $\mathcal{A}_S$ is contained in any other element of $\mathcal{A}_S$ by Theorem 2.2.20. By reverse implication the converse part follows. □

Theorem 3.2.22. *The space $\mathcal{A}_S$ is T_2 if and only if for any two distinct elements ρ_1, ρ_2 of $\mathcal{A}_S$, there exists two pairs $(a,b),(c,d)$ of elements of $S \times S$ such that $(a,b) \notin \rho_1$, $(c,d) \notin \rho_2$ and $(a\gamma c + b\gamma d, a\gamma d + b\gamma c) \in \rho$, for all $\rho \in \mathcal{A}_S$, where $[\gamma, f]$ is the strong right unity of the Γ-semiring S.*

Proof. Let the space $\mathcal{A}_S$ be T_2. Then for any two distinct congruences ρ_1, ρ_2 of $\mathcal{A}_S$, there exist two open sets $C\Delta(a,b)$ and $C\Delta(c,d)$ such that $\rho_1 \in C\Delta(a,b)$ and $\rho_2 \in C\Delta(c,d)$ and $C\Delta(a,b) \cap C\Delta(c,d) = \emptyset$. Therefore $(a,b) \notin \rho_1$ and $(c,d) \notin \rho_2$. Suppose if possible there exists ρ in $\mathcal{A}_S$ such that $(a\gamma c + b\gamma d, a\gamma d + b\gamma c) \notin \rho$. Then by Lemma 3.2.12,

$$\rho \in C\Delta(a\gamma c + b\gamma d, a\gamma d + b\gamma c) = C\Delta(a,b) \cap C\Delta(c,d) = \emptyset, \text{ a contradiction.}$$

Hence the given condition is proved.
Conversely, let the condition hold. Let ρ_1, ρ_2 be two distinct elements of $\mathcal{A}_S$. Then by our assumption there exist two elements $(a,b),(c,d) \in S \times S$ such that $(a,b) \notin \rho_1$, $(c,d) \notin \rho_2$ and $(a\gamma c + b\gamma d, a\gamma d + b\gamma c) \in \rho$ for all $\rho \in \mathcal{A}_S$. Therefore $\rho_1 \in C\Delta(a,b)$ and $\rho_2 \in C\Delta(c,d)$ and also there does not exist any congruence in

$$C\Delta(a\gamma c + b\gamma d, a\gamma d + b\gamma c) = C\Delta(a,b) \cap C\Delta(c,d),$$

i.e., $C\Delta(a,b) \cap C\Delta(c,d) = \emptyset$. Hence the space is T_2. □

The next result is required for the proof of Theorem 3.2.24.

Lemma 3.2.23.

$$C\Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]) = C\Delta(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma]),$$

where $[\gamma, f]$ *is the strong right unity of the* Γ*-semiring* S*.*

Proof. Let $\rho \in \Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j])$. Then $(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]) \in \rho$ which implies $(\sum_{i=1}^{m}[x_i,\alpha_i][f,\gamma],\sum_{j=1}^{n}[y_j,\beta_j][f,\gamma]) \in \rho$,
i.e., $(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma]) \in \rho$. Therefore $\rho \in \Delta(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma])$.
Hence $\Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]) \subseteq \Delta(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma])$,
i.e., $C\Delta(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma]) \subseteq C\Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j])$.
Again let $\sigma \in C\Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j])$.
Then $(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]) \notin \sigma$ which implies $(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma]) \notin \sigma$
i.e., $\sigma \in C\Delta(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma])$.
Therefore $C\Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]) \subseteq C\Delta(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma])$.
Hence $C\Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]) = C\Delta(\sum_{i=1}^{m}[x_i\alpha_i f,\gamma],\sum_{j=1}^{n}[y_j\beta_j f,\gamma])$. □

Theorem 3.2.24. *The space* $\mathcal{A}_S$ *is a regular space if and only if for any proper congruence* $\rho \in \mathcal{A}_S$ *and* $(a,b) \notin \rho$*, there exists a congruence* σ *on the* Γ*-semiring* S *and* $(c,d) \in S \times S$ *such that*

$$\rho \in C\Delta(c,d) \subseteq \Delta(\sigma) \subseteq C\Delta(a,b).$$

Proof. Let $[\gamma, f]$ be the strong right unity of S. Let $\mathcal{A}_S$ be regular. Then $\mathcal{A}_L$ is also regular (*cf.* Theorem 3.2.8). Now let $\rho \in \mathcal{A}_S$ and $(a,b) \in S \times S$ such that $(a,b) \notin \rho$. Therefore for $\rho^{+'} \in \mathcal{A}_L$ (*cf.* Theorem 2.2.20) and $([a,\gamma],[b,\gamma]) \notin \rho^{+'}$, there exists a congruence σ_1 on L and $(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]) \in L \times L$ such that

$$\rho^{+'} \in C\Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]) \subseteq \Delta(\sigma_1) \subseteq C\Delta([a,\gamma],[b,\gamma]).$$

Therefore $\rho \in (C\Delta(\sum_{i=1}^{m}[x_i,\alpha_i],\sum_{j=1}^{n}[y_j,\beta_j]))^{+} \subseteq (\Delta(\sigma_1))^{+} \subseteq (C\Delta([a,\gamma],[b,\gamma]))^{+}$.
Hence by Lemmas 3.2.9 (ii), 3.2.23 and Proposition 3.2.15,

$$\rho \in C\Delta(\sum_{i=1}^{m}x_i\alpha_i f,\sum_{j=1}^{n}y_j\beta_j f) \subseteq \Delta(\sigma_1^{+}) \subseteq C\Delta(a,b),$$

where σ_1^{+} is a congruence on S.
The converse part can be proved similarly by Theorem 3.1.10 (iv) and Theorem 3.2.8. □

Theorem 3.2.25. *Let A be a subset of $\mathcal{A}_S$. A is dense in $\mathcal{A}_S$ if and only if* $\bigcap_{\rho_i \in A} \rho_i = \bigcap_{\rho_i \in \mathcal{A}_S} \rho_i$.

Proof. Let A be a dense subset of $\mathcal{A}_S$. So $\overline{A} = \mathcal{A}_S$. Then using Lemma 3.2.2 (i) we have, $\overline{A^{+'}} = (\overline{A})^{+'} = (\mathcal{A}_S)^{+'} = \mathcal{A}_L$. Hence $A^{+'}$ is dense in $\mathcal{A}_L$. Therefore applying Theorem 3.1.15 we have, $\bigcap_{\rho_i^{+'} \in A^{+'}} \rho_i^{+'} = \bigcap_{\rho_i^{+'} \in \mathcal{A}_L} \rho_i^{+'}$. So $(\bigcap_{\rho_i \in A} \rho_i)^{+'} = (\bigcap_{\rho_i \in \mathcal{A}_S} \rho_i)^{+'}$ whence it follows that $\bigcap_{\rho_i \in A} \rho_i = \bigcap_{\rho_i \in \mathcal{A}_S} \rho_i$. Converse part follows by reversing the above argument.

□

Theorem 3.2.26. *Let A be a closed subset of $\mathcal{A}_S$. Then A is irreducible if and only if $\bigcap_{\rho_i \in A} \rho_i$ is a prime congruence on S.*

Proof. Let A be an irreducible subset of $\mathcal{A}_S$. To show $A^{+'}$ is irreducible in $\mathcal{A}_L$, let us suppose that for two closed sets X, Y in $\mathcal{A}_L$, $A^{+'} \subseteq X \cup Y$.
Then $A = (A^{+'})^{+} \subseteq (X \cup Y)^{+}$. So using Lemma 3.2.4 (ii) we have, $A \subseteq X^{+} \cup Y^{+}$.
Since A is irreducible in $\mathcal{A}_S$,

$$\text{either } A \subseteq X^{+} \text{ or } A \subseteq Y^{+}, \text{ i.e., either } A^{+'} \subseteq X \text{ or } A^{+'} \subseteq Y.$$

Hence $A^{+'}$ is irreducible in $\mathcal{A}_L$. Therefore in view of Theorem 3.1.14 we obtain that $\bigcap_{\rho_i \in A} \rho_i^{+'} = (\bigcap_{\rho_i \in A} \rho_i)^{+'}$ is a prime congruence on L.
So $\bigcap_{\rho_i \in A} \rho_i$ is a prime congruence on S.
The converse part of the proof follows by reversing the above argument.

□

Theorem 3.2.27. *The space $\mathcal{A}_S$ is compact if and only if for any collection of pairs $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ of elements in S, there exists a finite subcollection $\{(a_i, b_i) : i = 1, 2, ..., n\}$ in $S \times S$ such that for any $\rho \in \mathcal{A}_S$, there exists (a_i, b_i) from the subcollection such that $(a_i, b_i) \notin \rho$.*

Proof. Let the space $\mathcal{A}_S$ be compact and let $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ be a collection of pairs of elements in S and $\rho \in \mathcal{A}_S$. By Theorem 3.2.8, the space $\mathcal{A}_L$ is compact. Also $\{([a_\alpha, \gamma], [b_\alpha, \gamma])\}_{\alpha \in \Lambda}$ is a collection of pairs of elements in L, where $[\gamma, f]$ is the strong right unity of S. Then by Theorem 3.1.11, there exists a finite subcollection $\{([a_k, \gamma], [b_k, \gamma]) : k = 1, 2, ..., p\}$ of pairs of elements in L and there exists a member $([a_k, \gamma], [b_k, \gamma])$ of the subcollection such that $([a_k, \gamma], [b_k, \gamma]) \notin \rho^{+'}$, where $\rho^{+'} \in \mathcal{A}_L$. Therefore we obtain a finite subcollection $\{(a_k \gamma f, b_k \gamma f) = (a_k, b_k) : k = 1, 2, ..., p\}$ of pairs of elements in S such that for any $\rho \in \mathcal{A}_S$, there exists $(a_k, b_k) \in S \times S$ such that

$(a_k, b_k) \notin \rho$. Hence the condition holds. The converse part can be prove analogously by reversing the arguments above. □

Theorem 3.2.28. *The space $\mathcal{A}_S$ is disconnected if and only if there exists a congruence ρ on S and a collection of pairs $\{(a_\alpha, b_\alpha)\}_{\alpha\in\Lambda}$ of elements in S not belonging to ρ such that if $\rho' \in \mathcal{A}_S$ and $(a_\alpha, b_\alpha) \in \rho'$ for all $\alpha \in \Lambda$ then $\rho \setminus \rho' \neq \emptyset$.*

Proof. Let the space $\mathcal{A}_S$ be disconnected. Then by Theorem 3.2.8, $\mathcal{A}_L$ is disconnected. Therefore by Theorem 3.1.12, there exists a congruence σ on L and a collection $\{(x_\alpha, y_\alpha) : \alpha \in \Lambda\}$ of pairs of elements in L not belonging to σ such that for any $\sigma_1 \in \mathcal{A}_L$ and $(x_\alpha, y_\alpha) \in \sigma_1$ for all $\alpha \in \Lambda$, $\sigma \setminus \sigma_1 \neq \emptyset$. Then $\{(x_\alpha e, y_\alpha e) : \alpha \in \Lambda\}$ is a collection of pairs of elements in S not belonging to $\sigma^+ \in \mathcal{A}_S$, where $[e, \delta]$ is the strong left unity of S. Now let $\rho \in \mathcal{A}_S$ and $(x_\alpha e, y_\alpha e) \in \rho$ for all $\alpha \in \Lambda$. Then $\rho^{+'} \in \mathcal{A}_L$ and $([x_\alpha e, \delta], [y_\alpha e, \delta]) = (x_\alpha, y_\alpha) \in \rho^{+'}$ for all $\alpha \in \Lambda$. Therefore it follows $\sigma \setminus \rho^{+'} \neq \emptyset$. This implies $(\sigma \setminus \rho^{+'})^+ \neq \emptyset$ whence it follows that $\sigma^+ \setminus \rho \neq \emptyset$. Hence the condition holds. The converse part can be proved analogously reversing the above arguments.

□

In [18] we haven't been able to prove Theorems 3.2.27, 3.2.28 via operator then, so we took the approach of proving those in a straightforward way. Here we establish those results via operator.

Theorem 3.2.29. *$\mathcal{A}_S$ is a $T_{1/4}$-space if and only if every prime congruence on S is either maximal or minimal prime.*

Proof. By Theorem 3.2.8, $\mathcal{A}_S$ is a $T_{1/4}$-space if and only if $\mathcal{A}_L$ is a $T_{1/4}$-space. Now from Theorem 3.1.18 we obtain every prime congruence on L is either maximal or minimal prime which means in view of theorem 2.2.20 so is every prime congruence on S whenever $\mathcal{A}_S$ is a $T_{1/4}$-space. Therefore it follows that $\mathcal{A}_S$ is a $T_{1/4}$-space if and only if every prime congruence on S is either maximal or minimal prime. □

Analogous to the concept of absolutely prime-irreducible congruences on a semiring, we call a congruence ρ on S to be an *absolutely prime-irreducible* congruence if for every subset $\{\rho_i\}_{i\in\Lambda}$ of $\mathcal{A}_S$ such that $\bigcap_{i\in\Lambda} \rho_i \subseteq \rho$, there exists j such that $\rho_j \subseteq \rho$.

Remark 3.2.30. From the above definition and the order preserving bijection established in Theorem 3.2.8 it is clear that if ρ is an absolutely prime-irreducible congruence on S then so is $\rho^{+'}$ on L and vice versa.

Theorem 3.2.31. *$\mathcal{A}_S$ is a $T_{1/2}$-space if and only if every prime congruence on S is either maximal or absolutely prime-irreducible minimal prime.*

Proof. By Theorem 3.2.8, $\mathcal{A}_S$ is a $T_{1/2}$-space if and only if $\mathcal{A}_L$ is a $T_{1/2}$-space. Now from Theorem 3.1.20 we obtain that every prime congruence on L is either maximal or absolutely prime-irreducible minimal prime which means, in view of theorem 2.2.20 and Remark 3.2.30 that so is every prime congruence on S whenever $\mathcal{A}_S$ is a $T_{1/2}$-space. Therefore it follows that $\mathcal{A}_S$ is a $T_{1/2}$-space if and only if every prime congruence on S is either maximal or minimal prime. □

Remark 3.2.32. We observe that

$$(\mathcal{A}_S \setminus \{\rho\})^{+'} = (\{\sigma \in \mathcal{A}_S : \rho \neq \sigma\})^{+'} = \{\sigma^{+'} \in \mathcal{A}_L : \rho^{+'} \neq \sigma^{+'}\} = \mathcal{A}_L \setminus \{\rho^{+'}\}.$$

Theorem 3.2.33. *The following are equivalent.*

- (i) *$\mathcal{A}_S$ is a $T_{3/4}$-space.*
- (ii) *Every point ρ is either closed or satisfies $\mathcal{A}_S \setminus \{\rho\} = \overline{C\Delta(\rho)}$.*
- (iii) *Every point ρ is either closed or an isolated point such that*

$$\bigcap\{\sigma \in \mathcal{A}_S : \rho \neq \sigma\} = \bigcap C\Delta(\rho).$$

- (iv) *Every prime congruence ρ on S is either maximal or an absolutely prime-irreducible congruence such that $\bigcap\{\sigma \in \mathcal{A}_S : \rho \neq \sigma\} = \bigcap C\Delta(\rho)$.*

Proof. Before going to the main proof we observe the following facts which are necessitated.
By Lemma 3.2.2, $\overline{C\Delta(\sigma^{+'})} = \overline{(C\Delta(\sigma))^{+'}} = (\overline{C\Delta(\sigma)})^{+'}$..........$(a)$
$(\mathcal{A}_S \setminus \{\rho\})^{+'} = (\{\sigma \in \mathcal{A}_S : \rho \neq \sigma\})^{+'} = \{\sigma^{+'} \in \mathcal{A}_L : \rho^{+'} \neq \sigma^{+'}\} = \mathcal{A}_L \setminus \{\rho^{+'}\}$.......$(b)$
For any subset $\mathcal{A}$ of $\mathcal{A}_S$, $\bigcap_{(\rho_i)^{+'} \in (\mathcal{A})^{+'}} (\rho_i)^{+'} = \bigcap_{\rho_i \in \mathcal{A}} (\rho_i)^{+'} = (\bigcap_{\rho_i \in \mathcal{A}} \rho_i)^{+'}$......$(c)$

$(i) \Leftrightarrow (ii)$: By Theorem 3.2.8, $\mathcal{A}_S$ is a $T_{3/4}$-space if and only if $\mathcal{A}_L$ is a $T_{3/4}$-space. So let $\rho \in \mathcal{A}_S$. Then by Theorem 3.1.22, the point $\rho^{+'} \in \mathcal{A}_L$ is either closed in $\mathcal{A}_L$ or satisfies $\mathcal{A}_L \setminus \{\rho^{+'}\} = \overline{C\Delta(\rho^{+'})}$ in $\mathcal{A}_L$. We see that $(\mathcal{A}_L \setminus \{\rho^{+'}\})^{+} = \mathcal{A}_S \setminus \{\rho\}$ (by (b)) and $(\overline{C\Delta(\rho^{+'})})^{+} = \overline{C\Delta(\rho)}$ (by (a)). Hence ρ is either closed or $\mathcal{A}_S \setminus \{\rho\} = \overline{C\Delta(\rho)}$. Converse part can be proved analogously.

$(i) \Leftrightarrow (iii)$: Let $\mathcal{A}_S$ be a $T_{3/4}$-space. Then $\mathcal{A}_L$ is a $T_{3/4}$-space. So let $\rho \in \mathcal{A}_S$. Then by Theorem 3.1.22, the point $\rho^{+'} \in \mathcal{A}_L$ is either closed in $\mathcal{A}_L$ or an isolated point such

that $\bigcap\{\sigma \in \mathcal{A}_L : \sigma \neq \rho^{+'}\} = \bigcap C\Delta(\rho^{+'})$.
Now by (c), $(\bigcap C\Delta(\rho^{+'}))^+ = \bigcap(C\Delta(\rho))^{+'})^+ = \bigcap C\Delta(\rho)$ and by (b) and Theorem 3.2.8, $(\bigcap\{\sigma_1 \in \mathcal{A}_L : \sigma_1 \neq \rho^{+'}\})^+ = \bigcap\{\sigma_1^{+'} \in \mathcal{A}_S : \sigma_1^{+'} \neq \rho\} = \bigcap\{\sigma \in \mathcal{A}_S : \rho \neq \sigma\}$.
Also ρ is an isolated point in $\mathcal{A}_S$ if and only if $\rho^{+'}$ is an isolated point in $\mathcal{A}_L$ if and only if $\rho^{+'}$ is an absolutely prime-irreducible minimal prime congruence in $\mathcal{A}_L$ (by Lemma 3.1.19) if and only if ρ is an absolutely prime-irreducible minimal prime congruence in $\mathcal{A}_S$ (by Remark 3.2.30). Hence in view of the above arguments we obtain that ρ is either closed or an isolated point such that $\bigcap\{\sigma \in \mathcal{A}_S : \rho \neq \sigma\} = \bigcap C\Delta(\rho)$.
Converse part can be proved analogously.

$(i) \Leftrightarrow (iv)$: This can be proved similarly with the same arguments as above. □

We conclude this section by deducing some results of the particular Γ-semiring Z_0^- (the set of all non-positive integers) and its structure space.

Proposition 3.2.34. *The left operator semiring $L(Z_0^-)$ of the Γ-semiring of all non-positive integers Z_0^-, where $\Gamma = Z_0^-$, is isomorphic to the set of all non-negative integers Z_0^+ via the mapping*

$$\textstyle\sum_{i=1}^m [x_i, \alpha_i] \mapsto \sum_{i=1}^m x_i\alpha_i,$$

where $\sum_{i=1}^m [x_i, \alpha_i]$ is an element of the left operator semiring of Z_0^-.

Proof. Let us define a map $\phi : L(Z_0^-) \to Z_0^+$ by $\phi(\sum_{i=1}^m [x_i, \alpha_i]) := \sum_{i=1}^m x_i\alpha_i$.
Then $\sum_{i=1}^m [x_i, \alpha_i] = \sum_{j=1}^n [y_j, \beta_j]$ implies $\sum_{i=1}^m x_i\alpha_i s = \sum_{j=1}^n y_j\beta_j s$ for all $s \in Z_0^-$. In particular, if we take $s = -1$, it follows that

$$\textstyle\sum_{i=1}^m x_i\alpha_i = \sum_{j=1}^n y_j\beta_j, \text{ i.e., } \phi(\sum_{i=1}^m [x_i, \alpha_i]) = \phi(\sum_{i=1}^m [y_j, \beta_j]).$$

Therefore ϕ is well defined. Similarly we can prove that ϕ is one-one.
For $x \in Z_0^+$, $[-x, -1] \in L(Z_0^-)$ such that $\phi([-x, -1]) = x$. Hence the mapping is bijective. Clearly it is a semiring homomorphism. Therefore the semirings $L(Z_0^-)$ and Z_0^+ are isomorphic. □

Theorem 3.2.35. *The structure space $\mathcal{A}_{Z_0^-}$ of all prime congruences on Z_0^- and the structure space $\mathcal{A}_{Z_0^+}$ of all prime congruences on Z_0^+ are homeomorphic.*

Proof. By Proposition 3.2.34, $L(Z_0^-)$ and Z_0^+ are isomorphic via the mapping

$$\textstyle\sum_{i=1}^m [x_i, \alpha_i] \mapsto \sum_{i=1}^m x_i\alpha_i \text{ with inverse mapping } a \mapsto [-a, -1] \text{ for } a \in Z_0^+.$$

So by Theorem 3.1.24 and Theorem 1.3.38, $\mathcal{A}_{Z_0^+}$ and $\mathcal{A}_{L(Z_0^-)}$ are homeomorphic via the mapping $\rho \mapsto \rho'$, where $(a,b) \in \rho$ if and only if $([-a,-1],[-b,-1]) \in \rho'$. Again by Theorem 3.2.8, $\mathcal{A}_{Z_0^-}$ and $\mathcal{A}_{L(Z_0^-)}$ are homeomorphic via the mapping $\sigma \mapsto \sigma^{+'}$. Hence $\mathcal{A}_{Z_0^+}$ and $\mathcal{A}_{Z_0^-}$ are homeomorphic. □

Corollary 3.2.36. *$\mathcal{A}_{Z_0^-}$ is T_0, T_1, compact, connected but neither T_2 nor regular.*

Proof. $\mathcal{A}_{Z_0^+}$ is T_0, T_1, compact, connected but neither T_2 nor regular (*cf.* Example 3.1.26). So by Theorem 3.2.35, the result follows. □

From Theorems 3.1.27 and 3.2.35, we get the following theorem.

Theorem 3.2.37. *$\mathcal{A}_{Z_0^-}$ is a $T_{1/4}$, $T_{1/2}$, $T_{3/4}$ space.*

Corollary 3.2.38. *The prime congruences on the Γ-semiring Z_0^- are precisely of the form*

$$\rho'_p := \{(a,b) : a,b \in Z_0^- \text{ and } a-b \text{ is divisible by } p\}$$

for some prime number p.

Proof. Let us denote the homeomorphism $\mathcal{A}_{Z_0^+} \to \mathcal{A}_{L(Z_0^-)}$, obtained in Theorem 3.2.35 by ψ, i.e., $\psi(\rho) = \rho'$, where $(a,b) \in \rho$ if and only if $([-a,-1],[-b,-1]) \in \rho'$ (*cf.* proof of Theorem 3.2.35). Again by Theorem 2.2.20, $\phi : \mathcal{A}_{L(Z_0^-)} \to \mathcal{A}_{Z_0^-}$ is a bijective correspondence, where $\phi(\rho) = \rho^+$.
Let us define $f : \mathcal{A}_{Z_0^+} \to \mathcal{A}_{Z_0^-}$ by $f(\rho) := \phi(\psi(\rho)) = \phi(\rho') = (\rho')^+$. Then f being the composition of two bijections is a bijection. Let $\sigma \in \mathcal{A}_{Z_0^-}$. Then there exists $\rho \in \mathcal{A}_{Z_0^+}$ such that $f(\rho) = \sigma$, i.e., $(\rho')^+ = \sigma$.
By Theorem 3.1.25, $\rho = \{(a,b) : a,b \in Z_0^+ \textit{ and } a-b \text{ is divisible by some prime } p\}$.
So $\rho' = \{([-a,-1],[-b,-1]) : a,b \in Z_0^+ \textit{ and } a-b \text{ is divisible by some prime } p\}$.
Therefore $(\rho')^+ = \{(-a,-b) : a,b \in Z_0^+ \textit{ and } a-b \text{ is divisible by some prime } p\}$
$= \{(a,b) : a,b \in Z_0^- \textit{ and } a-b \text{ is divisible by some prime } p\}$ (see Definition 2.2.1).
Hence the proof is complete.

□

Note 3.2.39. It is well-known that the set of non-positive integers is an example of a ternary semiring with usual binary addition and ternary multiplication. S. Kar, in his paper [47], studied the ideal theory of this ternary semiring Z_0^- of non-positive integers and the structure space of the class of prime k-ideals of the ternary semiring

Z_0^-. Here in our work, we have taken the approach of studying the structure space of congruences on Z_0^- via operator, considering Z_0^-, as a Γ-semiring instead of a ternary semiring, where $\Gamma = Z_0^-$. As far as we are aware, there is no concept of operator semirings of a ternary semiring as such. So there are two reasons of studying Z_0^- as a Γ-semiring via its operator semiring and not as ternary semiring: (i) If we consider the structure as ternary semiring, we have to prove all the results in the straightforward way which is supposed to be lengthy comparatively. (ii) In our study we have found an interesting connection between Z_0^+ and Z_0^-, considering it as Γ-semiring. Using this connection, the proofs of the results on Z_0^- have not only been easy but also precise.

Now we will study the space of all maximal regular congruences which are prime congruences on a Γ-semiring. We have shown in previous chapter that none of the maximal congruence and the prime congruence on a Γ-semiring implies the other in general. Using Corollary 3.2.38 and Example 2.1.17, we deduce that on the Γ-semiring Z_0^- every prime congruence is maximal and regular. In this section we study the space of all maximal regular congruences which are prime congruences as a subspace of the space of all prime congruences on any commutative Γ-semiring with strong unities.

Let us denote the space of all maximal regular congruences which are prime congruences on a commutative Γ-semiring S with strong unities as $\mathcal{B}_S$. Also $\mathcal{B}_L$ denotes that on its operator semiring L and $\mathcal{B}_R$ denotes that on a semiring R.

Remarks 3.2.40. As $\mathcal{B}_S \subseteq \mathcal{A}_S$, taking the subspace topology $\tau_{\mathcal{B}_S}$ on $\mathcal{B}_S$, we have the following topological characteristics of the space $\mathcal{B}_S$, inherited from the space $\mathcal{A}_S$.

(i) Any closed set in $\mathcal{B}_S$ is of the form $\Delta(\rho) \cap \mathcal{B}_S$ denoted by $m(\rho)$, where ρ is a congruence on S and $m(\rho) = \{\rho' \in \mathcal{B}_S : \rho \subseteq \rho'\} = \Delta(\rho) \cap \mathcal{B}_S$.

(ii) $\{Cm(a,b) : (a,b) \in S \times S\}$ is an open base for $\mathcal{B}_S$, where $m(a,b) = \{\rho \in \mathcal{B}_S : (a,b) \in \rho\} = \Delta(a,b) \cap \mathcal{B}_S$.

(iii) $\mathcal{B}_S$ is T_0.

(iv) $\mathcal{B}_S$ is T_1 as no element of $\mathcal{B}_S$ is contained in any other element of $\mathcal{B}_S$.

Using Theorem 3.2.13 and the fact that $m(a,b) = \Delta(a,b) \cap \mathcal{B}_S$ we have the following.

Theorem 3.2.41.

$$Cm(a,b) \cap Cm(c,d) = Cm(a\gamma c + b\gamma d, a\gamma d + b\gamma c),$$

where $a, b, c, d \in S$ and $[\gamma, f]$ is the strong right unity of the Γ-semiring S.

Using Theorem 3.2.41, the proof of Theorem 3.2.42 is analogous to the proof of Theorem 3.2.22.

Theorem 3.2.42. *The space $\mathcal{B}_S$ is T_2 iff for any two distinct elements ρ_1, ρ_2 of $\mathcal{B}_S$, there exists two pairs $(a,b),(c,d)$ of elements of $S \times S$ such that $(a,b) \notin \rho_1$, $(c,d) \notin \rho_2$ and $(a\gamma c + b\gamma d, a\gamma d + b\gamma c) \in \rho$ for all $\rho \in \mathcal{B}_S$, where $[\gamma, f]$ is the strong right unity of the Γ-semiring S.*

Also by Theorems 2.2.16 and 2.2.20 and Lemma 3.2.2, we can prove the following Theorem.

Theorem 3.2.43. *$(\mathcal{B}_S, \tau_{\mathcal{B}_S})$ and $(\mathcal{B}_L, \tau_{\mathcal{B}_L})$ are homeomorphic, where $\tau_{\mathcal{B}_S}$ and $\tau_{\mathcal{B}_L}$ are the subspace topologies corresponding to the spaces $\mathcal{B}_S$ and $\mathcal{B}_L$ respectively.*

Theorem 3.2.44. *$\mathcal{B}_S$ is compact.*

Proof. By Theorems 3.2.43 and 1.3.36, the space $\mathcal{B}_S$ is compact. □

With similar arguments as in Theorems 3.2.24 and 3.1.12, we can prove the following results.

Theorem 3.2.45. *The space $\mathcal{B}_S$ is a regular space if and only if for any $\rho \in \mathcal{B}_S$ and $(a,b) \notin \rho$, there exists a congruence σ on S and $(c,d) \in S \times S$ such that*

$$\rho \in Cm(c,d) \subseteq m(\sigma) \subseteq Cm(a,b)$$

.

Theorem 3.2.46. *The space $\mathcal{B}_S$ is disconnected if and only if there exists a congruence ρ on S and a collection of pairs $\{(a_\alpha, b_\alpha)\}_{\alpha \in \Lambda}$ of elements in S not belonging to ρ such that if $\rho' \in \mathcal{B}_S$ and $(a_\alpha, b_\alpha) \in \rho'$ for all $\alpha \in \Lambda$ then $\rho \setminus \rho' \neq \emptyset$.*

Theorem 3.2.47. *If two Γ-semirings are isomorphic then their corresponding structure spaces of all maximal regular congruences which are prime congruences (structure spaces of all prime congruences) are homeomorphic.*

Proof. Let S_1 be a Γ_1-semiring and S_2 be a Γ_2-semiring and they are isomorphic and L_1 and L_2 be the corresponding left operator semirings respectively. So L_1 and L_2 are isomorphic semirings by Theorem 2.2.21. Therefore the spaces corresponding to L_1 and L_2, i.e., $\mathcal{B}_{L_1}$ (resp. $\mathcal{A}_{L_1}$) and $\mathcal{B}_{L_2}$ (resp. $\mathcal{A}_{L_2}$) are homeomorphic by Theorem 3.1.24. Since homeomorphism is a transitive relation, we can conclude that the required spaces are homeomorphic by Theorem 3.2.43 (resp. by Theorem 3.2.8). □

CHAPTER 4

A STUDY ON THE Γ-SEMIRING $C_{-}(X)$

4

A study on the $\Gamma-$semiring $C_{-}(X)$

In this chapter we mainly focus on the study of the Γ-semiring $C_{-}(X)$. Considering $C_{-}(X)$, the set of all non-positive valued continuous functions of this topological space X with usual pointwise addition and multiplication of functions, as a Γ-semiring (where $\Gamma = C_{-}(X)$), we have taken the approach of studying $C_{-}(X)$ via its operator semiring. Some results on the congruences and the topological properties of the structure space of maximal regular congruences on the Γ-semiring $C_{-}(X)$ over a Tychonoff space X have been studied as a natural extension to the studies on the semiring $C_{+}(X)$. We have found that the structure space of $C_{-}(X)$ is T_2 and compact and the real structure space of $C_{-}(X)$ is realcompact. Not only that we have been able to establish those spaces as another model of the Stone-Čech compactification and Hewitt realcompactification of a Tychonoff space X (other than the ring $C(X)$ and the semiring $C_{+}(X)$). Also the Γ-semiring analogue of the 'Banach-Stone Theorem' and 'Hewitt Isomorphism Theorem' have been obtained using these results. To accomplish our study, we have established the homeomorphism between the structure spaces of the Γ-semiring $C_{-}(X)$ and the semiring $C_{+}(X)$. For that we have established a relationship between the semiring $C_{+}(X)$ and the Γ-semiring $C_{-}(X)$ which have been instrumental in our present study. Lastly we have studied some results on the z-ideals and z^o-ideals of $C_{-}(X)$ characterizing the topological space X.

