STUDY OF FUNCTIONAL IDENTITIES INVOLVING GENERALIZED DERIVATIONS AND RELATED ADDITIVE MAPS IN PRIME RINGS

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CERTIFICATE FROM THE SUPERVISOR(S)

This is to certify that the thesis entitled "**STUDY OF FUNCTIONAL IDENTITIES INVOLVING GENERALIZED DERIVATIONS AND RELATED ADDITIVE MAPS IN PRIME RINGS**" submitted by me, Sri **Nripendu Bera** (registered on 22. 03. 2021) for the award of degree of DOCTOR OF PHILOSOPHY (Science) from Jadavpur University has been carried out under the guidance and supervision of Prof. Sukhendu Kar, Dept. of Mathematics, Jadavpur University and Dr. Basudeb Dhara, Dept. of Mathematics, Belda College, Paschim Medinipur. The results embodied herein is original and this work has not been submitted for any other Degree/Diploma or any other academic award anywhere before.

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Dedicated to

my parents

Sibaprasad Bera

and

Dipali Bera

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Preface

Motivated by ordinary derivative in several branches of mathematics the notion of derivation of ring was introduced long back. But the study of derivation got its momentum after the work of Posner [86]. For as long as rings and algebras have been studied for their own sakes, it has been a problem of interest to determine the consequences of various special identities and conversely, to find sufficient conditions on a given ring which ensure that a specified identity holds. It is well known that there is a strong relationship between the functional identities (FI) involving derivations, generalized derivations and the structure of the rings. Ring derivation is a branch of algebra in which we study about the structure of additive maps as well as structure of rings by analyzing some functional identities involving additive maps. These additive maps are derivation, skew derivation, generalized derivation, generalized skew derivation, X-generalized derivation, X-generalized skew derivation, X-generalized (α, β)-derivation, etc. The main objective of the thesis is to study some functional identities and generalized functional identities in prime and semiprime rings. Let R be an associative ring. Then a simple example of functional identity is the identity $[f(x), x] = 0$ for all $x \in R$, where $[x, y] = xy - yx$ and $f : R \to R$ is a mapping. In 1957, Posner [86] studied a special kind of functional identities in rings by taking the above function as a derivation. Much more recently, in [17], Brešar proved that if f is any additive mapping satisfying the identity studied by Posner [86] in prime ring R, then f must be in the form $f(x) = \lambda x + \xi(x)$, where $\lambda \in C$ and $\xi : R \to C$ is an additive mapping, C is the extended centroid of R.

This thesis contains eight chapters and in each chapter we study different type of identities under certain conditions. Chapter-wise brief information is given bellow:

Chapter 1 is basically an introduction to some basic definitions, preliminaries and prerequisites which are collected from other references and those are needed for the development of the subsequent chapters of this thesis.

Two additive maps $F, G : R \to R$ are said to be co-commuting (co-centralizing) on R if $F(x)x - xG(x) = 0$ for all $x \in R$ (resp. $F(x)x - xG(x) \in Z(R)$ for all $x \in R$).

In **Chapter 2**, we study an identity involving three generalized derivations in prime rings for the generalization of the concept of co-commuting maps.

There are many papers in literature which have studied mapping behave like a derivation as well as a homomorphism. EI Sofy [88] introduced the concept of homoderivation maps on a ring R. An additive mapping H from R into itself is called homoderivation if $H(xy) = H(x)H(y) + H(x)y + xH(y)$ for all $x, y \in R$. Now, we can define Jordan homoderivation maps in a ring R. An additive mapping H from R into itself is called Jordan homoderivation if $H(x^2) = H(x)H(x)+H(x)x+xH(x)$ for all $x \in R$.

In **Chapter 3**, we study the Jordan homoderivation behavior of three generalized derivations in prime rings. More precisely, we study the identity $F(x^2) = G(x)^2 + H(x)x + xH(x)$ for all $x \in f(R)$, where R is a prime ring and F, G, H are three generalized derivations.

Chapter 4 is devoted to the study of generalized derivations with annihilator conditions in prime rings.

In **Chapter 5**, we study an identity involving two generalized skew derivations in prime rings for generalization of commuting maps on prime rings.

In this line of investigation, De Filippis [32] introduced the new map *b-generalized skew derivation*. We note that b-generalized skew derivation generalizes the concept of generalized skew derivation as well as b-generalized derivation.

In **Chapter 6**, we study an identity involving two b-generalized skew derivations in prime rings.

Over the last few decades, several authors have investigated the relationship between the commutativity of the ring R and some specific types of maps of R. The first result in this context is due to Posner [86], who proved that if a prime ring R admits a nonzero centralizing derivation d, then R must be commutative.

In **Chapter 7**, we study some commutativity theorems involving multiplicative (generalized)-derivations in prime and semiprime rings. Some examples are given at the end of this chapter concluding that semiprimeness hypothesis in the theorems are not superfluous.

Lastly, in **Chapter 8**, we study some functional identities involving b-generalized (α, β)-derivations in prime rings to extend many known results in literature.

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List of Publications

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[2] V. De Filippis, B. Dhara and **N. Bera** : Generalized skew derivations and generalization of commuting maps on prime rings; *Beitr. Algebra Geom.*, 63 (2022), 599–620, DOI: 10.1007/s13366-021-00590-3.

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[5] **N. Bera** and B. Dhara : Jordan homoderivation behaviour of generalized derivations in prime rings; *Ukr. Math. J.*, 75 (2024), 1340-1360, DOI:10.1007/s11253-024-02265-3.

[6] **N. Bera** and B. Dhara : A note on b-generalized (α, β)-derivations in prime rings; *Georgian Math. J.*, DOI: 10.1515/gmj-2023-2121.

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Chapter 1

Introduction, Preliminaries and Prerequisites

Through out this thesis, by R we mean an associative ring and ideal of a ring means two sided ideal. I have worked on the functional identities involving some specific type of additive maps which are satisfied on prime rings or semiprime rings. This introductory Chapter includes some basic notions, preliminaries and some prerequisites that will be most frequently used in the subsequent Chapters. Mainly in this Chapter we discuss about some special classes of rings, some ring of quotients, some special type of additive maps and about generalized polynomial identities. Since all basic notations are not possible to mention here, we refer the books by Herstein [68] and Jacobson [71, 72] for more details about basic symbois, notions and results.

First we provide some basic well known ideas about several special classes of rings.

1.1 Some Special Classes of Rings

In trying to understand the theory of prime and semiprime rings, one quickly sees that it is very important to first understand the notions of prime and semiprime ideals.

Now we recall the definions of prime and semiprime ideals in rings.

Definition 1.1.1. *Let R be a ring. An ideal P of R is said to be a prime ideal if for any two ideals A, B of R, $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.*

Definition 1.1.2. *Let R be a ring. An ideal P' of R is said to be a semiprime ideal of R if for any ideal A of R, $A^2 \subseteq P'$ implies $A \subseteq P'$.*

The following well known results provide characterizations of prime ideal and semiprime ideal in a ring :

Theorem 1.1.1. *Let P be an ideal of a ring R. Then the following are equivalent :*

(i) P is a prime ideal of R;

(ii) If $a, b \in R$ such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$;

(iii) If $< a >$, $< b >$ are principal ideals of R such that $< a >< b > \subseteq P$ then either $a \in P$ or $b \in P$;

(iv) If U' and V' are right ideals of R, then $U'V' \subseteq P$ implies either $U' \subseteq P$ or $V' \in P$;

(v) If U' and V' are left ideals of R, then $U'V' \subseteq P$ implies either $U' \subseteq P$ or $V' \subseteq P$.

Theorem 1.1.2. *Let P' be an ideal of a ring R. Then the following are equivalent :*

(i) P' is a semiprime ideal of R;

(ii) If $a \in R$ be such that $aRa \subseteq P'$, then $a \in P'$;

(iii) If $< a >$ is a principal ideal of R such that $< a >^2 \subseteq P'$ then $a \in P'$;

(iv) If U' is a right ideal of R, then $U'^2 \subseteq P'$ implies $U' \subseteq P'$;

(v) If U' is a left ideal of R, then $U'^2 \subseteq P'$ implies $U' \subseteq P'$.

Definition 1.1.3. *A ring R is said to be a prime ring if the zero ideal is a prime ideal in R.*

Example 1.1.1.

1. Every domain is a prime ring.

2. Every simple ring is a prime ring.

3. Matrix ring over prime ring is a prime ring.

Now we mention a well known results of equivalent conditions of prime rings.

Theorem 1.1.3. *The following conditions are equivalent in a ring :*

1. R is a prime ring;

2. If A, B are two ideals of R such that $AB = (0)$, then either $A = (0)$ or $B = (0)$;

3. *If $a, b \in R$ such that $aRb = 0$, then either $a = 0$ or $b = 0$.*

Similarly, semiprime ring is defined as follows :

Definition 1.1.4. *A ring R is said to be a semiprime ring if the zero ideal is a semiprime ideal in R.*

Example 1.1.2.

1. *Every prime ring is a semiprime ring.*

2. *If R_1 and R_2 are nonzero prime rings, then $R_1 \bigoplus R_2$ is a semiprime ring.*

Now we mention a well known result of equivalent conditions of semiprime rings.

Theorem 1.1.4. *The following conditions are equivalent in a ring :*
(i) R is a semiprime ring;
(ii) If A is an ideal of R such that $A^2 = (0)$, then $A = (0)$;
(iii) If $a \in R$ such that $aRa = 0$, then $a = 0$.

Some Properties of Prime and Semiprime Rings:

In this section, we provide some well known properties of prime and semiprime rings. The more information about prime and semiprime rings can be found in the book of Herstein [69].

• Every prime ring is a semiprime ring but the converse is not true. For example, $\mathbb{Z} \oplus \mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers, is a semiprime ring but not a prime ring.

• Semiprime ring has no nonzero nilpotent ideal.

• For any subset S of R, we denote by $r_R(S)$ the right annihilator of S in R, that is, $r_R(S) = \{x \in R \ : \ Sx = 0\}$ and by $l_R(S)$ the left annihilator of S in R, that is, $l_R(S) = \{x \in R \ : \ xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of R and is written as $ann_R(S)$. We know that if R is a semiprime ring and I is an ideal of R, then $r_R(I) = l_R(I)$. Moreover, if R is semiprime and I is an ideal of R, then $I \bigcap ann_R(I) = 0$.

• Let R be a semiprime ring and ρ be a right ideal of R. Then $Z(\rho) \subseteq Z(R)$, where $Z(R)$ is the center of the ring R.

• Let R be a prime ring which contains a commutative one sided ideal. Then R must be commutative.

• If a prime ring R contains a nonzero one sided central ideal, then R must be a commutative ring.

• Center of a prime ring contains no divisor of zero.

• Let R be a prime ring with center $Z(R)$. If $zr \in Z(R)$ for some $0 \neq z \in Z(R)$ and $r \in R$, then $r \in Z(R)$.

It is well known that a ring may have a left identity which is not a right identity. But in case of semiprime rings, we have the following result :

• Let R be a semiprime ring and e be a left identity of R. Then e is a right identity of R and hence e is the identity of R.

Definition 1.1.5. *Let n be a positive integer. A ring R is said to be $n-$torsion free, if whenever $nx = 0$, with $x \in R$, then $x = 0$.*

We note that whenever a ring R is $n-$torsion free, $char(R) \neq n$. But converse of the above result is not true for all rings. The primeness is required for the converse statement to be true. Thus for a prime ring R, $char(R) \neq n$ if and only if R is $n-$torsion free.

1.2 Commutator Identities in Rings

For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ stands for the anti-commutator $xy + yx$.

We recall some basic commutator identities in a ring R as follows :
For all $x, y, z \in R$,

$$[xy, z] = x[y, z] + [x, z]y; \quad [x, yz] = y[x, z] + [x, y]z;$$

$$(xy \circ z) = x(y \circ z) - [x, z]y = x[y, z] + (x \circ z)y;$$

$$(x \circ yz) = (x \circ y)z - y[x, z] = [x, y]z + y(x \circ z).$$

Moreover,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The last identity is called as Jacobi Identity.

For $x, y \in R$, set $[x,y]_0 = x$, $[x,y]_1 = [x,y] = xy - yx$, and then an Engel type polynomial $[x,y]_k = [[x,y]_{k-1}, y]$, $k = 1, 2, \ldots$

Definition 1.2.1. *An additive subgroup L of R is said to be a Lie ideal of R, if $[u,r] \in L$ for all $u \in L$ and $r \in R$.*

Clearly, every ideal of a ring R is a Lie ideal of R. It is to be noted that a Lie ideal of a ring R may not be an ideal of R.

- A Lie ideal L of R is said to be square closed if $u^2 \in L$ for all $u \in L$.

1.3 Ring of Quotients

In the study of functional identities involving derivations in prime and semiprime rings we observe that rings of quotients play a crucial role. For us the most important ring of quotients is the *maximal right ring of quotients* or **Utumi ring of quotients**. It was first constructed by Y. Utumi [95]. Another important ring of quotients we use here, *two-sided ring of quotients* or **Martindale ring of quotients**. This ring of quotients was introduced in [89] as a tool to study prime rings satisfying a generalized polynomial identity.

1.3.1 Utumi Ring of Quotients

Suppose that R be a prime ring and $\mathcal{D} = \{J : J$ is a dense right ideals of $R\}$ and consider T to be the set of all R-homomorphisms $f : J_R \to R_R$ where J ranges over $\mathcal{D}$ and J and R are regarded as right R-modules. So $T = \{(f;J) |\ \ J \in \mathcal{D},\ \ f : J_R \to R_R\}$, where $(f;J)$ denotes f acting on J.

We define $(f;J) \sim (g;K)$ if there exists $L \subseteq J \cap K$ such that $L \in \mathcal{D}$ and $f = g$ on L. One readily check that ‘ $\sim$ ‘ is indeed an equivalence relation, let $[f;J]$ denote the equivalence class determined by $(f;J) \in \mathcal{D}$ and we let U denote the collection of all equivalence classes of T with respect to ' $\sim$ '. We define addition and multiplication of equivalence classes as follows :

$[f;J]+[g;K]=[f+g;J\cap K]$ and $[f;J][g;K]=[fg;g^{-1}(J)]$. One can easily check that the addition and multiplication is well-defined [14, pp. 55]. It is readily seen that U forms a ring with respect to above defined addition and multiplication. This ring U is called *Utumi ring of quotients.*

Some important properties are given in the following result:

Proposition 1.3.1. *[14, Proposition 2.1.7] For a semiprime ring R, the Utumi ring of quotients U satisfies the following properties :*

1. *R is a subring of U;*

2. *For all $q\in U$ there exists $J\in\mathcal{D}$ such that $qJ\subseteq R$;*

3. *For all $q\in U$ and $J\in\mathcal{D}$, $qJ=0$ if and only if $q=0$;*

4. *For all $J\in\mathcal{D}$ and $f:J_R\to R_R$, there exists $q\in U$ such that $f(x)=qx$ for all $x\in J$.*

Furthermore, the above properties $(1)-(4)$ characterize the Utumi ring of quotients U up to isomorphism.

1.3.2 Martindale Ring of Quotients

For a prime ring R, a nonzero two-sided ideal is obviously a dense right ideal of R. In the construction of Utumi ring of quotients if we consider only nonzero two-sided ideals instead of dense right ideals, then we obtain the Martindale ring of quotients (see [89]). Here we shall denote this ring by Q.

Now we have the following result:

Proposition 1.3.2. *[14, Proposition 2.2.1] Let R be a semiprime ring. Then the Martindale ring of quotients Q satisfies the following properties:*

1. *R is a subring of Q;*

2. *For all $q \in Q$ there exists $J \in \mathcal{D}$ such that $qJ \subseteq R$;*

3. *For all $q \in Q$ and $J \in \mathcal{D}$, $qJ = 0$ if and only if $q = 0$;*

4. *For all $J \in \mathcal{D}$ and $f : J_R \to R_R$, there exists $q \in Q$ such that $f(x) = qx$ for all $x \in J$.*

Furthermore, properties $(1)-(4)$ characterize the Martindale ring of quotients Q up to isomorphism.

Some important facts for Utumi ring of quotients and Martindale ring of quotients are as follows:

• Q can be naturally regarded as a subring of U [14, Proposition 2.2.2] and can be characterized as follows: For $a \in U$, $a \in Q$ if and only if $aI \subseteq R$ for some nonzero two-sided ideal I of R.

• Let R be a semiprime ring. Then the corresponding rings of quotients Q and U both are semiprime rings. Moreover, if R is prime, then the corresponding rings of quotients Q and U both are prime rings [55, p. 74].

• $Z(Q) = Z(U)$, where $Z(Q)$ and $Z(U)$ are centers of Q and U respectively [14, Remark 2.3.1].

Definition 1.3.1. *The center of the Martindale ring of quotients as well as the Utumi ring of quotients is called the extended centroid of R, it is denoted by C and $S = RC$ is called the central closure of R.*

It is very well known that the extended centroid C of R forms a field, when R is prime ring [14, p. 70]. In fact, S is a prime ring containing R. Further S is contained in $Q \subseteq U$. If R has a unit element then $C = Z(S)$. If R is a simple ring with unit element then $Q = S = R$ i.e., in this case R is its own central closure. We refer to [69, 89] for more details.

1.4 Some Special Type of Additive Maps in Rings

Let R be a ring. A mapping $f : R \to R$ is said to be additive if $f(x+y) = f(x)+f(y)$ holds for all $x, y \in R$. Let $S \subseteq R$. A mapping $f : R \to R$ is called commuting (resp. centralizing) on S, if $[f(x), x] = 0$ for all $x \in S$ (resp. $[f(x), x] \in Z(R)$ for all $x \in S$).

Definition 1.4.1. *Let R be a ring. An additive mapping $d : R \to R$ is called a derivation of R if*

$$d(xy) = d(x)y + xd(y)$$

holds for all $x, y \in R$.

Example 1.4.1.

1. An example of derivation is the usual derivation d on the polynomial ring $R = F[x]$ given by

$$d(\sum_{i=0}^{t} a_i x^i) = \sum_{i=1}^{t} i a_i x^{i-1} = \sum_{i=0}^{t} i a_i x^{i-1}, \text{ where } F \text{ is a field.}$$

2. Let us consider the ring,

$$R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \ : \ x, y, z \in \mathbb{Z} \right\},$$

where $\mathbb{Z}$ is the set of all integers. Let us consider a mapping $d : R \to R$ by

$$d \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & ny \\ 0 & 0 \end{pmatrix} \ : \ x, y, z \in \mathbb{Z},$$

where n is any fixed integer. Then it is obvious that d is a derivation on R.

3. There is another fundamental class of derivations. For a fixed $a \in R$ the mapping $d_a : R \to R$ defined by $d_a(x) = [a, x]$ for all $x \in R$, is a derivation of R. This kind of derivations are called as inner derivations of R.

Definition 1.4.2. *Let R be a ring. A Jordan derivation of R is an additive mapping $d : R \to R$ such that $d(x^2) = d(x)x + xd(x)$ for every $x \in R$.*

Every derivation of a ring R is a Jordan derivation but the converse is not true, in general.

A classical result of Herstein [67] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. Also if R is a 2-torsion free semiprime ring, then it is proved by M. Brešar [15] that every Jordan derivation of R is a derivation of R.

The notion of derivation was extended by Brešar [16] in 1991. He first introduced the concept of generalized derivation which was further studied algebraically by Hvala [70] in 1998.

Definition 1.4.3. *Let R be a ring. An additive mapping $F : R \to R$ is called a generalized derivation, if there exists a derivation $d : R \to R$ such that*

$$F(xy) = F(x)y + xd(y)$$

holds for all $x, y \in R$.

Thus it is evident from the above definitions that every derivation is a generalized derivation of R. If $d = 0$, then generalized derivation becomes $F(xy) = F(x)y$ for all $x, y \in R$, which is called a left multiplier mapping of R. Some authors used the notion of left centralizer instead of left multiplier in a ring. Thus generalized derivation of a ring covers the concept of derivation as well as the concept of left multiplier mapping in a ring.

Example 1.4.2.

1. Let R be a ring. For $a, b \in R$, the map $x \to ax + xb$ of R is a generalized derivation. This map is called inner generalized derivation of R.

2. Identity map on R is a generalized derivation associated with zero derivation. But it is not a derivation of R.

3. Let us consider the ring R ,

$$R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \ : \ x, y \in \mathbb{Z} \right\},$$

where $\mathbb{Z}$ is the set of all integers. Let us consider a mapping $d : R \to R$ and $F : R \to R$ by $d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $F\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & x+y \\ 0 & 0 \end{pmatrix}$. Then F is a generalized derivation of R with the derivation d of R.

4. Addition of two generalized derivations of a ring R is again a generalized derivation of R.

In [37], Dhara and Ali introduced the notion of multiplicative (generalized)-derivation.

Definition 1.4.4. *A mapping $F : R \to R$ (not necessarily additive) is said to be multiplicative (generalized)-derivation, if $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$, where g is any mapping (not necessarily a derivation nor an additive map).*

Hence, the concept of multiplicative (generalized)-derivation covers the concept of generalized derivation.

Definition 1.4.5. *Let R be a ring and $b \in R$. An additive mapping $F : R \to R$ is called b-generalized derivation of R, if there exists a derivation $d : R \to R$ such that*

$$F(xy) = F(x)y + bxd(y)$$

holds for all $x, y \in R$.

From the above definition we can easily see that generalized derivation is a 1-generalized derivation and we can define a map $F : R \to R$ by $F(x) = ax + bxc$ for all $x \in R$, where $a, b, c \in R$ which is called inner b-generalized derivation of R.

We now recall the following definitions:

Definition 1.4.6. *Let R be a ring and α be an automorphism of R. An additive mapping $d : R \to R$ is called a skew derivation of R if*

$$d(xy) = d(x)y + \alpha(x)d(y)$$

holds for all $x, y \in R$.

Definition 1.4.7. *Let R be a ring and α be an automorphism of R. An additive mapping $F : R \to R$ is called a generalized skew derivation of R if there exist a skew derivation d of R with associated automorphism α such that*

$$F(xy) = F(x)y + \alpha(x)d(y)$$

holds for all $x, y \in R$. Here d is called an associated skew derivation of F and α is called an associated automorphism of F.

Definition 1.4.8. *Let R be a prime ring, $b \in R$ and α be an automorphism of R. An additive mapping $F : R \to R$ is called a b-generalized skew derivation of R, if there exists a skew derivation d of R such that*

$$F(xy) = F(x)y + b\alpha(x)d(y)$$

holds for all $x, y \in R$.

We note that b-generalized skew derivation generalizes the concept of generalized skew derivation as well as b-generalized derivation. The mapping $x \mapsto ax + b\alpha(x)c$ *is an example of b-generalized skew derivation of* R *which is called inner b-generalized skew derivation of* R*.*

Let α, β *be two automorphisms of* R*. An additive mapping* $d : R \to R$ *is said to be a* (α, β)*-derivation if*

$$d(xy) = d(x)\alpha(y) + \beta(x)d(y)$$

holds for all $x, y \in R$*. Obviously* $(1, 1)$*-derivation is simply a derivation, where* 1 *denotes the identity map. For some fixed* $a \in R$*, the mapping* $x \mapsto a\alpha(x) - \beta(x)a$ *is an example of* (α, β)*-derivation which is called inner* (α, β)*-derivation.*

Being inspired by the definition of (α, β)*-derivation, the notion of generalized derivation is also extended to generalized* (α, β)*-derivation as follows :*

Definition 1.4.9. *Let* R *be a ring and* α, β *be any two automorphisms of* R*. An additive mapping* $F : R \to R$ *is said to be a generalized* (α, β)*-derivation, if there exists a* (α, β)*-derivation* d *of* R *such that*

$$F(xy) = F(x)\alpha(y) + \beta(x)d(y)$$

holds for all $x, y \in R$*.*

Of course every generalized $(1, 1)$*-derivation of a ring* R *is a generalized derivation of* R*, where* 1 *means the identity mapping of* R*. If* $d = 0$*, we have* $F(xy) = F(x)\alpha(y)$ *for all* $x, y \in R$*, which is called a left* α*-multiplier mapping of* R*. Thus generalized* (α, β)*-derivation generalizes both the concepts,* (α, β)*-derivation as well as left* α*-multiplier mapping of* R*.*

More recently, Filippis and Wei [66] introduced a new mapping b-generalized (α, β)*-derivation as follows:*

Definition 1.4.10. *Let* $b \in R$*,* d *an additive mapping of* R *and* α, β *be two automorphisms of* R*. A linear mapping* $F : R \to R$ *is called a b-generalized* (α, β)*-derivation of* R *with an associated word* (b, α, β, d) *if*

$$F(xy) = F(x)\alpha(y) + b\beta(x)d(y)$$

holds for all $x, y \in R$*.*

This mapping is a common generalization of generalized skew derivation and generalized (α, β)-derivation. The mapping of the form $x \mapsto a\alpha(x) + b\beta(x)c$ for some $a, b, c \in R$ is an example of b-generalized (α, β)-derivation of R with associated word (b, α, β, d), where $d(x) = \beta(x)c - c\alpha(x)$ for all $x \in R$. Such b-generalized (α, β)-derivation is called inner b-generalized (α, β)-derivation.

Let R be a prime ring, $\alpha, \beta \in Aut(R)$, $0 \neq b \in R$, $d : R \to R$ be an additive mapping of R and F be the b-generalized (α, β)-derivations of R associated to the mapping d. Then d is an (α, β)-derivation of R.

We now recall the definition of an another type of mapping:

An additive mapping $F : R \to R$ is called a homomorphism or anti-homomorphism on R if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in R$ respectively. The additive mapping F is called a Jordan homomorphism, if $F(x^2) = F(x)^2$ holds for all $x \in R$.

It is natural to consider a map which behave like a derivation as well as a homomorphism. In this point of view, EI Sofy [88] introduced the concept of homoderivation maps on a ring R.

Definition 1.4.11. *An additive mapping H from R into itself is called homoderivation if*

$$H(xy) = H(x)H(y) + H(x)y + xH(y)$$

for all $x, y \in R$.

An example of such mapping is $H(x) = f(x) - x$ for all $x \in R$, where f is an endomorphism on R.

1.5 Generalized Polynomial Identity (GPI) in Rings

Suppose that R be an associative ring and let $X = \{x_1, x_2, \ldots\}$ be an infinite set of non-commutative indeterminates. The classical approach to the theory of polynomial identities of a ring R was to consider identical relations in R of the form $p[x] = 0$, where $p[x] = \sum \alpha_{(i)} x_{i_1} x_{i_2} \cdots x_{i_n}$ is a polynomial in the x_j with coefficients $\alpha_{(i)}$ which are integers or belong to a commutative field F over which R is an algebra. The main result in the theory of these identities is due to Kaplansky [71, p. 226] which *states that a primitive ring satisfying a polynomial identity of degree d is a finite-dimensional algebra over its center, and its dimension is* $\leq [d/2]^2$.

The generalized polynomial identities to be dealt with are of the form:

$$P[x] = \sum \alpha_{i_1}\pi_{j_1}\alpha_{i_2}\pi_{j_2}\cdots\alpha_{i_k}\pi_{j_k}\alpha_{i_{k+1}} = 0,$$

where π_j *are monomials in the indeterminates* x_j *and the elements* a_{i_λ} *appear both as coefficients and between the monomials* π_j. *More precisely, one consider a prime ring* R *and* $S = RC$, *its central closure. Consider* $S < x >= S *_C \{X\}$, *the free product of* S *and* $\{X\}$ *over* C. *The elements of* $S < x >$ *are called the* ***generalized polynomials***. *By a nontrivial generalized polynomial, we mean a nonzero element of* $S < x >$. *An element* $m \in S < x >$ *of the form* $m = q_0y_1q_1y_2q_2\ldots y_nq_n$, *where* $\{q_0, q_1, \ldots, q_n\} \subseteq S$ *and* $\{y_1, y_2, \ldots, y_n\} \subseteq X$, *is called a monomial (some of the* q_i *can be* 1 *also);* $q_0, q_1, \ldots, q_n$ *are called the coefficients of* m. *Each* $f \in S < x >$ *can be represented as a finite sum of monomials. Such a representation is certainly not unique.*

Let B *be a set of* C*-independent vectors of* S. *By a* B*-monomial, we mean a monomial of the form* $u_0y_1u_1y_2u_2\ldots y_nu_n$, *where* $\{u_0, \ldots, u_n\} \subseteq B$ *and* $\{y_1, y_2, \ldots, y_n\} \subseteq X$. *Let* $V = BC$, *the* C*-subspace spanned by* B. *Then any* V*-generalized polynomial* f *can be written in the form* $\sum \alpha_i m_i$, *where* $\alpha_i \in C$ *and* m_i *are* B*-monomials, in the following manner: First fix a representation of* f *with all of its coefficients in* V *and express each coefficient of the given representation as a linear combination of elements of* B. *Then substitute these linear combinations into the representation of* f *and expand the resulting expression using the distributive law. Finally, we collect similar terms to get our desired form.*

It is also obvious that such representation of a given f *in terms of* B*-monomials is unique. If* B *is chosen to be a basis of* S *over* C, *the* B*-monomials span the whole* $S < x >$.

The uniqueness of representation in terms of B*-monomials gives a practical criterion to decide whether a given generalized polynomial* f *is trivial or not: Pick a basis* B *for the* C*-subspace spanned by the coefficients of a given representation of* f. *Express* f *as a linear combination of* B*-monomials in the way explained above. Let us say* $f = \sum \alpha_i m_i$, *where* $\alpha_i \in C$ *and* m_i *are* B*-monomials. Then* f *is trivial if and only if* $\alpha_i = 0$ *for each* i. *This simple criterion will be used frequently in several* *chapters.*

Remark 1.5.1. *As a consequence, if we consider $T = U *_C C\{X\}$, the free product of U and the free algebra $C\{X\}$ over C. If $a_1, a_2 \in U$ are linearly independent over C and $a_1 g_1(x_1, \dots, x_n) + a_2 g_2(x_1, \dots, x_n) = 0 \in T$, where*

$$g_1(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i h_i(x_1, \dots, x_n)$$

and

$$g_2(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i k_i(x_1, \dots, x_n)$$

for $h_i(x_1, \dots, x_n), k_i(x_1, \dots, x_n) \in T$, then both $g_1(x_1, \dots, x_n)$ and $g_2(x_1, \dots, x_n)$ are zero element of T.

Definition 1.5.1. *S is said to satisfy generalized polynomial identity if there exists an $0 \neq f \in S < x >$ such that $f(s_1, s_2, \dots, s_n) = 0$ for all $s_i \in S$.*

Definition 1.5.2. *The polynomial with n variables*

$$s_n(x_1, \dots, x_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)},$$

where $(-1)^{\sigma}$ is $+1$ or -1 according as σ being an even or odd permutation in symmetric group S_n, is called the standard polynomial of degree n.

Theorem 1.5.1. *(**Amitsur-Levitzki Theorem**): Let R be a commutative ring. Then $M_n(R)$ satisfies s_{2n}.*

Theorem 1.5.2. *[24, Theorem 2] Let R be a prime ring with its Utumi ring of quotients U. For any dense submodule M of U, the GPIs satisfied by M are the same as the GPIs satisfied by U.*

Theorem 1.5.3. *[24, Theorem 3] Let R be a prime ring with its Utumi ring of quotients U. Let M and N be two dense submodules of U. If M satisfies a GPI, then M satisfies a GPI of N.*

Let R be a prime ring with its Utumi ring of quotients U. Let $Der(U)$ be the set of all derivations of U. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 \dots d_n$ with each $d_i \in Der(U)$.

A differential polynomial is a generalized polynomial of the form $\Phi(\Delta_j(x_i))$ involving non-commutative indeterminates x_i which are acted by derivation words Δ_j as uniary operation and with coefficients from U. Then $\Phi(\Delta_j(x_i))$ is said to be differential identity on $S \subseteq U$, if $\Phi(\Delta_j(x_i))$ assumes the constant value 0 for any assignment of values from S to its indeterminates x_i.

Theorem 1.5.4. *[78, Theorem 3] Let R be a semiprime ring, U its Utumi ring of quotients and I_R a dense R-submodule of U_R. Then I and U satisfy the same differential identities.*

1.6 Some Important Theorems

Theorem 1.6.1. *[14, Proposition 2.5.1] Every derivation of a prime ring R can be uniquely extended to a derivation of the Utumi ring of quotients U.*

Theorem 1.6.2. *[79, Theorem 3] Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + \delta(x)$ for some $a \in U$ and a derivation δ on U.*

Theorem 1.6.3. *[62, Lemma 1.5] Suppose that $A_1, \ldots, A_k$ are non scalar matrices in $M_n(C)$, where $n \geq 2$ and C is infinite field. Then there exists an invertible matrix $P \in M_n(C)$ such that any matrices $PA_1P^{-1}, \ldots, PA_kP^{-1}$ have all nonzero entries.*

Theorem 1.6.4. *[75, Theorem 2] (**Kharchenko's Theorem**):*
Let R be a prime ring, U be its Utumi ring of quotients and I a ideal of R. Let $\Phi(\Delta_j(x_i)) = 0$ be a reduced differential identity for I. Then $\Phi(z_{ij}) = 0$ is GPI for U, where z_{ij} are distinct indeterminates.

In particular, we have the following result:
If d is a nonzero outer derivation of R and $\Phi(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)) = 0$ is a differential identity on R, then U satisfies GPI $\Phi(x_1, \ldots, x_n, z_1, \ldots, z_n) = 0$, where $x_1, \ldots, x_n, z_1, \ldots, z_n$ are distinct indeterminates.

Theorem 1.6.5. *[71] (**Jacobson Density Theorem**):*
Let R be a (left) primitive ring with R^V a faithful irreducible R-module and $D = End(R^V)$. Then for any natural number n, if $v_1, \ldots, v_n$ are D-independent in V and

$w_1, \ldots, w_n$ are arbitrary in V, then there exists $r \in R$ such that $rv_i = w_i, for i = 1, \ldots, n$.

Theorem 1.6.6. *[89, Theorem 3] (**Martindale Theorem**):*
Let R be a prime ring with its extended centroid C. Then $S = RC$ satisfies a GPI over C if and only if S contains a minimal right ideal eS(hence S is primitive) and eSe is a finite dimensional division algebra over C, where e is idempotent.

Theorem 1.6.7. *[14, 71] (**Litoff's Theorem**):*
Let R be a primitive ring with nonzero socle $H = Soc(R)$ and $b_1, \ldots, b_m \in H$. Then there exists an idempotent $e \in H$ such that $b_1, \ldots, b_m \in eRe$ and the ring eRe is isomorphic to $M_n(C)$.

Chapter 2

Generalized Derivations and Generalization of Co-commuting Maps in Prime Rings

2.1 Introduction

Throughout this Chapter R denotes a prime ring with center $Z(R)$, extended centroid C and U its Utumi quotient ring. This Chapter is devoted to study of generalized derivations in prime rings. More specifically in this Chapter, we investigate an identity containing three generalized derivations which generalize some of existing results in this literature. Let $S \subseteq R$. An additive map $F : R \to R$ is said to be commuting (centralizing) on S if $[F(x), x] = 0$ for all $x \in S$ (resp. $[F(x), x] \in Z(R)$ for all $x \in S$). Two additive maps $F, G : R \to R$ are said to be co-commuting (co-centralizing) on S if $F(x)x - xG(x) = 0$ for all $x \in S$ (resp. $F(x)x - xG(x) \in Z(R)$ for all $x \in S$).

In [62], De Filippis and De Vincenzo described the structure of additive mappings d and G satisfying $d(G(f(X))f(X) - f(X)G(f(X))) = 0$ for all $X = (x_1, \dots, x_n) \in R^n$, where f is a multilinear polynomial over extended centroid C and d is a nonzero derivation and G is a nonzero generalized derivation on prime ring R of char $(R) \neq 2$.

In [39], Dhara, Argac and Albas extended the above result by considering two generalized derivations. More precisely, they studied the situation $d(F(f(X))f(X) - f(X)G(f(X))) = 0$ for all $X = (x_1, \dots, x_n) \in R^n$, where f is a

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multilinear polynomial over extended centroid C and d is a nonzero derivation and F, G are two generalized derivations on prime ring R of char $(R) \neq 2$. In this paper authors determined all possible forms of the additive maps d, F and G.

On the other hand, Carini and De Filippis [18] proved that if R is a prime ring of characteristic different from 2, δ a nonzero derivation of R, G a nonzero generalized derivation of R, and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C such that $\delta(G(f(X))f(X)) = 0$ for all $X = (x_1, \ldots, x_n) \in R^n$, then there exist $a, b \in U$ such that $G(x) = ax$ and $\delta(x) = [b, x]$ for all $x \in R$, with $[b, a] = 0$ and $f(x_1, \ldots, x_n)^2$ is central-valued on R.

Further, Dhara and Argac [38] extended the above result replacing derivation δ with another generalized derivation F, that is, $F(G(f(X))f(X)) = 0$ for all $X = (x_1, \ldots, x_n) \in I^n$, and then gave the complete description of the additive maps F and G, where I is a non-zero two-sided ideal of R.

In another paper [10], Argaç and De Filippis studied the generalized derivations G and H co-commuting on $f(I) = \{f(x_1, \ldots, x_n) | x_i \in I\}$, that is, $G(u)u - uH(u) = 0$ for all $u \in f(I)$ and then obtained all possible forms of the maps F and G, where I is a non-zero two-sided ideal of R.

Motivated by the above results, in this Chapter we prove the following theorem.

Main Theorem. Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F, G and H are three nonzero generalized derivations of R such that

$$F\Big(G(f(X))f(X)\Big) = f(X)H(f(X))$$

for all $X = (x_1, \ldots, x_n) \in R^n$. Then one of the following holds:

1. there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = \lambda x$, $G(x) = xa$ and $H(x) = \lambda ax$ for all $x \in R$;

2. there exist $\lambda \in C$ and $p, q \in U$ such that $F(x) = \lambda x$, $G(x) = px + xq$ and $H(x) = \lambda(qx + xp)$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R;

3. there exist $\lambda \in C$ and $a, p \in U$ such that $F(x) = ax$, $G(x) = px$ and $H(x) = \lambda x$ for all $x \in R$ with $ap = \lambda$;

4. there exist $\lambda \in C$ and $a \in U$ such that $F(x) = xa$, $G(x) = \lambda x$ and $H(x) = \lambda xa$ for all $x \in R$;

5. there exist $a, b, p, v \in U$ such that $F(x) = ax + xb$, $G(x) = px$ and $H(x) = xv$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $ap + pb = v$.

In particular, when $G = H$, we have the following:

Corollary 1. Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F and G are two nonzero generalized derivations of R such that

$$F\Big(G(f(X))f(X)\Big) = f(X)G(f(X))$$

for all $X = (x_1, \ldots, x_n) \in R^n$. Then one of the following holds:

1. there exists $\mu \in C$ such that $F(x) = x$ and $G(x) = \mu x$ for all $x \in R$;

2. there exist $\alpha \in C$ and $p \in U$ such that $F(x) = x$ and $G(x) = px + xp + \alpha x$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R;

3. there exists $p \in U$ such that $F(x) = -x$ and $G(x) = [p, x]$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R;

4. there exist $a, b, p \in U$ such that $F(x) = ax + xb$ and $G(x) = px$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $F(p) = p$.

Proof. By Main Theorem, we have the following conclusions:

1. *there exists $\mu \in C$ such that $F(x) = x$ and $G(x) = \mu x$ for all $x \in R$. This is our conclusion (1).*

2. *there exist $\lambda, \alpha \in C$ and $p, q \in U$ such that $F(x) = \lambda x$, $G(x) = px + xq$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R and $q - \lambda p = \lambda q - p = \alpha \in C$. The last relation yields $(\lambda - 1)(p + q) = 0$. This yields either $\lambda = 1$ or*

$p+q=0$. (i) When $\lambda=1$, we have $q-p=\alpha\in C$ and hence $F(x)=x$ and $G(x)=px+xp+\alpha x$ for all $x\in R$. This gives conclusion (2). (ii) When $p+q=0$, we have $F(x)=\lambda x$, $G(x)=[p,x]$ for all $x\in R$ with $p+\lambda p=\alpha\in C$, i.e., $(1+\lambda)p\in C$. This implies $1+\lambda=0$, since $p\in C$ implies $G=0$, a contradiction. Thus $\lambda=-1$. This gives conclusion (3).

3. *there exist $\lambda\in C$ and $a\in U$ such that $F(x)=ax$, $G(x)=\lambda x$ for all $x\in R$ with $a\lambda=\lambda$. Since $G\neq 0$, $\lambda\neq 0$ and hence last relation gives $a=1$. This is conclusion (1).*

4. *there exist $\lambda\in C$ and $a\in U$ such that $F(x)=xa$, $G(x)=\lambda x$ for all $x\in R$ with $a\lambda=\lambda$. By the same argument as above, $a=1$, as desired in (1).*

5. *there exist $a,b,p\in U$ such that $F(x)=ax+xb$, $G(x)=px$ for all $x\in R$ with $f(x_1,\ldots,x_n)^2$ is central valued on R and $ap+pb=p$. This is conclusion (4).*

From conclusion (2) of Main Theorem, we conclude that when G is derivation then H also be a derivation. Thus following corollary is straightforward.

Corollary 2. Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F and H are two nonzero generalized derivations of R and d is a derivation of R such that

$$F\Big(d(f(X))f(X)\Big)=f(X)H(f(X))$$

for all $X=(x_1,\ldots,x_n)\in R^n$. Then there exist $\lambda\in C$ and $p,u\in U$ such that $F(x)=\lambda x$, $d(x)=[p,x]$ and $H(x)=-\lambda[p,x]$ for all $x\in R$ with $f(x_1,\ldots,x_n)^2$ is central valued in R.

Corollary 3. Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F and G are two nonzero generalized derivations of R and d is a derivation of R such that

$$F\Big(G(f(X))f(X)\Big)=f(X)d(f(X))$$

for all $X = (x_1, \ldots, x_n) \in R^n$. Then there exist $\lambda \in C$ and $p \in U$ such that $F(x) = \lambda x$, $G(x) = [p, x]$ and $d(x) = -\lambda[p, x]$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R.

In particular, when F is derivation, then we have last conclusion of Main theorem.

Corollary 4. Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that G and H are two nonzero generalized derivations of R and d is a nonzero derivation of R such that

$$d\Big(G(f(X))f(X)\Big) = f(X)H(f(X))$$

for all $X = (x_1, \ldots, x_n) \in R^n$. Then there exist $a, p, v \in U$ such that $d(x) = [a, x]$, $G(x) = px$ and $H(x) = xv$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $d(p) = v$.

Corollary 5. Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C. Suppose that d, δ and h are three nonzero derivations of R such that

$$d\Big(\delta(f(X))f(X)\Big) = f(X)h(f(X))$$

for all $X = (x_1, \ldots, x_n) \in R^n$. Then $f(x_1, \ldots, x_n)$ must be central valued.

Corollary 6. Let R be a prime ring of characteristic different from 2. Suppose that d, δ and h are three nonzero derivations of R such that

$$d\Big(\delta(x)x\Big) = xh(x)$$

for all $x \in R$. Then R must be commutative.

2.2 Main Results

Let $F(\neq 0)$, $G(\neq 0)$ and $H(\neq 0)$ be all inner generalized derivations of R. There exists some fixed $a, b, p, q, u, v \in U$ such that $F(x) = ax + xb$, $G(x) = px + xq$ and

$H(x) = ux + xv$ for all $x \in R$. Then by our hypothesis $F(G(x)x) = xH(x))$ for all $x \in f(R)$, we have

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0 \qquad (2.2.1)$$

for all $x \in f(R)$.

To investigate this generalized polynomial identity (GPI) in prime ring R, we recall the following:

Lemma 2.2.1. *[10, Lemma 3] Let R be a noncommutative prime ring with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that there exist $a, b, c, q \in U$ such that $(af(x)+f(x)b)f(x)-f(x)(cf(x)+f(x)q) = 0$ for all $x = (x_1, \ldots, x_n) \in R^n$. Then one of the following holds:*

(1) $a, q \in C$ and $q - a = b - c = \alpha \in C$;

(2) $f(x_1, \ldots, x_n)^2$ is central valued on R and there exists $\alpha \in C$ such that $q-a = b - c = \alpha$;

(3) char $(R) = 2$ and R satisfies s_4.

Now to investigate our generalized polynomial identity (GPI) (2.2.1), in all that follows, we assume R a noncommutative prime ring with extended centroid C, char $(R) \neq 2$. Moreover, we assume that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C which is not central valued on R.

Lemma 2.2.2. *If $a, b \in C$ and R satisfies (2.2.1), then one of the following holds:*

(1) $p, v \in C$ with $(a + b)(p + q) = u + v$;

(2) $f(x_1, \ldots, x_n)^2$ is central valued in R with $v-(a+b)p = (a+b)q-u = \alpha \in C$.

Proof. If $a, b \in C$, then by hypothesis

$$(a + b)(px + xq)x = x(ux + xv)$$

for all $x \in f(R)$. In this case by Lemma 2.2.1, one of the following holds:

(i) $(a+b)p, v \in C$ and $v-(a+b)p = (a+b)q-u = \alpha \in C$; Since $F \neq 0$, $a+b \neq 0$ and so $(a + b)p \in C$ implies $p \in C$;

(ii) $f(x_1, \ldots, x_n)^2$ is central valued on R and there exists $\alpha \in C$ such that $v - (a + b)p = (a + b)q - u = \alpha \in C$. □

Lemma 2.2.3. *If $q \in C$ and R satisfies (2.2.1), then one of the following holds:*

(1) $b, u, v \in C$ with $(a+b)(p+q) = v+u$;

(2) $a, p, u \in C$ with $(a+b)(p+q) = v+u$;

(3) $u \in C$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $a(p+q)+(p+q)b = u+v$.

Proof. If $q \in C$, then our hypothesis becomes

$$a(p+q)x^2 + px^2b + x^2(bq-v) - xux = 0$$

for all $x \in f(R)$. Then by Proposition 2.7 in [42], we conclude that $u \in C$. Then our hypothesis reduces to

$$a(p+q)x^2 + px^2b + x^2(bq-v-u) = 0$$

for all $x \in f(R)$. Then by applying Lemma 2.9 in [36], we conclude one of the following:

(i) $b, u, bq-v-u \in C$ with $a(p+q)+pb+(bq-v-u) = 0$ i.e., $(a+b)(p+q) = v+u$. Since $b, q, u \in C$, we have $v \in C$.

(ii) $a(p+q), u, p \in C$ with $a(p+q)+pb+(bq-v-u) = 0$ i.e., $(a+b)(p+q) = v+u$; In this case $G(x) = (p+q)x$ for all $x \in R$. As $G \neq 0$, thus $0 \neq p+q \in C$. Hence $a(p+q) \in C$ implies $a \in C$.

(iii) $u \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on R with $a(p+q)+pb+(bq-v-u) = 0$ i.e., $a(p+q)+(p+q)b = u+v$.

Thus the Lemma is proved. □

Lemma 2.2.4. *Let R be a prime ring with extended centroid C and $a, b, p, q, u, v \in R$. If*

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

for all $x \in f(R)$ is a trivial generalized polynomial identity, then either $a, b \in C$ or $q \in C$.

