STUDY OF COSMOLOGICAL SOLUTIONS IN DIFFERENT MODIFIED GRAVITY THEORIES AND THEIR PROPERTIES USING NOETHER SYMMETRY ANALYSIS

Dipankar Laya

THE THESIS IS PARTIAL FULFILMENT OF THE REQUIREDMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY IN SCIENCE

Department of Mathematics

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CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled "**Study of Cosmological Solutions in Different Modified Gravity Theories and Their Properties Using Noether Symmetry Analysis**" submitted by **Sri. Dipankar Laya**, who got his name registered on June 3, 2022 (Index No: 97/22/Maths./27) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own work under our supervision and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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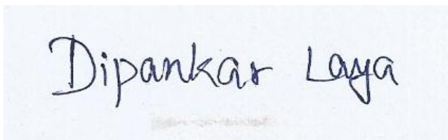
DECLARATION BY THE AUTHOR

I affirm that this thesis is the result of my independent efforts conducted at the Department of Mathematics, Jadavpur University, Kolkata-700032, India. Furthermore, I confirm that none of its content has been presented for the fulfillment of any degree, diploma, or other qualifications at any other academic institution.

The author created all figures in this thesis using Mathematica and Maple software. The thesis underwent rigorous scrutiny to eradicate discrepancies and typographical errors. Nevertheless, despite these efforts, vigilant readers may discern mistakes, and some sections may seem unwarranted or inaccurate. The author fully acknowledges responsibility for any such errors resulting from inadequate subject knowledge or oversight.

Finally, I state that, to the best of my knowledge, all the assistance taken to prepare this thesis has been properly cited and acknowledged.



Dipankar Laya

This thesis is dedicated to my maternal grandmother
Khanibala Mandal
and my maternal grandfather
***Nabadwip Mandal**.*

ACKNOWLEDGMENT

Completing this thesis, which is the culmination of several years of hard work, fills me with profound gratitude towards the many individuals who have provided inspiration and support throughout my Ph.D. journey at Jadavpur University and the writing process. Crafting a Ph.D. thesis in any field presents numerous challenges. Only those actively pursuing this degree can truly comprehend the intricate demands associated with the esteemed title of "Ph.D." Initially, as a novice Ph.D. scholar, I struggled to grasp why it was customary to acknowledge a group of individuals when submitting a Ph.D. thesis. However, after a few weeks of reflection, the significance of the "Acknowledgments" section became clear to me. Now, I feel compelled to express my heartfelt thanks and appreciation to the countless individuals who have offered invaluable assistance whenever I found myself in need.

While a doctoral thesis is commonly viewed as an individual pursuit, the extensive list that follows certainly contradicts that notion.

First and foremost, I extend my deepest gratitude to my esteemed supervisor, Subenoy Chakraborty, for his invaluable guidance, unwavering support, remarkable patience, generous assistance, and mentorship throughout my entire Ph.D. journey. His vast knowledge and extensive experience have been a constant source of encouragement both in my academic research and daily life. I will always remember his words of encouragement, **"Failure is the pillar of Success",** which motivated me to persevere during challenging tasks such as symmetry vector analysis, classical and quantum calculations, and graph plotting, even when I faced uncertainties. Without his guidance, I would not have been able to reach this milestone of completing my thesis. I would also like to express my gratitude to Mrs. Archana Chakraborty (affectionately known as cherish kakima) for her love, care, and for the delicious meals she graciously prepared for us. Additionally, I am thankful to Dr. Sumanta Chakraborty (Professor Chakraborty's son) for occasionally advising me on matters related to my research.

I am deeply grateful and would like to extend my heartfelt appreciation to my co-supervisor, Dr. Sourav Dutta, Dr. Meghnad Saha College in Malda, West Bengal, and fondly referred to as 'Dada.' His enlightening concepts and open-minded approach have been instrumental in shaping my perspective. Dr. Dutta not only offered me invaluable opportunities for critical thinking and shared useful ideas but also provided access to modern working facilities. Moreover, I am truly thankful for his generosity in ensuring that I had access to delicious meals throughout our collaboration.

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I wish to express my gratitude to the college teachers who have played a significant role in shaping my life. Under the guidance of Professor Pralay Poria, Head of Department at Sabang Sajanikant Mahavidyalay, I not only acquired knowledge in mathematics but also learned the importance of a rigorous scientific approach and a dedicated work ethic, particularly during mock test evaluations. His supervision and encouragement were indispensable in completing my graduation, and I offer him my deepest thanks and heartfelt respect. I also extend my appreciation to Professor Dr. Banshidhar Sahoo from Hiralal Bhakat College for his assistance and support during my undergraduate studies. Additionally, I am grateful to Dr. Gurupada Maity for his generous help. Special thanks are due to Nabin Sir for providing enjoyable tours and opportunities for exchanging ideas. I would also like to express my gratitude to all the professors in the Mathematics Department at Midnapore College Autonomous. Overall, I am thankful to all the teachers who have supported and guided me throughout my life.

I wish to thank all friends, from childhood to adulthood, including those from college, Masters degree, and Ph.D. colleagues. You all have made life much easier during the ups and downs. My thanks also go to the colleagues in these offices.

Lastly, I owe an immeasurable debt of gratitude to my family, without whom I would not be the person I am today. To my parents, who have also been my teachers, Shrihari and Shandhya, and to my siblings, Durga and Nanda, your unwavering belief in me and constant support have been my pillars through every significant decision in my life. Your presence has provided me with a sense of security, allowing me to focus on my studies even when far from home. I must also mention my extended family, especially Ambika (baro mama), Ankita (Kochi Mama), and all the other relatives who have played vital roles in my life. Parents, your patience, kindness, and support on all fronts exceed anything a son could ever hope for. Your faith in me has been a constant source of strength and motivation throughout this journey. Additionally, I want to express my heartfelt appreciation to my Dadabu for his encouragement, guidance, companionship, and emotional support. Above all, I want to acknowledge the immense sacrifices my parents, maa and baba, have made to provide me with an exceptional education. Your nurturing of my intellect and unwavering support of my aspirations have been truly remarkable. While there are many others deserving of acknowledgment, I hope I have not overlooked anyone.

Finally, I want to express my sincere apologies if I overlooked acknowledging the individuals who provided assistance to me in different ways.

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Dipankar Laya

Dipankar Laya

“Arise! Awake! And stop not until the goal is reached.”

Swami Vivekananda

A concise passage about Emmy Noether

Figure 0.1 – The German mathematician Amalie Emmy Noether.

Amalie Emmy Noether, born in 1882 in Germany, was a pioneering mathematician known for her groundbreaking contributions to abstract algebra and mathematical physics. She proved Noether's first and second theorems, fundamental in mathematical physics, and her work on group theory laid the foundation for modern algebra. Despite facing discrimination as a woman in academia, she became a leading mathematician and joined the prestigious mathematics department at the University of Göttingen in 1915. Noether's theorem, which explains the connection between symmetry and conservation laws, remains a cornerstone of physics. In 1933, due to Nazi policies, she moved to the United States and taught at Bryn Mawr College and the Institute for Advanced Study in Princeton. Her work has left an indelible mark on mathematics and physics, influencing generations of mathematicians and physicists worldwide.

Despite not initially distinguishing herself academically during her youth, Emmy Noether demonstrated a keen logical acumen and a passion for dancing. She was the eldest child of mathematician Max Noether and Ida Amalia Kaufmann, and she had three younger brothers: Alfred, who tragically passed away at a young age despite earning a chemistry doctorate; Fritz, who made contributions to applied mathematics but met an untimely end through execution in the Soviet Union; and Gustav, whose life was plagued by chronic illness, leading to an early demise.

In 1903, restrictions on women's enrollment in Bavarian universities were lifted. Emmy Noether returned to Erlangen University in October 1904, focusing solely on mathematics. She was one of six women in her year and the only one in her chosen school. Under Paul Gordan's guidance, she completed her dissertation on invariants for Ternary Biquadratic Forms in 1907, graduating summa cum laude. Despite its initial acclaim, Noether later criticized her thesis and subsequent papers as unsatisfactory.

From 1908 to 1915, Noether taught at Erlangen's Mathematical Institute without pay, sometimes filling in for her father when he was unwell. In 1910 and 1911, she expanded her thesis work from three variables to n variables. After Gordan's retirement in 1910, she worked under his successors, Erhard Schmidt and Ernst Fischer. Fischer introduced her to David Hilbert's work, which greatly influenced her. Between 1913 and 1916, she published several papers applying Hilbert's methods to mathematical concepts such as fields of rational functions and finite group invariants. This period marked her introduction to abstract algebra, where she would later excel. Noether and Fischer shared a passion for mathematics and often discussed lectures long after they ended; she even sent him postcards continuing their mathematical discussions.

Emmy Noether's work in abstract algebra revolutionized the field. She made significant contributions to algebraic structures, including rings, fields, and algebras. Noether's theorem, which establishes a profound connection between symmetries and conservation laws in physics, remains one of her most renowned achievements. Additionally, her research laid the groundwork for modern algebraic concepts, profoundly influencing diverse areas of mathematics and theoretical physics.

Emmy Noether's influential lectures attracted numerous graduate students who were captivated by her innovative approach to mathematics. Her lectures were known for their clarity, depth, and intellectual rigor, inspiring many aspiring mathematicians to pursue advanced studies in algebra and related fields. Through her mentorship and groundbreaking insights, Noether played a pivotal role in shaping the careers of numerous mathematicians, leaving a lasting impact on the field of abstract algebra.

Emmy Noether faced expulsion from Göttingen by Nazi Germany due to discriminatory policies against Jews in academia. Despite her significant contributions to mathematics and her esteemed position within the academic community, she was forced to leave her position at the University of Göttingen in 1933. This unjust expulsion was part of the broader persecution of Jewish intellectuals and professionals under the Nazi regime. Despite this adversity, Noether continued her mathematical work and teaching in the United States, where she made further contributions to the field until her untimely death in 1935.

Emmy Noether made profound contributions to both mathematics and physics throughout her career. In mathematics, she revolutionized abstract algebra, particularly in the areas of ring theory, field theory, and group theory. Her work on ideal theory and the structure of rings laid the foundation for modern algebraic concepts and greatly influenced subsequent developments in the field.

In physics, Noether's most famous contribution is her theorem linking symmetries and conservation laws. Known as Noether's theorem, it states that for every continuous symmetry in a physical system, there is a corresponding conservation law. This theorem has had a profound impact on theoretical physics, providing a fundamental understanding of the relationship between symmetries and the laws of nature.

Overall, Emmy Noether's work has left an indelible mark on both mathematics and physics, shaping the way we understand and approach these disciplines to this day.

Emmy Noether was diagnosed with a pelvic tumor in April 1935 and underwent surgery, during which doctors found an ovarian cyst and smaller tumors in her uterus. Despite initially recovering well, she suddenly fell unconscious and passed away on April 14, likely due to surgery complications or an infection. A memorial service was held at Bryn Mawr College, where tributes from notable figures like Albert Einstein were paid. Noether's ashes were interred under the walkway around the cloisters of the M. Carey Thomas Library at Bryn Mawr.

Emmy Noether's legacy endures as a testament to her profound contributions to mathematics and physics, despite facing numerous obstacles throughout her career. Her pioneering work in abstract algebra and mathematical physics, including Noether's theorem, continues to influence generations of mathematicians and physicists. Although her life was tragically cut short, her impact on the field remains indelible. Noether's enduring legacy serves as an inspiration for aspiring scientists and underscores the importance of perseverance and dedication in the pursuit of knowledge.

Abstract

The thesis consists of seven chapters. The first chapter contains a brief introductory overview of the modern cosmology, symmetry approaches, particularly the Noether symmetry, in the context of solving non-linear differential equations. Also in this introductory chapter, minisuperspace in quantum cosmology have been discussed in a particular way.

Chapter two deals with a multi-field cosmological model in a spatially flat FLRW space-time geometry. Cosmological solutions are obtained using symmetry analysis. The classical solutions are determined after simplifying the Lagrangian using cyclic variables. Finally, Wheeler-DeWitt (WD) equation in quantum cosmology has been formulated and conserved momenta corresponding to Noether symmetry will show the periodic part of the wave function and hence to have the complete integral for the wave function.

In chapter three, we work on the Einstein æther scalar-tensor gravity. The cosmological solutions are analyzed from the observational point of view. Finally, solution of WD equation has been formulated by identifying the periodic nature of the wave function using conserved (Noether) charge.

We have studied the classical and quantum cosmologies for teleparallel dark energy (DE) model in the fourth chapter. Using Noether symmetry analysis, we have determined not only the symmetry vector but also the potential function. Also the symmetry analysis have determined a transformation in the augmented space so that evolution equations become solvable. Finally, WD equation has been formulated in quantum domain.

Chapter five presents scalar tensor and the scalar torsion theories. Noether symmetry analysis has been used to determine the classical cosmological solution. Finally, the nature of the classical solution has been discussed from the observational point of view and the cosmological singularity has been examined both classically and quantum mechanically.

Chapter six deals with $f(T,T_G)$ gravity in the background of homogeneous and isotropic flat FLRW space-time model. The main aim of this work is to examine whether the model supports the observational data or not. Then the solutions are analyzed from the cosmological point of view.

Finally, a brief summary of the work presented in the thesis and a discussion about some possible future prospects has been given in chapter seven.

Preface

The work of this thesis has been carried out at the Department of Mathematics, Jadavpur University, Kolkata- 700032, India. The thesis is based on the following published papers:

• **Chapter 2** has been published as "Quantum Cosmology in Coupled Brans-Dicke Gravity: A Noether Symmetry Analysis", **D. Laya**, S. Dutta and S. Chakraborty, **Int. J. Mod. Phys. D 32**, 2350001 (2023).

• **Chapter 3** has been published as "Classical and Quantum Cosmology in Einstein-æther Scalar-tensor gravity: Noether Symmetry Analysis", **D. Laya**, R. Bhaumick, S. Dutta and S. Chakraborty, **Int. J. Mod. Phys. A 38**, 2350064 (2023).

• **Chapter 4** has been published as "A description of classical and quantum cosmology for a single scalar field torsion gravity" **D. Laya**, R. Bhaumick, S. Dutta and S. Chakraborty, **Mod. Phys. Lett. A. 38**, 2350109 (2023).

• **Chapter 5** has been published as "Noether symmetry analysis in scalar tensor cosmology: a study of classical and quantum cosmology", **D. Laya**, R. Bhaumick and S. Chakraborty, **Eur. Phys. Jour. C 83**, 701 (2023).

• **Chapter 6** has been accepted for publication as "Noether Symmetry Analysis in $f(T, T_G)$ Gravity Theory : A Study of Classical Cosmology", **D. Laya**, S. Dutta and S. Chakraborty, **Int. Jour. of Geom. methods in Modern Physics**.

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1 Introduction

1.1 A Short Introduction to Cosmology

Cosmology is the scientific study of the origin, evolution, and eventual fate of the universe. It seeks to understand the large-scale structure and dynamics of the cosmos, including its fundamental constituents such as galaxies, stars, dark matter, and dark energy (WEINBERG, 2008).

At its core, cosmology delves into profound questions about the universe's beginning, its expansion over billions of years, and its ultimate destiny. Key concepts in cosmology include the Big Bang theory, which proposes that the universe originated from an extremely dense and hot state about 13.8 billion years ago, and cosmic inflation, a period of rapid expansion in the universe's early moments.

Observational evidence from fields like astronomy, astrophysics, and particle physics provide insights into cosmological phenomena. Measurements of the cosmic microwave background radiation, the large-scale distribution of galaxies, and the redshift of distant celestial objects all contribute to our understanding of the universe's history and structure.

Cosmologists also investigate the composition and properties of the universe's contents, including dark matter, a mysterious substance that outweighs visible matter but does not emit light, and dark energy, a force driving the universe's accelerated expansion.

Overall, cosmology combines theoretical models, computational simulations, and observational data to paint a comprehensive picture of the universe's past, present, and future, advancing our understanding of the cosmos and our place within it.

We can start this section by posing the question: Where did the word "cosmology" originate from? The Greek words "Cosmos" and "Logia" together signify "cosmology", with "Cosmos" denoting "The Universe, the world" and "Logia" representing "Study of". Cosmology is a discipline spanning astrophysics, metaphysics, astronomy, and physics, focused on exploring the nature and evolution of the Universe. Specifically, this branch of astronomy encompasses the evolution of the Universe from the initial Big Bang singularity scenario to its future states, as well as the fundamental characteristics of the Universe.

According to NASA cosmology is “the scientific study of the large scale properties of the Universe as a whole”.

Since the end of the last century, the observational evidences (RIESS et al., 1998; RIESS et al., 2007; PERLMUTTER et al., 1999; BENNETT et al., 2003; EISENSTEIN et al., 2005; TEGMARK et al., 2004b; DAVIS et al., 2007; KOMATSU et al., 2009b; HINSHAW et al., 2007; PAGE et al., 2007; HICKEN et al., 2009; HINSHAW et al., 2009; KOMATSU et al., 2009a) from Supernovae type Ia (RIESS et al., 1998; PERLMUTTER et al., 1999; GARNAVICH et al., 1998), baryon acoustic oscillation (BAO) (PERCIVAL et al., 2007; SANCHEZ et al., 2012), cosmic microwave background (CMB), galaxy clustering, radiation along with the large scale structure (LSS) etc. suggest that at the present time the Universe is in an accelerating phase. Certainly! Standard cosmology faces a significant challenge in explaining why our Universe appears to be accelerating. To address this, cosmologists propose two main modifications. One group suggests altering the theory of gravity by adding an extra term to the Einstein-Hilbert action. The other group proposes a different approach within Einstein’s gravity framework, involving exotic matter known as dark energy. Dark energy behaves differently from normal matter—it has a large negative pressure. Just as a hypothetical scalar field (inflaton) drove the early accelerated expansion during inflation, these dark energy scalar fields are hypothesized to drive the current phase of accelerated expansion in the late universe.However very recent observational results predict the presence of some cosmological constant in the dark sector of the Universe and the ΛCDM cosmology has been favored.

In the realm of cosmology, the ΛCDM (Lambda Cold Dark Matter) model, along with the cosmological constant Λ, stands as the quintessential candidate for Dark Energy (DE) (PEEBLES; RATRA, 2003; PADMANABHAN, 2003). This model elegantly explains a Universe imbued with cold dark matter. The cosmological constant is the standard model of cosmology due to its simplicity and compliance with most of the observational data. But due to two extreme drawbacks the cosmological constant is not well agreed model of dark energy (DE) rather the dynamical DE models (AMENDOLA, 2010; CALDWELL; DAVE; STEINHARDT, 1998a; CALDWELL, 2002; YOO; WATANABE, 2012) are very widely used in the literature. These two drawbacks are commonly known as the coincidence problem (FITCH; MARLOW; DEMENTI, 2000; STEINHARDT, 2003) and fine-tuning problem (WEINBERG, 1989a; CARROLL, 2001). To resolve these problems, cosmologists have been found alternative DE model to describe the present accelerating phase. As a consequence we get different kinds of DE model such as Quintessence model (WETTERICH, 1988; RATRA; PEEBLES, 1988; CALDWELL; DAVE; STEINHARDT, 1998b; CARROLL, 1998), Phantom model (CALDWELL, 2002), Chaplygin gas (DEBNATH; BANERJEE; CHAKRABORTY, 2004; CHAKRABORTY; BANDYOPADHYAY, 2009; BISWAS et al., 2011; KAMENSHCHIK; MOSCHELLA; PASQUIER, 2001; BILIC; TUPPER; VIOLLIER, 2002;

BENTO; BERTOLAMI; SEN, 2002; KAMENSHCHIK; MOSCHELLA; PASQUIER, 2001), K-essence (ARMENDARIZ-PICON; DAMOUR; MUKHANOV, 1999; ARMENDARIZ-PICON; MUKHANOV; STEINHARDT, 2000; CHIBA; OKABE; YAMAGUCHI, 2000; ARMENDARIZ-PICON; MUKHANOV; STEINHARDT, 2001), Quintom (ELIZALDE; NOJIRI; ODINTSOV, 2004; FENG; WANG; ZHANG, 2005; CAI et al., 2010), Tachyon (comes from string energy) (PADMANABHAN, 2002; PADMANABHAN; CHOUDHURY, 2002), Holographic dark energy (WANG; WANG; LI, 2017a; WANG; WANG; LI, 2017b; MANOHARAN; SHAJI; MATHEW, 2023; NOJIRI; ODINTSOV; PAUL, 2021; LI, 2004b; LI, 2004a), teleparallel DE (CAI et al., 2016; BAMBA et al., 2012) etc. In this thesis, we have discussed some modified gravity models and found the classical solution on the basis of Noether symmetry analysis and also studied quantum cosmology by constructing Wheeler DeWitt (WD) equation of those models.

