Characterization of Wormholes and Cosmological Consequences: The Emergent Scenario

by

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Characterization of Wormholes and Cosmological Consequences: The Emergent Scenario**" submitted by **Sri Dhritimalya Roy** who got his name registered on 17.02.2022 (Index No. 47/22/Phys/27) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of Subenoy Chakraborty and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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I hereby affirm that this thesis is the result of my own work conducted at the Department of Mathematics, Jadavpur University, Kolkata - 700032, India. Furthermore, I declare that none of its content has been previously submitted for any degree, diploma, fellowship, or other qualification at any other university or institution.

All figures presented in this thesis were generated by the author using Mathematica software. Extensive checks have been conducted to ensure the accuracy and coherence of the content, yet vigilant readers may detect occasional errors or discrepancies. The author accepts full responsibility for any such oversights, which may arise from insufficient understanding of the subject matter or oversight.

Lastly, I confirm that all assistance received in the preparation of this thesis has been duly acknowledged and cited in accordance with academic standards.

Dhritimalya Roy
25/7/24

Dhritimalya Roy

To

My Father

Late Debabrata Roy

and

My Uncle

Late Dipak Kumar Roy

PREFACE

Wormholes, which are extraordinary occurrences originating from the principles of General Relativity, provide theoretical pathways across the fabric of spacetime, potentially enabling journeys between remote areas or even different universes. The recent groundbreaking finding of shadows cast by supermassive black holes and the direct observation of gravitational waves have sparked increased curiosity in pinpointing other extraordinary objects and entities that might imitate black holes, thus underscoring the significance of wormholes in contemporary astrophysics. Recent investigations have notably delved into the domain of wormholes composed of non-exotic matter, leading to a thorough exploration of their navigability and the nature of material at their entrances. This inquiry indicates a considerable likelihood of non-exotic matter wormholes with some transitioning from non-exotic to exotic matter as time progresses.

On the other hand, traditional big bang cosmology is confronted with challenges similar to those observed at the event horizon of a black hole, encompassing issues related to the horizon and singularity. Therefore, there exists a considerable interest in exploring cosmological hypotheses that do not exhibit these deficiencies. Ellis and Marteens introduced a model, the Emergent Universe (EU), which effectively tackles the singularity issue inherent in the framework of Einstein's general relativity.

Hence, this thesis explores the mathematical analysis of potential existence of wormhole solutions in different cosmological theories, investigating their viability and stability. Wormholes can also act as Closed timelike curves(CTC), hence, we also explore the particle motion and confinements in the context of cylindrical wormholes admitting to the presence of CTCs. Moreover, wormholes are free of singularity, so, the viability of dynamical wormholes in the context of Emergent Universe is also studied. Through rigorous derivations and analysis, it delves into the theoretical framework underpinning wormholes, shedding light on their physical properties, their validity of energy conditions and implications within the context of general relativity and modified theory of gravity.

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LIST OF PUBLICATIONS

The work of this thesis has been carried out at the Department of Mathematics, Jadavpur University, Kolkata- 700032, India. The thesis is based on the following papers:

- Chapter 2 has been published as *"Does violation of cosmic no-hair conjecture guarantee the existence of wormhole?"*, **D. Roy** , A. Dutta and S. Chakraborty, **EPL (Europhysics Letters) 140(1):19002**, DOI : 10.1209/0295-5075/ac969d.
- Chapter 3 has been communicated as *"Geodesic Motion and Particle Confinements in Cylindrical Wormhole Spacetime: Exploring Closed Timelike Curves."*, **D. Roy** , A. Dutta and S. Chakraborty, in **Mod. Phys. Lett. A** ,
- Chapter 4 has been published as *"Does Dynamical Wormhole Evolve From Emergent Scenario?"*, **D. Roy** , A. Dutta, B. Ghosh and S. Chakraborty, in **New Astronomy, Volume 111, October 2024, 102248**, DOI : https://doi.org/10.1016/j.newast.2024.102248.
- Chapter 5 has been communicated as *"Evolving Wormholes in $f(R,T)$ Gravity: An Analysis"*, **D. Roy** , A. Dutta, B. Ghosh and S. Chakraborty, in **IJGMMP**,

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Wesley Snipes rightly conforms, “Do not stop when you’re tired. Stop when you’re done.” Enveloped by the gruelling impediments and strenuous impediments of life, I have been able to comprehend with each passing day that besides hard work, determination, and commitment, what encourages an individual towards the zenith of triumph is encouragement by others, and the emboldening of oneself. There lies supreme power in being capable to life someone up, if you desire to lift yourself. Henceforth, there has not been a single pause- to my aspirations, my abilities, my strengths, my urge to know more. I am proud of myself for having being able to deal with the convolutions, intricacies, strengths, and shortcomings, and fossick through the facets of my concerned field of research with an unvacillating spirit of perseverance. The journey has significantly accorded to my personal enhancements, scholastic proficiency, thirst for knowledge, and paved the path to a pre-eminent future.

However, the burgeoning achievements, desire for success, and the littlest hope to keep going despite the chaotic humdrums of life, is not mine alone. It is a concerted endeavour of the paramount pillars of my life. My heart is infused with an unparalleled gratitude and indebtedness to the people who have extended their guiding hand and supremely encouraged me throughout the journey of pursuing of PhD degree. My thesis is beyond the unfluctuating efforts and my strife towards excellence- it is a book of belief- the belief that every individual held in me to achieve the desired slowly, but surely.

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Dhritimalya Roy

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कर्मण्येवाधिकारस्ते मा फलेषु कदाचन।
मा कर्मफलहेतुर्भूर्मा ते सङ्गोऽस्त्वकर्मणि।।
– श्रीमद् भगवद्गीता

CHAPTER 1

INTRODUCTION

1.1 Prelude

1.1.1 Gravity it is!

The narrative of gravity is closely connected to the development of our comprehension of the cosmos. The idea of gravity has profound origins in the annals of human history, tracing its roots to ancient civilizations. Nonetheless, it wasn't until the 17th century scientific revolution that a more organized and mathematical comprehension of gravity started to unfold.

The ancient Greeks, particularly Aristotle, proposed that objects fell towards the Earth because it was their natural place. This idea persisted for centuries until the time of Galileo Galilei in the early 17th century. Galileo's experiments with inclined planes and falling objects challenged the Aristotelian view, suggesting that all objects, regardless of their mass, fell at the same rate in the absence of air resistance.

In the Indian context, ancient Indian astronomers and mathematicians also contemplated the force that governs celestial bodies. Indian astronomy, documented in texts like the "Surya Siddhanta" and the "Brahmasphutasiddhanta," included discussions on the Earth's gravitational pull. The renowned mathematician and astronomer Brahmagupta (598–668 CE) mentioned the force of gravity in his works, recognizing that it pulls objects towards the Earth. The concept of gravity in Indian thought was not confined to astronomy; it also found its place in philosophical and religious discussions. The idea of a cosmic force influencing all things resonated with Indian cosmology and metaphysics.

Scientifically, both traditions recognized the gravitational force as a fundamental influence on celestial bodies. The universal applicability of gravitational principles was evident in Indian thought, where the concept of gravity extended beyond astronomy to integrate with philosophical and religious discussions.

The turning point in the history of gravity occurred through the contributions of Sir Isaac Newton. In 1687, Newton unveiled his revolutionary work, "Philosophiæ

Naturalis Principia Mathematica" (Mathematical Principles of Natural Philosophy). Within this monumental piece, Newton established the laws of motion and the law of universal gravitation. The universal gravitation law declared that each particle of matter in the universe exerts an attractive force on every other particle, proportional to the product of their masses and inversely proportional to the square of the distance between their centers. Described by the equation "$F = G*(m1*m2)/r^2$", this formula quantified gravity's force and elucidated the observed movements of celestial bodies.

Newton's laws of motion and universal gravitation transformed our comprehension of the physical realm, presenting a cohesive structure that accounted for both an apple falling on Earth and planetary orbits. Newton's contributions served as the cornerstone of classical physics for centuries.

It wasn't until the early 20th century that Albert Einstein's general relativity theory offered a more intricate and all-encompassing grasp of gravity. Einstein proposed that gravity emerges from the curvature of spacetime caused by the presence of mass and energy. This theory broadened our comprehension of gravity beyond Newtonian mechanics, particularly in extreme conditions such as proximity to massive objects or velocities nearing the speed of light. More than a century later, general relativity remains a remarkable theory that has transformed our understanding of gravity and the universe. General relativity's appeal lies in its courageous predictions, including the existence of black holes, the prediction of gravitational waves, the bending of light, the alteration of orbital precession angles, and the expansion of the universe, among others. The appreciation for general relativity over the past century arises not just from its theoretical elegance but also from its successful validation through various experimental tests [1–3]. Enthusiasm for general relativity has soared, especially following the inaugural direct detection of gravitational waves [4, 5].

1.2 The General Theory of Relativity

General theory of relativity (GR) [6] serves as the framework for understanding gravity. This theory is characterized by a geometric approach that correlates gravitational interactions with the geometry of spacetime. Before delving into the intricacies of GR, a first look is taken at the minimum mathematics required in the formulation of the field equations of GR.

1.2.1 Mathematical Preliminaries

In GR, the most used mathematical tool is Tensors. They are mathematical objects that generalize scalars, vectors, and matrices to higher dimensions. They are used to represent geometric entities such as scalars (0th-order tensors), vectors (1st-order tensors), matrices (2nd-order tensors), and higher-dimensional arrays (3rd-order tensors and beyond). Tensors obey certain transformation rules under coordinate transformations, making them valuable tools in various branches of mathematics and physics, including differential geometry, general relativity, and quantum mechanics. In GR , the geometry is expressed through the metric tensor $g_{\mu\nu}$, a second-rank tensor. This metric tensor plays a crucial role in the "first fundamental form" of differential geometry, also referred to as the line element given by:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{1.1}$$

where $x^\mu = (ct, \vec{x})$. In the special theory of relativity it appears as,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu, \tag{1.2}$$

where $\eta_{\mu\nu}$ is the metric tensor of the flat Minkowski spacetime.

As GR deals with curved spacetime, to use tensors as a tool in the formulation of GR, one should know the transformation of partial derivatives($\partial_\mu = \frac{\partial}{\partial x^\mu}$) of a tensor. The transformation is:

$$\partial'_c A'^a = \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\partial_d A^b + \frac{\partial^2 x'^a}{\partial x^b \partial x^d}\frac{\partial x^d}{\partial x'^c}A^b. \tag{1.3}$$

To preserve its tensorial form there is a need to generalise the partial derivative as covariant derivative, and for a vectorA^ν it is given by:

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda}A^\lambda, \tag{1.4}$$

$$\nabla_\mu A_\nu = \partial_\mu A_\nu + \Gamma^\lambda_{\mu\nu}A_\lambda. \tag{1.5}$$

The terms $\Gamma^\rho_{\mu\nu}$ are know as metric connections and in GR they take the form of Christoffel's symbols which may be expressed as:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}). \tag{1.6}$$

At this point a curvature tensor is defined which will be a very useful in the build up of GR. This curvature tensor is known as Riemann's curvature tensor and is given by:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{1.7}$$

Two valuable geometric measurements in general relativity, derivable from the Riemann tensor, include the Ricci tensor, and the other being Ricci scalar, characterized as:

$$R_{\mu\nu} = g^{\lambda\rho}R_{\lambda\mu\rho\nu} = R^\rho_{\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu}, \tag{1.8}$$

and, contacting the above equation one gets,

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{1.9}$$

With the aforementioned definitions at our disposal, one is now prepared to commence the formulation of General Relativity.

1.2.2 The Principles of General Relativity and its Formulation

General relativity is built upon several fundamental principles. These principles played an important role in Einstein's approach to formulate the robust theory of gravity that exists today. Given Einstein's recognition of its significance, it is crucial to acknowledge these principles [7–11]. They are the following:

- **The principle of Equivalence**:

In Newtonian theory, one can assign two different masses to a body to describe different properties, they are:

Inertial mass(m_i): Mass occurring in Newton's Second Law.

Gravitational mass(m_g): The mass occurring in the gravitational force.

Mathematically,

$$F = \frac{d(m_i \vec{v})}{dt}, \tag{1.10}$$

and,

$$F_g = -m_g \vec{\nabla}\phi. \tag{1.11}$$

Hence, $m_i\vec{a} = -m_g\vec{\nabla}\phi$, and from the experiments of Galileo one knows for free falling objects the gravitational mass is equal to the inertial mass $m_g = m_i$. Comparing these identities one may write,

$$\vec{a} = -\vec{\nabla}\phi. \tag{1.12}$$

In essence, the equivalence of masses is essentially the principle of equivalence. The Equivalence Principle's validity was initially assessed through experiments conducted by Eötvös [12], and subsequently by contemporary experiments [13–20]. Broadly speaking the Equivalence Principle may be summed up as [8]:

1. Test particle's motion in a gravitational field is unconstrained of its mass and composition. This may be interpreted as the strong equivalence principle.

2. From weak equivalence principle, it states the gravitational field is combined with everything.

3. There exist no local experiments capable of discerning between non-rotating free fall within a gravitational field and uniform motion in the absence of such a field.

4. A frame that undergoes linear acceleration concerning an inertial frame appears locally identical to a frame stationary within a gravitational field.

These conditions give rise to the notion of a space-time that is not flat but curved. In special relativity, within a coordinate system aligned with an inertial frame, such as Minkowski coordinates, the equation describing the motion of a test particle is,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0. \tag{1.13}$$

In non-inertial frame the more general form of this equation is:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{1.14}$$

One can easily notice that equation(1.14) is known as the Geodesic Equation. Within a curved spacetime, geodesic curves serve as the extension of straight lines; they represent the paths of minimal length connecting two separate points. While claiming the above equation to describes gravity field, keeping in mind the statement to be true, it should produce Newton's equation(1.12) at the weak field limit. This leads us to another important principle which is of great importance.

- **The correspondence principle:**

It states that GR needs to be consistent, firstly, with special relativity in scenarios lacking gravitational effects and, secondly, with Newtonian gravitational theory when gravitational fields are weak and velocities are low relative to the speed of light. Here we briefly review the Newtonian limit of GR, for detailed study one may refer [7–11]. In order to test GR in the Newtonian limit the weak field approximation is used. In this, the metric tensor may be written as,

$$g_{\mu\nu}(x^\mu) = \eta_{\mu\nu} + h_{\mu\nu}(x^\mu), \tag{1.15}$$

where $h_{\mu\nu}$ are small perturbations over the Minkowski space-time. The Christofel's Symbols reads as(for linear form of the perturbation $h_{\mu\nu}$),

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\rho}(h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho}), \tag{1.16}$$

where the notation with commas introduces $(A_{,\mu} \equiv \partial_\mu A)$. Within the linear approximation, the Minkowski metric $\eta_{\mu\nu}$ raises and lowers tensor indices. Additionally, the proper time,

$$d\tau^2 = -ds^2 = c^2dt^2\left(1 - \frac{1}{c^2}\frac{d\bar{r}^2}{dt^2}\right) \implies d\tau^2 = c^2\gamma^2dt^2, \tag{1.17}$$

and from the Geodesic equation for Newtonian limit one gets,

$$\frac{d^2x^\mu}{dt^2} = a^i \simeq -c^2\Gamma^i_{00}. \tag{1.18}$$

The right signifies 'gravitational force', responsible for imparting acceleration to the particle. Using, value of the connection coefficient one gets,

$$\frac{d^2x^\mu}{dt^2} = -\frac{c^2}{2}\nabla_\mu h_{00}. \tag{1.19}$$

The above equation is to be compared with the Newton's equation(1.12)

$$\frac{d^2\vec{X}}{dt^2} = \vec{a} = -\vec{\nabla}\phi, \tag{1.20}$$

where ϕ, is the gravitational potential. Comparing the above two equation one can deduce,

$$h_{00} = -\frac{2\phi}{c^2}, \tag{1.21}$$

where ϕ is the Newtonian gravitational potential. In terms of the metric tensor,

$$g_{00} = -\left(1 + \frac{2\phi}{c^2}\right). \tag{1.22}$$

Thus, it can be argued that the curvature of spacetime may describe the gravitational force at the Newtonian limit.

1.2.2.1 The Field Equation of GR:

To get to the full field equations of GR one must first arrive at the vacuum field equations of general relativity. The field equations to hold true in vacuum one must have the Ricci tensor, $R_{\mu\nu} = 0$. Now vanishing of the Ricci tensor suggests that the Einstein's Tensor to be zero too. Mathematically,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{1.23}$$

which is the vacuum field equation of GR. The complete field equations are the equations those apply in the presence of fields beyond gravity. As observed, these fields are characterized by the energy-momentum tensor ($T_{\mu\nu}$).Now, the equality of mass and energy as implied by special relativity proposes that all types of energy serve as origins for the gravitational field; this aligns with the essence of the weak form of the principle of equivalence . Hence, considering $T_{\mu\nu}$ as a source term in the field equations. In relativity, the energy-momentum tensor abides by the conservation equations given by:

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{1.24}$$

Through the Bianchi Identity the covariant derivative of Einstein's tensor also vanishes, hence consistently one can write,

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.25}$$

where, $\kappa = \frac{8\pi G}{C^4}$ and is called the coupling constant.Thus the above equation constitute the full field equation of General Relativity.

1.2.3 Lagrangian Formulation

The general theory of relativity can be articulated through the Lagrangian framework [7–11]. By incorporating a kinetic term and a potential, along with all probable interactions, it's typically feasible to formulate a Lagrangian for a field. In general relativity, the field is defined by the metric tensor. One knows, any tensor containing up to second-order derivatives of the metric tensor can be represented in terms of the Riemann tensor and its contractions. Consequently, the sole non-trivial scalar that could be employed as a Lagrangian density is the Ricci scalar, hence from the action principle

$$S = \int d^4x\sqrt{-g}R, \tag{1.26}$$

where g is the determinant of the metric tensor. The above form of the action is known as Einstein-Hilbert action. For the calculation of the field equations one has to vary the above action with respect to metric tensor $g^{\mu\nu}$. Using the relation $g_{\mu\alpha}g^{\nu\alpha} = \delta^\nu_\mu$ and the property of Kronecker delta, one finds $\delta_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}$. Using $R = g^{\mu\nu}R_{\mu\nu}$ one may write the variation of the action as:

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3, \tag{1.27}$$

where

$$\delta S_1 = \int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu}, \tag{1.28}$$

$$\delta S_2 = \int d^4x\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}, \tag{1.29}$$

$$\delta S_3 = \int d^4xR\delta\sqrt{-g}. \tag{1.30}$$

The term δS_2 is in the necessary standard format, thus one only needs to address the remaining two terms. To determine the variation of the Riemann tensor concerning the metric tensor, it is more convenient to initially ascertain its variation concerning the Christoffel symbols. One can define the variation of the Christoffel symbols as:

$$\Gamma'^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \delta\Gamma^\rho_{\mu\nu}. \tag{1.31}$$

Expressing the variation of the Christoffel symbols ($\delta\Gamma^\rho_{\mu\nu}$) as the disparity between two Christoffel symbols yields a tensor, allowing for the definition of its covariant derivative:

$$\nabla_\lambda(\delta\Gamma^\rho_{\mu\nu}) = \partial_\lambda(\delta\Gamma^\rho_{\mu\nu}) + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\lambda\nu}\delta\Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\lambda\mu}\delta\Gamma^\rho_{\nu\sigma}. \tag{1.32}$$

Thus covariant derivatives in the above equation are taken with respect to the metric tensor $g_{\mu\nu}$. After few mathematical steps one can find that variation of the Riemann tensor assumes the form:

$$\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\mu\nu}) - \nabla_\nu(\delta\Gamma^\rho_{\lambda\mu}). \tag{1.33}$$

Substituting the above equation in the first term of the action's variation gives:

$$\delta S_1 = \int d^4x\sqrt{-g}\nabla_\sigma[g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^\lambda_{\lambda\mu})]. \tag{1.34}$$

The variation depicted in the aforementioned equation adopts the structure of a total derivative, denoted as $\nabla_\mu A^\mu$. Employing Stoke's theorem, the impact of a total derivative in the field equations equates to a boundary contribution at infinity. By ensuring the variation diminishes at infinity, one can eliminate the boundary contribution. Hence, a total derivative doesn't influence the field equations, allowing us to set $\delta S_1 = 0$.

For the third term, δS_3, one necessitates the subsequent relation applicable to any square matrix M with a non-trivial determinant: $\ln(\det M) = \mathrm{Tr}(\ln M)$. The variation of this equation is,

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M). \tag{1.35}$$

Applying the equation mentioned above with $\det M = g$, one determines that $\delta g = -g g_{\mu\nu\delta} g^{\mu\nu}$. Consequently, it becomes straightforward to ascertain the variation of the $\sqrt{-g}$ term featured in δS_3 as:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{1.36}$$

The Newtonian gravitational potential (Φ) and the energy density(ρ) is related by the Poisson's equation, which is given by:

$$\nabla^2\Phi = 4\pi G\rho.$$

Making use of the above relation one can have:

$$\delta S = \frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \int d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}. \tag{1.37}$$

Using the equation above, one can deduce the vacuum field equations of General Relativity as $G_{\mu\nu} = 0$. Subsequent sections will illustrate the convenience of the Lagrangian formulation, particularly in its adaptability for modifying General Relativity. To derive the complete set of field equations, an additional term related to matter must be incorporated into the action:

$$S = \frac{1}{16\pi}S_H + S_M, \tag{1.38}$$

where S_H denotes the Hilbert action and with S_M the matter term. Using the definition of energy-momentum tensor, one can write the complete field equation as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}. \tag{1.39}$$

In the subsequent sections it will illustrated the convenience of the Lagrangian formulation, particularly in its adaptability for modifying General Relativity.

1.3 Modified Theories of Gravity

The pursuit of understanding the fundamental nature of gravity has been a cornerstone of theoretical physics for centuries. Einstein's General Theory of Relativity has stood as the pre-eminent framework for describing gravitational interactions, successfully predicting phenomena ranging from the bending of light by massive objects to the existence of black holes.

Modifying Einstein's General Relativity is not a recent endeavour. Einstein himself was among the earliest proponents of a slightly altered version of GR. During the era when Le-Maitre and Friedmann discovered precise solutions to Einstein's equations, foreseeing an expanding universe, the prevailing belief among most physicists, including Einstein, was in a static cosmos. Consequently, Einstein revised the original formulation of his theory, introducing the renowned cosmological constant Λ, with the aim of circumventing the anticipated expansion. As early as 1919, endeavors were initiated to explore a theory of gravity beyond the second order, diverging from the field equations of General Relativity (GR). Pioneered by A. Eddington and H. Weyl, these efforts were primarily driven by the pursuit of theoretical comprehensiveness [21,22]. Concurrently, in 1920, debates arose regarding the principal field associated with gravity: whether it should be the metric or the connection. Eddington further advanced these discussions in 1924 by presenting a purely affine rendition of GR in a vacuum. Subsequently, Schrödinger extended Eddington's theory to incorporate a non-symmetric metric [23].

General Relativity has encountered challenges in explaining certain observational discrepancies, such as the accelerated expansion of the universe [24–27]. On the largest spatial scales, these observations can be elucidated by the existence of an unusual entity known as dark energy or adjustments to the principles of General Relativity [28]. Currently, the ΛCDM model stands as the prevailing cosmological framework, widely accepted due to its alignment with a plethora of observations [29]. Inherent to the ΛCDM model is the assumption that General Relativity (GR) accurately characterizes gravitational interactions on cosmological scales. As its nomenclature suggests, the model posits the prevalence of dark matter and dark energy, symbolized by the cosmological constant Λ, as the dominant constituents shaping the energy distribution of the Universe. An inherent challenge within the ΛCDM model lies in the lack of a comprehensive understanding regarding the interactions among its constituents. For instance, while one possess insights into the interactions of approximately 5% of the matter as depicted in the standard model, along with the interactions of photons from the Cosmic Microwave Background (CMB), our understanding remains rudimentary. Consequently, it is difficult to fully embrace a model for which our knowledge regarding its fundamental mechanisms remains limited, even if it demonstrates efficacy in its predictions.

Instead of resorting to the cosmological constant as a solution to the universal acceleration attributed to dark energy, an alternative approach involves scrutinizing the left-hand side of the Einstein equations and contemplating modifications to gravity. The necessity for dark matter arises from the need to elucidate the rotation curves of galaxies, wherein the rotational speeds of stars exhibit minimal variation with distance from the galactic center. There are various modified gravity theories that have been proposed to account for dark matter, including concepts such as massive graviton geons, bimetric gravity, and even Modified Newtonian Dynamics (MOND) [30–32]. Modified Theory of Gravity (MoTG) offers a compelling framework for addressing the observed phenomena without recourse to exotic entities such as dark matter and dark energy. Instead, it posits modifications to the gravitational field equations, introducing additional degrees of freedom that capture the observed dynamics more accurately.

The primary requirement for a viable theory of modified gravity entails reproducing the outcomes of General Relativity (GR) within the range from micrometer to solar system scales, while also accounting for the observed weakening of gravitational interactions at universal scales to accommodate acceleration. Although there are numerous theories of modified gravity [33–47], here few of them which extends the action to incorporate higher orders of curvature invariants are briefly explored.

1.3.1 $f(R)$ Modified Theory of Gravity:

Interest in $f(R)$ theories stemmed from motivations such as inflationary scenarios, exemplified by the Starobinsky model where $f(R) = R - \Lambda + \alpha R^2$ was investigated [48]. Subsequently, it was demonstrated that $f(R)$ gravity can indeed account for the late-time cosmic acceleration [49]. Moreover, criteria for viable cosmological models have been established [50–55], and investigations into the explicit coupling of arbitrary functions of R with the matter Lagrangian density have been conducted [56–59]. Concerning the Solar system regime, stringent weak field constraints have largely invalidated many proposed models [60–64], though viable alternatives do exist [65–70]. Additionally, within the context of dark matter, the potential for understanding the galactic dynamics of massive test particles without invoking dark matter has been explored within $f(R)$ gravity models [71–73]. A straightforward expansion of General Relativity (GR) arises by substituting R in the Einstein-Hilbert action (1.26) with a generic function of R, thus giving rise to theories known as $f(R)$ gravity. The action for $f(R)$ modified theories of gravity is:

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M(g^{\mu\nu},\psi), \tag{1.40}$$

where $\kappa = 8\pi G$; throughout we consider $\kappa = 1$ for simplicity. $S_M(g^{\mu\nu},\psi)$ is the matter action, given by $S_M = \int d^4x\sqrt{-g}L_m(g_{\mu\nu},\psi)$, where L_m is the matter Lagrangian density, in which matter is minimally coupled to the metric $g_{\mu\nu}$ and ψ collectively denotes the matter fields. By varying the action with respect to $g_{\mu\nu}$, one obtains

$$FR_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F = T^m_{\mu\nu}, \tag{1.41}$$

where $F \equiv \frac{df}{dR}$. Contracting the above equation one obtains,

$$fR - 2f + 3\Box F = T. \tag{1.42}$$

Here, if $f(R) = R$ is considered, one retrieves the Einstein gravity. Generally, $\Box F(R) \neq 0$, implying the existence of certain propagating degrees of freedom. Hence, from equation(1.41) the explicit form of the field equations can be written as,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^M_{\mu\nu} + T^{eff}_{\mu\nu}, \tag{1.43}$$

where, $T^{eff}_{\mu\nu}$, is the effective energy momentum tensor.

At this point one must note that other than metric formalism there are other two alternative methodologies in the formalism of this modified gravity theory. One such approach is the Palatini formalism [74–77], which treats the metric and connections as distinct variables. Another method is the metric-affine formalism, wherein the matter portion of the action is contingent upon and modified in relation to the connection [78].

1.3.2 $f(T)$ Modified Theory of Gravity

The action for $f(T)$ gravity [79–89] is represented as follows:

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}\; f(T) \mathrm{d}^4x + \int \sqrt{-g} L_M \mathrm{d}^4x, \tag{1.44}$$

where T is torsion scalar, $f(T)$ is a differentiable function of torsion and L_M is the matter Lagrangian. Einstein himself suggested the inclusion of torsion as a term for gravitational interaction with the aim of harmonizing electromagnetism and gravity over the Weitzenbock non-Riemannian manifold. Consequently, the Weitzenbock connection replaces the Levi-Civita connection within the underlying Riemann-Cartan spacetime in this revised gravitational theory. The four linearly independent vierbein (tetrad) fields are the fundamental entities in this theory, forming orthogonal bases for the tangent space at every point in spacetime. These vierbeins, which function as parallel vector fields, give the theory the name "teleparallel." Similar to the extension of $f(R)$ gravity, an extension to teleparallel gravity has been developed by substituting the torsion scalar T with a general function $f(T)$, known as $f(T)$ gravity by Linder.

The torsion scalar is defined as

$$T = S_\sigma{}^{\mu\nu} T^\sigma{}_{\mu\nu}, \tag{1.45}$$

where super-potential, $S_\sigma{}^{\mu\nu}$, is given by

$$S_\sigma{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_\sigma + \delta^\mu_\sigma\, T^{\alpha\nu}{}_\alpha - \delta^\nu_\sigma\, T^{\alpha\mu}{}_\alpha\right), \tag{1.46}$$

the contortion tensor, $K^{\mu\nu}{}_\sigma$, is given by

$$K^{\mu\nu}{}_\sigma = -\frac{1}{2}\left(T^{\mu\nu}{}_\sigma - T^{\nu\mu}{}_\sigma - T_\sigma{}^{\mu\nu}\right), \tag{1.47}$$

and the torsion tensor, $T^\sigma{}_{\mu\nu}$ is defined as

$$T^\sigma{}_{\mu\nu} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu} = e^\sigma_A\left(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu\right), \tag{1.48}$$

the Weitzenböck connection $\Gamma^\sigma_{\mu\nu}$ is given by $\Gamma^\sigma_{\mu\nu} = e^\sigma_A \partial_\nu e^A_\mu$.

The orthogonal tetrad components denoted as $e_A(x^\mu)$ are viewed as dynamic entities and represent an orthonormal basis in a geometric sense within the tangent space at each specified point x^μ on the manifold. This is mathematically expressed as $e_A e_B = \eta_{AB} = diag(+1, -1, -1, -1)$. Furthermore, when utilizing a co-ordinate basis, it is possible to express e_A as $e^\mu_A \partial_\mu$ where e^μ_A are the components of e_A, with $\mu = 0, 1, 2, 3$ and $A = 0, 1, 2, 3$. It is important to note that uppercase letters pertain to the tangent space while Greek indices are used to denote coordinates on the manifold. As a result, the metric tensor can be derived from the dual vierbein according to the equation $g_{\mu\nu}(x) = \eta_{AB} e^A_\mu(x) e^B_\nu(x)$.

