SOME FURTHER STUDIES ON GENERALIZED CONVERGENCE AND RELATED CONTINUITY IN (L)-GROUPS AND RIESZ SPACES

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CERTIFICATE FROM THE SUPERVISORS

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Sagar Chakraborty.

Date: 28/03/2024
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(Sagar Chakraborty)

To

my parents

Sri Bidyut Chakraborty and Smt. Pramila Chakraborty

for their great interest in my education

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March, 2024 **Sagar Chakraborty**

Abstract

The subject of this thesis is to study different type of Convergence of Sequences and Nets in (l)-groups and Riesz spaces. The tool, we used in our discussions is Summability theory, which is one of the milestones of Real analysis and Topology during the last century.

Throughout the thesis, we shall denote by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}$ the set of all Real numbers, the set of all natural numbers and the set of all integers respectively.

This dissertation contains six chapters. Also, at the end of the thesis, we added one section containing the list of publications and bibliography.

In chapter 3, We consider the notion of generalized density, namely, the natural density of weight g recently introduced and primarily study some sufficient and almost converse necessary conditions for the generalized statistically convergent sequence under which the subsequence is also generalized statistically convergent. Also we consider similar types of results for the case of generalized statistically bounded sequence. Some results are further obtained in a more general form by using the notion of ideals. The entire investigation is performed in the setting of Riesz spaces.

In chapter 4, we introduce the notions of order quasi-Cauchy sequences, downward and upward order quasi-Cauchy sequences, order half Cauchy sequences. Next we consider an associated idea of continuity namely, ward order continuous functions and investigate certain interesting results. The entire investigation is performed in (l)-group setting to extend the recent results in this direction, observed by many authors.

In chapter 5, we introduce the notion of generalized relative order convergence in (l)-groups by using generalized density. We prove some results including a Cauchy-type criterion. Furthermore we present an idea of ideal order convergence of sequences and study some of its properties by using the mathematical tools of the theory of (l)-group.

In chapter 6, we study the concept of statistical order convergence and some generalized statistical convergence in (l)-group by using weighted density and modulus function. We study some sufficient and almost converse necessary conditions for the generalized statistically convergent sequences under which the subsequence is also generalized statistically convergent. Also we consider similar type of results for the case of generalized statistically bounded sequences. The entire investigation is performed in the setting of (l)-group extending the recent results.

In chapter 7, we introduce the ideas of rough convergence and rough ideal convergence of nets in a locally solid Riesz space endowed with a topology τ and investigate some of its consequences.

Almost all the results have already been published in the form of research papers in different journals of international repute.

List of Symbols

- $\mathbb{N}$: The set of natural numbers.
- $\mathbb{Q}$: The set of rational numbers.
- $\mathbb{R}$: The set of real numbers.
- $|A(n)|$ or $\sharp A(n)$: $|A \cap \{1, 2, 3..., n\}|$, $A \subseteq \mathbb{N}$
- $X \setminus A$: The complement of A in X.
- $d(A)$: Density of $A \subseteq \mathbb{N}$
- $d_f(A)$: f-density of $A \subseteq \mathbb{N}$, f being a unbounded modulus function.
- $K_{x'}$: The set $\{n_k : k \in \mathbb{N}\}$, $x' = (x_{n_k})$ is a subsequence of $x = (x_n)$.

For the following notations, let S be an arbitrary infinite set and (R, τ) be a locally solid Riesz space. $f : S \to R$ is a function. Also let $\mathcal{I}$ be an ideal of S.

- $L(f)$: The set of all limit points of f,
- $C_{\mathcal{I}}(f)$: The set of all $\mathcal{I}$-cluster points of f with respect to the ideal $\mathcal{I}$.

Let L be a (l)-group and E be a subset of L. We use the following notations :

- $C[L, L]$ = Set of all order continuous functions on L.
- $WC[L, L]$ = Set of all ward continuous functions on L.
- $UC[L, L]$ = Set of all uniform order continuous functions on L.
- $SWC[E, L]$= Set of all statistical ward continuous functions on E.
- $SOC[E, L]$= Set of all slowly oscillating continuous functions defined on E.

Contents

CHAPTER 1

Introduction

Chapter 1

Introduction

In Real Analysis, Metric space, Topology, one of most important thing is convergence. We can define convergence of sequence (or, Nets) in terms continuous functions. This usual convergence notion was further generalized in several ways by using asymptotic density (statistical convergence), ideal (ideal convergence), modulus function (f-statistical convergence), weighted density function (d_g-statistical convergence), Rough convergence in different spaces.

The term statistical convergence was first introduced by Fast [35] and Schoenberg [59] using natural density for Real numbers as: any sequence (x_n) in $\mathbb{R}$ is statistical convergent to the number L provided that for each $\epsilon > 0$, $\lim_{n\to\infty} \frac{d(k\leq n:|x_k-L|\geq\epsilon)}{n} = 0$. Equivalently, $|x_k - L| < \epsilon$ for almost all k. Its topological consequences were studied by Fridy [37] and $\check{S}al\acute{a}t$ [58].

The notion of ideal convergence was introduced by P. Kostyrko and T. $\check{S}$al$\acute{a}$t [39] in 2000. Using the notion of arbitrary ideal $\mathcal{I}$ of $\mathbb{N}$, $\mathcal{I}$-convergence of sequences is defined as follows: A sequence $(x_n)_{n\in\mathbb{N}}$ of points in a metric space (X, ρ) is said to be $\mathcal{I}$-convergent to l if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : \rho(x_k, l) \geq \varepsilon\} \in \mathcal{I}$.

The study of statistical convergence and its numerous extensions and, in particular, of the ideal convergence and its applications, has been one of the most active areas of research in the summability theory over the last few years.

Throughout this thesis is we will observe different Type of Convergence of Sequences and Nets in (l)-groups and Riesz spaces. Necessary things which are needed in this sequel will be discussed in the subsequent chapters.

CHAPTER 2

Preliminary

Chapter 2

Preliminary

The idea of usual convergence was further extended in several ways by using asymptotic density(statistical convergence), ideal(ideal convergence). In this chapter we will discuss different types od density functions and some spaces related to our dissertation.

2.1 Natural density and statistical convergence

Natural density or Asymptotic density of $A \subseteq \mathbb{N}$ is defined as,

$$d(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$$

where $|A(n)| = |\{k \leq n : k \in A\}| = |A \cap \{1, 2, 3..., n\}|$, the number of elements of A not exceeding n. For example, a finite subset of positive integers has natural density zero.

The notion of statistical convergence which is an extension of the idea of usual convergence, was introduced by H. Fast [35] and I. J. Schoenberg [59]. Any sequence (x_n) in $\mathbb{R}$ is statistical convergent to the number L provided that for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{d(k \leq n : |x_k - L| \geq \epsilon)}{n} = 0.$$

Equivalently, $|x_k - L| < \epsilon$ for almost all k.

Topological consequences of statistical convergence were studied by Fridy [37] and *Šalát* [58]. The study of statistical convergence and its numerous extensions and, in particular, of the ideal convergence and its applications has been one of the most active areas of research in the summability theory over the last 15 years.

2.2 Ideal convergence

The notion of $\mathcal{I}$-convergence of a sequence was introduced by using ideals of the set of positive integers as: $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$, $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \Phi$ and $\mathbb{N} \notin \mathcal{I}$. $\mathcal{I}$ is admissible if it contains all singletons. If $\mathcal{I}$ is a proper non-trivial ideal then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset X : M^c \in \mathcal{I}\}$ is a filter on X where c stand for the complement. It is called the filter associated with the ideal $\mathcal{I}$.

Using the notion of arbitrary ideal $\mathcal{I}$ of $\mathbb{N}$, $\mathcal{I}$-convergence of sequences is defined as follows: A sequence $(x_n)_{n\in\mathbb{N}}$ of points in a metric space (X, ρ) is said to be $\mathcal{I}$-convergent to l if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : \rho(x_k, l) \geq \varepsilon\} \in \mathcal{I}$. $(x_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-Cauchy if for $\varepsilon > 0$ there exists N such that the set $K(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x_N) \geq \varepsilon\} \in \mathcal{I}$. It is obvious that the notions of $\mathcal{I}$-convergence of sequences coincide with the usual convergence if $\mathcal{I} = \mathcal{I}_{fin}$, the ideal consisting of finite sets only. A real sequence (x_n) is said to be $\mathcal{I}$-convergent to l if and only if for each $\epsilon > 0$ the set $\{n \in \mathbb{N} : |x_n - l| \geq \epsilon\} \in \mathcal{I}$. Equivalently it can be written as $\{n \in \mathbb{N} : |x_n - l| < \epsilon\} \in F(\mathcal{I})$.

2.3 Modulus function and f-density

The idea of modulus function was introduced by Hidegoro Nakano in 1953 [50], as concave modular, and further studied by W. H. Ruckle [56], I. J. Maddox [48].

Let $f : [0, \infty) \to [0, \infty)$. f is called modulus function if it satisfies:

i) $f(x) = 0$ if and only if $x = 0$

ii) $f(x+y) \leq f(x) + f(y)$ for every $x, y \in [0, \infty)$

iii) $f(x)$ is increasing

iv) f is continuous from the right at 0.

Some examples of modulus functions are: $f(x) = x$, $x \in [0, \infty)$; $f(x) = \frac{x}{1+x}$, $x \in [0, \infty)$; $f(x) = \log(1+x)$, $x \in [0, \infty)$.

Modulus function is continuous on $[0, \infty)$. Modulus function may be bounded, e.g. $f(x) = \frac{x}{1+x}$, $x \in [0, \infty)$, may be unbounded e.g. $f(x) = x$, $x \in [0, \infty)$.

The notion of f-density (density via modulus function) as an extension of asymptotic density, was introduced by Aizpuru [2] as:

Definition 2.3.1. Let f be an unbounded modulus function. The f-density of a set $A \subseteq \mathbb{N}$ is defined by

$$d_f(A) = \limsup_{n \to \infty} \frac{f(|A(n)|)}{f(n)}$$

in case the limit exists.

Definition 2.3.2. The set $A \subset \mathbb{N}$ is said to be f-dense subset of $\mathbb{N}$ if $d_f(A) = 1$.

Note 2.3.1. We know that, $d(A) = 1 - d(\mathbb{N} \backslash A)$, this relation is true whenever one of the sides exists for any subset A of $\mathbb{N}$. But a similar relation is not true for f-density. If $A \subseteq \mathbb{N}$ and $d_f(A) = 0$ then $d_f(\mathbb{N} \backslash A) = 1 - d_f(A)$. But the converse relation is not true in general. Example 2.1.[2] serves our purpose.

Definition 2.3.3. A sequence (x_n) in R is said to be f-statistically order bounded if there exists an order interval $[a, b]$ such that $d_f\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$.

Definition 2.3.4. Two sequences $(x_n), (y_n) \in R$ are said to be f-statistically equivalent, if there is a f-dense set $A \subset \mathbb{N}$ such that $x_n = y_n$ for every $n \in A$.

Definition 2.3.5. Let $x = (x_n)$ be a sequence in R. If $(n(k))$ is an infinite strictly increasing sequence of natural numbers then $\tilde{x} = (x_{n(k)})$ is called a subsequence of x.

We denote $K_{\tilde{x}} = \{n(k) : k \in \mathbb{N}\}$ i.e. the set of indices of the subsequence $x_{n(k)}$. $\tilde{x}$ is called f-dense subsequence of x if $K_{\tilde{x}}$ is a f-dense subset of $\mathbb{N}$ i.e. $d_f(K_{\tilde{x}}) = 1$.

2.4 Weighted density

The notion of natural density was extended further in [7] as follows: Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined in [3] by the formula: for $A \subset \mathbb{N}$,

$$\overline{d_g}(A) = \limsup_{n\to\infty} \frac{|A(n)|}{g(n)}.$$

Similarly, lower density of weight g is $\underline{d_g}(A) = \liminf_{n\to\infty} \frac{|A(n)|}{g(n)}$. If $\overline{d_g}(A) = \underline{d_g}(A)$, then we say that weighted density of A exists and we denote it by $d_g(A)$. If we take g as identity function in $\mathbb{N}$, then clearly weighted density reduced to natural density.

Let $\mathcal{I}_g = \{A \subset \mathbb{N} : d_g(A) = 0\}$. Now clearly if $\frac{n}{g(n)} \nrightarrow 0$ then $\mathbb{N} \notin \mathcal{I}_g$. So if we assume that $\frac{n}{g(n)} \nrightarrow 0$ then $\mathcal{I}_g$ is an ideal of $\mathbb{N}$.

2.5 Nets

Definition 2.5.1. Let D be a non-empty set and "$\leq$" be a binary relation on D such that "$\leq$" is reflexive, transitive and for any two elements $m, n \in D$ there is an element $p \in D$ with $m \leq p$ and $n \leq p$. The pair $(D, \leq)$ is called a directed set.

Set of Natural number $\mathbb{N}$ is the basic example of directed set.
Any mapping $s : D \to X$ is called a net in X. It is denoted by $\{s_n : n \in D, \leq\}$ or $\{s_n : n \in D\}$ or $\{s_n\}$. The net $\{s_n\}$ is said to be eventually in the set A if there exists an element $n_0 \in D$ such that $s_n \in A$ for all $n \in D$ with $n_0 \leq n$. $\{s_n\}$ is said

to be frequently in the set A if for every element $m \in D$ there is an element $n \in D$ with $m \leq n$ such that $s_n \in A$. Let X be a topological space. A net $\{s_n : n \in D\}$ is said to converge to the element $x_0 \in X$ if $\{s_n : n \in D\}$ is eventually in every neighborhood of x_0 and we write, $s_n \to x_0$. A point $x_0 \in X$ is said to be cluster point of the net $\{s_n : n \in D\}$ if $\{s_n : n \in D\}$ is frequently in every neighborhood of x_0. Again a net $\{t_\alpha : \alpha \in E\}$ is said to be subnet of the net $\{s_n : n \in D\}$ if there is a mapping $i : E \to D$ such that: (a) $t = s \circ i$ and (b) for any $m \in D$ there is an element $\alpha_0 \in E$ with the property $m \leq i(\alpha)$ for all α in E with $\alpha_0 \leq \alpha$.
The following theorem is widely known.

Theorem 2.5.1. Let X be a topological space and $\{s_n : n \in D\}$ be a net in X. A point x_0 in X is a cluster point of the net $\{s_n : n \in D\}$ iff some subnet of $\{s_n : n \in D\}$ converges to x_0.

2.6 Riesz spaces

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by F. Riesz [55] in 1928. Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics (see [4])

Definition 2.6.1. Let L be a real vector space and "$\leq$" be a partial order on this space. L is said to be an ordered vector space if it satisfies the following properties:

(i) If $x, y \in L$ and $y \leq x$ then $y + z \leq x + z$ for each $z \in L$.

(ii) If $x, y \in L$ and $y \leq x$ then $\lambda y \leq \lambda x$ for each $\lambda \geq 0$.

Definition 2.6.2. A nonempty set L is said to a lattice with respect to the partial order $\leq$ if for each pair of elements $x, y \in L$, both the supremum and infimum of the set $\{x, y\}$ exists in L.

We shall write $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. For $x \in L$, we further define $|x| = x \vee (-x)$.

Definition 2.6.3. If L is an ordered vector space as well as a lattice, then we call L a Riesz space or a Vector lattice.

A subset S of a Riesz space L is said to be solid if $y \in S$ and $|x| \leq |y|$ imply that $x \in S$.

Definition 2.6.4. A topological vector space (L, τ) is a real vector space L which has a topology τ, such that, the mappings : $L \times L \longrightarrow L$ and : $\mathbb{R} \times L \longrightarrow L$ defined by $(x, y) \longrightarrow x + y$ and $(a, x) \longrightarrow ax$ are continuous. In this case, the topology is called linear topology.

A topological vector space L is said to be locally solid if τ has a base at zero (local base) consisting of solid sets.

Definition 2.6.5. A locally solid Riesz space (L, τ) is a Riesz space L equipped with a locally solid topology τ on L.

Every linear topology τ on a vector space L has a base N consisting of the neighborhoods of θ (zero) satisfying the following properties:

(a) Each $V \in N$ is a balanced set; that is, $\lambda x \in V$ holds for all $x \in V$ and every $\lambda \in \mathbb{R}$ with $| \lambda | \leq 1$.

(b) Each $V \in N$ is an absorbing set; that is, for every $x \in L$, there exists a $\lambda > 0$ such that $\lambda x \in V$.

(c) For each $V \in N$ there exists some $W \in N$ with $W + W \subseteq V$

In the sequel, by the symbol N_{sol} we will denote base at zero consisting of solid sets and satisfying the properties (a), (b) and (c) in a locally solid topology. Throughout the discussion, (L, τ) denotes a locally solid Riesz space and $\tilde{L}$ denotes the set of all sequences of points from L and D be an arbitrary directed set.

2.7 Lattice order group

Now we recall some concepts related to lattice, order convergence and lattice order group. A nonempty set L is said to be a lattice with respect to the partial order $\leq$ if for each pair of elements $x, y \in L$, both the supremum and infimum of the set $\{x, y\}$ exists in L. We shall write $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$.

Definition 2.7.1. An abelian group $(L, +)$ is said to be an (l)-group if it is lattice and $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in L$.

From now throughout this paper we will write L for (l)-group $(L, +)$ and θ denotes the identity element of the (l)-group $(L, +)$.

Let $x \in L$ be any element, we define $|x| = x \vee (-x)$ where $-x$ denotes the additive inverse of x. Also we use the notation $a \geq b$ equivalent as $b \leq a$, and $a > b$ as equivalent to $b \leq a$ with $b \neq a$.

A sequence (x_n) in L (i.e. a map : $\mathbb{N} \to L$) is said to be increasing (or, decreasing) if $x_1 \leq x_2 \leq ...$ (or, $x_1 \geq x_2 \geq ...$) and we write it symbolically as $x_n \uparrow$ (or $x_n \downarrow$).

A sequence (p_n) is called an order sequence if $p_n \downarrow$ and $\inf p_n = \theta$. In this case we write $p_n \downarrow \theta$. Some author use the term monotone sequences instead of order sequence. It is easy to observe that if (a_n) and (b_n) are two order sequences then the sequence $(a_n + b_n)$ is also an order sequence.

A sequence (x_n) in L is said to be order bounded if there exists an order interval $[a, b]$ such that $a \leq x_n \leq b$ for all $n \in \mathbb{N}$.

A sequence (a_n) in L is said to be convergent in order (or order convergence) to $a \in L$ if there exists an order sequence (p_n) such that $|a_n - a| \leq p_n$ holds for all n. We write it symbolically as $a_n \xrightarrow{ord} a$.

In the literature, there are two ways to define order convergence. Other than the above way one can define order convergence as: A sequence (a_n) in L is said to be order convergence to $a \in L$ if there exists an order sequence (p_n) such that for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|a_n - a| \leq p_{n_0}$ for all $n \geq m$.

The later definition is useful for defining order convergence in filter. The first definition is called 1-converging and the second one is called 2-converging. If the lattice is Dedekind complete then the two definitions are equivalent. Throughout the paper we use the second type definition of order convergence.

If a sequence (x_n) is order convergent to x_0 then we call the sequence $(x_n - x_0)$ as a null order sequence which converges to θ.

CHAPTER 3

Generalized statistical convergence in Riesz Spaces

Chapter 3

Generalized statistical convergence in Riesz Spaces

3.1 Introduction

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by F. Riesz [55] in 1928. Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics (see [4]).

Recall that a topology on a vector space that makes the operations of addition and scalar multiplication continuous is said to be a linear topology. A vector space equipped with a linear topology is called a topological vector space. A Riesz space equipped with a linear topology that has a base at zero consisting of solid sets is called a locally solid Riesz space.

The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [35] and Schoenberg [59] and its topological consequences were studied by Fridy [37] and $\check{S}al\acute{a}t$ [58]. The study of statistical convergence and its numerous extensions and, in particular, of the ideal convergence and its applications, has been one of the most active areas of research in the summability theory over the last 15 years. Subsequently, in a very recent development, the idea of statistical convergence of sequences was studied by Albayrak and Pehlivan [3] in locally solid Riesz spaces.

Naturally, it seems likely that the investigations of these generalized methods of convergence may provide a natural foundation for upbuilding of various tangent spaces to general metric spaces. The construction of tangent spaces in [10, 11, 29, 30, 31] is primarily based on the fundamental fact that, for a convergent sequence (x_n) in a metric space, each its subsequence (x_{n_k}) is also convergent. However, this is generally not true for the generalized methods of convergence mentioned above.

Very recently (see [42]), following the line of investigation in [47], the conditions have been studied for the density of a subsequence of a statistically bounded and also statistically convergent sequence under which the indicated subsequence is also statistically bounded or statistically convergent in the setting of metric space.

As a natural consequence, in [27] Das and Savas investigated the similar type problem that are proposed in [41, 42] for metric valued sequences by considering the notion of natural density of weight g, which was introduced in [3].

In the present chapter we continue the investigation proposed in [42] and study similar type problems for Riesz space-valued sequences by considering the notion of natural density of weight g as also the notion of f-density (introduced and studied in [3]).

3.2 Results for f-density

The idea of modulus function was introduced by Nakano in 1953 [50]. He used the term concave modular, and defined it on semi-ordered linear space. Several consequences were rather studied by Ruckle [56], Maddox [48] etc.

Let $f : [0, \infty) \to [0, \infty)$. f is called <u>modulus function</u> if it satisfies:

i) $f(x) = 0$ if and only if $x = 0$

ii) $f(x+y) \leq f(x) + f(y)$ for every $x, y \in [0, \infty)$

iii) $f(x)$ is increasing

iv) f is continuous from the right at 0.

Some examples are:

i) $f(x) = x$, $x \in [0, \infty)$

ii) $f(x) = \frac{x}{1+x}$, $x \in [0, \infty)$

iii) $f(x) = \log(1+x)$, $x \in [0, \infty)$

iv) $f(x) = x^p$ with $0 < p \leq 1$, $x \in [0, \infty)$.

From the condition that satisfied by a modulus function, it is clear that modulus function must be continuous on $\mathbb{R}^+$. Using f-density we now introduce a version of statistical convergence in locally solid Riesz space which we call it f-statistical convergence as in [3]. Throughout this section f denotes unbounded modulus function.

The notion of f-density (density function via modulus function) was introduced by Aizpuru [2] as follows:

Definition 3.2.1. Let f be an unbounded modulus function. The f-density of a set $A \subseteq \mathbb{N}$ is defined by

$$d_f(A) = \limsup_{n\to\infty} \frac{f(\mid A(n) \mid)}{f(n)}$$

in case the limit exists.

Definition 3.2.2. If $d_f(A) = 1$, then we say the set $A \subset \mathbb{N}$ is f-dense subset of $\mathbb{N}$.

Note 3.2.1. For density function it is clear that $d(A) = 1 - d(\mathbb{N}\backslash A)$, whenever one of the sides exists. If $A \subseteq \mathbb{N}$ and $d_f(A) = 0$ then $d_f(\mathbb{N}\backslash A) = 1 - d_f(A)$. On the other cases the relation is not true [2].

Definition 3.2.3. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L. Then we will say that (x_n) is f-statistical convergent to x_0 and write f-st lim $x_n = x$ if for any τ-neighborhood U of zero we have $d_f(\{n \in \mathbb{N} : x_n - x_0 \notin U\}) = 0$.
i.e.

$$\limsup_{n\to\infty} \frac{f(|\{n \in \mathbb{N} : x_n - x_0 \notin U\}|)}{f(n)} = 0.$$

Note 3.2.2. f-statistical convergence coincides with statistical convergence when we take the modulus function as the identity mapping.

Definition 3.2.4. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L. $\tilde{x}$ is called bounded if for every $x \in L$ there is a τ-neighborhood U of zero such that $x_n - x \in U$ for all n.

Definition 3.2.5. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L. $\tilde{x}$ is called f-statistically bounded if for every $x \in L$ there is a τ-neighborhood U of zero such that

$$\limsup_{n\to\infty} \frac{f(|\{k : k \leq n, x_n - x \notin U\}|)}{f(n)} = 0.$$

Note 3.2.3. It is clear from the definition that, a sequence $\tilde{x} = (x_n)$ in L is f-statistically bounded if and only if the sequence $(x_n - x)$ is f-statistical bounded for an arbitrary $x \in L$.

Definition 3.2.6. Two sequences $\tilde{x} = (x_n) \in \tilde{L}$ and $\tilde{y} = (y_n) \in \tilde{L}$ are f-statistically equivalent, if there is a f-dense set $A \subset \mathbb{N}$ such that $x_n = y_n$ for every $n \in A$.

Definition 3.2.7. Let $\tilde{x} = (x_n) \in \tilde{L}$. If (n_k) is an infinite strictly increasing sequence of natural number, then $\tilde{x}' = (x_{n_k})$ is called a subsequence of $\tilde{x}$. Let $K_{\tilde{x}'} = \{n_k : k \in \mathbb{N}\}$. $\tilde{x}'$ is called f-dense subsequence of $\tilde{x}$ if $K_{\tilde{x}'}$ is a f-dense subset of $\mathbb{N}$, i.e. $d_f(K_{\tilde{x}'}) = 1$.