This chapter is partially based on the work published in the following paper:
Soumi Basu, Sarbani Mukherjee (Goswami) and Sujit Kumar Sardar, *The structure space of* $C_{-}(X)$ *via that of* Γ*-semirings, Asian-European Journal of Mathematics,* 16 (8) (2023), 2350142 (20 pages).

In **section 1**, we first establish an isomorphism between the semiring $C_+(X)$ and the left operator semiring of the Γ-semiring $C_-(X)$ (*cf.* Theorem 4.1.2). Then we study some results on the congruences (viz. maximal regular, prime congruences) on $C_-(X)$ via operator (*cf.* Theorem 4.1.4, 4.1.7). It is found that the structure spaces of maximal regular congruences on the semiring $C_+(X)$ of non-negative real valued continuous functions and the Γ-semiring $C_-(X)$ are homeomorphic (*cf.* Theorem 4.1.6).

In **section 2**, we obtain some important results of the structure space of $C_-(X)$. It is found that the real structure spaces of the semiring $C_+(X)$ of non-negative real valued continuous functions and the Γ-semiring $C_-(X)$ are homeomorphic (*cf.* Theorem 4.2.4). Moreover it is shown that the structure space of maximal regular congruences on $C_-(X)$ is the Stone-Čech compactification of X (*cf.* Theorem 4.2.2) and the real structure space of maximal regular congruences that are real on $C_-(X)$ is the Hewitt realcompactification of X (*cf.* Theorem 4.2.6), where X is a Tychonoff space. Furthermore, the Γ-semiring analogue of the 'Banach-Stone Theorem' and 'Hewitt Isomorphism Theorem' is obtained (*cf.* Theorems 4.2.7, 4.2.8). Finally for two realcompact spaces X and Y, we establish an equivalence of isomorphism between the rings, semirings and Γ-semirings of continuous functions (*cf.* Theorem 4.2.9).

In **section 3**, we introduce the notions of the z-ideals and z^o-ideals of the Γ-semiring $C_-(X)$ and establish their correspondences with the z-ideals and z^o-ideals of the semiring $C_+(X)$. Then via operator we study the results on those ideals of $C_-(X)$ which characterizes the topological space X (for example, see Theorems 4.3.17, 4.3.23, 4.3.29, 4.3.32, 4.3.35).

4.1 The congruences on $C_-(X)$

In this section we study the results related to the congruences on the Γ-semiring $C_-(X)$.

Note 4.1.1. $C_-(X)$ is a Γ-semiring with the ternary operation:

$$C_-(X) \times C_-(X) \times C_-(X) \to C_-(X)$$
$$(f, \alpha, g) \mapsto f\alpha g$$

defined by $(f\alpha g)(x) = f(x)\alpha(x)g(x)$ for all $x \in X$, i.e., here $S = C_-(X)$ and $\Gamma = C_-(X)$ (see Example 1.4.3). Clearly $f\alpha h = h\alpha f$ for all $f, \alpha, h \in C_-(X)$. So $C_-(X)$ is a commutative Γ-semiring. Now the elements of the left operator semiring $L(C_-(X))$ of the Γ-semiring $C_-(X)$ are of the form $\sum_{i=1}^{m}[f_i, \alpha_i]$. Clearly $[-\mathbf{1}, -\mathbf{1}]$ is strong left unity as well as right unity of $C_-(X)$, where $-\mathbf{1}(x) = -1$ for all $x \in X$. It can be easily checked that every cancellative congruence is a regular congruence on $C_-(X)$.

We first establish the connection between the left operator semiring of the Γ-semiring $C_-(X)$ and the semiring $C_+(X)$.

Theorem 4.1.2. *The left operator semiring $L(C_-(X))$ of the Γ-semiring $C_-(X)$ is isomorphic to $C_+(X)$ via the mapping $\sum_{i=1}^{m}[f_i, \alpha_i] \mapsto \sum_{i=1}^{m} f_i\alpha_i$.*

Proof. Let us define a map $\phi : L(C_-(X)) \to C_+(X)$ by $\phi(\sum_{i=1}^{m}[f_i, \alpha_i]) := \sum_{i=1}^{m} f_i\alpha_i$. Then $\sum_{i=1}^{m}[f_i, \alpha_i] = \sum_{j=1}^{n}[h_j, \beta_j]$ implies $\sum_{i=1}^{m} f_i\alpha_i s = \sum_{j=1}^{n} h_j\beta_j s$ for all $s \in C_-(X)$. In particular, if we take $s = -\mathbf{1}$, it follows that $\sum_{i=1}^{m} f_i(x)\alpha_i(x) = \sum_{j=1}^{n} h_j(x)\beta_j(x)$, i.e., $\phi(\sum_{i=1}^{m}[f_i, \alpha_i]) = \phi(\sum_{i=1}^{m}[h_j, \beta_j])$. Therefore ϕ is well defined. Similarly we can prove that ϕ is one-one. For $f \in C_+(X)$, $[-f, -\mathbf{1}] \in L(C_-(X))$ such that $\phi([-f, -\mathbf{1}]) = f$. Hence the mapping is bijective. Clearly it is a semiring homomorphism. Therefore the semirings $L(C_-(X))$ and $C_+(X)$ are isomorphic. □

Remark: We could have considered $C_-(X)$ as a ternary semiring. But for the same reason as mentioned in Note 3.2.39, we have taken the same approach of studying the Γ-semiring $C_-(X)$ via its operator semiring.

Theorem 4.1.3. *If the semirings $C_+(X)$ and $C_+(Y)$ are isomorphic then the Γ-semirings $C_-(X)$ and $C_-(Y)$ are isomorphic and vice versa.*

Proof. Let the semirings $C_+(X)$ and $C_+(Y)$ be isomorphic and ϕ be the isomorphism. Let us define a map $\psi : C_-(X) \to C_-(Y)$ by $\psi(f) := -\phi(-f)$. By routine verification it can be obtained that ψ is a semigroup isomorphism between $C_-(X)$ and $C_-(Y)$, where $C_-(X)$ and $C_-(Y)$ are additive semigroups. Therefore (ψ, ψ) is a isomorphism between the Γ-semirings $C_-(X)$ and $C_-(Y)$.
Conversely, let the Γ-semirings $C_-(X)$ and $C_-(Y)$ be isomorphic. Then by Theorem 2.2.21, the left operator semirings $L_{C_-(X)}$ and $L_{C_-(Y)}$ are isomorphic. Again since $L_{C_-(X)}$ and $L_{C_-(Y)}$ are isomorphic to the semirings $C_+(X)$ and $C_+(Y)$ (*cf.* Theorem 4.1.2), it implies that $C_+(X)$ and $C_+(Y)$ are isomorphic. Hence the theorem. □

Notations:

$\mathcal{B}_1$:= The set of all maximal regular congruences on the Γ-semiring $C_-(X)$;
$\mathcal{B}_2$:= The set of all maximal regular congruences on the left operator semiring $L(C_-(X))$ of the Γ-semiring $C_-(X)$;
$\mathcal{B}_3$:= The set of all maximal regular congruences on the semiring $C_+(X)$;

We recall that every maximal regular congruence on the semiring $C_+(X)$ is a prime congruence on $C_+(X)$. Below we proof a similar result for $C_-(X)$.

Theorem 4.1.4. *Every maximal regular congruence on the* Γ*-semiring* $C_-(X)$ *is a prime congruence on* $C_-(X)$.

Proof. Let ρ be a maximal regular congruence on $C_-(X)$, i.e., $\rho \in \mathcal{B}_1$. By Theorem 2.2.16, $\rho^{+'}$ is a maximal regular congruence on $L(C_-(X))$, where $(a, b) \in \rho$ if and only if $([a, \gamma], [b, \gamma]) \in \rho^{+'}$ for all $\gamma \in \Gamma$. By Theorem 4.1.2 and Theorem 1.3.38 we obtain that $\mathcal{B}_2$ and $\mathcal{B}_3$ are in bijective correspondence via the mapping $\rho \mapsto \sigma$, where $(\sum_{i=1}^{m}[f_i, \alpha_i], \sum_{j=1}^{n}[g_j, \beta_j]) \in \rho$ if and only if $(\sum_{i=1}^{m} f_i\alpha_i, \sum_{j=1}^{n} g_j\beta_j) \in \sigma$. Therefore for some congruence $\sigma_1 \in \mathcal{B}_3$, $\rho^{+'}$ is mapped to σ_1. Therefore by Theorem 1.3.40, σ_1 is a prime congruence on $C_+(X)$. Again by Theorem 4.1.2 and Theorem 3.1.23, there is a bijection between the set of all prime congruences on $C_+(X)$ and $L(C_-(X))$ via the mapping $\rho \mapsto \rho_1$, where $(f, g) \in \rho$ if and only if $([-f, -\mathbf{1}], [-g, -\mathbf{1}]) \in \rho_1$. Therefore σ_1 is mapped to a prime congruence ρ_1 (say) on $L(C_-(X))$. So by Theorem 2.2.20, ρ_1^+ is a prime congruence on $C_-(X)$. Now let $(a, b) \in \rho$. Then $([a, \gamma], [b, \gamma]) \in \rho^{+'}$ for all $\gamma \in \Gamma$ which implies $(a\gamma, b\gamma) \in \sigma_1$ for all $\gamma \in \Gamma$. Therefore $([-a\gamma, -\mathbf{1}], [-b\gamma, -\mathbf{1}]) \in \rho_1$ for all $\gamma \in \Gamma$ whence it follows that $(a\gamma s, b\gamma s) \in \rho_1$ for all $\gamma \in \Gamma$, $s \in C_-(X)$. So $\rho \subseteq \rho_1^+$. Analogously the reverse inclusion can be obtained, i.e., $\rho_1^+ \subseteq \rho$. Therefore $\rho = \rho_1^+$. Hence it follows that ρ is a prime congruence on $C_-(X)$. $\square$

Remark 4.1.5. By Theorems 4.1.4, 1.3.40 we observe that the set of all maximal regular congruences coincides with the set $\mathcal{B}_{C_-(X)}$ ($\mathcal{B}_{C_+(X)}$) of all maximal regular congruences those are prime on $C_-(X)$ (resp. $C_+(X)$). So from now on, we will denote the spaces (equipped with Hull Kernel topology) $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ as $\mathcal{B}_{C_-(X)}$, $\mathcal{B}_{L(C_-(X))}$ and $\mathcal{B}_{C_+(X)}$ respectively.

Theorem 4.1.6. *The space* $\mathcal{B}_{C_-(X)}$ *of all maximal regular congruences on the* Γ*-semiring* $C_-(X)$ *and the space* $\mathcal{B}_{C_+(X)}$ *of all maximal regular congruences on the semiring* $C_+(X)$ *are homeomorphic.*

Proof. By Theorem 4.1.2, $L(C_-(X))$ and $C_+(X)$ are isomorphic via the mapping $\sum_{i=1}^{m}[f_i, \alpha_i] \mapsto \sum_{i=1}^{m} f_i\alpha_i$ with inverse mapping $f \mapsto [-f, -\mathbf{1}]$ for $f \in C_+(X)$. So by Theorem 3.1.24, their structure spaces $\mathcal{B}_{L(C_-(X))}$ and $\mathcal{B}_{C_+(X)}$ are homeomorphic. Again by Theorem 3.2.43, $\mathcal{B}_{C_-(X)}$ and $\mathcal{B}_{L(C_-(X))}$ are homeomorphic. Hence $\mathcal{B}_{C_-(X)}$ and $\mathcal{B}_{C_+(X)}$ are homeomorphic. $\square$

From the following result we obtain one example of a maximal regular congruence on the Γ-semiring $C_-(X)$.

Theorem 4.1.7. *For $x \in X$, $\rho'_x := \{(f,g) : f,g \in C_-(X)$ and $f(x) = g(x)\}$ is a maximal regular congruence on the Γ-semiring $C_-(X)$.*

Proof. Let us denote the homeomorphism $\mathcal{B}_{C_+(X)} \to \mathcal{B}_{L(C_-(X))}$, obtained in Theorem 4.1.6 by ψ, i.e., $\psi(\rho) = \sigma$, where $(f,g) \in \rho$ if and only if $([-f,-\mathbf{1}],[-g,-\mathbf{1}]) \in \sigma$ Again by Theorem 2.2.16, $\phi : \mathcal{B}_{L(C_-(X))} \to \mathcal{B}_{C_-(X)}$ is a bijective correspondence, where $\phi(\rho) = \rho^+$ where $\rho \in \mathcal{B}_{L(C_-(X))}$. Let us define $f : \mathcal{B}_{C_+(X)} \to \mathcal{B}_{C_-(X)}$ by $f(\rho) := \phi(\psi(\rho)) = \phi(\sigma) = \sigma^+$. Then f being the composition of two bijections is a bijection. By Theorem 1.3.41, for $x \in X$, $\rho_x \in \mathcal{B}_{C_+(X)}$.
Therefore $f(\rho_x) = \rho'_x = \{(-f,-g) : f,g \in C_+(X)$ *and* $f(x) = g(x)\}$ for $x \in X$.
Hence $\rho'_x = \{(f,g) : f,g \in C_-(X)$ *and* $f(x) = g(x)\}$ for $x \in X$ is a maximal regular congruence on $C_-(X)$. □

The Γ-semiring $C_-(X)/\rho'_x$ $(\Gamma = C_-(X))$ has some other nice properties. In this connection, we will deduced those properties here in this section. For this we first recall Theorem 2.3.5.

In view of the above discussion and the fact that the Γ-semiring $C_-(X)$ is a commutative Γ-semiring with both the left and right strong unities and ρ'_x is a prime congruence on $C_-(X)$, we have the following result.

Theorem 4.1.8. *The Γ-semiring $C_-(X)/\rho'_x$ is a Γ-semi-integral domain.*

Next, we will verify that $C_-(X)/\rho'_x$ is a Γ-semifield. For proving this, we will establish some results which are as important as the main result.

Theorem 4.1.9. *$C_-(X)/\rho'_x$ is isomorphic to the Γ-semiring R_0^- $(\Gamma = R_0^-)$ of all non-positive reals.*

Proof. Let us define the mapping $\phi_x : C_-(X) \to R_0^-$ by $\phi_x(f) := f(x)$, where $x \in X$. Clearly ϕ is a semigroup epimorphism. Therefore (ϕ,ϕ) is an epimorphism. Therefore by Theorem 1.4.30, $C_-(X)/\rho'_x$ is isomorphic to R_0^-, where $\rho'_x = \{(f,g) \in C_-(X) \times C_-(X) : f(x) = g(x)\}$. □

Lemma 4.1.10. *The left operator semiring $L(R_0^-)$ of the Γ-semiring of all non-positive real numbers R_0^-, where $\Gamma = R_0^-$, is isomorphic to the set of all non-negative reals R_0^+ via the mapping $\sum_{i=1}^{m}[x_i,\alpha_i] \mapsto \sum_{i=1}^{m} x_i\alpha_i$, where $\sum_{i=1}^{m}[x_i,\alpha_i]$ is an element of the left operator semiring of R_0^-.*

Proof. Let us define a map $\phi : L(R_0^-) \to R_0^+$ by $\phi(\sum_{i=1}^m [x_i, \alpha_i]) := \sum_{i=1}^m x_i\alpha_i$. Then $\sum_{i=1}^m [x_i, \alpha_i] = \sum_{j=1}^n [y_j, \beta_j]$ implies $\sum_{i=1}^m x_i\alpha_i s = \sum_{j=1}^n y_j\beta_j s$ for all $s \in R_0^-$. In particular, if we take $s = -1$, it follows that $\sum_{i=1}^m x_i\alpha_i = \sum_{j=1}^n y_j\beta_j$, i.e., $\phi(\sum_{i=1}^m [x_i, \alpha_i]) = \phi(\sum_{i=1}^m [y_j, \beta_j])$. Therefore ϕ is well defined. Similarly we can prove that ϕ is one-one. For $x \in R_0^+$, $[-x, -1] \in L(R_0^-)$ such that $\phi([-x, -1]) = x$. Hence the mapping is bijective. Clearly it is a semiring homomorphism. Therefore the semirings $L(R_0^-)$ and R_0^+ are isomorphic. □

Theorem 4.1.11. *The left operator semiring $L(C_-(X)/\rho'_x)$ of the Γ-semiring $C_-(X)/\rho'_x$ is isomorphic to the semiring R_0^+ of all non-negative reals.*

Proof. By Theorem 2.2.21, left operator semiring $L(C_-(X)/\rho'_x)$ of the Γ-semiring $C_-(X)/\rho'_x$ is isomorphic to the left operator semiring $L(R_0^-)$ of the Γ-semiring R_0^-. Again by Lemma 4.1.10, $L(R_0^-)$ is isomorphic to the semiring R_0^+. Hence we get that $L(C_-(X)/\rho'_x)$ is isomorphic to R_0^+. □

Theorem 4.1.12. *The Γ-semiring $C_-(X)/\rho'_x$ contains no proper nontrivial congruence.*

Proof. Using Theorem 1.3.38 and Theorem 4.1.11 it follows that $L(C_-(X)/\rho'_x)$ is a proper nontrivial congruence free semiring as R_0^+ contains no nontrivial congruence. Therefore $C_-(X)/\rho'_x$ contains no proper nontrivial congruence by Theorem 2.2.4. □

We have the following as a consequence of the above theorem.

Corollary 4.1.13. *ρ'_x is a maximal congruence on $C_-(X)$.*

Proof. From Theorem 2.1.27 we get that there exists an order preserving bijection between the set of all congruences on $C_-(X)/\rho'_x$ and the set of all congruences on $C_-(X)$ containing ρ'_x. Therefore by Theorem 4.1.12, there exists no proper congruence on $C_-(X)$ containing ρ'_x. Hence it follows from Theorem 2.1.28 that ρ'_x is a maximal congruence on $C_-(X)$. □

Hence we deduce the following.

Theorem 4.1.14. *The Γ-semiring $C_-(X)/\rho'_x$ is a Γ-semifield.*

Proof. By Theorem 4.1.8 $C_-(X)/\rho'_x$, being Γ-semi-integral domain, is a ZDF Γ-semiring. Also by Corollary 4.1.7 ρ'_x is a maximal congruence on $C_-(X)$. Therefore $C_-(X)/\rho'_x$ is a Γ-semifield. □

4.2 The structure space of $C_-(X)$

In this section we study the structure space of $C_-(X)$ (the set of all maximal regular congruences on $C_-(X)$ with Hull Kernel topology) for a Tychonoff space X and obtain among other things some important analogues of $C_+(X)$ viz. Theorems 4.2.2, 4.2.6, 4.2.7, 4.2.8. In this study Theorems 2.2.16, 2.2.20, Theorem 3.2.43, Theorem 1.3.38, Theorem 3.1.24 play a key role.

Recall that $\mathcal{B}_{C_+(X)}$ is a compact T_2 space, where X is a Tychonoff space. So in view of Theorems 4.1.6 and 1.3.42, we have the following result.

Theorem 4.2.1. *The structure space $\mathcal{B}_{C_-(X)}$ of the Γ-semiring $C_-(X)$ is a compact T_2 space, where X is a Tychonoff space.*

We recall that for any Tychonoff space X, an extension $(\alpha, \beta X)$ is the Stone-Čech compactification of X if for every continuous f from X into any compact T_2 space Y there exists a function F from βX to Y such that $F \circ \alpha = f$ and that $(\eta_X, \mathcal{B}_{C_+(X)})$ is the Stone-Čech compactification of X, where $\eta_X : X \to \mathcal{B}_{C_+(X)}$ is defined as $\eta_X(x) := \rho_x$ and $(f, g) \in \rho_x$ if and only if $f(x) = g(x)$ for $f, g \in C_+(X)$ and $\mathcal{B}_{C_+(X)}$ = the structure space of all maximal regular congruences on the semiring $C_+(X)$ (see Corollary 1.3.43).

Now we have the following theorem which establishes another representation of the Stone-Čech compactification of a Tychonoff space X as the structure space of all maximal regular congruences on the Γ-semiring $C_-(X)$.

Theorem 4.2.2. *If X is a Tychonoff space then $(\xi_X, \mathcal{B}_{C_-(X)})$ is the Stone-Čech compactification of X, where $\xi_X : X \to \mathcal{B}_{C_-(X)}$ is defined as $\xi_X(x) := \rho'_x$ and $\mathcal{B}_{C_-(X)}$ = the structure space of all maximal regular congruences on the Γ-semiring $C_-(X)$.*

Proof. Let us define a map $\xi_X := f \circ \eta_X$ from X onto $\mathcal{B}_{C_-(X)}$, where f is a homeomorphism between $\mathcal{B}_{C_+(X)}$ and $\mathcal{B}_{C_-(X)}$. Since composition of two homeomorphisms is again a homeomorphism, ξ_X is a homeomorphism between X and $\mathcal{B}_{C_-(X)}$. Also $\xi_X(x) = (f \circ \eta_X)(x) = f(\eta_X(x)) = f(\rho_x) = \rho'_x$, where $\rho_x = \{(f, g) : f, g \in C_+(X) \text{ and } f(x) = g(x)\} \in \mathcal{B}_{C_+(X)}$. We need to show that $\xi_X(X)$ is dense in $\mathcal{B}_{C_-(X)}$.
Indeed, $\overline{\xi_X(X)} = \overline{(f \circ \eta_X)(X)} = \overline{f(\eta_X(X))} = f(\overline{\eta_X(X)}) = f(\mathcal{B}_{C_+(X)}) = \mathcal{B}_{C_-(X)}$.
Now let $g : X \to Y$ be a continuous map, where Y is a T_2 compact space. Then in view of Corollary 1.3.43, there exists a map F from $\mathcal{B}_{C_+(X)}$ to Y such that $F \circ \eta_X = g$. Therefore we get a map $F \circ f^{-1}$ from $\mathcal{B}_{C_-(X)}$ to Y such that
$(F \circ f^{-1}) \circ \xi_X = (F \circ f^{-1}) \circ (f \circ \eta_X) = g$. This completes the proof. □

We again recall that for any Tychonoff space X, $(\alpha, \nu X)$ is the Hewitt realcompactification of X if for every continuous function f from X into any realcompact space Y there exists a function F from νX to Y such that $F \circ \alpha = f$ and that the extension $(\eta_X, \mathcal{B}^R_{C_+(X)})$ is the Hewitt realcompactification of X, where $\eta_X : X \to \mathcal{B}^R_{C_+(X)}$ is defined as $\eta_X(x) := \rho_x$ and $(f, g) \in \rho_x$ if and only if $f(x) = g(x)$ for $f, g \in C_+(X)$ and $\mathcal{B}^R_{C_+(X)}$ = the structure space of all maximal regular real congruences on the semiring $C_+(X)$.

We now introduce the notion of real congruences on the Γ-semiring $C_-(X)$ and real structure space of $C_-(X)$.

Let us assume that ρ is a maximal regular congruence on $C_-(X)$. We call ρ a *real congruence on* $C_-(X)$ whenever the corresponding congruence $\rho' = \{(-f, -g) : (f, g) \in \rho\}$ on $C_+(X)$ is a real congruence on $C_+(X)$ (see Definition 1.3.44). Now let us consider the space $\mathcal{B}^R_{C_-(X)}$ of all maximal regular real congruences on $C_-(X)$ which is a subspace of the space $\mathcal{B}_{C_-(X)}$. We call the space $\mathcal{B}^R_{C_-(X)}$, the *real structure space of* $C_-(X)$.

Remark 4.2.3. According to the definition of real congruence on $C_-(X)$ above and Theorem 4.1.6 we conclude that there exists a bijection between the set of all maximal regular congruence which are real on $C_+(X)$ and those on $C_-(X)$ via the mapping $\rho \mapsto \rho'$, where $\rho' = \{(-f, -g) : (f, g) \in \rho\}$ for a real maximal regular congruence ρ on $C_+(X)$.

Therefore by Theorem 4.1.6 and Remark 4.2.3 we have the following Theorem.

Theorem 4.2.4. *The spaces $\mathcal{B}^R_{C_+(X)}$ and $\mathcal{B}^R_{C_-(X)}$ are homeomorphic.*

Theorem 4.2.5. *The real structure space $\mathcal{B}^R_{C_-(X)}$ of $C_-(X)$ is a realcompact space.*

Proof. By Theorem 4.2.1, $\mathcal{B}_{C_-(X)}$ is compact. Then $\mathcal{B}_{C_-(X)}$ is realcompact. Also $\mathcal{B}^R_{C_-(X)}$ is a subspace of it. As every realclosed subset of a realcompact space is realcompact, we shall now show that $\mathcal{B}^R_{C_-(X)}$ is a realclosed subspace of $\mathcal{B}_{C_-(X)}$. To show that we need to prove that real closure of $\mathcal{B}^R_{C_-(X)} = \mathcal{B}^R_{C_-(X)}$. We denote real closure of $\mathcal{B}^R_{C_-(X)}$ as $rcl\mathcal{B}^R_{C_-(X)}$. Let $\rho \in rcl\mathcal{B}^R_{C_-(X)}$. Then ρ is a maximal regular congruence on $C_-(X)$ such that each G_δ set containing ρ meets $\mathcal{B}^R_{C_+(X)}$. Let $\bigcap_\alpha C\Delta(\rho_\alpha)$ be a G_δ set containing ρ such that $\bigcap_\alpha C\Delta(\rho_\alpha) \cap \mathcal{B}^R_{C_-(X)} \neq \emptyset$. Let ρ' (ρ'_α) be the corresponding maximal regular congruence of ρ (resp. ρ_α) on $C_+(X)$. Then $\rho \in \bigcap_\alpha C\Delta(\rho_\alpha)$ implies that $\rho' \in \bigcap_\alpha C\Delta(\rho'_\alpha)$, where $\bigcap_\alpha C\Delta(\rho'_\alpha)$ is a G_δ set in $\mathcal{B}_{C_+(X)}$ containing ρ'. Again

$\bigcap_\alpha C\Delta(\rho_\alpha) \cap \mathcal{B}^R_{C_-(X)} \neq \emptyset$. Now let $\sigma \in \bigcap_\alpha C\Delta(\rho_\alpha) \cap \mathcal{B}^R_{C_-(X)}$.
Then $\sigma' \in \bigcap_\alpha C\Delta(\rho'_\alpha) \cap \mathcal{B}^R_{C_+(X)}$, i.e., $\bigcap_\alpha C\Delta(\rho'_\alpha) \cap \mathcal{B}^R_{C_+(X)}$ is nonempty. Therefore by Theorem 1.3.45, $\rho' \in rcl\mathcal{B}^R_{C_+(X)} = \mathcal{B}^R_{C_+(X)}$ whence it follows that $\rho \in \mathcal{B}^R_{C_-(X)}$. Hence $\mathcal{B}^R_{C_-(X)}$ is realcompact. This completes the proof.

□

Next we have the following theorem which establishes another representation of the Hewitt realcompactification of a Tychonoff space X as the structure space of all maximal regular congruences those are real on the Γ-semiring $C_-(X)$.

Theorem 4.2.6. *If X is a Tychonoff space then $(\xi_X, \mathcal{B}^R_{C_-(X)})$ is the Hewitt realcompactification of X, where $\xi_X : X \to \mathcal{B}^R_{C_-(X)}$ is defined as $\xi_X(x) := \rho'_x$ and $\mathcal{B}_{C_-(X)} =$ the real structure space of $C_-(X)$.*

Proof. Let us define a map $\xi_X := f \circ \eta_X$ from X onto $\mathcal{B}^R_{C_-(X)}$, where f is a homeomorphism between $\mathcal{B}^R_{C_+(X)}$ and $\mathcal{B}^R_{C_-(X)}$ (see Theorem 4.2.4). Since composition of two homeomorphisms is again a homeomorphism, ξ_X is a homeomorphism between X and $\mathcal{B}^R_{C_-(X)}$. Also $\xi_X(x) = (f \circ \eta_X)(x) = f(\rho_x) = \rho'_x$. We need to show that $\xi_X(X)$ is dense in $\mathcal{B}_{C_-(X)}$.
Indeed, $\overline{\xi_X(X)} = \overline{(f \circ \eta_X)(X)} = \overline{f(\eta_X(X))} = f(\overline{\eta_X(X)}) = f(\mathcal{B}_{C_+(X)}) = \mathcal{B}_{C_-(X)}$.
Now let $g : X \to Y$ be a continuous map, where Y is a realcompact space. Then in view of Corollary 1.3.46, there exists a map F from $\mathcal{B}^R_{C_+(X)}$ to Y such that $F \circ \eta_X = g$. Therefore we get a map $F \circ f^{-1}$ from $\mathcal{B}^R_{C_-(X)}$ to Y such that
$(F \circ f^{-1}) \circ \xi_X = (F \circ f^{-1}) \circ (f \circ \eta_X) = g$. Hence the result follows. □

Now we prove the Γ-semiring analogue of the **'Banach-Stone Theorem'** using the semiring analogue of the same proved by Acharyya et al (Theorem 1.3.47).

Theorem 4.2.7. *If X and Y are compact T_2 spaces then the two Γ-semirings $C_-(X)$ and $C_-(Y)$ are isomorphic if and only if X and Y are homeomorphic.*

Proof. Let $\phi : X \to Y$ be a homeomorphism of X onto Y. Let us define a map $\phi' : C_-(Y) \to C_-(X)$ by $\phi'(f) := f \circ \phi$, where $f \in C_-(Y)$. Then for $f, g \in C_-(Y)$, $(f + g) \circ \phi = f \circ \phi + g \circ \phi$. Therefore ϕ' is a semigroup homomorphism. Now let $f, g \in C_-(Y)$ and $f \neq g$. Then there exists $y \in Y$ such that $f(y) \neq g(y)$ and hence $(f \circ \phi)(x) \neq (g \circ \phi)(x)$, where $x = \phi^{-1}(y)$. It implies that $\phi'(f) \neq \phi'(g)$. Thus ϕ' is one-one. Again for any $f \in C_-(X)$, $f \circ \phi^{-1} \in C_-(Y)$ and $\phi'(f \circ \phi^{-1}) = f$. Thus ϕ' is onto. Therefore ϕ' is a semigroup isomorphism between $C_-(Y)$ and $C_-(X)$. Also for

every $f, g \in C_-(Y)$ and $k \in C_-(Y)$, $\phi'(fkg) = \phi'(f)\phi'(k)\phi'(g)$ and $\phi'(\mathbf{0}) = \mathbf{0}$. Hence (ϕ', ϕ') is a Γ-semiring isomorphism of $C_-(X)$ onto $C_-(Y)$
Conversely, let the two Γ-semirings $C_-(X)$ and $C_-(Y)$ be isomorphic. Therefore their corresponding left operator semirings $L(C_-(X))$ and $L(C_-(Y))$ are isomorphic by Theorem 2.2.21. Then in view of Theorem 4.1.2 and semiring analogue of the 'Banach-Stone Theorem' (*cf.* Theorem 1.4.27) we conclude that X and Y are homeomorphic. □

In the following we obtain the Γ-semiring analogue of the **'Hewitt's Isomorphism Theorem'** using Theorem 4.1.3.