1.3.3 $f(R,T)$ Modified Theory of Gravity

$f(R,T)$ gravity represents an extension of the conventional $f(R)$ gravity framework, wherein the Lagrangian density is defined as an arbitrary function of both the Ricci scalar (R) and the trace of the energy-momentum tensor (T). Harko *et al.* [90]. have expanded upon $f(R)$ gravity by opting for the Lagrangian density to be an arbitrary function $f(R,T)$.

The $f(R,T)$ gravity theory starts from the action [91–93],

$$S = \frac{1}{16\pi}\int f(R,T)\sqrt{-g}d^4x + \int L_m\sqrt{-g}d^4x, \tag{1.49}$$

where $f(R,T)$ is an arbitrary generic function of R and T, the Ricci scalar and the trace of the energy momentum tensor $T_{\mu\nu}$, respectively. Moreover, g is the determinant of the metric $g_{\mu\nu}$, and L_m is the matter Lagrangian.

Varying the action w.r.t the metric $g_{\mu\nu}$, the field equation for $f(R,T)$ theory of gravity is given by:

$$\begin{aligned} f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T) \\ = 8\pi T_{\mu\nu} - f_T(R,T)(T_{\mu\nu} + \Theta_{\mu\nu}), \end{aligned} \tag{1.50}$$

where $f_R(R,T)$ and $f_T(R,T)$ are the differentiation of $f(R,T)$ with respect to R and T respectively, and $\Box f_R = g^{\mu\nu}\nabla_\mu\nabla_\nu f_R$

The energy-momentum tensor of matter is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}, \tag{1.51}$$

with the trace $T = g^{\mu\nu}T_{\mu\nu}$. Here the Lagrangian density L_m of matter depends only on the metric tensor components $g_{\mu\nu}$, and not its derivative. Thus, one has,

$$T_{\mu\nu} = g_{\mu\nu}L_m - \frac{2\partial(L_m)}{\partial g^{\mu\nu}}. \tag{1.52}$$

The variation of T with respect to $g^{\mu\nu}$, one obtains,

$$\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = T_{\mu\nu} + \Theta_{\mu\nu}. \tag{1.53}$$

It is noted that ∇_μ is associated with the Levi-Civita connection of metric tensor $g_{\mu\nu}$ and box operator $\Box$ is given by:

$$\Box \equiv \frac{\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)}{\sqrt{-g}}, \quad \text{and} \quad \Theta_{\mu\nu} = \frac{g^{\alpha\beta}\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}.$$

Hence from the above equations, the explicit form of field equations read:

$$G_{\mu\nu} = 8\pi G_{eff}T_{\mu\nu} + T^{eff}_{\mu\nu}, \tag{1.54}$$

where

$$G_{eff} = \frac{1}{f_R(R,T)}\left(1+\frac{f_T(R,T)}{8\pi}\right), \tag{1.55}$$

$$T^{eff}_{\mu\nu} = \frac{1}{f_R(R,T)}\Big[\frac{1}{2}(f(R,T) - Rf_R(R,T) \tag{1.56}$$
$$+ \; 2\rho f_T(R,T))g_{\mu\nu} - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T)\Big].$$

Note that, when $f_T(R,T) = 0$, one gets the usual $f(R)$. In these field equations, the energy-momentum tensor component $(T_{\mu\nu})$ of $f(R,T)$ gravity represents the interaction between matter and curvature, and one may interpret this as the curvature-matter coupling occurs due to the exchange of energy and momentum between the both.

1.4 Some Exotic Solutions of GTR

In 1915, Einstein gave his General Theory of Relativity, which till date is the most successful theory of gravity. The equation consist of differential equations which may have different solution. The solutions of these differential equations are commonly known as the solutions of general relativity. Karl Schwarzschild formulated the first precise solution [94] in the field of general relativity soon after the theory was introduced. This particular solution, recognized as the Schwarzschild solution, describes the gravitational field that envelops a stationary, non-rotating mass distribution in a spherical shape. Significantly, it offers a detailed representation of slowly spinning astronomical objects such as black holes, stars, and planets with exceptional precision. Kruskal [95] extended maximally the Schwarzschild metric. The solution proposed by Kruskal can be derived by concurrently rectifying the trajectories of both incoming and outgoing radial null geodesics for a gravitational mass equivalent to twice the Schwarzschild radius. The solutions are derived by initially examining a conceivable distribution of matter, represented by a stress-energy tensor. Subsequently, the Einstein field equation is employed to ascertain the spacetime metric governing the geometry. Conversely, one can approach the Einstein field equation in reverse, first contemplating an intriguing and unconventional spacetime metric and then identifying the corresponding matter source responsible for shaping the geometry. Some of the solutions obtained this way gives exotic solutions. Here, such solutions, namely, Wormhole and Closed Timelike Curves (CTC) are briefly reviewed.

1.5 Wormholes

Wormholes function as conduits that link distinct areas of spacetime, offering the possibility for individuals to transition between these regions without obstruction. Earliest traces of wormhole physics may be traced back to Flamm [96] where he analysed the Schwarzschild solution in 1916. The first ever wormhole type solution was proposed by Einstein himself together with Nathan Rosen [97] in 1935. They termed it the

"Einstein-Rosen Bridge". The domain remained dormant until Wheeler revived interest in the 1950s. Infact it was Wheeler and Missner [98] who coined the term "Wormhole". They considered wormholes, including Reissner-Nordström or Kerr wormholes, as integral parts of the quantum foam that connects separate spacetime regions and functions at the Planck scale [99, 100]. Subsequently, these Wheeler wormholes were reconceptualized as Euclidean wormholes by Hawking [101] and his peers. That being said, these initial Wheeler wormholes were not traversable and theoretically at risk of encountering some form of singularity[8]. After Wheeler, there was a big gap in the field of wormhole physics until in 1970s, Ellis [102,103] proposed his idea of "Drainhole" and classical wormhole and connected self-consistent solutions were studied by Kirill Bronnikov [104], followed by Takeshi Kodama [105], and Gérard Clément [106].

Travesability through wormholes was first ensured by Morris and Thorne [107] in 1988. They in their solution proposed horizon free solution of wormhole through which one may travel between two asymptotic flat universe or other universe. This solution opened completely a new area of research in Physics and it was the beginning of what today is known as "Modern Wormhole Physics". Their solution is a classic example of a reverse approach to solving the Einstein field equation. As a result, it has been established that these traversable wormholes exhibit a stress-energy tensor that contravenes the Null Energy Condition(NEC) the weakest of all energy conditions. [cite]. The other important energy conditions are Weak Energy Condition(WEC), Strong Energy Condition(SEC), Dominant Energy Condition(DEC). NEC being the weakest, violation of NEC signifies violation of all the other energy conditions. The classical matter obeys all the classical energy conditions [108], hence the concept of "Exotic Matter" arises to sustain the wormhole by violating the energy conditions [107,109]. While it is commonly understood that classical matter adhere to these energy conditions, it is also widely acknowledged that specific quantum fields contravene them. Examples include the Casimir effect and Hawking evaporation [110, 111]. All these outcomes over the years culminated in a book "Lorentzian Wormholes: From Einstein to Hawking" by Visser [109], where one may find a detailed review of the subject up to the year $1994 - 95$. One of the main aim of the research in wormhole physics is to avoid the violation of NEC and also limit the use of exotic matter as much as possible.

In terms of the energy conditions, there has been a significant shift, particularly with the demonstration that even classical systems, such as those constructed from scalar fields non-minimally coupled to gravity, contravene all energy conditions [112]. Notably, recent cosmological observations strongly indicate that the cosmological fluid breaches the strong energy condition (SEC) and suggest intriguing possibilities of the null energy condition (NEC) being infringed upon in classical settings [113–115]. Consequently, there is a gradual erosion of the conventional status of the weak and null energy conditions, along with other energy conditions, as laws [116]. Harris [117] revisited the wormhole solution proposed in [107], delving into a self-consistent Ellis wormhole facilitated by an exotic scalar field. Within a different line of investigation, Visser [118] aimed to minimize energy condition violations and ensure travellers do not encounter strange matter while passing through. This approach led to the proposal of polyhedral wormholes, which included cubic geometries. These particular shapes concentrated exotic matter at junctions and edges, enabling travellers to move across flat, matter-free surfaces. Visser expanded on Roman's idea of a two-wormhole system by introducing the Roman ring [119]. He later shifted his focus to generic, dynamically traversable wormhole throats in 1999 [120]. In the same year, Visser and Barcelo collaborated to identify classically consistent solutions utilizing scalar fields [112]. Alongside Dadhich, Kar, and Mukherjee, he also pinpointed self-dual solutions [121]. Certain studies have incorporated a cosmological constant into their analysis of wormhole construction. Thin-shell wormhole solutions featuring Λ, inspired by the work of Visser [109, 122], were scrutinized in [123, 124]. Roman [125] devised a wormhole solution that evolves over time to investigate the possibility of circumventing energy condition violations, while Delgaty and Mann [126] sought out new wormhole solutions incorporating Λ. The examination of the formation of wormhole solutions requires aligning an internal wormhole spacetime with an external vacuum solution at a connecting surface, a subject that has garnered significant interest [127–129]. Authors have delved into thin shells around traversable wormholes, exploring scenarios with zero surface energy density [127], as well as those with generic surface stresses [128]. Additionally, studies have examined plane symmetric cases with negative cosmological constants [130]. DeBenedictis and Das explored a broad class of wormhole geometries with cosmological constants and junction conditions [131], extending their analysis to higher dimensions [132]. Understanding the stability of these constructs against various perturbations is crucial. Studies have investigated the stability of thin-shell wormholes, which are created using the cut-paste method and taking into account particular equations of state [122, 123, 126, 133–135], or through the linearized radial perturbations around steady solutions [124, 136–139]. Pertaining to Ellis' drainhole [102, 103], Armendáriz-Picón found stability against linear perturbations [140], while Shinkai and Hayward [141] suggested instability to non-linear perturbations.Wormhole solution in alternative, higher order theories of gravity have also been studies by various authors [142–163].There also have been intensive studies of wormhole solutions in the modified theories of gravity where authors have investigated wormhole geometries in $f(R)$,$f(R,T)$,$f(T)$, etc theories of gravity [164–180].

Although, wormholes are considered hypothetical and exotic astrophysical objects which were considered very difficult to observe. The detection of black hole shadow [181, 182] and the detection of Gravitational Waves (GW) have turned the tide in the realm of wormhole observation.Wormholes are also believed to cast a shadow just like black holes with few differences in their characteristics. Several authors have investigated shadows of wormholes to find a way of observing them in the near future. Ohgami and Sakai [183] proposed that the contrast in intensity between the interior and exterior regions of the ring is notably distinct. Consequently, discerning between an Ellis wormhole and a black hole could become feasible with high-resolution very-long-baseline-interferometry observations in the foreseeable future. Rahaman *et al.* [184] examined the dimensions of the wormhole silhouette and determined that it displays an inclined orientation and can be modified based on various parameters within the wormhole spacetime. Additionally, they constrain the dimensions and rotational characteristics of the wormhole through analysis informed by observations of M87*. This involves assessing both the average diameter of the wormhole and its deviation from circularity relative to the throat size. Wang *et al.* [185] investigated novel shadows cast from thin shell wormhole. Saikh [186] investigated shadow of a rotating wormhole and concluded that as the spin values increase, the outlines of the wormhole shadows diverge significantly from those of black holes. This notable discrepancy, if observed in future studies, could potentially signal the existence of a wormhole. Superior works have been conducted in this field over the last decade, one may refer to [187–193] for elaborate and extensive reading for more works on wormhole shadows.

Gravitational Waves(GW) are detected when two astrophysical objects like black hole, neutron stars, etc collide and creates ripples in the fabric of space -time. With the detection of gravitational waves from the black hole mergers [4,5], the field of wormhole observation also experienced a huge surge. Since wormholes are black hole like object it is speculated to have some observational evidence with wormholes. Various authors have put forward different theories to observe wormholes with distinct GW signatures. Wormholes stand out because of their double-peak potential, which enables gravitational waves (GW) to oscillate between these peaks as they gradually depart from the wormhole. This process leads to a series of echoes [194, 195]. Chakraborty *et al.* [196] studied the GW in the context of wormholes using Bondi-Sachs formalism in the Brane-World scenario. Bao *et al.* [197] studied the phenomenon of GW scattering to search for wormholes and found within a specific range of wormhole masses, the transmitted wave exhibits a distinct, isolated chirp devoid of an inspiral waveform, while the reflected wave demonstrates anti-chirp behaviour, characterized by the absence of the chirping signal. For a deeper comprehension of the progress in this domain, one might refer [198–200].

Recent advancements in wormhole theory, particularly in the realm of quantum fields and strings, have sparked significant interest. Notably, a proposal by Maldacena and Susskind [201] in 2013 sought to address the “firewall paradox” in quantum black holes by establishing a link between quantum entanglement and wormholes. This concept, known as ER=EPR, suggests highly entangled states between distant black holes, forming a wormhole connection. Subsequent research has explored various aspects of wormholes, including their traversability and gravitational dynamics. A significant breakthrough occurred in 2022 when Jafferis *et al.* [202] proposed the potential traversability of a wormhole within a quantum computational environment using information teleportation. This innovative approach showcases the exciting possibilities in the field of wormhole physics.

1.6 Mathematics of Traversable Wormholes: A brief overview

1.6.1 Morris-Thorne Wormhole

1.6.1.1 The Spacetime:

The spherically symmetric metric of a wormhole can be written in Schwarzschild coordinates as:

$$ds^2 = -e^{2\phi(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.57}$$

Here, $\phi(r)$ and $b(r)$ represent two arbitrary functions dependent solely on radial co-ordinate. $\phi(r)$ governs the gravitational redshift [7] and is denoted as the 'redshift function', while $b(r)$ determines the spatial configuration of the wormhole [109]and is referred to as the 'shape function'. The metric must adhere to certain constraints to meet these conditions [109]:

I. For any static, asymptotically flat space time with a wormhole but containing an event horizon, implies $g_{00} \to 0$ at the event horizon. Therefore the absence of event horizon implies that $e^{2\phi(r)} \neq 0$, i.e. $\phi(r)$ must be everywhere finite.

II. The two asymptotically flat regions are assumed to be connected. Thus two coordinate patches are required, each one covering the range $[r_0, +\infty)$. Each patches covers one universe and the two patches join at the minimum value of $r = r_0$ which is known as the 'throat' of the wormhole. For simplicity one may assume that there is only one such minimum and it is an isolated minimum.

IV. Proper radial distance is related to the r coordinate by

$$l(r) = \pm \int_{r_0}^{r} \frac{dr'}{\sqrt{1-\frac{b(r')}{r'}}}. \tag{1.58}$$

It is essential for l to span the entire range $(-\infty, +\infty)$ while remaining finite across the spacetime, thereby necessitating that $\frac{b}{r} < 1$ throughout the spacetime and specifically at the throat where $l = 0$. Once again, r is a function of l, and for it to constitute a wormhole solution, the throat is identified by the location of minimum r, indicating an outward flare (expansion in space). Consequently, $\frac{dr}{dl} = 0$ at the throat $r = r_0$ of the wormhole, and $\frac{d^2r}{dl^2} > 0$ at or close to the throat. Note that

$$\frac{dr}{dl} = \pm\sqrt{1-\frac{b(r)}{r}}, \tag{1.59}$$

and this implies at the throat $b(r_0) = r_0$. Now

$$\frac{d^2r}{dl^2} = \frac{1}{2}\frac{d}{dr}\left[\left(\frac{dr}{dl}\right)^2\right] = \frac{1}{2r}\left(\frac{b}{r} - b'\right). \tag{1.60}$$

This implies that, at or near the throat $b'(r) < \frac{b}{r}$.

1.6.1.2 Embedding Mathematics

One utilizes embedding diagrams to enforce the requirement that the spacetime metric (1.57) represents a wormhole. Our focus lies on examining the geometry of the three-dimensional space at a specific time moment t. This spatial geometry exhibits spherical symmetry. Hence, one can effectively concentrate on an equatorial slice defined by $\theta = \pi/2$. The line element for this slice is derived by fixing t and $\theta = \pi/2$ in the equation of the metric.

$$ds^2 = \left(1 - \frac{b}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \tag{1.61}$$

Here, the aim is to create a two-dimensional surface in three-dimensional Euclidean space, representing the same geometry as this slice. Essentially, one seeks to depict this slice as if it were extracted from spacetime and embedded in Euclidean space. To achieve this, we introduce cylindrical coordinates z, r, and ϕ in the embedding Euclidean space. The Euclidean metric of the embedding space takes the form:

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2. \tag{1.62}$$

The embedded surface will exhibit axial symmetry, allowing it to be represented by a single function $z = z(r)$. On this surface, the line element takes the form:

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2. \tag{1.63}$$

Therefore, the line element will match that of our equatorial slice through the wormhole (1.61) if an identification between the coordinates $(r,\ \phi)$ of the embedding space and those of the wormhole's spacetime is established. Additionally, one needs to ensure that the function $z(r)$, which characterizes the embedded surface, adheres to the following condition:

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-\frac{1}{2}}. \tag{1.64}$$

For a geometry to qualify as a wormhole solution, it must possess a minimum radius denoted as the throat, located at $r = b(r) = r_0$. At the throat, the embedded surface becomes vertical, meaning $dz/dr \to \infty$, as illustrated in Figure(1.1). Away from the throat, in regions far from it, the space is assumed to approach asymptotic flatness, where $dz/dr \to 0$ as $r \to \infty$.

Equation (1.64) gives the expression of $z(r)$ in the following integral form

$$z(r) = \pm \int_{r_0}^{r} \left(\frac{r}{b(r)} - 1 \right)^{-\frac{1}{2}} dr. \tag{1.65}$$

To be classified as a wormhole, a solution must meet the criteria of the throat flaring out, as illustrated in Figure(1.1). From a mathematical perspective, this criterion necessitates that the second order derivative of the inverse embedding function $r(z)$, denoted as d^2r/dz^2, be positive at or in the vicinity of the throat r_0. The differentiation of $dr/dz = \pm(r/b(r) - 1)^{1/2}$ with respect to z yields:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}z^2} = \frac{b - b'r}{2b^2} > 0 \quad \text{at or near the throat.} \tag{1.66}$$

At the level of the throat, it is confirmed that the functional form meets the criterion $b'(r_0) < 1$.

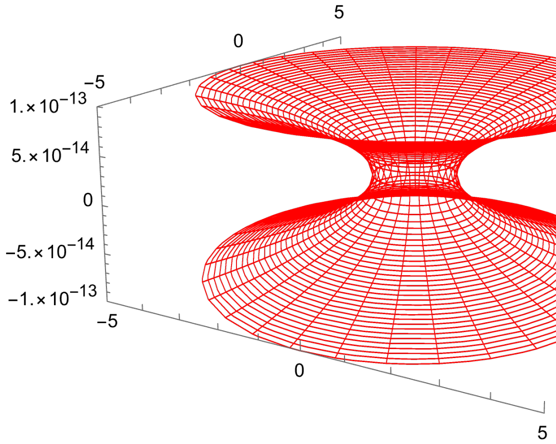

Figure 1.1: Embedding of a wormhole spacetime

1.6.1.3 The Field Equation:

The explicit form of the Einstein's field equation is given by:

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.67}$$

where, $G_{\mu\nu}$, is the Einstein tensor and $T_{\mu\nu}$ is called the stress-energy tensor. To obtain the complete set of field equations one must find the values of the Einstein tensor first. For the complete set of non zero values of the einstein tensor, one needs the Riemann curvature tensor, and the Ricci tensor. From the metric (1.57) and, using equations

(1.7) and (1.6) one obtains:

$$R^t_{rtr} = -\phi'' + \frac{(b'r-b)}{2r(r-b)}\phi' - (\phi')^2,\ R^t_{\theta t\theta} = -\phi' r\left(1-\frac{b}{r}\right),$$
$$R^t_{\phi t\phi} = -\phi' r\left(1-\frac{b}{r}\right)\sin^2\theta, R^r_{\theta r\theta} = \frac{b'r-b}{2r},\ R^r_{\phi r\phi} = \frac{b'r-b}{2r}\sin^2\theta,\ R^\theta_{\phi\theta\phi} = \frac{b}{r}\sin^2\theta. \tag{1.68}$$

Now, construct the orthonormal basis given by $\{e_{\hat{t}}, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}\}$ where $e_{\hat{t}} = e^{-\phi}e_t$, $e_{\hat{r}} = \left(1-\frac{b}{r}\right)^{\frac{1}{2}} e_r$, $e_{\hat{\theta}} = r^{-1}e_\theta$, $e_{\hat{\phi}} = (r\sin\theta)^{-1}e_\phi$ and the matrix coefficients take their standard special relativity form

$$g_{\hat{\alpha}\hat{\beta}} = e_{\hat{\alpha}}\dot{e}_{\hat{\beta}} = \eta_{\alpha\beta} = \text{diag}(-1,1,1,1).$$

Again if $\{e^{\hat{t}}, e^{\hat{r}}, e^{\hat{\theta}}, e^{\hat{\phi}}\}$ be the dual basis of $\{e_{\hat{t}}, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}\}$ from the formula of multilinear objects are given by

$$R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}} e_{\hat{t}} \otimes e^{\hat{r}} \otimes e^{\hat{t}} \otimes e^{\hat{r}} = R^t_{rtr} e_t \otimes e^r \otimes e^t \otimes e^r$$

which implies

$$R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}} = \left(1-\frac{b}{r}\right)R^t_{rtr} = \left(1-\frac{b}{r}\right)\left[-\phi'' + \frac{(b'r-b)}{2r(r-b)}\phi' - (\phi')^2\right].$$

Hence the non-zero components are:

$$R^{\hat{t}}_{\hat{r}\hat{t}\hat{r}} = -R^{\hat{r}}_{\hat{t}\hat{r}\hat{t}} = \left(1-\frac{b}{r}\right)R^t_{rtr} = \left(1-\frac{b}{r}\right)\left[-\phi'' + \frac{(b'r-b)}{2r(r-b)}\phi' - (\phi')^2\right] \tag{1.69}$$
$$R^{\hat{t}}_{\hat{\theta}\hat{t}\hat{\theta}} = R^{\hat{t}}_{\hat{\phi}\hat{t}\hat{\phi}} = -R^{\hat{\theta}}_{\hat{t}\hat{\theta}\hat{t}} = -R^{\hat{\phi}}_{\hat{t}\hat{\phi}\hat{t}} = -\frac{\phi'}{r}\left(1-\frac{b}{r}\right) \tag{1.70}$$
$$R^{\hat{r}}_{\hat{\theta}\hat{r}\hat{\theta}} = R^{\hat{r}}_{\hat{\phi}\hat{r}\hat{\phi}} = R^{\hat{\theta}}_{\hat{r}\hat{\theta}\hat{r}} = R^{\hat{\phi}}_{\hat{r}\hat{\phi}\hat{r}} = \frac{b'r-b}{2r^3} \tag{1.71}$$
$$R^{\hat{\phi}}_{\hat{\theta}\hat{\phi}\hat{\theta}} = R^{\hat{\theta}}_{\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{b}{r^3} \tag{1.72}$$

Using the above values, for Ricci tensor $R_{\hat{\mu}\hat{\nu}}$ and the Ricci scalar R, one can contract the above Riemann tensors by

$$R_{\hat{\mu}\hat{\nu}} = R^{\hat{\alpha}}_{\hat{\mu}\hat{\alpha}\hat{\nu}} \text{ and } R = g^{\hat{\mu}\hat{\nu}}R_{\hat{\mu}\hat{\nu}} = -R_{\hat{t}\hat{t}} + R_{\hat{r}\hat{r}} + R_{\hat{\theta}\hat{\theta}} + R_{\hat{\phi}\hat{\phi}}$$

and from this one obtains Einstein tensor as:

$$G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2}Rg_{\hat{\mu}\hat{\nu}}. \tag{1.73}$$

Hence, non-zero components of Einstein tensor are

$$\begin{aligned} G_{\hat{t}\hat{t}} &= \frac{b'}{r^2}, \\ G_{\hat{r}\hat{r}} &= 2\left(1-\frac{b}{r}\right)\frac{\phi'(r)}{r} - \frac{b}{r^3}, \\ G_{\hat{\theta}\hat{\theta}} &= G_{\hat{\phi}\hat{\phi}} = \left(1-\frac{b}{r}\right)\left[\phi'' + (\phi')^2 - \frac{b'r+b-2r}{2r(r-b)}\phi' - \frac{(b'r-b)}{2r^2(r-b)}\right]. \end{aligned} \tag{1.74}$$

For the EFE to be complete, one needs to account for the matter part which is given by the stress-energy tensor, is proportional to the Einstein tensor. In an orthonormal basis, the stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$ is required to exhibit the same algebraic structure as $G_{\hat{\mu}\hat{\nu}}$. According to [107, 109], the only non-zero components are $T_{\hat{t}\hat{t}}$, $T_{\hat{r}\hat{r}}$, $T_{\hat{\theta}\hat{\theta}}$, and $T_{\hat{\phi}\hat{\phi}}$. These components, when analyzed in the orthonormal basis, carry straightforward physical interpretations. i.e.,

$$T_{\hat{t}\hat{t}} = \rho(r), \quad T_{\hat{r}\hat{r}} = -\tau(r) \quad \text{and} \quad T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p(r). \tag{1.75}$$

Here, $\rho(r)$ is the energy density, $\tau(r)$ is the radial tension (i.e., the negative of the radial pressure) and $p(r)$ is the pressure measured in the cross-radial direction.

Therefore, after a bit of manipulation the complete set of Einstein's Field Equation, $G_{\hat{\alpha}\hat{\beta}} = 8\pi G T_{\hat{\alpha}\hat{\beta}}$ reads:

,

$$b' = 8\pi G r^2 \rho, \tag{1.76}$$

$$\phi' = \frac{(-8\pi G\tau r^3 + b)}{[2r(r-b)]}, \tag{1.77}$$

$$\tau' = (\rho-\tau)\phi' - 2(p+\tau)/r. \tag{1.78}$$

The field equations above comprise three differential equations that relate to five unknown functions of r: b, ϕ, ρ, τ, and p. Typically, the conventional method for solving these equations involves assuming a specific type of matter or fields as the source of the stress-energy tensor. From the physical characteristics of this source, one derives "equations of state" for the radial tension as a function of mass-energy density $\tau(\rho)$ and for the lateral pressure as a function of mass-energy density $p(\rho)$. These equations of state, combined with the three field equations, establish a system of five equations for the five unknown functions (b, ϕ, ρ, τ, and p) of r.

1.6.1.4 Energy Conditions:

To gain a knowledge on the Energy condition of the matter present within the wormhole, one must first have an insight about what type of matter is present in the wormhole. To acquire understanding regarding the matter interweaving the wormhole, Morris and Thorne introduced the dimensionless function $\xi = \dfrac{\tau-\rho}{|\rho|}$ [107], considering

(1.76) and (1.77) gives,

$$\xi = \frac{\tau - \rho}{|\rho|} = \frac{\frac{b}{r} - b' - 2r\left(1 - \frac{b}{r}\right)\phi'}{|b'|}. \tag{1.79}$$

After a few manipulation the function becomes:

$$\xi = \frac{2b^2}{r|b'|}\frac{d^2r}{dz^2} - 2r\left(1 - \frac{b}{r}\right)\frac{\phi'}{|b'|}. \tag{1.80}$$

Now, considering, the finite character of ρ, and consequently of b', and the fact that $\left(1 - \dfrac{b}{r}\right)\phi' \to 0$ at the throat,one has the following relationship:

$$\xi(r_0) = \frac{\tau_0 - \rho_0}{|\rho_0|} > 0, \tag{1.81}$$

where, the parameters τ_0 and ρ_0 are the values of τ and ρ at the throat radius(r_0). The requirement $\tau_0 > \rho_0$ is somewhat debatable and depends on one's viewpoint. It means that the radial tension at the throat must be greater than the energy density. Consequently, Morris and Thorne introduced the term "exotic matter" to describe material constrained by this condition [107]. Subsequent verification will demonstrate that this designation refers to matter that contravenes the NEC (indeed, it contravenes all energy conditions). Exotic matter poses significant challenges for observers travelling through the wormhole throat at radial velocities approaching the speed of light. Consider a Lorentz transformation,

$$x^{\hat{\mu}'} = A^{\hat{\mu}'}_{\hat{\nu}} x^{\hat{\nu}},$$

where $A^{\hat{\mu}}_{\hat{\alpha}'} A^{\hat{\alpha}'}_{\hat{\nu}} = \delta^{\hat{\mu}}_{\nu}$ and $A^{\hat{\mu}}_{\hat{\nu}'}$ is defined as,

$$A^{\hat{\mu}}_{\hat{\nu}'} = \begin{bmatrix} \gamma & 0 & 0 & \gamma\nu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\nu & 0 & 0 & \gamma \end{bmatrix}.$$

The energy density observed by these observers is given by $T_{\hat{0}'\hat{0}'} = A^{\hat{\mu}}_{\hat{0}'} A^{\hat{\nu}}_{\hat{0}'} T_{\hat{\mu}\hat{\nu}}$,

$$T_{\hat{0}'\hat{0}'} = \gamma^2(\rho_0 - \nu^2\tau_0),$$

where $\gamma = (1 - \nu^2)^{-1/2}$. For velocities sufficiently close to the speed of light, $\nu \to 1$, the observer will measure a negative energy density, $T_{\hat{0}'\hat{0}'} < 0$.

This phenomenon persists for any traversable, non-spherical, and non-static wormhole. One can ascertain that a bundle of null geodesics entering the wormhole at one mouth and exiting from the other must have a cross-sectional area that initially increases and then decreases. This change from decreasing to increasing is attributed to the gravitational repulsion of the matter through which the bundle of null geodesics traverses.

Considering that wormhole spacetimes are sustained by exotic matter, the classical point-wise energy conditions dictates:

$$(I) \quad \text{NEC} : \rho + p_r \geq 0,\ \ \rho + p_t \geq 0, \tag{1.82}$$
$$(II) \quad \text{WEC} : \rho \geq 0,\ \ \rho + p_r \geq 0,\ \ \rho + p_t \geq 0, \tag{1.83}$$
$$(III) \quad \text{SEC} : \rho + p_r \geq 0,\ \ \rho + p_t \geq 0,\ \ \rho + p_r + 2p_t \geq 0, \tag{1.84}$$
$$(IV) \quad \text{DEC} : \rho \geq 0,\ \ \rho - |p_r| \geq 0,\ \ \rho - |p_t| \geq 0, \tag{1.85}$$

where, ρ is the energy density,p_r is the radial pressure, and p_t is the transverse pressure. They are the components of $T_{\mu\nu}$.