We know that for any metric space if a sequence is convergent then it is bounded. We now show that similar type relation holds for f-statistical convergence.

Theorem 3.2.1. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L. The following statements hold:

(i) If $\tilde{x}$ is bounded then $\tilde{x}$ is f-statistically bounded.

(ii) If $\tilde{x}$ is f-statistically convergent to $x_0 \in L$ then $\tilde{x}$ is f-statistically bounded.

Proof. (i) Let $\tilde{x}$ is bounded. Then for every $x \in L$, there exists a τ-neighborhood U of zero such that $x_n - x \in U$ for all n. So, $|\{k : k \leq n, x_n - x \notin U\}| = 0$. This implies $f(| \{k : k \leq n, x_n - x \notin U\} |) = 0$. Hence $\tilde{x}$ is f-statistical bounded.

(ii) Let f-st lim $x_n = x_0$. For any arbitrary τ-neighborhood U of zero we can choose a τ-neighborhood W of zero such that $\{k : k \leq n,\ x_n - x_0 \notin W\} \subset \{n :\ \ x_n - x_0 \notin U\}$ i.e. $|\{k : k \leq n,\ x_n - x_0 \notin W\}| \leq |\{n :\ \ x_n - x_0 \notin U\}|$. As given $\tilde{x}$ is f-statistical convergent and f is unbounded modulus function so, $\limsup_{n\to\infty} \frac{f(|\{k : k \leq n, x_k - x_0 \notin W\}|)}{f(n)} = 0$. Therefore $\tilde{x}$ is f-statistically bounded. □

Converses of each cases of the theorem above are not true in general. The following example gives support about this.

Example 3.2.1. Consider real line $\mathbb{R}$ with usual metric and consider the sequence $\tilde{x} = (x_n)$ where $x_n = \{(-1)^n\}$. Take $f(x) = x, x \in [0, \infty)$ as a modulus function. Clearly this sequence $\tilde{x}$ is f-statistically bounded but it is not f-statistically convergent.

Example 3.2.2. Consider the sequence $\tilde{x} = (x_n)$ in real line $\mathbb{R}$ with usual metric where $x_n = 0$ if $n \neq 10^k$ and $x_n = k$ if $n = 10^k$. Also consider $f(x) = x, x \in [0, \infty)$ as a modulus function. Then $\tilde{x}$ is f-statistically bounded which is not a bounded sequence.

Theorem 3.2.2. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n)$ be f-statistical convergent sequence. Also let $\tilde{x}' = (x_{n_k})$ be any subsequence of $\tilde{x}$. Then $\tilde{x}'$ is f-statistically bounded.

Proof. Let $\tilde{x} = (x_n)$ be f-statistically converges to x_0. Then obviously $\tilde{x}$ is f-statistically bounded by above theorem. It is clear that for any τ neighborhood U

of zero, $\{n_k : n_k \leq n,\ x_{n_k} - x_0 \notin U\} \subset \{k : k \leq n,\ x_k - x_0 \notin U\}$. Hence $|\{n_k : n_k \leq n,\ x_{n_k} - x_0 \notin U\}| \leq |\{k : k \leq n,\ x_k - x_0 \notin U\}|$. As $\tilde{x}$ is f-statistically bounded so for any unbounded modulus function f, we have $0 \leq \limsup_{n\to\infty} \frac{f(|\{n_k : n_k \leq n, x_{n_k} - x_0 \in U\}|)}{f(n)} \leq 0$. That is $\tilde{x}'$ is f-statistically bounded. □

We know that, every subsequence of a convergent sequence is also convergent in a metric space. But this is not generally true for statistical convergence. In the articles [42, 47] some investigation done that under what condition subsequence of a statistical convergent sequence is also convergent. Following this line of investigation some work done in [27]. We prove the next results in this direction.

Theorem 3.2.3. Let (L, τ) be a locally solid Riesz space, let $\tilde{x} = (x_n) \in \tilde{L}$ and let $\tilde{x}' = (x_{n_k})$ be a subsequence of $\tilde{x}$ such that $\liminf_{n\to\infty} \frac{f(|K_{\tilde{x}'}(n)|)}{f(n)} > 0$. If $\tilde{x}$ is f-statistically convergent to $x_0 \in L$ then $\tilde{x}'$ is also f-statistically convergent to x_0.

Proof. Suppose that $\tilde{x}$ is f-statistically convergent to x_0. Let U be a τ-neighborhood of zero. Then from definition

$$\limsup_{n\to\infty} \frac{f(|\{m : m \leq n, x_n - x_0 \notin U\}|)}{f(n)} = 0.$$

In order to prove that $\tilde{x}'$ is f-statistically convergent to x_0 we have to show that

$$\limsup_{n\to\infty} \frac{f(|\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\}|)}{f(|K_{\tilde{x}'}(n)|)} = 0,$$

where $K_{\tilde{x}'}(n) = K_{\tilde{x}'} \cap \{1, 2, 3, ..., n\}$.

Then clearly, $\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\} \subseteq \{m : m \leq n, x_m - x_0 \notin U\}$. Thus we can write

$$\frac{f(|\{n_k : n_k \leq n, x_{n_k} - x_0 \notin U\}|)}{f(|K_{\tilde{x}'}(n)|)} \leq \frac{f(|\{m : m \leq n, x_m - x_0 \notin U\}|)}{f(|K_{\tilde{x}'}(n)|)}$$

We know for any two sequences (α_n) and (β_n) of nonnegative real numbers with $0 \neq \liminf_{n\to\infty} \alpha_n < \infty$, we have $(\liminf_{n\to\infty} \alpha_n)(\limsup_{n\to\infty} \beta_n) \leq \limsup_{n\to\infty} \alpha_n \beta_n$.

Taking $\alpha_n = \frac{f(|K_{\tilde{x}'}(n)|)}{f(n)}$, $\beta_n = \frac{f(\{m:\ m\leq n,\ x_m - x_0 \notin U\})}{f(|K_{\tilde{x}'}(n)|)}$ then, $\alpha_n\beta_n = \frac{f(\{m:\ m\leq n,\ x_m - x_0 \notin U\})}{f(n)}$. Therefore it follows that,

$$(\liminf_{n\to\infty}\frac{f(|K_{\tilde{x}'}(n)|)}{f(n)})(\limsup_{n\to\infty}\frac{f(|\{m : m\leq n, x_m - x_0 \notin U\}|)}{f(|K_{\tilde{x}'}(n)|)})$$
$$\leq \limsup_{n\to\infty}\frac{f(|\{m : m\leq n, x_m - x_0 \notin U\}|)}{f(n)}.$$

Since $\tilde{x}$ is f-statistically convergent to x_0, the right hand side of the above inequality is zero. Also by our assumption $\liminf_{n\to\infty}\frac{f(|K_{\tilde{x}'}(n)|)}{f(n)} > 0$, hence we find

$$\limsup_{n\to\infty}\frac{f(|\{m : m\leq n, x_m - x_0 \notin U\}|)}{f(|K_{\tilde{x}'}(n)|)} = 0.$$

Then by (1)

$$\limsup_{n\to\infty}\frac{f(|\{n_k : n_k\leq n, x_{n_k} - x_0 \notin U\}|)}{f(|K_{\tilde{x}'}(n)|)} = 0.$$

Hence $\tilde{x}'$ is f-statistically convergent to x_0. □

Theorem 3.2.4. Let (L,τ) be a locally solid Riesz space and let $\tilde{x},\tilde{y}$ be two f-statistically equivalent sequence in L. If K is a subset of $\mathbb{N}$ such that, $0 \leq \limsup_{n\to\infty}\frac{f(n)}{f(|K(n)|)} < +\infty$ and if $\tilde{x}' = (x_{n_k})$ and $\tilde{y}' = (y_{n_k})$ are subsequences of $\tilde{x},\tilde{y}$ respectively such that $K_{\tilde{x}'} = K_{\tilde{y}'} = K$, then $\tilde{x}'$ and $\tilde{y}$ are f-statistically equivalent.

Proof. Given that $\tilde{x}$, $\tilde{y}$ are f-statistically equivalent, so there exists f-dense set $M \subset \mathbb{N}$ such that $x_n = y_n$ for all $n \in \mathbb{M}$. Equivalently, $\frac{f(|\{n\in\mathbb{N}\ :\ x_n\neq y_n\ \ and\ \ n\leq m\}|)}{f(m)} = 0$.

We have to show that $\tilde{x}'$ and $\tilde{y}'$ are f-statistically equivalent, i.e. $x_{n_k} = y_{n_k}$ for all $n_k \in M$ with $d_f(M) = 1$. Equivalently we have to show that $d_f\{n_k : x_{n_k} \neq y_{n_k}\} = 0$. i.e.

$$\limsup_{m\to\infty}\frac{f(|\{n_k \in K : x_{n_k} \neq y_{n_k} and\ n_k \leq m\}|)}{f(|K(m)|)} = 0.$$

Now for any $m \in \mathbb{N}$, we have,

$$\{n_k \in K : x_{n_k} \neq y_{n_k} \ and \ n_k \leq m\} \leq \{n \in \mathbb{N} : x_n \neq y_n and \ n \leq m\}.$$

So for any unbounded modulus function f, we have

$$\begin{aligned} & \limsup_{m\to\infty} \frac{f(|\{n_k \in K : x_{n_k} \neq y_{n_k} \text{ and } n_k \leq m\}|)}{f(|K(m)|)} \\ \leq & \limsup_{m\to\infty} \frac{f(|\{n \in \mathbb{N} : x_n \neq y_n \text{ and } n \leq m\}|)}{f(|K(m)|)} \\ \leq & \limsup_{m\to\infty} \frac{f(|\{n \in \mathbb{N} \ : \ x_n \neq y_n \text{ and } n \leq m\}|)}{f(m)} \limsup_{m\to\infty} \frac{f(m)}{f(|K(m)|)} \end{aligned}$$

[We know for any two bounded sequences of non-negative reals (u_n) and (v_n), $\limsup(u_n v_n) \leq (\limsup u_n)(\limsup v_n)$ holds].

Now using the given assumption that $\tilde{x}, \tilde{y}$ are two f-statistically equivalent sequences in L and
$0 \leq \limsup_{n\to\infty} \dfrac{f(n)}{f(|K(n)|)} < +\infty$, we get

$$\limsup_{m\to\infty} \frac{f(|\{n_k \in K : x_{n_k} \neq y_{n_k} and \ n_k \leq m\}|)}{f(|K(m)|} = 0.$$

Hence $\tilde{x}'$ and $\tilde{y}$ are f-statistically equivalent. □

3.3 Results for weighted density

In [7], the notion of natural density was further extended as follows: Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined in [7] by the formula $\overline{d_g}(A) = \limsup_{n\to\infty} \dfrac{\text{card}(A \cap [1, n])}{g(n)}$ for $A \subset \mathbb{N}$. Let $\mathcal{I}_g = \{A \subset \mathbb{N} : \overline{d_g}(A) = 0\}$. Then $\mathcal{I}_g$ is an ideal of $\mathbb{N}$. Also $\mathbb{N} \in \mathcal{I}_g$ if and only if $\frac{n}{g(n)} \to 0$. Hence we assume that $\frac{n}{g(n)} \nrightarrow 0$. Hence $\mathbb{N} \notin \mathcal{I}_g$, and it was observed in [7] that $\mathcal{I}_g$ is a proper admissible P-ideal of $\mathbb{N}$. The collection of all functions g of this kind satisfying the above mentioned properties is denoted by G. As a natural consequence we can introduce the following definitions:

Definition 3.3.1. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L. $\tilde{x}$ is called bounded if for every $x \in L$ there is a τ-neighborhood U of zero such that $x_n - x \in U$ for all n.

Definition 3.3.2. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L. $\tilde{x}$ is called d_g-statistically bounded if for every $x \in L$ there is a τ-neighborhood U of zero such that

$$\limsup_{n\to\infty} \frac{|\{k : k \leq n, x_n - x \notin U\}|}{g(n)} = 0.$$

Definition 3.3.3. Let (L, τ) be a locally solid Riesz space and $(x_n) \in \tilde{L}$. Then (x_n) is said to be d_g-statistically convergent to $x_0 \in L$ if for any τ-neighborhood U of zero we have $d_g(A_U) = 0$, where $A_U = \{n \in \mathbb{N} : x_n - x_0 \notin U\}$.

In what follows, we present several more basic definitions required throughout the paper.

Definition 3.3.4. [26] A set $K \subset \mathbb{N}$ is called d_g-dense subset of $\mathbb{N}$ if $d_g(K^c) = 0$.

Definition 3.3.5. Let (L, τ) be a locally solid Riesz space. A sequence $(x_n) \in \tilde{L}$ is said to be $\mathcal{I}(\tau)$-convergent to an element $x_0 \in L$ if for each τ-neighborhood U of zero $\{k \in \mathbb{N} : x_k - x_0 \notin U\} \in \mathcal{I}$.

Definition 3.3.6. [27] A set $K \subset \mathbb{N}$ is called $\mathcal{I}$-dense subset of $\mathbb{N}$ if $K \in \mathcal{F}(\mathcal{I})$.

Definition 3.3.7. [27] If $(n(k))$ is an infinite strictly increasing sequence of natural numbers and $\tilde{x} = (x_n) \in \tilde{L}$, then we write $\tilde{x}' = (x_{n(k)})$ and $K_{\tilde{x}'} = \{n(k) : k \in \mathbb{N}\}$. A subsequence $\tilde{x}'$ is called an $\mathcal{I}$-dense subsequence of $\tilde{x}$, if $K_{\tilde{x}'}$ is an $\mathcal{I}$-dense subset of $\mathbb{N}$.

Definition 3.3.8. [27] Two sequences $\tilde{x} = (x_n) \in \tilde{L}$ and $\tilde{y} = (y_n) \in \tilde{L}$ are $\mathcal{I}$-equivalent,$\tilde{x} \asymp \tilde{y}$, if there is an $\mathcal{I}$-dense set $M \subset \mathbb{N}$ such that $x_n = y_n$ for every $n \in M$.

The following definitions are special case of the last two definitions:

Definition 3.3.9. If $(n(k))$ is an infinite strictly increasing sequence of natural numbers and $\tilde{x} = (x_n) \in \tilde{L}$, then we write $\tilde{x}' = (x_{n(k)})$ and $K_{\tilde{x}'} = \{n(k) : k \in \mathbb{N}\}$. A subsequence $\tilde{x}'$ is called an d_g-dense subsequence of $\tilde{x}$, if $K_{\tilde{x}'}$ is an d_g-dense subset of $\mathbb{N}$.

Definition 3.3.10. Two sequences $\tilde{x} = (x_n) \in \tilde{L}$ and $\tilde{y} = (y_n) \in \tilde{L}$ are d_g-statistically equivalent,$\tilde{x} \asymp \tilde{y}$($d_g$-statistically), if there is an d_g-dense set $M \subset \mathbb{N}$ such that $x_n = y_n$ for every $n \in M$.

The first result shows that there is a one-to-one correspondence between topologies on L and the subsets of $\tilde{L}$ consisting of all $\mathcal{I}$-convergent net for certain special types of ideals. The result is in line of Theorem 3.1 in [27].

Theorem 3.3.1. Let (L, τ_1) and (L, τ_2) be two locally solid Riesz spaces. Let $\mathcal{I}$ be a DP-ideal, which is not maximal. Then the following statement are equivalent:
(i) The set of all $\tau_1 - \mathcal{I}$-convergent nets coincides with the set of all $\tau_2 - \mathcal{I}$-convergent nets.
(ii) The set of all nets convergent in (L, τ_1) coincides with the set of all nets convergent in (L, τ_2).
(iii) The topologies τ_1 and τ_2 are homeomorphic on L.

Proof. $(ii) \Leftrightarrow (iii)$: The result is well known.
$(ii) \rightarrow (i)$: Let $\tilde{x} = (x_n)$ be $\tau_1 - \mathcal{I}$-convergent. Since $\mathcal{I}$ is a DP-ideal, we conclude that $\tilde{x}$ is $\tau_1 - \mathcal{I}^*$ convergent i.e. there is a set $M \in \mathcal{F}(\mathcal{I})$ such that $(\tilde{x})_M$ is τ_1-convergent [44]. By (ii) $(\tilde{x})_M$ is τ_2-convergent, and hence $\tilde{x}$ is $\tau_2 - \mathcal{I}^*$ convergent, which evidently implies that $\tilde{x}$ is $\tau_2 - \mathcal{I}$ convergent.
$(i) \rightarrow (iii)$: Assume that (i) holds. If possible suppose that the topologies are distinct. Then there exists $x_0 \in L$ and a τ_1-neighborhood U_0 of zero such that $\{x \in L : x - x_0 \in U_0\} \not\supseteq \{x \in L : x - x_0 \in U_2\}$ for all τ_2-neighborhood U_2 of zero or the opposite inclusion. Without loss of any generality we can assume that the first one holds. For any $n \in D$ we can choose $x_n \in L$ and a neighborhood U_n of zero such that $x_n - x_0 \in U_n$ and $x_n - x_0 \notin U_0$ for each $n \in D$. We choose a set $K \subset D$

such that $K \notin \mathcal{I}$ as well as $K^c \notin \mathcal{I}$ (because $\mathcal{I}$ is not maximal). Further we define a net $\tilde{y} = (y_n) \in \tilde{L}$ by

$$y_n = \begin{cases} x_n & \text{if } n \in K \\ x_0 & \text{if } n \notin K. \end{cases}$$

Clearly $\{n \in \mathbb{N} : y_n - x_0 \in U_0\} = K \notin \mathcal{I}$. We now observe that the net $\tilde{y} = (y_n)$ converges to x_0 in (L, τ_2) and therefore is $\tau_2{-}\mathcal{I}$-convergent. By virtue of (i), $\tilde{y} = (y_n)$ is also $\tau_1 - \mathcal{I}$-convergent. Note that $\tilde{y}$ must be $\tau_1 - \mathcal{I}$-convergent to x_0 because, otherwise if $\tilde{y}$ is $\tau_1 - \mathcal{I}$-convergent to $y_0 \neq x_0$, then taking a τ_1-neighborhood U' of zero with $x_0 - y_0 \notin U'$, we obtain that $\{n : y_n - y_0 \notin U'\} \supset K^c$. Since $K^c \notin \mathcal{I}$, we get $\{n : y_n - y_0 \notin U'\} \notin \mathcal{I}$. Which contradicts the fact that $\tilde{y} = (y_n)$ is $\tau_1 - \mathcal{I}$-convergent to y_0. However if $\tilde{y}$ is $\tau_1 - \mathcal{I}$-convergent to x_0 then we must have $\{n : y_n - x_0 \in U_0\} = K \in \mathcal{I}$. Which contradicts the fact $K \notin \mathcal{I}$. Thus $(i) \to (iii)$ holds. □

If the given sequence is d_g-statistical convergent, it is natural to ask how we can check that its subsequence is d_g-statistical convergent to the same limit. Also it is natural to ask when the converse assertion is true. We prove the next results in this direction, as also in the case of d_g-statistical bounded sequences.

Theorem 3.3.2. Let (L, τ) be a locally solid Riesz space, let $\tilde{x} = (x_n) \in \tilde{L}$ and let $\tilde{x}' = (x_{n(k)})$ be a subsequence of $\tilde{x}$ such that $\liminf_{n\to\infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} > 0$. If $\tilde{x}$ is d_g-statistically convergent to $x_0 \in L$, then $\tilde{x}'$ is also d_g-statistically convergent to x_0.

Proof. Suppose that $\tilde{x}$ is d_g-statistically convergent to x_0. Let U be a τ-neighborhood of zero. Then clearly $\{n(k) : n(k) \leq n, x_{n(k)} - x_0 \notin U\} \subseteq \{m : m \leq n, x_m - x_0 \notin U\}$. Thus we can write

$$\frac{1}{g(|K_{\tilde{x}'}(n)|)} |\{n(k) : n(k) \leq n, x_{n(k)} - x_0 \notin U\}| \leq \frac{|\{m : m \leq n, x_m - x_0 \notin U\}|}{g(|K_{\tilde{x}'}(n)|)}.$$

In order to prove that $\tilde{x}'$ is d_g-statistically convergent to x_0 we have to show that

$$\limsup_{n\to\infty}\frac{|\{n(k) : n(k) \leq n, x_{n(k)} - x_0 \notin U\}|}{g(|K_{\tilde{x}'}(n)|)} = 0.$$

Now we know for any two sequences (α_n) and (β_n) of nonnegative real numbers with $0 \neq \liminf_{n\to\infty}\alpha_n < \infty$, we have $(\liminf_{n\to\infty}\alpha_n)(\limsup_{n\to\infty}\beta_n) \leq \limsup_{n\to\infty}\alpha_n\beta_n$. Take $\alpha_n = \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)}$ and $\beta_n = \frac{\{m:m\leq n, x_m - x_0\notin U\}}{g(|K_{\tilde{x}'}(n)|)}$, then $\alpha_n\beta_n = \frac{\{m:m\leq n, x_m - x_0\notin U\}}{g(n)}$. Therefore it follows that

$$\begin{aligned}(\liminf_{n\to\infty}\frac{g(|K_{\tilde{x}'}(n)|)}{g(n)})(\limsup_{n\to\infty}\frac{\{m : m \leq n, x_m - x_0 \notin U\}}{g(|K_{\tilde{x}'}(n)|)}) \\ \leq \limsup_{n\to\infty}\frac{\{m : m \leq n, x_m - x_0 \notin U\}}{g(n)}.\end{aligned}$$

Since $\tilde{x}$ is d_g-statistically convergent to x_0, the right hand side of the above inequality is zero. Also by our assumption $\liminf_{n\to\infty}\frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} > 0$, hence we find

$$\limsup_{n\to\infty}\frac{\{m : m \leq n, x_m - x_0 \notin U\}}{g(|K_{\tilde{x}'}(n)|)} = 0.$$

Hence $\tilde{x}'$ is d_g-statistically convergent to x_0. □

The relation between bounded sequences and convergent sequences in an arbitrary metric space is known. How will it be for d_g-statistical boundedness and d_g-statistical convergence? The next results will answer this question and give some relations between d_g-statistical bondedness and d_g-statistical convergence of Riesz space-valued sequences.

Theorem 3.3.3. Let (L, τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be a sequence in L. Then the following statements hold:

(i) If $\tilde{x}$ is bounded, then $\tilde{x}$ is d_g-statistically bounded.

(ii) If $\tilde{x}$ is d_g-statistically convergent to $x_0 \in L$, then $\tilde{x}$ is d_g-statistically bounded.

Proof. (i) The proof is trivial.

(ii) For any arbitrary τ-neighborhood U of zero we can choose a τ-neighborhood U_0

of zero such that $\{k : k \leq n,\ x_k - x_0 \notin U_0\} \subset \{k : k \leq n,\ x_k - x_0 \in U\}$ i.e. $|\{k : k \leq n,\ x_k - x_0 \notin U_0\}| \leq |\{k : k \leq n,\ x_k - x_0 \in U\}|$. For this inequality we have $\limsup_{n\to\infty} \frac{|\{k : k \leq n, x_k - x_0 \notin U_0\}|}{g(n)} = 0$. Therefore $\tilde{x}$ is d_g-statistically bounded. □

Note 3.3.1. The converse of (i) and (ii) does not hold generally as can be seen from [41].

Theorem 3.3.4. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n) \in \tilde{L}$. Let $\tilde{x}' = (x_{n_k})$ be a subsequence of $\tilde{x}$ which is dense in (x_n). If $\tilde{x}$ is d_g-statistically bounded then $\tilde{x}'$ is also d_g-statistically bounded.

Proof. Suppose $\tilde{x}$ is d_g-statistically bounded. It is clear that there exists a τ-neighborhood U_0 of zero and $x_0 \in L$ such that $\{n_k : n_k \leq n,\ x_{n_k} - x_0 \notin U_0\} \subset \{k : k \leq n,\ x_k - x_0 \notin U_0\}$. Then since $|\{n_k : n_k \leq n,\ x_{n_k} - x_0 \notin U_0\}| \leq |\{k : k \leq n,\ x_k - x_0 \notin U_0\}|$ we have $0 \leq \limsup_{n\to\infty} \frac{|\{n_k : n_k \leq n, x_{n_k} - x_0 \in U_0\}|}{g(n)} \leq 0$. That is $\tilde{x}'$ is d_g-statistically bounded. □

Theorem 3.3.5. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n) \in \tilde{L}$. Then the following statements are equivalent:

(a) $\tilde{x}$ is d_g-statistically convergent;

(b) Every subsequence $\tilde{x}'$ of $\tilde{x}$ with $\liminf_{n\to\infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} > 0$ is d_g-statistically convergent;

(c) Every d_g-statistically dense subsequence $\tilde{x}'$ of $\tilde{x}$ is d_g-statistically convergent provided that $g \in G$ is such that $0 < \liminf_{n\to\infty} \frac{n}{g(n)} < \infty$.

Proof. $(a) \Rightarrow (b)$ follows from the Theorem 3.2. Since it is obvious that $\tilde{x}$ is a d_g-dense subsequence of itself, we conclude that $(c) \Rightarrow (a)$.

