DIFFERENT ASPECTS OF COMPLEX VALUED FUNCTIONS OF HIGHER DIMENSION

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BY

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CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled "**DIFFERENT ASPECTS OF COMPLEX VALUED FUNCTIONS OF HIGHER DIMENSION**" Submitted by Sri **Mukul Sk** who got his name registered on **07th April, 2022 (Index no.: 94/22/Maths./27)** for the award of Ph. D. (Science) Degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. (Dr.) Prakash Chandra Mali, Department of Mathematics, Professor, Jadavpur University, Kolkata-700032 and Prof. (Dr.) Sanjib Kumar Datta, Department of Mathematics, Professor, University of Kalyani, Nadia-741235** and that neither this thesis nor any part of it has been submitted for either any degree / diploma or any other academic award anywhere before.

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Dedicated to
my parents
Kader Sk & Meera Bibi
and my elder sister
Rekha Khatun
for their support, love and sacrifices.

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ABSTRACT

Title: Different aspects of complex valued functions of higher dimension
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The prime concern of the present thesis focuses on some investigation and exploration covering various aspects of higher dimensional complex valued functions.

The thesis consists of Eight Chapters.

Chapter One deals with the preliminaries of hybrid as well as hyperbolic hybrid numbers and also their flavour on certain types of probabilistic measurable spaces. Also the introductory theories concerning the influence of higher dimensional complex valued functions in Nevanllina's Value distribution theories mainly both p-adic analysis and bicomplex analysis are briefly discussed in this chapter.

In **Chapter Two**, we define a new notion of sets, termed as symmetric hybrid number & skew symmetric hybrid number and study some of their algebraic properties. The key result of this chapter is to establish that the set of non-lightlike hybrid number forms a non-abelian group under multiplication and also to find a normal subgroup of it. Moreover, the existance of subrings and also their ideals is the prime concern under some additional conditions.

The prime concern of **Chapter Three** is to introduce a notion of a hyperbolic hybrid valued probabilistic measurable space to generalize '*Kolmogorov's* system of axioms'. The probability which we define here may take values $\mathbf{e}_+$ and $\mathbf{e}_-$ for the certain event other than 1 which is the key difference from the probability in $\mathbb{R}$, where $\mathbf{e}_+$ and $\mathbf{e}_-$ are very special kind of zero divisors in the ring of hybrid numbers. In this work we also prove the usual properties of probability theory like extended addition theorem, Boole's inequality, continuity theorem, Bonferroni's inequalities etc. by this new measure.

Chapter Four focuses on the study of the conditional probability under the flavor of hyperbolic hybrid valued probabilistic measurable space, a generalization of '*Kolmogorov's* system of axioms'. We prove the well known result '*Multiplication Theorem*' in this probabilistic space and then generalize it in this context. In this work we also extend the *Bayes' theorem* and the law of total of probability by this new measure.

In **Chapter Five**, we consider $\mathbb{T}$ as an algebraically closed p-adic complete field of characteristic zero. We define L^*-order of growth $\rho^{L^*(\psi)}$ and L^*-type $\sigma^{L^*}(\psi)$ of an entire function $\psi(\omega) = \sum_{n=0}^{\infty} \mathfrak{c}_n \omega^n$ on $\mathbb{T}$ and show that $\rho^{L^*(\psi)}$ and $\sigma^{L^*}(\psi)$ satisfy the same relations as in complex analysis with regards to the coefficients $\mathfrak{c}_n$. We denoteL^*-cotype of ψ by $\psi^{L^*}(\psi)$ depending on the number of zeros inside the disks is very useful

and we show under certain wide hypothesis $\psi^{L^*}(\psi) \geqslant \rho^{L^*}(\psi).\sigma^{L^*}(\psi)$. We check that $\rho^{L^*}(\psi) = \rho^{L^*}(\psi')$, $\sigma^{L^*}(\psi) = \sigma^{L^*}(\psi')$, where ψ' is the derivative of ψ.

The main goal of **Chapter Six** is to prove the bicomplex version of Enström-Kakeya theorem [if $P(z) = \sum_(k = 0)^n a_k z^k$ is a polynomial of degree n with real coefficients satisfying $0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, then all zeros of $P(z)$ lie in $|z| \leqslant 1$] and some of its consequences. Some examples are provided to justify the results obtained.

Chapter Seven is about the investigation of some common fixed point theorems in bicomplex valued metric spaces under both rational type contraction mappings satisfying E. A. property and intimate mappings. Our results generalize some earlier results (Rajput & Singh, 2014; Meena, 2015) and extend some existing theorems (Azam et al., 2011; Rouzkard & Imdad, 2012) regarding common fixed point theorems in complex valued metric spaces. A few examples are also provided to justify the results obtained.

Chapter Eight is mainly based on future prospects including further course of work and also their applications of the works as carried out in the thesis.

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LIST OF PUBLICATIONS

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Contents

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List of Mathematical Symbols

The below list describes several symbols that will be used later within the body of the thesis

$\breve{p}$	complex valued metric
$\preccurlyeq$	Partial order relation on the set of hyperbolic hybrid numbers
$\varnothing$	The empty set
$\mathbb{C}_1$	The set of all complex numbers
$\mathbb{C}_2$	The set of all bicomplex numbers
$\mathbb{H}$	The set of all hyperbolic numbers
$\mathbb{N}$	The set of all natural numbers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{Z}$	The set of all integers
$\mathcal{P}^{\mathfrak{h}}$	The set of hyperbolic hybrid numbers
$\mathfrak{T}$	The set of all hybrid numbers
$\mathfrak{T}_{\mathcal{C}_1}$	The set of non-lightlike hybrid numbers of unit character
$\mathfrak{T}_{NL}$	The set of non-lightlike hybrid numbers
$\mathcal{L}$	Probabilistic measure
$\mathcal{L}_{\mathfrak{H}}$	Hyperbolic hybrid valued probabilistic measure
$\precsim$	Partial order relation on the set of complex numbers
$\precsim_{i_2}$	Partial order relation on the set of bicomplex numbers
$\Upsilon_{\wp}$	bicomplex valued metric

CHAPTER ONE

INTRODUCTION

Chapter 1

Introduction

1.1 Preliminaries.

1.1.1 Introduction to Hybrid Numbers.

In the last century, a lot of researchers works with some two-dimensional systems like complex, dual and hyperbolic which have the most significant roles in various aspects such as algebraic, geometric, physics, engineering, etc. The geometry of the Euclidean plane, the Minkowski plane and the Gallian plane can be described with the help of complex numbers

$$\mathbb{C}_1 = \{\mathfrak{k}_1 + \mathbf{i}\mathfrak{k}_2 : \mathfrak{k}_1, \mathfrak{k}_2 \in \mathbb{R}, \mathbf{i}^2 = -1\},$$

dual numbers

$$\mathbb{D} = \{\mathfrak{v}_1 + \varepsilon\mathfrak{v}_2 : \mathfrak{v}_1, \mathfrak{v}_2 \in \mathbb{R}, \varepsilon^2 = 0\}$$

and hyperbolic numbers

$$\mathbb{H} = \{\vartheta_1 + \mathbf{h}\vartheta_2 : \vartheta_1, \vartheta_2 \in \mathbb{R}, \mathbf{h}^2 = 1\}.$$

We know that the complex numbers, dual numbers and hyperbolic numbers can be described as the quotient of the polynomial ring $\mathbb{R}[\mathfrak{x}]$ by the ideal generated by the polynomials $\mathfrak{x}^2 + 1$, $\mathfrak{x}^2$ and $\mathfrak{x}^2 - 1$ respectively. i.e.,

$$\mathbb{C}_1 = \mathbb{R}[\mathfrak{x}]/ < \mathfrak{x}^2 + 1 >, \mathbb{D} = \mathbb{R}[\mathfrak{x}]/ < \mathfrak{x}^2 > \text{ and } \mathbb{H} = \mathbb{R}[\mathfrak{x}]/ < \mathfrak{x}^2 - 1 >.$$

S. Olariu {cf. [42],[43] & [44]} defined a different generalization of n-dimensional complex numbers terming them 'twocomplex numbers', 'threecomplex numbers', etc. Actually, Olariu used the name 'twocomplex numbers' instead of hyperbolic numbers. In these series of papers the geometrical and the algebraic properties of these numbers are thoroughly studied. The set of 'threecomplex' numbers is defined as

$$\mathbb{C}_3 = \{\breve{\mathfrak{Z}} = \mathfrak{k}_1 + \mathbf{h}\mathfrak{k}_2 + \mathbf{k}\mathfrak{k}_3 : \mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathbb{R} \textit{ and } \mathbf{h}^2 = \mathbf{k}, \mathbf{k}^2 = \mathbf{h}, \mathbf{hk} = 1\}.$$

Anthony Harkin and Joseph Harkin [28] generalized the two dimensional complex numbers as

$$\mathbb{C}_p = \{\mathfrak{Z} = \mathfrak{k}_1 + \mathbf{i}\mathfrak{k}_2 : \mathfrak{k}_1, \mathfrak{k}_2 \in \mathbb{R} \ and \ \mathbf{i}^2 = p\}.$$

Here they gave some trigonometric relations for this generalization. In [10] Catoni et.al. (2004) defined two dimensional hypercomplex numbers as

$$\mathbb{C}_{\xi,\eta} = \{\mathfrak{Z} = \mathfrak{k}_1 + \mathbf{i}\mathfrak{k}_2 : \mathbf{i}^2 = \xi + \mathbf{i}\eta, \mathfrak{k}_1, \mathfrak{k}_2, \xi, \eta \in \mathbb{R}, \mathbf{i} \notin \mathbb{R}\}$$

and extended the relationship amongst these numbers and Euclidean & semi-Euclidean geometry. This generalization is also expressible as a quotient ring $\mathbb{R}[\mathfrak{x}]/ < \mathfrak{x}^2 - \eta\mathfrak{x} - \xi >$.

A theory of commutative two dimensional conformal hyperbolic numbers as a generalization of the theory of hyperbolic numbers is presented by Zaripov [67].

In [45] Mustafa Özdemir (2018) defined a new generalization of set containing complex, hyperbolic and dual numbers as different from above generalizations. This new number system appears to be four-dimensional and it can be viewed as a two dimensional set of numbers, since it can be represented in a generalized two dimensional plane, called hybridian plane in $\mathbb{R}^4$.

The following definitions are due to Özdemir [45].

Definition 1.1.1 *[45] A number of the form* $Z = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$, *where* $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4 \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1$ *and satisfying the relation* $\mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon$ *is called hybrid number. The set of hybrid numbers is denoted by* $\mathfrak{T}$, *and defined by*

$$\mathfrak{T} = \{\mathfrak{Z} = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h} : \mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4 \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \\ \mathbf{ih} = -\mathbf{hi} = \mathbf{i} + \varepsilon\}.$$

The geometry corresponding to the hybrid numbers is called '*Hybridian plane geometry*' and it is a two-dimensional subspace of $\mathbb{R}^4$. The real part $'\mathfrak{d}_1'$ of the hybrid number $\mathfrak{Z} = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$ is called '*scalar part*' and is denoted by $S(\mathfrak{Z})$ whereas the remaining part $\mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$ is called '*vector part*' and is denoted by $V(\mathfrak{Z})$.

For any two hybrid numbers $\mathfrak{Z}_1 = \mathfrak{e}_0 + \mathfrak{e}_1\mathbf{i} + \mathfrak{e}_2\varepsilon + \mathfrak{e}_3\mathbf{h}$, and $\mathfrak{Z}_2 = \mathfrak{f}_0 + \mathfrak{f}_1\mathbf{i} + \mathfrak{f}_2\varepsilon + \mathfrak{f}_3\mathbf{h}$, $\mathfrak{e}_i, \mathfrak{f}_i \in \mathbb{R}, i = 0, 1, 2, 3$ the equality, addition, subtraction, multiplication by a scalar $s \in \mathbb{R}$ and multiplication of two hybrid numbers are defined as follows

$$\mathfrak{Z}_1 = \mathfrak{Z}_2 \iff \mathfrak{e}_0 = \mathfrak{f}_0, \mathfrak{e}_1 = \mathfrak{f}_1, \mathfrak{e}_2 = \mathfrak{f}_2, \mathfrak{e}_3 = \mathfrak{f}_3,$$

$$\mathfrak{Z}_1 \pm \mathfrak{Z}_2 = (\mathfrak{e}_0 \pm \mathfrak{f}_0) + (\mathfrak{e}_1 \pm \mathfrak{f}_1)\mathbf{i} + (\mathfrak{e}_2 \pm \mathfrak{f}_2)\varepsilon + (\mathfrak{e}_3 \pm \mathfrak{f}_3)\mathbf{h},$$

$$s\mathfrak{Z}_1 = s\mathfrak{e}_0 + s\mathfrak{e}_1\mathbf{i} + s\mathfrak{e}_2\varepsilon + s\mathfrak{e}_3\mathbf{h}$$

and

$$\begin{aligned}\mathfrak{z}_1.\mathfrak{z}_2 &= (\mathfrak{e}_0+\mathfrak{e}_1\mathbf{i}+\mathfrak{e}_2\varepsilon+\mathfrak{e}_3\mathbf{h}).(\mathfrak{f}_0+\mathfrak{f}_1\mathbf{i}+\mathfrak{f}_2\varepsilon+\mathfrak{f}_3\mathbf{h})\\ &= (\mathfrak{e}_0\mathfrak{f}_0-\mathfrak{e}_1\mathfrak{f}_1+\mathfrak{e}_2\mathfrak{f}_1-\mathfrak{e}_1\mathfrak{f}_2+\mathfrak{e}_3\mathfrak{f}_3)+(\mathfrak{e}_0\mathfrak{f}_2+\mathfrak{e}_1\mathfrak{f}_0+\mathfrak{e}_1\mathfrak{f}_3\\ &\quad -\mathfrak{e}_3\mathfrak{f}_1)\mathbf{i}+(\mathfrak{e}_0\mathfrak{f}_2+\mathfrak{e}_2\mathfrak{f}_0-\mathfrak{e}_2\mathfrak{f}_3+\mathfrak{e}_3\mathfrak{f}_2+\mathfrak{e}_1\mathfrak{f}_3-\mathfrak{e}_3\mathfrak{f}_1)\varepsilon\\ &\quad +(\mathfrak{e}_0\mathfrak{f}_3+\mathfrak{e}_3\mathfrak{f}_0+\mathfrak{e}_1\mathfrak{f}_2+\mathfrak{e}_2\mathfrak{f}_1)\mathbf{h}.\end{aligned}$$

The '+' operation is both commutative and associative. The null element is **0** and the inverse element of $\mathfrak{z}$ is $-\mathfrak{z}$. As a consequence of these properties, $(\mathfrak{T},+)$ forms an abelian group.

For the above multiplication of hybrid numbers we can use the following table of hybrid units.

.	1	$\mathbf{i}$	ε	$\mathbf{h}$
1	1	$\mathbf{i}$	ε	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\varepsilon+\mathbf{i}$
ε	ε	1+$\mathbf{h}$	0	-ε
$\mathbf{h}$	$\mathbf{h}$	-$\varepsilon-\mathbf{i}$	ε	1

Multiplication table of Hybrid units

From the above table it is clear that the multiplication operation in the hybrid number system is not commutative. But the multiplication is associative.

Definition 1.1.2 *[45] The* ***conjugate*** *of a hybrid number $\mathfrak{z}=\mathfrak{d}_1+\mathfrak{d}_2\mathbf{i}+\mathfrak{d}_3\varepsilon+\mathfrak{d}_4\mathbf{h}$ is defined by*

$$\overline{\mathfrak{z}}=S(\mathfrak{z})-V(\mathfrak{z})=\mathfrak{d}_1-\mathfrak{d}_2\mathbf{i}-\mathfrak{d}_3\varepsilon-\mathfrak{d}_4\mathbf{h}.$$

This conjugation of a hybrid number is additive, involutive and multiplicative operation on $\mathfrak{T}$. i.e., for any two hybrid numbers $\mathfrak{z}_1$ and $\mathfrak{z}_2$

a. $\overline{\mathfrak{z}_1+\mathfrak{z}_2}=\overline{\mathfrak{z}_1}+\overline{\mathfrak{z}_2}$

b. $\overline{(\overline{\mathfrak{z}_1})}=\mathfrak{z}_1$

c. $\overline{\mathfrak{z}_1.\mathfrak{z}_2}=\overline{\mathfrak{z}_1}.\overline{\mathfrak{z}_2}$.

Definition 1.1.3 *[45] The real number $\mathcal{C}(\mathfrak{z})=\mathfrak{z}\overline{\mathfrak{z}}=\mathfrak{d}_1^2+(\mathfrak{d}_2-\mathfrak{d}_3)^2-\mathfrak{d}_3^2-\mathfrak{d}_4^2$ is called the* ***character*** *of the hybrid number $\mathfrak{z}$.*

Since $\mathcal{C}(\mathfrak{z})\in\mathbb{R}$, so depending on the value of $\mathcal{C}(\mathfrak{z})$ a hybrid number can be categorized into three parts, **spacelike**, **timelike**, or **lightlike** according as the character is negative, positive or zero.

Also, the real number $\sqrt{|\mathcal{C}(\mathfrak{z})|}$ is called the **norm** of the hybrid number $\mathfrak{z}$ and is denoted by $\|\mathfrak{z}\|$.

Definition 1.1.4 *[45] The inverse of the hybrid number $\mathfrak{Z} = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$, $\|\mathfrak{Z}\| \neq 0$ is defined as*

$$\mathfrak{Z}^{-1} = \frac{\overline{\mathfrak{Z}}}{\mathcal{C}(\mathfrak{Z})}.$$

Therefore, we can conclude that a lightlike hybrid number never possesses an inverse.

Definition 1.1.5 *[45] For the hybrid number $\mathfrak{Z} = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$, the vector $\varepsilon_{\mathfrak{Z}} = ((\mathfrak{d}_2 - \mathfrak{d}_3), \mathfrak{d}_3, \mathfrak{d}_4)$ is called the* ***hybrid vector*** *of $\mathfrak{Z}$.*

Definition 1.1.6 *[45] The real number $\mathcal{C}_\varepsilon(\mathfrak{Z}) = -(\mathfrak{d}_2 - \mathfrak{d}_3)^2 + \mathfrak{d}_3^2 + \mathfrak{d}_4^2$ is called the* ***type*** *of the hybrid number $\mathfrak{Z}$.*

Depending on the real value of the type of a hybrid number, it is classified as '**complike** (**elliptic**)', '**hiperlike** (**hyperbolic**)' or '**duallike** (**parabolic**)' for $\mathcal{C}_\varepsilon(\mathfrak{Z}) < 0$, $\mathcal{C}_\varepsilon(\mathfrak{Z}) > 0$ or $\mathcal{C}_\varepsilon(\mathfrak{Z}) = 0$ respectively. Also, the real number $\sqrt{|\mathcal{C}_\varepsilon(\mathfrak{Z})|}$ is called the norm of the hybrid vector of $\mathfrak{Z}$ and is denoted by $\mathcal{N}(\mathfrak{Z})$.

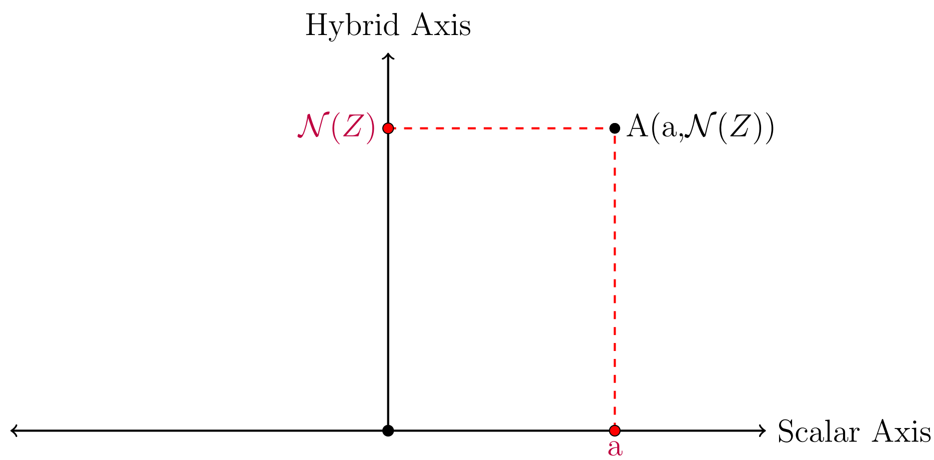

Figure 1.1: Hybridian Coordinate System

1.1.1.1 Idempotent Representation of hyperbolic Hybrid Numbers.

Let us denote the set of hyperbolic hybrid numbers as $\mathcal{P}^{\mathfrak{h}}$. Every hyperbolic hybrid number $\mathfrak{Z} = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$ can be written as

$$\mathfrak{Z} = s + \mathbf{H}v$$

where $\mathbf{H} = \dfrac{\mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}}{\mathcal{N}(\mathfrak{Z})}$ and $\mathbf{H}^2 = 1$.

Now, let us consider two numbers $\mathbf{e}_+ = \dfrac{1+\mathbf{H}}{2}$ and $\mathbf{e}_- = \dfrac{1-\mathbf{H}}{2}$. These two numbers are satisfying the equalities, $\|\mathbf{e}_+\| = \|\mathbf{e}_-\| = \mathbf{e}_+\mathbf{e}_- = 0, {\mathbf{e}_+}^2 = \mathbf{e}_+$, and ${\mathbf{e}_-}^2 = \mathbf{e}_-$ and are called idempotent elements. Since, $\|\mathbf{e}_+\| = \|\mathbf{e}_-\| = 0$, so $\mathbf{e}_+, \mathbf{e}_- \in \mathfrak{S}$ and both are mutually complementary idempotent elements. Thus, the two principal ideals $\mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_+} := \mathbf{e}_+.\mathcal{P}^{\mathfrak{h}}$ and $\mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_-} := \mathbf{e}_-.\mathcal{P}^{\mathfrak{h}}$ in the ring $\mathcal{P}^{\mathfrak{h}}$ have the following properties

$$\mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_+} \cap \mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_-} = \{0\} \quad \textit{and} \quad \mathcal{P}^{\mathfrak{h}} = \mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_+} \cup \mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_-} \tag{1.1}$$

and property (1.1) is known as the idempotent decomposition of $\mathcal{P}^{\mathfrak{h}}$. With the help of these idempotent elements $\mathbf{e}_+$ and $\mathbf{e}_-$, every hyperbolic hybrid number can be uniquely expressed as their linear combination

$$\mathfrak{Z} = (s+v)\mathbf{e}_+ + (s-v)\mathbf{e}_- = \mathfrak{z}_+\mathbf{e}_+ + \mathfrak{z}_-\mathbf{e}_- \tag{1.2}$$

where $\mathfrak{z}_+ = s+v \in \mathbb{R}$ and $\mathfrak{z}_- = s-v \in \mathbb{R}$ and this representation is called the idempotent representation [45] of a hyperbolic hybrid number.

1.1.2 Introduction to Probabilistic Measurable Space.

A useful mathematical model for describing the notion of uncertainty is the probability theory. The stochastic models that are frequently used in physics, biology and economics would be insignificant without probability theory. E. Borel (1871- 1956), S. N. Bernstein (1880- 1968) and A. N. Kolmogorov (1903- 1987) are the main contributors to the probability in the modern era.

In the year 1933, the Russian mathematician Andrei Nikolaevitch Kolmogorov proposed his modern perspective to study the probability theory by introducing the notion of a probability space. Actually, he introduced an alternative approach in formalizing the probability by three axioms known as 'Kolmogorov's axioms' [34], which opens a new direction of thoughts where one can explore the probability theory under the flavor of measure theory where the probability space is a measurable space with total mass equal to 1 and a random variable is a real valued measurable function {cf. [33], [47], [40], [41] & [61]}.

Definition 1.1.7 *A probabilistic measurable space* $(\Omega, \Sigma, \pounds)$ *is a triplet formed by a set* Ω *which has no structure but represents the sample space of a random experiment, a* σ*-field of subsets* Σ *of* Ω *and a measure* $\pounds$ *on the measurable space* (Ω, Σ) *which satisfies* $\pounds(\Omega) = 1$.

Example 1.1.1 *Let us consider the random experiment of throwing a die. Here the sample space* Ω *is* $\{1, 2, 3, 4, 5, 6\}$. *Then event space* Σ *will be the set of all subsets of* Ω. *Now define a measure or probability function* $\pounds$ *on the measurable space* (Ω, Σ) *as*

$$\pounds(\mathfrak{A}) = \frac{\textit{number of outcomes in the event } \mathfrak{A}}{6}$$

where $\mathfrak{A} \in \Sigma$. Then for the events $P = \{1,3,5\}, Q = \{2\}$ and $R = \Omega$ we have $\pounds(P) = \frac{1}{2}, \pounds(Q) = \frac{1}{6}$ and $\pounds(R) = 1$.

1.1.3 Introduction to Bicomplex numbers.

In this section, we give some basic ideas about bicomplex numbers.

Definition 1.1.8 *[38] The set $\mathbb{C}_2$ of bicomplex numbers is defined as $\mathbb{C}_2 = \{\eta : \eta = r_0 + \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3 : r_0, r_1, r_2, r_3 \in \mathbb{R}\}$ or equivalently $\mathbb{C}_2 = \{\eta : \eta = \eta_1 + \eta_2\mathbf{j} : \eta_1, \eta_2 \in \mathbb{C}_1\}$, where $\mathbb{C}_1$ is the set of complex numbers with imaginary unit i such that $\mathbf{i}^2 = \mathbf{j}^2 = -\mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{ji} = \mathbf{k}$. Thus we can think of bicomplex numbers as the complex numbers with complex coefficients.*

Definition 1.1.9 *[38] For any bicomplex number $\eta = \eta_1 + \eta_2\mathbf{j}$ the conjugation is defined in the following way:*

$$\overline{\eta}_i = \overline{\eta_1} + \overline{\eta_2}\mathbf{j}, \quad \overline{\eta}_j = \eta_1 - \eta_2\mathbf{j}, \quad \overline{\eta}_k = \overline{\eta_1} - \overline{\eta_2}\mathbf{j}.$$

Definition 1.1.10 *[38] Depending on the conjugation of bicomplex number $w = z_1 + z_2 j$ there are three types of modulus as follows:*

$$\begin{aligned} |\eta|_i^2 &:= \eta.\overline{\eta}_i = \eta_1^2 + \eta_2^2 \in \mathbb{C}(i) \\ |\eta|_j^2 &:= \eta.\overline{\eta}_j = |\eta_1|^2 - |\eta_2|^2 + 2Re(\eta_1\overline{\eta_2})\mathbf{j} \in \mathbb{C}(j) \\ |\eta|_k^2 &:= \eta.\overline{\eta}_k = |\eta_1|^2 + |\eta_2|^2 - 2Im(\eta_1\overline{\eta_2})\mathbf{k} \in \mathbb{D}. \end{aligned}$$

Definition 1.1.11 *[38] The norm function $\|.\| : \mathbb{C}_2 \to \mathbb{R}^+$ ($\mathbb{R}^+$ denote the set of all non negative real numbers) is defined as follows:*
If $\eta = \eta_1 + j\eta_2 = \xi_1\mathbf{e} + \xi_2\mathbf{e}^\dagger \in \mathbb{C}_2$, then

$$\|\eta\| = \sqrt{\{|\eta_1|^2 + |\eta_2|^2\}} = \sqrt{\left\{\frac{|\xi_1|^2 + |\xi_2|^2}{2}\right\}}.$$

From this, we can define the set of zero divisors $\mathcal{NC}$ of $\mathbb{C}_2$, called the $null - cone$, as

$$\mathcal{NC} = \{\eta = \eta_1 + \eta_2 j : \eta_1^2 + \eta_2^2 = 0\} = \{\eta(i \pm j) | \eta \in \mathbb{C}(i)\}.$$

Definition 1.1.12 *[21] The inverse of $\eta = \eta_1 + j\eta_2$ exists if $\|\eta\| \neq 0$ and it is defined as*

$$\eta^{-1} = \frac{1}{\eta} = \frac{\eta_1 - j\eta_2}{\eta_1^2 + \eta_2^2}.$$

If $\eta = \vartheta_0 + i\vartheta_1 + j\vartheta_2 + k\vartheta_3,\ \vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3 \in \mathbb{R}$, then

$$\frac{1}{\eta} = \left(\frac{\vartheta_0\mathfrak{g} + \vartheta_1\mathfrak{w}}{\mathfrak{l}}\right) + i\left(\frac{\vartheta_1\mathfrak{g} - \vartheta_0\mathfrak{w}}{\mathfrak{l}}\right) - j\left(\frac{\vartheta_2\mathfrak{g} + \vartheta_3\mathfrak{w}}{\mathfrak{l}}\right) - k\left(\frac{\vartheta_3\mathfrak{g} - \vartheta_3\mathfrak{w}}{\mathfrak{l}}\right),$$

where $\mathfrak{g} = \vartheta_0^2 - \vartheta_1^2 + \vartheta_2^2 - \vartheta_3^2,\ \mathfrak{w} = 2\vartheta_0\vartheta_1 + 2\vartheta_2\vartheta_3$ and $\mathfrak{l} = \mathfrak{g}^2 + \mathfrak{w}^2 = (\vartheta_0^2 + \vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2)^2 - 4(\vartheta_0\vartheta_3 - \vartheta_1\vartheta_2)^2$. Obviously $\frac{1}{\eta}$ exists if $\mathfrak{l} \neq 0$.

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CHAPTER TWO

A STUDY ON SOME ALGEBRAIC PROPERTIES OF HYBRID NUMBER

Chapter 2

A study on some algebraic properties of hybrid number

2.1 Introduction.

Definition 2.1.1 *A hybrid number is called symmetric if it's imaginary part and dual part both are same*
i.e. the hybrid number, $\mathfrak{N}_S = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$ *is called symmetric hybrid number if* $\mathfrak{d}_2 = \mathfrak{d}_3$.

The following example is a symmetric hybrid number.

Example 2.1.1 *The hybrid number* $2 + 3\mathbf{i} + 3\varepsilon + \mathbf{h}$ *is a symmetric.*

Definition 2.1.2 *The set of symmetric hybrid numbers with free hyperbolic unit is called null-hyperbolic and is denoted by* $\mathfrak{Z}_{S_{\bar{H}}}$.

Example 2.1.2 $-1 + i + \varepsilon$ *is an example of null-hyperbolic symmetric hybrid number.*

Definition 2.1.3 *A hybrid number is called skew-symmetric hybrid number if it's scalar, dual and hyperbolic coefficients are vanish.*

Example 2.1.3 $\mathfrak{Z} = 2\mathbf{i}$ *is an example of a skew-symmetric hybrid number.*

2.2 Lemmas.

Lemma 2.2.1 *For any two hybrid numbers* $\mathfrak{Z}_1, \mathfrak{Z}_2$ *the following equality holds*

$$\mathcal{C}(\mathfrak{Z}_1.\mathfrak{Z}_2) = \mathcal{C}(\mathfrak{Z}_1).\mathcal{C}(\mathfrak{Z}_2).$$

2.3 Theorems.

In this section, we present the main results of the chapter.

Theorem 2.3.1 *Symmetric hybrid numbers can never be elliptic.*

Proof. The type of any symmetric hybrid number $\mathfrak{N}_S = \mathfrak{d}_1 + \mathfrak{d}_3\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$ is

$$\mathcal{C}_\varepsilon(\mathfrak{N}_S) = -(\mathfrak{d}_3 - \mathfrak{d}_3)^2 + \mathfrak{d}_3^2 + \mathfrak{d}_4^2 = \mathfrak{d}_3^2 + \mathfrak{d}_4^2 \geqslant 0.$$

So, a symmetric hybrid number is either hyperbolic or parabolic, but it can never be elliptic. ■

Remark 2.3.1 *The character for any symmetric hybrid number* $\mathfrak{N}_S = \mathfrak{d}_1 + \mathfrak{d}_3\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h}$ *is* $\mathcal{C}(\mathfrak{N}_S) = \mathfrak{d}_1^2 - \mathfrak{d}_3^2 - \mathfrak{d}_4^2$.
Thus, $\mathfrak{N}_S$ *may be spacelike, timelike or lightlike for* $\mathfrak{d}_1^2 < \mathfrak{d}_3^2 + \mathfrak{d}_4^2$, $\mathfrak{d}_1^2 > \mathfrak{d}_3^2 + \mathfrak{d}_4^2$ *or* $\mathfrak{d}_1^2 = \mathfrak{d}_3^2 + \mathfrak{d}_4^2$ *respectively.*

Theorem 2.3.2 *A skew-symmetric hybrid number never be spacelike as well as hyperbolic.*

Proof. Character of a skew-symmetric hybrid number, $\mathfrak{Z} = \mathfrak{d}_2\mathbf{i}$ is $\mathcal{C}(\mathfrak{Z}) = \mathfrak{d}_2^2 \geqslant 0$ and therefore it is either timelike or lightlike, it never be spacelike.
Now, the type of a skew-symmetric hybrid number $\mathfrak{Z} = \mathfrak{d}_2\mathbf{i}$ is $\mathcal{C}_\varepsilon(\mathfrak{Z}) = -\mathfrak{d}_2^2 \leqslant 0$ and hence it is either elliptic or parabolic but never be hyperbolic. ■

Theorem 2.3.3 *The set* $\mathfrak{N}_S$ *is a subgroup of* $\mathfrak{T}$ *under* $'+'$.

Proof. The proof of the theorem is trivial. ■

Remark 2.3.2 *The group* $(\mathfrak{N}_S, +)$ *is abelian as hybrid addition is commutative.*

Theorem 2.3.4 *The set of null-hyperbolic symmetric hybrid numbers is a normal subgroup of symmetric hybrid numbers.*

Proof. Clearly, $\mathfrak{Z}_{S_{\bar{H}}}$ is a non empty subset of $\mathfrak{N}_S$ as $\mathbf{0} \in \mathfrak{Z}_{S_{\bar{H}}}$.
Let, $\mathfrak{Z}_1 = \mathfrak{e}_1 + \mathfrak{e}_3\mathbf{i} + \mathfrak{e}_3\varepsilon$, $\mathfrak{Z}_2 = \mathfrak{f}_1 + \mathfrak{f}_3\mathbf{i} + \mathfrak{f}_3\varepsilon \in \mathfrak{Z}_{S_{\bar{H}}}$.
Now, $\mathfrak{Z}_1 + \mathfrak{Z}_2 = (\mathfrak{e}_1 + \mathfrak{f}_1) + (\mathfrak{e}_3 + \mathfrak{f}_3)\mathbf{i} + (\mathfrak{e}_3 + \mathfrak{f}_3)\varepsilon \in \mathfrak{Z}_{S_{\bar{H}}}$ and $\mathfrak{Z}_1^{-1} = -\mathfrak{e}_1 - \mathfrak{e}_3\mathbf{i} - \mathfrak{e}_3\varepsilon \in \mathfrak{Z}_{S_{\bar{H}}}$.
Therefore, $(\mathfrak{Z}_{S_{\bar{H}}}, +)$ is a subgroup of $(\mathfrak{N}_S, +)$ and as $(\mathfrak{N}_S, +)$ is abelian implies $(\mathfrak{Z}_{S_{\bar{H}}}, +)$ is normal in $(\mathfrak{N}_S, +)$. ■
Theorem 2.3.4 leads us to the following remarks.