The exotic function (1.79) is intricately linked with the NEC, which stipulates that for any null vector k^μ, $T_{\mu\nu}k^\mu k^\nu \geq 0$. In the case of a diagonal stress-energy tensor, this implies $\rho - p_r \geq 0$ and $\rho + p_t \geq 0$. By utilizing the Einstein field equations at the throat r_0, and considering the finite nature of the redshift function such that $\left(1 - \frac{b}{r}\right)\phi'|_{r_0} \to 0$, one can confirm that the condition $(\rho - p_r)|_{r_0} < 0$ is violated, hence contravening the NEC.

1.6.1.5 Criteria for Traversability

Imagine a journey initiated radially by a traveller, commencing from a state of rest from the space station in the lower universe at $l = -l_1$ and concluding in a state of rest in the upper universe at $l = +l_2$. Suppose the traveller maintains a radial velocity $v(r)$, as observed by a static observer there. One can establish relationships between the proper radial distance travelled, the radius traversed dr, the coordinate time elapsed dt, and the proper time elapsed as measured by the traveller, as follows: Proper time is defined as

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = \left[1 - \left(\frac{v}{c}\right)^2\right] e^{2\phi}dt^2. \tag{1.86}$$

This gives

$$\gamma d\tau = e^\phi dt \text{ where, } \gamma^2 = \left[1 - \left(\frac{v}{c}\right)^2\right].$$

Again,

$$v(r) = \frac{1}{e^\phi}\frac{dl}{dt} = \pm \frac{1}{\left(1 - \frac{b}{r}\right)^{\frac{1}{2}} e^\phi}\frac{dr}{dt}, \tag{1.87}$$

which implies

$$\gamma v(r) = \frac{dl}{d\tau} = \pm \frac{1}{\left(1 - \frac{b}{r}\right)^{\frac{1}{2}} \gamma}\frac{dr}{d\tau}. \tag{1.88}$$

Here, the negative sign denotes the journey's first leg, i.e., through the lower universe, while the positive sign signifies the second leg, i.e., through the upper universe, after crossing the throat. Since the journey begins and ends at space stations, the velocity is $v = 0$ (at $l = -l_1$ and $l = +l_2$), and $v > 0$ for $-l_1 < l < +l_2$. Certain constraints need to be enforced at these two space stations:

I. The space stations are situated sufficiently far from the throat to ensure asymptotically flat spaces there. Thus,

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-\frac{1}{2}} \implies 0 \text{ i.e., } \frac{b}{r} << 1. \tag{1.89}$$

II.The gravitational redshift experienced by signals sent from the station to infinity should be negligible, meaning,

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\sqrt{g_{00}}} - 1 = e^{-\phi} - 1 \approx -\phi. \tag{1.90}$$

Thus, at the space stations, $\phi << 1$, yielding $d\tau = dt$, i.e., proper time equals coordinate time far from the throat.

III. The gravitational acceleration experienced at the stations should be less than that of Earth's gravitational acceleration. The traveller, remaining at rest relative to the space stations, is observed by a stationary observer. Therefore, the line element can be expressed as follows:

$$ds^2 = -e^{2\phi(r)}dt^2. \tag{1.91}$$

This yields $dt/d\tau = u^t = e^{-\phi}$, four velocity $\frac{dx^\mu}{d\tau} == (e^{-\phi}, 0, 0, 0)$, and acceleration components given by

$$A^\mu = u^\beta \nabla_\beta u^\mu = u^\beta \left(\frac{\partial u^\beta}{\partial x^\beta} + \Gamma^\mu_{\beta\alpha} u^\alpha\right) = u^t \left(\frac{\partial u^\beta}{\partial t} + \Gamma^\mu_{tt} u^t\right). \tag{1.92}$$

Thus, the only nonzero component of acceleration is $A^r = \Gamma^r_{tt}(u^t)^2 = \left(1 - \frac{b}{r}\right)\phi'$, and the four acceleration is $A^\mu = (0, (1 - b/r)\phi', 0, 0)$. Note that the magnitude of acceleration is:

$$A^2 = g_{\mu\nu}A^\mu A^\nu = \left(1 - \frac{b}{r}\right)(\phi')^2. \tag{1.93}$$

Hence, the acceleration of gravity measured by a static observer is

$$-\left(1 - \frac{b}{r}\right)^{\frac{1}{2}} \phi'.$$

At the space stations, i.e., far from the throat, $b/r << 1$, and acceleration should be less than or equal to Earth's gravitational acceleration, $g_\oplus$. Therefore, $|\phi'| < g_\oplus$ It is also crucial to impose certain restrictions for the journey through the wormhole with radial velocity $v(r)$ in $-l_1 < l < l_2$. These are:

The following points outline the necessary constraints for the journey:

1. The entire journey should be completed in a relatively short duration, as perceived by both the traveler and the stationary observer at the space station.

2. The acceleration experienced by the traveler within the region $-l_1 < l < l_2$ must not exceed Earth's gravitational acceleration.

3. The tidal acceleration affecting different parts of the traveler's body should also be less than Earth's gravitational acceleration.

1.6.1.6 Tidal gravitational forces and time to traverse the wormhole

Next, let's consider a hypothetical scenario where an individual embarks on a radial journey through a wormhole, commencing from a state of rest within a space station in the lower universe at $l = -l_1$, and concluding their journey at a space station in the upper universe at $l = +l_2$. As the traveler traverses a radius r with a radial velocity denoted as $v(r)$, as measured by a static observer, one can define $\gamma = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2}$, in accordance with the principles of special relativity. Subsequently, in terms of the distance traveled dl, radius traveled dr, coordinate time elapsed dt, and proper time elapsed as perceived by the traveler $d\tau_T$,

$$v = \frac{1}{e^{\phi}}\frac{dl}{dt} = \mp\frac{1}{(1-b/r)^{1/2}e^{\phi}}\frac{dr}{dt}, \tag{1.94}$$

$$v\gamma = \frac{v}{\left[1 - \left(\frac{v}{c}\right)^2\right]^{1/2}} = \frac{dl}{d\tau_T} = \mp\frac{1}{(1-b/r)^{1/2}}\frac{dr}{d\tau_T}. \tag{1.95}$$

The $(-)$ sign denotes the first half of the journey in the lower universe, while the $(+)$ sign signifies the second half in the upper universe. Since the trip starts and concludes at space stations that are at rest, we have:

$$\begin{aligned} &v = 0, \quad \text{at} \quad l = -l_1 \quad \text{and} \quad l = +l_2; \\ &v > 0 \quad \text{at} \quad -l_1 < l < l_2. \end{aligned} \tag{1.96}$$

The stations situated at $l = -l_1$ and $l = \pm l_2$ should be positioned sufficiently far from the throat to minimize the gravitational impact of the wormhole. Specifically, (i) the spatial geometry in those regions must approximate flatness, ensuring $\frac{b}{r} \ll 1$; (ii) the gravitational redshift of signals transmitted from the stations to infinity must be negligible, denoted as $|\phi| \ll 1$; (iii) the "gravitational acceleration" experienced at the stations, $g = -\left(1 - \frac{b}{r}\right)^{1/2}\phi' c^2 \approx -\phi' c^2$, should not exceed or be on the order of Earth's gravity, $g_e = 980$ cm$/s^2$. As, $|\phi| \ll 1$ at the stations, the proper time ticked by clocks there is equal to coordinate time t.

For wormhole travel to be feasible for human beings, the traveller's journey must adhere to three criteria: (i) the entire duration of the trip should be less than or approximately one year, according to both the traveller's perspective and that of individuals residing in the stations.

$$\Delta\tau_1 = \int_{-l_1}^{l_2} \frac{dl}{v\gamma} \lesssim 1 \ \text{ yr.}, \tag{1.97}$$

$$\Delta t = \int_{-l_1}^{l_2} \frac{dl}{ve^{\phi}} \lesssim 1 \ \text{ yr.}; \tag{1.98}$$

(ii)The acceleration experienced by the traveller should not surpass the gravitational force equivalent to that of Earth;
(iii) the variation in acceleration, or tidal acceleration, between different parts of the traveller's body should not exceed the gravitational force of Earth.
For the discussion of the traveller's accelerations, the orthonormal basis of the traveller's own reference frame, denoted as $e_{\hat{0}}$, $e_{\hat{1}}$, $e_{\hat{2}}$, and $e_{\hat{3}}$ are introduced. This basis is expressed in terms of the orthonormal basis of the static observer, represented as $e_{\hat{0}}$, $e_{\hat{1}}$, $e_{\hat{2}}$, and $e_{\hat{3}}$, using the standard Lorentz transformation from special relativity:

$$\begin{aligned} e_{\hat{0}'} &= u = \gamma e_t \mp \gamma \left(\frac{v}{c}\right) e_{\hat{r}}, \\ e_{\hat{1}'} &= \mp\gamma e_{\hat{r}} + \gamma \left(\frac{v}{c}\right) e_{\hat{t}}, \\ e_{\hat{2}'} &= e_{\hat{\theta}}, \\ e_{\hat{3}'} &= e_{\hat{\phi}}. \end{aligned}$$

Here, u represents the traveller's four-velocity, and $e_{\hat{1}'}$ points in the direction of travel (toward increasing l).

The four-accelerations of traveller, $a^{\hat{\alpha}'} = u^{\hat{\alpha}'}{}_{;\hat{\beta}'} u^{\hat{\beta}'} c^2$ represents the acceleration experienced by the traveller's body. Since the four-acceleration is always orthogonal to the four-velocity, $a \cdot u = a \cdot e_{\hat{0}'} = a_{\hat{0}'}$ vanishes. Due to the radial movement, the traveller's acceleration must be radial, so $a_{\hat{2}'} = a_{\hat{3}'} = 0$, and $a = ae_{\hat{1}'}$, where a denotes the magnitude of the acceleration.

To compute a, one can treat u_α as a function of the traveller's radial location r, evaluate $a_t/c^2 = u_{t;\alpha}u^\alpha = u_{t,r}u^r - \Gamma_{at\beta}u^\alpha u^\beta$ in the (ct, r, θ, ϕ) coordinate frame, and then observe that $a_t = a \cdot e_t = (ae_{\hat{1}'})(e_t) = -\gamma(v/c)e^{\phi}a$.

$$a = \mp \left(1 - \frac{b}{r}\right)^{1/2} e^{-\phi}(\gamma e^{\phi})' c^2 = e^{-\phi}\frac{d}{dl}(\gamma e^{\phi})c^2. \tag{1.99}$$

One demands that the traveller must not feel an acceleration larger than about 1 Earth gravity. Hence

$$\left| e^{-\phi}\frac{d(\gamma e^{\phi})}{dl} \right| \lesssim \frac{g_e}{c^2}. \tag{1.100}$$

Moving on to the gravitational forces exerted on the traveller due to tidal effects, let's designate the vector separation between two segments of the traveller's body as ξ (for instance, from head to feet). In the traveller's frame of reference, ξ is purely spatial, implying $\xi \cdot u = -\xi^{\hat{0}'} = 0$, where u represents the four-velocity. Consequently, the tidal acceleration between two segments of the traveller's body can be expressed as:

$$\Delta a^{\hat{\alpha}'} = -c^2 R^{\hat{\alpha}'}_{\hat{\beta}'\hat{\gamma}'\hat{\delta}'} u^{\hat{\beta}'} \xi^{\hat{\gamma}'} u^{\hat{\delta}'}, \tag{1.101}$$

where, $R^{\hat{\alpha}'}_{\hat{\beta}'\hat{\gamma}'\hat{\delta}'}$ are the components of Riemann curvature tensor.

Since $u^{\hat{\alpha}'} = \delta^{\hat{\alpha}'}_{\hat{0}'}$ and $\xi^{\hat{0}'} = 0$ in the traveller's frame, and since $R_{\hat{\alpha}'\hat{\beta}'\hat{\gamma}'\hat{\delta}'}$ is antisymmetric in its first two indices,

$$\Delta a^{\hat{j}'} = -c^2 R^{\hat{j}'}_{\hat{0}'\hat{k}'\hat{0}'} \xi^{\hat{k}'} = -c^2 R_{\hat{j}'\hat{0}'\hat{k}'\hat{0}'} \xi^{\hat{k}'}. \tag{1.102}$$

The transformation of components (1.69)-(1.72) of the Riemann tensor from observers' frame(static) $e_{\hat{t}}, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}$ to the traveller's frame $e_{\hat{0}'}, e_{\hat{1}'}, e_{\hat{2}'}, e_{\hat{3}'}$, one obtains,

$$R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'} = R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\left(1 - \frac{b}{r}\right)\left[-\phi'' + \frac{b'r - b}{2r(r-b)}\phi' - (\phi')^2\right], \tag{1.103}$$

$$\begin{aligned} R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'} &= R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'} \\ &= \gamma^2 R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} + \gamma^2 \left(\frac{v}{c}\right)^2 R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} \\ &= \frac{\gamma^2}{2r^2}\left[\left(\frac{v}{c}\right)^2 \left(b' - \frac{b}{r}\right) + 2(r-b)\phi'\right]. \end{aligned} \tag{1.104}$$

Given that these are the only non-zero components of $R_{\hat{j}'\hat{0}'\hat{k}'\hat{0}'}$ in the traveller's reference frame, the tidal acceleration (1.102) can be expressed in a simplified form as:

$$\Delta a^{\hat{1}'} = -c^2 R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'} \xi^{\hat{1}'}, \quad \Delta a^{\hat{2}'} = -c^2 R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'} \xi^{\hat{2}'}, \quad \Delta a^{\hat{3}'} = -c^2 R_{\hat{3}'\hat{0}'\hat{3}'\hat{0}'} \xi^{\hat{3}'}. \tag{1.105}$$

It is imperative that, for $|\xi| \sim 2\ m$ (the dimensions of the traveler's body) and with ξ oriented along any spatial direction in the traveler's frame, $|\Delta a| \lesssim$ (1 Earth gravity) $\equiv g_e$. Utilizing equation (1.105), this constraint can be expressed as:

$$\begin{aligned} |R_{\hat{1}'\hat{0}'\hat{1}'\hat{0}'}| &= \left|\left(1 - \frac{b}{r}\right)\left[-\phi'' + \frac{b'r - b}{2r(r-b)}\phi' - (\phi')^2\right]\right|, \\ &\lesssim \frac{g_e}{c^2 \times 2}\mathrm{m}^{-1} \approxeq \frac{1}{10^{10}}\mathrm{cm}^{-2}, \end{aligned} \tag{1.106}$$

$$\begin{aligned} |R_{\hat{2}'\hat{0}'\hat{2}'\hat{0}'}| &= \left|\frac{\gamma^2}{2r^2}\left[\left(\frac{v}{c}\right)^2 \left(b' - \frac{b}{r}\right) + 2(r-b)\phi'\right]\right| \\ &\lesssim \frac{g_e}{c^2 \times 2}\mathrm{m}^{-1} \approxeq \frac{1}{10^{10}}\mathrm{cm}^{-2}. \end{aligned} \tag{1.107}$$

The radial tidal constraint (1.106) serves to limit the metric coefficient ϕ. This constraint is most effectively met by ensuring $\phi' = 0$ universally, defining a category of wormhole solutions. On the other hand, the lateral tidal constraint (1.107) effectively limits the velocity v at which the traveler traverses the wormhole.

The overview presented here is very brief. There are more intricacies and subtleties in the mathematics of Morris Thorne wormholes. For elaborate and in-depth reading one may refer to [107].

1.6.2 Rotating wormholes

Considering the stationary and axially symmetric $(3+1)-$dimensional spacetime, Teo [203] investigated rotating traversable wormhole whose line element is given by,

$$ds^2 = -N^2 dt^2 + e^{\mu}\, dr^2 + r^2 K^2 [d\theta^2 + \sin^2\theta (d\phi - \omega\, dt)^2], \tag{1.108}$$

where N, K, ω, and μ are dependent on both r and θ. In this context, $\omega(r,\theta)$ can be understood as the angular velocity $\frac{d\phi}{dt}$ of a particle descending without restraint from infinity to the location (r,θ). Here, considering the definition [203]

$$e^{-\mu(r,\theta)} = 1 - \frac{b(r,\theta)}{r}\,, \tag{1.109}$$

Let's consider $K(r,\theta)$ as a positive, monotonically increasing function of r, dictating the proper radial distance R, where $R \equiv rK$ and $R_r > 0$ [203], as in the $(2+1)$-dimensional scenario. In line with convention, the subscripts $_r$ and $_\theta$ signify derivatives with respect to r and θ, respectively [203].

It's worth noting that an event horizon emerges whenever $N = 0$ [203]. Regularity conditions are applied to functions N, b, and K, ensuring that their derivatives with respect to θ vanish at the rotation axis, $\theta = 0, \pi$, thus guaranteeing a non-singular metric behavior on the rotation axis. The metric (1.108) converges to the Morris-Thorne spacetime metric under conditions of zero rotation and spherical symmetry.

$$N(r,\theta) \to \mathrm{e}^{\Phi(r)}, \quad b(r,\theta) \to b(r)\,, \quad K(r,\theta) \to 1\,, \quad \omega(r,\theta) \to 0\,. \tag{1.110}$$

The scalar curvature of the spacetime (1.108) appears intricate, yet at the throat $r = r_0$, it simplifies to:

$$\begin{aligned} R = & -\frac{1}{r^2K^2}\left(\mu_{\theta\theta} + \frac{1}{2}\mu_\theta^2\right) - \frac{\mu_\theta}{Nr^2K^2}\frac{(N\sin\theta)_\theta}{\sin\theta} - \frac{2}{Nr^2K^2}\frac{(N_\theta\sin\theta)_\theta}{\sin\theta} \\ & - \frac{2}{r^2K^3}\frac{(K_\theta\sin\theta)_\theta}{\sin\theta} + e^{-\mu}\,\mu_r\,\left[\ln(Nr^2K^2)\right]_r + \frac{\sin^2\theta\,\omega_\theta^2}{2N^2} + \frac{2}{r^2K^4}\left(K^2 + K_\theta^2\right)\,, \end{aligned} \tag{1.111}$$

where,

$$\mu_\theta = \frac{b_\theta}{(r-b)}\,, \qquad \mu_{\theta\theta} + \frac{1}{2}\mu_\theta^2 = \frac{b_{\theta\theta}}{r-b} + \frac{3}{2}\frac{{b_\theta}^2}{(r-b)^2}\,. \tag{1.112}$$

The above being the only complicated terms.

To avoid singularities at the throat one must impose $b_\theta = 0$ and $b_{\theta\theta} = 0$ implying throat being located at a constant value of r.

Therefore, one can deduce that the metric (1.108) defines a rotating wormhole configuration, with an angular velocity denoted as ω. The parameter K governs the proper radial distance, while N functions similarly to the redshift function in the Morris-Thorne wormhole, ensuring its finite and non-zero properties to avoid event horizons or curvature singularities. Concerning b, it denotes the shape function, meeting the requirement $b \leq r$ and remaining constant with respect to θ at the throat, specifically $b_\theta = 0$. Moreover, it satisfies the condition of flaring out, where $b_r < 1$.

The energy-momentum tensor being complicated is simplified, and has the form:

$$8\pi T_{\hat{t}\hat{t}} = -\frac{(K_\theta \sin\theta)_\theta}{r^2K^3\sin\theta} - \frac{\omega_\theta^2 \sin^2\theta}{4N^2} + e^{-\mu}\,\mu_r\,\frac{(rK)_r}{rK} + \frac{K^2+K_\theta^2}{r^2K^4}\,, \tag{1.113}$$

$$8\pi T_{\hat{r}\hat{r}} = \frac{(K_\theta \sin\theta)_\theta}{r^2K^3\sin\theta} - \frac{\omega_\theta^2 \sin^2\theta}{4N^2} + \frac{(N_\theta \sin\theta)_\theta}{Nr^2K^2\sin\theta} - \frac{K^2+K_\theta^2}{r^2K^4}\,, \tag{1.114}$$

$$8\pi T_{\hat{\theta}\hat{\theta}} = \frac{N_\theta(K \sin\theta)_\theta}{Nr^2K^3\sin\theta} + \frac{\omega_\theta^2 \sin^2\theta}{4N^2} - \frac{\mu_r\, e^{-\mu}(NrK)_r}{2NrK}\,, \tag{1.115}$$

$$8\pi T_{\hat{\phi}\hat{\phi}} = -\frac{\mu_r\, e^{-\mu}\,(NKr)_r}{2NKr} - \frac{3\sin^2\theta\,\omega_\theta^2}{4N^2} + \frac{N_{\theta\theta}}{Nr^2K^2} - \frac{N_\theta K_\theta}{Nr^2K^3}\,, \tag{1.116}$$

$$8\pi T_{\hat{t}\hat{\phi}} = \frac{1}{4N^2K^2r}\Big(6NK\,\omega_\theta\,\cos\theta + 2NK\,\sin\theta\,\omega_{\theta\theta} - \mu_r e^{-\mu} r^2 NK^3\,\sin\theta\,\omega_r + 4N\,\omega_\theta\,\sin\theta\,K_\theta - 2K\,\sin\theta\,N_\theta\,\omega_\theta\Big)\,. \tag{1.117}$$

The components $T_{\hat{t}\hat{t}}$ and $T_{\hat{i}\hat{j}}$ have usual interpretation, and particularly, the rotation of the matter distribution is given by $T_{\hat{t}\hat{\phi}}$. Hence, the NEC at the throat is given by

$$8\pi\, T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = \mathrm{e}^{-\mu}\mu_r\frac{(rK)_r}{rK} - \frac{{\omega_\theta}^2\sin^2\theta}{2N^2} + \frac{(N_\theta\sin\theta)_\theta}{(rK)^2N\sin\theta}\,. \tag{1.118}$$

At this point, instead of delving into further details, readers are directed to Ref. [203], where it was demonstrated that the NEC is violated in specific regions while upheld in others. Consequently, an observer falling into the wormhole might navigate around the throat to bypass the exotic matter supporting it. Nonetheless, it's crucial to underscore that complete avoidance of exotic matter is impossible.

1.6.3 Evolving wormholes in a cosmological background

The line element of a wormhole in a cosmological context is given by

$$ds^2 = \omega^2(t)\left[-e^{2\Phi(r)}\,dt^2 + \frac{dr^2}{1-kr^2-\frac{b(r)}{r}} + r^2\,\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\right]. \tag{1.119}$$

In the expression, $\omega^2(t)$ represents the conformal factor, which maintains finiteness and positive definiteness throughout the time domain t. An alternative representation of the metric (1.119) can be achieved using "physical time" instead of "conformal time". This substitution involves replacing t with $\tau = \int \omega(t)dt$ and consequently replacing $\omega(t)$ with $R(\tau)$, where $R(\tau)$ denotes the functional form of the metric in the τ coordinate system [204, 205]. When both the form function and the redshift function approach zero, such that $b(r) \to 0$ and $\Phi(r) \to 0$, respectively, the metric (1.119) transforms into the FRW metric. Moreover, as $\omega(t) \to$ const and $k \to 0$, it tends toward the static wormhole metric.

The Einstein field equation is expressed as:

$$G_{\hat{\mu}\hat{\nu}} = R_{\hat{\mu}\hat{\nu}} - \frac{1}{2} g_{\hat{\mu}\hat{\nu}} R = 8\pi T_{\hat{\mu}\hat{\nu}},$$

in an orthonormal reference frame, where any cosmological constant terms are integrated as part of the stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$. The components of the stress-energy tensor $T_{\hat{\mu}\hat{\nu}}$ are described by:

$$T_{\hat{t}\hat{t}} = \rho(r,t)\,, \qquad T_{\hat{r}\hat{r}} = -\tau(r,t)\,, \qquad T_{\hat{t}\hat{r}} = -f(r,t)\,, \qquad T_{\hat{\phi}\hat{\phi}} = T_{\hat{\theta}\hat{\theta}} = p(r,t)\,, \tag{1.120}$$

with

$$\rho(r,t) = \frac{1}{8\pi}\frac{1}{\omega^2}\left[3e^{-2\Phi}\left(\frac{\dot{\omega}}{\omega}\right)^2 + \left(3k + \frac{b'}{r^2}\right)\right], \tag{1.121}$$

$$\tau(r,t) = -\frac{1}{8\pi}\frac{1}{\omega^2}\left\{e^{-2\Phi(r)}\left[\left(\frac{\dot{\omega}}{\omega}\right)^2 - 2\frac{\ddot{\omega}}{\omega}\right] - \left[k + \frac{b}{r^3} - 2\frac{\Phi'}{r}\left(1 - kr^2 - \frac{b}{r}\right)\right]\right\}, \tag{1.122}$$

$$f(r,t) = -\frac{1}{8\pi}\left[2\frac{\dot{\omega}}{\omega^3}\, e^{-\Phi}\,\Phi'\left(1 - kr^2 - \frac{b}{r}\right)^{1/2}\right], \tag{1.123}$$

$$p(r,t) = \frac{1}{8\pi}\frac{1}{\omega^2}\left\{e^{-2\Phi(r)}\left[\left(\frac{\dot{\omega}}{\omega}\right)^2 - 2\frac{\ddot{\omega}}{\omega}\right] + \left(1 - kr^2 - \frac{b}{r}\right)\right.$$
$$\left.\times\left[\Phi'' + (\Phi')^2 - \frac{2kr^3 + b'r - b}{2r(r - kr^3 - b)}\Phi' - \frac{2kr^3 + b'r - b}{2r^2(r - kr^3 - b)} + \frac{\Phi'}{r}\right]\right\}. \tag{1.124}$$

The notation $\dot{\rho}(r,t)$, $\dot{\tau}(r,t)$, $\dot{f}(r,t)$, and $\dot{p}(r,t)$ signifies derivatives with respect to t, while the prime indicates derivatives with respect to r. The physical interpretation of $\rho(r,t)$, $\tau(r,t)$, $f(r,t)$, and $p(r,t)$ is as follows: $\rho(r,t)$ represents the energy density, $\tau(r,t)$ denotes the radial tension per unit area, $f(r,t)$ signifies the energy flux in the outward radial direction, and $p(r,t)$ indicates the lateral pressures, all as observed by stationary observers located at constant r, θ, and ϕ. It's worth noting that the stress-energy tensor exhibits a non-diagonal component due to the time variation of $\omega(t)$ and/or the dependence of the redshift function on the radial coordinate.

A particularly intriguing scenario arises when considering the metric (1.119) in the context of a wormhole embedded in a time-dependent inflationary background, as explored by Thomas Roman [125]. Roman's analysis primarily aimed to utilize inflationary dynamics to expand an initially small [125], potentially submicroscopic, wormhole. The functions $\Phi(r)$ and $b(r)$ are chosen such that they yield a viable wormhole configuration at $t = 0$, assumed to mark the onset of inflation. Roman [125] further delved into investigating intriguing properties of inflating wormholes, including analyzing constraints imposed on the initial size of the wormhole to ensure that the mouths remain causally connected throughout the inflationary epoch, and studying the maintenance of the wormhole during and after the decay of the false vacuum. Additionally, there's the prospect that the wormhole may continue to expand during the subsequent FRW phase of expansion. A similar analysis could be conducted by substituting the deSitter scale factor with an FRW scale factor $a(t)$ [204–206]. In particular, in Refs. [204, 205], specific instances of evolving wormholes existing for a finite duration were examined, along with an investigation into a special class of scale factors demonstrating 'flashes' of WEC violation.

1.7 Some Unconventional Wormholes Solutions

In this section two unconventional geometries of wormholes is briefly reviewed. Namely: Thin shell Wormholes and Cylindrical Wormholes.

1.7.1 Cylindrical Wormholes

The presence of wormholes and their adherence to energy conditions have been investigated extensively. It has been observed that the stress-energy tensor of matter violates the NEC in the vicinity of the throat. These conclusions are drawn under the assumption that the throat exhibits a 2-D geometry with finite area, particularly in a static scenario The Entrance of a Wormhole (owing to its description in the frame work of spherical symmetry) generally implied, as seen from outside to be a local object like black holes and stars. But our universe may contain structures which are extended along certain directions, like cosmic strings. So to describe the simplest string like structure or configuration one can use cylindrical symmetry. Wormholes in cylindrical symmetry along with its properties have been studied by authors [207–213].

1.7.1.1 The wormhole metric: A brief review

The simplest form of the cylindrically symmetric Wormhole metric is given as [207]

$$ds^2 = Adx^2 + Cdz^2 + Bd\phi^2 + Edtd\phi - Ddt^2. \tag{1.125}$$

One can rewrite (1.125) as,

$$ds^2 = -D[dt - \frac{E}{D}d\phi]^2 + Adx^2 + Cdz^2 + Bd\phi^2, \tag{1.126}$$

where,

$$D = e^{2\gamma}, A = e^{2\alpha}, C = e^{2\mu}, r(x) = e^{\beta}, r^2 = \frac{\delta}{D}, \delta = BD + E^2. \tag{1.127}$$

Hence, the metric becomes

$$ds^2 = -e^{2\gamma(x)}\left[dt - \frac{E}{e^{2\gamma(x)}}d\phi\right]^2 + e^{2\alpha(x)}dx^2 + e^{2\mu(x)}dz^2 + e^{2\beta(x)}d\phi^2. \tag{1.128}$$

where, $E(x)$ is the rotation term and putting $E(x) = 0$, simply gives us the static case.

Here,

$$e^{2\gamma(x)} = Q^2(x_0^2 - x^2), \tag{1.129}$$

$$r^2 = e^{2\beta} = \left(\frac{r_0^2}{Q^2(x_0^2 - x^2)}\right), \tag{1.130}$$

$$x_0 = \frac{\omega_0}{(\chi\rho_0 r_0)}, \qquad Q^2 = \chi\rho_0 r_0^2. \tag{1.131}$$

Here, Q and x_0 are dimensionless constants and r and e^α have the dimensions of length.

$$e^{2\mu} = e^{2mx}(x_0 - x)^{(1-\frac{x}{x_0})}(x_0 + x)^{(1+\frac{x}{x_0})}, \tag{1.132}$$

where m is a constant.

And lastly,

$$E = \frac{r_0(x_0^2 - x^2)}{2x_0^2}\left(\frac{2x_0 x}{(x_0^2 - x^2)} + \ln\frac{x_0 + x}{x_0 - x} + E_0\right). \tag{1.133}$$

where E_0 is a constant.

In terms of harmonic radial co-ordinate x,

$$\alpha = \beta + \gamma + \mu. \tag{1.134}$$

Equation (1.129), (1.130), (1.131), (1.132), (1.133) are obtained by simplifying and integrating the diagonal components of the Einstein equations. Readers may refer to [207, 210] for detailed calculation.