Proof. Let $a \notin C$ and $q \notin C$. By hypothesis, we have

$$\begin{aligned}\zeta(x_1, \ldots, x_n) &= apf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \\ &+pf(x_1, \ldots, x_n)^2b + f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n)b \\ &-f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2v = 0 \qquad (2.2.2)\end{aligned}$$

for all $x_1, \ldots, x_n \in R$. Since R and U satisfy the same generalized polynomial identity (see [24]), U satisfies $\zeta(x_1, \ldots, x_n) = 0$. By our assumption $\zeta(x_1, \ldots, x_n)$ is a trivial GPI for U. Let $T = U *_C C\{x_1, x_2, \ldots, x_n\}$, the free product of U and $C\{x_1, \ldots, x_n\}$, the free C-algebra in noncommuting indeterminates $x_1, x_2, \ldots, x_n$. Then, $\zeta(x_1, \ldots, x_n)$ is zero element in $T = U *_C C\{x_1, \ldots, x_n\}$. This implies that $\{ap, a, p, 1\}$ is linearly C-dependent. Then there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ such that $\alpha_1 ap + \alpha_2 a + \alpha_3 p + \alpha_4.1 = 0$. If $\alpha_1 = \alpha_3 = 0$, then $\alpha_2 \neq 0$ and so $a = -\alpha_2^{-1}\alpha_4 \in C$, a contradiction. Therefore, either $\alpha_1 \neq 0$ or $\alpha_3 \neq 0$. Without loss of generality, we assume that $\alpha_1 \neq 0$. Then $ap = \alpha a + \beta p + \gamma$, where $\alpha = -\alpha_1^{-1}\alpha_2, \beta = -\alpha_1^{-1}\alpha_3, \gamma = -\alpha_1^{-1}\alpha_4$. Then

$$\begin{aligned}&(\alpha a + \beta p + \gamma) f(x_1, \ldots, x_n)^2 + a f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n)\\&+p f(x_1, \ldots, x_n)^2 b + f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) b\\&-f(x_1, \ldots, x_n) u f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2 v = 0 \qquad (2.2.3)\end{aligned}$$

in T. This implies that $\{a, p, 1\}$ is linearly C-dependent. Then there exist $\beta_1, \beta_2, \beta_3 \in C$ such that $\beta_1 a + \beta_2 p + \beta_3 = 0$. By same argument as before, since $a \notin C$, we have $\beta_2 \neq 0$ and hence $p = \alpha' a + \beta'$ for some $\alpha', \beta' \in C$. Thus our identity becomes

$$\begin{aligned}&(\alpha a + \beta\alpha' a + \beta\beta' + \gamma) f(x_1, \ldots, x_n)^2 + a f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n)\\&+(\alpha' a + \beta') f(x_1, \ldots, x_n)^2 b + f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) b\\&-f(x_1, \ldots, x_n) u f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2 v = 0. \qquad (2.2.4)\end{aligned}$$

Since $\{a, 1\}$ is linearly C-independent, we have satisfies

$$\begin{aligned}&(\alpha + \beta\alpha') a f(x_1, \ldots, x_n)^2 + a f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n)\\&+\alpha' a f(x_1, \ldots, x_n)^2 b = 0, \qquad (2.2.5)\end{aligned}$$

that is

$$\begin{aligned}&a f(x_1, \ldots, x_n)\Big((\alpha + \beta\alpha' + q) f(x_1, \ldots, x_n) + \alpha' f(x_1, \ldots, x_n) b\Big)\\&= 0 \qquad (2.2.6)\end{aligned}$$

in T. Moreover, since $q \notin C$, the term $a f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n)$ can not be canceled and hence $a f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) = 0$ in T which implies $q = 0$, a contradiction. Thus either $a \in C$ or $q \in C$.

Similarly, we can prove that either $b \in C$ or $q \in C$. $\square$

Proposition 2.2.5. *Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the infinite field C and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C. If there exist $a, b, p, q, u, v \in R$ such that*

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

for all $x \in f(R)$, then either $a, b \in C.I_m$ or $q \in C.I_m$.

Proof. By our hypothesis, R satisfies the generalized polynomial identity

$$\begin{aligned} &apf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \\ &+pf(x_1, \ldots, x_n)^2b + f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n)b \\ &-f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2v = 0. \end{aligned} \tag{2.2.7}$$

We assume first that $a \notin Z(R)$ and $q \notin Z(R)$. Now we shall show that this case leads to a contradiction.
Since $a \notin Z(R)$ and $q \notin Z(R)$, by Theorem 1.6.3 there exists a C-automorphism ϕ of $M_m(C)$ such that $\phi(a)$, $\phi(q)$ have all non-zero entries. Clearly, R satisfies the generalized polynomial identity

$$\begin{aligned} &\phi(ap)f(x_1, \ldots, x_n)^2 + \phi(a)f(x_1, \ldots, x_n)\phi(q)f(x_1, \ldots, x_n) \\ &+\phi(p)f(x_1, \ldots, x_n)^2\phi(b) + f(x_1, \ldots, x_n)\phi(q)f(x_1, \ldots, x_n)b \\ &-f(x_1, \ldots, x_n)\phi(u)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2\phi(v) = 0. \end{aligned} \tag{2.2.8}$$

By e_{ij}, we mean the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. Since $f(x_1, \ldots, x_n)$ is not central valued, by [78] (see also [82]), there exist a sequence of matrices $v_1, \ldots, v_n$ in $M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(v_1, \ldots, v_n) = \gamma e_{pq}$, with $p \neq q$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n$ in $M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$. Hence by (2.2.8), we have

$$\phi(a)e_{ij}\phi(q)e_{ij} + e_{ij}\phi(q)e_{ij}b - e_{ij}\phi(u)e_{ij} = 0 \tag{2.2.9}$$

and then left multiplying by e_{ij}, it follows $e_{ij}\phi(a)e_{ij}\phi(q)e_{ij} = 0$, which is a contradiction, since $\phi(a)$ and $\phi(q)$ have all non-zero entries. Thus we conclude that either $a \in Z(R)$ or $q \in Z(R)$.

If we consider $b \notin Z(R)$ and $q \notin Z(R)$, then by same argument as above we have a contradiction with the fact $e_{ij}\phi(q)e_{ij}\phi(b)e_{ij} = 0$ obtained from (2.2.9). Thus we conclude either $b \in Z(R)$ or $q \in Z(R)$.

Thus, $q \notin Z(R)$ implies $a \in Z(R)$ and $b \in Z(R)$. Thus the conclusion follows.

□

Proposition 2.2.6. *Let $R = M_m(C)$ be the ring of all matrices over the field C with char $(R) \neq 2$ and $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C. If there exist $a, b, p, q, u, v \in R$ such that*

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

for all $x \in f(R)$, then either $a, b \in C.I_m$ or $q \in C.I_m$.

Proof. In case C is infinite, the conclusions follow by Proposition 2.2.5.
So we assume that C is finite. Let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \dots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned} &\Psi(x_1, \dots, x_n) = \\ &apf(x_1, \dots, x_n)^2 + af(x_1, \dots, x_n)qf(x_1, \dots, x_n) \\ &+pf(x_1, \dots, x_n)^2b + f(x_1, \dots, x_n)qf(x_1, \dots, x_n)b \\ &-f(x_1, \dots, x_n)uf(x_1, \dots, x_n) - f(x_1, \dots, x_n)^2 \end{aligned} \tag{2.2.10}$$

which is a generalized polynomial identity for R.
Moreover, it is a multi-homogeneous of multi-degree $(2, \dots, 2)$ in the indeterminates $x_1, \dots, x_n$.
Hence the complete linearization of $\Psi(x_1, \dots, x_n)$ yields a multilinear generalized polynomial $\Theta(x_1, \dots, x_n, y_1, \dots, y_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1, \dots, x_n, x_1, \dots, x_n) = 2^n\Psi(x_1, \dots, x_n).$$

Clearly the multilinear polynomial $\Theta(x_1, \dots, x_n, y_1, \dots, y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $char(C) \neq 2$ we obtain $\Psi(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in \overline{R}$ and then conclusion follows from Proposition 2.2.5. □

In particular, we have the following:

Corollary 2.2.7. *Let $R = M_m(C)$ be the ring of all matrices over the field C with char $(R) \neq 2$. If there exist $a, b, p, q, u, v \in R$ such that*

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0, \tag{2.2.11}$$

for all $x \in R$, then either $a, b \in C.I_m$ or $q \in C.I_m$.

Similarly, we have the following:

Corollary 2.2.8. *Let $R = M_m(C)$ be the ring of all matrices over the field C with char $(R) \neq 2$. If there exist $a', a, b, p, q, u, v \in R$ such that*

$$a'x^2 + axqx + px^2b + xqxb - xux - x^2v = 0, \tag{2.2.12}$$

for all $x \in R$, then either $a, b \in C.I_m$ or $q \in C.I_m$.

Lemma 2.2.9. *Let R be a primitive ring of char $(R) \neq 2$ with nonzero socle $Soc(R)$, which is isomorphic to a dense ring of linear transformations of a vector space V over C, such that $dim_C V = \infty$. Let $a', a, b, p, q, u, v \in R$. If*

$$a'x^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

for all $x \in R$, then either $a, b \in C$ or $q \in C$.

Proof. Recall that if any element $r \in R$ commutes the nonzero ideal $Soc(RC)$, i.e., $[r, Soc(RC)] = (0)$, then $r \in C$. Hence on contrary, we assume that there exist $h_0, h_1, h_2 \in Soc(R)$ such that

(i) either $[a, h_0] \neq 0$ or $[b, h_1] \neq 0$;

(ii) $[q, h_2] \neq 0$

and prove that a number of contradiction arises. Since V is infinite dimensional over C, for any $e = e^2 \in Soc(R)$, we have $eRe \cong M_k(C)$ with $k = dim_C Ve$. By Litoff's theorem (see Theorem 1.6.7), there exists an idempotent $e \in Soc(R)$ such that

- $h_0, h_1, h_2 \in eRe$;
- $h_0a, ah_0, h_1a, ah_1, h_2a, ah_2 \in eRe$;
- $h_0b, bh_0, h_1b, bh_1, h_2b, bh_2 \in eRe$;

- $h_0q, qh_0, h_1q, qh_1, h_2q, qh_2 \in eRe$.

where $eRe \cong M_k(C)$, $k = dim_C Ve$. Since R satisfies $e\{a'(exe)^2 + aexeqexe + p(exe)^2b + exeqexeb - exeuexe - (exe)^2v\}e = 0$, the subring eRe satisfies $ea'ex^2 + eaexeqex + epex^2ebe + xeqexebe - xeuex - x^2eve = 0$. By Corollary 2.2.8, we conclude that one of the following holds:

(i) $eae, ebe \in eC$ which contradicts with the choice of h_0 and h_1;

(ii) $eqe \in eC$ which contradicts with the choices of h_2. □

Lemma 2.2.10. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \dots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F, G and H are three nonzero inner generalized derivations of R such that $F(G(f(r))f(r)) = f(r)H(f(r))$ for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:*

1. *there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = \lambda x$, $G(x) = xa$ and $H(x) = bx$ for all $x \in R$ with $\lambda a = b$;*

2. *there exist $\lambda, \alpha \in C$ and $p, q, u, v \in U$ such that $F(x) = \lambda x$, $G(x) = px + xq$ and $H(x) = ux + xv$ for all $x \in R$ with $f(x_1, \dots, x_n)^2$ is central valued in R and $v - \lambda p = \lambda q - u = \alpha \in C$;*

3. *there exist $\lambda \in C$ and $a, p \in U$ such that $F(x) = ax$, $G(x) = px$ and $H(x) = \lambda x$ for all $x \in R$ with $ap = \lambda$;*

4. *there exist $\lambda \in C$ and $a, u \in U$ such that $F(x) = xa$, $G(x) = \lambda x$ and $H(x) = xu$ for all $x \in R$ with $a\lambda = u$;*

5. *there exist $a, b, p, v \in U$ such that $F(x) = ax + xb$, $G(x) = px$ and $H(x) = xv$ for all $x \in R$ with $f(x_1, \dots, x_n)^2$ is central valued on R and $ap + pb = v$.*

Proof. Suppose that for some $a, b, p, q, u, v \in U$, $F(x) = ax + xb$, $G(x) = px + xq$ and $H(x) = ux + xv$ for all $x \in R$. By hypothesis, we have

$$\begin{aligned} & a\Big((pf(x_1, \dots, x_n) + f(x_1, \dots, x_n)q)f(x_1, \dots, x_n)\Big) \\ & + \Big((pf(x_1, \dots, x_n) + f(x_1, \dots, x_n)q)f(x_1, \dots, x_n)\Big)b \\ & = f(x_1, \dots, x_n)(uf(x_1, \dots, x_n) + f(x_1, \dots, x_n)v), \end{aligned} \tag{2.2.13}$$

that is

$$\begin{aligned}&apf(x_1,\dots,x_n)^2 + af(x_1,\dots,x_n)qf(x_1,\dots,x_n)\\&+pf(x_1,\dots,x_n)^2b + f(x_1,\dots,x_n)qf(x_1,\dots,x_n)b\\&-f(x_1,\dots,x_n)uf(x_1,\dots,x_n) - f(x_1,\dots,x_n)^2v = 0\end{aligned}\qquad (2.2.14)$$

for all $x_1,\dots,x_n \in R$. Since R and U satisfy the same generalized polynomial identities (see [24]), therefore, U satisfies

$$\begin{aligned}&apf(x_1,\dots,x_n)^2 + af(x_1,\dots,x_n)qf(x_1,\dots,x_n)\\&+pf(x_1,\dots,x_n)^2b + f(x_1,\dots,x_n)qf(x_1,\dots,x_n)b\\&-f(x_1,\dots,x_n)uf(x_1,\dots,x_n) - f(x_1,\dots,x_n)^2v = 0.\end{aligned}\qquad (2.2.15)$$

If this is a trivial generalized polynomial identity for U, then by Lemma 2.2.4, either $a,b \in C$ or $q \in C$.

Next we assume that (2.2.15) is a non-trivial GPI for U.

Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [54, Theorems 2.5 and 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according as C finite or infinite. Then R is centrally closed over C and R satisfies (2.2.15). By Martindale's theorem (see Theorem 1.6.6), R is then a primitive ring with nonzero socle $soc(R)$ and with C as its associated division ring. Then, by Jacobson's theorem (see Theorem 1.6.5), R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1,\dots,r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Proposition 2.2.6, we get that $a,b \in C$ or $q \in C$. If V is infinite dimensional over C, then by Lemma 2.2.9, we conclude that either $a,b \in C$ or $q \in C$.

Thus up to now, we have proved that in any cases either $a,b \in C$ or $q \in C$.

Case 1: *When $a,b \in C$.*

In this case by Lemma 2.2.2, we have the following cases:

(i) $p,v \in C$ with $(a+b)(p+q) = u+v$; Thus $F(x) = ax+xb = (a+b)x$, $G(x) = px+xq = x(p+q)$ and $H(x) = ux+xv = (u+v)x$ for all $x \in R$. This is our conclusion (1).

(ii) $f(x_1,\dots,x_n)^2$ is central valued in R with $v-(a+b)p = (a+b)q-u = \alpha \in C$.

Thus $F(x) = ax + xb = (a+b)x$, $G(x) = px + xq$ and $H(x) = ux + xv$ for all $x \in R$. This is our conclusion (2).

Case 2: *When $q \in C$.*

In this case by Lemma 2.2.3, we have the following cases:

(i) $b, q, u, v \in C$ with $(a+b)(p+q) = v + u = \lambda \in C$. Thus $F(x) = (a+b)x$, $G(x) = (p+q)x$ and $H(x) = (u+v)x$ for all $x \in R$. This is our conclusion (3).

(ii) $a, u, p, q \in C$ with $(a+b)(p+q) = v+u$. Thus $F(x) = x(a+b)$, $G(x) = (p+q)x$ and $H(x) = x(u+v)$ for all $x \in R$. This is our conclusion (4).

(iii) $q, u \in C$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $a(p+q) + (p+q)b = u + v$. Thus $F(x) = ax + xb$, $G(x) = (p+q)x$ and $H(x) = x(u+v)$ for all $x \in R$. This is our conclusion (5). □

In particular we have

Corollary 2.2.11. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F is a nonzero inner generalized derivation of R such that $F([p, f(r)]f(r)) = f(r)[q, f(r)]$ for all $r = (r_1, \ldots, r_n) \in R^n$, then there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R and $\lambda p + q \in C$.*

Corollary 2.2.12. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F is a nonzero inner generalized derivation of R such that $F([p, f(r)]f(r)) = f(r)[p, f(r)]$ for all $r = (r_1, \ldots, r_n) \in R^n$, then there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R and $(\lambda + 1)p \in C$.*

Lemma 2.2.13. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that G and H are two generalized derivations of R and $F(x) = cx + xc'$ for all $x \in R$, for some $c, c' \in U$ is a nonzero inner generalized derivation of R, such that $F(G(f(x))f(x)) = f(x)H(f(x))$ for all $x = (x_1, \ldots, x_n) \in R^n$, then one of the following holds:*

1. *there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = \lambda x$, $G(x) = xa$ and $H(x) = bx$ for all $x \in R$ with $\lambda a = b$;*

2. *there exist $\lambda, \alpha \in C$ and $p, q, u, v \in U$ such that $F(x) = \lambda x$, $G(x) = px + xq$ and $H(x) = ux + xv$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued in R and $v - \lambda p = \lambda q - u = \alpha \in C$;*

3. *there exist $\lambda \in C$ and $a, p \in U$ such that $F(x) = ax$, $G(x) = px$ and $H(x) = \lambda x$ for all $x \in R$ with $ap = \lambda$;*

4. *there exist $\lambda \in C$ and $a, u \in U$ such that $F(x) = xa$, $G(x) = \lambda x$ and $H(x) = xu$ for all $x \in R$ with $a\lambda = u$;*

5. *there exist $a, b, p, v \in U$ such that $F(x) = ax + xb$, $G(x) = px$ and $H(x) = xv$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $ap + pb = v$.*

Proof. In view of [79, Theorem 3], we may assume that there exist $a, b \in U$ and derivations d', δ of U such that $G(x) = ax + d'(x)$ and $H(x) = bx + \delta(x)$. Since R and U satisfy the same generalized polynomial identities (see [24]) as well as the same differential identities (see [78]), we may assume that

$$\begin{aligned} c\Big\{af(x)^2 + d'(f(x))f(x)\Big\} + \Big\{af(x)^2 + d'(f(x))f(x)\Big\}c' \\ = f(x)bf(x) + f(x)\delta(f(x)) \end{aligned} \tag{2.2.16}$$

for all $x = (x_1, \ldots, x_n) \in U^n$, where d', δ are two derivations on U.

If G and H both are inner generalized derivations of R, then by Lemma 2.2.10 we obtain our conclusions (1) - (5). Thus we assume that not both of F and G are inner. Then d' and δ can not be both inner derivations of U. Now we consider the following two cases:

Case-I: Assume that d' and δ are C-dependent modulo inner derivations of U, say $\alpha d' + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in R$.

Subcase-i: *Let* $\alpha \neq 0$.

Then $d'(x) = \lambda\delta(x) + [p, x]$ for all $x \in U$, for some $\lambda \in C$ and $p \in U$.

Then δ can not be inner derivation of U. From (2.2.16), we obtain

$$c\left\{af(x)^2+\lambda\delta(f(x))f(x)+[p,f(x)]f(x)\right\}+\left\{af(x)^2+\lambda\delta(f(x))f(x)[p,f(x)]f(x)\right\}c'$$
$$=f(x)bf(x)+f(x)\delta(f(x)) \qquad (2.2.17)$$

for all $x=(x_1,\dots,x_n)\in U^n$.

As $f(x_1,\dots,x_n)$ is a multilinear polynomial over C, we have $\delta(f(x_1,\dots,x_n))$ $=f^\delta(x_1,\dots,x_n)+\sum_i f(x_1,\dots,\delta(x_i),\dots,x_n)$, where $f^\delta(x_1,\dots,x_n)$ be the polynomials obtained from $f(x_1,\dots,x_n)$ replacing each coefficients α_σ with $\delta(\alpha_\sigma)$. Thus by Kharchenko's theorem (see Theorem 1.6.4), we can replace $\delta(f(x_1,\dots,x_n))$ by $f^\delta(x_1,\dots,x_n)+\sum_i f(x_1,\dots,y_i,\dots,x_n)$ in (2.2.17) and then U satisfies blended components

$$c\left\{\lambda\sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)\right\}+\left\{\lambda\sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)\right\}c'$$
$$=f(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n). \qquad (2.2.18)$$

Replacing y_i with $[q,y_i]$ for some $q\notin C$ in (2.2.18), we obtain

$$c\lambda[q,f(x)]f(x)+[q,f(x)]f(x)\lambda c'=f(x)[q,f(x)].$$

By Corollary 2.2.12, $f(x_1,\dots,x_n)^2$ is central valued in R with $c\lambda, c'\lambda\in C$ and $(\lambda(c+c')+1)q\in C$. Since $q\notin C$, $(\lambda(c+c')+1)q\in C$ implies $(\lambda(c+c')+1)=0$ i.e., $\lambda(c+c')=-1$. Then by (2.2.18)

$$(c+c')\lambda\sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)$$
$$=f(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n)$$

which implies

$$f(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n)+\sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)=0.$$

In particular, for $y_1=x_1$ and $y_2=\cdots=y_n=0$, we have $2f(x_1,\dots,x_n)^2=0$ for all $x_1,\dots,x_n\in U$, implying $f(x_1,\dots,x_n)=0$ for all $x_1,\dots,x_n\in U$, a contradiction.

Subcase-ii: *Let* $\alpha=0$.

Then $\delta(x)=[q',x]$ for all $x\in U$, where $q'=\beta^{-1}q$. Since δ is inner, d' can not be inner derivation. From (2.2.16), we obtain

$$c\left\{af(x)^2+d'(f(x))f(x)\right\}+\left\{af(x)^2+d'(f(x))f(x)\right\}c'$$
$$=f(x)bf(x)+f(x)[q',f(x)] \qquad (2.2.19)$$

for all $x = (x_1, \ldots, x_n) \in U^n$.
Since $d'(f(x_1, \ldots, x_n)) = f^{d'}(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d'(x_i), \ldots, x_n)$, by Kharchenko's theorem (see Theorem 1.6.4), we can replace $d'(f(x_1, \ldots, x_n))$ by $f^{d'}(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)$ in (2.2.19) and then U satisfies blended component

$$c\sum_i f(x_1, \ldots, y_i, \ldots, x_n)f(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)f(x_1, \ldots, x_n)c' = 0. \quad (2.2.20)$$

Replacing y_i with $[a', x_i]$ for some $a' \notin C$, U satisfies

$$c[a', f(x_1, \ldots, x_n)]f(x_1, \ldots, x_n) + [a', f(x_1, \ldots, x_n)]f(x_1, \ldots, x_n)c' = 0. \quad (2.2.21)$$

Then by Corollary 2.2.11, $f(x_1, \ldots, x_n)^2$ is central valued in R with $c, c' \in C$ and $(c + c')a' \in C$. Since $a' \notin C$, $c + c' = 0$ implying $F = 0$, a contradiction.

Case-II: Assume next that d' and δ are C-independent modulo inner derivations of U. Then applying Kharchenko's theorem (see Theorem 1.6.4), we have from (2.2.16) that U satisfies blended components

$$c\sum_i f(x_1, \ldots, y_i, \ldots, x_n)f(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)f(x_1, \ldots, x_n)c' = f(x_1, \ldots, x_n)\sum_i f(x_1, \ldots, z_i, \ldots, x_n). \quad (2.2.22)$$

In particular, for $y_1 = \cdots = y_n = 0$, U satisfies $f(x_1, \ldots, x_n)\sum_i f(x_1, \ldots, z_i, \ldots, x_n) = 0$. In particular, $f(x_1, \ldots, x_n)^2 = 0$ for all $x_1, \ldots, x_n \in U$, implying $f(x_1, \ldots, x_n) = 0$, a contradiction. □

Lemma 2.2.14. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F and H are two generalized derivations of R and $G(x) = cx + xc'$ for all $x \in R$, for some $c, c' \in U$ is a nonzero inner generalized derivation of R, such that $F(G(f(x))f(x)) = f(x)H(f(x))$ for all $x = (x_1, \ldots, x_n) \in R^n$, then one of the following holds:*

1. *there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = \lambda x$, $G(x) = xa$ and $H(x) = bx$ for all $x \in R$ with $\lambda a = b$;*

2. *there exist* $\lambda, \alpha \in C$ *and* $p, q, u, v \in U$ *such that* $F(x) = \lambda x$, $G(x) = px + xq$ *and* $H(x) = ux + xv$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued in* R *and* $v - \lambda p = \lambda q - u = \alpha \in C$;

3. *there exist* $\lambda \in C$ *and* $a, p \in U$ *such that* $F(x) = ax$, $G(x) = px$ *and* $H(x) = \lambda x$ *for all* $x \in R$ *with* $ap = \lambda$;

4. *there exist* $\lambda \in C$ *and* $a, u \in U$ *such that* $F(x) = xa$, $G(x) = \lambda x$ *and* $H(x) = xu$ *for all* $x \in R$ *with* $a\lambda = u$;

5. *there exist* $a, b, p, v \in U$ *such that* $F(x) = ax + xb$, $G(x) = px$ *and* $H(x) = xv$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and* $ap + pb = v$.

Proof. In view of [79, Theorem 3], we may assume that there exist $a, b \in U$ and derivations d, δ of U such that $F(x) = ax + d(x)$ and $H(x) = bx + \delta(x)$. Since R and U satisfy the same generalized polynomial identities (see [24]) as well as the same differential identities (see [78]), we may assume that

$$a\{cf(x)^2 + f(x)c'f(x)\} + d\{cf(x)^2 + f(x)c'f(x)\} \\ = f(x)bf(x) + f(x)\delta(f(x)) \tag{2.2.23}$$

for all $x = (x_1, \ldots, x_n) \in U^n$, where d, δ are two derivations on U.

If F and H both are inner generalized derivations of R, then by Lemma 2.2.10 we obtain our conclusions (1) - (5). Thus we assume that not both of F and H are inner. Then d and δ can not be both inner derivations of U. Now we consider the following two cases:

Case-I: Assume that d and δ are C-dependent modulo inner derivations of U, say $\alpha d + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in R$. If $\beta = 0$, then $\alpha \neq 0$ and thus d is inner. In this case conclusion follows by Lemma 2.2.13. Next we assume that $\beta \neq 0$. Then there exist some $\lambda \in C$ and $p \in U$ such that $\delta(x) = \lambda d(x) + [p, x]$ for all $x \in U$. Then by (2.2.23), U satisfies

$$a\{cf(x)^2 + f(x)c'f(x)\} + d(c)f(x)^2 + cd(f(x))f(x) + cf(x)d(f(x)) \\ +d(f(x))c'f(x) + f(x)d(c')f(x) + f(x)c'd(f(x)) \\ = f(x)bf(x) + f(x)\lambda d(f(x)) + f(x)[p, f(x)]. \tag{2.2.24}$$

Since $f(x_1,\ldots,x_n)$ is a multilinear polynomial over C, we have $d(f(x_1,\ldots,x_n)) = f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,d(x_i),\ldots,x_n)$, where $f^d(x_1,\ldots,x_n)$ be the polynomials obtained from $f(x_1,\ldots,x_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$. Thus by Kharchenko's theorem (see Theorem 1.6.4), we can replace $d(f(x_1,\ldots,x_n))$ by $f^d(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)$ in (2.2.24) and then U satisfies blended components

$$\begin{aligned} c\sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n) + cf(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) \\ +\sum_i f(x_1,\ldots,y_i,\ldots,x_n)c'f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)c'\sum_i f(x_1,\ldots,y_i,\ldots,x_n) \\ = f(x_1,\ldots,x_n)\lambda\sum_i f(x_1,\ldots,y_i,\ldots,x_n) \end{aligned} \tag{2.2.25}$$

In particular, for $y_1 = x_1$ and $y_2 = \cdots = y_n = 0$, U satisfies

$$(2c-\lambda)f(x_1,\ldots,x_n)^2 + f(x_1,\ldots,x_n)(2c')f(x_1,\ldots,x_n) = 0, \tag{2.2.26}$$

which implies

$$\Big((2c-\lambda)f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)(2c')\Big)f(x_1,\ldots,x_n) = 0. \tag{2.2.27}$$

By Lemma 2.2.1, we conclude that $2c' = \lambda - 2c \in C$. Since char $(R) \neq 2$, $c, c' \in C$. Then by (2.2.25), U satisfies

$$\begin{aligned} (c+c')\sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n) \\ +(c+c'-\lambda)f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) = 0. \end{aligned} \tag{2.2.28}$$

Replacing y_i with $[q', x_i]$ for some $q' \notin C$, we have

$$\begin{aligned} (c+c')[q', f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n) \\ +(c+c'-\lambda)f(x_1,\ldots,x_n)[q', f(x_1,\ldots,x_n)] = 0, \end{aligned} \tag{2.2.29}$$

that is

$$\begin{aligned} [(c+c')q', f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n) \\ +f(x_1,\ldots,x_n)[(c+c'-\lambda)q', f(x_1,\ldots,x_n)] = 0. \end{aligned} \tag{2.2.30}$$

By Lemma 2.2.1, one of the following holds: (i) $(c+c')q', (c+c'-\lambda)q' \in C$; in this case as $q' \notin C$, $c+c' = 0$, implying $G = 0$, a contradiction. (ii) $f(x_1,\ldots,x_n)^2$ is central valued and $(c+c'-\lambda)q'-(c+c')q' \in C$ i.e., $\lambda q' \in C$. In this case as $q' \notin C$, $\lambda = 0$. Thus $\lambda = 2(c+c') = 0$ implying $c+c' = 0$. Hence $G = 0$, a contradiction.

Case-II: Assume that d and δ are C-independent modulo inner derivations of U.

Then applying Kharchenko's theorem (see Theorem 1.6.4) to (2.2.23), we can replace

$$d(f(x_1,\ldots,x_n)) = f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)$$

and

$$\delta(f(x_1,\ldots,x_n)) = f^\delta(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,t_i,\ldots,x_n).$$

Then U satisfies blended component $f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,t_i,\ldots,x_n) = 0$. In particular, $f(x_1,\ldots,x_n)^2 = 0$ implying $f(x_1,\ldots,x_n) = 0$, a contradiction. □

Lemma 2.2.15. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that F and G are two generalized derivations of R and $H(x) = bx + xb'$ for all $x \in R$, for some $b, b' \in U$ is a nonzero inner generalized derivation of R, such that $F(G(f(x))f(x)) = f(x)H(f(x))$ for all $x = (x_1,\ldots,x_n) \in R^n$, then one of the following holds:*

1. *there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = \lambda x$, $G(x) = xa$ and $H(x) = bx$ for all $x \in R$ with $\lambda a = b$;*
2. *there exist $\lambda, \alpha \in C$ and $p, q, u, v \in U$ such that $F(x) = \lambda x$, $G(x) = px + xq$ and $H(x) = ux + xv$ for all $x \in R$ with $f(x_1,\ldots,x_n)^2$ is central valued in R and $v - \lambda p = \lambda q - u = \alpha \in C$;*
3. *there exist $\lambda \in C$ and $a, p \in U$ such that $F(x) = ax$, $G(x) = px$ and $H(x) = \lambda x$ for all $x \in R$ with $ap = \lambda$;*
4. *there exist $\lambda \in C$ and $a, u \in U$ such that $F(x) = xa$, $G(x) = \lambda x$ and $H(x) = xu$ for all $x \in R$ with $a\lambda = u$;*
5. *there exist $a, b, p, v \in U$ such that $F(x) = ax + xb$, $G(x) = px$ and $H(x) = xv$ for all $x \in R$ with $f(x_1,\ldots,x_n)^2$ is central valued on R and $ap + pb = v$.*

Proof. In view of [79, Theorem 3], we may assume that there exist $a, b \in U$ and derivations d, δ of U such that $F(x) = cx + d(x)$ and $G(x) = ax + d'(x)$. Since R and U satisfy the same generalized polynomial identities (see [24]) as well as the same differential identities (see [78]), we may assume that

$$c\{af(x)^2 + d'(f(x))f(x)\} + d\{af(x)^2 + d'(f(x))f(x)\} \\ = f(x)bf(x) + f(x)^2b' \tag{2.2.31}$$

for all $x = (x_1, \ldots, x_n) \in U^n$, where d, d' are two derivations on U.

If d or d' is inner, then F or G is inner and then by Lemma 2.2.13 and 2.2.14, we obtain our conclusions (1) - (5). Thus we assume that both of d and d' are outer. Now we consider the following two cases:

Case-I: Assume that d and d' are C-dependent modulo inner derivations of U, then $d = \alpha d' + ad_{p'}$. Then (2.2.31) becomes

$$c\Big\{af(x)^2 + d'(f(x))f(x)\Big\} \\ +\alpha d'\Big\{af(x)^2 + d'(f(x))f(x)\Big\} + [p', af(x)^2 + d'(f(x))f(x)] \\ = f(x)bf(x) + f(x)^2b'. \tag{2.2.32}$$

We know that $d'(f(x_1, \ldots, x_n)) = f^{d'}(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d'(x_i), \ldots, x_n)$, and

$$d'^2(f(x_1, \ldots, x_n)) = f^{d'^2}(x_1, \ldots, x_n) + 2\sum_i f^{d'}(x_1, \ldots, d'(x_i), \ldots, x_n) \\ +\sum_i f(x_1, \ldots, d'^2(x_i), \ldots, x_n) + \sum_{i\neq j} f(x_1, \ldots, d'(x_i), \ldots, d'(x_j), \ldots, x_n).$$

By applying Kharchenko's theorem (see Theorem 1.6.4), we can replace $d(f(x_1, \ldots, x_n))$ with $f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)$ and $d'^2(f(x_1, \ldots, x_n))$ with

$$f^{d'^2}(x_1, \ldots, x_n) + 2\sum_i f^{d'}(x_1, \ldots, y_i, \ldots, x_n) \\ +\sum_i f(x_1, \ldots, t_i, \ldots, x_n) + \sum_{i\neq j} f(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n)$$

in (2.2.32) and then U satisfies blended component

$$\alpha \sum_i f(x_1, \ldots, t_i, \ldots, x_n)f(x_1, \ldots, x_n) = 0. \tag{2.2.33}$$

This implies $\alpha f(x_1,\ldots,x_n)^2 = 0$, implying $\alpha = 0$. Then d is inner, a contradiction.
Case-II: Assume that d and d' are C-independent modulo inner derivations of U.

Then applying Kharchenko's theorem (see Theorem 1.6.4) to (2.2.31), we can replace

$$d'(f(x_1,\ldots,x_n)) = f^{d'}(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n),$$

$$d(f(x_1,\ldots,x_n)) = f^{d}(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,t_i,\ldots,x_n),$$

and

$$\begin{aligned} dd'(f(x_1,\ldots,x_n)) &= f^{dd'}(x_1,\ldots,x_n) + \sum_i f^{\delta}(x_1,\ldots,y_i,\ldots,x_n) \\ &+ \sum_i f^{d'}(x_1,\ldots,t_i,\ldots,x_n) + \sum_{i\neq j} f(x_1,\ldots,y_i,\ldots,t_j\ldots,r_n) \\ &+ \sum_i f(x_1,\ldots,w_i',\ldots,x_n). \end{aligned}$$

Then U satisfies blended component $\sum_i f(x_1,\ldots,w_i',\ldots,x_n)f(x_1,\ldots,x_n) = 0$. In particular, $f(x_1,\ldots,x_n)^2 = 0$ implying $f(x_1,\ldots,x_n) = 0$, a contradiction. □

Proof of Main Theorem. *If F or G or H any one of them be inner, then conclusion follows by Lemma 2.2.13, Lemma 2.2.14, Lemma 2.2.15.*

Thus we assume that F, G and H all are outer generalized derivations of R. Then by [79], we have $F(x) = cx + d(x)$, $G(x) = ax + d'(x)$ and $H(x) = bx + \delta(x)$ for some $a,b,c \in U$ and d,d',δ are three derivations of U. By hypothesis, we have

$$\begin{aligned} &c\{af(x)^2 + d'(f(x))f(x)\} \\ &+d\{af(x)^2 + d'(f(x)\}f(x)) = f(x)bf(x) + f(x)\delta(f(x)) \end{aligned} \tag{2.2.34}$$

for all $x = (x_1,\ldots,x_n) \in U^n$. Now we consider the following two cases:

Case-1: Let d' and δ be C-dependent modulo inner derivations of U, i.e., $\alpha d' + \beta\delta = ad_{p'}$.
Now $\alpha = 0$ implies that δ is inner, a contradiction as H can not be inner. Thus $\alpha \neq 0$.

Then $d' = \lambda\delta + ad_p$, where $\lambda = -\beta\alpha^{-1} \in C$ and $p = p'\alpha^{-1} \in U$. Therefore, (2.2.34) gives

$$\begin{aligned} &c\{af(x)^2 + \lambda\delta(f(x))f(x) + [p,f(x)]f(x)\} \\ &+d\Big(af(x)^2 + \lambda\delta(f(x))f(x) + [p,f(x)]f(x)\Big) \\ &= f(x)bf(x) + f(x)\delta(f(x)) \end{aligned} \tag{2.2.35}$$

for all $x = (x_1, \ldots, x_n) \in U^n$*, that is*

$$c\Big(af(x)^2 + \lambda\delta(f(x))f(x) + [p, f(x)]f(x)\Big) \\ +d\Big(af(x)^2 + [p, f(x)]f(x)\Big) + d(\lambda)\delta(f(x))f(x) + \lambda(d\delta)(f(x))f(x) \\ +\lambda\delta(f(x))d(f(x)) = f(x)bf(x) + f(x)\delta(f(x)) \quad (2.2.36)$$

for all $x = (x_1, \ldots, x_n) \in U^n$*. We know that*

$$d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)$$

and

$$\delta d(f(x_1, \ldots, x_n)) = f^{\delta d}(x_1, \ldots, x_n) \\ + \sum_i f^d(x_1, \ldots, \delta(x_i), \ldots, x_n) + \sum_i f^\delta(x_1, \ldots, d(x_i), \ldots, x_n) \\ + \sum_i f(x_1, \ldots, \delta d(x_i), \ldots, x_n) + \sum_i f(x_1, \ldots, \delta(x_i), \ldots, d(x_j), \ldots, x_n).$$

Let δ *and* d *be* C*-independent modulo inner derivations of* U*. By applying Kharchenko's theorem (see Theorem 1.6.4) to (2.2.36), we can replace* $d(f(x_1, \ldots, x_n))$ *with* $f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)$ *and* $\delta d(f(x_1, \ldots, x_n))$ *with*

$$f^{\delta d}(x_1, \ldots, x_n) \\ + \sum_i f^d(x_1, \ldots, s_i, \ldots, x_n) + \sum_i f^\delta(x_1, \ldots, y_i, \ldots, x_n) \\ + \sum_i f(x_1, \ldots, t_i, \ldots, x_n) + \sum_i f(x_1, \ldots, s_i, \ldots, y_j, \ldots, x_n)$$

in (2.2.36) and then U *satisfies blended component*

$$\lambda \sum_i f(x_1, \ldots, t_i, \ldots, x_n) f(x_1, \ldots, x_n) = 0. \quad (2.2.37)$$

In particular, for $t_1 = x_1$ *and* $t_2 = \cdots = t_n = 0$ *in (2.2.37), we have* $\lambda f(x_1, \ldots, x_n)^2 = 0$*. If* $\lambda \neq 0$*, then* $f(x_1, \ldots, x_n)^2 = 0$ *which implies* $f(x_1, \ldots, x_n) = 0$ *for all* $x_1, \ldots, x_n \in U$ *(see [26]), a contradiction. Thus* $\lambda = 0$*. In this case* G *becomes inner, a contradiction.*

Now let δ *and* d *be* C*-dependent, i.e.,* $\alpha_1\delta + \beta_1 d = ad_{q'}$*. Now,* $\alpha_1 = 0$*, implies* d *is inner, a contradiction. Thus* $\alpha_1 \neq 0$ *and so* $\delta = \mu d + [q, x]$ *for some* $\mu \in C$ *and*

$q \in U$. Then by (2.2.36) U satisfies

$$c\Big(af(x)^2 + \lambda\mu d(f(x))f(x) + \lambda[q, f(x)]f(x) + [p, f(x)]f(x)\Big)$$
$$+d\Big(af(x)^2 + [p, f(x)]f(x)\Big) + d(\lambda)\mu d(f(x))f(x) + d(\lambda)[q, f(x)]f(x)$$
$$+\lambda d\Big(\mu d(f(x)) + [q, f(x)]\Big)f(x) + \lambda\Big(\mu d(f(x)) + [q, f(x)]\Big)d(f(x))$$
$$= f(x)bf(x) + f(x)\Big(\mu d(f(x)) + [q, f(x)]\Big) \tag{2.2.38}$$

for all $x = (x_1, \ldots, x_n) \in U^n$.

Since $d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n)$ and

$$d^2(f(x_1, \ldots, x_n)) = f^{d^2}(x_1, \ldots, x_n) + 2\sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n)$$
$$+\sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n) + \sum_{i \neq j} f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n),$$

by applying Kharchenko's theorem (see Theorem 1.6.4), we can replace $d(f(x_1, \ldots, x_n))$ with $f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)$ and $d^2(f(x_1, \ldots, x_n))$ with

$$d^2(f(x_1, \ldots, x_n)) = f^{d^2}(x_1, \ldots, x_n) + 2\sum_i f^d(x_1, \ldots, y_i, \ldots, x_n)$$
$$+\sum_i f(x_1, \ldots, t_i, \ldots, x_n) + \sum_{i \neq j} f(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n),$$

and then U satisfies blended component

$$\lambda\mu \sum_i f(x_1, \ldots, t_i, \ldots, x_n)f(x_1, \ldots, x_n) = 0. \tag{2.2.39}$$

In particular, $\lambda\mu f(x_1, \ldots, x_n)^2 = 0$. This implies $\lambda\mu = 0$ and so either $\lambda = 0$ or $\mu = 0$. Now $\lambda = 0$ gives G is inner, a contradiction. Again $\mu = 0$, gives H is inner, a contradiction.

Case-2: Let d' and δ be C-independent modulo inner derivations of U. We divide the proof into two subcases.

Subcase-i. Let d, d' and δ be C-independent modulo inner derivations of U. In this case we rewrite (2.2.34) as

$$c(af(x)^2 + d'(f(x))f(x))$$
$$+d(a)f(x)^2 + ad(f(x))f(x) + af(x)d(f(x)) + dd'(f(x))f(x) + d'(f(x))d(f(x))$$
$$= f(x)bf(x) + f(x)\delta(f(x)) \tag{2.2.40}$$

for all $x = (x_1, \ldots, x_n) \in U^n$.

By applying Kharchenko's theorem (see Theorem 1.6.4), we can replace $dd'(f(x_1, \ldots, x_n))$ *by*

$$f^{dd'}(x_1, \ldots, x_n) + \sum_i f^{d'}(x_1, \ldots, r_i, \ldots, x_n) + \sum_i f^d(x_1, \ldots, t_i, \ldots, x_n)$$
$$+ \sum_{i \neq j} f(x_1, \ldots, t_i, \ldots, r_j \ldots, x_n) + \sum_i f(x_1, \ldots, w_i, \ldots, x_n)$$

in above equality and then U *satisfies the blended component*

$$\sum_i f(x_1, \ldots, w_i, \ldots, x_n) f(x_1, \ldots, x_n) = 0. \tag{2.2.41}$$

In particular for $w_1 = x_1$ *and* $w_2 = \cdots = w_n = 0$, U *satisfies* $f(x_1, \ldots, x_n)^2 = 0$ *implying* $f(x_1, \ldots, x_n) = 0$, *a contradiction.*

Subcase-ii. Let d, d' *and* δ *be* C*-dependent modulo inner derivations of* U *i.e.,* $\alpha_1 d + \alpha_2 d' + \alpha_3 \delta = ad_{a'}$ *for some* $\alpha_1, \alpha_2, \alpha_3 \in C$. *Then* $\alpha_1 \neq 0$, *otherwise* d' *and* δ *be* C*-dependent modulo inner derivation of* U, *a contradiction. Then we can write* $d = \beta_1 d' + \beta_2 \delta + ad_{a''}$ *for some* $\beta_1, \beta_2 \in C$ *and* $a'' \in U$. *Then by (2.2.34), we have*

$$c\{af(x)^2 + d'(f(x))f(x)\} + \beta_1 d'\{af(x)^2 + d'(f(x))f(x)\}$$
$$+\beta_2 \delta\{af(x)^2 + d'(f(x))f(x)\} + [a'', af(x)^2 + d'(f(x))f(x)]$$
$$= f(x)bf(x) + f(x)\delta(f(x)) \tag{2.2.42}$$

for all $x = (x_1, \ldots, x_n) \in U^n$.

Using Kharchenko's theorem (see Theorem 1.6.4), we substitute the following values in (2.2.42)

$$d'(f(x_1, \ldots, x_n)) = f^{d'}(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n),$$
$$\delta(f(x_1, \ldots, x_n)) = f^{\delta}(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, t_i, \ldots, x_n),$$
$$\delta d'(f(x_1, \ldots, x_n)) = f^{\delta d'}(x_1, \ldots, x_n) + \sum_i f^{\delta}(x_1, \ldots, y_i, \ldots, x_n)$$
$$+ \sum_i f^{d'}(x_1, \ldots, t_i, \ldots, x_n) + \sum_{i \neq j} f(x_1, \ldots, y_i, \ldots, t_j \ldots, x_n)$$
$$+ \sum_i f(x_1, \ldots, w'_i, \ldots, x_n),$$
$$d'^2(f(x_1, \ldots, x_n)) = f^{d'^2}(x_1, \ldots, x_n) + 2\sum_i f^{d'}(x_1, \ldots, y_i, \ldots, x_n)$$
$$+ \sum_i f(x_1, \ldots, z'_i, \ldots, x_n) + \sum_{i \neq j} f(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n).$$

Therefore, U satisfies the blended components

$$\beta_1 \sum_i f(x_1, \ldots, z_i', \ldots, x_n) f(x_1, \ldots, x_n) = 0.$$

and

$$\beta_2 \sum_i f(x_1, \ldots, w_i', \ldots, x_n) f(x_1, \ldots, x_n) = 0.$$

If $\beta_1 \neq 0$, then from above, U satisfies

$$\sum_i f(x_1, \ldots, z_i', \ldots, x_n) f(x_1, \ldots, x_n) = 0.$$

This is same as (2.2.41) and hence by same argument as above, it leads to a contradiction. Thus we conclude that $\beta_1 = 0$. Similarly, from above relation, we conclude that $\beta_2 = 0$. Then d is inner, contradicting with the fact that F is outer. This completes the proof of the theorem. □

Chapter 3

Jordan Homoderivation Behaviour of Generalized Derivations in Prime Rings

3.1 Introduction

In this Chapter we consider that R is a prime ring of characteristic different from 2*. Also U is the Utumi quotient ring of R, $C = Z(U)$ is the extended centroid of R and $f(x_1, \ldots, x_n)$ is a noncentral multilinear polynomial over C.*

It is very clear that every homoderivation is Jordan homoderivation, but in general the converse need not be true. There are few papers in literature which studied the homoderivation maps in prime rings and obtained commutativity of ring under certain conditions (see [1], [3], [4], [85]).

In the spirit of consideration of above maps, in the present Chapter we consider three generalized derivations F, G, H which satisfy the situation

$$F(X^2) = G(X)^2 + H(X)X + XH(X)$$

for all $X \in f(R)$.

There are many papers in literature which studied the homomorphism or anti-homomorphism behaviour of generalized derivations in prime rings (see [2], [7], [34], [36], [53], [58], [97]).

In the present Chapter, we study the Jordan homoderivation behaviour of three generalized derivations in prime rings.