Remark 2.3.3 *The set of null-hyperbolic symmetric hybrid numbers is also a normal subgroup of hybrid numbers.*

Remark 2.3.4 *Let* $\mathfrak{A} = (\mathfrak{T}, +)$ *and* $\mathfrak{Y} = (\mathfrak{Z}_{S_{\bar{H}}}, +)$. *Then* $\mathfrak{A}/\mathfrak{Y}$ *forms the quotient group.*

Remark 2.3.5 *Let* $\mathfrak{L} = (\mathfrak{N}_S, +)$ *and* $\mathfrak{Y} = (\mathfrak{Z}_{S_{\bar{H}}}, +)$. *Then* $\mathfrak{L}/\mathfrak{Y}$ *forms the quotient group.*

Remark 2.3.6 *As* $\mathfrak{Y} = (\mathfrak{Z}_{S_{\bar{H}}}, +)$ *is a normal subgroup of* $\mathfrak{A} = (\mathfrak{T}, +)$ *and* $\mathfrak{L} = (\mathfrak{N}_S, +)$ *is a subgroup* $\mathfrak{A} = (\mathfrak{T}, +)$ *with* $\mathfrak{Y} \subset \mathfrak{L} \subset \mathfrak{A}$ *then* $\mathfrak{L}/\mathfrak{Y}$ *is a subgroup of* $\mathfrak{A}/\mathfrak{Y}$.

Theorem 2.3.5 *The set of all symmetric hybrid numbers* $\mathfrak{N}_S$ *does not form a group under the operation 'hybrid Multiplication'.*

Proof. Let, $\mathfrak{z}_1 = \mathfrak{e}_1 + \mathfrak{e}_3\mathbf{i} + \mathfrak{e}_3\varepsilon + \mathfrak{e}_4\mathbf{h}$ and $\mathfrak{z}_2 = \mathfrak{f}_1 + \mathfrak{f}_3\mathbf{i} + \mathfrak{f}_3\varepsilon + \mathfrak{f}_4\mathbf{h} \in \mathfrak{Z}_s$. Then

$$\begin{aligned}\mathfrak{z}_1.\mathfrak{z}_2 &= \mathfrak{e}_1\mathfrak{f}_1 + \mathfrak{e}_3\mathfrak{f}_1\mathbf{i} + \mathfrak{e}_3\mathfrak{f}_1\varepsilon + \mathfrak{e}_4\mathfrak{f}_1\mathbf{h} + \mathfrak{e}_1\mathfrak{f}_3\mathbf{i} - \mathfrak{e}_3\mathfrak{f}_3 + \mathfrak{e}_3\mathfrak{f}_3(\mathbf{h}+1)\\ &\quad + \mathfrak{e}_4\mathfrak{f}_3(-\mathbf{i}-\varepsilon) + \mathfrak{e}_1\mathfrak{f}_3\varepsilon + \mathfrak{e}_3\mathfrak{f}_3(1-\mathbf{h}) + \mathfrak{e}_4\mathfrak{f}_3\varepsilon + \mathfrak{e}_1\mathfrak{f}_4\mathbf{h}\\ &\quad + \mathfrak{e}_3\mathfrak{f}_4(\mathbf{i}+\varepsilon) - \mathfrak{e}_3\mathfrak{f}_4\varepsilon + \mathfrak{e}_4\mathfrak{f}_4\\ &= (\mathfrak{e}_1\mathfrak{f}_1 + \mathfrak{e}_3\mathfrak{f}_3 + \mathfrak{e}_4\mathfrak{f}_4) + (\mathfrak{e}_3\mathfrak{f}_1 + \mathfrak{e}_1\mathfrak{f}_3 - \mathfrak{e}_4\mathfrak{f}_3 + \mathfrak{e}_3\mathfrak{f}_4)\mathbf{i}\\ &\quad + (\mathfrak{e}_3\mathfrak{f}_1 + \mathfrak{e}_1\mathfrak{f}_3)\varepsilon + (\mathfrak{e}_4\mathfrak{f}_1 + \mathfrak{e}_1\mathfrak{f}_4)\mathbf{h}.\end{aligned}$$

Clearly, $\mathfrak{z}_1.\mathfrak{z}_2 \notin \mathfrak{N}_S$.
Hence, $(\mathfrak{N}_S, .)$ is not a group. ■
The following example ensures the above fact.

Example 2.3.1 *Let,* $\mathfrak{z}_1 = 1 + \mathbf{i} + \varepsilon + \mathbf{h}$, $\mathfrak{z}_2 = \mathbf{i} + \varepsilon + 2\mathbf{h} \in \mathfrak{N}_S$

$$\begin{aligned}Then, \mathfrak{z}_1.\mathfrak{z}_2 &= \mathbf{i} - 1 + \mathbf{h} + 1 - \varepsilon - \mathbf{i} + \varepsilon + 1 - \mathbf{h} + \varepsilon + 2\mathbf{h}\\ &\quad + 2(\mathbf{i}+\varepsilon) - 2\varepsilon + 2\\ &= 3 + 2\mathbf{i} + \varepsilon + 2\mathbf{h},\end{aligned}$$

not a symmetric hybrid number.

The following remark is immediate in the view above.

Remark 2.3.7 *The set of non-lightlike symmetric hybrid number whose dual and hyperbolic coefficients are in a constant ratio forms an abelian group under hybrid multiplication.*

Theorem 2.3.6 *The set of all non-lightlike hybrid numbers forms a group under the hybrid multiplication.*

Proof. Let $\mathfrak{T}_{NL} = \{\mathfrak{z} \in \mathfrak{T} : \mathcal{C}(\mathfrak{z}) \neq 0\}$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{T}_{NL}$.
Then, $\mathcal{C}(\mathfrak{z}_1) \neq 0, \mathcal{C}(\mathfrak{z}_2) \neq 0$.
Using Lemma 2.2.1, $\mathcal{C}(\mathfrak{z}_1.\mathfrak{z}_2) \neq 0$ and hence $\mathfrak{z}_1.\mathfrak{z}_2 \in \mathfrak{T}_{NL}$.
Here $\mathfrak{z}_e = 1 + 0.\mathbf{i} + 0.\varepsilon + 0.\mathbf{h} \in \mathfrak{T}_{NL}$ acts as the identity in $\mathfrak{T}_{NL}$.
Let $\mathfrak{z}^{-1}$ be the inverse of $\mathfrak{z} \in \mathfrak{T}_{NL}$.
Therefore, $\mathcal{C}(\mathfrak{z}.\mathfrak{z}^{-1}) = \mathcal{C}(\mathfrak{z}_e) = 1 \Rightarrow \mathcal{C}(\mathfrak{z}).\mathcal{C}(\mathfrak{z}^{-1}) = 1$
and $C(\mathfrak{z}^{-1}) = \frac{1}{\mathcal{C}(\mathfrak{z})} \neq 0$ [since $\mathcal{C}(\mathfrak{z}) \neq 0$].

Thus, $\mathfrak{z}^{-1} \in \mathfrak{T}_{NL}$.
As associative property is hereditary, it implies that the set of non-lightlike hybrid numbers $\mathfrak{T}_{NL}$ forms a group under multiplication. ■
The following corollary is immediate in view of above.

Corollary 2.3.1 *The group $\mathfrak{T}_{NL}$ is non-commutative.*

Proof. We know that $(\mathfrak{T}_{NL}, .)$ is a group.
Now, let $\mathfrak{z}_1 = \mathfrak{e}_1 + \mathfrak{e}_2\mathbf{i} + \mathfrak{e}_3\varepsilon + \mathfrak{e}_4\mathbf{h}, \mathfrak{z}_2 = \mathfrak{f}_1 + \mathfrak{f}_2\mathbf{i} + \mathfrak{f}_3\varepsilon + \mathfrak{f}_4\mathbf{h} \in \mathfrak{T}_{NL}$. Then

$$\begin{aligned}\mathfrak{z}_1.\mathfrak{z}_2 &= (\mathfrak{e}_1 + \mathfrak{e}_2\mathbf{i} + \mathfrak{e}_3\varepsilon + \mathfrak{e}_4\mathbf{h}).(\mathfrak{f}_1 + \mathfrak{f}_2\mathbf{i} + \mathfrak{f}_3\varepsilon + \mathfrak{f}_4\mathbf{h}) \\ &= (\mathfrak{f}_1\mathfrak{e}_1 - \mathfrak{f}_2\mathfrak{e}_2 + \mathfrak{f}_3\mathfrak{e}_2 + \mathfrak{f}_2\mathfrak{e}_3 + \mathfrak{f}_4\mathfrak{e}_4) + (\mathfrak{f}_2\mathfrak{e}_1 + \mathfrak{f}_1\mathfrak{e}_2 \\ &\quad + \mathfrak{f}_4\mathfrak{e}_2 - \mathfrak{f}_2\mathfrak{e}_4)\mathbf{i} + (\mathfrak{f}_3\mathfrak{e}_1 + \mathfrak{f}_4\mathfrak{e}_2 + \mathfrak{f}_1\mathfrak{e}_3 - \mathfrak{f}_4\mathfrak{e}_3 - \mathfrak{f}_2\mathfrak{e}_4 \\ &\quad + \mathfrak{f}_3\mathfrak{e}_4)\varepsilon + (\mathfrak{f}_4\mathfrak{e}_1 - \mathfrak{f}_3\mathfrak{e}_2 + \mathfrak{f}_2\mathfrak{e}_3 + \mathfrak{f}_1\mathfrak{e}_4)\mathbf{h}\end{aligned}$$

whereas

$$\begin{aligned}\mathfrak{z}_2.\mathfrak{z}_1 &= (\mathfrak{f}_1\mathfrak{e}_1 - \mathfrak{f}_2\mathfrak{e}_2 + \mathfrak{f}_3\mathfrak{e}_2 + \mathfrak{f}_2\mathfrak{e}_3 + \mathfrak{f}_4\mathfrak{e}_4) + (\mathfrak{f}_2\mathfrak{e}_1 + \mathfrak{f}_1\mathfrak{e}_2 \\ &\quad - \mathfrak{f}_4\mathfrak{e}_2 + \mathfrak{f}_2\mathfrak{e}_4)\mathbf{i} + (\mathfrak{f}_3\mathfrak{e}_1 - \mathfrak{f}_4\mathfrak{e}_2 + \mathfrak{f}_1\mathfrak{e}_3 + \mathfrak{f}_4\mathfrak{e}_3 + \mathfrak{f}_2\mathfrak{e}_4 \\ &\quad - \mathfrak{f}_3\mathfrak{e}_4)\varepsilon + (\mathfrak{f}_4\mathfrak{e}_1 + \mathfrak{f}_3\mathfrak{e}_2 - \mathfrak{f}_2\mathfrak{e}_3 + \mathfrak{f}_1\mathfrak{e}_4)\mathbf{h}.\end{aligned}$$

Thus, $\mathfrak{z}_1.\mathfrak{z}_2 \neq \mathfrak{z}_2.\mathfrak{z}_1$ and hence $(\mathfrak{T}_{NL}, .)$ is a non-abelian group. ■

Theorem 2.3.7 *The set of all hybrid numbers of unit character is normal in $\mathfrak{T}_{NL}$.*

Proof. Let $\mathfrak{T}_{\mathcal{C}_1} = \{\mathfrak{z} \in \mathfrak{T} : \mathcal{C}(\mathfrak{z}) = 1\}$,
which is non-empty as $\mathbf{1} = 1 + 0.\mathbf{i} + 0.\varepsilon + 0.\mathbf{h} \in \mathfrak{T}_{\mathcal{C}_1}$.
Let $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{T}_{\mathcal{C}_1}$, then $\mathcal{C}(\mathfrak{z}_1.\mathfrak{z}_2) = 1$.
Thus, $\mathfrak{z}_1.\mathfrak{z}_2 \in \mathfrak{T}_{\mathcal{C}_1}$.
Let $\mathfrak{z}_1^{-1}$ be the inverse of $\mathfrak{z}_1 \in \mathfrak{T}_{\mathcal{C}_1}$.
Now, $\mathcal{C}(\mathfrak{z}_1.\mathfrak{z}_1^{-1}) = \mathcal{C}(\mathfrak{z}_e) = 1$
i.e., $\mathcal{C}(\mathfrak{z}_1^{-1}) = \frac{1}{\mathcal{C}(\mathfrak{z}_1)} = 1$ [as $\mathcal{C}(\mathfrak{z}_1) = 1$]
Thus, $\mathfrak{z}_1^{-1} \in \mathfrak{T}_{\mathcal{C}_1}$.
Hence, $\mathfrak{T}_{\mathcal{C}_1}$ is a subgroup of $\mathfrak{T}_{NL}$.
Let $\aleph \in \mathfrak{T}_{NL}$ and $\mathfrak{z} \in \mathfrak{T}_{\mathcal{C}_1}$
Now, $\mathcal{C}(\aleph\mathfrak{z}\aleph^{-1}) = \mathcal{C}(\aleph)\mathcal{C}(\mathfrak{z})\mathcal{C}(\aleph^{-1}) = \mathcal{C}(\aleph)\mathcal{C}(\aleph^{-1}) = \mathcal{C}(\mathbf{1}) = 1$, non zero.
Therefore, $\aleph\mathfrak{z}\aleph^{-1} \in \mathfrak{T}_{\mathcal{C}_1}$ and hence $\mathfrak{T}_{\mathcal{C}_1}$ is a normal subgroup of $\mathfrak{T}_{NL}$.
This proves the theorem. ■

Remark 2.3.8 *The set of lightlike hybrid numbers does not form a group under hybrid multiplication as the inverse of any element does not exist.*

Theorem 2.3.8 *The set $\mathfrak{T}_{\mathbb{Q}} = \{\mathfrak{z} = \mathfrak{q}_1 + \mathfrak{q}_2\mathbf{i} + \mathfrak{q}_3\varepsilon + \mathfrak{q}_4\mathbf{h} : \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_4 \in \mathbb{Q}\}$ forms a subring of $\mathfrak{T}$.*

Proof. Clearly $\mathbf{0} \in \mathfrak{T}_{\mathbb{Q}}$ and let $\mathfrak{Z}_1 = \mathfrak{e}_1 + \mathfrak{e}_2\mathbf{i} + \mathfrak{e}_3\varepsilon + \mathfrak{e}_4\mathbf{h}$ &
$\mathfrak{Z}_2 = \mathfrak{f}_1 + \mathfrak{f}_2\mathbf{i} + \mathfrak{f}_3\varepsilon + \mathfrak{f}_4\mathbf{h} \in \mathfrak{T}_{\mathbb{Q}}$
Now, $\mathfrak{Z}_1 - \mathfrak{Z}_2 = (\mathfrak{e}_1 - \mathfrak{f}_1) + (\mathfrak{e}_2 - \mathfrak{f}_2)\mathbf{i} + (\mathfrak{e}_3 - \mathfrak{f}_3)\varepsilon + (\mathfrak{e}_4 - \mathfrak{f}_4)\mathbf{h} \in \mathfrak{T}_{\mathbb{Q}}$
and also
$\mathfrak{Z}_1.\mathfrak{Z}_2 = (\mathfrak{e}_1\mathfrak{f}_1 - \mathfrak{e}_2\mathfrak{f}_2 + \mathfrak{e}_3\mathfrak{f}_2 + \mathfrak{e}_2\mathfrak{f}_3 + \mathfrak{e}_4\mathfrak{f}_4) + (\mathfrak{e}_2\mathfrak{f}_1 + \mathfrak{e}_1\mathfrak{f}_2 - \mathfrak{e}_4\mathfrak{f}_2 + \mathfrak{e}_2\mathfrak{f}_4)\mathbf{i}$
$+ (\mathfrak{e}_3\mathfrak{f}_1 - \mathfrak{e}_4\mathfrak{f}_2 + \mathfrak{e}_1\mathfrak{f}_3 + \mathfrak{e}_4\mathfrak{f}_3 + \mathfrak{e}_2\mathfrak{f}_4 - \mathfrak{e}_3\mathfrak{f}_4)\varepsilon + (\mathfrak{e}_4\mathfrak{f}_1 + \mathfrak{e}_3\mathfrak{f}_2 - \mathfrak{e}_2\mathfrak{f}_3 + \mathfrak{e}_1\mathfrak{f}_4)\mathbf{h} \in \mathfrak{T}_{\mathbb{Q}}$
Therefore, $\mathfrak{T}_{\mathbb{Q}}$ is a subring of the ring $\mathfrak{T}$.
This completes the proof. ∎

Remark 2.3.9 *The set* $\mathfrak{T}_{\mathbb{Z}} = \{\mathfrak{Z} = \mathfrak{z}_1 + \mathfrak{z}_2\mathbf{i} + \mathfrak{z}_3\varepsilon + \mathfrak{z}_4\mathbf{h} : \mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4 \in \mathbb{Z}\}$ *is also a subring of* $\mathfrak{T}$.

Proof. The proof is omitted as it is similar to the previous one. ∎

Theorem 2.3.9 *The set* $\mathfrak{T}_{\mathfrak{R}} = \{\mathfrak{Z} = \frac{\varpi_1}{2} + \frac{\varpi_1+\varpi_2}{2}\mathbf{i} + \frac{\varpi_1}{2}\varepsilon + \frac{\varpi_2}{2}\mathbf{h} : \varpi_1, \varpi_2 \in \mathbb{Z}\}$ *is a right ideal of* $\mathfrak{T}$.

Proof. Clearly $\mathfrak{T}_{\mathfrak{R}}$ is a subring of $\mathfrak{T}$.
Let $\mathfrak{Z} = \frac{\varpi_1}{2} + \frac{\varpi_1+\varpi_2}{2}\mathbf{i} + \frac{\varpi_1}{2}\varepsilon + \frac{\varpi_2}{2}\mathbf{h} \in \mathfrak{T}_{\mathfrak{R}}$ and $\aleph = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h} \in \mathfrak{T}$. Then

$$\begin{aligned}
\mathfrak{Z}.\aleph = & \frac{\varpi_1\mathfrak{d}_1}{2} + \frac{\varpi_1\mathfrak{d}_1 + \varpi_2\mathfrak{d}_1}{2}\mathbf{i} + \frac{\varpi_1\mathfrak{d}_1}{2}\varepsilon + \frac{\varpi_2\mathfrak{d}_1}{2}\mathbf{h} + \frac{\varpi_1\mathfrak{d}_2}{2}\mathbf{i} - \frac{\varpi_1\mathfrak{d}_2 + \varpi_2\mathfrak{d}_2}{2} \\
& + \frac{\varpi_1\mathfrak{d}_2}{2}(\mathbf{h}+1) + \frac{\varpi_2\mathfrak{d}_2}{2}(-\varepsilon - \mathbf{i}) + \frac{\varpi_1\mathfrak{d}_3}{2}\varepsilon + \frac{\varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3}{2}(1 - \mathbf{h}) \\
& + \frac{\varpi_2\mathfrak{d}_3}{2}\varepsilon + \frac{\varpi_1\mathfrak{d}_4}{2}\mathbf{h} + \frac{\varpi_1\mathfrak{d}_4 + \varpi_2\mathfrak{d}_4}{2}(\varepsilon + \mathbf{i}) + \frac{\varpi_1\mathfrak{d}_4}{2}(-\varepsilon) + \frac{\varpi_2\mathfrak{d}_4}{2} \\
= & \frac{\varpi_1\mathfrak{d}_1 - \varpi_2\mathfrak{d}_2 + \varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_4 + \varpi_2\mathfrak{d}_3}{2} + \frac{\varpi_1\mathfrak{d}_1 + \varpi_1\mathfrak{d}_2 + \varpi_1\mathfrak{d}_4}{2} \\
& \frac{+\varpi_2\mathfrak{d}_4 + \varpi_2\mathfrak{d}_1 - \varpi_2\mathfrak{d}_2}{2}\mathbf{i} + \frac{\varpi_1\mathfrak{d}_1 - \varpi_2\mathfrak{d}_2 + \varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3 + \varpi_2\mathfrak{d}_4}{2}\varepsilon \\
& + \frac{\varpi_2\mathfrak{d}_1 + \varpi_1\mathfrak{d}_2 - \varpi_1\mathfrak{d}_3 - \varpi_2\mathfrak{d}_3 + \varpi_1\mathfrak{d}_4}{2}\mathbf{h}
\end{aligned}$$

Therefore, $\mathfrak{Z}.\aleph \in \mathfrak{T}_{\mathcal{RI}}$ and hence $\mathfrak{T}_{\mathfrak{R}}$ is a right ideal of $\mathfrak{T}$. ∎

Corollary 2.3.2 *The subring* $\mathfrak{T}_{\mathfrak{R}}$ *is not an left ideal of* $\mathfrak{T}$.

Proof. Let $\mathfrak{Z} = \frac{\varpi_1}{2} + \frac{\varpi_1+\varpi_2}{2}\mathbf{i} + \frac{\varpi_1}{2}\varepsilon + \frac{\varpi_2}{2}\mathbf{h} \in \mathfrak{T}_{\mathfrak{R}}$ and $\aleph = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + \mathfrak{d}_4\mathbf{h} \in \mathfrak{T}$. Then

$$
\begin{aligned}
\aleph.\mathfrak{Z} &= \frac{\mathfrak{d}_1\varpi_1}{2} + \frac{\varpi_1\mathfrak{d}_2}{2}\mathbf{i} + \frac{\varpi_1\mathfrak{d}_3}{2}\varepsilon + \frac{\varpi_1\mathfrak{d}_4}{2}\mathbf{h} + \frac{\varpi_1\mathfrak{d}_1 + \varpi_2\mathfrak{d}_2}{2}\mathbf{i} - \frac{\varpi_1\mathfrak{d}_2 + \varpi_2\mathfrak{d}_2}{2} \\
&\quad + \frac{\varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3}{2}(\mathbf{h}+1) + \frac{\varpi_1\mathfrak{d}_4 + \varpi_2\mathfrak{d}_4}{2}(-\varepsilon - \mathbf{i}) + \frac{\varpi_1\mathfrak{d}_1}{2}\varepsilon + \frac{\varpi_1\mathfrak{d}_2}{2}(1-\mathbf{h}) \\
&\quad + \frac{\varpi_1\mathfrak{d}_4}{2}\varepsilon + \frac{\mathfrak{d}_1\varpi_2}{2}\mathbf{h} + \frac{\mathfrak{d}_2\varpi_2}{2}(\varepsilon + \mathbf{i}) + \frac{\varpi_2\mathfrak{d}_3}{2}(-\varepsilon) + \frac{\varpi_2\mathfrak{d}_4}{2} \\
&= \frac{\varpi_1\mathfrak{d}_1 - \varpi_2\mathfrak{d}_2 + \varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3 + \varpi_2\mathfrak{d}_4}{2} + \frac{\varpi_1\mathfrak{d}_2 + \varpi_1\mathfrak{d}_1 + \varpi_2\mathfrak{d}_1}{2} \\
&\quad \frac{-\varpi_1\mathfrak{d}_4 - \varpi_2\mathfrak{d}_4 + \varpi_2\mathfrak{d}_2}{2}\mathbf{i} + \frac{\varpi_1\mathfrak{d}_3 - \varpi_2\mathfrak{d}_4 + \varpi_1\mathfrak{d}_1 + \varpi_2\mathfrak{d}_2 - \varpi_2\mathfrak{d}_3}{2}\varepsilon \\
&\quad + \frac{\varpi_1\mathfrak{d}_4 + \varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3 - \varpi_1\mathfrak{d}_2 + \varpi_2\mathfrak{d}_1}{2}\mathbf{h} \\
&= \frac{\nu_1}{2} + \frac{\nu_2}{2}\mathbf{i} + \frac{\nu_3}{2}\varepsilon + \frac{\nu_4}{2}\mathbf{h}\ (say)
\end{aligned}
$$

where $\nu_1 = \varpi_1\mathfrak{d}_1 - \varpi_2\mathfrak{d}_2 + \varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3 + \varpi_2\mathfrak{d}_4$; $\nu_2 = \varpi_1\mathfrak{d}_2 + \varpi_1\mathfrak{d}_1 + \varpi_2\mathfrak{d}_1 - \varpi_1\mathfrak{d}_4 - \varpi_2\mathfrak{d}_4 + \varpi_2\mathfrak{d}_2$; $\nu_3 = \varpi_1\mathfrak{d}_3 - \varpi_2\mathfrak{d}_4 + \varpi_1\mathfrak{d}_1 + \varpi_2\mathfrak{d}_2 - \varpi_2\mathfrak{d}_3$; and $\nu_4 = \varpi_1\mathfrak{d}_4 + \varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3 - \varpi_1\mathfrak{d}_2 + \varpi_2\mathfrak{d}_1$
Clearly, $\nu_1 \neq \nu_3$ and so $\aleph.\mathfrak{Z} \notin \mathfrak{T}_{\mathfrak{R}}$.
Therefore, $\mathfrak{T}_{\mathfrak{R}}$ is not left ideal. ■
The following example validates the above result.

Example 2.3.2 *Consider* $\mathfrak{Z} = \mathbf{i} + \mathbf{h} \in \mathfrak{T}$, $\aleph = (-\frac{1}{2})\mathbf{i} + \frac{1}{2}\mathbf{h} \in \mathfrak{T}_{\mathfrak{R}}$.
But, $\aleph.\mathfrak{Z} = 1 - \mathbf{i} - \varepsilon \notin \mathfrak{T}_{\mathfrak{R}}$.

Theorem 2.3.10 *The set* $\mathfrak{T}_{\mathcal{L}} = \{\mathfrak{Z} = \frac{\varpi_1}{2} + \frac{\varpi_1 - \varpi_2}{2}\mathbf{i} + \frac{\varpi_1}{2}\varepsilon + \frac{\varpi_2}{2}\mathbf{h} : \varpi_1, \varpi_2 \in \mathbb{Z}\}$ *is a left ideal of* $\mathfrak{T}$.

Proof. Let $\mathfrak{Z} = \frac{\varpi_1}{2} + \frac{\varpi_1 - \varpi_2}{2}\mathbf{i} + \frac{\varpi_1}{2}\varepsilon + \frac{\varpi_2}{2}\mathbf{h} \in \mathfrak{T}_{\mathcal{L}}$ and $\aleph = \mathfrak{d}_1 + \mathfrak{d}_2\mathbf{i} + \mathfrak{d}_3\varepsilon + s\mathbf{h} \in \mathfrak{T}$. Then

$$
\begin{aligned}
\aleph.\mathfrak{Z} &= \frac{\varpi_1\mathfrak{d}_1 + \varpi_2\mathfrak{d}_2 + \varpi_1\mathfrak{d}_3 - \varpi_2\mathfrak{d}_3 + \varpi_1\mathfrak{d}_2 + \varpi_2\mathfrak{d}_4}{2} + \frac{\varpi_1\mathfrak{d}_2 + \varpi_1\mathfrak{d}_1 - \varpi_2\mathfrak{d}_1 - \varpi_1\mathfrak{d}_4 + \varpi_2\mathfrak{d}_4 + \varpi_2\mathfrak{d}_2}{2}\mathbf{i} \\
&\quad + \frac{\varpi_1\mathfrak{d}_3 - \varpi_1\mathfrak{d}_4 + \varpi_2\mathfrak{d}_4 + \varpi_1\mathfrak{d}_1 + \varpi_1\mathfrak{d}_4 + \varpi_2\mathfrak{d}_2 - \varpi_2\mathfrak{d}_3}{2}\varepsilon \\
&\quad + \frac{\varpi_1\mathfrak{d}_4 + \varpi_1\mathfrak{d}_3 - \varpi_2\mathfrak{d}_3 - \varpi_1\mathfrak{d}_2 + \varpi_2\mathfrak{d}_1}{2}\mathbf{h}
\end{aligned}
$$

Therefore, $\aleph.\mathfrak{Z} \in \mathfrak{T}_{\mathcal{L}}$ and hence $\mathfrak{T}_{\mathcal{L}}$ is a left ideal of $\mathfrak{T}$.
This proves the theorem. ■

Corollary 2.3.3 *The subring* $\mathfrak{T}_{\mathcal{L}}$ *is not a right ideal of* $\mathfrak{T}$.

Proof.

$$
\begin{aligned}
\mathfrak{Z}.\aleph &= \frac{\varpi_1\mathfrak{d}_1 + \varpi_2\mathfrak{d}_2 + \varpi_1\mathfrak{d}_3 - \varpi_2\mathfrak{d}_3 + \varpi_2\mathfrak{d}_4}{2} + \frac{\varpi_1\mathfrak{d}_1 - \varpi_2\mathfrak{d}_1 + \varpi_1\mathfrak{d}_2 - \varpi_2\mathfrak{d}_2 + \varpi_1\mathfrak{d}_4 - \varpi_2\mathfrak{d}_4}{2}\mathbf{i} \\
&\quad + \frac{\varpi_1\mathfrak{d}_1 - \varpi_2\mathfrak{d}_2 + \varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3 - \varpi_2\mathfrak{d}_4}{2}\varepsilon + \frac{\varpi_2\mathfrak{d}_1 + \varpi_1\mathfrak{d}_2 - \varpi_1\mathfrak{d}_3 + \varpi_2\mathfrak{d}_3 + \varpi_1\mathfrak{d}_4}{2}\mathbf{h} \\
&= \frac{\delta_1}{2} + \frac{\delta_2}{2}\mathbf{i} + \frac{\delta_3}{2}\varepsilon + \frac{\delta_2}{2}\mathbf{h}\ (say).
\end{aligned}
$$

Clearly, $\delta_1 \neq \delta_3$ and so $\mathfrak{Z}.\aleph \notin \mathfrak{T}_{\mathcal{L}}$.
So, $\mathfrak{T}_{\mathcal{L}}$ is not right ideal. ■

Example 2.3.3 *Consider* $\mathfrak{Z} = \frac{1}{2} - \mathbf{i} + \frac{1}{2}\varepsilon + \frac{3}{2}\mathbf{h}$; $\aleph = \mathbf{i} + \mathbf{h}$, *then*
$\mathfrak{Z}.\aleph = 3 - 2\mathbf{i} - 3\varepsilon + \mathbf{h} \notin \mathfrak{T}_{\mathcal{L}}$.

2.4 Future Prospects

In the line of works as carried out in this chapter one may think of exploring some properties by taking into account some different kinds of ideals and also may try to investigate the results in higher dimensional system.

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CHAPTER THREE

HYPERBOLIC HYBRID VALUED PROBABILISTIC MEASURES UNDER THE FLAVOUR OF KOLMOGOROV'S AXIOMS

Chapter 3

Hyperbolic hybrid valued probabilistic measures under the flavour of Kolmogorov's axioms

3.1 Introduction, Definitions and Notations.

Definition 3.1.1 *Consider the set* $\mathcal{P}^{\mathfrak{h}} \cup \{0,1\}$ *as* $\mathfrak{H}$ *and let* (Ω, Σ) *be a measurable space, a function* $\pounds_{\mathfrak{H}} : \Sigma \mapsto \mathfrak{H}$ *with the properties:*

(i) for any event $\mathfrak{F} \in \Sigma$, $\pounds_{\mathfrak{H}}(\mathfrak{F}) \geqslant 0$,

(ii) for the certain event Ω, $\pounds_{\mathfrak{H}}(\Omega) = \mathfrak{p}$, *where* $\mathfrak{p}$ *is either* 1 *or* $\mathbf{e}_+$ *or* $\mathbf{e}_-$, *and*

(iii) for a given sequence $\{\mathfrak{F}_n\} \subset \Sigma$ *of pairwise disjoint events,*

$$\pounds_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty} \mathfrak{F}_n\right) = \sum_{n=1}^{\infty} \pounds_{\mathfrak{H}}\left(\mathfrak{F}_n\right)$$

is called a $\mathfrak{H}$*-valued probabilistic measure, or a* $\mathfrak{H}$*-valued probability, on the* σ*-algebra of events* Σ *and the triplet* $\left(\Omega, \Sigma, \pounds_{\mathfrak{H}}\right)$ *is called a* $\mathfrak{H}$*-probabilistic space.*

Since, $\pounds_{\mathfrak{H}}(\mathfrak{F}) \in \mathfrak{H}$. So, it can be expressed as

$$\pounds_{\mathfrak{H}}(\mathfrak{F}) = \Gamma_1(\mathfrak{F})\mathbf{e}_+ + \Gamma_2(\mathfrak{F})\mathbf{e}_-.$$

Now Property (i) of $\mathfrak{H}$-valued probabilistic measure implies that

$$\Gamma_1(\mathfrak{F}) \geqslant 0 \text{ and } \Gamma_2(\mathfrak{F}) \geqslant 0, \quad \forall \mathfrak{F} \in \Sigma.$$

From Property (ii) we get that

$$\pounds_{\mathfrak{H}}(\Omega) = \mathfrak{p} = \Gamma_1(\Omega)\mathbf{e}_+ + \Gamma_2(\Omega)\mathbf{e}_-,$$

i.e.,

(I) If $\mathfrak{p} = 1$ then $\Gamma_1(\Omega) = 1, \Gamma_2(\Omega) = 1$.

(II) If $\mathfrak{p} = \mathbf{e}_+$ then $\Gamma_1(\Omega) = 1, \Gamma_2(\Omega) = 0$.

(III) If $\mathfrak{p} = \mathbf{e}_-$ then $\Gamma_1(\Omega) = 0, \Gamma_2(\Omega) = 1$.

The Property (iii) of $\pounds_{\mathfrak{H}}$ implies that

$$P_i\left(\bigcup_{n=1}^{\infty} \mathfrak{F}_n\right) = \sum_{n=1}^{\infty} P_i(\mathfrak{F}_n) \quad \text{for } i = 1 \text{ and } 2.$$

Therefore one can say that, in general, the $\mathfrak{H}$-valued probabilistic measure is equivalent if we consider a pair of unrelated usual $\mathbb{R}$-valued measures on the same measurable space.

Here we can observe that Γ_1 is a probabilistic measure in Cases (I) and (II); Γ_2 is a probabilistic measure in Cases (I) and (III) whereas Γ_2 and Γ_1 are trivial measures for the Cases (II) and (III) respectively. Also, Cases (II) and (III) can be viewed as two different embedding of the $\mathbb{R}$-valued probabilistic measures into $\mathfrak{H}$-valued probabilistic measures. We can associate such $\mathbb{R}$-valued measures with our newly developed $\mathfrak{H}$-valued probabilistic measure which only takes zero divisors as its values.

3.2 Lemmas.

Lemma 3.2.1 *The sets* $\mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_+} = \{r_1\mathbf{e}_+ : r_1 \in \mathbb{R}\} = \mathbb{R}\mathbf{e}_+$ *and* $\mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_-} = \{r_2\mathbf{e}_- : r_2 \in \mathbb{R}\} = \mathbb{R}\mathbf{e}_-$ *satisfy the following properties*

a. $\mu \in \mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_+} \Longleftrightarrow \mu\mathbf{e}_+ = \mu$;

b. $\nu \in \mathcal{P}^{\mathfrak{h}}_{\mathbf{e}_-} \Longleftrightarrow \nu\mathbf{e}_- = \nu$.

Lemma 3.2.2 *Let us consider a set* $\mathcal{P}^{\mathfrak{h}+} = \{z_+\mathbf{e}_+ + z_-\mathbf{e}_- : z_+, z_- \geqslant 0\}$ *and a relation* $\preccurlyeq$ *on* $\mathcal{P}^{\mathfrak{h}}$ *in such a way that for any two hyperbolic hybrid numbers* ζ_1 *and* ζ_2, $\zeta_1 \preccurlyeq \zeta_2$ *if and only if* $\zeta_2 - \zeta_1 \in \mathcal{P}^{\mathfrak{h}+}$.

Clearly this relation is a reflexive, antisymmetric and transitive relation and hence it is a poset (partial order relation) on $\mathcal{P}^{\mathfrak{h}}$.

The poset $\preccurlyeq$ have the following properties:
For $\zeta_1, \zeta_2, \zeta_3\zeta_4 \in \mathcal{P}^{\mathfrak{h}}$ and $\xi \in \mathcal{P}^{\mathfrak{h}+}$,

a. If $\zeta_1 \preccurlyeq \zeta_2$ then $\xi\zeta_1 \preccurlyeq \xi\zeta_2$.

b. If $\zeta_1 \preccurlyeq \zeta_2$ and $\zeta_3 \preccurlyeq \zeta_4$ then $\zeta_1 + \zeta_3 \preccurlyeq \zeta_2 + \zeta_4$.

c. If $\zeta_1 \preccurlyeq \zeta_2$ then $-\zeta_2 \preccurlyeq -\zeta_1$.