1.7.1.2 Field Equations: A brief review

For the metric given by (1.128) describes a wormhole if [207, 210] $i)r(x) = e^{\beta(x)}$ has regular minimum and is large or finite away from this minimum. It is called the r-throat, or ii)for the area function the same condition holds i.e. $a(x) = e^{\mu+\beta}$, also has the regular minimum, called the a-throat. The term "$-E(x)$" is the spacetime rotation term and is characterised by $\omega(x)$ i.e. the angular velocity:

$$\omega = \frac{1}{2}\left(Ee^{-2\gamma}\right)' e^{\gamma-\beta-\alpha}. \tag{1.135}$$

The field equation along with the Ricci tensor components in the gauge $\alpha = \mu$ is given by [210](prime denotes $\frac{d}{du}$):-

$$R_1^1 = -e^{-2\mu}\left[\beta'' + \gamma'' + \mu'' + \beta'^2 + \gamma'^2 + \mu'(\beta' + \gamma')\right] + 2\omega^2; \tag{1.136}$$

$$R_2^2 = -e^{-2\mu}[\mu' + \mu'(\beta' + \gamma')]; \tag{1.137}$$

$$\sqrt{g}R_3^3 = -[\beta' e^{\beta+\gamma} - E\omega e^\mu]'; \tag{1.138}$$

$$\sqrt{g}R_4^4 = -\left(\omega e^{2\gamma+\mu}\right)'; \tag{1.139}$$

$$R_3^4 = 0; \tag{1.140}$$

The diagonal part of the Ricci tensor splits into two parts:- i) the static part ${}_sR_\mu^\nu$, and ii) for the rotational part ${}_\omega R_\mu^\nu$, where,

$${}_\omega R_\mu^\nu = \omega^2 dia(2,0,2,-2), coordinate\ order\ (x,z,\phi,t). \tag{1.141}$$

Similarly G_μ^ν also splits in two, i.e.

$$G_\mu^\nu = {}_sG_\mu^\nu + {}_\omega G_\mu^\nu, \quad where, \quad {}_\omega G_\mu^\nu = \omega^2 dia(1,-1,0,-3). \tag{1.142}$$

The Einstein's equation can be written as:

$$G_\mu^\nu = -8\pi G T_\mu^\nu, \tag{1.143}$$

and the stress energy tensor is given by:

$$R_\mu^\nu = -8\pi G\tilde{T}_\mu^\nu, \quad where, \quad \tilde{T}_\mu^\nu = T_\mu^\nu - \frac{1}{2}\delta_\mu^\nu T_\alpha^\alpha. \tag{1.144}$$

One may refer to [207, 208, 210] for the detailed study of the solution of EFE for cylindrical wormholes.

1.7.2 Thin shell Wormhole

Visser discovered a distinct category of traversable wormholes, known as thin-shell wormholes, which are obtained through a "cut-and-paste" method [109]. Initial stability analyses by Poisson and Visser for spherical perturbations around these wormholes revealed stable configurations contingent upon the equation of state of exotic matter located at the throat [139]. Building upon this work, investigations have delved into various extensions, including charged thin-shell wormholes [214], those constructed from Schwarzschild spacetimes with different masses [137], and those incorporating a cosmological constant [124]. Additionally, studies have explored thin-shell wormholes in cylindrically symmetric spacetimes [215, 216]. Garcia *et al.* provided a study on stability for generic static and spherically symmetric thin-shell wormholes [217], while Dias and Lemos examined stability in higher-dimensional Einstein gravity [218].

1.7.2.1 Junction conditions

Let's consider a hypersurface Σ within a spacetime. This hypersurface divides the spacetime into two distinct regions: $\mathcal{M}^+$, characterized by coordinates x_+^α and the metric $g_+^{\alpha\beta}$, and $\mathcal{M}^-$, with coordinates x_-^α and the metric $g_-^{\alpha\beta}$. Junction conditions enable the seamless connection of $\mathcal{M}^+$ and $\mathcal{M}^-$ at Σ, ensuring that both $g_+^{\alpha\beta}$ and $g_-^{\alpha\beta}$ satisfy the Einstein equations.

One can define the extrinsic curvatures as

$$K_{ab} := n_{\mu;\beta} e_a^\mu e_b^\beta. \tag{1.145}$$

Also defining h_{ab} as a metric of the line element on Σ. Then, for the smooth joint of spacetimes at their boundaries,

$$[h_{ab}] = 0 \quad \text{and} \quad [K_{ab}] = 0 \tag{1.146}$$

are required. The deviation of any tensorial function on both sides of the hypersurface as $[F] := (F^{+} - F^{-})|_{\Sigma}$, where F^{+} and F^{-} represent functions in $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$, respectively is introduced.

Alternatively, when $[K_{ab}] \neq 0$ alongside $[h_{ab}] = 0$, a stress-energy tensor S_{ab} must satisfy:

$$8\pi S_{ab} = -[K_{ab}] + [K]h_{ab}, \tag{1.147}$$

The matter with a non-zero S_{ab} is localized to the infinitesimally thin surface Σ, hence termed a "thin-shell." Equation (1.147) delineates the dynamics of the thin-shell. The associated constraints are provided by

$$S_{a\ \ |b}^{\ b} = -[T_{\alpha\beta}e^{\alpha}_{a}n^{\beta}] \tag{1.148}$$

and

$$\bar{K}^{ab}S_{ab} = [T_{\alpha\beta}n^{\alpha}n^{\beta}], \tag{1.149}$$

where $\bar{K}^{ab} := (K^{ab}_{+} + K^{ab}_{-})|_{\Sigma}/2$. One may refer [219] and Barrabes and Bressange [220] for the details of derivation.

1.7.2.2 Construction

It is important to note that the construction process is conducted within spherically, planarly, and hyperbolically symmetric spacetimes in d-dimensional ($d \geq 3$) Einstein gravity, incorporating an electromagnetic field and a cosmological constant in the bulk spacetimes. This comprehensive setup is essential for subsequent analysis.

The framework for maximally symmetric d-dimensional thin-shell wormholes was initially developed by Dias and Lemos [218], which one extends to encompass more generalized scenarios. Wormholes through three distinct steps utilizing junction conditions [219] is generated.

One can consider a pair of d-dimensional manifolds, $\mathcal{M}_{\pm}$, with $d \geq 3$. The d-dimensional Einstein equations are expressed as:

$$G_{\mu\nu\pm} + \frac{(d-1)(d-2)}{6}\Lambda_{\pm}g_{\mu\nu\pm} = 8\pi T_{\mu\nu\pm}, \tag{1.150}$$

Here, $G_{\mu\nu\pm}$, $T_{\mu\nu\pm}$, and $\Lambda_{\pm}$ represent the Einstein tensors, stress-energy tensors, and cosmological constants within the manifolds $\mathcal{M}_{\pm}$, respectively. The metrics on $\mathcal{M}_{\pm}$ are denoted by $g^{\pm}_{\mu\nu}(x^{\pm})$. These metrics pertain to static and spherically, planar, and hyperbolically symmetric spacetimes with $G_{(d-1)(d-2)/2}(d-2, S)$ symmetry on $\mathcal{M}_{\pm}$ are written as

$$ds^2_{\pm} = -f_{\pm}(r_{\pm})dt^2_{\pm} + f_{\pm}(r_{\pm})^{-1}dr^2_{\pm} + r^2_{\pm}(d\Omega^k_{d-2})^2_{\pm}, \tag{1.151}$$

respectively. $M_{\pm}$ and $Q_{\pm}$ represent the mass and charge parameters in $\mathcal{M}_{\pm}$, respectively. k is a constant determining the geometry of the $(d-2)$-dimensional space, taking values of ± 1 or 0. Specifically, $k = +1$, 0, and -1 correspond to a sphere, plane, and a hyperboloid, respectively.

It's important to note that, according to the generalized Birkhoff's theorem [221], the metric (1.151) is the unique solution of the Einstein equations of electrovacuum for $k = +1$, 0, and -1.

Next, one can construct a manifold $\mathcal{M}$ by joining $\mathcal{M}_\pm$ at their boundaries. The boundary hypersurfaces $\partial\mathcal{M}_\pm$ are defined as follows: $\partial\mathcal{M}_\pm := \{r_\pm = a \mid f_\pm(a) > 0\}$, where a denotes the thin-shell radius. Then, by connecting the two regions $\tilde{\mathcal{M}}_\pm$, defined as $\tilde{\mathcal{M}}_\pm := \{r_\pm \geq a \mid f_\pm(a) > 0\}$, and matching their boundaries $\partial\mathcal{M}_+ = \partial\mathcal{M}_- := \Sigma$, one can create a new manifold $\mathcal{M}$ with a wormhole throat at Σ. Σ should be a timelike hypersurface, on which the line element is given by:

$$s_\Sigma^2 = -d\tau^2 + a(\tau)^2 (d\Omega_{d-2}^k)^2. \tag{1.152}$$

The surface function defining Σ is $\Phi = r - a(\tau) = 0$. Here, τ represents the proper time on the junction surface Σ, which is described by the coordinates $x^\mu(y^a) = x^\mu(\tau, \theta_1, \theta_2, \ldots, \theta_{d-2}) = (t(\tau), a(\tau), \theta_1, \theta_2, \ldots, \theta_{d-2})$, where Greek indices range from 1 to d, and Latin indices range from 1 to $d-1$. The set $\{y^a\}$ represents the intrinsic coordinates on Σ.

In the third step, the junction conditions to obtain the Einstein equations for the submanifold Σ is utilised. To accomplish this, one can define the unit normals to the hypersurfaces $\partial\mathcal{M}_\pm$. These unit normals are determined by:

$$n_{\alpha\pm} := \pm \frac{\Phi_{,\alpha}}{|\Phi^{,\mu}\Phi_{,\mu}|^{\frac{1}{2}}}. \tag{1.153}$$

To fabricate thin-shell wormholes, one can assign different signs to the unit normals on $\partial\mathcal{M}_\pm$. Conversely, for constructing normal thin-shell models, the unit normals are selected to have the same signs. Tangent vectors e_a^α are determined by the partial derivatives $\partial x^\alpha / \partial y^a$. Additionally, one can define the four-velocity $u_\pm^\alpha$ of the boundary as:

$$u_\pm^\alpha := e_{\tau\pm}^\alpha = (\dot{t}_\pm, \dot{a}, 0, \ldots, 0) = \left(\frac{1}{f_\pm(a)} \sqrt{f_\pm(a) + \dot{a}^2}, \dot{a}, 0, \ldots, 0 \right), \tag{1.154}$$

where $\dot{} := \partial/\partial\tau$ and $u^\alpha u_\alpha = -1$ is satisfied. Where,

$$n_{\alpha\pm} = \pm \left(-\dot{a}, \frac{\sqrt{f_\pm + \dot{a}^2}}{f_\pm}, 0, \ldots, 0 \right) \tag{1.155}$$

and the unit normal satisfies $n_\alpha n^\alpha = 1$ and $u^\alpha n_\alpha = 0$. For more detailed derivation of the geometry one may refer to [222].

1.8 Closed Timelike Curves

Understanding the concept of time, both quantitatively and qualitatively, poses significant challenges and holds paramount importance. Special Relativity offers crucial insights into time dilation effects, while the General Theory of Relativity (GTR) delves deeply into how time behaves in the presence of strong and weak gravitational fields [223]. It's intriguing to note that within the framework of GTR, the fabric of spacetime can accommodate non-trivial geometries giving rise to "closed timelike curves" (CTCs) [109, 224–227]. Numerous solutions to the Einstein Field Equations (EFEs) featuring CTCs exist, with two notable features being the tipping of light cones due to rotation about a cylindrically symmetric axis and violations of the Energy Conditions of GTR [228]. These violations are fundamental in singularity theorems and classical black hole thermodynamics. Additionally, considerable attention has been devoted to the quantum aspects of closed timelike curves [229–231].

It's worth mentioning that the tilting of light cones appears to be a common characteristic in certain solutions exhibiting rotating cylindrical symmetry. The metric describing a stationary, axisymmetric solution with rotation can be found, as, [109, 232].

$$ds^2 = -F(r)\,dt^2 + H(r)\,dr^2 + L(r)\,d\phi^2 + 2\,M(r)\,d\phi\,dt + H(r)\,dz^2\,, \tag{1.156}$$

where z represents the distance along the axis of rotation, ϕ denotes the angular coordinate, r stands for the radial coordinate, and t indicates the temporal coordinate. The metric components solely depend on the radial coordinate r. It's noteworthy that the determinant, $g = \det(g_{\mu\nu}) = -(FL+M^2)H^2$, is Lorentzian given that $(FL+M^2) > 0$.

Since the angular coordinate ϕ is periodic, an azimuthal curve defined by $\gamma = \{t = \text{const}, r = \text{const}, z = \text{const}\}$ forms a closed curve of invariant length $s_\gamma^2 \equiv L(r)(2\pi)^2$. If $L(r)$ is negative, then the integral curve with fixed (t, r, z) constitutes a CTC. If $L(r) = 0$, the azimuthal curve becomes a closed null curve. Now, let's consider a null azimuthal curve, not necessarily a geodesic or closed, in the (ϕ, t) plane with (r, z) fixed. The null condition, $ds^2 = 0$, implies:

$$0 = -F + 2M\dot{\phi} + L\dot{\phi}^2\,, \tag{1.157}$$

with $\dot{\phi} = d\phi/dt$. Solving the quadratic, one can have

$$\frac{d\phi}{dt} = \dot{\phi} = \frac{-M \pm \sqrt{M^2 + FL}}{L}\,. \tag{1.158}$$

Given the Lorentzian signature constraint, $FL + M^2 > 0$, the roots are real. When $L(r) < 0$, the light cones are sufficiently tipped over to allow travel to the past. Circling once around the azimuthal direction results in the total backward time-jump for a null curve to be:

$$\Delta T = \frac{2\pi|L|}{-M + \sqrt{M^2 - F|L|}}\,. \tag{1.159}$$

When $L(r) < 0$ for any value of r, the region of chronology violation extends across the entire spacetime [109]. For an extensive review of the mathematics of CTCs for different spacetime one may refer [233].

1.9 Cosmological Scenarios

In this particular section a brief review about two cosmological scenarios, namely, the Cosmic No Hair Conjecture and the Emergent Universe Scenario is presented.

1.9.1 The Cosmic No Hair Conjecture

In broad terms, the Cosmic no-hair conjecture (CNHC) posits that "all expanding universe models with a positive cosmological constant eventually approach the de-Sitter solution." To explore whether the universe tends toward a homogeneous and isotropic state during an inflationary epoch, Gibbons *et al.* [234] and Hawking *et al.* [235] formulated this conjecture. Wald [236] provided a formal proof of it using normal matter that adheres to the dominant energy condition (DEC) and strong energy condition (SEC). Further, Kitada and Meada [237] studied the CNHC for Bianchi models in power-law inflation.They additionally demonstrated that if the initial proportion of vacuum energy to the utmost three curvature exceeds half, the assertion holds valid for Bianchi type IX as well. S. Cotsakis and J. Miritzis [238] established the validity of the cosmic no-hair conjecture (CNHC) across all orthogonal Bianchi cosmologies within the framework of the $R + \beta R^2$ theory, incorporating matter. Their proof utilized the conformally equivalent Einstein field equations, encompassing a scalar field with a complete self-interacting potential, alongside conformally related matter fields. Notably, we demonstrate that the Bianchi IX universe asymptotically converges towards de Sitter space, under the condition that initially, the scalar three-curvature remains below the potential of the scalar field linked to the conformal transformation. Subsequently, Chakraborty and his collegues investigated CNHC and has found extensive application in various contexts and within different gravitational theories. In their paper [239],they delved into the cosmic no-hair theorem concerning anisotropic Bianchi models that allow an inflationary resolution involving a scalar field. Their findings indicate that during the inflationary period, the potential's shape has no impact on the evolution, whereas the late-stage dynamics are governed by the constant additive term within the inflaton field's potential. In another paper [240], investigated the cosmic no-hair conjecture (CNHC) in the context of homogeneous anisotropic Bianchi models, considering a cosmological constant that varies within Randall-Sundrum braneworld-type scenarios. Their study demonstrated that in the initial scenario, the universe tends towards isotropy following power-law inflation, while exponential expansion characterizes the latter case. Chakraborty and Debnath [241] investigated energy conditions for the CNHC in braneworld scenarios using examples with realistic fluid models and concluded that strong and weak energy conditions are sufficient for for the CNHC in the considered scenarios.

Moreover, Chakraborty and Bandopadhyay [242] concluded that in brane scenarios, adherence to the cosmic no-hair conjecture (CNHC) does not necessitate standard matter in the bulk but imposes certain constraints contingent upon the brane tension. Subsequently, they demonstrated the applicability of CNHC for a Gauss-Bonnet dilatonic scalar coupled to Einstein gravity, with the coupling parameter increasing linearly over time [243]. In a separate publication [244], they endeavored to broaden Wald's findings within the framework of scalar tensor theory, contrasting the constraints for CNHC validity with those derived from general relativity.

CNHC have also been investigated by authors in different theories of gravity and cosmology in great extent. One may refer [245–247]

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1.9.2 The Emergent Universe Scenario

The conventional big bang cosmology encounters challenges related to horizon and singularity akin to those of a black hole's event horizon. Hence, there is a keen interest in exploring cosmological models devoid of these issues. The emergent universe scenario presents a cosmological model devoid of singularities, where time extends from negative infinity to positive infinity. It posits that the universe emerges from a quasi-static high-density phase, transitioning at a designated time (usually denoted as $t = 0$) to the expanding phase characteristic of Standard Big Bang cosmology. Ellis and Marteens [248],and Ellis *et al.* [249] proposed such a model, termed the Emergent Universe (EU), which resolves the singularity problem within Einstein's general relativity.They derived closed universes featuring a minimally coupled scalar field ϕ, characterized by a distinct form of self-interacting potential. Additionally, these universes might contain ordinary matter with an equation of state given by $p = \omega\rho$, where, ω ranges from -1 *to* 1. This nonsingular cosmology originates from Einstein's static universe, addressing horizon problems as well. The emergent scenario shares similarities with both inflationary cosmology and bouncing cosmology. Like inflationary cosmology, it starts with a hot and small universe. However, akin to bouncing cosmology, it extends time from negative infinity to positive infinity, and its evolution is non-singular.

The conditions for the universe to be emergent are:

- $a \to a_0$, $H \to 0$ as $t \to -\infty$
- $a \simeq a_0$, $H \simeq 0$, $t << t_0$

Since its inception, the EU model has garnered significant attention from cosmologists, leading to numerous studies in the field. Mukherjee *et al.* [250, 251] extended the emergent universe scenario to flat spacetime, presenting solutions within a relativistic framework. Their consideration of the equation of state allows for exotic matter violating the energy conditions of General Relativity, accommodating the universe's late-time acceleration. In the context of modified theories of gravity the emergent scenario has also been investigated. Shekh *et al.* [252] inspected the late time acceleration of the universe in $f(Q, T)$ gravity proposing a form of scale factor satisfying the conditions of emergent universe. Further, Shekh [253] investigated observational constraints in accelerated emergent $f(Q)$ gravity model and proposed an emergent scale factor that yields the deceleration parameter in redshift form, thereby determining the solution of the field equations in the FLRW Universe. Chakraborty [254] explored particle creation as a consequence of the emergent universe. Bhattacharya and Chakraborty [255] investigated an emergent universe model in inhomogeneous spacetime, noting its resemblance to features of Fred Hoyle's steady-state theory. Bannerjee *et al.* [256] investigated Emergent Universe in Brane World Scenario. Bose *et al.* [257] studied the Emergent scenario in Hořava–Lifshitz gravity. A great deal of investigation on consequences of emergent scenario has been investigated by Chakraborty and collaborators. One may refer [258–266].

In recent years extensive studies have been done on the alternative theories of the early universe scenarios, with particular emphasis on the emergent universe scenario. Emergent scenarios in Superstring Cosmology, String Gas cosmology, Matrix theory,etc have been investigated for theoretical and observational studies. One may refer [267–270]. Although few studies [271, 272] have questioned and put restrictions on the possible inflationary models [273–276] but cosmologies, such emergent models, maintain consistency with these conjectures and are readily fulfilled. Even though there's still a gap in our comprehensive understanding of the emergent phase, there are encouraging paths for exploration.

CHAPTER 2

COSMIC NO-HAIR CONJECTURE AND WORMHOLES

2.1 Prelude

In this chapter, the relationship between the cosmic no-hair conjecture's validity (or violation) and the presence (or absence) of wormholes, both within Einstein's Gravity and in modified gravity theories is explored. The investigation reveals that the existence of wormholes contradicts the cosmic no-hair conjecture, while the affirmation of the cosmic no-hair conjecture indicates the absence of wormholes. However, this relationship does not hold in reverse in both Einstein's Gravity and modified gravity theories. Additionally, the gravitational entropy predictions associated with wormholes is revisited and one can demonstrate their interconnectedness. Furthermore, the Hayward formalism for characterizing wormholes [277–279] is examined and one can illustrate how employing their approach to describe the wormhole's past outer trapping horizon allows us to validate the results.

2.2 Energy Conditions: CNHC and Wormholes

To analyze the cosmic evolution in Einstein's Gravity one can consider the Hamiltonian constraint arising from the Einstein equation and the Raychaudhuri equation, which is given by,

$$G_{\mu\nu}n^{\mu}n^{\nu} - \kappa^2 T_{\mu\nu}n^{\mu}n^{\nu} = 0. \tag{2.1}$$

$$R_{\mu\nu}n^{\mu}n^{\nu} - \kappa^2 (T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu})n^{\mu}n^{\nu} = 0. \tag{2.2}$$

One can now decompose the four dimensional manifold in (3+1) form so that the three space-like hypersurfaces are specified by the three metric h_{ab} given by

$$h_{ab} = g_{ab} + n_a n_b. \tag{2.3}$$

Here n^a be the unit normal vector to the hypersurface. The extrinsic curvature tensor $K_{ab} = \nabla_a n_b$ has the explicit form,

$$K_{ab} = \frac{1}{3}Kh_{ab} + \sigma_{ab}, \tag{2.4}$$

where $K = K_{ab}h^{ab}$ is the trace of the extrinsic curvature and σ_{ab} is the shear of the time-like geodesic congruence orthogonal to the hypersurfaces. Now using the Gauss-Codazzi equation that relates the space-time intrinsic curvature to the three curvature of the hypersurface ${}^3R$, one obtains $G_{\mu\nu}n^\mu n^\nu = \frac{1}{2}({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2)$. Rewriting the Ricci tensor in terms of the curvature tensor and then using some simple mathematical calculations, one obtains the dynamical equations (2.1) and (2.2) as

$$K^2 = 3\kappa^2 T_{\mu\nu}n^\mu n^\nu + \frac{3}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{3}{2}{}^3R, \tag{2.5}$$

$$\dot{K} = -\frac{1}{3}K^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - \kappa^2(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})n^\mu n^\nu, \tag{2.6}$$

where the overdot denotes the Lie derivative with respect to proper time.

It has been shown that the 3-space curvature scalar ${}^3R$ is negative definite for all Bianchi models except IX, for which ${}^3R$ is indefinite in sign. Thus using the idea of Wald and proceeding along the line of his approach one finds that for CNHC the matter should satisfy the weak and strong energy conditions, i.e.

$$a \geq 0 \qquad and \qquad b \geq 0, \tag{2.7}$$

where

$$a = T_{\mu\nu}n^\mu n^\nu, \tag{2.8}$$

$$b = (T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})n^\mu n^\nu = a + \frac{1}{2}T. \tag{2.9}$$

However, in modified gravity theories, the field equations can be written in the form of Einstein equations with an additional (hypothetical) matter term (known as geometric matter) as

$$G_{\mu\nu} = T_{\mu\nu} + T^{(g)}_{\mu\nu}. \tag{2.10}$$

As a result, equations (2.1) and (2.2) take the form

$$G_{\mu\nu}n^\mu n^\nu = T_{\mu\nu}n^\mu n^\nu + T^{(g)}_{\mu\nu}n^\mu n^\nu, \tag{2.11}$$

and

$$R_{\mu\nu}n^\mu n^\nu = (T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})n^\mu n^\nu + (T^{(g)}_{\mu\nu} - \frac{1}{2}T^{(g)}g_{\mu\nu})n^\mu n^\nu. \tag{2.12}$$

Hence after (3+1)-decomposition as before the above dynamical equations become (modifications of equations (2.5) and (2.6)) (with $\kappa^2 = 1$)

$$K^2 = -\frac{3}{2}{}^3R + 3\sigma^2 + 3a + 3c, \tag{2.13}$$

and

$$\dot{K} = -\frac{1}{3}K^2 - 2\sigma^2 - b - d, \tag{2.14}$$

where $\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}$ is the (square of) shear scalar, and the scalars c and d have the expressions

$$c = T^{(g)}_{\mu\nu} n^\mu n^\nu, \tag{2.15}$$

$$d = (T^{(g)}_{\mu\nu} - \frac{1}{2}T^{(g)} g_{\mu\nu}) n^\mu n^\nu = c + \frac{1}{2}T^{(g)}. \tag{2.16}$$

Thus for the validity of the CNHC in modified gravity theories one must have

$$a + c \geq 0 \qquad and \qquad b + d \geq 0, \tag{2.17}$$

i.e. the effective matter (equivalent geometric matter + normal matter) must obey the weak and the strong energy conditions.

On the other hand, for wormhole configuration one can consider the Raychaudhuri equation as

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 - R_{\mu\nu}k^\mu k^\nu, \tag{2.18}$$

where θ is the expansion scalar, $\omega^2 = \frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu}$ is the (square of) rotation scalar, and τ is the proper time. Also here the shear and rotation scalars are associated to the congruence defined by the null vector field k^μ (with $k^\mu k_\mu = 0$). As for any hypersurface orthogonal congruences $\omega_{\mu\nu} = 0$ so for attractive gravity, the above Raychaudhuri equation demands $R_{\mu\nu}k^\mu k^\nu \geq 0$. Hence the geodesic congruences focus within a finite value of the parameter labeling points on the geodesics. On the contrary, the role of wormhole throat is to defocus geodesic congruences (flaring-out condition). So for wormhole configuration it is essential to have $R_{\mu\nu}k^\mu k^\nu < 0$. However, the Raychaudhuri equation is a purely geometric statement, it is not related to any particular gravity theory. So the above flaring-out condition for the existence of a wormhole takes the form:

(i) Einstein's Gravity:

$$a_n \equiv T_{\mu\nu}k^\mu k^\nu < 0. \tag{2.19}$$

(violation of null energy condition)

(ii) Modified gravity theory:

$$T_{\mu\nu}k^\mu k^\nu + T^{(g)}_{\mu\nu}k^\mu k^\nu < 0 \qquad i.e. \quad a_n + c_n < 0. \tag{2.20}$$

Here a_n and c_n are the expressions for a and c for null geodesics. Thus in the modified gravity theory there are three possibilities for the existence of a wormhole, namely:

I) Both the usual matter and the equivalent geometric matter (EGM) violates null energy condition (NEC) and as a result, the resultant effective matter violates NEC (i.e. $a_n < 0$ and $c_n < 0$).

II) The usual matter violates NEC (i.e. $a_n < 0$) but the EGM does not (i.e. $c_n > 0$), still the inequality (2.20) holds, provided c_n has an upper bound $|a_n|$ (i.e. $c_n < |a_n|$).

III) The NEC is obeyed (i.e. $a_n \geq 0$) by the usual matter while it is violated by the EGM (i.e. $c_n < 0$) and a_n has an upper bound $|c_n|$ (i.e. $a_n < |c_n|$).

Further, if one considers a congruence of time-like geodesics instead of null one then the above Raychaudhuri equation becomes

$$\frac{d\theta_T}{d\tau} = -\frac{1}{3}\theta_T^2 - 2\sigma_T^2 + 2\omega_T^2 - R_{\mu\nu}v^\mu v^\nu, \tag{2.21}$$

where all the quantities with subscript T stand for the same quantities as above for the time-like geodesics and v^μ is a unit time-like vector (i.e. $v^\mu v_\mu = -1$). Thus the defocusing condition for the time-like geodesic congruence is

$$R_{\mu\nu}v^\mu v^\nu < 0, \tag{2.22}$$

which takes the following form in gravity theories,

(i) Einstein's Gravity:

$$(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})v^\mu v^\nu < 0 \qquad i.e. \quad b < 0. \tag{2.23}$$

(violation of strong energy condition)

(ii) Modified gravity theory:

$$\begin{aligned} &(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})v^\mu v^\nu + (T_{\mu\nu}^{(g)} - \frac{1}{2}T^{(g)}g_{\mu\nu})v^\mu v^\nu < 0, \\ &i.e. \quad b + d < 0 \qquad i.e. \quad a + c + \frac{1}{2}(T + T^{(g)}) < 0. \end{aligned} \tag{2.24}$$

As above, one again has three possibilities for the occurrence of the wormhole configuration in modified gravity theory.

A. Both the matter components violate NEC and the trace of the effective total matter energy-momentum tensor has an upper bound of $2|a + c|$ or equivalently both the matter components violate SEC.

B. The usual matter component violates SEC (i.e. $b < 0$) while EGM obeys SEC (i.e. $d \geq 0$) and one can have $0 \leq d < |b|$.

C. The SEC is satisfied by usual matter (i.e. $b \geq 0$) but not by the EGM (i.e. $d < 0$) and b is confined to the interval $(0, |d|)$.

Note that conditions III and C will be interesting in the present context. Condition III shows that in modified gravity theories it is possible to have a wormhole with normal matter but the null scalar a_n is restricted in the range $(0, |c_n|)$. Condition C shows that wormholes may be formulated with normal matter which satisfies the SEC in modified gravity theories but the scalar b should be restricted to $(0, |d|)$. Now both Einstein's Gravity and modified gravity theories is examined for how the validity/non-validity of CNHC is related to the existence/non-existence of wormhole configuration.

From the energy conditions, it is well known that:

i) Weak Energy Condition (WEC) implies null energy condition (NEC) but not the converse.

ii) SEC does not imply weak energy condition (WEC) and vice versa.

Thus in Einstein's Gravity if the space-time satisfies CNHC then both WEC and SEC are satisfied and hence NEC is satisfied. So it is not possible to form wormholes in space-time where CNHC is satisfied. On the other hand, if wormholes form in space-time then matter should be exotic (i.e violates NEC). As a result, WEC will not be satisfied and hence CNHC will be violated. Therefore, in Einstein's Gravity one can say that:

- validity of CNHC $\Rightarrow$ non-existence of wormhole
- existence of Wormhole $\Rightarrow$ violation of CNHC,

but not in a reverse way, i.e non-existence of wormholes only implies that NEC is satisfied which does not mean that WEC is satisfied. Similarly, a violation of CNHC may be due to a violation of SEC which does not imply a violation of NEC.

It should be noted that similar results also hold in modified gravity theories but with one basic difference. In Einstein's Gravity, the existence of a wormhole implies a violation of CNHC and hence the matter field must be exotic. On the other hand, in modified gravity theories one can have the same conclusion but the matter field may not have an exotic character.