1.1.1 Homogeneity and Isotropy

The cosmological principle states that the Universe is homogeneous and isotropic on a large scale, implying it lacks a preferred location and appears identical in all directions (ELLIS; STOEGER, 1987; BUCHERT, 2000; PARANJAPE; SINGH, 2006; CHUANG; GU; HWANG, 2008; KOLB et al., 2005). This concept is applied to the scales, approximately 100 million light-years per second (Mpc) (RYDEN, 2017) which indicating that the Universe exhibits uniformity and homogeneity on a large scale. The Universe is neither homogeneous

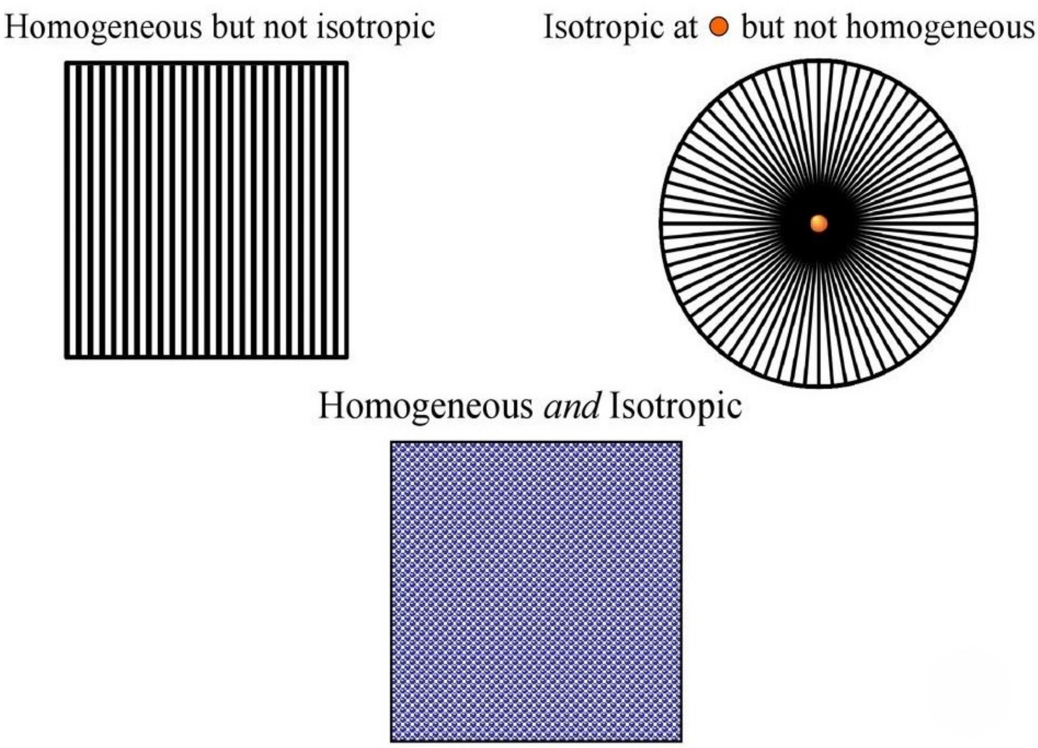

Figure 1.1 – Homogeneity and Isotropy

nor isotropic on small scales, and homogeneity does not necessarily imply isotropy.

In Figure (1.1) (top left), homogeneous stripes are evident on larger scales, yet isotropy is lacking. Conversely, Figure (1.1) (top right) displays an isotropic bullseye around the center, albeit lacking homogeneity, contrasting with the larger-scale stripes. However, Figure (1.1) (bottom) demonstrates both homogeneity and isotropy.

1.1.2 FLRW (Friedmann-Lemaître-Robertson-Walker) Universe:

In standard cosmology, our Universe adheres to the cosmological principle, it is homogeneous and isotropic on a large scale structure (ELLIS; STOEGER, 1987; BUCHERT, 2000; KOLB et al., 2005; PARANJAPE; SINGH, 2006; CHUANG; GU; HWANG, 2008). Homogeneity implies that the Universe appears identical on a large scale, indicating that the geometry of space-time and the properties of matter are the same at every point in space, making it homogeneous. Isotropy means that the Universe looks the same in every direction. When solved under certain assumptions, such as homogeneity and isotropy on large scales, they yield a family of solutions known as FLRW universes. Generally, the behavior of the Universe is either in a state of expansion or collapse. The explicit form of the Einstein field equations are

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.1}$$

where $\kappa = 8\pi G$ is the gravitational coupling constant, and $T_{\mu\nu}$ is the energy-momentum tensor containing matter distribution.

The Einstein tensor $G_{\mu\nu}$ is defined by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu}, \tag{1.2}$$

where the Ricci scalar R is defined as

$$R = R_{\mu\nu} g^{\mu\nu}. \tag{1.3}$$

The FLRW Universe model has revolutionized our understanding of cosmology by providing a comprehensive framework for describing the large-scale structure and evolution of the Universe. Its predictions have been validated by numerous observations, bolstering confidence in our current cosmological theories. Moving forward, further exploration of the FLRW model promises to unveil deeper insights into the nature of the cosmos and its origins.

1.1.3 Friedmann Equation:

1.1.3.1 History:

The Friedmann equations are a set of two equations in physical cosmology that govern the expansion and nature of the Universe within the framework of general relativity, assuming a homogeneous and isotropic model (SHEYKHI, 2013; ARIK; CALIK; SHEFTEL, 2008; MOSTAGHEL et al., 2016). Alexander Friedmann derived these equations in 1922, using the Einstein field equations of gravitation for the FLRW (Friedmann-Lemaître-Robertson-Walker) metric and a perfect fluid with pressure p and energy density ρ.

1.1.3.2 **Assumption:**

Let us start the Friedmann equations with a simple assumption: that our Universe is homogeneous and isotropic on a very large scale, of the order of 100 Mpc (RYDEN, 2017). This implies that the metric can describe large-scale geometry. This metric is known as the FLRW metric (FRIEDMAN, 1922; LEMAITRE, 1931; ROBERTSON, 1935; Walker, 1937), given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{1.4}$$

where $a(t)$ is the scale factor of the Universe, generally $a(t) = 1$ for the present time, and κ is the scalar curvature. The above four-dimensional metric ds^2 can be described by the nature of the Universe based on the scalar curvature κ as,

$$\kappa = \begin{cases} -1, & \text{closed Universe,} \\ +1, & \text{open Universe,} \\ 0, & \text{flat Universe.} \end{cases}$$

discuss the nature of the scale factor $a(t)$.

1.1.3.3 **Equation:**

The Einstein equations are related to the scale factor, thermodynamic pressure, and energy density of the matter in the Universe. Additionally, from the FLRW line element, we can derive the Christoffel symbols and the Ricci tensor.

Using the general theory of relativity, the field equations for the FLRW line element are given by:

$$3\left(H^2 + \frac{\kappa}{a^2}\right) = \frac{8\pi G}{c^4}\rho, \tag{1.5}$$

which arises from the 00 component of the Einstein equations, and

$$2\left(\dot{H} - \frac{\kappa}{a^2}\right) = -\frac{8\pi G}{c^4}(p+\rho), \tag{1.6}$$

where G is the Newtonian gravitational constant, c is the speed of light in a vacuum, and $\Lambda = 1$. H $(= \frac{\dot{a}}{a})$ is known as the Hubble parameter, where the dot indicates derivative with respect to cosmic time "t". Now, $H\left(=\frac{\left(\frac{da}{a}\right)}{dt}\right)$ denotes the rate of expansion or contraction of the Universe. Also, these equations are known as the first Friedmann equation and the second Friedmann equation, respectively. The virtue of Bianchi identities conserves the energy-momentum tensor, so the matter conservation relation

$$T^{\nu}_{\mu\ ,\nu} = 0, \tag{1.7}$$

has the explicit form

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.8}$$

The above equation is known as the energy conservation relation or conservation of mass-energy.

The above three equations are not independent for a homogeneous and isotropic Universe. If we eliminate $\frac{\kappa}{a^2}$ from (1.5) and (1.6) then we arrive at an important equation known as the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^4}(\rho + 3p), \tag{1.9}$$

which measures the acceleration. From this equation, one can conclude that our Universe is expanding when $(\rho + 3p) < 0$. This is discussed as follows:

$$\begin{aligned}
(i)\quad \text{if}(\rho + 3p) > 0 &\implies \frac{\ddot{a}}{a} < 0 \implies \ddot{a} < 0,\\
&\text{i.e., the Universe is decelerated.}\\
(ii)\quad \text{if}(\rho + 3p) < 0 &\implies \frac{\ddot{a}}{a} > 0 \implies \ddot{a} > 0,\\
&\text{i.e., the Universe is accelerated.}
\end{aligned}$$

If the matter satisfies the strong energy condition (SEC), i.e., $(\rho + 3p) > 0$, then this matter is called normal matter. On the other hand, If it does not satisfy the SEC then it is called an exotic matter.

1.1.4 Density Parameter:

Density parameter is a dimensionless quantity related to the density of matter. Generally, it is denoted by Ω. In other words, the density parameter is the ratio of the average density of matter (ρ) to the critical density of matter (ρ_c), i.e.,

$$\Omega = \frac{\rho}{\rho_c}.$$

From the 1st Friedmann equation (1.5), we get

$$
\begin{aligned}
3H^2 + \frac{\kappa}{a^2} &= \frac{8\pi G}{c^4}\rho \\
\implies 1 + \frac{\kappa}{3H^2a^2} &= \frac{\rho}{\left(\frac{c^4}{8\pi G}\right)3H^2} \\
\implies 1 + \frac{\kappa}{3H^2a^2} &= \frac{\rho}{\frac{3H^2}{\left(\frac{8\pi G}{c^4}\right)}} \\
\implies 1 + \Omega_k &= \Omega. \qquad (1.10)
\end{aligned}
$$

where the critical density is $\rho_c = \frac{3H^2c^4}{8\pi G}$, and $\Omega_k = \frac{\kappa}{3H^2a^2}$ is the density parameter for curvature. This equation is known as the Einstein field equation in terms of the density parameter for curvature. The density parameter plays an important role in describing the nature of the Universe. We know

$$
\Omega_k : \begin{cases} > 0, & \text{closed Universe,} \\ < 0, & \text{open Universe,} \\ = 0, & \text{flat Universe.} \end{cases}
$$

Using the above two relations, we find that for the flat Universe $\Omega = 1$, for the closed Universe $\Omega > 1$, and for the open Universe $\Omega < 1$ (KOMATSU et al., 2011). Moreover, the critical density determines the geometry of space-time. In the absence of the cosmological constant, the Universe will expand forever when $\Omega < 1$, whereas it will eventually collapse if $\Omega > 1$. The total density parameter at the critical density ρ_c would be 1, and the observation shows that the present density parameter is close to that value.

1.1.5 Dark Energy: The Suspects

Dark energy is a mysterious and enigmatic component of the universe that has intrigued scientists for decades. Unlike ordinary matter and dark matter, which exert gravitational attraction, dark energy is associated with a repulsive force that accelerates the expansion of the universe. Its existence was first suggested in the late 20th century based on observations of distant supernovae, which revealed that the universe's expansion rate is increasing over time. Despite its pervasive influence on cosmic evolution, the true nature of dark energy remains one of the most profound mysteries in modern astrophysics and cosmology.

1.1.5.1 Cosmological Constant:

In cosmology, the cosmological constant, often denoted by the Greek letter *Lambda*, represents the energy density of the vacuum in space (PEEBLES; RATRA, 2003; PADMAN-

ABHAN, 2003). Also known as Einstein's cosmological constant, it originally emerged as a constant coefficient introduced by Albert Einstein in 1917, to his field equations of general relativity (EMELYANOV, 2013; BENGOCHEA et al., 2020; HOBSON; EFSTATHIOU; LASENBY, 2006). Einstein proposed this addition to counteract gravity's attractive force, aiming to achieve a static Universe, a concept of interest at that time. Additionally, he sought to incorporate Mach's principle into his theory. However, Einstein's cosmological constant fell out of favor in 1929 when Edwin Hubble's discovery of the expanding Universe emerged. From 1929 to 1990, most cosmologists assumed the cosmological constant to be zero, as it was deemed to have no significant effect on the gravitational field.

After adding the cosmological constant, Einstein's equation can be written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.11}$$

where Λ is the cosmological constant (a new free parameter).

After this modification, the Friedmann equations become:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{1.12}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho) + \frac{\Lambda}{3}. \tag{1.13}$$

These equations have a static solution with positive spatial curvature. Here, all the parameters, namely, thermodynamic pressure, energy density, and the cosmological constant, are non-negative. This solution is known as the "Einstein static Universe". It should be noted that Λ plays an important role in equation (1.13) against gravity. The energy density and the thermodynamic pressure can be expressed in terms of the cosmological constant as:

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}, \quad p_\Lambda = -\frac{\Lambda}{8\pi G}, \tag{1.14}$$

i.e., $p_\Lambda + \rho_\Lambda = 0$. Then, for the cosmological constant, the equation of state parameter can be written as:

$$\omega_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -1 . \tag{1.15}$$

Clearly, from equation (1.15), one can conclude that it violates the strong energy condition (SEC), $3p + \rho \geq 0$. Hence, the cosmological constant may be a viable candidate for dark energy. Now, we discuss two major drawbacks of the cosmological constant.

There are two main mysteries/drawbacks of the cosmological constant, namely, the 'Fine Tuning problem' and the 'Coincidence problem', and many cosmologists believe that 'the vacuum holds the key to a full understanding of nature'. Now we have discussed those two cosmological problems (WEINBERG, 1989b; PADILLA, 2015; WANG; ZHU; UNRUH,

2017; PERCACCI, 2018).

1.1.5.2 Fine Tuning Problem:

"Fine-tuning" is a process in which parameters of a model are adjusted very accurately in order to fit with certain observations, which has led to the discovery that the fundamental constants and quantities fall into such an extraordinarily precise range that if they did not, the origin and evolution of conscious agents in the Universe would not be permitted (WEINBERG, 1989b; CARROLL, 2001).

According to the recent observations, cosmological constant is a nonzero quantity, and its value has been assumed to be (predicted value).

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = 10^{-47}\text{GeV}^4. \tag{1.16}$$

In the context of quantum field theory, the value of the cosmological constant should be very high ($\rho_{\text{vac}} \simeq 10^{74}$ GeV^4). As a result, there is a discrepancy between the measured value and the theoretical prediction (the order is almost 10^{121}). So the question arises: how is the value of the cosmological constant different between these two predictions? This is known as the cosmological constant problem or fine-tuning problem.

1.1.5.3 Coincidence Problem:

The recent observational evidence suggests that the vacuum energy density of matter (ρ_m) and the energy density of the cosmological constant (ρ_Λ) are of the same order, i.e., $\rho_m \propto \rho_\Lambda$. Hence, one question may arise: why do they become equal at the present epoch? In other words, why are we so privileged? This implies that we live in a special epoch of the evolution of the Universe when none of the energy densities are unequal. Usually, this problem is known as the "Coincidence Problem" (COPELAND; SAMI; TSUJIKAWA, 2006; WETTERICH, 1995; AMENDOLA, 2000a; AMENDOLA, 2000b; BILLYARD; COLEY, 2000; ZIMDAHL; PAVON, 2001; HERRERA; PAVON; ZIMDAHL, 2004; AMENDOLA; QUERCELLINI, 2003; HOFFMAN, 2003; CHIMENTO et al., 2003; AMENDOLA, 2004; GUO; CAI; ZHANG, 2005).

1.1.6 Cosmographic Parameter:

In standard cosmology, we utilize various parameters known as cosmographic parameters. They are highly useful in studying the different phases of the Universe across various cosmic timescales (MEHRABI; REZAEI, 2021). Let us now define these parameters. We assume that the Universe is homogeneous and isotropic on a large scale. Thus, the

FLRW line element of the Universe indicates its geometry. In the FLRW line element, $a(t)$ represents the scale factor of the Universe. Now we expand the scale factor "$a(t)$" about the present cosmic time "t_0" in Taylor series expansion and we have:

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t-t_0) + \frac{1}{2!}q_0H_0^2(t-t_0)^2 + \frac{1}{3!}J_0H_0^3(t-t_0)^3 + \frac{1}{4!}s_0H_0^4(t-t_0)^4 + \frac{1}{5!}l_0H_0^5(t-t_0)^5 + \frac{1}{6!}m_0H_0^6(t-t_0)^6 + ..., \tag{1.17}$$

where t_0 represents the present time, and the suffix zero indicates the present value of those parameters. The coefficients of the various powers of $(t-t_0)$ mentioned above can be defined as follows (SAHNI et al., 2003):

$$\begin{aligned} H &= \frac{\dot{a}}{a} = \text{Hubble Parameter.} \\ q &= -\frac{\ddot{a}}{aH^2} = \text{Deceleration Parameter.} \\ J &= \frac{\dddot{a}}{aH^3} = \text{Jerk Parameter.} \\ s &= \frac{\ddddot{a}}{aH^4} = \text{Snap Parameter.} \\ l &= \frac{a^{(5)}}{aH^5} = \text{Lerk Parameter.} \\ m &= \frac{a^{(6)}}{aH^6} = \text{m- Parameter.} \end{aligned}$$

Those parameters (VISSER, 2004b; VISSER, 2004a; SAHNI et al., 2003) are utilized to determine the distance-redshift relation and various distances within the Universe. The sign of the Hubble parameter indicates whether the Universe is expanding ($H > 0$) or contracting ($H < 0$). Similarly, the sign of the deceleration parameter q indicates whether the Universe is accelerating ($q < 0$) or decelerating ($q > 0$). Additionally, the deceleration parameter is defined as:

$$q = -\left(1 + \frac{\dot{H}}{H^2}\right) = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{1.18}$$

Similarly, a change in the sign of the Jerk parameter in the expanding Universe indicates whether the acceleration of the Universe is increasing or decreasing. Furthermore, the deceleration parameter q can be expanded in a Taylor series expansion in terms of the redshift parameter as:

$$q(z) = q_0 + (-q_0 - 2q_0^2 + J_0)z + \frac{1}{2}(2q_0 + 8q_0^2 - 7q_0J_0 + 8q_0^3 - 4J_0 - s_0)z^2 + O(z^3), \tag{1.19}$$

where the elementary relation $\frac{1}{a} = 1 + z$ holds between the canonical redshift and the scale factor of big-bang cosmology. The redshift parameter holds significant importance in the context of cosmology. There is no doubt that the expansion of the Universe has been described by the observational redshifts of distant galaxies. Additionally, the Hubble parameter in terms of the redshift parameter can be expanded in a Taylor series:

$$H(z) = H_0 + \frac{\dot{H}_0}{1!}z + \frac{\ddot{H}_0}{2!}z^2 + \frac{\dddot{H}_0}{3!}z^3 + \frac{\ddddot{H}_0}{4!}z^4 + ..., \tag{1.20}$$

where the dot indicates derivatives with respect to cosmic time. The redshift H_0 indicates the value at $z = 0$. Furthermore, the present values of the Hubble and deceleration parameters are as follows (RIESS et al., 2022; AGHANIM et al., 2020; RIESS et al., 1998; PERLMUTTER et al., 1999; SPERGEL et al., 2003; TEGMARK et al., 2004a; EISENSTEIN et al., 2005):

$$\begin{aligned} H_0 &= 73.04 + /-1.04 \mathrm{km/s/Mpc} \\ q_0 &= -0.55615 \end{aligned}$$

1.1.7 Modified Gravity theory

General Relativity (GR) stands as the conventional theory of gravity, offering a comprehensive understanding of the geometric properties of spacetime. Within a universe characterized by uniformity and isotropy, the Einstein field equations yield the Friedmann equations, which depict the universe's evolution. Notably, the conventional big-bang cosmology, marked by epochs dominated by radiation and matter, finds a coherent explanation within the framework of General Relativity. In this segment, we will provide a concise overview of the action principle, field equations, and crucial aspects of modified gravity theories. Hence, now is the opportune moment to delve into the fundamental theory of gravity, namely the $f(R)$ gravity theory (PEEBLES, 2020; DODELSON, 2003; CLIFTON et al., 2012; FARAONI; GUNZIG; NARDONE, 1999; LI; KOYAMA, 2020; PETROV; NASCIMENTO; PORFIRIO, 2023).