Now we are in a position to prove our main results.

3.3 Main Results.

In this section, we prove the main results of this chapter.

Theorem 3.3.1 *The $\mathfrak{H}$-valued probability of the null event and a complementary event of $\mathfrak{F} \in \Sigma$ are respectively, 0 and $\mathfrak{p} - \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})$.*

Proof. We know that $\mathfrak{F} \cup \mathfrak{F}^{\complement} = \Omega, \mathfrak{F} \cap \mathfrak{F}^{\complement} = \varnothing$ and so that

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) + \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}^{\complement}) = \mathcal{L}_{\mathfrak{H}}(\Omega) = \mathfrak{p}$$
$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}^{\complement}) = \mathfrak{p} - \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}).$$

Now, $\mathcal{L}_{\mathfrak{H}}(\varnothing) = \mathcal{L}_{\mathfrak{H}}(\Omega^{\complement}) = \mathfrak{p} - P(\Omega) = 0$.
Hence, the result follows. ■

Theorem 3.3.2 *If $\mathfrak{F}_1, \mathfrak{F}_2, ..., \mathfrak{F}_n \in \Sigma$ are n events, then*

$$\begin{aligned}\mathcal{L}_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} \mathfrak{F}_n\right) = \sum_{i=1}^{n} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i) - \sum_{1 \leqslant i < j \leqslant n} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j) \\ + \sum_{1 \leqslant i < j < k \leqslant n} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j \cap \mathfrak{F}_k) \\ - ... + (-1)^{n-1} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2 \cap .... \cap \mathfrak{F}_n).\end{aligned} \tag{3.1}$$

Proof. By the help of mathematical induction we can prove the theorem. Since for $n = 2$, $\mathfrak{F}_1 \cup \mathfrak{F}_2 = \mathfrak{F}_1 \cup (\mathfrak{F}_1{}^{\complement} \cap \mathfrak{F}_2)$ and also $\mathfrak{F}_2 = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \cup (\mathfrak{F}_1{}^{\complement} \cap \mathfrak{F}_2)$, then

$$\begin{aligned}\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1 \cup \mathfrak{F}_2) &= \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1) + \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1{}^{\complement} \cap \mathfrak{F}_2) \\ &= \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1) + \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_2) - \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2).\end{aligned}$$

Therefore, Equation (3.1) holds for n=2.
Now, suppose that Equation (3.1) is true for $n = r$.
i.e,.

$$\begin{aligned}\mathcal{L}_{\mathfrak{H}}\left(\bigcup_{i=1}^{r} \mathfrak{F}_i\right) = \sum_{i=1}^{r} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i) - \sum_{1 \leqslant i < j \leqslant r} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j) \\ + \sum_{1 \leqslant i < j < k \leqslant r} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j \cap \mathfrak{F}_k) \\ - ... + (-1)^{r-1} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2 \cap .... \cap \mathfrak{F}_r).\end{aligned}$$

Then, for $n = r + 1$

$$
\begin{aligned}
&\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cup \ldots \cup \mathfrak{F}_r \cup \mathfrak{F}_{r+1}) \\
=&\pounds_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{r}\mathfrak{F}_i\right] \cup \mathfrak{F}_{r+1}\right) \\
=&\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{r}\mathfrak{F}_i\right) + \pounds_{\mathfrak{H}}(\mathfrak{F}_{r+1}) - \pounds_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{r}\mathfrak{F}_i\right] \cap \mathfrak{F}_{r+1}\right) \\
=&\sum_{i=1}^{r+1}\pounds_{\mathfrak{H}}(\mathfrak{F}_i) - \sum_{1\leqslant i<j\leqslant r}\pounds_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j) + \sum_{1\leqslant i<j<k\leqslant r}\pounds_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j \cap \mathfrak{F}_k) - \cdots + \\
&\quad (-1)^{r-1}\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2 \cap \ldots. \cap \mathfrak{F}_r) - \left[\sum_{i=1}^{r}\pounds_{\mathfrak{H}}(\mathfrak{F}_i\mathfrak{F}_{r+1}) - \sum_{1\leqslant i<j\leqslant r}\pounds_{\mathfrak{H}}(\mathfrak{F}_i\mathfrak{F}_j\mathfrak{F}_{r+1})\right. \\
&\qquad \left. + \cdots + (-1)^{r-1}\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2 \cap \cdots \cap \mathfrak{F}_{r+1})\right] \\
=&\sum_{i=1}^{r+1}\pounds_{\mathfrak{H}}(\mathfrak{F}_i) - \sum_{1\leqslant i<j\leqslant r+1}\pounds_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j) + \sum_{1\leqslant i<j<k\leqslant r+1}\pounds_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j \cap \mathfrak{F}_k) \\
&\quad - \cdots + (-1)^n\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2 \cap \cdots \cap \mathfrak{F}_{r+1}).
\end{aligned}
$$

Thus, Equation (3.1) is also true for $n = r + 1$ and hence this proves the theorem. This results is the extension of 'Addition Theorem' in this probability space. ■

Theorem 3.3.3 *If $\mathfrak{F}, \mathfrak{U} \in \Sigma$ with $\mathfrak{F} \subset \mathfrak{U}$ then*

$$\pounds_{\mathfrak{H}}(\mathfrak{F}) \preccurlyeq \pounds_{\mathfrak{H}}(\mathfrak{U}).$$

i.e., $\pounds_{\mathfrak{H}}(\mathfrak{F})$ and $\pounds_{\mathfrak{H}}(\mathfrak{U})$ are comparable with respect to the partial order $\preccurlyeq$.

Proof. As $\mathfrak{U} = \mathfrak{U} \cap \Omega = \mathfrak{U} \cap (\mathfrak{F} \cup \mathfrak{F}^{\complement}) = (\mathfrak{U} \cap \mathfrak{F}) \cup (\mathfrak{U} \cap \mathfrak{F}^{\complement}) = \mathfrak{F} \cup (\mathfrak{F}^{\complement} \cap \mathfrak{U})$. Since, $\mathfrak{F} \cap (\mathfrak{F}^{\complement} \cap \mathfrak{U}) = \varnothing$, thus, $\pounds_{\mathfrak{H}}(\mathfrak{U}) = \pounds_{\mathfrak{H}}(\mathfrak{F}) + \pounds_{\mathfrak{H}}(\mathfrak{F}^{\complement} \cap \mathfrak{U})$, and also since $\pounds_{\mathfrak{H}}(\mathfrak{F}^{\complement} \cap \mathfrak{U}) \succcurlyeq 0$, so we get $\pounds_{\mathfrak{H}}(\mathfrak{F}) \preccurlyeq \pounds_{\mathfrak{H}}(\mathfrak{U})$. ■

Corollary 3.3.1 *Since, for given $\mathfrak{F} \in \Sigma, \mathfrak{F} \subseteq \Omega$, therefore $\pounds_{\mathfrak{H}}(\mathfrak{F})$ is always comparable with $\pounds_{\mathfrak{H}}(\Omega)$ and also, $\pounds_{\mathfrak{H}}(\mathfrak{F}) \preccurlyeq \pounds_{\mathfrak{H}}(\Omega) = \mathfrak{p}$. Hence, for any $\mathfrak{F} \in \Sigma, 0 \preccurlyeq \pounds_{\mathfrak{H}}(\mathfrak{F}) \preccurlyeq \mathfrak{p}$.*

Corollary 3.3.2 *If $\pounds_{\mathfrak{H}}(\Omega) = \mathbf{e}_+$ then for any random event $\mathfrak{F}$ there exists $\mathfrak{r} \in [0, 1]$ such that $\pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{r}\mathbf{e}_+$. Again, if $\pounds_{\mathfrak{H}}(\Omega) = \mathbf{e}_-$ then for any random events there also exists $\mathfrak{t} \in [0, 1]$ such that $\pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{t}\mathbf{e}_-$.*

Theorem 3.3.4 *Given n events $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ there follows:*

$$\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{n}\mathfrak{F}_i\right) \preccurlyeq \sum_{i=1}^{n}\pounds_{\mathfrak{H}}(\mathfrak{F}_i). \tag{3.2}$$

Proof. Since for any two events $\mathfrak{F}_1$ and $\mathfrak{F}_2$, we have

$$\begin{aligned}\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cup \mathfrak{F}_2) &= \pounds_{\mathfrak{H}}(\mathfrak{F}_1) + \pounds_{\mathfrak{H}}(\mathfrak{F}_2) - \pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2) \\ &\leqslant \pounds_{\mathfrak{H}}(\mathfrak{F}_1) + \pounds_{\mathfrak{H}}(\mathfrak{F}_2) \; [\because \pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2) \geqslant 0]\end{aligned}$$

Thus, the result (3.2) is true for $n = 2$.
Let us suppose that the result (3.2) is true for $n = m$.
Now, for $n = m + 1$,

$$\begin{aligned}\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \ldots \cup \mathfrak{F}_{m+1}) &\leqslant \pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \ldots \cup \mathfrak{F}_m) + \pounds_{\mathfrak{H}}(\mathfrak{F}_{m+1}) \\ &\leqslant \pounds_{\mathfrak{H}}(\mathfrak{F}_1) + \pounds_{\mathfrak{H}}(\mathfrak{F}_2) + \ldots + \pounds_{\mathfrak{H}}(\mathfrak{F}_{m+1}).\end{aligned}$$

Thus, the result (3.2) is true for all $n \in \mathbb{N}$.
This proves the result. ■

Remark 3.3.1 *Theorem 3.3.4 is analogous to 'Boole's inequality' in $\mathfrak{H}$-probabilistic space.*

Theorem 3.3.5 *For any n events $\mathfrak{F}_1$, $\mathfrak{F}_2$,...,$\mathfrak{F}_n$ the following relation holds*

$$\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{n} \mathfrak{F}_i\right) \geqslant \sum_{i=1}^{n} \pounds_{\mathfrak{H}}(\mathfrak{F}_i) - \sum_{1 \leqslant i < j \leqslant n} \pounds_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j). \tag{3.3}$$

Proof. From the addition theorem, we get that

$$\begin{aligned}\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3) &= \pounds_{\mathfrak{H}}(\mathfrak{F}_1) + \pounds_{\mathfrak{H}}(\mathfrak{F}_2) + \pounds_{\mathfrak{H}}(\mathfrak{F}_3) - \pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2) - \\ &\quad \pounds_{\mathfrak{H}}(\mathfrak{F}_2 \cap \mathfrak{F}_3) - \pounds_{\mathfrak{H}}(\mathfrak{F}_3 \cap \mathfrak{F}_1) + \pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2 \cap \mathfrak{F}_3) \\ \pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{3} \mathfrak{F}_i\right) &\geqslant \sum_{i=1}^{3} \pounds_{\mathfrak{H}}(\mathfrak{F}_i) - \sum_{1 \leqslant i < j \leqslant 3} \pounds_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j).\end{aligned}$$

Thus, the result (3.3) is true for $n = 3$.
Now, we will prove the result by the help of mathematical induction.
Let us suppose that the result (3.3) is true for $n = k$, i.e.,

$$\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k} \mathfrak{F}_i\right) \geqslant \sum_{i=1}^{k} \pounds_{\mathfrak{H}}(\mathfrak{F}_i) - \sum_{1 \leqslant i < j \leqslant k} \pounds_{\mathfrak{H}}(\mathfrak{F}_i \cap \mathfrak{F}_j).$$

Now, for $n = k + 1$

$$\begin{aligned}\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1}\mathfrak{F}_i\right) &= \pounds_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{k}\mathfrak{F}_i\right]\cup\mathfrak{F}_{k+1}\right)\\ &= \pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k}\mathfrak{F}_i\right)+\pounds_{\mathfrak{H}}(\mathfrak{F}_{k+1})-\pounds_{\mathfrak{H}}\left(\left[\bigcup_{i=1}^{k}\mathfrak{F}_i\right]\cap\mathfrak{F}_{k+1}\right)\\ &= \pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k}\mathfrak{F}_i\right)+\pounds_{\mathfrak{H}}(\mathfrak{F}_{k+1})-\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k}(\mathfrak{F}_i\cap\mathfrak{F}_{k+1})\right)\\ &\geqslant \left[\sum_{i=1}^{k}\pounds_{\mathfrak{H}}(\mathfrak{F}_i)-\sum_{1\leqslant i<j\leqslant k}\pounds_{\mathfrak{H}}(\mathfrak{F}_i\cap\mathfrak{F}_j)\right]+\pounds_{\mathfrak{H}}(\mathfrak{F}_{k+1})-\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k}(\mathfrak{F}_i\cap\mathfrak{F}_{k+1})\right).\end{aligned}$$

From Theorem 3.3.4, we get that

$$-\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k}(\mathfrak{F}_i\cap\mathfrak{F}_{k+1})\right)\succcurlyeq-\sum_{i=1}^{k}\pounds_{\mathfrak{H}}(\mathfrak{F}_i\cap\mathfrak{F}_{k+1})$$

and therefore,

$$\begin{aligned}\pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1}\mathfrak{F}_i\right) &\succcurlyeq \sum_{i=1}^{k+1}\pounds_{\mathfrak{H}}(\mathfrak{F}_i)-\sum_{1\leqslant i<j\leqslant k}\pounds_{\mathfrak{H}}(\mathfrak{F}_i\cap\mathfrak{F}_j)-\sum_{i=1}^{k}\pounds_{\mathfrak{H}}(\mathfrak{F}_i\cap\mathfrak{F}_{k+1})\\ \pounds_{\mathfrak{H}}\left(\bigcup_{i=1}^{k+1}\mathfrak{F}_i\right) &\succcurlyeq \sum_{i=1}^{k+1}\pounds_{\mathfrak{H}}(\mathfrak{F}_i)-\sum_{1\leqslant i<j\leqslant k+1}\pounds_{\mathfrak{H}}(\mathfrak{F}_i\cap\mathfrak{F}_j).\end{aligned}$$

Thus, the theorem is also true for $n = k + 1$ and hence this proves the theorem. ■

Theorem 3.3.6 *If* $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset ... \subset \mathfrak{F}_n \subset ...$ *and* $\mathfrak{F} := \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup ... \cup \mathfrak{F}_n \cup ...$ *then*

$$\lim_{n\to\infty}\pounds_{\mathfrak{H}}(\mathfrak{F}_n)=\pounds_{\mathfrak{H}}(\mathfrak{F})=\pounds_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty}\mathfrak{F}_n\right).$$

Proof. Since the given sequence of events is expanding, so we get that

$$\bigcup_{i=1}^{n}\mathfrak{F}_i=\mathfrak{F}_n \text{ and } \lim_{n\to\infty}\mathfrak{F}_n=\bigcup_{i=1}^{\infty}\mathfrak{F}_n. \tag{3.4}$$

Now if we take some events as $\mathfrak{U}_1 = \mathfrak{F}_1, \mathfrak{U}_2 = \mathfrak{F}_2 \cap \mathfrak{F}_1^{\complement}, ..., \mathfrak{U}_n = \mathfrak{F}_n \cap \mathfrak{F}_{n-1}^{\complement}$, then $\mathfrak{U}_i \cap \mathfrak{U}_j = \varnothing$, for all $i \neq j$ and $i, j = 1, 2, ..., n$ and hence

$$\bigcup_{n=1}^{\infty}\mathfrak{F}_n=\bigcup_{n=1}^{\infty}\mathfrak{U}_n \text{ and } \pounds_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty}\mathfrak{U}_n\right)=\sum_{n=1}^{\infty}\pounds_{\mathfrak{H}}(\mathfrak{U}_n). \tag{3.5}$$

Now, from (3.4) and (3.5), we obtain that

$$\begin{aligned}
\mathcal{L}_{\mathfrak{H}}\left(\lim_{n\to\infty}\mathfrak{F}_n\right) &= \mathcal{L}_{\mathfrak{H}}\left(\bigcup_{n=1}^{\infty}\mathfrak{F}_n\right)\\
&= \sum_{n=1}^{\infty}\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}_n)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}_i)\\
&= \lim_{n\to\infty}\mathcal{L}_{\mathfrak{H}}\left(\bigcup_{i=1}^{n}\mathfrak{U}_i\right)\\
&= \lim_{n\to\infty}\mathcal{L}_{\mathfrak{H}}\left(\bigcup_{i=1}^{n}\mathfrak{F}_i\right) = \lim_{n\to\infty}\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}).
\end{aligned}$$

This completes the proof. ■

Remark 3.3.2 *We call the theorem as 'Continuity theorem' of the $\mathfrak{H}$-probability.*

Theorem 3.3.7 *For any two events $\mathfrak{F}$ and $\mathfrak{U}$,*

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}^{\complement}\cap\mathfrak{U}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) - \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}).$$

Proof. Since, $\mathfrak{F}^{\complement}\cap\mathfrak{U}$ and $\mathfrak{F}\cap\mathfrak{U}$ are disjoint events and $(\mathfrak{F}^{\complement}\cap\mathfrak{U})\cup(\mathfrak{F}\cap\mathfrak{U}) = \mathfrak{U}$, then

$$\begin{aligned}
\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) &= \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}^{\complement}\cap\mathfrak{U}) + \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})\\
or, \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}^{\complement}\cap\mathfrak{U}) &= \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) - \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}).
\end{aligned}$$

This proves the theorem. ■

Corollary 3.3.3 *For any n events $\mathfrak{F}_1, \mathfrak{F}_2, ..., \mathfrak{F}_n$*

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1\cap\mathfrak{F}_2\cap...\cap\mathfrak{F}_n) \geqslant \mathfrak{p} - \sum_{i=1}^{n}\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i^{\complement}) \tag{3.6}$$

and

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1\cap\mathfrak{F}_2\cap...\cap\mathfrak{F}_n) \geqslant \sum_{i=1}^{n}\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i) - (n-1)\mathfrak{p}. \tag{3.7}$$

Proof. Applying the 'Boole's inequality' on the events $\mathfrak{F}_1^{\complement}, \mathfrak{F}_2^{\complement}, ..., \mathfrak{F}_n^{\complement}$, we get that

$$\begin{aligned}
\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1^{\complement}\cup\mathfrak{F}_2^{\complement}\cup...\cup\mathfrak{F}_n^{\complement}) &\leqslant \sum_{i=1}^{n}\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i^{\complement})\\
or,\ \mathcal{L}_{\mathfrak{H}}[(\mathfrak{F}_1\cap\mathfrak{F}_2\cap...\cap\mathfrak{F}_n)^{\complement}] &\leqslant \sum_{i=1}^{n}\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i^{\complement})\\
or,\ \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1\cap\mathfrak{F}_2\cap...\cap\mathfrak{F}_n) &\geqslant \mathfrak{p} - \sum_{i=1}^{n}\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_i^{\complement}).
\end{aligned} \tag{3.8}$$

Again,

$$\sum_{i=1}^{n} \pounds_{\mathfrak{H}}(\mathfrak{F}_i^{\complement}) = \mathfrak{p} - \pounds_{\mathfrak{H}}(\mathfrak{F}_1)\mathfrak{p} - \pounds_{\mathfrak{H}}(\mathfrak{F}_2) + \ldots\mathfrak{p} - \pounds_{\mathfrak{H}}(\mathfrak{F}_n)$$
$$= n\mathfrak{p} - \sum_{i=1}^{n} \pounds_{\mathfrak{H}}(\mathfrak{F}_i^{\complement}). \tag{3.9}$$

Now, combining Equations (3.8) and (3.9), we obtain that

$$\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2 \cap \ldots \cap \mathfrak{F}_n) \succcurlyeq \sum_{i=1}^{n} \pounds_{\mathfrak{H}}(\mathfrak{F}_i) - (n-1)\mathfrak{p}.$$

This proves the result. ■

Remark 3.3.3 *Equations (3.6) and (3.7) are the 'Bonferroni's inequalities' in $\mathfrak{H}$-valued probabilistic space.*

Corollary 3.3.4 *If $\mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \ldots \supset \mathfrak{F}_n \supset \ldots$ and $\mathfrak{F} := \mathfrak{F}_1 \cap \mathfrak{F}_2 \cap \ldots \cap \mathfrak{F}_n \cap \ldots$ then*

$$\lim_{n\to\infty} \pounds_{\mathfrak{H}}(\mathfrak{F}_n) = \pounds_{\mathfrak{H}}(\mathfrak{F}) = \pounds_{\mathfrak{H}}\left(\bigcap_{n=1}^{\infty} \mathfrak{F}_n\right).$$

3.4 Future Prospects

The works as carried out in this chapter can be extended from the view point of the conditional hyperbolic hybrid valued probabilistic measure in the next chapter.

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CHAPTER FOUR

SOME STUDIES ON THE CONDITIONAL HYPERBOLIC HYBRID VALUED PROBABILISTIC MEASURES

Chapter 4

Some studies on the conditional hyperbolic hybrid valued probabilistic measures

4.1 Preliminaries.

In 18th-century renowned British mathematician *Thomas Bayes* introduced a mathematical formulation known as *'Bayes' Theorem'* for determining conditional probability. The principle of conditional probability is the likelihood of an outcome occurring based on a outcome that has already been occured in similar circumstances.

Definition 4.1.1 *Conditional probability space is a set Ω equipped with a σ-finite measure $\pounds$ defined on a σ-algebra Σ of sets in Ω.*

In this chapter we prove the 'multiplication theorem' and its generalization in the hyperbolic hybrid valued probabilistic measurable space. Also, the extended version of Bayes' theorem and the law of total probability are deduced here in this context.

4.2 Theorems.

In this section, we present the main results of the chapter.
The following definition is relevant.

Definition 4.2.1 *Consider a probabilistic measurable space $\left(\Omega, \Sigma, \pounds_{\mathfrak{H}}\right)$. Let $\mathfrak{F}$ and $\mathfrak{U}$ be two events. The conditional probability $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U})$ of the event $\mathfrak{F}$ under the condition that the event $\mathfrak{U}$ has already been occurred is defined as follows:*

(a) $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})}$ *if* $\pounds_{\mathfrak{H}}(\mathfrak{U}) \succ 0$ *and* $\pounds_{\mathfrak{H}}(\mathfrak{U}) \notin \mathfrak{S}_{\mathfrak{H}}$*; where* $\mathfrak{S}_{\mathfrak{H}}$ *is the set of hyperbolic zero divisor*

(b) $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \pounds_{\mathfrak{H}}(\mathfrak{F})$ *if* $\pounds_{\mathfrak{H}}(\mathfrak{U}) = 0$;

(c) $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}_1}\mathbf{e}_+ + \pounds_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_-$ *if* $\pounds_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{r}_1\mathbf{e}_+, \mathfrak{r}_1 > 0$;

(d) $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \pounds_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_+ + \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}_2}\mathbf{e}_-$ *if* $\pounds_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{r}_2\mathbf{e}_-, \mathfrak{r}_2 > 0$.

Now we shall show that Item (c) and Item (d) are in a complete agreement with Item (a).
For any event $\mathfrak{F}$, we have

$$\pounds_{\mathfrak{H}}(\mathfrak{F}) = \Gamma_1(\mathfrak{F})\mathbf{e}_+ + \Gamma_2(\mathfrak{F})\mathbf{e}_-.$$

Therefore, $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})}$ can be written as

$$\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \frac{\Gamma_1(\mathfrak{F}\cap\mathfrak{U})}{\Gamma_1(\mathfrak{U})}\mathbf{e}_+ + \frac{\Gamma_2(\mathfrak{F}\cap\mathfrak{U})}{\Gamma_2(\mathfrak{U})}\mathbf{e}_- = \Gamma_1(\mathfrak{F}|\mathfrak{U})\mathbf{e}_+ + \Gamma_2(\mathfrak{F}|\mathfrak{U})\mathbf{e}_-,$$

whereas $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}_1}\mathbf{e}_+ + \pounds_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_-$ and $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \pounds_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_+ + \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}_2}\mathbf{e}_-$ can be written as

$$\begin{aligned}
\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) &= \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}_1}\mathbf{e}_+ + \pounds_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_- \\
&= \frac{\Gamma_1(\mathfrak{F}\cap\mathfrak{U})\mathbf{e}_+ + \Gamma_2(\mathfrak{F}\cap\mathfrak{U})\mathbf{e}_-}{\Gamma_1(\mathfrak{U})}\mathbf{e}_+ + (\Gamma_1(\mathfrak{F})\mathbf{e}_+ + \Gamma_2(\mathfrak{F})\mathbf{e}_-)\mathbf{e}_- \\
&= \frac{\Gamma_1(\mathfrak{F}\cap\mathfrak{U})}{\Gamma_1(\mathfrak{U})}\mathbf{e}_+ + \Gamma_2(\mathfrak{F})\mathbf{e}_- \\
&= \Gamma_1(\mathfrak{F}|\mathfrak{U})\mathbf{e}_+ + \Gamma_2(\mathfrak{F}|\mathfrak{U})\mathbf{e}_-
\end{aligned}$$

and

$$\begin{aligned}
\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) &= \pounds_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_+ + \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}_2}\mathbf{e}_- \\
&= (\Gamma_1(\mathfrak{F})\mathbf{e}_+ + \Gamma_2(\mathfrak{F})\mathbf{e}_-)\mathbf{e}_+ + \frac{\Gamma_1(\mathfrak{F}\cap\mathfrak{U})\mathbf{e}_+ + \Gamma_2(\mathfrak{F}\cap\mathfrak{U})\mathbf{e}_-}{\mathfrak{r}_2}\mathbf{e}_- \\
&= \Gamma_1(\mathfrak{F})\mathbf{e}_+ + \frac{\Gamma_2(\mathfrak{F}\cap\mathfrak{U})}{\Gamma_2(\mathfrak{U})}\mathbf{e}_- = \Gamma_1(\mathfrak{F}|\mathfrak{U})\mathbf{e}_+ + \Gamma_2(\mathfrak{F}|\mathfrak{U})\mathbf{e}_-.
\end{aligned}$$

Therefore, it is clear that (a) is completely agree with (c) and (d).

Now, we prove a theorem focusing the measurable space $(\mathfrak{U}, \Sigma_{\mathfrak{U}})$ as a $\mathfrak{H}$-valued probabilistic space.

Theorem 4.2.1 *For a fixed event* $\mathfrak{U}$*, with* $\pounds_{\mathfrak{H}}(\mathfrak{U}) \neq 0$*, the measurable space* $(\mathfrak{U}, \Sigma_{\mathfrak{U}})$ *where* $\Sigma_{\mathfrak{U}}$ *is the* σ*–algebra of the set of events of the form* $\mathfrak{F}\cap\mathfrak{U}$ *with* $\mathfrak{F}\in\Sigma$ *forms a* $\mathfrak{H}$*-valued probabilistic space under the measure* $\pounds_{\mathfrak{H}}(.|\mathfrak{U})$*.*

Proof. It is clear that $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) \geqslant 0$.
Next, we show that $\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{U}) = \mathfrak{p}$.
Case-I: If $\pounds_{\mathfrak{H}}(\mathfrak{U}) \notin \mathfrak{S}_{\mathfrak{H}}$, then

$$\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{U}) = \frac{\pounds_{\mathfrak{H}}(\mathfrak{U} \cap \mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})} = \frac{\pounds_{\mathfrak{H}}(\mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})} = 1.$$

Case-II: If $\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{U}) = \mathfrak{r}_1\mathbf{e}_+$, then

$$\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{U}) = \frac{\pounds_{\mathfrak{H}}(\mathfrak{U} \cap \mathfrak{U})}{\mathfrak{r}_1}\mathbf{e}_+ + \pounds_{\mathfrak{H}}(\mathfrak{U})\mathbf{e}_- = \frac{\pounds_{\mathfrak{H}}(\mathfrak{U})}{\mathfrak{r}_1}\mathbf{e}_+ = \mathbf{e}_+.$$

Case-III: If $\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{U}) = \mathfrak{r}_2\mathbf{e}_-$, then

$$\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{U}) = \pounds_{\mathfrak{H}}(\mathfrak{U})\mathbf{e}_+ + \frac{\pounds_{\mathfrak{H}}(\mathfrak{U} \cap \mathfrak{U})}{\mathfrak{r}_2}\mathbf{e}_- = \frac{\pounds_{\mathfrak{H}}(\mathfrak{U})}{\mathfrak{r}_2}\mathbf{e}_- = \mathbf{e}_-.$$

Now, let $\mathfrak{F} = \bigcup_{k=1}^{\infty} \mathfrak{F}_k$ with $\mathfrak{F}_i \cap \mathfrak{F}_j = \varnothing$ for $i \neq j$, then
Case-I: If $\pounds_{\mathfrak{H}}(\mathfrak{U}) \notin \mathfrak{S}_{\mathfrak{H}}$, then

$$\begin{aligned}\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) &= \frac{\pounds_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})} = \frac{\pounds_{\mathfrak{H}}(\bigcup_{k=1}^{\infty} \mathfrak{F}_k \cap \mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})} \\ &= \frac{\sum_{k=1}^{\infty} \pounds_{\mathfrak{H}}(\mathfrak{F}_k \cap \mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})} = \sum_{k=1}^{\infty} \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}_k \cap \mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})} \\ &= \sum_{k=1}^{\infty} \pounds_{\mathfrak{H}}(\mathfrak{F}_k|\mathfrak{U}).\end{aligned}$$

Case-II: If $\pounds_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{r}_1\mathbf{e}_+$, since for any k, $\mathfrak{F}_k \cap \mathfrak{U} \subset \mathfrak{U}$ and as $\mathfrak{F} \cap \mathfrak{U} \subset \mathfrak{U}$.
Now if $\pounds_{\mathfrak{H}}(\mathfrak{F}_k \cap \mathfrak{U}) = \mathfrak{s}_k\mathbf{e}_+$ then

$$\begin{aligned}\pounds_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) &= \mathfrak{s}\mathbf{e}_+ = \Gamma_1(\mathfrak{F} \cap \mathfrak{U})\mathbf{e}_+ = \Gamma_1\left(\bigcup_{k=1}^{\infty} \mathfrak{F}_k \cap \mathfrak{U}\right)\mathbf{e}_+ \\ &= \sum_{n=1}^{\infty} \Gamma_1(\mathfrak{F}_k \cap \mathfrak{U})\mathbf{e}_+ = \sum_{n=1}^{\infty} \mathfrak{s}_k\mathbf{e}_+.\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) &= \frac{\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}_1}\mathbf{e}_+ + \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_- = \frac{\mathfrak{s}}{\mathfrak{r}_1}\mathbf{e}_+ + \Gamma_2(\mathfrak{F})\mathbf{e}_- \\
&= \frac{1}{\mathfrak{r}_1}\sum_{k=1}^{\infty}\mathfrak{s}_k\mathbf{e}_+ + \sum_{k=1}^{\infty}\Gamma_2(\mathfrak{F}_k)\mathbf{e}_- \\
&= \sum_{k=1}^{\infty}\left(\frac{\mathfrak{s}_k}{\mathfrak{r}_1}\mathbf{e}_+ + \Gamma_2(\mathfrak{F}_k)\mathbf{e}_-\right) \\
&= \sum_{k=1}^{\infty}\big(\Gamma_1(\mathfrak{F}_k|\mathfrak{U})\mathbf{e}_+ + \Gamma_2(\mathfrak{F}_k)\mathbf{e}_-\big) \\
&= \sum_{k=1}^{\infty}\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_k|\mathfrak{U}).
\end{aligned}$$

Similarly, we can show for $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{r}_2\mathbf{e}_-$.

Therefore, $\big(\mathfrak{U}, \Sigma_{\mathfrak{U}}, \mathcal{L}_{\mathfrak{H}}(.|\mathfrak{U})\big)$ is a probabilistic space. ■

The following theorem ensures the multiplication rule for the conditional probability in a hyperbolic hybrid valued probabilistic space.

Theorem 4.2.2 *If $\mathfrak{F}$ and $\mathfrak{U}$ be two events of the probabilistic space $\big(\Omega, \Sigma, \mathcal{L}_{\mathfrak{H}}\big)$, then*

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{U})\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}).$$

Proof. To prove the theorem the following cases arise.
Case-I: If $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) \succ 0$ and $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) \notin \mathfrak{S}_{\mathfrak{H}}$, then

$$\begin{aligned}
\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) &= \frac{\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathcal{L}_{\mathfrak{H}}(\mathfrak{U})} \\
\textit{or,}\ \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}) &= \mathcal{L}_{\mathfrak{H}}(\mathfrak{U})\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}).
\end{aligned}$$

Case-II: Let $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) = 0$.
As $\mathfrak{F}\cap\mathfrak{U}\subset\mathfrak{U}$, then $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}) = 0$.

$$\therefore\ \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{U})\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}).$$

Case-III: Suppose $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{r}_1\mathbf{e}_+$ such that $\mathfrak{r}_1 > 0$, then

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \frac{\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}_1}\mathbf{e}_+ + \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_-,$$

therefore

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{U})\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \mathfrak{r}_1\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U})\mathbf{e}_+ = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})\mathbf{e}_+. \tag{4.1}$$

As $\mathfrak{F} \cap \mathfrak{U} \subset \mathfrak{U}$, we get that

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) = \Gamma_1(\mathfrak{F} \cap \mathfrak{U})\mathbf{e}_+ + 0.\mathbf{e}_- = \Gamma_1(\mathfrak{F} \cap \mathfrak{U})\mathbf{e}_+.$$

Therefore Equation (4.1) can be written as

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{U})\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \Gamma_1(\mathfrak{F} \cap \mathfrak{U})\mathbf{e}_+ = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}).$$

Case-IV: In a similar way one can conclude for $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{r}_2\mathbf{e}_-$ with $\mathfrak{r}_2 > 0$. ■
We call this theorem as "*Multiplication Theorem*" in $\mathfrak{H}$-valued probabilistic space. Now, we wish to extend Theorem 4.3.2 for n events.

Theorem 4.2.3 *If n random events $\mathfrak{F}_1, \mathfrak{F}_2, ..., \mathfrak{F}_n$ satisfy any one of the following conditions:*
(i) $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1 \cap \cdots \cap \mathfrak{F}_n)$ is not a zero-divisor.
(ii) (a) There exists $\tau_0 \in \{1, \ldots, n\}$ such that $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_{\tau_0}) = \mathfrak{t}_{\tau_0}\mathbf{e}_+$, with $\mathfrak{t}_{\tau_0} > 0$, it implies that, $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_{\tau_0})$ is a zero-divisor in $\mathfrak{H}^+_{\mathbf{e}_+}$ and also
(b) $\mathcal{L}_{\mathfrak{H}}\left(\bigcap_{l=1}^n \mathfrak{F}_l\right)$ belongs to $\mathfrak{H}^+_{\mathbf{e}_+}$.
(iii) (a) There exists $\tau_0 \in \{1, \ldots, n\}$ such that $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_{\tau_0}) = \mathfrak{r}_{\tau_0}\mathbf{e}_-$, with $\mathfrak{r}_{\tau_0} > 0$, it implies that, $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_{\tau_0})$ is a zero-divisor in $\mathfrak{H}^+_{\mathbf{e}_-}$ and also
(b) $\mathcal{L}_{\mathfrak{H}}\left(\bigcap_{l=1}^n \mathfrak{F}_l\right)$ belongs to $\mathfrak{H}^+_{e_-}$.
Then

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1 \cap \cdots \cap \mathfrak{F}_n) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_1)\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_2|\mathfrak{F}_1)\ldots\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_n|\mathfrak{F}_1 \cap \cdots \cap \mathfrak{F}_{n-1}). \tag{4.2}$$

Proof. Let us first consider Condition (i) holds. Since

$$\bigcap_{i=1}^{n-1} \mathfrak{F}_i \subset \bigcap_{i=1}^{n-1} \mathfrak{F}_i \subset \cdots \subset \mathfrak{F}_1 \tag{4.3}$$

and as

$$\mathcal{L}_{\mathfrak{H}}\Big(\bigcap_{l=1}^n \mathfrak{F}_l\Big) = \Gamma_1\Big(\bigcap_{l=1}^n \mathfrak{F}_l\Big)\mathbf{e}_+ + \Gamma_2\Big(\bigcap_{l=1}^n \mathfrak{F}_l\Big)\mathbf{e}_- \notin \mathfrak{S}_{\mathfrak{H},0}$$

we get $\Gamma_1\big(\bigcap_{l=1}^n \mathfrak{F}_l\big) > 0$ and $\Gamma_2\big(\bigcap_{l=1}^n \mathfrak{F}_l\big) > 0$ which implies $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}_l) \notin \mathfrak{S}_{\mathfrak{H},0}$ for all $l \in \{1, \ldots, n\}$ and also for all $k \in \{1, \ldots, n-1\}$, $\mathcal{L}_{\mathfrak{H}}\big(\bigcap_{i=1}^{n-k} \mathfrak{F}_i\big)$ is a strictly positive hyperbolic hybrid number.
Therefore, all conditional probabilities $\mathcal{L}_{\mathfrak{H}}\big(\mathfrak{F}_k|\bigcap_{i=1}^{k-1} \mathfrak{F}_i\big)$, where $k \in \{2, \ldots, n\}$ are well-defined, which leads to

$$\begin{aligned}\mathcal{L}_{\mathfrak{H}}\left(\bigcap_{l=1}^n \mathfrak{F}_l\right) &= \Gamma_1(\mathfrak{F}_1)\Gamma_1(\mathfrak{F}_2|\mathfrak{F}_1)\ldots\Gamma_1\left(\mathfrak{F}_n|\bigcap_{l=1}^{n-1} \mathfrak{F}_l\right)\mathbf{e}_+ \\ &\quad + \Gamma_2(\mathfrak{F}_1)\Gamma_2(\mathfrak{F}_2|\mathfrak{F}_1)\ldots\Gamma_2\left(\mathfrak{F}_n|\bigcap_{l=1}^{n-1} \mathfrak{F}_l\right)\mathbf{e}_-,\end{aligned}$$

and therefore (4.2) agrees.