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More precisely, we prove the following:

Theorem 3.1.1. *Let R be a prime ring with char $(R) \neq 2$ and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over $C(= Z(U))$ where U be the Utumi quotient ring of R. If F, G and H are three generalized derivations with associated derivations d, g and h respectively on R satisfying*

$$F(X^2) = G(X)^2 + H(X)X + XH(X)$$

for all $X \in f(R)$, then d, g and h are three inner derivations or g is inner, d, h are outer, d, h are linearly C-dependent modulo inner derivations of U. Moreover, the forms of the maps are as follows:

1. *there exist a derivation d' on R and $\lambda_1, \lambda_2, \lambda_3 \in C$ such that $F(x) = \lambda_1 x + d'(x)$, $G(x) = \lambda_2 x$ and $H(x) = \lambda_3 x + d'(x)$ for all $x \in R$ with $\lambda_1 = \lambda_2^2 + 2\lambda_3$;*

2. *there exist a derivation d' on R, $a_1 \in U$ and $\lambda_1, \lambda_2, \lambda_3 \in C$ such that $F(x) = \lambda_1 x + d'(x)$, $G(x) = \lambda_2 x$ and $H(x) = \lambda_3 x + [a_1, x] + d'(x)$ for all $x \in R$ with $f(R)^2 \in C$ and $\lambda_1 = \lambda_2^2 + 2\lambda_3$.*

If we consider $H = 0$ in our main theorem, then we get the following Corollary.

Corollary 3.1.2. *Let R be a prime ring of char $(R) \neq 2$ with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is a non-central valued on R. If F and G are two nonzero generalized derivations of R such that*

$$F(f(x)^2) = G(f(x))^2$$

for all $x = (x_1, \ldots, x_n) \in R^n$, then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda^2 x$, $G(x) = \lambda x$ for all $x \in R$;*

2. *there exist $\lambda \in C$ and $a \in U$ such that $F(x) = \lambda^2 x + [a, x], G(x) = \lambda x$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R.*

Example: *Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ be the set of all integers. Then it is clear that R is not prime ring, because $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0)$. Define* *the maps* $F, G, d, g : R \to R$ *by* $F\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$ *and* $G\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}$, $g\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ *on* R. *Then* F *and* G *are generalized derivations of* R *associated derivations* d *and* g *respectively. Now we consider a multilinear polynomial* $f(X, Y) = XY$, *which is not central valued on* R. *We see that* $F\big(f(X, Y)^2\big) = G\big(f(X, Y)\big)^2$ *for all* $X, Y \in R$ *but* $G(X) \neq \lambda X$ *where* $\lambda \in C$. *This example show that the primeness hypothesis is not superfluous in our above Corollary.*

3.2 The matrix ring case and results for inner generalized derivations

Lemma 3.2.1. *Suppose* $R = M_m(C)$, $m \geq 2$ *be the ring of all* $m \times m$ *matrices over the field* C *and* $f(R)$ *be the set of all evaluations of the polynomial* $f(x_1, \ldots, x_n)$ *in* R. *If* $A_1, A_2, A_3, a_3, a_4 \in R$ *such that*

$$A_1u^2 + u^2A_2 + a_3ua_3u + uA_3u + ua_4ua_4 + a_3u^2a_4 = 0$$

for all $u \in f(R)$, *then* a_3 *and* a_4 *are scalar matrices.*

Proof. **When C is infinite field.**

To prove this case of the Lemma, we assume on contrary that $a_3 \notin Z(R)$ and $a_4 \notin Z(R)$. We prove that this case leads to a contradiction. Since $a_3 \notin Z(R)$ and $a_4 \notin Z(R)$, by Theorem 1.6.3 there exists a C-automorphism θ of $M_m(C)$ for which $a_3' = \theta(a_3)$ and $a_4' = \theta(a_4)$ have all nonzero entries. Clearly $A_1' = \theta(A_1)$, $A_2' = \theta(A_2)$, $A_3' = \theta(A_3)$, a_3' and a_4' must satisfying the condition

$$\begin{aligned} A_1'f(x)^2 + f(x)^2A_2' + a_3'f(x)a_3'f(x) + f(x)A_3'f(x) \\ + f(x)a_4'f(x)a_4' + a_3'f(x)^2a_4' = 0. \end{aligned} \tag{3.2.1}$$

By e_{ij}, we consider a matrix unit with 1 in (i, j)-entry and zero elsewhere. Then it is obvious that $e_{ij}^2 = 0$. Since $f(x_1, \ldots, x_n)$ is not central valued, by [78] there exist some matrices $r_1, \ldots, r_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$, with $i \neq j$.

Therefore substituting $f(x) = \gamma e_{ij}$ in (3.2.1), we get

$$a_3'e_{ij}a_3'e_{ij} + e_{ij}A_3'e_{ij} + e_{ij}a_4'e_{ij}a_4' = 0. \tag{3.2.2}$$

Multiplying by e_{ij} in left side of the above relation, we have

$$e_{ij}a_3'e_{ij}a_3'e_{ij} = 0$$

which gives a contradiction, since $a_3' = \theta(a_3)$ have all nonzero entries. Thus we conclude that a_3 is scalar matrix. Again we multiplying by e_{ij} in right side of the above relation, we have

$$e_{ij}a_4'e_{ij}a_4'e_{ij} = 0$$

which gives a contradiction, since $a_4' = \theta(a_4)$ have all nonzero entries. Thus we conclude that a_4 is scalar matrix. Therefore when C is infinite field then a_3 and a_4 are scalar matrices.

When C is a finite field.

Suppose K be an infinite field which is an extension of C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Now the generalized polynomial identity is

$$\begin{gathered}\Psi(x_1, \ldots, x_n) = \\ A_1 f(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)^2 A_2 + a_3 f(x_1, \ldots, x_n) a_3 f(x_1, \ldots, x_n) \\ + f(x_1, \ldots, x_n) A_3 f(x_1, \ldots, x_n) \\ + f(x_1, \ldots, x_n) a_4 f(x_1, \ldots, x_n) a_4 + a_3 f(x_1, \ldots, x_n)^2 a_4 = 0. \end{gathered} \tag{3.2.3}$$

This is not only a generalized polynomial identity for R, but also a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $x_1, \ldots, x_n$.

Hence the complete linearization of $\Psi(x_1, \ldots, x_n)$ yields a multilinear generalized polynomial $\Theta(x_1, \ldots, x_n, t_1, \ldots, t_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1, \ldots, x_n, t_1, \ldots, t_n) = 2^n \Psi(x_1, \ldots, x_n).$$

Clearly the multilinear polynomial $\Theta(x_1, \ldots, x_n, t_1, \ldots, t_n) = 0$ is a generalized polynomial identity for R and $\overline{R}$ too. Since char $(C) \neq 2$ we obtain $\Psi(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in \overline{R}$ and then conclusion follows as above when C was infinite. □

Lemma 3.2.2. *Suppose that R is a prime ring of char $(R) \neq 2$ and $f(x_1, \ldots, x_n)$ is a non-central multilinear polynomial over C. If $A_1, A_2, A_3, a_3, a_4 \in R$ such that R satisfies $\Psi(x_1, \ldots, x_n)$ then $a_3 \in C$ and $a_4 \in C$.*

Proof. We will show this case by contradiction. Suppose both a_3 and a_4 are not central. By hypothesis, we have

$$\begin{aligned}\Psi(x_1,\ldots,x_n) = \\ A_1f(x_1,\ldots,x_n)^2 + f(x_1,\ldots,x_n)^2A_2 + a_3f(x_1,\ldots,x_n)a_3f(x_1,\ldots,x_n) \\ +f(x_1,\ldots,x_n)A_3f(x_1,\ldots,x_n) \\ +f(x_1,\ldots,x_n)a_4f(x_1,\ldots,x_n)a_4 + a_3f(x_1,\ldots,x_n)^2a_4 = 0 \qquad (3.2.4)\end{aligned}$$

for all $x_1,\ldots,x_n \in R$.

Since R and U satisfy the same generalized polynomial identities (see [24]) so, U satisfies $\Psi(x_1,\ldots,x_n) = 0$. Suppose that R does not satisfy any nontrivial GPI. Let $T = U *_C C\{x_1,x_2,\ldots,x_n\}$, the free product of U and $C\{x_1,\ldots,x_n\}$, the free C-algebra in noncommuting indeterminates $x_1,x_2,\ldots,x_n$. Then $\Psi(x_1,\ldots,x_n)$ is zero element in T. This gives that $\{A_1,a_3,1\}$ is linearly C-dependent, hence there exist $\alpha_1,\alpha_2,\alpha_3 \in C$ such that $\alpha_1A_1+\alpha_2a_3+\alpha_3.1 = 0$. If $\alpha_1 = 0$, then $\alpha_2 \neq 0$ and so $a_3 = -\alpha_2^{-1}\alpha_3 \in C$, a contradiction. Therefore, either $\alpha_1 \neq 0$. Then $A_1 = \alpha a_3 + \beta$, where $\alpha = -\alpha_1^{-1}\alpha_2, \beta = -\alpha_1^{-1}\alpha_3$. Then

$$\begin{aligned}(\alpha a_3+\beta)f(x_1,\ldots,x_n)^2 + f(x_1,\ldots,x_n)^2A_2 + a_3f(x_1,\ldots,x_n)a_3f(x_1,\ldots,x_n) \\ +f(x_1,\ldots,x_n)A_3f(x_1,\ldots,x_n) \\ +f(x_1,\ldots,x_n)a_4f(x_1,\ldots,x_n)a_4 + a_3f(x_1,\ldots,x_n)^2a_4 = 0. \qquad (3.2.5)\end{aligned}$$

Since $\{a_3,1\}$ is linearly C-independent, we have

$$\begin{aligned}\alpha a_3f(x_1,\ldots,x_n)^2 + a_3f(x_1,\ldots,x_n)a_3f(x_1,\ldots,x_n) \\ +a_3f(x_1,\ldots,x_n)^2a_4 = 0, \qquad (3.2.6)\end{aligned}$$

that is

$$\begin{aligned}a_3f(x_1,\ldots,x_n)\{\alpha f(x_1,\ldots,x_n) + a_3f(x_1,\ldots,x_n) \\ +f(x_1,\ldots,x_n)a_4\} = 0. \qquad (3.2.7)\end{aligned}$$

Since $a_3 \notin C$, the term $a_3f(x_1,\ldots,x_n)a_3f(x_1,\ldots,x_n)$ can not be canceled and hence $a_3f(x_1,\ldots,x_n)a_3f(x_1,\ldots,x_n) = 0$ in T which implies $a_3 = 0$, a contradiction.

Next suppose that U satisfies the non-trivial GPI $\Psi(x_1,\ldots,x_n) = 0$. Then by the well known theorem of Martindale (see Theorem 1.6.6), U is a primitive ring with nonzero socle S and with C as its associated division ring. By Jacobson's theorem (see Theorem 1.6.5), U is isomorphic to a dense ring of linear transformations of a vector space V over C.

If V is finite dimensional over C, that is, $\dim_C V = m$, then by density of U, we have $U \cong M_m(C)$. As U is noncommutative, $m \geq 2$. By Lemma 3.2.1, $a_3 \in C$ and $a_4 \in C$, a contradiction.

If V is infinite dimensional over C, then by [98, Lemma 2], the set $f(U)$ is dense on U. Thus U satisfies

$$A_1x^2 + x^2A_2 + a_3xa_3x + xA_3x + xa_4xa_4 + a_3x^2a_4 = 0.$$

Since $a_3 \notin C$ and $a_4 \notin C$, they can not commute any nonzero ideal of U, i.e., $[a_3, S] \neq (0)$ and $[a_4, S] \neq (0)$. Therefore, there exists $h_1, h_2 \in S$ such that $[a_3, h_1] \neq 0$ and $[a_4, h_2] \neq 0$. By [56], there exists idempotent $e \in S$ such that $a_3h_1, h_1a_3, a_4h_2, h_2a_4, h_1, h_2 \in eUe$. Since U satisfies generalized identity

$$e\{A_1(exe)^2 + (exe)^2A_2 + a_3(exe)a_3(exe) + (exe)A_3(exe) + (exe)a_4(exe)a_4 \\ +a_3(exe)^2a_4\}e = 0,$$

the subring eUe satisfies

$$(eA_1e)x^2+x^2(eA_2e)+(ea_3e)x(ea_3e)x+x(eA_3e)x+x(ea_4e)x(ea_4e)+(ea_3e)x^2(ea_4e) = 0.$$

Since $eUe \cong M_t(C)$ with $t = \dim_C Ve$, by above argument a_3 or a_4 are central elements of eUe. But then we have contradiction as $a_3h_1 = (ea_3e)h_1 = h_1ea_3e = h_1a_3$ and $a_4h_2 = (ea_4e)h_2 = h_2(ea_4e) = h_2a_4$. Therefore we get that a_3 and a_4 are central. □

Lemma 3.2.3. *Let R be a prime ring of char $(R) \neq 2$, C the extended centroid of R and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C. If $a_1, a_2, a_3 \in R$ such that*

$$a_1f(x)^2 + f(x)a_2f(x) + f(x)^2a_3 = 0$$

for all $x = (x_1, \ldots, x_n) \in R^n$, then a_2 is central.

Proof. By using the similar argument above as in Lemma 3.2.2, we get that a_2 is central. □

Lemma 3.2.4. *[36, Lemma 2.9] Let R be a noncommutative prime ring of characteristic different from 2 and $p(x_1, \ldots, x_n)$ be any polynomial over C which is not an identity for R. If there exist $a_1, a_2, a_3, a_4 \in U$ such that*

$$a_1p(x) + p(x)a_2 + a_3p(x)a_4 = 0$$

for all $x = (x_1, \ldots, x_n) \in R^n$, then one of the following holds:

(1) $a_2, a_4 \in C$ *and* $a_1 + a_2 + a_3a_4 = 0$;

(2) $a_1, a_3 \in C$ *and* $a_1 + a_2 + a_3a_4 = 0$;

(3) $a_1 + a_2 + a_3a_4 = 0$ *and* $p(x)$ *is central valued on* R.

Proposition 3.2.5. *Suppose that* R *is a prime ring of char*$(R) \neq 2$ *and* F, G, H *are three inner generalized derivations on* R. *If*

$$F(x^2) = G(x)^2 + H(x)x + xH(x)$$

for all $x \in f(R)$, *then one of the following holds:*

1. *there exist a derivation* d *on* R *and* $\lambda_1, \lambda_2, \lambda_3 \in C$ *such that* $F(x) = \lambda_1 x + d(x)$, $G(x) = \lambda_2 x$ *and* $H(x) = \lambda_3 x + d(x)$ *for all* $x \in R$, *with* $\lambda_1 = \lambda_2^2 + 2\lambda_3$;
2. *there exist* $a_1, a_2 \in U$ *and* $\lambda_1, \lambda_2, \lambda_3 \in C$ *such that* $F(x) = \lambda_1 x + [a_1, x]$, $G(x) = \lambda_2 x$ *and* $H(x) = \lambda_3 x + [a_2, x]$ *for all* $x \in R$, *with* $f(R)^2 \in C$ *and* $\lambda_1 = \lambda_2^2 + 2\lambda_3$.

Proof. By our hypothesis, generalized derivations F, G and H all are inner. Then there exist $a_1, a_2, a_3, a_4, a_5, a_6 \in U$ such that $F(x) = a_1x + xa_2$, $G(x) = a_3x + xa_4$ and $H(x) = a_5x + xa_6$ for all $x \in R$. Now $F(x^2) = G(x)^2 + H(x)x + xH(x)$ for all $x \in f(R)$ gives

$$\begin{aligned}(a_5 - a_1)f(x)^2 + f(x)^2(a_6 - a_2) + a_3f(x)a_3f(x) + f(x)(a_5 + a_4a_3 + a_6)f(x)\\ +f(x)a_4f(x)a_4 + a_3f(x)^2a_4 = 0 \qquad (3.2.8)\end{aligned}$$

for all $x = (x_1, \ldots, x_n) \in R^n$. By Lemma 3.2.2, we get a_3 and a_4 are central. Then (3.2.8) reduces to

$$\begin{aligned}(a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1)f(x)^2 + f(x)^2(a_6 - a_2)\\ +f(x)(a_5 + a_3a_4 + a_6)f(x) = 0 \qquad (3.2.9)\end{aligned}$$

for all $x = (x_1, \ldots, x_n) \in R^n$. Now applying the Lemma 3.2.4 we get $a_5 + a_3a_4 + a_6 \in C$, that is $a_5 + a_6 \in C$ and (3.2.9) reduces to

$$(a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1)f(x)^2 + f(x)^2(2a_6 - a_2 + a_5 + a_3a_4) = 0 \quad (3.2.10)$$

for all $x = (x_1, \ldots, x_n) \in R^n$. By application of Lemma 3.2.4, we get

(i) $a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1 \in C$ that is $a_5 - a_1 \in C$, $2a_6 - a_2 + a_5 + a_3a_4 \in C$ that is $a_6 - a_2 \in C$ and $a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1 + 2a_6 - a_2 + a_5 + a_3a_4 = 0$ that is, $(a_3 + a_4)^2 + 2(a_5 + a_6) = a_1 + a_2$. Since $a_5 - a_1 \in C$ and $a_6 - a_2 \in C$, so $a_1 + a_2 \in C$.

Therefore in this case we get $F(x) = a_1x + xa_2 = a_1x + x(a_1 + a_2) - xa_1 = [a_1, x] + (a_1 + a_2)x$, where $a_1 + a_2 \in C$ and $G(x) = a_3x + xa_4 = (a_3 + a_4)x$, where $a_3 + a_4 \in C$. Also $H(x) = a_5x + xa_6 = (a_5 - a_1)x + a_1x + x(a_6 - a_2) + xa_2 = (a_5 + a_6 - a_1 - a_2)x + (a_1x + xa_2) = (a_5 + a_6 - a_1 - a_2)x + [a_1, x] + (a_1 + a_2)x = (a_5 + a_6)x + [a_1, x]$, where $a_5 + a_6 \in C$, and $(a_3 + a_4)^2 + 2(a_5 + a_6) = a_1 + a_2$, this is our conclusion (1).

(ii) $f(x_1, \dots, x_n)^2$ is central valued and $a_3^2 + a_4^2 + a_3a_4 + a_5 - a_1 + 2a_6 - a_2 + a_5 + a_3a_4 = 0$ that is $(a_3 + a_4)^2 + 2(a_5 + a_6) = a_1 + a_2$ implying $a_1 + a_2 \in C$.

Therefore in this case we get $F(x) = a_1x + xa_2 = a_1x + x(a_1 + a_2) - xa_1 = [a_1, x] + (a_1 + a_2)x$, where $a_1 + a_2 \in C$ and $G(x) = a_3x + xa_4 = (a_3 + a_4)x$, where $a_3 + a_4 \in C$. Also $H(x) = a_5x + xa_6 = a_5x + x(a_5 + a_6) - xa_5 = [a_5, x] + (a_5 + a_6)x$ where $a_5 + a_6 \in C$, also $(a_3 + a_4)^2 + 2(a_5 + a_6) = a_1 + a_2$ with $f(x_1, \dots, x_n)^2$ is central valued on R. This is our conclusion (2). □

3.3 Proof of Main Theorem

Here R is a prime ring and U the Utumi quotient ring of R and $C = Z(U)$(see [14] for more details). It is well known that any derivation of R can be uniquely extended to a derivation of U. Now we consider $f(x_1, \dots, x_n)$ be a noncentral multilinear polynomial over the field C and d be a derivation on R.

We shall use the notation

$$f(x_1, \dots, x_n) = x_1x_2 \cdots x_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$, and S_n denotes the symmetric group of degree n.

Then we have

$$d(f(x_1, \dots, x_n)) = f^d(x_1, \dots, x_n) + \sum_i f(x_1, \dots, d(x_i), \dots, x_n),$$

where $f^d(x_1, \dots, x_n)$ be the polynomials obtained from $f(x_1, \dots, x_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$.

By [79, Theorem 3], every generalized derivation g of R can be uniquely extended to a generalized derivation of U and its form will be $g(x) = ax + d(x)$ for all $x \in U$, *where $a \in U$ and d is a derivation of U. Thus we can assume that $F(x) = ax + d(x)$, $G(x) = bx + g(x)$ and $H(x) = cx + h(x)$ for all $x \in R$ with some fixed $a, b, c \in U$ and d, g, h are derivations on U. Thus by [24] and [78], our hypothesis yields*

$$\begin{aligned} & af(x)^2 + d\big(f(x)\big)f(x) + f(x)d\big(f(x)\big) \\ = \; & bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)g(f(x)) + g(f(x))bf(x) \\ & + g\big(f(x)\big)^2 + h\big(f(x)\big)f(x) + f(x)h\big(f(x)\big) \end{aligned} \tag{3.3.1}$$

for all $x = (x_1, \ldots, x_n) \in U^n$. If d, g and h are three inner derivations, then F, G and H are three inner generalized derivations and in this case conclusions follow by Proposition 3.2.5. Therefore, to prove our Theorem 3.1.1, we need to consider the following cases:

1. *d, g are inner and h is outer;*

2. *g, h are inner and d is outer;*

3. *d, h are inner and g is outer;*

4. *h is inner and d, g are outer;*

5. *d is inner and g, h are outer;*

6. *g is inner and d, h are outer;*

7. *d, g, h all are outer.*

We divide these 7 cases into the following cases:

Case-1. *d, g are inner and h is outer.*

Let $d(x) = [p, x]$ and $g(x) = [q, x]$ for all $x \in R$ and for some $p, q \in U$. By (3.3.1), we get

$$\begin{aligned} & af(x)^2 + [p, f(x)]f(x) + f(x)[p, f(x)] \\ & = bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)[q, f(x)] + [q, f(x)]bf(x) \\ & + \Big([q, f(x)]\Big)^2 + h\big(f(x)\big)f(x) + f(x)h\big(f(x)\big) \end{aligned} \tag{3.3.2}$$

for all $x = (x_1, \ldots, x_n) \in U^n$. Since h is outer derivation, by Kharchenko's theorem (see Theorem 1.6.4) we may replace $h(f(x_1, \ldots, x_n))$ by $f^h(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, z_i, \ldots, x_n)$ in (3.3.2). Then U satisfies blended component

$$\sum_i f(x_1, \ldots, z_i, \ldots, x_n)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)\sum_i f(x_1, \ldots, z_i, \ldots, x_n) = 0.$$

In particular, for $z_1 = x_1$ *and* $z_2 = \cdots = z_n = 0$ *we get that* $2f(x_1,\ldots,x_n)^2 = 0$. *Since char* $(R) \neq 2$ *so* $f(x_1,\ldots,x_n)^2 = 0$ *which implies* $f(x_1,\ldots,x_n) = 0$, *a contradiction.*

Case-2. g, h *are inner and* d *is outer.*
Let $g(x) = [p, x]$ *and* $h(x) = [q, x]$ *for all* $x \in R$ *and for some* $p, q \in U$. *By (3.3.1), we get*

$$\begin{aligned} &af(x)^2 + d\big(f(x)\big)f(x) + f(x)d\big(f(x)\big) \\ &= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)[p, f(x)] + [p, f(x)]bf(x) \\ &+\Big([p, f(x)]\Big)^2 + [q, f(x)]f(x) + f(x)[q, f(x)] \end{aligned} \tag{3.3.3}$$

for all $x = (x_1,\ldots,x_n) \in U^n$. *Since* d *is outer derivation, by Kharchenko's theorem (see Theorem 1.6.4) we may replace* $d(f(x_1,\ldots,x_n))$ *by* $f^d(x_1,\ldots,x_n)$ $+\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n)$ *in (3.3.3). Then* U *satisfies blended component*

$$\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n)f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n) = 0.$$

In particular, for $\xi_1 = x_1$ *and* $\xi_2 = \cdots = \xi_n = 0$ *we get that* $2f(x_1,\ldots,x_n)^2 = 0$ *which implies* $f(x_1,\ldots,x_n) = 0$, *a contradiction.*

Case-3. d, h *are inner and* g *is outer.*
Let $d(x) = [p, x]$ *and* $h(x) = [q, x]$ *for all* $x \in R$ *and for some* $p, q \in U$. *By (3.3.1), we get*

$$\begin{aligned} &af(x)^2 + [p, f(x)]f(x) + f(x)[p, f(x)] \\ &= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)g(f(x)) + g(f(x))bf(x) \\ &+g(f(x))^2 + [q, f(x)]f(x) + f(x)[q, f(x)] \end{aligned} \tag{3.3.4}$$

for all $x = (x_1,\ldots,x_n) \in U^n$. *Since* g *is outer derivation, by Kharchenko's theorem (see Theorem 1.6.4) we may replace* $g(f(x_1,\ldots,x_n))$ *by* $f^g(x_1,\ldots,x_n)$ $+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)$ *in (3.3.4). Then* U *satisfies blended component*

$$\begin{aligned} &bf(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)bf(x_1,\ldots,x_n) \\ &+f^g(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n)f^g(x_1,\ldots,x_n) \\ &+\Big(\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big)^2 = 0. \end{aligned} \tag{3.3.5}$$

Putting $y_i = -y_i$ we get

$$-bf(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) - \sum_i f(x_1,\ldots,y_i,\ldots,x_n)bf(x_1,\ldots,x_n)$$
$$-f^g(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) - \sum_i f(x_1,\ldots,y_i,\ldots,x_n)f^g(x_1,\ldots,x_n)$$
$$+\Big(\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big)^2 = 0. \tag{3.3.6}$$

Now adding (3.3.5) and (3.3.6) we get $2\big(\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\big)^2 = 0$. *In particular, for* $y_1 = x_1$ *and* $y_2 = \cdots = y_n = 0$ *we have* $2f(x_1,\ldots,x_n)^2 = 0$. *Since char(R)* $\neq 2$ *so* $f(x_1,\ldots,x_n)^2 = 0$ *which implies* $f(x_1,\ldots,x_n) = 0$, *a contradiction.*

Case-4. *h is inner and d, g are outer.*
Let $h(x) = [p,x]$ *for all* $x \in R$ *and for some* $p \in U$. *By (3.3.1), we get*

$$af(x)^2 + d\big(f(x)\big)f(x) + f(x)d\big(f(x)\big)$$
$$= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)g(f(x)) + g(f(x))bf(x)$$
$$+g\big(f(x)\big)^2 + [p,f(x)]f(x) + f(x)[p,f(x)] \tag{3.3.7}$$

for all $x = (x_1,\ldots,x_n) \in U^n$.

<u>*Sub-case-i*</u>*: d, g are linearly $C-$independent modulo inner derivations of U. Since d and g are outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace* $d(f(x_1,\ldots,x_n))$ *by*

$$f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,\xi_i,\ldots,x_n),$$

and $g(f(x_1,\ldots,x_n))$ *by*

$$f^g(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,y_i,\ldots,x_n).$$

Then U satisfies blended component

$$\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n)f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n) = 0.$$

In particular, for $\xi_1 = x_1$ *and* $\xi_2 = \cdots = \xi_n = 0$ *we get that*
$2f(x_1,\ldots,x_n)^2 = 0$ *which implies* $f(x_1,\ldots,x_n) = 0$, *a contradiction.*

<u>Sub-case-ii</u>: d, g are linearly $C-$dependent modulo inner derivations of U.
In this subcase we have $d(x) = \alpha g(x) + [q, x]$ for all $x \in U$, for some $q \in U$ and $0 \neq \alpha \in C$. Then from (3.3.7) we get

$$\begin{aligned} &af(x)^2 + \Big(\alpha g(f(x)) + [q, f(x)]\Big)f(x) + f(x)\Big(\alpha g(f(x)) + [q, f(x)]\Big) \\ &= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)g(f(x)) + g(f(x))bf(x) \\ &\qquad + g\big(f(x)\big)^2 + [p, f(x)]f(x) + f(x)[p, f(x)] \end{aligned} \tag{3.3.8}$$

for all $x = (x_1, \ldots, x_n) \in U^n$. Since g is outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace $g(f(x_1, \ldots, x_n))$ by

$$f^g(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n).$$

Then U satisfies blended component

$$\begin{aligned} &\alpha \sum_i f(x_1, \ldots, y_i, \ldots, x_n)f(x_1, \ldots, x_n) + \alpha f(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) \\ &= bf(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)bf(x_1, \ldots, x_n) \\ &+ f^g(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)f^g(x_1, \ldots, x_n) \\ &\qquad + \Big(\sum_i f(x_1, \ldots, y_i, \ldots, x_n)\Big)^2. \end{aligned} \tag{3.3.9}$$

Putting $y_i = -y_i$ we get

$$\begin{aligned} &-\alpha \sum_i f(x_1, \ldots, y_i, \ldots, x_n)f(x_1, \ldots, x_n) - \alpha f(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) \\ &= -bf(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) - \sum_i f(x_1, \ldots, y_i, \ldots, x_n)bf(x_1, \ldots, x_n) \\ &- f^g(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) - \sum_i f(x_1, \ldots, y_i, \ldots, x_n)f^g(x_1, \ldots, x_n) \\ &\qquad + \Big(\sum_i f(x_1, \ldots, y_i, \ldots, x_n)\Big)^2. \end{aligned} \tag{3.3.10}$$

Now adding (3.3.9) and (3.3.10) we get $2\big(\sum_i f(x_1, \ldots, y_i, \ldots, x_n)\big)^2 = 0$. In particular, for $y_1 = x_1$ and $y_2 = \cdots = y_n = 0$ we have $2f(x_1, \ldots, x_n)^2 = 0$. Since char$(R) \neq 2$ so $f(x_1, \ldots, x_n)^2 = 0$ which implies $f(x_1, \ldots, x_n) = 0$, a contradiction.

Case-5. *d is inner and g, h are outer.*
Let $d(x) = [p, x]$ for all $x \in R$ and for some $p \in U$. By (3.3.1), we get

$$\begin{aligned} &af(x)^2 + [p, f(x)]f(x) + f(x)[p, f(x)] \\ &= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)g(f(x)) + g(f(x))bf(x) \\ &+ g(f(x))^2 + h\big(f(x)\big)f(x) + f(x)h\big(f(x)\big) \end{aligned} \tag{3.3.11}$$

for all $x = (x_1, \ldots, x_n) \in U^n$*.*

<u>*Sub-case-i*</u>*:* g, h *are linearly* $C-$*independent modulo inner derivations of* U*. Since* g *and* h *are outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace* $g(f(x_1, \ldots, x_n))$ *by*

$$f^g(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)$$

and $h(f(x_1, \ldots, x_n))$ *by*

$$f^h(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, z_i, \ldots, x_n).$$

Then U *satisfies a blended component*

$$\sum_i f(x_1, \ldots, z_i, \ldots, x_n) f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, z_i, \ldots, x_n) = 0.$$

In particular, for $z_1 = x_1$ *and* $z_2 = \cdots = z_n = 0$ *we get* $2f(x_1, \ldots, x_n)^2 = 0$ *which implies* $f(x_1, \ldots, x_n) = 0$*, a contradiction.*

<u>*Sub-case-ii*</u>*:* g, h *are linearly* $C-$*dependent modulo inner derivations of* U*. Here we have* $h(x) = \alpha g(x) + [q, x]$ *for all* $x \in U$ *for some* $q \in U$ *and* $0 \neq \alpha \in C$*. Then from (3.3.11) we get*

$$\begin{aligned} & af(x)^2 + [p, f(x)]f(x) + f(x)[p, f(x)] \\ & = bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)g(f(x)) + g(f(x))bf(x) + g(f(x))^2 \\ & + \Big(\alpha g(f(x)) + [q, f(x)]\Big) f(x) + f(x)\Big(\alpha g(f(x)) + [q, f(x)]\Big) \end{aligned} \quad (3.3.12)$$

for all $x = (x_1, \ldots, x_n) \in U^n$*. Since* g *is outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace* $g(f(x_1, \ldots, x_n))$ *by*

$$f^g(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n).$$

Then U *satisfies blended component*

$$\begin{aligned} & bf(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n) bf(x_1, \ldots, x_n) \\ & + f^g(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n) f^g(x_1, \ldots, x_n) \\ & + \Big(\sum_i f(x_1, \ldots, y_i, \ldots, x_n) \Big)^2 + \alpha \sum_i f(x_1, \ldots, y_i, \ldots, x_n) f(x_1, \ldots, x_n) \\ & + \alpha f(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) = 0. \end{aligned} \quad (3.3.13)$$

Putting $y_i = -y_i$ we get

$$
\begin{aligned}
&-bf(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) - \sum_i f(x_1,\ldots,y_i,\ldots,x_n)bf(x_1,\ldots,x_n)\\
&-f^g(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) - \sum_i f(x_1,\ldots,y_i,\ldots,x_n)f^g(x_1,\ldots,x_n)\\
&\quad +\Big(\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\Big)^2 - \alpha\sum_i f(x_1,\ldots,y_i,\ldots,x_n)f(x_1,\ldots,x_n)\\
&\qquad\qquad -\alpha f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,y_i,\ldots,x_n) = 0. \qquad (3.3.14)
\end{aligned}
$$

Now adding (3.3.13) and (3.3.14) we get $2\big(\sum_i f(x_1,\ldots,y_i,\ldots,x_n)\big)^2 = 0$. *In particular, for* $y_1 = x_1$ *and* $y_2 = \cdots = y_n = 0$ *we have* $2f(x_1,\ldots,x_n)^2 = 0$. *Since* $char(R) \neq 2$ *so* $f(x_1,\ldots,x_n)^2 = 0$ *which implies* $f(x_1,\ldots,x_n) = 0$, *a contradiction.*

Case-6. *g is inner and d, h are outer.*
Let $g(x) = [p, x]$ *for all* $x \in R$ *and for some* $p \in U$. *By (3.3.1),* U *satisfies*

$$
\begin{aligned}
&af(x)^2 + d\big(f(x)\big)f(x) + f(x)d\big(f(x)\big)\\
&= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)[p, f(x)] + [p, f(x)]bf(x)\\
&\quad +\Big([p, f(x)]\Big)^2 + h\big(f(x)\big)f(x) + f(x)h\big(f(x)\big). \qquad (3.3.15)
\end{aligned}
$$

Sub-case-i: d, h are linearly $C-$independent modulo inner derivations of U.
Since d and h are outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace $d(f(x_1,\ldots,x_n))$ *by*

$$f^d(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,\xi_i,\ldots,x_n)$$

and $h(f(x_1,\ldots,x_n))$ *by*

$$f^h(x_1,\ldots,x_n) + \sum_i f(x_1,\ldots,z_i,\ldots,x_n).$$

Then U satisfies a blended component

$$\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n)f(x_1,\ldots,x_n) + f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n) = 0.$$

In particular, for $\xi_1 = x_1$ *and* $\xi_2 = \cdots = \xi_n = 0$ *we get that* $2f(x_1,\ldots,x_n)^2 = 0$ *which implies* $f(x_1,\ldots,x_n) = 0$, *a contradiction.*

<u>Sub-case-ii</u>: d, h are linearly $C-$dependent modulo inner derivations of U. Here we have $h(x) = \alpha d(x) + [q, x]$ for all $x \in U$ for some $q \in U$ and $0 \neq \alpha \in C$. Then from (3.3.15) we get

$$\begin{aligned} &af(x)^2 + d\big(f(x)\big)f(x) + f(x)d\big(f(x)\big) \\ &= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)[p, f(x)] + [p, f(x)]bf(x) \\ &+ \Big([p, f(x)]\Big)^2 + \Big(\alpha d(f(x)) + [q, f(x)]\Big)f(x) \\ &+ f(x)\Big(\alpha d(f(x)) + [q, f(x)]\Big) \end{aligned} \tag{3.3.16}$$

for all $x = (x_1, \ldots, x_n) \in U^n$. Since d is outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace $d(f(x_1, \ldots, x_n))$ by

$$f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, \xi_i, \ldots, x_n).$$

Then U satisfies blended component

$$\begin{aligned} &\sum_i f(x_1, \ldots, \xi_i, \ldots, x_n)f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)\sum_i f(x_1, \ldots, \xi_i, \ldots, x_n) \\ &= \alpha\Big(\sum_i f(x_1, \ldots, \xi_i, \ldots, x_n)\Big)f(x_1, \ldots, x_n) \\ &+ f(x_1, \ldots, x_n)\alpha\sum_i f(x_1, \ldots, \xi_i, \ldots, x_n) = 0. \end{aligned} \tag{3.3.17}$$

In particular, for $\xi_1 = x_1$ and $\xi_2 = \cdots = \xi_n = 0$ we get that

$$(2\alpha - 2)f(x_1, \ldots, x_n)^2 = 0, \tag{3.3.18}$$

implying $\alpha = 1$. Then (3.3.16), reduces to

$$\begin{aligned} &af(x)^2 = \Big((b+p)f(x) - f(x)p\Big)^2 + \Big(cf(x) \\ &+ [q, f(x)]\Big)f(x) + f(x)\Big(cf(x) + [q, f(x)]\Big). \end{aligned} \tag{3.3.19}$$

Then by Lemma 3.2.2 we get $b+p, p \in C$, that is, $b, p \in C$. Then the equation (3.3.19) becomes

$$(a - b^2 - c - q)f(x)^2 = f(x)cf(x) - f(x)^2 q. \tag{3.3.20}$$

By Lemma 3.2.3 we get $c \in C$, so (3.3.20) reduces to

$$(a - b^2 - c - q)f(x)^2 + f(x)^2(q - c) = 0. \tag{3.3.21}$$

By applying Lemma 3.2.4, we obtain

(1) $a-b^2-c-q, q-c \in C$ *and* $a-b^2-c-q+q-c=0$, *that is* $a=b^2+2c$. *Now* $b,p,c \in C$ *and* $a-b^2-c-q, q-c \in C$ *implies that* $a,b,c,p,q \in C$ *with* $a=b^2+2c$.

Therefore in this section we get $F(x)=ax+d(x)$, *where* $a \in C$ *and* $G(x)= bx+[p,x]=bx$, *where* $b \in C$. *Also* $H(x)=cx+\alpha d(x)+[q,x]=cx+d(x)$, *where* $c \in C$ *with* $a=b^2+2c$. *This is our conclusion (1) of Theorem 3.1.1.*

(2) $f(x)^2 \in C$ *and* $a-b^2-c-q+q-c=0$, *that is* $a=b^2+2c$. *Now* $b,p,c \in C$ *and* $a=b^2+2c$ *implies that* $a,b,p,c \in C$ *with* $a=b^2+2c$.

Therefore in this section we get $F(x)=ax+d(x)$, *where* $a \in C$ *and* $G(x)= bx+[p,x]=bx$, *where* $b \in C$. *Also* $H(x)=cx+\alpha d(x)+[q,x]=cx+d(x)+[q,x]$, *where* $c \in C$ *with* $a=b^2+2c$. *This is our conclusion (2) of Theorem 3.1.1.*

Case-7. d,g,h *all are outer.*

Sub-case-i: d,g,h *are linearly* $C-$*independent modulo inner derivations of* U. *Since* d,g *and* h *are all outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace* $d(f(x_1,\ldots,x_n))$ *by*

$$f^d(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n),$$

$g(f(x_1,\ldots,x_n))$ *by*

$$f^g(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,y_i,\ldots,x_n)$$

and $h(f(x_1,\ldots,x_n))$ *by*

$$f^h(x_1,\ldots,x_n)+\sum_i f(x_1,\ldots,z_i,\ldots,x_n).$$

Then U *satisfies a blended component*

$$\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n)f(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)\sum_i f(x_1,\ldots,\xi_i,\ldots,x_n)=0.$$

In particular, for $\xi_1=x_1$ *and* $\xi_2=\cdots=\xi_n=0$ *we get that* $2f(x_1,\ldots,x_n)^2=0$ *which implies* $f(x_1,\ldots,x_n)=0$, *a contradiction.*

Sub-case-ii: d,g,h *are linearly* $C-$*dependent modulo inner derivations of* U. *This implies there exist* $\alpha,\beta,\gamma \in C$ *and* $q \in U$ *such that*

$$\alpha d(x)+\beta g(x)+\gamma h(x)=[q,x] \tag{3.3.22}$$

for all $x \in R$. If we consider $\beta = \gamma = 0$, then inevitably $\alpha \neq 0$, which implies the contradiction that d is inner. So to move forward we have to consider $(\beta, \gamma) \neq (0, 0)$. Without loss of generality we assume that $\gamma \neq 0$. By (3.3.22) we obtain

$$h(x) = \alpha' d(x) + \beta' g(x) + [q', x] \tag{3.3.23}$$

for all $x \in R$, where $\alpha' = -\gamma^{-1}\alpha$, $\beta' = -\gamma^{-1}\beta$, $q' = \gamma^{-1}q$.
By (3.3.1), we get

$$\begin{aligned} af(x)^2 + d\big(f(x)\big)f(x) + f(x)d\big(f(x)\big) &= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) \\ +bf(x)g(f(x)) + g(f(x))bf(x) + g\big(f(x)\big)^2 &+ \big(\alpha' d(f(x)) + \beta' g(f(x)) + [q', f(x)]\big)f(x) \\ &+ f(x)\big(\alpha' d(f(x)) + \beta' g(f(x)) + [q', f(x)]\big) \end{aligned} \tag{3.3.24}$$

for all $x = (x_1, \ldots, x_n) \in U^n$.

Now we have to consider the following two cases:

(I) *We consider that d, g are linearly $C-$independent modulo inner derivations of U. Since d and g are outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace $d(f(x_1, \ldots, x_n))$ by*

$$f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, \xi_i, \ldots, x_n)$$

and $g(f(x_1, \ldots, x_n))$ by

$$f^g(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n)$$

in (3.3.24). Then U satisfies blended component

$$\begin{aligned} bf(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n) bf(x_1, \ldots, x_n) \\ + f^g(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n) f^g(x_1, \ldots, x_n) \\ + \Big(\sum_i f(x_1, \ldots, y_i, \ldots, x_n)\Big)^2 + \beta' \sum_i f(x_1, \ldots, y_i, \ldots, x_n) f(x_1, \ldots, x_n) \\ + \beta' f(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) = 0. \end{aligned} \tag{3.3.25}$$

taking $y_i = -y_i$ we get

$$\begin{aligned} -bf(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) - \sum_i f(x_1, \ldots, y_i, \ldots, x_n) bf(x_1, \ldots, x_n) \\ - f^g(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) - \sum_i f(x_1, \ldots, y_i, \ldots, x_n) f^g(x_1, \ldots, x_n) \\ + \Big(\sum_i f(x_1, \ldots, y_i, \ldots, x_n)\Big)^2 - \beta' \sum_i f(x_1, \ldots, y_i, \ldots, x_n) f(x_1, \ldots, x_n) \\ - \beta' f(x_1, \ldots, x_n) \sum_i f(x_1, \ldots, y_i, \ldots, x_n) = 0. \end{aligned} \tag{3.3.26}$$

Adding (3.3.25) and (3.3.26) we get $2\big(\sum_i f(x_1,\dots,y_i,\dots,x_n)\big)^2 = 0$. *In particular, for* $y_1 = x_1$ *and* $y_2 = \cdots = y_n = 0$ *we have* $2f(x_1,\dots,x_n)^2 = 0$. *Since char*$(R) \neq 2$ *so* $f(x_1,\dots,x_n)^2 = 0$ *which implies* $f(x_1,\dots,x_n) = 0$, *a contradiction.*

(II) *We consider that* d, g *are linearly* $C-$*dependent modulo inner derivations of* U *and* $d(x) = \alpha_1 g(x) + [q_1, x]$ *for all* $x \in U$ *and some* $\alpha_1 \in C$. *Then (3.3.24) reduces to*

$$\begin{aligned} &af(x)^2 + \big(\alpha_1 g(f(x)) + [q_1, f(x)]\big)f(x) + f(x)\big(\alpha_1 g(f(x)) + [q_1, f(x)]\big)\\ &= bf(x)bf(x) + cf(x)^2 + f(x)cf(x) + bf(x)g(f(x)) + g(f(x))bf(x)\\ &+g\big(f(x)\big)^2 + \big(\alpha'\alpha_1 g(f(x)) + \alpha'[q_1, f(x)] + \beta' g(f(x)) + [q', f(x)]\big)f(x)\\ &\quad +f(x)\big(\alpha'\alpha_1 g(f(x)) + \alpha'[q_1, f(x)] + \beta' g(f(x)) + [q', f(x)]\big) \end{aligned} \tag{3.3.27}$$

for all $x = (x_1,\dots,x_n) \in U^n$. *Since* g *is a outer derivations, by Kharchenko's theorem (see Theorem 1.6.4) we may replace* $g(f(x_1,\dots,x_n))$ *by*

$$f^g(x_1,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n).$$

Then U *satisfies blended component*

$$\begin{aligned} &\alpha_1 \sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n) + \alpha_1 f(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n)\\ &= bf(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n)bf(x_1,\dots,x_n)\\ &+f^g(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n) + \sum_i f(x_1,\dots,y_i,\dots,x_n)f^g(x_1,\dots,x_n)\\ &+\big(\sum_i f(x_1,\dots,y_i,\dots,x_n)\big)^2 + (\alpha'\alpha_1 + \beta')\sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)\\ &\quad +(\alpha'\alpha_1 + \beta')f(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n) = 0. \end{aligned} \tag{3.3.28}$$

taking $y_i = -y_i$ *we get*

$$\begin{aligned} &-\alpha_1 \sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n) - \alpha_1 f(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n)\\ &= -bf(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n) - \sum_i f(x_1,\dots,y_i,\dots,x_n)bf(x_1,\dots,x_n)\\ &-f^g(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n) - \sum_i f(x_1,\dots,y_i,\dots,x_n)f^g(x_1,\dots,x_n)\\ &+\big(\sum_i f(x_1,\dots,y_i,\dots,x_n)\big)^2 - (\alpha'\alpha_1 + \beta')\sum_i f(x_1,\dots,y_i,\dots,x_n)f(x_1,\dots,x_n)\\ &\quad -(\alpha'\alpha_1 + \beta')f(x_1,\dots,x_n)\sum_i f(x_1,\dots,y_i,\dots,x_n) = 0. \end{aligned} \tag{3.3.29}$$

Adding (3.3.28) and (3.3.29) we get $2\big(\sum_i f(x_1,\dots,y_i,\dots,x_n)\big)^2 = 0$. *In particular, for* $y_1 = x_1$ *and* $y_2 = \cdots = y_n = 0$ *we have* $2f(x_1,\dots,x_n)^2 = 0$. *Since char*$(R) \neq$

2 *so* $f(x_1, \ldots, x_n)^2 = 0$ *which implies* $f(x_1, \ldots, x_n) = 0$, *a contradiction. This completes the proof of the theorem.* $\square$

Chapter 4

Annihilators on Generalized Derivations in Prime Rings

4.1 Introduction

In this Chapter, we consider an annihilated identity involving two generalized derivations of prime ring and determine the structure of generalized derivations involved in the identity. Let R be always a prime ring with center $Z(R)$. Let U be the Utumi quotient ring of R and C be the extended centroid of R that is, $C = Z(U)$.

In [19], Carini et al. proved for a prime ring R of char $(R) \neq 2$ that if H and G are two nonzero generalized derivations of R such that

$$a(H(u)u - uG(u)) = 0$$

for all $u \in L$ and for some $0 \neq a \in R$, where L is a noncentral Lie ideal of R, then one of the following holds:

(1) there exist $b, c \in U$ such that $G(x) = cx$ and $H(x) = bx + xc$ with $ab = 0$;

(2) R satisfies s_4 and there exist $b, c, q \in U$ such that $G(x) = cx + xq$ and $H(x) = bx + xc$ with $a(b - q) = 0$.

Argaç and De Filippis [10] proved the following:

Let K be a commutative ring with unity, R be a noncommutative prime K-algebra with center $Z(R)$ and I a nonzero ideal of R. Let $f(x_1, \dots, x_n)$ be a noncentral multilinear polynomial over K. If G and H are two nonzero generalized derivations of R such that

$$G(f(X))f(X) - f(X)H(f(X)) = 0$$

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for all $X = (x_1, \ldots, x_n) \in I^n$*, then one of the following holds:*

1. $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exist* $a, b \in U$ *such that* $G(x) = ax + xb$ *for all* $x \in R$*,* $H(x) = bx + xa$ *for all* $x \in R$*;*

2. *there exists* $a \in U$ *such that* $G(x) = xa$ *for all* $x \in R$*,* $H(x) = ax$ *for all* $x \in R$*;*

3. $char(R) = 2$ *and* R *satisfies* s_4.