However for the wormhole, to make it traversable there are restrictions on the geometry which defines the throat. The wormhole throat is obtained by reconditioning two asymptotically flat space-times, such that the location of the throat is at their minimum value, and as one moves away from the throat there is a flare out of the space-time that has to be satisfied by the radial coordinate. This essentially means that congruence of geodesics that one obtains moving towards the focal point of an asymptotically flat metric is never achieved on either side of the restructured point (in this case the point of minimum r is called the throat), rather they tend to diverge. Hence from the perspective of the Ricci tensor this would mean $R_{\mu\nu}n^{\mu}n^{\nu} < 0$. The breach in attaining geodesic congruence at the throat can be manifested in terms of the violation of DEC as one can write the Ricci tensor as $R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}$. In fact, for the wormhole metric at the throat, the matter essentially violates the NEC. Rewriting the dynamic Raychaudhuri equation (2.6) using the transform $K = 3\frac{\dot{\epsilon}}{\epsilon}$ one can obtain the following equation

$$\ddot{\epsilon} = \frac{1}{3}(-\sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}n^{\mu}n^{\nu})\epsilon \tag{2.25}$$

With $R_{\mu\nu}n^{\mu}n^{\nu} < 0$ there is no way of an unambiguous conclusion that the right-hand side of (2.25) will be negative and hence no way of ascertaining that expansions will converge. Thus essentially CNHC will not hold for such space-time. In other words, one can say that in Einstein's Gravity, regions of space-time with geometry like wormholes, that is where NEC is violated, are examples of possible regions of CNHC violation too.

2.3 CNHC and Hayward Formalism of Wormhole

The Hayward formalism is actually the Morris Thorne wormhole, with the metric being, not necessarily asymptotically flat. The $(3+1)$ dimensional wormhole metric in terms of the dual null coordinate ξ^{+}, ξ^{-} is

$$ds^2 = 2g_{+-}d\xi^{+}d\xi^{-} + r^2 d\Omega^2, \tag{2.26}$$

defining the variation along the null vector as $\partial_\pm = \frac{\partial}{\partial \xi_\pm}$ one can obtain the expansion as $K_\pm = \frac{2}{r}\partial_\pm r$. The notion of a trapped, untrapped and marginal sphere could be formalised based on the sign invariance of the expansion.

If the signs of K_+ and K_- are invariant then essentially $K_+K_- > 0$ and hence one will obtain a trapped sphere.

One obtains expansion concerning an untrapped sphere if $K_+K_- < 0$. In this case, there are two possibilities:

- $K_+ > 0$ *and* $K_- < 0$, this means that the variation with respect to the null vector is outgoing for ∂_+ and incoming for ∂_-.
- $K_+ < 0$ and $K_- > 0$, this means that the variation with respect to null vector is outgoing for ∂_- and incoming for ∂_+.

The sphere is marginal if $K_+K_- = 0$. Then again this might be due to several possibilities listed under:

- $K_+ = 0$: Then $K_- < 0$ gives a future pointing marginal sphere, while $K_- > 0$ implies a past marginal sphere. Also, the marginal sphere is a maximal sphere if $\partial_- K_+ < 0$ which is classified as the outer marginal sphere. While $\partial_- K_+ > 0$ gives a minimal sphere that is called the inner marginal sphere.
- $K_+ = 0, K_- = 0$: The marginal sphere is bifurcating that is outer and inner as $\partial_- K_+ < 0$ or $\partial_- K_+ > 0$ respectively. It is degenerate for $\partial_- K_+ = 0$.
- $K_- = 0$: Then $K_+ < 0$ gives a future pointing marginal sphere, while $K_+ > 0$ implies a past marginal sphere.

The foliation of the space-time manifold into a three-dimensional hypersurface by the marginal sphere is called the *trapping horizon*. These trapping horizons are apparent horizons that are quasi-local causal in nature. They are classified as past/future and inner/outer just as the classification of the marginal sphere. The corresponding Einstein's equations concerning the metric (2.26) and energy-momentum tensor $T_\pm$ are given by:

$$\partial_\pm\partial_\pm r - \partial_\pm log(-g_{+-})\partial_\pm r = -4\pi r T_{\pm\pm}, \tag{2.27}$$

$$r\partial_+\partial_- r + \partial_+ r\partial_- r - \frac{1}{2}g_{+-} = 4\pi r^2 T_{+-}, \tag{2.28}$$

$$r^2\partial_+\partial_- log(-g_{+-}) - 2\partial_+ r\partial_- r + g_{+-} = 8\pi r^2(g_{+-}T^0_0 - T_{+-}). \tag{2.29}$$

Using the definition for $K_\pm$ and evaluating the first and second equations, that is equations (2.27) and (2.28), one can obtain the dynamical equations for the expansion of the null congruences as

$$\partial_\pm K_\pm = -\frac{1}{2}(K_\pm)^2 - K_\pm\partial_\pm log(-g_{+-}) - 8\pi T_{\pm\pm} \tag{2.30}$$

$$\partial_\pm K_\mp = -K_+K_- + \frac{1}{r^2}g_{+-} + 8\pi T_{+-} \tag{2.31}$$

Thus for the trapping horizon fixing $K_+ = 0$ on the horizon one arrives at

$$\partial_+ K_+ = -8\pi T_{++}, \tag{2.32}$$

$$\partial_- K_+ = \frac{1}{r^2} g_{+-} + 8\pi T_{+-}. \tag{2.33}$$

Therefore from equation (2.33) one classifies the future trapping horizon to be inner and outer as $\partial_- K_+ > 0$ and $\partial_- K_+ < 0$. In the case of a wormhole $T_{+-} > 0$ and hence one would obtain a future, inner trapping horizon. The equation (2.32) would give us $\partial_+ K_+ = -8\pi T_{++}$. Since for wormhole one can have $T_{++} < 0$, one ends up with $\partial_+ K_+ > 0$ showing that the expansion rate is positive and hence CNHC holds.

In case one fixes $K_- = 0$ on the horizon,one arrives at

$$\partial_- K_- = -8\pi T_{--}, \tag{2.34}$$

$$\partial_+ K_- = \frac{1}{r^2} g_{+-} + 8\pi T_{+-}. \tag{2.35}$$

Thus the trapping horizon in this case is a past outer trapping horizon with $\partial_- K_- > 0$ since $T_{--} < 0$. Since the expansion rate is the opposite, CNHC cannot hold here.

2.4 Discussion

The authors of [279] and [280] argued that the horizon for a wormhole would be essentially ‘past outer’ type. Referring to this, it is evident that a past outer trapping horizon where CNHC is violated, allows the formation of a wormhole. Thus it verifies the results obtained in *“Energy Conditions: CNHC and Wormholes”*.

Therefore, the importance of the present study is that in Einstein’s Gravity it is not possible to have wormholes with normal matter but in modified gravity one may have wormholes with normal matter. Finally, one may conclude that the validity of CNHC or existence of wormhole configuration implies the non-existence of wormhole or violation of CNHC respectively but not in a reverse way both in Einstein’s Gravity as well as in modified gravity theories.

CHAPTER 3

GEODESIC MOTION AND PARTICLE CONFINEMENTS IN CYLINDRICAL WORMHOLE SPACETIME: EXPLORING CLOSED TIMELIKE CURVES

3.1 Prelude

In this chapter, the geodesic motion of a test particle along with its confinement is investigated within Cylindrically Symmetric Wormhole spacetime admitting to Closed Timelike Curves. The confinement of particles with or without angular momentum is also investigated. It is found that particles with positive angular momentum that co-rotates with the spacetime can only pass through the causality violating region. Particles with only non-zero angular momentum are present in the vicinity of the Closed Timelike Curves.

3.2 Geodesic Formulation

For the formulation of the geodesic equation one can use the cylindrically symmetric metric given by (1.128), i.e:

$$ds^2 = -e^{2\gamma(x)}\left[dt - \frac{E}{e^{2\gamma(x)}}d\phi\right]^2 + e^{2\alpha(x)}dx^2 + e^{2\mu(x)}dz^2 + e^{2\beta(x)}d\phi^2. \tag{3.1}$$

The geodesics of the spacetime can be calculated using the Euler-Lagrange equation of motion. From the metric given by equation, (1.128), the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}(-e^{2\gamma(x)}\dot{t}^2 + \left[e^{2\beta(x)} - e^{-2\gamma(x)}E^2(x)\right]\dot{\phi}^2 + 2E(x)\dot{\phi}\dot{t} + e^{2\alpha(x)}\dot{x}^2 + e^{2\mu(x)}\dot{z}^2). \quad (3.2)$$

From the Lagrangian equation of motion the geodesic equations are obtained as follows:

$$\dot{t} = -\frac{-Ae^{2\beta(x)}e^{2\gamma(x)} - p_\phi e^{2\gamma(x)}E + AE^2}{e^{2\beta(x)}(e^{2\gamma(x)})^2}, \quad (3.3)$$

$$\dot{\phi} = -\frac{-p_\phi e^{2\gamma(x)} + AE}{e^{2\beta(x)}e^{2\gamma(x)}}, \quad (3.4)$$

$$p_z = e^{2\mu(x)}\dot{z}, \quad (3.5)$$

and,

$$\dot{x}^2 = \frac{-\epsilon + e^{2\gamma(x)}\dot{t}^2 - \left[e^{2\beta(x)} - e^{-2\gamma(x)}E^2(x)\right]\dot{\phi}^2 - 2E(x)\dot{\phi}\dot{t} - e^{2\mu(x)}\dot{z}^2}{e^{2\alpha(x)}}. \quad (3.6)$$

Here A, p_ϕ, p_z are integration constants and are termed as Energy of the particle, angular momentum and momentum along the z-direction respectively. ϵ, here can take the values 1,0,-1 for timelike, null and spacelike signature of metric.

In equation(3.4), it can be seen that it depends on the angular momentum of the test particles. If the angular momentum of the test particle is considered to be zero i.e $p_\phi = 0$, equation becomes:

$$\dot{\phi} = -\frac{AE}{e^{2\beta(x)}e^{2\gamma(x)}}. \quad (3.7)$$

The above expression may be termed as the equation of "*Space-time dragging*". It is proportional to the rotation term of the spacetime. As a result of the above equation, it can be deduced that the zero angular momentum particles also co-rotates with the rotating spacetime. Thus there are two types of co-rotating particles with $p_\phi = 0$, and $p_\phi > 0$. Particles with negative angular momentum i.e $p_\phi < 0$ are considered counter rotating particles.

Bronnikov *et. al* in their paper [207,209]mentioned that their is a causality violation in the form of CTC. It emerges from the solution of the described spacetime. It is shown that if the $g_{\phi\phi}$ of the metric given by equation (1.128), is less than zero, it contains a closed timelike curve. From the inspection of the $g_{\phi\phi}$, which is given by:

$$g_{\phi\phi} = \frac{r_0}{(x_0^2 - x^2)}\left[-1 + (y + \frac{1-y^2}{2}\ln\frac{(1+y)}{(1-y)})^2\right]. \quad (3.8)$$

shows, that the $g_{\phi\phi}$ component becomes negative and there occurs a CTC at $|y| > 0.564$ and there also occurs singularity at $|y| = \pm 1$, here $y = x/x_0$.

Hence, it is of interest to investigate the trajectories of the particles in the vicinity of the CTC and also look for their confinements.

3.3 Particle motion and Confinements

Geodesic motion of a particle is allowed only when it satisfies the condition [211],

$$\left(\frac{dx}{d\phi}\right)^2 \geq 0. \tag{3.9}$$

The co-rotating particles satisfy the above condition while the counter rotating particles violates the condition for the positive radial distance. Hence from here, only the co-rotating particles are considered for investigation. The motion of the test particles in the vicinity of Closed Timelike Curves are obtained by solving the (dt/dx) equation for null and time-like particles numerically, as it is quite difficult to obtain analytic solutions for the first order geodesic equations.

The solutions for the Radial Null Geodesic and Radial Timelike Geodesic is plotted along the radial distance (maintaining co-rotation of particles with the spacetime) to get the spacetime diagram to study their confinements.

3.3.1 Radial Null Geodesic

The radial null geodesic equation is obtained by simply using $\epsilon = 0$ in equation(3.6). Hence, one gets,

$$\dot{x}^2 = \frac{e^{2\gamma(x)}\dot{t}^2 - \left[e^{2\beta(x)} - e^{-2\gamma(x)}E^2(x)\right]\dot{\phi}^2 - 2E(x)\dot{\phi}\dot{t} - e^{2\mu(x)}\dot{z}^2}{e^{2\alpha(x)}}. \tag{3.10}$$

The trajectories of the null particles or photons are obtained by solving the equation,

$$\frac{dt}{dx} = \frac{\dot{t}}{\dot{x}}, \tag{3.11}$$

From equations (3.3), (3.4), and (3.10) one obtains,

$$\frac{\dot{t}}{\dot{x}} = \frac{e^{\mu}}{e^{\beta}e^{2\gamma}} \frac{-\left(-Ae^{2\beta}e^{2\gamma} - p_\phi e^{2\gamma} + E + AE^2\right)}{\left(\frac{2Ap_\phi E}{e^{2\beta}} - \left(\frac{p_\phi^2 e^{2\gamma}}{e^{2\beta}} + A^2\left(-1 + \frac{E^2}{e^{2\gamma}e^{2\beta}}\right) + \frac{p_z^2 e^{2\gamma}}{e^{2\mu}}\right)\right)^{\frac{1}{2}}}. \tag{3.12}$$

Here, $\dot{x}^2$ has two roots, and for $\dot{x}$ has to be real for the solution to make sense.This can be readily achieved by carefully choosing the values of the constants. The values so chosen must not alter any conditions that allows the formation of the wormhole and the CTC. The values of m and E_0 need to be Zero to maintain the signature $g_{\phi\phi} < 0$ and also the value of $r_0 < 1.041$ for causality violation to occur. The value of A needs to be greater than 1, otherwise it may encounter $\frac{1}{0}$ form during computation.

It is of significance to note here that the study of both the co-rotating type of particles i.e. with zero angular momentum and non zero angular momentum is important, because of the presence of the space-time dragging term, as the zero angular momentum particles also co-rotates with the rotating spacetime. Consequently, it is of interest to see the confinements of both type of test particles. Here only the movements of the particles in constant z-plane is considered, ergo '$p_z = 0$'.

- Zero Angular Momentum Particles ($p_\phi = 0$)

  Equation(3.12) for $p_\phi = 0$ becomes:

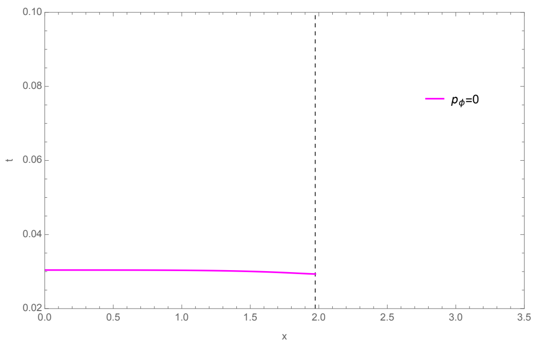
$$\frac{\dot{t}}{\dot{x}} = \frac{e^{\mu}}{e^{\beta}e^{2\gamma}} \frac{-\left(-Ae^{2\beta}e^{2\gamma} + AE^2\right)}{\left(-A^2\left(-1+\frac{E^2}{e^{2\gamma}e^{2\beta}}\right)\right)^{\frac{1}{2}}}. \tag{3.13}$$

- Non-Zero Angular Momentum Particles ($p_\phi \neq 0$)

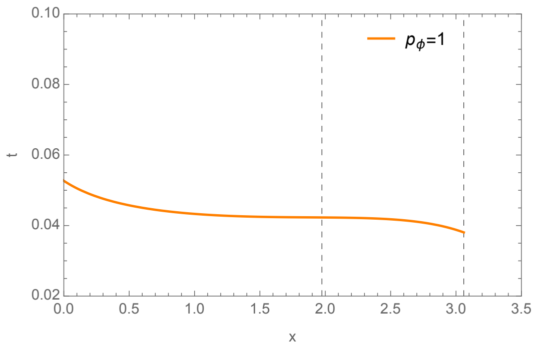
  For $p_\phi \neq 0$ equation (3.12) remains same, i.e.:

$$\frac{\dot{t}}{\dot{x}} = \frac{e^{\mu}}{e^{\beta}e^{2\gamma}} \frac{-\left(-Ae^{2\beta}e^{2\gamma} - p_\phi e^{2\gamma} + E + AE^2\right)}{\left(\frac{2Ap_\phi E}{e^{2\beta}} - \left(\frac{p_\phi^2 e^{2\gamma}}{e^{2\beta}} + A^2\left(-1+\frac{E^2}{e^{2\gamma}e^{2\beta}}\right) + \frac{p_z^2 e^{2\gamma}}{e^{2\mu}}\right)\right)^{\frac{1}{2}}}.$$

Now using values from equations (1.129), (1.130), (1.131), (1.132), (1.133), in equations (3.12 and 3.13), and after numerically solving the equation, and maintaining $\dot{x}$ remains real. The space-time diagram for photons with $p_\phi = 0$ and $p_\phi = 1$ is obtained.



(a)



(b)

Figure 3.1: Space-time diagram for radial null geodesic with (a) $p_\phi = 0$ and (b) $p_\phi = 1$. Here the values used are: $x_0 = 3.5, \rho_0 = .4, r_0 = 1, \chi = .5, m = E_0 = 0, A = 3$. For $p_\phi = 0$, particles are confined within $x = 1.974$.

Figure (3.1) depict the spacetime diagrams for particles with Zero AM and Non-zero AM, respectively. It is evident from the figures that both types of test particles exhibit distinct trajectories, as indicated by the dashed lines. Zero AM particles are confined within the region $x = 1.974$, coinciding with the boundary of the CTC (for specific constant values). In contrast, particles with Non-zero AM are confined within $x = 3.059$, surpassing the CTC boundary. This phenomenon can be interpreted as a consequence of the Total Effective AM (i.e., the sum of the spacetime's AM and the particle's AM), allowing the particle to breach the CTC boundary and traverse the CTC region. The velocity profile of these particles for the same constant values further corroborates these observations.

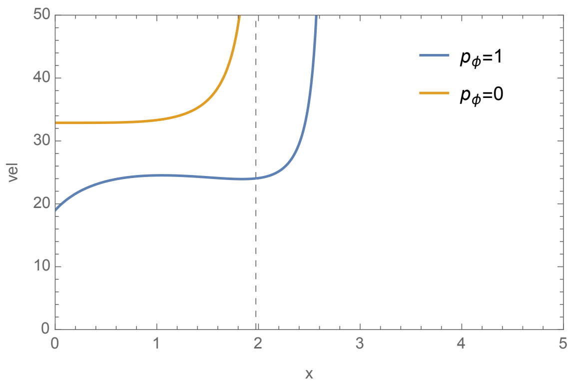


Figure 3.2: Velocity profile graph for Null particles with $p_\phi = 0$, and $p_\phi = 1$. Particles with AM can only cross the CTC boundary denoted by the dashed lines.

3.3.2 Radial time like geodesic

One can obtain the radial timelike geodesic from equation (3.6) by choosing $\epsilon = 1$, i.e.

$$\dot{x}^2 = \frac{-1 + e^{2\gamma(x)}\dot{t}^2 - \left[e^{2\beta(x)} - e^{-2\gamma(x)}E^2(x)\right]\dot{\phi}^2 - 2E(x)\dot{\phi}\dot{t} - e^{2\mu(x)}\dot{z}^2}{e^{2\alpha(x)}}. \tag{3.14}$$

Again proceeding as before to get the spacetime diagram one can use

$$\frac{\dot{t}}{\dot{x}} = \frac{e^{\mu}}{e^{\beta}e^{2\gamma}} \frac{-\left(-Ae^{2\beta}e^{2\gamma} - p_\phi e^{2\gamma} + E + AE^2\right)}{\left(\frac{2Ap_\phi E}{e^{2\beta}} - \left(\frac{p_\phi^2 e^{2\gamma}}{e^{2\beta}} + A^2\left(-1 + \frac{E^2}{e^{2\gamma}e^{2\beta}}\right) + \frac{p_z^2 e^{2\gamma}}{e^{2\mu}} + e^{2\gamma}\right)\right)^{\frac{1}{2}}}. \tag{3.15}$$

As for the previously discussed reason in this subsection above, for the radial time-like particle also the confinements of the particles without AM and also with AM is investigated.

- Zero Angular Momentum Particles ($p_\phi = 0$)

  Equation(3.15) for $p_\phi = 0$ becomes:

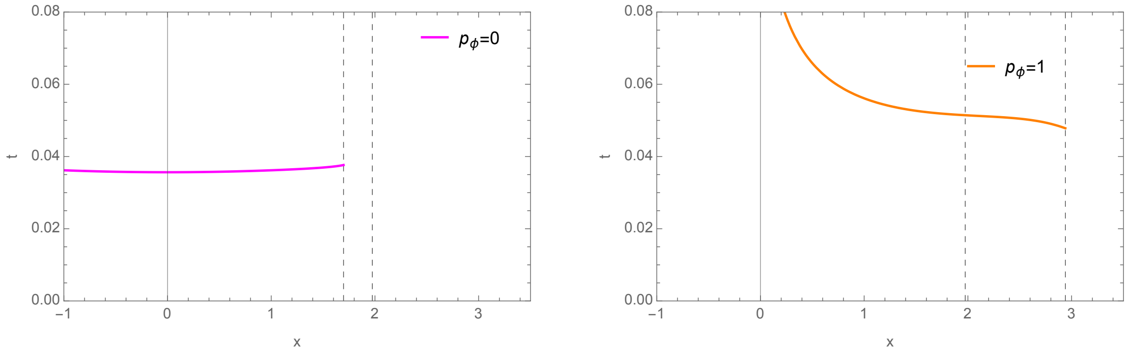
$$\frac{\dot{t}}{\dot{x}} = \frac{e^{\mu}}{e^{\beta}e^{2\gamma}} \frac{-\left(-Ae^{2\beta}e^{2\gamma} + AE^2\right)}{\left(-A^2\left(-1 + \frac{E^2}{e^{2\gamma}e^{2\beta}}\right) + e^{2\gamma}\right)^{\frac{1}{2}}}. \tag{3.16}$$

- Non-Zero Angular Momentum Particles ($p_\phi \neq 0$)

  For $p_\phi \neq 0$ equation (3.15) remains same, i.e.:

$$\frac{\dot{t}}{\dot{x}} = \frac{e^{\mu}}{e^{\beta}e^{2\gamma}} \frac{-\left(-Ae^{2\beta}e^{2\gamma} - p_\phi e^{2\gamma} + E + AE^2\right)}{\left(\frac{2Ap_\phi E}{e^{2\beta}} - \left(\frac{p_\phi^2 e^{2\gamma}}{e^{2\beta}} + A^2\left(-1 + \frac{E^2}{e^{2\gamma}e^{2\beta}}\right) + \frac{p_z^2 e^{2\gamma}}{e^{2\mu}} + e^{2\gamma}\right)\right)^{\frac{1}{2}}}.$$

Using the values from equations (1.129), (1.130), (1.131), (1.132), (1.133) in (3.16 and 3.15), the space-time diagram for radial timelike particles with $p_\phi = 0$ and $p_\phi = 1$



(a) (b)

Figure 3.3: Space-time diagram for radial time-like geodesic with (a) $p_\phi = 0$ and (b) $p_\phi = 1$. Here the values used are: $x_0 = 3.5, \rho_0 = .4, r_0 = 1, \chi = .5, m = E_0 = 0, A = 3$. For $p_\phi = 0$, particles are confined within $x = 1.697$ and for $p_\phi = 1$, particles are confined within $x = 2.940$.

are obtained. The numerical values of the constants used here are the same as used in the previous section.

Interestingly, similar confinement characteristics are seen in the space-time diagram and also in velocity profile of the radial time-like particles which supports only particles with AM crossing the CTC boundary.

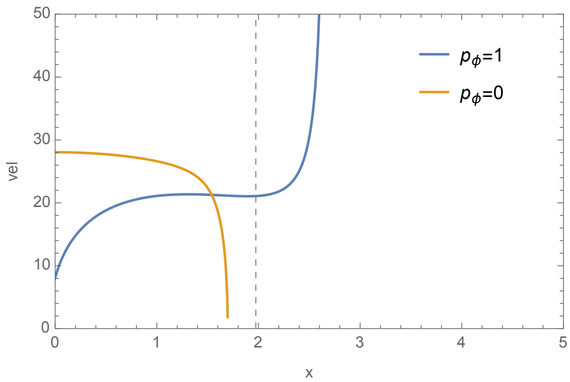


Figure 3.4: Velocity profile graph for radial time-like particles with $p_\phi = 0$, and $p_\phi = 1$. Particles with AM can only cross the CTC boundary denoted by the dashed lines.

Another effective way to study the dynamics of the motion of the radial time-like particles in the constant Z-plane, is considering the effective potential. The motion of the test particles here is considered as the motion of a classical particle in one dimension in the effective potential $V(x)$. The effective potential for the geodesics (considering no motion in the Z- coordinate) is given by [7, 281]

$$V(x) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.17}$$

The above expression can be easily derived from the radial time-like geodesic equation. The values of a, b and c in this case is given by:

$$\begin{aligned} a &= -\left(-1+\frac{E^2}{e^{2\gamma}e^{2\beta}}\right), \\ b &= -\left(\frac{2p_\phi E}{e^{2\beta}}\right), \\ c &= -\left(\frac{p_\phi^2 e^{2\gamma}}{e^{2\beta}}+\frac{p_z^2 e^{2\gamma}}{e^{2\mu}}+e^{2\gamma}\right). \end{aligned} \tag{3.18}$$

Using equations(1.129), (1.130), (1.131), (1.132), (1.133) for the values of a, b and c one may obtain the effective potential for the radial time-like particles from equation (3.17).

Effective Potential is plotted against the radial distance to investigate the dynamics of the time-like particles.

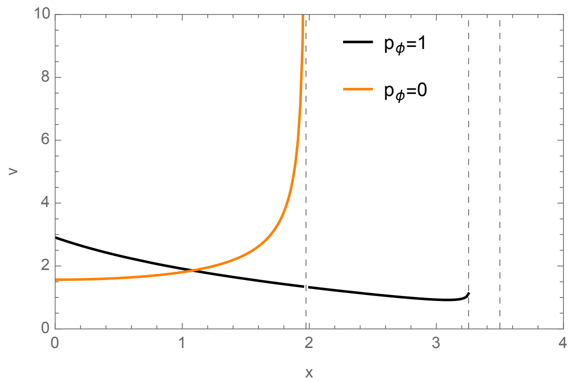


Figure 3.5: Effective potential v/s radial distance graph for particles with $p_\phi = 0$ and $p_\phi = 1$. The vertical dashed line (at $x = 1.974$)represents the boundary of the CTC.

Figure (3.3) depict the spacetime diagrams for particles exhibiting Zero Angular Momentum (AM) and particles with Non-zero AM. Notably, these particles are confined within significantly smaller values of 'x' compared to Photon confinement, aligning with expectations. Specifically, Zero AM particles are confined within the region $x = 1.697$, whereas particles with Non-zero AM are confined within $x = 2.940$. This behavior can be attributed to the Total Effective AM, enabling particles to breach the Closed Timelike Curve (CTC) boundary and traverse within that region.

Additionally, this characteristic can be elucidated through the effective potential of the particles. Zero AM particles exhibit an attractive potential, whereas particles with AM demonstrate a repulsive potential, as evident from Figure (3.5). The attractive potential of Zero AM particles acts as a barrier near their confinement, impeding their motion and causing them to oscillate within the specified region. This behavior is further illustrated in the velocity profile (Figure 3.4), where velocity decreases with increasing radial distance but does not reach zero. Moreover, this potential prevents particles from reaching the singularity region, as it asymptotically approaches the confinement region. The velocity profile of these particles supports these findings.

The repulsive nature of the potential for particles with AM is evident in Figure (3.5). This potential facilitates particles in crossing the boundary and traversing within the CTC region. Initially, the particle velocity increases with radial distance, aiding in crossing over to the causality-violating region, but it ceases to increase after a certain point. This suggests that despite the repulsive potential, particles are restricted from any motion near the singularity.

Furthermore, Closed Timelike Curves typically manifest when coordinates other than the ϕ coordinate (such as t, r, z) are held constant [109, 233].This allows the particle to only have equatorial motion. Now using this equation our metric reduces to:

$$ds^2 = \left[-\frac{E^2}{e^{2\gamma(x)}} + e^{2\beta(x)}\right] d\phi^2. \tag{3.19}$$

Using the Euler-Lagrange equation one may obtain the canonical momentum of the ϕ coordinate, which is:

$$\dot{\phi} = \frac{p_\phi}{\left[-\frac{E^2}{e^{2\gamma(x)}} + e^{2\beta(x)}\right]}. \tag{3.20}$$

From equation (3.20) it is clear that in CTC particle with zero angular momentum cannot exist as, if p_ϕ becomes zero, $\dot{\phi}$ becomes zero which in turn makes 'ϕ' a constant. This violates the basic condition for CTC to form as all other coordinates except ϕ needs to be constant. Hence zero AM particles are forbidden in the causality violating region.

3.4 Discussion

The findings in this chapter deepen our understanding of particle dynamics in confined spaces and near causality-challenging regions, shedding light on traversing Cylindrically Symmetric Wormholes with Closed Timelike Curves (CTCs).

One finds that particles lacking AM are confined outside the CTC boundary, while those with AM can breach it and traverse within the closed orbit. This is attributed to the Total Effective AM, enhancing particle acceleration and enabling traversal of the causality-violating barrier. Geodesic equation solutions yield spacetime diagrams, showing that only particles with non-zero AM can traverse within the CTC.

Additionally, the effective potential of particles is analysed, revealing that AM-less particles face an inward pull, hindering their passage beyond the CTC confinement. Conversely, non-zero AM particles experience a repulsion, facilitating their traversal beyond the CTC confinement.

In conclusion, the study emphasizes that spacetime admitting CTCs cannot accommodate particles lacking AM within the curves. The presence of zero-angular momentum particles violates conditions necessary for CTC formation, thus precluding their existence within the region.