Now, let Condition (ii) holds.

From the hypothesis that $\pounds_{\mathfrak{H}}\left(\bigcap_{l=1}^{n}\mathfrak{F}_l\right)$ is a positive zero-divisor, let us assume $\pounds_{\mathfrak{H}}\left(\bigcap_{l=1}^{n}\mathfrak{F}_l\right) = \mathfrak{t}\mathbf{e}_+, \mathfrak{t} > 0$, which implies $\Gamma_1\left(\bigcap_{l=1}^{n}\mathfrak{F}_l\right) = \mathfrak{t} > 0$.

Suppose that $\tau_0 = \min\{0, 1, \dots, n\}$ such that $\pounds_{\mathfrak{H}}(\mathfrak{F}_{\tau_0})$ is a zero divisor which implies $\pounds_{\mathfrak{H}}\left(\bigcap_{l=1}^{k}\mathfrak{F}_l\right) = \mathfrak{s}_k\mathbf{e}_+$, then

$$\pounds_{\mathfrak{H}}\left(\mathfrak{F}_{k+1}|\bigcap_{l=1}^{k}\mathfrak{F}_l\right) = \frac{\pounds_{\mathfrak{H}}\left(\bigcap_{l=1}^{k+1}\mathfrak{F}_l\right)}{\mathfrak{s}_k}\mathbf{e}_+ + \pounds_{\mathfrak{H}}(\mathfrak{F}_{k+1})\mathbf{e}_- \ for\ k \geqslant \tau_0.$$

Also, we have

$$\pounds_{\mathfrak{H}}\left(\mathfrak{F}_{\tau_0}|\bigcap_{l=1}^{\tau_0-1}\mathfrak{F}_l\right) = \frac{\pounds_{\mathfrak{H}}\left(\bigcap_{l=1}^{\tau_0}\mathfrak{F}_l\right)}{\pounds_{\mathfrak{H}}\left(\bigcap_{l=1}^{\tau_0-1}\mathfrak{F}_l\right)} = \frac{\mathfrak{s}_k\mathbf{e}_+}{\pounds_{\mathfrak{H}}\left(\mathfrak{F}_{\tau_0}|\bigcap_{l=1}^{\tau_0-1}\mathfrak{F}_l\right)} = \frac{\mathfrak{s}_k}{\Gamma_1\left(\bigcap_{l=1}^{\tau_0-1}\mathfrak{F}_l\right)}\mathbf{e}_+.$$

Hence

$$\begin{aligned}
&\pounds_{\mathfrak{H}}(\mathfrak{F}_1)\pounds_{\mathfrak{H}}(\mathfrak{F}_2|\mathfrak{F}_1)\dots\pounds_{\mathfrak{H}}\left(\mathfrak{F}_{\tau_0}|\bigcap_{l=1}^{\tau_0-1}\mathfrak{F}_l\right)\dots\pounds_{\mathfrak{H}}\left(\mathfrak{F}_n|\bigcap_{l=1}^{n-1}\mathfrak{F}_l\right)\\
=&\Gamma_1(\mathfrak{F}_1)\Gamma_1(\mathfrak{F}_2|\mathfrak{F}_1)\dots\Gamma_1\left(\mathfrak{F}_{\tau_0}|\bigcap_{l=1}^{\tau_0-1}\mathfrak{F}_l\right)\dots\Gamma_1\left(\mathfrak{F}_n|\bigcap_{l=1}^{n-1}\mathfrak{F}_l\right)\mathbf{e}_+\\
=&\Gamma_1(\mathfrak{F}_1\cap\mathfrak{F}_2\cap\dots\cap\mathfrak{F}_n)\mathbf{e}_+\\
=&\pounds_{\mathfrak{H}}(\mathfrak{F}_1\cap\mathfrak{F}_2\cap\dots\cap\mathfrak{F}_n).
\end{aligned}$$

By similar argument, (4.2) follows for the Condition (iii). ∎

Remark 4.2.1 *The above result is the generalization of 'Multipication Theorem' in hyperbolic hybrid valued probabilistic space.*

Definition 4.2.2 *Let $\mathfrak{F}$ and $\mathfrak{U}$ be two random events.*

1. *Event $\mathfrak{F}$ is called independent of the event $\mathfrak{U}$ if $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \pounds_{\mathfrak{H}}(\mathfrak{F})$.*
2. *Event $\mathfrak{U}$ is called independent of the event $\mathfrak{F}$ if $\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{F}) = \pounds_{\mathfrak{H}}(\mathfrak{U})$.*
3. *$\mathfrak{F}$ and $\mathfrak{U}$ are called mutually independent if $\mathfrak{F}$ is independent of $\mathfrak{U}$ and $\mathfrak{U}$ is independent of $\mathfrak{F}$.*

Let us take into account all possible cases.

(i) Suppose that at least one of these two probabilities $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F})$ and $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U})$ equals zero, $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) = 0$ (say) then $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) = 0$, which leads to

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{F}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) \textit{ and } \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = 0 = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}).$$

Therefore, the events $\mathfrak{F}$ and $\mathfrak{U}$ are mutually independent.
If $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) = 0$, then by definition, $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})$ and $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{F}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{U})$, thus $\mathfrak{F}$ and $\mathfrak{U}$ are mutually independent.
Therefore, in any scenario it agrees with $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathcal{L}_{\mathfrak{H}}(\mathfrak{U})$.

(ii) Assume that both probabilities $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F})$ and $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U})$ are not in $\mathfrak{S}_{\mathfrak{H},0}$.

Now the event $\mathfrak{F}$ is independent of the event $\mathfrak{U}$ implying

$$\begin{aligned} \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) &= \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) = \frac{\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U})}{\mathcal{L}_{\mathfrak{H}}(\mathfrak{U})} \\ \textit{i.e.}, \ \mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) &= \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}).\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) \\ \textit{i.e.}, \ \mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) &= \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}).\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}). \end{aligned}$$

From the above, one can conclude that event $\mathfrak{F}$ is independent of event $\mathfrak{U}$ if and only if event $\mathfrak{U}$ is independent of event $\mathfrak{F}$ and therefore both events $\mathfrak{F}$ and $\mathfrak{U}$ are mutually independent.

(iii) Let us consider both probabilities $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F})$ and $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U})$ be zero–divisors and both of which belong to $\mathfrak{H}_{\mathbf{e}_+}$. So, there exist two positive real numbers $\mathfrak{r}$ and $\mathfrak{t}$ such that $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{r}\mathbf{e}_+$ and $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{t}\mathbf{e}_+$, therefore we get $\Gamma_1(\mathfrak{F}) = \mathfrak{r}$ and $\Gamma_1(\mathfrak{U}) = \mathfrak{t}$ whereas $\Gamma_2(\mathfrak{F}) = \Gamma_2(\mathfrak{U}) = 0$ which implies that $0 \leqslant \Gamma_1(\mathfrak{F} \cap \mathfrak{U}) = \mathfrak{s}$ and $\Gamma_2(\mathfrak{F} \cap \mathfrak{U}) = 0$, thus $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) = \mathfrak{s}\mathbf{e}_+$.
Let $\mathfrak{F}$ is independent of $\mathfrak{U}$; then

$$\begin{aligned} \mathfrak{r}\mathbf{e}_+ = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) &= \frac{\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U})}{\mathfrak{t}}\mathbf{e}_+ + \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_- \\ &= \frac{\Gamma_1(\mathfrak{F} \cap \mathfrak{U})}{\mathfrak{t}}\mathbf{e}_+ = \frac{\mathfrak{s}}{\mathfrak{t}}\mathbf{e}_+; \end{aligned} \tag{4.4}$$

therefore, $\mathfrak{F}$ is independent of $\mathfrak{U}$ if and only if $\mathfrak{r} = \frac{\mathfrak{s}}{\mathfrak{t}}$.

Suppose, $\mathfrak{r} = \frac{\mathfrak{s}}{\mathfrak{t}}$. Then we get that

$$\begin{aligned} \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{F}) &= \frac{\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U})}{\mathfrak{r}}\mathbf{e}_+ + \mathcal{L}_{\mathfrak{H}}(\mathfrak{U})\mathbf{e}_- = \frac{\mathfrak{s}}{\mathfrak{r}}\mathbf{e}_+ = \frac{\mathfrak{s}}{\mathfrak{s}/\mathfrak{t}}\mathbf{e}_+ \\ &= \mathfrak{t}\mathbf{e}_+ = \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}); \end{aligned} \tag{4.5}$$

this leads us to the fact that $\mathfrak{U}$ is independent of $\mathfrak{F}$, and hence events $\mathfrak{F}$ and $\mathfrak{U}$ are mutually independent.

Now from any one of (4.4) and (4.5), we can say that for independent events $\mathfrak{F}$ and $\mathfrak{U}$

$$\begin{aligned}\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}) =& \pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})\mathbf{e}_{+} = \mathfrak{r}\mathfrak{t}\mathbf{e}_{+} = (\mathfrak{r}\mathbf{e}_{+})(\mathfrak{t}\mathbf{e}_{+})\\ =& \pounds_{\mathfrak{H}}(\mathfrak{F})\pounds_{\mathfrak{H}}(\mathfrak{U}).\end{aligned}$$

The case where both probabilities $\pounds_{\mathfrak{H}}(\mathfrak{F})$ and $\pounds_{\mathfrak{H}}(\mathfrak{U})$ are in $\mathfrak{H}_{\mathbf{e}_{-}}$ is also similar.

(iv) Let $\pounds_{\mathfrak{H}}(\mathfrak{F})$ and $\pounds_{\mathfrak{H}}(\mathfrak{U})$ be both zero-divisors such that $\pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{r}\mathbf{e}_{+} \neq 0$ and $\pounds_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{t}\mathbf{e}_{-} \neq 0$, and conversely.

As $\mathfrak{F}\cap\mathfrak{U} \subset \mathfrak{F}$ and $\mathfrak{F}\cap\mathfrak{U} \subset \mathfrak{U}$, we have $\Gamma_1(\mathfrak{F}\cap\mathfrak{U}) = \Gamma_2(\mathfrak{F}\cap\mathfrak{U}) = 0$.

Thus, $\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}) = 0$.

Now,

$$\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{t}}\mathbf{e}_{-} + \pounds_{\mathfrak{H}}(\mathfrak{F})\mathbf{e}_{-} = \mathfrak{r}\mathbf{e}_{+} = \pounds_{\mathfrak{H}}(\mathfrak{F}),$$

which implies that $\mathfrak{F}$ is independent of $\mathfrak{U}$. Again

$$\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{F}) = \frac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}}\mathbf{e}_{+} + \pounds_{\mathfrak{H}}(\mathfrak{U})\mathbf{e}_{-} = \pounds_{\mathfrak{H}}(\mathfrak{U})\mathbf{e}_{-} = \mathfrak{t}\mathbf{e}_{-} = \pounds_{\mathfrak{H}}(\mathfrak{U}),$$

Hence, event $\mathfrak{U}$ is independent of event $\mathfrak{F}$.
Therefore events $\mathfrak{F}$ and $\mathfrak{U}$ are always mutually independent with

$$\pounds_{\mathfrak{H}}(\mathfrak{F})\pounds_{\mathfrak{H}}(\mathfrak{U}) = (\mathfrak{r}\mathbf{e}_{+})(\mathfrak{t}\mathbf{e}_{-}) = 0 = \pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U}).$$

(v) Finally, assumes that any one of the events $\pounds_{\mathfrak{H}}(\mathfrak{F})$ and $\pounds_{\mathfrak{H}}(\mathfrak{U})$is zero–divisor and another is invertable hyperbolic hybrid number.

Let $\pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{r}\mathbf{e}_{+} \neq 0, \pounds_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{t}_1\mathbf{e}_{+} + \mathfrak{t}_2\mathbf{e}_{-} \notin \mathfrak{S}_{\mathfrak{H},0}$.

Therfore

$$\mathfrak{s} := \Gamma_1(\mathfrak{F}\cap\mathfrak{U}) \geqslant 0 \; and \; \Gamma_2(\mathfrak{F}\cap\mathfrak{U}) = 0.$$

Thus, $\mathfrak{F}$ is independent of $\mathfrak{U}$ which implies

$$\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathfrak{U}) = \tfrac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\pounds_{\mathfrak{H}}(\mathfrak{U})} = \tfrac{\mathfrak{s}\mathbf{e}_{+}}{\mathfrak{t}_1\mathbf{e}_{+}+\mathfrak{t}_2\mathbf{e}_{-}} = \tfrac{\mathfrak{s}}{\mathfrak{t}_1}\mathbf{e}_{+} = \pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{r}\mathbf{e}_{+},$$

$$\implies \mathfrak{r} = \frac{\mathfrak{s}}{\mathfrak{t}_1}.$$

Now,

$$\pounds_{\mathfrak{H}}(\mathfrak{U}|\mathfrak{F}) = \tfrac{\pounds_{\mathfrak{H}}(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}}\mathbf{e}_{+} + \pounds_{\mathfrak{H}}(\mathfrak{U})\mathbf{e}_{-} = \tfrac{\Gamma_1(\mathfrak{F}\cap\mathfrak{U})}{\mathfrak{r}}\mathbf{e}_{+} + \Gamma_2(\mathfrak{U})\mathbf{e}_{-}$$

$$= \frac{\mathfrak{s}}{\mathfrak{r}}\mathbf{e}_+ + \Gamma_2(\mathfrak{U})\mathbf{e}_- = \mathfrak{t}_1\mathbf{e}_+ + \mathfrak{t}_2\mathbf{e}_- = \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}),$$

therefore, $\mathfrak{U}$ is independent of $\mathfrak{F}$.
Actually, $\mathfrak{F}$ and $\mathfrak{U}$ are mutually independent if and only if one of them is independent of the another one.
Thus from the above we can say that one has for independent events that $\mathfrak{s} = \mathfrak{r}\mathfrak{t}_1$, therefore

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) = \mathfrak{s}\mathbf{e}_+ = (\mathfrak{r}\mathbf{e}_+)(\mathfrak{t}_1\mathbf{e}_+) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}).$$

From the above, we can state the following corollaries.

Corollary 4.2.1 *Let $\mathfrak{F}$ and $\mathfrak{U}$ be two random events. Then $\mathfrak{F}$ is independent of $\mathfrak{U}$ if and only if $\mathfrak{U}$ is independent of $\mathfrak{F}$.*

Corollary 4.2.2 *For any two mutually independent events $\mathfrak{F}$ and $\mathfrak{U}$, the multiplication theorem takes the form*

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}).$$

Theorem 4.2.4 *If $\mathfrak{F}$ and $\mathfrak{U}$ are mutually independent events then $\mathfrak{F}$ and $\mathfrak{U}^{\complement}$, $\mathfrak{F}^{\complement}$ and $\mathfrak{U}$, $\mathfrak{F}^{\complement}$ and $\mathfrak{U}^{\complement}$ are also.*

Proof. We prove it for $\mathfrak{F}$ and $\mathfrak{U}^{\complement}$ elsewhere other cases can be proved in similar manner.
As $\mathfrak{F} = (\mathfrak{F} \cap \mathfrak{U}) \cup (\mathfrak{F} \cap \mathfrak{U}^{\complement})$, $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) + \mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}^{\complement})$ and thus,

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}^{\complement}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) - \mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) - \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}). \tag{4.6}$$

Also, we can factorize $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F})$ but the consequent computation depends on the value of $\mathcal{L}_{\mathfrak{H}}(\Omega)$ and therefore, the following cases may arise:

(1) If $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) \notin \mathfrak{S}_{\mathfrak{H},0}$ then $\mathcal{L}_{\mathfrak{H}}(\Omega) \notin \mathfrak{S}_{\mathfrak{H},0}$ and therefore $\mathfrak{p} = 1$. Thus we have

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})(1 - \mathcal{L}_{\mathfrak{H}}(\mathfrak{U})) = \mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}),$$

which implies $\mathfrak{F}$ and $\mathfrak{U}^{\complement}$ are mutually independent.

(2) Let $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F})\mathfrak{r}\mathbf{e}_+ \in \mathfrak{S}_{\mathfrak{H},0}$, as $\mathfrak{F} \cap \mathfrak{U}^{\complement} \subset \mathfrak{F}$, then $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}^{\complement}) = \mathfrak{s}\mathbf{e}_+$ for some $\mathfrak{s} \geqslant 0$; consider $\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{t}_1\mathbf{e}_+ + \mathfrak{t}_2\mathbf{e}_-$, which leads us to the following subcases:

(a) If $\mathcal{L}_{\mathfrak{H}}(\Omega) = 1 \notin \mathfrak{S}_{\mathfrak{H},0}$ then

$$\mathcal{L}_{\mathfrak{H}}(\mathfrak{U}^{\complement}) = 1 - \mathcal{L}_{\mathfrak{H}}(\mathfrak{U}) = (1 - \mathfrak{t}_1)\mathbf{e}_+ + (1 - \mathfrak{t}_2)\mathbf{e}_-,$$

thus,

$$\mathfrak{s}\mathbf{e}_+ = \pounds_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}^{\complement}) = \mathfrak{r}\mathbf{e}_+ - \mathfrak{r}\mathbf{e}_+(\mathfrak{t}_1\mathbf{e}_+ + \mathfrak{t}_2\mathbf{e}_-) = \mathfrak{r}\mathbf{e}_+ - \mathfrak{r}\mathbf{e}_+(\mathfrak{t}_1\mathbf{e}_+)$$
$$= \mathfrak{r}(1 - \mathfrak{t}_1)\mathbf{e}_+ = (\mathfrak{r}\mathbf{e}_+)((1 - \mathfrak{t}_1)\mathbf{e}_+) = \pounds_{\mathfrak{H}}(\mathfrak{F})\pounds_{\mathfrak{H}}(\mathfrak{U}^{\complement}).$$

Therefore, $\mathfrak{F}$ and $\mathfrak{U}^{\complement}$ are mutually independent.

(b) Let $\pounds_{\mathfrak{H}}(\Omega) = \mathbf{e}_+$ (It is obvious, $\pounds_{\mathfrak{H}}(\Omega) = \mathbf{e}_-$ is impossible) then it is necessarily $\pounds_{\mathfrak{H}}(\mathfrak{U}) = \mathfrak{t}_1\mathbf{e}_+$ and $\pounds_{\mathfrak{H}}(\mathfrak{U}^{\complement}) = (1 - \mathfrak{t}_1)\mathbf{e}_+$, hence

$$\pounds_{\mathfrak{H}}(\mathfrak{F} \cap \mathfrak{U}^{\complement}) = \mathfrak{r}\mathbf{e}_+ - \mathfrak{r}\mathbf{e}_+\mathfrak{t}_1\mathbf{e}_+ = (\mathfrak{r}\mathbf{e}_+)(1 - \mathfrak{t})\mathbf{e}_+ = \pounds_{\mathfrak{H}}(\mathfrak{F})\pounds_{\mathfrak{H}}(\mathfrak{U}^{\complement}).$$

Therefore, $\mathfrak{F}$ & $\mathfrak{U}^{\complement}$ are mutually independent.

(3) In similar way the case $\pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{r}_2\mathbf{e}_-$ can be treated easily. ■

Definition 4.2.3 *Let $\mathfrak{F}_1, \mathfrak{F}_2 ..., \mathfrak{F}_n$ be n random events, are said to be mutually (or jointly) independent if for any subset of indices $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ such that $1 \leqslant i_1 < i_2 < .... < i_m \leqslant n$ ($m \in \{2, \ldots, n\}$) the following equality holds:*

$$\pounds_{\mathfrak{H}}(\mathfrak{F}_{i_1} \cap ... \cap \mathfrak{F}_{i_m}) = \pounds_{\mathfrak{H}}(\mathfrak{F}_{i_1}) ... \pounds_{\mathfrak{H}}(\mathfrak{F}_{i_m}).$$

If the equality holds for $m = 2$ only, then the random events are called pair–wise independent. In general, pairwise independence and joint independence of events are different notions.

If $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_n$ are mutually independent events then the general multiplication theorem will be

$$\pounds_{\mathfrak{H}}(\mathfrak{F}_1 \cap \mathfrak{F}_2 \cap \cdots \cap \mathfrak{F}_n) = \pounds_{\mathfrak{H}}(\mathfrak{F}_1)\pounds_{\mathfrak{H}}(\mathfrak{F}_2) \ldots \pounds_{\mathfrak{H}}(\mathfrak{F}_n).$$

Definition 4.2.4 *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ be pairwise disjoint random events with positive (not necessarily strictly) probabilities in a $\mathfrak{H}$-valued probabilistic space $(\Omega, \Sigma, \pounds_{\mathfrak{H}})$ such that $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_n = \Omega$. Then $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n\}$ is called a fundamental (or complete) system of events (FSE).*

Theorem 4.2.5 *Let $\mathfrak{F}$ be a random event and $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n\}$ be a fundamental system of events in $\mathfrak{H}$-valued probabilistic space $(\Omega, \Sigma, \pounds_{\mathfrak{H}})$. Then*

$$\pounds_{\mathfrak{H}}(\mathfrak{F}) = \textstyle\sum_{i=1}^{n} \pounds_{\mathfrak{H}}(H_i)\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_i).$$

Proof. Since $\mathfrak{F} = \mathfrak{F} \cap \Omega = \mathfrak{F} \cap \Big(\cup_{i=1}^{n} \mathcal{F}_i \Big) = \cup_{i=1}^{n} \mathfrak{F} \cap \mathcal{F}_i$ and the events $\mathfrak{F} \cap \mathcal{F}_i$ are pairwise disjoint then

$$\pounds_{\mathfrak{H}}(\mathfrak{F}) = \sum_{i=1}^{n} \pounds_{\mathfrak{H}}(\mathfrak{F} \cap \mathcal{F}_i).$$

Now we can apply Theorem 4.2.2 (*'Multiplication Theorem'*) which leads us to the required result. ■

Remark 4.2.2 *The above result is the 'Law of Total Probability' in Hyperbolic Hybrid valued probabilistic space or it can also be called 'Complete hyperbolic hybrid valued probability formula'.*

Now, we state and prove the *'Bayes' theorem'* in hyperbolic hybrid valued probabilistic space.

Theorem 4.2.6 *Let us consider the hyperbolic hybrid valued probabilistic space* $(\Omega, \Sigma, \pounds_{\mathfrak{H}})$, *a random event* $\mathfrak{F}$ *and a fundamental system of events* $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n\}$. *Now,*

(1) if $\pounds_{\mathfrak{H}}(\mathfrak{F})$ *is an invertible hyperbolic hybrid number, then*

$$\pounds_{\mathfrak{H}}(\mathcal{F}_k|\mathfrak{F}) = \frac{\pounds_{\mathfrak{H}}(\mathcal{F}_k).\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_k)}{\sum_{i=1}^{n} \pounds_{\mathfrak{H}}(\mathcal{F}_i).\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_i)} = \frac{\pounds_{\mathfrak{H}}(\mathcal{F}_k).\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_k)}{\pounds_{\mathfrak{H}}(\mathfrak{F})}; \tag{4.7}$$

(2) if $\pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{r}\mathbf{e}_+$, *where* $\mathfrak{r} > 0$, *then*

$$\Big(\pounds_{\mathfrak{H}}(\mathcal{F}_k).\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_k) - \pounds_{\mathfrak{H}}(\mathcal{F}_k|\mathfrak{F}).\sum_{i=1}^{n} \pounds_{\mathfrak{H}}(\mathcal{F}_i).\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_i)\Big)\mathbf{e}_+ = 0. \tag{4.8}$$

(3) if $\pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{t}\mathbf{e}_-$, *where* $\mathfrak{t} > 0$, *then*

$$\Big(\pounds_{\mathfrak{H}}(\mathcal{F}_k).\pounds_{\mathcal{F}}(\mathfrak{F}|\mathcal{F}_k) - \pounds_{\mathfrak{H}}(\mathcal{F}_k|\mathfrak{F}).\sum_{i=1}^{n} \pounds_{\mathfrak{H}}(\mathcal{F}_i).\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_i)\mathbf{e}_-\Big) = 0. \tag{4.9}$$

Proof.

(1) Let us consider an invertible hyperbolic hybrid number $\pounds_{\mathfrak{H}}(\mathfrak{F})$. Then by Theorem 4.2.2, we get that

$$\pounds_{\mathfrak{H}}(\mathfrak{F} \cap \mathcal{F}_k) = \pounds_{\mathfrak{H}}(\mathcal{F}_k)\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_k) = \pounds_{\mathfrak{H}}(\mathfrak{F})\pounds_{\mathfrak{H}}(\mathcal{F}_k|\mathfrak{F}),$$

which proves one part of Equation (4.7).
The other part of Equation (4.7) can be proved by using Theorem 4.2.5.

(2) Let $\pounds_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{r}\mathbf{e}_+$, where $\mathfrak{r}$ is a positive real number. Then Theorem 4.2.2 gives that

$$\pounds_{\mathfrak{H}}(\mathfrak{F})\pounds_{\mathfrak{H}}(\mathcal{F}_k|\mathfrak{F}) = \pounds_{\mathfrak{H}}(\mathcal{F}_k)\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_k). \tag{4.10}$$

Clearly, the R.H.S. of (4.10) is an element of $\mathfrak{H}_{\mathbf{e}_+}$ as L.H.S. of (4.10) belongs to $\mathfrak{H}_{\mathbf{e}_+}$. Since the definition of $\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_k)$ contains the factor $\pounds_{\mathfrak{H}}(\mathfrak{F} \cap \mathcal{F}_k)$ and also $\mathfrak{F} \cap \mathcal{F}_k \subset \mathfrak{F}$, we get $\pounds_{\mathfrak{H}}(\mathfrak{F} \cap \mathcal{F}_k) \in \mathfrak{H}_{\mathbf{e}_+}$.
Now, using Lemma 3.2.1, Equation (4.10) becomes

$$\begin{aligned} 0 &= \pounds_{\mathfrak{H}}(\mathcal{F}_k)\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_k) - \pounds_{\mathfrak{H}}(\mathcal{F}_k|\mathfrak{F})\pounds_{\mathfrak{H}}(\mathfrak{F}) \\ &= (\pounds_{\mathfrak{H}}(\mathcal{F}_k)\pounds_{\mathfrak{H}}(\mathfrak{F}|\mathcal{F}_k) - \pounds_{\mathfrak{H}}(\mathcal{F}_k|\mathfrak{F})\pounds_{\mathfrak{H}}(\mathfrak{F}))\mathbf{e}_+. \end{aligned}$$

Therefore, we can obtain Equation (4.8) by using Theorem 4.2.5.

(3) Similarly we can prove Equation (4.9) for $\mathcal{L}_{\mathfrak{H}}(\mathfrak{F}) = \mathbf{te}_{-}$ where $\mathbf{t} > 0$.

■

4.3 Future Prospects

The works under the umbrella of Chapter 3 and Chapter 4, the exploration of probability distribution in the hyperbolic hybrid valued probabilistic measurable space may be an open problem for the future workers in this branch. Also, under this flavor the techniques of expectation are still virgin.

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CHAPTER FIVE

SOME COMPARATIVE GROWTH PROPERTIES OF P-ADIC ENTIRE FUNCTIONS AND THEIR DERIVATIVES

Chapter 5

Some comparative growth properties of p-adic entire functions and their derivatives

5.1 Introduction, Definitions and Notations.

Let us consider $\mathbb{T}$ to be an algebraically closed field of zero characteristic which is complete with respect to a p-adic absolute value $|.|$ (for an example $\mathbb{C}_p$). For any $\theta \in \mathbb{T}$ and $R \in]0, +\infty[$, the closed and open discs are defined by $\bar{d}(\theta, R) = \{\varrho \in \mathbb{T} : |\varrho - \theta| \leqslant R\}$ and $d(\theta, R) = \{\varrho \in \mathbb{T} : |\varrho - \theta| < R\}$ respectively. Also circle of radius R and center at θ is $C(\theta, R) = \{\varrho \in \mathbb{T} : |\varrho - \theta| = R\}$. Furthermore, the set of power series with an infinite radius of convergence $\mathfrak{I}(\mathbb{T})$ is the $\mathbb{T}$-algebra of analytic functions in $\mathbb{T}$. In non-Archimedean analysis Rodriganez [48] defined an entire function on $\mathbb{T}$ as a Taylor series

$$\sum_{n=0}^{\infty} \mathfrak{c}_n \omega^n$$

with convergence on all of $\mathbb{T}$; an analytic function as a Laurent series which convergence on a certain domain $\mathcal{D}$. Also, $\mathbf{M}(\mathbb{T})$ denote the field of meromorphic functions in $\mathbb{T}$ (i.e., field of fractions of $\mathfrak{I}(\mathbb{T})$) i.e., a meromorphic function is a quotient of two entire functions. Given $\hbar \in \mathbf{M}(\mathbb{T})$, we will denote by $q(\hbar, r)$, the number of zeros of $\hbar$ in $\bar{d}(0, r)$, taking multiplicity into account.

Many of the series that appear in non-Archimedean analysis have small domain of convergence. For example,

$$\begin{aligned} \exp_p(\mathfrak{z}) &= \sum_{n=0}^{\infty} \frac{\mathfrak{z}^n}{\lfloor n} \text{ converges for } |\mathfrak{z}| < p^{\frac{-1}{(p-1)}}; \\ \log_p(1+\mathfrak{z}) &= \sum (-1)^{n+1} \frac{\mathfrak{z}^n}{n} \text{ converges for } |\mathfrak{z}| < 1. \end{aligned}$$

Analytic functions inside disc or in the whole field $\mathbb{T}$ were introduced in [1], [23] and [24].

Let $\hbar \in \eth(\mathbb{T})$ and $r > 0$, then we denote by $|\hbar|(r)$ the number

$$\sup\{|\hbar(\varrho)| : |\varrho| = r\},$$

where $|.|(r)$ is a multiplicative norm on $\eth(\mathbb{T})$. Moreover if $\hbar$ is not constant the $|\hbar|(r)$ is a strictly increasing function of r and tends to $+\infty$ with r. So there exists its inverse function

$$\widehat{|\hbar|} : (|\hbar(0)|, \infty) \to (0, \infty) \text{ with } \lim_{s\to\infty} \widehat{|\hbar|}(s) = \infty.$$

Therefore for any two entire functions $\hbar_1 \in \eth(\mathbb{T})$ and $\hbar_2 \in \eth(\mathbb{T})$ the ratio $\frac{|\hbar_1|(r)}{|\hbar_2|(r)}$ as $r \to \infty$ is called the growth of $\hbar_1$ with respect to $\hbar_2$ in terms of their multiplicative norm.

Somasundaram et.al [55] introduced the notion L-order and L-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(\delta r) \sim L(r)$ as $r \to \infty$ for every positive constant δ. The most generalized concept for L-order (lower order) for entire function are L^*-order (lower order) which is introduced by Sato [53].

For $\mathfrak{t} \in [0, \infty)$ and $k \in \mathbb{N}$, we define

$$\log^{[k]} \mathfrak{t} = \log(\log^{[k-1]} \mathfrak{t}) \text{ and } \exp^{[k]} \mathfrak{t} = \exp(\exp^{[k-1]} \mathfrak{t}),$$

where $\mathbb{N}$ be the set of all positive integers. We also denote as

$$\log^{[0]} \mathfrak{t} = \mathfrak{t} \text{ and } \exp^{[0]} \mathfrak{t} = \mathfrak{t}.$$

Throughout this chapter, log denotes the Neperian logarithm.

Here we mean to introduce and study the notion of L^* order of growth and L^* type of growth for function of order t. We will also introduce a new notion of L^* cotype of growth in relation with the distribution of zeros in disks which plays an important role in processes that are quite different from those in complex analysis.

Definition 5.1.1 *[37] Let $\psi \in \eth(\mathbb{T})$,the order of growth of ψ or the order of ψ in brief, denoted by $\rho(\psi)$ is defined as*

$$\rho(\psi) = \lim_{r\to\infty} \sup \frac{\log(\log(|\psi|(r)))}{\log r}.$$

We say that ψ has finite order if $\rho(\psi) < +\infty$.

Definition 5.1.2 *[8] Let $\psi \in \eth(\mathbb{T})$ of order t, where $t \in [0, +\infty[$. Then the cotype of ψ, denoted by $\Psi(\psi)$ is defined as*

$$\Psi(\psi) = \lim_{r\to\infty} \sup \frac{q(\psi, r)}{r^t}.$$

Definition 5.1.3 *[8] Let $\psi \in \eth(\mathbb{T})$ of order t. Then the type of growth of ψ or the type of ψ in brief denoted by $\sigma(\psi)$ is defined as*

$$\sigma(\psi) = \lim_{r\to\infty} \sup \frac{\log(|\psi|(r))}{r^t}.$$

Definition 5.1.4 *[20] The L^*-order $\rho^{L^*}(\psi)$ of an entire function $\psi \in \eth(\mathbb{T})$ is defined as*

$$\rho^{L^*}(\psi) = \lim_{r\to\infty} \sup \frac{\log^{[2]} |\psi|(r)}{\log\left[re^{L(r)}\right]}.$$

Definition 5.1.5 *Let $\psi \in \eth(\mathbb{T})$. Then L^*-cotype of ψ, denoted by $\Psi^{L^*}(\psi)$ is defined as*

$$\Psi^{L^*}(\psi) = \lim_{r\to\infty} \sup \frac{q(\psi, r)}{\left\{re^{L(r)}\right\}^{\rho^{L^*}(\psi)}}.$$

Definition 5.1.6 *Let $\psi \in \eth(\mathbb{T})$. Then L^*-type of ψ, denoted by $\sigma^{L^*}(\psi)$ is defined as*

$$\sigma^{L^*}(\psi) = \lim_{r\to\infty} \sup \frac{\log(|\psi|(r))}{\left\{re^{L(r)}\right\}^{\rho^{L^*}(\psi)}}.$$

Definition 5.1.7 *[8] An entire function $\sum_{n=1}^{+\infty} \mathfrak{c}_n \omega^n \in \eth(\mathbb{T})$ satisfies Hypothesis L when the sequence $\left(\frac{|\mathfrak{c}_{n-1}|}{|\mathfrak{c}_n|}\right)_{n\in\mathbb{N}}$ is strictly increasing.*

5.2 Lemmas.

In this section we present the following lemma which will be needed in the sequel.