In [60], De Filippis et al. studied a situation with left annihilator condition. They proved the following:

Let K *be a commutative ring with unity,* R *be a noncommutative prime* K*-algebra of char* $(R) \neq 2$*. Let* $f(x_1, \ldots, x_n)$ *be a noncentral multilinear polynomial over* K*. If* G *and* H *are two nonzero generalized derivations of* R *and there exists* $0 \neq a \in R$ *such that*

$$a(G(f(X))f(X) - f(X)H(f(X))) = 0$$

for all $X = (x_1, \ldots, x_n) \in R^n$*, then one of the following holds:*

1. $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exist* $b', c', q' \in U$ *such that* $G(x) = b'x + xc'$ *for all* $x \in R$*,* $H(x) = c'x + xq'$ *for all* $x \in R$ *with* $a(b' - q') = 0$*;*

2. *there exist* $b', c' \in U$ *such that* $G(x) = b'x + xc'$ *for all* $x \in R$*,* $H(x) = c'x$ *for all* $x \in R$ *with* $ab' = 0$.

More recently, in [91], Tiwari considers the situation, when

$$F^2(f(X))f(X) - G(f(X)^2) = 0$$

for all $X = (x_1, \ldots, x_n) \in R^n$ *and then all possible forms of the maps are obtained.*

Carini and Scudo in [22] already proved the following:

Let R *be a non-commutative prime ring of characteristic different from* 2*,* U *its Utumi quotient ring,* C *its extended centroid,* F *a generalized derivation on* R*, and* $f(r_1, \ldots, r_n)$ *a noncentral multilinear polynomial over* C*. If there exists* $a \in R$ *such that, for all* $X = (x_1, \ldots, x_n) \in R^n$*,*

$$a(F^2(f(X))f(X) - f(X)F^2(f(X))) = 0$$

then one of the following statements hold:

1. *$a = 0$;*

2. *there exists $\lambda \in C$ such that $F(x) = \lambda(x)$, for all $x \in R$;*

3. *there exists $c \in U$ such that $F(x) = cx$, for all $x \in R$, with $c^2 \in C$;*

4. *there exists $c \in U$ such that $F(x) = xc$, for all $x \in R$, with $c^2 \in C$.*

In [41], Dhara and De Filippis considered

$$F^2(f(X))f(X) - f(X)G^2(f(X)) = 0$$

for all $X = (x_1, \ldots, x_n) \in R^n$ and then authors obtained all possible forms of the maps.

In this line of study, in the present Chapter, we study the following situation with left annihilator condition. More precisely we prove the following Theorem:

Main Theorem. *Let R be a prime ring of char $(R) \neq 2$ with Utumi quotient ring U and extended centroid C. Suppose that $0 \neq p \in R$, $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C and F, G are two nonzero generalized derivations of R. If*

$$p\Big(F\big(f(X)\big)f(X) - f(X)G^2\big(f(X)\big)\Big) = 0$$

for all $X = (x_1, \ldots, x_n) \in R^n$, then one of the following holds:

1. *there exist $\lambda \in C$ and $a \in U$ such that $F(x) = \lambda x$ and $G(x) = xa$ for all $x \in R$, with $\lambda = a^2$;*

2. *there exists $b \in U$ such that $F(x) = xb^2$ and $G(x) = bx$ for all $x \in R$;*

3. *there exist $a, b \in U$ such that $F(x) = ax$ and $G(x) = xb$ for all $x \in R$, with $b^2 \in C$ and $p(a - b^2) = 0$;*

4. *there exist $a, b \in U$ such that $F(x) = ax$ and $G(x) = xb$ for all $x \in R$, with $f(R)^2 \in C$ and $p(a - b^2) = 0$;*

5. *there exist $a, b, c \in U$ such that $F(x) = ax + xb$ and $G(x) = cx$ for all $x \in R$, with $b - c^2 \in C$ and $p(a + b - c^2) = 0$.*

The following corollary is immediate consequence of our main result:

Corollary 4.1.1. *Let R be a prime ring of char $(R)\neq 2$ with Utumi quotient ring U and extended centroid C. Suppose that $f(x_1,\dots,x_n)$ be a noncentral multilinear polynomial over C and F, G are two nonzero generalized derivations of R. If*

$$F\big(f(X)\big)f(X)-f(X)G^2\big(f(X)\big)=0$$

for all $X=(x_1,\dots,x_n)\in R^n$, then one of the following holds:

1. *there exist $\lambda\in C$ and $a\in U$ such that $F(x)=\lambda x$ and $G(x)=xa$ for all $x\in R$, with $\lambda=a^2$;*

2. *there exists $b\in U$ such that $F(x)=xb^2$ and $G(x)=bx$ for all $x\in R$;*

3. *there exists $b\in U$ such that $F(x)=b^2x$ and $G(x)=xb$ for all $x\in R$, with $f(R)^2\in C$.*

4.2 Main Results

Proposition 4.2.1. *Let R be a prime ring of char $(R)\neq 2$, U be its Utumi quotient ring and $C(=Z(U))$ be its extended centroid. Suppose that $0\neq p\in R$ and $f(x_1,\dots,x_n)$ be a noncentral multilinear polynomial over C, $F(x)=a_1x+xa_2$ and $G(x)=a_3x+xa_4$ for some $a_1,a_2,a_3,a_4\in U$ such that*

$$p\Big(F\big(f(X)\big)f(X)-f(X)G^2\big(f(X)\big)\Big)=0$$

for all $X=(x_1,\dots,x_n)\in R^n$, then one of the following holds:

1. *there exist $\lambda\in C$ and $a\in U$ such that $F(x)=\lambda x$ and $G(x)=xa$ for all $x\in R$, with $\lambda=a^2$;*

2. *there exists $b\in U$ such that $F(x)=xb^2$ and $G(x)=bx$ for all $x\in R$;*

3. *there exist $a,b\in U$ such that $F(x)=ax$ and $G(x)=xb$ for all $x\in R$, with $b^2\in C$ and $p(a-b^2)=0$;*

4. *there exist $a,b\in U$ such that $F(x)=ax$ and $G(x)=xb$ for all $x\in R$, with $f(R)^2\in C$ and $p(a-b^2)=0$;*

5. *there exist $a,b,c\in U$ such that $F(x)=ax+xb$ and $G(x)=cx$ for all $x\in R$, with $b-c^2\in C$ and $p(a+b-c^2)=0$.*

By our hypothesis

$$p\Big(F\big(f(x)\big)f(x) - f(x)G^2\big(f(x)\big)\Big) = 0$$

for all $x = (x_1, \ldots, x_n) \in R^n$ *and* $F(x) = a_1x + xa_2$, $G(x) = a_3x + xa_4$ *for all* $x \in R$ *we have*

$$pa_1f(x)^2 + pf(x)(a_2 - a_3^2)f(x) - 2pf(x)a_3f(x)a_4 - pf(x)^2a_4^2 = 0 \qquad (4.2.1)$$

for all $x = (x_1, \ldots, x_n) \in R^n$*. To investigate the above generalized polynomial identity (GPI) in prime ring* R*, we need the following Lemmas. Throughout we assume that* R *be a prime ring of char* $(R) \neq 2$*,* U *be its Utumi quotient ring and* C *be its extended centroid.*

Lemma 4.2.2. *[59, Lemma 2.4] Let* R *be a prime ring of char* $(R) \neq 2$*,* C *the extended centroid of* R *and* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C*. If* $a_1, a_2, a_3, a_4, a_5 \in R$ *such that*

$$a_1u^2 + ua_2u + ua_3ua_4 + u^2a_5 = 0$$

for all $u \in f(R)$*, then either* a_3 *or* a_4 *is central.*

Lemma 4.2.3. *[59, Lemma 2.3] Let* R *be a prime ring of char* $(R) \neq 2$*,* C *the extended centroid of* R *and* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C*. If* $a_1, a_2, a_3, a_5 \in R$ *such that*

$$a_1u^2 + a_2ua_3u + ua_5u = 0$$

for all $u \in f(R)$*, then either* a_2 *or* a_3 *is central.*

Lemma 4.2.4. *[42, Proposition 2.7] Let* R *be a prime ring of char* $(R) \neq 2$*,* C *the extended centroid of* R *and* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C*. If* $a_1, a_2, a_3, a_4, a_5 \in R$ *such that*

$$a_1u^2 + ua_2u + a_3u^2a_4 + u^2a_5 = 0$$

for all $u \in f(R)$*, then* a_2 *is central.*

Lemma 4.2.5. *[36, Lemma 2.9] Let* R *be a noncommutative prime ring of char* $(R) \neq 2$ *and* $p(r_1, \ldots, r_n)$ *be any polynomial over* C *which is not an identity for* R*. If there exist* $a, b, c, v \in U$ *such that* $ap(r) + p(r)b + cp(r)v = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$*, then one of the following holds:*

(1) $b, v \in C$ *and* $a + b + cv = 0$*;*

(2) $a, c \in C$ *and* $a + b + cv = 0$*;*

(3) $a + b + cv = 0$ *and* $p(r)$ *is central valued on* R*.*

Lemma 4.2.6. *Let* R *be a prime ring of char*$(R) \neq 2$*,* $f(x_1, \ldots, x_n)$ *be a noncentral multilinear polynomial over* C *and there exist* $0 \neq p, a_1, a_2, a_3, a_4 \in R$ *such that*

$$pa_1 f(x)^2 + pf(x)(a_2 - a_3^2)f(x) - 2pf(x)a_3 f(x)a_4 - pf(x)^2 a_4^2 = 0$$

for all $x = (x_1, \ldots, x_n) \in R^n$*.*

If $p \in C$*, then one of the following holds:*

1. *there exist* $\lambda \in C$ *and* $a \in U$ *such that* $F(x) = \lambda x$ *and* $G(x) = xa$ *for all* $x \in R$*, with* $\lambda = a^2$*;*

2. *there exists* $b \in U$ *such that* $F(x) = xb^2$ *and* $G(x) = bx$ *for all* $x \in R$*;*

3. *there exists* $b \in U$ *such that* $F(x) = b^2 x$ *and* $G(x) = xb$ *for all* $x \in R$*, with* $f(R)^2 \in C$*.*

Proof. As $p \in C$, the given identity becomes

$$p\{a_1 f(x)^2 + f(x)(a_2 - a_3^2)f(x) - 2f(x)a_3 f(x)a_4 - f(x)^2 a_4^2\} = 0, \quad (4.2.2)$$

that is

$$a_1 f(x)^2 + f(x)(a_2 - a_3^2)f(x) - 2f(x)a_3 f(x)a_4 - f(x)^2 a_4^2 = 0 \quad (4.2.3)$$

for all $x = (x_1, \ldots, x_n) \in R^n$.

Then by Lemma 4.2.2, either $a_3 \in C$ or $a_4 \in C$.

Case-1: *When* $a_3 \in C$*.*

Then from above equation we get

$$a_1 f(x)^2 + f(x)(a_2 - a_3^2)f(x) - f(x)^2(2a_3 a_4 + a_4^2) = 0 \quad (4.2.4)$$

for all $x = (x_1, \ldots, x_n) \in R^n$. Then by applying Lemma 4.2.4 we get $a_2 - a_3^2 \in C$, that is $a_2 \in C$. Hence the above equation becomes

$$(a_1 + a_2 - a_3^2)f(x)^2 - f(x)^2(2a_3 a_4 + a_4^2) = 0 \quad (4.2.5)$$

for all $x = (x_1, \dots, x_n) \in R^n$. Then by using Lemma 4.2.5, we have the following results:

• $a_1 + a_2 - a_3^2 \in C$, $2a_3a_4 + a_4^2 \in C$ and $a_1 + a_2 - a_3^2 - 2a_3a_4 - a_4^2 = 0$. Now $a_1 + a_2 - a_3^2 \in C$ and $a_2 - a_3^2 \in C$ implies $a_1 \in C$. Also $a_1 + a_2 - a_3^2 - 2a_3a_4 - a_4^2 = 0$ implies $a_1 + a_2 = (a_3 + a_4)^2$. Therefore in this case $F(x) = a_1x + xa_2 = (a_1 + a_2)x$ where $a_1 + a_2 \in C$ and $G(x) = a_3x + xa_4 = x(a_3 + a_4)$ with $a_1 + a_2 = (a_3 + a_4)^2$. This is our conclusion (1).

• $f(R)^2 \in C$ and $a_1 + a_2 - a_3^2 - 2a_3a_4 - a_4^2 = 0$, that is $a_1 + a_2 = (a_3 + a_4)^2$. Therefore in this case $F(x) = a_1x + xa_2 = (a_1 + a_2)x$, $G(x) = a_3x + xa_4 = x(a_3 + a_4)$ with $a_1 + a_2 = (a_3 + a_4)^2$ and $f(R)^2 \in C$. This is our conclusion (2).

<u>Case-2</u>: *When $a_4 \in C$.*

In this case (4.2.3) yields

$$a_1f(x)^2 + f(x)(a_2 - 2a_3a_4 - a_3^2)f(x) - f(x)^2a_4^2 = 0 \tag{4.2.6}$$

for all $x = (x_1, \dots, x_n) \in R^n$. Then by applying Lemma 4.2.4, we get $a_2 - 2a_3a_4 - a_3^2 \in C$. Hence the equation becomes

$$(a_1 + a_2 - 2a_3a_4 - a_3^2)f(x)^2 - f(x)^2a_4^2 = 0 \tag{4.2.7}$$

for all $x = (x_1, \dots, x_n) \in R^n$. Then by using Lemma 4.2.5, we have the following results:

• $a_1 + a_2 - 2a_3a_4 - a_3^2 \in C$, $a_4^2 \in C$ and $a_1 + a_2 - a_3^2 - 2a_3a_4 - a_4^2 = 0$. Now $a_1 + a_2 - 2a_3a_4 - a_3^2 \in C$ and $a_2 - 2a_3a_4 - a_3^2 \in C$ implies $a_1 \in C$. Also $a_1 + a_2 - a_3^2 - 2a_3a_4 - a_4^2 = 0$ implies $a_1 + a_2 = (a_3 + a_4)^2$. Therefore in this case $F(x) = a_1x + xa_2 = x(a_1 + a_2)$ and $G(x) = a_3x + xa_4 = (a_3 + a_4)x$ with $a_1 + a_2 = (a_3 + a_4)^2$. This is our conclusion (3).

• $f(R)^2 \in C$ and $a_1 + a_2 - a_3^2 - 2a_3a_4 - a_4^2 = 0$, that is $a_1 + a_2 = (a_3 + a_4)^2$. Now $a_2 - 2a_3a_4 - a_3^2 \in C$ and $a_1 + a_2 - a_3^2 - 2a_3a_4 - a_4^2 = 0$ implies $a_1 \in C$. Therefore in this case $F(x) = a_1x + xa_2 = x(a_1 + a_2)$, $G(x) = a_3x + xa_4 = (a_3 + a_4)x$ with $a_1 + a_2 = (a_3 + a_4)^2$. This is our conclusion (3). □

Lemma 4.2.7. *Let R be a prime ring of char$(R) \neq 2$, $f(x_1, \dots, x_n)$ be a noncentral multilinear polynomial over C and there exist $0 \neq p, a_1, a_2, a_3, a_4 \in R$ such that*

$$pa_1f(x)^2 + pf(x)(a_2 - a_3^2)f(x) - 2pf(x)a_3f(x)a_4 - pf(x)^2a_4^2 = 0$$

for all $x = (x_1, \dots, x_n) \in R^n$.

If $a_3 \in C$, then one of the following holds:

1. *there exist* $\lambda \in C$ *and* $a \in U$ *such that* $F(x) = \lambda x$ *and* $G(x) = xa$ *for all* $x \in R$*, with* $\lambda = a^2$*;*

2. *there exists* $b \in U$ *such that* $F(x) = xb^2$ *and* $G(x) = bx$ *for all* $x \in R$*;*

3. *there exist* $a, b \in U$ *such that* $F(x) = ax$ *and* $G(x) = xb$ *for all* $x \in R$*, with* $b^2 \in C$ *and* $p(a - b^2) = 0$*;*

4. *there exist* $a, b \in U$ *such that* $F(x) = ax$ *and* $G(x) = xb$ *for all* $x \in R$*, with* $f(R)^2 \in C$ *and* $p(a - b^2) = 0$*.*

Proof. As $a_3 \in C$, the given identity becomes

$$pa_1 f(x)^2 + pf(x)(a_2 - a_3^2)f(x) - pf(x)^2(2a_3a_4 + a_4^2) = 0 \tag{4.2.8}$$

for all $x = (x_1, \ldots, x_n) \in R^n$. Then by Lemma 4.2.3, we get either $p \in C$ or $a_2 - a_3^2 \in C$.

If we take $p \in C$, then the conclusions (1)-(3) hold from Lemma 4.2.6. On the other hand if we consider $a_2 - a_3^2 \in C$, then the equation becomes

$$p(a_1 + a_2 - a_3^2)f(x)^2 - pf(x)^2(2a_3a_4 + a_4^2) = 0 \tag{4.2.9}$$

for all $x = (x_1, \ldots, x_n) \in R^n$. Then by using Lemma 4.2.5, we have the following results:

- $2a_3a_4 + a_4^2 \in C$ and $p(a_1 + a_2 - a_3^2) - p(2a_3a_4 + a_4^2) = 0$, that is $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$. Now $a_3 \in C$ and $a_2 - a_3^2 \in C$ implies $a_2 \in C$. Also $2a_3a_4 + a_4^2 \in C$ and $a_3 \in C$ implies $(a_3 + a_4)^2 \in C$. Therefore in this case $F(x) = a_1x + xa_2 = (a_1 + a_2)x$, $G(x) = a_3x + xa_4 = x(a_3 + a_4)$ where $(a_3 + a_4)^2 \in C$ with $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$. This is our conclusion (3).
- $p \in C$, $p(a_1 + a_2 - a_3^2) \in C$ and $p(a_1 + a_2 - a_3^2) - p(2a_3a_4 + a_4^2) = 0$. Since $p \in C$ so conclusion follows from Lemma 4.2.6.
- $f(R)^2 \in C$ and $p(a_1 + a_2 - a_3^2) - p(2a_3a_4 + a_4^2) = 0$, that is $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$. Now $a_3 \in C$ and $a_2 - a_3^2 \in C$ implies $a_2 \in C$. Therefore in this case $F(x) = a_1x + xa_2 = (a_1 + a_2)x$, $G(x) = a_3x + xa_4 = x(a_3 + a_4)$ with $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$ and $f(R)^2 \in C$. This is our conclusion (4). □

Lemma 4.2.8. *Let R be a prime ring of char$(R) \neq 2$, $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C and there exist $0 \neq p, a_1, a_2, a_3, a_4 \in R$ such that*

$$pa_1 f(x)^2 + pf(x)(a_2 - a_3^2)f(x) - 2pf(x)a_3 f(x)a_4 - pf(x)^2 a_4^2 = 0$$

for all $x = (x_1, \ldots, x_n) \in R^n$.

If $a_4 \in C$, then one of the following holds:

1. *there exist $\lambda \in C$ and $a \in U$ such that $F(x) = \lambda x$ and $G(x) = xa$ for all $x \in R$, with $\lambda = a^2$;*
2. *there exists $b \in U$ such that $F(x) = xb^2$ and $G(x) = bx$ for all $x \in R$;*
3. *there exists $b \in U$ such that $F(x) = b^2 x$ and $G(x) = xb$ for all $x \in R$, with $f(R)^2 \in C$;*
4. *there exist $a, b, c \in U$ such that $F(x) = ax + xb$ and $G(x) = cx$ for all $x \in R$, with $b - c^2 \in C$ and $p(a + b - c^2) = 0$.*

Proof. As $a_4 \in C$, the given identity becomes

$$pa_1 f(x)^2 + pf(x)(a_2 - a_3^2 - 2a_3 a_4)f(x) - pf(x)^2 a_4^2 = 0 \tag{4.2.10}$$

for all $x = (x_1, \ldots, x_n) \in R^n$. By Lemma 4.2.3, we get either $p \in C$ or $a_2 - a_3^2 - 2a_3 a_4 \in C$.

If we take $p \in C$, then the conclusions (1)-(3) hold from Lemma 4.2.6. On the other hand if we consider $a_2 - a_3^2 - 2a_3 a_4 \in C$, then the equation becomes

$$p(a_1 + a_2 - a_3^2 - 2a_3 a_4)f(x)^2 - pf(x)^2 a_4^2 = 0 \tag{4.2.11}$$

for all $x = (x_1, \ldots, x_n) \in R^n$. Then by using Lemma 4.2.5, we have the following results:

• $a_4^2 \in C$ and $p(a_1 + a_2 - a_3^2 - 2a_3 a_4) - pa_4^2 = 0$, that is $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$. Now $a_4 \in C$ and $a_2 - a_3^2 - 2a_3 a_4 \in C$ implies $a_2 - (a_3 + a_4)^2 \in C$. Therefore in this case $F(x) = a_1 x + xa_2$, $G(x) = a_3 x + xa_4 = (a_3 + a_4)x$ where $a_2 - (a_3 + a_4)^2 \in C$ with $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$. This is our conclusion (4).

• $p \in C$, $p(a_1 + a_2 - a_3^2 - 2a_3 a_4) \in C$ and $p(a_1 + a_2 - a_3^2 - 2a_3 a_4) - pa_4^2 = 0$, that is $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$. Since $p \in C$ so conclusions follows from Lemma 4.2.6.

- $f(R)^2 \in C$ and $p(a_1 + a_2 - a_3^2 - 2a_3a_4) - pa_4^2 = 0$, that is $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$. Now $a_4 \in C$ and $a_2 - a_3^2 - 2a_3a_4 \in C$ implies $a_2 - (a_3 + a_4)^2 \in C$. Therefore in this case $F(x) = a_1x + xa_2$, $G(x) = a_3x + xa_4 = (a_3 + a_4)x$ with $p\{(a_1 + a_2) - (a_3 + a_4)^2\} = 0$. This is our conclusion (4). □

Lemma 4.2.9. *Let $R = M_m(C)$, $m \geq 2$ be the ring of all $m \times m$ matrices over the field C and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C. If there exist $A, B, C, p, a_3, a_4 \in R$ such that*

$$Af(x)^2 + pf(x)Bf(x) - 2pf(x)a_3f(x)a_4 - pf(x)^2C = 0$$

for all $x = (x_1, \ldots, x_n) \in R^n$, then either p or a_3 or a_4 are scalar matrices.

Proof. First we consider C is infinite field.

To prove our conclusion, we assume on contrary that $p \notin Z(R)$, $a_3 \notin Z(R)$ and $a_4 \notin Z(R)$. We prove that this case leads to a contradiction. Since $p \notin Z(R)$, $a_3 \notin Z(R)$ and $a_4 \notin Z(R)$, by [62, Lemma 1], there exists an invertible matrix Q for which QpQ^{-1}, Qa_3Q^{-1} and Qa_4Q^{-1} have all nonzero entries. Now we consider the automorphism $\phi(x) = QxQ^{-1}$ for all $x \in R$. Since $\{f(x_1, \ldots, x_n) : x_1, \ldots, x_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, by hypothesis

$$\begin{aligned} \phi(A)f(x)^2 + \phi(p)f(x)\phi(B)f(x) - 2\phi(p)f(x)\phi(a_3)f(x)\phi(a_4) \\ -\phi(p)f(x)^2\phi(C) = 0 \end{aligned} \tag{4.2.12}$$

for all $x = (x_1, \ldots, x_n) \in R$. By e_{ij}, we consider the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. Then it is obvious that $e_{ij}^2 = 0$. Since $f(x_1, \ldots, x_n)$ is not central valued, by [78] there exist some matrices $r_1, \ldots, r_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$, with $i \neq j$. Therefore substituting $f(x_1, \ldots, x_n) = \gamma e_{ij}$ in (4.2.12), we get

$$\phi(p)e_{ij}\phi(B)e_{ij} - 2\phi(p)e_{ij}\phi(a_3)e_{ij}\phi(a_4) = 0$$

Multiplying by e_{ij} in right side and left side of the above relation, we have

$$2e_{ij}\phi(p)e_{ij}\phi(a_3)e_{ij}\phi(a_4)e_{ij} = 0$$

which gives a contradiction, since $\phi(p)$, $\phi(a_3)$ and $\phi(a_4)$ have all nonzero entries. Thus we conclude that either p or a_3 or a_4 are scalar matrices, when C is infinite field.

Now we consider C is finite field. Let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \dots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Now the GPI is

$$\begin{gathered}\Gamma(x_1, \dots, x_n) = \\ Af(x_1, \dots, x_n)^2 + pf(x_1, \dots, x_n)Bf(x_1, \dots, x_n) \\ -2pf(x_1, \dots, x_n)a_3 f(x_1, \dots, x_n)a_4 - pf(x_1, \dots, x_n)^2 C = 0. \end{gathered} \tag{4.2.13}$$

which is a GPI for R.
Also, it is a multi-homogeneous of multi-degree $(2, \dots, 2)$ in the indeterminates $x_1, \dots, x_n$.

Hence the complete linearization of $\Gamma(x_1, \dots, x_n)$ yields a multilinear generalized polynomial $\Theta(x_1, \dots, x_n, s_1, \dots, s_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1, \dots, x_n, x_1, \dots, x_n) = 2^n \Gamma(x_1, \dots, x_n).$$

Clearly the multilinear polynomial $\Theta(x_1, \dots, x_n, s_1, \dots, s_n) = 0$ is a generalized polynomial identity for R and $\overline{R}$ too. Since char $(C) \neq 2$ we obtain $\Gamma(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in \overline{R}$ and then conclusion follows as above when C was infinite. □

Lemma 4.2.10. *Let R be a prime ring of char $(R) \neq 2$ and $f(x_1, \dots, x_n)$ be a non-central multilinear polynomial over C. If there exist $A, B, C, p, a_3, a_4 \in R$ such that*

$$Af(x)^2 + pf(x)Bf(x) - 2pf(x)a_3 f(x)a_4 - pf(x)^2 C = 0$$

for all $x = (x_1, \dots, x_n) \in R^n$, then either $p \in C$ or $a_3 \in C$ or $a_4 \in C$.

Proof. Let $p \notin C$, $a_3 \notin C$ and $a_4 \notin C$. Now we show that this assumption leads to a contradiction. By hypothesis, we have

$$\begin{gathered}\Gamma(x_1, \dots, x_n) = \\ Af(x_1, \dots, x_n)^2 + pf(x_1, \dots, x_n)Bf(x_1, \dots, x_n) \\ -2pf(x_1, \dots, x_n)a_3 f(x_1, \dots, x_n)a_4 - pf(x_1, \dots, x_n)^2 C = 0 \end{gathered} \tag{4.2.14}$$

for all $x_1, \dots, x_n \in R$.

Since R and U satisfy the same GPIs (see [24]) so, U satisfies $\Gamma(x_1, \dots, x_n) = 0$. Suppose that R does not satisfy any nontrivial GPI. Let $T = U *_C C\{x_1, x_2, \dots, x_n\}$, the free product of U and $C\{x_1, \ldots, x_n\}$, the free C-algebra in noncommuting indeterminates $x_1, x_2, \ldots, x_n$. Then $\Gamma(x_1, \ldots, x_n)$ is zero element in T. This gives that $\{A, p\}$ is linearly C-dependent, hence there exist $\alpha \in C$ such that $A = \alpha p$. Then

$$\alpha p f(x_1, \ldots, x_n)^2 + p f(x_1, \ldots, x_n) B f(x_1, \ldots, x_n)$$
$$-2p f(x_1, \ldots, x_n) a_3 f(x_1, \ldots, x_n) a_4 - p f(x_1, \ldots, x_n)^2 C = 0,$$

that is

$$p f(x_1, \ldots, x_n)\{\alpha(x_1, \ldots, x_n) + B f(x_1, \ldots, x_n)$$
$$-2a_3 f(x_1, \ldots, x_n) a_4 - f(x_1, \ldots, x_n) C\} = 0. \quad (4.2.15)$$

for all $x_1, \ldots, x_n \in R$. Since $p, a_3, a_4 \notin C$, the term $-2p f(x_1, \ldots, x_n) a_3 f(x_1, \ldots, x_n) a_4$ can not be canceled and hence we arrive at a contradiction.

Next suppose that U satisfies the non-trivial GPI $\Gamma(x_1, \ldots, x_n) = 0$. Then by the well known theorem of Martindale (see Theorem 1.6.6), U is a primitive ring with nonzero socle H and with C as its associated division ring. By Jacobson's theorem (see Theorem 1.6.5), U is isomorphic to a dense ring of linear transformations of a vector space V over C.

If V is finite dimensional over C, that is, $\dim_C V = m$, then by density of U, we have $U \cong M_m(C)$. As U is noncommutative, $m \geq 2$. By Lemma 4.2.9, $P \in C$ or $a_3 \in C$ or $a_4 \in C$, a contradiction.

If V is infinite dimensional over C, then by [98, Lemma 2], the set $f(U)$ is dense on U. Thus U satisfies

$$Ax^2 + pxBx - 2pxa_3xa_4 - px^2C = 0. \quad (4.2.16)$$

Since $p \notin C$, $a_3 \notin C$ and $a_4 \notin C$, they can not commute any nonzero ideal of U, i.e., $[p, H] \neq (0)$, $[a_3, H] \neq (0)$ and $[a_4, H] \neq (0)$. Therefore, there exists $h_1, h_2, h_3 \in H$ such that $[p, h_1] \neq 0$, $[a_3, h_2] \neq 0$ and $[a_4, h_3] \neq 0$. By [56], there exists idempotent $e \in H$ such that $ph_1, h_1p, a_3h_2, h_2a_3, a_4h_3, h_3a_4, h_1, h_2, h_3 \in eUe$. Since U satisfies generalized identity

$$e\{A(exe)^2 + pexeBexe - 2pexea_3exea_4 - p(exe)^2C\}e = 0,$$

the subring eUe satisfies

$$(eAe)x^2 + (epe)x(eBe)x - 2(epe)x(ea_3e)x(ea_4e) - (epe)x^2(eCe) = 0.$$

Since $eUe \cong M_t(C)$ with $t = \dim_C Ve$, by above argument p or a_3 or a_4 are central elements of eUe. But then we have contradiction as $ph_1 = (epe)h_1 = h_1epe = h_1p$ or $a_3h_2 = (ea_3e)h_2 = h_2(ea_3e) = h_2a_3$ or $a_4h_3 = (ea_4e)h_3 = h_3(ea_4e) = h_3a_4$. □

Proof of the Proposition 4.2.1. *By hypothesis, we have*

$$pa_1f(x)^2 + pf(x)(a_2 - a_3^2)f(x) - 2pf(x)a_3f(x)a_4 - pf(x)^2a_4^2 = 0$$

for all $x = (x_1, \ldots, x_n) \in R^n$. Then by Lemma 4.2.10, we get either p or a_3 or a_4 are central. Therefore from Lemma 4.2.6, Lemma 4.2.7 and Lemma 4.2.8 we get the conclusions (1)-(5).

4.3 Proof of Main Theorem

In light of the notion in [79, Theorem 3], we may assume that there exist $a, b \in U$ and derivations d, δ of U such that $F(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$.

We shall use the notation

$$f(x_1, \ldots, x_n) = x_1x_2\cdots x_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$, and S_n denotes the symmetric group of degree n.

Then we have

$$d(f(x_1, \ldots, x_n)) = f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n),$$

where $f^d(x_1, \ldots, x_n)$ be the polynomials obtained from $f(x_1, \ldots, x_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$. Similarly, by calculation, we have

$$\begin{aligned} d^2(f(x_1, \ldots, x_n)) &= f^{d^2}(x_1, \ldots, x_n) + 2\sum_i f^d(x_1, \ldots, d(x_i), \ldots, x_n) \\ &\quad + \sum_i f(x_1, \ldots, d^2(x_i), \ldots, x_n) \\ &\quad + \sum_i f(x_1, \ldots, d(x_i), \ldots, d(x_j), \ldots, x_n). \end{aligned}$$

By hypothesis

$$p\Big(F\big(f(x)\big)f(x) - f(x)G^2\big(f(x)\big)\Big) = 0$$

for all $x = (x_1, \ldots, x_n) \in R^n$.

Since R and U satisfy the same generalized polynomial identities as well as the same differential identities(see [24] and [82]), without loss of generality, we may assume

$$p\Big(F\big(f(x)\big)f(x) - f(x)G^2\big(f(x)\big)\Big) = 0$$

for all $x = (x_1, \ldots, x_n) \in U^n$. Hence U satisfies

$$\begin{aligned} p\Big\{ \Big(af(x) + d\big(f(x)\big)\Big)f(x) - f(x)b\Big(bf(x) + \delta\big(f(x)\big)\Big) \\ -f(x)\delta\Big(bf(x) + \delta\big(f(x)\big)\Big)\Big\} = 0 \end{aligned} \tag{4.3.1}$$

for all $x = (x_1, \ldots, x_n) \in U^n$, where d, δ are two derivations on U not both inner.

Case-1: When d and δ are linearly C-independent modulo inner derivations of U.

Now from (4.3.1), U satisfies

$$\begin{aligned} & p\Big\{af(x_1, \ldots, x_n)^2 - f(x_1, \ldots, x_n)b^2 f(x_1, \ldots, x_n) \\ & +\Big(f^d(x_1, \ldots, x_n) + \sum_i f\big(x_1, \ldots, d(x_i), \ldots, x_n\big)\Big)f(x_1, \ldots, x_n) \\ & -2f(x_1, \ldots, x_n)b\Big(f^\delta(x_1, \ldots, x_n) + \sum_i f\big(x_1, \ldots, \delta(x_i), \ldots, x_n\big)\Big) \\ & -f(x_1, \ldots, x_n)\delta(b)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)\Big(f^{\delta^2}(x_1, \ldots, x_n) \\ & +2\sum_i f^\delta\big(x_1, \ldots, \delta(x_i), \ldots, x_n\big) + \sum_i f\big(x_1, \ldots, \delta^2(x_i), \ldots, x_n\big) \\ & +\sum_{i \neq j} f\big(x_1, \ldots, \delta(x_i), \ldots, \delta(x_j), \ldots, x_n\big)\Big)\Big\} = 0 \end{aligned} \tag{4.3.2}$$

for all $x_1, \ldots, x_n \in U$. Since d and δ are not inner, by Kharchenko's theorem (see Theorem 1.6.4), U satisfies

$$\begin{aligned} & p\Big\{af(x_1, \ldots, x_n)^2 - f(x_1, \ldots, x_n)b^2 f(x_1, \ldots, x_n) \\ & +\Big(f^d(x_1, \ldots, x_n) + \sum_i f\big(x_1, \ldots, x_i, \ldots, x_n\big)\Big)f(x_1, \ldots, x_n) \\ & -2f(x_1, \ldots, x_n)b\Big(f^\delta(x_1, \ldots, x_n) + \sum_i f\big(x_1, \ldots, y_i, \ldots, x_n\big)\Big) \\ & -f(x_1, \ldots, x_n)\delta(b)f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)\Big(f^{\delta^2}(x_1, \ldots, x_n) \\ & +2\sum_i f^\delta\big(x_1, \ldots, y_i, \ldots, x_n\big) + \sum_i f\big(x_1, \ldots, t_i, \ldots, x_n\big) \\ & +\sum_{i \neq j} f\big(x_1, \ldots, y_i, \ldots, y_j, \ldots, x_n\big)\Big)\Big\} = 0. \end{aligned} \tag{4.3.3}$$

In particular U satisfies the blended component

$$pf(x_1,\ldots,x_n)\sum_i f\big(x_1,\ldots,t_i,\ldots,x_n\big)=0.$$

For $t_1=x_1$ and $t_2=\cdots=t_n=0$ we get

$$pf(x_1,\ldots,x_n)^2=0, \tag{4.3.4}$$

implying either $p=0$ or $f(x_1,\ldots,x_n)^2=0$ for all $x_1,\ldots,x_n\in U$, that is either $p=0$ or $f(x_1,\ldots,x_n)=0$ for all $x_1,\ldots,x_n\in U$ (see [26]), a contradiction.

Case-2: When d and δ are linearly C-dependent modulo inner derivations of U. *In this case we get $\alpha,\beta\in C$ and $q\in U$ such that $\alpha d+\beta\delta=ad_q$. It is clear from the context that $(\alpha,\beta)\neq(0,0)$. So without loss of generality we arrive the following two subcases:*

Subcase-1: When $\alpha=0$.

Then we get $\delta(x)=[p_1,x]$, where $p_1=\beta^{-1}q$. Obviously d will not be an inner derivation of U. Now from (4.3.1) U satisfies

$$\begin{aligned}&p\Big\{af(x_1,\ldots,x_n)^2-f(x_1,\ldots,x_n)b^2f(x_1,\ldots,x_n)\\&+\Big(f^d(x_1,\ldots,x_n)+\sum_i f\big(x_1,\ldots,d(x_i),\ldots,x_n\big)\Big)f(x_1,\ldots,x_n)\\&-2f(x_1,\ldots,x_n)b\Big[p_1,f(x_1,\ldots,x_n)\Big]-f(x_1,\ldots,x_n)\Big[p_1,b\Big]f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big[p_1,[p_1,f(x_1,\ldots,x_n)]\Big]\Big\}=0\end{aligned} \tag{4.3.5}$$

for all $x_1,\ldots,x_n\in U$. Since d is not inner, by Kharchenko's theorem (see Theorem 1.6.4), U satisfies

$$\begin{aligned}&p\Big\{af(x_1,\ldots,x_n)^2-f(x_1,\ldots,x_n)b^2f(x_1,\ldots,x_n)\\&+\Big(f^d(x_1,\ldots,x_n)+\sum_i f\big(x_1,\ldots,r_i,\ldots,x_n\big)\Big)f(x_1,\ldots,x_n)\\&-2f(x_1,\ldots,x_n)b\Big[p_1,f(x_1,\ldots,x_n)\Big]-f(x_1,\ldots,x_n)\Big[p_1,b\Big]f(x_1,\ldots,x_n)\\&-f(x_1,\ldots,x_n)\Big[p_1,[p_1,f(x_1,\ldots,x_n)]\Big]\Big\}=0.\end{aligned} \tag{4.3.6}$$

In particular U satisfies the blended component

$$p\sum_i f\big(x_1,\ldots,r_i,\ldots,x_n\big)f(x_1,\ldots,x_n)=0.$$

For $r_1 = x_1$ and $r_2 = \cdots = r_n = 0$ we get $pf(x_1,\dots,x_n)^2 = 0$ which is same as (4.3.4)*, so we get a contradiction.*

Subcase-2: When $\alpha \neq 0$.

Then we have $d = \mu\delta + ad_c$, for some $\mu \in C$ and $c \in U$. Here δ never be an inner derivation, otherwise both d and δ will be inner, a contradiction. Then from (4.3.1) we have

$$\begin{aligned}
&p\Big\{af(x_1,\dots,x_n)^2 - f(x_1,\dots,x_n)b^2 f(x_1,\dots,x_n)\\
&+\mu\Big(f^\delta(x_1,\dots,x_n) + \sum_i f\big(x_1,\dots,\delta(x_i),\dots,x_n\big)\Big)f(x_1,\dots,x_n)\\
&+\Big[c, f(x_1,\dots,x_n)\Big]f(x_1,\dots,x_n)\\
&-2f(x_1,\dots,x_n)b\Big(f^\delta(x_1,\dots,x_n) + \sum_i f\big(x_1,\dots,\delta(x_i),\dots,x_n\big)\Big)\\
&-f(x_1,\dots,x_n)\delta(b)f(x_1,\dots,x_n) - f(x_1,\dots,x_n)\Big(f^{\delta^2}(x_1,\dots,x_n)\\
&+2\sum_i f^\delta\big(x_1,\dots,\delta(x_i),\dots,x_n\big) + \sum_i f\big(x_1,\dots,\delta^2(x_i),\dots,x_n\big)\\
&+\sum_{i\neq j} f\big(x_1,\dots,\delta(x_i),\dots,\delta(x_j),\dots,x_n\big)\Big)\Big\} = 0. \qquad (4.3.7)
\end{aligned}$$

The above differential identity is involved with the derivation words appear of the type δ and δ^2. Now by Kharchenko's theorem (see Theorem 1.6.4) we may replace $\delta(x_i) = y_i$ and $\delta^2(x_i) = t_i$, then U satisfies

$$\begin{aligned}
&p\Big\{af(x_1,\dots,x_n)^2 - f(x_1,\dots,x_n)b^2 f(x_1,\dots,x_n)\\
&+\mu\Big(f^\delta(x_1,\dots,x_n) + \sum_i f\big(x_1,\dots,y_i,\dots,x_n\big)\Big)f(x_1,\dots,x_n)\\
&+\Big[c, f(x_1,\dots,x_n)\Big]f(x_1,\dots,x_n)\\
&-2f(x_1,\dots,x_n)b\Big(f^\delta(x_1,\dots,x_n) + \sum_i f\big(x_1,\dots,y_i,\dots,x_n\big)\Big)\\
&-f(x_1,\dots,x_n)\delta(b)f(x_1,\dots,x_n) - f(x_1,\dots,x_n)\Big(f^{\delta^2}(x_1,\dots,x_n)\\
&+2\sum_i f^\delta\big(x_1,\dots,y_i,\dots,x_n\big)\\
&+\sum_i f\big(x_1,\dots,t_i,\dots,x_n\big) + \sum_{i\neq j} f\big(x_1,\dots,y_i,\dots,y_j,\dots,x_n\big)\Big)\Big\} = 0 \qquad (4.3.8)
\end{aligned}$$

In particular U satisfies the blended component

$$pf(x_1,\dots,x_n)\sum_i f\big(x_1,\dots,t_i,\dots,x_n\big) = 0.$$

For $t_1 = x_1$ *and* $t_2 = \cdots = t_n = 0$ *we get* $pf(x_1, \ldots, x_n)^2 = 0$ *which is same as* (4.3.4)*, so we get a contradiction. Thus the proof of the theorem is completed.* □

Chapter 5

Generalized Skew Derivations and Generalization of Commuting Maps in Prime Rings

5.1 Introduction

Let R be a prime ring of characteristic different from 2. Also Q_r is the right Martindale quotient ring of R, $C = Z(Q_r)$ is the extended centroid of R and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C. Let α be an automorphism of R.

Let S be a nonzero subset of R and $F : R \to R$, $G : R \to R$ are two additive mappings of R. The map F is said to be commuting on S (centralizing on S), if $F(x)x - xF(x) = 0$ for all $x \in S$ (resp. $F(x)x - xF(x) \in Z(R)$ for all $x \in S$). We generalize the concept of commuting maps by considering another map G such that $F(x)x - G(x)F(x) = 0$ for all $x \in S$.

In [20], Carini, De Filippis and Scudo studied the case when $F(x)G(x) = 0$ for all $x \in f(R)$, where F and G are generalized skew derivations of R associated to the same automorphism and then describe all possible forms of F and G.

In [21], Carini, De Filippis and Wei investigated the situation when generalized skew derivations F and G of R are co-commuting on $f(R)$, that is, $F(x)x - xG(x) = 0$ for all $x \in f(R)$ and then obtain all possible forms of the maps F and G.

In a recent paper, Tiwari [92] studied the situation $F(x)G(x) - xH(x) = 0$ for all $x \in f(R)$ and obtain all possible forms of the maps F and G, where F and G are generalized derivations of R.

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In the present Chapter, we shall study the case $F(x)x - G(x)F(x) = 0$ for all $x \in f(R)$ and F, G two nonzero generalized skew-derivations of R. More precisely, we prove the following:

Theorem 5.1.1. *Let R be a prime ring of characteristic different from 2, Q_r be the right Martindale quotient ring of R and $C = Z(Q_r)$ be the extended centroid of R. Suppose that $f(x_1, \ldots, x_n)$ is a noncentral multilinear polynomial over C and F, G are two nonzero generalized skew-derivations of R. If*

$$F(x)x - G(x)F(x) = 0$$

for all $x \in f(R)$, then one of the following holds:

1. *$F = G = Id$, the identity map on R;*
2. *there exist $a, p \in Q_r$ such that $F(x) = ax$ and $G(x) = pxp^{-1}$ for all $x \in R$, with $p^{-1}a \in C$;*
3. *there exist $a, c, p \in Q_r$ such that $F(x) = ax + pxp^{-1}c$ and $G(x) = pxp^{-1}$ for all $x \in R$, with $f(R)^2 \in C$ and $p^{-1}(a - c) \in C$;*
4. *there exist $a, p \in Q_r$ such that $F(x) = ax - pxp^{-1}a$ and $G(x) = -pxp^{-1}$ for all $x \in R$, with $f(R)^2 \in C$.*

5.2 Some reductions in case of inner generalized skew derivations.

Here we look attentively at the case of inner generalized skew derivations associated to the same inner automorphism and collect a number of results which can be deduced by others present in the literature.
We assume that $F(x) = ax + pxp^{-1}c$ and $G(x) = bx + pxp^{-1}q$ for all $x \in R$ and for some $a, b, c, p, q \in Q_r$. Now $F(x)x - G(x)F(x) = 0$ for all $x \in f(R)$ gives

$$(ax + pxp^{-1}c)x - (bx + pxp^{-1}q)(ax + pxp^{-1}c) = 0$$

for all $x \in f(R)$. Left multiplying by p^{-1} to the above expression, we get

$$(p^{-1}ax + xp^{-1}c)x - (p^{-1}bx + xp^{-1}q)(ax + pxp^{-1}c) = 0,$$

that is

$$(p^{-1}ax + xp^{-1}c)x - p^{-1}bx(ax + pxp^{-1}c) - xp^{-1}qax$$
$$-xp^{-1}qpxp^{-1}c = 0 \tag{5.2.1}$$

for all $x \in f(R)$. *Therefore, we consider the generalized polynomial identity*

$$a_1x^2 + xa_2x + a_3xa_4x + a_3xa_5xa_6 + xa_7xa_2 = 0 \tag{5.2.2}$$

for all $x \in f(R)$, *where* $a_1 = p^{-1}a$, $a_2 = p^{-1}(c - qa)$, $a_3 = -p^{-1}b$, $a_4 = a$, $a_5 = p$, $a_6 = p^{-1}c$, $a_7 = -p^{-1}qp$.

We firstly state the following result, which is a direct consequence of [44, Lemma 2.4]:

Lemma 5.2.1. *Let* R *be a noncommutative prime ring of char* $(R) \neq 2$, Q_r *be its right Martindale quotient ring and also* C *the extended centroid of* R. *If* $f(x_1, \ldots, x_n)$ *is a non-central multilinear polynomial over* C *and* $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in R$ *such that* R *satisfies (5.2.2), then either* $a_3 \in C$ *or* $a_5 \in C$ *or* $a_6 \in C$.

Moreover, we also get the following five Lemmas:

Lemma 5.2.2. *Let* R *be a prime ring of char* $(R) \neq 2$, C *the extended centroid of* R *and* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C.
If $a_1, a_2, a_3, a_4 \in R$ *such that*

$$a_1x^2 + xa_2x + xa_3xa_4 = 0$$

for all $x \in f(R)$, *then either* a_3 *or* a_4 *is central.*

Lemma 5.2.3. *Let* R *be a prime ring of char* $(R) \neq 2$, C *the extended centroid of* R *and* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C. *If* $a_1, a_2, a_3, a_4 \in R$ *such that*

$$a_1x^2 + a_2xa_3x + xa_4x = 0$$

for all $x \in f(R)$, *then either* a_2 *or* a_3 *is central.*

Lemma 5.2.4. *Let* R *be a prime ring of char* $(R) \neq 2$, C *the extended centroid of* R *and* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C.
If $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \in R$ *such that*

$$a_1x^2 + xa_2x + a_3xa_4x + xa_5xa_6 + a_7x^2a_8 = 0$$

for all $x \in f(R)$, *then either* a_3 *or* a_4 *are central and either* a_5 *or* a_6 *are central.*

Lemma 5.2.5. *Let R be a prime ring of char $(R) \neq 2$, C the extended centroid of R and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C.*
If $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in R$ such that

$$a_1x^2 + xa_2x + a_3xa_4x + x^2a_5 + a_6x^2a_7 = 0$$

for all $x \in f(R)$, then either a_3 or a_4 are central.