$(b) \Rightarrow (c)$ The proof is similar to the proof of Theorem 3.3 [27]. □

The next results are given in the more general version in terms of ideals.

Lemma 3.3.1. Let (L, τ) be a locally solid Riesz space with $|L| > 2$, let $\tilde{x} = (x_n) \in \tilde{L}$ and let $\tilde{x}' = (x_{n(k)})$ be an infinite subsequence of $\tilde{x}$ such that $K_{\tilde{x}'} \in \mathcal{I}$. Then there exists a sequence $\tilde{y} \in \tilde{L}$ and a subsequence $\tilde{y}'$ of $\tilde{y}$ such that $K_{\tilde{x}'} = K_{\tilde{y}'}$, where $\tilde{y}'$ is not $\mathcal{I}$-convergent provided that $\mathcal{I}$ is not a maximal ideal.

Proof. Choose a and b be two disjoint elements from L and a subset $M \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and in addition $M \notin \mathcal{F}(\mathcal{I})$. Now let us define a sequence $\tilde{y} = (y_n) \in \tilde{L}$ as:

$$y_n = \begin{cases} x_n & \text{if } n \in \mathbb{N} \setminus K_{\tilde{x}'} \\ a & \text{if } n = n(k) \in K_{\tilde{x}'}, \text{ where } k \in M \\ b & \text{if } n = n(k) \in K_{\tilde{x}'}, \text{ where } k \notin M. \end{cases}$$

Since $K_{\tilde{x}'} \in \mathcal{I}$, we get $\mathbb{N} \setminus K_{\tilde{x}'} \in \mathcal{F}(\mathcal{I})$, which shows that $\tilde{x} \asymp \tilde{y}$ (ideally). Obviously taking $\tilde{y}' = (y_{n(k)})$ we have $K_{\tilde{x}'} = K_{\tilde{y}'}$. Hence for any $c \in L$ choose a τ-neighborhood U of zero with $(a-c) \vee (b-c) \in U$, we observe that $\{k : y_{n(k)} - c \notin U\} \supset M$ or M^c and thus cannot belong to $\mathcal{I}$. This shows that $\tilde{y}'$ is not $\mathcal{I}$-convergent. □

Lemma 3.3.2. Let (L, τ) be a locally solid Riesz space, let $a \in L$ and $\tilde{x} = (x_n), \tilde{y} = (y_n) \in \tilde{L}$. If $\tilde{x}$ is $\mathcal{I}$-convergent to a and $\tilde{x} \asymp \tilde{y}$ (ideally), then $\tilde{y}$ is also $\mathcal{I}$-convergent to a.

Proof. Since $\tilde{x} \asymp \tilde{y}$ (ideally), there is $M \in \mathcal{F}(\mathcal{I})$ such that $x_n = y_n$ for all $n \in M$. Hence clearly for any τ-neighborhood U of zero $\{n : y_n - a \notin U\} \subset M^c \cup \{n : x_n - a \notin U\}$. Since $\tilde{x}$ is $\mathcal{I}$-convergent to a, $\{n : x_n - a \notin U\} \in \mathcal{I}$. Which implies that $\{n : y_n - a \notin U\} \in \mathcal{I}$ and hence $\tilde{y}$ is also $\mathcal{I}$-convergent to a. □

Using these Lemma 3.3.1 and Lemma 3.3.2 we can formulate the following theorem:

Theorem 3.3.6. Let (L, τ) be a locally solid Riesz space with $|L| > 2$, let $a \in L$ and $\mathcal{I}$ be not maximal. Also let $\tilde{x} = (x_n)$ be $\mathcal{I}$-convergent to a. Then for every infinite subsequence $\tilde{x}'$ of $\tilde{x}$ with $K_{\tilde{x}'} \in \mathcal{I}$ there exists a sequence $\tilde{y}$ in L and a subsequence $\tilde{y}'$ of $\tilde{y}$ such that:

(i) $\tilde{x} \asymp \tilde{y}$ (ideally) and $K_{\tilde{x}'} = K_{\tilde{y}'}$

(ii) $\tilde{y}$ is $\mathcal{I}$-convergent to a.

(iii) $\tilde{y}'$ is not $\mathcal{I}$-convergent.

Lemma 3.3.3. Let (L,τ) be a locally solid Riesz space and $\tilde{x},\tilde{y} \in \tilde{L}$ with $\tilde{x} \asymp \tilde{y}$ (d_g-statistically). If K is a subset of $\mathbb{N}$ such that $\liminf_{n\to\infty} \frac{g(|K(n)|)}{g(n)} > 0$ and if $\tilde{x}' = (x_{n(k)})$ and $\tilde{y}' = (y_{n(k)})$ are subsequences of $\tilde{x},\tilde{y}$ respectively such that $K_{\tilde{x}'} = K_{\tilde{y}'} = K$ then the relation $\tilde{x}' \asymp \tilde{y}(d_g$-statistically) is true.

Proof. The proof is similar to the usual case with some trivial modification and so we omit it. □

Theorem 3.3.7. Let (L,τ) be a locally solid Riesz space and $\tilde{x} = (x_n)$ be d_g-statistically convergent to a. Suppose that $\tilde{x}' = (x_{n(k)})$ is a subsequence of $\tilde{x}$ for which there are $\tilde{y} = (y_n)$ and $\tilde{y}'$ such that (i) $\tilde{x} \asymp \tilde{y}$ (d_g-statistically) and $K_{\tilde{x}'} = K_{\tilde{y}'}$. (ii) $\tilde{y}'$ is not d_g-statistically convergent. Then $\liminf_{n\to\infty} \frac{|K_{\tilde{x}}(n)|}{g(n)} = 0$ provided that $g : \mathbb{N} \to [0,\infty)$ satisfying the inequalities $0 < \liminf_{n\to\infty} \frac{n}{g(n)}$ and $\limsup_{n\to\infty} \frac{n}{g(n)} < \infty$.

Proof. If possible, suppose that $\liminf_{n\to\infty} \frac{|K_{\tilde{x}}(n)|}{g(n)} > 0$. Then

$$\liminf_{n\to\infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} \geq (\liminf_{n\to\infty} \frac{g(|K_{\tilde{x}'}(n)|)}{|K_{\tilde{x}'}(n)|})(\liminf_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{g(n)}) > 0.$$

Let $\tilde{y} \in \tilde{L}$ and $\tilde{y}'$ be a subsequence of $\tilde{y}$ such that (i) and (ii) hold. Then we have $K_{\tilde{x}'} = K_{\tilde{y}'}$ and $\tilde{x} \asymp \tilde{y}$ (d_g-statistically). Thus it follows from Lemma 3.2 that $\tilde{x}' \asymp \tilde{y}'$ (d_g-statistically). Now applying Theorem 3.4 we conclude that $\tilde{x}'$ is d_g-statistically convergent to a. Since $\tilde{x}' \asymp \tilde{y}'$ (d_g-statistically), by Lemma 3.2, $\tilde{y}'$ is also d_g-statistically convergent to a. Which contradicts (ii). Hence the theorem. □

Now we will give some relations between d_g-statistical boundedness with usual boundedness.

Lemma 3.3.4. Let (L,τ) be a locally solid Riesz space and let $\tilde{x} = (x_n) \in \tilde{L}$. Then the sequence $\tilde{x}$ is d_g-statistically bounded if and only if the sequence $(x_n - x)$ is d_g-statistical bounded for an arbitrary $x \in L$.

Proof. It follows from the definition. □

Theorem 3.3.8. Let (L, τ) be a locally solid Riesz space and let $\tilde{x} = (x_n) \in \tilde{L}$ be a d_g-statistically bounded sequence. Then the sequence has at least one bounded subsequence.

Proof. From the Lemma 3.4 the sequence $(x_n - x)$ is d_g-statistical bounded for an arbitrary $x \in L$. Thus there exists a τ-neighborhood U_0 of zero such that $d_g(A) = 1$ and $d_g(B) = 0$. Where, $A = \{k : x_k - x \in U_0\}$ and $B = \{k : x_k - x \notin U_0\}$. Let $k_1 \in \mathbb{N}$ be the minimal element of A and $x_{k_1} - x \in U_0$. Since $d_g(A) = 1$, it can be chosen $k_2 \geq k_1$ such that the minimal element of the set $\{k : k > k_1, k \in A\}$ satisfying $x_{k_2} - x \in U_0$. In the n-th step we can chose $k_n \geq k_{n-1}$ which is the minimal element of the set $\{k : k > k_{n-1}, k \in A\}$ such that $x_k - x \in U_0$. So we obtain a non-decreasing sequence (k_n) such that $\tilde{x}' = (x_{n_k})$ is the subsequence of $\tilde{x}$ satisfying $x_{k_n} - x \in U_0$ for all $k_n \in \mathbb{N}$. This shows that the subsequence $\tilde{x}'$ is bounded. □

Theorem 3.3.9. Let (L, τ) be a locally solid Riesz space. Also let $\tilde{x} = (x_n), \tilde{y} = (y_n) \in \tilde{L}$ and $\tilde{x}$ is d_g-statistically bounded. If $\tilde{x} \asymp \tilde{y}$ (d_g-statistically), then $\tilde{y}$ is also d_g-statistical bounded.

Proof. The proof is parallel to the proof in [41]. □

Note 3.3.2. Let (L, τ) be a locally solid Riesz space and $\tilde{x} \in \tilde{L}$. If every subsequence $\tilde{x}'$ of $\tilde{x}$ with
$\liminf_{n\to\infty} \frac{g(|K_{\tilde{x}'(n)}|)}{g(n)} > 0$ is d_g-statistically bounded then $\tilde{x}$ must be d_g-statistical bounded.

3.4 $\mathcal{I}$-convergence of Riesz space valued functions

In this section we investigate certain aspects of ideal convergence of Riesz space valued functions, in a very general context. We consider the situation when an $\mathcal{I}$-convergent function f from S to a Riesz space L will have a F-subfunction which is

K- convergent to the same limit. Further we also introduce the notion of $\mathcal{I}$-cluster points of Riesz space valued functions and make some observations.

Let S be an arbitrary infinite set and (L, τ) be a Locally solid Riesz space. Let $\mathcal{I}$, K be ideals of S. $F(\mathcal{I})$ denotes filter associated with ideal $\mathcal{I}$. We recall that a topological space (L, τ) is called finitely generated space (Alexandroff space) if any intersection of open subsets of L, is open set. L is finitely generated if and only if each point of L has a smallest neighborhood.

The notion of $\mathcal{I}^K$-convergence was first introduced by Ma$\hat{c}$aj [47].

Definition 3.4.1. Any function $f : S \to L$ is said to be $\mathcal{I}$-convergent to $x \in L$ if for every τ neighborhood U of zero, the set $\{s \in S : f(s) - x \notin U\} \in \mathcal{I}$.

In this case we say $\mathcal{I}_\tau - \lim f = x$.

Definition 3.4.2. Any function $f : S \to L$ is said to be $\mathcal{I}^*_\tau$-convergent to $x \in L$ if there exists set $M \in F(\mathcal{I})$ such that g is defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M. \end{cases}$$

is convergent to x.

In this case we say $\mathcal{I}^*_\tau - \lim f = x$.

Definition 3.4.3. Any function $f : S \to L$ is said to be $\mathcal{I}^K_\tau$-convergent to $x \in L$ if there exists set $M \in F(\mathcal{I})$ such that g is defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in M \\ x & \text{if } s \notin M. \end{cases}$$

is K-convergent to x.

In this case we say $\mathcal{I}^K_\tau - \lim f = x$.

Note 3.4.1. If $S = \mathbb{N}$, then we obtain usual $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}^K$-convergence.

From the above definitions the theorem easily follows.

Theorem 3.4.1. Let $\mathcal{I}$ and K be two ideals of S, and $f : S \to L$ be a function such that $K_\tau - \lim f = x$ then $\mathcal{I}_\tau^K - \lim f = x$.

Definition 3.4.4. Suppose $f : S \to X$ is an arbitrary function. Let $F \subseteq 2^S$. A nonempty function g is defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ x & \text{if } s \notin A. \end{cases}$$

where $A \in F$ and $x \in L$ is called a F-subfunction of f with respect to x.

Theorem 3.4.2. Let $\mathcal{I}$ and K be two ideals on a set S such that $K \subset \mathcal{I}$, and $F \subset 2^S$. Let (L, τ) be Riesz space where the topology τ on L is first countable and not finitely generated. Then the following two conditions are equivalent :
(i) for any $\mathcal{I}$-convergent function $f : S \to L$ has a F-subfunction which is K-convergent to the same limit.
(ii) for any sequence of sets $(A_n)_{n\in\mathbb{N}}$ from $\mathcal{I}$ there exist $A \in F$ such that $A \cap A_n \in K$ for all $n \in \mathbb{N}$.

Proof. $(i) \Rightarrow (ii)$: As L is not finitely generated, so we get $x \in L$, such that there is a sequence of distinct points x_n in X convergent to x. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets from $\mathcal{I}$. We have to prove that there exist $A \in F$ such that $A \cap A_n \in K$ for all $n \in \mathbb{N}$.
Let $U_n = \bigcup_{k=1}^{n} A_k$, then clearly $A_n \subset U_n$ and each U_n belongs to $\mathcal{I}$.
Now we define $f : S \to L$ by

$$f(s) = \begin{cases} x_n & \text{when } s \in U_{n+1} \setminus U_n \\ x & \text{when } s \notin \bigcup_{n\in\mathbb{N}} A_n. \end{cases}$$

Take any τ neighborhood U of zero, then there exists $m \in \mathbb{N}$ such that $x_n - x \in U$ for all $n \geq m$. Hence $\{s \in S : f(s) - x \notin U\} \subset U_{m+1} \in \mathcal{I}$. Hence f is $\mathcal{I}$-convergent to x.

Now from the given condition f has a F-subfunction g which is K-convergent to x. i.e., there is $A \in F$ such that the function g defined by

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ x & \text{if } s \notin A. \end{cases}$$

is K-convergent to x. Which implies $A \cap U_n \in K$ for all n.
Consequently $A \cap A_n \subset A \cap U_n \in K$.

$(ii) \Rightarrow (i)$: Let $f : S \to L$ be a function such that $\mathcal{I}_\tau - \lim f = x$, where $x \in L$. As (L, τ) is first countable space, let U_n be the monotonically decreasing τ-neighborhoods of zero. Now define, $V_n = \{s \in S : f(s) - x \notin U_n\}$, clearly V_n is sequence of sets from $\mathcal{I}$. So from the given condition there exists $A \in F$ such that $A \cap V_n \in K$ for all $n \in \mathbb{N}$. We define g by,

$$g(s) = \begin{cases} f(s) & \text{if } s \in A \\ x & \text{if } s \notin A. \end{cases}$$

Clearly g is F-subfunction of f. We have to show that $K_\tau - \lim g = x$.
Consider any τ-neighborhood U of zero, as U_n is the monotonically decreasing τ-neighborhoods of zero. So we get a $n_0 \in \mathbb{N}$ such that $U_{n_0} \subset U$. Then $\{s \in S : g(s) - x \notin U\} \subset \{s \in S : g(s) - x \notin U_{n_0}\} \subset A \cap V_n \subset K$. Hence g is K-convergent to x. □

Let S be an arbitrary infinite set and (L, τ) be a locally solid Riesz space. Also let $\mathcal{I}$ be an ideal of S.

Definition 3.4.5. Let $f : S \to L$ be a function. $x \in L$ is said to be limit point of f, if for every τ neighborhood U of x , the set $\{s \in S : f(s) \in U\}$ is infinite.

By $L(f)$ we denote the set of all limit points of f.

Definition 3.4.6. $x \in L$ is said to be $\mathcal{I}$-cluster point of a function $f : S \to L$ if for every τ neighborhood U of x, $\{s \in S : f(s) \in U\} \notin \mathcal{I}$.

By $C_{\mathcal{I}}(f)$ we denote the set of all $\mathcal{I}$-cluster points of f with respect to the ideal $\mathcal{I}$.

Definition 3.4.7. A function $f : S \to L$ is said to be $\mathcal{I}$-maximal if for any set $A \subset L$, either $\{s \in S : f(s) \notin A\} \in \mathcal{I}$ or $\{s \in S : f(s) \notin L \setminus A\} \in \mathcal{I}$.

Theorem 3.4.3. Suppose the function $f : S \to L$ be $\mathcal{I}$-maximal. If $x_0 \in C_{\mathcal{I}}(f)$ then f is $\mathcal{I}$-convergent to x_0.

Proof. Let $x_0 \in C_{\mathcal{I}}(f)$. Let U be any τ-neighborhood of x_0. By our assumption f is $\mathcal{I}$-maximal. So either $\{s \in S : f(s) \notin U\} \in \mathcal{I}$ or $\{s \in S : f(s) \notin L \setminus U\} \in \mathcal{I}$. If $\{s \in S : f(s) \notin L \setminus U\} \in \mathcal{I}$ then this implies $\{s \in S : f(s) \in U\} \in \mathcal{I}$. Which contradicts the fact that $x_0 \in C_{\mathcal{I}}(f)$. So $\{s \in S : f(s) \notin U\} \in \mathcal{I}$. Hence f is $\mathcal{I}$-convergent to x_0. $\square$

Theorem 3.4.4. If $f : S \to L$ and $g : S \to L$ be two functions such that $\{s \in S : f(s) \neq g(s)\} \in \mathcal{I}$, then $C_{\mathcal{I}}(f) = C_{\mathcal{I}}(g)$.

Proof. Let $x_0 \in C_{\mathcal{I}}(f)$ then for any τ-neighborhood U of x_0, $\{s \in S : f(s) \in U\} \notin \mathcal{I}$. If possible let $x_0 \notin C_{\mathcal{I}}(g)$ then there exist τ-neighborhood W of x_0 such that $\{s \in S : g(s) \in W\} \in \mathcal{I}$. Now,

$$\{s \in S : f(s) \in W\} \subset \{s \in S : g(s) \in W\} \bigcup \{s \in S : f(s) \neq g(s)\} \in \mathcal{I}.$$

So we get a τ-neighborhood W of x_0, that $\{s \in S : f(s) \in W\} \in \mathcal{I}$. This contradicts the assumption that $x_0 \in C_{\mathcal{I}}(f)$. Hence $x_0 \in C_{\mathcal{I}}(g)$. This implies $C_{\mathcal{I}}(f) \subseteq C_{\mathcal{I}}(g)$. Similarly $C_{\mathcal{I}}(g) \subseteq C_{\mathcal{I}}(f)$. Hence $C_{\mathcal{I}}(f) = C_{\mathcal{I}}(g)$. $\square$

Theorem 3.4.5. Let $f : S \to L$ be a function. Let $x_0 \in L$. Then the following two conditions are equivalent

(i) $x_0 \in C_{\mathcal{I}}(f)$

(ii) $x_0 \in \overline{f(M)}$, for every $M \in F(\mathcal{I})$.

Here $f(M) = \{f(m) : m \in M\}$, $F(\mathcal{I})$ is filter associated with the ideal $\mathcal{I}$ and $\overline{f(M)}$ denotes the closure of $f(m)$.

Proof. $(i) \Rightarrow (ii)$: suppose $x_0 \in C_{\mathcal{I}}(f)$. Let U be any τ-neighborhood of x_0. So $\{s \in S : f(s) \in U\} \notin \mathcal{I}$. This implies for any $M \in F(\mathcal{I})$, $M \not\subset \{s \in S : f(s) \in L \setminus U\}$. Hence there exists $m \in M$ such that $f(m) \in U$. Which implies $U \cap f(M) \neq \phi$, this is true for any arbitrary τ-neighborhood U of x_0. Hence $x_0 \in \overline{f(M)}$, for every $M \in F(\mathcal{I})$.
$(ii) \Rightarrow (i)$: Let $x_0 \in \overline{f(M)}$, for every $M \in F(\mathcal{I})$. If possible let $x_0 \notin C_{\mathcal{I}}(f)$. Then we get a τ-neighborhood V of x_0 such that $\{s \in S : f(s) \in V\} \in \mathcal{I}$. Hence for $M = \{s \in S : f(s) \in L \setminus V\} \in F(\mathcal{I})\}$ and $x_0 \in \overline{f(M)}$. So $V \cap f(M) \neq \phi$. Then there exists $y_0 \in V$ such that $y_0 = f(m_0)$ for some $m_0 \in M \subseteq F(\mathcal{I})$, which implies $y_0 = f(m_0) \in L \setminus V$. We get a contradiction. Hence $x_0 \in C_{\mathcal{I}}(f)$. □

Theorem 3.4.6. Let $f : S \to L$ be a function. Then for any compact subset G of L if $\{s \in S : f(s) \in G\} \notin I$ then $G \cap C_I(f) \neq \phi$.

Proof. If possible let $G \cap C_I(f) = \phi$. Then for every $x \in G$, $x \notin C_I(f)$. So we get a τ-neighborhood U_x of x such that $\{s \in S : f(s) \in U_x\} \in I$. Clearly $\{U_x : x \in G\}$ is an τ-open cover of compact set G hence $\{s \in S : f(s) \in G\} \subset \bigcup\limits_{i=1}^{n} \{s \in S : f(s) \in U_{x_i}\} \in I$. Which contradicts the given assumption. Hence $G \cap C_I(f) \neq \phi$. □

The matter of this chapter is based on the following research paper:

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CHAPTER 4

Summability methods of Nets in locally Solid Riesz Spaces

Chapter 4

Summability methods of Nets in Locally Solid Riesz Spaces

4.1 Introduction

In a very recent development, the idea of statistical convergence of sequences was studied by Albayrak and Pehlivan [3] in locally solid Riesz spaces. However if one considers the concept of nets instead of sequences, which is undoubtedly plays a more important and natural role in general structures like topological spaces, uniform spaces and Riesz spaces. The above approach does not seem to be appropriate because of the absence of any idea of density in arbitrary directed sets. Instead it seems more appropriate to follow the more general approach of [39] where the notion of $\mathcal{I}$-convergence of a sequence was introduced by using ideals of the set of positive integers (recall that $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$, $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \Phi$ and $\mathbb{N} \notin \mathcal{I}$. $\mathcal{I}$ is admissible if it contains all singletons. If $\mathcal{I}$ is a proper non-trivial ideal, then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset X : M^c \in \mathcal{I}\}$ is a filter on X where c stand for the complement. It is called the filter associated with the ideal $\mathcal{I}$). One may consider an arbitrary ideal $\mathcal{I}$ of $\mathbb{N}$ and define $\mathcal{I}$-convergence of sequences as follows:

A sequence $(x_n)_{n \in \mathbb{N}}$ of points in a metric space (X, ρ) is said to be $\mathcal{I}$-convergent to l if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : \rho(x_k, l) \geq \varepsilon\} \in \mathcal{I}$. $(x_n)_{n \in \mathbb{N}}$ is said to be $\mathcal{I}$-Cauchy if for $\varepsilon > 0$ there exists N such that the set $K(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x_N) \geq \varepsilon\} \in \mathcal{I}$. The notions of $\mathcal{I}$-convergence of sequences coincide with the usual convergence if $\mathcal{I} = \mathcal{I}_{fin}$, the ideal consisting of finite sets only. One can see [24, 40, 44, 45] for more works in this direction where many more references can be found.

The idea of rough convergence was first developed by H.X. Phu [53] in 2001 for normed linear spaces as following: Let $x = (x_i)$ be a sequence in a finite dimensional normed space X and r be a nonnegative real number. Suppose that for every $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that $||x_i - y|| < r + \epsilon$ for every $i \geq m$. Then the sequence (x_i) is said to be r-convergent to y. Here we will adopt the definitions and notations in [53]. In this chapter initially we introduce the idea of rough convergence of nets in a locally solid Riesz space and study some of its properties by using the mathematical tools of the theory of topological vector spaces. Further we extend this idea to rough ideal convergence and investigate its basic nature in Locally solid Riesz space.

4.2 Rough convergence of nets

In this section, we introduce the concept of rough convergence of nets in locally solid Riesz space (X, τ), and we study some results. We first introduce our main definition:

Definition 4.2.1. Let (X, τ) be a locally solid Riesz space. Let D be a directed set and $\{s_\alpha : \alpha \in D\}$ be a net in X. $\{s_\alpha : \alpha \in D\}$ is said to be rough convergent (r-convergent) to x_0 (with roughness degree $r \in X$) if for every τ-neighborhood V of 0, there exists a $k \in D$ such that $s_n - x_0 \in \ r + V$ for all $n \geq k$.

We write it symbolically as $s_\alpha \xrightarrow{r} x_0$ in X. We say x_0 is a r-limit of $\{s_\alpha : \alpha \in D\}$, that is, $r\text{-}\lim s_\alpha = s_0$. Clearly for $r = 0$, we get the classical convergency. So we always take $r > 0$.

Example 4.2.1. Let $X = \mathbb{R}$ and we consider usual topology on $\mathbb{R}$. Consider the sequence $s_\alpha = (-1)^\alpha, \alpha \in \mathbb{N}$. Clearly the sequence is non-convergent in classical sense. But if we take $r = 2$, then $s_\alpha \xrightarrow{2} x_0$ where $x_0 \in (-1, 1)$.