Lemma 5.2.1 *[24] Let*

$$\psi(\omega) = \sum_{n=0}^{+\infty} \mathfrak{c}_n \omega^n \in \eth(\mathbb{T}).$$

Then for all $s > 0$ we have

$$|\psi|(s) = \sup_{n\geqslant 0} |\mathfrak{c}_n| s^n = \left|\mathfrak{c}_{q(\psi,r)}\right| s^{q(\psi,r)} > |\mathfrak{c}_n| s^n$$

for all $n > q(\psi, s)$. Moreover if ψ is not a constant, the function in $s : |\psi|(s)$ is strictly increasing and tend to $+\infty$ with s. If ψ is transcendental, the function in $s : \frac{|\psi|(s)}{s^m}$ tends to $+\infty$ with s, whenever $m > 0$.

Lemma 5.2.2 $\{$*cf. [8], [24]*$\}$ *Let $\psi \in \eth(\mathbb{T})$ be non-identically zero and let $s', s'' \in]0, +\infty[$ with $s' < s''$. Then*

$$\left(\frac{s''}{s'}\right)^{q(\psi,s)} \geqslant \frac{|\psi|(s'')}{|\psi|(s')} \geqslant \left(\frac{s''}{s'}\right)^{q(\psi,s')}.$$

Lemma 5.2.3 *[8] Let*

$$\psi(\omega) = \sum_{n=0}^{+\infty} \mathfrak{c}_n \omega^n \in \eth^0(\mathbb{T}).$$

such that $\rho(\psi) \in \left]0, +\infty\right[$. *Then*

$$\sigma^L(\psi)\,\rho^L(\psi)e = \lim_{n\to\infty} \sup \left(n \left(|\mathfrak{c}_n| \right)^{\frac{1}{n}} \right).$$

Lemma 5.2.4 *Let*

$$\psi(\omega) = \sum_{n=0}^{+\infty} \mathfrak{c}_n \omega^n \in \eth^0(\mathbb{T}).$$

such that $\rho(\psi) \in \left]0, +\infty\right[$. *Then*

$$\sigma^{L^*}(\psi)\,\rho^{L^*}(\psi)e = \lim_{n\to\infty} \sup \left(n \left(|\mathfrak{c}_n| \, e^{L(|\mathfrak{c}_n|)} \right)^{\frac{1}{n}} \right).$$

Proof. The proof of Lemma 5.2.4 can be carried out in the line of Lemma 5.2.3 and therefore its proof is omitted. ■

5.3 Main Results.

In this section we state the main results of the chapter.

Theorem 5.3.1 *Let* $\psi \in \eth(\mathbb{T})$ *be not identically zero. If there exists* $\theta \geqslant 0$ *such that*

$$\lim_{r\to\infty} \sup \left\{ \frac{q(\psi, r)}{\left(re^{L(r)} \right)^{\theta}} \right\} < +\infty$$

then $\rho^{L^*}(\psi)$ *is the lower bound of the set of* $\theta \in \left[0, +\infty\right[$ *such that*

$$\lim_{r\to\infty} \sup \left\{ \frac{q(\psi, r)}{\left(re^{L(r)} \right)^{\theta}} \right\} = 0.$$

Moreover, if

$$\lim_{r\to\infty} \sup \left\{ \frac{q(\psi, r)}{\left(re^{L(r)} \right)^{t}} \right\} = b \in \left]0, +\infty\right[,$$

then $\rho^{L^*}(\psi) = t$. *If there exists no* θ *such that*

$$\lim_{r\to\infty} \sup \left\{ \frac{q(\psi, r)}{\left(re^{L(r)} \right)^{\theta}} \right\} < +\infty$$

then $\rho^{L^*}(\psi) = +\infty$.

Proof. At first we will prove that given $\psi \in \eth(\mathbb{T})$ be non constant and if for some $t \geqslant 0$,

$$\lim_{r\to\infty} \sup \left\{ \frac{q(\psi, r)}{\left(re^{L(r)}\right)^t} \right\}$$

is finite, then $\rho^{L^*}(\psi) \leqslant t$.

Consider,

$$\lim_{r\to\infty} \sup \left\{ \frac{q(\psi, r)}{\left(re^{L(r)}\right)^t} \right\} = b \in [0, +\infty[.$$

Let us take $\varepsilon > 0$, then we can find a positive real number $\mathcal{G} > 1$ such that $|\psi|(\mathcal{G}) > e^2$ and

$$\frac{q(\psi, r)}{\left(re^{L(r)}\right)^t} \leqslant b + \varepsilon \forall r \geqslant \mathcal{G}$$

and hence by Lemma 5.2.2

$$\frac{|\psi|(r)}{|\psi|(\mathcal{G})} \leqslant \left(\frac{r}{\mathcal{G}}\right)^{q(\psi,r)} \leqslant \left(\frac{r}{\mathcal{G}}\right)^{r^t(b+\varepsilon)}.$$

As $\mathcal{G} > 1$, we have

$$\log(|\psi|(r)) \leqslant \log(|\psi|(\mathcal{G})) + r^t(b+\varepsilon)(\log(r)).$$

When $p > 2$, $q > 2$, we have

$$\log(p+q) \leqslant \log(p) + \log(q).$$

Now, applying the above inequality with $p = \log(|\psi|(\mathcal{G}))$ and $q = r^t(b+\varepsilon)(\log(r))$ when $r^t(b+\varepsilon)(\log(r)) > 2$ that gives

$$\log(\log(|\psi|(r))) \leqslant \log(\log|\psi|(\mathcal{G})) + t\log(r) + \log(b+\varepsilon) + \log(\log(r))$$

i.e.,

$$\frac{\log(\log(|\psi|(r)))}{\log\left(re^{L(r)}\right)} \leqslant \frac{\log(\log|\psi|(\mathcal{G})) + t\log(r) + \log(b+\varepsilon) + \log(\log(r))}{\log\left(re^{L(r)}\right)}.$$

Hence,

$$\lim_{r\to\infty} \sup \frac{\log(\log(|\psi|(r)))}{\log\left(re^{L(r)}\right)} \leqslant \lim_{r\to\infty} \frac{t\log(r)}{\log\left(re^{L(r)}\right)} \leqslant \lim_{r\to\infty} \frac{t\log(r)}{\log(r)} = t.$$

$$\therefore \rho^{L^*}(\psi) \leqslant t. \tag{5.1}$$

This proves the first part of the theorem.

Conversely, we will show that $\rho^{L^*}(\psi) \geqslant t$.

First we shall prove that given $\psi \in \eth(\mathbb{T})$ is not identically zero and if for some $t \geqslant 0$, we have

$$\lim_{r\to\infty} \frac{q(\psi, r)}{r^t} > 0,$$

and

$$\lim_{r\to\infty}\frac{q(\psi,r)}{\left(re^{L(r)}\right)^t}>0,$$

then $\rho^{L^*}(\psi)\geqslant t$.

By hypothesis, there exists a sequence $(\pi_n)_{n\in N}$ such that

$$\lim_{r\to\infty}\frac{q(\psi,r)}{\left(\pi_n e^{L(\pi_n)}\right)^t}>0.$$

Thus there exists $b>0$ such that

$$\lim_{r\to\infty}\frac{q(\psi,r)}{\left(\pi_n e^{L(\pi_n)}\right)^t}\geqslant b.$$

Now assuming $|\psi|(r_0)\geqslant 1$, therefore by Lemma 5.2.1 we get that

$$|\psi|(\pi_n)\geqslant 1\forall n.$$

Taking $\varpi>1$ and using Lemma 5.2.2

$$\frac{|\psi|(\varpi\pi_n)}{|\psi|(\pi_n)}\geqslant(\varpi)^{q(\psi,\pi_n)}\geqslant(\varpi)^{\left[b\left\{\pi_n e^{L(\pi_n)}\right\}^t\right]}$$

i.e.,

$$\log(|\psi|(\varpi\pi_n))\geqslant\log(|\psi|(\pi_n))+b\left\{\pi_n e^{L(\pi_n)}\right\}^t\cdot\log(\varpi).$$

Since $|\psi|(\pi_n)\geqslant 1$, we have

$$\log(\log(|\psi|(\varpi\pi_n)))\geqslant\log(b\log(\varpi))+t\log\left(\pi_n e^{L(\pi_n)}\right).$$

Therefore,

$$\frac{\log(\log(|\psi|(\varpi\pi_n)))}{\log\left(\pi_n e^{L(\pi_n)}\right)}\geqslant t+\frac{\log(b\log(\varpi))}{\log(\pi_n)}$$

for all $n\in\mathbb{N}$. So,

$$\lim_{r\to\infty}\sup\frac{\log(\log(|\psi|(\varpi\pi_n)))}{\log\left(re^{L(r)}\right)}\geqslant t$$

i.e.,

$$\rho^{L^*}(\psi)\geqslant t. \tag{5.2}$$

Combining Equations (5.1) and (5.2) we can say that

$$\rho^{L^*}(\psi)=t.$$

Thus the theorem is proved. ■

Example 5.3.1 *Suppose that for each $r > 0$ we have*

$$\begin{aligned} q(\psi, r) &\in \left[r^t \log r, r^t \log r + 1\right], \\ L(r) &= \log(\log r)^{\frac{1}{\theta}}. \end{aligned}$$

If we take

$$q(\psi, r) = r^t \log r$$

then of course for every $\theta > t$, we have

$$\begin{aligned} \lim_{r\to\infty} \frac{q(\psi, r)}{\{re^{L(r)}\}^{\theta}} &= \lim_{r\to\infty} \frac{r^t \log r}{r^{\theta}\left\{(\log r)^{\frac{1}{\theta}}\right\}^{\theta}} \\ &= \lim_{r\to\infty} \frac{1}{r^{\theta-t}} = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{r\to\infty} \frac{q(\psi, r)}{\{re^{L(r)}\}^{t}} &= \lim_{r\to\infty} \frac{r^t \log r}{r^{t}\left\{(\log r)^{\frac{1}{\theta}}\right\}^{t}} \\ &= \lim_{r\to\infty} (\log r)^{1-\frac{t}{\theta}} = \infty. \end{aligned}$$

So, there exists no $t > 0$ such that

$$\limsup_{r\to\infty} \frac{q(\psi, r)}{\{re^{L(r)}\}^{t}} < +\infty.$$

Theorem 5.3.2 *Let $\psi(\omega) = \sum_{n=0}^{+\infty} \mathfrak{c}_n \omega^n \in \eth(\mathbb{T})$. Then*

$$\rho^{L^*}(\psi) = \lim_{n\to\infty} \left(\frac{n \log n}{-\log|\mathfrak{c}_n| \cdot e^{L(|\mathfrak{c}_n|)}} \right).$$

Proof. If $\rho^{L^*}(\psi) < +\infty$, the proof is identical in the complex context.

Let us suppose that $\rho^{L^*}(\psi) = +\infty$. If possible let,

$$\limsup_{n\to\infty} \left(\frac{n \log n}{-\log|\mathfrak{c}_n| \cdot e^{L(|\mathfrak{c}_n|)}} \right) < +\infty. \tag{5.3}$$

Let us take $s \in \mathbb{N}$ such that

$$\frac{n \log n}{-\log|\mathfrak{c}_n| \cdot e^{L(|\mathfrak{c}_n|)}} < s \forall n \in N \tag{5.4}$$

and

$$\limsup_{r\to\infty} \frac{q(\psi, r)}{\{re^{L(r)}\}^{s}} = +\infty.$$

So we take a sequence $(\pi_m)_{m\in N}$ such that

$$\lim_{m\to\infty} \frac{q(\psi,\pi_m)}{\{\pi_m e^{L(\pi_m)}\}^s} = +\infty. \tag{5.5}$$

Set $u_m = q(\psi,\pi_m)\,, m \in \mathbb{N}$. So,

$$\begin{aligned} u_m \log(u_m) &< s\left(-\log|\mathfrak{c}_{u_m}|\, e^{L(|\mathfrak{c}_{u_m}|)}\right) \\ &= se^{L(|\mathfrak{c}_{u_m}|)} \cdot \log\left(\frac{1}{|\mathfrak{c}_{u_m}|}\right). \end{aligned}$$

Hence

$$(u_m)^{u_m} < \frac{1}{|a_{u_m}|^{se^{L(|\mathfrak{c}_{u_m}|)}}}$$

i.e.,

$$\frac{1}{(u_m)^{u_m}} > |\mathfrak{c}_{u_m}|^{se^{L(|\mathfrak{c}_{u_m}|)}}.$$

Therefore,

$$|\mathfrak{c}_{u_m}|^{se^{L(|\mathfrak{c}_{u_m}|)}} \left(\pi_m e^{L(\pi_m)}\right)^{su_m} < \frac{\left(\pi_m e^{L(\pi_m)}\right)^{su_m}}{(u_m)^{u_m}}$$

i.e.,

$$(|\psi|(\pi_m))^{se^{L(|a_{u_m}|)}} < \left\{\frac{\left(\pi_m e^{L(\pi_m)}\right)^s}{u_m}\right\}^{u_m},$$

whereas Lemma 5.2.1 gives that

$$\lim_{\pi_m\to\infty} |\psi|(\pi_m) = +\infty,$$

hence,

$$\left(\pi_m e^{L(\pi_m)}\right)^s > u_m$$

where $m \in \mathbb{N}$ (a large number)

$$\lim_{m\to\infty} \frac{q(\psi,\pi_m)}{\{\pi_m e^{L(\pi_m)}\}^s} \leqslant 1,$$

which contradicts Equation (5.5).
Therefore Equation (5.4) is impossible and hence,

$$\lim_{n\to\infty}\left(\frac{n\log n}{-\log|\mathfrak{c}_n| \cdot e^{L(|\mathfrak{c}_n|)}}\right) = +\infty = \rho^{L^*}(\psi).$$

■

Theorem 5.3.3 *Let $\psi \in \eth(\mathbb{T})$ be not identically zero. Then*

$$\rho^{L^*}(\psi) = \rho^{L^*}(\psi^{/}).$$

Proof. From Theorem 5.3.2,

$$\rho^{L^*}(\psi^{/}) = \lim_{n\to\infty}\left(\frac{n\log n}{-\log|(n+1)\,\mathfrak{c}_{n+1}|\cdot e^{L(|(n+1)\mathfrak{c}_{n+1}|)}}\right).$$

But since $\frac{1}{n} \leqslant |n| \leqslant 1$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{n\log n}{-\log|(n+1)\,\mathfrak{c}_{n+1}|\cdot e^{L(|(n+1)\mathfrak{c}_{n+1}|)}}\right)\\
=&\lim_{n\to\infty}\left(\frac{n\log n}{-\log(|\mathfrak{c}_{n+1}|)\cdot e^{L(|\mathfrak{c}_{n+1}|)}}\right)\\
=&\limsup_{n\to\infty}\left(\frac{(n+1)\log(n+1)}{-\log(|\mathfrak{c}_{n+1}|)\cdot e^{L(|\mathfrak{c}_{n+1}|)}}\right) = \rho^{L^*}(\psi).
\end{aligned}$$

■

Corollary 5.3.1 *The derivation on $\eth(\mathbb{T})$ restricted to the algebra $\eth(\mathbb{T},t)$ (resp. $\eth^0(\mathbb{T})$) provides the algebra with a derivation.*

Theorem 5.3.4 *Let $\psi \in \eth(\mathbb{T})$ such that $\rho^{L^*}(\psi) \in]0,+\infty[$. If*

$$\sigma^{L^*}(\psi) = \lim_{n\to\infty}\frac{\log(|\psi|(r))}{\{re^{L(r)}\}^{\rho^{L^*}(\psi)}}$$

then

$$\Psi^{L^*}(\psi) \geqslant \rho^{L^*}(\psi)\,\sigma^{L^*}(\psi).$$

Proof. Let

$$\psi(\omega) = \sum_{m=0}^{+\infty}\mathfrak{c}_m\omega^m.$$

Without loss of generality, we consider that $\psi(0) = 1$. Let $t = \rho^{L^*}(\psi)$, $l = \frac{1}{\rho^{L^*}(\psi)}$ and $(C(0,S_m))_{m\in\mathbb{N}}$ be the sequence of circles having at least one zero of ψ, where $S_m < S_{m+1}$. Suppose that the Hypothesis L satisfied by ψ. By Lemma 5.2.1, each circle $C(0,S_m), m \in \mathbb{N}$ contains a unique zero of ψ and also ψ has no other zero in $\mathbb{T}$. Furthermore, for each $m \in \mathbb{N}$, we have $S_m = \frac{|\mathfrak{c}_{m-1}|}{|\mathfrak{c}_m|}$ and consequently $q(\psi, S_m) = m, m \in \mathbb{N}$. Now, when $r \in [S_m, S_{m+1}[$, ψ admits exactly m zeros in $d(0,r)$ and therefore when $r = S_m$, $\frac{q(\psi,r)}{\{re^{L(r)}\}^t}$ is maximum in $[S_m, S_{m+1}[$. Consequently, we have

$$\Psi^{L^*}(\psi) = \limsup_{n\to\infty}\frac{m}{\{S_m e^{L(S_m)}\}^t}.$$

So, for $\varepsilon_m > 0$ and

$$\lim_{n\to+\infty} \sup \varepsilon_m = 0,$$

we get that

$$\frac{m}{\{S_m e^{L(S_m)}\}^t} = \Psi^{L^*}(\psi) + \varepsilon_m.$$

Therefore,

$$\begin{aligned} S_m &= \left(\frac{m}{\Psi^{L^*}(\psi) + \varepsilon_m}\right)^{\frac{1}{t}} \\ &= \left(\frac{m}{\Psi^{L^*}(\psi) + \varepsilon_m}\right)^{l}. \end{aligned} \tag{5.6}$$

Now, let $(\varphi(m))_{m\in\mathbb{N}}$ be a strictly increasing sequence of integers and also consider the expression

$$E(m) = \log\left(|\psi|\left(S_{\varphi(m)}\right)\right) = \sum_{k=1}^{\varphi(m)} \log\left(S_{\varphi(m)}\right) - \log S_k.$$

By Equation (5.6) we have

$$\begin{aligned} E(m) &= l\left\{\begin{array}{c} \varphi(m)\cdot\log\varphi(m) - \varphi(m)\cdot\log(\Psi^{L^*}(\psi) + \varepsilon_{\varphi(m)}) - \sum_{k=1}^{\varphi(m)}\log(k) + \\ \sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi) + \varepsilon_k\right) \end{array}\right\} \\ &= l\left\{\begin{array}{c} \varphi(m)\cdot\log\varphi(m) - \varphi(m)\cdot\log\varphi(m) + \varphi(m) + O(1) - \\ \varphi(m)\log(\Psi^{L^*}(\psi) + \varepsilon_{\varphi(m)}) + \sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi) + \varepsilon_k\right) \end{array}\right\}. \end{aligned}$$

Hence,

$$E(m) = l\left\{\varphi(m) + O(1) - \varphi(m)\log(\Psi^{L^*}(\psi) + \varepsilon_{\varphi(m)}) + \sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi) + \varepsilon_k\right)\right\}. \tag{5.7}$$

We assume that

$$\sigma^{L^*}(\psi) = \lim_{r\to\infty} \frac{\log(|\psi|(r))}{\{re^{L(r)}\}^t}$$

and let us consider a sequence $\{\varphi(m)\}_{m\in\mathbb{N}}$ such that

$$\lim_{m\to+\infty} \frac{q\left(\psi, S_{\varphi(m)}\right)}{\left\{S_{\varphi(m)} e^{L\left(S_{\varphi(m)}\right)}\right\}^t} = \Psi^{L^*}(\psi)$$

i.e.,

$$\lim_{m\to+\infty}\frac{\varphi(m)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}=\Psi^{L^*}(\psi).$$

Obviously, we can check that

$$\begin{aligned}\sigma^{L^*}(\psi) &= \lim_{m\to\infty}\frac{\log\left(|\psi|\left(S_{\varphi(m)}\right)\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\\ &= \lim_{m\to\infty}\frac{E(m)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}.\end{aligned}$$

From Equation (5.7) we get that

$$\begin{aligned}&\sigma^{L^*}(\psi)\\ &=\lim_{m\to\infty}\frac{l\left\{\varphi(m)+O(1)-\varphi(m)\log\left(\Psi^{L^*}(\psi)+\varepsilon_{\varphi(m)}\right)+\sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)\right\}}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\\ &=\lim_{m\to\infty}\frac{l\varphi(m)\left\{1+\frac{O(1)}{\varphi(m)}-\left(\log\left(\Psi^{L^*}(\psi)+\varepsilon_{\varphi(m)}\right)\right)\right\}}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}+\lim_{m\to\infty}\frac{l\sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\end{aligned}$$

i.e.,

$$\begin{aligned}\sigma^{L^*}(\psi) &= l\cdot\lim_{m\to\infty}\frac{\varphi(m)\left\{1-\log\left(\Psi^{L^*}(\psi)+\varepsilon_{\varphi(m)}\right)\right\}}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}+\lim_{m\to\infty}\frac{l\sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\\ &= l\cdot\Psi^{L^*}(\psi)\cdot\left(1-\log\Psi^{L^*}(\psi)\right)+\lim_{m\to\infty}\frac{l\sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}. \qquad (5.8)\end{aligned}$$

Hence, from (5.8) $\frac{\sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}$ admits a limit when m tends to ∞ which is

$$\sigma^{L^*}(\psi)-l\cdot\Psi^{L^*}(\psi)\cdot\left(1-\log\Psi^{L^*}(\psi)\right).$$

Since

$$\lim_{m\to\infty}\sup\left(\varepsilon_{\varphi(m)}\right)=0,$$

we can check that

$$\lim_{m\to\infty}\frac{\sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\leqslant\lim_{m\to\infty}\frac{\varphi(m)\log\Psi^{L^*}(\psi)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}$$
$$=\Psi^{L^*}(\psi)\log\Psi^{L^*}(\psi). \tag{5.9}$$

Indeed, let us fix $\omega>0$ and let $M\in\mathbb{N}$ be such that $\varepsilon_k\leqslant\omega$ for all $k>M$ and

$$\frac{\sum_{k=1}^{M}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\leqslant\omega\quad\forall\quad\varphi(m)>M.$$

Then

$$\frac{\sum_{k=M+1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\leqslant\log\left(\Psi^{L^*}(\psi)+\omega\right)\frac{\varphi(m)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}$$
$$\leqslant\log\left(\Psi^{L^*}(\psi)+\omega\right)\Psi^{L^*}(\psi). \tag{5.10}$$

So, from Equation (5.9) and (5.10) we get that

$$\lim_{m\to\infty}\frac{\sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\leqslant\omega+\log\left(\Psi^{L^*}(\psi)+\omega\right)\Psi^{L^*}(\psi).$$

This is true for each $\omega>0$.Hence from Equation (5.8) we have

$$\sigma^{L^*}(\psi)=l\left\{\Psi^{L^*}(\psi)-\Psi^{L^*}(\psi)\log\Psi^{L^*}(\psi)+\lim_{m\to\infty}\frac{\sum_{k=1}^{\varphi(m)}\log\left(\Psi^{L^*}(\psi)+\varepsilon_k\right)}{\left\{S_{\varphi(m)}e^{L\left(S_{\varphi(m)}\right)}\right\}^t}\right\}.$$

By using Equation (5.9) we obtain from above that

$$\sigma^{L^*}(\psi)\leqslant l\Psi^{L^*}(\psi),$$

i.e.,

$$\sigma^{L^*}(\psi)\leqslant\frac{1}{\rho^{L^*}(\psi)}\Psi^{L^*}(\psi),$$

i.e.,

$$\rho^{L^*}(\psi)\,\sigma^{L^*}(\psi)\leqslant\Psi^{L^*}(\psi).$$

Hence the theorem is proved. ■

Theorem 5.3.5 *Let $\psi \in \mathfrak{O}(\mathbb{T})$ be not identically zero, of order $t \in]0, +\infty[$. Then*

$$\sigma^{L^*}(\psi) = \sigma^{L^*}(\psi^/).$$

Proof. By Lemma 5.2.4 we have

$$\begin{aligned}
\sigma^{L^*}(\psi^/)\,\rho^{L^*}(\psi/)e &= \limsup_{n\to\infty}\left(n\left(|n+1|\,|\mathfrak{c}_{n+1}|\,e^{L(|(n+1)\mathfrak{c}_{n+1}|)}\right)^{\frac{t}{n}}\right) \\
&= \limsup_{n\to\infty}\left(\left((n+1)\left(|n+1|\,|\mathfrak{c}_{n+1}|\,e^{L(|(n+1)\mathfrak{c}_{n+1}|)}\right)^{\frac{t}{n}}\right)^{\frac{n}{n+1}}\right) \\
&= \limsup_{n\to\infty}\left(\left((n+1)\left(|n+1|\,|\mathfrak{c}_{n+1}|\,e^{L(|(n+1)\mathfrak{c}_{n+1}|)}\right)^{\frac{t}{n+1}}\right)^{\frac{n}{n+1}}\right) \\
&= \sigma^{L^*}(\psi)\,\rho^{L^*}(\psi)e.
\end{aligned}$$

From Theorem 5.3.3 we get that

$$\rho^{L^*}(\psi) = \rho^{L^*}(\psi^/)$$

and since $\rho^{L^*}(\psi) \neq 0$, we can see that

$$\sigma^{L^*}(\psi) = \sigma^{L^*}(\psi^/).$$

This proves the theorem. ■

5.4 Future Prospects

In the line of work as carried out in this chapter, one may think of finding out various problem on relative $(p, q, t)\,L^{th}$ Ψorder (lower order)[22], type in the p-adic ground where $\Psi : [0, \infty) \to (0, \infty)$ be a non-decreasing unbounded function, satisfying the following two conditions:

$$\lim_{r\to\infty}\frac{log^{[p]}r}{log^{[q]}\Psi(r)} = 0$$

and

$$\lim_{r\to\infty}\frac{log^{[q]}\Psi(\alpha r)}{log^{[q]}\Psi(r)} = 1$$

for some $\alpha > 1$ and this treatment can be done under the flavour of bicomplex analysis. As a consequence, the derivation of relevant results is still open to the future workers of this branch.

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The works of the chapter have been **communicated**.

CHAPTER SIX

A NOTE ON THE BICOMPLEX VERSION OF ENSTRÖM-KAKEYA THEOREM

Chapter 6

A note on the bicomplex version of Enstr$\ddot{o}$m-Kakeya theorem

6.1 Preliminary Definitions and Notations.

In this section, we give some basic definitions about bicomplex numbers which will be needed in this sequel.

Definition 6.1.1 *[46] Idempotent representation is one of the important presentations of a bicomplex number. The bicomplex numbers $\mathbf{e} := \frac{1+ij}{2}, \mathbf{e}^{\dagger} := \frac{1-ij}{2}$ are linearly independent in the $\mathbb{C}_1$-linear space $\mathbb{C}_2$ and $\mathbf{e}+\mathbf{e}^{\dagger} = 1, \mathbf{e}-\mathbf{e}^{\dagger} = ij, \mathbf{e}.\mathbf{e}^{\dagger} = 0, \mathbf{e}^2 = \mathbf{e}, \mathbf{e}^{\dagger 2} = \mathbf{e}^{\dagger}$. Any number $\eta = \eta_1 + j\eta_2 \in \mathbb{C}_2$ can be uniqely expressed as $\eta = (\eta_1 - i\eta_2)\mathbf{e} + (\eta_1 + i\eta_2)\mathbf{e}^{\dagger}$. This representation is named as idempotent representation of η.*

Definition 6.1.2 *[46] The complex spaces $\mathfrak{X}_1 = \{\eta_1 - i\eta_2 : \eta_1, \eta_2 \in \mathbb{C}_1\}$ and $\mathfrak{X}_2 = \{\eta_1 + i\eta_2 : \eta_1, \eta_2 \in \mathbb{C}_1\}$ are called the auxiliary complex spaces. Each point $\eta_1 + j\eta_2 = (\eta_1 - i\eta_2)\mathbf{e} + (\eta_1 + i\eta_2)\mathbf{e}^{\dagger} \in \mathbb{C}_2$ associates the points $\eta_1 - i\eta_2 \in \mathfrak{X}_1$ and $\eta_1 + i\eta_2 \in \mathfrak{X}_2$. Also to each pair of points $(\eta_1 - i\eta_2, \eta_1 + i\eta_2) \in \mathfrak{X}_1 \times \mathfrak{X}_2$ there is a uniqe point in $\mathbb{C}_2$.*

Definition 6.1.3 *[46] An open discus $D(\xi; \tau_1, \tau_2)$ with centre $\xi = \xi_1\mathbf{e} + \xi_2\mathbf{e}^{\dagger}$ and radii $\tau_1 > 0, \tau_2 > 0$ is defined as*

$$\boldsymbol{D}(\xi; \tau_1, \tau_2) = \{\omega_1\mathbf{e} + \omega_2\mathbf{e}^{\dagger} \in \mathbb{C}_2 : |\omega_1 - \xi_1| < \tau_1, |\omega_2 - \xi_2| < \tau_2\}.$$

Definition 6.1.4 *[46] A closed discus $\bar{D}(\xi; \tau_1, \tau_2)$ with centre $\xi = \xi_1\mathbf{e} + \xi_2\mathbf{e}^{\dagger}$ and radii $\tau_1 > 0, \tau_2 > 0$ is defined by*

$$\bar{\boldsymbol{D}}(\xi; \tau_1, \tau_2) = \{\omega_1\mathbf{e} + \omega_2\mathbf{e}^{\dagger} \in \mathbb{C}_2 : |\omega_1 - \xi_1| \leqslant \tau_1, |\omega_2 - \xi_2| \leqslant \tau_2\}.$$

Definition 6.1.5 *[46] If $\tau_1 > 0$, $\tau_2 > 0$ both are equal to $\tilde{\tau}$, then the discus is called a $\mathbb{C}_2$- disc and is denoted by $\boldsymbol{D}(\xi; \tau, \tau) = \boldsymbol{D}(\xi; \tau)$.*

6.2 Lemmas.

In this section we present lemmas which will be needed in the sequel.

Lemma 6.2.1 *[46] Let* $\widetilde{\mathfrak{B}} = \widetilde{\mathfrak{B}_1}\mathbf{e}+\widetilde{\mathfrak{B}_2}\mathbf{e}^\dagger := \{\varrho_1\mathbf{e}+\varrho_2\mathbf{e}^\dagger : \varrho_1 \in \widetilde{\mathfrak{B}_1}, \varrho_2 \in \widetilde{\mathfrak{B}_2}\}$ *be a domain in* $\mathbb{C}_2$*. A bicomplex function* $\mathfrak{K} = \mathfrak{K}_1\mathbf{e} + \mathfrak{K}_2\mathbf{e}^\dagger : \widetilde{\mathfrak{B}} \to \mathbb{C}_2$ *is holomorphic if and only if both the component function* $\mathfrak{K}_1$ *and* $\mathfrak{K}_2$ *are holomorphic in* $\widetilde{\mathfrak{B}_1}$ *and* $\widetilde{\mathfrak{B}_2}$ *respectively .*

Lemma 6.2.2 *[46] Let* $\mathfrak{K}$ *be a bicomplex holomorphic function defined in a domain* $\widetilde{\mathfrak{B}} = \widetilde{\mathfrak{B}_1}\mathbf{e}+\widetilde{\mathfrak{B}_2}\mathbf{e}^\dagger := \{\varrho_1\mathbf{e}+\varrho_2\mathbf{e}^\dagger : \varrho_1 \in \widetilde{\mathfrak{B}_1}, \varrho_2 \in \widetilde{\mathfrak{B}_2}\}$ *such that* $\mathfrak{K}(\varrho) = \mathfrak{K}_1(\varrho_1)\mathbf{e}+\mathfrak{K}_2(\varrho_2)\mathbf{e}^\dagger$*, for all* $\varrho = \varrho_1\mathbf{e} + \varrho_2\mathbf{e}^\dagger \in \widetilde{\mathfrak{B}}$*. Then,* $\mathfrak{K}(\varrho)$ *has zero in* $\widetilde{\mathfrak{B}}$ *if and only if* $\mathfrak{K}_1(\varrho_1)$ *and* $\mathfrak{K}_2(\varrho_2)$ *both have zero at* ϱ_1 *in* $\widetilde{\mathfrak{B}_1}$ *and at* ϱ_2 *in* $\widetilde{\mathfrak{B}_2}$ *respectively.*

The following lemma is termed as Schwarz's lemma in $\mathbb{C}_1$.

Lemma 6.2.3 *[9] If* $\mathfrak{h}(\omega)$ *is holomorphic in* $|\omega| \leqslant \lambda$ *in* $\mathbb{C}_1$*,* $\mathfrak{h}(0) = 0$ *and* $|\mathfrak{h}(\omega)| \leqslant M$ *for* $|\omega| = \lambda$*, then*

$$|\mathfrak{h}(\omega)| \leqslant \frac{M|\omega|}{\lambda}.$$

6.3 Theorems.

In this section we present the main results of this chapter.

Theorem 6.3.1 *Let* $\mathfrak{K}(\varrho) = \sum_{j=0}^{\infty} \mu_j \varrho^j$ *be a bicomplex entire function with real positive coefficients and for some* $\mathfrak{c} \geqslant 1, t > 0$

$$\mathfrak{c}\mu_0 \geqslant t\mu_1 \geqslant t^2\mu_2 \geqslant \ldots\ .$$

Then $\mathfrak{K}(\varrho)$ *does not vanish in the open discus* $\boldsymbol{D}(0;\tau_0,\tau_0)$ *where* $\tau_0 = \frac{t}{2\mathfrak{c}-1}$ *.*

Proof. Since $\mu_j = \mu_j\mathbf{e} + \mu_j\mathbf{e}^\dagger$ and $\varrho = \varrho_1\mathbf{e} + \varrho_2\mathbf{e}^\dagger$, then $\mathfrak{K}(\varrho)$ can be expressed as

$$\begin{aligned}\mathfrak{K}(\varrho) &= \sum_{j=0}^{\infty}(\mu_j\mathbf{e} + \mu_j\mathbf{e}^\dagger)(\varrho_1\mathbf{e} + \varrho_2\mathbf{e}^\dagger)^j \\ &= \sum_{j=0}^{\infty}(\mu_j\mathbf{e} + \mu_j\mathbf{e}^\dagger)(\varrho_1^j\mathbf{e} + \varrho_2^j\mathbf{e}^\dagger) \\ &= \sum_{j=0}^{\infty}\mu_j\varrho_1^j\mathbf{e} + \sum_{j=0}^{\infty}\mu_j\varrho_2^j\mathbf{e}^\dagger \\ &= \mathfrak{K}_1(\varrho_1)\mathbf{e} + \mathfrak{K}_2(\varrho_2)\mathbf{e}^\dagger.\end{aligned}$$

Since $\mathfrak{K}(\varrho)$ is holomorphic in any closed discus $\bar{\boldsymbol{D}}(0;t,t) \subset \mathbb{C}_2, 0 < t < \infty$, in view of Lemma 6.2.1, $\mathfrak{K}_1(\varrho_1)$ and $\mathfrak{K}_2(\varrho_2)$ both are holomorphic respectively in $\widetilde{\mathfrak{B}_1} = \{\varrho_1 \in \mathfrak{X}_1 :|$

$\varrho_1 \mid\leqslant t\} \subset \mathbb{C}_1$ and $\widetilde{\mathfrak{B}}_2 = \{\varrho_2 \in \mathfrak{X}_2 :\mid \varrho_2 \mid\leqslant t\} \subset \mathbb{C}_1$.

Clearly, $\lim_{j\to\infty} \mu_j t^j = 0$.