CHAPTER 4

ON THE EVOLUTION OF DYNAMICAL WORMHOLE FROM EMERGENT SCENARIO

4.1 Prelude

Introducing a time-dependent metric allows for the derivation of non-static configurations, yielding dynamic wormhole structures. Visser [136] developed a class of traversable wormholes, encompassing both static and dynamic types, employing the cut-and-paste technique with suitable junction convergence criteria. Hochberg [282] explored dynamic wormholes within higher-order R^2 gravity. Addressing the horizon problem, other authors [283] employed modified dynamic wormhole solutions during the early inflationary universe. Various authors have explored similar lines of solutions [125, 284]. Kar [204, 205] introduced the concept of non-static Lorentzian wormholes, demonstrating the feasibility of obtaining non-static wormhole geometries with throat matter satisfying energy conditions. They showed that dynamic wormholes in FRW spacetime can satisfy all energy conditions. Subsequently, Hochberg, Visser [285], and Hayward [286] demonstrated that the throat can act as an anti-trapped surface, typically violating the NEC. Dynamical wormholes with phantom matter and as well as two fluids system as the matter content has been studied by authors [287–290]. The study of evolving wormholes has extended to higher-order gravity theories and modified gravity theories by several authors [174, 291–304].

In this chapter, a dynamic wormhole solution featuring a two-fluid system at the throat, with one fluid being isotropic and homogeneous, and the other exhibiting inhomogeneous and anisotropic characteristics is examined. Here two distinct forms of Equation of State (EoS) is considered and two solutions for the wormhole geometry is explored. The analysis encompasses the examination of properties ensuring the existence and traversability of the wormhole. Additionally, the possibility of the dynamic wormhole in the Emergent Universe (EU) model within a cosmological framework is assessed. Finally, for the dynamical wormholes so obtained, Null Energy Condition(NEC) has been examined near the throat.

4.2 Inhomogeneous FLRW model: An Overview

The line element for inhomogeneous FLRW model is given by:

$$ds^2 = -e^{2\phi(r,t)}dt^2 + a^2(t)\left[\frac{dr^2}{1-\frac{b(r)}{r}-Kr^2} + r^2 d{\Omega_2}^2\right], \tag{4.1}$$

where, $\phi(r,t)$ is the redshift function, $a(t)$ is the scale factor of the spacetime geometry, $d{\Omega_2}^2 = d\theta^2 + sin^2\theta d\phi^2$, and K is a constant.

For the distribution of matter, the analysis involves the consideration of two non-interacting fluids, specifically labeled as Fluid 1 and Fluid 2. Fluid 1 is characterized by its dissipative nature and homogeneity, while Fluid 2 exhibits anisotropic properties along with inhomogeneity.

The energy momentum tensor for the Fluid 1 has the expression:

$$T^{I}_{\mu\nu} = (\rho_1 + p_1 + \Gamma)u_\mu u_\nu + (p_1 + \Gamma)g_{\mu\nu}, \tag{4.2}$$

where, $\rho_1 = \rho_1(t)$ is the energy density, $p_1 = p_1(t)$ is the isotropic pressure, Γ is the pressure due to dissipation and u_μ is the four velocity of the fluid. Further, for the Fluid 2, the energy momentum tensor is given by:

$$T^{II}_{\mu\nu} = (\rho_2 + p_t)v_\mu v_\nu + p_t g_{\mu\nu} + (p_r + p_t)\xi_\mu\xi_\nu. \tag{4.3}$$

Here, $\rho_2 = \rho_2(r,t)$ is the energy density of the inhomogeneous fluid, $p_r = p_r(r,t)$ is the radial pressure, $p_t = p_t(r,t)$ is the transverse pressure of the anisotropic Fluid 2. v_μ and ξ_μ are unit timelike and spacelike vector respectively, i.e. $v_\mu v^\nu = \xi_\mu\xi^\nu = -1$ and $\xi^\mu v_\mu = 0$.

The explicit form of Einstein's field equations

$$G_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{4.4}$$

for the above spacetime model are given by [305]

$$3e^{-2\phi(r,t)}H^2 + \frac{b'}{a^2r^2} + \frac{3K}{a^2} = \kappa\rho_1 + \kappa\rho_2 - \Lambda, \tag{4.5}$$

$$-e^{-2\phi(r,t)}\left(\frac{2\ddot{a}}{a} + H^2\right) + \frac{K}{a^2} - \frac{b}{a^2r^3} + 2e^{-2\phi(r,t)}H\frac{\partial\phi}{\partial t} + \frac{2}{a^2r^2}(r-b)\frac{\partial\phi}{\partial r} = \kappa(p_1+\Gamma) + \kappa p_r - \Lambda, \tag{4.6}$$

$$e^{-\phi(r,t)}\left(\frac{2\ddot{a}}{a} + H^2\right) + \frac{K}{a^2} + \frac{b-rb'}{2a^2r^3} + 2e^{-2\phi(r,t)}H\frac{\partial\phi}{\partial t} + \frac{2r-b-rb'}{2a^2r^2}\frac{\partial\phi}{\partial r} + \frac{r-b'}{a^2r}\left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{\partial^2\phi}{\partial^2 r}\right] = \kappa(p_1+\Gamma) + \kappa p_t - \Lambda, \tag{4.7}$$

$$2\dot{a}e^{-\phi(r,t)}\left(\sqrt{\frac{r-b(r)}{r}}\right)\frac{\partial\phi(r,t)}{\partial r}=0, \tag{4.8}$$

where $\kappa = 8\pi\text{G}$, $H = \dfrac{\dot{a}}{a}$ is the Hubble parameter, an 'overdot' denotes the differentiation w.r.t time t and 'prime' denotes the differentiation w.r.t. to radial coordinate r.

From the field equation (4.8),there are two possibilities, namely: $\dot{a} = 0$ (static case) and $\dfrac{\partial\phi}{\partial r} = 0$(non static case). In this chapter one deals with a non-static spacetime geometry therefore one can opt for the 2nd possibility with $\phi = 0$ without the loss of generality (re-scaling of the time coordinate).

Thus the metric simplifies to

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\frac{b(r)}{r}-Kr^2}+r^2 d{\Omega_2}^2\right]. \tag{4.9}$$

Also the above field equations simplify to,

$$3H^2+\frac{b'}{a^2r^2}+\frac{3K}{a^2}=\kappa\rho_1-\Lambda, \tag{4.10}$$

$$-\left(\frac{2\ddot{a}}{a}+H^2\right)+\frac{K}{a^2}-\frac{b}{a^2r^3}=\kappa(p_1+\Gamma)+\kappa p_r-\Lambda, \tag{4.11}$$

$$-\left(\frac{2\ddot{a}}{a}+H^2\right)+\frac{K}{a^2}+\frac{b-rb'}{2a^2r^3}=\kappa(p_1+\Gamma)+\kappa p_t-\Lambda. \tag{4.12}$$

Since the fluids are non interacting in nature so the conservation equations take the form:

$$\frac{\partial\rho_1}{\partial t}+3H(\rho_1+p_1+\Gamma)=0, \tag{4.13}$$

$$\frac{\partial\rho_2}{\partial t}+H(3\rho_2+p_r+2p_t)=0, \tag{4.14}$$

$$\frac{\partial p_r}{\partial r}=\frac{2}{r}(p_t-p_r). \tag{4.15}$$

The conservation equations for Fluid 2 and Fluid 1 are represented by equations (4.14) and (4.15), respectively. It is evident that the relativistic Euler equation is denoted by equation (4.15). It is crucial to highlight the anisotropic characteristic of Fluid 2 ($p_r \neq p_t$) as it plays a significant role. Without this anisotropy, the fluids would become non-interacting homogeneous fluids, leading to a state of physical uninterest.

4.3 Potential solutions for wormholes: An analysis from a mathematical perspective.

In order to solve the field equations given by equations (4.10)- (4.12), one assumes that the EoS for the anisotropic Fluid 2 is of barotropic in nature and in the present work the following two choices are considered, namely: $i)p_r = \alpha\rho_2$ and $ii)p_r = \alpha{\rho_2}^n$, where, α and $n(\neq 1)$ are arbitrary constants.

4.3.1 Case I: $p_r = \alpha\rho_2$

Further assuming that the two components of the anisotropic pressure are linearly related *i.e.*

$$p_t = \omega_t p_r. \tag{4.16}$$

Using this assumption the Euler equation(4.15), has the solution

$$p_r(r,t) = p_{r0}(t) r^{2(\omega_t - 1)}, \tag{4.17}$$

with p_{r0} as an arbitrary integration function.

Similarly, using equation(4.16) in equation(4.14), gives the energy density as:

$$\rho_2(r,t) = \rho_0(r) a^{(3+\alpha(1+2\omega_t))}, \tag{4.18}$$

with $\rho_0(r)$ an arbitrary function.

However due to barotropic EoS one may estimate the above arbitrary(integration) functions in equation(4.17) and (4.18)as,

$$p_{r0}(t) = \alpha a^{\mu} \qquad and \qquad \rho_0(r) = r^{2(\omega_t - 1)}, \tag{4.19}$$

with, $\mu = (3 + \alpha(1 + 2\omega_t))$.

Combining the above relations, one gets,

$$\rho_2(r,t) = r^{2(\omega_t-1)} a^{\mu} \qquad and \qquad p_r(r,t) = \alpha r^{2(\omega_t-1)} a^{\mu}. \tag{4.20}$$

At this point, after obtaining the values of the energy density and radial pressure for the Fluid 2, one can now try to obtain a viable solution for the shape function using the field equation.

Now to obtain the shape like function $b(r)$, one can take the difference between equation(4.11) and equation(4.12), and using equations(4.16),(4.17), and (4.19) one gets the differential equation in b as,

$$\frac{b - rb'}{2r^3} + \frac{b}{r^3} = \kappa(\omega_t - 1)\alpha r^{2(\omega_t-1)} a^{\mu+2}. \tag{4.21}$$

Now for consistency, the above equation demands,

$$\mu + 2 = 0, \tag{4.22}$$

and the differential equation for b becomes,

$$rb' - 3b = 2\kappa\alpha(1-\omega_t)r^{(2\omega_t+1)}, \tag{4.23}$$

which has a solution,

$$b(r) = b_0 r^3 - \kappa\alpha r^{(2\omega_t+1)}, \tag{4.24}$$

where b_0 is the integration constant.

4.3.2 Case II: $p_r = \alpha{\rho_2}^n$, $n \neq 1$

This choice of EoS together with the relation(4.16), one gets the differential equation for ρ_2 from the energy conservation equation(4.14) as,

$$\frac{\partial\rho_2}{\partial t} + H(3\rho_2 + \alpha{\rho_2}^n(1+2\omega_t)) = 0, \tag{4.25}$$

which has a solution of the form,

$$\rho_2^{(n-1)} = 3\rho_{20}(r)/[a^{9(n-1)} - \alpha\rho_{20}(r)(1+2\omega_t)]. \tag{4.26}$$

with $\rho_{20}(r)$ an arbitrary integration function.

Now using equations (4.17) and (4.26) in EoS(*i.e.* $p_r = \alpha{\rho_2}^n$), the arbitrary functions $\rho_{20}(r)$ and $p_{r0}(t)$ can be determined as,

$$p_{r0}(t)r^{2(\omega_t-1)} = \alpha[3\rho_{20}(r)]^{(\frac{n}{n-1})}/[a^{9(n-1)} - \alpha_0]^{(\frac{n}{n-1})}. \tag{4.27}$$

From the equation above one may write,

$$p_{r0}(t) = \alpha/[a^{9(n-1)} - \alpha_0]^{(\frac{n}{n-1})}, \tag{4.28}$$

and

$$\rho_{20(r)} = \left[\frac{r^{2(\omega_t-1)}}{3}\right]^{(\frac{n-1}{n})}, \tag{4.29}$$

with, $\alpha_0 = \alpha\rho_0(1+2\omega_t)$. Thus,

$$\rho_2(r,t) = 3^{[\frac{1}{n(n-1)}]} r^{\frac{2(\omega_t-1)}{n}}/[a^{9(n-1)} - \alpha_0]^{(\frac{1}{n-1})}, \tag{4.30}$$

and,

$$p_r(r,t) = 3^{[\frac{1}{(n-1)}]}\alpha r^{2(\omega_t-1)}/[a^{9(n-1)} - \alpha_0]^{(\frac{n}{n-1})}. \tag{4.31}$$

Then, similar to the previous case subtracting equation(4.11) from equation(4.12), one gets:

$$b - rb' + 2b = [\kappa 2a^2r^3(\omega_t-1)\alpha r^{2(\omega_t-1)}]/[a^{9(n-1)} - \alpha_0]^{(\frac{n}{n-1})}, \tag{4.32}$$

where relation(4.16) and the expression for p_r from equations(4.17) and (4.28) has been used. Now for consistency of the above differential equation in $'b'$ one should restrict the arbitrary constants as: $\alpha_0 = 0$ *i.e.* $\omega_t = 1/2$ and $n = 2/9$. As a result differential equation for 'b' simplifies to,

$$rb' - 3b = 3\kappa\alpha, \tag{4.33}$$

having solution,

$$b(r) = b_1 r^3 - \kappa\alpha, \tag{4.34}$$

where, b_1 is the constant of integration.

It is important to mention that these two solutions of the shape function obtained must satisfy the flaring out condition for the wormhole to be traversable. Hence in the following section the condition required for these shape functions to satisfy the flaring out condition is analysed.

4.4 WH Geometry: The Flaring Out Condition

In this section, the constraints established by Morris-Thorne regarding the traversability of a Wormhole (WH) will be examined for the validity and restrictions on the shape functions obtained in the previous section. These constraints include:

- The throat radius is a global minimum r $= r_0$, so the radial coordinate, r is in the interval [r_0,∞) and the wormhole is constructed by connecting two asymptotic flat regions at the throat.
- The redshift function $\phi(r)$ must be finite everywhere in order to avoid the presence of horizons and singularities. So, $e^{\phi(r)} > 0$ everywhere for $r > r_0$.
- $b(r_0) = r_0$ and $b'(r) < 1$ at $r = r_0$. Also for $r > r_0, b(r) < r$.
- For traversability of the wormhole the flaring out condition is: $b(r) - rb'(r) > 0$
- Asymptotic flatness:It implies that $\phi(r) \to 0$ and $b(r)/r \to 0$ as $r \to \infty$.

Hence,the above solutions for the shape function must satisfy the above conditions for the wormhole to be viable and traversable in nature.

4.4.1 Case I

The shape function in equation(4.24), for this barotropic EoS is restricted by the value of b_0, form $b(r_0) = r_0$, (r_0 is the throat radius) as,

$$b_0 = \frac{[r_0 + \kappa\alpha r_0^{(2\omega_t+1)}]}{r_0^3}, \tag{4.35}$$

i.e. the form of the shape function becomes,

$$b(r) = \frac{[r_0 + \kappa\alpha r_0^{(2\omega_t+1)}]}{r_0^3} r^3 - \kappa\alpha r^{(2\omega_t+1)}. \tag{4.36}$$

The flaring out condition (in the 4^{th} bullet) restricts the throat radius as,

$$r_0 \geq \left[\frac{1}{\alpha\kappa(2+\omega_t)}\right]^{\frac{1}{2\omega_t}}, \tag{4.37}$$

for $r \geq r_0$. Hence, the functional form of $b(r)$ stated in equation (4.36) represents the geometric profile of a traversable wormhole, subject to the condition that r_0 complies with the constraint (4.37). It is important to emphasize that the presence of an exotic anisotropic fluid is not a prerequisite for the existence of a traversable wormhole in this context. Additionally, it is crucial to acknowledge that ω_t must not equal 1 throughout the entire analysis in order to uphold the anisotropic characteristics of the secondary fluid. Further, the wormhole is infinitely extended over $[r_0, \infty)$.

4.4.2 Case II

For this solution the throat condition $b(r_0) = r_0$ gives the integration constant b_1 as,

$$b_1 = \frac{[r_0 + \kappa\alpha]}{r_0^3}. \tag{4.38}$$

Hence, the shape function takes the form,

$$b(r) = \frac{[r_0 + \kappa\alpha]}{r_0^3} r^3 - \kappa\alpha. \tag{4.39}$$

Now due to the flaring out condition the throat radius r_0 is restricted as,

$$r_0 > \frac{\kappa|\alpha|}{2}, \tag{4.40}$$

provided α is a negative real number.

When this restriction on r_0 is obeyed, a proper, viable shape function can be constructed for the given form of EoS. Thus for the second choice of the equation of state it is possible to have traversable wormhole configuration provided the anisotropic fluid must be exotic in nature ($\omega_t < -1/2$) and as in the first choice this wormhole is infinitely extended.

4.5 The Emergent Universe Scenario

The universe is said to be emergent if it satisfies certain conditions so as to avoid the big bang singularity. The conditions are [254–260]:

- $a \to a_0$, $H \to 0$ as $t \to -\infty$
- $a \simeq a_0$, $H \simeq 0$, $t << t_0$

Inhomogeneous spacetime also exhibits emergent universe scenario which was investigated by Bhattacharya and Chakraborty [255]. Hence in the present work it is examined whether EU scenario is possible for the above dynamic wormhole configurations

From the line element (4.9) one has,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2 - \frac{b(r)}{r}} + r^2 d{\Omega_2}^2\right], \tag{4.41}$$

Using the value of $b(r)$ as obtained in equation(4.24) one can get,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2 - b_0 r^2 + \kappa\alpha r^{2\omega_t}} + r^2 d{\Omega_2}^2\right]. \tag{4.42}$$

Here one can assume, $K + b_0 = \kappa_0$ and, by proper scaling of the radial coordinate one may have, $\kappa_0 = 0, \pm 1$.

Thus without loss of generality,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - \kappa_0 r^2 + \kappa\alpha r^{2\omega_t}} + r^2 d{\Omega_2}^2\right]. \tag{4.43}$$

One important point to note here is that for the present wormhole solution one has to choose, $\kappa_0 = 0$, $\alpha < 0$, $\omega_t \neq 0, 1$. Thus $\alpha < 0$ implies, the anisotropic fluid for the first WH solution is also exotic in nature. Now, using this value of $b(r)$, ρ_2, and p_r, p_t from the first case, in the field equations and from the Friedmann equations, one gets,

$$3(H^2 + \frac{\kappa_0}{a^2}) = \kappa\rho_1 - \Lambda, \tag{4.44}$$

and,

$$-(2\dot{H} + 3H^2 + \frac{\kappa_0}{a^2}) = \kappa(p_1 + \Gamma) - \Lambda. \tag{4.45}$$

With equation(4.13) as the conservation equation, choosing, $p_1 = \omega\rho_1$ and $\Gamma = -(\delta/\rho_1^n)$, then the conservation equation(4.13) becomes,

$$\frac{\partial\rho_1}{\partial t} + 3H(\rho_1 + \omega\rho_1 - \delta/\rho_1^n) = 0. \tag{4.46}$$

Here the constant δ may be termed as dissipation coefficient.

The above first order non linear differential equation(4.46) has a solution:

$$\rho_1 = \left(\frac{\delta}{(1+\omega)} + \frac{Z_0 a^{-\rho}}{(1+\omega)}\right)^{1/(1+n)}, \ for \ \ \omega \neq -1 \tag{4.47}$$

and,

$$\rho_1 = \left(3\delta(1+n)\ln(\frac{a}{a_0})\right)^{(1/1n+1)}, \ for \ \ \omega = -1, \tag{4.48}$$

where, $\rho = 3(1+\omega)(1+n)$, Z_0 and a_0 are the integration constants.

In equation (4.44) if, $\kappa_0 = 0$ and $\Lambda = 0$, then the first Friedmann equation is,

$$3H^2 = \kappa\rho_1. \tag{4.49}$$

Now using the above solution for ρ_1 in equation(4.49) gives the scale factor as,

$$\sqrt{3}/2(1+\omega)Z_0^{\beta}(t-t_0) = a^{\frac{\sqrt{3}}{2}(1+\omega)}\,{}_2F_1[\beta,\beta,\beta+1,-\frac{\delta}{Z_0(1+\omega)}a^{(\frac{3(1+\omega)}{2\beta})}], \quad for\ \omega \neq -1, \tag{4.50}$$

and,

$$a = a_0 \exp\left[\Sigma_0(t-t_0)^{[(n+1)/n]}\right], \quad for\ \omega = -1, \tag{4.51}$$

where, $\Sigma_0 = \left(\frac{\sqrt{\kappa}}{\sqrt{3}}(\frac{n}{n+1})\right)^{[\frac{n+1}{n}]}[3\delta(n+1)]^{1/n}$, and ${}_2F_1$ is the usual hypergeometric function.

In the above solution(4.51) for the scale factor if $-1 < n < -1/2$, *i.e.* $\beta > 1$ then $a \to a_0$ as $t \to -\infty$. Thus from the above conditions of Emergent Scenario and the graph of $a(t)$ in fig(4.1), shows that the solution (4.51) of the scale factor provides a possible Emergent Universe scenario.

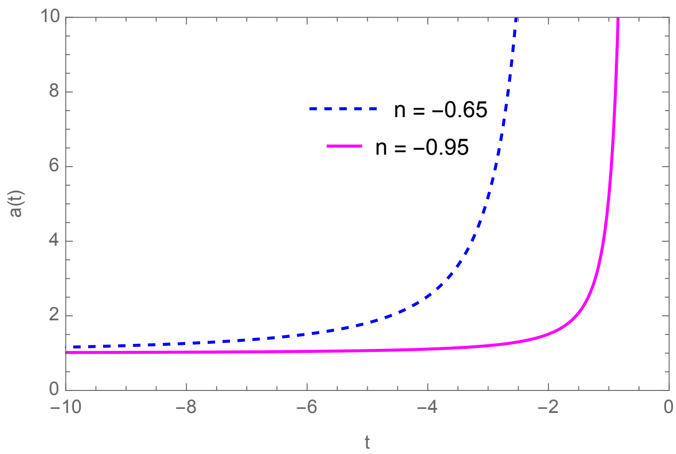


Figure 4.1: Graphical representation of '$a(t)$' v/s 't' for $\omega = -1$.

4.6 Energy Conditions

The establishment of traversable wormholes exhibits a notably captivating and somewhat unique property, which involves the requirement of exotic material that goes against classical energy principles. This exotic substance plays a crucial role in upholding the passage through the wormhole and thus ensuring its navigability. As described in the initial section, thorough investigation has been carried out regarding this aspect within the realm of developing traversable wormholes and Morris-Thorne type wormholes, examined through different iterations of adjusted hypotheses. Intriguingly, certain hypotheses endorse varieties of material that adhere to the energy principles in various manners. Of particular note is the fact that even if conventional matter meets the criteria, the inclusion of associated matter becomes necessary to serve as exotic energy, meeting the tarversability criteria. The energy criteria, specifically the Null Energy Condition (NEC), Weak Energy Condition (WEC), Dominant Energy Condition (DEC), and Strong Energy Condition (SEC), are fulfilled in a dual-fluid system when the subsequent criteria are satisfied:

- (i)NEC: $\rho_T + (p_r)_T \geq 0,\ \rho_T + (p_t)_T \geq 0;$
- (ii)WEC: $\rho_T \geq 0, \rho_T + (p_r)_T \geq 0,\ \rho_T + (p_t)_T \geq 0;$
- (iii)SEC: $\rho_T + (p_r)_T \geq 0,\ \rho_T + (p_t)_T \geq 0;\ \rho_T + (p_r)_T + 2(p_t)_T \geq 0;$
- (iv)DEC: $\rho_T \geq 0, \rho_T - |(p_r)_T| \geq 0,\ \rho_T - |(p_t)_T| \geq 0.$

where, $\rho_T = \rho_1 + \rho_2$ is the total energy density, $(p_r)_T = p_1 + \Gamma + p_r$ is the total radial pressure and $(p_t)_T = p_1 + \Gamma + p_t$ is the total transverse pressure for the two fluids combined.

For the first case the expressions for combined energy conditions for both fluids are:

$$\rho_T + (p_r)_T = (1+\omega)\rho_1 - \delta\rho_1^{-n} + (1+\alpha)\rho_2, \tag{4.52}$$

$$\rho_T + (p_t)_T = (1+\omega)\rho_1 - \delta\rho_1^{-n} + (1+\omega_t\alpha)\rho_2, \tag{4.53}$$

$$\rho_T + (p_r)_T + 2 + (p_t)_T = (1+3\omega)\rho_1 - 3\delta\rho_1^{-n} + [1+\alpha(1+2\omega_t)]\rho_2, \tag{4.54}$$

$$\rho_T - (p_r)_T = (1-\omega)\rho_1 + \delta\rho_1^{-n} + (1-\alpha)\rho_2, \tag{4.55}$$

$$\rho_T - (p_t)_T = (1-\omega)\rho_1 + \delta\rho_1^{-n} + (1-\omega_t\alpha)\rho_2, \tag{4.56}$$

similarly for the second case,

$$\rho_T + (p_r)_T = (1+\omega)\rho_1 - \delta\rho_1^{-n} + (1+\alpha\rho_2^{(n-1)})\rho_2, \tag{4.57}$$

$$\rho_T + (p_t)_T = (1+\omega)\rho_1 - \delta\rho_1^{-n} + (1+\omega_t\alpha\rho_2^{(n-1)})\rho_2, \tag{4.58}$$

$$\rho_T + (p_r)_T + 2 + (p_t)_T = (1+3\omega)\rho_1 - 3\delta\rho_1^{-n} + [1+\alpha\rho_2^{(n-1)}(1+2\omega_t)]\rho_2, \tag{4.59}$$

$$\rho_T - (p_r)_T = (1-\omega)\rho_1 + \delta\rho_1^{-n} + (1-\alpha\rho_2^{(n-1)})\rho_2, \tag{4.60}$$

$$\rho_T - (p_t)_T = (1-\omega)\rho_1 + \delta\rho_1^{-n} + (1-\omega_t\alpha\rho_2^{(n-1)})\rho_2. \tag{4.61}$$

From the above equations one can can easily analyse the validation or violation of the energy conditions.

As one deals with dynamic wormhole in EU scenario so we consider $\omega = -1$. Thus the above energy conditions can be written in a compact way in the following table.

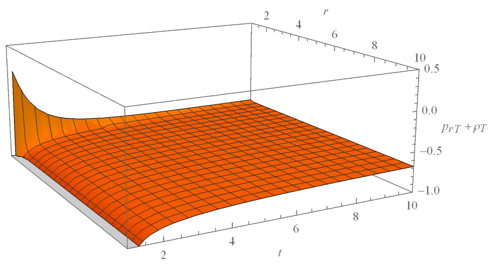
Further, to have a clear picture both the conditions for NEC has been plotted graphically in figures (4.2) and (4.3). For the case I wormhole, $\rho_T + (p_r)_T$ is positive very close to the throat while $\rho_T + (p_t)_T$ is negative, and this is true for possitive and negative values of α. On the other hand for Case II wormhole both the NECs are satisfied for $\alpha < 2$ but $\rho_T + (p_t)_T$ is violated for $\alpha \geq 2$.

Energy Conditions	Observations
NEC	Satisfied if, $\alpha \geq \frac{\delta\rho_1^{-n}}{\rho_2} - 1 = \alpha_1$ and $\omega_t \geq \omega_{t1} = \frac{\alpha_1}{\alpha}$
DEC	Satisfied if, NEC holds, also $\alpha \leq 1+\frac{\rho_1}{\rho_2}[2+\delta\rho_1^{(-n-1)}] = \alpha_2$ and $\omega_t \leq \omega_{t2} = \frac{\alpha_2}{\alpha}$
SEC	Satisfied if, NEC holds along with $\alpha \geq [\frac{\rho_1}{\rho_2}(3\delta\rho_1^{-(n+1)} + 2) - 1]/(1+2\omega_t) = \alpha_3$
WEC	Satisfied if, condition of NEC holds along with $\rho_T \geq 0$

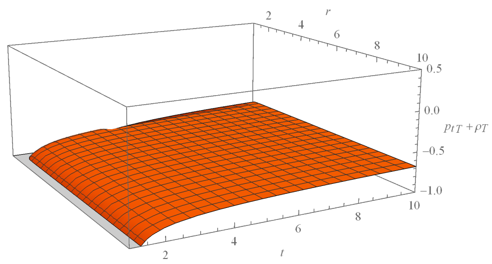
Table 4.1: Range of α, ω_t considering $\omega = -1$ for the Energy conditions to be satisfied for case I.

Energy Conditions	Observations
NEC	Satisfied if, $\alpha \geq \frac{\alpha_1}{\rho_2^{(n-1)}} = \alpha_4$ and $\omega_t \geq \omega_{t3} = \frac{\alpha_4}{\alpha}$
DEC	Satisfied if, NEC is obeyed, also $\alpha \leq \frac{\alpha_2}{\rho_2^{(n-1)}} = \alpha_5$ and $\omega_t \leq \omega_{t4} = \frac{\alpha_5}{\alpha}$
SEC	Satisfied if, NEC holds along with $\alpha \geq \frac{\alpha_3}{\rho_2^{(n-1)}} = \alpha_6$
WEC	Satisfied if, condition of NEC holds along with $\rho_T \geq 0$

Table 4.2: Range of α, ω_t considering $\omega = -1$ for the Energy conditions to be satisfied for case II.



(a)



(b)

Figure 4.2: Plots demonstrating the variation of NEC($\rho_T + (p_r)_T$, $\rho_T + (p_t)_T$) with radial distance r and scale factor a for Case I. The nature of the plots are obtained considering $\alpha = 0.7$, $a_0 = 1$, $n = 2/9$, $\delta = 1$, $\omega = -1$.

4.7 Discussion

The results obtained in the chapter may be concluded as:

• Wormhole solutions are obtained with a linear relationship between radial and tangential pressures, accounting for the anisotropic nature of Fluid 2.

• Throat conditions and traversability are examined for both wormhole solutions, leading to the estimation of arbitrary integration constants.