Example 4.2.2. [5] Let us consider the sequence $s_\alpha = 0.5 + 2\frac{(-1)^\alpha}{\alpha}$ in $\mathbb{R}$ with usual topology and $\alpha \in \mathbb{N}$. It is obvious that s_α converges to 0.5 in ordinary sense. But s_α cannot be calculated exactly for large α. But it can be rounded to some machine number, i.e., to the nearest one. Let, t_α be a sequence defined as $t_\alpha =$ round $s_\alpha = z$, where z is an integer lies $z - 0.5 \leq s_\alpha < z + 0.5$. Then $t_1 = -1$, $t_2 = 2$, $t_{2j-1} = 0$ and $t_{2j} = 1$ for $j = 2, 3, ...$ Obviously the sequence t_α does not converge in ordinary sense. but $t_\alpha \xrightarrow{r} 0$ for $r < 0.5$ and $t_\alpha \xrightarrow{r} [1-r, r]$ for $0.5 \leq r$.

From the above examples, we see that for r-convergent net, r-limit is not unique. For this, we define r-limit set, we denote it by $\text{LIM}^r S_\alpha$.

Definition 4.2.2. Let $\{s_\alpha : \alpha \in D\}$ be a net in X.Then $\text{LIM}^r s_\alpha := \{x_0 \in X : s_\alpha \xrightarrow{r} x_0\}$.

For example 4.2.1, $\text{LIM}^r s_\alpha = (-1, 1)$.

Note 4.2.1. Clearly $\{s_\alpha\}$ is r-convergent if $\text{LIM}^r_{s_\alpha} \neq \phi$.

Definition 4.2.3. [6] Let (X, τ) be a locally solid Riesz space. A net $\{s_\alpha : \alpha \in D\}$ of X is said to be bounded in X if for each τ-neighborhood V of 0, there exists $p \in \mathbb{R}$ such that $S_\alpha \in pV$ for all $\alpha \in D$.

We can easily prove that,

Theorem 4.2.1. If a Net $\{s_\alpha\}$ in (X, τ) is bounded then *LIM*$^r s_\alpha \neq \phi$ for some $r \in X$.

Theorem 4.2.2. Let (X, τ) be a locally solid Riesz space and $\{s_\alpha : \alpha \in D\}$ be a net in X. If $\text{LIM}^r s_\alpha \neq \phi$ for some $r \in X$ then $\{s_\alpha\}$is bounded.

Proof. If $LIM^r s_\alpha \neq \phi$ for some $r \in X$, then we get atleast one $x_0 \in LIM^r s_\alpha$. Let U be any τ-neighborhood of zero. We always get $V \in N_{sol}$ such that $V + V \subset U$ and for this V we get $W \in N_{sol}$ such that $W + W \subset V$. Now as $x_0 \in LIM^r_{s_\alpha}$ there exists $k \in D$ such that $s_\alpha - x_0 \in r + W$ for all $\alpha \geq k$. Also for $k \geq \alpha$ we get $q \in K$ such that $s_\alpha \in qU$.

Now for $\alpha \geq k,\ s_\alpha = (s_\alpha - x_0) + x_0 \in (r+W) + (x_0+W) = (r+x_0) + (W+W) \subset (r+x_0) + V \subset pV + pV \subset pU$. (We choose $p \in X$ such that $r + x_0 \in pV$ and also $pV \subset V$). So for $\alpha \geq k,\ s_\alpha \in pU$ and for $k \geq \alpha,\ s_\alpha \in qU$ which shows that $s_\alpha \in tU$ for all $\alpha \in D$, where $t = p \vee q$. □

Note 4.2.2. Every r-convergent net in locally solid Riesz space (X, τ) is bounded.

Theorem 4.2.3. Let (X, τ) be a locally solid Riesz space and $\{s_\alpha : \alpha \in D\}$ be a net in X. Then the set *LIM*$^r s_\alpha$ is closed.

Proof. Let U be any τ-neighborhood of zero. Let $\{t_\alpha\}$ be arbitrary net in LIM$^r s_\alpha$ which converges to some t^*. Let U be any τ-neighborhood of zero, we always get $V \in N_{sol}$ such that $V + V \subset U$. As $t_\alpha \to t^*$, so $\{t_\alpha\}$ is eventually in any neighborhood of t^*. i.e. $t_\alpha - t^* \in V$, for all $\alpha \geq k$. Also $s_\alpha - t_\alpha \in r + W$ (we choose W such that $W \subset V$).
Now, for all $\alpha \geq k$, $s_\alpha - t^* = (s_\alpha - t_\alpha) + (t_\alpha - t^*) \in (r+W) + V \subset r + (V+V) \subset r + U$. Which shows that $s_\alpha - t^* \in r + U$, for all $\alpha \geq k$ and hence $s_\alpha \xrightarrow{r} t^*$. So $t^* \in$ LIM$^r s_\alpha$. Thus LIM$^r s_\alpha$ is closed. □

Theorem 4.2.4. Let (X, τ) be a locally solid Riesz space and $\{s_\alpha : \alpha \in D\}$ be a net in X. Then *LIM*$^{r_1} s_\alpha \subseteq$ *LIM*$^{r_2} s_\alpha$ for $r_1 < r_2$.

Proof. Let $x \in$ LIM$^{r_1} s_\alpha$, then for any τ-neighborhood V of zero, $s_\alpha - x \in r_1 + V$, which is clearly subset of $r_2 + V$ as $r_1 < r_2$. So $x \in$ LIM$^{r_2} s_\alpha$. Hence LIM$^{r_1} s_\alpha \subseteq$ LIM$^{r_2} s_\alpha$. □

Theorem 4.2.5. Let (X, τ) be a locally solid Riesz space and $\{s_\alpha : \alpha \in D\}$ be a net in X. Then
(a) $y_0 \in$ *LIM*$^{r_0} s_\alpha$ and $y_1 \in$ *LIM*$^{r_1} s_\alpha$, then $y_\lambda = (1-\lambda) y_0 + \lambda y_1 \in$ *LIM*$^{(1-\lambda) r_0 + \lambda r_1} s_\alpha$, for $\lambda \in [0, 1]$.
(b) The set *LIM*$^r s_\alpha$ is convex.

Proof. : (a) Let $y_0 \in$ LIM$^{r_0} s_\alpha$ and $y_1 \in$ LIM$^{r_1} s_\alpha$. i.e. $s_\alpha \xrightarrow{r_0} y_0$ and $s_\alpha \xrightarrow{r_1} y_1$. For every τ-neighborhood V of zero, there exists k_0 and k_1 such that $s_\alpha - y_0 \in r_0 + V$ for all $\alpha \geq k_0$ and $s_\alpha - y_1 \in r_1 + V$ for all $\alpha \geq k_1$. Take $k = k_0 \vee k_1$.

Now, $y_\lambda = (1-\lambda)y_0 + \lambda y_1$. So $s_\alpha - y_\lambda = s_\alpha - (1-\lambda)y_0 - \lambda y_1 = (1-\lambda)(s_\alpha - y_0) + \lambda(s_\alpha - y_1) \in (1-\lambda)(r_0+V) + \lambda(r_1+V) = (1-\lambda)r_0 + \lambda r_1 + V$. Which shows that $s_\alpha - y_\lambda \in (1-\lambda)r_0 + \lambda r_1 + V$. So $s_\alpha \xrightarrow{(1-\lambda)r_0+\lambda r_1} y_\lambda = (1-\lambda)y_0 + \lambda y_1$. Hence $y_\lambda \in \text{LIM}^{(1-\lambda)r_0+\lambda r_1} s_\alpha$.

(b) If we take $r_0 = r_1 = r$ then $s_\alpha \xrightarrow{r} y_0$ and $s_\alpha \xrightarrow{r} y_1$ and also $s_\alpha \xrightarrow{r} (1-\lambda)y_0 + \lambda y_1$ (From (a)). So If $y_0, y_1 \in \text{LIM}^r s_\alpha$ then $(1-\lambda)y_0 + \lambda y_1 \in \text{LIM}^r s_\alpha$. Hence $\text{LIM}^r s_\alpha$ is convex. □

Theorem 4.2.6. Let (X, τ) be a locally solid Riesz space and $\{s_\alpha\}$, $\{t_\alpha\}$ be two nets in X. If r_1-$\lim s_\alpha = s_0$ and r_2-$\lim t_\alpha = t_0$ then (r_1+r_2)-$\lim(s_\alpha + t_\alpha) = s_0 + t_0$.

Proof. Let U be a τ-neighborhood of zero. Choose $V \in N_{sol}$ such that $V \subset U$ and $W \in N_{sol}$ such that $W + W \subset V$. As r_1-$\lim s_\alpha = s_0$ and r_2-$\lim t_\alpha = t_0$, there exists $m_1, m_2 \in D$ such that $s_\alpha - s_0 \in r_1 + W$ for all $\alpha \geq m_1$ and $t_\alpha - t_0 \in r_2 + W$ for all $\alpha \geq m_2$. Let us take $m = m_1 \vee m_2$, then for all $\alpha \geq m$, $s_\alpha - s_0 \in r_1 + W$ and $t_\alpha - t_0 \in r_2 + W$. Clearly for all $\alpha \geq m$,

$(s_\alpha + t_\alpha) - (s_0 + t_0) = (s_\alpha - s_0) + (t_\alpha - t_0) \in (r_1 + W) + (r_2 + W) = (r_1 + r_2) + W + W \subset (r_1 + r_2) + U$. Hence $(r_1 + r_2)$-$\lim(s_\alpha + t_\alpha) = s_0 + t_0$. □

Theorem 4.2.7. Let $\{s_\alpha\}$, $\{t_\alpha\}$ and $\{v_\alpha\}$ be three nets in X such that $s_\alpha \leq t_\alpha \leq v_\alpha$ for each $\alpha \in D$. If r_1-$\lim s_\alpha = r_2$-$\lim v_\alpha = x_0$ then $(r_1 + r_1)$-$\lim t_\alpha = x_0$.

Proof. Let U be any arbitrary τ-neighborhood of zero. Choose $V, W \in N_{sol}$ such that $W + W \subset V \subset U$. As r_1-$\lim s_\alpha = r_2$-$\lim v_\alpha = x_0$, there exists $m \in D$ such that $s_\alpha - x_0 \in r_1 + W$ for all $\alpha \geq m$ and $v_\alpha - x_0 \in r_2 + W$ for all $\alpha \geq m$. Also $s_\alpha - x_0 \leq t_\alpha - x_0 \leq v_\alpha - x_0$ and so

$|t_\alpha - x_0| \leq |s_\alpha - x_0| + |v_\alpha - x_0| \in (r_1 + W) + (r_2 + W) \subset (r_1 + r_2) + V + W \subset (r_1 + r_2) + V$. Since V is solid $t_\alpha - x_0 \in (r_1 + r_2) + V \subset (r_1 + r_2) + U$. This implies that $(r_1 + r_1)$-$\lim t_\alpha = x_0$. □

Theorem 4.2.8. Suppose $r_1 \geq 0$ and $r_2 > 0$. A net $\{s_\alpha\}$ in X is $(r_1 + r_2)$ convergent to s_0 if there exists a net $\{t_\alpha\}$ in X such that $t_\alpha \xrightarrow{r_1} s_0$ and $s_\alpha - t_\alpha \in r_2 + W$ for some τ-neighborhood W of zero.

Proof. Let $\{t_\alpha\}$ be a net in X such that $t_\alpha \xrightarrow{r_1} s_0$. Let U be any arbitrary τ-neighborhood of zero. Choose $V, W \in N_{sol}$ such that $W + W \subset V \subset U$ and $s_\alpha - t_\alpha \in r_2 + W$, there exists $k \in D$ such that $t_\alpha - s_0 \in r_1 + W$, for all $k \geq \alpha$. Now $s_\alpha - s_0 = s_\alpha - t_\alpha + t_\alpha - s_0 \in (r_1 + r_2) + (W + W) \subset (r_1 + r_2) + U$. Therefore $s_\alpha \xrightarrow{r_1+r_2} s_0$. □

Note 4.2.3. In particular if we take $r_1 = 0$ and $r_2 = r > 0$ then the above theorem simplifies as : A net $\{s_\alpha\}$ in X is r-convergent to s_0 if there exists a net $\{t_\alpha\}$ in X such that $t_\alpha \longrightarrow s_0$ and $s_\alpha - t_\alpha \in r + W$ for some τ-neighborhood W of zero.

4.3 Rough Cauchy Net

Definition 4.3.1. Let (X, τ) be a locally solid Riesz space and $\{s_\alpha : \alpha \in D\}$ be a net in X. $\{s_\alpha : \alpha \in D\}$ is said to be rough Cauchy with roughness degree r if for each τ-neighborhood V of zero there exists $k \in D$ such that $s_\alpha - s_\beta \in r + V$ for all $\alpha, \beta \geq k$.

Theorem 4.3.1. Every r-convergent sequence is always $2r$-Cauchy.

Proof. Let s_α be r-convergent to $s_0 \in X$. Let U be any arbitrary τ-neighborhood of zero. Choose $V \in N_{sol}$ such that $V + V \subset U$. As $s_\alpha \xrightarrow{r} s_0$ there exist $k \in D$ such that $s_\alpha - s_0 \in r + V$ for all $\alpha \geq k$. Now clearly $s_\alpha - s_\beta \in 2r + W$. Hence s_α is $2r$-Cauchy net. □

Definition 4.3.2. For a net $\{s_\alpha\}$ we say that y is an r-cluster point if there exists a subnet $\{t_\alpha\}$ of $\{s_\alpha\}$ such that $t_\alpha \xrightarrow{r} y$.

Theorem 4.3.2. If an r-Cauchy net $\{s_\alpha\}$ has a r-cluster point s^* then $\{s_\alpha\}$ is $2r$-convergent to s^*.

Proof. Let U be a τ-neighborhood of zero. Choose $V, W \in N_{sol}$ such that, $W + W \subset V \subset U$. As s^* is r-cluster point of $\{s_\alpha\}$ there exists a subnet $\{t_\alpha\}$ of $\{s_\alpha\}$ such that $t_\alpha \xrightarrow{r} s^*$. So there exists $k_1 \in D$ such that for all $\alpha \geq k_1$, $t_\alpha - s^* \in r + V$. Also it is given that s_α is r-Cauchy net so there exists $k_2 \in D$ such that $s_\alpha - t_\alpha \in r + W$ for all $\alpha \geq k_2$. Now take $k = k_1 \vee k_2$, then for all $\alpha \geq k$,

$$s_\alpha - x^* = (s_\alpha - t_\alpha) + (t_\alpha - x_*) \in (r + W) + (r + W) \subset 2r + V.$$

Hence $\{s_\alpha\}$ is $2r$-convergent to s^*. □

4.4 Rough ideal convergence of nets

In this section we introduce the concept of rough ideal convergence and rough ideal Cauchy condition of nets in locally solid Riesz space endowed with the topology τ and studied some properties of these concepts. There are some cases where the properties of rough convergence and rough ideal convergence are different. We study some of these properties.

Recently, in [25] Das and Savas introduced the notion of ideal convergnce in locally solid Riesz spaces as follows:

Definition 4.4.1. Let (X, τ) be a locally solid Riesz space. Let D be a directed set and $\{s_\alpha : \alpha \in D\}$ be a net in X. $\{s_\alpha : \alpha \in D\}$ is said to be ideal convergent ($\mathcal{I}_\tau$-convergent) to $s_0 \in X$ if for every τ-neighborhood U of zero, $\{\alpha \in D : s_\alpha - s_0 \notin U\} \in \mathcal{I}$.

We write it symbolically as $s_\alpha \xrightarrow{\mathcal{I}_\tau} s_0$ in X or $\mathcal{I}_\tau\text{-}\lim s_\alpha = s_0$.

Now we give the definitions of rough ideal convergence in locally solid Riesz spaces.

Definition 4.4.2. Let (X, τ) be a locally solid Riesz space. Let $\{s_\alpha : \alpha \in D\}$ be a net in X. $\{s_\alpha : \alpha \in D\}$ is said to be rough ideal convergent (r-$\mathcal{I}_\tau$-convergent) to $s_0 \in X$ if for every τ-neighborhood U of zero, $\{\alpha \in D : s_\alpha - s_0 \notin r + U\} \in \mathcal{I}$ or equivalently $\{\alpha \in D : s_\alpha - s_0 \in r + U\} \in F(\mathcal{I})$.

We write it symbolically as $s_\alpha \xrightarrow{r-\mathcal{I}_\tau} s_0$ in X. We say s_0 is a r-$\mathcal{I}_\tau$ limit of $\{s_\alpha : \alpha \in D\}$, i.e. r-$\mathcal{I}_\tau$-$\lim s_\alpha = s_0$. Clearly for $r = 0$, we get the notion of ideal convergence. So we always take $r > 0$.

Definition 4.4.3. Let (X, τ) be a locally solid Riesz space. Let $\{s_\alpha : \alpha \in D\}$ be a net in X. $\{s_\alpha : \alpha \in D\}$ is said to be $\mathcal{I}(\tau)$-bounded in X if for each τ-neighborhood V of zero there is some $a > 0$, $\{k \in D : as_k \notin V\} \in \mathcal{I}$.

Definition 4.4.4. Let (X, τ) be a locally solid Riesz space. Let $\{s_\alpha : \alpha \in D\}$ be a net in X. Then we define $\mathcal{I} - \mathsf{LIM}^r s_\alpha := \{s_0 \in X : s_\alpha \xrightarrow{r-\mathcal{I}_\tau} s_0\}$.

Clearly $\{s_\alpha\}$ is r-$\mathcal{I}_\tau$ convergent if $\mathcal{I} - \mathsf{LIM}^r s_\alpha \neq \phi$.

Theorem 4.4.1. The r-$\mathcal{I}_\tau$ limit set $\mathcal{I} - \mathsf{LIM}^r s_\alpha$ of a net $\{s_\alpha\}$ is closed set.

Proof. Let $\{t_\alpha\}$ be arbitrary net in $\mathcal{I} - \mathsf{LIM}^r s_\alpha$ which converges to some t_0. Let U be any τ-neighborhood of zero, we always get $V \in N_{sol}$ such that $V + V \subset U$. As $t_\alpha \to t_0$, $\{t_\alpha\}$ is eventually in any neighborhood of t_0. i.e. $t_\alpha - t_0 \in V$ for all $\alpha \geq k$. That is $A = \{\alpha \in D : t_\alpha - t_0 \in V\} \in F(\mathcal{I})$. Also since $t_\alpha \in \mathcal{I} - \mathsf{LIM}^r s_\alpha$, $B = \{\alpha \in D : s_\alpha - t_\alpha \in r + V\} \in F(\mathcal{I})$.

Let $\alpha \in A \cap B$, then $s_\alpha - t_0 = (s_\alpha - t_\alpha) + (t_\alpha - t_0) \in r + V + V \subset r + U$. Now $(A \cap B) \subset \{\alpha \in D : s_\alpha - t_0 \in r + U\} \in F(\mathcal{I})$. Which shows that $t_0 \in \mathcal{I} - \mathsf{LIM}^r s_\alpha$. Hence $\mathcal{I} - \mathsf{LIM}^r s_\alpha$ is closed. □

It can be readily seen that:

Theorem 4.4.2. $\mathcal{I} - \mathsf{LIM}^{r_1} s_\alpha \subseteq \mathcal{I} - \mathsf{LIM}^{r_2} s_\alpha$, for $r_1 < r_2$.

Theorem 4.4.3. r-τ limit set of a net $\{s_\alpha\}$, $\mathcal{I} - \mathsf{LIM}^r s_\alpha$ is convex.

Theorem 4.4.4. $\{s_\alpha\}$, $\{t_\alpha\}$ and $\{v_\alpha\}$ be three nets in X such that $s_\alpha \leq t_\alpha \leq v_\alpha$ for each $\alpha \in D$.

If r_1-$\mathcal{I}_\tau \lim s_\alpha = r_2$–$\mathcal{I}_\tau \lim\, v_\alpha = x_0$ then $(r_1 + r_1)$–$\mathcal{I}_\tau \lim t_\alpha = x_0$.

Proof. Let U be any arbitrary τ-neighborhood of zero. Choose $V, W \in N_{sol}$ such that $W + W \subset V \subset U$. As r_1-$I_\tau \lim s_\alpha = r_2$-$I_\tau \lim\, v_\alpha = x_0$, so,
$A = \{\alpha \in D : s_\alpha - x_0 \in r + W\} \in F(I)$, and $B = \{\alpha \in D : v_\alpha - x_0 \in r + W\} \in F(I)$. Let us choose $\alpha \in A \cap B$, for this α, as $s_\alpha \leq t_\alpha \leq v_\alpha$, this implies $s_\alpha - x_0 \leq t_\alpha - x_0 \leq v_\alpha - x_0$ and so $|t_\alpha - x_0| \leq |s_\alpha - x_0| + |v_\alpha - x_0| \in$

$(r_1 + W) + (r_2 + W) \subset (r_1 + r_2) + V + W \subset (r_1 + r_2) + V$, since V is solid so, $t_\alpha - x_0 \in (r_1 + r_2) + V \subset (r_1 + r_2) + U$. This implies that $(r_1 + r_1)\text{-}\lim t_\alpha = x_0$. □

Theorem 4.4.5. (X, τ) be a locally solid Riesz space and $\{s_\alpha\}$ and $\{t_\alpha\}$ be two nets in X. If $r_1\text{-}\lim s_\alpha = s_0$ and $r_2\text{-}\lim t_\alpha = t_0$, then $(r_1 + r_2)\text{-}\lim(s_\alpha + t_\alpha) = s_0 + t_0$.

Theorem 4.4.6. If a net $\{s_\alpha : \alpha \in D\}$ in a locally solid Riesz space (X, τ) is $r\text{-}\mathcal{I}_\tau$convergent then it is $\mathcal{I}(\tau)$-bounded.

Proof. Let U be any arbitrary τ-neighborhood of zero. Choose $W \in N_{sol}$ such that $W + W \subset U$. Let s_α be $r\text{-}\mathcal{I}_\tau$convergent to $s_0 \in X$. So $A = \{\alpha \in D : s_\alpha - s_0 \in r + W\} \in F(\mathcal{I})$. As $W \in N_{sol}$ so $r + W$ is absorbing set. Let $s_0 \in X$, so there exists $\lambda \in \mathbb{R}$ such that $\lambda s_0 \in r + W$. Now $s_\alpha - s_0 \in r + W$, also $r + W$ balanced set, so $\lambda(s_\alpha - s_0) \in r + W$.
Thus $\lambda s_\alpha = \lambda(s_\alpha - s_0) + \lambda s_0 \in r + W + W \subset r + U$ for $\alpha \in D - A$. Hence $\{\alpha \in D : \lambda s_\alpha \notin r + U\} \subset \mathcal{I}$. This implies $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}(\tau)$-bounded. □

4.5 Rough Ideal Cauchy Nets

Definition 4.5.1. A net $\{s_\alpha : \alpha \in D\}$ in a locally solid Riesz space (X, τ) is said to be $r\text{-}\mathcal{I}_\tau$ Cauchy in X if for each τ-neighborhood U of zero, there exists $\beta \in D$ such that $\{\alpha \in D : s_\alpha - s_\beta \notin r + U\} \in \mathcal{I}$.

Theorem 4.5.1. If a net $\{s_\alpha : \alpha \in D\}$ in a locally solid Riesz space (X, τ) is $r\text{-}\mathcal{I}_\tau$convergent then it is $2r\text{-}\mathcal{I}_\tau$ Cauchy.

Proof. : Let U be any arbitrary τ-neighborhood of zero. Choose $W \in N_{sol}$ such that $W + W \subset U$. Let s_α be $r\text{-}\mathcal{I}_\tau$convergent to $s_0 \in X$. So $A = \{\alpha \in D : s_\alpha - s_0 \in r + W\} \in F(\mathcal{I})$. Let $\alpha, \beta \in D$. Then, $s_\alpha - s_\beta = (s_\alpha - s_0) + (s_0 - s_\beta) \in (r + W) + (r + W) \subset 2r + U$. Hence $A \subset \{\alpha \in D : s_\alpha - s_\beta \in 2r + U\}$, where $\beta \in A$ is fixed. This shows the existence of $\beta \in D$ for which $\{\alpha \in D : s_\alpha - s_\beta \in 2r + U\} \in F(\mathcal{I})$ and this holds for every τ-neighborhood U of zero. Therefore $\{s_\alpha : \alpha \in D\}$ is $2r\text{-}\mathcal{I}_\tau$ Cauchy. □

Definition 4.5.2. For a net $\{s_\alpha\}$ we say that y is an r-$\mathcal{I}_\tau$cluster point if there exists a subnet $\{t_\alpha\}$ of $\{s_\alpha\}$ such that $t_\alpha \xrightarrow{r-\mathcal{I}_\tau} y$.

Theorem 4.5.2. If an r-$\mathcal{I}_\tau$Cauchy net $\{s_\alpha\}$ has a r-$\mathcal{I}_\tau$cluster point s_0 then $\{s_\alpha\}$ is $2r$-$\mathcal{I}_\tau$convergent to s_0.