Now, let us consider

$$\begin{aligned}\mathfrak{F}(\varrho_1) &= (\varrho_1 - t)\mathfrak{K}_1(\varrho_1),\\ &= -t\mu_0 + (\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \ldots\\ &= -t\mu_0 + (\mu_0 - \mathfrak{c}\mu_0 + \mathfrak{c}\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \ldots\\ &= -t\mu_0 + (1-\mathfrak{c})\mu_0\varrho_1 + (\mathfrak{c}\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \ldots\end{aligned}$$

i.e, $\mathfrak{F}(\varrho_1) = -t\mu_0 + (1-\mathfrak{c})\mu_0\varrho_1 + \mathfrak{G}(\varrho_1)$, where $\mathfrak{G}(\varrho_1) = (\mathfrak{c}\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \ldots$.

For $|\varrho_1| = t$, we have

$$\begin{aligned}|\mathfrak{G}(\varrho_1)| &= |(\mathfrak{c}\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \ldots|\\ &\leqslant |\mathfrak{c}\mu_0 - t\mu_1||\varrho_1| + |\mu_1 - t\mu_2||\varrho_1|^2 + \ldots\\ &= (\mathfrak{c}\mu_0 - t\mu_1)t + (\mu_1 - t\mu_2)t^2 + \ldots\\ &= \mathfrak{c}\mu_0 t.\end{aligned}$$

As $\mathfrak{G}(\varrho_1)$ is holomorphic in $\mid \varrho_1 \mid\leqslant t$, $\mathfrak{G}(0) = 0$ and $\mid \mathfrak{G}(\varrho_1) \mid\leqslant \mathfrak{c}\mu_0 t$ for $\mid \varrho_1 \mid= t$, applying Lemma 6.2.3, we get that

$$\begin{aligned}\mid \mathfrak{G}(\varrho_1) \mid &\leqslant \frac{\mathfrak{c}\mu_0 t|\varrho_1|}{t}\\ &= \mathfrak{c}\mu_0|\varrho_1|, \text{ for } |\varrho_1| \leqslant t.\end{aligned}$$

Now, for $|\varrho_1| < t$, it follows that

$$\begin{aligned}|\mathfrak{F}(\varrho_1)| &\geqslant |-t\mu_0 + (1-\mathfrak{c})\mu_0\varrho_1| - |\mathfrak{G}(\varrho_1)|\\ &\geqslant |t\mu_0 + (\mathfrak{c}-1)\mu_0\varrho_1| - \mathfrak{c}\mu_0|\varrho_1|\\ &\geqslant t\mu_0 - (\mathfrak{c}-1)\mu_0|\varrho_1| - \mathfrak{c}\mu_0|\varrho_1|\\ &= t\mu_0 - (2\mathfrak{c}-1)\mu_0|\varrho_1| > 0 \text{ if } |\varrho_1| < \frac{t}{2\mathfrak{c}-1}.\end{aligned}$$

Hence for both $|\varrho_1| < t$ and $|\varrho_1| < \tau_0$, $|\mathfrak{K}_1(\varrho_1)| > 0$ where $\tau_0 = \frac{t}{2\mathfrak{c}-1}$.

Similarly, $|\mathfrak{K}_2(\varrho_2)| > 0$ if $|\varrho_2| < \tau_0$.

Therefore $\mathfrak{K}_1(\varrho_1)$ and $\mathfrak{K}_2(\varrho_2)$ do not vanish respectively in $\widetilde{\mathfrak{B}}_1{}' = \{\varrho_1 \in \widetilde{\mathfrak{B}}_1 :\mid \varrho_1 \mid< \tau_0\}$ and $\widetilde{\mathfrak{B}}_2{}' = \{\varrho_2 \in \widetilde{\mathfrak{B}}_2 :\mid \varrho_2 \mid< \tau_0\}$.

Hence by Lemma 6.2.3, $\mathfrak{K}(\varrho) = \mathfrak{K}_1(\varrho_1)\mathbf{e} + \mathfrak{K}_2(\varrho_2)\mathbf{e}^\dagger$ does not vanish in $\widetilde{\mathfrak{B}}_1{}'\mathbf{e} + \widetilde{\mathfrak{B}}_2{}'\mathbf{e}^\dagger = \boldsymbol{D}(0;\tau_0,\tau_0)$.
This proves the theorem. ∎

Remark 6.3.1 *Theorem 6.3.1 is in fact the bicomplex version of Enström-Kakeya theorem* $\{cf.[27]\}$.

Remark 6.3.2 *The following example with related figure ensures the validity of Theorem 6.3.1.*

Example 6.3.1 *Let* $\mathfrak{K}(\varrho) = e^{\varrho} - \frac{1}{2}$.

Then, $\mathfrak{K}(\varrho) = \frac{1}{2} + \varrho + \frac{\varrho^2}{2!} + \dots$.

Here, $\mu_0 = \frac{1}{2}, \mu_j = \frac{1}{j!}, j = 1, 2, \dots$.

We see that all the coefficients are positive real numbers and for $\mathfrak{c} = 1, t = \frac{1}{2}$,

$$\mathfrak{c}\mu_0 \geqslant t\mu_1 \geqslant t^2\mu_2 \geqslant \dots .$$

Hence by Theorem 6.3.1, $\mathfrak{K}(\varrho) = e^{\varrho} - \frac{1}{2}$ *does not vanish in* $\boldsymbol{D}(0; .5, .5)$.

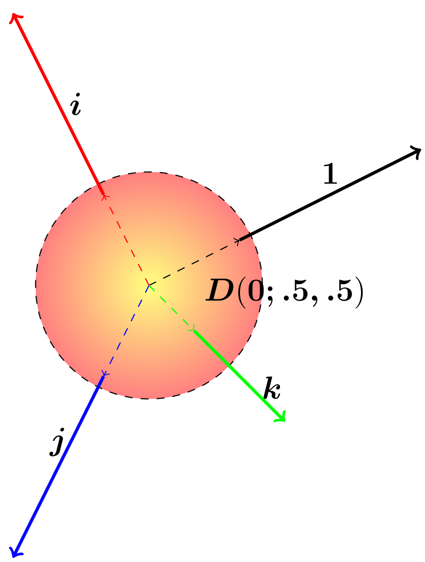

Figure 6.1: Zero free region of $\mathfrak{K}(z) = e^{\varrho} - \frac{1}{2}$

Theorem 6.3.2 *Let* $\mathfrak{K}(\varrho) = \sum_{j=0}^{\infty} \mu_j \varrho^j$ *be an entire function in* $\mathbb{C}_2$ *with real positive coefficients such that for some* $\mathfrak{c} \leqslant 1, t > 0$ *and* $\gamma \geqslant 1$

$$\mathfrak{c}\mu_0 \leqslant t\mu_1 \leqslant t^2\mu_2 \leqslant \dots \leqslant t^{\gamma}\mu_{\gamma} \geqslant t^{\gamma+1}\mu_{\gamma+1} \geqslant \dots .$$

Then $\mathfrak{K}(\varrho)$ *does not vanish in the open discus* $\boldsymbol{D}(0; \tau_0, \tau_0)$ *where* $\tau_0 = \frac{t\mu_0}{(1-2\mathfrak{c})\mu_0 + 2\mu_{\gamma}t^{\gamma}}$.

Proof. As in Theorem 6.3.1, we have

$$\begin{aligned}\mathfrak{K}(\varrho) &= \sum_{j=0}^{\infty}\mu_j\varrho_1^j\mathbf{e} + \sum_{j=0}^{\infty}\mu_j\varrho_2^j\mathbf{e}^{\dagger}\\ &= \mathfrak{K}_1(\varrho_1)\mathbf{e} + \mathfrak{K}_2(\varrho_2)\mathbf{e}^{\dagger}.\end{aligned}$$

Clearly, $\mathfrak{K}(\varrho)$ is holomorphic in any closed discus $\bar{\boldsymbol{D}}(0;t,t) \subset \mathbb{C}_2, 0 < t < \infty$ and so by Lemma 6.2.1, $\mathfrak{K}_1(\varrho_1)$ and $\mathfrak{K}_2(\varrho_2)$ both are holomorphic respectively in $\widetilde{\mathfrak{B}_1} = \{\varrho_1 \in \mathfrak{X}_1 :\mid \varrho_1 \mid\leqslant t\} \subset \mathbb{C}_1$ and $\widetilde{\mathfrak{B}_2} = \{\varrho_2 \in \mathfrak{X}_2 :\mid \varrho_2 \mid\leqslant t\} \subset \mathbb{C}_1$.

Also, $\lim\limits_{j\to\infty}\mu_j t^j = 0$.

Let

$$\begin{aligned}\mathfrak{F}(\varrho_1) &= (\varrho_1 - t)\mathfrak{K}_1(\varrho_1)\\ &= (\varrho_1 - t)(\mu_0 + \mu_1\varrho_1 + \mu_2\varrho_1^2 + \cdots + \mu_\gamma\varrho_1^\gamma + \mu_{\gamma+1}\varrho_1^{\gamma+1} + \dots)\\ &= -t\mu_0 + (\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \cdots + (\mu_{\gamma-1} - t\mu_\gamma)\varrho_1^\gamma +\\ &\qquad (\mu_\gamma - t\mu_{\gamma+1})\varrho_1^{\gamma+1} + \dots\\ &= -t\mu_0 + (\mu_0 - \mathfrak{c}\mu_0 + \mathfrak{c}\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \cdots + (\mu_{\gamma-1} - t\mu_\gamma)\varrho_1^\gamma +\\ &\qquad (\mu_\gamma - t\mu_{\gamma+1})\varrho_1^{\gamma+1} + \dots\\ &= -t\mu_0 + (1-\mathfrak{c})\mu_0\varrho_1 + (\mathfrak{c}\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \cdots + (\mu_{\gamma-1} - t\mu_\gamma)\varrho_1^\gamma +\\ &\qquad (\mu_\gamma - t\mu_{\gamma+1})\varrho_1^{\gamma+1} + \dots\end{aligned}$$

$$\text{i.e.,}\mathfrak{F}(\varrho_1) = -t\mu_0 + (1-\mathfrak{c})\mu_0\varrho_1 + \mathfrak{G}(\varrho_1), \text{ where}$$

$$\mathfrak{G}(\varrho_1) = (\mathfrak{c}\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \cdots + (\mu_{\gamma-1} - t\mu_\gamma)\varrho_1^\gamma + (\mu_\gamma - t\mu_{\gamma+1})\varrho_1^{\gamma+1} + \dots.$$

Also, for $|\varrho_1| = t$,

$$\begin{aligned}|\mathfrak{G}(\varrho_1)| &\leqslant |\mathfrak{c}\mu_0 - t\mu_1||\varrho_1| + |\mu_1 - t\mu_2||\varrho_1|^2 + \cdots + |\mu_{\gamma-1} - t\mu_\gamma||\varrho_1|^\gamma + |\mu_\gamma - t\mu_{\gamma+1}||\varrho_1|^{\gamma+1} + \dots.\\ &= (t\mu_1 - \mathfrak{c}\mu_0)t + (t\mu_2 - \mu_1)t^2 + \cdots + (t\mu_\gamma - \mu_{\gamma-1})t^\gamma + (\mu_\gamma - t\mu_{\gamma+1})t^{\gamma+1} + \dots\\ &= t^{\gamma+1}\mu_\gamma - \mathfrak{c}\mu_0 t + t^{\gamma+1}\mu_\gamma\\ &= (2t^\gamma\mu_\gamma - \mathfrak{c}\mu_0)t\ .\end{aligned}$$

Now, $\mathfrak{G}(\varrho_1)$ is holomorphic in $\mid \varrho_1 \mid\leqslant t$. Also, $\mathfrak{G}(0) = 0$ and $|\mathfrak{G}(\varrho_1)| \leqslant (2t^\gamma\mu_\gamma - \mathfrak{c}\mu_0)t$ for $\mid \varrho_1 \mid= t$. So using Lemma 6.2.3, we get that

$$|\mathfrak{G}(\varrho_1)| \leqslant \frac{(2t^\gamma\mu_\gamma - \mathfrak{c}\mu_0)t|\varrho_1|}{t}$$

$$= (2t^{\gamma}\mu_{\gamma} - \mathfrak{c}\mu_0)|\varrho_1| \text{ , for } |\varrho_1| \leqslant t.$$

For $|\varrho_1| < t$, we see that

$$\begin{aligned} |\mathfrak{F}(\varrho_1)| &\geqslant |-t\mu_0 + (1-\mathfrak{c})\mu_0\varrho_1| - |\mathfrak{G}(\varrho_1)| \\ &\geqslant t\mu_0 - (1-\mathfrak{c})\mu_0|\varrho_1| - (2t^{\gamma}\mu_{\gamma} - \mathfrak{c}\mu_0)|\varrho_1| \\ &= t\mu_0 - \{(1-2\mathfrak{c})\mu_0 + 2t^{\gamma}\mu_{\gamma}\}|\varrho_1| > 0 \text{ if } |\varrho_1| < \tfrac{t\mu_0}{(1-2\mathfrak{c})\mu_0 + 2t^{\gamma}\mu_{\gamma}} \ . \end{aligned}$$

Therefore for $|\varrho_1| < t$,

$|\mathfrak{K}_1(\varrho_1)| > 0$ if $|\varrho_1| < \tau_0$ where $\tau_0 = \frac{t\mu_0}{(1-2\mathfrak{c})\mu_0 + 2t^{\gamma}\mu_{\gamma}}$.

Similarly, $|\mathfrak{K}_2(\varrho_2)| > 0$ if $|\varrho_2| < \tau_0$.

Hence both $\mathfrak{K}_1(\varrho_1)$ and $\mathfrak{K}_2(\varrho_2)$ have no zeros respectively in $\widetilde{\mathfrak{B}_1}' = \{\varrho_1 \in \widetilde{\mathfrak{B}_1} :| \varrho_1 |< \tau_0\}$ and $\widetilde{\mathfrak{B}_2}' = \{\varrho_2 \in \widetilde{\mathfrak{B}_2} :| \varrho_2 |< \tau_0\}$.

Consequently, by Lemma 6.2.2, $\mathfrak{K}(\varrho) = \mathfrak{K}_1(\varrho_1)\mathbf{e} + \mathfrak{K}_2(\varrho_2)\mathbf{e}^{\dagger}$ has no zero in $\widetilde{\mathfrak{B}_1}'\mathbf{e} + \widetilde{\mathfrak{B}_2}'\mathbf{e}^{\dagger} = \boldsymbol{D}(0; \tau_0, \tau_0)$.
Thus the theorem is established. ■

Remark 6.3.3 *Theorem 6.3.2 can be regarded as the bicomplex version of Theorem 1 of A. [2].*

Remark 6.3.4 *The following example with given figure justifies the validity of Theorem 6.3.2.*

Example 6.3.2 *Let us consider* $\mathfrak{K}(\varrho) = e^{\varrho} + \frac{\varrho^2}{2} - \frac{\varrho}{2} + 2$.

Then, $\mathfrak{K}(\varrho) = 3 + \frac{\varrho}{2} + \varrho^2 + \frac{\varrho^3}{3!} + \dots$.

Here, $\mu_0 = 3, \mu_1 = \frac{1}{2}, \mu_2 = 1, \mu_j = \frac{1}{j!}, j = 3, 4, \dots$.

So it follows that all the coefficients are positive real numbers and for $\mathfrak{c} = \frac{1}{6}, t = 1$ *and* $\gamma = 2$,

$$\mathfrak{c}\mu_0 \leqslant t\mu_1 \leqslant t^2\mu_2 \leqslant \dots \leqslant t^{\gamma}\mu_{\gamma} \geqslant t^{\gamma+1}\mu_{\gamma+1} \geqslant \dots$$

where $\tau_0 = \frac{t\mu_0}{(1-2\mathfrak{c})\mu_0 + 2t^{\gamma}\mu_{\gamma}} = .75$.

Hence by Theorem 6.3.2, we obtain that $\mathfrak{K}(\varrho) = e^{\varrho} + \frac{\varrho^2}{2} - \frac{\varrho}{2} + 2$ *does not vanish in* $\boldsymbol{D}(0; .75, .75)$.

The bicomplex version of Theorem B of Aziz & Mohammad [3] can be seen in the next theorem.

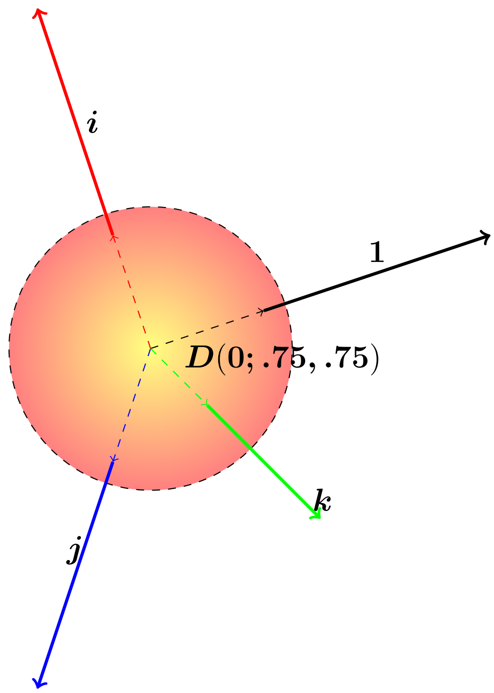

Figure 6.2: Zero free region of $\mathfrak{K}(\varrho) = e^{\varrho} + \frac{\varrho^2}{2} - \frac{\varrho}{2} + 2$

Theorem 6.3.3 *Let* $\mathfrak{K}(\varrho) = \sum_{j=0}^{\infty} \mu_j \varrho^j$ *be a bicomplex entire function with complex coefficients such that* $\mu_0 \neq 0$ *and for some* $t > 0$

$$|\mu_0| \geqslant t|\mu_1| \geqslant t^2|\mu_2| \geqslant \dots \ .$$

Then no zero of $\mathfrak{K}(\varrho)$ *lie in the open discus* $\boldsymbol{D}(0;\tau_0,\tau_0)$
where $\tau_0 = \frac{t|\mu_0|}{|\mu_0|+||\mu_0|-\mu_0|+2\sum_{j=1}^{\infty}||\mu_j|-\mu_j|t^j}$.

Proof. We can write $\mathfrak{K}(\varrho)$ as

$$\begin{aligned}\mathfrak{K}(\varrho) &= \sum_{j=0}^{\infty} \mu_j \varrho_1^j \mathbf{e} + \sum_{j=0}^{\infty} \mu_j \varrho_2^j \mathbf{e}^{\dagger} \\ &= \mathfrak{K}_1(\varrho_1)\mathbf{e} + \mathfrak{K}_2(\varrho_2)\mathbf{e}^{\dagger}.\end{aligned}$$

Since $\mathfrak{K}(\varrho)$ is holomorphic in any closed discus $\bar{\boldsymbol{D}}(0;t,t) \subset \mathbb{C}_2, 0 < t < \infty$, by Lemma 6.2.1, $\mathfrak{K}_1(\varrho_1)$ and $\mathfrak{K}_2(\varrho_2)$ both are holomorphic respectively in $\widetilde{\mathfrak{B}}_1 = \{\varrho_1 \in \mathfrak{X}_1 :| \varrho_1 |\leqslant t\} \subset \mathbb{C}_1$ and $\widetilde{\mathfrak{B}}_2 = \{\varrho_2 \in \mathfrak{X}_2 :| \varrho_2 |\leqslant t\} \subset \mathbb{C}_1$.

Also, $\lim_{j\to\infty} \mu_j t^j = 0$.

Let

$$\begin{aligned}\mathfrak{F}(\varrho_1) &= (\varrho_1 - t)\mathfrak{K}_1(\varrho_1)\\ &= -t\mu_0 + (\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + ...\\ &= -t\mu_0 + \mathfrak{G}(\varrho_1), \text{ where } \mathfrak{G}(\varrho_1) = \sum_{j=1}^{\infty}(\mu_{j-1} - t\mu_j)\varrho_1^j.\end{aligned}$$

For $|\varrho_1| = t$,

$$\begin{aligned}|\mathfrak{G}(\varrho_1)| &=| \textstyle\sum_{j=1}^{\infty}(\mu_{j-1} - t\mu_j)\varrho_1^j|\\ &=| \textstyle\sum_{j=1}^{\infty}\{(|\mu_{j-1}| - t|\mu_j|) + (\mu_{j-1} - |\mu_{j-1}|) + t(|\mu_j| - \mu_j)\}\varrho_1^j|\\ &\leqslant \textstyle\sum_{j=1}^{\infty}||\mu_{j-1}| - t|\mu_j||t^j + \sum_{j=1}^{\infty}||\mu_{j-1}| - \mu_{j-1}|t^j + \sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^{j+1}\\ &= \textstyle\sum_{j=1}^{\infty}(|\mu_{j-1}| - t|\mu_j|)t^j + \sum_{j=1}^{\infty}||\mu_{j-1}| - \mu_{j-1}|t^j + \sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^{j+1}\\ &= t(|\mu_0| + ||\mu_0| - \mu_0| + 2\textstyle\sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^j)\ .\end{aligned}$$

Since $\mathfrak{G}(\varrho_1)$ is holomorphic in $|\ \varrho_1\ | \leqslant t, \mathfrak{G}(0) = 0$ and $|\mathfrak{G}(\varrho_1)| \leqslant t(|\mu_0| + ||\mu_0| - \mu_0| + 2\sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^j)$ for $|\ \varrho_1\ |= t$, by Lemma 6.2.3, we get that

$$\begin{aligned}|\mathfrak{G}(\varrho_1)| &\leqslant \frac{t(|\mu_0| + ||\mu_0| - \mu_0| + 2\sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^j)|\varrho_1|}{t}\\ &= (|\mu_0| + ||\mu_0| - \mu_0| + 2\textstyle\sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^j)|\varrho_1| \text{ for } |\varrho_1| \leqslant t\ .\end{aligned}$$

Therefore for $|\varrho_1| < t$,

$$\begin{aligned}|\mathfrak{F}(\varrho_1)| &\geqslant t|\mu_0| - |\mathfrak{G}(\varrho_1)|\\ &\geqslant t|\mu_0| - (|\mu_0| + ||\mu_0| - \mu_0| + 2\sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^j)|\varrho_1| > 0\\ &\qquad \text{if } |\varrho_1| < \frac{|\mu_0|t}{|\mu_0| + ||\mu_0| - \mu_0| + 2\sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^j}\end{aligned}$$

Hence for $|\varrho_1| < t$,

$$|\mathfrak{K}_1(\varrho_1)| > 0 \text{ if } |\varrho_1| < \tau_0 \text{ where } \tau_0 = \frac{|\mu_0|t}{|\mu_0| + ||\mu_0| - \mu_0| + 2\sum_{j=1}^{\infty}||\mu_j| - \mu_j|t^j}\ .$$

Similarly for $|\varrho_1| < t$, $|\mathfrak{K}_2(\varrho_2)| > 0$ if $|\varrho_2| < \tau_0$.

Thus both $\mathfrak{K}_1(\varrho_1)$ and $\mathfrak{K}_2(\varrho_2)$ have no zeros respectively in $\widetilde{\mathfrak{B}_1}' = \{\varrho_1 \in \widetilde{\mathfrak{B}_1} :| \varrho_1 |< \tau_0\}$ and $\widetilde{\mathfrak{B}_2}' = \{\varrho_2 \in \widetilde{\mathfrak{B}_2} :| \varrho_2 |< \tau_0\}$.

Consequently, by Lemma 6.2.2, $\mathfrak{K}(\varrho) = \mathfrak{K}_1(\varrho_1)\mathbf{e}+\mathfrak{K}_2(\varrho_2)\mathbf{e}^\dagger$ has no zero in $\widetilde{\mathfrak{B}_1}'\mathbf{e}+\widetilde{\mathfrak{B}_2}'\mathbf{e}^\dagger = \boldsymbol{D}(0;\tau_0,\tau_0)$.
This completes the proof of the theorem. ■

Remark 6.3.5 *The following example with related figure ensures the validity of Theorem 6.3.3.*

Example 6.3.3 *Let* $\mathfrak{K}(\varrho) = 6 + (2 + 3i)\varrho + (3 - i)\varrho^2 + \varrho^3$.

Here, $\mu_0 = 6, \mu_1 = 2 + 3i, \mu_2 = 3 - i, \mu_3 = 1, \mu_j = 0, j = 4, 5, \ldots$.

For $t = 1$*, the condition of Theorem 6.3.3 is satisfied.*

Now, $\tau_0 = \frac{|\mu_0|t}{|\mu_0|+||\mu_0|-\mu_0|+2\sum_{j=1}^{\infty}||\mu_j|-\mu_j|t^j} \approx .4$.

Hence by Theorem 6.3.3, the polynomial $\mathfrak{K}(\varrho)$ *has no zero in* $\boldsymbol{D}(0;.4,.4)$.

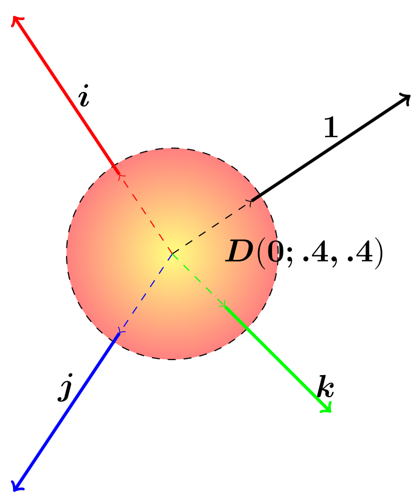

Figure 6.3: Zero free region of $\mathfrak{K}(\varrho) = 6 + (2 + 3i)\varrho + (3 - i)\varrho^2 + \varrho^3$

The following theorem can be deduced analogously to Theorem 4 of Aziz & Mohammad [3] under the treatment of bicomplex analysis.

Theorem 6.3.4 *Let* $\mathfrak{K}(\varrho) = \sum_{j=0}^{\infty}\mu_j\varrho^j$ *be an entire function in* $\mathbb{C}_2$ *with each* $\mu_j \in \mathbb{C}_1$ *and* $\mu_0 \neq 0$ *. Also, let for some* $t > 0, \mathfrak{c} \geqslant 1$

$$t^{\mathfrak{c}}|\mu_{\mathfrak{c}}| \geqslant t^{\mathfrak{c}+1}|\mu_{\mathfrak{c}+1}| \geqslant t^{\mathfrak{c}+2}|\mu_{\mathfrak{c}+2}| \geqslant \ldots .$$

Then $\mathfrak{K}(\varrho)$ *has no zero in the open discus* $\boldsymbol{D}(0;\tau_0,\tau_0)$,

where $\tau_0 = \frac{t|\mu_0|}{|\mu_0|+2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}}+||\mu_{\mathfrak{c}}|-\mu_{\mathfrak{c}}|t^{\mathfrak{c}}+2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j+2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j|-\mu_j|t^j}$.

Proof. $\mathfrak{K}(\varrho)$ can be written as

$$\begin{aligned}\mathfrak{K}(\varrho) &= \sum_{j=0}^{\infty}\mu_j\varrho_1^j\mathbf{e} + \sum_{j=0}^{\infty}\mu_j\varrho_2^j\mathbf{e}^{\dagger}\\ &= \mathfrak{K}_1(\varrho_1)\mathbf{e} + \mathfrak{K}_2(\varrho_2)\mathbf{e}^{\dagger}.\end{aligned}$$

As $\mathfrak{K}(\varrho)$ is holomorphic in any closed discus $\bar{\boldsymbol{D}}(0;t,t) \subset \mathbb{C}_2, 0 < t < \infty$, So by Lemma 6.2.1, $\mathfrak{K}_1(\varrho_1)$ and $\mathfrak{K}_2(\varrho_2)$ both are holomorphic respectively in $\widetilde{\mathfrak{B}_1} = \{\varrho_1 \in \mathfrak{X}_1 :| \varrho_1 |\leqslant t\} \subset \mathbb{C}_1$ and $\widetilde{\mathfrak{B}_2} = \{\varrho_2 \in \mathfrak{X}_2 :| \varrho_2 |\leqslant t\} \subset \mathbb{C}_1$.

Clearly, $\lim_{j\to\infty}\mu_j t^j = 0$.

Let

$$\begin{aligned}\mathfrak{F}(\varrho_1) &= (\varrho_1 - t)\mathfrak{K}_1(\varrho_1)\\ &= -t\mu_0 + (\mu_0 - t\mu_1)\varrho_1 + (\mu_1 - t\mu_2)\varrho_1^2 + \ldots\\ \text{i.e., } \mathfrak{F}(\varrho_1) &= -t\mu_0 + \mathfrak{G}(\varrho_1), \text{ where } \mathfrak{G}(\varrho_1) = \sum_{j=1}^{\infty}(\mu_{j-1} - t\mu_j)\varrho_1^j.\end{aligned}$$

Now, for $|\varrho_1| = t$,

$$\begin{aligned}|\mathfrak{G}(\varrho_1)| &=|\sum_{j=1}^{\infty}(\mu_{j-1} - t\mu_j)\varrho_1^j|\\ &\leqslant|\sum_{j=1}^{\mathfrak{c}}(\mu_{j-1} - t\mu_j)\varrho_1^j|+|\sum_{j=\mathfrak{c}+1}^{\infty}(\mu_{j-1} - t\mu_j)\varrho_1^j|\\ &\leqslant \sum_{j=1}^{\mathfrak{c}}|\mu_{j-1} - t\mu_j|t^j+|\sum_{j=\mathfrak{c}+1}^{\infty}\{(|\mu_{j-1}| - t|\mu_j|) + (\mu_{j-1}-|\mu_{j-1}|) + t(|\mu_j| - \mu_j)\}\varrho_1^j|\\ &\leqslant \sum_{j=1}^{\mathfrak{c}}(|\mu_{j-1}| + t|\mu_j|)t^j + \sum_{j=\mathfrak{c}+1}^{\infty}||\mu_{j-1}| - t|\mu_j||t^j + \sum_{j=\mathfrak{c}+1}^{\infty}||\mu_{j-1}| - \mu_{j-1}|t^j +\\ &\qquad \sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^{j+1}\end{aligned}$$

$$= (|\mu_0|+|\mu_{\mathfrak{c}}|t^{\mathfrak{c}})t + 2t\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j + \sum_{j=\mathfrak{c}+1}^{\infty}(|\mu_{j-1}| - t|\mu_j|)t^j + ||\mu_{\mathfrak{c}}| - \mu_{\mathfrak{c}}|t^{\mathfrak{c}+1} + 2t\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^j$$

$$= t(|\mu_0| + 2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}} + ||\mu_{\mathfrak{c}}| - \mu_{\mathfrak{c}}|t^{\mathfrak{c}} + 2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j + 2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^j).$$

Clearly, $\mathfrak{G}(\varrho_1)$ is holomorphic in $|\ \varrho_1\ | \leqslant t, \mathfrak{G}(0) = 0$ and $|\mathfrak{G}(\varrho_1)| \leqslant t(|\mu_0| + 2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}} + ||\mu_{\mathfrak{c}}| - \mu_{\mathfrak{c}}|t^{\mathfrak{c}} + 2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j + 2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^j)$ for $|\varrho_1| = t$.
Hence by Lemma 6.2.3, we get that

$$|\mathfrak{G}(\varrho_1)| \leqslant \frac{t(|\mu_0| + 2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}} + ||\mu_{\mathfrak{c}}| - \mu_{\mathfrak{c}}|t^{\mathfrak{c}} + 2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j + 2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^j)|\varrho_1|}{t}$$

$$= (|\mu_0| + 2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}} + ||\mu_{\mathfrak{c}}| - \mu_{\mathfrak{c}}|t^{\mathfrak{c}} + 2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j + 2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^j)|\varrho_1| \text{ for } |\varrho_1| \leqslant t.$$

Therefore for $|\varrho_1| < t$, we get that

$$|\mathfrak{F}(\varrho_1)| \geqslant t|\mu_0| - |\mathfrak{G}(\varrho_1)|$$

$$\geqslant t|\mu_0| - (|\mu_0| + 2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}} + ||\mu_{\mathfrak{c}}| - \mu_{\mathfrak{c}}|t^{\mathfrak{c}} + 2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j + 2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^j)|\varrho_1| > 0$$

$$\text{if } |\varrho_1| < \frac{t|\mu_0|}{|\mu_0| + 2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}} + ||\mu_{\mathfrak{c}}| - \mu_{\mathfrak{c}}|t^{\mathfrak{c}} + 2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j + 2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^j}.$$

Consequently, for $|\varrho_1| < t$,

$|\mathfrak{K}_1(\varrho_1)| > 0$ if $|\varrho_1| < \tau_0$ where $\tau_0 = \frac{t|\mu_0|}{|\mu_0| + 2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}} + ||\mu_{\mathfrak{c}}| - \mu_{\mathfrak{c}}|t^{\mathfrak{c}} + 2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j + 2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j| - \mu_j|t^j}$.

Similarly, for $|\varrho_1| < t$, $|\mathfrak{K}_2(\varrho_2)| > 0$ if $|\varrho_2| < \tau_0$.

Therefore $\mathfrak{K}_1(\varrho_1)$ and $\mathfrak{K}_2(\varrho_2)$ both have no zeros respectively in $\widetilde{\mathfrak{B}_1}' = \{\varrho_1 \in \widetilde{\mathfrak{B}_1} :| \varrho_1 |< \tau_0\}$ and $\widetilde{\mathfrak{B}_2}' = \{\varrho_2 \in \widetilde{\mathfrak{B}_2} :| \varrho_2 |< \tau_0\}$.

Finally by Lemma 6.2.2, $\mathfrak{K}(\varrho) = \mathfrak{K}_1(\varrho_1)\mathbf{e} + \mathfrak{K}_2(\varrho_2)\mathbf{e}^{\dagger}$ has no zero in $\widetilde{\mathfrak{B}_1}'\mathbf{e} + \widetilde{X_2}'\mathbf{e}^{\dagger} = \boldsymbol{D}(0; \tau_0, \tau_0)$.
Thus the theorem is proved. ■

Remark 6.3.6 *The following example with respective figure justifies the validity of Theorem 6.3.4.*

Example 6.3.4 *Let* $\mathfrak{K}(\varrho) = e^{\varrho} + \frac{\varrho^2}{2} + (-\frac{1}{2} + \frac{1}{2}i)\varrho + 2 + 4i$.

Then, $\mathfrak{K}(\varrho) = 3 + 4i + (\frac{1}{2} + \frac{1}{2}i)\varrho + \varrho^2 + \frac{\varrho^3}{3!} + \ldots$.

Here, $\mu_0 = 3 + 4i, \mu_1 = \frac{1}{2} + \frac{1}{2}i, \mu_2 = 1, \mu_j = \frac{1}{j!}, j = 3, 4, \ldots$.

For $t = 1, \mathfrak{c} = 2$,

$$t^{\mathfrak{c}}|\mu_{\mathfrak{c}}| \geqslant t^{\mathfrak{c}+1}|\mu_{\mathfrak{c}+1}| \geqslant t^{\mathfrak{c}+2}|\mu_{\mathfrak{c}+2}| \geqslant \ldots .$$

Now, $\tau_0 = \frac{t|\mu_0|}{|\mu_0|+2|\mu_{\mathfrak{c}}|t^{\mathfrak{c}}+||\mu_{\mathfrak{c}}|-\mu_{\mathfrak{c}}|t^{\mathfrak{c}}+2\sum_{j=1}^{\mathfrak{c}-1}|\mu_j|t^j+2\sum_{j=\mathfrak{c}+1}^{\infty}||\mu_j|-\mu_j|t^j} \approx .59$.

Hence by Theorem 6.3.4 ,

$$\mathfrak{K}(\varrho) = e^{\varrho} + \frac{\varrho^2}{2} + (-\frac{1}{2} + \frac{1}{2}i)\varrho + 2 + 4i$$

has no zero in $\boldsymbol{D}(0; .59, .59)$.