1.1.7.1 **f(R) Gravity:**

$f(R)$ gravity serves as a modification of Einstein's General Relativity, providing a wider scope. It encompasses a range of theories, each distinguished by a distinct function, f, of the Ricci scalar, R. The simplest method to return to General Relativity within $f(R)$ gravity is by equating the function to the scalar itself. Introducing this variable function offers the potential to explain the accelerated expansion and structure formation of the Universe without relying on enigmatic concepts such as dark energy or dark matter (FELICE; TSUJIKAWA, 2010; SOTIRIOU; FARAONI, 2010). Some of these functions may take inspiration from adjustments derived from a quantum theory of gravity. Initially proposed by Hans Adolph Buchdahl in 1970 (BUCHDAHL, 1970), $f(R)$ gravity has evolved significantly, particularly propelled by Starobinsky's contributions to cosmic inflation.

The Einstein-Hilbert action lays the groundwork for general relativity, acting as the basis from which the Einstein field equations arise through the principle of least action. Utilizing the metric signature $(-,+,+,+)$ to describe spacetime, the gravitational component of this action is formulated as:

$$S_{EH} = \frac{1}{2\kappa}\int R\sqrt{-g}d^4x + \int \sqrt{-g}d^4x\mathcal{L}_m(g_{\mu\nu},\varphi), \tag{1.21}$$

where $g = \det(g_{\mu\nu})$, $R = R_{\mu\nu}g^{\mu\nu}$ is the Ricci scalar, $\mathcal{L}_m$ is the matter Lagrangian that depends on the metric tensor $g_{\mu\nu}$ and the matter fields φ, and $\kappa = \frac{8\pi G}{c^4}$ is the Einstein gravitational constant, with G and c stand for the gravitational constant and the speed of light in a vacuum, respectively. It is noted that if the integral converges, it is taken over the entire spacetime. If it does not converge, S is no longer well-defined. However, a modified definition, where one integrates over arbitrarily large but relatively compact domains, still yields the Einstein equation as the Euler–Lagrange equation of the Einstein–Hilbert action.

One of the most straightforward adjustments to General Relativity is the $f(R)$ gravity theory, which is recognized as one of the well-established modified gravity theories employed in explaining the ongoing accelerated period. In this theory, f represents any arbitrary function of R, where R is replaced by $f(R)$.

$$S = \frac{1}{2\kappa}\int f(R)\sqrt{-g}d^4x + \int \sqrt{-g}d^4x\mathcal{L}_m(g_{\mu\nu},\Psi), \tag{1.22}$$

where all other quantities remain unchanged. To derive the field equations, they are obtained by varying the action concerning both the metric tensor $g_{\mu\nu}$ and the Ricci scalar R.

We won't delve into examining $f(R)$ gravity theory through two distinct methods: the metric formalism and the Palatini formalism. These methods have been extensively investigated in existing literature. Instead, we'll offer a concise introduction to Scalar-tensor theory and Brans-Dicke theory.

1.1.7.2 Scalar tensor Gravity Theory:

The scalar–tensor theory, developed by researcher John Moffat, is among the modified gravitational theories (BRANS, 2005; FUJII; MAEDA, 2007; SOTIRIOU, 2006; BOISSEAU et al., 2000). It is rooted in the action principle and suggests that the existence of a vector field while transforming the three constants of the theory into scalar fields. Widely accepted, this theory offers an explanation for the accelerating Universe, where gravitational action encompasses both metric and scalar fields. It's worth noting that the Brans-Dicke theory is a specific instance of the Scalar-tensor theory, featuring a constant coupling parameter. While it has a longstanding presence in cosmology, starting from the Brans-Dicke theory (BRANS; DICKE, 1961a), it has recently garnered attention from several researchers in the context of the current era of acceleration (AMENDOLA, 1999; UZAN, 1999; BARTOLO; PIETRONI, 2000; BOISSEAU et al., 2000; ESPOSITO-FARESE; POLARSKI, 2001). Additionally, one advantage of this theory is that it can resolve the fine-tuning as well as the coincidence problem.

The action integral of this gravity theory is given by:

$$S = \frac{1}{2\kappa}\int \left[f(\varphi,R) - \xi(\varphi)(\Delta\varphi)^2\right]\sqrt{-g}d^4x + \int \sqrt{-g}d^4x\mathcal{L}_m(g_{\mu\nu},\Psi), \tag{1.23}$$

where f represents a flexible function of both the scalar field φ and the Ricci scalar R. $\mathcal{L}_m$ denotes the Lagrangian for matter, while ξ is a function specifically associated with φ. Additionally, κ is defined as $\frac{8\pi G}{c^4}$.

The action integral transforms into $f(R)$ gravity if we select $f(\varphi,R) = f(R)$ with $\xi(\varphi) = 0$. Similarly, it adopts the Brans-Dicke (BD) theory guise when $f(\varphi,R) = \varphi R$ and $\xi(\varphi) = \frac{\omega_{BD}}{\varphi}$, where ω_{BD} denotes the Brans-Dicke parameter (GASPERINI; VENEZIANO, 2003). Next, we can derive the two field equations by varying the action concerning φ and the metric tensor $g_{\mu\nu}$. I won't delve further into explaining the BD theory as I have already covered it in the subsequent section.

In this context, it is important to note that a connection can be established between the Brans-Dicke (BD) theory and the $f(R)$ theory using both metric and Palatini formalisms. To illustrate this connection, we examine the following correspondence:

$$U(\varphi) = \frac{1}{2}\left\{R(\varphi)F - f(R(\varphi))\right\}, \tag{1.24}$$

$$\varphi = F(R), \tag{1.25}$$

where $R = R(T)$ in the Palatini formalism and $R = R(g)$ in the metric formalism. At this point, we can contrast the field equations of this theory with the equations mentioned earlier.

1.1.7.3 **Brans-Dicke Theory:**

In 1961, Charles H. Brans and Robert H. Dicke proposed a cosmological theory of gravity distinct from Einstein's general theory of relativity. Currently, both theories are generally consistent with observations. In physics, Brans-Dicke (BD) gravity theory stands as one of the competitors to Einstein's general theory of relativity (BRANS; DICKE, 1961a; BERRY, 1989; FARAONI, 1999; CAMPANELLI; LOUSTO, 1993; KOFINAS; TSOUKALAS, 2016). Since the end of the last century, standard cosmology has faced a significant challenge: explaining observational data suggesting that our Universe is undergoing accelerated expansion. In this cosmological context, researchers are exploring possible modifications. One group proposes adding an extra term to the Einstein-Hilbert action, while another favors incorporating exotic matter within the framework of Einstein gravity. Thus, two theories of gravity emerge: modified gravity theory and dark energy theory. Scalar fields play a crucial role in describing the evolution of the Universe within cosmological frameworks. The BD theory serves as an example of a relativistic classical field theory of gravitation. In this theory, spacetime is represented by a metric tensor $g_{\mu\nu}$, and the gravitational field is described by the Riemannian curvature tensor R_{abcd}, which can be determined by the metric tensor. In addition to these fields, there is a scalar field φ, which effectively changes the gravitational constant across space. The BD equation, containing a parameter commonly referred to as the BD parameter and denoted by ω, is a dimensionless constant chosen to fit observations. Such parameters are often called tunable parameters. Additionally, BD theory predicts light deflection and the precision of planetary perihelia orbits around the Sun, similar to general relativity (LEE et al., 2000; ACQUAVIVA et al., 2003; TSUJIKAWA, 2013; PERIVOLAROPOULOS, 2008).

The scalar-tensor theory stands out as a prominent choice for elucidating the accelerating expansion of the Universe. Within this framework, gravitational action incorporates both a scalar field and a metric. It's worth noting that the Brans-Dicke (BD) theory is a specific instance of the scalar-tensor theory with a constant coupling parameter (BERGMANN, 1968; DAMOUR; NORDTVEDT, 1993; MAGNANO; SOKOLOWSKI, 1994; FARAONI, 2004). Despite its long-standing presence in cosmology, stemming from the BD theory, the scalar-tensor theory has recently garnered attention from numerous researchers for its ability to address the current acceleration and tackle challenges such as fine-tuning and coincidence.

Action principle:

Suppose the scalar field φ adheres to the cosmological principle and is minimally linked to gravity. Within this framework, the action integral can be described as follows:

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\varphi R - \frac{\omega_{BD}}{\varphi}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right) + \int d^4x\sqrt{-g}\mathcal{L}_m, \tag{1.26}$$

where $g = |g_{\mu\nu}|$ denotes the determinant of the metric tensor, $\mathcal{L}_m$ stands for the matter Lagrangian density, and $\sqrt{-g}d^4x$ represents the fundamental volume element. Typically, the matter Lagrangian term is zero in the vacuum region. To obtain the field equations for this model, we must vary the gravitational term in the Lagrangian with respect to $g_{\mu\nu}$ and φ.

Field Equations:

The field equations of this gravity theory are as follows:

$$G_{\mu\nu} = \frac{8\pi}{\varphi}T_{\mu\nu} + \frac{\omega}{\varphi^2}\left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_a\varphi\partial^a\varphi\right) + \frac{1}{\varphi}\left(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Box\varphi\right), \tag{1.27}$$

$$\Box\varphi = \frac{8\pi}{3+2\omega}T, \tag{1.28}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ represents the Einstein tensor, with $g_{\mu\nu}$ being the metric tensor, $T_{\mu\nu}$ the stress-energy tensor, and $R_{\mu\nu} = R^a_{\mu a\nu}$ with the Ricci scalar $R = R^{\mu\nu}_{\mu\nu}$. $T = T^\mu_\mu$ is the trace of the stress tensor. The first equation describes how space-time curvature is affected by both the stress-energy tensor $T_{\mu\nu}$ and the scalar field φ. The second equation shows that the trace of the stress-energy tensor acts as the source of the scalar field φ. In this theory, the simultaneous presence of the mass-energy tensor and the long-range scalar field φ determines the Einstein tensor. The vacuum field equations of the BD theory are determined by setting the stress-energy tensor to zero.

1.2 Role of symmetry on Differential Equation:

In this introductory chapter, we have discussed the basic ideas of cosmology and modified gravity theory. However, when we write down the field equations for a particular gravity theory, they become highly nonlinear and coupled in nature. The general method used to solve these equations in this thesis is through symmetry analysis. The remaining part of this thesis will focus on discussing symmetry analysis.

One of the most popular and well-established methods for finding exact solutions of ordinary and partial differential equations is the classical symmetry method (CSM), also known simply as the symmetry method. This famous method, also referred to as the group analysis method, was introduced by the renowned mathematician Sophus Lie in the late 19th century (1881), as a result of pioneering work. Numerous excellent books have been written on this topic and its generalizations. On the other hand, Noether symmetry is one of the most important methods for solving the differential equations of motion. Noether symmetries and conserved quantities are useful for revealing the inherent physical properties of dynamical systems (OLVER, 1993; BLUMAN; SK, 1989; BASKARAN; ANDERSON, 1988; OVSIANNIKOV, 2014; HYDON, 2000).

Symmetry is an exact congruence in shape or position with respect to a point, line, or plane, originating from the Latin word "Symmetria". It is an operation (or transformation) that permits an action upon it. The concept of symmetry is linked with mathematicians Felix Klein and Sophus Lie, who developed important mathematical tools to explain it.

Generally, nonlinear differential equations play a crucial role in solving physical problems, but sometimes they are not easy to solve. Symmetry analysis methods play an important role in solving them. In 1888, his pioneering work was published with the title "Theory of Transformation Groups", which led to a new path in mathematics: symmetries (LIE; MERKER; ENGEL, 2015; LÜCK, 2006; KAWAKUBO; KAWAKUBO, 1991; DIECK, 2011). It is noted that the theory of transformation groups forms a bridge connecting group theory with differential geometry. Beyond a shadow of doubt, symmetry analysis is an essential mathematical tool in modern science. It involves determining a canonical coordinate system for a differential equation and transforming it into a new system in such a way that makes it easier to solve or greatly simplifies it.

The General Theory of Relativity (GTR) consists of ten nonlinear second-order partial differential equations, which pose a great challenge to solve precisely. We can use symmetry analysis methods to reduce the number of independent variables or equations. Lie symmetries are limited to the space of independent variables, while continuous transformations may exist in the space of dependent variables. As a consequence, the field equations exhibit invariance.

Since the last century, symmetry analysis has dominated the field, encompassing global continuous symmetries, internal symmetries of space-time, gauge symmetries, permutation symmetry in Quantum Field Theory, etc. On the other hand, Noether symmetry analysis uses conserved charges to identify physical processes, simplifying differential equations and opening new doors in the study of Quantum Cosmology. Furthermore, symmetry analysis has played a crucial role in various branches of physics, including condensed matter physics, particle physics, and quantum mechanics, providing valuable insights into the fundamental principles governing these fields.

In this thesis, our main focus on the Noether's theorem and its applications. Using Noether symmetry analysis, we have determined the classical cosmological solutions for various models, such as coupled Brans-Dicke (BD) gravity, Einstein-æther scalar-tensor gravity, single scalar field torsion gravity, scalar-tensor and scalar-torsion, and $f(T,T_G)$ gravity. Additionally, quantum cosmology, physical metric, and symmetry analysis have been extensively covered for those modified gravity models, solving the Wheeler-DeWitt (WD) equation to find the wave function of the Universe using the formulation of the Hamiltonian.

1.2.1 Symmetry Group

Symmetry group is a set of all invariant transformations of a geometric object (i.e., the object remains invariant under those transformations) forming a group. Generally, in physical space-time, the symmetry group (HUNGERFORD, 2003; SAGAN, 2013; MILLER, 1973; ARMSTRONG, 2013) coincides with the conformal group. This type of transformation goes from the object to itself. Usually, the symmetry group of an object N is denoted by $G = \text{Sym}(N)$. Such a transformation is an invertible mapping of the ambient space that preserves all the relevant structure of the object. The ambient space is the space surrounding a mathematical object along with the object itself. Now, let's understand this with an example.

Example:

I). When we examine $\{a\}$ independently, we designate the ambient space as $\{a\}$.

II). Considering $\{a\}$ as a subset of $\mathcal{R}$, we conclude that the ambient space encompasses the entirety of $\mathcal{R}$ (LUDWIG; FALTER, 2012).

1.2.2 One-parameter groups of point transformations

One-parameter groups were introduced by Sophus Lie, a Norwegian mathematician, in 1893 to define infinitesimal transformations (OLVER, 1993; BLUMAN; SK, 1989; IBRAGIMOV, 1993; FUSHCHYCH; ZHDANOV, 1997; BASKARAN; ANDERSON, 1988; OVSIANNIKOV, 2014; HYDON, 2000). He was inspired by lectures on Galois theory and Abel's related works given by Sylow at Oslo (then known as Christiania). According to Lie, an infinitesimal transformation is an infinitely small transformation of the one-parameter group that it generates. He showed that the order of an ordinary differential equation (ODE) could be reduced by one constructively if it is invariant under a one-parameter Lie group of point transformations. These infinitesimal transformations generate a Lie algebra used to describe a Lie group of any dimension. As stated in the first section of this chapter, our main aim is to utilize symmetries of differential equations and their solution. Now, I would like to delve into the topic of one-parameter group transformations in detail.

To simplify a given differential equation, it is a common procedure to transform the dependent and independent variables into new variables. Let us consider x and y as the independent and dependent variables, respectively, of the given differential equation. We perform a point transformation from (x, y) to $(\tilde{x}, \tilde{y})$, where $\tilde{x}$ and $\tilde{y}$ are both some functions of x and y, i.e., it can be written as

$$\bar{x} = \bar{x}(x,y), \quad \bar{y} = \bar{y}(x,y). \tag{1.29}$$

In the context of symmetry analysis, the point transformation depends on one parameter, ε, i.e.,

$$\bar{x} = \bar{x}(x,y,\varepsilon), \quad \bar{y} = \bar{y}(x,y,\varepsilon). \tag{1.30}$$

Moreover, this transformation should be invertible, and repeated application should yield a transformation belonging to the same family,

$$\bar{\bar{x}} = \bar{\bar{x}}(\bar{x},\bar{y},\bar{\varepsilon}) = \bar{\bar{x}}(\bar{x},\bar{y},\bar{\bar{\varepsilon}}), \quad \bar{\bar{y}} = \bar{\bar{y}}(\bar{x},\bar{y},\bar{\varepsilon}) = \bar{\bar{y}}(\bar{x},\bar{y},\bar{\bar{\varepsilon}}), \tag{1.31}$$

with $\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}(\bar{\varepsilon},\varepsilon)$. Also, we have $x = \bar{x}(x, y, 0)$ and $y = \bar{y}(x, y, 0)$, which is known as the identity transformation.

Now, we come to the definition of a one-parameter group of transformations. When the set of these transformations forms a group, then the group is said to be a one-parameter group of point transformations (STEPHANI; MACCALLUM, 1989). Let's understand this with the following examples.

Example:

I). Suppose the rotation is given by $\bar{x} = x\cos\theta - y\sin\theta$ and $\bar{y} = x\sin\theta + y\cos\theta$, where θ is a parameter. This is an example of a one-parameter group of transformation.

II). Another example of a one-parameter group of transformation is translation, where $\tilde{x} = x + l$ and $\tilde{y} = y + l$.

III). The reflection is a group of transformation without parameter.

IV). Screw motions: $(x,y,z) \longrightarrow (x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon, z + \varepsilon)$.

V). Scaling transformations: $(x,y,z) \longrightarrow (\lambda x,\lambda y,\lambda^{-1}z)$.

VI). Rotation around a fixed axis: $(x,y,z) \longrightarrow (x\cos\varepsilon - z\sin\varepsilon, y, x\sin\varepsilon + z\cos\varepsilon)$.

The action of a one-parameter group on a set is known as a flow. A smooth vector field on a manifold induces a local flow at a point, one-parameter group of local diffeomorphisms, which sends points along integral curves of the vector field. The local flow of a vector field is used to define the Lie derivative of tensor fields along the vector field.

1.2.3 Invariance

Invariance is one of the most important concepts in physics, and many theories are defined in terms of their symmetries and invariants (STEPHANI; MACCALLUM, 1989). In theoretical physics, an invariant is an observable concept (or some mathematical object) of a physical system that remains unaltered under certain transformations or operations. A function $f : A \longrightarrow B$ is called invariant when it remains unchanged under group transformations

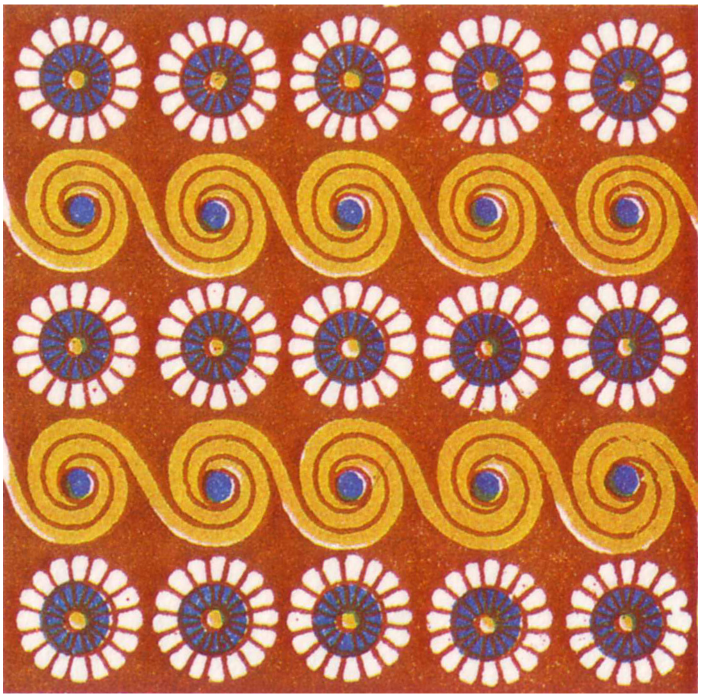

Figure 1.2 – It is an invariant under some transformation. It is invariant under vertical and horizontal translation, as well as rotation by 180°. Also one can note that it not invariant under reflection.

or group actions.

$$f(g \circ a) = f(a) \tag{1.32}$$

where $g \in G$, is a symmetry group of the system and $a \in A$. We have to mention one examples of one-parameter point transformations (STEPHANI; MACCALLUM, 1989).