Proof. Let U be a τ-neighborhood of zero. Choose $V, W \in N_{sol}$ such that $W+W \subset V \subset U$. As s_0 is r-$\mathcal{I}_\tau$cluster point of $\{s_\alpha\}$ so there exists a subnet $\{t_\alpha\}$ of $\{s_\alpha\}$ such that $t_\alpha \xrightarrow{r-\mathcal{I}_\tau} s_0$. Thus $A = \{\alpha \in D : t_\alpha - s_0 \in r + W\} \in F(\mathcal{I})$. Also $\{s_\alpha\}$ is r-$\mathcal{I}_\tau$Cauchy net so $B = \{\alpha \in D : s_\alpha - t_\alpha \in r + W\} \in F(\mathcal{I})$. Let $\alpha \in (A \cap B)$. Then α, $s_\alpha - s_0 = (s_\alpha - t_\alpha) + (t_\alpha - s_0) \in (r+W) + (r+W) \subset 2r + U$. Therefore $\{\alpha \in D : s_\alpha - s_0 \in 2r + U\} \in F(\mathcal{I})$. Hence $\{s_\alpha\}$ is $2r$-$\mathcal{I}_\tau$convergent to s_0. □

Theorem 4.5.3. Let (X, τ) be a locally solid Riesz space. Let $\{s_\alpha\}$ be a r-$\mathcal{I}_\tau$ Cauchy net in X. Then the following conditions holds-

(a) for every τ neighborhood U of zero there exists $A = \{\alpha : \alpha \in D\} \in \mathcal{I}$ such that $s_\alpha - s_\beta \in 2r + U$ for all $\alpha, \beta \notin A$.

(b) for every τ neighborhood U of zero, $\{\beta \in D : F_\beta(U) \notin \mathcal{I}\} \in \mathcal{I}$ where $F_\beta(U) = \{\alpha \in D : s_\alpha - s_\beta \in 2r + U\}$.

Proof. (a) Let U be a τ-neighborhood of zero. Choose $V, W \in N_{sol}$ such that $W + W \subset V \subset U$. As $\{s_\alpha\}$ is r-$\mathcal{I}_\tau$ Cauchy net, so there exists $\beta \in D$ such that $\{\alpha \in D : s_\alpha - s_\beta \notin r + W\} \in \mathcal{I}$. We choose $A = \{\alpha \in D : s_\alpha - s_\beta \notin r + W\}$. Clearly $A \in \mathcal{I}$. Let $\gamma, \alpha \notin A$ this implies $s_\alpha - s_\beta \in r + W$ and $s_\gamma - s_\beta \in r + W$. so $s_\alpha - s_\gamma = (s_\alpha - s_\beta) + (s_\beta - s_\gamma) \in (r+W) + (r+W) \subset 2r + U$.

(b) Let U be a τ-neighborhood of zero then by (a) there exists $A \in \mathcal{I}$ such that $\alpha, \gamma \notin A$. Then $s_\gamma - s_\alpha \in 2r + U$. Let $\beta \in D$ such that $F_\beta(U) \notin \mathcal{I}$, that is $\{\alpha \in D : s_\alpha - s_\beta \notin 2r + U\} \notin \mathcal{I}$. We will show $\{\beta \in D : F_\beta(U) \notin \mathcal{I}\} \subset A \in \mathcal{I}$. If possible let $\beta \notin A$. Since $A \in \mathcal{I}$ and $F_\beta(U) \notin \mathcal{I}$ so $E_\beta(U) \nsubseteq A$. Choose $\alpha \in F_\beta(U) \setminus A$. As $\alpha \in F_\beta(U)$ so $s_\alpha - s_\beta \notin 2r + U$ also since $\alpha, \beta \notin A$ so $s_\alpha - s_\beta \in 2r + U$, which is clearly a contradiction. Therefore $\beta \in A$ which shows that $\{\beta \in D : F_\beta(U) \notin \mathcal{I}\} \subset A \in \mathcal{I}$. □

Theorem 4.5.4. Let $\{s_\alpha\}$ be a net in a locally solid Riesz space (X, τ). If for every τneighborhood U of zero, $\{\beta \in D : E_\beta(U) \notin \mathcal{I}\} \in \mathcal{I}$ where $E_\beta(U) = \{\alpha \in D : s_\alpha - s_\beta \in r + U\}$ then $\{s_\alpha\}$ is $2r$-$\mathcal{I}_\tau$Cauchy net.

Proof. Let U be any τ-neighborhood of zero. Then $\{\beta \in D : E_\beta(U) \notin \mathcal{I}\} \in \mathcal{I}$ this implies $A = \{\beta \in D : E_\beta(U) \in \mathcal{I}\} \in F(\mathcal{I})$. As $\phi \notin F(\mathcal{I})$ so $A \neq \phi$. Let $\gamma \in A \subset D$ this implies $\gamma \in D$ such that $E_\gamma(U) \in \mathcal{I}$. So there exists $\gamma \in D$ such that $\{\alpha \in D : s_\alpha - s_\gamma \notin 2r + U\} \in \mathcal{I}$.This implies $\{s_\alpha\}$ is $2r$-$\mathcal{I}_\tau$ Cauchy net. □

This gives an Idea about when a net becomes Cauchy net.

The matter of this chapter is based on the following research paper:

Sagar Chakraborty, Sudip Kumar Pal, On some generalized summability method of Nets in locally solid Riesz spaces, Southeast Asian Bulletin of Mathematics, Vol.45(3) (2021), pp. 317-327. (ESCI)

CHAPTER 5

Order Cauchy sequences in (l)-group

Chapter 5

Order Cauchy sequences in (l)-group

5.1 Introduction

The concept of continuity and any concept involving continuity play a very important role not only in pure mathematics but also in other branches of sciences especially in computer science, information theory, biological science. In 2010, Burton and Coleman first introduce the term quasi-Cauchy sequence which is weaker than Cauchy sequence but interesting in their own right. They defined the term quasi-Cauchy sequence as: any sequence of real numbers (x_n) is quasi-Cauchy if given any $\epsilon > 0$ there exists an integer $K > 0$ such that $n \geq K$ implies $|x_{n+1} - x_n| < \epsilon$. Evidently Cauchy sequences are quasi-Cauchy but the converse is not true in general as the counter example is provided by the sequence of partial sums of the harmonic series. This and several such examples establish the important fact that the class of quasi-Cauchy sequences is much bigger than the class of Cauchy sequences, taking in the process more sequences under the preview. Understandably mathematical consequence are not analogous to the already existing notions bases on Cauchy sequences, like the usual idea of compactness. In a current development of the study of generalized metric space, the term ward continuity comes remembering the definition of continuity in sequential sense. The concepts of ward continuity of real valued function and ward compactness of subsets of $\mathbb{R}$ are introduced by Cakalli [14]. A real valued function f is called ward continuous on E if for every quasi-Cauchy sequence in E, the corresponding f-image sequence is also quasi-Cauchy.

In this chapter we introduce the notion of order quasi-Cauchy sequences and some weaker versions of it [13]. We primarily investigate several features of this new notion. Finally, a new concept, namely the concept of order statistical ward continuity of a function is introduced and investigated [17]. In this investigation we have obtained theorems related to order statistical ward continuity, order statistical ward compactness, compactness, and uniform order continuity. We also introduced and studied some other continuities involving statistical order quasi-Cauchy sequences and order convergent sequences of points in (l)-group [18].

Throughout $\mathbb{R}$ and $\mathbb{N}$ stand for the sets of all real numbers and natural numbers respectively and our topological terminologies and notations are as in the book [32] from where the notions (undefined inside the article) can be found. All spaces in the sequel are (l)-group.

5.2 Ward Continuity in (l)-group

In this section, we introduce the concept of order quasi-Cauchy sequences and ward continuity in (l)-group L, and we study some results related to it.

We know that any sequence of real numbers (x_n) is said to be Cauchy if given any $\epsilon > 0$ there exists an integer $K > 0$ such that $m, n \geq K$ implies $|x_m - x_n| < \epsilon$.

In 2010, Burton and Coleman first use the term quasi-Cauchy sequence. They defined the term quasi-Cauchy sequence as: any sequence of real numbers (x_n) is quasi-Cauchy if given any $\epsilon > 0$ there exists an integer $K > 0$ such that $n \geq K$ implies $|x_{n+1} - x_n| < \epsilon$. Using this idea we first introduce two definitions.

Definition 5.2.1. Any sequence (x_n) in a (l)-group L is said to be an order-Cauchy sequence if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|x_i - x_j| \leq p_{n_0}$ for all $i, j \geq m$.

Definition 5.2.2. Any sequence (x_n) in a (l)-group L is said to be an order quasi-Cauchy sequence if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists

$m \in \mathbb{N}$ such that $|x_{n+1} - x_n| \leq p_{n_0}$ for all $n \geq m$.

Remark 5.2.1. It is easy to verify that every order-Cauchy sequence is order quasi-Cauchy but the converse is not true in general. For counter example we take (l)-group $(\mathbb{R}, +)$ and consider the sequence (x_n) where $x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$. Clearly this sequence is order quasi-Cauchy but not order-Cauchy.

Remark 5.2.2. Every order convergent sequence is also order quasi-Cauchy.

Remark 5.2.3. Also every subsequence of order-Cauchy sequence is order-Cauchy. But the analogous property fails for quasi-Cauchy sequences. For instance we take the sequence (x_n) in $\mathbb{R}$, $x_n = \sqrt{n}$. (x_n) is order quasi-Cauchy but the subsequence (x_{n^2}) is not order quasi-Cauchy.

We know that if a function preserves Cauchy sequences, then it is called Cauchy continuous function. Similarly we define quasi-Cauchy continuous function in (l)-group. Some author rename it as 'ward continuous function'. Throughout this chapter we use the name ward continuous.

Definition 5.2.3. Let L be a (l)-group. A function $f : L \to L$ is said to be Ward Continuous on L if the sequence $(f(x_n))$ is order quasi-Cauchy whenever (x_n) is order quasi-Cauchy in L.

Definition 5.2.4. A subset E of L is said to be ward compact if any sequence in E has an order quasi-Cauchy subsequence.

The following theorems are obvious.

Theorem 5.2.1.

1. Every finite subset of L is ward compact.
2. Union of any two ward compact subsets of L is ward compact.
3. Intersection of any family of ward compact sets is ward compact.
4. Any subset of ward compact set is ward compact.

We see that for any metric space, continuity can be described by using sequence. Remembering this idea, we introduce continuity in (l)-group L. We call it order continuity.

Definition 5.2.5. A function $f : L \to L$ is said to be order continuous at x_0 if for any sequence (x_n) in L, which is order convergent to x_0, the corresponding image sequence $(f(x_n))$ is order convergent to $f(x_0)$.

In the next theorem we will investigate the relationship between ward continuity and order continuity [19].

Theorem 5.2.2. Let $f : R \to L$ be a ward continuous function, where $R \subset L$ then it is order continuous on R.

Proof. Suppose that $f : R \to L$ is ward continuous on $R \subset L$. Let (x_n) be a sequence in R such that $x_n \xrightarrow{ord} x_0$. Now we define a new sequence (y_n) as :

$$y_n = \begin{cases} x_0, & \text{if } n \; is \; even \\ x_k, & \text{if } n = 2k-1,\ k \in \mathbb{N}. \end{cases}$$

So,

$$y_n - x_0 = \begin{cases} \theta, & \text{if } n \; is \; even \\ x_k - x_0, & \text{if } n = 2k-1,\ k \in \mathbb{N}. \end{cases}$$

As (x_n) is order convergent to x_0 so for any order sequence (p_n) and $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|x_n - x_0| \leq p_{n_0}$ for all $n \geq m$. Now using this (p_n) and $n_0 \in \mathbb{N}$ with some suitable changes of m, from the construction of $y_n - x_0$, we can easily conclude that (y_n) is order convergent. Hence it is an order quasi-Cauchy sequence. As f is ward continuous so it preserves order quasi Cauchy sequence.

Hence $(f(y_n))$ is also order quasi-Cauchy, which is given by:

$$f(y_n) = \begin{cases} f(x_0), & \text{if } n \text{ } is \text{ } even \\ f(x_k), & \text{if } n = 2k-1, \ k \in \mathbb{N}. \end{cases}$$

Now for any order sequence (q_n) and $n'_0 \in \mathbb{N}$, there exists $m' \in \mathbb{N}$ such that $|f(y_{n+1}) - f(y_n)| \leq p_{n'_0}$ for all $n \geq m'$ which implies $|f(x_k) - f(x_0)| \leq p_{n'_0}$, for all $k \geq M$, where $M(\in \mathbb{N})$ depends on m'. So $f(x_n) \xrightarrow{ord} f(x_0)$. Hence f is order continuous. □

Converse of the above theorem is not true in general, which follows from the next example.

Example 5.2.1. Conisder the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$ and consider the sequence (x_n) given by $x_n = \sqrt{n}$.

Theorem 5.2.3. Ward continuous function preserves ward compact set.

Proof. Let $f : L \to L$ be a ward continuous map and $E \subseteq L$ be ward compact set. Let (x_n) be any sequence in E, as E is ward compact so we get subsequence (y_n) of (x_n) such that (y_n) is order quasi-Cauchy sequence. Now as f is ward continuous function, $(f(y_n))$ is order quasi-Cauchy subsequence of the sequence $(f(x_n))$ in $f(E)$. This completes the proof of the theorem. □

Definition 5.2.6. A function $f : L \to L$ is said to be uniformly order continuous on a subset E of L if for any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$, depending on this we get another order sequence (δ_n) and $m \in \mathbb{N}$, such that $|f(x) - f(y)| < \epsilon_{n_0}$ whenever $|x - y| < \delta_m$.

Theorem 5.2.4. If $f : L \to L$ is an uniform order continuous map on $E \subset L$ then it is ward continuous on E.

Proof. Let (x_n) be any order quasi-Cauchy sequence of points in E. As f is uniform order continuous so any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$, depending on this we get another order sequence (δ_n) and $m \in \mathbb{N}$, such that $|f(x) - f(y)| < \epsilon_{n_0}$ whenever $|x - y| < \delta_m$. This implies for this δ_n and m, n_0, we get suitably N, depends on δ_n, m, n_0 such that $|x_{n+1} - x_n| < \delta_n$ for all $n > N$. So $|f(x_n) - f(x_{n+1})| < \epsilon_{n_o}$, for all $n > N$. □

From the above theorem we can easily conclude that uniform order continuous functions are also order continuous.

Remark 5.2.4. Uniform order continuous image of ward compact set is ward compact.

We use the following notations :

$C[L, L] =$ Set of all order continuous functions on L.

$WC[L, L] =$ Set of all ward continuous functions on L.

$UC[L, L] =$ Set of all uniform order continuous functions on L.

Now from the above discussion we can easily conclude that,

Remark 5.2.5. $UC[L, L] \subseteq WC[L, L] \subseteq C[L, L]$.

We see that, a sequence (x_n) in L is said to be order convergent to $x_0 \in L$ if there exists an order sequence (p_n) such that for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|x_n - x_0| \leq p_{n_0}$ for all $n \geq m$. In this case choice of m depends on x_0. To deal this type of situation we introduce the concept of uniform order convergence in (l)-group.

Definition 5.2.7. A sequence (x_n) in $E \subset L$ is said to be uniform order convergent to $x \in E$ if there exists an order sequence (p_n) in L such that for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|x_n - x| \leq p_{n_0}$ for all $n \geq m$ and for all $x \in E$.

Theorem 5.2.5. Let (f_n) be a sequence of uniform order continuous functions on $E \subset L$ and (f_n) is uniformly order convergent to a function f then f is uniform order continuous on E.

Proof. Suppose that (f_n) is uniformly order convergent to some function f. Then for a given order sequence (p_n) in L and for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$

satisfying $|f_n(x)-f(x)| \leq p_{n_0}$ for all $n \geq m$ and for all $x \in E$. Now each $f_n : L \to L$ is uniform order continuous on E. Hence for this order sequence (p_n), $n' \in \mathbb{N}$, there exists order sequence q_n and $m' \in \mathbb{N}$ such that $|f_n(x) - f_n(y)| < p_{n'}$, for all $n \geq m'$, whenever $|x-y| \leq q_n$. Now $f(x)-f(y) = f(x)-f_n(x)+f_n(x)-f_n(y)+f_n(y)-f(y)$, sum of three null sequences, hence $f(x)$ is uniform order continuous on E.

□

Theorem 5.2.6. Let (f_n) be a sequence of ward continuous functions defined on $E \subset L$ and (f_n) is uniformly order convergent to a function f, then f is ward continuous on E.

Proof. Let (x_n) be an order quasi-Cauchy sequence of points on E. As (f_n) is uniformly order convergent to f, given order sequence (p_n) in L such that for each $n_0 \in \mathbb{N}$, there exists some $m \in \mathbb{N}$ satisfying $|f_n(x) - f(x)| \leq p_{n_0}$ for all $n \geq m$ and for all $x \in E$. Now each f_n is ward continuous on E. So for this (p_n) and $n' \in \mathbb{N}$ there exists $m' \in \mathbb{N}$ with $|f_m(x_{n+1}) - f_m(x_n)| < p_{n'}$ for all $n \geq m'$. Now

$$f(x_{n+1}) - f(x_n) = f(x_{n+1}) - f_m(x_{n+1}) + f_m(x_{n+1}) - f_m(x_n) + f_m(x_n) - f(x_n).$$

This implies $f(x_{n+1}) - f(x_n)$ is sum of three null sequences, so we easily conclude that f is ward continuous on E. □

5.3 Statistical Ward Continuity in (l)-group

Recently, it has been proved that a real-valued function defined on an interval A of the set of real numbers, is uniformly continuous on A if and only if it preserves quasi-Cauchy sequences of points in A. In this section we call a real-valued function order statistically ward continuous if it preserves statistical order quasi-Cauchy sequences. It turns out that any order statistically ward continuous function on a statistically ward order compact subset A of an (l) -group is uniformly order continuous on A. We prove theorems related to order statistical ward compactness, order statistical compactness, order continuity, statistical order continuity, ward order continuity, and uniform order continuity.

Definition 5.3.1. We call a sequence (x_n) of points in (l)-group L statistically order quasi-Cauchy if for any order sequence (p_n) and $n_0 \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{d(k \leq n : |x_{k+1} - x_k| \geq p_{n_0})}{n} = 0.$$

Where $d(A)$ denotes the cardinality of the set A.

It is clear that, any order quasi-Cauchy sequence is statistically order quasi-Cauchy.

Definition 5.3.2. A subset E of L is said to be statistically ward compact if any sequence of points in E has a statistically order quasi-Cauchy subsequence.

Definition 5.3.3. A function $f : E \to L$ is said to be statistically order continuous on $E \subseteq L$ if it preserves statistically order convergent sequences.

Definition 5.3.4. Let $E \subseteq L$. A function $f : E \to L$ is said to be statistically ward continuous if it preserves statistically order quasi-Cauchy sequences.

Theorem 5.3.1. Every statistically ward continuous functions are also statistically order continuous.

Proof. Let $f : E \to L$ be a statistically ward continuous function and (x_n) be any statistically order convergent sequence which converges to x_0. For any order sequence (p_n), $n_0 \in \mathbb{N}$, $\lim_{n\to\infty} \frac{d(k\leq n:|x_k - x_0|\geq p_{n_0})}{n} = 0$. Hence the sequence $(x_1, x_0, x_2, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$ is also statistically order convergent to x_0. Hence it is statistically order quasi-Cauchy. As f is statistically ward continuous so, $(f(x_1), f(x_0), f(x_2), f(x_0), f(x_3), ...)$ is also statistically order quasi-Cauchy. As the even terms of the sequence are $f(x_0)$, odd terms are nothing but the sequence $(f(x_n))$, we can easily conclude that $(f(x_n))$ is statistically order convergent to $f(x_0)$. This completes the proof. □

The converse is not true in general. For counter example we take the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and consider the sequence $(\sqrt{n})$.

We know that any continuous function on a compact set is uniformly continuous. Similarly for statistically ward continuous function defined on a statistically ward compact subset of an (l)-group, we have the following result :

Theorem 5.3.2. Let E be a statistically ward compact subset of an (l)-group L and $f : E \to L$ be a statistically ward continuous function on E. Then it is uniform order continuous.

Proof. If possible suppose that f is not uniformly order continuous on E. Then there exists order sequence (ϵ_n) and $n_0 \in \mathbb{N}$ such that for any δ_n and $m \in \mathbb{N}$ with $|x-y| \leq \delta_m$, $|f(x)-f(y)| > \epsilon_{n_0}$. Now for each $n \in \mathbb{N}$ fix $|x_n-y_n| < \delta_n$ and $|f(x_n)-f(y_n)| \geq \epsilon_{n_0}$. Since E is statistically ward compact so (x_n) has a subsequence (x_{n_k}) which is statistically order quasi-Cauchy. Now $y_{n_{k+1}} - y_{n_k} = (y_{n_{k+1}} - x_{n_{k+1}}) + (x_{n_{k+1}} - x_{n_k}) + (x_{n_k} - y_{n_k})$ which is clearly sum of three null sequences hence (y_{n_k}) is statistically order quasi-Cauchy subsequence of (y_n). As $x_{n_{k+1}} - y_{n_k} = (x_{n_{k+1}} - y_{n_{k+1}}) + (y_{n_{k+1}} - y_{n_k})$ so the sequence $(x_{n_{k+1}} - y_{n_k})$ is statistically order convergent to θ. Hence the sequence $(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, ..., x_{n_k}, y_{n_k}, ...)$ is statistically order quasi-Cauchy. Which implies that the sequence $(f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), ..., f(x_{n_k}), f(y_{n_k}), ...)$ is also statistically order quasi-Cauchy in $f(E)$. But this contradicts the fact that $|f(x) - f(y)| > \epsilon_{n_0}$. Thus f is uniformly order continuous on E. □

Theorem 5.3.3. Statistically ward continuous image of any statistically ward compact subset of L is statistically ward compact.

Proof. Let $f : A \to L$ be a statistically ward continuous function defined on a subset A of L and E be a statistically ward compact subset of A. We want to show $f(E)$ is statistically ward compact subset of L. Let (y_n) be any sequence of points in $f(E)$. Then clearly $y_n = f(x_n)$ for some sequence (x_n) of points in E. As E is statistically ward compact set so there exists subsequence $(z_k) = (x_{n_k})$ of (x_n) such that (z_k) is statistically ward compact. Now as f is ward continuous function so

$f(z_k)$ is statistically order quasi-Cauchy sequence. Thus we get a order quasi-Cauchy subsequence $(f(z_k))$ of the sequence (y_n) of $f(E)$. Hence $f(E)$ is a ward compact set. □

Theorem 5.3.4. If (f_n) is a sequence of statistically ward continuous functions on a subset E of L and (f_n) is uniformly order convergent to a function f then f is statistically order ward continuous on E.

Proof. Suppose that (f_n) is uniform order convergence to f. For any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon_{n_0}$, for all $n \geq m$ and for all $x \in E$. Consider any statistically order quasi-Cauchy sequence (x_n) of points in E. As f_m is statistically ward continuous on E so it preserves statistically order quasi-Cauchy sequence. Hence $\lim_{n\to\infty} \frac{d(k\leq n:|f_m(x_{k+1})-f_m(x_k)|\geq\epsilon_{n_0})}{n} = 0$. Now

$f(x_{k+1})-f(x_k) = [f(x_{k+1})-f_m(x_{k+1})]+[f_m(x_{k+1})-f_m(x_k)]+[f_m(x_k)-f(x_k)]$. Hence using the fact that f_m is statistically ward continuous and f_m is uniform convergent to f we can easily conclude that

$$\lim_{n\to\infty} \frac{d(k \leq n : |f(x_{k+1}) - f(x_k)| \geq \epsilon_{n_0})}{n} = 0.$$

This completes the proof. □

The following result follows immediately:

Theorem 5.3.5. The set $SWC[E, L]$, the set of all statistical ward continuous functions is a closed set.

From the above discussion we see that $UC[L, L] \subseteq WC[L, L] \subseteq C[L, L]$. Now the obvious question is when these sets are equal. In [2], Burton and Coleman gives some partial idea about the equality of $UC[L, L] \subseteq WC[L, L]$.

Theorem 5.3.6. Let $I \subseteq \mathbb{R}$ be any interval. Then $UC[I, \mathbb{R}] = WC[I, \mathbb{R}]$.

Now we introduce another type of convergence in (l)-group called slowly oscillating order convergence.

Definition 5.3.5. A sequence (x_n) of points in (l)-group L is called slowly oscillating order convergence if for any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $|x_i - x_j| < \epsilon_{n_0}$ for all $i \geq m$ and $1 \leq \frac{i}{j}$ and $\frac{i}{j} \to 1$ as $i, j \to \infty$.

From Definition it is clear that order Cauchy sequences are obviously slowly oscillating and every slowly oscillating sequence is order quasi-Cauchy.

Definition 5.3.6. [14, 62] A function $f : E \to L$ is said to be slowly oscillating continuous if it preserves slowly oscillating order sequences.

By $SOC[E, L]$ we denote set of all slowly oscillating continuous functions defined on E.

Now we introduce the concept of Connectedness in (l)-group L.