Lemma 5.2.6. *(see [42, Proposition 2.7]) Let R be a prime ring of char $(R) \neq 2$, C the extended centroid of R and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over C. If $a_1, a_2, a_3, a_4, a_5 \in R$ such that*

$$a_1x^2 + xa_2x + a_3x^2a_4 + x^2a_5 = 0$$

for all $x \in f(R)$, then a_2 is central.

For sake of brevity, we omit the proofs of Lemmas 5.2.2, 5.2.3, 5.2.4, 5.2.5 and 5.2.6. To be more precise, those results can be proved by using the same techniques and computations contained in [44, Lemma 2.4].
We finally recall the following result:

Lemma 5.2.7. *[10, Lemma 3] Let R be a noncommutative prime ring with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, which is not central valued on R. Suppose that there exist $a, b, c, q \in U$ such that $(ax + xb)x - x(cx + xq) = 0$ for all $x \in f(R)$. Then one of the following holds:*

(1) $a, q \in C$ and $q - a = b - c = \alpha \in C$;

(2) $f(x_1, \ldots, x_n)^2$ is central valued on R and there exists $\alpha \in C$ such that $q - a = b - c = \alpha$;

(3) char $(R) = 2$ and R satisfies s_4.

5.3 The proof of Main Theorem for inner generalized skew derivations.

We start with the following:

Lemma 5.3.1. *Let R be a noncommutative prime ring of char$(R) \neq 2$, Q_r be its right Martindale quotient ring and C be its extended centroid. Also let $f(x_1, \ldots, x_n)$*

be a noncentral multilinear polynomial over C and $a, b, c, p, q \in Q_r$ such that R satisfies (5.2.1).

If $p^{-1}b \in C$, then one of the following holds:

1. *$p^{-1}a, p^{-1}c(b+q), b+q \in C$ with $(a+c)(b+q-1) = 0$;*

2. *$f(R)^2 \in C$, $b+q \in C$ and $p^{-1}\{c(b+q) - a\} = p^{-1}\{c - (b+q)a\} \in C$;*

3. *$p^{-1}c \in C$ with $p^{-1}a = -p^{-1}(c - (b+q)(a+c)) \in C$.*

Proof. As $p^{-1}b \in C$, (5.2.1) becomes

$$(p^{-1}ax + xp^{-1}c)x - x(p^{-1}bax + p^{-1}bpxp^{-1}c) - xp^{-1}qax$$
$$-xp^{-1}qpxp^{-1}c = 0, \qquad (5.3.1)$$

that is

$$p^{-1}ax^2 + xp^{-1}(c - ba - qa)x - xp^{-1}(b+q)pxp^{-1}c = 0 \qquad (5.3.2)$$

for all $x \in f(R)$.

Then by Lemma 5.2.2, either $p^{-1}(b+q)p \in C$ or $p^{-1}c \in C$.

Case-1: *When $p^{-1}(b+q)p \in C$.*

Then $p^{-1}(b+q)p \in C$ implies $b+q \in C$ and hence from above

$$p^{-1}ax^2 + xp^{-1}(c - ba - qa)x - x^2p^{-1}c(b+q) = 0 \qquad (5.3.3)$$

for all $x \in f(R)$, which can be re-written as

$$\{p^{-1}ax + xp^{-1}c)\}x - x\{p^{-1}(b+q)ax + xp^{-1}c(b+q)\} = 0. \qquad (5.3.4)$$

Then by applying Lemma 5.2.7, we have the following results:

• $p^{-1}a \in C$, $p^{-1}c(b+q) \in C$ with $\{p^{-1}c(b+q) - p^{-1}a\} = \{p^{-1}c - p^{-1}(b+q)a\} = \alpha \in C$.

Now $\{p^{-1}c(b+q) - p^{-1}a\} = \{p^{-1}c - p^{-1}(b+q)a\}$ gives $(a+c)(b+q-1) = 0$. Thus we obtain conclusion (1).

• $f(R)^2 \in C$ and $\{p^{-1}c(b+q) - p^{-1}a\} = \{p^{-1}c - p^{-1}(b+q)a\} = \alpha \in C$.

Case-2: *When $p^{-1}c \in C$.*

In this case (5.3.2) yields

$$p^{-1}ax^2 + xp^{-1}(c - ba - qa)x - xp^{-1}(b+q)cx = 0, \qquad (5.3.5)$$

that is

$$\Big\{p^{-1}ax + xp^{-1}(c-(b+q)(a+c))\Big\}x = 0 \tag{5.3.6}$$

for all $x \in f(R)$. Then by Lemma 5.2.7, $p^{-1}a = -p^{-1}(c-(b+q)(a+c)) \in C$. This is our conclusion (3). □

Lemma 5.3.2. *Let R be a noncommutative prime ring of $char(R) \neq 2$, Q_r be its right Martindale quotient ring and C be its extended centroid. Also let $f(x_1,\ldots,x_n)$ be a noncentral multilinear polynomial over C and $a,b,c,p,q \in Q_r$ such that R satisfies (5.2.1).*

If $p^{-1}c \in C$, then one of the following holds:

1. $p^{-1}b, p^{-1}a, p^{-1}c(b+q), b+q \in C$ *with* $(a+c)(b+q-1) = 0$*;*
2. $f(R)^2 \in C$, $b+q, p^{-1}b \in C$ *and* $p^{-1}\{c(b+q)-a\} = p^{-1}\{c-(b+q)a\} \in C$*;*
3. $p^{-1}b \in C$ *with* $p^{-1}a = -p^{-1}(c-(b+q)(a+c)) \in C$;
4. $a+c \in C$ *and* $p^{-1}(a+c)(1-b) = p^{-1}(a+c)q \in C$.

Proof. By (5.2.1),

$$p^{-1}(a+c)x^2 - p^{-1}bx(a+c)x - xp^{-1}q(a+c)x = 0 \tag{5.3.7}$$

for all $x \in f(R)$. Then by Lemma 5.2.3, either $p^{-1}b \in C$ or $a+c \in C$. If $p^{-1}b \in C$, we have our conclusions (1)-(3) by Lemma 5.3.1. Thus we consider the case when $a+c \in C$. By (5.3.7),

$$\{p^{-1}(a+c)(1-b)x - xp^{-1}(a+c)q\}x = 0 \tag{5.3.8}$$

for all $x \in f(R)$. By Lemma 5.2.7, $p^{-1}(a+c)(1-b) = p^{-1}(a+c)q \in C$. This is our conclusion (4). □

Lemma 5.3.3. *Let R be a noncommutative prime ring of char $(R) \neq 2$, Q_r be its right Martindale quotient ring and C be its extended centroid. Also let $f(x_1,\ldots,x_n)$ be a noncentral multilinear polynomial over C and $a,b,c,p,q \in Q_r$ such that R satisfies (5.2.1).*

If $p \in C$, then one of the following holds:

1. $a, b, c(b+q), q \in C$ *with* $(a+c)(b+q-1) = 0$*;*
2. $f(R)^2 \in C$, $q, b \in C$ *and* $c(b+q)-a = c-(b+q)a \in C$*;*

3. *$a, b, c \in C$ with $a = -c + (b+q)(a+c) \in C$;*
4. *$a, c \in C$ and $(a+c)(1-b) = (a+c)q \in C$.*

Proof. By (5.2.1),

$$(ax + xc)x - bx(ax + xc) - xqax - xqxc = 0, \tag{5.3.9}$$

that is

$$ax^2 + x(c - qa)x - bxax - xqxc - bx^2c = 0 \tag{5.3.10}$$

for all $x \in f(R)$. Then by Lemma 5.2.4, either $q \in C$ or $c \in C$. In case $c \in C$, we have $p^{-1}c \in C$ and so we get conclusions (1)-(4) by Lemma 5.3.2. So $q \in C$ and hence by (5.3.10),

$$ax^2 + x(c - qa)x - bxax - x^2qc - bx^2c = 0 \tag{5.3.11}$$

for all $x \in f(R)$. Again by Lemma 5.2.5, either $b \in C$ or $a \in C$. Now $b \in C$ implies $p^{-1}b \in C$ and then conclusions (1)-(3) follows by Lemma 5.3.1. Thus we consider the case only $a \in C$. Then (5.3.11) becomes

$$(a - aq - ab)x^2 + xcx - bx^2c - x^2qc = 0 \tag{5.3.12}$$

for all $x \in f(R)$. This implies by Lemma 5.2.6, $c \in C$ and hence $p^{-1}c \in C$ and so we are done again by Lemma 5.3.2. □

Lemma 5.3.4. *Let R be a noncommutative prime ring of characteristic different from 2, Q_r be its right Martindale quotient ring and C be its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, $F(x) = ax + pxp^{-1}c$ and $G(x) = bx + pxp^{-1}q$ for some $a, b, c, p, q \in Q_r$ such that*

$$F(x)x - G(x)F(x) = 0$$

for all $x \in f(R)$. Then one of the following holds:

1. *$F(x) = (a+c)x$ and $G(x) = pxp^{-1}$ for all $x \in R$, with $p^{-1}(a+c) \in C$;*
2. *$F(x) = ax + pxp^{-1}c$ and $G(x) = pxp^{-1}$ for all $x \in R$, with $f(R)^2 \in C$ and $p^{-1}(a-c) \in C$;*
3. *$F(x) = ax - pxp^{-1}a$ and $G(x) = -pxp^{-1}$ for all $x \in R$, with $f(R)^2 \in C$.*

Proof. By hypothesis, we have

$$(ax + pxp^{-1}c)x - (bx + pxp^{-1}q)(ax + pxp^{-1}c) = 0$$

for all $x \in f(R)$. Left multiplying by p^{-1} to the above expression, we get

$$(p^{-1}ax + xp^{-1}c)x - (p^{-1}bx + xp^{-1}q)(ax + pxp^{-1}c) = 0,$$

that is

$$\begin{aligned}(p^{-1}ax + xp^{-1}c)x - p^{-1}bxax - p^{-1}bxpxp^{-1}c - xp^{-1}qax \\ -xp^{-1}qpxp^{-1}c = 0 \qquad (5.3.13)\end{aligned}$$

for all $x \in f(R)$. Then by Lemma 5.2.1, either $p^{-1}b \in C$ or $p \in C$ or $p^{-1}c \in C$. Thus we consider the three cases:

Case-1: *When* $p^{-1}b \in C$.

Then by Lemma 5.3.1 one of the following holds:

• $p^{-1}a, p^{-1}c(b+q), b+q \in C$ with $(a+c)(b+q-1) = 0$. Since $b+q-1 \in C$, we have either $b+q = 1$ or $a+c = 0$. If $a+c = 0$, we have $p^{-1}F(x) = p^{-1}ax + xp^{-1}c = xp^{-1}(a+c) = 0$ and hence $F = 0$, a contradiction. Thus we have $b+q = 1$ and so $p^{-1}a, p^{-1}c \in C$. Therefore, $F(x) = ax + pxp^{-1}c = (a+c)x$ and $p^{-1}G(x) = p^{-1}bx + xp^{-1}q = xp^{-1}(b+q) = xp^{-1}$ and so $G(x) = pxp^{-1}$ for all $x \in R$ with $p^{-1}(a+c) \in C$. This is our conclusion (1).

• $f(R)^2 \in C$, $b+q \in C$ and $p^{-1}\{c(b+q) - a\} = p^{-1}\{c - (b+q)a\} \in C$. The last case implies $(b+q-1)(a+c) = 0$, which again implies either $b+q = 1$ or $a+c = 0$. If $b+q = 1$, then $p^{-1}(c-a) \in C$ and hence $F(x) = ax + pxp^{-1}c$ and $p^{-1}G(x) = p^{-1}bx + xp^{-1}q = xp^{-1}(b+q)$ and so $G(x) = pxp^{-1}(b+q)$ for all $x \in R$, as desired in conclusion (2).

On the other hand, if $a+c = 0$, then $F(x) = ax + pxp^{-1}c = ax - pxp^{-1}a$ and $p^{-1}G(x) = p^{-1}bx + xp^{-1}q = xp^{-1}(b+q)$ and so $G(x) = (b+q)pxp^{-1}$ for all $x \in R$, with $b+q \in C$ and $p^{-1}\{c(b+q) - a\} \in C$ i.e., $p^{-1}a(b+q+1) \in C$. As $b+q+1 \in C$, we have either $b+q+1 = 0$ or $p^{-1}a \in C$. Since $p^{-1}a \in C$ implies $F(x) = ax - pxp^{-1}a = 0$, a contradiction. Hence $b+q = -1$. This is our conclusion (3).

• $p^{-1}c \in C$ with $p^{-1}a = -p^{-1}(c - (b+q)(a+c)) \in C$. Therefore, $p^{-1}F(x) = p^{-1}ax + xp^{-1}c = p^{-1}(a+c)x$ and $p^{-1}G(x) = p^{-1}bx + xp^{-1}q = xp^{-1}(b+q)$ and so $F(x) = (a+c)x$ and $G(x) = pxp^{-1}(b+q)$ for all $x \in R$, with $p^{-1}(a+c) \in C$ i.e., $(a+c)p^{-1} \in C$ and $p^{-1}a = -p^{-1}(c - (b+q)(a+c))$, i.e, $p^{-1}(b+q-1)(a+c) = 0$, i.e., $(b+q-1)(a+c)p^{-1}=0$. This implies either $b+q-1=0$ or $(a+c)p^{-1}=0$. In the last case $(a+c)p^{-1}=0$ implies $a+c=0$ and then $F=0$, a contradiction. Thus $b+q=1$ and so conclusion (1) is obtained.

Case-2: *When* $p^{-1}c \in C$.

By Lemma 5.3.2 one of the following holds:

• $p^{-1}b, p^{-1}a, p^{-1}c(b+q), b+q \in C$ with $(a+c)(b+q-1)=0$. Since $p^{-1}b \in C$, conclusions follow by Case-1.

• $f(R)^2 \in C$, $b+q, p^{-1}b \in C$ and $p^{-1}\{c(b+q)-a\} = p^{-1}\{c-(b+q)a\} \in C$. Here also $p^{-1}b \in C$ and hence conclusions follow by Case-1.

• $p^{-1}b \in C$ with $p^{-1}a = -p^{-1}(c-(b+q)(a+c)) \in C$. Again conclusions follow by Case-1 as $p^{-1}b \in C$.

• $a+c \in C$ and $p^{-1}(a+c)(1-b) = p^{-1}(a+c)q \in C$. Thus $(a+c)(1-b-q)=0$.

If $0 \neq a+c \in C$, then these relation yields $b+q=1$ and $p^{-1}(1-b) = p^{-1}q \in C$ and hence $F(x) = ax + pxp^{-1}c = (a+c)x$ and $G(x) = bx + pxp^{-1}q = (b+q)x = x$, which is a particular case of conclusion (1), when $p \in C$.

If $a+c=0$, then $F(x) = ax + pxp^{-1}c = (a+c)x = 0$, a contradiction.

Case-3: *When* $p \in C$.

By Lemma 5.3.3 one of the following holds:

• $a, b, c(b+q), q \in C$ with $(a+c)(b+q-1)=0$.

• $f(R)^2 \in C$, $q, b \in C$ and $c(b+q)-a = c-(b+q)a \in C$.

• $a, b, c \in C$ with $a = -c + (b+q)(a+c) \in C$.

• $a, c \in C$ and $(a+c)(1-b) = (a+c)q \in C$.

In the first three cases, as $p, b \in C$, we have $p^{-1}b \in C$ and so conclusions follow by Case-1.

The last case gives conclusion (1) as discussed in the last case of Case-2. □

In particular, for $0 \neq p \in C$*, we have the following Corollary from above.*

Corollary 5.3.5. *Let* R *be a noncommutative prime ring of characteristic different from 2,* Q_r *be its right Martindale quotient ring and* C *be its extended centroid. Suppose that* $f(x_1,\ldots,x_n)$ *be a noncentral multilinear polynomial over* C*,* $F(x) = ax + xc$ *and* $G(x) = bx + xq$ *for some* $a, b, c, q \in Q_r$ *such that*

$$F(x)x - G(x)F(x) = 0$$

for all $x \in f(R)$*. Then one of the following holds:*

1. $F(x) = (a + c)x$ *and* $G(x) = x$ *for all* $x \in R$*, with* $a + c \in C$*;*

2. $F(x) = ax + xc$ *and* $G(x) = x$ *for all* $x \in R$*, with* $f(R)^2 \in C$ *and* $a - c \in C$*;*

3. $F(x) = ax - xa$ *and* $G(x) = -x$ *for all* $x \in R$*, with* $f(R)^2 \in C$*.*

We now consider the case of inner generalized skew-derivations having associated automorphism α*, which is not necessarily* X*-inner.*

Lemma 5.3.6. *Let* R *be a noncommutative prime ring of characteristic different from 2,* Q_r *be its right Martindale quotient ring and* C *be its extended centroid. Suppose that* $f(x_1, \ldots, x_n)$ *be a noncentral multilinear polynomial over* C*,* $F(x) = ax + \alpha(x)c$ *and* $G(x) = bx + \alpha(x)q$ *for some* $a, b, c, q \in Q_r$ *and* $\alpha \in Aut(R)$*, such that*

$$F(x)x - G(x)F(x) = 0$$

for all $x \in f(R)$*. Then one of the following holds:*

1. $F = G = Id$*, the identity map on* R*;*

2. *there exist* $a, p \in Q_r$ *such that* $F(x) = ax$ *and* $G(x) = pxp^{-1}$ *for all* $x \in R$*, with* $p^{-1}a \in C$*;*

3. *there exist* $a, c, p \in Q_r$ *such that* $F(x) = ax + pxp^{-1}c$ *and* $G(x) = pxp^{-1}$ *for all* $x \in R$*, with* $f(R)^2 \in C$ *and* $p^{-1}(a - c) \in C$*;*

4. *there exist* $a, p \in Q_r$ *such that* $F(x) = ax - pxp^{-1}a$ *and* $G(x) = -pxp^{-1}$ *for all* $x \in R$*, with* $f(R)^2 \in C$*.*

Proof. If α is X-inner, we are done by Lemma 5.3.4. Moreover, in case α is the identity map on R, then both F and G are generalized derivations of R and then we are done by Corollary 5.3.5. Therefore we assume that α is X-outer and it is not the identity map on R.

Under these assumptions we show that a number of contradictions follows.

By [25, Theorem 1] that R and Q_r satisfy the same generalized polynomial identities with automorphisms. Therefore Q_r satisfies

$$\begin{aligned}\Phi(x_1, \ldots, x_n) = \Big(af(x_1, \ldots, x_n) + \alpha(f(x_1, \ldots, x_n))c\Big)f(x_1, \ldots, x_n)\\ -\Big(bf(x_1, \ldots, x_n) + \alpha(f(x_1, \ldots, x_n))q\Big)\\ \Big(af(x_1, \ldots, x_n) + \alpha(f(x_1, \ldots, x_n))c\Big) = 0. \qquad (5.3.14)\end{aligned}$$

Moreover, Q_r is a centrally closed prime C-algebra. If $c = q = 0$, then we are done again by Lemma 5.3.4 as particular case and hence we assume that either $c \neq 0$ or $q \neq 0$. Then it follows by [25, Main Theorem] that $\Phi(x_1, \ldots, x_n)$ is a non-trivial generalized identity for Q_r. By [74, Theorem 1], Q_r is primitive with non-zero socle. Since α is an outer automorphism and any $(x_i)^\alpha$-word degree in $\Phi(x_1, \ldots, x_n)$ is equal to 2 and $char(R) = 0$ or $char(R) = p > 2$, then by [25, Theorem 3], Q_r satisfies the generalized polynomial identity

$$\begin{aligned} &\Big(af(x_1, \ldots, x_n) + f^\alpha(y_1, \ldots, y_n)c\Big)f(x_1, \ldots, x_n) \\ -&\Big(bf(x_1, \ldots, x_n) + f^\alpha(y_1, \ldots, y_n)q\Big)\Big(af(x_1, \ldots, x_n) + f^\alpha(y_1, \ldots, y_n)c\Big) \\ &= 0 \end{aligned} \tag{5.3.15}$$

where $f^\alpha(x_1, \ldots, x_n)$ is the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient γ_σ with $\alpha(\gamma_\sigma)$. Notice that $f^\alpha(x_1, \ldots, x_n)$ is not central valued on R.

From (5.3.15), Q_r satisfies (in particular when $y_1, \ldots, y_n = 0$)

$$af(x_1, \ldots, x_n)^2 - bf(x_1, \ldots, x_n)af(x_1, \ldots, x_n) = 0 \tag{5.3.16}$$

and (in particular when $x_1, \ldots, x_n = 0$)

$$f^\alpha(y_1, \ldots, y_n)qf^\alpha(y_1, \ldots, y_n)c = 0. \tag{5.3.17}$$

Applying Corollary 5.3.5, (5.3.16) implies that $a \in C$ and $b = 1$.

Moreover, (5.3.17) implies that $c = 0$ or $q = 0$.

Let $c = 0$ and then $q \neq 0$. Then by (5.3.15) (as $a \in C$, $b = 1$ and $c = 0$), Q_r satisfies

$$f^\alpha(y_1, \ldots, y_n)qaf(x_1, \ldots, x_n) = 0$$

which implies $qa = 0$, i.e., $a = 0$. Then $F(x) = ax + \alpha(x)c = 0$, a contradiction.

Next let $q = 0$ and then $c \neq 0$.

Then by (5.3.15) (as $a \in C$, $b = 1$ and $q = 0$), Q_r satisfies

$$f^\alpha(y_1, \ldots, y_n)cf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)f^\alpha(y_1, \ldots, y_n)c = 0,$$

that is

$$[c', f(x_1, \ldots, x_n)] = 0 \quad \forall x_1, \ldots, x_n \in Q_r$$

where $c' = f^{\alpha}(y_1, \ldots, y_n)c$. This implies $c' \in C$. Notice that, in this situation and since $f^{\alpha}(y_1, \ldots, y_n)$ cannot be central valued on Q_r, it must be $c \notin C$. On the other hand, since $[c', c] = 0$, it follows that

$$[f^{\alpha}(y_1, \ldots, y_n), c]c = 0$$

is a generalized identity for Q_r. But, in such eventuality, [57, Theorem 1] implies a contradiction. □

5.4 The proof of Main Theorem in the general case.

In light of the results obtained in the previous Section, Theorem 5.1.1 is proved if any one of the following cases holds:

1. $d = \delta = 0$*, that is both* F *and* G *are centralizers of* R*;*

2. α *is the identity map on* R*, that is both* F *and* G *are generalized derivations of* R*;*

3. *both* d *and* δ *are inner skew derivations of* R*, that is both* F *and* G *are inner generalized skew derivations of* R*.*

Thus in all that follows, we assume that

1. *either* $d \neq 0$ *or* $\delta \neq 0$*;*

2. α *is not the identity map on* R*;*

3. d *and* δ *are not simultaneously inner skew derivations of* R*.*

In consideration of what has just been said, in the last part of our paper we prove that these assumption drive us to a number of contradictions. To do this we need the following:

Fact: *Let* R *be a prime ring,* D *be an* X*-outer skew derivation of* R *and* α *be an* X*-outer automorphism of* R*. If* $\Phi(x_i, D(x_i), \alpha(x_i))$ *is a generalized polynomial identity for* R*, then* R *also satisfies the generalized polynomial identity* $\Phi(x_i, y_i, z_i)$*, where* x_i*,* y_i *and* z_i *are distinct indeterminates ([27, Theorem 1]).*

The Proof of Theorem 5.1.1

By [23], we have $F(x) = ax + d(x)$ *and* $G(x) = bx + \delta(x)$*, for all* $x \in R$*, where* $a, b \in Q_r$ *and* d, δ *are skew derivations of* R*. Thus* R *satisfies*

$$\begin{aligned}\Psi(x_1,\ldots,x_n,d(x_1),\ldots,d(x_n),\delta(x_1),\ldots,\delta(x_n)) = \\ \Big(af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n))\Big)f(x_1,\ldots,x_n) \\ -\Big(bf(x_1,\ldots,x_n)+\delta(f(x_1,\ldots,x_n))\Big)\Big(af(x_1,\ldots,x_n)+d(f(x_1,\ldots,x_n))\Big) \\ = 0. \end{aligned} \tag{5.4.1}$$

The action of the skew derivation d *on any monomial of* $f(x_1,\ldots,x_n)$ *can be described as follows:*

$$\begin{aligned} d\Big(\gamma_\sigma\cdot x_{\sigma(1)}\cdots x_{\sigma(n)}\Big) = d(\gamma_\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)} \\ +\alpha(\gamma_\sigma)\textstyle\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}. \end{aligned}$$

So we have

$$\begin{aligned} d(f(x_1,\ldots,x_n)) = f^d(x_1,\ldots,x_n) \\ +\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}. \end{aligned}$$

Since R *satisfies (5.4.1), then it satisfies*

$$\begin{aligned} \Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n) \\ +\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n) \\ -\Big(bf(x_1,\ldots,x_n)+f^\delta(x_1,\ldots,x_n) \\ +\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) \\ .\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n) \\ +\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) = 0. \end{aligned} \tag{5.4.2}$$

First we prove that none of d *and* δ *could be inner.*

Assume that $d(x) = ux - \alpha(x)u$, for some $u \in Q_r$. Then, by [27, Theorem 2]

$$\Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big)f(x_1,\dots,x_n) - \Big(bf(x_1,\dots,x_n)$$
$$+f^{\delta}(x_1,\dots,x_n) + \textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$.\Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big) = 0 \qquad (5.4.3)$$

is also satisfied by Q_r. Since, δ is not inner, then (5.4.3) implies that Q_r satisfies the following

$$\Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big)f(x_1,\dots,x_n) - \Big(bf(x_1,\dots,x_n)$$
$$+f^{\delta}(x_1,\dots,x_n) + \textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$\Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big) = 0. \qquad (5.4.4)$$

In particular Q_r satisfies

$$\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$\Big((a+u)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))u\Big) = 0. \qquad (5.4.5)$$

In case there is an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$, then we write (5.4.5) as follows

$$\Big(\textstyle\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}qx_{\sigma(1)}\cdots x_{\sigma(j)}q^{-1}y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$.\Big((a+u)f(x_1,\dots,x_n) - qf(x_1,\dots,x_n)q^{-1}u\Big) = 0 \qquad (5.4.6)$$

and replacing any $y_{\sigma(j+1)}$ by $qy_{\sigma(j+1)}$, it follows that Q_r satisfies

$$q\Big(\textstyle\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}x_{\sigma(1)}\cdots x_{\sigma(j)}y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$\Big((a+u)f(x_1,\dots,x_n) - qf(x_1,\dots,x_n)q^{-1}u\Big) = 0. \qquad (5.4.7)$$

In particular, Q_r satisfies blended component

$$qf(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_n)\Big((a+u)f(x_1,\dots,x_n) - qf(x_1,\dots,x_n)q^{-1}u\Big) = 0,$$

that is (left multiplying by q^{-1} and replacing y_i by x_i)

$$f(x_1,\dots,x_n)(a+u)f(x_1,\dots,x_n) - f(x_1,\dots,x_n)qf(x_1,\dots,x_n)q^{-1}u = 0.$$

By Lemma 5.2.2, either $q \in C$ or $q^{-1}u \in C$. Since α is not the identity map, then $q \notin C$. Thus $q^{-1}u \in C$, $d = 0$ and Q_r satisfies $f(x_1, \ldots, x_n)af(x_1, \ldots, x_n)$. This implies $a = 0$. Thus $F = 0$, a contradiction.

On the other hand, if α is outer, by (5.4.5) Q_r satisfies

$$\left(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} z_{\sigma(1)} \cdots z_{\sigma(j)} y_{\sigma(j+1)} x_{\sigma(j+2)} \cdots x_{\sigma(n)}\right) \left((a+u)f(x_1, \ldots, x_n) - f^\alpha(z_1, \ldots, z_n)u\right) = 0. \tag{5.4.8}$$

For $x_1 = 0$ and $y_k = 0$ in (5.4.8), with $k = 2, \ldots, n$, one has that Q_r satisfies

$$\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) z_{\sigma(1)} \cdots z_{\sigma(j)} y_1 x_{\sigma(j+2)} \cdots x_{\sigma(n)} f^\alpha(z_1, \ldots, z_n)u = 0$$

implying that $f^\alpha(x_1, \ldots, x_n)^2 u$ is an identity for Q_r. This gives $u = 0$ (i.e., $d = 0$). Then (5.4.8) reduces to

$$\left(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} z_{\sigma(1)} \cdots z_{\sigma(j)} y_{\sigma(j+1)} x_{\sigma(j+2)} \cdots x_{\sigma(n)}\right) af(x_1, \ldots, x_n) = 0 \tag{5.4.9}$$

which is satisfied by Q_r.

In particular, Q_r satisfies blended component

$$\left(\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} z_{\sigma(1)} \cdots z_{\sigma(n-1)} y_{\sigma(n)}\right) af(x_1, \ldots, x_n) = 0. \tag{5.4.10}$$

Replacing z_i by $\alpha(z_i)$ and y_i by $\alpha(z_i)$ for $i = 1, 2, \ldots, n$, from above relation Q_r satisfies

$$\left(\alpha(f(z_1, \ldots, z_n))\right) af(x_1, \ldots, x_n) = 0,$$

that is $c'af(x_1, \ldots, x_n) = 0$, where $c' = \alpha(f(z_1, \ldots, z_n))$. This implies $c'a = 0$, that is, $\alpha(f(z_1, \ldots, z_n))a = 0$ for all $z_1, \ldots, z_n \in Q_r$. This implies $a = 0$. Thus $F = 0$, a contradiction.

Conversely, assume now $\delta(x) = vx - \alpha(x)v$*, for some* $v \in Q_r$*. Hence, as above,*

$$\Big(af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n) - \Big((b+v)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))v\Big)\Big(af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) = 0 \quad (5.4.11)$$

is also a generalized differential identity for Q_r*. Also in this case, d is not inner, so that* Q_r *satisfies the generalized identity*

$$\Big(af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n) - \Big((b+v)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))v\Big)\Big(af(x_1,\dots,x_n) + f^d(x_1,\dots,x_n) + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) = 0. \quad (5.4.12)$$

Here Q_r *satisfies the blended component*

$$\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n) - \Big((b+v)f(x_1,\dots,x_n) - \alpha(f(x_1,\dots,x_n))v\Big)\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) = 0. \quad (5.4.13)$$

If $\alpha(x) = qxq^{-1}$*, for all* $x \in R$*, then we write (5.4.13) as follows*

$$\Big(\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}qx_{\sigma(1)}\cdots x_{\sigma(j)}q^{-1}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n) - \Big((b+v)f(x_1,\dots,x_n) - qf(x_1,\dots,x_n)q^{-1}v\Big)\Big(\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}qx_{\sigma(1)}\cdots x_{\sigma(j)}q^{-1}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) = 0 \quad (5.4.14)$$

and replacing $z_{\sigma(j+1)}$ *by* $qz_{\sigma(j+1)}$*, it follows that* Q_r *satisfies*

$$q\Bigg(\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}x_{\sigma(1)}\cdots x_{\sigma(j)}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Bigg)f(x_1,\ldots,x_n)$$
$$-\Bigg((b+v)f(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}v\Bigg)$$
$$q\Bigg(\sum_{\sigma\in S_n}\gamma_\sigma\sum_{j=0}^{n-1}x_{\sigma(1)}\cdots x_{\sigma(j)}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Bigg)=0. \qquad (5.4.15)$$

In particular, Q_r *satisfies any blended component*

$$qf(x_1,\ldots,x_{i-1},z_i,x_{i+1},\ldots,x_n)f(x_1,\ldots,x_n)$$
$$-\Bigg((b+v)f(x_1,\ldots,x_n)-qf(x_1,\ldots,x_n)q^{-1}v\Bigg)$$
$$.qf(x_1,\ldots,x_{i-1},z_i,x_{i+1},\ldots,x_n)=0, \qquad (5.4.16)$$

that is (left multiplying by q^{-1} *and replacing* z_i *by* x_i*)*

$$f(x_1,\ldots,x_n)^2-\Bigg(q^{-1}(b+v)f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)q^{-1}v\Bigg)qf(x_1,\ldots,x_n)=0.$$

By Lemma 5.2.3, and again by the fact that α *is not the identity map, it follows that* $q^{-1}(b+v)\in C$*. Thus* Q_r *satisfies*

$$\Bigg(f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)q^{-1}bq\Bigg)f(x_1,\ldots,x_n)=0.$$

This implies $b=1$ *(see [10, Lemma 3]). Then left multiplying by* q^{-1}*, (5.4.16) yields*

$$f(x_1,\ldots,x_{i-1},z_i,x_{i+1},\ldots,x_n)f(x_1,\ldots,x_n)$$
$$-f(x_1,\ldots,x_n)f(x_1,\ldots,x_{i-1},z_i,x_{i+1},\ldots,x_n)=0. \qquad (5.4.17)$$

Replacing z_i *by* $[p,x_i]$ *for some* $p\notin C$*,* Q_r *satisfies*

$$[p,f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)[p,f(x_1,\ldots,x_n)]=0. \qquad (5.4.18)$$

By [77], $p\in C$*, a contradiction.*

Hence we may assume that α *is not inner. Thus, by (5.4.13),* Q_r *satisfies*

$$\Bigg(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}y_{\sigma(1)}\cdots\cdot y_{\sigma(j)}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Bigg)f(x_1,\ldots,x_n)$$
$$-\Bigg((b+v)f(x_1,\ldots,x_n)-f^{\alpha}(y_1,\ldots,y_n)v\Bigg)$$
$$.\Bigg(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}y_{\sigma(1)}\cdots\cdot y_{\sigma(j)}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Bigg)=0. \qquad (5.4.19)$$

In particular, for $x_1 = 0$, Q_r satisfies

$$f^\alpha(y_1,\dots,y_n)v\left(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1} y_{\sigma(1)}\cdots y_{\sigma(j)}z_1x_{\sigma(j+2)}\cdots x_{\sigma(n)}\right) = 0, \qquad (5.4.20)$$

that gives $f^\alpha(y_1,\dots,y_n)vf^\alpha(y_1,\dots,y_n) = 0$. This implies $v = 0$ (i.e., $\delta = 0$). Then by (5.4.19), Q_r satisfies

$$\left(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1} y_{\sigma(1)}\cdots y_{\sigma(j)}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\right)f(x_1,\dots,x_n)$$
$$-bf(x_1,\dots,x_n).\left(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1} y_{\sigma(1)}\cdots y_{\sigma(j)}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\right) = 0. \qquad (5.4.21)$$

In particular, Q_r satisfies blended component

$$\left(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1} y_{\sigma(1)}\cdots y_{\sigma(n-1)}z_{\sigma(n)}\right)f(x_1,\dots,x_n)$$
$$-bf(x_1,\dots,x_n).\left(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1} y_{\sigma(1)}\cdots y_{\sigma(n-1)}z_{\sigma(n)}\right) = 0. \qquad (5.4.22)$$

Replacing y_i by $\alpha(y_i)$ and z_i by $\alpha(y_i)$ for $i = 1, 2, \dots, n$, from above relation Q_r satisfies

$$\alpha(f(y_1,\dots,y_n))f(x_1,\dots,x_n) - bf(x_1,\dots,x_n)\alpha(f(y_1,\dots,y_n)) = 0, \qquad (5.4.23)$$

that is $c'f(x_1,\dots,x_n) - bf(x_1,\dots,x_n)c' = 0$, where $c' = \alpha(f(y_1,\dots,y_n))$. This implies $c' \in C$ (see [38, Lemma 2.1]), that is, $f(y_1,\dots,y_n)$ is central valued, a contradiction.

By the above argument, in all that follows we may assume that both d and δ are not inner skew derivations of R.

In order to prove the final part of the Theorem we need to divide the argument in several cases.

Case 1: $d = 0$

In this case, $0 \neq \delta$ is not an inner skew derivation. Hence, by [27, Theorem 2] *and (5.4.2), Q_r satisfies the following:*

$$af(x_1,\dots,x_n)^2 - \Big(bf(x_1,\dots,x_n) + f^{\delta}(x_1,\dots,x_n)$$
$$+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$af(x_1,\dots,x_n) = 0. \tag{5.4.24}$$

This last relation leads to (5.4.3) as a particular case, and we get a contradiction as above.

Case 2: $\delta = 0$

In this case, $0 \neq d$ is not an inner skew derivation and we may reduce (5.4.2) to the following identity for Q_r:

$$\Big(af(x_1,\dots,x_n) + f^{d}(x_1,\dots,x_n)$$
$$+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n)$$
$$-bf(x_1,\dots,x_n).\Big(af(x_1,\dots,x_n) + f^{d}(x_1,\dots,x_n)$$
$$+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) = 0. \tag{5.4.25}$$

Since d is not inner, by (5.4.25) it follows that Q_r satisfies the generalized identity

$$\Big(af(x_1,\dots,x_n) + f^{d}(x_1,\dots,x_n)$$
$$+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n)$$
$$-bf(x_1,\dots,x_n).\Big(af(x_1,\dots,x_n) + f^{d}(x_1,\dots,x_n)$$
$$+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big) = 0. \tag{5.4.26}$$

Here Q_r satisfies the blended component

$$\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n)$$
$$-bf(x_1,\dots,x_n)\Big(\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$= 0. \tag{5.4.27}$$

which is particular case of (5.4.13). By the same argument as above, it leads to a contradiction.

Case 3: $\{d, \delta\}$ ***is linearly*** C***-independent modulo inner skew derivations of*** R***.***

In this case, since d *and* δ *are associated with the same automorphism, by relation (5.4.2) it follows that* R *satisfies the generalized identity*

$$\Big(af(x_1,\dots,x_n)+f^d(x_1,\dots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\dots,x_n)$$
$$-\Big(bf(x_1,\dots,x_n)+f^\delta(x_1,\dots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$.\Big(af(x_1,\dots,x_n)+f^d(x_1,\dots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)=0. \quad (5.4.28)$$

Thus R *satisfies blended component*

$$\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)$$
$$.\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)=0. \quad (5.4.29)$$

In relation (5.4.29), for any $\sigma\in S_n$ *and for any* $j=0,\dots,n-1$*, we replace* $y_{\sigma(j+1)}$ *with* $z_{\sigma(j+1)}$*. Hence* R *satisfies*

$$\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)^2=0. \quad (5.4.30)$$

Notice that, if $\alpha(x)=pxp^{-1}$*, for any* $x\in R$*, (5.4.30) reduces to*

$$\Big(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}px_{\sigma(1)}\cdots x_{\sigma(j)}p^{-1}z_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)^2=0. \quad (5.4.31)$$

Here, for any $\sigma\in S_n$ *and for any* $j=0,\dots,n-1$*, we replace* $z_{\sigma(j+1)}$ *with* $px_{\sigma(j+1)}$*. Therefore* R *satisfies*

$$\Big(pf^\alpha(x_1,\dots,x_n)\Big)^2=0. \quad (5.4.32)$$

In this case, as a consequence of [96, Theorem 2] and since $f(x_1,\dots,x_n)$ *is not central valued on* R*, it follows the contradiction* $p=0$*.*

On the other hand, if α is not X-inner, (5.4.30) implies that

$$\left(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)z_{\sigma(1)}\cdots z_{\sigma(n)}\right)^2=0$$

is an identity for R and, as above, a contradiction follows.

***Case 4:* $\{d,\delta\}$ *is linearly* C*-dependent modulo inner skew derivations of* R**

Under our last assumption, there exist $\lambda,\mu\in C$, $q\in Q_r$ and $\gamma\in Aut(R)$ such that $\lambda d(x)+\mu\delta(x)=qx-\gamma(x)q$, for any $x\in R$. Recall that d and δ are not inner skew derivations, so that both $\lambda\neq 0$ and $\mu\neq 0$.
Here we write $\delta(x)=px-\gamma(x)p+\eta d(x)$, where $p=\mu^{-1}q$ and $\eta=-\lambda\mu^{-1}\neq 0$. Hence R satisfies

$$\begin{aligned}&\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n)\\&-\Big((b+p)f(x_1,\ldots,x_n)-\gamma(f(x_1,\ldots,x_n))p+\eta f^d(x_1,\ldots,x_n)\\&+\eta\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)\\&.\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)=0.\end{aligned}\qquad(5.4.33)$$

Since d is outer, by [27, Theorem 1] and relation (5.4.33), it follows that R satisfies

$$\begin{aligned}&\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)f(x_1,\ldots,x_n)\\&-\Big((b+p)f(x_1,\ldots,x_n)-\gamma(f(x_1,\ldots,x_n))p+\eta f^d(x_1,\ldots,x_n)\\&+\eta\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)\\&.\Big(af(x_1,\ldots,x_n)+f^d(x_1,\ldots,x_n)\\&+\textstyle\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)}\Big)=0.\end{aligned}\qquad(5.4.34)$$

In particular, For $x_1 = 0$ *in (5.4.34) and since* $\eta \neq 0$, R *satisfies*

$$\left(\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_1x_{\sigma(j+2)}\cdots x_{\sigma(n)}\right)^2=0$$

which has the same flavour of the identity (5.4.30), exchanging the role that the variables $z_{\sigma(j+1)}$ *assumed in (5.4.30) with the one that* y_1 *assumes in the latter relation. Thus, by using the same above argument, we get a contradiction. Thus the proof of the theorem is completed.* □

Chapter 6

A Result Concerning b-generalized Skew Derivations on Multilinear Polynomials in Prime Rings

6.1 Introduction

In this Chapter, unless otherwise stated, R always denotes a prime ring with center $Z(R)$. Q_r denotes its right Martindale quotient ring. The center of Q_r is denoted by C and is called the extended centroid of the ring R.

In [10], Argac and De Filippis have studied the identity

$$F(f(X))f(X) - f(X)G(f(X)) = 0$$

for all $X = (x_1, \ldots, x_n) \in R^n$ and descried the forms of the maps, where $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C and F, G are two generalized derivations on prime ring R. In [90], Tiwari also studied the generalization of the work of Argac and De Filippis [10]. In [62], De Filippis and Di Vincenzo studied the following:

Let K be a commutative ring with unity, R be a prime K-algebra and $f(x_1, \ldots, x_n)$ a multilinear polynomial over K, not central valued on R. If d is a nonzero derivation and F is a nonzero generalized derivation on R such that

$$d([F(f(X)), f(X)]) = 0$$

for all $X = (x_1, \ldots, x_n) \in R^n$, then all possible forms of the maps are described.

From the definition of b-generalized derivation we can easily see that generalized derivation is a 1-generalized derivation and we can define a map $F : R \to Q_r$

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by $F(x) = ax + bxc$ *for all* $x \in R$*, where* $a, b, c \in Q_r$ *which is called inner b-generalized derivation. The b-generalized derivations appeared canonically in [28] and were studied recently in [76, 80, 83].*

In this line of investigation Dhara [35], generalized the above result of [62] by replacing F *with a b-generalized derivation.*

Many researchers have investigated generalized skew derivations from various points of view.

It is natural to consider a map which will be common generalization of these above maps. In view of this idea De Filippis [32] introduced the new map b-generalized skew derivation

It is very easy to check that b-generalized skew derivation generalizes the concept of generalized skew derivation as well as b-generalized derivation. The map $x \mapsto ax + b\alpha(x)c$ *is an example of b-generalized skew derivation of* R *which is called inner b-generalized skew derivation of* R*. There are few papers which recently studied the b-generalized skew derivations (viz. [63], [64], [65]). In the present Chapter our motivation is to study b-generalized skew derivation in prime rings.*

In [39], Dhara, Argac and Albas described the structure of additive maps d, F *and* G *satisfying* $d(F(f(X))f(X) - f(X)G(f(X))) = 0$ *for all* $X = (x_1, \ldots, x_n) \in R^n$*, where* $f(x_1, \ldots, x_n)$ *is a multilinear polynomial over extended centroid* C *and* d *is a nonzero derivation and* F, G *are two generalized derivations on prime ring* R *with char* $(R) \neq 2$*.*

On the other hand, Carini and De Filippis [18] proved that if R *be a prime ring of char* $(R) \neq 2$*,* U *the Utumi ring of quotients,* δ *a nonzero derivation of* R*,* G *a nonzero generalized derivation of* R*, and* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C *such that* $\delta(G(f(X))f(X)) = 0$ *for all* $X = (x_1, \ldots, x_n) \in R^n$*, then there exist* $a, b \in U$ *such that* $G(x) = ax$ *and* $\delta(x) = [b, x]$ *for all* $x \in R$*, with* $[b, a] = 0$ *and* $f(x_1, \ldots, x_n)^2$ *is central-valued on* R*.*

Further, Dhara and Argac [38] extended the above result replacing derivation δ *with another generalized derivation* F*, that is,* $F(G(f(X))f(X)) = 0$ *for all* $X = (x_1, \ldots, x_n) \in I^n$*, and then gave the complete description of the additive maps* F *and* G*, where* I *is a non-zero two-sided ideal of* R*.*

In [41], Dhara and De Filippis gave the complete description of generalized derivations F *and* G *when* $F^2(f(X))f(X) - f(X)G^2(f(X)) = 0$ *for all* $X =$

$(x_1, \ldots, x_n) \in I^n$, *where* I *is a non-zero two-sided ideal of* R. *Then in [61], De Filippis, Scudo and Wei studied the situation replacing* F^2 *and* G^2 *with two b-generalized skew derivations in prime ring.*

In [47], Dhara, Kar and Kuila studied the case $d(F^2(f(X))f(X)) = f(X)G^2(f(X))$ *for all* $X = (x_1, \ldots, x_n) \in I^n$ *and obtained the forms of the maps* F *and* G, *where* F *and* G *are two generalized derivations. In the present Chapter our main goal is to study the above situation replacing* F^2 *and* G^2 *with two different b-generalized skew derivations and* d *with an inner derivation in prime ring. In this line of investigation recently, Prajapati and Gupta [87] studied* $[a, F(f(X))f(X)] \in C$ *for all* $X = (x_1, \ldots, x_n) \in R^n$, *where* F *is a b-generalized skew derivation of* R.

More precisely we prove the following Theorem.

Theorem 6.1.1. *Let* R *be a prime ring of characteristic different from* 2, $Z(R)$ *its center,* Q_r *its right Martindale quotient ring,* C *its extended centroid,* $f(x_1, \ldots, x_n)$ *a non-central multilinear polynomial over* C, $b \in Q_r$ *and* F, G *two b-generalized skew derivations of* R. *If there exists* $a \in R\backslash Z(R)$ *such that*

$$\big[a, F(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n)\big] = f(x_1, \ldots, x_n)G\big(f(x_1, \ldots, x_n)\big)$$

for any $x_1, \ldots, x_n \in R$, *then* $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exist* $p, q \in Q_r$ *such that* $F(x) = px$, $G(x) = xq$ *for all* $x \in R$ *with* $[a, p] = q$.

In particular, if $G = 0$, *we have the following corollary.*

Corollary 6.1.2. *Let* R *be a prime ring of char* $(R) \neq 2$, Q_r *its right Martindale quotient ring,* C *its extended centroid and* $b \in Q_r$. *Also let* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C *which is not central valued on* R. *Suppose that* F *is a b-generalized skew derivation of* R *and* $a \notin Z(R)$ *such that*

$$\big[a, F(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n)\big] = 0$$

for all $x_1, \ldots, x_n \in R$. *Then there exist* $p \in Q_r$ *such that* $F(x) = px$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and* $[a, p] = 0$.