Theorem 4.2.8. *If X and Y are realcompact spaces then the two Γ-semirings $C_-(X)$ and $C_-(Y)$ are isomorphic if and only if X and Y are homeomorphic.*

Therefore we have the following theorem.

Theorem 4.2.9. *Let X and Y be realcompact spaces. Then the following are equivalent.*

(i) X and Y are realcompact spaces.

(ii) The rings $C(X)$ and $C(Y)$ are isomorphic.

(iii) The semirings $C_+(X)$ and $C_+(Y)$ are isomorphic.

(iv) The Γ-semirings $C_-(X)$ and $C_-(Y)$ are isomorphic.

4.3 The z-ideals and z^o-ideals of $C_-(X)$

In this section, we introduce the notions of z-ideals and z^o-ideals of $C_-(X)$ and on those study the results characterizing the topological space X.
Note that throughout this section we consider the topological space X as a Tychonoff space.

Recall that if R and S are two isomorphic semirings then the sets of all k-ideals, prime ideals, prime k-ideals (maximal, maximal k-ideals) of R and S are in order preserving bijective correspondences (resp. bijective correspondences).

So there exists an order preserving bijection $\phi_I : Id(C_+(X)) \to Id(L(C_-(X)))$, where $Id(C_+(X)), Id(L(C_-(X)))$ are the sets of ideals of the isomorphic semirings $C_+(X)$ and $L(C_-(X))$ respectively which maps the k-ideals, prime ideals, prime k-ideals, maximal, maximal k-ideals of $C_+(X)$ to those of $L(C_-(X))$ respectively.

Finally by Theorems 4.1.2 and 1.4.19 we get the following.

Theorem 4.3.1. *There exists an order preserving bijection (a bijection) between the sets of all k-ideals, prime ideals, prime k-ideals (resp. maximal, maximal k-ideals) of the semiring $C_+(X)$ and the Γ-semiring $C_-(X)$.*

Remark 4.3.2. Let ϕ be the isomorphism between $C_+(X)$ and $L(C_-(X))$ via the definition of mapping $f \mapsto [-f, -\mathbf{1}]$.
Let K be an ideal of $C_+(X)$ and $f \in K$.
Then $\phi(f) = [-f, -\mathbf{1}] \in \phi_I(K)$ which implies $(-f) \in (\phi_I(K))^+$.
We denote $(\phi_I(K))^+$ as K' and call K' *the corresponding ideal* of K in $C_-(X)$.
Again let L be an ideal of $C_-(X)$ and $g \in L$. Then $[g, -\mathbf{1}] \in L^{+'}$ which implies $(-g) \in \phi_I^{-1}(L^{+'}) = L_+$.
We denote $\phi_I^{-1}(L^{+'})$ as L_+ and call L_+ *the corresponding ideal* of L in $C_+(X)$.
Also we observe that for any ideal I of $C_+(X)$, $(I')_+ = I$ and for any ideal J of $C_-(X)$, $(J_+)' = J$.
In addition if I is a k-ideal (prime ideal, prime k-ideal, maximal ideal, maximal k-ideal) in $C_+(X)$ then I' is a k-ideal (resp. prime ideal, prime k-ideal, maximal ideal, maximal k-ideal) in $C_-(X)$. Also if J is a k-ideal (prime ideal, prime k-ideal, maximal ideal, maximal k-ideal) in $C_-(X)$ then J_+ is a k-ideal (resp. prime ideal, prime k-ideal, maximal ideal, maximal k-ideal) in $C_+(X)$.

We note the following result which will be used in the sequel.

Theorem 4.3.3. *Let K be an ideal of $C_+(X)$. $K = P \cap C_+(X)$ if and only if $K' = P \cap C_-(X)$ for any ideal P of the ring $C(X)$. Also let L be an ideal of $C_-(X)$. $L = Q \cap C_-(X)$ if and only if $L_+ = Q \cap C_+(X)$ for any ideal Q of the ring $C(X)$.*

Though we may not always mention explicitly but frequent use of Theorems 4.3.1 and 4.3.3 and Remark 4.3.2 is implicit in the deduction of the following results in this section.

In the following, we observe some interesting facts about the prime and maximal ideals of the Γ-semiring $C_-(X)$:

(i) Any maximal ideal of $C_-(X)$ is of the form $M \cap C_-(X)$ for various maximal ideals M of the ring $C(X)$.

(ii) Any maximal ideal of $C_-(X)$ has the following structure:

$$M_p^- = \{f \in C_-(X) : p \in Cl_{\beta X}(Z(f))\}, p \in \beta X.$$

(*iii*) Any prime ideal of $C_-(X)$ is of the form $P \cap C_-(X)$ for various prime ideals P of the ring $C(X)$.

(*iv*) Any minimal prime ideal in $C_-(X)$ is of the form $P_m \cap C_-(X)$ for some minimal prime ideal P_m of $C(X)$.

Recall that a proper ideal I in a semiring is called *strong* if and only if $a + b \in I$ implies $a, b \in I$ and evidently strong ideals are k-ideal.

Definition 4.3.4. A proper ideal I in a Γ-semiring S is called *strong* if and only if $a + b \in I$ implies $a, b \in I$.

Theorem 4.3.5. *If I is a strong ideal of $C_-(X)$ then the corresponding ideal I_+ is strong ideal of $C_+(X)$. Also for a strong ideal J of $C_+(X)$, the corresponding ideal J' is strong ideal of $C_-(X)$.*

Proof. Let $a + b \in I_+$. Then $(-a) + (-b) = -(a + b) \in I = (I_+)'$. Since I is a strong ideal of $C_-(X)$, $-a, -b \in I$. Therefore $a, b \in I_+$. Hence I_+ is strong ideal of $C_+(X)$. Similarly we can prove the other part of the theorem. □

Theorem 4.3.6. *Prime ideals of $C_-(X)$ are strong ideals.*

Proof. Let P be a prime ideal in $C_-(X)$. Then P_+ is a prime ideal in $C_+(X)$ which is a strong ideal in $C_+(X)$, by Lemma 1.3.52. Therefore by Theorem 4.3.5, $P = (P_+)'$ is a strong ideal in $C_-(X)$. □

Theorem 4.3.7. *For every ring ideal I in $C(X)$, the corresponding Γ-semiring ideal $I \cap C_-(X)$ is a k-ideal of $C_-(X)$.*

Proof. Let I be a ring ideal in $C(X)$. Then by Theorem 1.3.53 $I \cap C_+(X)$ is a k-ideal in $C_+(X)$. Hence $I \cap C_-(X)$ is a k-ideal of $C_-(X)$. □

Definition 4.3.8. A proper ideal I in $C_-(X)$ is called a *z-ideal* if $Z(f) \subseteq Z(g)$ and $f \in I$ implies $g \in I$.

Theorem 4.3.9. *If I is a z-ideal of $C_-(X)$ then the corresponding ideal I_+ is z-ideal of $C_+(X)$. Also for a z-ideal J of $C_+(X)$, the corresponding ideal J' is z-ideal of $C_-(X)$.*

Proof. Let I be a z-ideal of $C_-(X)$. Let $Z(f) \subseteq Z(g)$ and $f \in I_+$.
Then $Z(-f) \subseteq Z(-g)$ and $-f \in (I_+)' = I$. Since I is a z-ideal of $C_-(X)$, $-g \in I$ which implies $g \in I_+$. Hence I_+ is z-ideal of $C_+(X)$. The other part can be proved analogously. □

Let us consider for any function $f \in C_-(X)$, M_f^- to be the intersection of all maximal ideals of $C_-(X)$ containing f.

Recall that for $a \in C_+(X)$, the intersection of all maximal ideals of $C_+(X)$ containing $a= M_a^+ = \{g \in C_+(X) : Z(a) \subseteq Z(g)\}$ (see Definition 1.3.54).

Then the following result is clear.

Proposition 4.3.10. *For $a \in C_+(X)$, $(M_a^+)' = M_{-a}^-$.*

Theorem 4.3.11. *I is a z-ideal of $C_-(X)$ if and only if $f \in I$ implies $M_f^- \subseteq I$.*

Proof. Let I be a z-ideal of $C_-(X)$ and $f \in I$. Then by Theorem 4.3.9, I_+ is a z-ideal of $C_+(X)$ and also $-f \in I_+$. So by Definition 1.3.54 $M_{-f}^+ \subseteq I_+$ which implies $M_f^- = (M_{-f}^+)' \subseteq (I_+)' = I$, i.e., $M_f^- \subseteq I$.

Conversely, let us suppose that for an ideal I of $C_-(X)$, $f \in I$ implies $M_f^- \subseteq I$. Let $Z(f) \subseteq Z(g)$ and $f \in I$. Then $Z(-f) \subseteq Z(-g)$ and $-f \in I_+$. So by Proposition 1.3.55, $-g \in M_{-f}^+$. Therefore $g \in (M_{-f}^+)' = M_f^- \subseteq I$. Hence I is a z-ideal of $C_-(X)$. □

Theorem 4.3.12. *Any z-ideal of $C_-(X)$ is of the form $I \cap C_-(X)$ for various z-ideals I of $C(X)$.*

Proof. Let I be a z-ideal in $C_-(X)$. Then I_+ is z-ideal in $C_+(X)$. So by Theorem 1.3.56, $I_+ = J \cap C_+(X)$ for a z-ideal J in $C(X)$. Then by Theorem 4.3.3,

$I = (I_+)' = J \cap C_-(X)$ for the z-ideal J in $C(X)$. This completes the proof. □

Theorem 4.3.13. *The following are true for any z-ideal of $C_-(X)$.*

(*i*) *z-ideals of $C_-(X)$ are strong.*

(*ii*) *The minimal prime ideals of $C_-(X)$ are z-ideals.*

Proof. (*i*) Let I be a z-ideal of $C_-(X)$. Then I_+ is a z-ideal of $C_+(X)$ which is, by Proposition 1.3.57, a strong ideal in $C_+(X)$. Then by Theorem 4.3.5, $I = (I_+)'$ is a strong ideal in $C_-(X)$.

(*ii*) Let I be a minimal prime ideal in $C_-(X)$. Then I_+ is a minimal prime ideal of $C_+(X)$. Therefore by Theorem 1.3.58, I_+ is a z-ideal of $C_+(X)$. Hence $I = (I_+)'$ is a z-ideal in $C_-(X)$.

□

Below we prove a result for its immediate use.

Lemma 4.3.14. *The following conditions are equivalent for a z-ideal I in $C_-(X)$.*

(*i*) *For all* $g, h \in C_-(X)$ *and* $\gamma \in C_-(X) \setminus \{\mathbf{0}\}$, *if* $g\gamma h = \mathbf{0}$ *then* $g \in I$ *or* $h \in I$.

(*ii*) *For all* $k, l \in C_+(X)$, *if* $kl = \mathbf{0}$ *then* $k \in I_+$ *or* $l \in I_+$.

Proof. Suppose for any z-ideal I in $C_-(X)$ (i) holds. Then I_+ is a z-ideal in $C_+(X)$.

$(i) \Rightarrow (ii)$: Let for all $k, l \in C_+(X)$, $kl = \mathbf{0}$. So for all $\gamma \in C_-(X) \setminus \{\mathbf{0}\}$, $(-k)\gamma(-l) = \mathbf{0}$ which implies $-k \in I$ or $-l \in I$, i.e., $k \in I_+$ or $l \in I_+$.

$(ii) \Rightarrow (i)$: Let us suppose that for all $g, h \in C_-(X)$ and $\gamma \in C_-(X) \setminus \{\mathbf{0}\}$, $g\gamma h = \mathbf{0}$. Then $(-g)(-h) = \mathbf{0}$ which implies $-g \in I_+$ or $-h \in I_+$ by condition (ii). Hence $g \in I$ or $h \in I$.

This completes the proof. □

Theorem 4.3.15. *For any z-ideal I in $C_-(X)$, the following are equivalent.*

(*i*) *I is a prime ideal.*

(*ii*) *I contains a prime ideal.*

(*iii*) *For all* $g, h \in C_-(X)$ *and* $\gamma \in C_-(X) \setminus \{\mathbf{0}\}$, *if* $g\gamma h = \mathbf{0}$ *then* $g \in I$ *or* $h \in I$.

Proof. I is a z-ideal in $C_-(X)$ if and only if I_+ is a z-ideal in $C_+(X)$.

$(i) \Leftrightarrow (ii)$: I is a prime ideal in $C_-(X)$ if and only if I_+ is a prime ideal in $C_+(X)$ if and only if I_+ contains a prime ideal P (say) (by Theorem 1.3.59). This implies that I contains prime ideal P' and conversely. Hence the proof is complete.

$(i) \Leftrightarrow (iii)$: Let I be a prime z-ideal in $C_-(X)$. From Lemma 4.3.14 and Theorem 1.3.59 we deduce that if (iii) holds then I_+ is a prime z-ideal in $C_+(X)$ which again implies I is a prime ideal in $C_-(X)$ and conversely if I is a prime ideal in $C_-(X)$ then I_+ is a prime z-ideal in $C_+(X)$ which implies by the above lemma that (iii) holds.

This completes the proof. □

Recall that a Γ-semiring S is called a *regular* Γ*-semiring* if for any $a \in S$, $a \in a\Gamma S\Gamma a$.

Theorem 4.3.16. *$C_+(X)$ is a regular semiring if and only if $C_-(X)$ is a regular Γ-semiring.*

Proof. Let $C_+(X)$ be a regular semiring. Let $a \in C_-(X)$. Then $-a \in C_+(X)$. Since $C_+(X)$ is regular, there exists $b \in C_+(X)$ such that $(-a) = (-a)b(-a)$, i.e.,

$$a = a(-\mathbf{1})(-b)(-\mathbf{1})a \in a\Gamma S\Gamma a, \text{ where } S = \Gamma = C_-(X).$$

Since a is an arbitrary element of $C_-(X)$, it follows that $C_-(X)$ is regular.
Conversely, let $C_-(X)$ be a regular Γ-semiring. Let $x \in C_+(X)$. Then $-x \in C_-(X)$.
Since $C_-(X)$ is regular, $-x \in (-x)\Gamma S\Gamma(-x)$ for $S = \Gamma = C_-(X)$.
Therefore $-x = (-x)\sum_i \sum_j \sum_k \alpha_i s_j \beta_k(-x)$. So

$$x = x(-\sum_i \sum_j \sum_k \alpha_i s_j \beta k)x, \text{ where } (-\sum_i \sum_j \sum_k \alpha_i s_j \beta k) \in C_+(X).$$

Since x is an arbitrary element of $C_+(X)$, it follows that $C_+(X)$ is regular. □

Theorem 4.3.17. *The following are equivalent.*

(*i*) *X is a P-space.*

(*ii*) *Each ideal of $C(X)$ is a z-ideal.*

(*iii*) *Each ideal of $C_+(X)$ is a z-ideal.*

(*iv*) *Each ideal of $C_-(X)$ is a z-ideal.*

(*v*) *Each strong ideal is a z-ideal in $C_+(X)$.*

(*vi*) *Each strong ideal is a z-ideal in $C_-(X)$.*

(*vii*) *Each prime ideal is a z-ideal in $C_+(X)$.*

(*viii*) *Each prime ideal is a z-ideal in $C_-(X)$.*

(*ix*) *$C_+(X)$ is a regular semiring.*

(*x*) *$C_-(X)$ is a regular Γ-semiring.*

Proof. $(iii) \Leftrightarrow (iv)$: Let (iii) hold. Let I be an ideal of $C_-(X)$. Then I_+ being an ideal in $C_+(X)$ is a z-ideal in $C_+(X)$ which implies that $I = (I_+)'$ is a z-ideal in $C_-(X)$ (by Theorem 4.3.9). Again if (iv) holds then for an ideal J of $C_+(X)$, J' is a z-ideal in $C_-(X)$ which implies that $(J')_+ = J$ is a z-ideal in $C_+(X)$. Therefore each ideal of $C_+(X)$ is a z-ideal if and only if each ideal of $C_-(X)$ is a z-ideal.

$(v) \Leftrightarrow (vi)$: Let (v) hold. Let I be a strong ideal of $C_-(X)$. Then I_+ being a strong ideal in $C_+(X)$ is a z-ideal in $C_+(X)$ which implies that $I = (I_+)'$ is a z-ideal in $C_-(X)$ (see Theorem 4.3.9 and Theorem 4.3.5). Again if (vi) holds then for a strong ideal J of $C_+(X)$, J' is a strong z-ideal in $C_-(X)$ which implies that $(J')_+ = J$ is a z-ideal in $C_+(X)$. Therefore each strong ideal is a z-ideal in $C_+(X)$ if and only if each strong ideal is a z-ideal in $C_-(X)$.

$(vii) \Leftrightarrow (viii)$: Let (vii) hold. Let I be a prime ideal of $C_-(X)$. Then I_+ being a prime ideal in $C_+(X)$ is a z-ideal in $C_+(X)$ which implies that $I = (I_+)'$ is a z-ideal in $C_-(X)$ (by Theorem 4.3.9). Again if $(viii)$ holds then for a prime ideal J of $C_+(X)$, J' is a prime z-ideal in $C_-(X)$ which implies that $(J')_+ = J$ is a z-ideal in $C_+(X)$. Therefore each prime ideal is a z-ideal in $C_+(X)$ if and only if each prime ideal is a z-ideal in $C_-(X)$.

$(ix) \Leftrightarrow (x)$: Already proved in Theorem 4.3.16.

Other equivalences are already proved in Theorem 1.3.61. □

Definition 4.3.18. An ideal of a Γ-semiring is called *essential* if it intersects every nonzero ideal nontrivially.

Theorem 4.3.19. *If I is an essential ideal of $C_+(X)$ then the corresponding ideal I' is an essential ideal of $C_-(X)$. Also if J is an essential ideal of $C_-(X)$ then the corresponding ideal J_+ is an essential ideal of $C_+(X)$.*

Proof. If I is an essential ideal of $C_+(X)$ then the corresponding ideal I' is an ideal of $C_-(X)$. Now let K be a nonzero ideal in $C_-(X)$. So K_+ is a nonzero ideal in $C_+(X)$. Therefore I intersects K_+ which implies that I' intersects $K = (K_+)'$ nontrivially. Since K is an arbitrary nonzero ideal in $C_-(X)$, I' is an essential ideal of $C_-(X)$. Analogously we can prove the other part as well. □

Proposition 4.3.20. *For any essential ideal E of $C_-(X)$, $(Ann^+(E_+))' = Ann^-(E)$, where $Ann^-(E) = \{g \in C_-(X) : E\Gamma g = (\mathbf{0})\}$.*

Proof. Let $x \in Ann^-(E)$. Then $E\Gamma x = (\mathbf{0})$, where $\Gamma = C_-(X)$.
So for all $g \in E$, $\gamma \in C_-(X)$, $g\gamma x = \mathbf{0}$ which implies $f(-x) = \mathbf{0}$ for all $f \in E_+$. Therefore $-x \in Ann^+(E_+)$, i.e., $x \in (Ann^+(E_+))'$. Hence $Ann^-(E) \subseteq (Ann^+(E_+))'$. Again let $y \in (Ann^+(E_+))'$. Then $-y \in Ann^+(E_+)$ which implies that $a(-y) = \mathbf{0}$, i.e., $ay = \mathbf{0}$ for all $a \in E_+$. So $E\Gamma y$ must be $(\mathbf{0})$. Indeed, if $h \in E\Gamma y$ then by the argument above, $h = \sum_i \sum_j y_i\gamma_j y = \mathbf{0}$ as $\sum_i \sum_j y_i\gamma_j = \sum_i \sum_j (-y_i)(-\gamma_j) \in E_+$. Therefore $y \in Ann^-(E)$ and hence $(Ann^+(E_+))' \subseteq Ann^-(E)$. So $(Ann^+(E_+))' = Ann^-(E)$.

□

Theorem 4.3.21. *The following are equivalent for any essential ideal E of the Γ-semiring $C_-(X)$.*

(i) E intersects every nonzero z-ideals nontrivially.

(ii) *E intersects every ideal nontrivially.*

(iii) $Ann^-(E) = (\mathbf{0})$.

(iv) $\bigcap Z[E]$ *is nowhere dense.*

Proof. (i) $\Leftrightarrow$ (ii) : By Theorems 1.3.63, 4.3.19, 4.3.9 we have the following arguments that, for any essential ideal E of $C_-(X)$, E intersects every nonzero z-ideals of $C_-(X)$ nontrivially if and only if E_+ intersects every nonzero z-ideals of $C_+(X)$ nontrivially if and only if E_+ intersects every ideal of $C_+(X)$ nontrivially if and only if E intersects every ideal of $C_-(X)$ nontrivially. Hence the proof is complete.

(ii) $\Leftrightarrow$ (iii) : For any essential ideal E of $C_-(X)$, E intersects every ideal of $C_-(X)$ nontrivially if and only if E_+ intersects every ideal of $C_+(X)$ nontrivially if and only if $Ann^+(E_+) = (\mathbf{0})$ (by Theorem 1.3.63). Since $(Ann^+(E_+))' = Ann^-(E)$, $Ann^+(E_+) = (\mathbf{0})$ if and only if $Ann^-(E) = (\mathbf{0})$. This completes the proof.

(iii) $\Leftrightarrow$ (iv) : Let us first note that $(Ann^+(E_+))' = Ann^-(E)$ for any essential ideal E of $C_-(X)$. This implies that $Ann^-(E) = (\mathbf{0})$ if and only if $Ann^+(E_+) = (\mathbf{0})$, where E_+ is an essential ideal of $C_+(X)$. So by Theorem 1.3.63, $\bigcap Z[E_+]$ is nowhere dense. Now By the two facts that $f \in E$ if and only if $-f \in E_+$ and $Z(f) = Z(-f)$, we deduce that $Z[E] = Z[E_+]$ for any essential ideal E of $C_-(X)$. Hence $\bigcap Z[E]$ is nowhere dense. By reversing the arguments above, we can prove the converse. □

Theorem 4.3.22. *Any essential z-ideal in $C_-(X)$ is of the form $I \cap C_-(X)$, where I is an essential z-ideal of $C(X)$.*

Proof. Let I be an essential z-ideal in $C_-(X)$. Then by Theorems 4.3.9, 4.3.19, I_+ is an essential z-ideal in $C_+(X)$. So by Theorem 1.3.64, $I_+ = J \cap C_+(X)$ for an essential z-ideal J in $C(X)$. Then $I = (I_+)' = J \cap C_-(X)$ for the essential z-ideal J in $C(X)$. This completes the proof.

□

Theorem 4.3.23. *The following are equivalent.*

(i) *X is a P-space.*

(ii) *Every essential ideal is a z-ideal in $C_+(X)$.*

(iii) *Every essential ideal is a z-ideal in $C_-(X)$.*

Proof. $(i) \Leftrightarrow (ii)$: Already proved in Theorem 1.3.65.

$(ii) \Leftrightarrow (iii)$: Let (ii) hold. Let I be an essential ideal of $C_-(X)$. Then I_+ being an essential ideal in $C_+(X)$ is a z-ideal in $C_+(X)$ which implies that $I = (I_+)'$ is a z-ideal in $C_-(X)$ (see Theorem 4.3.9 and Theorem 4.3.19). Again if (iii) holds then for an essential ideal J of $C_+(X)$, J' is an essential z-ideal in $C_-(X)$ which implies that $(J')_+ = J$ is a z-ideal in $C_+(X)$. Therefore every essential ideal is a z-ideal in $C_+(X)$ if and only if every essential ideal is a z-ideal in $C_-(X)$. □

Let us now introduce the notion of z^o-ideals in $C_-(X)$.

Definition 4.3.24. An ideal I in $C_-(X)$ is called z^o-*ideal* if $a \in I$ implies $P_a^- \subseteq I$, where P_a^- is the intersection of all minimal prime ideals of $C_-(X)$ containing a.

Recall that P_f^+ is the intersection of all minimal prime ideals of $C_+(X)$ containing f. Then the following proposition is easy to observe.

Proposition 4.3.25. *(i) For $a \in C_-(X)$, $P_a^- = (P_{-a}^+)'$.*

(ii) For $x \in C_+(X)$, $P_x^+ = (P_{-x}^-)_+$.

Using the above proposition we prove the following theorem.

Theorem 4.3.26. *If I is a z^o-ideal of $C_-(X)$ then the corresponding ideal I_+ is a z^o-ideal of $C_+(X)$. Also for a z^o-ideal J of $C_+(X)$, the corresponding ideal J' is a z^o-ideal of $C_-(X)$.*

Proof. Let I be a z^o-ideal of $C_-(X)$. Then I_+ is an ideal of $C_+(X)$.
Let $x \in I_+$. So $-x \in (I_+)' = I$. Since I is a z^o-ideal of $C_-(X)$, $P_{-x}^- \subseteq I$ which implies $P_{-x}^- = (P_x^+)' \subseteq I$. That means $P_x^+ = ((P_x^+)')_+ \subseteq I_+$. Therefore I_+ is a z^o-ideal of $C_+(X)$ (see Definition 1.3.66).
Next let J be a z^o-ideal of $C_+(X)$. Then J' is an ideal of $C_-(X)$. Let $a \in J'$.
So $-a \in (J')_+ = J$. Since J is a z^o-ideal of $C_+(X)$, $P_{-a}^+ \subseteq J$ which implies $(P_a^-)_+ \subseteq J$.
That means $P_a^- = ((P_a^-)_+)' \subseteq J'$. Therefore J' is a z^o-ideal of $C_-(X)$. □

The following theorem is the $C_-(X)$-analogue of Proposition 1.3.67.

Theorem 4.3.27. *z^o-ideals in $C_-(X)$ are strong.*

Proof. Let I be a z^o-ideal in $C_-(X)$. Then I_+ is a z^o-ideal in $C_+(X)$. So by Proposition 1.3.67, I_+ is a strong ideal in $C_+(X)$. Therefore $I = (I_+)'$ is a strong ideal in $C_-(X)$. □

Theorem 4.3.28. *Any z^o-ideal in $C_-(X)$ is of the form $J \cap C_-(X)$ for various z^o-ideals J in $C(X)$.*

Proof. Let I be a z^o-ideal in $C_-(X)$. Then I_+ is z^o-ideal in $C_+(X)$. So by Theorem 1.3.68, $I_+ = J \cap C_+(X)$ for a z^o-ideal J in $C(X)$. Then by Theorem 4.3.3, $I = (I_+)' = J \cap C_-(X)$ for the z^o-ideal J in $C(X)$. This completes the proof. □

Theorem 4.3.29. *The following are equivalent.*

(i) *X is an almost P-space.*

(ii) *Every z-ideal in $C(X)$ is a z^o-ideal.*

(iii) *Every maximal ideal in $C(X)$ is a z^o-ideal.*

(iv) *Every maximal ideal in $C(X)$ consists entirely of zero-divisors.*

(v) *For each nonunit element $f \in C(X)$, there exists $0 \neq g \in C(X)$ with $P_f \subseteq Ann(g)$.*

(vi) *Every z-ideal in $C_+(X)$ is a z^o-ideal.*

(vii) *Every maximal ideal in $C_+(X)$ is a z^o-ideal.*

(viii) *Every maximal ideal in $C_+(X)$ consists entirely of zero-divisors.*

(ix) *For each nonunit element $f \in C_+(X)$, there exists $0 \neq g \in C_+(X)$ with $P_f^+ \subseteq Ann^+(g)$.*

(x) *Every z-ideal in the Γ-semiring $C_-(X)$ is a z^o-ideal.*

(xi) *Every maximal ideal in the Γ-semiring $C_-(X)$ is a z^o-ideal.*

(xii) *Every maximal ideal in the Γ-semiring $C_-(X)$ consists entirely of zero-divisors.*

(xiii) *For each nonunit function $x \in C_-(X)$, there exists $0 \neq y \in C_-(X)$ with $P_x^- \subseteq Ann^-(y)$.*

Proof. $(vi) \Leftrightarrow (x)$: Let I be a z-ideal in $C_-(X)$. Then I_+ is a z-ideal in $C_+(X)$. By condition (vi), I_+ is a z^o-ideal in $C_+(X)$. Therefore $I = (I_+)'$ is a z^o-ideal in $C_+(X)$. Again let J be a z-ideal in $C_+(X)$. Then J' is a z-ideal in $C_-(X)$. By condition (x), J' is a z^o-ideal in $C_-(X)$. Therefore $J = (J')_+$ is a z^o-ideal in $C_+(X)$. Therefore every z-ideal in $C_+(X)$ is a z^o-ideal if and only if every z-ideal in $C_-(X)$ is a z^o-ideal.

$(vii) \Leftrightarrow (xi)$: Let M be a maximal ideal in $C_-(X)$. Then M_+ is a maximal ideal in $C_+(X)$. By condition (vii), M_+ is a z^o-ideal in $C_+(X)$. Therefore $M = (M_+)'$ is a z^o-ideal in $C_+(X)$. Again let N be a maximal ideal in $C_+(X)$. Then N' is a maximal ideal in $C_-(X)$. By condition (xi), N' is a z^o-ideal in $C_-(X)$. Therefore $N = (N')_+$ is a z^o-ideal in $C_+(X)$. So every maximal ideal in $C_+(X)$ is a z^o-ideal if and only if every maximal ideal in $C_-(X)$ is a z^o-ideal.

$(viii) \Leftrightarrow (xii)$: Let us suppose that every maximal ideal in $C_+(X)$ consists entirely of zero-divisors of $C_+(X)$. Let M be a maximal ideal of $C_-(X)$ and $\mathbf{0} \neq f \in M$. Then M_+ be a maximal ideal of $C_+(X)$ and $-f \in M_+$ is a zero-divisor of $C_+(X)$. Therefore there exists $\mathbf{0} \neq g \in C_+(X)$ such that $(-f)g = \mathbf{0}$. So for all $\alpha \in \Gamma$, $f\alpha(-g) = \mathbf{0}$. Hence f is a zero-divisor of $C_-(X)$ which implies that every maximal ideal in $C_-(X)$ consists entirely of zero-divisors of $C_-(X)$.
Conversely, let us suppose that every maximal ideal in $C_-(X)$ is entirely made up of zero-divisors of $C_-(X)$. Let M_1 be a maximal ideal of $C_+(X)$ and $\mathbf{0} \neq x \in M_1$. Then M_1' be a maximal ideal of $C_-(X)$ and $-x \in M_1'$ is a zero-divisor of $C_-(X)$. So there exists $\gamma \neq \mathbf{0}$ and $y \neq \mathbf{0}$ in $C_-(X)$ such that $(-x)\gamma y = 0$. Therefore $x(-y) = 0$ for some $-y \neq \mathbf{0}$ in $C_+(X)$ which implies that x is a zero-divisor of $C_+(X)$. Hence it follows that every maximal ideal in $C_+(X)$ consists entirely of zero-divisors of $C_+(X)$.