Example: Consider the linear heat equation is

$$u_t = \alpha u_{xx},$$

where α is a constant. It undergoes a one-parameter point transformation under the translation $x' = x + \varepsilon_1, t' = t + \varepsilon_2$.

While the other pertains to functional analysis, with G being the group of unitary operators on a Hilbert space. Differential equations remain invariant under the approach of Lie symmetry. One can study non-linear differential equations using the Lie symmetry approach. Additionally, we may determine the analytic solutions of systems of differential equations using the Lie symmetry approach. These solutions are called invariant solutions. Now, we have understood this with a graphic (1.2).

Example:

The wave function equation $\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0$ (with c being the speed of light) is invariant under the following transformation

$$\begin{aligned}\tilde{t} &= t+\varepsilon, \tilde{x}=x, \tilde{f}=f,\\ \tilde{t} &= t, \tilde{x}=x+\varepsilon, \tilde{f}=f,\\ \tilde{t} &= t, \tilde{x}=x, \tilde{f}=f+\varepsilon f(t,x).\end{aligned}$$

1.2.4 Infinitesimal Generator

An infinitesimal generator, often referred to simply as a generator, is a mathematical operator that generates a continuous transformation. It represents an infinitesimal change or transformation in a system. In group theory, an infinitesimal generator corresponds to the tangent vector to a curve in the group manifold that describes the transformation. These generators are fundamental in understanding symmetries and transformations in various mathematical and physical contexts, such as Lie groups and differential equations (HALL, 2015; BLUMAN; SK, 1989; ARNOLD; KOZLOV; NEISHTADT, 1985; VINOGRADOV, 2012; CHEVALLEY, 2018). Let, (a,b) is an arbitrary point. We can write

$$\begin{aligned}\tilde{a}(a,b,\varepsilon) = a+\varepsilon\xi(a,b)+\ldots = a+\varepsilon Xa+\ldots,\\ \tilde{b}(a,b,\varepsilon) = b+\varepsilon\eta(a,b)+\ldots = b+\varepsilon Xb+\ldots,\end{aligned} \tag{1.33}$$

where, $\xi(a,b) = \frac{\partial\tilde{a}}{\partial\varepsilon}, \eta(a,b) = \frac{\partial\tilde{b}}{\partial\varepsilon}$ at $\varepsilon = 0$ and

$$X = \xi(a,b)\frac{\partial}{\partial a} + \eta(a,b)\frac{\partial}{\partial b}. \tag{1.34}$$

In simpler terms, the operator X mentioned above is typically referred to as the infinitesimal generator of the transformation. The terms $\xi(a,b)$ and $\eta(a,b)$ represent the coefficients or components of this generator. The term 'generator' suggests that if we apply the transformation repeatedly, we'll eventually achieve a finite transformation. In other words, integrating the infinitesimal changes leads to the final transformation.

$$\begin{aligned}\frac{\partial\tilde{a}}{\partial\varepsilon} &= \xi(\tilde{a},\tilde{b})\\ \text{and}\quad \frac{\partial\tilde{b}}{\partial\varepsilon} &= \eta(\tilde{a},\tilde{b})\end{aligned} \tag{1.35}$$

with the initial condition (a,b) at $\varepsilon = 0$. Therefore, we can infer that we obtain the final transformation.

Now, let's consider the scenario where ε is expressed as $\varepsilon = g(\hat{\varepsilon})$, under the condition that $g(0) = 0$ and $g'(0) \neq 0$. From this, we can derive expressions for ξ and η

$$\hat{\xi} = \left.\frac{\partial\tilde{a}}{\partial\hat{\varepsilon}}\right|_{\varepsilon=0} = g'(0)\xi. \tag{1.36}$$

In a similar manner, we can obtain $\hat{\eta}$ as $g'(0)\eta$.

1.2.5 Law of transformations

The generalized form of the infinitesimal generator which can be expressed as follows (for more than two variables) (STEPHANI; MACCALLUM, 1989; ELLIS, 1971; MUKHANOV, 2005)

$$X = u^j(a^m)\frac{\partial}{\partial a^j}, \ j = 1,2,3,...,M. \tag{1.37}$$

Now we make a point transformation

$$a^{j'} = a^{j'}(a^j) \text{ with } \left|\frac{\partial a^{j'}}{\partial a^j}\right|. \tag{1.38}$$

From the above transformation we get

$$\frac{\partial}{\partial a^j} = \frac{\partial a^{j'}}{\partial a^j}\frac{\partial}{\partial a^{j'}}. \tag{1.39}$$

Now the infinitesimal generator (1.37) transforms into

$$X = u^j(a^m)\frac{\partial a^{j'}}{\partial a^j}\frac{\partial}{\partial a^{j'}} = u^{j'}\frac{\partial}{\partial j'}, \tag{1.40}$$

where $u^{j'} = u^j(a^m)\frac{\partial a^{j'}}{\partial a^j}$.

Now one can write as

$$Xa^m = \left(u^j\frac{\partial}{\partial a^j}\right)(a^m) = u^m \tag{1.41}$$

Therefore the infinitesimal generator can be written as

$$X = (Xa^j)\frac{\partial}{\partial a^j} = (Xa^{j'})\frac{\partial}{\partial a^{j'}} \tag{1.42}$$

The previous discussion indicates that identifying the infinitesimal generator in the $a^{j'}$ co-ordinate is straightforward if one knows the infinitesimal generator in that co-ordinate.

Hence we may conclude that the suitable coordinates can be found for an arbitrary number M of co-ordinates a^j, with the generator taking the simple form

$$X = \frac{\partial}{\partial u}, \tag{1.43}$$

which is the normal form of the Infinitesimal generator.

Example:

Let's consider a generator given by

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{1.44}$$

Now, our aim is to represent this generator in terms of u and v, where

$$u = \frac{y}{x}, \; v = xy.$$

From equation (1.42), we find

$$Xu = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u = 0, \;\; Xv = 2xy = 2v. \tag{1.45}$$

Therefore, the generator can be expressed as

$$X = 2v\frac{\partial}{\partial v} \tag{1.46}$$

for the symmetry vector.

1.2.6 Extension of Transformation

In the context of symmetry analysis, the "extension of transformation" refers to the process of extending or generalizing a transformation to include additional variables or parameters. When analyzing the symmetries of a system, sometimes it's beneficial to consider transformations that involve not only the original variables but also additional ones. These additional variables can provide insights into the behavior of the system under more general conditions or in more complex scenarios. For example, if a transformation originally involves only spatial coordinates, extending it to include time or other physical parameters could reveal symmetries that are not apparent in the original analysis. This extension allows for a more comprehensive understanding of the symmetries and dynamics of the system under study.

Suppose, a differential equation is

$$I(x,y,y',y'',...,y^n) = 0 \;\text{ where }\; y' = \frac{dy}{dx}. \tag{1.47}$$

To perform a point transformation on the differential equation, we must determine the transformation of y^n. Now, the point transformations of the prolongation of derivatives can be written as (STEPHANI; MACCALLUM, 1989; BLUMAN; ANCO, 2008)

$$\begin{aligned} \tilde{y}' &= \frac{d\tilde{y}}{d\tilde{x}} = \frac{d\tilde{y}(x,y;\varepsilon)}{d\tilde{x}(x,y;\varepsilon)} \\ &= \frac{y'\frac{\partial\tilde{y}}{\partial y} + y'\frac{\partial\tilde{y}}{\partial x}}{y'\frac{\partial\tilde{x}}{\partial y} + y'\frac{\partial\tilde{x}}{\partial x}} \\ &= \tilde{y}'(x,y,y',\varepsilon). \end{aligned} \tag{1.48}$$

Similarly, for $\tilde{y''} = \tilde{y''}(x, y, y', y'', \varepsilon)$ and so on. Next, we express the extension of the infinitesimal generator X as follows:

$$
\begin{aligned}
\tilde{x}(x,y,\varepsilon) &= x + \varepsilon\xi(x,y) + ... = x + \varepsilon X x + ..., \\
\tilde{y}(x,y,\varepsilon) &= y + \varepsilon\eta(x,y) + ... = y + \varepsilon X y + ..., \\
\tilde{y'}(x,y,\varepsilon) &= y' + \varepsilon\eta'(x,y,y') + ... = y' + \varepsilon X y' + ..., \\
& \quad ... \quad ... \quad ... \quad ... \\
\tilde{y}^n(x,y,\varepsilon) &= y^n + \varepsilon\eta^n(x,y,y',...,y^n) + ... = y^n + \varepsilon X y^n + ...,
\end{aligned}
\tag{1.49}
$$

where η'^j are of the form

$$
\eta'^j = \frac{\partial \tilde{y}^j}{\partial \varepsilon}\bigg|_{\varepsilon=0}. \tag{1.50}
$$

From equations (1.49) and (1.50) one can obtain,

$$
\begin{aligned}
\tilde{y}' &= y' + \varepsilon\eta' + ... = \frac{d\tilde{y}}{d\tilde{x}} \\
&= y' + \varepsilon\left(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right) + ...,
\end{aligned}
\tag{1.51}
$$

similarly

$$
\tilde{y'}^n = y'^n + \varepsilon\left(\frac{d\eta^{(n-1)}}{dx} - y^n\frac{d\xi}{dx}\right) + ... \tag{1.52}
$$

From these, we can obtain

$$
\begin{aligned}
\eta' &= \frac{d\eta}{dx} - y'\frac{d\xi}{dx} \\
&= \frac{d\eta}{dx} + y'\left(\frac{d\eta}{dy} - \frac{d\xi}{dx}\right) - y'^2\frac{d\xi}{dy},
\end{aligned}
\tag{1.53}
$$

$$
\eta'^n = \frac{d\eta^{(n-1)}}{dx} - y^n\frac{d\xi}{dx}, \tag{1.54}
$$

Now, using mathematical induction, one can express it as

$$
\eta'^n = \frac{d^n}{dx^n}(\eta - y'\xi) + y^{(n+1)}\xi. \tag{1.55}
$$

The above discussion demonstrates that if the infinitesimal generator of a point transformation has the form

$$
X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{1.56}
$$

then its extension or prolongation (up to the nth order) can be expressed as

$$
X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta'(x,y,y')\frac{\partial}{\partial y'} + ... + \eta^n(x,y,y',...,y^n)\frac{\partial}{\partial y^n}. \tag{1.57}
$$

1.2.7 Multiple-parameter groups of transformations

Utilizing multiple-parameter groups in symmetry analysis enhances our understanding of system symmetries, facilitating the exploration of complex transformations involving multiple variables. They play a crucial role in characterizing symmetries within intricate physical systems, offering a versatile framework across various domains. In this section, we delve into the concept of the multiple-parameter group of transformation (OLVER, 1993; BLUMAN; SK, 1989; BLUMAN; ANCO, 2008). To execute a transformation dependent on at least two parameters, denoted as ε, we proceed as follows:

$$\tilde{a} = \tilde{a}(a,b,\varepsilon_n), \;\; \tilde{b} = \tilde{b}(a,b,\varepsilon_n), \tag{1.58}$$

where n varies from 0 to r. Here, we consider each ε_n to be independent of each other. If this transformation includes the identity and allows for their repeated application (potentially with different ε_n), then it is termed an r-parameter group G_r of transformations.

So, corresponding to the parameter ε_n, the infinitesimal generator can be written as:

$$X_n = \xi_n(a,b)\frac{\partial}{\partial a} + \eta_n(a,b)\frac{\partial}{\partial b}, \tag{1.59}$$

here

$$\xi_n(a,b) = \frac{\partial \tilde{a}}{\partial \varepsilon_n}\bigg|_{\varepsilon_m}, \;\; \eta_n(a,b) = \frac{\partial \tilde{b}}{\partial \varepsilon_n}\bigg|_{\varepsilon_m} \tag{1.60}$$

when all ε_m is equal to 0.

1.2.8 The definition of symmetry

Symmetry essentially involves a transformation from one solution set to another. Consider $\tilde{a} = \tilde{a}(a,b)$, $\tilde{b} = \tilde{b}(a,b)$, a point transformation that may or may not depend on parameters. If this transformation maps one solution to another, it is termed a symmetry transformation of an ordinary differential equation (ODE), meaning the image of any solution is again a solution of the ODE (STEWART, 2013).

Let's consider an nth-order differential equation

$$F(a,b,b',b'',...,b^n) = 0. \tag{1.61}$$

If it doesn't change under certain symmetry conditions, then it can be expressed as:

$$F(\tilde{a},\tilde{b},\tilde{b}',\tilde{b}'',...,\tilde{b}^n) = 0. \tag{1.62}$$

So, the existence of symmetry doesn't depend on the choice of variables in the given differential equation (DE). From this, one can infer that a simple differential equation may have more than one symmetry (perhaps infinite).

Symmetries that don't conform to a Lie group structure can be highly beneficial for investigating differential equations. However, discovering these symmetries lacks a practical methodology. Let's consider a symmetry transformation that includes at least one parameter (ε) capable of forming a Lie point symmetry (STEPHANI; MACCALLUM, 1989) like

$$\tilde{a} = \tilde{a}(a,b,\varepsilon), \; \tilde{b} = \tilde{b}(a,b,\varepsilon), \; \tilde{b}' = \tilde{b}'(a,b,b'\varepsilon) \; \text{ etc.} \tag{1.63}$$

1.2.9 Canonical Coordinates

Canonical coordinates are essential in simplifying the analysis of physical systems, particularly in classical mechanics and Hamiltonian dynamics. They are chosen to preserve the symplectic structure, ensuring that fundamental properties like Poisson brackets remain unchanged. The use of canonical coordinates allow for the simplification of equations of motion, revealing underlying symmetries and conserved quantities. Overall, canonical coordinates provide a systematic and powerful framework for analyzing physical systems in both classical and quantum contexts (GOLDSTEIN; POOLE; SAFKO, 2002; VOGTMANN; WEINSTEIN; ARNOL'D, 2013; LANDAU; LIFSHITZ, 1982).

Canonical coordinates refer to a set of coordinates in phase space that fully describe the state of a dynamical system. They are chosen to simplify the formulation of the system's equations of motion, typically by making them directly compatible with Hamilton's equations. Let $\overrightarrow{X}$ be a vector field in the augmented space, given by:

$$\overrightarrow{X} = \xi(a,b)\frac{\partial}{\partial a} + \eta(a,b)\frac{\partial}{\partial b}. \tag{1.64}$$

A coordinate (p,q) is said to be a canonical coordinate of $\overrightarrow{X}$ if a point transformation from (a,b) to (p,q) satisfies:

$$\overrightarrow{X}_p = 0 \;\text{ and }\; \overrightarrow{X}_q = 1. \tag{1.65}$$

Example:

Consider the scaling group defined as follows:

$$\tilde{x} = e^t x, \; \tilde{y} = e^{4t} y \tag{1.66}$$

and the corresponding infinitesimal generator $\overrightarrow{X} = x\frac{\partial}{\partial x} + 4y\frac{\partial}{\partial y}$. Then, from equation (1.65), we obtain:

$$\overrightarrow{X}_p = 0 \implies x\frac{\partial p}{\partial x} + 4y\frac{\partial p}{\partial y} = 0.$$

Then the characteristic differential equation becomes:

$$\frac{dx}{x} = \frac{dy}{4y} = \frac{dp}{0}. \tag{1.67}$$

After solving, one obtains:

$$p(x,y) = \frac{x}{y^4}. \tag{1.68}$$

Similarly, from the other condition (1.65), we obtain:

$$q(x,y) = \log x. \tag{1.69}$$

Hence, we can say the canonical coordinates are $(p,q) = (\frac{x}{y^4}, \log x)$.

1.2.10 Noether Symmetry Approach:

The mathematician Emmy Noether's analysis offers an intriguing method for creating new cosmic models and associated structures in alternative gravitational theories. This method provides the first integrals of motion, which are helpful in obtaining exact solutions. It is based on a result due to Emmy Noether, known as Noether's first theorem. Additionally, the symmetry constraints lead to solvable evolution equations or the evolution equations in a much simplified form.

Noether's 1st theorem: "For every differentiable symmetry of the action of a physical system, there corresponds a conservation law. Specifically, it connects continuous symmetries of a Lagrangian with conserved quantities.".

Noether's first theorem establishes a crucial connection between symmetries and conservation laws in physics. It states that for each continuous symmetry observed in a physical system, there corresponds a conserved quantity. Symmetry in this context refers to transformations that leave the system unchanged, like translations in time or space, rotations, or certain gauge transformations. These conserved quantities, stemming from symmetries, remain constant over time and are pivotal for comprehending the dynamics of physical systems. Examples include the conservation of energy, momentum, and angular momentum in classical mechanics. Noether's first theorem holds immense significance across physics, offering a powerful tool for deducing and understanding conservation laws rooted in the inherent symmetries of physical systems.

Noether's 2nd theorem: "There always exists a non-trivial differential relation between an infinite dimensional variational symmetry group depending on an arbitrary function and the associated non-trivial differential relation among its Euler-Lagrange equations".

Noether's second theorem is significant because it extends the powerful connection between symmetries and conservation laws established by her first theorem to systems with gauge symmetries. This extension is crucial in modern theoretical physics, particularly in the study of fundamental forces and elementary particles, where gauge theories play a central role. By identifying conserved currents associated with gauge symmetries, the second theorem provides deeper insights into the dynamics of these systems and enhances our understanding of fundamental interactions in nature.

According to the first theorem (IBRAGIMOV, 1993; BLUMAN; SK, 1989), this conserved quantity is known as the Noether current or conserved current. Mathematically, if

$$\mathcal{L}_{\vec{\mathcal{X}}} L = \vec{\mathcal{X}} L = 0, \tag{1.70}$$

then there exists a conserved current (referred to as the Noether conserved current) (DUTTA; CHAKRABORTY, 2016; DUTTA; LAKSHMANAN; CHAKRABORTY, 2019; DUTTA; PANJA; CHAKRABORTY, 2016a)

$$\vec{Q^i} = \eta^\alpha \frac{\partial L}{\partial\left(\partial_j q^\alpha\right)}, \alpha = 1,2,3,...,N. \tag{1.71}$$

where L is a generic Lagrangian, $\mathcal{L}$ be the Lie derivative along the flux of the vector $\vec{\mathcal{X}}$.

The Euler-Lagrange equations, in the context of a point-like Lagrangian denoted as $L = L[q^\alpha(x^i), \partial_j q^\alpha(x^i)]$ with $q^\alpha(x^i)$ as generalized coordinates, manifest in the following manner:

$$\partial_j \left(\frac{\partial L}{\partial(\partial_j q^\alpha(x^i))} \right) = \frac{\partial L}{\partial q^\alpha}, \alpha = 1,2,3,...,N. \tag{1.72}$$

Now, when we contract the Euler-Lagrangian equations mentioned above with an unspecified quantity $\eta^\alpha(q^\beta)$ i.e.,

$$\eta^\alpha \left[\partial_j \left(\frac{\partial L}{\partial(\partial_j q^\alpha(x^i))} \right) - \frac{\partial L}{\partial q^\alpha} \right] = 0, \tag{1.73}$$

after simplification, one can get,

$$\eta^\alpha \frac{\partial L}{\partial q^\alpha} + (\partial_j \eta^\alpha) \frac{\partial L}{\partial(\partial_j q^\alpha)} = \partial_j \left(\eta^\alpha \frac{\partial L}{\partial(\partial_j q^\alpha)} \right). \tag{1.74}$$

Alternatively, one can express this as

$$\mathcal{L}_{\vec{\mathcal{X}}} L = \vec{\mathcal{X}} L = \eta^\alpha \frac{\partial L}{\partial q^\alpha} + (\partial_j \eta^\alpha) \frac{\partial L}{\partial(\partial_j q^\alpha)} = \partial_j \left(\eta^\alpha \frac{\partial L}{\partial(\partial_j q^\alpha)} \right), \tag{1.75}$$

which ensures that equation (1.74) is conserved, i.e., $\partial(Q^j) = 0$ using (1.70).