Definition 5.3.7. Let L be a (l)-group. Suppose that $x \in U \subseteq L$. U is called order sequential neighborhood of $x \in L$ if any sequence (x_n) which is order convergent to x, then $\{x_n : n \geq m\} \subset U$ for some $m \in \mathbb{N}$.

Definition 5.3.8. U is said to be an order sequential open subset of L if for each $x \in U$, U is order sequential neighborhood of x.

Definition 5.3.9. A is said to be an order sequential closed subset of L if $L \setminus A$ is an order sequential open subset of L.

Definition 5.3.10. Let $A \subseteq L$, by $\bar{A}$ we denote closure of A, defined as intersection of all sequentially closed sets containing A.

Definition 5.3.11. An (l)-group L is said to be order connected if there do not exists any non empty subsets A, B such that $X = A \cup B$ with $\bar{A} \cap \bar{B} = \phi$.

Now we modify the Lemma 1 in [13] which is given in metric space setting.

Lemma 5.3.1. Let $((a_n, b_n))$ be a sequence of ordered pair of points in a connected subset $E \subseteq L$ such that given any ordered sequence (ϵ_n) there exists $n_0, m \in \mathbb{N}$, we have $|a_n - b_n| \leq \epsilon_{n_0}$ for all $n \geq m$. Then there exists an order quasi-Cauchy sequence (t_n) with the property that for any positive integer i there exists a positive integer k such that $(a_i, b_i) = (t_{j-1}, t_j)$.

It turns out that a function defined on a connected subset E of a metric space is uniformly continuous if and only if it preserves either quasi-Cauchy sequences or slowly oscillating sequences of points in E.

Now we are in the position of most desired result:

Theorem 5.3.7. Let E be an order connected subset of a (l)-group L then the three sets $UC[E, L], WC[E, L]$ and $SOC[E, L]$ are equivalent.

Proof. $\underline{UC[E, L] \subseteq WC[E, L]}$: Let $f : E \to L$ be any uniformly order continuous function on E. Let (x_n) be any order quasi-Cauchy sequence of points in E. As f is uniform order continuous so any order sequence (ϵ_n) and $n_0 \in \mathbb{N}$, depending on this we get another order sequence (δ_n) and $m \in \mathbb{N}$ such that $|f(x) - f(y)| < \epsilon_{n_0}$ whenever $|x-y| < \delta_m$. This implies for this δ_n and m, n_0, we get suitably N, depends on δ_n, m, n_0 such that $|x_{n+1} - x_n| < \delta_n$ for all $n > N$. So, $|f(x_n) - f(x_{n+1})| < \epsilon_{n_0}$, for all $n > N$.

$\underline{UC[E, L] \subseteq SOC[E, L]}$: Let $f : E \to L$ be uniform order continuous. We take slowly oscillating sequence (x_n) of points on E. Let (ϵ_n) be any order sequence. We get another order sequence (δ_n) and $n_0, m_0 \in \mathbb{N}$ such that $|f(x_i) - f(x_j)| \leq \epsilon_{n_0}$ whenever $x_i, x_j \in E$ and $|x_i - x_j| \leq \delta_{m_0}$. As (x_n) is slowly oscillating so $|x_i - x_j| \leq \delta_k$, for all $i \geq m$ and $1 \leq \frac{i}{j}$ and $\frac{i}{j} \to 1$ as $i, j \to \infty$. Now using uniform continuity of f, $|f(x_i) - f(x_j)| \leq \epsilon_k$, for all $i \geq m$ and $1 \leq \frac{i}{j}$ and $\frac{i}{j} \to 1$ as $i, j \to \infty$. This implies $(f(x_n))$ is slowly oscillating. Hence $f \in SOC[E, L]$.

$\underline{SOC[E, L] \subseteq UC[E, L]}$: Let $f : E \to L$ be not uniformly continuous on E. Now each $n \in \mathbb{N}$, we fixed $|x_n - y_n| < \delta_n$ then as f is not uniformly order continuous so there exists order sequence ϵ_n such that $|f(x_n) - f(y_n) \geq \epsilon(k)|$. Now As it is given that E is order connected so by the Lemma 5.3.1, from (x_n) we can construct a slowly oscillating sequence (t_n) but as f is not uniformly continuous, so the transformed sequence $(f(t_n))$ is not slowly oscillating. Hence $f \notin SOC[E, L]$. This implies $SOC[E, L] \subseteq UC[E, L]$.

$\underline{WC[E, L] \subseteq UC[E, L]}$: Suppose f is not uniformly order continuous on E. Since we know that slowly oscillating sequences are also order quasi-Cauchy hence the sequence (t_n) constructed on previous case is also order quasi-Cauchy, but as f is not uniformly order continuous so $f(t_n)$ is not order quasi-Cauchy. So $WC[E, L] \subseteq UC[E, L]$.

This completes the proof of the Theorem. □

5.4 Downward and upward order Continuity in (l)-group

In this section, we introduce and investigate the concepts of down order continuity and down order compactness. A real valued function f on a subset E of the set of real numbers is down continuous if it preserves downward half Cauchy sequences, i.e. the sequence $(f(a_n))$ is downward half Cauchy whenever (a_n) is a downward half Cauchy sequence of points in E. A sequence (a_k) of points in $\mathbb{R}$ is called downward half Cauchy if for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $a_m - a_n < e$ for $m \geq n \geq n_0$. It turns out that the set of all down continuous functions is a proper subset of the set of all continuous functions. First we introduce the following definition:

Definition 5.4.1. Let (x_n) be a sequence in (l)-group L. Then (x_n) is called downward order quasi-Cauchy if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $x_{n+1} - x_n \leq p_{n_0}$, for all $n \geq m$.

It is clear that every order quasi-Cauchy sequence is also downward order quasi-Cauchy but the converse is not true in general. For example we take the lattice order group $(\mathbb{R}, +)$ and the sequence (x_n), where $x_n = -n$. This sequence is downward order quasi-Cauchy but not order quasi-Cauchy.

Any order Cauchy sequence is obviously order quasi-Cauchy and hence downward order quasi-Cauchy.

Definition 5.4.2. A sequence (x_n) of points in L is said to be downward order half Cauchy if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $x_m - x_n \leq p_{n_0}$ where $m, n \in \mathbb{N}$ with $m \geq n > k$.

It is obvious that downward order half Cauchy sequences are also downward order quasi-Cauchy and any subsequence of downward order half Cauchy sequence are same type. But for the downward order quasi Cauchy sequences the situation is different. We take the sequence (x_n) in $(\mathbb{R}, +)$ such that $x_n = \sqrt{n}$. Clearly (x_n) is downward order half Cauchy but one of it's subsequence, namely (x_{n_k}) is not downward order half Cauchy.

In [13], authors proved that a sequence of real numbers is Cauchy if and only if every subsequence is quasi-Cauchy. In the next theorem we present similar type result for downward order half Cauchy sequences in (l)-group.

Theorem 5.4.1. A sequence (x_n) in L is downward order half Cauchy if and only if every subsequence of (x_n) is downward order quasi-Cauchy.

Proof. If (x_n) is downward order half Cauchy then every subsequence of (x_n) is downward half Cauchy so is downward order quasi-Cauchy.

To prove the converse part, we use contrapositive statement. Let (x_n) be not downward order half Cauchy. Then there exists order sequence (p_n) and $n_0 \in \mathbb{N}$ such that for every positive integer m, $x_{n_i} - x_{n_j} > p_{n_0}$, $n_i > n_j \geq m$.

Now for $m = 1$ we get such n_i, n_j, we rename it as k_1, k_2. So $k_2 > k_1 > 1$ and $x_{k_2} - x_{k_1} > p_{n_0}$. Similarly for $m = 2, 3, \ldots$. Inductively we get $x_{k_{n+1}} - x_{k_n} > p_{n_0}$, where $k_{n+1} > k_n > k_{n-1} > \ldots$. Which shows that the subsequence (x_{k_n}) is not downward order quasi-Cauchy. Hence the proof. □

Now we study the sequential compactness like property. First of all we introducing the idea of downward order compact set in a lattice order group as:

Definition 5.4.3. A subset E of L is called downward order compact if any sequence of points in E has a downward order quasi-Cauchy subsequence.

We know that a real valued function of real variables is continuous if it preserves convergent sequences. In a similar way we already defined order continuity. Now if a function preserves downward order quasi-Cauchy sequences then we get a new type of continuity, we call it downward order continuity.

Definition 5.4.4. A function $f : E \to L$ is called downward order continuous on a subgroup E of L if it preserves downward order quasi-Cauchy sequences.

Theorem 5.4.2. Sum of two downward order continuous functions is downward order continuous.

Proof. Suppose that $f : E \to L$ and $g : E \to L$ are two downward order continuous functions on E, a subgroup of L. Let (x_n) be a downward order quasi-Cauchy sequence in E. As f, g both are downward order continuous so the sequences $(f(x_n))$ and $(g(x_n))$ both are downward order quasi-Cauchy. We know that sum of two order sequences is also an order sequence. Take any order sequence (p_n) then $(2p_n)$ is also order sequence. So for order sequence $2p_n$ and for $n_0 \in \mathbb{N}$ there exists positive integers m_1 and m_2 such that $f(x_{n+1}) - f(x_n) \leq p_{n_0}$ for all $n \geq m_1$ and $g(x_{n+1}) - g(x_n) \leq p_{n_0}$ for all $n \geq m_2$. Take $m = \max\{m_1, m_2\}$. Then $f(x_{n+1}) + g(x_{n+1}) - g(x_n) - f(x_n) \leq 2p_{n_0}$ for all $n \geq m$. This proves the theorem. □

From definition it is quite obvious that every ward order continuous function is downward order continuous. The following theorem make the link between ward order continuity and order continuity.

Theorem 5.4.3. Every downward order continuous function is order continuous.

Proof. Let $f : E \to L$ be downward order continuous on E and (x_n) be a sequence which order converges to x_0. Now we construct a sequence

$$(x_1, x_0, x_1, x_0, x_2, x_0, x_2, x_0, ...).$$

As (x_n) is order converges to x_0 so the new sequence is also order converges to x_0. Also from the construction it is clear that the new sequence is also downward order quasi-Cauchy. As f is downward order continuous so

$$(f(x_1), f(x_0), f(x_2), f(x_0), f(x_2), f(x_0), ...)$$

is downward order quasi-Cauchy. From here we can easily conclude that $f(x_n)$ order converges to $f(x_0)$. Hence f is order continuous. □

Theorem 5.4.4. Let E be a downward compact subgroup of L and $f : E \to L$ be downward order continuous function. Then $f(E)$ is also downward order compact.

Proof. Suppose that E is a downward compact subgroup of L. Let us take any sequence (y_n) in $F(E)$. So $y_n = f(x_n)$ where $x_n \in A$ for each n. As E is downward order compact and (x_n) is any sequence in E so (x_n) has a downward order quasi-Cauchy sub-sequence, say, (z_k). As $f : E \to L$ is a downward order continuous function, $f(z_k)$ is a downward order quasi-Cauchy sequence. Hence we get $f(z_k)$ is a downward order quasi-Cauchy sub-sequence of (y_n). This completes the proof. □

Now the question arise that does the downward continuous function preserves uniform limit? The following theorem gives the answer. The technique of the proof is almost same as word continuous function. So we just state the theorem.

Theorem 5.4.5. If (f_n) be a sequence of downward continuous functions defined on a subgroup E of L and (f_n) is uniform order convergent to a function f then f is downward order continuous on E.

If we change the position of x_{n+1} and x_n in the definition of downward order quasi-Cauchy sequence we get a new type of sequence, we call it upward order quasi-Cauchy sequence. The results are similar. We just state the result.

Definition 5.4.5. Let (x_n) be a sequence in (l)-group L. Then (x_n) is called upward order quasi-Cauchy if for any order sequence (p_n) and for each $n_0 \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $x_n - x_{n+1} \leq p_{n_0}$.

It is clear that every order quasi-Cauchy sequence is also upward order quasi-Cauchy. But the converse is not true in general. For example, we take the lattice order group $(\mathbb{R}, +)$ and the sequence (x_n), where $x_n = n$. This sequence is upward order quasi-Cauchy but not order quasi-Cauchy.

Any order Cauchy sequence is obviously order quasi-Cauchy and hence upward order quasi-Cauchy.

The matter of this chapter is based on the following research paper:
Sudip Kumar Pal, Sagar Chakraborty, On certain new notion of Order Cauchy sequences, continuity in (l)-group, Applied General Topology Vol.23(1) (2022), pp 55-68 (SCOPUS, ESCI)

CHAPTER 6

Statistical order convergence in (l)-group

Chapter 6

Statistical order convergence in (l)-group

6.1 Introduction

In metric space for a convergent sequence (x_n), each its subsequence (x_{n_k}) is also convergent. However, this is generally not true for the generalized methods of convergence. Very recently, following the line of investigation in [41], the conditions have been studied for the density of a subsequence of a statistically bounded and also statistically convergent sequence under which the indicated subsequence is also statistically bounded or statistically convergent in the setting of metric space. As a natural consequence, in [27] Das and Şavaş investigated the similar type problem that are proposed in [41] for metric valued sequences by considering the notion of natural density of weight g, which was introduced in [7].

In recent time some ideas in (l)-group, namely order convergence, order Cauchyness of nets in (l)-group using ideals developed by Boccuto and others [12]. As a continuation, P. Das and E. Şavaş was introduce the notion of $\mathcal{I}^K$-order convergence in (l)-group [26].

In the present chapter we continue the investigation posed in [41] for the sequences in (l)-group considering the notion of natural density of weight g as also the notion f-density (introduced and studied in [2]).

6.2 f-statistical order convergence

Using f-density we now introduce statistical order convergence in (l)-group. We call it f-statistical order convergence. Throughout this section f denotes unbounded modulus function.

Definition 6.2.1. A sequence (x_n) in R is said to be f-statistical order convergent to x_0 if there exists a sequence (y_n) in R such that $y_n \downarrow \theta$ and $d_f(\{n \in \mathbb{N} : |x_n - x_0| > y_n\}) = 0$.
i.e.

$$\limsup_{n\to\infty} \frac{f(|\{n \in \mathbb{N} : |x_n - x_0| > y_n\}|)}{f(n)} = 0.$$

We write it as (st_f-ord.)lim $x_n = x_0$.

Note 6.2.1. It is obvious that f-statistical order convergence and statistical order convergence are same when the modulus function is the identity mapping.

Definition 6.2.2. A sequence (x_n) in R is said to be f-statistically order bounded if there exists an order interval $[a, b]$ such that $d_f\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$.

Definition 6.2.3. Two sequences $(x_n), (y_n)$ in R are said to be f-statistically equivalent, if there is a f-dense set $A \subset \mathbb{N}$ such that $x_n = y_n$ for every $n \in A$.

Definition 6.2.4. Let $\tilde{x} = (x_n)$ be a sequence in R. If (n_k) is an infinite strictly increasing sequence of natural number then $\tilde{x}' = (x_{n_k})$ is called a subsequence of $\tilde{x}$.

We denote $K_{\tilde{x}'} = \{n_k : k \in \mathbb{N}\}$ i.e. the set of indices of the subsequence (x_{n_k}). $\tilde{x}'$ is called f-dense subsequence of $\tilde{x}$ if $K_{\tilde{x}'}$ is a f-dense subset of $\mathbb{N}$, i.e. $d_f(K_{\tilde{x}'}) = 1$

Theorem 6.2.1. Let $(R, +)$ be a (l)-group and (x_n) be a sequence in R. Then
(i) If (x_n) is bounded then it is also f-statistically order bounded.
(ii) If (x_n) is f-statistically order convergent to $x_0 \in R$ then it is f-statistically order bounded.

Proof. (i) Follows from the definition. (ii) Let (x_n) be f-statistically order convergent to $x_0 \in R$. So, there exists a sequence (y_n) in R such that $y_n \downarrow \theta$ and

$d_f(\{n \in \mathbb{N} : |x_n - x_0| > y_n\}) = 0$. Chose $a = x_0 - y_1$ and $b = x_0 + y_1$ then clearly for this order interval $[a, b]$, $d_f(\{n \in \mathbb{N} : x_n \notin [a, b]\}) = 0$. Therefore (x_n) is f-statistically order bounded. □

Converse of the above theorems is not true in general. Considering the sequence $x_n = (-1)^n$ in $\mathbb{R}$ we can easily verify it.

Theorem 6.2.2. Let $\tilde{x} = (x_n)$ be f-statistical order convergent sequence and $\tilde{x}' = (x_{n_k})$ be any subsequence of $\tilde{x}$. Then, $\tilde{x}'$ is f-statistically order bounded.

Proof. Let $\tilde{x} = (x_n)$ be f-statistically order converges to x_0. Then $\tilde{x}$ is f-statistically order bounded. Hence we get a order interval $[a, b]$, such that $d_f\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$. Now $|\{n_k \in \mathbb{N} : x_{n_k} \notin [a, b]\}| \leq |\{n \in \mathbb{N} : x_n \notin [a, b]\}|$. Hence $d_f\{n_k \in \mathbb{N} : x_{n_k} \notin [a, b]\} = 0$. Therefore $\tilde{x}'$ is f-statistically bounded. □

In statistical convergence, we see that every subsequence of a statistical convergent sequence may not be statistical convergent. One can see the reference [42] for supporting example. Using the same example we can verify that, every subsequence of a f-statistically order convergent sequence may not be f-statistically order convergent. The following theorem gives idea about under what condition the subsequences also be convergent.

Theorem 6.2.3. Let $(R, +)$ be an (l)-group, $\tilde{x} = (x_n)$ be a sequence in R and $\tilde{x}' = (x_{n_k})$ be a subsequence of $\tilde{x}$ such that $\liminf\limits_{n\to\infty} \dfrac{f(|K_{\tilde{x}'}(n)|)}{f(n)} > 0$. If $\tilde{x}$ is f-statistically order convergent to $x_0 \in R$, then $\tilde{x}'$ is also f-statistically order convergent to x_0.

Proof. Suppose that $\tilde{x} = (x_n)$ is f-statistically order convergent to x_0. Then, there exist a sequence (y_n) in R such that $y_n \downarrow \theta$ and $d_f\{n \in \mathbb{N} : |x_n - x_0| > y_n\} = 0$. Let $\tilde{x}' = (x_{n_k})$ be a subsequence of $\tilde{x}$ such that $\liminf\limits_{n\to\infty} \dfrac{f(|K_{\tilde{x}'}(n)|)}{f(n)} > 0$ holds.
Now $\{n_k \in \mathbb{N} : |x_{n_k} - x_0| > y_{n_k}\} \subseteq \{n \in \mathbb{N} : |x_n - x_0| > y_n\}$.

Thus, we can write following inequalities

$$\frac{|\{n_k \in \mathbb{N} : |x_{n_k} - x_0| > y_{n_k)}\}|}{f(|K_{\tilde{x}'}(n)|)} \leq \frac{|\{n \in \mathbb{N} : |x_n - x_0| > y_n\}|}{f(|K_{\tilde{x}'}(n)|)}.$$

Now we know from [42] that, for any two sequences (α_n) and (β_n) of non-negative real numbers with $0 \neq \liminf_{n\to\infty} \alpha_n < \infty$, we have $(\liminf_{n\to\infty} \alpha_n)(\limsup_{n\to\infty} \beta_n) \leq \limsup_{n\to\infty} \alpha_n\beta_n$.

Now taking $\alpha_n = \frac{f(|K_{\tilde{x}'}(n)|)}{f(n)}$ and $\beta_n = \frac{|\{n\in\mathbb{N}:|x_n - x_0|>y_n\}|}{f(|K_{\tilde{x}'}(n)|)}$ and using first inequality with the given condition, we easily conclude that

$$\limsup_{n\to\infty} \frac{|\{n_k \in \mathbb{N} : |x_{n_k} - x_0| > y_{n_k}\}|}{f(|K_{\tilde{x}'}(n)|)} = 0,$$

holds. Hence $\tilde{x}'$ is f-statistically order convergent to x_0. □

Theorem 6.2.4. Let $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ be two f-statistically equivalent sequences in an (l)-group $(R, +)$ and K be a subset of $\mathbb{N}$ such that, $0 < \limsup_{n\to\infty} \dfrac{f(n)}{f(|K(n)|)} < +\infty$. If $\tilde{x}' = (x_{n_k})$ and $\tilde{y'} = (y_{n_k})$ are subsequences of $\tilde{x}, \tilde{y}$ respectively such that $K_{\tilde{x}'} = K_{\tilde{y'}} = K$, then $\tilde{x}'$ and $\tilde{y'}$ are f-statistically equivalent.

Proof. Let $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ be two f-statistically equivalent, so there exists a f-dense set $M \subset \mathbb{N}$ such that $x_n = y_n$ for all $n \in \mathbb{M}$. Equivalently,

$$\limsup_{m\to\infty} \frac{f(|\{n \in \mathbb{N} \ : \ x_n \neq y_n \ and \ n \leq m\}|)}{f(m)} = 0,$$

holds. We want to prove that $\tilde{x}'$ and $\tilde{y'}$ are f-statistically equivalent, i.e. $x_{n_k} = y_{n_k}$ for all $n_k \in M$ with $d_f(M) = 1$. Equivalently we have to show that,

$$\limsup_{m\to\infty} \frac{f(|\{n_k \in K : x_{n_k} \neq y_{n_k} and \ n_k \leq m\}|)}{f(|K(m)|)} = 0.$$

Now, for any $m \in \mathbb{N}$, we have

$$\{n_k \in K : x_{n_k} \neq y_{n(k)} \ and \ n_k \leq m\} \subseteq \{n \in \mathbb{N} : x_n \neq y_n and \ n \leq m\}$$

So for any unbounded modulus function f, we have

$$\begin{aligned}
&\limsup_{m\to\infty}\frac{f(|\{n_k\in K:x_{n_k}\neq y_{n_k}\text{ and }n_k\leq m\}|)}{f(|K(m)|)}\\
\leq\ &\limsup_{m\to\infty}\frac{f(|\{n\in\mathbb{N}:x_n\neq y_n\text{ and }n\leq m\}|)}{f(|K(m)|)}\\
\leq\ &\limsup_{m\to\infty}\frac{f(|\{n\in\mathbb{N}\ :\ x_n\neq y_n\text{ and }n\leq m\}|)}{f(m)}.\limsup_{m\to\infty}\frac{f(m)}{f(|K(m)|)}
\end{aligned}$$

Now, $\tilde{x}=(x_n)$ and $\tilde{y}=(y_n)$ are f-statistically equivalent sequences in R and $0<\limsup_{n\to\infty}\dfrac{f(n)}{f(|K(n)|)}<+\infty$, we get

$$\limsup_{m\to\infty}\frac{f(|\{n_k\in K:x_{n_k}\neq y_{n_k} and\ n_k\leq m\}|)}{f(|K(m)|)}=0.$$

Hence $\tilde{x}'$ and $\tilde{y}'$ are f-statistically equivalent. □

6.3 d_g-statistical order convergence

The concept of statistical order convergence was introduced in [60]. In a similar way we define statistical order convergence in (l)-group as follows:

Definition 6.3.1. Let $(R,+)$ be an (l)-group. A sequence (x_n) in R is said to be statistically order convergent to $x_0\in R$ if there exists a sequence (y_n) in R such that $y_n\downarrow\theta$ (i.e. (y_n) is decreasing sequence and converges to θ) and a set $K=\{n_1<n_2<...\}\subset\mathbb{N}$ with $d(K)=1$ such that $|\ x_n-x_0\ |\leq y_n$ for every $n\in K$. In this case we have $d\{n\in N:|\ x_n-x_0\ |>y_n\}=0$.We write it symbolically as $x_n\xrightarrow{st.ord.}x_0$.

Let $g:\mathbb{N}\to[0,\infty)$ be a function with $\lim_{n\to\infty}g(n)=\infty$ and $\lim_{n\to\infty}\frac{n}{g(n)}\neq 0$. The density of weight g was defined in [7] by the formula,

$$d_g(A)=\limsup_{n\to\infty}\frac{\text{card}(A\cap[1,n])}{g(n)}$$

for $A \subseteq \mathbb{N}$. Using the weighted density we define the weighted statistical order convergence as follows:

Definition 6.3.2. Let $(R, +)$ be an (l)-group. A sequence (x_n) in R is said to be statistical order convergent of weight g or d_g-statistically order convergent to $x_0 \in R$ if there exists a sequence (y_n) in R such that $y_n \downarrow \theta$ and $d_g\{n \in N :| x_n - x_0 |> y_n\} = 0$. We write it symbolically as $x_n \xrightarrow{d_g - st.ord.} x_0$.

Definition 6.3.3. A sequence (x_n) in R is said to be d_g-statistically order bounded if there exists an order interval $[a, b]$ such that $d_g\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$

From the above definitions we can easily conclude that:

Theorem 6.3.1. Let R be an (l)-group and (x_n) be any sequence in R:
i. if (x_n) is order bounded, then it is d_g-statistically order bounded.
ii. if (x_n) is d_g-statistically order convergent , then it is d_g-statistically order bounded.