• The scale factor is determined by solving the first Friedmann equation, resulting

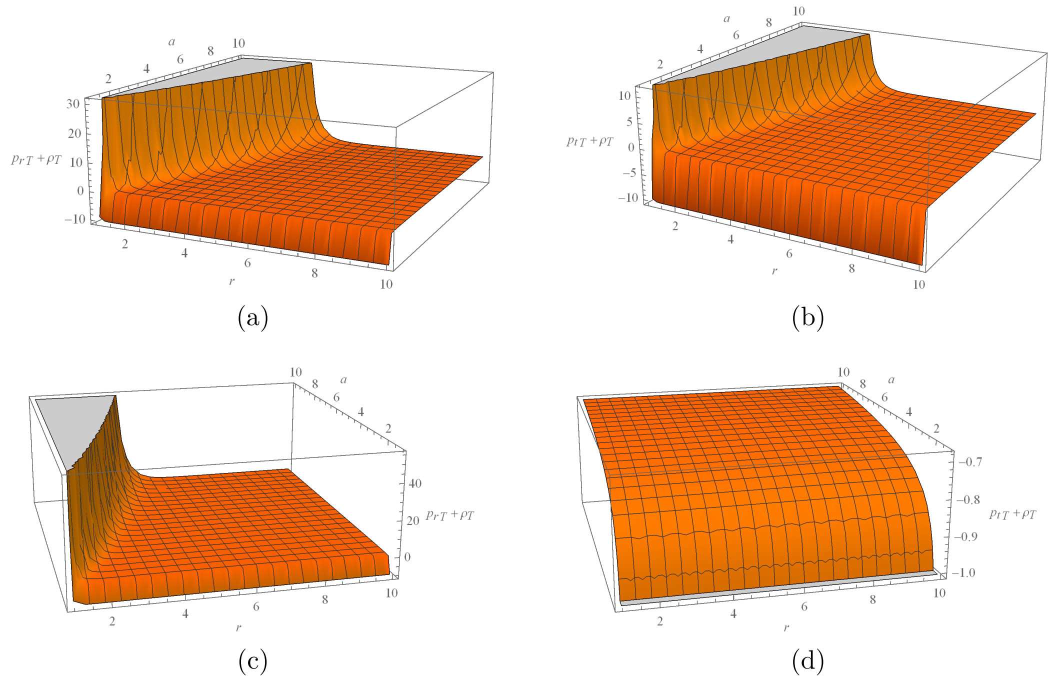


Figure 4.3: Plots demonstrating the variation of NEC($\rho_T + (p_r)_T,\ \rho_T + (p_t)_T$) with radial distance r and scale factor a for Case II. The nature of the plots are obtained considering $\alpha = 0.7,\ a_0 = 1,\ n = 2/9,\ \delta = 1\ \omega_t = -0.5,\ \omega = -1$ for plots (a) and (b). For plots (c) and (d), $\alpha = 2,\ a_0 = 1,\ n = 2/9,\ \delta = 1,\ \omega_t = -0.5,\ \omega = -1$

in two distinct solutions for $\omega \neq -1$ and $\omega = -1$, where ω is the barotropic constant for Fluid 1.

• Asymptotic analysis reveals that the scale factor behaves as an emergent scenario at infinite past for $\omega = -1$, subject to specific parameter restrictions.

• Energy conditions can be met for both types of wormhole setups by setting suitable limits on the parameters α and ω_t. Graphical analysis of NEC has been conducted and findings have been presented. Consequently, this study demonstrates the feasibility of dynamic wormholes with an emergent phase during the early cosmic era. Moreover, in the current wormhole arrangement, it is feasible to maintain non-violation of NEC in proximity to the throat, distinguishing it from static wormhole solutions.

• From a cosmological perspective, this dynamical wormhole configuration related to an emergent scenario may represent the early evolution of the universe, potentially circumventing the big-bang singularity.

CHAPTER 5

EVOLVING WORMHOLES IN $f(R, T)$ GRAVITY: AN ANALYTICAL STUDY

5.1 Prelude

In the last few years, static wormhole configurations in various modified theories of gravity had been investigated by various authors [306–312]. To more accurately explain the expansion of the universe, modified theories of gravity, such as the $f(R, T)$ theory of gravity [90], were developed. These modified gravity theories allow for adjustments to the effective stress-energy tensor, providing an alternative approach to address the exotic matter problem while adhering to the energy conditions. Among these theories, $f(R, T)$ incorporates terms involving the Ricci scalar and the trace of the energy-momentum tensor in its gravitational action. The inclusion of terms proportional to 'T' is motivated by the potential presence of imperfect fluids in the universe. Therefore, considering these terms is particularly relevant when describing the matter content of wormholes, which are characterized by an imperfect anisotropic fluid. The $f(R, T)$ gravity theory has already been demonstrated as a promising alternative. Various researchers have employed $f(R, T)$ gravity to derive numerous wormhole solutions [313–317]. Sahoo *et al.* [318] investigated wormholes in R^2-gravity within the $f(R, T)$ framework, concluding that the quadratic geometric and linear material corrections in this theory enable the wormhole's matter content to comply with energy conditions. T. Azizi [319] examined wormhole geometries in $f(R, T)$ gravity by considering a specific equation of state (EoS) for the matter field, showing that the effective stress-energy tensor causes the violation of the NEC. Zubair *et al.* [320] explored static spherically symmetric wormholes in $f(R, T)$ gravity and demonstrated that wormhole solutions could be constructed without exotic matter in certain space-time regions. Moraes *et al.* [321] investigated charged wormhole solutions in $f(R, T)$ modified gravity. Sahoo *et al.* [92] studied phantom fluid wormholes in $f(R, T)$ gravity, using an EoS associated with phantom dark energy. Elizalde *et al.* [93] presented wormhole solutions using specific wormhole models in $f(R, T)$ gravity and examined all the energy conditions. Additionally, spherical and hyperbolic wormholes in $f(R,T)$ gravity have been studied by Zubair *et al.* [291], while Lu *et al.* [165] investigated the physical properties of traversable wormholes in this gravity theory.

In this chapter, we explore potential evolving wormhole solutions with anisotropic fluid distribution within the framework of $f(R,T)$ gravity theory, considering an inhomogeneous FLRW background geometry. The anisotropic fluid components are evaluated from the field equations for possible wormhole solutions considering viable shape functions with the scale factor being either in power law form or in exponential form. The energy conditions are examined and their variations are presented graphically. The graphical analysis show that all the energy conditions are satisfied (near the throat) for all three the choices of the shape functions with the power law form as well as the exponential form scale factor. Finally, the possibility of emergent scenario at early cosmic evolution has been mentioned.

These findings shed light on the interplay between evolving wormhole solutions and the energy conditions across different cosmological scenarios.

The Chapter is organized as follows: Begining with a brief review of the action and field equation followed by the geometry of the wormhole in the mentioned framework. Further, the energy conditions considering few well known shape functions is investigated with the scale factor represented using the power law and exponential form. The evolution of the wormhole geometry in the Emergent Universe(EU) scenario in the framework of $f(R,T)$ gravity theory is also examined.

5.2 Evolving Wormholes in $f(R,T)$ Gravity

5.2.1 Action and the Field Equations

In the $f(R,T)$ gravity model, the action is expressed as [313,314],

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(R,T) + \int d^4x\sqrt{-g}\mathcal{L}_m, \tag{5.1}$$

where,$f(R,T)$ represents an arbitrary function involving the Ricci scalar(R), and T denotes the trace term of the energy-momentum tensor.Here, g stands for the metric determinant, $\mathcal{L}_m$ represents the matter Lagrangian density.

Varying the action w.r.t the metric $g_{\mu\nu}$, the field equation for $f(R,T)$ theory of gravity is given by

$$\begin{aligned} &f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T) \\ &= 8\pi T_{\mu\nu} - f_T(R,T)(T_{\mu\nu} + \Theta_{\mu\nu}), \end{aligned} \tag{5.2}$$

where $f_R(R,T)$ and $f_T(R,T)$ are the differentiation of $f(R,T)$ with respect to R and T respectively, and $\Box f_R = g^{\mu\nu}\nabla_\mu\nabla_\nu f_R$. The variation of trace of energy-momentum tensor of the matter field, $T = g^{\mu\nu}T_{\mu\nu}$ is given by:

$$\frac{\delta(g^{\alpha\beta}T_{\alpha\beta})}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu}, \tag{5.3}$$

where $\Theta_{\mu\nu}$ and $T_{\mu\nu}$ are given by

$$\Theta_{\mu\nu} \equiv g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}, \tag{5.4}$$

$$T_{\mu\nu} \equiv g_{\mu\nu}\mathcal{L}_m - \frac{2\partial(\mathcal{L}_m)}{\partial g^{\mu\nu}}. \tag{5.5}$$

Here, $T_{\mu\nu}$ can also be written as:

$$T_{\mu\nu} = (\rho + p_t)u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t)\chi_\mu \chi_\nu, \tag{5.6}$$

where u_μ is the timelike unit vector, χ_μ is a spacelike unit vector orthogonal to timelike unit vector, such that $u_\mu u^\mu = -1$, $\chi_\mu \chi^\mu = 1$ and $u_\mu \chi^\mu = 0$. Assuming the universal choice of the matter Lagrangian density $\mathcal{L}_m = \rho$, we have $\Theta_{\mu\nu} = -2T_{\mu\nu} + \rho g_{\mu\nu}$.

Further, from Eq. (5.6), final field equation reeds:

$$G_{\mu\nu} = 8\pi G_{eff} T_{\mu\nu} + T^{eff}_{\mu\nu}, \tag{5.7}$$

where

$$G_{eff} = \frac{1}{f_R(R,T)}\left(1 + \frac{f_T(R,T)}{8\pi}\right), \tag{5.8}$$

$$T^{eff}_{\mu\nu} = \frac{1}{f_R(R,T)}\Big[\frac{1}{2}(f(R,T) - Rf_R(R,T) + 2\rho f_T(R,T))g_{\mu\nu} - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T)\Big]. \tag{5.9}$$

Note that, when $f_T(R,T) = 0$, one gets the usual $f(R)$. In this field equation, the energy-momentum tensor component ($T_{\mu\nu}$) of $f(R,T)$ gravity represents the interaction between matter and curvature, and one may interpret this as the curvature-matter coupling occurs due to the exchange of energy and momentum between the both. The combined energy-momentum tensor has the form:

$$T^{tot}_{\mu\nu} = T_{\mu\nu} + \frac{1}{\left(1 + \frac{f_T(R,T)}{8\pi}\right)}T^{eff}_{\mu\nu}. \tag{5.10}$$

However, the $f(R,T)$ gravity model introduced by Harko *et al.* [90] raises a significant concern regarding the violation of the energy conservation principle, resulting in deviations from geodesic trajectories for test particles. Conversely, an alternative formulation of $f(R,T)$ gravity proposed by Chakraborty [322] demonstrates that by considering the conservation of the electromagnetic tensor, the field equation retains its structure. Consequently, test particles follow geodesic paths, and the selection of

Lagrangian is not entirely arbitrary. When applied to a homogeneous and isotropic universe model, this approach yields a field equation that is analogous to Einstein's gravity but with a non-interacting 2-fluid system. One of these components corresponds to the conventional perfect fluid within modified gravity, while the other exhibits exotic characteristics.

5.2.2 The Metric

The metric ansatz of the dynamical wormhole space-time is given by [294]:

$$\mathrm{d}s^2 = -e^{2\phi(r,t)}\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\frac{b(r)}{r}} + r^2\mathrm{d}{\Omega_2}^2\right], \tag{5.11}$$

where, $\phi(r,t)$ is the redshift function, $a(t)$ is the scale factor of the wormhole universe, $b(r)$ is the usual shape function of the wormhole, $\mathrm{d}{\Omega_2}^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$.

The shape function present in the metric has to fulfil certain conditions and restrictions for it to correspond to a wormhole solution. The conditions imposed by Morris-Thorne are as follows:

- $b(r_0) = r_0$, where r_0 is the throat radius and is also a radial global minimum
- $b'(r_0) < 1$ at $r = r_0$ and
- the Flaring out condition: $b(r) - rb'(r) > 0$.

In order to maintain traversability, one must also consider: the redshift function $\phi(r)$ must be finite everywhere in order to avoid the presence of horizons and singularities. So, $e^{\phi(r)} > 0$ everywhere for $r > r_0$.

5.2.3 The Space-time Geometry

For the evolving wormhole geometry, combining equations (5.7),(5.9) and the metric given by equation (5.11), one can get,

$$3e^{-2\phi(r,t)}H^2 + \frac{b'}{a^2r^2} = \frac{1}{f_R}\Bigg[8\pi\rho - \frac{1}{2}fe^{2\phi(r,t)} + \frac{1}{2}Re^{2\phi(r,t)}f_R + \rho f_T(1-e^{2\phi(r,t)}) -$$
$$3H\dot{f}_R + e^{2\phi(r,t)}a^{-2}\left(1-\frac{b(r)}{r}\right)f''_R + e^{2\phi(r,t)}f'_R\frac{5b(r) - r\prime b(r) - 4r}{2a(t)^2r^2}\Bigg] \tag{5.12}$$

$$-e^{-2\phi(r,t)}\left(\frac{2\ddot{a}}{a} + H^2\right) - \frac{b}{a^2r^3} + 2e^{-2\phi(r,t)}H\frac{\partial\phi}{\partial t} + \frac{2}{a^2r^2}(r-b)\frac{\partial\phi}{\partial r} = \frac{1}{f_R}\Bigg[8\pi p_r + p_r f_T +$$
$$\frac{a^2(t)}{(1-\frac{b(r)}{r})}\left[(f/2 - 3\dot{H}f_R - 6H^2f_R - \frac{b'(r)}{a^2(t)r^2}f_R + \rho f_T) - e^{-2\phi(r,t)}(\dot{f}_R\frac{\partial\phi}{\partial t} - 2H\dot{f}_R - \ddot{f}_R)\right]$$
$$+f'_R\left(\frac{\partial\phi}{\partial r} + \frac{2}{r}\right)\Bigg] \tag{5.13}$$

$$e^{-\phi(r,t)}\left(\frac{2\ddot{a}}{a}+H^2\right)+\frac{b-rb'}{2a^2r^3}+2e^{-2\phi(r,t)}H\frac{\partial\phi}{\partial t}+\frac{2r-b-rb'}{2a^2r^2}\frac{\partial\phi}{\partial r}$$
$$+\frac{r-b'}{a^2r}\left[\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{\partial^2\phi}{\partial^2 r}\right]=\frac{1}{f_R}\Bigg[8\pi p_t+p_tf_T+a^2(t)\times$$
$$r^2\Bigg[\left(f/2-3\dot{H}f_R-6H^2f_R-\frac{b'(r)}{a^2(t)r^2}f_R+\rho f_T+\frac{1}{2}f'_R\frac{rb'(r)-b(r)}{r(r-b(r))}\right)$$
$$-e^{-2\phi(r,t)}\left(\dot{f}_R\frac{\partial\phi}{\partial t}-2H\dot{f}_R-\ddot{f}_R\right)\Bigg]-r^2\left(1-\frac{b(r)}{r}\right)\left[f''_R+f'_R\frac{\partial\phi}{\partial r}\right]-f'_R(b(r)-r)\Bigg] \tag{5.14}$$

$$2\dot{a}e^{-\phi(r,t)}\left(\sqrt{\frac{r-b(r)}{r}}\right)\frac{\partial\phi(r,t)}{\partial r}=0, \tag{5.15}$$

where, $H=\frac{\dot{a}}{a}$ is the Hubble parameter, an 'overdot' denotes the differentiation w.r.t time t and 'prime' denotes the differentiation w.r.t. to radial coordinate r,f_R and f_T are the differentiation of $f(R,T)$ with respect to R and T respectively. Overdot on f_R denotes differentiation w.r.t to t.

From the field equation(5.15), ϕ should be independent of r. Here, we have chosen ϕ to be time independent, the reason for this choice is as follows:

- As the field equations in $f(R,T)$ gravity are very complicated, so to have a little bit simpler field equations $\phi=0$ is assumed.
- $\phi=0$ ensures that there is no horizon which is an important necessary condition for existence of wormholes.
- As we are considering wormhole geometry characterized by the shape function $b(r)$ and time dependence of ϕ will not effect the wormhole geometry.
- $\Phi=0$, gives a family of evolving wormhole conformally related to another family of zero tidal force evolving wormhole.

Hence, the gravitational field equations simplifies to:

$$3H^2+\frac{b'}{a^2r^2}=\frac{1}{f_R}\left[8\pi\rho-\frac{1}{2}f+\frac{1}{2}Rf_R-3H\dot{f}_R+a^{-2}\left(1-\frac{b(r)}{r}\right)f''_R+\right.$$
$$\left.f'_R\frac{5b(r)-r\prime b(r)-4r}{2a(t)^2r^2}\right], \tag{5.16}$$

$$-\left(\frac{2\ddot{a}}{a}+H^2\right)-\frac{b}{a^2r^3}=\frac{1}{f_R}\Bigg[8\pi p_r+p_rf_T+\frac{a^2(t)}{\left(1-\frac{b(r)}{r}\right)}\times$$
$$\left[\left(f/2-3\dot{H}f_R-6H^2f_R-\frac{b'(r)}{a^2(t)r^2}f_R+\rho f_T\right)-\left(\dot{f}_R\frac{\partial\phi}{\partial t}-2H\dot{f}_R-\ddot{f}_R\right)\right]+f'_R\frac{2}{r}\Bigg], \tag{5.17}$$

$$
-\left(\frac{2\ddot{a}}{a}+H^2\right)+\frac{b-rb'}{2a^2r^3}
$$
$$
=\frac{1}{f_R}\Bigg[8\pi p_t+p_tf_T+a^2(t)r^2\Bigg[\Bigg(f/2-3\dot{H}f_R-6H^2f_R-\frac{b'(r)}{a^2(t)r^2}f_R+\rho f_T+
$$
$$
\frac{1}{2}f'_R\frac{rb'(r)-b(r)}{r(r-b(r))}\Bigg)-(-2H\dot{f}_R-\ddot{f}_R)\Bigg]-r^2\left(1-\frac{b(r)}{r}\right)f''_R-f'_R(b(r)-r)\Bigg]. \tag{5.18}
$$

From the aforementioned field equations, it is feasible to derive the components of the energy-momentum tensor, specifically the energy density and the radial as well as transverse pressure components, in order to facilitate subsequent analysis. Upon contemplation of the linear format exhibited by the function $f(R,T)$, characterized by $f(R,T)=\alpha R+\beta T$, with α and β denoting constants. If one considers $\alpha=1$ and $\beta=2\lambda$, then it brings forth the very well known form of $f(R,T)$=$R+2\lambda T$. This known linear form was proposed by Harko [90] in their paper and is well established in $f(R,T)$ cosmology. As described in Chapter 2, wormhole solutions in general relativity are constrained to be necessary with exotic matter. However, with a coupled geometric matter in terms of modified gravity can enable the possibility of non-exotic matter at the wormhole throat. The linear $f(R,T)$ theory is significantly simple and proposes a minimal modification on GR. Non-exotic traversable wormhole solutions in linear $f(R,T)$ gravity has been examined by Rosa and Kull [323], they found the geometry satisfying all energy conditions in the entire spacetime. Therefore, in this article a minimal coupling of geometric matter is considered to study its impact on the evolving wormhole geometry and its impact on the energy conditions considering a simple linear form without any higher order corrections to the Ricci scalar or the trace of the energy-momentum tensor. Using this linear choice for $f(R,T)$, the expression for the energy density, and the pressure components take the form:

$$
\begin{aligned}
\rho=&\Bigg[2a^2(t)r^3(8\pi+\beta)\Bigg\{256\pi^2(-b(r)+r)\\
&+16\pi\Big(-b(r)(3+2a^2(t)r^2)+r(3+a^2(t)(1+2r^2))\Big)\beta\\
&+\Big(-b(r)(3+2a^2(t)r^2)+r(2+a^2(t)(2+3r^2))\Big)\beta\Bigg\}\Bigg]^{-1}\\
&\times\Bigg[\alpha\Big\{2b(r)(b(r)-r)\beta(8\pi+\beta)\\
&+6a^2(t)\dot{a}^2(t)r^4\beta\Big(16\pi(1+2r(-b(r)+r))+3(1-b(r)r+r^2)\beta\Big)\\
&+a^2(t)r\Big(1536a^{-2}(t)\dot{a}^2(t)\pi^2r^2(-b(r)+r)\\
&+192a^{-2}(t)\dot{a}^2(t)\pi r^2(-b(r)+r)\beta+b(r)(-1+2(b(r)-r)r)\beta^2\Big)\\
&+r\Big(-2b(r)(8\pi+\beta)\Big(32\pi+(3+4a^2(t)r^2)\beta\Big)\\
&+r\Big(512\pi^2+16\pi(7+a^2(t)(2+4r^2))\beta+(6+a^2(t)(5+8r^2))\beta^2\Big)\Big)b'(r)\\
&+4a^2(t)r^3\beta\Big(32\pi(b(r)-r)+b(r)(-4-a^2(t)(-1+r^2))\beta\Big)\\
&\Big(a(t)\ddot{a}(t)-\dot{a}^2(t)\Big)a^{-2}(t)\Big\}\Bigg],
\end{aligned} \tag{5.19}
$$

$$
\begin{aligned}
p_r =& \Bigg[2a^2(t)r^3(8\pi+\beta)\Bigg\{256\pi^2(-b(r)+r) \\
&+16\pi\Big(-b(r)(3+2a^2(t)r^2)+r(3+a^2(t)(1+2r^2))\Big)\beta \\
&+\Big(-b(r)(3+2a^2(t)r^2)+r(2+a^2(t)(2+3r^2))\Big)\beta\Bigg\}\Bigg]^{-1} \\
&\times\Bigg[\alpha\Bigg\{512\pi^2(b(r)-r)(b(r)+3\dot{a}^2(t)r^3) \\
&+32\pi\Big(b^2(r)(3+2a^2(t)r^2)+3\dot{a}^2(t)r^4(-3+a^2(1-2r^2)) \\
&+b(r)r\Big(-3+a^2(t)(-1+r^2(-2+\dot{a}^2(t)a^{-2}(t)(9+6a^2(t)r^2)))\Big)\Big)\beta \\
&+\Big(b^2(r)(4+6a^2(t)r^2)+6\dot{a}^2(t)r^4(-2+a^2(t)(1-3r^2)) \\
&+b(r)r\Big(-4+3a^2(t)(-1+2r^2(-1+a^{-2}(t)\dot{a}^2(t)(2+3a^2(t)r^2)))\Big)\Big)\beta^2 \\
&+a^2(t)r^2\Big(-\beta^2b'(r)+4r\Big(256\pi^2(b(r)-r) \\
&+16\pi\Big(b(r)(3+2a^2(t)r^2)+r(-3-2a^2(t)(-1+r^2))\Big)\beta \\
&+\Big(b(r)(2+3a^2(t)r^2)+r(-2-3a^2(t)(-1+r^2))\Big)\beta^2\Big)(a(t)\ddot{a}(t)-\dot{a}^2(t))a^{-2}(t)\Bigg\}\Bigg],
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
p_t =& \Bigg[a^2(t)r^3\Bigg\{256\pi^2(-b(r)+r) \\
&+16\pi\Big(-b(r)(3+2a^2(t)r^2)+r(3+a^2(t)(1+2r^2))\Big)\beta \\
&+\Big(-b(r)(3+2a^2(t)r^2)+r(2+a^2(t)(2+3r^2))\Big)\beta\Bigg\}\Bigg]^{-1} \\
&\times\Bigg[\alpha\Bigg\{-16\pi(b(r)-r)(b(r)-6\dot{a}^2(t)r^3) \\
&+\Big(-6a^2(t)(1+a^2(t))\dot{a}^2(t)a^{-2}(t)r^4-b^2(r)(1+2a^2(t)r^2) \\
&+b(r)r(1+a^2(t)(1+(2+6\dot{a}^2(t)a^{-2}(t))r^2))\Big)\beta \\
&-r\Big(-b(r)(16\pi+\beta+2a^2(t)r^2\beta)+r(16\pi+\beta+a^2(t)(1+2r^2)\beta)\Big)b'(r) \\
&-4a^2(t)r^3\Big(16\pi(-b(r)+r) \\
&+(r+a^2(t)r-a^2(t)r^3+b(r)(-1+a^2(t)r^2))\beta\Big)(a(t)\ddot{a}(t)-\dot{a}^2(t))a^{-2}(t)\Bigg\}\Bigg].
\end{aligned}
\tag{5.21}
$$

The equations governing the components of the energy momentum tensor provide a means for examining the potential existence of evolving wormholes within the $f(R,T)$ theory of gravity. In order for wormholes to persist and remain traversable in the framework of Einstein gravity, it is necessary for the Null Energy Condition (NEC) to be breached at the throat. However, a recent investigation conducted by Ilyas *et al.* [317] showcased that a wormhole could indeed form under the umbrella of $f(R,T)$

gravity utilizing non-exotic materials, while adhering to the conventional Electromagnetic (EM) tensor that upholds the NEC. The adjustments in curvature induced by the modified gravitational effects lead to a transformation in NEC violation, ultimately resulting in an overarching violation of the NEC by the collective EM tensor. A brief overview regarding the plausibility of wormhole existence with usual matter in altered gravitational models is succinctly outlined in Chapter 2. Furthermore, Banerjee *et al.* [324] present another scenario of a wormhole supported by non-exotic matter within the realm of $f(R,T)$ gravity.

The above mentioned studies were all done on static wormhole geometry. Hence, whether an evolving wormhole can exist in $f(R,T)$ gravity by violating the NEC or by satisfying the energy conditions for our choice of the $f(R,T)$ gravity model is examined.

5.3 Energy Conditions

The development of traversable wormholes exhibits a particularly fascinating and somewhat unique aspect, involving the need for exotic matter, which contradicts classical energy conditions. This exotic matter plays a crucial role in maintaining the wormhole throat, ensuring its traversability. As discussed in the introduction, extensive investigations have been carried out on this characteristic within the realm of evolving traversable wormholes and Morris-Thorne type wormholes, explored through various iterations of modified theories. Notably, certain theories within this framework support types of matter that adhere to energy conditions in diverse manners. Interestingly, even if standard matter satisfies these conditions, the presence of coupled matter becomes necessary to serve as exotic energy, fulfilling the requirement for traversability.The energy conditions, namely the Null Energy Condition (NEC), Weak Energy Condition (WEC), Dominant Energy Condition (DEC), and Strong Energy Condition (SEC), are satisfied when the following inequalities are met:

- (i)NEC: $\rho + (p_r) \geq 0$, $\rho + (p_t) \geq 0$;
- (ii)WEC: $\rho \geq 0$,$\rho + (p_r) \geq 0$, $\rho + (p_t) \geq 0$;
- (iii)SEC: $\rho + (p_r) \geq 0$, $\rho + (p_t) \geq 0$, $\rho + (p_r) + 2(p_t) \geq 0$;
- (iv)DEC: $\rho \geq 0$,$\rho - |(p_r)| \geq 0$, $\rho - |(p_t)| \geq 0$.

In the analysis three different well known shape functions are used and they are given as [325, 326]:

$$b(r) = \frac{r_0 \ln(1+r)}{\ln(1+r_0)} \quad \text{(Model I)} \tag{5.22}$$

$$b(r) = r/(e^{r-r_0}) \quad \text{(Model II)} \tag{5.23}$$

$$b(r) = r/(1+r-r_0) \quad \text{(Model III)} \tag{5.24}$$

where r_0 is the throat radius. All the above shape functions satisfy all the necessary conditions like the flaring out, asymptotic flatness and more. We now use these shape functions to investigate the energy condition using the equations (5.19), (5.20) and (5.21). We investigate all the energy conditions, stated by the above bullet points. The above shape functions will be treated as Model I, Model II and Model III respectively, throughout the analysis.

The Ricci scalar is given by:

$$R = 6(\dot{H} + 2H^2) + \frac{2b'(r)}{a^2 r^2}, \tag{5.25}$$

where, H is the Hubble parameter and a is the scale factor. For the study three known shape functions are used and investigated whether they satisfy or violate the energy conditions in the framework of this model of $f(R,T)$ gravity.

We analyse some of the viable models of evolving wormholes and check the validity of the energy conditions. For the investigation we consider two forms of the scale factor($a(t)$):

- Power law form:
$$a(t) = a_0 t^n, \tag{5.26}$$
- Exponential form:
$$a(t) = [a_0 + e^{\mu t^n}], \tag{5.27}$$

with a_0, μ and n as constants.

We will now evaluate whether the energy conditions are satisfied for the specified choices of the scale factor (in equations (5.26) and (5.27)) and for the shape functions in equations (5.22-5.24). In the initial model, which assumes power law expansion, there are five parameters: a_0, r_0, α, β, and n. The values of a_0 and r_0 are somewhat arbitrary, as they can be used to rescale the solutions without loss of generality.

Given that $f(R,T)$ gravity theory encompasses GR as a limiting case, it is logical to set $\alpha = 1$. Furthermore, the literature [90, 321, 322] frequently adopts a linear form $f(R,T) = R + 2\lambda T$. For the power law form of the scale factor, the chosen parameter values are $\alpha = 1, \beta = 2, a_0 = 2.5, n = 0.8$ and $r_0 = 0.5$ (throat radius). For the exponential form of the scale factor, the parameter values are $\alpha = 1, \beta = 2, a_0 = 1, n = 0.2, \mu = 0.2$ and $r_0 = 0.5$ (throat radius).

The figures(5.1)-(5.3), represent the components of the energy conditions considering the power law form, and figures(5.4)-(5.6), represents the components of the energy conditions considering exponential form of scale factor. From the graphs in figures (5.1)-(5.6) it is clear that all the energy conditions are satisfied for both the choices of the scale factor.

Further from (5.27), we see that as $t \to -\infty (\mu > 0)$, then $a \to a_0$. Also the Hubble parameter becomes vanishingly small as $t \to -\infty (\mu > 0)$. The universe is said to be emergent if it satisfies certain conditions so as to avoid the big bang singularity.The conditions for emergent scenario are [254–260]:

- $a \to a_0$, $H \to 0$ as $t \to -\infty$.
- $a \simeq a_0$, $H \simeq 0$, $t << t_0$.

So we may say that the universe evolves from an emergent scenario at infinite past.