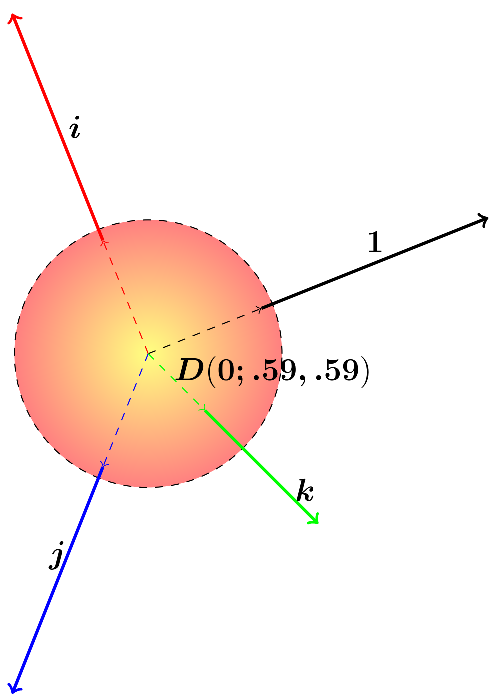

Figure 6.4: Zero free region of $\mathfrak{K}(\varrho) = e^{\varrho} + \frac{\varrho^2}{2} + (-\frac{1}{2} + \frac{1}{2}i)\varrho + 2 + 4i$

6.4 Future Prospects.

In the line of the works as carried out in this chapter one may think of the extension of the results obtained dealing with n-dimensional bicomplex numbers with the help of the idempotents $0, 1, \frac{1\pm i_1 i_2}{2}, \frac{1\pm i_2 i_3}{2}, ..., \frac{1\pm i_{n-1} i_n}{2}$ in $\mathbb{C}_n$. As a consequence, the problem of taking the coeffiecients of the power series in $\mathbb{C}_n$ is still virgin and may be considered as an open problem to the future researchers of this branch.

On the other hand, generalization of the Enstr$\ddot{o}$m Kakeya theorem can be thought of through polynomial coefficients relaxation. In fact this study introduces the generalizations of this theorem selectively relaxing polynomial coefficients. Analysing these relationships classical theorems understanding, revealing fresh insights and mathematical opportunities within the Enstr$\ddot{o}$m-Kakeya context are enriched. This approach provides valuable insights into the distribution of zeros across a broader spectrum of polynomials, offering new avenues for understanding and characterizing polynomial behaviour.

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CHAPTER SEVEN

COMMON FIXED POINT THEOREMS IN BICOMPLEX VALUED METRIC SPACES UNDER BOTH RATIONAL TYPE CONTRACTION MAPPINGS SATISFYING E. A. PROPERTY AND INTIMATE MAPPINGS

Chapter 7

Common fixed point theorems in bicomplex valued metric spaces under both rational type contraction mappings satisfying E. A. property and intimate mappings

7.1 Existing Literature

During the last fifty years, fixed point theories in complex valued metric spaces are emerging areas of works in the field of the complex as well as functional analysis.The fixed point theorem, generally known as the Banach's contraction mapping principle [5] appeared in explicit form in Banach's thesis in 1922. The famous theorem states that Let $(\mathfrak{X}, p)$ be a complete metric space and T be a mapping of $\mathfrak{X}$ into itself satisfying $p(Tu, Tv) \leqslant kp(u, v), \forall u, v \in \mathfrak{X}$, where k is a constant in $(0, 1)$. Then, T has a unique fixed point $x^* \in \mathfrak{X}$. Banach's fixed point theorem plays a major role in the fixed point theory. Rajput & Singh [51] and Meena [39] respectively proved some common fixed point theorems under rational type contraction mappings and intimate mappings in the set bicomplex numbers. Many properties on the set bicomplex numbers $\mathbb{C}_2$ are scattered over a number of books and articles $\{$*cf.*[25],[46]& [49]$\}$. Searching for special algebras, Segre published a paper [52] in which he treated an algebra whose elements were bicomplex numbers. Rochon and Shapiro [49] presented some varieties of algebraic properties of both bicomplex numbers and hyperbolic numbers in a unified manner. Elena *et al.* [25] showed how to introduce elementary functions such as polynomials, exponentials and trigonometric functions in this algebra as well as their inverses which is not possible in the case of quaternions incidentally. They showed how these elementary functions enjoy the properties that are very similar to those enjoyed by their complex counterparts. Some interesting results on fixed point theory are established by Tripathy

et al.{*cf.* [62], [63] & [64]} using fuzzy metric. Azam *et al.* [4] made a generalization by introducing a complex valued metric spaces using some contractive type conditions. Although there are several number of generalization such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, D-metric spaces and cone metric spaces including bicomplex valued metric spaces, yet the area of research in bicomplex valued metric spaces is not expanded to a remarkable stage compared to the other metric spaces till now. Jebril *et al.* {[31]&[32]} and Choi *et al.* [13] respectively investigated some fixed point theorems under rational contractions for a pair of mappings and with two weakly compatible mappings in $\mathbb{C}_2$. Recently Rouzkard & Imdad [50] extended and improved the common fixed point theorems more general than the result of Azam *et al.* [4].
We write regular complex number as $w = w_1 + \mathbf{i}w_2$ where w_1 and w_2 are real and $\mathbf{i}^2 = -1$. Let $\mathbb{R}$ and $\mathbb{C}_1$ be the set of real and complex numbers respectively and w_1 , $w_2 \in \mathbb{C}_1$. The partial order relation $\precsim$ on $\mathbb{C}_1$ is defined as follows:

$$w_1 \precsim w_2 \text{ if and only if } Re\,(w_1) \leqslant Re\,(w_2) \text{ and } Im\,(w_1) \leqslant Im\,(w_2)\,.$$

Thus $w_1 \precsim w_2$ if one of the following conditions is satisfied $(i)\, Re\,(w_1) = Re\,(w_2)$ and $Im\,(w_1) = Im\,(w_2)$, $(ii)\, Re\,(w_1) < Re\,(w_2)$ and $Im\,(w_1) = Im\,(w_2)$, $(iii)\, Re\,(w_1) = Re\,(w_2)$ and $Im\,(w_1) < Im\,(w_2)$ and $(iv)\, Re\,(w_1) < Re\,(w_2)$ and $Im\,(w_1) < Im\,(w_2)$.

We write $w_1 \precnsim w_2$ if $w_1 \precsim w_2$ and $w_1 \neq w_2$ i.e., one of $(ii)\,,(iii)$ and (iv) is satisfied and we write $w_1 \prec w_2$ if only (iv) is satisfied.

7.2 Definitions and Notations

Azam *et al.* [4] defined the complex valued metric space in the following way.

Definition 7.2.1 *Let $\mathfrak{C}$ be a non empty set whereas $\mathbb{C}_1$ be the set of complex numbers. Suppose that the mapping $\breve{p} : \mathfrak{C} \times \mathfrak{C} \to \mathbb{C}_1$, satisfies the following conditions*

(d_1) $0 \precsim \breve{p}\,(u, w)$, *for all* $u, w \in \mathfrak{C}$ *and* $\breve{p}\,(u, w) = 0$ *if and only if* $u = w$;

(d_2) $\breve{p}\,(u, w) = \breve{p}\,(w, u)$ *for all* $u, w \in \mathfrak{C}$;

(d_3) $\breve{p}\,(u, w) \precsim \breve{p}\,(u, t) + \breve{p}\,(t, w)$, *for all* $u, t, w \in \mathfrak{C}$.

Then $\breve{p}$ is called a complex valued metric on $\mathfrak{C}$ and $(\mathfrak{C}, \breve{p})$ is called a complex valued metric space.

The space $\mathbb{C}_2$ is the first in an infinite sequence of multicomplex spaces which are generalization of $\mathbb{C}_1$.

A bicomplex number $\eta = r_0 + \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3$ $(u_k \in \mathbb{R};\ k = 0,\ 1,\ 2,\ 3)$ is said to be degenerated if the matrix $\begin{pmatrix} r_0 & r_1 \\ r_2 & r_3 \end{pmatrix}$ is degenerated i.e., if the determinant

$$\Delta(u) = \begin{vmatrix} r_0 & r_1 \\ r_2 & r_3 \end{vmatrix} = r_0 r_3 - r_1 r_2 = 0.$$

The partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ was defined by Choi *et al.* [13] as $u \precsim_{i_2} v$ if and only if $u_1 \precsim u_2$ and $v_1 \precsim v_2$, where $u_1, u_2, v_1, v_2 \in \mathbb{C}_1$. The bicomplex valued metric $\Upsilon_{\wp} : \mathfrak{B} \times \mathfrak{B} \to \mathbb{C}_2$ on a non-empty set $\mathfrak{B}$ and the structure $(\mathfrak{B}, \Upsilon_{\wp})$ on $\mathbb{C}_2$ were defined by Choi *et al.* [13] accordingly.

Considering $\eta = r_0 + \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3$ $(r_k \in \mathbb{R};\ k = 0, 1, 2, 3)$ and by the Definition 1.1.11, this can be defined as

$$\|w\| = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}.$$

One can easily verify that

$$0 \precsim_{i_2} u \precsim_{i_2} v \Rightarrow \|u\| \leqslant \|v\|\,;\ \|u+v\| \leqslant \|u\| + \|v\|\,;\ \|\alpha\, u\| = \alpha\, \|u\|\,; \|u\| \leqslant \|1+u\|\,,$$

for any $u,\ v \in \mathbb{C}_2$ and $\alpha \in \mathbb{R}$. If we consider the case that $0 \precnsim_{i_2} u = r_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3$ $(p_k \in \mathbb{R};\ k = 0, 1, 2, 3)$ (i.e., at least one of p_k's is positive) and u is degenerated, then $\gamma = (p_0^2 + p_1^2 + p_2^2 + p_3^2)^2 - 4\,(p_0p_3 - p_1p_2)^2$ will be positive and therefore u willbe invertible.

By the deduction of Rochon & Shapiro[49], we get the following results

(i) $\|uv\| \leqslant \sqrt{2}\,\|u\|\,\|v\|$for any $u,\ v \in \mathbb{C}_2$; .

(ii) $\|uv\| = \|u\|\,\|v\|$for any $u,\ v \in \mathbb{C}_2$ with at least one of them is degenerated;

(iii) $\left\|\frac{1}{u}\right\| = \frac{1}{\|u\|}$ for any degenerated bicomplex number u with $0 \precnsim_{i_2} u$.

Definition 7.2.2 *[4] Let $\{\pi_n\}$ be sequence in $\mathfrak{B}$ and $\pi \in \mathfrak{B}$. If for every $c \in \mathbb{C}_2$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $\Upsilon_{\wp}(\pi_n, \pi) \prec c$ for all $n > n_0$, then π is called the limit of $\{\pi_n\}$ and we write $\lim_{n\to\infty} \pi_n = \pi$ or $\pi_n \to \pi$ as $n \to \infty$.*

Definition 7.2.3 *[4] If every Cauchy sequence is convergent in $\mathbb{C}_2$ then the space is called a complete bicomplex valued metric space.*

Definition 7.2.4 *[7] Let $\mathfrak{f}$ and $\mathfrak{g}$ be two self-maps defined on a set $\mathfrak{B}$. Then $\mathfrak{f}$ and $\mathfrak{g}$ are said to be weakly compatible if they commute at their coincidence points.*

Definition 7.2.5 *[65] Let $\mathfrak{h}_1, \mathfrak{h}_2 : \mathfrak{B} \to \mathfrak{B}$ be two self mappings of a bicomplex valued metric space $(\mathfrak{B}, \Upsilon_{\wp})$. The pair $(\mathfrak{h}_1, \mathfrak{h}_2)$ are said to satisfy E. A. property if there exists a sequence $\{\pi_n\}$ in $\mathfrak{B}$ such that $\lim_{n\to\infty} \mathfrak{h}_1\pi_n = \lim_{n\to\infty} \mathfrak{h}_2\pi_n = t$ for some $t \in \mathfrak{B}$.*

Definition 7.2.6 *[57] The self mappings $\mathfrak{h}_1, \mathfrak{h}_2 : \mathfrak{B} \to \mathfrak{B}$ are said to satisfy the common limit in the range of $\mathfrak{h}_1$ property ($CLRs$ property) if there exists a sequence $\{\pi_n\}$ in $\mathfrak{B}$ such that $\lim_{n\to\infty} \mathfrak{h}_1\pi_n = \lim_{n\to\infty} \mathfrak{h}_2\pi_n = u$ for some $u \in \mathfrak{h}_1(\mathfrak{B})$.*

Definition 7.2.7 *[65] Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be self maps on a bicomplex valued metric space $(\mathfrak{B}, \Upsilon_{\wp})$. Then the pair $\{\mathfrak{h}_1, \mathfrak{h}_2\}$ is said to be $\mathfrak{h}_2-$intimate if and only if $\alpha\Upsilon_{\wp}(\mathfrak{h}_2\mathfrak{h}_1\pi_n, \mathfrak{h}_2\pi_n)$ $\precsim \alpha\Upsilon_{\wp}(\mathfrak{h}_1\mathfrak{h}_1\pi_n, \mathfrak{h}_1\pi_n)$, where $\alpha = \limsup\{\pi_n\}$ or $\liminf\{\pi_n\}$ is a sequence in $\mathfrak{B}$ such that $\lim_{n\to\infty} \mathfrak{h}_1\pi_n = \lim_{n\to\infty} \mathfrak{h}_2\pi_n = t$ for some t in $\mathfrak{B}$.*

Some common fixed point results are established by Rajput & Singh [51] for rational type contraction mapping in $\mathbb{C}_1$ on which they have proved the following theorem.

Theorem 7.2.1 *[51] Let $(\mathfrak{C}, \breve{p})$ be a complex valued metric space and $F_1, F_2, F_3, F_4 : \mathfrak{C} \to \mathfrak{C}$ be four self mappings satisfying the following conditions*

(i) $F_1(\mathfrak{C}) \subseteq F_4(\mathfrak{C})$, $F_2(\mathfrak{C}) \subseteq F_3(\mathfrak{C})$,

(ii) For all $u, v \in \mathfrak{C}$ and $0 < \alpha < 1$,

$$\breve{p}(F_1u, F_2v) \precsim \alpha \frac{[\breve{p}(F_1u, F_3u)\,\breve{p}(F_1u, F_4v) + \breve{p}(F_2v, F_4v)\,\breve{p}(F_2v, F_3u)]}{\breve{p}(F_1u, F_4v) + \breve{p}(F_2v, F_3u)} + \beta \frac{\left[\{\breve{p}(F_1u, F_4v)\}^2 + \{\breve{p}(F_2v, F_3u)\}^2\right]}{\breve{p}(F_1u, F_4v) + \breve{p}(F_2v, F_3u)},$$

(iii) The pairs (F_1, F_3) and (F_2, F_4) are weakly compatible and

(iv) The pair (F_1, F_3) or (F_2, F_4) satisfies E. A. property if the range of mappings $F_3(\mathfrak{C})$ or $F_4(\mathfrak{C})$ is closed subspace of $\mathfrak{C}$ then F_1, F_2, F_3 and F_4 have a unique common fixed point in $\mathfrak{C}$.

Meena [39] investigated a common fixed point for intimate mappings in $\mathbb{C}_1$. He proved the following theorem in his paper.

Theorem 7.2.2 *[25] Let F_1, F_2, F_3 and F_4 be the four mappings from a complex valued metric space $(\mathfrak{C}, \breve{p})$ into itself, such that*

(a) $F_1(\mathfrak{C}) \subset F_4(\mathfrak{C})$ and $F_2(\mathfrak{C}) \subset F_3(\mathfrak{C})$,

$$(b)\, \breve{p}(F_1u, F_2v) \precsim \alpha \breve{p}(F_3u, F_4v) + \beta \frac{\breve{p}(F_1u, F_3u) \cdot \breve{p}(F_2v, F_4v)}{\breve{p}(F_1u, F_4v) + \breve{p}(F_3u, F_2v) + \breve{p}(F_3u, F_4v)},$$

for all $u, v \in \mathfrak{C}$ and $\breve{p}(F_1u, F_4v) + \breve{p}(F_3u, F_2v) + \breve{p}(F_3u, F_4v) \neq 0$, where α, β are non-negative real numbers with $\alpha + \beta < 1$,

(c) (F_1, F_3) is F_3-intimate and (F_2, F_4) is F_4-intimate and

(d) $F_3(\mathfrak{C})$ is complete.

Then F_1, F_2, F_3 and F_4 have a unique common fixed point in $\mathfrak{C}$.

Our results are the generalizations and as well as the extensions of the above theorems which are established by Rajput & Singh [51] and Meena [39]. Here the results of Rochon *et al.*[49] and Elena *et al.* [25] have helped us. Also we have taken some concepts from the papers of Choi *et al.* [13] and Jebril *et al.* [31].

7.3 Main Result

In this section we prove some theorems with supporting lemma and examples.

Theorem 7.3.1 *Let* $(\mathfrak{B}, \Upsilon_{\wp})$ *be a bicomplex valued metric space and* $\Psi_1, \Psi_2, \Psi_3, \Psi_4 : \mathfrak{B} \to \mathfrak{B}$ *are four self mappings on* $\mathfrak{B} \subseteq \mathbb{C}_2$ *satisfying the following conditions*

(i) $\Psi_1(\mathfrak{B}) \subseteq \Psi_4(\mathfrak{B}), \Psi_2(\mathfrak{B}) \subseteq \Psi_3(\mathfrak{B});$

(ii) For all $\tau, \tau' \in \mathfrak{B}$ *and* $0 < \alpha, \beta, \alpha + \beta < 1$,

$$\Upsilon_{\wp}(\Psi_1\tau, \Psi_2\tau') \precsim_{i_2} \alpha \frac{[\Upsilon_{\wp}(\Psi_1\tau, \Psi_3\tau)\Upsilon_{\wp}(\Psi_1\tau, \Psi_4\tau') + \Upsilon_{\wp}(\Psi_2\tau', \Psi_4\tau')\Upsilon_{\wp}(\Psi_2\tau', \Psi_3\tau)]}{\Upsilon_{\wp}(\Psi_1\tau, \Psi_4\tau') + \Upsilon_{\wp}(\Psi_2\tau', \Psi_3\tau)}$$
$$+\beta \frac{\left[\{\Upsilon_{\wp}(\Psi_1\tau, \Psi_4\tau')\}^2 + \{\Upsilon_{\wp}(\Psi_2\tau', \Psi_3\tau)\}^2\right]}{\Upsilon_{\wp}(\Psi_1\tau, \Psi_4\tau') + \Upsilon_{\wp}(\Psi_2\tau', \Psi_3\tau)};$$

(iii) The pairs (Ψ_1, Ψ_3) *and* (Ψ_2, Ψ_4) *are weakly compatible and*

(iv) The pair (Ψ_1, Ψ_3) *or* (Ψ_2, Ψ_4) *satisfies E. A. property.*

If the range of mappings $\Psi_3(\mathfrak{B})$ *or* $\Psi_4(\mathfrak{B})$ *is closed subspace of* $\mathbb{C}_2$ *then* Ψ_1, Ψ_2, Ψ_3 *and* Ψ_4 *have a unique common fixed point in* $\mathbb{C}_2$.

Proof. Suppose that the pair (Ψ_2, Ψ_4) satisfies E. A. property in $\mathbb{C}_2$. Then there exists a sequence $\{\pi_n\}$ in $\mathbb{C}_2$ such that $\lim_{n\to\infty} \Psi_2\pi_n = \lim_{n\to\infty} \Psi_4\pi_n = t$ for some $t \in \mathbb{C}_2$. Further since $\Psi_2(\mathfrak{B}) \subseteq \Psi_3(\mathfrak{B})$, there exists a sequence $\{\pi'_n\}$ in $\mathbb{C}_2$ such that $\Psi_2\pi_n = \Psi_3\pi'_n$. Therefore $\lim_{n\to\infty} \Psi_3\pi'_n = t$. Now we claim that $\lim_{n\to\infty} \Psi_1\pi'_n = t$. If possible,let $\lim_{n\to\infty} \pi'_n = r \neq t$. Then by putting $\tau = \pi'_n,\ \tau' = \pi_n$ in the condition (ii) we have

$$\Upsilon_{\wp}(\Psi_1\pi'_n, \Psi_2\pi_n) \precsim_{i_2}$$
$$\alpha \frac{[\Upsilon_{\wp}(\Psi_1\pi'_n, \Psi_3\pi'_n)\Upsilon_{\wp}(\Psi_1\pi'_n, \Psi_4\pi_n) + \Upsilon_{\wp}(\Psi_2\pi_n, \Psi_4\pi_n)\Upsilon_{\wp}(\Psi_2\pi_n, \Psi_3\pi'_n)]}{\Upsilon_{\wp}(\Psi_1\pi'_n, \Psi_4\pi_n) + \Upsilon_{\wp}(\Psi_2\pi_n, \Psi_3\pi'_n)}$$
$$+\beta \frac{\left[\{\Upsilon_{\wp}(\Psi_1\pi'_n, \Psi_4\pi_n)\}^2 + \{\Upsilon_{\wp}(\Psi_2\pi_n, \Psi_3\pi'_n)\}^2\right]}{\Upsilon_{\wp}(\Psi_1\pi'_n, \Psi_4\pi_n) + \Upsilon_{\wp}(\Psi_2\pi_n, \Psi_3\pi'_n)}.$$

By taking limit as $n \to \infty$, we get that

$$\Upsilon_{\wp}(r,t) \precsim_{i_2} \alpha \frac{[\Upsilon_{\wp}(r,t)\Upsilon_{\wp}(r,t) + \Upsilon_{\wp}(t,t)\Upsilon_{\wp}(t,t)]}{\Upsilon_{\wp}(r,t) + \Upsilon_{\wp}(t,t)} + \beta \frac{\left[\{\Upsilon_{\wp}(r,t)\}^2 + \{\Upsilon_{\wp}(t,t)\}^2\right]}{\Upsilon_{\wp}(r,t) + \Upsilon_{\wp}(t,t)}$$

i.e, $\Upsilon_{\wp}(r,t) \precsim_{i_2} \alpha \frac{\{\Upsilon_{\wp}(r,t)\}^2}{\Upsilon_{\wp}(r,t)} + \beta \frac{\{\Upsilon_{\wp}(r,t)\}^2}{\Upsilon_{\wp}(r,t)}$

i.e., $\Upsilon_{\wp}(r,t) \precsim_{i_2} (\alpha + \beta)\Upsilon_{\wp}(r,t),$

which implies

$$(1 - \alpha - \beta)\Upsilon_{\wp}(r,t) \precsim_{i_2} 0.$$

Therefore we have $\|\Upsilon_{\wp}(r,t)\| \leqslant 0$. Hence $r = t$. i.e., $\lim_{n\to\infty} \Psi_1\pi'_n = \lim_{n\to\infty} \Psi_2\pi_n = t$. Now suppose that $\Psi_3(\mathbb{C}_2)$ is a closed subspace of $\mathbb{C}_2$. Then $t = \Psi_3 u$ for some $u \in \mathbb{C}_2$.

Subsequently we have $\lim_{n\to\infty} \Psi_1 \pi'_n = \lim_{n\to\infty} \Psi_2 \pi_n = \lim_{n\to\infty} \Psi_3 \pi'_n = \lim_{n\to\infty} \Psi_4 \pi_n = t = \Psi_3 u$. Now we prove that $\Psi_1 u = \Psi_3 u$ i.e., $\Psi_1 u = t$. By putting $\tau = u$ and $\tau' = \pi_n$ in condition (ii) we get that

$$\begin{aligned}\Upsilon_\wp\left(\Psi_1 u, \Psi_2 \pi_n\right) \precsim_{i_2} & \\ & \alpha \frac{\left[\Upsilon_\wp\left(\Psi_1 u, \Psi_3 u\right) \Upsilon_\wp\left(\Psi_1 u, \Psi_4 \pi_n\right) + \Upsilon_\wp\left(\Psi_2 \pi_n, \Psi_4 \pi_n\right) \Upsilon_\wp\left(\Psi_2 \pi_n, \Psi_3 u\right)\right]}{\Upsilon_\wp\left(\Psi_1 u, \Psi_4 \pi_n\right) + \Upsilon_\wp\left(\Psi_2 \pi_n, \Psi_3 u\right)} \\ & + \beta \frac{\left[\left\{\Upsilon_\wp\left(\Psi_1 u, \Psi_4 \pi_n\right)\right\}^2 + \left\{\Upsilon_\wp\left(\Psi_2 \pi_n, \Psi_3 u\right)\right\}^2\right]}{\Upsilon_\wp\left(\Psi_1 u, \Psi_4 \pi_n\right) + \Upsilon_\wp\left(\Psi_2 \pi_n, \Psi_3 u\right)}.\end{aligned}$$

Taking limit as $n \to \infty$ in the above, we have

$$\Upsilon_\wp\left(\Psi_1 u, t\right) \precsim_{i_2} \alpha \frac{\left\{\Upsilon_\wp\left(\Psi_1 u, t\right)\right\}^2}{\Upsilon_\wp\left(\Psi_1 u, t\right)} + \beta \frac{\left\{\Upsilon_\wp\left(\Psi_1 u, t\right)\right\}^2}{\Upsilon_\wp\left(\Psi_1 u, t\right)},$$

which implies $(1 - \alpha - \beta) \Upsilon_\wp\left(\Psi_1 u, t\right) \precsim_{i_2} 0$. Therefore we get $\left\|\Upsilon_\wp\left(\Psi_1 u, t\right)\right\| \leqslant 0$. So $\Psi_1 u = t = \Psi_3 u$. Hence u is a coincidence point of (Ψ_1, Ψ_3). Now the weak compatibility of pair (Ψ_1, Ψ_3) implies that $\Psi_1 \Psi_3 u = \Psi_3 \Psi_1 u$ or $\Psi_1 t = \Psi_3 t$. On the other hand since $\Psi_1\left(\mathbb{C}_2\right) \subseteq \Psi_4\left(\mathbb{C}_2\right)$, there exists a point v in $\mathbb{C}_2$ such that $\Psi_1 u = \Psi_4 v = t$. Now we show that v is a coincidence point of $\left(\Psi_2, \Psi_4\right)$. i.e., $\Psi_2 v = \Psi_4 v = t$. So by putting $\tau = u, \tau' = v$ in condition (ii) we have

$$\begin{aligned}\Upsilon_\wp\left(\Psi_1 u, \Psi_2 v\right) \precsim_{i_2} & \ \alpha \frac{\left[\Upsilon_\wp\left(\Psi_1 u, \Psi_3 u\right) \Upsilon_\wp\left(\Psi_1 u, \Psi_4 v\right) + \Upsilon_\wp\left(\Psi_2 v, \Psi_4 v\right) \Upsilon_\wp\left(\Psi_2 v, \Psi_3 u\right)\right]}{\Upsilon_\wp\left(\Psi_1 u, \Psi_4 v\right) + \Upsilon_\wp\left(\Psi_2 v, \Psi_3 u\right)} \\ & + \beta \frac{\left[\left\{\Upsilon_\wp\left(\Psi_1 u, t v\right)\right\}^2 + \left\{\Upsilon_\wp\left(\Psi_2 v, \Psi_3 u\right)\right\}^2\right]}{\Upsilon_\wp\left(\Psi_1 u, \Psi_4 v\right) + \Upsilon_\wp\left(\Psi_2 v, \Psi_3 u\right)}.\end{aligned}$$

Now by taking limit as $n \to \infty$, we obtain that

$$\begin{aligned}\Upsilon_\wp\left(t, \Psi_2 v\right) \precsim_{i_2} & \ \alpha \frac{\left[\Upsilon_\wp(t, t)\, \Upsilon_\wp(t, t) + \Upsilon_\wp\left(\Psi_2 v, \Psi_4 v\right) \Upsilon_\wp\left(\Psi_2 v, t\right)\right]}{\Upsilon_\wp\left(t, \Psi_4 v\right) + \Upsilon_\wp\left(\Psi_2 v, t\right)} \\ & + \beta \frac{\left[\left\{\Upsilon_\wp\left(t, \Psi_4 v\right)\right\}^2 + \left\{\Upsilon_\wp((\Psi_2 v, t)\right\}^2\right]}{\Upsilon_\wp\left(t, \Psi_4 v\right) + \Upsilon_\wp\left(\Psi_2 v, t\right)} \\ \precsim_{i_2} & \ \alpha \frac{\Upsilon_\wp\left(\Psi_2 v, \Psi_4 v\right) \Upsilon_\wp\left(\Psi_2 v, t\right)}{\Upsilon_\wp(t, t) + \Upsilon_\wp\left(\Psi_2 v, \Psi_4 v\right)} + \beta \frac{\left[\left\{\Upsilon_\wp(t, t)\right\}^2 + \left\{\Upsilon_\wp((\Psi_2 v, t)\right\}^2\right]}{\Upsilon_\wp(t, t) + \Upsilon_\wp\left(\Psi_2 v, t\right)} \\ \precsim_{i_2} & \ \alpha \frac{\Upsilon_\wp\left(\Psi_2 v, \Psi_4 v\right) \Upsilon_\wp\left(\Psi_2 v, t\right)}{\Upsilon_\wp\left(\Psi_2 v, \Psi_4 v\right)} + \beta \frac{\left\{\Upsilon_\wp\left(\Psi_2 v, t\right)\right\}^2}{\Upsilon_\wp\left(\Psi_2 v, t\right)},\end{aligned}$$

which implies $\Upsilon_\wp\left(t, \Psi_2 v\right) \precsim_{i_2} \alpha \Upsilon_\wp\left(\Psi_2 v, t\right) + \beta \Upsilon_\wp\left(\Psi_2 v, t\right)$ i.e., $(1 - \alpha - \beta) \Upsilon_\wp\left(t, \Psi_2 v\right) \precsim_{i_2} 0$. Therefore, we have $\left\|\Upsilon_\wp\left(t, \Psi_2 v\right)\right\| \leqslant 0$. Hence $t = \Psi_2 v$. So $\Psi_2 v = \Psi_4 v = t$ and v is the coincidence point of Ψ_2 and Ψ_4. Also the weak compatibility of pair (Ψ_2, Ψ_4) implies that $\Psi_2 \Psi_4 v = \Psi_4 \Psi_2 v$ or $\Psi_2 t = \Psi_4 t$. Therefore t is a common coincidence point of Ψ_1, Ψ_2, Ψ_3 and Ψ_4. Now we have to show that t is a common fixed point of Ψ_1, Ψ_2, Ψ_3 and Ψ_4. Putting $\tau = u, \tau' = t$ in condition (ii) we have

$$\begin{aligned}\Upsilon_\wp(t, \Psi_2 t) &= \Upsilon_\wp(\Psi_1 u, \Psi_2 t)\\ &\precsim_{i_2} \alpha \frac{[\Upsilon_\wp(\Psi_1 u, \Psi_3 u)\Upsilon_\wp(\Psi_1 u, \Psi_4 t) + \Upsilon_\wp(\Psi_2 t, \Psi_4 t)\Upsilon_\wp(\Psi_2 t, \Psi_3 u)]}{\Upsilon_\wp(\Psi_1 u, \Psi_4 t) + \Upsilon_\wp(\Psi_2 t, \Psi_3 u)}\\ &\quad + \beta \frac{\left[\{\Upsilon_\wp(\Psi_1 u, \Psi_4 t)\}^2 + \{\Upsilon_\wp(\Psi_2 t, \Psi_3 u)\}^2\right]}{\Upsilon_\wp(\Psi_1 u, \Psi_4 t) + \Upsilon_\wp(\Psi_2 t, \Psi_3 u)}.\end{aligned}$$

By putting $\Psi_4 t = \Psi_2 t$ and $\Psi_3 u = \Psi_1 u$ in the above inequality we obtain that

$$\begin{aligned}\Upsilon_\wp(t, \Psi_2 t) &= \Upsilon_\wp(\Psi_1 u, \Psi_2 t)\\ &\precsim_{i_2} \alpha.0 + \beta \frac{\left[\{\Upsilon_\wp(\Psi_1 u, \Psi_2 t)\}^2 + \{\Upsilon_\wp(\Psi_2 t, \Psi_1 u)\}^2\right]}{\Upsilon_\wp(\Psi_1 u, \Psi_2 t) + \Upsilon_\wp(\Psi_2 t, \Psi_1 u)}\\ &\precsim_{i_2} \beta \frac{2\{\Upsilon_\wp(\Psi_2 t, \Psi_1 u)\}^2}{2\Upsilon_\wp(\Psi_1 u, \Psi_2 t)},\end{aligned}$$

which implies that

$$\Upsilon_\wp(t, \Psi_2 t) = \Upsilon_\wp(\Psi_1 u, \Psi_2 t) \precsim_{i_2} \beta \Upsilon_\wp(\Psi_1 u, \Psi_2 t)$$
$$\text{i.e., } (1-\beta)\Upsilon_\wp(\Psi_1 u, \Psi_2 t) \precsim_{i_2} 0.$$

Therefore $\|\Upsilon_\wp(\Psi_1 u, \Psi_2 t)\| \leqslant 0$. Hence $\Psi_2 t = \Psi_1 u = t$. But $\Psi_2 t = \Psi_4 t = t$. Therefore, we get $\Psi_1 t = \Psi_2 t = \Psi_3 t = \Psi_4 t = t$. i.e., t is a common fixed point. If we take $\Psi_4(\mathbb{C}_2)$ is closed then similar argument arises and if we take E. A. property of the pair (Ψ_1, Ψ_3) then also similar result is obtained.

Uniqueness:

Let us assume that $\bar{t}$ be another common fixed point of Ψ_1, Ψ_2, Ψ_3 and Ψ_4. i.e., $\Psi_1\bar{t} = \Psi_2\bar{t} = \Psi_3\bar{t} = \Psi_4\bar{t} = \bar{t}$. Then by putting $\tau = \bar{t}$ and $\tau' = t$ in the condition (ii) we have

$$\begin{aligned}\Upsilon_\wp(\Psi_1\bar{t}, \Psi_2 t) &\precsim_{i_2} \alpha \frac{[\Upsilon_\wp(\Psi_1\bar{t}, \Psi_3\bar{t})\Upsilon_\wp(\Psi_1\bar{t}, \Psi_4 t) + \Upsilon_\wp(\Psi_2 t, \Psi_4 t)\Upsilon_\wp(\Psi_2 t, \Psi_3 t)]}{\Upsilon_\wp(\Psi_1\bar{t}, \Psi_4 t) + \Upsilon_\wp(\Psi_2 t, \Psi_3\bar{t})}\\ &\quad + \beta \frac{\left[\{\Upsilon_\wp(\Psi_1\bar{t}, \Psi_4 t)\}^2 + \{\Upsilon_\wp(\Psi_2 t, \Psi_3\bar{t})\}^2\right]}{\Upsilon_\wp(\Psi_1\bar{t}, \Psi_4 t) + \Upsilon_\wp(\Psi_2 t, \Psi_3\bar{t})}\\ &\precsim_{i_2} \alpha.0 + \beta \frac{\left[\{\Upsilon_\wp(\bar{t}, t)\}^2 + \{\Upsilon_\wp(t, \bar{t})\}^2\right]}{\Upsilon_\wp(\bar{t}, t) + \Upsilon_\wp(t, \bar{t})}\\ \text{i.e., } \Upsilon_\wp(\bar{t}, t) &\precsim_{i_2} \beta \frac{2\{\Upsilon_\wp(t, \bar{t})\}^2}{2\Upsilon_\wp(\bar{t}, t)} = \beta.\Upsilon_\wp(\bar{t}, t),\end{aligned}$$

which gives that $(1-\beta)\Upsilon_\wp(\bar{t}, t) \precsim_{i_2} 0$. Therefore we have $\|\Upsilon_\wp(\bar{t}, t)\| \leqslant 0$ which implies that $\bar{t} = t$. Hence $\Psi_1 t = \Psi_2 t = \Psi_3 t = \Psi_4 t = t$ is the unique common fixed point of Ψ_1, Ψ_2, Ψ_3 and Ψ_4.