$(ix) \Leftrightarrow (xiii)$: Let us first recall that for any function $f \in C_+(X)$, $(P_f^+)' = P_{-f}^-$ and also note that

$$(Ann^+(f))' = Ann^-(-f).$$

Indeed, $a \in (Ann^+(f))'$ implies $-a \in Ann^+(f)$ which implies $(-a)f = \mathbf{0}$ whence it follows that $a\gamma(-f) = \mathbf{0}$ for all $\gamma \in C_-(X)$.
So $a \in Ann^-(-f)$, i.e., $(Ann^+(f))' \subseteq Ann^-(-f)$. By reversing the arguments above we get, $Ann^-(-f) \subseteq (Ann^+(f))'$. Therefore $Ann^-(-f) = (Ann^+(f))'$.
Now let the condition (ix) hold. Let x be a nonunit function of $C_-(X)$. Then $-x$ is a nonunit function of $C_+(X)$. So there exists $z \neq \mathbf{0}$ in $C_+(X)$ such that $P_{-x}^+ \subseteq Ann^+(z)$ which implies that

$$P_x^- = (P_{-x}^+)' \subseteq (Ann^+(z))' = Ann^-(-z), \text{ i.e., } P_x^- \subseteq Ann^-(-z).$$

Therefore the condition $(xiii)$ holds.
Conversely, let the condition $(xiii)$ hold. Let f be a nonunit function of $C_+(X)$. Then $-f$ is a nonunit function of $C_-(X)$. So there exists $h \neq \mathbf{0}$ in $C_-(X)$ such that $P_{-f}^- \subseteq Ann^-(h)$ which implies that

$$P_f^+ = (P_{-f}^-)_+ \subseteq (Ann^-(h))_+ = Ann^+(-h), \text{ i.e., } P_f^+ \subseteq Ann^+(-h).$$

Therefore the condition (ix) holds.
Other implications are already proved in Theorem 1.3.70. Therefore this completes the proof. □

Though we haven't been able to introduce the concept of almost regular Γ-semiring analogous to concept of the almost regular semirings, here we define the notion of almost regular $C_-(X)$. The *nonunit elements of* $C_-(X)$ ($\subset C(X)$) are the functions in $C_-(X)$ which are zero at some point of X.

Definition 4.3.30. $C_-(X)$ is called *almost regular* if for every nonunit function f in $C_-(X)$, there exists γ and s in $C_-(X)$ such that $f = f\gamma s$, where $\gamma s \neq \mathbf{1}$.

Theorem 4.3.31. *$C_+(X)$ is almost regular if and only if so is $C_-(X)$.*

Proof. Let $C_+(X)$ be almost regular and f be a nonunit function of $C_-(X)$. Then $-f$ is a nonunit function of $C_+(X)$. Since $C_+(X)$ is almost regular, there exists $\mathbf{1} \neq g \in C_+(X)$ such that $(-f)g = -f$.
So $f = f(-\mathbf{1})(-g)$ and this implies that $C_-(X)$ is almost regular.
Conversely, let $C_-(X)$ be almost regular and x be a nonunit function of $C_+(X)$. Then $(-x)$ is a nonunit function of $C_-(X)$. Since $C_-(X)$ is almost regular, there exists $\gamma, s \in C_-(X)$ such that $-x = (-x)\gamma s$, where $\gamma s \neq \mathbf{1}$. So $x = x(\gamma s)$. Therefore $C_+(X)$ is almost regular. □

We get the following theorem combining Theorems 1.3.72 and 4.3.31.

Theorem 4.3.32. *The following are equivalent.*

- (i) *X is an almost P-space.*
- (ii) *$C_+(X)$ is almost regular.*
- (iii) *$C_-(X)$ is almost regular.*

Theorem 4.3.33. *$C_+(X)$ is Noetherian (k-Noetherian) if and only if $C_-(X)$ is Noetherian (resp. k-Noetherian).*

Proof. $C_-(X)$ is Noetherian (k-Noetherian) if and only if $L(C_-(X))$ is Noetherian (resp. k-Noetherian) (see Theorem 1.4.22). Since $L(C_-(X))$ and $C_+(X)$ are isomorphic semirings then by Theorem 1.3.51 we deduce that $C_+(X)$ is a Noetherian (resp. k-Noetherian) semiring if and only if $L(C_-(X))$ is Noetherian (resp. k-Noetherian). Therefore we get that $C_+(X)$ is Noetherian (k-Noetherian) if and only if $C_-(X)$ is Noetherian (resp. k-Noetherian). $\square$

In the similar fashion as of Theorem 4.3.33, we can prove the following.

Theorem 4.3.34. *$C_+(X)$ is Artinian (k-Artinian) if and only if $C_-(X)$ is Artinian (resp. k-Artinian).*

Proof. Using Theorems 1.4.22 and 1.3.51, the proof is analogous to that of the above theorem. $\square$

Combining Theorems 1.3.75, 4.3.33 and 4.3.34 we have the following.

Theorem 4.3.35. *The following are equivalent.*

(i) *X is finite.*

(ii) *$C_+(X)$ is Noetherian.*

(iii) *$C_-(X)$ is Noetherian.*

(iv) *$C_+(X)$ is k-Noetherian.*

(v) *$C_-(X)$ is k-Noetherian.*

(vi) *$C_+(X)$ is Artinian.*

(vii) *$C_-(X)$ is Artinian.*

(viii) *$C_+(X)$ is k-Artinian.*

(ix) *$C_-(X)$ is k-Artinian.*

CHAPTER 5

A STUDY ON PRIME ΓS-SUBSEMIMODULES OF A ΓS-SEMIMODULE VIA ITS ASSOCIATED L-SEMIMODULE

5

A study on prime ΓS-subsemimodules of a ΓS-semimodule via its associated L-semimodule

In this chapter we have first studied the notions of prime ΓS-subsemimodules, prime $k\Gamma S$-subsemimodules of a ΓS-semimodule as a natural generalization of the concept of the prime subsemimodule of a semimodule. An inclusion preserving one-to-one correspondence between the set of all prime ΓS-subsemimodules ($k\Gamma S$-subsemimodules) of a ΓS-semimodule and the set of all prime subsemimodules (resp. k-subsemimodules) of the associated L-semimodule has been established. We have also established an one-to-one correspondence between the set of all maximal ΓS-subsemimodules (finitely generated ΓS-subsemimodules and k-finitely generated ΓS-subsemimodules) of a ΓS-semimodule and the set of all maximal L-subsemimodules (resp. finitely generated L-subsemimodules and k-finitely generated L-subsemimodules) of the associated L-semimodule. Using all these correspondences, we have obtained that ΓS-semimodules are finitely (k-finitely) generated whenever their associated L-semimodules are finitely (resp. k-finitely) generated and vice versa. Also it has been proved that a ΓS-semimodules is a multiplication ΓS-semimodule if and only if the associated L-semimodule is a multiplication L-semimodule. All these correspondences mentioned above have been used to obtain some properties of the prime ΓS-subsemimodules (es-

This chapter is mainly based on the work of the following paper:
Soumi Basu, Sarbani Mukherjee (Goswami) and Sujit Kumar Sardar, A study on prime ΓS-subsemimodules of a ΓS-semimodule via its associated L-semimodule, communicated

pecially on a multiplication ΓS-semimodule, finitely generated ΓS-semimodule and k-finitely generated ΓS-semimodule). Some results of the prime subsemimodules (k-subsemimodules) of a semimodule and the radical of a subsemimodule have been generalized in the context of ΓS-semimodules via those of its associated L-semimodule.

In **section 1**, we define the notion of prime ΓS-subsemimodules of a ΓS-semimodule (*cf.* Definition 5.1.5) and give examples of prime ΓS-subsemimodules and prime ΓS-subsemimodules (*cf.* Examples 5.1.7 5.1.8) and also we find equivalent criteria of prime ΓS-subsemimodules (*cf.* Theorem 5.1.15). We establish the correspondence between the set of all prime ΓS-subsemimodules ($k\Gamma S$-subsemimodules) of a ΓS-semimodule and that of all prime subsemimodules (resp. k-subsemimodules) of the associated L-semimodule (*cf.* Theorem 5.1.14). We studied the relation between the maximal ΓS-subsemimodules ($k\Gamma S$-subsemimodules) and the prime ΓS-subsemimodules (resp. $k\Gamma S$-subsemimodules) of a ΓS-semimodule (*cf.* Theorem 5.1.19).

In **section 2**, we establish that the set of all finitely (k-finitely) generated ΓS-subsemimodules of a ΓS-semimodule and the set of all finitely (resp. k-finitely) generated subsemimodules of its associated L-semimodule are in bijective correspondence (*cf.* Theorems 5.2.4, 5.2.9 respectively). Then we study a few results of prime ΓS-subsemimodules on finitely (k-finitely) generated ΓS-semimodules (*cf.* Theorems 5.2.11, 5.2.13).

In **section 3**, we study the notions of multiplication and k-multiplication ΓS-semimodules and its correspondence with the multiplication associated L-semimodule (*cf.* Definition 5.3.1, Theorem 5.3.3) and proved some interesting properties of prime ΓS-subsemimodules on a multiplication ΓS-semimodule such as we prove the irreducibility property of prime ΓS-subsemimodules (Theorem 5.3.13), we also prove that the associated ideal of a prime ΓS-subsemimodule is a prime ideal of the Γ-semiring S and vice versa (Theorem 5.3.8) etc. In addition we deduce some results on the radical (for instance, *cf.* Theorems 5.3.16, 5.3.19).

5.1 Prime ΓS-subsemimodule of a ΓS-semimodule

In this section we introduce some notions such as prime ΓS-subsemimodule, $k\Gamma S$-subsemimodule of an unitary ΓS-semimodule and study some useful properties of prime ΓS-subsemimodules and prime $k\Gamma S$-subsemimodules.

First we look into the definition of prime subsemimodules of a semimodule.

Definition 5.1.1. A proper subsemimodule P of a R-semimodule M is said to be *prime* in M if for an ideal I of R and for a subsemimodule N of M,

$$NI \subseteq P \text{ implies } N \subseteq P \text{ or } I \subseteq (P : M),$$

where $(P : M) = \{r \in R : Mr \subseteq P\}$.

Observation 5.1.2. It is easy to see that for any ideal I of R and for any subsemimodule N of an unitary R-semimodule M,

$$NI \subseteq P \text{ implies } N \subseteq P \text{ or } I \subseteq (P : M) \text{ if and only if}$$
$$mRx \subseteq P \text{ with } x \in R \text{ and } m \in M \text{ implies } m \in P \text{ or } x \in (P : M).$$

Therefore we have the following result.

Proposition 5.1.3. *Let M be an unitary R-semimodule and P be a proper subsemimodule of M. Then the following are equivalent.*

(*i*) *P is prime in M.*

(*ii*) *$mRx \subseteq P$ with $x \in R$ and $m \in M$ implies that $m \in P$ or $x \in (P : M)$.*

Note that the equivalent criteria in Proposition 5.1.3 is considered as the definition of prime subsemimodules of an unitary semimodule in [34] and its references.

Definition 5.1.4. Let N be a ΓS-subsemimodule of a ΓS-semimodule M. Then we define the set $(N : M) = \{x \in S : M\Gamma x \subseteq N\}$.
It is easy to verify that the set $(N : M)$ is an ideal of the Γ-semiring S and we call it the *associated ideal* of N.
Also $(N : M)$ is a k-ideal of S if N is a $k\Gamma S$-subsemimodule of M.

Below we introduce some notations:

(*i*) Let N be a ΓS-subsemimodule of a ΓS-semimodule M and I be a right ideal of S. Then the set $N\Gamma I = \{\Sigma_{i=1}^{n} a_i\alpha_i x_i : a_i \in N, \alpha_i \in \Gamma, x_i \in I, n$ any positive integer$\}$ is a ΓS-subsemimodule of M.

(*ii*) $N\Gamma x = \{\Sigma_{i=1}^{n} a_i\alpha_i x : a_i \in N, \alpha_i \in \Gamma, n$ any positive integer$\}$.

Taking M as an unitary ΓS-semimodule over a Γ-semiring S throughout this chapter, we now give the definition of prime ΓS-subsemimodules of M.

Definition 5.1.5. Let P be a proper ΓS-subsemimodule of a ΓS-semimodule M. P is said to be a *prime* ΓS*-subsemimodule* of M if for an ideal I of S and for a ΓS-subsemimodule N of M, $N\Gamma I \subseteq P$ implies either $N \subseteq P$ or $I \subseteq (P : M)$. A prime ΓS-subsemimodule which is also a $k\Gamma S$-subsemimodule M is called a *prime* $k\Gamma S$*-subsemimodule* of M.

The following theorem gives a characterization of prime ΓS-subsemimodules of a ΓS-semimodule.

Theorem 5.1.6. *Let M be a ΓS-semimodule. A ΓS-subsemimodule P of M is prime if and only if $\langle m\rangle\Gamma\langle x\rangle \subseteq P$ implies $m \in P$ or $x \in (P : M)$, where $\langle m\rangle$ is the cyclic ΓS-subsemimodule of M generated by $m \in M$ and $\langle x\rangle$ is the ideal of S generated by $x \in S$.*

Proof. If P is a prime ΓS-subsemimodule of M then the result follows from Definition 5.1.5. Now suppose that $\langle m\rangle\Gamma\langle x\rangle \subseteq P$ implies $m \in P$ or $x \in (P : M)$. Let us assume that for any ideal I of S and for any ΓS-subsemimodule N of M, $N\Gamma I \subseteq P$ and $N \nsubseteq P$. Then there exists $a \in N$ such that $a \notin P$. So

$$\langle a\rangle\Gamma\langle s\rangle \subseteq P \text{ for any } s \in I \text{ implies } s \in (P : M).$$

Thus $I \subseteq (P : M)$. Therefore P is a prime ΓS-subsemimodule of M. □

Below we give one example of prime ΓS-subsemimodule and one example of ΓS-subsemimodule which is not prime.

Example 5.1.7. For any prime number p, let K_p be the additive commutative semigroup $\{n : n \text{ is an non-negative integer divisible by } p\}$. Then K_p is a ΓS-subsemimodule of the ΓS-semimodule Z_0^+, where p is a prime number and $S = \Gamma = Z_0^-$. Let us assume that $\langle m\rangle\Gamma\langle x\rangle \subseteq K_p$ where $\langle m\rangle$ is the cyclic ΓS-subsemimodule of Z_0^+ generated by $m \in Z_0^+$ and $\langle x\rangle$ is the ideal of the Γ-semiring Z_0^- generated by $x \in Z_0^-$. Then $m\gamma x \in K_p$ for all $\gamma \in Z_0^-$ which implies that $p|m$ or $p|x$. Therefore $m \in K_p$ or $x \in (K_p : Z_0^+)$. Hence K_p is a prime ΓS-subsemimodule of the ΓS-semimodule Z_0^+, where p is a prime number. Again K_p is a $k\Gamma S$-subsemimodule of M. Therefore K_p is a prime $k\Gamma S$-subsemimodule of Z_0^+ for any prime p.

Example 5.1.8. In view of Example 5.1.7, it follows that for $p = 2, 3$, K_2 and K_3 are prime ΓS-subsemimodules of Z_0^+ though intersection of K_2 and K_3 is not a prime ΓS-subsemimodule of Z_0^+. For example, we see that $2 \notin K_2 \cap K_3$ and $-3 \notin (K_2 \cap K_3 : Z_0^+)$ but $\langle 2\rangle\Gamma\langle -3\rangle \subseteq K_2 \cap K_3$.

From Example 5.1.8, the following observation is clear.

Observation 5.1.9. Intersection of two prime ΓS-subsemimodules of a ΓS-semimodule may not be a prime ΓS-subsemimodule. So the set of all prime ΓS-subsemimodules of a ΓS-semimodule does not form a sublattice (with respect to set inclusion) of the lattice of all ΓS-subsemimodules of a ΓS-semimodule.

Below we find a condition under which intersection of prime ΓS-subsemimodules is a prime ΓS-subsemimodule.

Proposition 5.1.10. *Let M be a ΓS-semimodule and $\{P_i : i \in \Lambda\}$ be the collection of prime ΓS-subsemimodules (prime $k\Gamma S$-subsemimodules) of M such that $\{P_i : i \in \Lambda\}$ forms a chain of prime ΓS-subsemimodules (resp. prime $k\Gamma S$-subsemimodules) of M. Then $\bigcap_{i\in\Lambda} P_i$ is a prime ΓS-subsemimodule (resp. prime $k\Gamma S$-subsemimodule) of M.*

Proof. In view of Remark 1.6.7, $\bigcap_{i\in\Lambda} P_i$ is a ΓS-subsemimodule ($k\Gamma S$-subsemimodule) of M. Clearly $\bigcap_{i\in\Lambda} P_i \neq M$ Let us assume that for an ideal I of S and for a ΓS-subsemimodule N of M, $N\Gamma I \subseteq \bigcap_{i\in\Lambda} P_i$ which implies $N\Gamma I \subseteq P_i$ for all $i \in \Lambda$. if possible let $N \nsubseteq \bigcap_{i\in\Lambda} P_i$ and $I \nsubseteq (\bigcap_{i\in\Lambda} P_i : M) = \bigcap_{i\in\Lambda}(P_i : M)$. Then there exist j, k such that $N \nsubseteq P_j$ and $I \nsubseteq (P_k : M)$. Since $\{P_i : i \in \Lambda\}$ is a chain of prime ΓS-subsemimodules (resp. prime $k\Gamma S$-subsemimodules) of M, then either $P_j \subseteq P_k$ or $P_k \subseteq P_j$. If $P_j \subseteq P_k$ then $I \nsubseteq (P_j : M)$ and if $P_k \subseteq P_j$ then $N \nsubseteq P_k$. But, for all $i \in \Lambda$, $N\Gamma I \subseteq P_i$. Since P_i is a prime ΓS-subsemimodule of M for all $i \in \Lambda$, we must have either $N \subseteq P_i$ or $I \subseteq (P_i : M)$ for all $i \in \Lambda$. So if $P_j \subseteq P_k$ then $N \subseteq P_j$ and if $P_k \subseteq P_j$ then $I \subseteq (P_k : M)$. We arrive at this contradiction because of our assumption that $N \nsubseteq \bigcap_{i\in\Lambda} P_i$ and $I \nsubseteq (\bigcap_{i\in\Lambda} P_i : M)$. Therefore either $N \subseteq \bigcap_{i\in\Lambda} P_i$ or $I \subseteq (\bigcap_{i\in\Lambda} P_i : M)$. Hence $\bigcap_{i\in\Lambda} P_i$ is a prime ΓS-subsemimodule (resp. prime $k\Gamma S$-subsemimodule) of M. □

Now we will study the correspondence between the set of all prime ΓS-subsemimodules of a ΓS-semimodule M and the set of all prime subsemimodules of the associated L-semimodule $M^{\#}$ of M via the mapping $P \mapsto P^{+'}$, where P is a ΓS-subsemimodule of M. Note that unless otherwise mentioned, we denote the left operator semiring of a Γ-semiring S as L.

We recall the following:

For any ΓS-subsemimodule P of M,

$P^{+'} = \{\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle : (\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle)S \subseteq P\}$ is an L-subsemimodule of $M^{\#}$ and

$(P^{+'} : M^{\#})$ is an ideal of L (see Definition 1.5.3).
For any L-subsemimodule Q of $M^{\#}$, $Q^{+} = \{m \in M : \langle m, \Gamma\rangle \subseteq Q\}$ is a ΓS-subsemimodule of M and $(Q^{+} : M)$ is an ideal S.

Proposition 5.1.11. *Let $M^{\#}$ be the associated L-semimodule of the ΓS-semimodule M. Then*

(i) *For a ΓS-subsemimodule P of M, $(P : M)^{+'} = (P^{+'} : M^{\#})$.*

(ii) *For a L-subsemimodule Q of $M^{\#}$, $(Q : M^{\#})^{+} = (Q^{+} : M)$.*

Proof. (i) Suppose P is a ΓS-subsemimodule of M. Then $(P : M)$ is an ideal of S. So $(P : M)^{+'}$ is an ideal of L (by Theorem 1.4.19). Let $\sum_{i=1}^{m}[x_i, \alpha_i] \in (P : M)^{+'}$. Then $\sum_{i=1}^{m} x_i\alpha_i s \in (P : M)$ for all $s \in S$. Therefore $M\Gamma(\sum_{i=1}^{m} x_i\alpha_i s) \subseteq P$ for all $s \in S$. So

$$\text{for all } s \in S \text{ and } m_j \in M,\ \beta_j \in \Gamma,\ \textstyle\sum_{i,j} m_j\beta_j x_i\alpha_i s \in P$$

which implies $\sum_{j=1}^{n}\langle m_j, \beta_j\rangle \sum_{i=1}^{m}[x_i, \alpha_i] \in P^{+'}$ for all $m_j \in M$, $\beta_j \in \Gamma$. Since $\sum_{j=1}^{n}\langle m_j, \beta_j\rangle$ is an arbitrary element of $M^{\#}$, $M^{\#}\sum_{i=1}^{m}[x_i, \alpha_i] \subseteq P^{+'}$. It implies $\sum_{i=1}^{m}[x_i, \alpha_i] \in (P^{+'} : M^{\#})$. Therefore $(P : M)^{+'} \subseteq (P^{+'} : M^{\#})$. Again let $\sum_{j=1}^{n}[y_j, \beta_j] \in (P^{+'} : M^{\#})$. Then

$$M^{\#}\textstyle\sum_{j=1}^{n}[y_j, \beta_j] \subseteq P^{+'} \text{ i.e., for all } n_i \in M,\ \alpha_i \in \Gamma,\ \sum_{i=1}^{m}\langle n_i, \alpha_i\rangle \sum_{j=1}^{n}[y_j, \beta_j] \in P^{+'}.$$

Therefore for all $s \in S$ and for $n_i \in M$, $\alpha_i \in \Gamma$, $\sum_{i,j} n_i\alpha_i y_j\beta_j s \in P$ whence it follows that $M\Gamma(\sum_{j=1}^{n} y_j\beta_j s) \subseteq P$ for all $s \in S$.
This implies $\sum_{j=1}^{n} y_j\beta_j s \in (P : M)$ for all $s \in S$.
Therefore $\sum_{j=1}^{n}[y_j, \beta_j] \in (P : M)^{+'}$ and therefore $(P^{+'} : M^{\#}) \subseteq (P : M)^{+'}$.
Hence $(P : M)^{+'} = (P^{+'} : M^{\#})$.

(ii) Suppose Q is a L-subsemimodule of $M^{\#}$. Then $(Q : M^{\#})$ is an ideal of L. So $(Q : M^{\#})^{+}$ is an ideal of the Γ-semiring S (by Theorem 1.4.19). Let $a \in (Q : M^{\#})^{+}$. Then $[a, \gamma] \in (Q : M^{\#})$ for all $\gamma \in \Gamma$. So $M^{\#}[a, \gamma] \subseteq Q$ for all $\gamma \in \Gamma$. Then

$$\text{for all } \gamma \in \Gamma \text{ and for } m_j \in M,\ \beta_j \in \Gamma,\ \textstyle\sum_{j=1}^{n}\langle m_j\beta_j a, \gamma\rangle \in Q \text{ i.e., } \sum_{j=1}^{n} m_j\beta_j a \in Q^{+}.$$

Then $M\Gamma a \subseteq Q^{+}$. Therefore $a \in (Q^{+} : M)$ which implies $(Q : M^{\#})^{+} \subseteq (Q^{+} : M)$.
Again let $x \in (Q^{+} : M)$. Then $M\Gamma x \subseteq Q^{+}$. Therefore

$$\text{for all } \gamma \in \Gamma \text{ and for all } n_i \in M,\ \alpha_i \in \Gamma,\ \textstyle\sum_{i=1}^{m}\langle n_i\alpha_i x, \gamma\rangle = \sum_{i=1}^{m}\langle n_i, \alpha_i\rangle[x, \gamma] \in Q,$$
$$\text{i.e., for all } \gamma \in \Gamma,\ M^{\#}[x, \gamma] \subseteq Q,$$

since $\sum_{i=1}^{m}\langle n_i, \alpha_i\rangle \in M^{\#}$ is arbitrary.
So $[x, \Gamma] \subseteq (Q : M^{\#})$ which implies $x \in (Q : M^{\#})^{+}$.
Therefore $(Q^{+} : M) \subseteq (Q : M^{\#})^{+}$. Hence $(Q : M^{\#})^{+} = (Q^{+} : M)$. □

We first prove the result for its immediate use.

Proposition 5.1.12. *For any ideal I of the left operator semiring L of the Γ-semiring S and for any proper subsemimodule N of $M^{\#}$, $(NI)^{+} = N^{+}\Gamma I^{+}$.*

Proof. Let I be an ideal of L and let $\sum_{i=1}^{m}[x_i, \alpha_i] \in I = (I^{+})^{+'}$.
Then $(\sum_{i=1}^{m}[x_i, \alpha_i])S \subseteq I^{+}$ by Theorem 1.4.19. So

$$N^{+}\Gamma(\textstyle\sum_{i=1}^{m}[x_i, \alpha_i])S \subseteq N^{+}\Gamma I^{+} \text{ which implies } \langle N^{+}, \Gamma\rangle \textstyle\sum_{i=1}^{m}[x_i, \alpha_i] \subseteq (N^{+}\Gamma I^{+})^{+'}.$$

It is easy to see that $\langle N^{+}, \Gamma\rangle = N$. Then $N(\sum_{i=1}^{m}[x_i, \alpha_i]) \subseteq (N^{+}\Gamma I^{+})^{+'}$.
Therefore $NI \subseteq (N^{+}\Gamma I^{+})^{+'}$, i.e., $(NI)^{+} \subseteq N^{+}\Gamma I^{+}$.
Let $\sum_{j=1}^{n}[y_j, \beta_j] \in I$. Then

$$N(\textstyle\sum_{j=1}^{n}[y_j, \beta_j]) \subseteq NI, \text{ i.e., } \langle N^{+}, \Gamma\rangle \textstyle\sum_{j=1}^{n}[y_j, \beta_j] \subseteq NI.$$

So $N^{+}\Gamma(\sum_{j=1}^{n}[y_j, \beta_j])S \subseteq (NI)^{+}$. Since $\sum_{j=1}^{n}[y_j, \beta_j] \in I$ is equivalent to saying that $(\sum_{j=1}^{n}[y_j, \beta_j])S \subseteq I^{+}$ and $\sum_{j=1}^{n}[y_j, \beta_j] \in I$ is arbitrary then $N^{+}\Gamma I^{+} \subseteq (NI)^{+}$. Therefore $N^{+}\Gamma I^{+} = (NI)^{+}$. □

Proposition 5.1.13. *Let $M^{\#}$ be the associated L-semimodule of the ΓS-semimodule M, where L is the left operator semiring of the Γ-semiring S.*

(i) If P is a prime ΓS-subsemimodule of M then $P^{+'}$ is a prime L-subsemimodule of $M^{\#}$, where $P^{+'} = \{\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle : (\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle)S \subseteq P\}$.

(ii) If Q is a prime L-subsemimodule of $M^{\#}$ then Q^{+} is a prime ΓS-subsemimodule of M, where $Q^{+} = \{m \in M : \langle m, \Gamma\rangle \subseteq Q\}$.

Proof. (i) Let P be a prime ΓS-subsemimodule of M. Then by Theorem 1.6.11, $P^{+'}$ is a proper L-subsemimodule of $M^{\#}$. Let us assume that for an ideal I of the left operator semiring L of the Γ-semiring S and a subsemimodule N of $M^{\#}$, $NI \subseteq P^{+'}$. Then by Proposition 5.1.12, $N^{+}\Gamma I^{+} = (NI)^{+} \subseteq (P^{+'})^{+} = P$. Since P is a prime ΓS-subsemimodule of M,

$$N^{+}\Gamma I^{+} \subseteq P \text{ implies that either } N^{+} \subseteq P \text{ or } I^{+} \subseteq (P : M).$$

Therefore by Theorem 1.6.11, either $N \subseteq P^{+'}$ or $I \subseteq (P : M)^{+'} = (P^{+'} : M^{\#})$ (by Proposition 5.1.11 (i)). Therefore $P^{+'}$ is a prime L-subsemimodule of $M^{\#}$.

(ii) Let Q be a prime L-subsemimodule of $M^{\#}$. Then by Theorem 1.6.11, Q^{+} is a proper ΓS-subsemimodule of M. Let us assume that for an ideal J of the Γ-semiring

S and a ΓS-subsemimodule N_1 of M, $N_1\Gamma J \subseteq Q^+$. Then $(N_1\Gamma J)^{+'} \subseteq (Q^+)^{+'} = Q$. Now using Proposition 5.1.12 we get,

$$N_1\Gamma J = (N_1^{+'})^+\Gamma(J^{+'})^+ = (N_1^{+'}J^{+'})^+.$$

So $N_1^{+'}J^{+'} = (N_1\Gamma J)^{+'} \subseteq Q$. Since Q is a prime subsemimodule of $M^\#$ then by Definition 5.1.1, either $N_1^{+'} \subseteq Q$ or $J^{+'} \subseteq (Q : M^\#)$,
i.e., either $N_1 \subseteq Q^+$ or $J \subseteq (Q : M^\#)^+ = (Q^+ : M)$ (by Proposition 5.1.11 (ii)). Therefore Q^+ is a prime ΓS-subsemimodule of M. $\square$

In view of Theorem 1.6.11 and Proposition 5.1.13, we have the following.

Theorem 5.1.14. *The set of all prime ΓS-subsemimodules ($k\Gamma S$-subsemimodules) of a ΓS-semimodule M and the set of all prime subsemimodules (resp. k-subsemimodules) of its associated L-semimodule $M^\#$ are in order preserving bijective correspondence via the mapping $P \mapsto P^{+'}$, where P is a ΓS-subsemimodule (resp. $k\Gamma S$-subsemimodule) of M.*

The following result characterizes prime ΓS-subsemimodules of a ΓS-semimodule.

Theorem 5.1.15. *If P is a proper ΓS-subsemimodule of a ΓS-semimodule M then the following conditions are equivalent:*

(i) *P is a prime ΓS-subsemimodule of M.*

(ii) *If $\langle m\rangle\Gamma\langle x\rangle \subseteq P$ then $m \in P$ or $x \in (P : M)$, where $\langle m\rangle$ is the cyclic subsemimodule of M generated by $m \in M$ and $\langle x\rangle$ is the principal ideal of S generated by $x \in S$.*

(iii) *If $m\Gamma S\Gamma x \subseteq P$ then either $m \in P$ or $x \in (P : M)$, where $m \in M$, $x \in S$.*

Proof. $(i) \Leftrightarrow (ii)$: This equivalence has already been proved in Theorem 5.1.6.

$(i) \Rightarrow (iii)$: Let P be a prime ΓS-subsemimodule of M. Then by Theorem 5.1.14, $P^{+'}$ is a prime L-subsemimodule of $M^\#$. Let us assume that for $m \in M$, $m\Gamma S\Gamma x \subseteq P$. Then $\langle m\Gamma S\Gamma x, \Gamma\rangle \subseteq P^{+'}$. So $\langle m\alpha s\alpha x, \alpha\rangle \in P^{+'}$ for all $s \in S$ and $\alpha \in \Gamma$ which implies $\langle m, \alpha\rangle[s, \alpha][x, \alpha] \in P^{+'}$ for all $s \in S$ and $\alpha \in \Gamma$, i.e.,
$\langle m, \alpha\rangle L[x, \alpha] \in P^{+'}$ for all $\alpha \in \Gamma$. Since $P^{+'}$ is a prime L-subsemimodule of $M^\#$ then by Proposition 5.1.3, we have

$$\text{either } \langle m, \alpha\rangle \in P^{+'} \text{ or } [x, \alpha] \in (P^{+'} : M^\#) = (P : M)^{+'} \text{ for all } \alpha \in \Gamma.$$

Therefore either $m \in P$ or $x \in (P : M)$. Hence the proof is complete.

$(iii) \Rightarrow (ii)$: Let for $m \in M$, $x \in S$, $\langle m\rangle\Gamma\langle x\rangle \subseteq P$. Then we observe that $m\Gamma S\Gamma x \subseteq P$. So by the virtue of (iii), either $m \in P$ or $x \in (P : M)$. Hence (ii) is proved. □

Theorem 5.1.16. *Let M be a ΓS-semimodule and P be a prime ΓS-subsemimodule ($k\Gamma S$-subsemimodule) of M. Then $(P : M)$ is a prime ideal (resp. k-ideal) of S.*

Proof. Let M be a ΓS-semimodule and P be a prime ΓS-subsemimodule of M. Then by Theorem 5.1.14, $P^{+'}$ is a prime L-subsemimodule of $M^{\#}$. So $(P^{+'} : M^{\#}) = (P : M)^{+'}$ is a prime ideal of $M^{\#}$ (*cf.* Proposition 5.1.11 and Remark 1.5.7). Then by Theorem 1.4.19 we find that $(P : M)$ is a prime ideal of S. With the similar argument as above and using Remark 1.5.8 we get that if P is a prime $k\Gamma S$-subsemimodule of M then $(P : M)$ is a prime k-ideal of S. □

Definitions 5.1.17. A proper ΓS-subsemimodule N of a ΓS-semimodule M is said to be *maximal* in M if for each ΓS-subsemimodule P of M, $N \subseteq P \subseteq M$ implies that $P = N$ or $P = M$.
A proper $k\Gamma S$-subsemimodule N of a ΓS-semimodule M is said to be a *maximal $k\Gamma S$-subsemimodule* in M if for each $k\Gamma S$-subsemimodule P of M, $N \subseteq P \subseteq M$ implies that $P = N$ or $P = M$.