Remark 6.1.1. *We notice that, in case* F *and* G *are generalized derivations of* R, *Theorem 6.1.1 is a direct consequence of [40, Corollary 1.5].*

In light of this, in all that follows we will assume that at least one of F *and* G *is not a generalized derivation of* R.

6.2 F and G be inner b-generalized skew derivations

First we consider that F and G are two inner b-Generalized skew derivations with associated inner automorphisms.

Then there exists some fixed $b, c, p, q, u, v \in Q_r$ with invertible p such that $F(x) = cx + bpxp^{-1}u$, $G(x) = qx + bpxp^{-1}v$ for all $x \in R$. Then by our hypothesis

$$[a, F(x)x] = xG(x)$$

for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$, we have

$$acx^2 + abpxp^{-1}ux - cx^2a - bpxp^{-1}uxa - xqx - xbpxp^{-1}v = 0 \qquad (6.2.1)$$

for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$.
This can be written as

$$Ax^2 + PxDx - cx^2a - BxDxa - xqx - xBxE = 0 \qquad (6.2.2)$$

for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$, where $A = ac, P = abp, D = p^{-1}u, B = bp, E = p^{-1}v$. In light of Remark 6.1.1, in all that follows we may assume that $B = bp \notin C$.

Proposition 6.2.1. *Let $R = M_m(C)$, $m \geq 2$, be the ring of all $m \times m$ matrices over the infinite field C of characteristic different from 2. If R satisfies (6.2.2) and $a, B \notin Z(R)$, then $D, E \in Z(R)$.*

Proof. By our hypothesis, R satisfies the generalized polynomial identity

$$\begin{aligned} &Af(x_1, \ldots, x_n)^2 + Pf(x_1, \ldots, x_n)Df(x_1, \ldots, x_n) \\ &-cf(x_1, \ldots, x_n)^2a - Bf(x_1, \ldots, x_n)Df(x_1, \ldots, x_n)a \\ &-f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)Bf(x_1, \ldots, x_n)E = 0. \qquad (6.2.3) \end{aligned}$$

Firstly suppose $D \notin Z(R)$. Now we show that this assumption leads to a contradiction.
Since $a \notin Z(R)$, $B \notin Z(R)$ and $D \notin Z(R)$ by [62, Lemma 1] there exists a C-automorphism ϕ of $M_m(C)$ such that $\phi(a)$, $\phi(B)$ and $\phi(D)$ have all non-zero entries.

Clearly, R satisfies the generalized polynomial identity

$$\begin{aligned}\phi(A)f(x_1,\ldots,x_n)^2+\phi(P)f(x_1,\ldots,x_n)\phi(D)f(x_1,\ldots,x_n)\\-\phi(c)f(x_1,\ldots,x_n)^2\phi(a)-\phi(B)f(x_1,\ldots,x_n)\phi(D)f(x_1,\ldots,x_n)\phi(a)\\-f(x_1,\ldots,x_n)\phi(q)f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)\phi(B)f(x_1,\ldots,x_n)\phi(E)\\=0.\end{aligned} \tag{6.2.4}$$

By e_{ij}, we mean the usual matrix unit with 1 in (i,j)-entry and zero elsewhere. Since $f(x_1,\ldots,x_n)$ is not central valued, by [78] (see also [82]), there exist $u_1,\ldots,u_n \in M_m(C)$ and $\gamma \in C-\{0\}$ such that $f(u_1,\ldots,u_n)=\gamma e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(x_1,\ldots,x_n) : x_1,\ldots,x_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $x_1,\ldots,x_n \in M_m(C)$ such that $f(x_1,\ldots,x_n)=e_{ij}$. Hence by (6.2.4), we have

$$\begin{aligned}\phi(P)e_{ij}\phi(D)e_{ij}-\phi(B)e_{ij}\phi(D)e_{ij}\phi(a)-e_{ij}\phi(q)e_{ij}-e_{ij}\phi(B)e_{ij}\phi(E)\\=0.\end{aligned} \tag{6.2.5}$$

Then left and right multiplying by e_{ij}, it follows $e_{ij}\phi(B)e_{ij}\phi(D)e_{ij}\phi(a)e_{ij}=0$, which is a contradiction, since $\phi(a)$, $\phi(B)$ and $\phi(D)$ have all non-zero entries. Thus we conclude that $D \in Z(R)$.

Since $D \in Z(R)$, then (6.2.3) reduces to

$$\begin{aligned}Af(x_1,\ldots,x_n)^2+PDf(x_1,\ldots,x_n)^2\\-cf(x_1,\ldots,x_n)^2a-BDf(x_1,\ldots,x_n)^2a\\-f(x_1,\ldots,x_n)qf(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)Bf(x_1,\ldots,x_n)E=0.\end{aligned} \tag{6.2.6}$$

Replacing $f(x_1,\ldots,x_n)$ with e_{ij} in (6.2.6) yields

$$-e_{ij}qe_{ij}-e_{ij}Be_{ij}E=0. \tag{6.2.7}$$

Right multiplying by e_{ij} yields $e_{ij}Be_{ij}Ee_{ij}=0$ implying $B \in Z(R)$ or $E \in Z(R)$. Since $B \notin Z(R)$, $E \in Z(R)$. □

Proposition 6.2.2. *Let $R=M_m(C)$, $m \geq 2$, be the ring of all $m \times m$ matrices over the field C of characteristic different from 2. If R satisfies (6.2.2) and $a, B \notin Z(R)$, then $D, E \in Z(R)$.*

Proof. In case C is infinite, the conclusions follow by Proposition 6.2.1.
Now we assume that C is a finite field. Consider that K be an infinite field which is an extension of the field C, also $\overline{R} = M_m(K) \cong R \otimes_C K$. Then the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned} \Psi(x_1, \ldots, x_n) \quad = \quad & A f(x_1, \ldots, x_n)^2 + P f(x_1, \ldots, x_n) D f(x_1, \ldots, x_n) \\ & - c f(x_1, \ldots, x_n)^2 a - B f(x_1, \ldots, x_n) D f(x_1, \ldots, x_n) a \\ & - f(x_1, \ldots, x_n) q f(x_1, \ldots, x_n) \\ & - f(x_1, \ldots, x_n) F f(x_1, \ldots, x_n) E. \end{aligned} \tag{6.2.8}$$

which is also a generalized polynomial identity for R. Also, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $x_1, \ldots, x_n$. Hence the complete linearization of $\Psi(x_1, \ldots, x_n)$ gives a multilinear generalized polynomial $\Omega(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $2n$ indeterminates, moreover

$$\Omega(x_1, \ldots, x_n, x_1, \ldots, x_n) = 2^n \Psi(x_1, \ldots, x_n).$$

Clearly the multilinear polynomial $\Omega(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $char(C) \neq 2$ we obtain $\Psi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and then conclusion follows from Proposition 6.2.1. □

Lemma 6.2.3. *Let R be a primitive ring of characteristic different from 2, with nonzero socle $Soc(R)$, which is isomorphic to a dense ring of linear transformations of a vector space V over C, such that $dim_C V = \infty$. If R satisfies (6.2.2) and $a, B \notin Z(R)$, then $D, E \in Z(R)$.*

Proof. Suppose either $D \notin C$ or $E \notin C$. Since $a, B \notin C$ and $f(x_1, \ldots, x_n)$ is not central valued, there exist $h_0, \ldots, h_{n+2} \in Soc(R)$ such that

$[a, h_0] \neq 0, \quad [B, h_1] \neq 0,$

either $[D, h_2] \neq 0$ or $[E, h_2] \neq 0,$

and $f(h_3, \ldots, h_{n+2}) \notin C.$

Since V is infinite dimensional over C, by Martindale's theorem (see Theorem 1.6.6) we get for any $e^2 = e \in soc(R)$, $eRe \cong M_t(C)$ with $t = dim_C Ve$. By Litoff's theorem (see Theorem 1.6.7), there exist an idempotent element $e \in soc(R)$ such that

$$h_0, \ldots, h_{n+2} \in eRe;$$

and

$$ah_0, h_0a, Bh_1, h_1B, Dh_2, h_2D, Eh_2, h_2E \in eRe.$$

Since R satisfies generalized identity

$$e\{A(exe)^2 + P(exe)D(exe) - c(exe)^2a - B(exe)D(exe)a \\ -(exe)q(exe) - (exe)B(exe)E\}e = 0, \tag{6.2.9}$$

the subring eRe satisfies

$$(eAe)x^2 + (ePe)x(eDe)x - (ece)x^2(eae) - (eBe)x(eDe)x(eae) \\ -xeqex - x(eBe)x(eEe) = 0. \tag{6.2.10}$$

By Proposition 6.2.2, we get one of the following:

(i) $eae \in eC$, which contradicts with the choice of h_0;

(ii) $eBe \in eC$, which contradicts with the choice of h_1;

(iv) $eDe, eEe \in eC$, which contradicts with the choice of h_2. □

Lemma 6.2.4. *Let R be a prime ring of char $(R) \neq 2$, Q_r be its right Martindale quotient ring and C be its extended centroid. Also consider $f(x_1, \dots, x_n)$ be a non-central multilinear polynomial over C and $b, c, p, q, u, v \in Q_r$ with invertible p. Let $F(x) = cx + bpxp^{-1}u$ and $G(x) = qx + bpxp^{-1}v$ for all $x \in R$. Let $0 \neq a \in R\backslash Z(R)$ be such that $[a, F(x)x] = xG(x)$, for all $x \in f(R)$. If $bp \notin C$, then $p^{-1}u, p^{-1}v \in C$, $f(x_1, \dots, x_n)^2$ central valued on R and $[a, u] = v$.*

Proof. By our hypothesis, R satisfies the generalized polynomial

$$[a, F(x)x] = xG(x)$$

for all $x \in \{f(x_1, \dots, x_n) | x_1, \dots, x_n \in R\}$, that is

$$acx^2 + abpxp^{-1}ux - cx^2a - bpxp^{-1}uxa - xqx - xbpxp^{-1}v = 0 \tag{6.2.11}$$

for all $x \in \{f(x_1, \dots, x_n) | x_1, \dots, x_n \in R\}$.

We re-write it as

$$\Psi(x_1, \dots, x_n) = acf(x_1, \dots, x_n)^2 + abpf(x_1, \dots, x_n)p^{-1}uf(x_1, \dots, x_n) \\ -cf(x_1, \dots, x_n)^2a - bpf(x_1, \dots, x_n)p^{-1}uf(x_1, \dots, x_n)a \\ -f(x_1, \dots, x_n)qf(x_1, \dots, x_n) - f(x_1, \dots, x_n)bpf(x_1, \dots, x_n)p^{-1}v \\ = 0 \tag{6.2.12}$$

for all $x_1, \ldots, x_n \in R$. Since R and Q_r satisfy same generalized polynomial identity (GPI) (see [24]), Q_r satisfies $\Psi(x_1, \ldots, x_n) = 0$ and we may assume $a \notin C$.

Case-I. *Let R does not satisfy any nontrivial GPI.*

Then $\Psi(x_1, \ldots, x_n) = 0$ is a trivial GPI for Q_r. Let $T = Q_r *_C C\{x_1, x_2, \ldots, x_n\}$, the free product of Q_r and $C\{x_1, \ldots, x_n\}$, the free C-algebra in noncommuting indeterminates $x_1, x_2, \ldots, x_n$. Then, $\Psi(x_1, \ldots, x_n)$ is zero element in $T = Q_r *_C C\{x_1, x_2, \ldots, x_n\}$.

Sub-case-i. *Let $\{1, a, p^{-1}v\}$ is linearly C- independent.*

In this case by (6.2.12),

$$f(x_1, \ldots, x_n)bpf(x_1, \ldots, x_n)p^{-1}v = 0 \in T$$

and

$$-cf(x_1, \ldots, x_n)^2 a - bpf(x_1, \ldots, x_n)p^{-1}uf(x_1, \ldots, x_n)a = 0 \in T.$$

Since $bp \notin C$, first relation gives $p^{-1}v = 0$, i.e., $v = 0$.

The second relation gives

$$\{cf(x_1, \ldots, x_n) + bpf(x_1, \ldots, x_n)p^{-1}u\}f(x_1, \ldots, x_n)a = 0 \in T$$

which implies $p^{-1}u \in C$.

Thus, in any case, F and G are generalized derivations of R and the conclusion follows from Remark 6.1.1.

Sub-case-ii. *Let $\{1, a, p^{-1}v\}$ is linearly C- dependent.*

In this case, since $a \notin C$, there exist $\alpha_1, \alpha_2 \in C$ such that $p^{-1}v = \alpha_1 a + \alpha_2$. Then by (6.2.12),

$$\begin{aligned} acf(x_1, \ldots, x_n)^2 + abpf(x_1, \ldots, x_n)p^{-1}uf(x_1, \ldots, x_n) \\ -cf(x_1, \ldots, x_n)^2 a - bpf(x_1, \ldots, x_n)p^{-1}uf(x_1, \ldots, x_n)a \\ -f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \\ -f(x_1, \ldots, x_n)bpf(x_1, \ldots, x_n)(\alpha_1 a + \alpha_2) = 0 \in T. \end{aligned} \tag{6.2.13}$$

Since $a \notin C$, $\{1, a\}$ is linearly C-independent. Hence by (6.2.13), we have

$$\begin{aligned} -cf(x_1, \ldots, x_n)^2 a - bpf(x_1, \ldots, x_n)p^{-1}uf(x_1, \ldots, x_n)a \\ -f(x_1, \ldots, x_n)bpf(x_1, \ldots, x_n)\alpha_1 a = 0 \in T. \end{aligned} \tag{6.2.14}$$

This can be re-written as

$$\begin{aligned} \{cf(x_1, \ldots, x_n) + bpf(x_1, \ldots, x_n)p^{-1}u \\ +f(x_1, \ldots, x_n)bp\}f(x_1, \ldots, x_n)\alpha_1 a = 0 \in T. \end{aligned} \tag{6.2.15}$$

This relation implies that $\{1, c, bp\}$ is linearly C-dependent. Since $bp \notin C$, there exists $\beta_1, \beta_2 \in C$ such that $c = \beta_1 bp + \beta_2$. Then by (6.2.15), we have

$$\{(\beta_1 bp + \beta_2)f(x_1, \ldots, x_n) + bpf(x_1, \ldots, x_n)p^{-1}u$$
$$+f(x_1, \ldots, x_n)bp\}f(x_1, \ldots, x_n)\alpha_1 a = 0 \in T. \quad (6.2.16)$$

Since $bp \notin C$, $\{1, bp\}$ is linearly C-independent and hence by (6.2.16)

$$bpf(x_1, \ldots, x_n)\{\beta_1 + p^{-1}u\}f(x_1, \ldots, x_n)\alpha_1 a = 0 \in T \quad (6.2.17)$$

and

$$f(x_1, \ldots, x_n)\{\beta_2 + bp\}f(x_1, \ldots, x_n)\alpha_1 a = 0 \in T. \quad (6.2.18)$$

In particular (6.2.18) implies that $\beta_2 + bp = 0$, which is a contradiction.

Case-II. *Let R satisfies a nontrivial GPI.*

In this case $\Psi(x_1, \ldots, x_n) = 0$ be a non-trivial generalized polynomial identity for Q_r. By Martindale's theorem (see Theorem 1.6.6) we get Q_r be a primitive ring with nonzero socle $\mathrm{soc}(R)$ with C as its associated division ring. Therefore by Jacobson's theorem (see Theorem 1.6.5) we get Q_r is isomorphic to a dense ring of linear transformations of a vector space V over C.

Then by Proposition 6.2.2 and Lemma 6.2.3, it follows $p^{-1}u, p^{-1}v \in C$ and we obtain the required conclusion by applying Remark 6.1.1. □

Lemma 6.2.5. *[36, Lemma 2.9] Let R be a non-commutative prime ring of characteristic different from 2, $a, b, c, v \in Q_r$, and $p(r_1, \ldots, r_n)$ be any polynomial over C which is not an identity for R. If $ap(r) + p(r)b + cp(r)v = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

1. *$b, v \in C$ and $a + b + cv = 0$;*
2. *$a, c \in C$ and $a + b + cv = 0$;*
3. *$a + b + cv = 0$ and $p(r_1, \ldots, r_n)$ is central valued on R.*

Proposition 6.2.6. *Let R be a prime ring of char $(R) \neq 2$ with right Martindale quotient ring Q_r and extended centroid C. Also $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that $F(x) = cx + b\alpha(x)u$, $G(x) = qx + b\alpha(x)v$ for some $b, c, q, u, v \in Q_r$, $\alpha \in Aut(R)$ and $a \notin Z(R)$ such that*

$$[a, F(x)x] = xG(x)$$

for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$. *Then there exist* $c, q \in Q_r$ *such that* $F(x) = cx$, $G(x) = qx$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and* $[a, c] = q \in C$.

Proof. By the hypothesis $F(x) = cx + b\alpha(x)u$ and $G(x) = qx + b\alpha(x)v$ for all $x \in f(R)$ and for some fixed $b, c, q, u, v \in Q_r$. If α is inner automorphism, then by Lemma 6.2.4, we obtain our conclusions. Moreover, it is clear that, if $b = 0$ then the conclusion follows from Remark 6.1.1. Thus we assume that α is an outer automorphism and $b \neq 0$.

Now

$$[a, F(f(x))f(x)] = f(x)G(f(x))$$

gives

$$\begin{aligned}&[a, (cf(x_1, \ldots, x_n) + b\alpha(f(x_1, \ldots, x_n))u)f(x_1, \ldots, x_n)] \\ &= f(x_1, \ldots, x_n)(qf(x_1, \ldots, x_n) + b\alpha(f(x_1, \ldots, x_n))v)\end{aligned} \tag{6.2.19}$$

for all $x_1, \ldots, x_n \in R$.

By [24], we know that R and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r. Thus

$$\begin{aligned}&[a, (cf(x_1, \ldots, x_n) + b\alpha(f(x_1, \ldots, x_n))u)f(x_1, \ldots, x_n)] \\ &= f(x_1, \ldots, x_n)(qf(x_1, \ldots, x_n) + b\alpha(f(x_1, \ldots, x_n))v)\end{aligned} \tag{6.2.20}$$

for all $x_1, \ldots, x_n \in Q_r$.

Since $\alpha \in Aut(R)$, an outer automorphism of R, by [74], we get from above

$$\begin{aligned}&[a, (cf(x_1, \ldots, x_n) + bf^{\alpha}(y_1, \ldots, y_n)u)f(x_1, \ldots, x_n)] \\ &= f(x_1, \ldots, x_n)(qf(x_1, \ldots, x_n) + bf^{\alpha}(y_1, \ldots, y_n)v)\end{aligned} \tag{6.2.21}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in Q_r$. In particular, Q_r satisfies the blended component

$$[a, cf(x_1, \ldots, x_n)^2] = f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \tag{6.2.22}$$

and

$$[a, bf^{\alpha}(y_1, \ldots, y_n)uf(x_1, \ldots, x_n)] = f(x_1, \ldots, x_n)bf^{\alpha}(y_1, \ldots, y_n)v. \tag{6.2.23}$$

Application of Remark 6.1.1 to (6.2.22) gives $f(x_1, \ldots, x_n)^2$ is central valued and $[a, c] = q \in C$. The equation (6.2.23) gives

$$ab'f(x_1, \ldots, x_n) - b'f(x_1, \ldots, x_n)a - f(x_1, \ldots, x_n)b'' = 0 \tag{6.2.24}$$

where $b' = bf^{\alpha}(y_1, \ldots, y_n)u$ and $b'' = bf^{\alpha}(y_1, \ldots, y_n)v$. By Lemma 6.2.5, one of the following holds:

(i) $b'', a \in C$ with $ab' - b'a - b'' = 0$; which gives contradiction, as $a \notin C$.

(ii) $ab', b' \in C$ with $ab' - b'a - b'' = 0$; In this case, since $a \notin C$, we must have $b' = 0$ and $b'' = 0$. Thus $bf^{\alpha}(y_1, \ldots, y_n)u = 0$ and $bf^{\alpha}(y_1, \ldots, y_n)v = 0$ for all $y_1, \ldots, y_n \in Q_r$. Since $f^{\alpha}(y_1, \ldots, y_n)$ is not central valued, by applying Lemma 6.2.5 and since $b \neq 0$, we have $u = 0 = v$. Then F and G both becomes generalized derivations, and hence conclusion follows again by Remark 6.1.1. □

Corollary 6.2.7. *Let R be a prime ring of char $(R) \neq 2$ with right Martindale quotient ring Q_r and extended centroid C. Also $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that for some $c \in Q_r$, $\alpha \in Aut(R)$ and $a \notin Z(R)$*

$$[a, c(x - \alpha(x))x] = 0$$

for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$. Then $\alpha = Id$ (identity map).

Corollary 6.2.8. *Let R be a prime ring of char $(R) \neq 2$ with right Martindale quotient ring Q_r and extended centroid C. Also $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that for some $c, c' \in Q_r$, $\alpha \in Aut(R)$ and $a \notin Z(R)$*

$$[a, c(x - \alpha(x))x] = xc'(x - \alpha(x))$$

for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$. Then $\alpha = Id$ (identity map).

6.3 Proof of Main Theorem.

To prove our Main Theorem, we need the following Lemma.

Lemma 6.3.1. *[42, Proposition 2.4] Let R be a noncommutative prime ring of characteristic different from 2 and C be its extended centroid. Also let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. If $a_1, a_2, a_3, a_4, a_5, a_6 \in R$ such that*

$$a_1r^2 + a_2ra_3r + a_4r^2a_5 + a_6ra_3ra_5 = 0$$

for all $r \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in R\}$, then either $a_3 \in C$ or $a_5 \in C$ or $a_6 = 0$.

More over we state the following remark.

Remark. *Suppose that $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C, then*

$$f(x_1,\ldots,x_n)=\sum_{\sigma\in S_n}\gamma_\sigma x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)},$$

where $\gamma_\sigma\in C$ and S_n be the symmetric group of n symbols. If d be a skew derivation associated with an automorphism α then

$$\begin{aligned}d(\gamma_\sigma x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}) &= d(\gamma_\sigma)x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}\\ &\quad+\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)}\end{aligned}$$

so we get

$$\begin{aligned}&d(f(x_1,\ldots,x_n))=f^d(x_1,\ldots,x_n)\\ &+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)},\end{aligned}\tag{6.3.1}$$

where $f^d(x_1,\ldots,x_n)$ be a multilinear polynomial given from $f(x_1,\ldots,x_n)$ by replacing each coefficients γ_σ with $d(\gamma_\sigma)$. Also we use $f^\alpha(x_1,\ldots,x_n)$ to denote a multilinear polynomial given from $f(x_1,\ldots,x_n)$ by replacing each coefficients γ_σ with $\alpha(\gamma_\sigma)$.

Proof of Theorem 1.1. *Since F and G are b-generalized skew derivations, so by [65, Lemma 3.3] we have $F(x)=cx+bd(x)$, $G(x)=qx+b\delta(x)$ for all $x\in R$ and $b,c,q\in Q_r$, where d,δ are two skew derivations on R. If d and δ both are skew inner derivations, then the result follows from Proposition 6.2.6. Also when $b=0$ then F and G becomes generalized derivations and hence conclusion follows by [40, Corollary 1.5]. So we assume that $b\neq 0$ and d,δ both are not skew inner derivations.*

Since any skew derivations of R can be uniquely extended in Q_r and by [27, Theorem 2] R and Q_r satisfy the same generalized polynomial identities with a single skew derivation, Q_r satisfies

$$\begin{aligned}&\Big[a,\Big(cf(x_1,\ldots,x_n)+bd(f(x_1,\ldots,x_n))\Big)f(x_1,\ldots,x_n)\Big]\\ &=f(x_1,\ldots,x_n)\Big(qf(x_1,\ldots,x_n)+b\delta(f(x_1,\ldots,x_n))\Big).\end{aligned}\tag{6.3.2}$$

Now we have the following cases.

Case I: When d be skew inner and δ be outer.

Then we can write $F(x) = cx + b\alpha(x)u$ and $G(x) = qx + b\delta(x)$. By the hypothesis

$$\begin{aligned}&\Big[a, \Big(cf(x_1, \ldots, x_n) + b\alpha\big(f(x_1, \ldots, x_n)\big)u\Big)f(x_1, \ldots, x_n)\Big]\\&= f(x_1, \ldots, x_n)\Big(qf(x_1, \ldots, x_n) + b\delta\big(f(x_1, \ldots, x_n)\big)\Big).\end{aligned} \tag{6.3.3}$$

Using the value of $\delta(f(x_1, \ldots, x_n))$ from (6.3.1) in the above equation we get

$$\begin{aligned}&\Big[a, \Big(cf(x_1, \ldots, x_n) + b\alpha\big(f(x_1, \ldots, x_n)\big)u\Big)f(x_1, \ldots, x_n)\Big]\\&= f(x_1, \ldots, x_n)\Big(qf(x_1, \ldots, x_n) + b\big(f^{\delta}(x_1, \ldots, x_n)\\&+ \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)} \ldots x_{\sigma(n)}\big)\Big).\end{aligned} \tag{6.3.4}$$

As δ is outer, using [27] we get

$$\begin{aligned}&\Big[a, \Big(cf(x_1, \ldots, x_n) + b\alpha\big(f(x_1, \ldots, x_n)\big)u\Big)f(x_1, \ldots, x_n)\Big]\\&= f(x_1, \ldots, x_n)\Big(qf(x_1, \ldots, x_n) + b\big(f^{\delta}(x_1, \ldots, x_n)\\&+ \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)} \ldots x_{\sigma(n)}\big)\Big),\end{aligned} \tag{6.3.5}$$

where $z_{\sigma(j+1)} = \delta(x_{\sigma(j+1)})$. In particular, Q_r satisfies the blended component

$$\begin{aligned}&f(x_1, \ldots, x_n)b \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)} \ldots x_{\sigma(n)}\\&= 0.\end{aligned} \tag{6.3.6}$$

Let α be an inner automorphism. Then there exists $p \in Q_r$ such that $\alpha(x) = pxp^{-1}$ for all $x \in R$ and hence it follows that

$$\begin{aligned}&f(x_1, \ldots, x_n)b \sum_{\sigma \in S_n} \gamma_\sigma \sum_{j=0}^{n-1} p(x_{\sigma(1)}x_{\sigma(2)} \ldots x_{\sigma(j)})p^{-1}z_{\sigma(j+1)}x_{\sigma(j+2)} \ldots x_{\sigma(n)}\\&= 0.\end{aligned} \tag{6.3.7}$$

Replacing z_i with px_i, we have

$$f(x_1, \ldots, x_n)bpf(x_1, \ldots, x_n) = 0 \tag{6.3.8}$$

which implies $bp = 0$, i.e., $b = 0$, contradiction (since we consider that $b \neq 0$).

Let α be an outer automorphism. Then from (6.3.6)

$$f(x_1,\ldots,x_n)b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}y_{\sigma(1)}y_{\sigma(2)}\ldots y_{\sigma(j)}z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)} = 0. \tag{6.3.9}$$

In particular Q_r satisfies blended component

$$f(x_1,\ldots,x_n)b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}y_{\sigma(1)}y_{\sigma(2)}\ldots y_{\sigma(j)}\ldots y_{\sigma(n-1)}z_{\sigma(n)} = 0. \tag{6.3.10}$$

Replacing y_i with $\alpha(y_i)$ and z_i with $\alpha(y_i)$, we have

$$f(x_1,\ldots,x_n)b\alpha(f(y_1,\ldots,y_n)) = 0. \tag{6.3.11}$$

This implies $b\alpha(f(y_1,\ldots,y_n)) = 0$, since $f(x_1,\ldots,x_n)$ is noncentral valued so $b = 0$, contradiction.

Case II: When d be outer and δ be skew inner.

Then we can write $F(x) = cx + bd(x)$ and $G(x) = qx + b\alpha(x)v$. By our hypothesis

$$\Big[a,\Big(cf(x_1,\ldots,x_n)+bd\big(f(x_1,\ldots,x_n)\big)\Big)f(x_1,\ldots,x_n)\Big] = f(x_1,\ldots,x_n)\Big(qf(x_1,\ldots,x_n)+b\alpha\big(f(x_1,\ldots,x_n)\big)v\Big). \tag{6.3.12}$$

Using the value of $d(f(x_1,\ldots,x_n))$ from (6.3.1) in the above equation we get

$$\Big[a,\Big(cf(x_1,\ldots,x_n)+b\big(f^d(x_1,\ldots,x_n) + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)}\big)\Big)f(x_1,\ldots,x_n)\Big] = f(x_1,\ldots,x_n)\Big(qf(x_1,\ldots,x_n)+b\alpha\big(f(x_1,\ldots,x_n)\big)v\Big).$$

As d is outer, by using [27] we get

$$\Big[a,\Big(cf(x_1,\ldots,x_n)+b\big(f^d(x_1,\ldots,x_n) + \sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\big)\Big)f(x_1,\ldots,x_n)\Big] = f(x_1,\ldots,x_n)\Big(qf(x_1,\ldots,x_n)+b\alpha\big(f(x_1,\ldots,x_n)\big)v\Big).$$

In particular, Q_r satisfies

$$\Big[a,\Big(b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\Big).f(x_1,\ldots,x_n)\Big] = 0. \tag{6.3.13}$$

Replacing y_i *with* $x - \alpha(x)$*, we have*

$$\Big[a, \Big(b(f(x_1,\ldots,x_n) - \alpha(f(x_1,\ldots,x_n)))\Big) f(x_1,\ldots,x_n)\Big] = 0.$$

By Corollary 6.2.7, $\alpha = Id$ *(identity map). Then assuming* $y_i = x_i$ *in (6.3.13) we have*

$$\Big[a, bf(x_1,\ldots,x_n)^2\Big] = 0, \qquad (6.3.14)$$

that is

$$abf(x_1,\ldots,x_n)^2 - bf(x_1,\ldots,x_n)^2 a = 0. \qquad (6.3.15)$$

Then by Lemma 6.2.5, since $a \notin C$*, one of the following holds:*

(i) $ab, b \in C$*;*

(ii) $ab - ba = 0$ *with* $f(x_1,\ldots,x_n)^2$ *is central valued.*

In the first case when $b \in C$*, then* F *and* G *becomes generalized derivations and hence conclusion follows by [40, Corollary 1.5].*

Thus we consider the second case, i.e., $[a, b] = 0$ *with* $f(x_1,\ldots,x_n)^2$ *is central valued. Now for some* $p \notin C$*, we replace* $y_i = [p, x_i]$ *in (6.3.13) and then obtain*

$$\Big[a, b[p, f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n)\Big] = 0. \qquad (6.3.16)$$

Since $[a, b] = 0$*, this yields*

$$b\Big[a, [p, f(x_1,\ldots,x_n)]f(x_1,\ldots,x_n)\Big] = 0. \qquad (6.3.17)$$

This can be written as

$$\begin{aligned} &bapf(x_1,\ldots,x_n)^2 - baf(x_1,\ldots,x_n)pf(x_1,\ldots,x_n) \\ &-bpf(x_1,\ldots,x_n)^2 + bf(x_1,\ldots,x_n)pf(x_1,\ldots,x_n)a = 0. \end{aligned} \qquad (6.3.18)$$

Then by Lemma 6.3.1, either $b = 0$ *or* $p \in C$ *or* $a \in C$*. In any case a contradiction follows.*

Case III: When d ***and*** δ ***both be outer.***

We have $F(x) = cx + bd(x)$ *and* $G(x) = qx + b\delta(x)$*, where* d, δ *both are outer. We divide this case into two sub-cases.*

Sub-case i: $\{d,\delta\}$ ***is linearly*** C***-independent modulo inner skew derivations of*** R.

In this subcase our hypothesis gives

$$\Big[a,\Big(cf(x_1,\ldots,x_n)+bd\big(f(x_1,\ldots,x_n)\big)\Big)f(x_1,\ldots,x_n)\Big]$$
$$=f(x_1,\ldots,x_n)\Big(qf(x_1,\ldots,x_n)+b\delta\big(f(x_1,\ldots,x_n)\big)\Big). \qquad (6.3.19)$$

Using the value of $d(f(x_1,\ldots,x_n))$ *and* $\delta(f(x_1,\ldots,x_n))$ *from (6.3.1) in the above equation we get*

$$\Big[a,\Big(cf(x_1,\ldots,x_n)+b\big(f^d(x_1,\ldots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})d(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)}\big)\Big)f(x_1,\ldots,x_n)\Big]$$
$$=f(x_1,\ldots,x_n)\Big(qf(x_1,\ldots,x_n)+b\big(f^d(x_1,\ldots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\ldots x_{\sigma(n)}\big)\Big).$$

As d, δ both are outer derivations, so by using [27], Q_r *satisfies*

$$\Big[a,\Big(cf(x_1,\ldots,x_n)+b\big(f^d(x_1,\ldots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\big)\Big)f(x_1,\ldots,x_n)\Big]$$
$$=f(x_1,\ldots,x_n)\Big(qf(x_1,\ldots,x_n)+b\big(f^d(x_1,\ldots,x_n)$$
$$+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}\big)\Big).$$

In particular, Q_r *satisfies the blended component*

$$f(x_1,\ldots,x_n)b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\ldots x_{\sigma(n)}$$
$$=0. \qquad (6.3.20)$$

This is same as (6.3.6), so by the same argument we leads to a contradiction.

Sub-case ii: $\{d,\delta\}$ ***is linearly*** C***-dependent modulo inner skew derivations of*** R.

Under the assumption of this subcase there exist $\lambda,\mu\in C$ *and* $q_1\in Q_r$ *such that*

$$\lambda d(x)+\mu\delta(x)=q_1x-\alpha(x)q_1$$

for all $x \in R$. *Since* d, δ *both are not inner skew derivations, so* λ, μ *both are nonzero. We can write*

$$d(x) = \beta\delta(x) + p_1 x - \alpha(x)p_1$$

where $\beta = -\lambda^{-1}\mu$ *and* $p_1 = \lambda^{-1}q_1$.
Hence R *satisfies*

$$\begin{aligned}&\Big[a, \Big(cf(x_1,\dots,x_n) + b\beta\delta(f(x_1,\dots,x_n)) + bp_1 f(x_1,\dots,x_n)\\&-b\alpha(f(x_1,\dots,x_n))p_1\Big)f(x_1,\dots,x_n)\Big]\\&= f(x_1,\dots,x_n)\Big(qf(x_1,\dots,x_n) + b\delta\big(f(x_1,\dots,x_n)\big)\Big). \qquad (6.3.21)\end{aligned}$$

Using the value of $\delta(f(x_1,\dots,x_n))$ *from (6.3.1) in the above equation we get*

$$\begin{aligned}&\Big[a, \Big(cf(x_1,\dots,x_n) + b\beta\big(f^{\delta}(x_1,\dots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}\big)\\&+bp_1 f(x_1,\dots,x_n) - b\alpha(f(x_1,\dots,x_n))p_1\Big)f(x_1,\dots,x_n)\Big]\\&= f(x_1,\dots,x_n)\Big(qf(x_1,\dots,x_n) + b\big(f^{\delta}(x_1,\dots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})\delta(x_{\sigma(j+1)})x_{\sigma(j+2)}\dots x_{\sigma(n)}\big)\Big).\end{aligned}$$

As δ *is an outer derivation so by using [27] we get*

$$\begin{aligned}&\Big[a, \Big(cf(x_1,\dots,x_n) + b\beta\big(f^{\delta}(x_1,\dots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}\big) + bp_1 f(x_1,\dots,x_n)\\&-b\alpha(f(x_1,\dots,x_n))p_1\Big)f(x_1,\dots,x_n)\Big]\\&= f(x_1,\dots,x_n)\Big(qf(x_1,\dots,x_n) + b\big(f^{\delta}(x_1,\dots,x_n)\\&+\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}\big)\Big).\end{aligned}$$

In particular, Q_r *satisfies the blended component*

$$\begin{aligned}&\Big[a, \Big(b\beta\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}\Big)f(x_1,\dots,x_n)\Big]\\&= f(x_1,\dots,x_n)b\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}x_{\sigma(2)}\dots x_{\sigma(j)})z_{\sigma(j+1)}x_{\sigma(j+2)}\dots x_{\sigma(n)}. \qquad (6.3.22)\end{aligned}$$

Replacing y_i with a skew-derivation $h(x) = x - \alpha(x)$, we have

$$\Big[a, b\beta h\big(f(x_1,\ldots,x_n)\big)f(x_1,\ldots,x_n)\Big] \\ = f(x_1,\ldots,x_n)bh\big(f(x_1,\ldots,x_n)\big), \qquad (6.3.23)$$

that is

$$\Big[a, b\beta\Big(f(x_1,\ldots,x_n) - \alpha(f(x_1,\ldots,x_n)\Big)f(x_1,\ldots,x_n)\Big] \\ = f(x_1,\ldots,x_n)b\Big(f(x_1,\ldots,x_n) - \alpha(f(x_1,\ldots,x_n))\Big). \qquad (6.3.24)$$

By Corollary 6.3.2 and since $b \neq 0$, it follows $\alpha = Id$. Then F and G are b-generalized derivations. In this case (6.3.22) reduces to

$$\Big[a, b\beta\sum_{i=0}^{n} f(x_1,\ldots,z_i,\ldots,x_n)f(x_1,\ldots,x_n)\Big] \\ = f(x_1,\ldots,x_n)b\sum_{i=0}^{n} f(x_1,\ldots,z_i,\ldots,x_n). \qquad (6.3.25)$$

In particular,

$$\Big[a, b\beta f(x_1,\ldots,x_n)^2\Big] = f(x_1,\ldots,x_n)bf(x_1,\ldots,x_n) \qquad (6.3.26)$$

By Remark 6.1.1 $b \in C$ with $[a, b\beta] = b$ and $f(x_1,\ldots,x_n)^2$ is central valued implying $b = 0$, which is again a contradiction. This completes the proof of the theorem. □

Corollary 6.3.2. *Let R be a prime ring of char $(R) \neq 2$, Q_r its right Martindale quotient ring, C its extended centroid and $b \in Q_r$. Also let $f(x_1,\ldots,x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that F and G are two b-generalized skew derivations of R such that*

$$\big[F(u)u, G(v)v\big] = 0$$

for all $u, v \in f(R)$, where $f(R) = \{f(x_1,\ldots,x_n) | x_1,\ldots,x_n \in R\}$. Then there exist $p, q \in Q_r$ such that $F(x) = px$, $G(x) = qx$ for all $x \in R$ with $f(x_1,\ldots,x_n)^2$ is central valued on R and $[p, q] = 0$.

Proof. Let $a = F(u)u$. Then by hypothesis

$$\big[a, G(f(x_1,\ldots,x_n))f(x_1,\ldots,x_n)\big] = 0$$

for all $x_1, \ldots, x_n \in R$. By Corollary 6.2.7, there exists some $q \in Q_r$ such that $G(x) = qx$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $[a, q] = 0$. Now $[a, q] = 0$ implies that

$$\big[F(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n), q\big] = 0$$

for all $x_1, \ldots, x_n \in R$. Again applying Corollary 6.2.7, we conclude that there exists some $p \in Q_r$ such that $F(x) = px$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $[p, q] = 0$, as desired. □

Chapter 7

Some Identities Related to Multiplicative (generalized)-derivations in Prime and Semiprime Rings

7.1 Introduction

In this Chapter, we develop the idea of generalized derivation of a ring R. Note that a mapping $F : R \to R$ is a generalized derivation if the following conditions are satisfied:

1. $F(x + y) = F(x) + F(y)$ *for all* $x, y \in R$;

2. $F(xy) = F(x)y + xd(y)$ *for all* $x, y \in R$; *for some derivation* d *of* R.

Now the question arises what will happen if the additivity conditions or some other conditions be removed from the definition. In this regard many authors introduce several related maps in this literature.

In this Chapter R denotes an associative ring with the center $Z(R)$. In [31], Daif and Tammam-El-Sayiad introduced the notion of the multiplicative generalized derivation. It is obvious that every generalized derivation is a multiplicative generalized derivation of R, but the converse need not be true in general. Thus multiplicative generalized derivations are the large number of maps covering the concept of derivations, generalized derivations and left multiplier maps etc. One can find an

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example of multiplicative generalized derivation, which is neither a derivation, nor a generalized derivation.

Example 7.1.1.
Let $R = \begin{pmatrix} 0 & S & S \\ 0 & 0 & S \\ 0 & 0 & 0 \end{pmatrix}$, *where S is the set of all integers. Define the mappings* $F, d : R \longrightarrow R$ *by* $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ *and* $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & ac \\ 0 & 0 & 0 \end{pmatrix}$.
Then it is easy to verify that d is additive, but F is not additive map in R. Moreover, $F(xy) = F(x)y + xd(y)$ *and* $d(xy) = d(x)y + xd(y)$ *holds for all* $x, y \in R$. *Hence, F is a multiplicative generalized derivation associated with a derivation d, but F is not a generalized derivation of R (since F is not additive map).*

Further, if d be any map (not necessarily a derivation), then F becomes a multiplicative (generalized)-derivation associated with the map d. In [37], Dhara and Ali introduced the notion of multiplicative (generalized)-derivation. The concept of multiplicative (generalized)-derivation covers the concept of multiplicative generalized derivation as well as generalized derivation.

Example 7.1.2.
Let $R = C[0,1]$, *the ring of all continuous (real or complex valued) functions. Define the mappings* $F, D : R \longrightarrow R$ *by*

$$F(f)(x) = \begin{cases} af(x) + f(x)\log|f(x)|, & \text{when } f(x) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $a \in C$ *is a fixed element and*

$$D(f)(x) = \begin{cases} f(x)\log|f(x)|, & \text{when } f(x) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is straightforward to check that F satisfies the condition $F(fg) = F(f)g + fD(g)$ *for all* $f, g \in C[0,1]$, *but F and D are not additive. Thus F is a multiplicative (generalized)-derivation, but F is neither a generalized derivation nor a multiplicative generalized derivation.*

Many papers in literature have investigated about the commutativity of prime and semiprime rings satisfying certain functional identities involving derivations and generalized derivations (see [2], [6], [11], [12], [13], [29], [33], [48], [50], [93]).

In [11], Asharf et al. proved for a prime ring R having a nonzero ideal I and F a generalized derivation on R associated with nonzero derivation d that if R satisfies any one of the following: (1) $F(xy) - xy \in Z(R)$ for all $x, y \in I$, (2) $F(xy) + xy \in Z(R)$ for all $x, y \in I$, (3) $F(xy) - yx \in Z(R)$ for all $x, y \in I$, (4) $F(xy) + yx \in Z(R)$ for all $x, y \in I$, (5) $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, (6) $F(x)F(y) + xy \in Z(R)$ for all $x, y \in I$; then R must be commutative.

From above it is natural to consider the situation $F(x)F(y) \pm yx \in Z(R)$ which has been studied by B. Dhara in [48].

In [34], B. Dhara has studied the situations when a generalized derivation F acts as homomorphism or anti-homomorphism in a nonzero left ideal of a semiprime ring R. Albas [2] consider the situations with central values, that is, (1) $F(xy) - F(x)F(y) \in Z(R)$, (2) $F(xy) + F(x)F(y) \in Z(R)$, (3) $F(xy) - F(y)F(x) \in Z(R)$, (4) $F(xy) + F(y)F(x) \in Z(R)$; for all x, y in some suitable subset of R.

Recently, in [50], Dhara et al. studied these situations of Albas [2] in semiprime rings.

Combining all these above relations, Tiwari et al. [93] considered the following situations:

(i) $G(xy) \pm F(x)F(y) \pm xy \in Z(R)$ for all x, $y \in I$;

(ii) $G(xy) \pm F(x)F(y) \pm yx \in Z(R)$ for all x, $y \in I$;

(iii) $G(xy) \pm F(y)F(x) \pm xy \in Z(R)$ for all x, $y \in I$;

(iv) $G(xy) \pm F(y)F(x) \pm yx \in Z(R)$ for all x, $y \in I$;

(v) $G(xy) \pm F(y)F(x) \pm [x, y] \in Z(R)$ for all x, $y \in I$;;

(vi) $G(xy) \pm F(x)F(y) \pm [\alpha(x), y] \in Z(R)$ for all x, $y \in I$;

where I is a non-zero ideal in prime ring R and $\alpha : R \to R$ is any mapping.

There is also ongoing interest to study above relations replacing generalized derivations F, G with multiplicative (generalized)-derivations or multiplicative generalized derivations. For example, we refer the reader to [5], [9], [37], [46], [51], [52], [94]; where further references can be found. Let F be a multiplicative (generalized)-derivation of R with associated map g. In [37], Dhara and Ali studied the following identities in semiprime ring: (1) $F(xy) \pm xy \in Z(R)$, (2) $F(xy) \pm yx \in Z(R)$, (4) $F(x)F(y) \pm xy \in Z(R)$, (6) $F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in \lambda$; where λ is a nonzero left sided ideal in R.

Dhara and Mozumder investigated in [51] the cases when a multiplicative generalized derivation F *satisfies the identities: (1)* $F(xy) - F(x)F(y) \in Z(R)$*, (2)* $F(xy)+F(x)F(y) \in Z(R)$*, (3)* $F(xy)-F(y)F(x) \in Z(R)$*, (4)* $F(xy)+F(y)F(x) \in Z(R)$*, (5)* $F(xy) - g(y)F(x) \in Z(R)$*; for all* x, y *in some suitable subset of* R*.*

Recently, in [46], Dhara et al. studied the following identities in semiprime rings: (1) $[d(x), F(y)] = \pm[x, y]$*, (2)* $[d(x), F(y)] = \pm x \circ y$*, (3)* $[d(x), F(y)] = 0$*, (4)* $F([x, y]) \pm [\delta(x), \delta(y)] \pm [x, y] = 0$*, (5)* $d'([x, y]) \pm [\delta(x), \delta(y)] \pm [x, y] = 0$*, (6)* $d'([x, y]) \pm [\delta(x), \delta(y)] = 0$*, (7)* $F(x \circ y) \pm \delta(x) \circ \delta(y) \pm x \circ y = 0$*, (8)* $d'(x \circ y) \pm \delta(x) \circ \delta(y) \pm x \circ y = 0$*, (9)* $d'(x \circ y) \pm \delta(x) \circ \delta(y) = 0$*, for all* $x, y \in \lambda$*, where* F *is a multiplicative (generalized)-derivation of* R *associated to the map* d*, and* δ*,* d' *are multiplicative derivations of* R*.*

In the present Chapter, our motivation is to study the identities in semiprime rings:

(1) $F([x, y]) + G(yx) + d(x)F(y) + xy \in Z(R)$,

(2) $F(x \circ y) + G(yx) + d(x)F(y) + xy \in Z(R)$,

(3)$F(xy) + G(yx) + d(x)F(y) \pm [x, y] \in Z(R)$,

(4)$F([x, y]) + G(xy) + d(x)F(y) + yx \in Z(R)$,

(5)$F(x \circ y) + G(xy) + d(x)F(y) + yx \in Z(R)$,

(6)$F([x, y]) + G(yx) + d(y)F(x) - xy \in Z(R)$,

(7)$F(x)F(y) - G(yx) - xy + yx \in Z(R)$*; for all* $x, y \in \lambda$*, where* λ *is a nonzero left ideal of* R *and* F, G *are multiplicative (generalized)-derivations of* R *associated to the maps* d *and* g *respectively.*

7.2 Preliminary Results

Throughout this paper we shall make use of the following identities without any specific mention :

$$[xy, z] = x[y, z] + [x, z]y$$

$$[x, yz] = y[x, z] + [x, y]z$$

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z.$$

We shall use the following facts to prove our theorems:

Fact-1 *[30, Lemma 2] (a) Let* R *be a semiprime ring and the center of a nonzero* *one-sided ideal is contained in the center of R; in particular, any commutative one-sided ideal is contained in the center of R.*

(b) If R is a prime ring with a nonzero central ideal, then R must be commutative.