Thus the proof of the theorem is established. ■

Lemma 7.3.1 *Let Ψ_3 and Ψ_4 be self maps on a bicomplex valued metric space $(\mathfrak{B}, \Upsilon_\wp)$. If the pair $\{\Psi_3, \Psi_4\}$ is Ψ_4-intimate and $\Psi_3 t = \Psi_4 t = p \in \mathbb{C}_2$ for some t in $\mathbb{C}_2$ then $\Upsilon_\wp(\Psi_4 p, p) \precsim_{i_2} \Upsilon_\wp(\Psi_3 p, p)$.*

Proof. We consider the sequence $\pi_n = t$ for all $n \geqslant 1$. So $\lim_{n\to\infty} \Psi_3 \pi_n = \lim_{n\to\infty} \Psi_4 \pi_n = \Psi_3 t = \Psi_4 t = p \in \mathbb{C}_2$. Since the pair $\{\Psi_3, \Psi_4\}$ is Ψ_4-intimate, we have

$$\begin{aligned}\Upsilon_\wp(\Psi_4\Psi_3 t, \Psi_4 t) &= \lim_{n\to\infty} \Upsilon_\wp(\Psi_4\Psi_3\pi_n, \Psi_4\pi_n) \\ &\precsim_{i_2} \lim_{n\to\infty} \Upsilon_\wp(\Psi_3\Psi_3\pi_n, \Psi_3\pi_n) = \Upsilon_\wp(\Psi_3\Psi_3 t, \Psi_3 t),\end{aligned}$$

which implies $\Upsilon_\wp(\Psi_4 p, p) \precsim_{i_2} \Upsilon_\wp(\Psi_3 p, p)$.
This completes the proof of the lemma. ■

Theorem 7.3.2 *Let $(\mathfrak{B}, \Upsilon_\wp)$ be a bicomplex valued metri space and Ψ_1, Ψ_2, Ψ_3 and Ψ_4 be four self mappings on $\mathfrak{B} \subseteq \mathbb{C}_2$ such that*

(a) $\Psi_1(\mathfrak{B}) \subset \Psi_4(\mathfrak{B})$ and $\Psi_2(\mathfrak{B}) \subset \Psi_3(\mathfrak{B})$,

(b) For all $\tau, \tau' \in \mathbb{C}_2$,

$$\begin{aligned}\Upsilon_\wp(\Psi_1\tau, \Psi_2\tau') &\precsim_{i_2} \alpha\Upsilon_\wp(\Psi_3\tau, \Psi_4\tau') \\ &\quad + \beta\frac{\Upsilon_\wp(\Psi_1\tau, \Psi_3\tau).\Upsilon_\wp(\Psi_2\tau', \Psi_4\tau')}{\Upsilon_\wp(\Psi_1\tau, \Psi_4\tau') + \Upsilon_\wp(\Psi_3\tau, \Psi_2\tau') + \Upsilon_\wp(\Psi_3\tau, \Psi_4\tau')},\end{aligned}$$

and $\Upsilon_\wp(\Psi_1\tau, \Psi_4\tau') + \Upsilon_\wp(\Psi_3\tau, \Psi_2\tau') + \Upsilon_\wp(\Psi_3\tau, \Psi_4\tau') \neq 0$, where α, β are non-negative real numbers with $\alpha + \sqrt{2}\beta < 1$;

(c) (Ψ_1, Ψ_3) is Ψ_3-intimate and (Ψ_2, Ψ_4) is Ψ_4-intimate and

(d) $\Psi_3(\mathfrak{B})$ is complete.

Then Ψ_1, Ψ_2, Ψ_3 and Ψ_4 have a unique common fixed point in $\mathbb{C}_2$.

Proof. Let π_0 be any arbitrary point in $\mathbb{C}_2$. Then by condition (a), there exists a point $\pi_1 \in \mathbb{C}_2$ such that $\Psi_1\pi_0 = \Psi_4\pi_1$. Also for $\pi_1 \in \mathbb{C}_2$ we can choose a point $\pi_2 \in \mathbb{C}_2$ such that $\Psi_2\pi_1 = \Psi_3\pi_2$ and so on. Inductively we can define a sequence $\{\pi_n\}$ in $\mathbb{C}_2$ such that $\pi_{2n} = \Psi_1\pi'_{2n} = \Psi_4\pi'_{2n+1}$ and $\pi_{2n+1} = \Psi_2\pi'_{2n+1} = \Psi_3\pi'_{2n+2}$. Then by (b) we have

$$\begin{aligned}\Upsilon_\wp(\pi_{2n}, \pi_{2n+1}) &= \Upsilon_\wp\left(\Psi_1\pi'_{2n}, \Psi_2\pi'_{2n+1}\right) \\ &\precsim_{i_2} \alpha\Upsilon_\wp\left(\Psi_3\pi'_{2n}, \Psi_4\pi'_{2n+1}\right) \\ &\quad + \beta\frac{\Upsilon_\wp(\Psi_1\pi'_{2n}, \Psi_3\pi'_{2n}).\Upsilon_\wp\left(\Psi_2\pi'_{2n+1}, \Psi_4\pi'_{2n+1}\right)}{\Upsilon_\wp\left(\Psi_1\pi'_{2n}, \Psi_4\pi'_{2n+1}\right) + \Upsilon_\wp\left(\Psi_3\pi'_{2n}, \Psi_2\pi'_{2n+1}\right) + \Upsilon_\wp\left(\Psi_3\pi'_{2n}, \Psi_4\pi'_{2n+1}\right)} \\ &\precsim_{i_2} \alpha\Upsilon_\wp(\pi_{2n-1}, \pi_{2n}) \\ &\qquad + \beta\frac{\Upsilon_\wp(\pi_{2n}, \pi_{2n-1}).\Upsilon_\wp(\pi_{2n+1}, \pi_{2n})}{\Upsilon_\wp(\pi_{2n}, \pi_{2n}) + \Upsilon_\wp(\pi_{2n-1}, \pi_{2n+1}) + \Upsilon_\wp(\pi_{2n-1}, \pi_{2n})},\end{aligned}$$

which implies that

$$\begin{aligned}\left\|\Upsilon_{\wp}\left(\pi_{2n},\pi_{2n+1}\right)\right\| &\leqslant \alpha\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\| + \beta\left\|\frac{\Upsilon_{\wp}\left(\pi_{2n},\pi_{2n-1}\right).\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)}{\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n+1}\right)+\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)}\right\| \\ &\leqslant \alpha\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\| + \beta\frac{\left\|\Upsilon_{\wp}\left(\pi_{2n},\pi_{2n-1}\right).\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\|}{\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n+1}\right)+\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\|} \\ &\leqslant \alpha\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\| + \beta\frac{\sqrt{2}\left\|\Upsilon_{\wp}\left(\pi_{2n},\pi_{2n-1}\right)\right\|.\left\|\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\|}{\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n+1}\right)+\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\|}.\end{aligned}$$

We know that
$\left\|\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\| < \left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n+1}\right)+\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\|$

$$\text{i.e., } \frac{\left\|\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\|}{\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n+1}\right)+\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\|} < 1.$$

Therefore, we get that

$$\begin{aligned}\left\|\Upsilon_{\wp}\left(\pi_{2n},\pi_{2n+1}\right)\right\| &\leqslant \alpha\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\| + \beta\sqrt{2}\left\|\Upsilon_{\wp}\left(\pi_{2n},\pi_{2n-1}\right)\right\| \\ &\leqslant \left(\alpha+\sqrt{2}\beta\right)\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\|\end{aligned}$$

$$\text{i.e., } \left\|\Upsilon_{\wp}\left(\pi_{2n},\pi_{2n+1}\right)\right\| \leqslant \gamma\left\|\Upsilon_{\wp}\left(\pi_{2n-1},\pi_{2n}\right)\right\|, \text{ where } \gamma = \left(\alpha+\sqrt{2}\beta\right).$$

Also, we have

$$\begin{aligned}&\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n+1}\right) \\ &= \Upsilon_{\wp}\left(\Psi_1\pi'_{2n+2},\Psi_2\pi'_{2n+1}\right) \\ &\precsim_{i_2} \alpha\Upsilon_{\wp}\left(\Psi_3\pi'_{2n+2},\Psi_4\pi'_{2n+1}\right) \\ &\quad + \beta\frac{\Upsilon_{\wp}\left(\Psi_1\pi'_{2n+2},\Psi_3\pi'_{2n+2}\right).\Upsilon_{\wp}\left(\Psi_2\pi'_{2n+1},\Psi_4\pi'_{2n+1}\right)}{\Upsilon_{\wp}\left(\Psi_1\pi'_{2n+2},\Psi_4\pi'_{2n+1}\right)+\Upsilon_{\wp}\left(\Psi_3\pi'_{2n+2},\Psi_2\pi'_{2n+1}\right)+\Upsilon_{\wp}\left(\Psi_3\pi'_{2n+2},\Psi_4\pi'_{2n+1}\right)} \\ &\precsim_{i_2} \alpha\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right) + \beta\frac{\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n+1}\right).\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)}{\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n}\right)+\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n+1}\right)+\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)} \\ &= \alpha\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right) + \beta\frac{\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n+1}\right).\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)}{\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n}\right)+\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)},\end{aligned}$$

which implies that

$$\begin{aligned}\left\|\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n+1}\right)\right\| &\leqslant \alpha\left\|\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\| + \left\|\beta\frac{\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n+1}\right).\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)}{\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n}\right)+\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)}\right\| \\ &\leqslant \alpha\left\|\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\| + \beta\frac{\left\|\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n+1}\right).\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\|}{\left\|\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n}\right)+\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\|} \\ &\leqslant \alpha\left\|\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\| + \beta\frac{\sqrt{2}\left\|\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n+1}\right)\right\|.\left\|\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\|}{\left\|\Upsilon_{\wp}\left(\pi_{2n+2},\pi_{2n}\right)+\Upsilon_{\wp}\left(\pi_{2n+1},\pi_{2n}\right)\right\|}.\end{aligned}$$

Again since $\|\Upsilon_\wp(\pi_{2n+2},\pi_{2n+1})\| \leqslant \|\Upsilon_\wp(\pi_{2n+2},\pi_{2n}) + \Upsilon_\wp(\pi_{2n},\pi_{2n+1})\|$, therefore we get that

$$\begin{aligned}\|\Upsilon_\wp(\pi_{2n+2},\pi_{2n+1})\| &\leqslant \alpha\|\Upsilon_\wp(\pi_{2n+1},\pi_{2n})\| + \beta\sqrt{2}\|\Upsilon_\wp(\pi_{2n+1},\pi_{2n})\| \\ &\leqslant \left(\alpha+\sqrt{2}\beta\right)\|\Upsilon_\wp(\pi_{2n+1},\pi_{2n})\| \\ \text{i.e., } \|\Upsilon_\wp(\pi_{2n+2},\pi_{2n+1})\| &\leqslant \gamma\|\Upsilon_\wp(\pi_{2n+1},\pi_{2n})\|.\end{aligned}$$

Thus we have

$$\|\Upsilon_\wp(\pi_{n+1},\pi_{n+2})\| \leqslant \gamma\|\Upsilon_\wp(\pi_n,\pi_{n+1})\| \leqslant \gamma^2\|\Upsilon_\wp(\pi_{n-1},\pi_n)\| \leqslant \ldots \leqslant \gamma^{n+1}\|\Upsilon_\wp(\pi_0,\pi_1)\|.$$

So for any $m > n$, we obtain that

$$\begin{aligned}\|\Upsilon_\wp(\pi_n,\pi_m)\| &\leqslant \|\Upsilon_\wp(\pi_n,\pi_{n+1})\| + \|\Upsilon_\wp(\pi_n,\pi_{n+1})\| + \ldots + \|\Upsilon_\wp(\pi_{m-1},\pi_m)\| \\ &\leqslant \gamma^n\|\Upsilon_\wp(\pi_0,\pi_1)\| + \gamma^{n+1}\|\Upsilon_\wp(\pi_0,\pi_1)\| + \ldots + \gamma^{m-1}\|\Upsilon_\wp(\pi_0,\pi_1)\|,\end{aligned}$$

which implies $\|\Upsilon_\wp(\pi_n,\pi_m)\| \leqslant \frac{\gamma^n}{1-\gamma}\|\Upsilon_\wp(\pi_0,\pi_1)\| \to 0$ as $m,n \to \infty$. So the sequence $\{\pi_n\} = \{\Psi_3\pi'_{2n}\}$ is a Cauchy sequence in $\Psi_3(\mathbb{C}_2)$. Again since $\Psi_3(\mathbb{C}_2)$ is complete, therefore the sequence $\{\pi_n\}$ converges to a point $p = \Psi_3 u$ for some $u \in \mathbb{C}_2$. Thus $\Psi_1\pi'_{2n}, \Psi_3\pi'_{2n}, \Psi_2\pi'_{2n+1}, \Psi_4\pi'_{2n+1} \to p$ as $n \to \infty$. Now

$$\begin{aligned}\Upsilon_\wp\left(\Psi_1 u, \Psi_2\pi'_{2n+1}\right) \precsim_{i_2} &\ \alpha\Upsilon_\wp\left(\Psi_3 u, \Psi_4\pi'_{2n+1}\right) \\ &+ \beta\frac{\Upsilon_\wp(\Psi_1 u,\Psi_3 u).\Upsilon_\wp\left(\Psi_2\pi'_{2n+1},\Psi_4\pi'_{2n+1}\right)}{\Upsilon_\wp\left(\Psi_1 u,\Psi_4\pi'_{2n+1}\right)+\Upsilon_\wp\left(\Psi_3 u,\Psi_2\pi'_{2n+1}\right)+\Upsilon_\wp\left(\Psi_3 u,\Psi_4\pi'_{2n+1}\right)}\end{aligned}$$

implies that

$$\begin{aligned}&\left\|\Upsilon_\wp\left(\Psi_1 u, \Psi_2\pi'_{2n+1}\right)\right\| \\ &\leqslant \alpha\left\|\Upsilon_\wp\left(\Psi_3 u,\Psi_4\pi'_{2n+1}\right)\right\| \\ &\qquad + \left\|\beta\frac{\Upsilon_\wp(\Psi_1 u,\Psi_3 u).\Upsilon_\wp\left(\Psi_2\pi'_{2n+1},\Psi_4\pi'_{2n+1}\right)}{\Upsilon_\wp\left(\Psi_1 u,\Psi_4\pi'_{2n+1}\right)+\Upsilon_\wp\left(\Psi_3 u,\Psi_2\pi'_{2n+1}\right)+\Upsilon_\wp\left(\Psi_3 u,\Psi_4\pi'_{2n+1}\right)}\right\| \\ &\leqslant \alpha\left\|\Upsilon_\wp\left(\Psi_3 u,\Psi_4\pi'_{2n+1}\right)\right\| \\ &\qquad + \beta\frac{\left\|\Upsilon_\wp(\Psi_1 u,\Psi_3 u).\Upsilon_\wp\left(\Psi_2\pi'_{2n+1},\Psi_4\pi'_{2n+1}\right)\right\|}{\left\|\Upsilon_\wp\left(\Psi_1 u,\Psi_4\pi'_{2n+1}\right)+\Upsilon_\wp\left(\Psi_3 u,\Psi_2\pi'_{2n+1}\right)+\Upsilon_\wp\left(\Psi_3 u,\Psi_4\pi'_{2n+1}\right)\right\|} \\ &\leqslant \alpha\left\|\Upsilon_\wp\left(\Psi_3 u,\Psi_4\pi'_{2n+1}\right)\right\| \\ &\qquad + \beta\frac{\|\Upsilon_\wp(\Psi_1 u,\Psi_3 u)\|.\left\|\Upsilon_\wp\left(\Psi_2\pi'_{2n+1},\Psi_4\pi'_{2n+1}\right)\right\|}{\left\|\Upsilon_\wp\left(\Psi_1 u,\Psi_4\pi'_{2n+1}\right)+\Upsilon_\wp\left(\Psi_3 u,\Psi_2\pi'_{2n+1}\right)+\Upsilon_\wp\left(\Psi_3 u,\Psi_4\pi'_{2n+1}\right)\right\|}.\end{aligned}$$

Taking limit as $n \to \infty$, we have $\|\Upsilon_\wp(\Psi_1 u,p)\| \leqslant \alpha\|\Upsilon_\wp(\Psi_3 u,p)\|$, therefore we get $\|\Upsilon_\wp(\Psi_1 u,p)\| = 0$. i.e., $\Psi_1 u = p = \Psi_3 u$. Again since $\Psi_1(\mathbb{C}_2) \subset \Psi_4(\mathbb{C}_2)$, therefore there

exists a $v \in \mathbb{C}_2$ such that $\Psi_1 u = \Psi_4 v = p$.
Now applying (b) we have

$$\begin{aligned}\Upsilon_{\wp}(p, \Psi_2 v) = \Upsilon_{\wp}(\Psi_1 u, \Psi_2 v) \precsim_{i_2} \alpha \Upsilon_{\wp}(\Psi_3 u, \Psi_4 v) \\ + \beta \frac{\Upsilon_{\wp}(\Psi_1 u, \Psi_3 u) . \Upsilon_{\wp}(\Psi_2 v, \Psi_4 v)}{\Upsilon_{\wp}(\Psi_1 u, \Psi_4 v) + \Upsilon_{\wp}(\Psi_3 u, \Psi_2 v) + \Upsilon_{\wp}(\Psi_3 u, \Psi_4 v)},\end{aligned}$$

which implies that

$$\begin{aligned}\|\Upsilon_{\wp}(p, \Psi_2 v)\| \leqslant \alpha \|\Upsilon_{\wp}(\Psi_3 u, \Psi_4 v)\| \\ + \beta \frac{\|\Upsilon_{\wp}(\Psi_1 u, \Psi_3 u) . \Upsilon_{\wp}(\Psi_2 v, \Psi_4 v)\|}{\|\Upsilon_{\wp}(\Psi_1 u, \Psi_4 v) + \Upsilon_{\wp}(\Psi_3 u, \Psi_2 v) + \Upsilon_{\wp}(\Psi_3 u, \Psi_4 v)\|} \\ \leqslant \alpha \|\Upsilon_{\wp}(\Psi_3 u, \Psi_4 v)\| \\ + \beta \frac{\sqrt{2}\|\Upsilon_{\wp}(\Psi_1 u, \Psi_3 u)\| . \|\Upsilon_{\wp}(\Psi_2 v, \Psi_4 v)\|}{\|\Upsilon_{\wp}(\Psi_1 u, \Psi_4 v) + \Upsilon_{\wp}(\Psi_3 u, \Psi_2 v) + \Upsilon_{\wp}(\Psi_3 u, \Psi_4 v)\|}.\end{aligned}$$

Thus $\|\Upsilon_{\wp}(p, \Psi_2 v)\| = 0$ and this gives that $p = \Psi_2 v = \Psi_4 v = \Psi_1 u = \Psi_3 u$. Now since $\Psi_1 u = \Psi_3 u = p$ and (Ψ_1, Ψ_3) is Ψ_3-intimate, therefore by Lemma 7.3.1, we have $\|\Upsilon_{\wp}(\Psi_3 p, p)\| \leqslant \|\Upsilon_{\wp}(\Psi_1 p, p)\|$. Also by (b)we have

$$\begin{aligned}\Upsilon_{\wp}(\Psi_1 p, p) = \Upsilon_{\wp}(\Psi_1 p, \Psi_2 v) \precsim_{i_2} \alpha \Upsilon_{\wp}(\Psi_3 p, \Psi_4 v) \\ + \beta \frac{\Upsilon_{\wp}(\Psi_1 p, \Psi_3 p) . \Upsilon_{\wp}(\Psi_2 v, \Psi_4 v)}{\Upsilon_{\wp}(\Psi_1 p, \Psi_4 v) + \Upsilon_{\wp}(\Psi_3 p, \Psi_2 v) + \Upsilon_{\wp}(\Psi_3 p, \Psi_4 v)}.\end{aligned}$$

i.e., $\|\Upsilon_{\wp}(\Psi_1 p, p)\| \leqslant \alpha \|\Upsilon_{\wp}(\Psi_3 p, p)\|$, which yields that $\|\Upsilon_{\wp}(\Psi_1 p, p)\| = 0$. Therefore $\Psi_1 p = p$ and $\Psi_3 p = p$. Similarly, we can show that $\Psi_2 p = \Psi_4 p = p$.

Uniqueness

Let us consider that p and q are two common fixed points of Ψ_1, Ψ_2, Ψ_3 and Ψ_4 such that $p \neq q$. Then using (b) we get that

$$\begin{aligned}\Upsilon_{\wp}(p, q) = \Upsilon_{\wp}(\Psi_1 p, \Psi_2 q) \precsim_{i_2} \alpha \Upsilon_{\wp}(\Psi_3 p, \Psi_4 q) \\ + \beta \frac{\Upsilon_{\wp}(\Psi_1 p, \Psi_3 p) . \Upsilon_{\wp}(\Psi_2 q, \Psi_4 q)}{\Upsilon_{\wp}(\Psi_1 p, \Psi_4 q) + \Upsilon_{\wp}(\Psi_3 p, \Psi_2 q) + \Upsilon_{\wp}(\Psi_3 p, \Psi_4 q)} \\ \precsim_{i_2} \alpha \Upsilon_{\wp}(p, q).\end{aligned}$$

i.e.,$\|\Upsilon_{\wp}(p, q)\| \leqslant \alpha \|\Upsilon_{\wp}(p, q)\|$, which implies that $p = q$.

This completes the the proof of the theorem. ■

Example 7.3.1 *Let* $\mathbf{H} = \{\pi_1, \pi_2\} \subset \mathbb{C}_1$ *with* $\Upsilon_{\wp} : \mathbf{H} \times \mathbf{H} \to \mathbb{C}_{12}$ *is defined by*

$$\Upsilon_{\wp}(\pi_1, \pi_2) = \begin{cases} 1, \text{ if } \pi_1 \neq \pi_2 \\ 0, \text{ if } \pi_1 = \pi_2. \end{cases}$$

Then $(\mathbf{H}, \Upsilon_{\wp})$ *is a complete bicomplex valued metric space. Define* $\Psi_1, \Psi_2, \Psi_3, \Psi_4 : \mathbf{H} \to \mathbf{H}$ *by* $\Psi_1 \tau = \pi_1'$ *for all* $\tau \in \mathbf{H}$, $\Psi_3 \pi_1' = \Psi_4 \pi_2' = \pi_2'$ *and* $\Psi_3 \pi_2' = \Psi_4 \pi_2' = \pi_1'$. *Then all the conditions of above theorem are satisfied except intimate condition. We see that* $\|\Upsilon_{\wp}(\Psi_3 \Psi_1 \pi_2', \Psi_3 \pi_2')\| = \|\Upsilon_{\wp}(\pi_2', \pi_1')\| > 0 = \|\Upsilon_{\wp}(\Psi_1 \Psi_1 \pi_2', \Psi_1 \pi_2')\|$, *where* $\{\pi_2'\}$ *is a constant sequence in* $\mathbb{C}_2$ *such that* $\Psi_1 \pi_2' = \Psi_3 \pi_2' = \pi_1'$. *Thus the pair* (Ψ_1, Ψ_3) *is not* Ψ_3-*intimate. Therefore* Ψ_1, Ψ_3 *and* Ψ_4 *do not have a common fixed point.*

Example 7.3.2 *Let us define* $\Upsilon_{\wp} : \mathbb{C}_1 \times \mathbb{C}_1 \to \mathbb{C}_2$ *by* $\Upsilon_{\wp}(\pi_1, \pi_2) = i_2 \|\pi_1 - \pi_2\|$ *where* $\pi_1 = x_1 + i_1 y_1$ *and* $\pi_2 = x_2 + i_1 y_2$. *Then* $(\mathbb{C}_1, \Upsilon_{\wp})$ *is a complete bicomplex valued metric space. Define* $\Psi_1, \Psi_2, \Psi_3, \Psi_4 : \mathbb{C}_2 \to \mathbb{C}_2$ *as* $\Psi_1 \tau = 0, \Psi_2 \tau = 0$, $\Psi_3 \tau = \tau$ *and* $\Psi_4 \tau = \frac{\tau}{2}$. *Clearly* $\Psi_1(\mathbb{C}_2) \subset \Psi_4(\mathbb{C}_2)$ *and* $\Psi_2(\mathbb{C}_2) \subset \Psi_3(\mathbb{C}_2)$. *Now considering the sequence* $\left\{\pi_n = \frac{1}{n}, n \in \mathbb{N}\right\}$ *in* $\mathbb{C}_2$ *we get* $\lim_{n\to\infty} \Psi_1 \pi_n = \lim_{n\to\infty} \Psi_3 \pi_n = 0$. *Also we have* $\lim_{n\to\infty} \Upsilon_{\wp}(\Psi_3 \Psi_1 \pi_n, \Psi_3 \pi_n) \precsim_{i_2} \lim_{n\to\infty} \Upsilon_{\wp}(\Psi_1 \Psi_1 \pi_n, \Psi_1 \pi_n)$. *Thus the pair* (Ψ_1, Ψ_3) *is* Ψ_3-*intimate. Again since* $\lim_{n\to\infty} \Upsilon_{\wp}(\Psi_4 \Psi_2 \pi_n, \Psi_4 \pi_n) \precsim_{i_2} \lim_{n\to\infty} \Upsilon_{\wp}(\Psi_2 \Psi_2 \pi_n, \Psi_2 \pi_n)$, *therefore the pair* (Ψ_2, Ψ_4) *is* Ψ_4-*intimate. Further, the mappings satisfy all the conditions of the above theorem. Hence* Ψ_1, Ψ_2, Ψ_3 *and* Ψ_4 *have a unique common fixed point in* $\mathbb{C}_2$.

7.4 Future Prospects

In the line of the works as carried out in this chapter one may think of the deduction of fixed point theorems using common limit in the range (CLR) property, expansive metric and other different types of metrices under the flavour of bicomplex analysis. This may be an active area of research to the future workers in this branch.

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CHAPTER EIGHT

FUTURE COURSE OF WORK

Chapter 8

Future course of work

8.1 Introduction

Quasi-ideals in a semiring are the generalization of one-sided right ideals and left ideals. Bi-ideals are generalized forms of the quasi-ideals. Steinfeld [54] initially defined the quasi-ideals for semigroups and rings. Iseki [30] introduced this concept for semirings without zero and proved some results. Shabir *et al.* [56] characterized semirings by the properties of their Quasi-ideals. For different aspects of bi-ideals one may also view {cf. [14], [15], [16], [17] & [18]}.

Definition 8.1.1 *A nonempty set $\mathcal{S}$ is said to form a semiring with respect to two binary operations called Addition (+) and Multiplication (∗), if the following condition are satisfied.*

(i) $(\mathcal{S}, +)$ *is a commutative semigroup,*

(ii) $(\mathcal{S}, *)$ *is a non-commutative semigroup and*

(iii) $\varsigma_1 * (\varsigma_2 + \varsigma_3) = \varsigma_1 * \varsigma_2 + \varsigma_1 * \varsigma_3$ *and* $(\varsigma_1 + \varsigma_2) * \varsigma_3 = \varsigma_1 * \varsigma_3 + \varsigma_2 * \varsigma_3,\ \forall\ \varsigma_1,\ \varsigma_2,\ \varsigma_3 \in \mathcal{S}$.

Example 8.1.1 *The set of hybrid numbers $\mathfrak{T}$ is a semiring as it forms a commutative semigroup under addition and also forms a non-commutative semigroup under hybrid multiplication with distributive laws.*

Definition 8.1.2 *A subsemiring of a semiring $(\mathcal{S}, +, *)$ is a nonempty subset $\mathcal{T}$ provided it is itself a semiring under the operation of $\mathcal{S}$.*

Example 8.1.2 *The set of non-lightlike hybrid numbers $\mathfrak{T}_{NL}$ is a subsemiring of the semiring $\mathfrak{T}$.*

Definition 8.1.3 *A nonempty subset $\mathcal{I}$ of a semiring $(\mathcal{S}, +, *)$, is called a right(left) ideal of $\mathcal{S}$ if it satisfies the conditions that $\varsigma_1 + \varsigma_2 \in \mathcal{I},\ \varsigma_1\varsigma_2 \in \mathcal{I}\ \forall \varsigma_1, \varsigma_2 \in \mathcal{S}$, and $\varsigma_1 \in \mathcal{I}\ (a \in \mathcal{I}),\ \forall \varsigma_1 \in \mathcal{I},\ a \in \mathcal{S}$. I is called an ideal of $\mathcal{S}$ if it is both left and right ideal.*

Definition 8.1.4 *[30] Let $(\mathcal{S}, +, *)$ be a semiring. A quasi-ideal $\mathcal{Q}$ of $\mathcal{S}$ is a subsemigroup $(\mathcal{Q}, +)$ of $\mathcal{S}$ such that $\mathcal{S}\mathcal{Q} \cap \mathcal{Q}\mathcal{S} \subseteq \mathcal{Q}$.*

Each quasi-ideal of a semiring $\mathcal{S}$ is its subsemiring. Every one-sided ideal of $\mathcal{S}$ is its quasi-ideal. Since intersection of any family of quasi-ideals of $\mathcal{S}$ is its quasi-ideal [56], so intersection of a right ideal $\mathcal{I}_{\mathcal{R}}$ and a left-ideal $\mathcal{I}_{\mathcal{L}}$ of $\mathcal{S}$ is a quasi-ideal of $\mathcal{S}$. Both the sum and the product of two or more quasi-ideals of $\mathcal{S}$ need not be its quasi-ideal [56].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two arbitrary subsets of a ring $\mathcal{R}$. The product $\mathcal{X}\mathcal{Y}$ is defined as the additive subgroup of $\mathcal{R}$ which is generated by the set of all products $\xi_1\xi_2$ where $\xi_1 \in \mathcal{X}$ and $\xi_2 \in \mathcal{Y}$.

Definition 8.1.5 *A bi-ideal $\mathcal{B}^\flat$ of a ring $\mathcal{R}$ is defined as a subring $\mathcal{B}^\flat$ of $\mathcal{R}$ satisfying the following condition*

$$\mathcal{B}^\flat\mathcal{R}\mathcal{B}^\flat \subseteq \mathcal{B}^\flat.$$

Definition 8.1.6 *Let $\mathfrak{D}_s$ be subring of a ring $\mathcal{R}$. For a positive integer m, $\mathfrak{D}_s^m$ is defined as follows:*

$$\mathfrak{D}_s^1 = \mathfrak{D}_s,\ \mathfrak{D}_s^m = \mathfrak{D}_s^{m-1}\mathfrak{D}_s.$$

$\mathfrak{D}_s^0$ is defined as an operator element such that $\mathfrak{D}_s^0\mathcal{R} = \mathcal{R}\mathfrak{D}_s^0 = \mathcal{R}$.

A subring $\mathfrak{D}_s$ of a ring $\mathcal{R}$ is said to be an (m, n) ideal of $\mathcal{R}$ if $\mathfrak{D}_s^m\mathcal{R}\mathfrak{D}_s^n \subseteq \mathfrak{D}_s$ where m and n are non-negative integers.

8.2 Lemmas

Lemma 8.2.1 *Every one sided (left or right) ideal of a ring $\mathcal{R}$ is a bi-ideal of $\mathcal{R}$.*

Proof. The proof is omitted. ■

Lemma 8.2.2 *The intersection of a left ideal and a right ideal of a ring $\mathcal{R}$ is also a bi-ideal of $\mathcal{R}$.*

Proof. The proof is omitted. ■

Lemma 8.2.3 *The intersection of a bi-ideal $\mathcal{B}^\flat$ of a ring $\mathcal{R}$ and of a subring $\mathfrak{D}_s$ of $\mathcal{R}$ is a bi-ideal of the ring $\mathfrak{D}_s$.*

Proof. Let $\mathcal{C} = \mathcal{B}^\flat \cap \mathfrak{D}_s$. Since $\mathcal{B}^\flat$ is a bi-ideal of $\mathcal{R}$, therefore $\mathcal{B}^\flat$ is a subring of $\mathcal{R}$ satisfying $\mathcal{B}^\flat\mathcal{R}\mathcal{B}^\flat \subseteq \mathcal{B}^\flat$. We know that the intersection of two subrings of a ring is again a subring of that ring. So $\mathcal{C}$ is a subring of the ring $\mathcal{R}$. Since, $\mathfrak{D}_s$ is a subring of $\mathcal{R}$ and $\mathcal{C} \subseteq \mathfrak{D}_s$ so $\mathcal{C}\mathfrak{D}_s\mathcal{C} \subseteq \mathfrak{D}_s\mathfrak{D}_s\mathfrak{D}_s \subseteq \mathfrak{D}_s$.
Also, $\mathcal{C}\mathfrak{D}_s\mathcal{C} \subseteq \mathcal{B}^\flat\mathfrak{D}_s\mathcal{B}^\flat \subseteq \mathcal{B}^\flat\mathcal{R}\mathcal{B}^\flat \subseteq \mathcal{B}^\flat$.
Combining the above two relations, $\mathcal{C}\mathfrak{D}_s\mathcal{C} \subseteq \mathcal{B}^\flat \cap \mathfrak{D}_s = \mathcal{C}$.
Hence $\mathcal{C}$ is bi-ideal of the ring $\mathfrak{D}_s$. ■

8.3 Theorems

In this section we present the main results of this chapter.

Theorem 8.3.1 *The subring* $\mathfrak{T}_{\mathfrak{R}} = \{\mathfrak{z} = \frac{\varpi_1}{2} + \frac{\varpi_1+\varpi_2}{2}\mathbf{i} + \frac{\varpi_1}{2}\varepsilon + \frac{\varpi_2}{2}\mathbf{h} : \varpi_1, \varpi_2 \in \mathbb{Z}\}$ *of the ring* $\mathfrak{T}$ *is a bi-ideal of* $\mathfrak{T}$.

Proof. From Theorem 2.2.9, we have $\mathfrak{T}_{\mathfrak{R}}$ is a right ideal of $\mathfrak{T}$.
As, $\mathfrak{T}_{\mathfrak{R}}.\mathfrak{T} \subseteq \mathfrak{T}_{\mathfrak{R}}$, so $\mathfrak{T}_{\mathfrak{R}}.\mathfrak{T}.\mathfrak{T}_{\mathfrak{R}} \subseteq \mathfrak{T}_{\mathfrak{R}}.\mathfrak{T}_{\mathfrak{R}} \subseteq \mathfrak{T}_{\mathfrak{R}}$.
Hence $\mathfrak{T}_{\mathfrak{R}}$ is a bi-ideal of $\mathfrak{T}$. ■

Theorem 8.3.2 *The subring* $\mathfrak{T}_{\mathcal{L}} = \{\mathfrak{z} = \frac{\varpi_1}{2} + \frac{\varpi_1-\varpi_2}{2}\mathbf{i} + \frac{\varpi_1}{2}\varepsilon + \frac{\varpi_2}{2}\mathbf{h} : \varpi_1, \varpi_2 \in \mathbb{Z}\}$ *of the ring* $\mathfrak{T}$ *is a bi-ideal of* $\mathfrak{T}$.

Proof. The proof is omitted as it is similar to the previous one. ■

Remark 8.3.1 *The subring* $\mathbb{I} = \{\mathfrak{z} = \frac{\varpi}{2} + \frac{\varpi}{2}\mathbf{i} + \frac{\varpi}{2}\varepsilon : \varpi \in \mathbb{Z}\}$ *of* $\mathfrak{T}$ *is a bi-ideal of* $\mathfrak{T}$.

Theorem 8.3.3 *The intersection of the bi-ideal* $\mathfrak{T}_{\mathfrak{R}}$ *and the subring* $\mathfrak{T}_{NL}$ *of* $\mathfrak{T}$ *is a bi-ideal of* $\mathfrak{T}$.

Proof. The proof of the theorem is trivial and it follows from Lemma 8.2.3 ■

Remark 8.3.2 *The relationship between quasi-ideal and bi-ideal also ensures the validity of the above results in quasi-ideal.*

Remark 8.3.3 *In view of Definition 8.1.6, a bi-ideal* $\mathcal{B}^{\flat}$ *of a ring* $\mathcal{R}$ *can be regarded as* $(1,1)$*-ideal in* $\mathcal{R}$*. Therefore the results 8.3.1, 8.3.2 and 8.3.3 can be respectively extended for (m,n)-ideal of* $\mathfrak{T}$.

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