From the order preserving bijection between the set of all ΓS-subsemimodules ($k\Gamma S$-subsemimodules) of a ΓS-semimodule and the set of all subsemimodules (resp. k-subsemimodules) of its associated L-semimodule (*cf.* Theorem 1.6.11), we obtain the following.

Theorem 5.1.18. *The set of all maximal ΓS-subsemimodules ($k\Gamma S$-subsemimodules) of a ΓS-semimodule M and the set of all maximal subsemimodules (resp. k-subsemimodules) of its associated L-semimodule $M^{\#}$ are in bijective correspondence via the mapping $K \mapsto K^{+'}$, where K is a ΓS-subsemimodule (resp. $k\Gamma S$-subsemimodule) of M.*

Proof. (i) Let P be a maximal ΓS-subsemimodule of M. Then $P^{+'}$ is a proper subsemimodule of $M^{\#}$. if possible let $P^{+'}$ be not maximal. Then there exists a proper subsemimodule N of $M^{\#}$ properly containing $P^{+'}$. Now $P^{+'} \subset N$ implies $P = (P^{+'})^{+} \subset N^{+}$ which is a contradiction to the fact that P is maximal. Therefore $P^{+'}$ is a maximal L-subsemimodule of $M^{\#}$.

(ii) Let Q be a maximal L-subsemimodule of $M^{\#}$. Then Q^{+} is a proper ΓS-subsemimodule of M. if possible let Q^{+} be not maximal. Then there exists a proper ΓS-subsemimodule K of M properly containing Q^{+}. Now $Q^{+} \subset K$ implies $Q = (Q^{+})^{+'} \subset K^{+'}$ which is a contradiction to the fact that Q is maximal. Therefore Q^{+} is a maximal L-subsemimodule of M. □

Theorem 5.1.19. *Every maximal ΓS-subsemimodule ($k\Gamma S$-subsemimodule) of a ΓS-semimodule M is a prime ΓS-subsemimodule (resp. $k\Gamma S$-subsemimodule).*

Proof. Let K be a maximal ΓS-subsemimodule of M. Therefore by Theorem 5.1.18, $K^{+'}$ is a maximal subsemimodule of $M^{\#}$. Then by Lemma 1.5.9, $K^{+'}$ is a prime subsemimodule of $M^{\#}$. So by Theorem 5.1.14, $(K^{+'})^{+} = K$ is a prime ΓS-subsemimodule of M. Hence the proof is complete.
Similarly using Lemma 1.5.10, we can prove the result for a maximal $k\Gamma S$-subsemimodule. □

5.2 Prime ΓS-subsemimodules on a finitely (k-finitely) generated ΓS-semimodule

In this section we study some results on finitely (k-finitely) generated ΓS-semimodule. A ΓS-semimodule M is called *finitely generated ΓS-semimodule* if M is generated by a finite subset of it.

Proposition 5.2.1. *Let M be a ΓS-semimodule and N be a proper ΓS-subsemimodule and $\sum_{i=1}^{m}[e_i, \delta_i]$ be the left unity of S. Then $N^{+'} = \{\sum_{i=1}^{m}\langle a_i, \delta_i\rangle : a_i \in N\}$.*

Proof. If $a_1, a_2, ..., a_m \in N$ then $\sum_{i=1}^{m} a_i\delta_i s \in N$ for all $s \in S$. So $\sum_{i=1}^{m}\langle a_i, \delta_i\rangle \in N^{+'}$. Let $\sum_{j=1}^{n}\langle x_j, \alpha_j\rangle \in N^{+'}$. Now

$$\sum_{j=1}^{n}\langle x_j, \alpha_j\rangle = (\sum_{j=1}^{n}\langle x_j, \alpha_j\rangle)(\sum_{i=1}^{m}[e_i, \delta_i]) = \sum_{j,i}\langle x_j\alpha_j e_i, \delta_i\rangle = \sum_{i=1}^{m}\langle\sum_{j=1}^{n} x_j\alpha_j e_i, \delta_i\rangle.$$

Since $\sum_{j=1}^{n} x_j\alpha_j e_i \in N$ for all $i = 1, 2, ..., m$, the result follows. □

Proposition 5.2.2. *If N is a finitely generated ΓS-subsemimodule of M then $N^{+'}$ is a finitely generated L-subsemimodule of $M^{\#}$.*

Proof. Let N be a finitely generated ΓS-subsemimodule and $\{a_1, a_2, ..., a_t\}$ be the set of generators of N, where t is a positive integer. Then $N^{+'}$ is a proper ΓS-subsemimodule of $M^{\#}$. Let us assume that $a \in N^{+'}$. Then by Proposition 5.2.1, there exists $x_1, x_2, ..., x_m \in N$ such that $a = \sum_{i=1}^{m}\langle x_i, \delta_i\rangle$, where $\sum_{i=1}^{m}[e_i, \delta_i]$ is the left unity of S. So there exist $z_{ik} \in [\Gamma, S]$ for $k = 1, 2, ..., t$ and $i = 1, 2, ..., m$ such that $x_i = \sum_{k=1}^{i} a_k z_{ik}$, $i = 1, 2, ..., m$. Hence

$$a = \sum_{i=1}^{m}\langle x_i, \delta_i\rangle = \sum_{i=1}^{m}\langle\sum_{k=1}^{i} a_k z_{ik}, \delta_i\rangle = \sum_{i,k}\langle a_k z_{ik}, \delta_i\rangle = \sum_{i,k}\langle a_k(\sum_{j=1}^{n}[\gamma_j, f_j])z_{ik}, \delta_i\rangle$$

$$= \sum_{i,k,j}\langle a_k, \gamma_j\rangle[f_j z_{ik}, \delta_i] = \sum_{k,j}\langle a_k, \gamma_j\rangle(\sum_{i}[f_j z_{ik}, \delta_i]),$$

where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of S. Hence $N^{+'}$ is finitely generated by the set $\{\langle a_k, \gamma_j\rangle : 1 \leq k \leq t, 1 \leq j \leq n, t, n \in Z^+\}$. □

Proposition 5.2.3. *If K is a finitely generated L-subsemimodule of $M^\#$ then K^+ is a finitely generated ΓS-subsemimodule of M.*

Proof. Let K be a finitely generated L-subsemimodule of $M^\#$ generated by the subset $\{a_1, a_2, ..., a_r\}$ of K. Let $a \in K^+$. Then $\langle a, \gamma_j\rangle \in K$ for all $j = 1, 2, ..., n$, where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of S. So there exist $x_{jk} \in L$, for $j = 1, 2, ..., n$ and $k = 1, 2, ..., r$ such that $\langle a, \gamma_j\rangle = \sum_{k=1}^{r} a_k x_{jk} = \sum_{k=1}^{r} a_k\{(\sum_{i=1}^{m}[e_i, \delta_i])x_{jk}\}$
for $j = 1, 2, ..., n$. Hence

$$a = \sum_{j=1}^{n} a\gamma_j f_j = \sum_{j,k,i}(a_k e_i)\delta_i(x_{jk} f_j) = \sum_{k,i}(a_k e_i)\delta_i(\sum_{j=1}^{n} x_{jk} f_j).$$

Hence K^+ is finitely generated by the subset $\{a_k e_i : 1 \leq k \leq r, 1 \leq i \leq m\}$ of M. □

Therefore combining propositions 5.2.2 and 5.2.3, we have the following theorem.

Theorem 5.2.4. *The set of all finitely generated ΓS-subsemimodules of a ΓS-semimodule M and the set of all finitely generated subsemimodules of its associated L-semimodule $M^\#$ are in bijective correspondence via the mapping $K \mapsto K^{+'}$, where K is a ΓS-subsemimodule of M.*

As every ΓS-semimodule (semimodule) is a ΓS-subsemimodule (resp. subsemimodule) of itself, by virtue of Theorem 5.2.4, we have the following theorem.

Theorem 5.2.5. *M is a finitely generated ΓS-semimodule if and only if $M^\#$ is a finitely generated L-semimodule.*

Lemma 5.2.6. *Let N be a proper ΓS-subsemimodule of M and T be a proper subsemimodule of $M^\#$. Then:*

(i) $(\overline{N}^k)^{+'} = \overline{(N^{+'})}^k$.

(ii) $(\overline{T}^k)^{+} = \overline{(T^{+})}^k$.

Proof. (i) Let N be a proper ΓS-subsemimodule of M. $N \subseteq \overline{N}^k$ which implies $N^{+'} \subseteq (\overline{N}^k)^{+'}$. Also $(\overline{N}^k)^{+'}$ is a k-subsemimodule of $M^{\#}$. Since $\overline{(N^{+'})}^k$ is the smallest k-subsemimodule containing $N^{+'}$, $\overline{(N^{+'})}^k \subseteq (\overline{N}^k)^{+'}$. Again $N^{+'} \subseteq \overline{(N^{+'})}^k$. This implies $N = (N^{+'})^{+} \subseteq (\overline{(N^{+'})}^k)^{+}$ which is a $k\Gamma S$-subsemimodule of M. Then $\overline{N}^k \subseteq (\overline{(N^{+'})}^k)^{+}$. So $(\overline{N}^k)^{+'} \subseteq \overline{(N^{+'})}^k$. Hence $(\overline{N}^k)^{+'} = \overline{(N^{+'})}^k$.

(ii) The proof is analogous to that of (i). □

Recall that a semimodule K is *k-finitely generated* if there exists a finite subset L of K such that $\overline{\langle L\rangle}^k = K$ [35].

Definition 5.2.7. A ΓS-semimodule M is called *k-finitely generated* if there exists a finitely generated ΓS-subsemimodule F of M such that $\overline{F}^k = M$.

Proposition 5.2.8. *(i) If N is a k-finitely generated ΓS-subsemimodule of M then $N^{+'}$ is a k-finitely generated L-subsemimodule of $M^{\#}$.*

(ii) If K is a k-finitely generated L-subsemimodule of $M^{\#}$ then K^{+} is a k-finitely generated ΓS-subsemimodule of M.

Proof. (i) Let N be a k-finitely generated ΓS-subsemimodule. Then there exists a finitely generated ΓS-subsemimodule $N_1 \subseteq N$ such that $\overline{N_1}^k = N$. By Theorem 5.2.4, $N_1^{+'}$ is a finitely generated subsemimodule of $M^{\#}$ and by Lemma 5.2.6 (i), $\overline{(N_1^{+'})}^k = (\overline{N_1}^k)^{+'} = N^{+'}$. Therefore $N^{+'}$ is a k-finitely generated subsemimodule of $M^{\#}$.

(ii) Let K be a k-finitely generated subsemimodule. Then there exists a finitely generated subsemimodule $K_1 \subseteq K$ such that $\overline{K_1}^k = K$. By Theorem 5.2.4, K_1^{+} is a finitely generated subsemimodule of M and by Lemma 5.2.6 (ii), $\overline{(K_1^{+})}^k = (\overline{K_1}^k)^{+} = K^{+}$. Therefore K^{+} is a k-finitely generated ΓS-subsemimodule of M. □

From Proposition 5.2.8, we obtain the following theorem.

Theorem 5.2.9. *The set of all k-finitely generated ΓS-subsemimodules of a ΓS-semimodule M and the set of all k-finitely generated subsemimodules of its associated L-semimodule $M^{\#}$ are in bijective correspondence via the mapping $K \mapsto K^{+'}$, where K is a ΓS-subsemimodule of M.*

As a particular case of Theorem 5.2.9, we have the following theorem.

Theorem 5.2.10. *M is a k-finitely generated ΓS-semimodule if and only if $M^{\#}$ is a k-finitely generated L-semimodule.*

Theorem 5.2.11. *Let M be a finitely generated ΓS-semimodule. If N is a proper ΓS-subsemimodule of M then there exists a maximal ΓS-subsemimodule of M containing N.*

Proof. Let M be a finitely generated ΓS-semimodule and N be a proper ΓS-subsemimodule of M. Then $M^{\#}$ is finitely generated and $N^{+'}$ is a proper subsemimodule of $M^{\#}$. Therefore by Lemma 1.5.11, there exists a maximal subsemimodule K of $M^{\#}$ such that $K \subseteq N^{+'}$. So by Theorem 5.1.18, K^{+} is a maximal ΓS-subsemimodule of M containing N. Hence the proof is complete. □

In view of Theorems 5.2.11 and 5.1.19, we obtain the following result.

Theorem 5.2.12. *If N is a proper ΓS-subsemimodule of a finitely generated ΓS-semimodule M then there exists a prime ΓS-subsemimodule of M containing N.*

Theorem 5.2.13. *Let M be a k-finitely generated ΓS-semimodule. If N is a proper $k\Gamma S$-subsemimodule of M then there exists a maximal $k\Gamma S$-subsemimodule of M containing N.*

Proof. Let M be a k-finitely generated ΓS-semimodule and N be a proper $k\Gamma S$-subsemimodule of M. Then $M^{\#}$ is k-finitely generated and $N^{+'}$ is a proper k-subsemimodule of $M^{\#}$. Therefore by Theorem 1.5.13, there exists a maximal k-subsemimodule K of $M^{\#}$ such that $K \subseteq N^{+'}$. So by Theorem 5.1.18, K^{+} is a maximal $k\Gamma S$-subsemimodule of M containing N. Hence the proof is complete. □

In view of Theorems 5.2.13 and 5.1.19, we obtain the following result.

Theorem 5.2.14. *If N is a proper $k\Gamma S$-subsemimodule of a k-finitely generated ΓS-semimodule M then there exists a prime $k\Gamma S$-subsemimodule of M containing N.*

5.3 Prime ΓS-subsemimodules on a multiplication ΓS-semimodule

We recall that an R-semimodule M is said to be *multiplication semimodule* if for every subsemimodule N of M there exists an ideal I of R such that $N = MI$. Also

$N = M(N : M)$, where $(N : M) = \{r \in R : Mr \subseteq N\}$ [34]. We begin this section introducing the notion of multiplication ΓS-semimodule.

Definition 5.3.1. A ΓS-semimodule M is called *multiplication ΓS-semimodule* if for any ΓS-subsemimodule N of M, there exists an ideal I of S such that $N = M\Gamma I$.

Proposition 5.3.2. *(i) For any ideal J of a Γ-semiring S, $(M\Gamma J)^{+'} = M^{\#}J^{+'}$.*

(ii) For any ideal I of the left operator semiring L of a Γ-semiring S, $(M^{\#}I)^{+} = M\Gamma I^{+}$.

Proof. (i) Let J be an ideal of a Γ-semiring S. Then by Theorem 1.4.19, there exists an ideal I of the left operator semiring L of S such that $I^+ = J$. So let $\sum_{i=1}^{m}[x_i, \alpha_i] \in I$. Then $\sum_{i=1}^{m} x_i\alpha_i s \in I^+$ for all $s \in S$. So

$$M\Gamma(\textstyle\sum_{i=1}^{m} x_i, \alpha_i s) \subseteq M\Gamma I^+ \text{ for all } s \in S.$$

It implies that for all $s \in S$ and $m_j \in M$, $\beta_j \in \Gamma$,

$$\textstyle\sum_{i,j} m_j\beta_j x_i\alpha_i s \in (M\Gamma I^+), \text{ i.e., } \sum_{j=1}^{n}\langle m_j, \beta_j\rangle \sum_{i=1}^{m}[x_i, \alpha_i] \in (M\Gamma I^+)^{+'}.$$

Hence $M^{\#}\sum_{i=1}^{m}[x_i, \alpha_i] \subseteq (M\Gamma I^+)^{+'}$. Therefore $M^{\#}I \subseteq (M\Gamma I^+)^{+'}$.
Let $\sum_{j=1}^{n}[y_j, \beta_j] \in I$. So $\sum_{j=1}^{n} y_j\beta_j s \in I^+$ for all $s \in S$. Then

$$M^{\#}\textstyle\sum_{j=1}^{n}[y_j, \beta_j] \subseteq M^{\#}I, \text{ i.e., } \langle M, \Gamma\rangle \sum_{j=1}^{n}[y_j, \beta_j] \subseteq M^{\#}I.$$

Then for all $s \in S$, $M\Gamma(\sum_{j=1}^{n} y_j\beta_j s) \subseteq (M^{\#}I)^+$ which implies $M\Gamma I^+ \subseteq (M^{\#}I)^+$. Hence $(M\Gamma I^+)^{+'} \subseteq M^{\#}I$. Therefore $(M\Gamma I^+)^{+'} = (M^{\#}I)$, i.e., $(M\Gamma J)^{+'} = M^{\#}J^{+'}$.

(ii) Let I be an ideal of L. Then by Theorem 1.4.19, there exists an ideal J of S such that $J^{+'} = I$. Therefore by Proposition 5.3.2 (i), $(M\Gamma J)^{+'} = M^{\#}J^{+'}$ which implies $M\Gamma I^+ = M\Gamma J = (M^{\#}J^{+'})^+ = (M^{\#}I)^+$. □

Theorem 5.3.3. *If M is a multiplication ΓS-semimodule then the associated semimodule $M^{\#}$ of M is a multiplication L-semimodule and vice versa.*

Proof. Let M be a multiplication ΓS-semimodule. Let us take a subsemimodule N of $M^{\#}$. Then by Theorem 1.6.11, N^+ is a ΓS-subsemimodule of M. Since M is a multiplication ΓS-semimodule, there exists an ideal I of S such that $N^+ = M\Gamma I$. So by Proposition 5.3.2 (i), $N = (N^+)^{+'} = (M\Gamma I)^{+'} = M^{\#}I^{+'}$, where $I^{+'}$ is an ideal of L. Hence $M^{\#}$ is a multiplication L-semimodule.
Conversely, let $M^{\#}$ be a multiplication L-semimodule and let N be a ΓS-subsemimodule of M. Then $N^{+'}$ is a subsemimodule of $M^{\#}$. So there exists an ideal I of L such that

$N^{+'} = M^{\#}I$. Therefore by Proposition 5.3.2 (ii), $N = (N^{+'})^{+} = (M^{\#}I)^{+} = M\Gamma I^{+}$, where I^{+} is an ideal of S. It implies that M is a multiplication ΓS-semimodule. □

Theorem 5.3.4. *Let M be a multiplication ΓS-semimodule and N be a ΓS-subsemimodule of M. Then $N = M\Gamma(N : M)$.*

Proof. Let M be a multiplication ΓS-semimodule and N be a ΓS-subsemimodule of M. Then by Theorem 5.3.3, $M^{\#}$ is a multipliccation L-semimodule and also $N^{+'}$ is a L-subsemimodule of $M^{\#}$. Therefore $N^{+'} = M^{\#}(N^{+'} : M^{\#})$ which implies
$N = (N^{+'})^{+} = (M^{\#}(N^{+'} : M^{\#}))^{+} = (M^{\#}(N : M)^{+'})^{+}$ by Proposition 5.1.11 (i).
Again $(M^{\#}(N : M)^{+'})^{+} = M\Gamma(N : M)$ by Proposition 5.3.2 (ii).
Hence $N = M\Gamma(N : M)$. □

Theorem 5.3.5. *A ΓS-semimodule M is a multiplication ΓS-semimodule if and only if there exists an ideal I of the Γ-semiring S such that $m\Gamma S = M\Gamma I$ for each $m \in M$.*

Proof. Let M be a multiplication ΓS-semimodule. Then by Theorem 5.3.3, $M^{\#}$ is a multiplication L-semimodule. So by Proposition 1.5.15, there exists an ideal J of L such that $\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle L = M^{\#}J$ for all $\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle \in M^{\#}$.
Now $\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle L = \sum_{i=1}^{m}\langle m_i, \alpha_i\rangle[S, \Gamma] = \sum_{i=1}^{m}\langle m_i\alpha_i S, \Gamma\rangle$ for all $m_i \in M$ and $\alpha_i \in \Gamma$.
Also by Proposition 5.3.2 (ii), $(M^{\#}J)^{+} = M\Gamma J^{+}$. This implies $\sum_{i=1}^{m} m_i\alpha_i s \in M\Gamma J^{+}$ for all $m_i \in M$, $\alpha_i \in \Gamma$ and $s \in S$.
Hence $m\Gamma S = M\Gamma I$ for each $m \in M$, where $I = J^{+}$ is an ideal of S.
Conversely, let I be an ideal of the Γ-semiring S such that $m\Gamma S = M\Gamma I$ for each $m \in M$. Then for the ideal $I^{+'}$ of L, $(M\Gamma I)^{+'} = M^{\#}I^{+'}$ (by Proposition 5.3.2 (i)). Therefore $\sum_{j=1}^{n}\langle m, \alpha_j\rangle L = M^{\#}I^{+'}$ for each $m \in M$, $\alpha_j \in \Gamma$. So by Proposition 1.5.15, $M^{\#}$ is a multiplication L-semimodule. Hence by Theorem 5.3.3, M is a multiplication ΓS-semimodule. □

Now we give one example of a multiplication ΓS-semimodule.

Example 5.3.6. Let us consider, for any prime number p, the additive commutative semigroup $pZ_0^{+} = \{n : n$ is an non-negative integer divisible by $p\}$. Clearly for any prime number p, pZ_0^{+} is a ΓS-semimodule, where $S = \Gamma = Z_0^{-}$. We observe that pZ_0^{+} is a multiplication ΓS-semimodule, where $S = \Gamma = Z_0^{-}$. Indeed, for a prime number p, every element of pZ_0^{+} is of the form $k \cdot p$, where k is a non-negative integer. Therefore for all $m \in pZ_0^{+}$, $m\Gamma S = kpZ_0^{-}Z_0^{-} = pZ_0^{+}Z_0^{-}kZ_0^{-} = pZ_0^{+}\Gamma I$, where $I = kZ_0^{-}$ is an ideal of the Γ-semiring Z_0^{-} for a non-negative integer k. So by Theorem 5.3.5, pZ_0^{+} is a multiplication ΓS-semimodule, where $S = \Gamma = Z_0^{-}$.

Theorem 5.3.7. *Let M be a multiplication ΓS-semimodule and P be a ΓS-subsemimodule of M. Then $(P : M)$ is a prime ideal of S implies P is a prime ΓS-subsemimodule of M.*

Proof. Let M be a multiplication ΓS-semimodule and P be a ΓS-subsemimodule of M such that $(P : M)$ is a prime ideal of S. Then $M^{\#}$ is a multiplication L-semimodule and $(P : M)^{+'} = (P^{+'} : M^{\#})$ is a prime ideal of L (by Proposition 5.1.11 (i) and Theorem 1.4.19). It implies that $P^{+'}$ is a prime L-subsemimodule of $M^{\#}$ (*cf.* Theorem 1.5.14). Therefore by Theorem 5.1.14, $(P^{+'})^{+} = P$ is a prime ΓS-subsemimodule of M. □

Combining Theorems 5.1.16, 5.3.7, we have the following result.

Theorem 5.3.8. *Let M be a multiplication ΓS-semimodule. Then a ΓS-subsemimodule P is a prime ΓS-subsemimodule of M if and only if $(P : M)$ is a prime ideal of S.*

Let us now define k-multiplication ΓS-semimodule.

Definition 5.3.9. A ΓS-semimodule M is called *k-multiplication ΓS-semimodule* if for any ΓS-subsemimodule N of M, there exists a k-ideal I of S such that $N = M\Gamma I$.

Note that every k-multiplication ΓS-semimodule is a multiplication ΓS-semimodule but not the converse.

Theorem 5.3.10. *If M is a k-multiplication ΓS-semimodule then the associated semimodule $M^{\#}$ of M is a k-multiplication L-semimodule and vice versa.*

Proof. Let M be a k-multiplication ΓS-semimodule. Let us take a subsemimodule N of $M^{\#}$. Then by Theorem 1.6.11, N^{+} is a ΓS-subsemimodule of M. Since M is a k-multiplication ΓS-semimodule, there exists a k-ideal I of S such that $N^{+} = M\Gamma I$. So by Proposition 5.3.2 (i), $N = (N^{+})^{+'} = (M\Gamma I)^{+'} = M^{\#}I^{+'}$, where $I^{+'}$ is an k-ideal of L. Hence $M^{\#}$ is a k-multiplication L-semimodule.
Conversely, let $M^{\#}$ be a k-multiplication L-semimodule and let N be a ΓS-subsemimodule of M. Then $N^{+'}$ is a subsemimodule of $M^{\#}$. So there exists a k-ideal I of L such that $N^{+'} = M^{\#}I$. Therefore by Proposition 5.3.2 (ii), $N = (N^{+'})^{+} = (M^{\#}I)^{+} = M\Gamma I^{+}$, where I^{+} is a k-ideal of S. It implies that M is a k-multiplication ΓS-semimodule. □

Theorem 5.3.11. *Let M be a k-multiplication ΓS-semimodule. Then a $k\Gamma S$-subsemimodule P of M is prime if and only if $(P : M)$ is a prime k-ideal of S.*

Proof. Let M be a k-multiplication ΓS-semimodule and P be a $k\Gamma S$-subsemimodule of M such that $(P : M)$ is a prime k-ideal of S. Then $M^{\#}$ is a multiplication L-semimodule and $(P : M)^{+'} = (P^{+'} : M^{\#})$ is a prime k-ideal of L (by Theorems 5.3.10 and 1.4.19, Proposition 5.1.11 (i) respectively). It implies that $P^{+'}$ is a prime k-subsemimodule of $M^{\#}$ (*cf.* Theorem 1.5.14). Therefore by Theorem 5.1.14, P is a prime $k\Gamma S$-subsemimodule of M. The converse part is already proved in Theorem 5.1.16. This completes the proof. □

We observe that prime ΓS-subsemimodules do not exhibit the strong irreducibility property analogous to that of the prime ideals of a Γ-semiring in general. Even this is not true in general for modules. For example, let us consider Q-module $Q \bigoplus Q$ where Q is the set of all rationals. It is well known that the proper submodules of $Q \bigoplus Q$ are of the form $\langle (a, b) \rangle$, where $a, b \in Q$, Also it is easy to observe that $\langle (0, 0) \rangle$ is a prime submodule of $Q \bigoplus Q$ and $\langle (0, 1) \rangle \cap \langle (1, 0) \rangle = \langle (0, 0) \rangle$. This implies that the prime submodule $\langle (0, 0) \rangle$ of $Q \bigoplus Q$ doesn't have the property.
In the following Theorem 5.3.13, we see that prime ΓS-subsemimodules possess the strong irreducibility property under certain condition. Before that let us recall the following.

Remark 5.3.12. [28] For any two ΓS-subsemimodules P_1, P_2 of M,
$P_1^{+'} \cap P_2^{+'} = (P_1 \cap P_2)^{+'}$.

Theorem 5.3.13. *Let M be a multiplication ΓS-semimodule and P be a prime ΓS-subsemimodule of M and P_1, P_2 be ΓS-subsemimodules of M. If $P_1 \cap P_2 \subseteq P$ then either $P_1 \subseteq P$ or $P_2 \subseteq P$.*

Proof. Let M be a multiplication ΓS-semimodule and P be a prime ΓS-subsemimodule of M such that $P_1 \cap P_2 \subseteq P$, where P_1, P_2 are ΓS-subsemimodules of M. Then $M^{\#}$ is a multiplication L-semimodule and $P^{+'}$ is a prime L-subsemimodule of $M^{\#}$ such that $(P_1 \cap P_2)^{+'} \subseteq P^{+'}$ by Theorem 5.1.14. Now by Remark 5.3.12, $P_1^{+'} \cap P_2^{+'} = (P_1 \cap P_2)^{+'}$ whence it follows that $P_1^{+'} \cap P_2^{+'} \subseteq P^{+'}$. It implies that either $P_1^{+'} \subseteq P^{+'}$ or $P_2^{+'} \subseteq P^{+'}$ (*cf.* Proposition 1.5.16) which implies that $P_1 \subseteq P$ or $P_2 \subseteq P$ (see Theorem 5.1.14). □

Theorem 5.3.14. *Let M be a multiplication ΓS-semimodule. If N is a proper ΓS-subsemimodule ($k\Gamma S$-subsemimodule) of M then there exists a prime ΓS-subsemimodule (resp. $k\Gamma S$-subsemimodule) of M containing N.*

Proof. Let M be a multiplication ΓS-semimodule and N be a proper ΓS-subsemimodule ($k\Gamma S$-subsemimodule) of M. Then by Theorem 5.3.3 and Theorem 1.6.11, $M^{\#}$ is a multiplication L-semimodule and $N^{+'}$ is a subsemimodule (resp. k-subsemimodule) of $M^{\#}$ respectively. So by Lemma 1.5.23 (resp. by Theorem 1.5.24) it follows that there is a prime subsemimodule (resp. k-subsemimodule) P (say) of $M^{\#}$ containing $N^{+'}$. Therefore P^{+} is a prime ΓS-subsemimodule (resp. $k\Gamma S$-subsemimodule) of M containing N. Hence the proof is complete. □

We denote the intersection of prime ΓS-subsemimodules of a ΓS-semimodule M containing a ΓS-subsemimodule N of M, *the radical of* N, as $rad(N)$. Also the intersection of prime $k\Gamma S$-subsemimodules of a ΓS-semimodule M containing a ΓS-subsemimodule N of M, *the* k*-radical of* N, is denoted as $rad_k(N)$.

The following result is easily observed.

Lemma 5.3.15. *Let M be a ΓS-semimodule and N, K be a ΓS-subsemimodule of M. Then the following holds:*

(*i*) $N \subseteq rad(N)$.

(*ii*) $rad(N \cap K) \subseteq rad(N) \cap rad(K)$.

Proof. (*i*) The proof is clear by the definition of $rad(N)$.

(*ii*) It is clear that the collection of all prime ΓS-subsemimodules containing $N \cap K$ contains both the collection of all prime ΓS-subsemimodules containing N and the collection of all prime ΓS-subsemimodules containing K. So taking intersection over the collections mentioned above we obtain that $rad(N \cap K) \subseteq rad(N)$ and $rad(N \cap K) \subseteq rad(K)$. Hence $rad(N \cap K) \subseteq rad(N) \cap rad(K)$. □

Theorem 5.3.16. *Let M be a multiplication ΓS-semimodule and N, K be ΓS-subsemimodules of M. Then $rad(N \cap K) = rad(N) \cap rad(K)$.*

Proof. Let M be a multiplication ΓS-semimodule. Then Theorem 5.3.14 ensures that ΓS-subsemimodule $N \cap K$ is contained in atleast one prime ΓS-subsemimodule P of M. Let P be a prime ΓS-subsemimodule of M such that $N \cap K \subseteq P$. Since M is a multiplication ΓS-semimodule, it implies $N \subseteq P$ or $K \subseteq P$ by Theorem 5.3.13. So $rad(N) \subseteq P$ or $rad(K) \subseteq P$ which implies $rad(N) \cap rad(K) \subseteq P$. Since it is true for all such prime ΓS-subsemimodule of M containing $N \cap K$. Therefore $rad(N) \cap rad(K) \subseteq rad(N \cap K)$.
So in view of Lemma 5.3.15 (*ii*), $rad(N \cap K) = rad(N) \cap rad(K)$. □

Proposition 5.3.17. *Suppose N is a ΓS-subsemimodule of M and K is a subsemimodule of $M^{\#}$. Then*

(i) $(rad(N))^{+'} = rad(N^{+'})$,

(ii) $(rad(K))^{+} = rad(K^{+})$.