According to the Noether theorem, if $\mathcal{L}_{\vec{X}} L = \vec{X} L = 0$, then the vector field

$$\vec{X} = \eta^{\alpha} \frac{\partial}{\partial q^{\alpha}} + (\partial_j \eta^{\alpha}) \frac{\partial}{\partial(\partial_j q^{\alpha})}, \tag{1.76}$$

is referred to as the Noether symmetry vector or infinitesimal generator of the Noether symmetry (DIALEKTOPOULOS; CAPOZZIELLO, 2018; BASILAKOS et al., 2013). An important note is that, due to the Euler-Lagrange equation, the conservation of J^i is assured. Noether symmetry analysis plays a significant role in determining the conserved quantity of any physical system. We can infer from (1.75) that there is a constant of motion associated with this symmetry criterion for the system.

Furthermore, if the Lagrangian of a physical system is explicitly time-independent, then the energy function can be expressed as

$$E = \dot{q}^{\alpha} \frac{\partial L}{\partial \dot{q}^{\alpha}} - L = \text{Conserved}. \tag{1.77}$$

The above equation is known as the Hamiltonian of the system and also serves as a constant of motion (DUTTA; CHAKRABORTY, 2016; DUTTA; LAKSHMANAN; CHAKRABORTY, 2019; DUTTA; PANJA; CHAKRABORTY, 2016a). In addition, if the conserved quantity due to the symmetry has some physical analog, then this symmetry approach can identify a reliable model. By imposing these symmetry constraints on any physical system, the evolution equations of the system become solvable or simpler. The Noether current (defined in equation (1.71)) coincides with the Noether charge, given that all the variables are only time-dependent in the present homogeneous geometry. Moreover, the conserved current, (i.e., conserved charge in this case) can be written in compact form as i.e.,

$$\vec{Q} = i_{\vec{X}} \Theta_L, \tag{1.78}$$

where the symbol $i_{\vec{X}}$ indicates the inner product with the vector field $\vec{X}$, and the one-form Θ_L (known as Cartan one-form) is defined as follows:

$$\Theta_L = \frac{\partial L}{\partial q^{\alpha}} dq^{\alpha}. \tag{1.79}$$

Furthermore, the above geometric inner product is absolutely fit to find the cyclic variables in the augmented space. For a transformation $q^{\alpha} \longrightarrow s^{\alpha}$ in the augmented space, the symmetry vector becomes

$$\vec{X}_T = (i_{\vec{X}} ds^{\alpha}) \frac{\partial}{\partial s^{\alpha}} + \left\{ \frac{d}{dt} (i_{\vec{X}} ds^{\alpha}) \right\} \frac{d}{ds^{\alpha}}, \tag{1.80}$$

where the transformed symmetry vector $\vec{X}_T$ is nothing but a lift of a vector X in the augmented space. Now, without loss of generality, if the above point like transformation is restricted to

$$\begin{aligned} i_{\vec{X}_T} ds^\alpha &= 1 \text{ for } \alpha = m \\ i_{\vec{X}_T} ds^\alpha &= 0 \text{ for } \alpha \notin m, \end{aligned} \tag{1.81}$$

then

$$\vec{X}_T = \frac{\partial}{\partial s^m} \text{ and } \frac{\partial L_T}{\partial s^m} = 0, \tag{1.82}$$

where s^m is a cyclic variable in the augmented space. The above geometric process is useful for identifying a cyclic vector along the direction of the symmetry vector $\vec{X}$.

Moreover, if the Lagrangian of any physical system is explicitly time-independent, then the above energy function indicates both the Hamiltonian of the system as well as the constant of motion. The Hamiltonian formulation is highly applicable within the framework of quantum cosmology. Then the modified form of the Noether theorem can be written as:

$$\mathcal{L}_{\vec{X}_H} H = 0, \tag{1.83}$$

where the symmetry vector $\vec{X}_H = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \ddot{q}^\alpha \frac{\partial}{\partial \dot{q}^\alpha}$.

Example:

Suppose we have a system of ordinary differential equations given by $\ddot{a} = f(t,a,\dot{a},\varphi,\dot{\varphi},\psi,\dot{\psi})$ and $\ddot{\varphi} = g(t,a,\dot{a},\varphi,\dot{\varphi},\psi,\dot{\psi})$, with the point-like Lagrangian in 4D augmented space (t,a,φ,ψ) defined as $L = L(a,\varphi,\psi,\dot{a},\dot{\varphi},\dot{\psi})$.

Here, we may consider the infinitesimal generator as

$$\vec{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \varphi} + \gamma\frac{\partial}{\partial \psi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\varphi}} + \dot{\gamma}\frac{\partial}{\partial \dot{\psi}}, \tag{1.84}$$

where α,β,γ are coefficients of the infinitesimal generator and they are some function of (a,φ,ψ). The terms $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ are defined as

$$\begin{aligned} \dot{\alpha} &= \frac{d\alpha}{dt} = \frac{d}{dt}(\alpha(a,\varphi,\psi)) = \dot{a}\frac{\partial\alpha}{\partial a} + \dot{\varphi}\frac{\partial\alpha}{\partial\varphi} + \dot{\psi}\frac{\partial\alpha}{\partial\psi} \\ \dot{\beta} &= \frac{d\beta}{dt} = \frac{d}{dt}(\beta(a,\varphi,\psi)) = \dot{a}\frac{\partial\beta}{\partial a} + \dot{\varphi}\frac{\partial\beta}{\partial\varphi} + \dot{\psi}\frac{\partial\beta}{\partial\psi} \\ \dot{\gamma} &= \frac{d\gamma}{dt} = \frac{d}{dt}(\gamma(a,\varphi,\psi)) = \dot{a}\frac{\partial\gamma}{\partial a} + \dot{\varphi}\frac{\partial\gamma}{\partial\varphi} + \dot{\psi}\frac{\partial\gamma}{\partial\psi}. \end{aligned} \tag{1.85}$$

Now, based on the above discussion, the existence of the Noether symmetry of the point-like Lagrangian requires that $\mathcal{L}_{\vec{X}} L = \vec{X} L = 0$, resulting in a set of partial differential equations. Subsequently, we can determine the values of α, β, and γ using the method of separation of variables applied to this set of partial differential equations.

If explicit values for α, β, and γ are available, we can then use them to determine the infinitesimal generator of the Noether symmetry.

1.2.11 Symmetry and Laws of Conservation

Symmetry principles in physics, such as Noether's theorem, establish a profound link between symmetries and conservation laws. They demonstrate that for every continuous symmetry in a physical system, there exists a corresponding conserved current. That symmetry leads to the law of conservation:

- Symmetry under space translations $\implies$ Conservation of linear momentum.
- Symmetry under rotations $\implies$ Conservation of angular momentum.
- Symmetry under time translations $\implies$ Conservation of energy.
- Symmetry under boosts (moving coordinates) $\implies$ Linear motion of the center of mass.

1.2.12 Noether symmetries of ODEs

In the preceding sections, we examined how functions remain unchanged when subjected to point transformations. Now, we shift our focus to explore how ordinary differential equations (ODEs) maintain their form under the influence of a one-parameter point transformation. Consider the 'N'-dimensional system of ODEs

$$x^{(n)i} = \omega^i \left(t, x^k, \dot{x}^k, \ddot{x}^k, \ldots, \text{(n-1)th derivative of } x^k\right) \tag{1.86}$$

where $\dot{x}^i = \frac{dx^i}{dt}$, $x^{(n)} = \frac{d^n x}{dt^n}$. The infinitesimal point transformation is

$$\tilde{t} = t + \varepsilon\xi(t, x^k), \tag{1.87}$$

$$\tilde{x}^i = x^i + \varepsilon\eta^i(t, x^k). \tag{1.88}$$

In Analytical Mechanics, the Lagrangian, denoted as $L = L(t, x^k, \dot{x}^k)$, serves as a function describing the dynamics of a system. The equations governing the motion of the dynamical system arise from the action of the Euler-Lagrange operator, represented as E_i, on the Lagrangian L. In other words,

$$E_i(L) = 0 \tag{1.89}$$

with

$$E_i = \frac{d}{dt}\frac{\partial}{\partial \dot{x}^k} - \frac{\partial}{\partial x^k}. \tag{1.90}$$

If the Lagrangian remains unchanged when subjected to the transformation described by equations (1.87)-(1.88), then it becomes evident that the Euler-Lagrange equations (1.89) also remain unaffected by this transformation. Emmy Noether demonstrated that if a one-parameter point transformation preserves the Euler-Lagrange equations (1.89), then there exists a conserved quantity associated with that transformation.

Theorem-I: Let

$$X = \xi(t,x^k)\partial_t + \eta^i(t,x^k)\partial_i \tag{1.91}$$

be the infinitesimal generator of transformation (1.87)-(1.88) and

$$L = L(t,x^k,\dot{x}^k) \tag{1.92}$$

be a Lagrangian describing the dynamical system (1.89). The transformation (1.87)-(1.88) acting on (1.92) preserves the invariance of the Euler-Lagrange equations (1.89) if and only if there exists a function $f = f(t,x^k)$ satisfying the following condition (MIRON, 1995):

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt} \tag{1.93}$$

where $X^{[1]} = \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial x^i} + \eta^i{}_{[1]}\frac{\partial}{\partial \dot{x}^i}$, ($\eta^i_{[1]} = \left[\eta^i_{,t} + \dot{x}^i(\eta^i_{,x} - \eta^i_{,t}) - \dot{x}^{i2}\xi_{,x}\right]$, the 1st prolongation of η) is the first prolongation of (1.91).

If the generator (1.91) satisfies condition (1.93), it qualifies as a Noether symmetry of the dynamical system described by the Lagrangian (1.92). These Noether symmetries collectively constitute a Lie algebra known as the Noether algebra. In cases where the dynamical system (1.89) possesses Lie symmetries which spanning a Lie algebra G_m with a dimension $m \geq 1$, then Noether symmetries of (1.89) form another Lie algebra, denoted as G_h, where $h \geq 0$. This G_h algebra is a subalgebra of G_m, specifically $G_h \subseteq G_m$.

Theorem-II: For every Noether point symmetry (1.91) of the Lagrangian (1.92), there exists a function $\varphi(t,x^k,\dot{x}^k)$ defined as:

$$\varphi = \xi\left(\dot{x}^i\frac{\partial L}{\partial \dot{x}^i} - L\right) - \eta^i\frac{\partial L}{\partial \dot{x}^i} + f \tag{1.94}$$

which is a first integral i.e., $\frac{d\varphi}{dt} = 0$ of the equations of motion, is called a Noether integral (first integral) of (1.89).

To elaborate, a Noether symmetry, by preserving the differential equations (1.89) also qualifies as a Lie symmetry for (1.94). Consequently, as per $\left[X^{[n]},A\right] = \lambda A$ (where $X^{[n]}$ is the nth prolongation of infinitesimal generator X, λ is a function and $A = \frac{\partial}{\partial t} + \dot{x}_i\frac{\partial}{\partial x_i} + \ldots + \omega_i\left(t, x_k, \dot{x}_k, \ddot{x}_k, \ldots, x_i^{(n-1)}\right)\frac{\partial}{\partial x_i^{(n)}}$.), we can assert that (1.94) meets the criterion $X(\varphi) = 0$, signifying that Noether integrals remain unchanged as functions under the Noether symmetry vector X.

From the existence of Noether symmetries we are able to determine the characteristic of a dynamical system. When a dynamical system (1.89) with n degrees of freedom possesses (at least) n linearly independent first integrals φ_J, for $J = 1$ to n, which are in involution i.e.,

$$\{\varphi_J, \varphi_K\} = 0 \tag{1.95}$$

where $\{,\}$ denotes the Poisson bracket, then the solution of the dynamical system can be obtained by quadratures.

We proceed to calculate the Noether symmetries of the harmonic oscillator.

Example: Determine the Noether symmetries for the Lagrangian of a one-dimensional harmonic oscillator given by

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2. \tag{1.96}$$

Solution: To obtain the Noether symmetries for equation (1.96), we utilize theorem-I. Let $X = \xi(t,x)\frac{\partial}{\partial t} + \eta(t,x)\frac{\partial}{\partial x}$ be the infinitesimal generator. The first prolongation $X^{[1]}$ is given by

$$X^{[1]} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \eta_{[1]}\frac{\partial}{\partial \dot{x}}$$

where $\eta_{[1]} = \eta_{,t} + \dot{x}(\eta_{,x} - \xi_{,t}) - \dot{x}^2\xi_{,x}$. For the terms of (1.93) we have

$$\frac{df}{dt} = f_{,t} + \dot{x}f_{,x}.$$

$$X^{[1]}L = \dot{x}\eta_{,t} + \dot{x}^2(\eta_{,x} - \xi_{,t}) - \dot{x}^3\xi_{,x} - x\eta,$$

$$L\frac{d\xi}{dt} = \frac{1}{2}\dot{x}^2\xi_t - \frac{1}{2}x^2\xi_t + \frac{1}{2}\dot{x}^3\xi_x - \frac{1}{2}x^2\dot{x}\xi_{,x}.$$

Substituting into (1.93), we obtain

$$0 = -\left[x\eta + \frac{1}{2}x^2\xi_{,t} + f_{,t}\right] + \dot{x}\left[\eta_{,t} - \frac{1}{2}x^2\xi_{,x} - f_{,x}\right] + \dot{x}^2\left[\eta_{,x} - \frac{1}{2}\xi_{,t}\right] + \dot{x}^3[\xi_{,x}].$$

The determining equations are

$$x\eta + \frac{1}{2}x^2\xi_{,t} + f_{,t} = 0, \quad \eta_{,t} - \frac{1}{2}x^2\xi_{,x} - f_{,x} = 0, \tag{1.97}$$

$$\xi_{,x} = 0, \quad \eta_{,x} - \frac{1}{2}\xi_{,t} = 0. \tag{1.98}$$

The solution to the system of equations (1.97)-(1.98) provides the generic Noether symmetry (WULFMAN; WYBOURNE, 1976)

$$\begin{aligned} X &= (a_1 + a_4\cos(2t) + a_5\sin(2t))\frac{\partial}{\partial t} \\ &+ (a_2\sin(t) + a_3\cos(t) - a_4 x\sin(2t) + a_5 x\cos(2t))\frac{\partial}{\partial x} \end{aligned} \tag{1.99}$$

with the corresponding gauge function

$$f(x,t) = a_2 x\cos(t) - a_3\sin(t) - a_3 x^2\cos(2t) - a_5 x^2\sin(2t). \tag{1.100}$$

It's important to emphasize that both the free particle and the harmonic oscillator possess an equal number of Noether symmetries. Notably, it becomes evident that the Lie algebras (the generic Noether symmetry of the free particle is $X = (a_1 + 2a_4 t + a_5 t^2)\frac{\partial}{\partial t} + (a_2 + a_3 t + a_4 x + a_5 t x)\frac{\partial}{\partial x}$) and (1.99) represent the same algebra albeit in distinct representations.

1.2.13 Noether symmetries of PDE:

When dealing with partial differential equations

$$H = H(x^i, y, y_i, y_{ij}) \tag{1.101}$$

derived from a variational principle, the following theorem applies.

Theorem: The action of the transformation (1.49) on the Lagrangian

$$L_P = L_P\left(x^k, y, y_k\right) \tag{1.102}$$

leaves (1.101) invariant if there exists a vector field $F^i = F^i(x^i, y)$ such that

$$X^{[1]}L_P + L_P\frac{d\xi^i}{dx^i} = \frac{dF^i}{dx^i}. \tag{1.103}$$

The point transformation generator referred to in equation (1.49) is termed a Noether symmetry. The corresponding Noether flow is

$$\Phi_i = \xi^k\left(y_k\frac{\partial L_P}{\partial y_i} - L_P\right) - \eta\frac{\partial L}{\partial y_i} + F^i \tag{1.104}$$

and satisfies the condition $\frac{d\Phi}{dx^i} = 0$.

The approach to implementing Noether flows in PDEs differs from that in ODEs. The conservation flow referenced in (1.104) is employed to reduce the order of (1.101) by introducing a new dependent variable $v(x^i)$. It has been proved that the solution of the system

$$v_{,i}(x^k) = \Phi_i(x^k, y, y_k) \tag{1.105}$$

is also a solution of (1.101). Moreover, it's possible for additional symmetries to emerge from the system (1.105) that are not present in (1.101). These newly discovered symmetries are referred to as potential symmetries (BLUMAN; REID; KUMEI, 1988).

1.2.14 Movements within Riemannian spaces

Earlier, we explored situations where a function remains unchanged under a point transformation. In the subsequent sections, we delve into scenarios where a function exhibits invariance under such transformations. To facilitate this exploration, the geometric objects that will be considered are the metric tensor g_{ij} and the connection coefficients Γ^i_{jk} in a Riemannian space.

Definition: In an n-dimensional space denoted as A^n, which belongs to the class C^p, an object qualifies as a geometric entity of class r (where $r \leq p$) if it exhibits the following characteristics.

I) In every coordinate system $\{x_i\}$, it holds a clearly defined collection of components $\Omega^a(x^k)$.

II) During a coordinate transformation $x^{i'} = J^i(x^k)$, the new components $\Omega^{a'}$ of the object in the new coordinate system $\{x^{i'}\}$ are expressed as well-defined functions belonging to the class $r' = p - r$ of the old components Ω^a in the original coordinates $\{x^i\}$, of the functions J^i, and of their sth derivatives ($1 \leq s \leq p$). In simpler terms, the new components $\Omega^{a'}$ of the object can be represented by equations of the form:

$$\Omega^{a'} = \Phi^a\left(\Omega^k, x^k, x^{k'}\right). \tag{1.106}$$

III) The functions Φ^a have the group properties, that is, they satisfy the following relations

$$\Phi^a\left(\Phi\left(\Omega, x^k, x^{k'}\right), x^k, x^{k'}\right) = \Phi^a\left(\Omega, x^k, x^{k'}\right), \tag{1.107}$$

$$\Phi^a\left(F\left(\Omega, x^k, x^{k'}\right), x^k, x^{k'}\right) = \Omega^a. \tag{1.108}$$

The rule of coordinate transformation $\Phi\left(\Omega, x^k, x^{k'}\right)$ defines the nature of the geometric object. If the function Φ solely involves Ω and the partial derivatives of J^k with respect to x^k, the geometric object is termed a differential geometric object.

Furthermore, we say that a geometric object is linear if for the transformation law $\Phi\left(\Omega, x^k, x^{k'}\right)$ it holds

$$\Phi\left(\Omega, x^k, x^{k'}\right) = J^a_b\left(x^k, x^{k'}\right)\Omega^b + C\left(x^k, x^{k'}\right). \tag{1.109}$$

When the transformation law is

$$\Phi\left(\Omega, x^k, x^{k'}\right) = J^a_b\left(x^k, x^{k'}\right)\Omega^b \tag{1.110}$$

we say that the geometric object is a linear homogeneous geometric object. One important calss of linear homogeneous geometric objects are the tensors.

Collineations: A linear differential geometric object $\Omega(x^i)$ remains unchanged under a one-parameter point transformation:

$$\bar{x}^i = \bar{x}^i(x^k,\varepsilon) \tag{1.111}$$

if and only if $\bar{\Omega}(\bar{x}^i) = \Omega(x^i)$ at every point where the transformation occurs. Alternatively, the geometric object Ω experiences no change under the action of the infinitesimal generator of (1.111), expressed as $L_\xi\Omega = 0$.

A straightforward implication stemming from the definition of the Lie derivative and the transformation law of linear differential geometric objects is that if such an object Ω

remains unchanged under the transformation (1.111), then there exists a coordinate system in which the components of Ω are unaffected by one of the coordinates.

One can extend the notion of symmetry by not necessarily requiring the Lie derivative to be zero but rather equal to another tensor. In this case, the Lie derivative of the linear differential geometric object Ω with respect to the infinitesimal generator ξ is denoted as $L_\xi \Omega = \Psi$, where Ψ possesses the same number of components and symmetries of the indices as Ω. When this occurs, the infinitesimal generator ξ is termed a collineation of Ω, and the specific type of collineations is determined by Ψ. Collineations serve as a potent tool in examining the geometric characteristics of Riemannian manifolds.