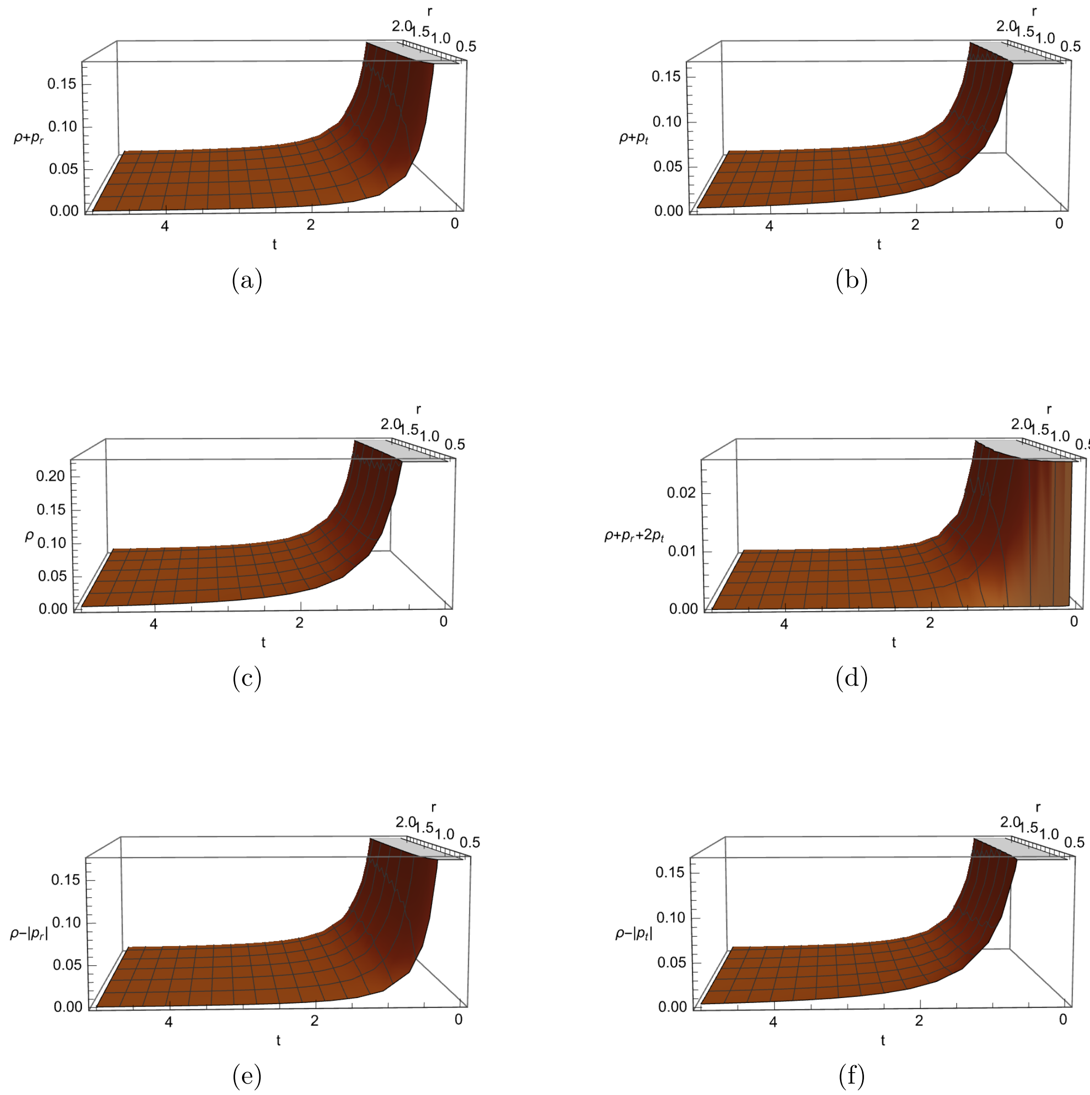


Figure 5.1: Plots demonstrating the variation of energy condition components with radial distance r and time t for Model I. The nature of the plots are obtained considering $\alpha = 1,\ \beta = 2,\ a_0 = 2.5,\ n = 0.8$ and $r_0 = 0.5$ (throat radius).

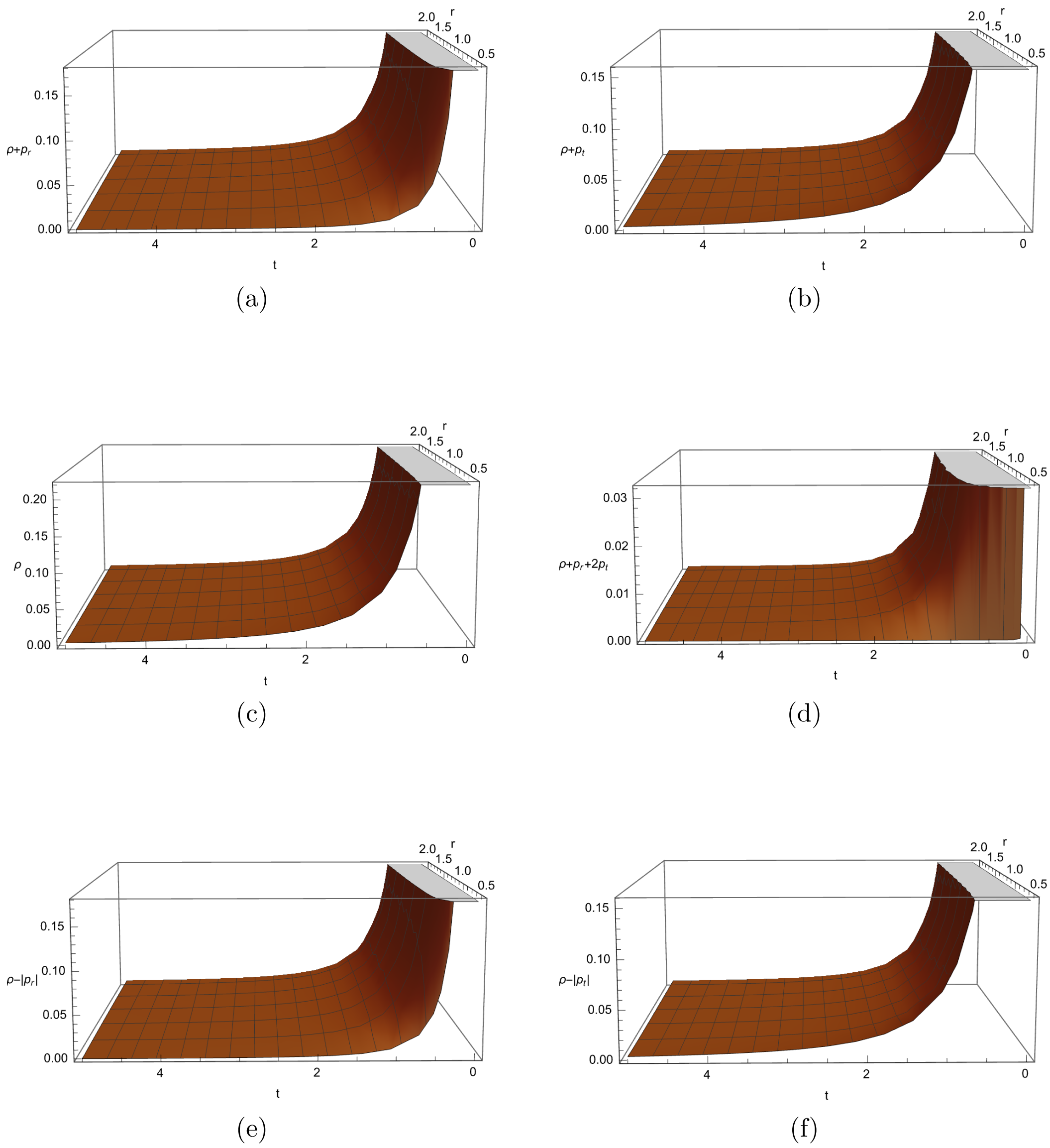


Figure 5.2: Plots demonstrating the variation of energy condition components with radial distance r and time t for Model II. The nature of the plots are obtained considering $\alpha = 1,\ \beta = 2,\ a_0 = 2.5,\ n = 0.8$ and $r_0 = 0.5$ (throat radius).

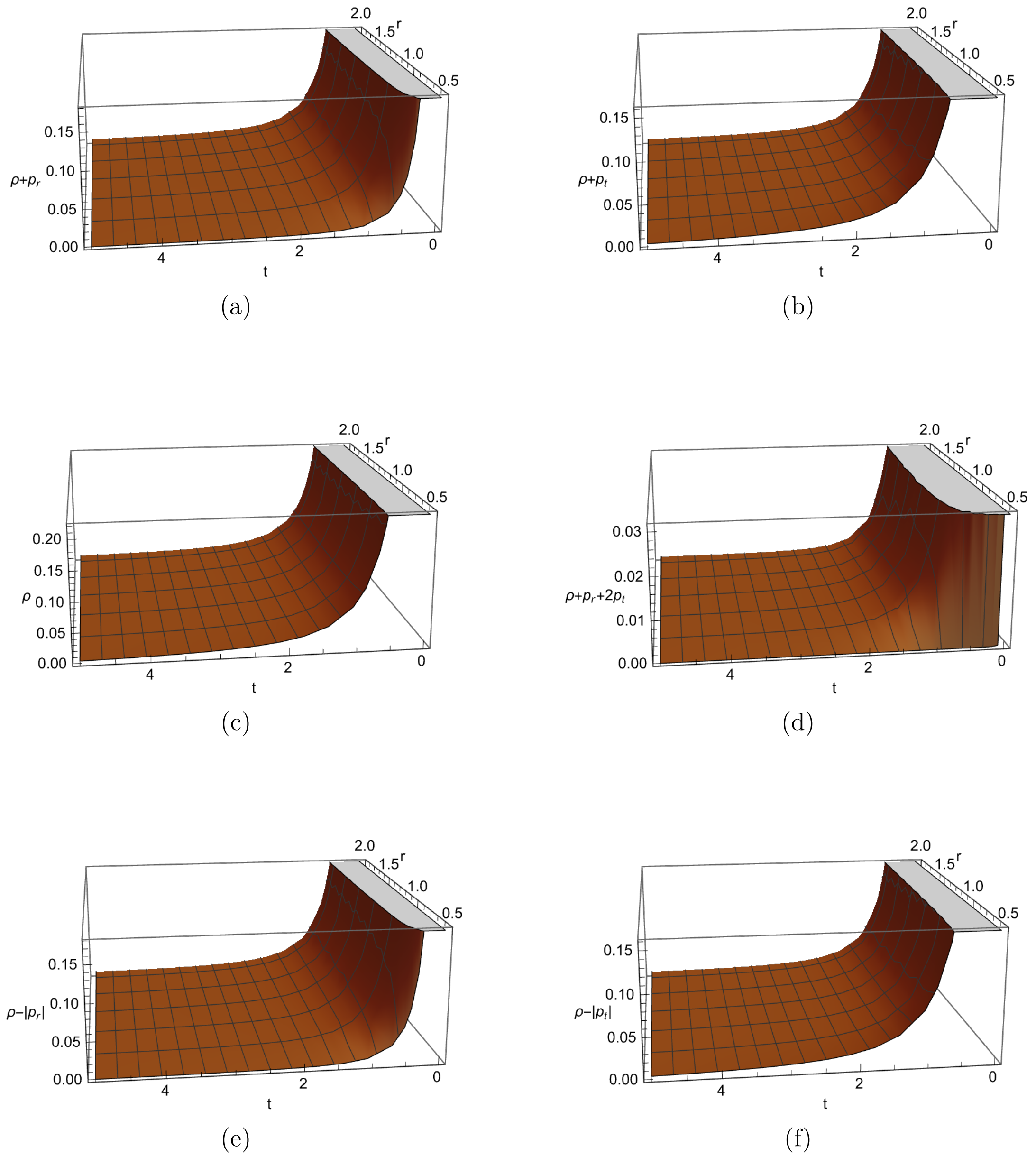


Figure 5.3: Plots demonstrating the variation of energy condition components with radial distance r and time t for Model III. The nature of the plots are obtained considering $\alpha = 1,\ \beta = 2,\ a_0 = 2.5,\ n = 0.8$ and $r_0 = 0.5$ (throat radius).

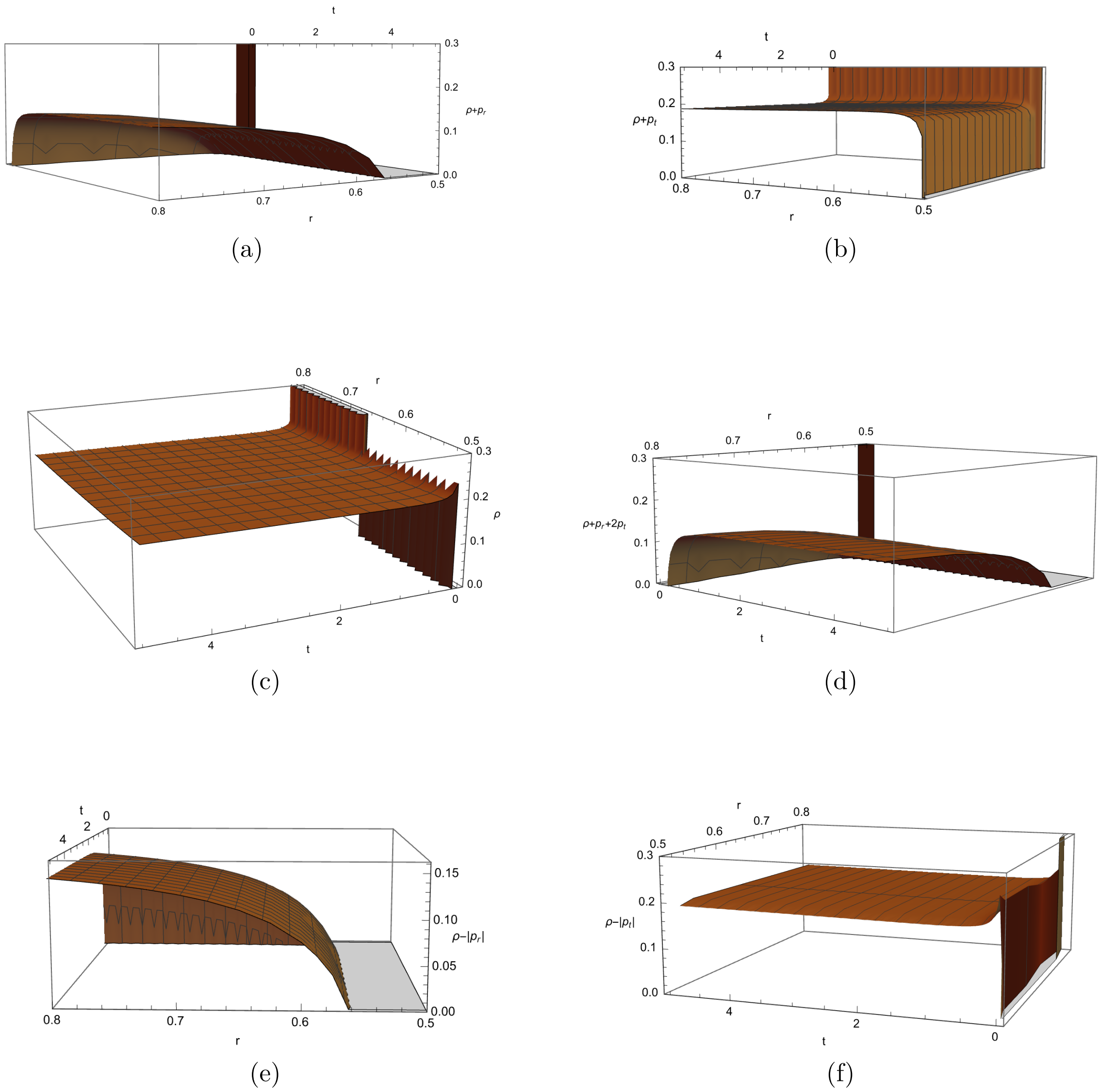


Figure 5.4: Plots demonstrating the variation of energy condition components with radial distance r and time t for Model I considering exponential form of scale factor. The nature of the plots are obtained considering $\alpha = 1,\ \beta = 2,\ a_0 = 1,\ n = 0.2,$ $\mu = 0.2$ and $r_0 = 0.5$ (throat radius).

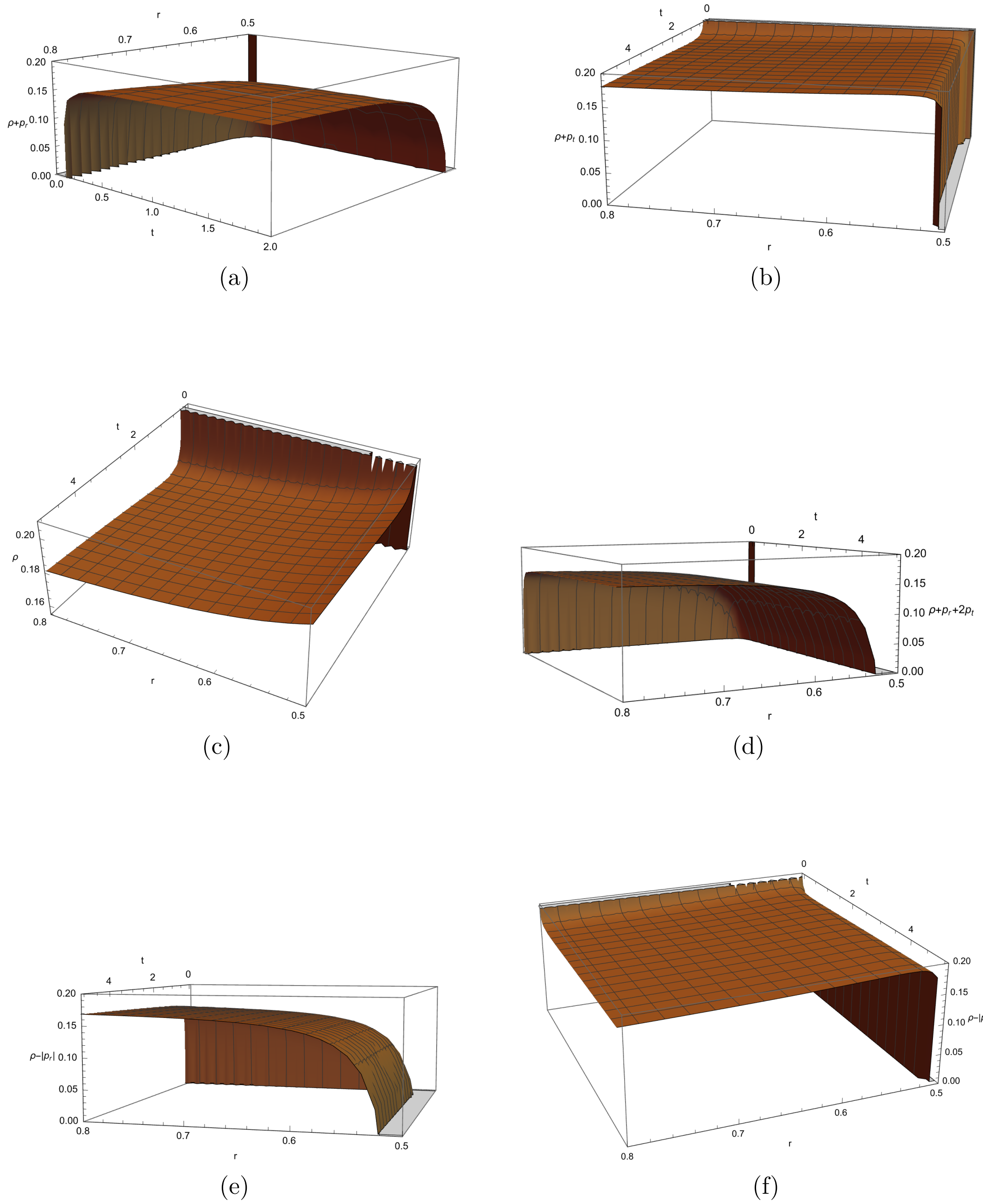


Figure 5.5: Plots demonstrating the variation of energy condition components with radial distance r and time t for Model II considering exponential form of scale factor. The nature of the plots are obtained considering $\alpha = 1$, $\beta = 2$, $a_0 = 1$, $n = 0.2$, $\mu = 0.2$ and $r_0 = 0.5$ (throat radius).

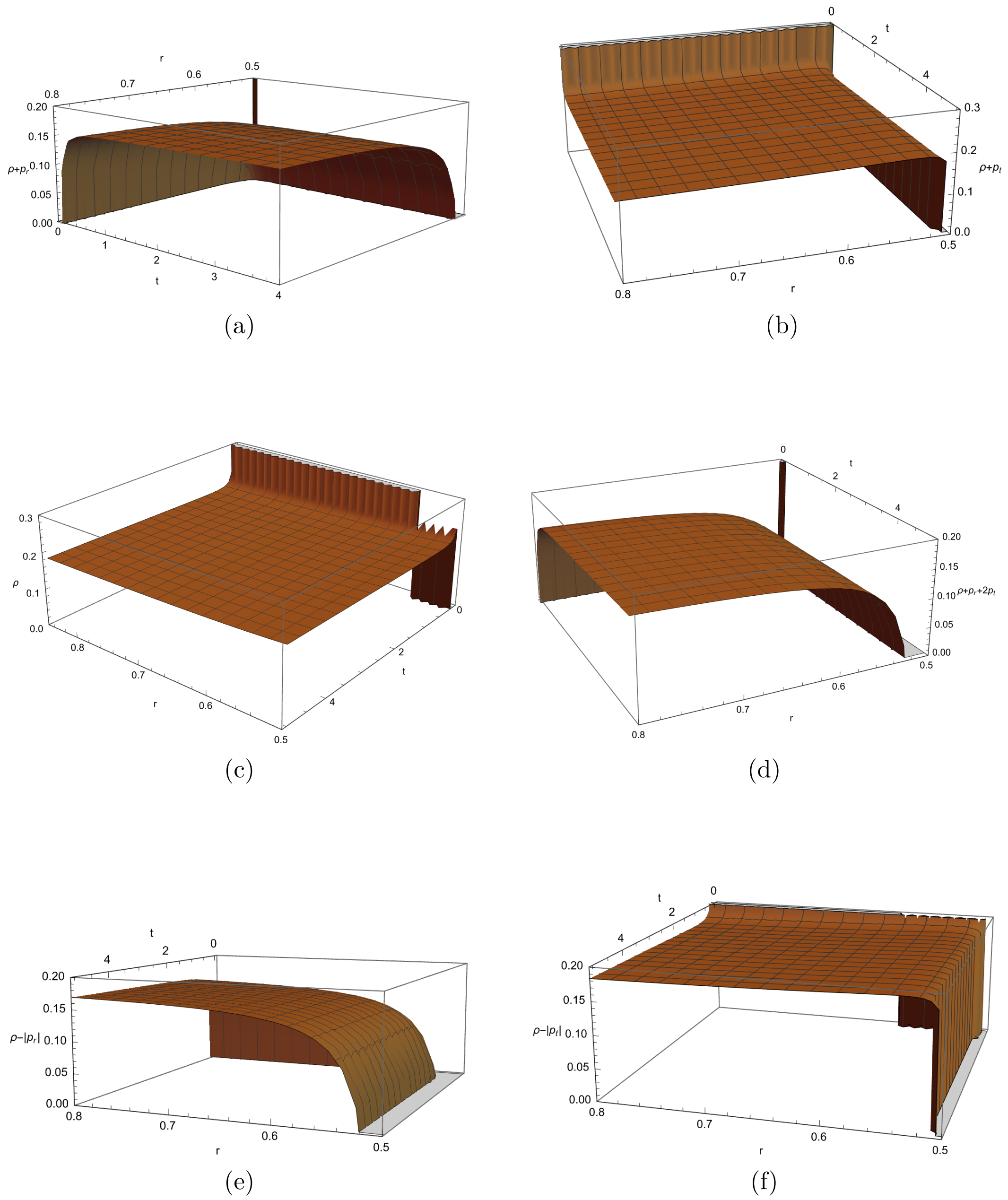


Figure 5.6: Plots demonstrating the variation of energy condition components with radial distance r and time t for Model III considering exponential form of scale factor. The nature of the plots are obtained considering $\alpha = 1,\ \beta = 2,\ a_0 = 1,\ n = 0.2,$ $\mu = 0.2$ and $r_0 = 0.5$ (throat radius).

5.4 Discussion

In the present chapter we have investigated evolving wormhole solutions in the realm of $f(R,T)$ gravity. Due to the very complicated and coupled form of the field equations we assume (i) the redshift function $\phi = 0$ and (ii) $f(R,T) = \alpha R + \beta T$. For wormhole configuration we have chosen three models of the shape functions (which are well known in the literature) and are named as Model I, Model II and Model III respectively. For the scale factor $a(t)$ we have restricted ourselves to (i) power law form (ii) exponential form. Subsequently we have examined the energy conditions namely the NEC, WEC, SEC and DEC. Due to the complicated form of the matter component it is not possible to analyse the energy conditions analytically only graphical analysis is possible.

The graphical analysis of the validity of energy conditions are presented in the figures(5.1)-(5.6) for both the choice of scale factor for some fixed values of the parameters..It is found that all the energy conditions are satisfied considering both the choices of scale factor. Hence, we can say that traversable evolving wormholes can exist in the $f(R,T)$ theory of gravity for both the choices of the scale factor. In fact time dependent wormholes exist in $f(R)$ theories too, but there is a need for squared Ricci scalar correction [304] for the wormhole to exist with ordinary matter and be traversable. In the present work we were able to obtain physically viable and traversable evolving wormhole geometry considering the linear form of $f(R,T)$ without any corrections to the geometry nor consideration of any exotic matter.

Hence, having a solution of an evolving wormhole with the in a modified theory of gravity which explains the acceleration of the universe also accelerates the understanding of how exotic objects like wormholes evolve with the expansion of the universe. Thus, the present study shows that dynamical wormhole is possible in modified theories of gravity without any need of exotic matter.

Finally, the exponential choice of the scale factor for the present dynamical wormhole scenario has a singularity free emergent configuration at infinite past.

CHAPTER 6

CONCLUSION AND FUTURE PROSPECT

This thesis contains 6 chapters, begining with a brief introduction, followed by the chapters describing works undertaken during the course, and ending with this particular chapter intended to conclude the works in a presice manner along with some discussions on future prospects in the field.

To begin the first chapter deals with an short and precise introduction of Einstein's General Relativity along with its mathematical review and its extension into Modified Theories of Gravity. A brief review of $f(R)$, $f(R,T)$ and $f(T)$ theories of modified gravity is presented. Further, to gain an understanding of wormholes, with a specific focus on the properties of Morris-Thorne's wormhole structure is reviewed. Our discussion avoided delving into technical intricacies and instead focused on their assumptions regarding a spherically symmetric and static metric to describe a wormhole, as well as certain shape functions that influence the desired structure. Also few other types traversable, evolving and unconventional wormhole geometries are discussed, followed by a brief review of Closed Timelike Curves (CTCs).

In the second chapter, a relation between CNHC and existence of Wormholes is explored, and concluded that the presence of a past outer trapping horizon where the CNHC is violated indicates the potential formation of a wormhole.Further, it is established that within Einstein's Gravity, the existence of wormholes with normal matter is not feasible, whereas in modified gravity, such wormholes may be plausible. Additionally, it is concluded that the validity of the CNHC or the presence of a wormhole configuration implies the non-existence of a wormhole or the violation of CNHC, respectively. However, this relationship does not hold in reverse, both within Einstein's Gravity and in modified gravity theories.

In the third chapter, an extensive examination of the trajectory of test particles within the Cylindrical Wormhole spacetime is conducted . The analysis delved into the intricacies of particle motion near Closed Timelike Curves (CTCs) and their constraints. Our findings underscore the significant role of angular momentum in particle confinement, emphasizing its relevance in understanding particle dynamics. The detailed analysis demonstrates that particles lacking Angular Momentum (AM) remain confined outside the CTC boundary, while those with AM can penetrate the boundary and travel within the closed orbit. This is attributed to the Total Effective AM, which boosts the particle's effective acceleration, thereby increasing its velocity and enabling it to surpass the barrier where the causality-violating region occurs. The spacetime diagram for radial null and radial time-like particles, along with their respective confinements, clearly indicates that only particles with non-zero AM can traverse within the CTC. Furthermore, it can be inferred that the potential for a particle lacking Angular Momentum (AM) is attractive, preventing it from crossing the CTC confinement and reaching the singularity region. Instead, the particle oscillates within the confines of its region. This assertion is reinforced by the velocity profile, which demonstrates a decrease in velocity with increasing radial distance, although the velocity of AM-less particles never reaches zero. In contrast, the potential for particles with non-zero AM is repulsive, enabling them to traverse the CTC confinement but preventing them from reaching the singularity region. The velocity of these particles supports this observation, as it increases with radial distance but asymptotes before reaching the singularity.

Chapter four delves into the investigation of inhomogeneous FLRW spacetime, aiming to investigate the feasibility of a dynamical wormhole solution with non-interacting two-fluid system as the matter content. Fluid 1, homogeneous yet dissipative, is contrasted with the inhomogeneous and anisotropic Fluid 2. Two scenarios are explored: one with a barotropic equation of state for Fluid 1 and another with a polytropic equation of state for Fluid 2. Both scenarios yield wormhole solutions exhibiting a linear relationship between radial and tangential pressures. Energy conditions are satisfied within certain parameter bounds and one may conclude that there is no need of any exotic matter near the throat - a distinct feature for dynamical wormhole compared to the static wormhole solutions.. This dynamical wormhole configuration, with emergent behavior at infinite past, could serve as a model for the early universe, potentially circumventing the big bang singularity.

In the following chapter five, evolving wormhole solutions within the framework of $f(R,T)$ gravity is presented. Three models of shape functions are explored and analysed for two forms of the scale factor $a(t)$. All the energy conditions were examined graphically due to the intricate nature of the matter component. Our analysis, depicted in figures reveals that all three models satisfy the all the energy conditions for both forms of scale factor, indicating the potential existence of traversable evolving wormholes in $f(R,T)$ gravity without the need for exotic matter. This stands in contrast to $f(R)$ theories, which typically require Ricci scalar corrections for traversable wormholes with ordinary matter. Notably, the exponential choice of the scale factor for the present dynamical wormhole scenario has a singularity free emergent configuration at infinite past. Overall, our findings shed light on the evolution of exotic objects like wormholes within a modified theory of gravity, advancing our understanding of the universe's accelerated expansion. Thus, the study shows that dynamical wormhole is possible in modified theories of gravity without any need of exotic matter.

To conclude, the works presented in this thesis extensively explores static wormholes as well as evolving wormholes. From their mere existence, practicle confinements in the context of CTCs, their stability to traversability is comprehensively explored.

Though wormholes remain purely theoretical exotic objects, the future prospects of research in wormhole physics remains wide open. Future prospects in wormhole research hold promising avenues for further exploration and discovery. Here are some potential directions for future investigation:

1. Possible Substitutes for Exotic Matter: Delving deeper into frameworks that could enable the creation of traversable wormholes without relying on exotic matter has the potential to drive significant progress. Exploring adjustments to theories of gravity or fresh perspectives on spacetime geometry might unveil perspectives.

2. Implications in Cosmology: examination of how wormholes could impact our understanding of the universe and its development holds promise for valuable insights. Investigating how wormholes fit into models including their links to dark energy or cosmic inflation could enrich our comprehension of essential cosmological concepts.

3. Exploring Quantum Aspects: Investigating the quantum properties of wormholes and their interactions with quantum fields may reveal phenomena at the intersection of relativity and quantum mechanics. Researching quantum adjustments to wormhole solutions and their implications for information transmission and quantum entanglement could be research avenues.

4. Detecting Wormholes: Developing techniques, for observing or disproving the presence of wormholes in the universe remains a compelling objective. Advancing methods like using wave detection or astrophysical imaging could offer valuable insights into confirming or disproving the existence and characteristics of wormholes.

5. Interstellar Travel: Understanding the stability and navigability of wormholes in situations and contexts is crucial, for evaluating their use as pathways for interstellar travel or cosmic shortcuts. Utilizing simulations and theoretical examinations to study how matter and spacetime interact near wormholes could provide insights, into their practicality and viability.

Overall, the field of wormhole research holds immense potential for further exploration and discovery, with the potential to revolutionize our understanding of the universe and unlock new frontiers in science and technology. Continued interdisciplinary collaboration and innovation will be key to realizing these future prospects in wormhole research.

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