Proof. i. Let (x_n) be an order bounded sequence in R, so there exists an order interval $[a, b]$ such that $a \leq x_n \leq b$ for all n. This implies $d_g\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$. Hence x_n is d_g-statistically order bounded.
ii. Let (x_n) is d_g-statistically order convergent then there exists sequence (y_n) in R such that $d_g\{n \in \mathbb{N} : |x_n - x_0| > y_n\} = 0$. Now chose $a := \inf\limits_{n \in \mathbb{N}} \{x_0 - y_n\}$ and $b := \sup\limits_{n \in \mathbb{N}} \{y_n + x_0\}$. Clearly for this a, b we get order interval $[a, b]$ for which $d_g\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$. Hence (x_n) is d_g-statistically order bounded. □

Theorem 6.3.2. Let R be an (l)-group and (x_n) be d_g-statistically order bounded sequence in R. Then, $x_n = a_n + b_n$, where (a_n) is order bounded sequence and $d_g\{n \in \mathbb{N} : b_n \neq \theta\} = 0$.

Proof. Let (x_n) be d_g-statistically order bounded sequence in R. Then, there exists an order interval $[a, b]$ such that $d_g\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$. Now, we define,

$$a_n := \begin{cases} x_n & \text{if } x_n \in [a, b] \\ \theta & otherwise. \end{cases}$$

and

$$b_n := \begin{cases} \theta & \text{if } x_n \in [a,b] \\ x_n & otherwise. \end{cases}$$

then clearly from the construction $x_n = a_n + b_n$, where (a_n) is order bounded, and $d_g\{n \in \mathbb{N} : b_n \neq \theta\} = 0$. □

Theorem 6.3.3. Let $(x_n), (y_n), (z_n)$ be sequences in R such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$. If $x_n \xrightarrow{d_g-st.ord.} x_0$, and $z_n \xrightarrow{d_g-st.ord.} x_0$, then $y_n \xrightarrow{d_g-st.ord.} x_0$ holds.

Proof. Given that $x_n \xrightarrow{d_g-st.ord.} x_0$ and $z_n \xrightarrow{d_g-st.ord.} x_0$, so there exists order sequence (p_n) and (q_n) such that $d_g\{n \in \mathbb{N} : |x_n - x_0| > p_n\} = 0$ and $d_g\{n \in \mathbb{N} : |z_n - x_0| > q_n\} = 0$. That is $|x_n - x_0| \leq p_n$ and $|z_n - x_0| \leq q_n$ for almost all n . Now clearly $p_n + q_n$ is also a order sequence and $|y_n - x_0| \leq p_n + q_n$. Hence $d_g\{n \in \mathbb{N} : |y_n - x_0| > p_n + q_n\} = 0$. So, $y_n \xrightarrow{d_g-st.ord.} x_0$. □

From the definition of d_g-statistically order boundedness the following theorem is obvious:

Theorem 6.3.4. Let (x_n) be a d_g-statistically order bounded sequence in (l)-group. Then, every subsequence of (x_n) which is dense in (x_n) is also d_g-statistically order bounded.

Definition 6.3.4. A set $M \subset \mathbb{N}$ is said to be d_g-dense subset of N if $d_g(M^c) = 0$.

Definition 6.3.5. If (n_k) is an strictly increasing sequence of natural numbers and $\tilde{x} = (x_n)$ be a sequence in R , then we write $\tilde{x}' = (x_{n_k})$ be a subsequence of $\tilde{x}$ and $K_{\tilde{x}'} = \{n_k : k \in \mathbb{N}\}$. A subsequence $\tilde{x}'$ is called an d_g-dense subsequence of $\tilde{x}$, if $K_{\tilde{x}'}$ is an d_g-dense subset of $\mathbb{N}$.

For a d_g-statistically order convergent sequence, there exist some subsequence which may not be d_g-statistically order convergent. Then the obvious question is which subsequence will be d_g-statistically order convergent ? The following theorem gives the answer.

Theorem 6.3.5. Let $(R,+)$ be a (l)-group, let $\tilde{x}=(x_n)$ be sequence in R and let $\tilde{x}'=(x_{n_k})$ be a subsequence of $\tilde{x}$ such that $\liminf\limits_{n\to\infty}\frac{g(|K_{\tilde{x}}(n)|)}{g(n)}>0$. If $\tilde{x}$ is d_g-statistically order convergent to $x_0\in R$, then $\tilde{x}'$ is also d_g-statistically order convergent to x_0.

Proof. Suppose that $\tilde{x}=(x_n)$ is d_g-statistically order convergent to x_0. Then there exist e sequence (y_n) in R such that $y_n\downarrow\theta$ and $d_g\{n\in\mathbb{N}:|x_n-x_0|>y_n\}=0$. let $\tilde{x}'=(x_{n_k})$ be a subsequence of $\tilde{x}$ such that $\liminf\limits_{n\to\infty}\frac{g(|K_{\tilde{x}'}(n)|)}{g(n)}>0$.
Now $\{n_k\in\mathbb{N}:|x_{n_k}-x_0|>y_{n_k}\}\subseteq\{n\in\mathbb{N}:|x_n-x_0|>y_n\}$.
Thus, we can write,

$$\frac{|\{n_k\in\mathbb{N}:|x_{n_k}-x_0|>y_{n_k}\}|}{g(|K_{\tilde{x}'}(n)|)}\leq\frac{|\{n\in\mathbb{N}:|x_n-x_0|>y_n\}|}{g(|K_{\tilde{x}'}(n)|)}.$$

Now we know for any two sequences (α_n) and (β_n) of non-negative real numbers with $0\neq\liminf\limits_{n\to\infty}\alpha_n<\infty$, we have $(\liminf\limits_{n\to\infty}\alpha_n)(\limsup\limits_{n\to\infty}\beta_n)\leq\limsup\limits_{n\to\infty}\alpha_n\beta_n$.

Now taking $\alpha_n=\frac{g(|K_{\tilde{x}'}(n)|)}{g(n)}$ and $\beta_n=\frac{\{n\in\mathbb{N}:|x_n-x_0|>y_n\}}{g(|K_{\tilde{x}'}(n)|)}$ and using above two inequalities and the given condition, we easily conclude that

$$\limsup_{n\to\infty}\frac{|\{n_k\in\mathbb{N}:|x_{n_k}-x_0|>y_{n_k}\}|}{g(|K_{\tilde{x}'}(n)|)}=0.$$

Hence $\tilde{x}'$ is d_g-statistically order convergent to x_0. □

Theorem 6.3.6. Let R be a (l)-group and let $\tilde{x}=(x_n)$ be a sequence in R. Then the following statements are equivalent:

(a) $\tilde{x}$ is d_g-statistically order convergent;

(b) Every subsequence $\tilde{x}'=(x_{n_k})$ of $\tilde{x}$ with $\liminf\limits_{n\to\infty}\frac{g(|K_{\tilde{x}'}(n)|)}{g(n)}>0$ is also d_g-statistically order convergent;

(c) Every d_g-statistically dense subsequence $\tilde{x}'$ of $\tilde{x}$ is d_g-statistically convergent provided that $g\in G$ is such that $0<\liminf\limits_{n\to\infty}\frac{n}{g(n)}<\infty$.

Proof. From theorem 6.3.5, we can say that $a \Rightarrow b$. Now let $\tilde{x}' = (x_{n_k})$ be d_g-statistically dense subsequence of $\tilde{x}$, so $K_{\tilde{x}'}$ is an d_g-dense subset of $\mathbb{N}$. Using the given condition we can say that

$$\limsup_{n\to\infty} \frac{|\{n_k \in \mathbb{N} : |x_{n_k} - x_0| > y_{n_k}\}|}{g(|K_{\tilde{x}'}(n)|)} = 0.$$

Hence $\tilde{x}'$ is d_g-statistically order convergent to x_0. So, $b \Rightarrow c$. Using Theorem 6.3.5 and the definition 6.3.4 and 6.3.5 we can say $c \Rightarrow a$. □

Definition 6.3.6. Two sequences $\tilde{x} = (x_n)$ and $\tilde{y} = (y_n)$ in a (l)-group R are said to be d_g-statistically equivalent if there exist d_g-dense subset M of $\mathbb{N}$ such that $x_n = y_n$ for all $n \in M$. We write it symbolically as $\tilde{x} \asymp \tilde{y}$.

Lemma 6.3.1. Let R be a (l)-group, and $\tilde{x} = (x_n)$, $\tilde{y} = (y_n)$ be two sequences in R. If $\tilde{x}$ is d_g- statistically order convergent to x_0 and $\tilde{x} \asymp \tilde{y}$, then $\tilde{y}$ is also d_g-statistically order convergent to x_0.

Proof. Since $\tilde{x} \asymp \tilde{y}$, there is d_g-dense set $M \subset \mathbb{N}$ such that $x_n = y_n$ for all $n \in M$. Hence clearly for any sequence (t_n) in R with $t_n \downarrow \theta$ we have $\{n : |y_n - x_0| > t_n\} \subset M^c \cup \{n : |x_n - x_0| > t_n\}$. Since $\tilde{x}$ is d_g- statistically order convergent to x_0 so, $d_g(\{n : |x_n - x_0| > t_n\}) = 0$. Also M is d_g-dense i.e. $d_g(M^c) = 0$. This implies that $d_g(\{n : |y_n - x_0| > t_n\}) = 0$. Hence $\tilde{y}$ is also d_g- statistically order convergent to x_0. □

Lemma 6.3.2. Let R be a (l)-group and $\tilde{x} = (x_n), \tilde{y} = (y_n)$ be two sequences in R such that $\tilde{x} \asymp \tilde{y}$ (d_g-statistically). If K is a subset of $\mathbb{N}$ such that $\liminf_{n\to\infty} \frac{g(|K(n)|)}{g(n)} > 0$ and if $\tilde{x}' = (x_{n_k})$ and $\tilde{y}' = (y_{n_k})$ are subsequences of $\tilde{x}, \tilde{y}$ respectively such that $K_{\tilde{x}'} = K_{\tilde{y}'} = K$ then the relation $\tilde{x}' \asymp \tilde{y}'$ (d_g-statistically) is true.

Proof. The proof is similar to the usual case with some trivial modification and so we omit it. □

Theorem 6.3.7. Let R be a (l)-group and $\tilde{x} = (x_n)$ be d_g-statistically order convergent to x_0. Suppose that $\tilde{x}' = (x_{n_k})$ is a subsequence of $\tilde{x}$ for which there are

$\tilde{y} = (y_n)$ and $\tilde{y}' = (y_{n_k})$ subsequences of $\tilde{y}$ such that (i) $\tilde{x} \asymp \tilde{y}$ (d_g-statistically) and $K_{\tilde{x}'} = K_{\tilde{y}'}$. (ii) $\tilde{y}'$ is not d_g-statistically order convergent. Then $\liminf\limits_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{g(n)} = 0$ provided that $g : \mathbb{N} \to [0, \infty)$ satisfying the inequalities $0 < \liminf\limits_{n\to\infty} \frac{n}{g(n)}$ and $\limsup\limits_{n\to\infty} \frac{n}{g(n)} < \infty$.

Proof. If possible suppose that $\liminf\limits_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{g(n)} > 0$. Then

$$\liminf_{n\to\infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} \geq (\liminf_{n\to\infty} \frac{g(|K_{\tilde{x}'}(n)|)}{|K_{\tilde{x}'}(n)|})(\liminf_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{g(n)}) > 0.$$

Let $\tilde{y} = (y_n)$ be a sequence in R and $\tilde{y}' = (y_{n_k})$ be a subsequence of $\tilde{y}$ such that (i) and (ii) holds. Then we have $K_{\tilde{x}'} = K_{\tilde{y}'}$ and $\tilde{x} \asymp \tilde{y}$ (d_g-statistically). Thus it follows from Lemma 6.3.2 that $\tilde{x}' \asymp \tilde{y}'$ (d_g-statistically). Now as $\tilde{x} = (x_n)$ be d_g-statistically order convergent to x_0 and $\liminf\limits_{n\to\infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} \geq 0$, by applying Theorem 3.10 we conclude that $\tilde{x}'$ is also d_g-statistically convergent to x_0. Since $\tilde{x}' \asymp \tilde{y}'$ (d_g-statistically), by Lemma 3.1, $\tilde{y}'$ is also d_g-statistically convergent to x_0. Which contradicts (ii). Hence $\liminf\limits_{n\to\infty} \frac{|K_{\tilde{x}'}(n)|}{g(n)} = 0$. □

Theorem 6.3.8. Let R be a (l)-group and $\tilde{x} = (x_n), \tilde{y} = (y_n)$ be two sequences in R. If $\tilde{x}$ is d_g-statistically order bounded and $\tilde{x} \asymp \tilde{y}$, then $\tilde{y}$ is also d_g-statistically order bounded.

Proof. Let $\tilde{x} = (x_n)$ is d_g-statistically order bounded then there exists an order interval $[a, b]$ such that $d_g\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$. As $\tilde{x} \asymp \tilde{y}$, there exist d_g-dense subset M of $\mathbb{N}$ such that $x_n = y_n$ for all $n \in M$. Equivalently $d_g\{n \in \mathbb{N} : x_n \neq y_n\} = 0$. Now $\{n \in \mathbb{N} : y_n \notin [a, b]\} \subseteq \{n \in N : x_n \neq y_n\} \cup \{n \in \mathbb{N} : x_n \notin [a, b]\}$. This implies $d_g\{n \in \mathbb{N} : y_n \notin [a, b]\} = 0$. Hence $\tilde{y}'$ is also d_g-statistically order bounded. □

Note 6.3.1. Let R be a (l)-group and $\tilde{x} = (x_n)$ be a sequence in R. If every subsequence $\tilde{x}' = (x_{n_k})$ of $\tilde{x}$ with $\liminf\limits_{n\to\infty} \frac{g(|K_{\tilde{x}'}(n)|)}{g(n)} > 0$ is d_g-statistically order bounded then $\tilde{x}$ must be d_g-statistically order bounded.

Theorem 6.3.9. Let $\tilde{x} = (x_n)$ be a d_g-statistically order bounded sequence in a (l)-group R. Then the sequence $\tilde{x}$ has at least one order bounded subsequence.

Proof. As $\tilde{x} = (x_n)$ is a d_g-statistically order bounded sequence in a (l)-group R, then there exists an order interval $[a, b]$ in R such that $d_g\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$. Let $A = \{n \in \mathbb{N} : x_n \in [a, b]\}$ and $B = \{n \in \mathbb{N} : x_n \notin [a, b]\}$, then $d_g(A) = 1$ and $d_g(B) = 0$. Let $n_1 \in \mathbb{N}$ be the minimal element of A and $x_{n_1} \in [a, b]$. Since $d_g(A) = 1$, it can be chosen $n_2 \geq n_1$ such that n_2 is the minimal element of the set $\{n : n > n_1, n \in A\}$ satisfying $x_{n_2} \in [a, b]$. Continuing this process, after k-th step we can chose $n_k \geq n_{k-1}$ which is the minimal element of the set $\{n : n > n_{k-1}, n \in A\}$ such that $x_{n_k} \in [a, b]$. So we obtain a non-decreasing sequence n_k such that $\tilde{x}' = (x_{n_k})$ is the subsequence of x and satisfying $x_{n_k} \in [a, b]$ for all $n_k \in \mathbb{N}$. So the subsequence $\tilde{x}$ is order bounded. □

So, Every d_g-statistically order bounded sequence in a (l)-group has at least one order bounded subsequence.

The matter of this chapter is based on the following research paper:

Sudip Kumar Pal, Sagar Chakraborty, Some Further Results On Order Convergence of Sequences in (l)-groups, Southeast Asian Bulletin of Mathematics, Accepted, (ESCI)

CHAPTER 7

Generalized Order convergence in (l)-group

Chapter 7

Generalized Order convergence in (l)-group

7.1 Introduction

Very recent time some applications of ideal convergence was studied in Riesz spaces [3]. In 2012, Boccuto and others introduced some important ideas in (l)-group [12]. They extend the idea of $\mathcal{I}$-convergence, namely, order convergence, order Cauchy condition. Continuing the same direction, in present chapter in first we introduce the notion of f-statistical relative order convergence which are more general than the previous. We examine some of its consequences. We will adopt the definitions and notations in [54] and studying some results of [3, 12, 25, 53] in (l)-group context. One can see [5, 6, 8, 9] for more works in this direction where many more references can be found.

The aim of the second section is to continue the investigation discussed in [22, 23, 63] and study similar problems in more general form. We also deal with some generalized relative order convergence notion and f-statistical uniform Cauchy criterion in (l)-groups context. Further we study some properties of d_g-(e)uniform pre-Cauchy sequence.

7.2 f-statistical relative order convergence

Using f-density we can generalize the notion of statistical relative order convergence in (l)-group. We call it f-statistical relative order convergence. Throughout this discussion f denotes unbounded modulus function and $\sharp A$ denotes the cardinality of the set A, we use this symbol to overcome the confusion with usual modulus, as both are used in this chapter.

Definition 7.2.1. Let $e \in R^+$. A sequence (x_n) in R is said to be converges e-uniformly to $x_0 \in R$ if there exists a sequence (y_n) in R such that $y_n \downarrow \theta$ with $y_n \geq \theta$ and $|x_n - x_0| \leq e \cdot y_n$, for all n.

We say that a sequence (x_n) in R converges relatively uniformly to $x_0 \in R$ if it is converges e-uniformly to x_0 for some $e \in R^+$.

Definition 7.2.2. Let $e \in R^+$. A sequence (x_n) in R is said to be f-statistical e-order convergent to $x_0 \in R$ if there exists a sequence (y_n) in R such that $y_n \downarrow \theta$ with $y_n \geq \theta$ and $d_f(\{n \in \mathbb{N} : |x_n - x_0| > e \cdot y_n\}) = 0$.
i.e.

$$\limsup_{n\to\infty} \frac{f(\sharp\{n \in \mathbb{N} : |x_n - x_0| > e \cdot y_n\})}{f(n)} = 0.$$

We say that a sequence (x_n) in R converges f-statistical relatively order convergent to $x_0 \in R$ if it is f-statistical e-order convergent to x_0 for some $e \in R^+$.

Note 7.2.1. f-statistical relatively order convergence and statistical order convergence coincide when we take $e = I$ (identity element with respect to the operation) and the modulus function as the identity mapping.

In the line of [63], a characterization result for f-statistical e-order convergent sequences is as follows:

Theorem 7.2.1. Let R be a (l)-group and $e \in R^+$. Any sequence (x_n) in R is f-statistical e-order convergent to $x_0 \in R$ if and only if there exist a sequence (y_n) in R such that $y_n \downarrow \theta$ with $y_n \geq \theta$ such that $|x_n - x_0| \leq e \cdot y_n$ for a.a.n.

Theorem 7.2.2. Let (x_n) be a f-statistical relative order convergent sequence in R. Then (x_n) is f-statistically order bounded.

Proof. Let (x_n) be a f-statistical relative order convergent to $x_0 \in R$. Hence there exist $e \in R^+$ and a sequence (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$ such that $d_f(\{n \in \mathbb{N} : |x_n - x_0| > e \cdot y_n\} = 0$. Choose $a = x_0 - e \cdot y_n$ and $b = x_0 + e \cdot y_n$, then clearly for the order interval $[a, b]$, $d_f\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$. Therefore (x_n) is f-statistically order bounded. □

In usual cases every subsequence of a convergent sequence must be convergent. But in statistical convergence, this is not true in general. From the following example it is clear that a subsequence of a f-statistical relative order convergent sequence need not be f-statistical relative order convergent.

Example 7.2.1. Let us consider the following sequence in $\mathbb{R}$ defined as, for $n \neq k^2 (k \in \mathbb{N})$ take, $x_n = \frac{1}{n}$ otherwise, $x_n = n$. Then (x_n) is f-statistical relatively order convergent to 0. But the subsequence $(x_{k^2})_{k \in \mathbb{N}}$ is not order bounded, hence not f-statistical relative order convergent.

In the following theorem we show that every subsequence of a f-statistical relative order convergent sequence must be f-statistically order bounded.

Theorem 7.2.3. Let (x_n) be a f-statistical relative order convergent sequence and $(x_{n(k)})$ be any subsequence of (x_n). Then $(x_{n(k)})$ is f-statistically order bounded.

Proof. Let (x_n) be a f-statistical relative order convergent to x_0. Then the sequence (x_n) is f-statistically order bounded. Hence we get an order interval $[a, b]$, such that $d_f\{n \in \mathbb{N} : x_n \notin [a, b]\} = 0$. Now, $|\{n(k) \in \mathbb{N} : x_{n(k)} \notin [a, b]\}| \leq |\{n \in \mathbb{N} : x_n \notin [a, b]\}|$. Thus $d_f\{n(k) \in \mathbb{N} : x_{n(k)} \notin [a, b]\} = 0$. Therefore (x_n) is f-statistically bounded. □

In the following theorem we will discuss under what condition the subsequence is also convergent.

Theorem 7.2.4. Let $x = (x_n)$ be a sequence in R and let $\tilde{x} = (x_{n(k)})$ be a subsequence of x, with $\liminf_{n\to\infty} \frac{f(\sharp K_{\tilde{x}}(n))}{f(n)} > 0$. If x is f-statistical e-order convergent to $x_0 \in R$, then $\tilde{x}$ is also f-statistical e-order convergent to x_0.

Proof. Suppose that x is f-statistical e-order convergent to x_0. Then there exist a sequence (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$ such that $d_f\{n \in \mathbb{N} : |x_n - x_0| > e \cdot y_n\} = 0$. Let $\tilde{x} = (x_{n(k)})$ be a subsequence of x such that $\liminf_{n\to\infty} \frac{f(\sharp K_{\tilde{x}}(n))}{f(n)} > 0$. Now $\{n(k) \in \mathbb{N} : |x_{n(k)} - x_0| > e \cdot y_{n(k)}\} \subseteq \{n \in \mathbb{N} : |x_n - x_0| > e \cdot y_n\}$.
Thus we can write,

$$\frac{\sharp\{n(k) \in \mathbb{N} : |x_{n(k)} - x_0| > e \cdot y_{n(k)}\}}{f(\sharp K_{\tilde{x}}(n))} \leq \frac{\sharp\{n \in \mathbb{N} : |x_n - x_0| > e \cdot y_n\}}{f(\sharp K_{\tilde{x}}(n))}.$$

Again we know for any two sequences (α_n) and (β_n) of non-negative real numbers with $0 \neq \liminf_{n\to\infty} \alpha_n < \infty$, we have $(\liminf_{n\to\infty} \alpha_n)(\limsup_{n\to\infty} \beta_n) \leq \limsup_{n\to\infty} \alpha_n\beta_n$.

Now, taking $\alpha_n = \frac{f(\sharp K_{\tilde{x}}(n))}{f(n)}$ and $\beta_n = \frac{\{n\in\mathbb{N}:|x_n - x_0|>e\cdot y_n\}}{f(\sharp K_{\tilde{x}}(n))}$ and using above two inequalities with the given condition, we can easily conclude that

$$\limsup_{n\to\infty} \frac{\sharp\{n(k) \in \mathbb{N} : |x_{n(k)} - x_0| > e \cdot y_{n(k)}\}}{f(\sharp K_{\tilde{x}}(n))} = 0.$$

Hence $\tilde{x}$ is f-statistical e-order convergent to x_0. □

Theorem 7.2.5. Let (x_n) be a sequence in R and $e \in R^+$. The following are equivalent : (1) (x_n) in R is f-statistical e-order convergent to $x_0 \in R$.
(2) There exists a sequence (t_n) in R with $(x_n), (t_n)$ are f-statistically equivalent and the sequence (t_n) converges e-uniformly to $x_0 \in R$.

Proof. Let (x_n) in R be f-statistical e-order convergent to $x_0 \in R$. Then there exists a subset $K = \{n(k) : k \in N\}$ such that $(x_{n(k)})$ converges e-uniformly to x_0. We set $t_n = x_n$ if $n \in K$, otherwise $t_n = x_0$. Then clearly (x_n) and (t_n) are f-statistically equivalent. Also the sequence (t_n) in R converges e-uniformly to $x_0 \in R$. □

7.3 Generalized uniform Cauchy sequences

In this section we discuss about the concept of generalized uniform Cauchy sequence using f-density and weighted density. First we consider the following definitions.

Definition 7.3.1. Let R be a (l)-group and $e \in R^+$. A sequence (x_n) in R is said to be e-uniform Cauchy sequence if for any (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$ there exists a natural number N such that $|x_n - x_m| \leq e \cdot y_n$ holds for all $m, n \geq N$.

We say that a sequence (x_n) is uniform Cauchy if it is e-uniform Cauchy for some $e \in R^+$.

Definition 7.3.2. Let $e \in R^+$. A sequence (x_n) in R is said to be f-statistical e-uniform Cauchy sequence if for any (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$ there exists a natural number m such that $d_f(\{n \in \mathbb{N} : |x_n - x_m| > e \cdot y_n\} = 0$.

We say that a sequence (x_n) is f-statistical uniform Cauchy if it is f-statistically e-uniform Cauchy for some $e \in R^+$.

Definition 7.3.3. Let (x_n) be sequence in R and $e \in R^+$. (x_n) is said to be d_g- (e) uniform Cauchy sequence if for any (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$ there exists a natural number m such that $d_g(\{n \in \mathbb{N} : |x_n - x_m| > e \cdot y_n\} = 0$.