In the realm of Riemannian geometry, the geometric object Ω could be either the metric tensor g_{ij} or any other derived geometric object (such as the connection coefficients) that originates from it.

Definition: Collineations involving geometric objects Ω derived from the metric g_{ij} of a Riemannian space are termed geometric collineations. Specifically, the collineation defined by the metric $L_\xi g_{ij}$ is referred to as the generic collineation, as all other geometric collineations can be expressed in terms of it. Moreover, these geometric collineations can be expressed in terms of irreducible parts denoted as ψ and H_{ij} as follows

$$L_\xi g_{ij} = 2\psi g_{ij} + 2H_{ij} \tag{1.112}$$

where the function ψ is called the conformal factor and H_{ij} is a symmetric traceless tensor.

The significance of the quantities Ψ and H_{ij} lies in their utility as variables for investigating geometric collineations. To delve into this, one needs to represent how any metric tensor changes under the influence of these symmetry variables and their derivatives, using the Lie derivative. In the subsequent discussion, our focus will be on exploring geometric collineations, especially those affecting the metric and connection coefficients within a Riemannian space.

1.2.15 Motions in Riemannian spaces

Let us consider a space V^n (say), which has n dimensions and follows Riemannian geometry principles, in this space, we measure distances using a formula called the line element, which is written as

$$ds^2 = g_{ij} dx^i dx^j, \tag{1.113}$$

where g_{ij} represents the components of the metric tensor.

Definition: The transformation described in equation (1.111) is considered a motion of V^n, when applied, it leaves the line element unaltered. Another way to express this condition is that the metric tensor g_{ij} does not change when subjected to the action of the transformation. This condition is mathematically represented by stating that the Lie derivative of g_{ij} with respect to the infinitesimal generator ξ of the transformation is zero, i.e.,

$$L_\xi g_{ij} = 0. \tag{1.114}$$

The transformations described by equation (1.111), when they preserve V^n, collectively constitute a group known as the group motions. Because g_{ij} represents a metric, condition (1.114) can alternatively be expressed as:

$$L_\xi g_{ij} = 2\xi_{(i;j)} = 0. \tag{1.115}$$

This equation is referred to as the Killing equation, and the vector field ξ is termed an isometry or a Killing Vector (KV). The set of Killing Vectors for a given metric constitutes a Lie algebra, known as the Killing algebra.

Motions play a significant role in physics. For instance, in Euclidean space, we have two important motion groups: translations and rotations, denoted as $T(3) \otimes SO(3)$. These motions are associated with the conservation of linear and angular momentum, respectively. Another example is in cosmology, where assuming that the universe appears the same in all directions and locations leads to the concept of Friedmann-Robertson-Walker (FRW) spacetime, which describes a homogeneous and isotropic universe.

Theorem-I: In an n-dimensional Riemannian space V^n, if there's a set of Killing Vectors forming a Killing algebra G_{KV}, then the dimension of G_{KV} falls within the range $0 \leq \dim G_{KV} \leq \frac{1}{2}n(n+1)$.

A Riemannian space with a Killing algebra of dimension $\frac{1}{2}n(n+1)$ is known as a maximally symmetric space. Examples of such spaces include Euclidean space E^3 and Minkowski spacetime M^4.

Among the Killing Vectors, there's a special category called gradient KVs. A KV ξ is considered a gradient iff its covariant derivative satisfies $\xi_{i;j} = 0$, indicating both $\xi_{(i;j)} = 0$ and $\xi_{[i;j]} = 0$.

For every gradient KV ξ, there exists a function S such that $S_{,k}g^{ik} = \xi^i$ and $S_{;ij} = 0$.

Theorem-II: If V^n has p gradient Killing Vectors (where $p \leq n$), it's termed as p-decomposable space. In such cases, there's a coordinate system where the line element (1.113) can be expressed as:

$$ds^2 = M_{\alpha\beta}dz^\alpha dz^\beta + h_{AB}(y^A)dx_A dx_B. \tag{1.116}$$

In this context, $M_{\alpha\beta}$ stands for the components associated with the p directions linked to the gradient Killing Vectors. These components take on the form $M_{\alpha\beta} = \text{diag}(c_1, c_2, ..., c_p)$, where each c_p is a constant. On the other hand, h_{AB} represents the components corresponding to the remaining $n-p$ directions. Here, α and β range from 1 to p, while A and B range from $p+1$ to n.

Example: Compute the Killing Vectors (KVs) of the Euclidean sphere with a given line element

$$ds^2 = d\varphi^2 + \sin^2\varphi d\theta^2. \tag{1.117}$$

To find the Killing Vectors, we need to solve the Killing equation (1.115), which results in a system of equations

$$\begin{aligned} \xi^{\theta}{}_{,\theta} &= 0, \\ 2\xi^{\varphi}{}_{,\varphi} + 2\xi^{\theta}\sin\theta\cos\theta &= 0, \\ \xi^{\theta}{}_{,\varphi} + \xi^{\varphi}{}_{,\theta} - 2\xi^{\varphi}\cot\theta &= 0. \end{aligned}$$

whose solutions are the elements of the $\mathfrak{so}(3)$ Lie algebra.

The 2D Euclidean sphere (1.117) possesses a three-dimensional Killing algebra, indicating it is a maximally symmetric space. Furthermore, all spaces with constant curvature are also considered maximally symmetric spaces.

Conformal motion: When the transformation described by equation (1.113) maintains the angle between two directions at a point, it's termed a conformal motion. In technical terms, we define it as follows:

The infinitesimal displacement ξ associated with the point transformation (1.113) earns the title of Conformal Killing Vector (CKV) when its effect on the metric g_{ij}, as measured by the Lie derivative, results in a scalar multiple of the original metric. This condition signifies that

$$L_{\xi}g_{ij} = 2\psi g_{ij} \tag{1.118}$$

where $\psi = \frac{1}{n}\xi^{k}{}_{;k}$. In the case where $\psi_{;ij} = 0$, ξ is a special Conformal Killing Vector (sp.CKV), and if ψ is constant, ξ is a Homothetic Killing Vector (HV).

The Conformal Killing Vectors (CKVs) associated with a metric compose a Lie algebra known as the conformal algebra, denoted as G_{CV}. It's evident that both Killing Vectors (KVs) and homothetic vectors belong to this conformal algebra, G_{CV}. If G_{HV} represents the algebra containing Homothetic Vectors (HVs), which also encompasses the algebra G_{KV} of KVs, then the theorem below holds true.

Theorem-III: Consider an n-dimensional Riemannian space, where n is greater than or equal to 2, equipped with a conformal algebra denoted as G_{CK}, a homothetic algebra denoted as G_{HV}, and a Killing algebra denoted as G_{KV}. In this scenario, the following statement holds:

1. $G_{KV} \subseteq G_{HV} \subseteq G_{CV}$.
2. For arbitrary n, $G_{H-K} = G_{HV} - (G_{HV} \cap G_{KV})$, then $0 \leq \dim G_{H-K} \leq 1$; that is, V^n admits at most one HV.
3. V^2 admits an infinite-dimensional conformal algebra G_{CV}.
4. For $n > 2$, $0 \leq \dim G_{CV} \leq \frac{1}{2}(n+1)(n+2)$.

1.2.16 Symmetries of the connection

Suppose ξ serves as the generator of an infinitesimal transformation pertaining to equation (1.113). Within a Riemannian space characterized by the metric g_{ij}, an identity emerges:

$$L_\xi \Gamma^i_{jk} = g^{ir} \left[(L_\xi g_{rk})_{;j} + (L_\xi g_{rj})_{;k} - (L_\xi g_{jk})_{;r} \right]. \tag{1.119}$$

If ξ is a hypersurface-orthogonal (HV) or a Killing vector (KV), then according to equation (1.119), it follows that the Lie derivative of the Christoffel symbols Γ^i_{jk} with respect to ξ vanishes. This implies that the Christoffel symbols Γ^i_{jk} remain invariant under the action of the transformation described in equation (1.113).

Definition: The infinitesimal generator, denoted by ξ, associated with the point transformation (1.113), transforms a geodesic into another geodesic and maintains the affine parameter's preservation if and only if the Lie derivative of the connection coefficients, denoted by Γ^i_{jk}, with respect to ξ equals zero. This condition is met when

$$L_\xi \Gamma^i_{jk} = 0. \tag{1.120}$$

The infinitesimal generator, ξ, is called an Affine Killing vector or an Affine collineation (AC).

Affine collineations (ACs) of V^n create a Lie algebra, known as the Affine algebra, G_{AC}. It's clear that the homothetic algebra, G_{HV}, is a subset of G_{AC}, denoted as $G_{HV} \subseteq G_{AC}$. We determine that a spacetime allows proper ACs when the dimension of G_{HV} is less than that of G_{AC}.

For example, in flat space, when condition (1.120) simplifies to $\xi_{i,jk} = 0$, the solution takes the form $\xi_i = A_{ij} x^j + B_i$, where A_{ij} and B_i are constants determined by $n(n+1)$ parameters. Consequently, flat space possesses an Affine algebra of dimension $n(n+1)$, encompassing both Killing Vectors (KVs) and Hypersurface-Orthogonal Vectors (HV). This finding yields the opposite outcome.

Theorem: If an n-dimensional Riemannian space V^n possesses an Affine algebra G_{AC} with a dimension of $n(n+1)$, then V^n constitutes a flat space.

Another form of affine symmetry that holds interest is the Projective collineation.

Definition: The infinitesimal generator, denoted by ξ, of the point transformation (1.113) earns the title of a Projective Collineation (PC) if a function, denoted by φ, exists such that

$$L_\xi \Gamma^i_{jk} = \varphi_{,j} \delta^i_k + \varphi_{,k} \delta^i_j \tag{1.121}$$

or equivalently

$$\xi_{(i;j);k} = 2 g_{ij} \varphi_{,k} + 2 g_{k(i} \varphi_{,j)}. \tag{1.122}$$

The function φ is termed the projective function. When this projective function satisfies the condition $\varphi_{;ij} = 0$, we label ξ as a special PC (sp.PC). Projective transformations alter the arrangement of geodesics (autoparallel curves) within V^n, maintaining their form but not conserving the affine parameter.

The Projective Collineations (PCs) of V^n assemble to form a Lie algebra named the Projective algebra, G_{PC}. It's notable that the Affine algebra, G_{AC}, operates as a subset of G_{PC}, symbolized by $G_{AC} \subset G_{PC}$. When examining flat space, condition (1.121) outlines the fundamental structure of projective collineations.

In flat space condition (1.121) gives the generic projective collineation

$$\xi_i = A_{ij}x^j + \left(B_j x^j\right) x_i + C_i \tag{1.123}$$

where A_{ij},B_j,C_i are arbitrary constant.

Theorem: When a Riemannian space V^n of dimension n possesses a projective algebra G_{PC}, the maximum dimension of G_{PC} is limited to $n(n+2)$. If G_{PC} achieves this maximal dimension, then V^n qualifies as a maximally symmetric space (BARNES, 1993; HALL; COSTA, 1988; HALL; ROY, 1997).

Proposition: Let V^n be an n dimensional Riemannian space, then

i) If V^n accommodates a Lie algebra of special Projective Collineations (sp. PCs) with dimension p (where $p \leq n$), then it also accommodates p gradient Killing Vectors (KVs) and a gradient HV. When p equals n, the space is flat. This statement holds true in reverse as well.

ii) A maximally symmetric space that allows for a proper Affine Collineation (AC) or a special Projective Collineation (sp.PC) is inherently flat.

A Riemannian space can potentially accommodate broader types of collineations, such as Curvature collineations. The complete categorization of collineations for a Riemannian space, regardless of whether it has a definite or indefinite metric, is detailed in reference (KATZIN; LEVINE; DAVIS, 1969). A condensed overview of the definitions mentioned above is provided in Table-(1.1). In the subsequent chapters, this assertion will be validated and applied across different domains of Physical metric and symmetry analysis.

1.3 Quantum Cosmology

In the context of cosmology, the Hamiltonian formulation provides a powerful framework for describing the dynamics of the Universe. Named after the physicist William Rowan Hamilton, this approach characterizes the evolution of a system using Hamilton's equations, which express the time evolution of variables in terms of a Hamiltonian function. In cosmology, the Hamiltonian formulation is particularly useful for studying the dynamics of gravitational fields and matter content on large scales. By casting Einstein's equations of general relativity into Hamiltonian form, researchers can analyze the behavior of the Universe's expansion, structure formation, and other key phenomena. This approach offers

Collineation $L_\xi A = B$	**A**	**B**
Killing Vector (KV)	g_{ij}	0
Homothetic vector (HV)	g_{ij}	$2\psi g_{ij},\ \psi_{,i} = 0$
Conformal Killing vector (CKV)	g_{ij}	$2\psi g_{ij},\ \psi_{,i} \neq 0$
Affine Collineation (AC)	Γ^i_{jk}	0
Proj. Collineations (PC)	Γ^i_{jk}	$2\varphi_{(,j}\delta^i_{k)},\ \varphi_{,i} = 0$
Sp. Proj. collineation (sp.PC)	Γ^i_{jk}	$2\varphi_{(,j}\delta^i_{k)},\ \varphi_{;jk} = 0$

Table 1.1 – Collineations in a Riemannian space

insights into the fundamental properties of space-time, the nature of cosmic inflation, and the formation of galaxies and large-scale structures. Moreover, the Hamiltonian formulation facilitates the study of quantum cosmology, where it provides a framework for investigating the quantum behavior of the Universe from its earliest moments. Overall, the Hamiltonian formulation plays a crucial role in advancing our understanding of the cosmos and its evolution.

Quantum cosmology is an intriguing branch of theoretical physics that delves into the application of quantum mechanics to the vast expanse of the universe. It embarks on a quest to unravel the mysteries surrounding the origin, evolution, and ultimate destiny of our cosmos, employing the principles of quantum mechanics as its guiding light.

At its core, quantum cosmology grapples with the fundamental questions that have intrigued humanity for centuries: What triggered the birth of the universe? How has it evolved over billions of years? What forces govern its trajectory? These queries prompt physicists to delve into the realm of the very small and the very early universe, exploring the fabric of space-time and the behavior of matter and energy on cosmological scales (STEINWACHS, 2014; HALLIWELL, 2002; HAWKING, 1996; ASHTEKAR; SINGH, 2011; ASHTEKAR et al., 2007; ASHTEKAR; PAWLOWSKI; SINGH, 2006c).

One of the key approaches in quantum cosmology is the concept of the "minisuperspace approximation", which simplifies the complexity of the universe by focusing on a finite number of degrees of freedom. These may include variables such as the scale factor, describing the expansion of the universe, and scalar fields representing various forms of matter and energy. By reducing the problem to a more manageable form akin to a system of interacting particles, researchers strive to uncover the underlying quantum nature of the cosmos (CALCAGNI et al., 2012; ASHTEKAR; PAWLOWSKI; SINGH, 2006a; ASHTEKAR; PAWLOWSKI; SINGH, 2006b).

However, quantum cosmology is not without its challenges. The transition from classical to quantum descriptions of the universe presents profound theoretical and conceptual hurdles. Issues such as the treatment of singularities, the definition of time, and the interpretation of quantum states pose formidable obstacles that demand innovative solutions.

Despite these challenges, quantum cosmology holds immense promise. It offers a tantalizing glimpse into the workings of the early universe, potentially unlocking secrets that have eluded us thus far. As our understanding of quantum gravity advances and new theoretical frameworks emerge, quantum cosmology continues to captivate the imagination of scientists and enthusiasts alike, pushing the boundaries of human knowledge to ever greater heights.

Quantum cosmology is a branch of theoretical physics that seeks to apply the principles of quantum mechanics to the study of the Universe on cosmological scales. It aims to address fundamental questions about the origin, evolution, and ultimate fate of the cosmos using quantum mechanical frameworks. By investigating the behavior of the Universe at the smallest scales and earliest times, quantum cosmology endeavors to understand phenomena such as the initial singularity, the nature of space-time, and the emergence of cosmic structures. Through mathematical models and theoretical approaches, quantum cosmologists seek to provide insights into the underlying quantum nature of the Universe and its implications for our understanding of cosmic evolution.

The link between quantum cosmology and Noether symmetry analysis is rooted in their shared goal of understanding the fundamental laws governing the universe. Noether's theorem provides a powerful framework for uncovering symmetries and corresponding conservation laws, which are essential for describing the dynamics of quantum cosmological models. By applying Noether symmetry analysis to quantum cosmology, researchers aim to reveal hidden symmetries underlying the evolution of the universe and gain deeper insights into its fundamental nature.

A quantum theory of gravity aims to resolve singularities and avoid divergences present in classical cosmology. This motivates the application of quantum gravity to the homogeneous, isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime with small perturbations, as it accurately describes the observed Universe on large scales. Similar simplifications are anticipated in quantum theory, rendering a comprehensive understanding of quantum gravity in the nonlinear regime unnecessary for drawing observationally relevant conclusions.

The study of quantum cosmology begins with the "minisuperspace approximation" and the development of a quantum theory of FLRW Universes. This involves transitioning from field theory to a finite-dimensional dynamical system, similar to a system of interacting particles. The Universe's degrees of freedom are the scale factor a, the matter scalar field φ, and the spatial curvature parameter κ. A cosmological constant Λ may be present, absorbed into the scalar field potential energy $V(\varphi)$. The Hamiltonian viewpoint states that the total Hamiltonian must vanish, posing technical and interpretational issues. These include ordering ambiguities, no unique inner product defining Hilbert space, no time evolution in the system, too many solutions for dynamical equations, and no observer to the Universe. These issues would be addressed by a more complete theory of quantum gravity, making quantum cosmology a simplified case for addressing these problems.

1.3.1 Formulation of the Wheeler-DeWitt equation

The Wheeler-DeWitt equation serves as a cornerstone in the field of quantum cosmology, aiming to provide a quantum description of the entire Universe. It represents a key equation in the canonical quantization of general relativity, incorporating the principles of quantum mechanics into the framework of cosmology. The WD equation encapsulates the dynamics of the Universe's wave function, allowing researchers to explore its behavior across different cosmological epochs. Its solutions offer insights into the fundamental nature of space, time, and matter, shedding light on the origin and evolution of the cosmos. Ultimately, the WD equation is instrumental in addressing fundamental questions about the Universe's quantum nature and the possibility of a singular origin.

In the realm of quantum cosmology, the Hamiltonian formulation holds greater significance compared to the Lagrangian formulation. It's noteworthy that the canonically conjugate momenta associated with the conserved current, a result of Noether symmetry, remains constant, or the canonically conjugate momenta corresponding to the conserved current due to Noether symmetry is constant i.e.,

$$\Pi_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} = \Sigma_\beta, \tag{1.124}$$

where α is varies from 0 to r, and Σ_β is a constant. In the quantization scheme, the operator version of the conserved momentum mentioned earlier takes the following form:

$$-i\partial_{q^l}|\psi >= \Sigma_l \, |\psi\rangle \,. \tag{1.125}$$

So, for the real Σ_β, the aforementioned differential equation possesses an oscillatory solution, which is expressed as:

$$|\psi>=\sum_{l=1}^{r} e^{ip_{0\rho}q^{\rho}|\psi(\varphi^k)\rangle}, k<n. \tag{1.126}$$

where "k" represents the directions along which there are no symmetry and "n" represents the dimension of the minisuperspace. In accordance with Hartle's perspective (CARTER; HARTLE, 2012; CARTER, 1986), the presence of oscillations within the universe's wave function indicates the existence of Noether symmetry. This symmetry entails the conservation of conjugate momenta along the symmetry vector, and conversely, the presence of such conservation implies the existence of Noether symmetry. To put it differently, the symmetry vector assists in recognizing the oscillatory nature of the physical system, aiding in identifying the periodic aspects of solutions to the Wheeler-DeWitt equation.

1. **Facilitating Investigation into Quantum Mechanical Approaches:** The method discussed appears to provide a means to investigate whether quantum mechanical theories, derived from solutions to the Wheeler-DeWitt (WD) equation, can avoid or bypass the singularity problem associated with the big bang. The WD equation is a key equation in quantum cosmology, attempting to describe the quantum state of the entire universe.