Proof. (i) Let $\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle \in (rad(N))^{+'}$. Then $\sum_{i=1}^{m} m_i\alpha_i s \in rad(N)$ for all $s \in S$. So for all prime ΓS-subsemimodule P_i containing N,

$$\sum_{i=1}^{m} m_i\alpha_i s \in P_i \text{ which implies } \sum_{i=1}^{m}\langle m_i\alpha_i s, \gamma\rangle \in P_i^{+'} \text{ for all } s \in S, \gamma \in \Gamma.$$

In particular, $\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle \in P_i^{+'}$ for all $P_i^{+'} \supseteq N^{+'}$ (take $s = e_k$ and $\gamma = \delta_k$ for $k = 1, 2, ..., l$, where $\sum_{k=1}^{l}[e_k, \delta_k]$ is the left unity of S).
Therefore $\sum_{i=1}^{m}\langle m_i, \alpha_i\rangle \in rad(N^{+'})$. So $(rad(N))^{+'} \subseteq rad(N^{+'})$.
The other inclusion follows by reversing the above argument.
(ii) The proof is analogous to that of (i). □

The following proposition is the k-analogue of Proposition 5.3.17.

Proposition 5.3.18. *Suppose N is a $k\Gamma S$-subsemimodule of M and K is a k-subsemimodule of $M^{\#}$. Then*

(i) $(rad_k(N))^{+'} = rad_k(N^{+'})$,

(ii) $(rad_k(K))^{+} = rad_k(K^{+})$.

Using Proposition 5.3.17, in the following we generalize Proposition 1.5.20, Theorem 1.5.21 and Corollary 1.5.22.
It is to be noted that for an ideal I of S, $Rad(I)$ is the radical of I, the intersection of all prime ideals of S containing I.

Theorem 5.3.19. *For any ΓS-subsemimodule N of a multiplication ΓS-semimodule M, $rad(N) = M\Gamma rad(N : M)$.*

Proof. In view of Proposition 5.3.2 (i) we have,

$$(M\Gamma rad(N : M))^{+'} = M^{\#}(rad(N : M))^{+'}.$$

Applying Propositions 5.3.17 and 5.1.11 we get,

$$(rad(N : M))^{+'} = rad((N : M)^{+'}) = rad(N^{+'} : M^{\#}).$$

$$\text{So } M^{\#}(rad(N : M))^{+'} = M^{\#}rad((N : M)^{+'}) = M^{\#}rad(N^{+'} : M^{\#})....(*)$$

Now by Proposition 1.5.20, $M^{\#} rad(N^{+'} : M^{\#}) = rad(N^{+'})$.
Again by Proposition 5.3.17, $rad(N^{+'}) = (rad(N))^{+'}$.
Therefore by Theorem 1.6.11 and equation $(*)$,

$$rad(N) = ((rad(N))^{+'})^{+} = (M^{\#}(rad(N : M))^{+'})^{+} = M\Gamma rad(N : M).$$

□

Theorem 5.3.20. *For any $k\Gamma S$-subsemimodule N of a multiplication ΓS-semimodule M, $rad(N) = rad_k(N)$.*

Proof. Using Proposition 5.3.17 we have, $rad(N) = ((rad(N))^{+'})^{+} = (rad(N^{+'}))^{+}$. Then by Theorem 1.5.21, Theorem 5.3.3 and Proposition 5.3.18, $rad(N^{+'}) = rad_k(N^{+'})$ in $M^{\#}$. Therefore by Proposition 5.3.18,

$$(rad(N^{+'}))^{+} = (rad_k(N^{+'}))^{+} = ((rad_k(N))^{+'})^{+} = rad_k(N).$$

Hence $rad(N) = rad_k(N)$. □

In view of Theorems 5.3.19 and 5.3.20, we obtain the following result.

Theorem 5.3.21. *For any $k\Gamma S$-subsemimodule N of a multiplication ΓS-semimodule M, $rad_k(N) = M\Gamma rad(N : M)$.*

CHAPTER 6

STRUCTURE SPACES OF SEMIMODULES AND ΓS-SEMIMODULES

6

Structure spaces of semimodules and ΓS-semimodules

There are several works on the topology defined on the prime spectrum of modules (semimodules) over rings (resp. semirings). The works in [7], [75], [38], [35], [34] give impetus to study the topological structure of the space consisting of ΓS-subsemimodules of a ΓS-semimodule as a natural generalization. So the topological space formed by the collection of prime ΓS-subsemimodules of a ΓS-semimodule endowed with the Hull Kernel topology has been studied in the setting of a Γ-semiring with unities in this chapter. At first, with the objective to study the topological properties of the space formed by prime ΓS-subsemimodules of a ΓS-semimodule with the Hull Kernel topology, we have first topologized the space of all prime subsemimodules over a multiplication semimodule with the Hull Kernel topology, calling it as the structure space of a semimodule. Then we investigated for the members of the topology, i.e., the open sets, closed set, open base and observed that they are the same as those of the said space with Zariski topology. Also we have studied some topological properties of that space over semimodules. After that we have defined the structure space of all prime ΓS-subsemimodules over an unitary multiplication ΓS-semimodule, where S is a Γ-semiring. In order to study the topological space of prime ΓS-subsemimodules via the topological space of prime L-subsemimodules of its associated L-semimodule,

This chapter is mainly based on the work of the following paper:
Soumi Basu, Sarbani Mukherjee (Goswami) and Sujit Kumar Sardar, *Structure space of a ΓS-semimodule over a Γ-semiring,* To appear in Bulletin of Calcutta Mathematical Society, 116 (5) (2024).

we have proved that the two spaces are homeomorphic, where L is the left operator semiring of S. Thereafter an open base, form of closed sets of the spaces of all prime ΓS-subsemimodules over a multiplication ΓS-semimodule have been characterized and necessary and sufficient conditions for the separation properties viz., T_1, T_2, regular etc. and compactness, connectedness etc. and also density and irreducibility of a subset have been studied. In addition to that some topological properties of the space of all prime subsemimodules of a semimodule, proved in [34] has been generalized (for example, Theorem 4.1, Theorem 4.2, Corollary 4.2, Corollary 4.3, Theorem 4.3 of [34]). The whole study of the topological properties mentioned above have been examined for the space of all prime $k\Gamma S$-semimodules of a ΓS-semimodule, considering it as a subspace of the space of all prime ΓS-subsemimodules of a ΓS-semimodule.

In **section 1**, we define the structure space of prime subsemimodules over a multiplication semimodule with the Hull Kernel topology and characterize the members of the topology, i.e., the open sets, closed set, open base (see Propositions 6.1.3, 6.1.4, 6.1.5). Also we study the topological properties of the structure space of semimodules (see for instance, Theorems 6.1.7, 6.1.9, 6.1.10, 6.1.11).

In **section 2**, we study the topological properties of the structure space of prime ΓS-subsemimodule over a multiplication ΓS-semimodule with the Hull Kernel topology via that of its associated L-semimodule (*cf.* Theorems 6.2.22, 6.2.23, 6.2.24, 6.2.26, 6.2.31, 6.2.32, 6.2.33, 6.2.28, 6.2.29) .

In **section 3**, we study the topological properties of the space of all prime $k\Gamma S$-semimodules of a ΓS-semimodule as a subspace of the space of all prime ΓS-subsemimodules of a ΓS-semimodule.

6.1 Structure space of prime subsemimodules of a semimodule

In this section we topologize the set of all prime subsemimodules of an unitary multiplication semimodule with the Hull Kernel topology.

Throughout the section, we denote the set of all prime subsemimodules of an unitary R-semimodule V as X_V. Below we define the closure and prove the closure axioms on X_V.

Definition 6.1.1. Let R be a semiring and V be a R-semimodule. Let X_V be the collection of all prime subsemimodule of V. For any subset $\mathcal{A}$ of X_V, we define,

$$\text{closure of } \mathcal{A} = \overline{\mathcal{A}} = \{P \in X_V : \bigcap_{P_i \in \mathcal{A}} P_i \subseteq P\}.$$

Evidently, $\overline{\emptyset} = \emptyset$.

Theorem 6.1.2. *Let V be a multiplication R-semimodule and X_V be the collection of all prime subsemimodules of V. Then the mapping from $\mathcal{A} \mapsto \overline{\mathcal{A}}$ is a Kuratowski closure operator on X_V, where $\mathcal{A} \subseteq X_V$.*

Proof. Let $\mathcal{A}, \mathcal{B} \subseteq X_V$.

(i) Let $P_\alpha \in \mathcal{A}$. $\bigcap_{P_\alpha \in \mathcal{A}} P_\alpha \subseteq P_\alpha$ for each $P_\alpha \in \mathcal{A}$ implies that $P_\alpha \in \overline{\mathcal{A}}$. Hence $\mathcal{A} \subseteq \overline{\mathcal{A}}$.

(ii) From (i) it is clear that $\overline{\mathcal{A}} \subseteq \overline{\overline{\mathcal{A}}}$. Let $P_\beta \in \overline{\overline{\mathcal{A}}}$. Then $\bigcap_{P_\alpha \in \overline{\mathcal{A}}} P_\alpha \subseteq P_\beta$. Again $\bigcap_{P_\gamma \in \mathcal{A}} P_\gamma \subseteq P_\alpha$ for all $P_\alpha \in \overline{\mathcal{A}}$. Then

$$\bigcap_{P_\gamma \in \mathcal{A}} P_\gamma \subseteq \bigcap_{P_\alpha \in \overline{\mathcal{A}}} P_\alpha \subseteq P_\beta \text{ implies } P_\beta \in \overline{\mathcal{A}}.$$

Thus $\overline{\overline{\mathcal{A}}} \subseteq \overline{\mathcal{A}}$. Therefore $\overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}$.

(iii) Let us suppose that $\mathcal{A} \subseteq \mathcal{B}$. Let $P_\beta \in \overline{\mathcal{A}}$. Then $\bigcap_{P_\alpha \in \mathcal{A}} P_\alpha \subseteq P_\beta$. Since $\mathcal{A} \subseteq \mathcal{B}$, it follows that $\bigcap_{P_\alpha \in \mathcal{B}} P_\alpha \subseteq \bigcap_{P_\alpha \in \mathcal{A}} P_\alpha \subseteq P_\beta$. This implies that $P_\beta \in \overline{\mathcal{B}}$ and hence $\overline{\mathcal{A}} \subseteq \overline{\mathcal{B}}$.

(iv) Clearly $\overline{\mathcal{A}} \cup \overline{\mathcal{B}} \subseteq \overline{\mathcal{A} \cup \mathcal{B}}$. Now let $P_\beta \in \overline{\mathcal{A} \cup \mathcal{B}}$. Then $\bigcap_{P_\alpha \in \mathcal{A} \cup \mathcal{B}} P_\alpha \subseteq P_\beta$. It can be easily seen that

$$\bigcap_{P_\alpha \in \mathcal{A} \cup \mathcal{B}} P_\alpha = (\bigcap_{P_\alpha \in \mathcal{A}} P_\alpha) \bigcap (\bigcap_{P_\alpha \in \mathcal{B}} P_\alpha).$$

Since $(\bigcap_{P_\alpha \in \mathcal{A}} P_\alpha)$ and $(\bigcap_{P_\alpha \in \mathcal{B}} P_\alpha)$ are R-subsemimodules on V and P_β is a prime R-subsemimodule on V then by Proposition 1.5.16,

$$(\bigcap_{P_\alpha \in \mathcal{A}} P_\alpha) \bigcap (\bigcap_{P_\alpha \in \mathcal{B}} P_\alpha) \subseteq P_\beta \text{ implies either } \bigcap_{P_\alpha \in \mathcal{A}} P_\alpha \subseteq P_\beta \text{ or } \bigcap_{P_\alpha \in \mathcal{B}} P_\alpha \subseteq P_\beta.$$

Therefore $P_\beta \in \overline{\mathcal{A}}$ or $P_\beta \in \overline{\mathcal{B}}$. Hence $P_\beta \in \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$. Consequently, $\overline{\mathcal{A} \cup \mathcal{B}} \subseteq \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$. Therefore $\overline{\mathcal{A} \cup \mathcal{B}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$.

Therefore combining $(i), (ii), (iii), (iv)$, it follows that the mapping is a Kuratowski closure operator on X_V. □

For a multiplication R-semimodule V, the topology τ_V induced by the Kuratowski closure operator on X_V is called Hull Kernel topology on X_V and the topological space (X_V, τ_V) is called the *structure space* of the semimodule V. Therefore throughout the section, V is a multiplication R-semimodule and we denote the space as X_V without mentioning the topology explicitly.

Notations: Let N be a R-subsemimodule on V and $m \in V$. We define,

$$\Delta_V(N) = \{P \in X_V : N \subseteq P\}; C\Delta_V(N) = X_V \setminus \Delta_V(N) = \{P \in X_V : N \nsubseteq P\};$$
$$\Delta_V(m) = \{P \in X_V : m \in P\}; C\Delta_V(m) = X_V \setminus \Delta_V(m) = \{P \in X_V : m \notin P\}.$$

The following three results describe the form of closed set, open set and open base of X_V.

Proposition 6.1.3. *Any closed set in X_V is of the form $\Delta_V(N)$, where N is a R-subsemimodule on V.*

Proof. Let $\mathcal{A}$ be any closed set in X_V. Then $\overline{\mathcal{A}} = \mathcal{A}$. Let $\mathcal{A} = \{P_\alpha : \alpha \in \Lambda\}$ and $P = \bigcap_{P_\alpha \in \mathcal{A}} P_\alpha$. Then P is a R-subsemimodule on V. Let $P' \in \overline{\mathcal{A}}$. Then $\bigcap_{P_\alpha \in \mathcal{A}} P_\alpha \subseteq P'$, i.e., $P \subseteq P'$. Consequently, $P' \in \Delta_V(P)$. So $\overline{\mathcal{A}} \subseteq \Delta_V(P)$. By reversing the above argument, we obtain that $\Delta_V(P) \subseteq \overline{\mathcal{A}}$. Thus $\overline{A} = \Delta_V(P)$. □

The following result follows from Theorem 6.1.3.

Corollary 6.1.4. *Any open set in X_V is of the form $C\Delta_V(N)$, where N is a R-subsemimodule on V.*

Proposition 6.1.5. *$\{C\Delta_V(m) : m \in V\}$ is an open base for X_V.*

Proof. Let $\mathcal{U}$ be an open set in X_V. Then $\mathcal{A} = X_V \setminus \mathcal{U}$ is a closed set in X_V. By Proposition 6.1.3, $\mathcal{A} = \Delta_V(N)$ for some R-subsemimodule N on V. Then $K \in \mathcal{U}$ implies $K \notin \mathcal{A}$, i.e., $N \nsubseteq K$. Then there exists $m \in N$ such that $m \notin K$. Hence $K \in C\Delta_V(m)$. Now let $K' \in C\Delta_V(m)$. Then $m \notin K'$. This implies that $N \nsubseteq K'$ whence it follows that $K' \in C\Delta_V(N) = X_V \setminus \mathcal{A} = \mathcal{U}$. Hence $C\Delta_V(m) \subseteq \mathcal{U}$. Consequently, $K \in C\Delta_V(m) \subseteq \mathcal{U}$. Thus $\{C\Delta_V(m) : m \in V\}$ is an open base for X_V. □

Remark 6.1.6. From propositions 6.1.3, 6.1.4, 6.1.5 we observe that for a multiplication semimodule V, the collections of open sets, closed sets, open bases of the space of all prime subsemimodules of V with the Hull Kernel topology are same as those of the space with the Zariski topology which has been intensively studied by Han et al. in [34] (in [34], the space is called the prime spectrum of V and is denoted as $Spec(V)$) and in its references. So all the results related to the space $Spec(V)$ obtained in [34] and in its references hold true for the space X_V. So subsequently we will use those results to study the ΓS-semimodule version of those.

Now we shall the study necessary and sufficient conditions for the properties viz. T_2, regular, connectedness and for dense subsets of the space X_V (the topological properties like T_0, T_1, compactness, irreducibility of subsets etc. of the prime spectrum has already been studied by the authors in [34]).

Theorem 6.1.7. *The space X_V is T_2 if and only if for any distinct pair of elements N_1, N_2 of X_V, there exist $m_1, m_2 \in V$ such that $m_1 \notin N_2$, $m_2 \notin N_1$ and there does not exist any element $N \in X_V$ such that $m_1 \notin N$ and $m_2 \notin N$.*

Proof. Let X_V be a T_2 space. Then for any two distinct elements N_1, N_2 of X_V, there exist basic open sets $C\Delta_V(m_1)$ and $C\Delta_V(m_2)$ such that

$$N_1 \in C\Delta_V(m_2),\ N_2 \in C\Delta_V(m_1) \text{ and } C\Delta_V(m_1) \cap C\Delta_V(m_2) = \emptyset.$$

Thus we have $m_1 \notin N_2$, $m_2 \notin N_1$. Now if possible let $N \in X_V$ such that $m_1 \notin N$ and $m_2 \notin N$. Then $N \in C\Delta_V(m_1) \cap C\Delta_V(m_2)$ which is a contradiction to the fact that $C\Delta_V(m_1)$ and $C\Delta_V(m_2)$ are disjoint. Thus there does not exist any element $N \in X_V$ such that $m_1 \notin N$ and $m_2 \notin N$.

Conversely, let us suppose that the given condition holds and N_1, N_2 are two distinct elements of X_V. Then by our hypothesis there exist $m_1, m_2 \in V$ such that $m_1 \notin N_2$, $m_2 \notin N_1$ and there does not exist any element $N \in X_V$ such that $m_1 \notin N$ and $m_2 \notin N$. Thus we have,

$$N_1 \in C\Delta_V(m_2),\ N_2 \in C\Delta_V(m_1) \text{ and } C\Delta_V(m_1) \cap C\Delta_V(m_2) = \emptyset.$$

Hence X_V is a T_2 space. □

Theorem 6.1.8. *If the space X_V is T_2 and X_V contains more than one element then there exist $m_1, m_2 \in V$ such that $X_V = C\Delta_V(m_1) \cup C\Delta_V(m_2) \cup \Delta_V(N)$, where N is the subsemimodule generated by m_1 and m_2.*

Proof. Let N_1, N_2 be a distinct pair of elements of X_V. Since X_V is a T_2 space, there exist basic open sets $C\Delta_V(m_1)$ and $C\Delta_V(m_2)$ for $m_1, m_2 \in V$ such that

$$N_1 \in C\Delta_V(m_2),\ N_2 \in C\Delta_V(m_1) \text{ and } C\Delta_V(m_1) \cap C\Delta_V(m_2) = \emptyset.$$

Let N be the subsemimodule generated by m_1, m_2. Then N is the smallest subsemimodule containing m_1, m_2. Let $N' \in X_V$. Then either $m_1 \in N'$, $m_2 \notin N'$ or $m_1 \notin N'$, $m_2 \in N'$ or $m_1, m_2 \in N'$ as the case $m_1, m_2 \notin N'$ is not possible because of the fact that $C\Delta_V(m_1) \cap C\Delta_V(m_2) = \emptyset$. Therefore it implies that

$$\begin{gathered}\text{either } N' \in C\Delta_V(m_1) \subseteq C\Delta_V(m_1) \cup C\Delta_V(m_2) \cup \Delta_V(N)\\ \text{or } N' \in C\Delta_V(m_2) \subseteq C\Delta_V(m_1) \cup C\Delta_V(m_2) \cup \Delta_V(N)\\ \text{or } N \subseteq N', \text{ i.e., } N' \in \Delta_V(N) \subseteq C\Delta_V(m_1) \cup C\Delta_V(m_2) \cup \Delta_V(N) \text{ respectively.}\end{gathered}$$

Hence $X_V \subseteq C\Delta_V(m_1) \cup C\Delta_V(m_2) \cup \Delta_V(N)$.
Again clearly $C\Delta_V(m_1) \cup C\Delta_V(m_2) \cup \Delta_V(N) \subseteq X_V$.
Hence $X_V = C\Delta_V(m_1) \cup C\Delta_V(m_2) \cup \Delta_V(N)$. □

Theorem 6.1.9. *The space X_V is a regular space if and only if for any $N \in X_V$ and $m \notin N$, there exists a subsemimodule N' of V and $m' \in V$ such that $N \in C\Delta_V(m') \subseteq \Delta_V(N') \subseteq C\Delta_V(m)$.*

Proof. Let the space X_V be regular. Let $N \in X_V$ and $m \notin N$. Then $N \in C\Delta_V(m)$ and $X_V \setminus C\Delta_V(m)$ is a closed set not containing N. Since X_V is a regular space, there exist two disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ such that $N \in \mathcal{U}$ and $X_V \setminus C\Delta_V(m) \subseteq \mathcal{V}$, i.e., $X_V \setminus \mathcal{V} \subseteq C\Delta_V(m)$. $X_V \setminus \mathcal{V}$ is a closed set which means $X_V \setminus \mathcal{V} = \Delta_V(N') \subseteq C\Delta_V(m)$ for some subsemimodule N' of V (*cf.* Proposition 6.1.3). ... (1)
Since $\mathcal{U} \cap \mathcal{V} = \emptyset$, $\mathcal{V} \subseteq X_V \setminus \mathcal{U}$ and $X_V \setminus \mathcal{U}$ being a closed set, is of the form $X_V \setminus \mathcal{U} = \Delta_V(N'')$ for some subsemimodule N'' of V.
Since $N \in \mathcal{U}$ then $N \notin X_V \setminus \mathcal{U} = \Delta_V(N'')$ which implies $N'' \nsubseteq N$. Therefore there exists $m' \in N''$ such that $m' \notin N$ whence it follows that $N \in C\Delta_V(m')$. ... (2)
Now we are to show that $\mathcal{V} \subseteq \Delta_V(m')$. Let $N_1 \in \mathcal{V}$. Then $\mathcal{V} \subseteq \Delta_V(N'')$ implies $N'' \subseteq N_1$. Since $m' \in N''$, $m' \in N_1$ and hence $N_1 \in \Delta_V(m')$. Thus $\mathcal{V} \subseteq \Delta_V(m')$.
Consequently, $C\Delta_V(m') \subseteq X_V \setminus \mathcal{V} = \Delta_V(N')$. ... (3)
Thus combining (1), (2), (3), we find that $N \in C\Delta_V(m') \subseteq \Delta_V(N') \subseteq C\Delta_V(m)$.
Conversely, let the given condition hold and let $N \in X_V$ and $\mathcal{A}$ be a closed set not containing N. Then $\mathcal{A} = \Delta_V(N'')$ for some subsemimodule N'' of V.
Since $N \notin \Delta_V(N'')$, we have $N'' \nsubseteq N$. This implies that there exists $m \in N''$ such that $m \notin N$. Now by the given condition, there exists a subsemimodule N' of V and $m' \in V$ such that $N \in C\Delta_V(m') \subseteq \Delta_V(N') \subseteq C\Delta_V(m)$. Since $m \in N''$, $C\Delta_V(m) \cap \Delta_V(N'') = \emptyset$. Indeed, if $C\Delta_V(m) \cap \Delta_V(N'') \neq \emptyset$ then $P \in C\Delta_V(m) \cap \Delta_V(N'')$ would imply that $m \notin P$ and $N'' \subseteq P$ which is a contradiction to the fact that $m \in N''$.
Hence $\Delta_V(N'') \subseteq X_V \setminus C\Delta_V(m) \subseteq X_V \setminus \Delta_V(N')$. Therefore $X_V \setminus \Delta_V(N')$ is an open set containing $\Delta_V(N'')$. It is clear that $C\Delta_V(m') \cap (X_V \setminus \Delta_V(N')) = \emptyset$. So we find that $C\Delta_V(m')$ and $X_V \setminus \Delta_V(N')$ are two disjoint open sets containing N and $\Delta_V(N'')$ respectively. Hence the space X_V is a regular space. □

Theorem 6.1.10. *Let $\mathcal{A}$ be a nonempty subset of X_V. $\mathcal{A}$ is dense in X_V if and only if $\bigcap_{P_i \in \mathcal{A}} P_i = \bigcap_{P_i \in X_V} P_i$.*

Proof. Let $\mathcal{A}$ be a subset of X_V which is dense in X_V. Obviously, $\bigcap_{P_i \in X_V} P_i \subseteq \bigcap_{P_i \in \mathcal{A}} P_i$. Since $\mathcal{A}$ is dense subset of X_V then by the definition of $\overline{\mathcal{A}}$, $\bigcap_{P_i \in \mathcal{A}} P_i \subseteq \bigcap_{P_i \in X_V} P_i$. Therefore $\bigcap_{P_i \in A} P_i = \bigcap_{P_i \in X_V} P_i$.
To prove the converse, let us assume that $X_V \setminus \overline{\mathcal{A}} \neq \emptyset$. Then there exists a prime subsemimodule P of V such that $P \in X_V \setminus \overline{\mathcal{A}}$. Therefore there exists an open neighbourhood $\mathcal{U}$ of P in X_V such that $\mathcal{U} \cap \mathcal{A} = \emptyset$, where $\mathcal{U} = C\Delta_V(x)$ for some $x \in V$, i.e., $C\Delta_V(x) \cap \mathcal{A} = \emptyset$ for some $x \in V$. Then $\mathcal{A} \subseteq \Delta_V(x)$ implies $x \in \bigcap_{P_i \in \mathcal{A}} P_i$. Now if possible let $x \in \bigcap_{P_i \in X_V} P_i$. Then $P_i \in \Delta_V(x)$ for each $P_i \in X_V$. It implies that $X_V = \Delta_V(x)$, i.e., $C\Delta_V(x) = \emptyset$ which is a contradiction to the fact that it is an open neighbourhood of P. That means $x \notin \bigcap_{P_i \in X_V} P_i$. Therefore $\bigcap_{P_i \in X_V} P_i \subsetneqq \bigcap_{P_i \in \mathcal{A}} P_i$ which is a contradiction to our assumption. Hence $X_V \setminus \overline{\mathcal{A}} = \emptyset$ which implies that $\mathcal{A}$ is dense in X_V. □

Theorem 6.1.11. *The space X_V is disconnected if and only if there exists a subsemimodule N of V and a collection $\{a_\alpha : \alpha \in \Lambda\}$ of elements in V not belonging to N such that if $N' \in X_V$ and $a_\alpha \in N'$ for all $\alpha \in \Lambda$ then $N \setminus N' \neq \emptyset$.*

Proof. Let X_V be disconnected. Then there exists a nontrivial clopen subset of X_V. Let N be a subsemimodule of V for which $\Delta_V(N)$ is closed as well as open. Then $\Delta_V(N) = \bigcup_{\alpha \in \Lambda} C\Delta_V(a_\alpha)$, where $\{a_\alpha\}_{\alpha \in \Lambda}$ is a collection of elements in V. Also if $N' \in X_V$ and $a_\alpha \in N'$ for all $\alpha \in \Lambda$ then we have $N' \notin C\Delta(a_\alpha)$ for all $\alpha \in \Lambda$ which implies $N' \notin \Delta_V(N)$. So $N \nsubseteq N'$, i.e., $N \setminus N' \neq \emptyset$.
Conversely let the given condition hold. Clearly $\bigcup_{\alpha \in \Lambda} C\Delta_V(a_\alpha) \subseteq \Delta_V(N)$. Now let $B \in \bigcap_{\alpha \in \Lambda} \Delta_V(a_\alpha)$. Then $a_\alpha \in B$ for all α. So B does not contain N, i.e., $B \in C\Delta_V(N)$. Therefore $\Delta_V(N) \subseteq \bigcup_{\alpha \in \Lambda} C\Delta_V(a_\alpha)$.
Hence $\Delta_V(N) = \bigcup_{\alpha \in \Lambda} C\Delta_V(a_\alpha)$ which is a clopen subset of X_V.
Then X_V is disconnected. □

In [34] authors proved that the space is compact if and only if the semimodule is finitely generated. Here we shall study another necessary and sufficient condition of compactness.

Theorem 6.1.12. *The space X_V is compact if and only if for any collection $\{a_\alpha\}_{\alpha \in \Lambda}$ of elements in V, there exists a finite subcollection $\{a_i : i = 1, 2, ..., n\}$ in V such that for any $N \in X_V$, there exists a_i from the subcollection such that $a_i \notin N$.*

Proof. Let X_V be compact. Then the open cover $\{C\Delta_V(a_\alpha) : a_\alpha \in V\}$ of X_V has a finite subcover $\{C\Delta_V(a_i) : i = 1, 2, ..., n\}$. Then for any $N \in X_V$, $N \in C\Delta_V(a_i)$ for some $a_i \in V$. This implies that $a_i \notin N$. Hence $\{a_i : i = 1, 2, ..., n\}$ is the required finite subcollection of elements of V such that for any $N \in X_V$, there exists some a_i for $i = 1, 2, ..., n$ such that $a_i \notin N$.

Conversely, let us suppose that the given condition holds. Let $\{C\Delta_V(a_i) : a_i \in V\}$ be an open cover of X_V. Suppose to the contrary that no finite subcollection of $\{C\Delta_V(a_i) : a_i \in V\}$ covers X_V. This means that for any finite set $\{a_i : i = 1, 2, ..., n\}$ of elements of V, $\bigcup_{i=1}^{n} C\Delta_V(a_i) \neq X_V$ whence $\bigcap_{i=1}^{n} \Delta_V(a_i) \neq \emptyset$. Then there exists $P \in X_V$ such that $P \in \bigcap_{i=1}^{n} \Delta_V(a_i)$ which implies $a_i \in P$ for $i = 1, 2, ..., n$ and this leads to a contradiction. So the open cover $\{C\Delta_V(a_i) : a_i \in V\}$ has a finite subcover and hence X_V is compact. □

Remark 6.1.13. In view of Remark 6.1.6 we observe that for a multiplication semimodule V, the collections of open sets, closed sets, open bases of the subspace (induced with the subspace topology) of all prime k-subsemimodules Y_V are same as those of the space with the Zariski topology which has been intensively studied by Han et al. in [35] (in [35], the space is called the subtractive prime spectrum of V and is denoted as $Spec_k(V)$) and in its references. So all the results related to the space $Spec_k(V)$ obtained in [35] hold true for the space Y_V and subsequently we will use the results to prove the ΓS-semimodule version of those.

Now we shall study the necessary and sufficient conditions for the properties viz. T_2, regular, connectedness and for dense subset of the space Y_V of all prime k-subsemimodules of a semimodule (T_0, T_1, compactness, irreducibility of a subset etc. of the space has already been studied by the authors in [35]).

We skip the proofs of Theorems 6.1.14, 6.1.16, 6.1.17, 6.1.18, 6.1.19, Theorem 6.1.15, as they are the counterparts of Theorems 6.1.7, 6.1.9, 6.1.10, 6.1.11, 6.1.12, Theorem 6.1.8 respectively.