Fact-2 *(see [81, Theorem 2(ii)] or [43]) Let R be a prime ring and λ be a non-zero left ideal of R. If there exist a derivation d of R such that $[x^p d(x^q), x^r]_k = 0$ for all $x \in \lambda$ and k, p, q, r are fixed positive integers, then either $\lambda d(\lambda) = 0$ or $\lambda[\lambda, \lambda] = (0)$.*

Fact-3 *Let R be a prime ring and I be a non-zero ideal of R. If there exist a derivation d of R such that $d(I) = 0$, then $d(R) = 0$.*

Proof. By the hypothesis, $d(x) = 0$ for all $x \in I$. Taking $x = xr$, $r \in R$ we get $xd(r) = 0$ for all $x \in I$. Therefore $d(r) = 0$, for all $r \in R$ implying $d(R) = 0$. □

Fact-4 *Let R be a prime ring and I be a non-zero ideal of R. If $[I, I] = 0$ then R is commutative.*

Proof. By the hypothesis, $[x, y] = 0$, for all $x, y \in I$. Taking $x = xr$, $r \in R$ we get $x[r, y] = 0$ for all $x, y \in I$. Again taking $y = ys$, $s \in R$ we get $xy[r, s] = 0$ for all $x, y \in I$ implying $[r, s] = 0$ for all $r, s \in R$, that is R is commutative. □

7.3 Main Theorem

Theorem 7.3.1. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative (generalized)-derivations of R associated with maps d and g on R respectively. If*

$$F([x, y]) + G(yx) + d(x)F(y) + xy \in Z(R)$$

for all $x, y \in \lambda$, then $\lambda[d(x), x]_2 = (0)$ and $\lambda[xg(x), x]_2 = (0)$ for all $x \in \lambda$.

Proof. By the hypothesis, we have

$$F([x, y]) + G(yx) + d(x)F(y) + xy \in Z(R) \tag{7.3.1}$$

for all $x, y \in \lambda$. Replacing y by yx in (7.3.1) we obtain

$$\begin{aligned} &F([x, y])x + [x, y]d(x) + G(yx)x + yxg(x) \\ &+d(x)F(y)x + d(x)yd(x) + xyx \in Z(R) \end{aligned} \tag{7.3.2}$$

for all $x, y \in \lambda$, which gives

$$(F([x,y]) + G(yx) + d(x)F(y) + xy)x$$
$$[x,y]d(x) + yxg(x) + d(x)yd(x) \in Z(R) \tag{7.3.3}$$

for all $x, y \in \lambda$. Since $F([x,y]) + G(yx) + d(x)F(y) + xy \in Z(R)$ for all $x, y \in \lambda$, we have $[(F([x,y]) + G(yx) + d(x)F(y) + xy)x, x] = 0$ for all $x, y \in \lambda$. Commuting both side of (7.3.3) with x we get

$$[[x,y]d(x), x] + [yxg(x), x] + [d(x)yd(x), x] = 0 \tag{7.3.4}$$

for all $x, y \in \lambda$. Left multiply by x to (7.3.4), we get

$$x[[x,y]d(x), x] + x[yxg(x), x] + [xd(x)yd(x), x] = 0 \tag{7.3.5}$$

for all $x, y \in \lambda$. Replacing y by xy in (7.3.4), we obtain

$$x[[x,y]d(x), x] + x[yxg(x), x] + [d(x)xyd(x), x] = 0 \tag{7.3.6}$$

for all $x, y \in \lambda$. Subtracting (7.3.5) from (7.3.6), we obtain

$$[[d(x), x]yd(x), x] = 0 \tag{7.3.7}$$

for all $x, y \in \lambda$. This implies

$$[[d(x), x]y[d(x), x], x] = 0, \tag{7.3.8}$$

that is

$$[d(x), x]y[d(x), x]x - x[d(x), x]y[d(x), x] = 0 \tag{7.3.9}$$

for all $x, y \in \lambda$. Replacing y by $y[d(x), x]u$ in (7.3.9), where $u \in \lambda$ we obtain

$$[d(x), x]y[d(x), x]u[d(x), x]x - x[d(x), x]y[d(x), x]u[d(x), x] = 0 \tag{7.3.10}$$

for all $x, y, u \in \lambda$. By using (7.3.9), (7.3.10) yields

$$[d(x), x]yx[d(x), x]u[d(x), x] - [d(x), x]y[d(x), x]xu[d(x), x] = 0 \tag{7.3.11}$$

for all $x, y, u \in \lambda$. This gives

$$[d(x), x]y[[d(x), x], x]u[d(x), x] = 0 \tag{7.3.12}$$

for all $x, y, u \in \lambda$. Thus we have $[[d(x), x], x]y[[d(x), x], x]u[[d(x), x], x] = 0$ for all $x, y, u \in \lambda$, that is $(\lambda[[d(x), x], x])^3 = (0)$ for all $x \in \lambda$. Since R is semiprime ring, it contains no nonzero nilpotent left sided ideals and hence $\lambda[[d(x), x], x] = (0)$ for all $x \in \lambda$.

Replacing y by yx in (7.3.4), we have

$$[[x, y]xd(x), x] + [yx^2g(x), x] + [d(x)yxd(x), x] = 0 \tag{7.3.13}$$

for all $x, y \in \lambda$. Right multiplying (7.3.4) by x and then subtracting from (7.3.13), we get

$$[[x, y][d(x), x], x] + [yx[g(x), x], x] + [d(x)y[d(x), x], x] = 0 \tag{7.3.14}$$

for all $x, y \in \lambda$. Again replacing y by yx in (7.3.14) and right multiplying (7.3.14) by x, we get two equations. Subtracting one of them from another yields

$$[[x, y][d(x), x]_2, x] + [yx[g(x), x]_2, x] + [d(x)y[d(x), x]_2, x] = 0 \tag{7.3.15}$$

for all $x, y \in \lambda$. Since $\lambda[[d(x), x], x] = (0)$ for all $x \in \lambda$, $[yx[g(x), x]_2, x] = 0$ for all $x, y \in \lambda$. Replacing y by ty, we have $0 = [tyx[g(x), x]_2, x] = [t, x]yx[g(x), x]_2$ for all $x, y \in \lambda$ and $t \in R$. We can replace t with $x[g(x), x]$ and then obtain $x[g(x), x]_2yx[g(x), x]_2 = 0$ i.e., $x[g(x), x]_2Ryx[g(x), x]_2 = (0)$ for all $x, y \in \lambda$. Since R is semiprime ring, $\lambda[xg(x), x]_2 = (0)$ for all $x \in \lambda$.

Thereby, the proof is completed. □

Corollary 7.3.2. *Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If*

$$F([x, y]) + G(yx) + d(x)F(y) + xy \in Z(R)$$

for all $x, y \in \lambda$, then either $\lambda d(\lambda) = (0)$ and $\lambda g(\lambda) = (0)$ or $\lambda[\lambda, \lambda] = (0)$.

Proof. By Theorem 7.3.1, we have $\lambda[d(x), x]_2 = (0)$ and $\lambda[xg(x), x]_2 = (0)$ for all $x \in \lambda$. By Fact 2, $\lambda d(\lambda) = (0)$ and $\lambda g(\lambda) = (0)$ or $\lambda[\lambda, \lambda] = (0)$. □

Corollary 7.3.3. *Let R be a prime ring and I a nonzero ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If $F([x, y]) + G(yx) + d(x)F(y) + xy = 0$ for all $x, y \in I$, then one of the following holds:*

1. *$F(x) = G(x) = -x$ for all $x \in R$;*
2. *R is commutative, $d = 0$ and $G(x) = -x$ for all $x \in R$.*

Proof. By Corollary 7.3.2 and Fact-3, Fact-4, we get either $d = g = 0$ or R is commutative.

Case 1: *When* $d = g = 0$.

Then the identity reduces to

$$F(x)y - F(y)x + G(y)x + xy = 0 \tag{7.3.16}$$

for all $x, y \in I$. Replacing x by xt we obtain

$$F(x)ty - F(y)xt + G(y)xt + xty = 0 \tag{7.3.17}$$

for all $x, y, t \in I$. Now right multiply by t to (7.3.16), and then subtracting from (7.3.17) we get

$$(F(x) + x)[t, y] = 0 \tag{7.3.18}$$

for all $x, y, t \in I$. Putting $x = xr$, $r \in R$, we have $(F(x) + x)R[t, y] = (0)$ implying $F(x) + x = 0$ for all $x \in I$ or $[I, I] = (0)$. The case $[I, I] = (0)$ implies R is commutative by Fact-4 and then conclusion follows by Case-2. On the other hand, when $F(x) = -x$ for all $x \in I$, putting $x = rx, r \in R$, we get $(F(r) + r)x = 0$ for all $x \in I$, implying $F(r) = -r$ for all $r \in R$. We have from (7.3.16), $(G(y) + y)x = 0$ for all $x, y \in I$, implying $G(y) + y = 0$ for all $y \in I$. Putting $y = ry, r \in R$, we get $G(r) + r = 0$ for all $r \in R$, thus we obtain our conclusion (1).

Case 2: *When R is commutative.*

Then the identity reduces to

$$G(xy) + d(x)F(y) + xy = 0 \tag{7.3.19}$$

for all $x, y \in I$. Replacing y by yz we obtain

$$G(xy)z + xyg(z) + d(x)F(y)z + d(x)yd(z) + xyz = 0 \tag{7.3.20}$$

for all $x, y, z \in I$. Now right multiplying by z to (7.3.19), and then subtracting from (7.3.20) we get $xyg(z) + d(x)yd(z) = 0$, that is $y(xg(z) + d(x)d(z)) = 0$ for all $x, y, z \in I$. Since R is prime ring, $xg(z) + d(x)d(z) = 0$ for all $x, y, z \in I$. Replacing x by tx, we obtain $0 = txg(z) + td(x)d(z) + d(t)xd(z) = d(t)xd(z)$ for all $x, t, z \in I$. This implies $d(I) = (0)$ so by Fact-3, $d = 0$ on R. Then $xg(z) + d(x)d(z) = 0$ for all $x, y, z \in I$ implies $g = 0$. Thus by (7.3.19), we have $(G(x) + x)y = 0$ for all $x, y \in I$, implying $G(x) + x = 0$ for all $x \in I$. Putting $x = rx, r \in R$, we get $(G(r) + r)x = 0$ for all $x \in I$, implying $G(r) = -r$ for all $r \in R$. This is our conclusion (2). □

Similarly, following theorem can be proved easily.

Theorem 7.3.4. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative (generalized)-derivations of R associated with the maps d and g on R respectively. If*

$$F(x \circ y) + G(yx) + d(x)F(y) + xy \in Z(R)$$

for all $x, y \in \lambda$, then $\lambda[d(x), x]_2 = (0)$ and $\lambda[xg(x), x]_2 = (0)$ for all $x \in \lambda$.

Corollary 7.3.5. *Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If*

$$F(x \circ y) + G(yx) + d(x)F(y) + xy \in Z(R)$$

for all $x, y \in \lambda$, then either $\lambda d(\lambda) = (0)$ and $\lambda g(\lambda) = (0)$ or $\lambda[\lambda, \lambda] = (0)$.

Corollary 7.3.6. *Let R be a prime ring and I a nonzero ideal of R. Suppose that F and G are two multiplicative generalized derivations on R associated with derivations d and g on R respectively. If $F(x \circ y) + G(yx) + d(x)F(y) + xy = 0$ for all $x, y \in I$, then one of the following holds:*

1. *$G(x) = -F(x) = x$ for all $x \in R$;*
2. *R is commutative with $2F(x) + G(x) + x = 0$ for all $x \in R$.*

Proof. By Corollary 7.3.5, we get either $d = g = 0$ or R is commutative.

Case 1: *When $d = g = 0$.*

Then the identity reduces to

$$F(x)y + F(y)x + G(y)x + xy = 0 \tag{7.3.21}$$

for all $x, y \in I$. Replacing x by xt we obtain

$$F(x)ty + F(y)xt + G(y)xt + xty = 0 \tag{7.3.22}$$

for all $x, y, t \in I$. Now right multiply by t to (7.3.21), and then subtracting from (7.3.22) we get $(F(x) + x)[t, y] = 0$ for all $x, y, t \in I$. Replacing x with xr, where $r \in R$, we have $(F(x) + x)R[t, y] = (0)$ for all $x, y, t \in I$. By primeness of R, either $F(x) = -x$ for all $x \in I$ or $[I, I] = (0)$. The first case gives $F(x) = -x$ for all $x \in I$ and by Fact-4 the second case implies R is commutative. If $F(x) = -x$ for all $x \in I$, then from (7.3.21) we get $(G(y)-y)x=0$ for all $x,y\in I$ implying $G(x)=x$ for all $x\in I$ and therefore $G(x)=x$ for all $x\in R$ and $F(x)=-x$ for all $x\in R$. In the second case when R is commutative, conclusion follows by Case-2.

Case 2: *When R is commutative.*

Then the identity reduces to

$$2F(xy)+G(xy)+d(x)F(y)+xy=0 \tag{7.3.23}$$

for all $x,y\in I$. Replacing y by yz we obtain

$$\begin{aligned}&2F(xy)z+2xyd(z)+G(xy)z+xyg(z)\\&+d(x)F(y)z+d(x)yd(z)+xyz=0\end{aligned} \tag{7.3.24}$$

for all $x,y,z\in I$. Now right multiply by z to (7.3.23), and then subtracting from (7.3.24) we get

$$2xyd(z)+xyg(z)+d(x)yd(z)=0 \tag{7.3.25}$$

for all $x,y,z\in I$, that is $2xd(z)+xg(z)+d(x)d(z)=0$ for all $x,z\in I$. Replacing x by xt we obtain $t(2xd(z)+xg(z)+d(x)d(z))+xd(t)d(z)=0$ that gives $xd(t)d(z)=0$ for all $x,t,z\in I$. By primeness of R, it yields $d(I)=(0)$, so by Fact-3 $d(R)=(0)$. Then from (7.3.25), $xyg(z)=0$ for all $x,y,z\in I$. Again by primeness of R, it gives $g(I)=(0)$ and hence $g(R)=(0)$. Therefore from (7.3.23) we get $2F(x)+G(x)+x=0$ for all $x\in I$. Putting $x=rx$, $r\in R$, we get $(2F(r)+G(r)+r)x=0$ for all $x\in I$ which implies $2F(r)+G(r)+r=0$ for all $r\in R$. □

Theorem 7.3.7. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative (generalized)-derivations of R associated with the maps d and g on R respectively. If*

$$F(xy)+G(yx)+d(x)F(y)\pm[x,y]\in Z(R)$$

for all $x,y\in\lambda$, then $\lambda[d(x),x]_2=(0)$ and $\lambda[xg(x),x]_2=(0)$ for all $x\in\lambda$.

Proof. By the hypothesis, we have

$$F(xy)+G(yx)+d(x)F(y)+[x,y]\in Z(R) \tag{7.3.26}$$

for all $x,y\in\lambda$. Replacing y by yx in (7.3.26) we obtain

$$\begin{aligned}&F(xy)x+xyd(x)+G(yx)x+yxg(x)\\&+d(x)F(y)x+d(x)yd(x)+[x,y]x\in Z(R)\end{aligned} \tag{7.3.27}$$

for all $x, y \in I$. This gives

$$\begin{aligned}(F(xy) + G(yx) + d(x)F(y) + [x, y])x \\ +xyd(x) + yxg(x) + d(x)yd(x) + \in Z(R)\end{aligned} \tag{7.3.28}$$

for all $x, y \in \lambda$. Commuting both side with x and then using the fact $F(xy) + G(yx) + d(x)F(y) + [x, y] \in Z(R)$ for all $x, y \in \lambda$, we have

$$[xyd(x), x] + [yxg(x), x] + [d(x)yd(x), x] = 0 \tag{7.3.29}$$

for all $x, y \in \lambda$. Replacing y by xy and left multiplying by x in (7.3.29), we get two equations; subtracting one from another yields

$$[[d(x), x]yd(x), x] = 0 \tag{7.3.30}$$

for all $x, y \in \lambda$. This is same as (7.3.7), in Theorem 7.3.1. Therefore by the same argument of Theorem 7.3.1, we can conclude $\lambda[d(x), x]_2 = (0)$ for all $x \in \lambda$.

Replacing y by yx in (7.3.29), we have

$$[xyxd(x), x] + [yx^2g(x), x] + [d(x)yxd(x), x] = 0 \tag{7.3.31}$$

for all $x, y \in \lambda$. Right multiplying (7.3.29) by x and then subtracting from (7.3.31), we get

$$[xy[d(x), x], x] + [yx[g(x), x], x] + [d(x)y[d(x), x], x] = 0 \tag{7.3.32}$$

for all $x, y \in \lambda$. Again replacing y by yx in (7.3.32) and right multiplying (7.3.32) by x, we get two equations. Subtracting one of them from another yields

$$[xy[d(x), x]_2, x] + [yx[g(x), x]_2, x] + [d(x)y[d(x), x]_2, x] = 0 \tag{7.3.33}$$

for all $x, y \in \lambda$. Since $\lambda[[d(x), x], x] = (0)$ for all $x \in \lambda$, $[yx[g(x), x]_2, x] = 0$ for all $x, y \in \lambda$. Replacing y by ty, we have $0 = [tyx[g(x), x]_2, x] = [t, x]yx[g(x), x]_2$ for all $x, y \in \lambda$ and $t \in R$. We can replace t with $x[g(x), x]$ and then obtain $x[g(x), x]_2yx[g(x), x]_2 = 0$ i.e., $x[g(x), x]_2Ryx[g(x), x]_2 = (0)$ for all $x, y \in \lambda$. Since R is semiprime ring, $\lambda[xg(x), x]_2 = (0)$ for all $x \in \lambda$.

Thereby, the proof is completed.

By similar manner, the same conclusion hold for $F(xy) + G(yx) + d(x)F(y) - [x, y] \in Z(R)$ for all $x, y \in I$. □

Corollary 7.3.8. *Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If*

$$F(xy) + G(yx) + d(x)F(y) \pm [x, y] \in Z(R)$$

for all $x, y \in \lambda$, then either $\lambda d(\lambda) = (0)$ and $\lambda g(\lambda) = (0)$ or $\lambda[\lambda, \lambda] = (0)$.

Corollary 7.3.9. *Let R be a prime ring and I a nonzero ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If*

$$F(xy) + G(yx) + d(x)F(y) + [x, y] = 0$$

for all $x, y \in I$, then one of the following holds:

1. *$-F(x) = G(x) = x$ for all $x \in R$;*
2. *R is commutative, $d = 0$ and $F(x) + G(x) = 0$ for all $x \in R$.*

Proof. By Corollary 7.3.8, we get either $d = g = 0$ or R is commutative.

Case 1: *When $d = g = 0$.*

Then the identity reduces to

$$F(x)y + G(y)x + [x, y] = 0 \tag{7.3.34}$$

for all $x, y \in I$. Replacing x by xt we obtain

$$F(x)ty + G(y)xt + [xt, y] = 0 \tag{7.3.35}$$

for all $x, y, t \in I$. Now right multiply by t to (7.3.34), and then subtracting from (7.3.35) we get

$$(F(x) + x)[t, y] = 0 \tag{7.3.36}$$

for all $x, y, t \in I$. Replacing x with xr, where $r \in R$, we have $(F(x)+x)R[t, y] = (0)$ for all $x, y, t \in I$. By primeness of R, either $F(x) = -x$ for all $x \in I$ or $[I, I] = (0)$. If $[I, I] = (0)$ then R is commutative, conclusion follows by Case-2. When $F(x) = -x$ then from (7.3.34) we get $G(x) = -F(x)$ for all $x \in I$ which is the conclusion (1).

Case 2: *When R is commutative.*

Then the identity reduces to

$$F(xy) + G(xy) + d(x)F(y) = 0 \tag{7.3.37}$$

for all $x, y \in I$. Replacing y by yz we obtain

$$F(xy)z + xyd(z) + G(xy)z + xyg(z) + d(x)F(y)z + d(x)yd(z) = 0 \tag{7.3.38}$$

for all $x, y, z \in I$. Now right multiply by z to (7.3.37), and then subtracting from (7.3.38) we get

$$xyd(z) + xyg(z) + d(x)yd(z) = 0 \tag{7.3.39}$$

for all $x, y, z \in I$, that is $xd(z) + xg(z) + d(x)d(z) = 0$ for all $x, z \in I$. Replacing x by xt we have $t(xd(z) + xg(z) + d(x)d(z)) + xd(t)d(z) = 0$ for all $x, t, z \in I$, that is $xd(t)d(z) = 0$ for all $x, t, z \in I$. By primeness of R we get $d(I) = (0)$ so by Fact-3 $d(R) = (0)$. Then from (7.3.39) we get $xyg(z) = 0$ for all $x, y, z \in I$. Again by primeness of R we get $g(I) = (0)$ hence by Fact-3 $g(R) = (0)$. Therefore from (7.3.37) we get $F(x) + G(x) = 0$, for all $x \in I$. Putting $x = rx, r \in R$, we get $(F(r) + G(r))x = 0$, for all $x \in I$. This implies $F(r) + G(r) = 0$, for all $r \in R$, our conclusion (2). □

Similarly, following corollary is straightforward.

Corollary 7.3.10. *Let R be a prime ring and I a nonzero ideal of R. Suppose that F and G are two multiplicative generalized derivations on R associated with derivations d and g on R respectively. If*

$$F(xy) + G(yx) + d(x)F(y) - [x, y] = 0$$

for all $x, y \in I$, then one of the following holds:

1. *$F(x) = -G(x) = x$ for all $x \in R$;*
2. *R is commutative, $d = 0$ and $F(x) + G(x) = 0$ for all $x \in R$.*

Theorem 7.3.11. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative (generalized)-derivations of R associated with the maps d and g on R respectively. If*

$$F([x, y]) + G(xy) + d(x)F(y) + yx \in Z(R)$$

for all $x, y \in \lambda$, then $\lambda[d(x), x]_2 = (0)$ and $x^2[g(x), x]_3 = 0$ for all $x \in \lambda$.

Proof. By the hypothesis, we have

$$F([x,y]) + G(xy) + d(x)F(y) + yx \in Z(R) \tag{7.3.40}$$

for all $x, y \in \lambda$. Replacing y by yx in (7.3.40) we obtain

$$\begin{aligned} &F([x,y])x + [x,y]d(x) + G(xy)x + xyg(x) \\ &+d(x)F(y)x + d(x)yd(x) + yx^2 \in Z(R) \end{aligned} \tag{7.3.41}$$

for all $x, y \in \lambda$. This gives

$$\begin{aligned} &(F([x,y]) + G(xy) + d(x)F(y) + yx)x \\ &+[x,y]d(x) + xyg(x) + d(x)yd(x) \in Z(R) \end{aligned} \tag{7.3.42}$$

for all $x, y \in \lambda$. Commuting both side with x and then using $F([x,y]) + G(xy) + d(x)F(y) + yx \in Z(R)$ for all $x, y \in \lambda$, we get

$$[[x,y]d(x), x] + [xyg(x), x] + [d(x)yd(x), x] = 0 \tag{7.3.43}$$

for all $x, y \in \lambda$. Replacing y by xy and left multiplying by x in (7.3.43) respectively, we obtain two equations. Subtracting one from another yields

$$[[d(x), x]yd(x), x] = 0 \tag{7.3.44}$$

for all $x, y \in \lambda$. This is same as (7.3.7) in Theorem 7.3.1. Therefore by the same argument of Theorem 7.3.1, we can conclude that $\lambda[d(x), x]_2 = (0)$ for all $x \in \lambda$.

Replacing y by yx in (7.3.43) and right multiplying (7.3.43) by x, we obtain two equations. Subtracting one from another gives

$$[[x,y][d(x),x],x] + [xy[g(x),x],x] + [d(x)y[d(x),x],x] = 0 \tag{7.3.45}$$

for all $x, y \in \lambda$. Again, replacing y by yx in (7.3.45) and right multiplying (7.3.45) by x, we obtain two equations. Subtracting one from another we obtain

$$[[x,y][d(x),x]_2,x] + [xy[g(x),x]_2,x] + [d(x)y[d(x),x]_2,x] = 0 \tag{7.3.46}$$

for all $x, y \in \lambda$. Since $\lambda[d(x), x]_2 = (0)$ for all $x \in \lambda$, we have from above

$$[xy[g(x),x]_2,x] = 0 \tag{7.3.47}$$

for all $x, y \in \lambda$. In particular, $x^2[g(x), x]_3 = 0$ for all $x \in \lambda$. □

Corollary 7.3.12. *Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If*

$$F([x,y]) + G(xy) + d(x)F(y) + yx \in Z(R)$$

for all $x, y \in \lambda$, then either $\lambda d(\lambda) = (0)$ and $\lambda g(\lambda) = (0)$ or $\lambda[\lambda, \lambda] = (0)$.

Corollary 7.3.13. *Let R be a prime ring and I a nonzero ideal of R. Suppose that F and G are two multiplicative generalized derivations on R associated with derivations d and g on R respectively. If*

$$F([x,y]) + G(xy) + d(x)F(y) + yx = 0$$

for all $x, y \in I$, then one of the following holds:

1. *$F(x) = -G(x) = x$ for all $x \in R$;*
2. *R is commutative, $d = 0$ and $G(x) = -x$ for all $x \in R$.*

Proof. By Corollary 7.3.12, we get either $d = g = 0$ or R is commutative.

Case 1: *When $d = g = 0$.*

Then the identity reduces to

$$F(x)y - F(y)x + G(x)y + yx = 0 \tag{7.3.48}$$

for all $x, y \in I$. Replacing y by yt we obtain

$$F(x)yt - F(y)tx + G(x)yt + ytx = 0 \tag{7.3.49}$$

for all $x, y, t \in I$. Now right multiply by t to (7.3.48), and then subtracting from (7.3.49) we get

$$(F(y) - y)[x,t] = 0 \tag{7.3.50}$$

for all $x, y, t \in I$. Putting $y = yr$, $r \in R$ we have $(F(y) - y)R[x,t] = (0)$ implying either $F(y) = y$ for all $y \in I$ or $[I, I] = (0)$. If $[I, I] = (0)$ then by Fact-4 we get R is commutative and conclusion follows by case-2. On the other hand when $F(y) = y$ for all $y \in I$, putting $y = ry$, $r \in R$, we get $\big(F(r) - r\big)y = 0$ for all $y \in I$, implying $F(r) = r$ for all $r \in R$. Putting this value of $F(x)$ in (7.3.48) we get $(G(x) + x)y = 0$ for all $x, y \in I$. Taking $x = xr, r \in R$ we get $(G(x) + x)Ry = 0$, implying $G(x) + x = 0$ for all $x \in I$. Again taking $x = rx, r \in R$ we get $G(r) = -r$ for all $r \in R$, which is our conclusion (1).

Case 2: *When R is commutative.*

Then the identity reduces to

$$G(xy) + d(x)F(y) + xy = 0 \qquad (7.3.51)$$

for all $x, y \in I$. Then by Corollary 7.3.3 case-2, we have our conclusion (2). □

Similarly, we can prove the following theorem.

Theorem 7.3.14. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative (generalized)-derivations of R associated with the maps d and g on R respectively. If $F(x \circ y) + G(xy) + d(x)F(y) + yx \in Z(R)$ for all $x, y \in \lambda$, then $\lambda[d(x), x]_2 = (0)$ and $x^2[g(x), x]_3 = 0$ for all $x \in \lambda$.*

Corollary 7.3.15. *Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If*

$$F(x \circ y) + G(xy) + d(x)F(y) + yx \in Z(R)$$

for all $x, y \in \lambda$, then either $\lambda d(\lambda) = (0)$ and $\lambda g(\lambda) = (0)$ or $\lambda[\lambda, \lambda] = (0)$.

Corollary 7.3.16. *Let R be a prime ring and I a nonzero ideal of R. Suppose that F and G are two multiplicative generalized derivations on R associated with derivations d and g on R respectively. If*

$$F(x \circ y) + G(xy) + d(x)F(y) + yx = 0$$

for all $x, y \in I$, then one of the following holds:

1. *$F(x) = -G(x) = -x$ for all $x \in R$;*

2. *R is commutative with $2F(x) + G(x) + x = 0$ for all $x \in R$.*

Proof. By Corollary 7.3.15, we get either $d = g = 0$ or R is commutative.

Case 1: *When $d = g = 0$.*

Then the identity reduces to

$$F(x)y + F(y)x + G(x)y + yx = 0 \tag{7.3.52}$$

for all $x, y \in I$. Replacing y by yt we obtain

$$F(x)yt + F(y)tx + G(x)yt + ytx = 0 \tag{7.3.53}$$

for all $x, y, t \in I$. Now right multiply by t to (7.3.52), and then subtracting from (7.3.53) we get

$$(F(y) + y)[t, x] = 0 \tag{7.3.54}$$

for all $x, y, t \in I$. Putting $y = yr$, $r \in R$, we have $(F(y) + y)R[t, x] = (0)$ implying $F(y) + y = 0$ for all $y \in I$ or $[I, I] = (0)$. The case $[I, I] = (0)$ implies R is commutative and then conclusion follows by Case-2. On the other hand, when $F(y) = -y$ for all $y \in I$, we have from (7.3.52), $(G(x) - x)y = 0$ for all $y \in I$, implying $G(x) - x = 0$ for all $y \in I$. Thus we obtain our conclusion (1).

Case 2: *When R is commutative.*

Then the identity reduces to

$$2F(xy) + G(xy) + d(x)F(y) + xy = 0 \tag{7.3.55}$$

for all $x, y \in I$. which is same as (7.3.23), in Corollary 7.3.6. Therefore by the same argument of Corollary 7.3.6, we get conclusion (2). □

Theorem 7.3.17. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative (generalized)-derivations of R associated with derivations d and g on R respectively. If*

$$F([x, y]) + G(yx) + d(y)F(x) - xy \in Z(R)$$

for all $x, y \in \lambda$, then $\lambda[d(x), x]_2 = (0)$ and $x^2[g(x), x]_3 = 0$ for all $x \in \lambda$.

Proof. By the hypothesis, we have

$$F([x, y]) + G(yx) + d(y)F(x) - xy \in Z(R) \tag{7.3.56}$$

for all $x, y \in \lambda$. Replacing x by xy in (7.3.56) we obtain

$$\begin{aligned} &F([x, y])y + [x, y]d(y) + G(yx)y + yxg(y) \\ &+d(y)F(x)y + d(y)xd(y) - xy^2 \in Z(R) \end{aligned} \tag{7.3.57}$$

for all $x, y \in \lambda$. This gives

$$\begin{aligned}&(F([x,y]) + G(yx) + d(y)F(x) - xy)y \\ &+[x,y]d(y) + yxg(y) + d(y)xd(y) \in Z(R)\end{aligned} \tag{7.3.58}$$

for all $x, y \in \lambda$. Since $F([x,y]) + G(yx) + d(y)F(x) - xy \in Z(R)$ for all $x, y \in \lambda$, we have from above

$$[[x,y]d(y), y] + [yxg(y), y] + [d(y)xd(y), y] = 0 \tag{7.3.59}$$

for all $x, y \in \lambda$. This is same as (7.3.43) in Theorem 7.3.11. Then by same argument conclusions follows. □

Corollary 7.3.18. *Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If $F([x,y]) + G(yx) + d(y)F(x) - xy \in Z(R)$ for all $x, y \in \lambda$, then either $\lambda d(\lambda) = (0)$ and $\lambda g(\lambda) = (0)$ or $\lambda[\lambda, \lambda] = (0)$.*

Corollary 7.3.19. *Let R be a prime ring and I a nonzero ideal of R. Suppose that F and G are two multiplicative generalized derivations on R associated with derivations d and g on R respectively. If $F([x,y]) + G(yx) + d(y)F(x) - xy = 0$ for all $x, y \in I$, then one of the following holds:*

1. *$F(x) = G(x) = x$ for all $x \in R$;*
2. *R is commutative, $d = 0$ and $G(x) = x$ for all $x \in R$.*

Proof. By Corollary 7.3.18, we get either $d = g = 0$ or R is commutative.

Case 1: *When $d = g = 0$.*

Then the identity reduces to

$$F(x)y - F(y)x + G(y)x - xy = 0 \tag{7.3.60}$$

for all $x, y \in I$. Replacing x by xt we obtain

$$F(x)ty - F(y)xt + G(y)xt - xty = 0 \tag{7.3.61}$$

for all $x, y, t \in I$. Now right multiply by t to (7.3.60), and then subtracting from (7.3.61) we get

$$(F(x) - x)[t, y] = 0 \tag{7.3.62}$$

for all $x, y, t \in I$. Putting $x = xr$, $r \in R$, we have $(F(x) - x)R[t, y] = (0)$ implying $F(x) - x = 0$ for all $x \in I$ or $[I, I] = (0)$. The case $[I, I] = (0)$ implies R is commutative and then conclusion follows by Case-2. On the other hand, when $F(x) = x$ for all $x \in I$, we have from (7.3.60), $(G(y) - y)x = 0$ for all $x \in I$, implying $G(y) - y = 0$ for all $y \in I$. Thus we obtain our conclusion (1).

Case 2: *When R is commutative.*

Then the identity reduces to

$$G(yx) + d(y)F(x) - yx = 0 \tag{7.3.63}$$

for all $x, y \in I$. Replacing x by xz we obtain

$$G(yx)z + yxg(z) + d(y)F(x)z + d(y)xd(z) - yxz = 0 \tag{7.3.64}$$

for all $x, y, z \in I$. Now right multiply by z to (7.3.63), and then subtracting from (7.3.64) we get

$$yxg(z) + d(y)xd(z) = 0 \tag{7.3.65}$$

for all $x, y, z \in I$, that is $yg(z) + d(y)d(z) = 0$. Replacing y by yt we obtain $(yg(z) + d(y)d(z))t + yd(t)d(z) = 0$, this gives $yd(t)d(z) = 0$ for all $y, z, t \in I$. By primeness of R we get $d(I) = (0)$ so by using Fact-3 $d(R) = (0)$. Then from (7.3.65), $xyg(z) = 0$ for all $x, y, z \in I$. Again by primeness of R we get $g(I) = (0)$ hence by Fact-3 we get $g(R) = (0)$. Therefore from (7.3.63), we get $G(x) - x = 0$ for all $x \in I$, which is the conclusion (2). □

Theorem 7.3.20. *Let R be a semiprime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative (generalized)-derivations of R associated with the maps d and g on R respectively. If*

$$F(x)F(y) - G(yx) - xy + yx \in Z(R)$$

for all $x, y \in \lambda$, then $\lambda x[d(x), x]_2 = (0)$ and $x^4[g(x), x]_3 = 0$ for all $x \in \lambda$.

Proof. By the hypothesis, we have

$$F(x)F(y) - G(yx) - xy + yx \in Z(R) \tag{7.3.66}$$

for all $x, y \in \lambda$. Putting $y = yz$ in (7.3.66), we obtain

$$F(x)F(yz) - G(yzx) - [x, yz] \in Z(R) \tag{7.3.67}$$

for all $x, y, z \in \lambda$, this gives

$$F(x)F(y)z + F(x)yd(z) - G(yzx) - y[x,z] - [x,y]z \in Z(R) \qquad (7.3.68)$$

for all $x, y, z \in \lambda$. Commuting both sides with z, we have

$$\Big[F(x)F(y)z + F(x)yd(z) - G(yzx) - y[x,z] - [x,y]z, z\Big] = 0, \qquad (7.3.69)$$

that is

$$[F(x)F(y)z - [x,y]z, z] + [F(x)yd(z) - G(yzx) - y[x,z], z] = 0 \qquad (7.3.70)$$

for all $x, y, z \in \lambda$. By using (7.3.66), this relation reduces to

$$[G(yx)z, z] + [F(x)yd(z) - G(yzx), z] - [y[x,z], z] = 0 \qquad (7.3.71)$$

for all $x, y, z \in \lambda$. Putting $x = z^2$ in (7.3.71), we have

$$[G(yz^2)z, z] + [F(z^2)yd(z), z] - [G(yz^3), z] = 0 \qquad (7.3.72)$$

for all $x, y, z \in \lambda$, which gives

$$[G(yz^2)z, z] + [F(z)zyd(z) + zd(z)yd(z), z] - [G(yz^2)z + yz^2g(z), z] \\ = 0. \qquad (7.3.73)$$

This can be written as

$$[F(z)zyd(z), z] + [zd(z)yd(z), z] - [yz^2g(z), z] = 0 \qquad (7.3.74)$$

for all $x, y, z \in \lambda$. Putting $y = zy$ in (7.3.71), we have

$$[G(zyx)z, z] + [F(x)zyd(z) - G(zyzx), z] - [zy[x,z], z] = 0 \qquad (7.3.75)$$

for all $x, y, z \in \lambda$. Assuming $x = z$, we have

$$[G(zyz)z, z] + [F(z)zyd(z) - G(zyz^2), z] = 0 \qquad (7.3.76)$$

for all $y, z \in \lambda$. This gives

$$[G(zyz)z, z] + [F(z)zyd(z), z] - [G(zyz)z + zyzg(z), z] = 0, \qquad (7.3.77)$$

that is

$$[F(z)zyd(z), z] = [zyzg(z), z] \qquad (7.3.78)$$

for all $y, z \in \lambda$. From (7.3.74) and (7.3.78), we get

$$[zyzg(z), z] + [zd(z)yd(z), z] - [yz^2g(z), z] = 0 \tag{7.3.79}$$

for all $y, z \in \lambda$. Now putting $y = zy$ in (7.3.79), we have

$$z[zyzg(z), z] + [zd(z)zyd(z), z] - z[yz^2g(z), z] = 0 \tag{7.3.80}$$

for all $y, z \in \lambda$. Left multiplying (7.3.79) by z and then subtracting from (7.3.80), we get

$$[[zd(z), z]yd(z), z] = 0 \tag{7.3.81}$$

for all $y, z \in \lambda$. Again putting $y = yz$ in above relation, we get

$$[[zd(z), z]yzd(z), z] = 0 \tag{7.3.82}$$

for all $y, z \in \lambda$. Now right multiplying (7.3.81) by z and then subtracting from (7.3.82), we get

$$[[zd(z), z]y[d(z), z], z] = 0 \tag{7.3.83}$$

for all $y, z \in \lambda$ and hence

$$[z[d(z), z]yz[d(z), z], z] = 0 \tag{7.3.84}$$

for all $y, z \in \lambda$. This implies

$$z[d(z), z]yz[d(z), z]z - z^2[d(z), z]yz[d(z), z] = 0 \tag{7.3.85}$$

for all $y, z \in \lambda$. In above relation replacing y with $yz[d(z), z]u$, we get

$$z[d(z), z]yz[d(z), z]uz[d(z), z]z - z^2[d(z), z]yz[d(z), z]uz[d(z), z] = 0 \tag{7.3.86}$$

for all $y, z, u \in \lambda$. Using (7.3.85), (7.3.86) gives

$$z[d(z), z]yz^2[d(z), z]uz[d(z), z] - z[d(z), z]yz[d(z), z]zuz[d(z), z] = 0 \tag{7.3.87}$$

for all $y, z, u \in \lambda$, that is

$$z[d(z), z]y[z[d(z), z], z]uz[d(z), z] = 0 \tag{7.3.88}$$

for all $y, z, u \in \lambda$. This implies $[z[d(z), z], z]y[z[d(z), z], z]u[z[d(z), z], z] = 0$ for all $y, z, u \in \lambda$, that is, $(\lambda[z[d(z), z], z])^3 = (0)$ for all $z \in R$. Since R is a semiprime ring, we can conclude that $\lambda z[d(z), z]_2 = (0)$ for all $z \in R$.

Now we replace y with yz in (7.3.78) and then obtain

$$[F(z)zyzd(z), z] = [zyz^2g(z), z] \tag{7.3.89}$$

for all $y, z \in \lambda$. Right multiplying by z in (7.3.78) and then subtracting from (7.3.89), we get

$$[F(z)zy[d(z), z], z] = [zyz[g(z), z], z] \tag{7.3.90}$$

for all $y, z \in \lambda$. Again applying same argument,

$$[F(z)zy[d(z), z]_2, z] = [zyz[g(z), z]_2, z] \tag{7.3.91}$$

for all $y, z \in \lambda$. Replacing y by yz and using the fact $\lambda z[d(z), z]_2 = (0)$ for all $z \in R$, we get $[zyz^2[g(z), z]_2, z] = 0$ for all $y, z \in \lambda$. This implies that $z^4[g(z), z]_3 = 0$ for all $z \in \lambda$. □

Corollary 7.3.21. *Let R be a prime ring with center $Z(R)$ and λ a nonzero left ideal of R. Suppose that F and G are two multiplicative generalized derivations of R associated with derivations d and g on R respectively. If*

$$F(x)F(y) - G(yx) - xy + yx \in Z(R)$$

for all $x, y \in \lambda$, then either $\lambda d(\lambda) = (0)$ and $\lambda g(\lambda) = (0)$ or $\lambda[\lambda, \lambda] = (0)$.

Corollary 7.3.22. *Let R be a prime ring with center $Z(R)$. Suppose that F and G are two multiplicative generalized derivations on R associated with derivations d and g on R respectively. If*

$$F(x)F(y) - G(yx) - xy + yx = 0$$

for all $x, y \in R$, then one of the following holds:

1. *$d = 0$ that is F is a left centralizer mapping and $G(x) = x$ for all $x \in R$;*
2. *R is commutative, $d = g = 0$, that is F and G both are left centralizer mapping.*

Proof. By Corollary 7.3.21, we get either $d(R) = g(R) = (0)$ or R is commutative.

Case 1: *When $d(R) = g(R) = (0)$.*

Then the identity reduces to

$$F(x)F(y) - G(y)x - [x, y] = 0 \tag{7.3.92}$$

for all $x, y \in R$. Replacing y by yt we obtain

$$F(x)F(y)t - G(y)tx - y[x,t] - [x,y]t = 0 \tag{7.3.93}$$

for all $x, y, t \in R$. Now right multiply by t to (7.3.92), and then subtracting from (7.3.93) we get

$$(G(y) - y)[x,t] = 0$$

for all $x, y, t \in R$. By primeness of R we get, either $G(y) - y = 0$ for all $y \in R$ or $[x,t] = 0$ for all $x, t \in R$, that is R is commutative, conclusion follows by Case-2. On the other hand, $G(y) = y$ for all $y \in R$. thus we obtain our conclusion (1).

Case 2: *When R is commutative.*

Then the identity reduces to

$$F(x)F(y) - G(xy) = 0 \tag{7.3.94}$$

for all $x, y \in R$. Replacing y by yz we obtain

$$F(x)F(y)z + F(x)yd(z) - G(xy)z - xyg(z) = 0 \tag{7.3.95}$$

for all $x, y, z \in R$, this gives

$$F(x)yd(z) - xyg(z) = 0. \tag{7.3.96}$$

Again taking $x = xz$ we get, $F(x)zyd(z) + xd(z)yd(z) - xzyg(z) = 0$, that is $\big(F(x)yd(z) - xyg(z)\big)z + xd(z)yd(z) = 0$ implying $xd(z)yd(z) = 0$ for all $x, y, z \in R$. By primeness of R we get $d(z) = 0$ for all $z \in R$. Putting $d(z) = 0$ in (7.3.96), we get $xyg(z) = 0$. By primeness of R we get $g(z) = 0$ for all $z \in R$. Thus we obtain our conclusion (2). □

Example 7.3.1.

Let us consider the ring $R = \begin{pmatrix} 0 & GF(2) & GF(2) \\ 0 & 0 & GF(2) \\ 0 & 0 & 0 \end{pmatrix}$. *Then R is not a prime ring, as* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Define the mappings $F, G, d, g : R \longrightarrow R$ *by* $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c^2 \\ 0 & 0 & 0 \end{pmatrix}$,
$G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & ac \\ 0 & 0 & 0 \end{pmatrix}$ *and also*

$$d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = g\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that d *and* g *are derivations.* F *and* G *are not additive maps in* R*, but* $F(xy) = F(x)y + xd(y)$ *and* $G(xy) = G(x)y + xg(y)$ *hold for all* $x, y \in R$*. Hence,* F *and* G *are multiplicative generalized derivations associated with the derivations* d *and* g *respectively. Assuming both sided ideal* $I = R$*, we have* $F([x,y]) + G(yx) + d(x)F(y) + xy = 0$, $F(x \circ y) + G(yx) + d(x)F(y) + xy = 0$, $F(xy) + G(yx) + d(x)F(y) + [x,y] = 0$, $F(xy) + G(yx) + d(x)F(y) - [x,y] = 0$, $F([x,y]) + G(xy) + d(x)F(y) + yx = 0$ *for all* $x, y \in I$*. Since* $F(x) \neq -x$ *and* $G(x) \neq -x$ *for all* $x \in R$*,* R *is noncommutative,* $d \neq 0$*, the primeness hypothesis is not superfluous in Corollary 7.3.3, Corollary 7.3.6, Corollary 7.3.9, Corollary 7.3.10, Corollary 7.3.13.*

Example 7.3.2.
Let $R = \begin{pmatrix} 0 & GF(2) & GF(2) \\ 0 & 0 & GF(2) \\ 0 & 0 & 0 \end{pmatrix}$. *As above,* R *is not prime.*

Define the mappings $F, G, d, g : R \longrightarrow R$ *by* $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$,
$G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c^2 \\ 0 & 0 & 0 \end{pmatrix}$ *and also*
$$d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = g\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that F and G are multiplicative generalized derivations associated with the derivations d and g respectively. Assuming both sided ideal $I = R$, we have $F(x)F(y) - G(yx) - xy + yx = 0$ for all $x, y \in I$. Since $d \neq 0$ and $G(x) \neq x$ for all $x \in R$, R is noncommutative, the primeness hypothesis is not superfluous in Corollary 7.3.22.

Chapter 8

A Note on b-generalized (α, β)-derivations in Prime Rings

8.1 Introduction

Let R be a prime ring with center $Z(R)$ and extended centroid C. For any $x, y \in R$, set $[x, y]_{\alpha,\beta} = x\alpha(y) - \beta(y)x$ and $(x \circ y)_{\alpha,\beta} = x\alpha(y) + \beta(y)x$. Obviously $(1, 1)$-derivation is a simply derivation, where 1 denotes the identity map. For some fixed $a \in R$, the map $x \mapsto a\alpha(x) - \beta(x)a$ is an example of (α, β)-derivations which is called inner (α, β)-derivations.

Every (α, β)-derivations are generalized (α, β)-derivations. The maps of the form $x \mapsto a\alpha(x) + \beta(x)b$ for some $a, b \in R$ is an example of generalized derivation which is said to be an inner generalized (α, β)-derivation.

Many papers in the literature have investigated about the commutativity of prime and semiprime rings satisfying some functional identities involving derivations and generalized derivations (see [2], [6], [11], [12], [13], [29], [33], [34], [45], [48], [50], [93]).

In [11], Ashraf et al. proved for a prime ring R having a nonzero ideal I and a generalized derivation F on R associated with a nonzero derivation d that if R satisfies any one of the following: (1) $F(xy) - xy \in Z(R)$ for all $x, y \in I$, (2) $F(xy) + xy \in Z(R)$ for all $x, y \in I$, (3) $F(xy) + yx \in Z(R)$ for all $x, y \in I$, (4) $F(xy) - yx \in Z(R)$ for all $x, y \in I$, (5) $F(x)F(y) + xy \in Z(R)$ for all $x, y \in I$, (6) $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$; then R must be commutative.

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There is ongoing interest to study the identities replacing generalized derivations with generalized (α, β)-derivations. Several authors studied the commutativity in prime and semiprime rings admitting (α, β)-derivations and generalized (α, β)-derivations which satisfy appropriate algebraic conditions on appropriate subsets of the rings (see [8], [20], [49], [73], [84]).