2. **Linking Universe's Wave Function to Classical Trajectories:** It suggests a way to connect the wave function of the universe, derived from quantum mechanical principles, with classical trajectories. This connection allows for the derivation of solutions that describe the classical behavior of the cosmos. Essentially, it's trying to bridge the gap between quantum and classical descriptions of the universe.

3. **Utilizing Noether Symmetry for Hypothetical Lagrangians:** The passage mentions the use of Noether symmetry, a concept from theoretical physics related to conservation laws, to consider a wide range of hypothetical Lagrangians. Lagrangians describe the dynamics of physical systems, and by identifying specific invariants through Noether's theorem, one can understand the symmetries and conservation laws governing these systems. In the context of cosmology, this approach might help explore different theoretical frameworks and their implications for the evolution of the universe.

Overall, these sentences outline a methodological approach that seeks to integrate quantum mechanics and classical cosmology, leveraging concepts such as the WD equation, classical trajectories, and Noether symmetry to explore solutions to fundamental questions about the origin and evolution of the universe.

2 Quantum Cosmology in Coupled Brans-Dicke Gravity: A Noether Symmetry Analysis

2.1 Prelude

The Brans-Dicke (BD) scalar field theory is a modified theory of gravity, first introduced by Brans and Dicke (BRANS; DICKE, 1961b; FARAONI, 2004; HOHMANN, 2018a) using Mach's principle(BRANS; DICKE, 1961b) in the gravity theory. In this picture the gravitational field is characterized not only geometrically but also by a non-minimally coupled scalar, known as Brans-Dicke(BD) scalar field. The coupling parameter (known as BD parameter) is considered as an effective Newton's constant and it measures the strength of the coupling between the scalar field and gravity. An extension of this scalar field model is the chiral model (CHERVON, 2013; PERELOMOV, 1987; CHERVON, 1995) where two scalar fields are minimally coupled to gravity but the two scalar fields interact in the kinetic term. So one can define a second-rank tensor from the Kinetic components of the scalar fields and for the chiral model it is a 2D hyperbolic sphere. Hence there does not exist any co-ordinate choice in which the two scalar fields are non-interacting. Such coupled two scalar field models are widely used in the literature (CHRISTODOULIDIS; ROEST; SFAKIANAKIS, 2019a; CHRISTODOULIDIS; ROEST; SFAKIANAKIS, 2019b; CHEN; SODA, 2021) and by suitable extension it is possible to have the negative energy density of the scalar field (PALIATHANASIS; LEON, 2021a; PALIATHANASIS; LEON, 2022). On the other hand, due to complexity of the modified Einstein field equations it become almost impossible to have an analytic solution. So normally, one has to use qualitative tools in the theory of differential equations to analyzed these physical theories. In particular, the group invariant transformations method is suitable in the present problem. More precisely, geometric symmetries related to space-time i.e., Noether symmetry is very much relevant to analyze the BD cosmological model. The advantage of using Noether symmetry is of two fold : the associated conserved quantities (charge) can be used as a selection criterion to determine similar physical processes and the symmetry analysis either simplify the evolution equations or determine the integrability of the system. Further, Noether symmetry analysis examines the self consistency of the phenomenological physical models and the physical parameters (involved in the physical system) are contained by this symmetry analysis (TSAMPARLIS; PALIATHANASIS, 2011; LEACH, 2009; BLUMAN; KUMEI, 1989; STEPHANI, 1989; OLVER, 1986; Aminova, 1995; AMINOVA; AMINOV, 2000; FEROZE T., 2006; TSAMPARLIS; PALIATHANASIS, 2010). The symmetry vector identifies a transformation in the augmented space so that one of the variables becomes cyclic and consequently the Lagrangian simplifies to a great extend (DUTTA; LAKSHMANAN; CHAKRABORTY, 2019; DUTTA; LAKSHMANAN; CHAKRABORTY, 2021) and the evolution equation becomes solvable. In quantum cosmology the Wheeler-DeWitt(WD) equation has been formed and Noether charges identify the oscillatory part of the solution. Then the WD equation becomes easily solvable.

In cosmology, scalar field models may describe the early inflationary era (as inflaton field) or may be responsible for the late time accelerating phase (as a DE candidate). The motivation of the present two scalar fields model is to examine whether the model is in accord with the cosmological observations. Due to complicated nature of the model it is not possible to have analytic classical solution by solving the usual field equations. Noether symmetry analysis provides the tool for solving the field equations. For quantum cosmology, the motivation is to examine whether the initial singularity may be avoided by quantum description. The chapter deals with both classical and quantum cosmological study of two scalar fields coupled BD theory of gravity by Noether symmetry analysis. The plan of this chapter is as follows: Section-**2.2** deals with an overview of the Noether symmetry analysis. Section-**2.3** presents the mathematical description of the coupled two scalar field BD theory and Noether symmetry has been imposed to have the symmetric vector. Also the coupled Lagrangian and the evolution equations are simplified by identifying the cyclic variable in the augmented space. Quantum cosmology of the present model has been formulated in section-**2.4**. The conserved Noether charge identifies the oscillatory part of the wave function and as a result the WD equation has been solved in this section. The chapter ends with a brief discussion and concluding remarks in section-**2.5**.

2.2 Noether Symmetry Analysis : An overview

According to Noether's 1st (DUTTA; LAKSHMANAN; CHAKRABORTY, 2019; DUTTA; LAKSHMANAN; CHAKRABORTY, 2021; TSAMPARLIS; PALIATHANASIS, 2012) theorem every physical system is associated to some conserved quantities provided the Lagrangian of the system is invariant with respect to the Lie derivative along the appropriate vector field i.e., Mathematically

$$\mathcal{L}_{\overrightarrow{X}} L = \overrightarrow{X}(L) = 0 \tag{2.1}$$

Thus for a point like canonical Lagrangian $L = L[q^{\alpha}(x^i), \partial_j q^{\alpha}(x^i)]$, $q^{\alpha}(x^i)$ being the generalized co-ordinate, the Euler-Lagrange equations are

$$\partial_j \left(\frac{\partial L}{\partial(\partial_j q^{\alpha})} \right) - \frac{\partial L}{\partial q^{\alpha}} = 0 \tag{2.2}$$

Now contracting with some unknown function $\lambda^\alpha(q^\beta)$ i.e.,

$$\lambda^\alpha\left\{\partial_j\left(\frac{\partial L}{\partial(\partial_j q^\alpha)}\right)-\frac{\partial L}{\partial q^\alpha}\right\}=0$$

gives after simplification

$$\lambda^\alpha\frac{\partial L}{\partial q^\alpha}+(\partial_j\lambda^\alpha)\frac{\partial L}{\partial(\partial_j q^\alpha)}=\partial_j\left(\lambda^\alpha\frac{\partial L}{\partial(\partial_j q^\alpha)}\right)$$

Thus

$$\mathcal{L}_{\overrightarrow{X}}L=\lambda^\alpha\frac{\partial L}{\partial q^\alpha}+(\partial_j\lambda^\alpha)\frac{\partial L}{\partial(\partial_j q^\alpha)}=\partial_j\left(\lambda^\alpha\frac{\partial L}{\partial(\partial_j q^\alpha)}\right) \tag{2.3}$$

will provided (DUTTA; CHAKRABORTY, 2016; DUTTA; PANJA; CHAKRABORTY, 2016a)

$$Q^j=\lambda^\alpha\frac{\partial L}{\partial(\partial_j q^\alpha)} \tag{2.4}$$

to be conserved (i.e., $\partial_j Q^j=0$) and

$$\overrightarrow{X}=\lambda^\alpha\frac{\partial}{\partial q^\alpha}+(\partial_j\lambda^\alpha)\frac{\partial}{\partial(\partial_j q^\alpha)} \tag{2.5}$$

is the symmetry vector. Now the energy function associated with the system can be written as

$$E=\dot{q}^\alpha\frac{\partial L}{\partial\dot{q}^\alpha}-L \tag{2.6}$$

Thus associated with Noether symmetry there is a conserved current Q^j (given in equation (2.4)). The scalar quantity obtained by integrating the time component of Q^j over the spatial volume is termed as (conserved) Noether charge (Q). As in the present homogeneous geometry all the variables are only time dependent so the Noether current defined in equation (2.4) coincides with the Noether charge. Further, the Noether charge can be geometrically interpreted as the inner product of the infinitesimal generator with cartan one form (CAPOZZIELLO; STABILE; TROISI, 2007) as

$$Q=i_{\overrightarrow{X}}\theta_L \tag{2.7}$$

where $i_{\overrightarrow{X}}$ indicates the inner product with vector field $\overrightarrow{X}$ and

$$\theta_L=\frac{\partial L}{\partial q^\alpha}dq^\alpha \tag{2.8}$$

is the cartan one form.

Further, this geometric inner product representation is suitable to identify cyclic variables in the augmented space . If $q^\alpha\longrightarrow s^\alpha$ is a transformation in the augmented space then the symmetry vector transforms to

$$\overrightarrow{X}_T=(i_{\overrightarrow{X}}ds^\alpha)\frac{\partial}{\partial s^\alpha}+\left\{\frac{d}{dt}(i_{\overrightarrow{X}}ds^s)\right\}\frac{d}{ds^\alpha} \tag{2.9}$$

This transformed symmetry vector $\overrightarrow{X}_T$ is nothing but a lift of the vector field X on the augmented space. Now, without loss of generality if the above point like transformation is restricted to

$$i_{\overrightarrow{X}_T} ds^\alpha = 1, \text{ for } \alpha = m \text{ (say)}$$
$$i_{\overrightarrow{X}_T} ds^\alpha = 0, \text{ for } \alpha \neq m \tag{2.10}$$

then

$$\overrightarrow{X}_T = \frac{\partial}{\partial s^m} \text{ and } \frac{\partial L_T}{\partial s^m} = 0 \tag{2.11}$$

Thus augmented variable s^m is a cyclic variable . The above geometric process can be interpreted as to choose the transformed infinitesimal generator along the co-ordinate line (identified as the cyclic variable).

Moreover, if the Lagrangian of the system has no explicit time dependence then the above energy function (given in equation (2.6)) is nothing but the Hamiltonian of the system and it is also a constant of motion (DUTTA; LAKSHMANAN; CHAKRABORTY, 2021) . The Hamiltonian formulation is very suitable in the context of quantum cosmology. Then the Noether symmetry condition modifies to

$$\mathcal{L}_{\overrightarrow{X}_H} H = 0 \tag{2.12}$$

With $\overrightarrow{X}_H = \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + \ddot{q}^\alpha \frac{\partial}{\partial \dot{q}^\alpha}$, as the symmetry vector. The conjugate momenta which are conserved due to Noether symmetry can be written as

$$\pi_\beta = \frac{\partial L}{\partial q^\beta} = \Sigma_\beta, \quad \beta = 1,2,...,r \tag{2.13}$$

where r is the number of symmetries. In quantization scheme the operator version of the above conserved momentum takes the form

$$-i\partial_{q^\beta}|\Psi >= \Sigma_\beta|\Psi > \tag{2.14}$$

So for real Σ_β, the differential equation (2.14) has the solution

$$|\Psi >= \sum_{\beta=1}^{r} e^{i\Sigma_\beta q^\beta}|\varphi(q^k) > , \ k < n \tag{2.15}$$

where 'k' is the direction along which there is no symmetry and n is the dimension of the minisuperspace. Thus according to Hartle (CARTER, 1986) the oscillatory part of the wave function implies the existence of the Noether symmetry and the conjugate momentum along the symmetry direction is conserved. Therefore, using Noether symmetry it is possible to consider entire class of hypothetical Lagrangians with given invariant for description of a physical system.

2.3 Multifield Cosmological Model and Noether Symmetry

In multifield cosmological model, we consider two scalar fields which are minimally coupled to gravity but the two scalar fields interact in the kinetic term. So the action integral for this model is given by

$$\mathcal{A} = \int d^4x\sqrt{-g}\Big[\frac{\varphi R}{2} + \frac{\omega}{2\varphi}g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} + \frac{1}{2}F^2(\varphi)g^{\mu\nu}\psi_{;\mu}\psi_{;\nu} + V(\varphi)\Big] \tag{2.16}$$

where $\varphi(x^\alpha)$ is the usual BD scalar field, ω is the BD coupling parameter, $V(\varphi)$ is the potential function and $\psi(x^\alpha)$ is the other scalar field minimally coupled to gravity and $F(\varphi)$ is the coupling function indicating interaction between the two scalar fields. In the background of homogeneous and isotropic flat FLRW space-time model the Lagrangian of the above cosmological model is given by (PALIATHANASIS, 2022a)

$$L(a,\dot{a},\varphi,\dot{\varphi},\psi,\dot{\psi}) = 3a\dot{a}^2\varphi + 3a^2\dot{a}\dot{\varphi} - \frac{\omega}{2\varphi}a^3\dot{\varphi}^2 - \frac{a^3}{2}F^2(\varphi)\dot{\psi}^2 + a^3V(\varphi) \tag{2.17}$$

So the field equations for the present cosmological model (obtained by Euler- Lagrange equations) are given by

$$3H^2\Big(1 + \frac{\dot{\varphi}}{H\varphi}\Big) = \frac{\omega}{2}\Big(\frac{\dot{\varphi}}{\varphi}\Big)^2 + \frac{F^2(\varphi)}{2\varphi}\dot{\psi}^2 - \frac{V(\varphi)}{\varphi} \tag{2.18}$$

$$2\dot{H} = -2H\frac{\dot{\varphi}}{\varphi} - 3H^2 - \frac{\omega}{2}\Big(\frac{\dot{\varphi}}{\varphi}\Big)^2 - \frac{F^2(\varphi)}{2\varphi}\dot{\psi}^2 - \frac{\ddot{\varphi}}{\varphi} + \frac{V(\varphi)}{\varphi} \tag{2.19}$$

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{2}\Big(\frac{\dot{\varphi}}{\varphi}\Big)^2 + \frac{1}{\omega}\Big[6H^2\varphi + 3\dot{H}\varphi + \varphi V_{,\varphi} - \frac{1}{2}(F^2)_{,\varphi}\varphi\dot{\psi}^2\Big] = 0 \tag{2.20}$$

and

$$\ddot{\psi} + 3H\dot{\psi} + (\ln(F^2))_{,\varphi}\dot{\varphi}\dot{\psi} = 0 \tag{2.21}$$

Note that equation (2.18) is known as the Hamiltonian constraint for the present model and ψ is a cyclic co-ordinate for the above Lagrangian.

The infinitesimal generator corresponding to Noether symmetry is given by

$$\overrightarrow{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial\varphi} + \dot{\alpha}\frac{\partial}{\partial\dot{a}} + \dot{\beta}\frac{\partial}{\partial\dot{\varphi}} + \dot{\gamma}\frac{\partial}{\partial\dot{\psi}} \tag{2.22}$$

with $\alpha = \alpha(a,\varphi,\psi)$, $\beta = \beta(a,\varphi,\psi)$, $\gamma = \gamma(\psi)$, $\dot{\alpha} = \frac{\partial\alpha}{\partial a}\dot{a} + \frac{\partial\alpha}{\partial\varphi}\dot{\varphi} + \frac{\partial\alpha}{\partial\psi}\dot{\psi}$, and similarly for $\dot{\beta}$ and $\dot{\gamma}$. As due to Noether symmetry $\mathcal{L}_{\overrightarrow{X}}L = 0 \implies \overrightarrow{X}L = 0$, so α, β, and γ will satisfy the following set of equations

$$\begin{aligned}
\varphi\alpha + a\beta + 2a\varphi\frac{\partial\alpha}{\partial a} + a^2\frac{\partial\beta}{\partial a} &= 0 &(2.23)\\
3a^2\varphi\omega\alpha - a^3\omega\beta - 6a^2\varphi^2\frac{\partial\alpha}{\partial\varphi} + 2a^3\omega\varphi\frac{\partial\beta}{\partial\varphi} &= 0 &(2.24)\\
6a\varphi\alpha + 3a^2\varphi\frac{\partial\alpha}{\partial a} + 6a\varphi^2\frac{\partial\alpha}{\partial\varphi} + 3a^2\varphi\frac{\partial\beta}{\partial\varphi} - a^3\omega\frac{\partial\beta}{\partial a} &= 0 &(2.25)\\
3F(\varphi)\alpha + 2a\beta F'(\varphi) + 2aF(\varphi)\frac{\partial\gamma}{\partial\psi} &= 0 &(2.26)\\
2\varphi\frac{\partial\alpha}{\partial\psi} + a\frac{\partial\beta}{\partial\psi} &= 0 &(2.27)\\
3\varphi\frac{\partial\alpha}{\partial\psi} - a\omega\frac{\partial\beta}{\partial\psi} &= 0 &(2.28)\\
3\alpha V(\varphi) + a\beta V'(\varphi) &= 0 &(2.29)
\end{aligned}$$

The above set of seven equations (2.23–2.29) are first order partial differential equations, to determine the infinitesimal generator (2.22) of the above Noether symmetry and also the unknown coupling function $F(\varphi)$ and potential function $V(\varphi)$, we use the method of separation of variables and the solution takes the form

$$\alpha = \alpha_0 a,\ \beta = \beta_0\varphi,\ \gamma = \gamma_0,\ V = V_0\varphi,\ F = F_0\varphi^{\frac{1}{2}} \tag{2.30}$$

where α_0, β_0 are related with the relation $3\alpha_0 + \beta_0 = 0$, γ_0, V_0, F_0 are integration constants.

Now, one makes a transformation $(a,\varphi,\psi) \longrightarrow (P,M,R)$ in the augmented space so that one of the variables becomes cyclic. Using the technique of inner product with cartan one form (defined in the previous section: equation (2.10)) one gets

$$\begin{aligned}
\ln\left(\frac{a}{\varphi}\right) &= (\alpha_0 - \beta_0)P\\
\ln(a^3\varphi) &= M\\
\psi &= \gamma_0 R
\end{aligned} \tag{2.31}$$

As a result, the Lagrangian of the system simplifies to

$$L = e^M\left\{A\dot{P}^2 + B\dot{M}^2 + C\dot{P}\dot{M} + V_0 - D\dot{R}^2\right\} \tag{2.32}$$

where the constants A,B,C,D are connected with ω by the following relations

$$\begin{aligned}
A &= (\alpha_0 - \beta_0)^2\left(-\frac{6}{16} - \frac{9\omega}{32}\right)\\
B &= \left(\frac{6}{16} - \frac{\omega}{32}\right)\\
C &= (\alpha_0 - \beta_0)\frac{6\omega}{32}\\
D &= \frac{F_0^2\gamma_0^2}{2}
\end{aligned}$$

Now solving the evolution equations corresponding to this transformation (2.32) one can get:

$$P = \frac{k_1}{bA\Delta\sqrt{\Gamma}}F\left[\frac{\pi-2b(t+c_2)}{4}\middle|\frac{2V_0\Delta}{V_0\Delta\pm 1}\right] \mp C_1\ln\{1-\Delta\sin\{b(t+c_2)\}\}+c_4 \quad (2.33)$$

$$M = \frac{1}{2}\ln\left\{\Gamma(1\mp\Delta\sin b(t+c_2)\right\} \quad (2.34)$$

$$R = \frac{-2k_2}{b\sqrt{\Gamma(V_0\Delta\mp 1)}}F\left[\frac{\pi-2b(t+c_2)}{4}\middle|\frac{2V_0\Delta}{V_0\Delta\pm 1}\right]+c_3 \quad (2.35)$$

where $C_1 = \frac{c}{4Ab\Delta\Gamma}\sqrt{4c_1-2V_0\Gamma^2b^2}$, k_1, Δ, Γ, b, c_1, c_2, c_3, c_4, k_2 are constants and $F[x|y]$ is the elliptic integral of the first kind with parameter $y = n^2$ (where n being any real number) or in terms of the old variables the solution becomes

$$a = \exp\left[\frac{1}{8}\ln\left(\Gamma\mp\Gamma\Delta\sin[b(t+c_2)]\right)+(\alpha_0-\beta_0)\left\{\frac{k_1}{b\Delta A\sqrt{\Gamma}}F\left[\frac{\pi-2b(t+c_2)}{4}\middle|\frac{2V_0\Delta}{V_0\Delta\mp 1}\right]\right.\right.$$
$$\left.\left.\mp C_1\ln(1-\Delta\sin(b(t+c_2)))+c_4\right\}\right] \quad (2.36)$$

$$\varphi = \exp\left[\frac{1}{8}\ln\left(\Gamma\mp\Gamma\Delta\sin[b(t+c_2)]\right)-3(\alpha_0-\beta_0)\left\{\frac{k_1}{b\Delta A\sqrt{\Gamma}}F\left[\frac{\pi-2b(t+c_2)}{4}\middle|\frac{2V_0\Delta}{V_0\Delta\mp 1}\right]\right.\right.$$
$$\left.\left.\mp C_1\ln(1-\Delta\sin(b(t+c_2)))+c_4\right\}\right] \quad (2.37)$$

$$\psi = -\frac{2k\gamma_0}{b\Gamma\sqrt{V_0\Delta\mp 1}}F\left[\frac{\pi-2b(t+c_2)}{4}\middle|\frac{2\Delta V_0}{\Delta V_0\mp 1}\right]+c_3 \quad (2.38)$$

The solutions for the scale factor a and the cosmological parameters namely $H = \frac{\dot{a}}{a}$ and the acceleration parameter $\frac{\ddot{a}}{a}$ have been plotted in FIG. (2.1) for various choices for the parameters involved. Form the figures we see that the present cosmological model is an expanding model of the Universe and the Hubble parameter is positive but decreases with time. The graph of the acceleration parameter show that the Universe was in an accelerated era of expansion at the early phase then there was an epoch of decelerated expansion and finally the Universe accelerated again. So the cosmic evolution matches with the qualitative evolution as predicted by observations.