Theorem 6.1.14. *The space Y_V is T_2 if and only if for any distinct pair of elements N_1, N_2 of Y_V, there exist $m_1, m_2 \in V$ such that $m_1 \notin N_2$, $m_2 \notin N_1$ and there does not exist any element $N \in Y_V$ such that $m_1 \notin N$ and $m_2 \notin N$.*

Theorem 6.1.15. *If the space Y_V is T_2 and Y_V contains more than one element then there exist $m_1, m_2 \in V$ such that $Y_V = C\Delta_V^k(m_1) \cup C\Delta_V^k(m_2) \cup \Delta_V^k(N)$, where N is the k-subsemimodule generated by m_1, m_2.*

Theorem 6.1.16. *The space Y_V is a regular space if and only if for any $N \in Y_V$ and $m \notin N$, there exists a k-subsemimodule N' of V and $m' \in V$ such that $N \in C\Delta_V^k(m') \subseteq \Delta_V^k(N') \subseteq C\Delta_V^k(m)$.*

Theorem 6.1.17. *Let $\mathcal{A}$ be a nonempty subset of Y_V. $\mathcal{A}$ is dense in Y_V if and only if $\bigcap_{P_i \in \mathcal{A}} P_i = \bigcap_{P_i \in Y_V} P_i$.*

Theorem 6.1.18. *The space Y_V is disconnected if and only if there exists a k-subsemimodule N of V and a collection $\{a_\alpha : \alpha \in \Lambda\}$ of elements in V not belonging to N such that if $N' \in Y_V$ and $a_\alpha \in N'$ for all $\alpha \in \Lambda$ then $N \setminus N' \neq \emptyset$.*

Theorem 6.1.19. *The space Y_V is compact if and only if for any collection $\{a_\alpha\}_{\alpha\in\Lambda}$ of elements in V, there exists a finite subcollection $\{a_i : i = 1, 2, ..., n\}$ in V such that for any $N \in Y_V$, there exists $a_i \in V$ such that $a_i \notin N$.*

6.2 Structure space of prime ΓS-subsemimodules of a ΓS-semimodule

In this section we define the structure space X_M of all prime ΓS-subsemimodules of a ΓS-semimodule M equipped with the Hull Kernel topology and investigate several topological properties of that space via those of its associated L-semimodule. We denote the space of all prime L-subsemimodules of the corresponding associated L-semimodule as $X_{M^\#}$.
Note that rest of the chapter, unless otherwise mentioned, S is a Γ-semiring with both the unities and $\sum_{i=1}^{m}[e_i, \delta_i]$ is the left unity, $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of S and L is the left operator semiring of S.

We note the following result which assures that the space we are going to topologize is always nonempty.

Theorem 6.2.1. *Let M be a multiplication ΓS-semimodule. Then the space X_M is nonempty.*

Proof. Let M be a multiplication ΓS-semimodule and N be a proper ΓS-subsemimodule of M. Then by Theorem 5.3.14, there exists a prime ΓS-subsemimodule of M containing N. Therefore X_M is nonempty. □

In order to construct the structure space X_M of the ΓS-semimodule M, let us first define the closure in X_M.

Definition 6.2.2. Let S be a Γ-semiring and M be a ΓS-semimodule. Let X_M be the collection of all prime ΓS-subsemimodule of M. For any subset $\mathcal{A}$ of X_M, we define closure of $\mathcal{A} = \overline{\mathcal{A}} = \{P \in X_M : \bigcap_{P_i \in \mathcal{A}} P_i \subseteq P\}$.
Evidently, $\overline{\emptyset} = \emptyset$.

To study its topological properties via those of the structure space of its associated semimodule, we first prove few results that are necessitated.

Notations: For $A \subseteq X_M$, $A^{+'} = \{N^{+'} \in X_{M\#} : N \in A\}$.
For $B \subseteq X_{M\#}$, $B^{+} = \{K^{+} \in X_M : K \in B\}$.

Lemma 6.2.3. (i) *For any subset A of X_M , $\overline{A^{+'}} = (\overline{A})^{+'}$.*
(ii) *For any subset B of $X_{M\#}$, $\overline{B^{+}} = (\overline{B})^{+}$.*

Proof. (i) Let $N \in \overline{A^{+'}}$. Then $N \in X_{M\#}$ and $\bigcap_{K_i \in A} K_i^{+'} \subseteq N$.
$\bigcap_{K_i \in A} K_i^{+'} = (\bigcap_{K_i \in A} K_i)^{+'}$. It implies that

$$\bigcap_{K_i \in A} K_i = \bigcap_{K_i \in A} (K_i^{+'})^{+} = ((\bigcap_{K_i \in A} K_i)^{+'})^{+} \subseteq N^{+}$$

whence it follows that $N^{+} \in \overline{A}$. Hence $N \in (\overline{A})^{+'}$. So $\overline{A^{+'}} \subseteq (\overline{A})^{+'}$.
The reverse inclusion follows similarly. Consequently, $\overline{A^{+'}} = (\overline{A})^{+'}$.

(ii) The proof is analogous to that of (i). □

Theorem 6.2.4. *There exists an inclusion preserving bijection between $\wp(X_M)$ and $\wp(X_{M\#})$ via the mapping $A \mapsto A^{+'}$, where $A \in \wp(X_M)$ and $\wp(X_M), \wp(X_{M\#})$ are the power sets of X_M and $X_{M\#}$ respectively.*

Proof. Let $A \in \wp(X_{M\#})$. Then $A^{+} \in \wp(X_M)$. We shall now prove that $(A^{+})^{+'} = A$.
Let $N^{+'} \in (A^{+})^{+'}$. Then $N \in A^{+}$. So there exists $K \in A$ such that $N = K^{+}$.
Therefore $N^{+'} = (K^{+})^{+'} = K \in A$ implies that $(A^{+})^{+'} \subseteq A$.
Again let $K_1 \in A$. Then $K_1 = (K_1^{+})^{+'} \in (A^{+})^{+'}$. Therefore $A \subseteq (A^{+})^{+'}$. Hence $(A^{+})^{+'} = A$. Similarly it can be proved that $B = (B^{+'})^{+}$ for all $B \in \wp(X_M)$. Therefore the mapping $A \mapsto A^{+'}$ is bijective. Now let $A, B \in \wp(X_M)$ such that $A \subseteq B$. Then $K^{+'} \in A^{+'}$ implies $K \in A \subseteq B$. So $K^{+'} \in B^{+'}$ and hence $A^{+'} \subseteq B^{+'}$. This completes the proof. □

Lemma 6.2.5. (i) *For $A, B \subseteq X_M$, $A^{+'} \cap B^{+'} = (A \cap B)^{+'}$.*
(ii) *For $A, B \subseteq X_{M\#}$, $A^{+} \cap B^{+} = (A \cap B)^{+}$.*

Proof. (i) $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then $(A \cap B)^{+'} \subseteq A^{+'}$ and $(A \cap B)^{+'} \subseteq B^{+'}$ which implies $(A \cap B)^{+'} \subseteq A^{+'} \cap B^{+'}$. Let $N^{+'} \in A^{+'} \cap B^{+'}$. Then $N \in A$ and $N \in B$. Therefore $N^{+'} \in (A \cap B)^{+'}$ which implies $A^{+'} \cap B^{+'} \subseteq (A \cap B)^{+'}$. Hence $A^{+'} \cap B^{+'} = (A \cap B)^{+'}$.

(ii) The proof is analogous to that of (i). □

Lemma 6.2.6. (i) *For* $A, B \subseteq X_M$, $A^{+'} \cup B^{+'} = (A \cup B)^{+'}$.

(ii) *For* $A, B \subseteq X_{M\#}$, $A^{+} \cup B^{+} = (A \cup B)^{+}$.

Proof. (i) $A, B \subseteq A \cup B$. Then $A^{+'}, B^{+'} \subseteq (A \cup B)^{+'}$ implies $A^{+'} \cup B^{+'} \subseteq (A \cup B)^{+'}$. Let $N^{+'} \in (A \cup B)^{+'}$. Then either $N \in A$ or $N \in B$. Therefore $N^{+'} \in A^{+'} \cup B^{+'}$ which implies $(A \cup B)^{+'} \subseteq A^{+'} \cup B^{+'}$. Hence $A^{+'} \cup B^{+'} = (A \cup B)^{+'}$.

(ii) The proof is analogous to that of (i). □

Using Theorems 6.1.2, 6.2.4 and Lemmas 6.2.3, 6.2.6, and Theorem 5.3.3, we prove the kuratowski closure axioms in the following Theorem.

Theorem 6.2.7. *Let* M *be a multiplication* ΓS*-semimodule and* $A, B \subseteq X_M$. *Then*

(i) $A \subseteq \overline{A}$

(ii) $\overline{\overline{A}} = \overline{A}$

(iii) $A \subseteq B$ *implies* $\overline{A} \subseteq \overline{B}$

(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (i) $A^{+'} \subseteq \overline{A^{+'}} = (\overline{A})^{+'}$. This implies $(A^{+'})^{+} \subseteq ((\overline{A})^{+'})^{+}$. Therefore $A \subseteq \overline{A}$.

(ii) $\overline{\overline{A}} = ((\overline{\overline{A}})^{+'})^{+} = (\overline{(\overline{A})^{+'}})^{+} = (\overline{(\overline{A^{+'}})})^{+} = (\overline{A^{+'}})^{+} = ((\overline{A})^{+'})^{+} = \overline{A}$.

(iii) $A \subseteq B$ implies $A^{+'} \subseteq B^{+'}$. Hence $\overline{A^{+'}} \subseteq \overline{B^{+'}}$ from which it follows that $(\overline{A})^{+'} \subseteq (\overline{B})^{+'}$. It implies that $((\overline{A})^{+'})^{+} \subseteq ((\overline{B})^{+'})^{+}$. Therefore $\overline{A} \subseteq \overline{B}$.

(iv) $\overline{A \cup B} = ((\overline{A \cup B})^{+'})^{+} = (\overline{(A \cup B)^{+'}})^{+} = (\overline{A^{+'} \cup B^{+'}})^{+} = (\overline{A^{+'}} \cup \overline{B^{+'}})^{+} = (\overline{A^{+'}})^{+} \cup (\overline{B^{+'}})^{+} = ((\overline{A})^{+'})^{+} \cup ((\overline{B})^{+'})^{+} = \overline{A} \cup \overline{B}$.

□

Now we have the following from Theorem 6.2.7.

Theorem 6.2.8. *Let* M *be a multiplication* ΓS*-semimodule. The mapping from* $A \mapsto \overline{A}$ *is a Kuratowski closure operator on* X_M*, where* $A \subseteq X_M$.

Suppose M is a multiplication ΓS-semimodule. Then the topology τ, induced by the Kuratowski closure operator on X_M is called the Hull Kernel topology on X_M and we call the topological space (X_M, τ) as the *structure space* of the ΓS-semimodule M. Also the structure space of the associated L-semimodule $M^{\#}$ is denoted as $(X_{M^{\#}}, \tau')$. Therefore throughout the section, M is a unitary multiplication ΓS-semimodule and we denote the spaces as X_M and $X_{M^{\#}}$ without mentioning the topologies explicitly. It is to be noted that $M^{\#}$ is a multiplication L-semimodule whenever M is a multiplication ΓS-semimodule and vice versa (see Theorem 5.3.3).

Using Lemma 6.2.3 we will now prove that the structure spaces of a ΓS-semimodule and its associated L-semimodule are homeomorphic.

Theorem 6.2.9. *(X_M, τ) and $(X_{M^{\#}}, \tau')$ are homeomorphic.*

Proof. Let us define a mapping $f : X_M \to X_{M^{\#}}$ by $f(N) = N^{+'}$, where $N \in X_M$. f is a bijective map by Theorem 6.2.4. For any subset A of X_M, $f(A) = A^{+'}$. Let A be any closed set in X_M,. Then $f(A) = f(\overline{A}) = (\overline{A})^{+'} = \overline{A^{+'}}$ which is a closed set in $X_{M^{\#}}$. Again let B be any closed set in $X_{M^{\#}}$. Then $f^{-1}(B) = f^{-1}(\overline{B}) = (\overline{B})^{+} = \overline{B^{+}}$ which is a closed set in X_M. Therefore f is a homeomorphism. □

Before characterizing the form of closed set, open set and open base of X_M, we introduce the following notions and prove some necessary propositions.

Notations: Let N be a ΓS-subsemimodule on M and $m \in M$. We define,

$$\Delta_M(N) = \{P \in X_M : N \subseteq P\}; C\Delta_M(N) = X_M \setminus \Delta_M(N) = \{P \in X_M : N \nsubseteq P\};$$

$$C\Delta_M(m) = \{P \in X_M : m \in P\}; C\Delta_M(m) = X_M \setminus \Delta_M(m) = \{P \in X_M : m \notin P\}.$$

Proposition 6.2.10. *Let N be a ΓS-subsemimodule on the ΓS-semimodule M. Then $(\Delta_M(N))^{+'} = \Delta_{M^{\#}}(N^{+'})$.*

Proof. Let $K^{+'} \in (\Delta_M(N))^{+'} \subseteq X_{M^{\#}}$. Then $K \in \Delta_M(N) \subseteq X_M$, i.e., $K \in X_M$ and $N \subseteq K$. Thus $N^{+'} \subseteq K^{+'} \in X_{M^{\#}}$. So $K^{+'} \in \Delta_{M^{\#}}(N^{+'})$.
Therefore $(\Delta_M(N))^{+'} \subseteq \Delta_{M^{\#}}(N^{+'})$. The reverse inclusion follows with a similar argument. Hence $(\Delta_M(N))^{+'} = \Delta_{M^{\#}}(N^{+'})$. □

Proposition 6.2.11. *Let K be a subsemimodule of the L-semimodule $M^{\#}$. Then $(\Delta_{M^{\#}}(K))^{+} = \Delta_M(K^{+})$.*

Proof. $\Delta_{M^{\#}}(K) = \Delta_{M^{\#}}((K^{+})^{+'}) = (\Delta_M(K^{+}))^{+'}$, by Proposition 6.2.10. Therefore $(\Delta_{M^{\#}}(K))^{+} = \Delta_M(K^{+})$. □

From Propositions 6.2.10, 6.2.11, we clearly have the following.

Proposition 6.2.12. *Let N be a ΓS-subsemimodule of M. Then*

(i) $(C\Delta_M(N))^{+'} = C\Delta_{M^\#}(N^{+'})$.

(ii) $(C\Delta_{M^\#}(K))^{+} = C\Delta_M(K^{+})$.

Now in the following results, we obtain the closed sets and open sets of the space X_M via the space $X_{M^\#}$ of the associated L-semimodule.

Proposition 6.2.13. *Any closed set in X_M is of the form $\Delta_M(N)$, where N is a ΓS-subsemimodule on M.*

Proof. Let A be any closed set in X_M. Then $A^{+'}$ is a closed set in $X_{M^\#}$ by Theorem 6.2.9. So $A^{+'} = \Delta_M(K)$ for some subsemimodule K of $M^\#$ by Theorem 6.1.3. It implies $(A^{+'})^{+} = (\Delta_M(K))^{+}$. Thus by Proposition 6.2.11,
$A = \Delta_M(K^{+}) = \Delta_M(N)$, where $N = K^{+}$ is a ΓS-subsemimodule of M. □

The following result is easy to observe from Proposition 6.2.13.

Corollary 6.2.14. *Any open set in X_M is of the form $C\Delta_M(N)$, where N is a ΓS-subsemimodule on M.*

Below in Lemma 6.2.15 and Theorem 6.2.16 we discuss few properties of closed sets of X_M.

Lemma 6.2.15. *Let N, K be two ΓS-subsemimodules of M. Then*

(i) *$N \subseteq K$ implies $\Delta_M(K) \subseteq \Delta_M(N)$.*

(ii) $\bigcap_{\alpha\in\Lambda} \Delta_M(N_\alpha) = \Delta_M(\sum_{\alpha\in\Lambda} N_\alpha)$.

(iii) $\Delta_M(N) = \Delta_M(rad(N))$.

(iv) *If $\Delta_M(N) \subseteq \Delta_M(K)$ then $K \subseteq rad(N)$.*

Proof. The proofs of $(i), (ii)$ are easy.

(iii) Using Lemma 1.5.26, Propositions 6.2.10, 6.2.11, Proposition 5.3.17, we have,

$$\Delta_M(N) = ((\Delta_M(N))^{+'})^{+} = (\Delta_{M^\#}(N^{+'}))^{+} = (\Delta_{M^\#}(rad(N^{+'})))^{+} = (\Delta_{M^\#}((rad(N))^{+'}))^{+} = ((\Delta_M(rad(N)))^{+'})^{+} = \Delta_M(rad(N)).$$

(*iv*) Let $\Delta_M(N) \subseteq \Delta_M(K)$.
Then $(\Delta_M(N))^{+'} = \Delta_M(N^{+'}) \subseteq \Delta_M(K^{+'}) = (\Delta_M(K))^{+'}$ (using Theorem 6.2.4). Therefore from Lemma 1.5.27, it follows that $K^{+'} \subseteq rad(N^{+'})$. Hence by Proposition 5.3.17,

$$K = (K^{+'})^{+} \subseteq (rad(N^{+'}))^{+} = ((rad(N))^{+'})^{+} = rad(N).$$

□

Theorem 6.2.16. *Let N, K be two ΓS-subsemimodules of M. Then*

$$\Delta_M(N) \cup \Delta_M(K) = \Delta_M(N \cap K).$$

Proof. Clearly $\Delta_M(N) \cup \Delta_M(K) \subseteq \Delta_M(N \cap K)$. Let $P \in \Delta_M(N \cap K)$. So $N \cap K \subseteq P$. Since M is a multiplication ΓS-semimodule then by Theorem 5.3.13, $N \subseteq P$ or $K \subseteq P$ which implies that

$$P \in \Delta_M(N) \text{ or } P \in \Delta_M(K), \text{ i.e., } P \in \Delta_M(N) \cup \Delta_M(K).$$

Therefore $\Delta_M(N \cap K) \subseteq \Delta_M(N) \cup \Delta_M(K)$. Hence $\Delta_M(N) \cup \Delta_M(K) = \Delta_M(N \cap K)$. □

The following result describe form of an open base of the space X_M.

Proposition 6.2.17. *$\{C\Delta_M(m) : m \in M\}$ is an open base for X_M.*

Proof. Let $\mathcal{U}$ be an open set in X_M. Then $\mathcal{A} = X_M \setminus \mathcal{U}$ is a closed set in X_M. By Proposition 6.2.13, $\mathcal{A} = \Delta_M(N)$ for some ΓS-subsemimodule N of M. Then $K \in \mathcal{U}$ implies $K \notin \mathcal{A}$, i.e., $N \nsubseteq K$. Then there exists $m \in N$ such that $m \notin K$.
Hence $K \in C\Delta_M(m)$. Now let $K' \in C\Delta_M(m)$. Then $m \notin K'$. This implies that $N \nsubseteq K'$ whence it follows that $K' \in C\Delta_M(N) = \mathcal{U}$. Hence $C\Delta_M(m) \subseteq \mathcal{U}$.
Consequently, $K \in C\Delta_M(m) \subseteq \mathcal{U}$. Thus $\{C\Delta_M(m) : m \in M\}$ is an open base for X_M. □

Next we are going to prove a few lemmas and subsequently we will use them.

Lemma 6.2.18. *Let $m \in M$. Then*

(*i*) $(\Delta_M(m))^{+'} = \Delta_{M^{\#}}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle)$ *and*

(*ii*) $(\Delta_{M^{\#}}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle))^{+} = \Delta_M(m)$,

where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of the Γ-semiring S.

Proof. Let $m \in M$ and $\sum_{j=1}^{n}[\gamma_j, f_j]$ be the right unity of the Γ-semiring S.
(i) Let $N \in \Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle)$. Then $\sum_{j=1}^{n}\langle m, \gamma_j\rangle \in N$ implies for all $s \in S$, $\sum_{j=1}^{n} m\gamma_j s \in N^+$, in particular, $m = \sum_{j=1}^{n} m\gamma_j f_j \in N^+$. Therefore $N^+ \in \Delta_M(m)$.
Hence $N = (N^+)^{+'} \in (\Delta_M(m))^{+'}$ implies $\Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle) \subseteq (\Delta_M(m))^{+'}$.
Again let $K^{+'} \in (\Delta_M(m))^{+'}$. Then $K \in \Delta_M(m)$ implies $m \in K = (K^{+'})^+$. Therefore for all $\alpha \in \Gamma$, $\langle m, \alpha\rangle \in K^{+'}$, in particular, $\sum_{j=1}^{n}\langle m, \gamma_j\rangle \in K^{+'}$ which implies $K^{+'} \in \Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle)$. Hence $(\Delta_M(m))^{+'} \subseteq \Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle)$.
So $(\Delta_M(m))^{+'} = \Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle)$.

(ii) $\Delta_M(m) = ((\Delta_M(m))^{+'})^+ = (\Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle))^+$, by (i). □

From the above lemma the following is easily derived.

Lemma 6.2.19. *Let $m \in M$. Then*

(i) $(C\Delta_M(m))^{+'} = C\Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle)$ *and*

(ii) $(C\Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle))^+ = C\Delta_M(m)$,

where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of the Γ-semiring S.

Lemma 6.2.20. $C\Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle) = C\Delta_{M^\#}(\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle)$, *where $\sum_{i=1}^{m}[e_i, \delta_i]$, $\sum_{j=1}^{n}[\gamma_j, f_j]$ are the left, right unities of S respectively.*

Proof. Let $N \in \Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle)$.
Then $\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle \in N$ which implies $\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle \sum_{j,i}[e_i, \gamma_j] \in N$, i.e., $\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle \in N$.
Then $N \in \Delta_{M^\#}(\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle)$. Therefore

$$\Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle) \subseteq \Delta_{M^\#}(\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle)$$

$$i.e., C\Delta_{M^\#}(\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle) \subseteq C\Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle).$$

Again let $K \in C\Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle)$. Then $\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle \notin K$.
This implies $\sum_{k,i} x_k\alpha_k e_i \notin K^+$, where $\sum_{i=1}^{m}[e_i, \delta_i]$ is the left unity of S.
Again since $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of S,

$$\textstyle\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle \notin K, \text{ i.e., } K \in C\Delta_{M^\#}(\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle).$$

Therefore $C\Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle) \subseteq C\Delta_{M^\#}(\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle)$.
Hence $C\Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle) = C\Delta_{M^\#}(\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle)$. □

Corollary 6.2.21. *$\{C\Delta_{M\#}(\sum_{j=1}^{n}\langle m,\gamma_j\rangle) : m \in M\}$ is an open base for $X_{M\#}$, where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of the Γ-semiring S.*

Proof. Let $\mathcal{V}$ be any open set in $X_{M\#}$ and $N \in \mathcal{V}$. So $N^+ \in \mathcal{V}^+$, where $\mathcal{V}^+$ is an open set in X_M. It implies that there exists some $m \in M$ such that $N^+ \in C\Delta_M(m) \subseteq \mathcal{V}^+$, by Proposition 6.2.17. Therefore

$$N = (N^+)^{+'} \in (C\Delta_M(m))^{+'} \subseteq (\mathcal{V}^+)^{+'} = \mathcal{V}.$$

This implies that $N \in C\Delta_{M\#}(\sum_{j=1}^{n}\langle m,\gamma_j\rangle) \subseteq \mathcal{V}$ (see Lemma 6.2.19). □

Now we prove the separation properties of the space X_M via the space $X_{M\#}$ of its associated semimodule using the homeomorphism proved in Theorem 6.2.9.

Theorem 6.2.22. *The space X_M is T_0.*

Proof. By Theorem 1.5.28 and Theorem 6.2.9, it follows that X_M is T_0. □

Theorem 6.2.23. *The space X_M is T_1 if and only if no prime ΓS-subsemimodule of M is contained in any other prime ΓS-subsemimodule of M.*

Proof. Let the space X_M be T_1. Then $X_{M\#}$ is a T_1-space by Theorem 6.2.9. Therefore by Theorem 1.5.29, it follows that no element of $X_{M\#}$ is contained in any other element of $X_{M\#}$. So by Theorem 5.1.14, no element of X_M is contained in any other element of X_M. The converse part follows by reversing the above argument. □

Using Theorem 6.1.7, Theorem 6.1.8 and Propositions 6.2.10, 6.2.11, 6.2.19 we prove the following two results.

Theorem 6.2.24. *The space X_M is T_2 if and only if for any distinct pair of elements N_1, N_2 of X_M, there exist $m_1, m_2 \in M$ such that $m_1 \notin N_2$, $m_2 \notin N_1$ and there does not exist any element $N \in X_M$ such that $m_1 \notin N$ and $m_2 \notin N$.*

Proof. Let X_M be T_2. Therefore by Theorem 6.2.9, $X_{M\#}$ is a T_2-space. Let N_1, N_2 be two distinct elements of X_M. So $N_1^{+'}, N_2^{+'}$ are two distinct elements of $X_{M\#}$. Since $X_{M\#}$ is T_2, by Theorem 6.1.7 there exists $\sum_{i=1}^{m}\langle x_i,\alpha_i\rangle, \sum_{j=1}^{n}\langle y_j,\beta_j\rangle \in M^{\#}$ such that $\sum_{i=1}^{m}\langle x_i,\alpha_i\rangle \notin N_2^{+'}$, $\sum_{j=1}^{n}\langle y_j,\beta_j\rangle \notin N_1^{+'}$ and there does not exist any element $N \in X_{M\#}$ such that $\sum_{i=1}^{m}\langle x_i,\alpha_i\rangle \notin N$ and $\sum_{j=1}^{n}\langle y_j,\beta_j\rangle \notin N$. This implies for some $s_1, s_2 \in S$, $\sum_{i=1}^{m} x_i\alpha_i s_1 \notin N_2$, $\sum_{j=1}^{n} y_j\beta_j s_2 \notin N_1$. Also if $K \in X_M$ such that $\sum_{i=1}^{m} x_i\alpha_i s_1 \notin K$, $\sum_{j=1}^{n} y_j\beta_j s_2 \notin K$ then for $K^{+'} \in X_{M\#}$, $\sum_{i=1}^{m}\langle x_i,\alpha_i\rangle \notin K^{+'}$, $\sum_{j=1}^{n}\langle y_j,\beta_j\rangle \notin K^{+'}$

which is a contradiction to the fact that there is no element of $X_{M\#}$ not containing $\sum_{i=1}^{m}\langle x_i,\alpha_i\rangle, \sum_{j=1}^{n}\langle y_j,\beta_j\rangle$. Therefore there does not exist any element $K \in X_M$ such that $\sum_{i=1}^{m} x_i\alpha_i s_1 \notin K$, $\sum_{j=1}^{n} y_j\beta_j s_2 \notin K$. Hence the condition holds.
Conversely, let the condition hold. Let us suppose that K_1, K_2 be two distinct elements of $X_{M\#}$. Then K_1^+, K_2^+ are two distinct elements of X_M. So by the condition given, there exists $m_1, m_2 \in M$ such that $m_1 \notin K_2^+$, $m_2 \notin K_1^+$ and there does not exist any element $K \in X_M$ such that $m_1 \notin K$ and $m_2 \notin K$. Therefore $\sum_{j=1}^{n}\langle m_1,\gamma_j\rangle \notin K_2$, $\sum_{j=1}^{n}\langle m_2,\gamma_j\rangle \notin K_1$, where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of S. Also if $L \in X_{M\#}$ such that $\sum_{j=1}^{n}\langle m_1,\gamma_j\rangle \notin L$, $\sum_{j=1}^{n}\langle m_2,\gamma_j\rangle \notin L$ then for $L^+ \in X_M$, $m_1 \notin L^+$ and $m_2 \notin L^+$ which is a contradiction to the fact that there is no element of X_M such that not containing m_1, m_2. Therefore there exist no $L \in X_{M\#}$ such that $\sum_{j=1}^{n}\langle m_1,\gamma_j\rangle \notin L$ and $\sum_{j=1}^{n}\langle m_2,\gamma_j\rangle \notin L$. So by Theorem 6.1.7 it follows that $X_{M\#}$ is a T_2-space. Hence by Theorem 6.2.9, X_M is T_2. □

Corollary 6.2.25. *If the space X_M is T_2 and X_M contains more than one element then there exist $m_1, m_2 \in M$ such that $X_M = C\Delta_M(m_1) \cup C\Delta_M(m_2) \cup \Delta_M(N)$, where N is the ΓS-subsemimodule generated by m_1, m_2.*

Proof. Let X_M be T_2 and contain more than one element. Then $X_{M\#}$ contains more than one element and by Theorem 6.2.9, $X_{M\#}$ is T_2. Therefore using Theorem 6.1.8 we get that there exists $\sum_{k=1}^{p}\langle m_k,\alpha_k\rangle, \sum_{l=1}^{q}\langle n_l,\beta_l\rangle \in M^\#$ such that

$$X_{M\#} = C\Delta_{M\#}(\sum_{k=1}^{p}\langle m_k,\alpha_k\rangle) \cup C\Delta_{M\#}(\sum_{l=1}^{q}\langle n_l,\beta_l\rangle) \cup \Delta_{M\#}(N),$$

where N is the L-subsemimodule of $M^\#$ generated by $\sum_{k=1}^{p}\langle m_k,\alpha_k\rangle$,
$\sum_{l=1}^{q}\langle n_l,\beta_l\rangle$. So we have $\sum_{k,i} m_k\alpha_k e_i, \sum_{l,i} n_l\beta_l e_i \in M$, where $\sum_{i=1}^{m}[e_i,\delta_i]$ is the left unity of S and using Lemma 6.2.6 (ii),

$$X_M = (C\Delta_{M\#}(\sum_{k=1}^{p}\langle m_k,\alpha_k\rangle))^+ \cup (C\Delta_{M\#}(\sum_{l=1}^{q}\langle n_l,\beta_l\rangle))^+ \cup (\Delta_{M\#}(N))^+.$$

Again by Lemma 6.2.20,

$$X_M = (C\Delta_{M\#}(\sum_{k,i,j}\langle m_k\alpha_k e_i,\gamma_j\rangle))^+ \cup (C\Delta_{M\#}(\sum_{l,i,j}\langle n_l\beta_l e_i,\gamma_j\rangle))^+ \cup (\Delta_{M\#}(N))^+.$$

Then using Lemma 6.2.19 and Proposition 6.2.10 we have,

$$X_M = C\Delta_M(\sum_{k,i} m_k\alpha_k e_i) \cup C\Delta_M(\sum_{l,i} n_l\beta_l e_i) \cup \Delta_M(N^+)$$

and also N^+ is the ΓS-subsemimodule generated by $\sum_{k,i} m_k\alpha_k e_i, \sum_{l,i} n_l\beta_l e_i$ (see Proposition 5.2.3). Therefore the condition holds. We can proof the converse by reversing the argument accordingly. This completes the proof. □

Using Theorem 6.1.9 and Propositions 6.2.10, 6.2.11, 6.2.19 and Lemma 6.2.20, we prove Theorem 6.2.26.

Theorem 6.2.26. *The space X_M is a regular space if and only if for any $N \in X_M$ and $m \notin N$, there exists a ΓS-subsemimodule N' of M and $m' \in M$ such that $N \in C\Delta_M(m') \subseteq \Delta_M(N') \subseteq C\Delta_M(m)$.*

Proof. Let X_M be regular. Then by Theorem 6.2.9, $X_{M^\#}$ is regular. Let us suppose that $N \in X_M$ and $m \notin N$. So $N^{+'} \in X_{M^\#}$ and $\sum_{j=1}^{n}\langle m, \gamma_j\rangle \notin N^{+'}$, where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of the Γ-semiring S. The by Theorem 6.1.9 it implies that there exists a subsemimodule N_1 of $M^\#$ and
$\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle \in M^\#$ such that

$$N^{+'} \in C\Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle) \subseteq \Delta_{M^\#}(N_1) \subseteq C\Delta_{M^\#}(\sum_{j=1}^{n}\langle m, \gamma_j\rangle).$$

By Lemma 6.2.20 we have, $C\Delta_{M^\#}(\sum_{k=1}^{m}\langle x_k, \alpha_k\rangle) = C\Delta_{M^\#}(\sum_{k,j,i}\langle x_k\alpha_k e_i, \gamma_j\rangle)$. Hence by Lemma 6.2.19 and Proposition 6.2.11

$$N = (N^{+'})^+ \in C\Delta_M(\sum_{k,i} x_k\alpha_k e_i) \subseteq \Delta_M(N_1^+) \subseteq C\Delta_M(m).$$

Hence the condition holds.

Conversely, let the condition hold.