Let R be a prime ring and F, G be two generalized (α, β)-derivations with associated (α, β)-derivations d and g respectively. Recently, in [20] Garg studied the following identities in prime rings

(i) $G(xy) + d(x)F(y) = 0$;

(ii) $G(xy) + d(x)F(y) + \alpha(yx) = 0$;

(iii) $G(xy) + d(y)F(x) = 0$;

(iv) $G(xy) + d(y)F(x) + \alpha(yx) = 0$;

(v) $G(xy) + F(y)F(x) = 0$;

(vi) $G(xy) + F(x)F(y) = 0$;

for all $x, y \in L$, where L is a square closed Lie ideal of R.

More recently, in [66] Filippis and Wei introduced a new mapping b-generalized (α, β)-derivation.

This mapping is a common generalization of generalized skew derivation and generalized (α, β)-derivation. The maps of the form $x \mapsto a\alpha(x) + b\beta(x)c$ for some $a, b, c \in R$ is an example of b-generalized (α, β)-derivation of R with associated word (b, α, β, d), where $d(x) = \beta(x)c - c\alpha(x)$ for all $x \in R$. Such b-generalized (α, β)-derivations is called inner b-generalized (α, β)-derivations.

The aim of the present Chapter is to extend many known results for b-generalized (α, β)-derivations. More precisely, we study the following algebraic identities involving b-generalized (α, β)-derivation in prime rings:

1. $F(xy) + \alpha(xy) + \alpha(yx) = 0$;

2. $F(xy) + G(x)\alpha(y) + \alpha(yx) = 0$;

3. $F(xy) + G(yx) + \alpha(xy) + \alpha(yx) = 0$;

4. $F(x)F(y) + G(x)\alpha(y) + \alpha(yx) = 0$;

5. $F(xy) + d(x)F_1(y) + \alpha(xy) = 0$;

6. $F(xy) + d(x)F_1(y) = 0$;

7. $F(xy) + d(x)F_1(y) + \alpha(yx) = 0$;

8. $F(xy) + d(x)F_1(y) + \alpha(xy) + \alpha(yx) = 0$;

9. $F(xy) + d(x)F_1(y) + \alpha(yx) - \alpha(xy) = 0$;

10. $\Big[F(x), x\Big]_{\alpha,\beta} = 0$; 11. $\Big(F(x) \circ x\Big)_{\alpha,\beta} = 0$;

12. $F\Big([x, y]\Big) = \Big[x, y\Big]_{\alpha,\beta}$; 13. $F\Big(x \circ y\Big) = \Big(x \circ y\Big)_{\alpha,\beta}$; *for all* x, y *in some suitable subset of* R*, where* F, F_1 *and* G *are three b-generalized* (α, β)*-derivations of* R *associated to the mapping* d, d *and* g *respectively for* $0 \neq b \in R$.

8.2 Preliminary Results

Throughout this paper, α *and* β *will denote automorphisms of* R*. We shall use without explicit mention the following basic identities:*

1. $[xy, z] = x[y, z] + [x, z]y$;

2. $[x, yz] = y[x, z] + [x, y]z$;

3. $(x \circ yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$;

4. $(xy \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$;

5. $[xy, z]_{\alpha,\beta} = x[y, z]_{\alpha,\beta} + [x, \beta(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha,\beta}y$;

6. $[x, yz]_{\alpha,\beta} = \beta(y)[x, z]_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$;

7. $(x \circ yz)_{\alpha,\beta} = (x \circ y)_{\alpha,\beta}\alpha(z) - \beta(y)[x, z]_{\alpha,\beta} = \beta(y)(x \circ z)_{\alpha,\beta} + [x, y]_{\alpha,\beta}\alpha(z)$;

8. $(xy \circ z)_{\alpha,\beta} = x(y \circ z)_{\alpha,\beta} - [x, \beta(z)]y = (x \circ z)_{\alpha,\beta}y + x[y, \alpha(z)]$;

for all $x, y, z \in R$.

Fact-1 *[66]: Let* R *be a prime ring,* $\alpha, \beta \in Aut(R)$*,* $0 \neq b \in R$*,* $d : R \to R$ *be an additive mapping of* R *and* F *be the b-generalized* (α, β)*-derivation of* R *associated to the mapping* d*. Then* d *is an* (α, β)*-derivation of* R.

8.3 Main Results

Theorem 8.3.1. *Let R be a prime ring, λ a nonzero left ideal of R, $0 \neq b \in R$ and F be a b-generalized (α,β)-derivation of R associated with the (α,β) derivation d. If $F(xy)+\alpha(xy)+\alpha(yx)=0$ for all $x,y \in \lambda$, then R is commutative, $\beta(\lambda)d(\lambda)=(0)$ and $F(x)=-2\alpha(x)$ for all $x \in \lambda$.*

Proof. By the hypothesis, we have

$$F(xy)+\alpha(xy)+\alpha(yx)=0 \tag{8.3.1}$$

for all $x,y \in \lambda$. Replacing x^2 in place of x in (8.3.1), we obtain

$$F(x^2y)+\alpha(x^2y)+\alpha(yx^2)=0 \tag{8.3.2}$$

for all $x,y \in \lambda$. Again replacing xy in place of y in (8.3.1), we obtain

$$F(x^2y)+\alpha(x^2y)+\alpha(xyx)=0 \tag{8.3.3}$$

for all $x,y \in \lambda$. Now subtracting (8.3.2) from (8.3.3), we obtain

$$\alpha(yx^2)-\alpha(xyx)=0 \tag{8.3.4}$$

for all $x,y \in \lambda$, that is $\alpha(yx^2-xyx)=0$, implying $[x,y]x=0$ for all $x,y \in \lambda$. Replacing y by ry for $r \in R$, we obtain $[x,ry]x=r[x,y]x+[x,r]yx=0$, that is $[x,r]yx=0$. Since λ is nonzero so $[x,r]=0$ for all $r \in R$. Replacing x by tx we get $[tx,r]=t[x,r]+[t,r]x=0$ for all $t,r \in R$, that is $[t,r]x=0$. Again since λ is nonzero left ideal of R so $[t,r]=0$ for all $t,r \in R$, so R is commutative.

From (8.3.1), by using commutativity of R we get

$$F(xy)+2\alpha(xy)=0 \tag{8.3.5}$$

for all $x,y \in \lambda$. Replacing y by yz for $z \in \lambda$, we obtain

$$F(xy)\alpha(z)+b\beta(xy)d(z)+2\alpha(xy)\alpha(z)=0 \tag{8.3.6}$$

for all $x,y,z \in \lambda$. Right multiply by $\alpha(z)$ to (8.3.5), we get

$$F(xy)\alpha(z)+2\alpha(xy)\alpha(z)=0 \tag{8.3.7}$$

for all $x,y,z \in \lambda$. Subtracting (8.3.6) from (8.3.7), we obtain

$$b\beta(x)\beta(y)d(z)=0 \tag{8.3.8}$$

for all $x, y, z \in \lambda$. Replacing y by ry for $r \in R$, we obtain

$$b\beta(x)\beta(ry)d(z) = 0 \tag{8.3.9}$$

for all $x, y, z \in \lambda$ and $r \in R$, that is $b\beta(x)\beta(r)\beta(y)d(z) = 0$. Now replacing r by $\beta^{-1}\big(d(z)r\big)$, we get $b\beta(x)d(z)r\beta(y)d(z) = 0$ for all $r \in R$. Thus, by primeness of R we get either $\beta(x)d(z) = 0$ or $b\beta(x)d(z) = 0$ for all $x, z \in \lambda$, therefore in both conditions we get $b\beta(x)d(z) = 0$ for all $x, z \in \lambda$. Again taking $x = rx$ for $r \in R$ we get $b\beta(r)\beta(x)d(z) = 0$, that is $bR\beta(x)d(z) = (0)$. By primeness of R and using $b \neq 0$ we get $\beta(x)d(z) = 0$ for all $x, z \in \lambda$, that is $\beta(\lambda)d(\lambda) = (0)$. Thereby, the proof is completed. □

Corollary 8.3.2. *Let R be a prime ring, $0 \neq b \in R$ and F be a b-generalized (α,β)-derivation of R associated with the (α,β) derivation d. If $F(xy)+\alpha(xy)+\alpha(yx) = 0$ for all $x, y \in R$, then R is commutative, $d = 0$ and $F(x) = -2\alpha(x)$ for all $x \in R$.*

Proof. By Theorem 8.3.1 and considering $\lambda = R$ we get the result. □

Theorem 8.3.3. *Let R be a prime ring, λ a nonzero left ideal of R, $0 \neq b \in R$ and F, G are two b-generalized (α,β)-derivations of R associated with the (α,β) derivations d, g respectively. If $F(xy) + G(x)\alpha(y) + \alpha(yx) = 0$ for all $x, y \in \lambda$, then one of the following holds:*

1. $\beta(\lambda)d(\lambda) = (0)$, $\lambda[\lambda, \lambda] = (0)$ *and* $\alpha(\lambda)b = (0)$;
2. $\beta(\lambda)d(\lambda) = (0)$, $\lambda[\lambda, \lambda] = (0)$ *and* $\beta(\lambda)g(\lambda) = (0)$.

Proof. By the hypothesis, we have

$$F(xy) + G(x)\alpha(y) + \alpha(yx) = 0 \tag{8.3.10}$$

for all $x, y \in \lambda$. Replacing xz in place of x in (8.3.10), we obtain

$$F(xzy) + G(xz)\alpha(y) + \alpha(yxz) = 0 \tag{8.3.11}$$

for all $x, y, z \in \lambda$. Again replacing zy in place of y in (8.3.10), we obtain

$$F(xzy) + G(x)\alpha(zy) + \alpha(zyx) = 0 \tag{8.3.12}$$

for all $x, y, z \in \lambda$. Subtracting (8.3.11) from (8.3.12), we obtain

$$b\beta(x)g(z)\alpha(y) + \alpha(yxz - zyx) = 0 \tag{8.3.13}$$

for all $x, y, z \in \lambda$, which gives

$$b\beta(x)g(z)\alpha(y) + \alpha[yx, z] = 0 \tag{8.3.14}$$

for all $x, y, z \in \lambda$.

Now replace yz in place of y in (8.3.10), we obtain

$$F(xy)\alpha(z) + b\beta(xy)d(z) + G(x)\alpha(y)\alpha(z) + \alpha(yzx) = 0 \tag{8.3.15}$$

for all $x, y, z \in \lambda$. Right multiplying (8.3.10) by $\alpha(z)$ and then subtracting from (8.3.15), we get

$$b\beta(xy)d(z) + \alpha(y[z, x]) = 0 \tag{8.3.16}$$

for all $x, y, z \in \lambda$. Taking $z = x$ we get $b\beta(xy)d(x) = 0$, that is $b\beta(x)\beta(y)d(x) = 0$ for all $x, y \in \lambda$. Replacing y by ry for $r \in R$ we obtain $b\beta(x)\beta(r)\beta(y)d(x) = 0$. Now replacing r by $\beta^{-1}(rb)$ we get $b\beta(x)rb\beta(y)d(x) = 0$ for all $r \in R$. Thus, by primeness of R we get either $b\beta(y)d(x) = 0$ or $b\beta(x) = 0$ for all $x, y \in \lambda$. We consider two sets $T_1 = \{x \in \lambda : b\beta(\lambda)d(x) = (0)\}$ and $T_2 = \{x \in \lambda : b\beta(x) = 0\}$, then T_1 and T_2 form two additive subgroups of λ such that $T_1 \cup T_2 = \lambda$. Therefore either $T_1 = \lambda$ or $T_2 = \lambda$, that is, either $b\beta(y)d(x) = 0$ for all $x, y \in \lambda$ or $b\beta(x) = 0$ for all $x \in \lambda$. Both of these two conditions together imply that $b\beta(y)d(x) = 0$ for all $x, y \in \lambda$. Replacing y by ry for $r \in R$ we obtain $b\beta(r)\beta(y)d(x) = 0$, that is $bR\beta(y)d(x) = (0)$. Thus, by primeness of R and using $b \neq 0$ we get $\beta(\lambda)d(\lambda) = (0)$.

Taking $\beta(\lambda)d(\lambda) = (0)$, then from (8.3.16) we get $\alpha(y[z, x]) = 0$ for all $x, y, z \in \lambda$, that is $\lambda[\lambda, \lambda] = (0)$. Again left multiplying (8.3.14) by $\alpha(t)$ and using $\lambda[\lambda, \lambda] = (0)$, we get $\alpha(t)b\beta(x)g(z)\alpha(y) = 0$ for all $x, y, z, t \in \lambda$. Replacing y by ry, $r \in R$ in the last expression and using primeness of R, we conclude that $\alpha(t)b\beta(x)g(z) = 0$. Again taking x by rx, $r \in R$ we get $\alpha(t)b\beta(r)\beta(x)g(z) = 0$ for all $r \in R$. Thus by primeness of R we get either $\alpha(\lambda)b = (0)$ or $\beta(\lambda)g(\lambda) = (0)$. Thus, the proof of the theorem is completed. □

Corollary 8.3.4. *Let R be a prime ring, $0 \neq b \in R$ and F, G are two b-generalized (α,β)-derivations of R associated with the (α,β) derivations d, g respectively. If $F(xy) + G(x)\alpha(y) + \alpha(yx) = 0$ for all $x, y \in R$, then $d = g = 0$, R is commutative and $F(x) + G(x) = -\alpha(x)$ for all $x \in R$.*

Proof. By Theorem 8.3.3 and considering $\lambda = R$, we get one of the following:

(1) $d = 0$, R is commutative and $b = 0$, contradiction;

(2) $d = g = 0$ and R is commutative.
When R is commutative and $d = 0$ the identity $F(xy) + G(x)\alpha(y) + \alpha(yx) = 0$ becomes $\{F(x) + G(x) + \alpha(x)\}\alpha(y) = 0$ for all $x, y \in R$. Therefore by primeness of R we get $F(x) + G(x) = -\alpha(x)$ for all $x \in R$. □

Theorem 8.3.5. *Let R be a prime ring, λ a nonzero left ideal of R, $0 \neq b \in R$ and F, G are two b-generalized (α,β)-derivations of R associated with the (α,β) derivations d, g respectively. If $F(xy) + G(yx) + \alpha(xy) + \alpha(yx) = 0$ for all $x, y \in \lambda$, then one of the following holds:*

1. $b\big[\beta(\lambda), b\big] = (0)$;
2. $\beta(\lambda)d(\lambda) = (0)$ *and* $\beta(\lambda)g(\lambda) = (0)$.

Proof. By the hypothesis, we have

$$F(xy) + G(yx) + \alpha(xy) + \alpha(yx) = 0 \tag{8.3.17}$$

for all $x, y \in \lambda$.

Case-I: Replacing yx in place of y in (8.3.17), we obtain

$$\begin{aligned} F(xy)\alpha(x) + b\beta(xy)d(x) + G(yx)\alpha(x) + b\beta(yx)g(x) \\ +\alpha(xy)\alpha(x) + \alpha(yx)\alpha(x) = 0 \end{aligned} \tag{8.3.18}$$

for all $x, y \in \lambda$. Right multiplying (8.3.17) by $\alpha(x)$ and then subtracting from (8.3.18), we get

$$b\beta(xy)d(x) + b\beta(yx)g(x) = 0 \tag{8.3.19}$$

for all $x, y \in \lambda$. Taking $y = \beta^{-1}(b)y$, we obtain

$$b\beta(x)b\beta(y)d(x) + b^2\beta(y)\beta(x)g(x) = 0 \tag{8.3.20}$$

for all $x, y \in \lambda$. Left multiplying (8.3.19) by b and then subtracting from (8.3.20), we get

$$\{b\beta(x)b - b^2\beta(x)\}\beta(y)d(x) = 0, \tag{8.3.21}$$

that is $b[\beta(x), b]\beta(y)d(x) = 0$ for all $x, y \in \lambda$. Replacing y by ry for $r \in R$, we obtain $b[\beta(x), b]\beta(r)\beta(y)d(x) = 0$ for all $r \in R$. Thus, by primeness of R we get either $b[\beta(x), b] = 0$ or $\beta(\lambda)d(x) = (0)$ for all $x \in \lambda$. We consider two sets

$T_1 = \{x \in \lambda : b[\beta(x), b] = 0\}$ and $T_2 = \{x \in \lambda : \beta(\lambda)d(x) = (0)\}$, then T_1 and T_2 form two additive subgroups of λ such that $T_1 \cup T_2 = \lambda$. Therefore either $T_1 = \lambda$ or $T_2 = \lambda$, that is, either $b[\beta(x), b] = 0$ for all $x \in \lambda$ or $\beta(\lambda)d(x) = (0)$ for all $x \in \lambda$, hence either $b[\beta(\lambda), b] = (0)$ or $\beta(\lambda)d(\lambda) = (0)$.

Case-II: Replacing xy in place of x in (8.3.17), we obtain

$$F(xy)\alpha(y) + b\beta(xy)d(y) + G(yx)\alpha(y) + b\beta(yx)g(y) \\ +\alpha(xy)\alpha(y) + \alpha(yx)\alpha(y) = 0 \tag{8.3.22}$$

for all $x, y \in \lambda$. Right multiplying (8.3.17) by $\alpha(y)$ and then subtracting from (8.3.22), we get

$$b\beta(xy)d(y) + b\beta(yx)g(y) = 0 \tag{8.3.23}$$

for all $x, y \in \lambda$. Taking $x = \beta^{-1}(b)x$, we obtain

$$b^2\beta(xy)d(y) + b\beta(y)b\beta(x)g(y) = 0 \tag{8.3.24}$$

for all $x, y \in \lambda$. Left multiplying (8.3.23) by b and then subtracting from (8.3.24), we get

$$\{b\beta(y)b - b^2\beta(y)\}\beta(x)g(y) = 0, \tag{8.3.25}$$

that is $b[\beta(y), b]\beta(x)g(y) = 0$ for all $x, y \in \lambda$. Replacing x by rx for $r \in R$, we obtain $b[\beta(y), b]\beta(r)\beta(x)g(y) = 0$ for all $r \in R$. Thus, by primeness of R we get either $b[\beta(y), b] = 0$ or $\beta(\lambda)g(y) = (0)$ for all $y \in \lambda$. We consider two sets $T_1 = \{y \in \lambda : b[\beta(y), b] = 0\}$ and $T_2 = \{y \in \lambda : \beta(\lambda)g(y) = (0)\}$, then T_1 and T_2 form two additive subgroups of λ such that $T_1 \cup T_2 = \lambda$. Therefore either $T_1 = \lambda$ or $T_2 = \lambda$, that is, either $b[\beta(y), b] = 0$ for all $y \in \lambda$ or $\beta(\lambda)g(y) = (0)$ for all $y \in \lambda$, hence either $b[\beta(\lambda), b] = (0)$ or $\beta(\lambda)g(\lambda) = (0)$.

Therefore, from above two results we can conclude that either $b[\beta(\lambda), b] = (0)$ or $\beta(\lambda)d(\lambda) = (0)$ and $\beta(\lambda)g(\lambda) = (0)$. Thus, the proof of the theorem is completed. □

Corollary 8.3.6. *Let R be a prime ring, $0 \neq b \in R$, F and G are two b-generalized (α,β)-derivations of R associated with the (α,β) derivations d and g respectively. If $F(xy) + G(yx) + \alpha(xy) + \alpha(yx) = 0$ for all $x, y \in R$, then one of the following holds:*

1. *$b \in Z(R)$, that is F and G are two generalized (α, β) derivations;*

2. *$d = g = 0$ and $F = G = -\alpha$;*

3. *$d = g = 0$, R is commutative and $F + G = -2\alpha$.*

Proof. By Theorem 8.3.5 and considering $\lambda = R$, we get one of the following: (1) $b[\beta(x), b] = 0$ for all $x \in R$; (2) $d = g = 0$.

Case I: When $b[\beta(x), b] = 0$ for all $x \in R$. Taking $x = xt$, we get $b[\beta(xt), b] = 0$, that is $b\beta(x)[\beta(t), b] = 0$ for all $x, t \in R$. By primeness of R and using $b \neq 0$, we get $[\beta(t), b] = 0$ for all $t \in R$, that is $b \in Z(R)$. Thus, F, G are two generalized (α, β)-derivations of R, this is our conclusion (1).

Case II: When $d = g = 0$, then $F(xy) + G(yx) + \alpha(xy) + \alpha(yx) = 0$ implies that

$$\{F(x) + \alpha(x)\}\alpha(y) + \{G(y) + \alpha(y)\}\alpha(x) = 0 \tag{8.3.26}$$

for all $x, y \in R$. Taking xt in place of x for $t \in R$, we obtain

$$\{F(x) + \alpha(x)\}\alpha(t)\alpha(y) + \{G(y) + \alpha(y)\}\alpha(x)\alpha(t) = 0 \tag{8.3.27}$$

for all $x, y, t \in R$. Right multiply (8.3.26) by $\alpha(t)$ and then subtracting from (8.3.27), we get $\{F(x) + \alpha(x)\}\big[\alpha(t), \alpha(y)\big] = 0$ for all $x, y, t \in R$. By primeness of R we get either R is commutative or $F(x) = -\alpha(x)$ for all $x \in R$. When $F(x) = -\alpha(x)$ for all $x \in R$, then from (8.3.26) we get $\{G(y) + \alpha(y)\}\alpha(x) = 0$ for all $x, y \in R$. Again by primeness of R we conclude that $G(y) = -\alpha(y)$ for all $y \in R$, this is our conclusion (2). On the other hand when R is commutative then $F(xy)+G(yx)+\alpha(xy)+\alpha(yx) = 0$, implies $\{F(x) + G(x) + 2\alpha(x)\}\alpha(y) = 0$ for all $x, y \in R$. By primeness of R we get $F(x) + G(x) = -2\alpha(x)$ for all $x \in R$, this is our conclusion (3). □

Theorem 8.3.7. *Let R be a prime ring, λ a nonzero left ideal of R, $0 \neq b \in R$, F and G are two b-generalized (α,β)-derivations of R associated with the (α,β) derivations d and g respectively. If $F(x)F(y) + G(x)\alpha(y) + \alpha(yx) = 0$ for all $x, y \in \lambda$, then one of the following holds:*

1. *$\beta(\lambda)d\big(\alpha^{-1}(b)\lambda\big)b = (0)$ and $\lambda[\lambda, \lambda] = (0)$;*

2. *$\beta(\lambda)d(\lambda) = (0)$ and $\lambda[\lambda, \lambda] = (0)$.*

Proof. By the hypothesis, we have

$$F(x)F(y) + G(x)\alpha(y) + \alpha(yx) = 0 \tag{8.3.28}$$

for all $x, y \in \lambda$. Replacing yz in place of y, we obtain

$$F(x)F(y)\alpha(z) + F(x)b\beta(y)d(z) + G(x)\alpha(y)\alpha(z) + \alpha(yzx) = 0 \quad (8.3.29)$$

for all $x, y, z \in \lambda$. Right multiplying (8.3.28) by $\alpha(z)$ and then subtracting from (8.3.29), we get

$$F(x)b\beta(y)d(z) + \alpha(yzx - yxz) = 0,$$

that is

$$F(x)b\beta(y)d(z) + \alpha(y[z, x]) = 0 \quad (8.3.30)$$

for all $x, y, z \in \lambda$. Assuming $z = x$, we get

$$F(x)b\beta(y)d(x) = 0 \quad (8.3.31)$$

for all $x, y \in \lambda$. Replacing y by ry for $r \in R$, we obtain $F(x)b\beta(r)\beta(y)d(x) = 0$. Thus, by primeness of R we get either $F(x)b = 0$ or $\beta(y)d(x) = 0$ for all $x, y \in \lambda$. We consider two sets $T_1 = \{x \in \lambda : F(x)b = 0\}$ and $T_2 = \{x \in \lambda : \beta(\lambda)d(x) = (0)\}$, then T_1 and T_2 form two additive subgroups of λ such that $T_1 \cup T_2 = \lambda$. Therefore either $T_1 = \lambda$ or $T_2 = \lambda$, that is, either $F(x)b = 0$ for all $x \in \lambda$ or $\beta(\lambda)d(x) = (0)$ for all $x \in \lambda$, that is either $F(x)b = 0$ for all $x \in \lambda$ or $\beta(\lambda)d(\lambda) = (0)$.

Case I: When $F(x)b = 0$ for all $x \in \lambda$, taking $x = x\alpha^{-1}(b)t$ for $t \in \lambda$ we get $b\beta(x)d\big(\alpha^{-1}(b)t\big)b = 0$ for all $x, t \in \lambda$. Replacing x by rx for $r \in R$ we obtain $b\beta(r)\beta(x)d\big(\alpha^{-1}(b)t\big)b = 0$. Thus, by primeness of R and using $b \neq 0$ we get $\beta(x)d\big(\alpha^{-1}(b)t\big)b = 0$ for all $x, t \in \lambda$, that is $\beta(\lambda)d\big(\alpha^{-1}(b)\lambda\big)b = (0)$. Also from (8.3.30) we get $\alpha(y[z, x]) = 0$, that is $\lambda[\lambda, \lambda] = (0)$, this is our conclusion (1).

Case II: When $\beta(\lambda)d(\lambda) = (0)$, then from (8.3.30) we get $\alpha(y[z, x]) = 0$, that is $\lambda[\lambda, \lambda] = (0)$, this is our conclusion (2). □

Corollary 8.3.8. *Let R be a prime ring, $0 \neq b \in R$, F and G are two b-generalized (α,β)-derivations of R associated with the (α,β) derivations d and g respectively. If $F(x)F(y) + G(x)\alpha(y) + \alpha(yx) = 0$ for all $x, y \in R$, then R is commutative and one of the following holds:*

1. *$d = g = 0$, that is F and G are two left α multiplier mapping of R;*

2. *$d\big(\alpha^{-1}(b)\big) = 0$.*

Proof. By Theorem 8.3.7 and taking $\lambda = R$, we get one of the following: (1) $d\big(\alpha^{-1}(b)R\big)b = (0)$ and R is commutative; (2) $d = 0$ and R is commutative.

Case I: When $d = 0$ and R is commutative, the identity $F(x)F(y)+G(x)\alpha(y)+\alpha(yx) = 0$ reduces to

$$F(x)F(y) + G(x)\alpha(y) + \alpha(xy) = 0 \tag{8.3.32}$$

for all $x, y \in R$. Replacing x by xz, we get

$$F(x)F(y)\alpha(z) + G(x)\alpha(y)\alpha(z) + b\beta(x)g(z)\alpha(y) + \alpha(xy)\alpha(z) = 0 \tag{8.3.33}$$

for all $x, y, z \in R$. Right multiplying (8.3.32) by $\alpha(z)$ and then subtracting from (8.3.33), we get $b\beta(x)g(z)\alpha(y) = 0$ for all $x, y, z \in R$. By primeness of R and using $b \neq 0$ we can conclude that $g = 0$, this is our conclusion (1).

Case II: When R is commutative and $d\big(\alpha^{-1}(b)R\big)b = (0)$, that is $d\big(\alpha^{-1}(b)t\big)b = 0$ for all $t \in R$, then replacing t by $t\alpha^{-1}(b)$ we get $\beta\big(\alpha^{-1}(b)\big)\beta(t)d\big(\alpha^{-1}(b)\big) = 0$ for all $t \in R$. By primeness of R we get either $\beta\big(\alpha^{-1}(b)\big) = 0$ or $d\big(\alpha^{-1}(b)\big) = 0$. If $\beta\big(\alpha^{-1}(b)\big) = 0$ then $b = 0$, contradiction. On the other hand $d\big(\alpha^{-1}(b)\big) = 0$, thus the proof is completed. □

Theorem 8.3.9. *Let R be a prime ring, $0 \neq b \in R$ and F, F_1 are two b-generalized (α,β)-derivations of R associated with the same (α,β) derivation d. If $F(xy) + d(x)F_1(y) + \alpha(xy) = 0$ for all $x, y \in R$, then $d = 0$, F_1 is a left α multiplier mapping of R and $F = -\alpha$.*

Proof. By the given condition

$$F(xy) + d(x)F_1(y) + \alpha(xy) = 0 \tag{8.3.34}$$

for all $x, y \in R$. Substituting $y = yz$, we obtain

$$F(xyz) + d(x)F_1(yz) + \alpha(xyz) = 0, \tag{8.3.35}$$

that is

$$\begin{aligned} F(xy)\alpha(z) + b\beta(xy)d(z) + d(x)\big(F_1(y)\alpha(z) + b\beta(y)d(z)\big) \\ +\alpha(xy)\alpha(z) = 0 \end{aligned} \tag{8.3.36}$$

for all $x, y, z \in R$. Right multiplying (8.3.34) by $\alpha(z)$ and then subtracting from (8.3.36), we get

$$b\beta(xy)d(z) + d(x)b\beta(y)d(z) = 0 \tag{8.3.37}$$

for all $x, y, z \in R$. Above relation implies

$$\{b\beta(x) + d(x)b\}\beta(y)d(z) = 0 \tag{8.3.38}$$

for all $x, y, z \in R$. Since R is prime, either $b\beta(x) + d(x)b = 0$ for all $x \in R$ or $d(R) = (0)$.

Case I: In this case we consider

$$b\beta(x) + d(x)b = 0 \tag{8.3.39}$$

for all $x \in R$. Replacing x with $\beta^{-1}(b)x$, we get $b^2\beta(x) + d(\beta^{-1}(b)x)b = 0$, that is $b^2\beta(x) + d(\beta^{-1}(b))\alpha(x)b + bd(x)b = 0$ for all $x \in R$. By using $b\beta(x) + d(x)b = 0$ for all $x \in R$, this relation yields $d(\beta^{-1}(b))\alpha(x)b = 0$ for all $x \in R$. By primeness of R and using $b \neq 0$ we get $d(\beta^{-1}(b)) = 0$. Now replacing x with $\beta^{-1}(b)$ in $b\beta(x) + d(x)b = 0$ for all $x \in R$, we get $b^2 = 0$. Thus right multiplying by b, $b\beta(x) + d(x)b = 0$ for all $x \in R$ yields $b\beta(x)b = 0$ for all $x \in R$ and hence $b = 0$, a contradiction.

Case II: Next assume that $d = 0$. Then by hypothesis, we have $\{F(x) + \alpha(x)\}\alpha(y) = 0$ for all $x, y \in R$, implying $F(x) + \alpha(x) = 0$ for all $x \in R$ and $F_1(xy) = F_1(x)\alpha(y)$, that is F_1 is a left multiplier mapping of R, as desired. □

- *Since F is a b-generalized (α, β)-derivation on R associated to the (α, β)-derivation d of R, $F - \alpha$ is also a b-generalized (α, β)-derivation on R associated to the (α, β)-derivation d of R. Thus replacing F with $F - \alpha$ in Theorem 8.3.9, the following Theorem is straightforward.*

Theorem 8.3.10. *Let R be a prime ring, $0 \neq b \in R$ and F, F_1 are two b-generalized (α, β)-derivations on R associated with the same (α, β)-derivation d. If*

$$F(xy) + d(x)F_1(y) = 0$$

for all $x, y \in R$, then $F = 0$ and F_1 is left multiplier mapping of R.

Theorem 8.3.11. *Let R be a prime ring, $0 \neq b \in R$ and F, F_1 are two b-generalized (α,β)-derivations of R associated with the same (α,β)-derivation d. If $F(xy) + d(x)F_1(y) + \alpha(yx) = 0$ for all $x, y \in R$, then R is commutative, $d = 0$, F_1 is a left multiplier mapping of R and $F(x) = -\alpha(x)$ for all $x \in R$.*

Proof. By the hypothesis, we have

$$F(xy) + d(x)F_1(y) + \alpha(yx) = 0 \tag{8.3.40}$$

for all $x, y \in R$. Replacing yz in place of y in (8.3.40), we obtain

$$F(xyz) + d(x)F_1(yz) + \alpha(yzx) = 0, \tag{8.3.41}$$

that is

$$F(xy)\alpha(z) + b\beta(xy)d(z) + d(x)F_1(y)\alpha(z) + d(x)b\beta(y)d(z) + \alpha(yzx) = 0 \tag{8.3.42}$$

for all $x, y, z \in R$. Right multiplying (8.3.40) by $\alpha(z)$ and then subtracting from (8.3.42), we get

$$b\beta(xy)d(z) + d(x)b\beta(y)d(z) + \alpha(yzx - yxz) = 0 \tag{8.3.43}$$

for all $x, y, z \in R$, which gives

$$b\beta(x)\beta(y)d(z) + d(x)b\beta(y)d(z) + \alpha(y[z, x]) = 0 \tag{8.3.44}$$

for all $x, y, z \in R$.

Now replacing z by x in (8.3.44), we obtain

$$\{b\beta(x) + d(x)b\}\beta(y)d(x) = 0 \tag{8.3.45}$$

for all $x, y \in R$. Since R is prime, either $b\beta(x) + d(x)b = 0$ or $d(x) = 0$ for all $x \in R$. We consider two sets $T_1 = \{x \in R : d(x) = 0\}$ and $T_2 = \{x \in R : b\beta(x) + d(x)b = 0\}$, then T_1 and T_2 form two additive subgroups of R such that $T_1 \cup T_2 = R$. Therefore either $T_1 = R$ or $T_2 = R$, that is, either $d(x) = 0$ for all $x \in R$ or $b\beta(x) + d(x)b = 0$ for all $x \in R$.

Case I: When $b\beta(x) + d(x)b = 0$ for all $x \in R$, this is same as (8.3.39) in Theorem 8.3.9. Therefore by same argument of Case-1 in Theorem 8.3.9 we have a contradiction.

Case II: When $d = 0$, then from (8.3.44) we get $\alpha(y[z, x]) = 0$ for all $x, y, z \in R$, that is $y[z, x] = 0$ for all $x, y, z \in R$ implying R is commutative. Applying R is commutative and $d = 0$, equation (8.3.40) yields $\{F(x) + \alpha(x)\}\alpha(y) = 0$ for all $x, y \in R$. By primeness of R we get $F(x) + \alpha(x) = 0$ for all $x \in R$. Also in this case $F_1(xy) = F_1(x)\alpha(y)$, that is F_1 is a left α multiplier mapping of R, as desired. □

• *Since F is a b-generalized (α, β)-derivation on R associated to the (α, β)-derivation d of R, $F + \alpha$ and $F - \alpha$ are also b-generalized (α, β)-derivations on R associated to the (α, β)-derivation d of R. Thus replacing F with $F + \alpha$ and $F - \alpha$ in Theorem 8.3.11 respectively, the following Theorems are straightforward.*

Theorem 8.3.12. *Let R be a prime ring, $0 \neq b \in R$ and F, F_1 are two b-generalized (α, β)-derivations on R with associated the same (α, β) derivation d of R. If*

$$F(xy) + d(x)F_1(y) + \alpha(yx) + \alpha(xy) = 0$$

for all $x, y \in R$, then R is commutative, $d = 0$, F_1 is a left multiplier mapping of R and $F = -2\alpha$.

Theorem 8.3.13. *Let R be a prime ring, $0 \neq b \in R$ and F, F_1 are two b-generalized (α, β)-derivations on R with associated the same (α, β) derivation d. If*

$$F(xy) + d(x)F_1(y) + \alpha(yx) - \alpha(xy) = 0$$

for all $x, y \in R$, then R is commutative, $d = 0$, F_1 is a left multiplier mapping of R and $F = 0$.

Theorem 8.3.14. *Let R be a prime ring, $0 \neq b \in R$ and F be a b-generalized (α,β)-derivation of R associated with the (α,β) derivation d. If $\left[F(x), x\right]_{\alpha,\beta} = 0$ for all $x \in R$, then one of the following holds:*

1. *R is commutative;*

2. *$F(x) = \mu\alpha(x)$ for some $\mu \in C$.*

Proof. By the hypothesis, we have

$$\left[F(x), x\right]_{\alpha,\beta} = 0 \tag{8.3.46}$$

for all $x \in R$. Linearizing (8.3.46), we obtain

$$\left[F(x), y\right]_{\alpha,\beta} + \left[F(y), x\right]_{\alpha,\beta} = 0$$

for all $x, y \in R$. Replacing yx in place of y, we obtain

$$\begin{aligned} &\left[F(x), y\right]_{\alpha,\beta}\alpha(x) + \beta(y)\left[F(x), x\right]_{\alpha,\beta} \\ &+\left[F(y), x\right]_{\alpha,\beta}\alpha(x) + \left[b\beta(y)d(x), x\right]_{\alpha,\beta} = 0 \end{aligned}$$

for all $x, y \in R$, that is

$$\left[b\beta(y)d(x), x\right]_{\alpha,\beta} = 0$$

for all $x, y \in R$. Taking $y = \beta^{-1}(b)y$, we obtain

$$\left[b^2\beta(y)d(x), x\right]_{\alpha,\beta} = 0$$

for all $x, y \in R$. This implies

$$b\left[b\beta(y)d(x), x\right]_{\alpha,\beta} + \left[b, \beta(x)\right]b\beta(y)d(x) = 0$$

for all $x, y \in R$, that is

$$\left[b, \beta(x)\right]b\beta(y)d(x) = 0 \tag{8.3.47}$$

for all $x, y \in R$. By primeness of R we get either $\left[b, \beta(x)\right]b = 0$ or $d(x) = 0$ for all $x \in R$. We consider two sets $T_1 = \{x \in R : d(x) = 0\}$ and $T_2 = \{x \in R : \left[b, \beta(x)\right]b = 0\}$, then T_1 and T_2 form two additive subgroups of R such that $T_1 \cup T_2 = R$. Therefore either $T_1 = R$ or $T_2 = R$, that is, either $d(x) = 0$ for all $x \in R$ or $\left[b, \beta(x)\right]b = 0$ for all $x \in R$.

Case I: When $d = 0$, then $F(xy) = F(x)\alpha(y)$ for all $x, y \in R$, that is $(\alpha^{-1}F)(xy) = (\alpha^{-1}F)(x)y$ for all $x, y \in R$. By using [70, Lemma 2] we can conclude that, there exist $\lambda \in C$ such that $(\alpha^{-1}F)x = \lambda x$ for all $x \in R$, that is $F(x) = \alpha(\lambda)\alpha(x) = \mu\alpha(x)$ where $\mu = \alpha(\lambda) \in C$, this is our conclusion (2).

Case II: When $\left[b, \beta(x)\right]b = 0$ for all $x \in R$. Taking $x = tx$, we obtain $\left[b, \beta(t)\beta(x)\right]b = \beta(t)[b, \beta(x)]b + [b, \beta(t)]\beta(x)b = [b, \beta(t)]\beta(x)b = 0$ for all $x, t \in R$. Thus by primeness of R and using $b \neq 0$, we get $\left[b, \beta(t)\right] = 0$ for all $t \in R$, that is $b \in Z(R)$, hence F is a generalized (α,β)-derivations of R. From [84, Corollary 3.7], we get either $d = 0$ or R is commutative. When $d = 0$, by Case-I we arrive to our conclusion (2), on the other hand we get our conclusion (1). □

Theorem 8.3.15. *Let R be a prime ring, $0 \neq b \in R$ and F be a b-generalized (α,β)-derivation of R associated with the (α,β) derivation d. If $\left(F(x) \circ x\right)_{\alpha,\beta} = 0$ for all $x \in R$, then one of the following holds:*

1. *$b \in Z(R)$, that is F is a generalized (α,β)-derivation of R;*
2. *$F(x) = \mu\alpha(x)$ for some $\mu \in C$.*

Proof. By the hypothesis, we have

$$\left(F(x) \circ x\right)_{\alpha,\beta} = 0 \tag{8.3.48}$$

for all $x \in R$. Linearizing (8.3.48), we obtain

$$\big(F(x) \circ y\big)_{\alpha,\beta} + \big(F(y) \circ x\big)_{\alpha,\beta} = 0 \tag{8.3.49}$$

for all $x, y \in R$. Replacing yx in place of y, we obtain

$$\big(F(x) \circ yx\big)_{\alpha,\beta} + \big((F(y)\alpha(x) + b\beta(y)d(x)) \circ x\big)_{\alpha,\beta} = 0 \tag{8.3.50}$$

for all $x, y \in R$. Since $\big(F(y)\alpha(x) \circ x\big)_{\alpha,\beta} = \big(F(y) \circ x\big)_{\alpha,\beta}\alpha(x)$, we have from above

$$\begin{aligned} &\big(F(x) \circ y\big)_{\alpha,\beta}\alpha(x) - \beta(y)\big[F(x), x\big]_{\alpha,\beta} \\ &+\big(F(y) \circ x\big)_{\alpha,\beta}\alpha(x) + \big(b\beta(y)d(x) \circ x\big)_{\alpha,\beta} = 0 \end{aligned} \tag{8.3.51}$$

for all $x, y \in R$. By (8.3.49), above relation yields

$$\big(b\beta(y)d(x) \circ x\big)_{\alpha,\beta} - \beta(y)\big[F(x), x\big]_{\alpha,\beta} = 0 \tag{8.3.52}$$

for all $x, y \in R$. Taking $y = \beta^{-1}(b)y$, we obtain

$$\big(b^2\beta(y)d(x) \circ x\big)_{\alpha,\beta} - b\beta(y)\big[F(x), x\big]_{\alpha,\beta} = 0 \tag{8.3.53}$$

for all $x, y \in R$. Left multiplying (8.3.52) by b and then subtracting from (8.3.53), we get

$$\big(b^2\beta(y)d(x) \circ x\big)_{\alpha,\beta} - b\big(b\beta(y)d(x) \circ x\big)_{\alpha,\beta} = 0 \tag{8.3.54}$$

for all $x, y \in R$. This implies

$$b\big(b\beta(y)d(x) \circ x\big)_{\alpha,\beta} - \big[b, \beta(x)\big]b\beta(y)d(x) - b\big(b\beta(y)d(x) \circ x\big)_{\alpha,\beta} = 0$$

for all $x, y \in R$. By cancelation of first and third terms, above relation gives

$$\big[b, \beta(x)\big]b\beta(y)d(x) = 0 \tag{8.3.55}$$

for all $x, y \in R$. This is same as (8.3.47) in Theorem 8.3.14. Thus, by same argument we can obtain either $F(x) = \mu\alpha(x)$ for all $x \in R$, where $\mu \in C$ or $b \in Z(R)$, i.e., F is a generalized (α, β)-derivation of R. □

Theorem 8.3.16. *Let R be a prime ring, $0 \neq b \in R$ and F be a b-generalized (α,β)-derivation of R associated with the (α,β) derivation d. If $F\big([x, y]\big) = \big[x, y\big]_{\alpha,\beta}$ for all $x, y \in R$, then one of the following holds:*

1. *R is commutative;*

2. $F(x) = \mu\alpha(x)$ *for some* $\mu \in C$.

Proof. By the hypothesis, we have

$$F\big([x,y]\big) = \big[x,y\big]_{\alpha,\beta} \tag{8.3.56}$$

for all $x, y \in R$. Replacing yx in place of y, we obtain

$$F\big([x,y]\big)\alpha(x) + b\beta([x,y])d(x) = [x,y]_{\alpha,\beta}\alpha(x) + \beta(y)[x,x]_{\alpha,\beta}$$

for all $x, y \in R$, that is

$$b\beta([x,y])d(x) = 0 \tag{8.3.57}$$

for all $x, y \in R$. Taking $y = \beta^{-1}(b)y$, we obtain

$$b\beta([x,\beta^{-1}(b)y])d(x) = 0 \tag{8.3.58}$$

for all $x, y \in R$. This implies

$$b^2\beta([x,y])d(x) + b\beta([x,\beta^{-1}(b)]y)d(x) = 0 \tag{8.3.59}$$

for all $x, y \in R$, that is

$$b\beta([x,\beta^{-1}(b)])\beta(y)d(x) = 0 \tag{8.3.60}$$

for all $x, y \in R$. By primeness of R we get either $b\beta([x,\beta^{-1}(b)]) = 0$ or $d(x) = 0$ for all $x \in R$. We consider two sets $T_1 = \{x \in R : d(x) = 0\}$ and $T_2 = \{x \in R : b\beta([x,\beta^{-1}(b)]) = 0\}$, then T_1 and T_2 form two additive subgroups of R such that $T_1 \cup T_2 = R$. Therefore either $T_1 = R$ or $T_2 = R$, that is, either $d(x) = 0$ for all $x \in R$ or $b\beta([x,\beta^{-1}(b)]) = 0$ for all $x \in R$.

Case I: When $d = 0$, then by applying similar process of Case-I in Theorem 8.3.14, we get $F(x) = \mu\alpha(x)$ where $\mu \in C$, this is our conclusion (2).

Case II: When $b\beta([x,\beta^{-1}(b)]) = 0$ for all $x \in R$, then $\beta^{-1}(b)[x,\beta^{-1}(b)] = 0$ for all $x \in R$. Therefore $\beta^{-1}(b) \in Z(R)$, that is $b \in Z(R)$, hence F is a generalized (α,β)-derivation of R. From [84, Corollary 3.7], we get either $d = 0$ or R is commutative. When $d = 0$, by Case-I we can conclude our conclusion (2), on the other hand we get our conclusion (1). □

Theorem 8.3.17. *Let* R *be a prime ring,* $0 \neq b \in R$ *and* F *be a* b*-generalized* (α,β)*-derivation of* R *associated with the* (α,β) *derivation* d*. If* $F\big(x \circ y\big) = \big(x \circ y\big)_{\alpha,\beta}$ *for all* $x, y \in R$*, then one of the following holds:*

1. *R is commutative;*
2. $F(x) = \mu\alpha(x)$ *for some* $\mu \in C$.

Proof. By the hypothesis, we have

$$F\big(x \circ y\big) = \big(x \circ y\big)_{\alpha,\beta} \tag{8.3.61}$$

for all $x, y \in R$. Replacing yx in place of y, we obtain

$$F\big(x \circ y\big)\alpha(x) + b\beta(x \circ y)d(x) = (x \circ y)_{\alpha,\beta}\alpha(x) - \beta(y)[x, x]_{\alpha,\beta}$$

for all $x, y \in R$, that is

$$b\beta(x \circ y)d(x) + \beta(y)[x, x]_{\alpha,\beta} = 0 \tag{8.3.62}$$

for all $x, y \in R$. Taking $y = \beta^{-1}(b)y$, we obtain

$$b\beta(x \circ \beta^{-1}(b)y)d(x) + b\beta(y)[x, x]_{\alpha,\beta} = 0 \tag{8.3.63}$$

for all $x, y \in R$. This implies

$$b^2\beta(x \circ y)d(x) + b\beta([x, \beta^{-1}(b)]y)d(x) + b\beta(y)[x, x]_{\alpha,\beta} = 0 \tag{8.3.64}$$

for all $x, y \in R$, that is

$$b\beta([x, \beta^{-1}(b)])\beta(y)d(x) = 0 \tag{8.3.65}$$

for all $x, y \in R$. This is same as (8.3.60) in Theorem 8.3.16. Thus, by same argument of Theorem 8.3.16 and using [84, Corollary 3.7], we can obtain our results. □

Example 8.3.1.

Let $\mathbb{Z}$ *be the set of all integers. Next, we consider a ring* $R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | x, y \in \mathbb{Z} \right\}$. *Then it is clear that R is not prime ring, because* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = (0)$. *Define the maps* $F, d, \alpha, \beta : R \to R$ *by* $F\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ *and* $d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix}$. *Also* $\alpha\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$ *and* $\beta\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix}$. *Then F is a b-generalized* (α, β)*-derivation on R associated with* (α, β)

derivation d, where $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a nonzero fixed element of R and α, β are two automorphisms of R. Let us consider a nonzero left ideal of R, $\lambda = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} | x \in \mathbb{Z} \right\}$ Then we see that $F(XY) + \alpha(XY) + \alpha(YX) = 0$ for all $X, Y \in \lambda$ but R is not commutative, $b \neq 0$ and $F \neq -2\alpha$. Therefore the primeness hypothesis in Theorem 8.3.1 is not superfluous.

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