We say that a sequence (x_n) is d_g- uniform Cauchy if it is d_g-(e) uniform Cauchy for some $e \in R^+$.

Note 7.3.1. From the definition it is obvious that every e-uniform Cauchy sequence is also f-statistical e-uniform Cauchy and d_g-(e) uniform Cauchy.

Theorem 7.3.1. Let $e \in R^+$ and (x_n) be a sequence in R. Then (x_n) is a f-statistically e-uniform Cauchy sequence if and only if there exists a subset $A = \{n(k) : k \in \mathbb{N}\}$ of $\mathbb{N}$ with $d_f(A) = 1$ such that $(x_n)_{n \in A}$ is an e-uniform Cauchy sequence.

Proof. Let there exist a subset $A = \{n(k) : k \in \mathbb{N}\}$ of $\mathbb{N}$ with $d_f(A) = 1$ and $(x_n)_{n \in A}$ be a e-uniform Cauchy sequence. Then for (y_n) in R, with $y_n \downarrow \theta$ and

$y_n \geq \theta$ there exists a natural number N_0 such that $|x_{n(i)} - x_{n(j)}| \leq e \cdot y_n$ holds for $i, j, n \geq N_0$. Now $\{n(k) \in A : k \geq N_0\} \subseteq \{n \in \mathbb{N} : |x_n - x_{n(N_0)}| \leq e \cdot y_n\}$. This implies $|x_n - x_{n(N_0)}| \leq e \cdot y_n$ a.a.n. So for (y_n) in R we get a natural number $m = n(N_0)$ such that $d_f(n \in \mathbb{N} : |x_n - x_m| > e \cdot y_n) = 0$. Hence (x_n) is a f-statistically e-uniform Cauchy.

Conversely, let (x_n) be a f-statistically e-uniform Cauchy sequence, then for each (y_n) in R, with $y_n \downarrow \theta$ and $y_n \geq \theta$, we get a natural number m, such that $d_f(A_m) = 1$, where $A_m = \{n \in \mathbb{N} : |x_n - x_m| \leq e \cdot y_n\}$. So we get a countable collection A_m of subsets of $\mathbb{N}$ such that $d_f(A_m) = 1$. Hence there exist a set $A \subseteq \mathbb{N}$ such that $A \setminus A_m$ is finite and $d_f(A) = 1$. Clearly $(x_n)_{n \in A}$ is an e-uniform Cauchy sequence. □

A similar result hold for weighted density. We just state the theorem.

Theorem 7.3.2. Let $e \in R^+$ and (x_n) be a sequence in R. Then (x_n) is a d_g-(e) uniform Cauchy sequence if and only if there exists a subset $A = \{n(k) : k \in \mathbb{N}\}$ of $\mathbb{N}$ with $d_g(A) = 1$ such that $(x_n)_{n \in A}$ is an e-uniform Cauchy sequence.

Theorem 7.3.3. Every monotone f-statistical e-uniform Cauchy sequence in R is e-uniform Cauchy for some $e \in R^+$.

Proof. Let (x_n) be an increasing f-statistical e-uniform Cauchy sequence. So we get a subset $A = \{n(k) : k \in \mathbb{N}\}$ of $\mathbb{N}$ with $d_f(A) = 1$ such that $(x_n)_{n \in A}$ is an e-uniform Cauchy sequence. Then for (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$ we get a natural number N_0 such that $|x_{n(i)} - x_{n(j)}| \leq e \cdot y_n$ holds for all $i, j \geq N_0$. Now for all $m > n > n(N_0)$, if $n(k) \leq n \leq m \leq n(k+1)$ for some $k > N_0$ we have $0 \leq x_m - x_n \leq x_{n(k+1)} - x_{n(k)} \leq e \cdot y_n$. If $n(k) \leq n \leq n(k+1)$ and $n(l) \leq m \leq n(l+1)$ for $k, l > N_0$ then also $0 \leq x_m - x_n \leq x_{n(l+1)} - x_{n(k)} \leq e \cdot y_n$. Hence (x_n) is e-uniform Cauchy. □

7.4 Generalized pre-Cauchy sequence

In paper [23], the authors introduce statistical pre-Cauchy sequence and also investigate relationship with Cauchy sequence. In line of [23], [33], [63] we introduce

d_g-uniform pre-Cauchy sequence using weighted density proposed in [7].

Definition 7.4.1. Let R be a (l)-group and $e \in R^+$. A sequence (x_n) in R is said to be d_g-(e) uniform pre-Cauchy if for any (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$,

$$\lim_{n\to\infty} \frac{\sharp\{(i,j) : i,j \leq n, |x_i - x_j| > e \cdot y_n\}}{(g(n))^2} = 0.$$

We say that a sequence (x_n) in R is d_g-uniform pre-Cauchy if it is d_g-(e) uniform pre-Cauchy for some $e \in R^+$.

Theorem 7.4.1. Every d_g-(e) uniform Cauchy sequence in R is d_g-(e) uniform pre-Cauchy.

Proof. Let (x_n) be a d_g-(e) uniform Cauchy sequence in R. So there exists a subset $A = \{n(k) : k \in \mathbb{N}\}$ with $d_g(A) = 1$ such that $(x_{n(k)})$ is e-uniform Cauchy. Then for (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$, there exists $B \subseteq A$ such that $A \setminus B$ is finite and $B \times B \subset \{(i,j) : |x_i - x_j| \leq e \cdot y_n\}$.
Now for each $n \in \mathbb{N}$, we have

$$(\sharp B_n)^2 = \sharp\{(i,j) \in B \times B : i,j \leq n\}.$$

Hence,
$$\frac{(\sharp B_n)^2}{(g(n))^2} \leq \frac{\sharp\{(i,j) : i,j \leq n, |x_i - x_j| \leq e \cdot y_n\}}{(g(n))^2}.$$
Since, $\lim_{n\to\infty} d_g(B_n) = 1$, it follows that

$$\lim_{n\to\infty} \frac{|\{(i,j) : i,j \leq n, |x_i - x_j| \leq e \cdot y_n\}|}{(g(n))^2} = 1.$$

This ends the proof. □

The converse is not true in general [Example 8, [50]].

Theorem 7.4.2. Let R be a (l)-group and $e \in R^+$. Also let (x_n) be a d_g-(e)uniform pre-Cauchy sequence in R. If (x_n) has a subsequence $\tilde{x} = (x_{n(k)})$ which converges

(e)-uniformly to $x_0 \in R$ and $\liminf_{n\to\infty} \frac{\sharp K_{\tilde{x}}(n)}{g(n)} > 0$ then (x_n) converges d_g-(e) uniformly to $x_0 \in R$.

Proof. Let (x_n) be a d_g-(e) uniform pre-Cauchy sequence and $\tilde{x} = (x_{n(k)})$ converges (e)-uniformly to $x_0 \in R$. Then for (y_n) in R with $y_n \downarrow \theta$ and $y_n \geq \theta$, there exists a $A \subseteq K_{\tilde{x}}$ such that $K_{\tilde{x}} \setminus A$ is finite and $|x_i - x_0| \leq e \cdot y_n$ for all $i \in A$. Now we take $B = \{j \in \mathbb{N} : |x_j - x_0| > e \cdot y_n\}$.

Then for each $n \in \mathbb{N}$ we have $A(n) = \{i \in A : i \leq n, |x_i - x_0| \leq e \cdot y_n\}$ and similar result for $B(n)$. As $\liminf_{n\to\infty} \frac{\sharp K_{\tilde{x}}(n)}{g(n)} > 0$ so, $\liminf_{n\to\infty} \frac{\sharp A(n)}{g(n)} > 0$. Now

$$\frac{\sharp A(n)}{g(n)} \frac{\sharp B(n)}{g(n)} \leq \frac{\sharp\{(i,j) : i,j \leq n, |x_i - x_j| > e \cdot y_n\}}{(g(n))^2}.$$

As (x_n) is d_g-(e) uniform pre-Cauchy sequence in R and using the fact that $\liminf_{n\to\infty} \frac{\sharp A(n)}{g(n)} > 0$, taking $n \to \infty$ we get $d_g(B) = 0$. Hence (x_n) converges d_g-(e) uniformly to $x_0 \in R$. □

7.5 Rough ideal order convergence

In this section we introduce the concept of rough ideal order convergence of sequences in (l)-group R and we study some results related to it. We first introduce our main definition:

Definition 7.5.1. A sequence (x_n) in R is said to be rough ideal order convergent(r-$\mathcal{I}$-order convergent) to $x \in R$ if there exists an order sequence (y_n) with roughness degree $r \in R$ provided $\{n : |x_n - x| \leq r + y_n\} \in F(\mathcal{I})$.

In symbol we write it as $x_n \xrightarrow{r-\mathcal{I}-order} x$ in R. We call x as r-$\mathcal{I}$-order limit of (x_n), that is, (r-$\mathcal{I}$-order)$\lim x_n = x$. Clearly if $r = 0$, we get the ideal order convergence. So we always take $r > 0$.

Example 7.5.1. Let us consider usual order on additive group $\mathbb{R}$. Consider the sequence $x_n = (-1)^n, n \in \mathbb{N}$. This is a divergent sequence in classical sense. But if we take $r = 2$, and consider the order sequence (y_n) as $y_n = \frac{1}{n}$, then clearly $x_n \xrightarrow{2-\mathcal{I}-order} x$, where x is any real number between -1 and 1.

In usual convergence we see that convergent sequence has unique limit. But in r-$\mathcal{I}$-order convergence, the case is different. From the above example we see that r-$\mathcal{I}$-order limit is not unique. For this we define r-$\mathcal{I}$-order limit set. We denote it by $\mathcal{I} - \text{LIM}^r x_n$. It is evident that (x_n) is r-$\mathcal{I}$-order convergent if and only if $\mathcal{I} - \text{LIM}^r x_n \neq \phi$.

Definition 7.5.2. Let (x_n) be a sequence in R. Then $\mathcal{I} - \text{LIM}^r x_n := \{x \in R : x_n \xrightarrow{r-\mathcal{I}-order} x\}$.

For above example $\mathcal{I} - \text{LIM}^r x_n = (-1, 1)$.

Definition 7.5.3. A sequence (x_n) in R is said to be $\mathcal{I}$-order bounded if there exists an order interval $[a, b]$ such that $\{n \in \mathbb{N} : x_n \in [a, b]\} \in F(\mathcal{I})$.

In usual convergence we see that every convergent sequence is bounded. In r-$\mathcal{I}$-order convergence the sequence is $\mathcal{I}$-order bounded.

Next we will discuss some properties of r-$\mathcal{I}$-order convergent sequences.

Theorem 7.5.1. Let (x_n) be a r-$\mathcal{I}$-order convergent sequence in R. Then (x_n) is $\mathcal{I}$-order bounded.

Proof. Let (x_n) be a r-$\mathcal{I}$-order convergent sequence in R. So there exists an order sequence (y_n) with roughness degree $r \in R$ such that $\{n : |x_n - x| \leq r + y_n\} \in F(\mathcal{I})$. We know that $|x_n - x| = (x_n - x) \vee -(x_n - x)$. Choose $a = x + r + y_n$ and $b = x - r - y_n$, then $\{n \in \mathbb{N} : x_n \in [a, b]\} \in F(\mathcal{I})$. Hence (x_n) is $\mathcal{I}$-order bounded. □

The following two theorems are straightforward. We just sketch the proof.

Theorem 7.5.2. If $(r_1$-$\mathcal{I}$-order$)$ $\lim x_n = a$ and $(r_2$-$\mathcal{I}$-order$)$ $\lim y_n = b$ then $((r_1 + r_2)$-$\mathcal{I}$-order$)$ $\lim(x_n + y_n) = a + b$, where (x_n) and (y_n) are two sequences in R and $r_1, r_2 \in R$.

Proof. Given, (r_1-$\mathcal{I}$-order) $\lim x_n = a$ and (r_2-$\mathcal{I}$-order) $\lim y_n = b$. Hence for any order sequence (t_n), $\{n \in \mathbb{N} : |x_n - a| \leq r_1 + t_n\}$, $\{n \in \mathbb{N} : |y_n - b| \leq r_2 + t_n\} \in F(\mathcal{I})$. Now $(x_n + y_n) - (a + b) = (x_n - a) + (y_n - b)$. Using the basic properties of filter we can conclude that $(r_1 + r_2)$-ord $\lim(x_n + y_n) = a + b$. □

Theorem 7.5.3. If $(x_n), (y_n), (z_n)$ are three sequences in R such that $x_n \leq y_n \leq z_n$ and (r-$\mathcal{I}$-order) $\lim x_n =$ (r-$\mathcal{I}$-order) $\lim z_n = a$ then (r-$\mathcal{I}$-order) $\lim y_n = a$.

Proof. Given that (r-$\mathcal{I}$-order) $\lim x_n =$ (r-$\mathcal{I}$-order) $\lim z_n = a$. So for order sequence (t_n), $A = \{n \in \mathbb{N} : |x_n - a| \leq r + t_n\} \in F(\mathcal{I})$ and $B = \{n \in \mathbb{N} : |z_n - a| \leq r + t_n\} \in F(\mathcal{I})$. Now $x_n \leq y_n \leq z_n$ implies $x_n - a \leq y_n - a \leq z_n - a$, hence $|y_n - a| \leq |x_n - a| + |z_n - a|$. Thus $A \cap B \subseteq \{n \in \mathbb{N} : |y_n - a| \leq r + t_n\}$. This ends the proof. □

We now have the following theorem whose proof is rather simple.

Theorem 7.5.4. A sequence (x_n) in R is $(r_1 + r_2)$-$\mathcal{I}$-order convergent to x if there exists a sequence (y_n) in R such that $y_n \xrightarrow{r_1-\mathcal{I}-order} x$ and $|x_n - y_n| \leq r_2$.

Proof. Suppose that $y_n \xrightarrow{r_1-\mathcal{I}-order} x$. Hence for an order sequence (p_n), $\{n \in \mathbb{N} : |x_n - x| \leq r_1 + p_n\} \in F(\mathcal{I})$. Also $|x_n - y_n| \leq r_2$. Which implies $\{n \in \mathbb{N} : |x_n - x| \leq (r_1 + r_2) + p_n\} \in F(\mathcal{I})$. Hence (x_n) is $(r_1 + r_2)$-$\mathcal{I}$-order convergent to x. □

Using the above theorem (taking $r_1 = 0$ and $r_2 = r > 0$) we can get a relation between r-$\mathcal{I}$-order convergence and order convergence as :

Theorem 7.5.5. A sequence (x_n) in R is r-$\mathcal{I}$-order convergent to x if there exists a sequence (y_n) in R such that y_n is order convergent to x and $|x_n - y_n| \leq r$.

We see that for our usual convergence limit set contains atmost one point, but for r-order convergence limit set is not a singleton set. So it is something different from our classical sense. Now we will study some interesting properties of r-$\mathcal{I}$-limit set.

Theorem 7.5.6. For a r-$\mathcal{I}$-order convergent sequence (x_n), the set $\mathcal{I}$-*LIM*$^r x_n$ is closed and convex.

Proof. Let (y_n) be a sequence in $\mathcal{I}$-*LIM*$^r x_n$ which is order convergent to $y \in R$. So there exists an order sequence (z_n) in R such that $A = \{n \in \mathbb{N} : |y_n - y| \le z_n\} \in F(I)$. Also (y_n) is a sequence in $\mathcal{I}$-*LIM*$^r x_n$, so for each m, $x_n \xrightarrow{r-\mathcal{I}-order} y_m$. Hence there exists an order sequence (w_n) such that $B = \{n \in \mathbb{N} : |x_n - y_m| \le r + w_n\} \in F(\mathcal{I})$.

Now for $n \in A \cap B$, $|x_n - y| \le r + (z_n + w_n)$.
We know that if $A, B \in F(\mathcal{I})$ then $A \cap B \in F(\mathcal{I})$ and sum of two order sequences is also an order sequence. Which implies $y \in \mathcal{I} - \textit{LIM}^r x_n$. Hence $\mathcal{I}$-*LIM*$^r x_n$ is a closed set.

To prove $\mathcal{I}$-*LIM*$^r x_n$ is convex, first of all, we will prove that if $a \in \mathcal{I} - \textit{LIM}^{r_1} x_n$ and $b \in \mathcal{I} - \textit{LIM}^{r_2} x_n$, then $(1-t)a + tb \in \mathcal{I} - \textit{LIM}^{(1-t)r_1 + tr_2} x_n$, for all $t \in [0,1]$. As $x_n \xrightarrow{r_1-\mathcal{I}-order} a$ and $x_n \xrightarrow{r_2-\mathcal{I}-order} b$, so there exist two order sequences (a_n) and (b_n) such that $A = \{n \in \mathbb{N} : |x_n - a| \le r_1 + a_n\} \in F(\mathcal{I})$ and $B = \{n \in \mathbb{N} : |x_n - b| \le r_2 + b_n\} \in F(\mathcal{I})$. Now $x_n - (1-t)a - tb = (1-t)(x_n - a) + t(x_n - b)$. From here we can easily conclude that $x_n \xrightarrow{(1-t)r_1+tr_2)-\mathcal{I}-order} (1-t)a + tb$. That is $(1-t)a + tb \in \mathcal{I} - \textit{LIM}^{(1-t)r_1+tr_2} x_n$.

Now putting $r_1 = r_2 = r$ we easily observe that $\mathcal{I} - \textit{LIM}^r x_n$ is a convex set. □

Theorem 7.5.7. Let (x_n) be a sequence in R. Then $\mathcal{I} - \textit{LIM}^{r_1} x_n \subseteq \mathcal{I} - \textit{LIM}^{r_2} x_n$ for $r_1 < r_2$.

Proof. Let $a \in \mathcal{I} - \mathrm{LIM}^{r_1} x_n$ then there exists an order sequence (y_n) in R such that $A = \{n : |x_n - a| < r_1 + y_n\} \in F(\mathcal{I})$. As $r_1 < r_2$, this implies $r_1 + y_n < r_2 + y_n$. Let $B = \{n : |x_n - a| < r_2 + y_n\}$. Clearly $B \in F(\mathcal{I})$. Hence $a \in \mathcal{I} - \mathrm{LIM}^{r_2} x_n$. $\mathcal{I} - \mathrm{LIM}^{r_1} x_n \subseteq \mathcal{I} - \mathrm{LIM}^{r_2} x_n$. □

Definition 7.5.4. A sequence (x_n) in R is said to be r-$\mathcal{I}$-order Cauchy with roughness degree r, if for each order sequence (y_n) there exists $m \in \mathbb{N}$ such that $\{n \in \mathbb{N} : |x_n - x_m| \le r + y_n\} \in F(\mathcal{I})$.

Sometimes we denote the set $\{n \in \mathbb{N} : |x_n - x_m| \leq r + y_n\}$ by $r - \mathcal{I} - K^m_{\{y_n\}}$.

Definition 7.5.5. Let (x_n) be a sequence in R. We say that a is an r-$\mathcal{I}$-order cluster point of (x_n) if there exists a subsequence $(x_{n(k)})$ of (x_n) such that $(x_{n(k)})$ is r-$\mathcal{I}$-order convergent to a.

Theorem 7.5.8. If a sequence (x_n) in R is $r - \mathcal{I}$-order convergent then it is $2r$-$\mathcal{I}$-order Cauchy.

Proof. : Given $x_n \xrightarrow{r-\mathcal{I}-order} x$ in R then there exists an order sequence (y_n) in R such that $A = \{n \in \mathbb{N} : |x_n - x| \leq r + y_n\} \in F(\mathcal{I})$. Then for any fixed $m \in A$ and arbitrary $n \in A$, $|x_n - x_m| \leq 2r + y_n + y_n$. So (x_n) is $2r$-$\mathcal{I}$-order Cauchy. □

Theorem 7.5.9. For a r-$\mathcal{I}$-order Cauchy sequence (x_n), if a is r-$\mathcal{I}$-order cluster point then (x_n) is $2r$-$\mathcal{I}$-order convergent to a.

Proof. : Let (x_n) be a r-$\mathcal{I}$-order Cauchy sequence with r-$\mathcal{I}$-order cluster point a. So we get a subsequence $(x_{n(k)})$ of (x_n) such that $(x_{n(k)})$ is r-$\mathcal{I}$-order convergent to a. So there exists an order sequence (y_n) such that $A = \{n \in \mathbb{N} : |x_{n(k)} - a| \leq r + y_n\} \in F(\mathcal{I})$. Also as (x_n) is r-$\mathcal{I}$-order Cauchy, so for fixed $n(k) \in \mathbb{N}$, $B = \{n \in \mathbb{N} : |x_n - x_{n(k)}| \leq r + y_n\} \in F(\mathcal{I})$. Now for $n \in A \cap B$, clearly $|x_n - a| \leq 2r + 2y_n$. Hence (x_n) is $2r$-$\mathcal{I}$-order convergent to a. □

Theorem 7.5.10. Let (x_n) be a r-$\mathcal{I}$-order Cauchy sequence in R. Then for every order sequence (y_n) there exists $A \in \mathbb{N}$, where $A \in F(\mathcal{I})$ such that $|x_a - x_b| \leq 2r + 2y_n$, for all $a, b \in A$.

Proof. Let (x_n) be a r-$\mathcal{I}$-order Cauchy sequence in R. So for the order sequence (y_n), we get $m \in \mathbb{N}$ such that $A = \{n \in \mathbb{N} : |x_n - x_m| \leq r + y_n\} \in F(\mathcal{I})$. Now for $a, b \in A$, $|x_a - x_m| \leq r + y_n$ and $|x_b - x_m| \leq r + y_n$. Hence clearly $|x_a - x_b| \leq 2r + 2y_n$. □

Theorem 7.5.11. Let (x_n) be a r-$\mathcal{I}$-order Cauchy sequence in R. Then for every order sequence (y_n), $\{m \in \mathbb{N} : 2r - \mathcal{I} - K^m_{\{y_n\}} \in \mathcal{I}\} \in F(\mathcal{I})$.

Proof. Let (x_n) be a r-$\mathcal{I}$-order Cauchy sequence in R. Also let (y_n) be any order sequence. So by the above theorem there exists $A \in F(\mathcal{I})$, where $A \subseteq \mathbb{N}$, such that for all $a, b \in A$, $|x_a - x_b| \leq 2r + 2y_n$. Let $m \in \mathbb{N}$ be such that $2r - \mathcal{I} - K^m_{\{y_n\}} \in \mathcal{I}$, that is, $\{n \in \mathbb{N} : |x_n - x_m| \leq 2r + y_n\} \in \mathcal{I}$. Now clearly $A \subset \{m \in \mathbb{N} : 2r - \mathcal{I} - K^m_{\{y_n\}} \in \mathcal{I}\}$. This ends the proof. □

Now the obvious question is what about converse ? For converse part the condition will be slightly different.

Theorem 7.5.12. Let (x_n) be a sequence in R. If for every order sequence (y_n), $\{m \in \mathbb{N} : r - \mathcal{I} - K^m_{\{y_n\}} \in \mathcal{I}\} \in F(\mathcal{I})$, then (x_n) is $2r$-$\mathcal{I}$-order Cauchy.

Proof. : Let $A = \{m \in \mathbb{N} : r - \mathcal{I} - K^m_{\{y_n\}} \in \mathcal{I}\}$. By the given condition $A \in F(\mathcal{I})$. Hence $A \neq \phi$. Let $p \in A$, so $r - \mathcal{I} - K^p_{\{y_n\}} \in \mathcal{I}$. Which implies $\{n \in \mathbb{N} : |x_n - x_p| \leq r + y_n\} \in F(\mathcal{I})$. Hence (x_n) is $2r$-$\mathcal{I}$-order Cauchy sequence. □

This gives some idea about the converse part.

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Sudip Kumar Pal, Sagar Chakraborty, on certain generalized summability methods of order convergence in (l)-groups, Journal of Analysis, (Published online 08 February, 2022) (SCOPUS, ESCI).

List of Publications

1. Sudip Kumar Pal, **Sagar Chakraborty**, On Generalized Statistical Convergence and Boundedness of Riesz Space-Valued Sequences, **Filomat** Vol.33, No.15(2019), pp 4989-5002. (SCOPUS, SCIE)

2. **Sagar Chakraborty**, Sudip Kumar Pal, On some generalized summability method of Nets in locally solid Riesz spaces, **Southeast Asian Bulletin of Mathematics**, Vol.45(3) (2021), pp 317-327. (ESCI)

3. Sudip Kumar Pal, **Sagar Chakraborty**, On certain new notion of Order Cauchy sequences, continuity in (l)-group, **Applied General Topology** Vol.23(1) (2022), pp 55-68 (SCOPUS, ESCI)

4. Sudip Kumar Pal, **Sagar Chakraborty**, on certain generalized summability methods of order convergence in (l)-groups, **Journal of Analysis** (Published online 08 February, 2022) (SCOPUS, ESCI).

5. Sudip Kumar Pal, **Sagar Chakraborty**, Some Further Results On Order Convergence of Sequences in (l)-groups, **Southeast Asian Bulletin of Mathematics** Accepted, (ESCI)

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