Let us suppose that $N_1 \in X_{M^\#}$ and $\sum_{k=1}^{m}\langle m_k, \alpha_k\rangle \notin N_1$. Then $N_1^+ \in X_M$ and $\sum_{k,i} m_k\alpha_k e_i \notin N_1^+$, where $\sum_{i=1}^{m}[e_i, \delta_i]$ is the left unity of S. So by the condition, there exists a ΓS-subsemimodule N_2 of M and $m' \in M$ such that

$$N_1^+ \in C\Delta_M(m') \subseteq \Delta_M(N_2) \subseteq C\Delta_M(\sum_{k,i} m_k\alpha_k e_i).$$

Therefore by Lemma 6.2.19 and Proposition 6.2.10,

$$N_1 = (N_1^+)^{+'} \in C\Delta_{M^\#}(\sum_{j=1}^{n}\langle m', \gamma_j\rangle) \subseteq \Delta_{M^\#}(N_2^{+'}) \subseteq C\Delta_{M^\#}(\sum_{k,j,i}\langle m_k\alpha_k e_i, \gamma_j\rangle).$$

So by Lemma 6.2.20,

$$N_1 = (N_1^+)^{+'} \in C\Delta_{M^\#}(\sum_{j=1}^{n}\langle m', \gamma_j\rangle) \subseteq \Delta_{M^\#}(N_2^{+'}) \subseteq C\Delta_{M^\#}(\sum_{k}\langle m_k, \alpha_k\rangle).$$

So by Theorem 6.1.9, $X_{M^\#}$ is regular. Hence by Theorem 6.2.9, X_M is regular. □

Before going to the next results, we first note the following lemma which is easy to verify.

Lemma 6.2.27. *$(\bigcap_{P_i \in \mathcal{A}} P_i)^{+'} = \bigcap_{P_i^{+'} \in \mathcal{A}^{+'}} P_i^{+'}$ for any subset $\mathcal{A}$ of X_M.*

Now we study the necessary and sufficient condition for a subset to be irreducible and dense in X_M.

Theorem 6.2.28. *Let $\mathcal{A}$ be a nonempty subset of X_M. $\mathcal{A}$ is dense in X_M if and only if $\bigcap_{P_i \in \mathcal{A}} P_i = \bigcap_{P_i \in X_M} P_i$.*

Proof. Let $\mathcal{A}$ be a dense subset of X_M. So $\overline{\mathcal{A}} = X_M$. Then using Lemma 6.2.3 (i) we have, $(\overline{\mathcal{A}})^{+'} = \overline{(\mathcal{A}^{+'})} = X_{M\#}$. Hence $\mathcal{A}^{+'}$ is dense in $X_{M\#}$. Therefore applying Theorem 6.1.10 we have, $\bigcap_{P_i^{+'} \in \mathcal{A}^{+'}} P_i^{+'} = \bigcap_{P_i^{+'} \in X_{M\#}} P_i^{+'}$. So from Lemma 6.2.27, it implies that $(\bigcap_{P_i \in \mathcal{A}} P_i)^{+'} = (\bigcap_{P_i \in X_M} P_i)^{+'}$. Hence it follows that $\bigcap_{P_i \in \mathcal{A}} P_i = \bigcap_{P_i \in X_M} P_i$. Converse part follows by reversing the above argument. □

Theorem 6.2.29. *Let $\mathcal{A}$ be a nonempty subset of X_M. $\mathcal{A}$ is irreducible in X_M if and only if $\bigcap_{P_i \in \mathcal{A}} P_i$ is a prime ΓS-subsemimodule of M.*

Proof. Let $\mathcal{A}$ be an irreducible subset of X_M. To show, $\mathcal{A}^{+'}$ is irreducible in $X_{M\#}$, let us suppose that for two closed sets X, Y, $\mathcal{A}^{+'} \subseteq X \cup Y$. Then $\mathcal{A} = (\mathcal{A}^{+'})^{+} \subseteq (X \cup Y)^{+}$. So using Lemma 6.2.6 (ii) we have, $\mathcal{A} \subseteq X^{+} \cup Y^{+}$. Since $\mathcal{A}$ is irreducible in X_M,

$$\text{either } \mathcal{A} \subseteq X^{+} \text{ or } \mathcal{A} \subseteq Y^{+}, \text{ i.e., either } \mathcal{A}^{+'} \subseteq X \text{ or } \mathcal{A}^{+'} \subseteq Y.$$

Hence $\mathcal{A}^{+'}$ is irreducible in $X_{M\#}$. Therefore in view of Lemma 6.2.27 and Theorem 1.5.30 we obtain that $\bigcap_{P_i^{+'} \in \mathcal{A}^{+'}} P_i^{+'} = (\bigcap_{P_i \in \mathcal{A}} P_i)^{+'}$ is a prime subsemimodule of $M^{\#}$. So by Theorem 5.1.14, $\bigcap_{P_i \in \mathcal{A}} P_i$ is a prime ΓS-subsemimodule of M.
The converse part of the proof follows by reversing the above argument. □

Theorem 6.2.30. *Let N be a ΓS-subsemimodule of M. Then the following are equivalent to one another.*

(i) *$\Delta_M(N)$ is irreducible subset of X_M.*

(ii) *$\bigcap_i \{P_i \in X_M : N \subseteq P_i\} = rad(N)$ is a prime ΓS-subsemimodule of M.*

(iii) *$rad(N)$ is a generic point of $\Delta_M(N)$ in X_M.*

Proof. $(i) \Leftrightarrow (ii)$: It directly follows from Theorem 6.2.29.

$(ii) \Rightarrow (iii)$: Let $rad(N)$ be a prime ΓS-subsemimodule of M. By Lemma 6.2.15 (iii),

$$\overline{\{rad(N)\}} = \{P \in X_M : rad(N) \subseteq P\} = \Delta_M(rad(N)) = \Delta_M(N).$$

Hence $rad(N)$ is a generic point of $\Delta_M(N)$ in X_M.

$(iii) \Rightarrow (ii)$: It is obvious. □

Now we present some necessary and sufficient condition for compactness, connectedness etc. of the space X_M.

The following Theorem is the ΓS-semimodule analogue of Theorem 1.5.32 which states that for a multiplication semimodule M, $Spec(M)$ is compact if and only if M is finitely generated.

Theorem 6.2.31. *The space X_M is compact if and only if M is finitely generated.*

Proof. In view of Theorem 6.2.9, Theorem 1.5.32 and Theorem 5.2.5, it follows that X_M is compact if and only if $X_{M^\#}$ is compact if and only if $M^\#$ is finitely generated if and only if M is finitely generated. □

Below we give another necessary and sufficient condition for compactness of the space X_M.

Theorem 6.2.32. *The space X_M is compact if and only if for any collection $\{a_\alpha\}_{\alpha\in\Lambda}$ of elements in M, there exists a finite subcollection $\{a_i : i = 1, 2, ..., n\}$ in M such that for any $N \in X_M$, there exists a_i from the subcollection such that $a_i \notin N$.*

Proof. Let the space X_M be compact and let $\{m_\alpha\}_{\alpha\in\Lambda}$ be a collection of elements in M and $N \in X_M$. By Theorem 6.2.9, the space $X_{M^\#}$ is compact. Also $\{\sum_{j=1}^{n}\langle m_\alpha, \gamma_j\rangle\}_{\alpha\in\Lambda}$ is a collection of elements in $M^\#$, where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of the Γ-semiring S. Then by Theorem 6.1.12, there exists a finite subcollection $\{\sum_{j=1}^{n}\langle m_k, \gamma_j\rangle : k = 1, 2, ..., p\}$ of elements in $M^\#$ and there exists $\sum_{j=1}^{n}\langle m_k, \gamma_j\rangle \in M^\#$ such that $\sum_{j=1}^{n}\langle m_k, \gamma_j\rangle \notin N^{+'}$, where $N^{+'} \in X_{M^\#}$. Therefore we obtain a finite subcollection $\{\sum_j m_k\gamma_j f_j = m_k : k = 1, 2, ..., p\}$ of elements in M such that for any $N \in X_M$, there exists $m_k \in M$ such that $m_k \notin N$. Hence the condition holds. The converse part can be prove analogously by reversing the arguments above. □

Theorem 6.2.33. *The space X_M is disconnected if and only if there exists a ΓS-subsemimodule L of M and a collection $\{m_\alpha : \alpha \in \Lambda\}$ of elements in M not belonging to L such that if $L' \in X_M$ and $m_\alpha \in L'$ for all $\alpha \in \Lambda$ then $L \setminus L' \neq \emptyset$.*

Proof. Let the space X_M be disconnected. Then by Theorem 6.2.9, $X_{M^\#}$ is disconnected. Therefore by Theorem 6.1.11, there exists a L-subsemimodule N of $M^\#$ and a collection $\{a_\alpha : \alpha \in \Lambda\}$ of elements in $M^\#$ not belonging to N such that for any $N' \in X_{M^\#}$ and $a_\alpha \in N'$ for all $\alpha \in \Lambda$, $N \setminus N' \neq \emptyset$. Then $\{\sum_{i=1}^m a_\alpha e_i : \alpha \in \Lambda\}$ is a collection of elements in M not belonging to $N^+ \in X_M$, where $\sum_{i=1}^m [e_i, \delta_i]$ is the left unity of S. Now let $K \in X_M$ and $\sum_{i=1}^m a_\alpha e_i \in K$ for all $\alpha \in \Lambda$. Then $K^{+'} \in X_{M^\#}$ and $\sum_{i=1}^m \langle a_\alpha e_i, \delta_i \rangle = a_\alpha \in K^{+'}$ for all $\alpha \in \Lambda$. Therefore it follows that $N \setminus K^{+'} \neq \emptyset$. This implies that $(N \setminus K^{+'})^+ \neq \emptyset$ whence it follows that $N^+ \setminus K \neq \emptyset$. Hence the condition holds. The converse part can be proved analogously reversing the above arguments. $\square$

Following Theorems 6.2.34, 6.2.35, 6.2.36, 6.2.37 generalize Theorem 1.5.32, Corollary 1.5.34, Corollary 1.5.35, Theorem 1.5.36.

Theorem 6.2.34. *Every basic open set of X_M is compact.*

Proof. Let $C\Delta_M(m)$ be a basic open set of X_M for $m \in M$. Then by Lemma 6.2.19, $(C\Delta_M(m))^{+'} = C\Delta_M(\sum_{j=1}^n \langle m, \gamma_j \rangle)$ which is a basic open set of $X_{M^\#}$ by Corollary 6.2.21, where $\sum_{j=1}^n [\gamma_j, f_j]$ is the right unity of the Γ-semiring S. So by Theorem 1.5.33, $(C\Delta_M(m))^{+'}$ is compact in $X_{M^\#}$. Therefore from Theorem 6.2.9, it follows that $C\Delta_M(m)$ is compact in X_M. $\square$

Theorem 6.2.35. *An open set of X_M is compact if and only if it is a union of a finite number of basic open sets.*

Proof. Let $C\Delta_M(N)$ be an open compact set in X_M for some ΓS-subsemimodule N of M. Therefore by Proposition 6.2.12, $(C\Delta_M(N))^{+'} = C\Delta_{M^\#}(N^{+'})$ which is an open compact set in $X_{M^\#}$ by Theorem 6.2.9. So by Corollary 1.5.34, $(C\Delta_{M^\#}(N))^{+'}$ is a union of a finite number of basic open sets in $X_{M^\#}$, i.e., $(C\Delta_{M^\#}(N))^{+'} = \bigcup_{k=1}^p U_k$ (say), where each U_k is a basic open set in $X_{M^\#}$. Now by applying Lemma 6.2.6 (ii) finite times we obtain that $C\Delta_M(N) = (\bigcup_{k=1}^p U_k)^+ = \bigcup_{k=1}^p (U_k^+)$. So by Theorem 6.2.9 it follows that each U_k^+ is surely a basic open set in X_M whence it follows that $C\Delta_M(N)$ can be represented as a union of a finite number of basic open sets in X_M. Converse part follows from Theorem 6.2.34. $\square$

Theorem 6.2.36. *If N is a finitely generated ΓS-subsemimodule of a ΓS-semimodule M then $C\Delta_M(N)$ is compact in X_M.*

Proof. Let N be a finitely generated ΓS-subsemimodule of a ΓS-semimodule M. Then by Theorem 5.2.4, $N^{+'}$ is a finitely generated subsemimodule of the semimodule $M^{\#}$. Therefore by Corollary 1.5.35, $C\Delta_{M^{\#}}(N^{+'})$ is compact in $X_{M^{\#}}$.
Also $C\Delta_{M^{\#}}(N^{+'}) = (C\Delta_M(N))^{+'}$ (see Proposition 6.2.12 (i)).
Hence by Theorem 6.2.9, $C\Delta_M(N)$ is compact in X_M. □

Theorem 6.2.37. *The intersection of finitely many basic open sets is compact in X_M.*

Proof. Let us consider a finite family of basic open sets $\{C\Delta_M(m_i) : i = 1, 2, ..., m\}$ of X_M. Therefore $\{C\Delta_{M^{\#}}(\sum_{j=1}^{n}\langle m_i, \gamma_j\rangle) : i = 1, 2, ..., m\}$ is a finite family of basic open sets of $X_{M^{\#}}$, where $\sum_{j=1}^{n}[\gamma_j, f_j]$ is the right unity of the Γ-semiring S. So by Theorem 1.5.36, the intersection $\bigcap_{i=1}^{m} C\Delta_{M^{\#}}(\sum_{j=1}^{n}\langle m_i, \gamma_j\rangle)$ is compact subset of $X_{M^{\#}}$. Now applying Lemma 6.2.5 (ii) finite times and by Lemma 6.2.19 (ii) we have,

$$(\bigcap_{i=1}^{m} C\Delta_{M^{\#}}(\sum_{j=1}^{n}\langle m_i, \gamma_j\rangle))^{+} = \bigcap_{i=1}^{m}(C\Delta_{M^{\#}}(\sum_{j=1}^{n}\langle m_i, \gamma_j\rangle))^{+} = \bigcap_{i=1}^{m} C\Delta_M(m_i).$$

Therefore in view of Theorem 6.2.9, it follows that $\bigcap_i C\Delta_M(m_i)$ is compact in X_M. □

6.3 Structure space of prime $k\Gamma S$-subsemimodules of a ΓS-semimodule

In this section we study the topological properties of the space Y_M of all prime $k\Gamma S$-subsemimodules of a multiplication ΓS-semimodule. As $Y_M \subseteq X_M$, we consider the topological space (Y_M, τ^k) taking the topology τ^k as the subspace topology induced by the topology τ on X_M. Also we denote the space of all prime k-subsemimodules of $M^{\#}$ as $Y_{M^{\#}}$ which is a subspace of $X_{M^{\#}}$.

We have the following result which assures that the space is nonempty.

Theorem 6.3.1. *Let M be a multiplication ΓS-semimodule. Then the space Y_M is nonempty.*

Proof. Let M be a multiplication ΓS-semimodule and N be a proper $k\Gamma S$-subsemimodule of M. Then by Theorem 5.3.14, there exists a prime $k\Gamma S$-subsemimodule of M containing N. Hence the proof is complete. □

In view of Theorem 6.2.9 and Theorem 5.1.14 we have the following.

Theorem 6.3.2. *The spaces Y_M and $Y_{M^\#}$ are homeomorphic.*

Now let us observe the following topological properties of the subspace Y_M, inherited from the space X_M.

(i) For any subset $\mathcal{B}$ of Y_M, closure of $\mathcal{B}$ in $Y_M = \overline{\mathcal{B}} \cap Y_M = \{P \in Y_M : \bigcap_{P_i \in \mathcal{B}} P_i \subseteq P\}$, where $\overline{\mathcal{B}}$ is the closure of $\mathcal{B}$ in X_M.

(ii) Any closed set in Y_M is of the form $\Delta_M^k(N) = \Delta_M(N) \cap Y_M$, where N is a ΓS-subsemimodule on M.

(iii) Any open set in Y_M is of the form $C\Delta_M^k(N) = C\Delta_M(N) \cap Y_M$, where N is a ΓS-subsemimodule on M.

(iv) $\{C\Delta_M^k(m) : m \in M\} = \{C\Delta_M(m) \cap Y_M : m \in M\}$ is an open base for Y_M.

(v) The space Y_M is T_0.

The following results can be obtained with a similar argument as 6.2.10, 6.2.11, 6.2.19, Lemma 6.2.20.

Lemma 6.3.3. *Let N be a ΓS-subsemimodule of M and K be a L-subsemimodule of $M^\#$ and m be an element of M. Then*

(i) $(\Delta_M^k(N))^{+'} = \Delta_{M^\#}^k(N^{+'})$.

(ii) $(\Delta_{M^\#}^k(K))^{+} = \Delta_M^k(K^{+})$.

(iii) $(C\Delta_M^k(N))^{+'} = C\Delta_{M^\#}^k(N^{+'})$.

(iv) $(C\Delta_{M^\#}^k(K))^{+} = C\Delta_M^k(K^{+})$.

(v) $(C\Delta_M^k(m))^{+'} = C\Delta_{M^\#}^k(\sum_{j=1}^n \langle m, \gamma_j \rangle)$.

(vi) $(C\Delta_{M^\#}^k(\sum_{j=1}^n \langle m, \gamma_j \rangle))^{+} = C\Delta_M^k(m)$.

(vii) $C\Delta_{M^\#}^k(\sum_{k=1}^m \langle x_k, \alpha_k \rangle) = C\Delta_{M^\#}^k(\sum_{k,j,i} \langle x_k \alpha_k e_i, \gamma_j \rangle)$.
where $\sum_{j=1}^n [\gamma_j, f_j]$ is the right unity of S.

We have the following properties of the closed sets of the space Y_M analogous to X_M.

Lemma 6.3.4. *Let N, K be two ΓS-subsemimodules of M. Then*

(i) $N \subseteq K$ *implies* $\Delta_M^k(K) \subseteq \Delta_M^k(N)$.

(ii) $\Delta_M^k(N) \cup \Delta_M^k(K) \subseteq \Delta_M^k(N \cap K)$.

(iii) $\bigcap_{\alpha\in\Lambda} \Delta_M^k(N_\alpha) = \Delta_M^k(\sum_{\alpha\in\Lambda} N_\alpha)$.

(iv) $\Delta_M^k(N) = \Delta_M^k(\overline{N}^k)$ *for any* ΓS*-subsemimodule* N *on* M*,* $\overline{N}^k$ *being the* k*-closure of* N*.*

(v) $\Delta_M^k(N) = \Delta_M(rad_k(N))$.

(vi) *If* $\Delta_M^k(N) \subseteq \Delta_M^k(K)$ *then* $K \subseteq rad_k(N)$.

Using Lemmas 6.3.3, 6.3.4, Theorem 6.3.2 we can derive the same topological properties as earlier for Y_M and prove the results similarly by just replacing the members of X_M with those of the subspace topology (i.e. the closed set, open set, open base of Y_M). So we omit most of the proofs and mention the statements of the results only. The following result follows from Theorem 1.5.39 and Theorem 5.1.14 and Theorem 6.3.2.

Theorem 6.3.5. *The space* Y_M *is* T_1 *if and only if no prime* $k\Gamma S$*-subsemimodule of* M *is contained in any other prime* $k\Gamma S$*-subsemimodule of* M*.*

The following two results generalize Theorem 6.1.14 and Theorem 6.1.15.

Theorem 6.3.6. *The space* Y_M *is* T_2 *if and only if for any distinct pair of elements* N_1, N_2 *of* Y_M*, there exist* $m_1, m_2 \in M$ *such that* $m_1 \notin N_2$*,* $m_2 \notin N_1$ *and there does not exist any element* $N \in Y_M$ *such that* $m_1 \notin N$ *and* $m_2 \notin N$*.*

Corollary 6.3.7. *If the space* Y_M *is* T_2 *and* Y_M *contains more than one element then there exist* $m_1, m_2 \in M$ *such that* $Y_M = C\Delta_M^k(m_1) \cup C\Delta_M^k(m_2) \cup \Delta_M^k(N)$*, where* N *is the* $k\Gamma S$*-subsemimodule generated by* m_1, m_2*.*

The following result generalizes Theorem 6.1.16.

Theorem 6.3.8. *The space* Y_M *is a regular space if and only if for any* $N \in Y_M$ *and* $m \notin N$*, there exists a* $k\Gamma S$*-subsemimodule* N' *of* M *and* $m' \in M$ *such that* $N \in C\Delta_M^k(m') \subseteq \Delta_M^k(N') \subseteq C\Delta_M^k(m)$*.*

The following result generalizes Theorem 6.1.17.

Theorem 6.3.9. *Let $\mathcal{A}$ be a nonempty subset of Y_M. $\mathcal{A}$ is dense in Y_M if and only if $\bigcap_{P_i \in \mathcal{A}} P_i = \bigcap_{P_i \in Y_M} P_i$.*

The following two consecutive theorems are generalizations of Theorems 3.8 and 3.9 of [35] respectively.

Theorem 6.3.10. *Let $\mathcal{B}$ be a nonempty subset of Y_M. Then following are equivalent.*

(i) *$\mathcal{B}$ is irreducible in Y_M.*

(ii) *$\mathcal{B}$ is irreducible in X_M.*

(iii) *$\bigcap_{P_i \in \mathcal{B}} P_i$ is a prime ΓS-subsemimodule of M.*

(iv) *$\bigcap_{P_i \in \mathcal{B}} P_i$ is a prime $k\Gamma S$-subsemimodule of M.*

Proof. $(i) \Leftrightarrow (ii)$: Let $\mathcal{B}$ be irreducible in Y_M. Then in view of Theorem 6.3.2, it implies that $\mathcal{B}^{+'}$ is irreducible in $Y_{M\#}$. Now from Theorem 1.5.40, it follows that $\mathcal{B}^{+'}$ is irreducible in $X_{M\#}$. Again by Theorem 6.2.9, $\mathcal{B} = (\mathcal{B}^{+'})^{+}$ is irreducible in X_M. The converse part of the proof follows by reversing the above argument.

$(ii) \Leftrightarrow (iii)$: It follows from Theorem 6.2.29.

$(iii) \Rightarrow (iv)$: It follows from the fact that intersection of $k\Gamma S$-subsemimodules of M is a $k\Gamma S$-subsemimodule of M.

$(iv) \Rightarrow (iii)$: This is obvious. □

Theorem 6.3.11. *Let N be a ΓS-subsemimodule of M. Then the following are equivalent to one another.*

(i) *$\Delta_M^k(N)$ is irreducible subset of Y_M.*

(ii) *$\bigcap_i \{P_i \in Y_M : N \subseteq P_i\} = rad^{(k)}(N)$ is a prime $k\Gamma S$-subsemimodule of M.*

(iii) *$rad^{(k)}(N)$ is a generic point of $\Delta_M^k(N)$ in Y_M.*

Proof. $(i) \Leftrightarrow (ii)$: It directly follows from Theorem 6.3.10.

$(ii) \Rightarrow (iii)$: Let $rad^{(k)}(N)$ be a prime $k\Gamma S$-subsemimodule of M. By Lemma 6.3.4 (v)),

$$\overline{\{rad^{(k)}(N)\}} = \{P \in Y_M : rad^{(k)}(N) \subseteq P\} = \Delta_M^k(rad^{(k)}(N)) = \Delta_M(N).$$

Hence $rad^{(k)}(N)$ is a generic point of $\Delta_M^k(N)$ in Y_M.

$(iii) \Rightarrow (ii)$: It is obvious. □

The following theorem generalizes Theorem 6.1.18.

Theorem 6.3.12. *The space Y_M is disconnected if and only if there exists a $k\Gamma S$-subsemimodule N of M and a collection $\{a_\alpha : \alpha \in \Lambda\}$ of elements in M not belonging to N such that if $N' \in Y_M$ and $a_\alpha \in N'$ for all $\alpha \in \Lambda$ then $N \setminus N' \neq \emptyset$.*

The following theorem generalizes Theorem 4.7 of [35].

Theorem 6.3.13. *The space Y_M is compact if and only if M is k-finitely generated.*

Proof. In view of Theorem 6.3.2, Theorem 1.5.42 and Theorem 5.2.10, it follows that Y_M is compact if and only if $Y_{M\#}$ is compact if and only if $M^{\#}$ is k-finitely generated if and only if M is k-finitely generated. □

The following theorem generalizes Theorem 6.1.19.

Theorem 6.3.14. *The space Y_M is compact if and only if for any collection $\{a_\alpha\}_{\alpha\in\Lambda}$ of elements in M, there exists a finite subcollection $\{a_i : i = 1, 2, ..., n\}$ in M such that for any $N \in Y_M$, there exists a_i from the subcollection such that $a_i \notin N$.*

The following results generalize Theorem 1.5.43, Corollary 1.5.44, Corollary 1.5.45 and Theorem 1.5.46.

Theorem 6.3.15. *Every basic open set of Y_M is compact.*

Proof. Let $C\Delta_M^k(m)$ be a basic open set of Y_M for $m \in M$. Then by Lemma 6.3.3 (v), $(C\Delta_M^k(m))^{+'} = C\Delta_M^k(\sum_{j=1}^n \langle m, \gamma_j \rangle)$ which is a basic open set of $Y_{M\#}$, where $\sum_{j=1}^n [\gamma_j, f_j]$ is the right unity of the Γ-semiring S. So by Theorem 1.5.43, $(C\Delta_M^k(m))^{+'}$ is compact in $Y_{M\#}$. Therefore from Theorem 6.3.2, it follows that $C\Delta_M^k(m)$ is compact in Y_M.

□

Theorem 6.3.16. *An open set of Y_M is compact if and only if it is a union of a finite number of basic open sets.*

Proof. Let $C\Delta_M^k(N)$ be an open compact set in Y_M for some ΓS-subsemimodule N of M. Therefore by Lemma 6.3.3 (iii), $(C\Delta_M^k(N))^{+'} = C\Delta_{M\#}^k(N^{+'})$ which is an open compact set in $Y_{M\#}$ by Theorem 6.3.2. So by Corollary 1.5.44, $(C\Delta_{M\#}^k(N))^{+'}$ is a union of a finite number of basic open sets in $Y_{M\#}$, i.e., $(C\Delta_{M\#}^k(N))^{+'} = \bigcup_{k=1}^p U_k$ (say), where each U_k is a basic open set in $Y_{M\#}$. Now by applying Lemma 6.2.6 (ii) finite times we obtain that $C\Delta_M^k(N) = (\bigcup_{k=1}^p U_k)^+ = \bigcup_{k=1}^p (U_k^+)$. Also by Theorem

6.3.2, it follows that each U_k^+ is surely a basic open set in Y_M whence it follows that $C\Delta_M^k(N)$ can be represented as a union of a finite number of basic open sets in Y_M. Converse part follows from Theorem 6.3.15 and the fact that union of a finite number of compact sets is compact.

□

Theorem 6.3.17. *If N is a k-finitely generated ΓS-subsemimodule of a ΓS-semimodule M then $C\Delta_M^k(N)$ is compact in Y_M.*

Proof. Let N be a k-finitely generated ΓS-subsemimodule of a ΓS-semimodule M. Then by Theorem 5.2.9, $N^{+'}$ is a k-finitely generated subsemimodule of the semimodule $M^{\#}$. Therefore by Corollary 1.5.45, $C\Delta_{M^{\#}}^k(N^{+'})$ is compact in $Y_{M^{\#}}$. Also $C\Delta_{M^{\#}}^k(N^{+'}) = (C\Delta_M^k(N))^{+'}$ (see Lemma 6.3.3 (iii)).
Hence by Theorem 6.3.2, $C\Delta_M^k(N)$ is compact in Y_M.

□

Theorem 6.3.18. *The intersection of finitely many basic open sets is compact in Y_M.*

Proof. Let us consider a finite family of basic open sets $\{C\Delta_M^k(m_i) : i = 1, 2, ..., k\}$ of Y_M. Therefore $\{C\Delta_{M^{\#}}^k(\sum_{j=1}^n \langle m_i, \gamma_j \rangle) : i = 1, 2, ..., k\}$ is a finite family of basic open sets of $Y_{M^{\#}}$, where $\sum_{j=1}^n [\gamma_j, f_j]$ is the right unity of the Γ-semiring S. So by Theorem 1.5.46, the intersection $\bigcap_{i=1}^k C\Delta_{M^{\#}}^k(\sum_{j=1}^n \langle m_i, \gamma_j \rangle)$ is compact subset of $Y_{M^{\#}}$. Now applying Lemma 6.2.5 (ii) finite times and by Lemma 6.3.3 (vi) we have,

$$(\bigcap_{i=1}^k C\Delta_{M^{\#}}^k(\sum_{j=1}^n \langle m_i, \gamma_j \rangle))^+ = \bigcap_{i=1}^k (C\Delta_{M^{\#}}^k(\sum_{j=1}^n \langle m_i, \gamma_j \rangle))^+ = \bigcap_{i=1}^k C\Delta_M^k(m_i).$$

Therefore in view of Theorem 6.3.2, it follows that $\bigcap_i C\Delta_M^k(m_i)$ is compact in Y_M.

□

Some Remarks and Scope of Further Study

We list below some remarks and observations which are mainly related with some possible extension of the research work undertaken in this thesis.

1. In Chapter 2, we could not provide the converse of Theorem 2.3.4 or any counterexample to illustrate that the said converse does not hold. It would be nice if one can find the same.

   Also it would have been better if Theorem 2.3.5 and Theorem 2.3.8 could have been obtained with less restrictions.

2. It would be good if one can characterize the structure space of the Γ-semiring $C_-(X)$ and extend the study of this space investigating various topological properties. Also the $C_-(X)$ analogue of many results of $C_+(X)$ and $C(X)$ characterizing the topological spaces can be obtained via the correspondences achieved in our work in Chapter 4.

3. In [9] [60] and many other papers, graded semirings, its ideal theory, topological properties of the graded spectrum etc. has been studied. Not only that graded prime spectrum of a module has also been studied. So motivated by these works, as an extension to our study, the notion of graded prime congruences can be introduced on a graded semiring. The algebraic properties of the graded prime congruences can be investigated and structure space of graded semirings consisting of those congruences can be studied. Also so far as our knowledge is concerned, the area of graded semimodule is still open to explore.

4. In [8] the survey of the studies on fuzzy congruences on semirings, fuzzy prime subsemimodules of a semimodule etc. can be found. Also in [56], fuzzy struc-

ture spaces of semirings and Γ-semirings have been studied. In [64] properties of fuzzy Γ-semimodules have been investigated. Adopting all these concepts mentioned above, one can study the fuzzification of prime congruences and prime ΓS-subsemimodules. Fuzzy prime congruences on semirings as well as Γ-semiring can be defined and their properties can be studied. On the other hand, the notions of the fuzzy prime subsemimodules and fuzzy prime ΓS-subsemimodules can be studied. Further the fuzzy structure spaces of those on the semirings, Γ-semirings as well as the related semimodules can be studied.

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List of Publications Based on the Thesis

A list of publications resulted from the work of this thesis has been appended below.

(1) **Sarbani Mukherjee (Goswami), Soumi Basu and Sujit Kumar Sardar, A study on congruences on a Γ-semiring via its operator semirings, Communicated**.

(2) **Soumi Basu, Sarbani Mukherjee (Goswami) and Kamalika Chakraborty, On the structure space of prime congruences on semirings, *Discussiones Mathematicae - General Algebra and Applications,* 43 (2) (2023) 389-401, DOI: 10.7151/dmgaa.1429**.

(3) **Soumi Basu, Sarbani Mukherjee (Goswami) and Sujit Kumar Sardar, The structure space of $C_-(X)$ via that of Γ-semirings, *Asian-European Journal of Mathematics,* 16 (8) (2023), 2350142 (20 pages), DOI: 10.1142/S1793557123501425**.

(4) **Soumi Basu, Sarbani Mukherjee (Goswami) and Sujit Kumar Sardar, A study on prime ΓS-subsemimodules of a ΓS-semimodule via its associated L-semimodule, Communicated**.

(5) **Soumi Basu, Sarbani Mukherjee (Goswami) and Sujit Kumar Sardar, Structure space of a ΓS-semimodule over a Γ-semiring, To appear in Bulletin of Calcutta Mathematical Society, 116 (5) (2024)**.

Index of Symbols

Subject Index