On the other hand, the Lagrangian for the present model defines the kinetic metric as

$$dS^2_{(k)} = -6a\varphi da^2 - 6a^2 da d\varphi + \frac{\omega}{\varphi}a^3 d\varphi^2 + a^3\{F(\varphi)\}^2 d\psi^2 \quad (2.39)$$

with effective potential $a^3V(\varphi)$. Using the change of variables as

$$e^u = a \ , \ e^v = \varphi \ \ i.e., \ \frac{\dot{a}}{a} = \dot{u} \ , \ \frac{\dot{\varphi}}{\varphi} = \dot{v},$$

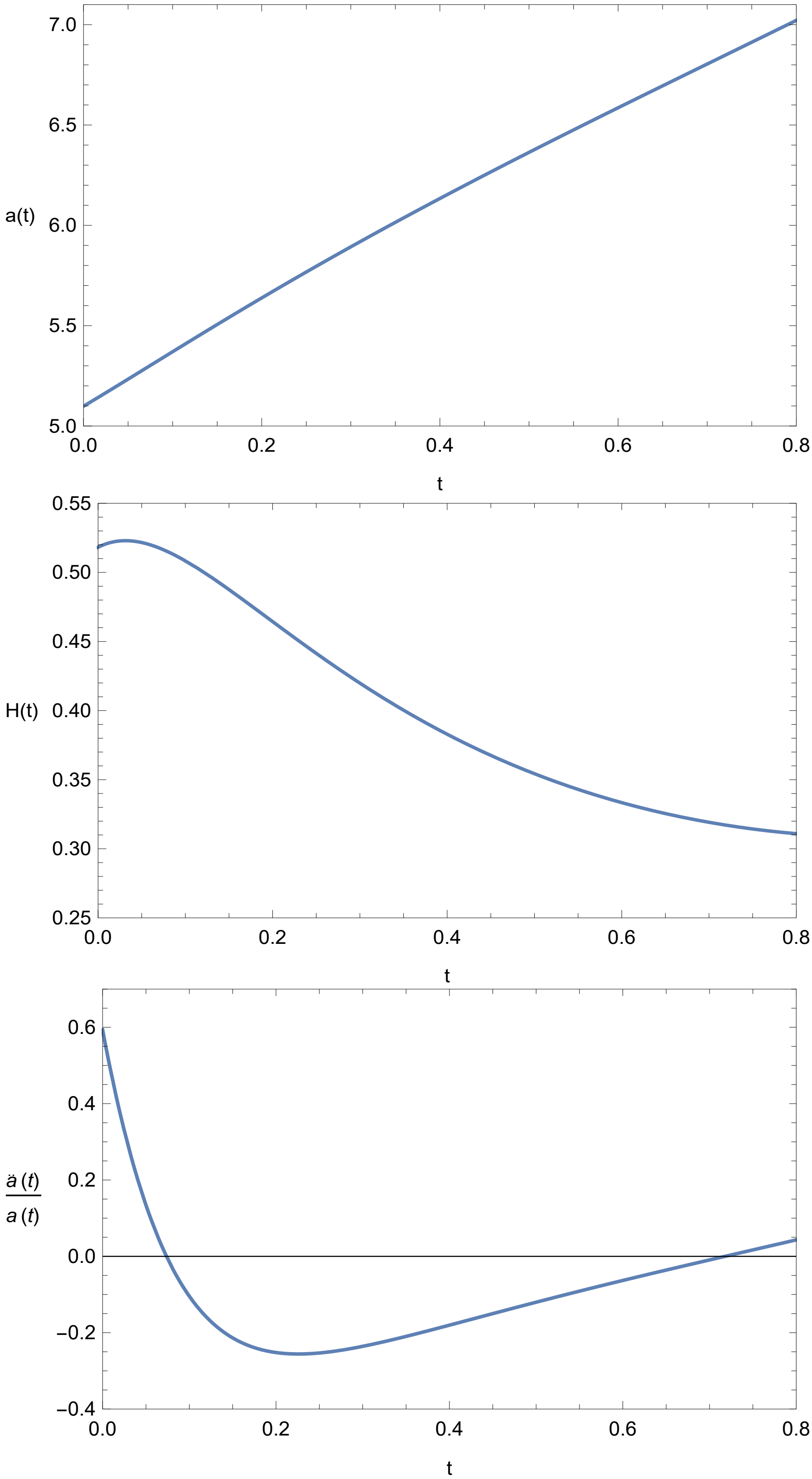

Figure 2.1 – The graphical representation of scale factor $a(t)$ (top left), Hubble parameter $H(t)$ (top right) and acceleration parameter $\frac{\ddot{a}(t)}{a(t)}$ (bottom) with respect to cosmic time t.

the above kinetic metric becomes

$$\begin{aligned} dS^2_{(k)} &= a^3\varphi\Big[-6du^2 - 6dudv + \omega dv^2 + F_0^2 d\psi^2\Big] \\ &= e^{(3u+v)}\Big[-6d\big(u+\frac{v}{2}\big)^2 + (\omega - \frac{3}{2})dv^2 + F_0^2 d\psi^2\Big] \end{aligned}$$

Substituting $\xi = 12(u+\frac{v}{2}),\ 2\eta = (\omega - \frac{3}{2})v,\ 2\delta = F_0\psi$ one gets

$$dS^2_{(k)} = e^{(3u+v)}\big[-d\xi^2 + d\eta^2 + d\delta^2\big] \tag{2.40}$$

The kinetic metric is conformal to flat 3D metric. This 3D metric admits a gradient homothetic vector field $H_V = \frac{2}{3}a\partial_a$, $\psi_{H_V} = 1$.

Note that this homothetic vector field does not generate a Noether point symmetry for the Lagrangian. However, additional Noether point symmetries are possible for the present flat case due to three killing vectors of the 2D metric and these killing vector fields span the $E(2)$ group. Further, the transformed Lagrangian takes the form

$$L = -\frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\eta}^2 + \frac{1}{2}\dot{\delta}^2 + V_0, \tag{2.41}$$

which is nothing but the Lagrangian for the quintom dark energy model, Further written $\Phi = \eta + i\delta$, the action can be equivalent to a single complex scalar field.

2.4 Quantum Cosmology and Noether Symmetry Analysis

In this section Noether symmetry analysis has been used to formulate quantum cosmology of the present cosmological model. In particular, the conserved quantities (namely Noether charge) corresponding to this symmetry determine the oscillatory part of the wave function of the Universe and as a result the WD equation simplifies to a great extend and may be solvable.

Usually in cosmology, one uses the simplest and widely used minisuperspace (BHAUMIK; DUTTA; CHAKRABORTY, 2022b) models corresponding to homogeneous and isotropic space-time metrics. So the Lapse function $N(= N(t))$ is homogeneous and the shift function vanishes identically. As a result, the 4D manifold has the metric in (3+1)-decomposition as

$$ds^2 = -N(t)dt^2 + q_{ab}(x,t)dx^a dx^b \ , \ a,b = 1,2,3. \tag{2.42}$$

In this (3+1) decomposition the Einstein-Hilbert action can be written as

$$I(q_{ab},N) = \int dt d^3x N\sqrt{q}[K_{ab}K^{ab} - K^2 + {}^{(3)}R] \tag{2.43}$$

with K_{ab}, the extrinsic curvature of the three space and $3_{(R)}$, the curvature scalar of the three space.

Now as the three space is homogeneous, so the three metric q_{ab} can be described by a finite number of time functions, namely $q^{\alpha}(t)$, $\alpha = 0,1,2,...,n-1$. As a consequence, the above action can be written as that of a relativistic point particle having self interacting potential in nD curved space-time as (BANERJEE; DAS; GANGULY, 2010; SHEIKHAHMADI et al., 2019)

$$I(q^{\alpha},N) = \int_0^1 N dt \Big[\frac{1}{2N^2}\mu_{\alpha\beta}(q)\dot{q}^{\alpha}\dot{q}^{\beta} - U(q)\Big], \tag{2.44}$$

with equation of motion of the relativistic particle as

$$\frac{1}{N}\frac{d}{dt}\Big(\frac{\dot{q}^{\alpha}}{N}\Big) + \frac{1}{N^2}\Gamma^{\alpha}_{\beta\gamma}\dot{q}^{\beta}\dot{q}^{\gamma} + \mu^{\alpha\beta}\frac{\partial U}{\partial q^{\beta}} = 0. \tag{2.45}$$

Also the variation of the action with respect to the Lapse function gives the constraint equation (Hamiltonian constraints)

$$\frac{1}{2N^2}\mu_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta} + V(q) = 0. \tag{2.46}$$

Using the momenta conjugate to q^{α} *i.e.*, $p_{\alpha} = \frac{\partial L}{\partial \dot{q}_{\alpha}} = \mu_{\alpha\beta}\frac{\dot{q}^{\beta}}{N}$ the Hamiltonian of the system turns out to be

$$H = p_{\alpha}\dot{q}^{\alpha} - L = N\Big[\frac{1}{2}\mu^{\alpha\beta}p_{\alpha}p_{\beta} + V(q)\Big] \equiv NH \tag{2.47}$$

Now combining equations (2.46) and (2.47) one gets

$$H(q^{\alpha},p_{\alpha}) \equiv \frac{1}{2}\mu^{\alpha\beta}p_{\alpha}p_{\beta} + V(q) = 0 \tag{2.48}$$

In quantization scheme, if one writes the operator version of the the above Hamiltonian constraint then it becomes the Wheeler-DeWitt(WD) equation in quantum cosmology as the form

$$H\Big(q^{\alpha} - ih\frac{\partial}{\partial q^{\alpha}}\Big)\Psi(q^{\alpha}) = 0 \tag{2.49}$$

This is a second order hyperbolic type partial differential equation. The problem related to formulation of these equations is the operator ordering problem as the minisuperspace metric in general depends on q^{α}. A possible resolution of this problem is to impose the covariant nature of the minisuperspace quantization.

Further, in the quantization scheme of minisuperspace there exists a conserved current for probability measure namely

$$\overrightarrow{J} = \frac{i}{2}(\Psi^{\star}\bigtriangledown\Psi - \Psi\bigtriangledown\Psi^{\star}) \tag{2.50}$$

with $\vec{\nabla}.\vec{J} = 0$. Here Ψ satisfies the above WD equation and probability measure can be express as

$$dp = |\Psi(q^{\alpha})|^2 dV \tag{2.51}$$

with dV representing a volume element on minisuperspace.

In the present cosmological model the minisuperspace is a $3D$ space $\{a,\varphi,\psi\}$ (or $\{P,M,R\}$) and the associated conjugate momenta to the variables are

$$\begin{aligned} p_{_P} &= \frac{\partial L}{\partial \dot{P}} = e^M \left(2A\dot{P} + C\dot{M}\right) = \text{Conserved } (\Sigma_P,\ \text{say}) \\ p_{_M} &= \frac{\partial L}{\partial \dot{Q}} = e^M \left(C\dot{P} + 2B\dot{M}\right) \\ p_{_R} &= \frac{\partial L}{\partial \dot{R}} = -2D\dot{R}e^M = \text{Conserved } (\Sigma_R,\ \text{say}). \end{aligned} \tag{2.52}$$

So, the Hamiltonian of the system (also known as Hamiltonian's constraint) takes the form,

$$H = e^{-M}\left[A_1 p_{_P}^2 + A_2 p_{_M}^2 + A_3 p_{_P} p_{_M} - A_4 p_{_R}^2 + V_0 e^{2M}\right] \tag{2.53}$$

where A_1, A_2, A_3, A_4 are constant and they are connected with A, B, C. Then the WD equation (which is the operator version of the above Hamiltonian (51)) has the explicit form

$$\left[-A_1 e^{-M}\frac{\partial^2}{\partial P^2} - A_2 e^{-M}\frac{\partial^2}{\partial M^2} - A_3 e^{-M}\frac{\partial^2}{\partial P \partial M} + A_4 e^{-M}\frac{\partial^2}{\partial R^2} + V_0 e^M\right]\Psi(P,M,R) = 0 \tag{2.54}$$

where Ψ is termed as wave function of the universe. In the context of WKB approximation the above wave function can be written as $\Psi(q^{\alpha})e^{i\delta_{\alpha}(q^{\alpha})}$ and consequently the WD equation becomes first order non-linear partial differential equations which is the usual (null) Hamilton-Jacobi(H-J) equation in the minisuperspace.

Further, the general solution of WD equation can be obtained as the superposition of eigen function of the above WD operator as (CARTER; HARTLE, 1987)

$$\Psi(P,M,R) = \int W(Q)\Psi(P,M,R,Q)dQ \tag{2.55}$$

here Ψ is an eigen function of the WD operator and $W(Q)$ is weight function of the conserved charge Q. Usually, it is preferable that the wave function in quantum cosmology should be consistent with the classical theory i.e., there should be a coherent wave packet with the good asymptotic behavior in the minisuperspace and have a maximum around classical trajectory.

In order to solve the above WD equation, we write the wave function in the form

$$\Psi(P,M,R) = \psi_1(P)\psi_2(M)\psi_3(R) \tag{2.56}$$

as P,R are the cyclic variables so operator version of the associated conserved charge take the form.

$$-i\frac{\partial\psi_1(P)}{\partial P} = \Sigma_P\psi_1(P) \tag{2.57}$$

$$-i\frac{\partial\psi_3(R)}{\partial R} = \Sigma_R\psi_1(R) \tag{2.58}$$

and the solution of the above equations (2.57), (2.58) are given below

$$\psi_1(P) = \exp(i\Sigma_P P) \tag{2.59}$$

$$\psi_3(R) = \exp(i\Sigma_R R) \tag{2.60}$$

then by substituting the value of (2.59) and (2.60) in the WD equation (2.54), explicit form of the wave function becomes

$$\implies \Psi = e^{i(\Sigma_P P+\Sigma_R R)}\left\{M_1 e^{-\frac{B^*M}{A^*}} J_1\left[\frac{\sqrt{(B^*)^2-4A^*C^*}}{2A^*}, \sqrt{\frac{V_0}{A^*}}e^M\right]\right. \\ \left. +M_2 e^{-\frac{B^*M}{2A^*}} J_2\left[\frac{\sqrt{(B^*)^2-4A^*C^*}}{A^*}, \sqrt{\frac{V_0}{A^*}}e^M\right]\right\} \tag{2.61}$$

where A^*, B^*, C^*, D^*, E^* are given below as

$$\begin{aligned} A^* &= \left(4AB^2C^2+16A^2B^3-8AB^2C^2\right) \\ B^* &= \left(AC^4+B(4AB-C+2AC)^2-C^3(4AB-C+2AC)\right) \\ C^* &= -\frac{C^2(4AB-C)^2}{4D} \\ D^* &= \left(-4BC^3A-(8AB^2+c^3)(4AB-C+2AC)-8AB^2C^2\right) \\ E^* &= V_0C^2(4AB-C)^2 \end{aligned} \tag{2.62}$$

and J_1 , J_2 are Bessel functions of 1st and 2nd kind.

Usually a consistent quantum cosmological model should predict the classical solution at late time, while at early time the quantum model should be distinct from the classical solution (i.e., singularity) to have a well defined era of evolution. For the present problem we have plotted $|\Psi|^2$ against a, φ in FIG.-(2.2). From the figure it is clear that there is finite non-zero probability at zero volume . Hence the initial classical singularity (i.e., big-bang singularity) can not be remove by quantum description.

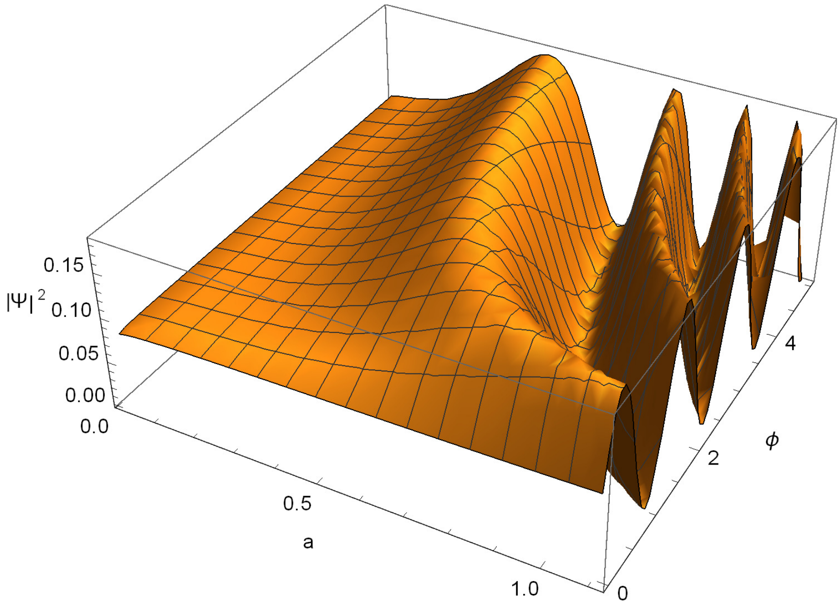

Figure 2.2 – 3D graphical presentation of $|\Psi|^2$ versus the scale factor a and φ .

2.5 Brief Discussion and Conclusion

This chapter is an example where the Noether symmetry analysis has been extensively used in a very complicated cosmological model namely coupled multiscalar field model. In the context of classical cosmology, the symmetry analysis not only determines the coupling function (instead of phenomenological choice) but also gives the classical solution by choosing the cyclic variable in the augmented space. The graphical representation of the cosmological parameters show that the evolution of the present model agrees qualitatively with the observational predictions. The geometric symmetry of the physical space shows that there is a homothetic vector field and there is no Noether symmetry along the homothetic vector field. Further, the physical space is conformally flat and the model is equivalent to a quintom model. In quantum cosmology, the WD equation is highly coupled non-linear second order (hyperbolic) partial differential equation for this complex cosmological model. The operator version of the conserved charge equations identify the oscillatory part of the wave function and as a result the WD equation becomes solvable. The graphical representation of the wave function has been shown in FIG.-(2.2). for various choices of the parameters involve and from the figure we see that the wave function has a finite value at $a = 0$ i.e., there is a finite probability for the system to have zero volume. So the wave function does not vanish at the big bang singularity. Therefore, the quantum version of the present cosmological model can not remove the big-bang singularity rather the initial classical singularities continue to be present at the quantum level. Finally, for this strange behavior an alternative interpretation of quantum mechanics: the de Bohm–Broglie interpretation of quantum mechanics may be useful (CHAKRABORTY, 2001b; COLISTETE JR.; FABRIS; PINTO-NETO, 1998).

3 Classical and Quantum Cosmology in Einstein-æther Scalar-tensor gravity: Noether Symmetry Analysis

3.1 Prelude

A generally covariant modification of Einstein gravity is the Einstein–æther theory (BLANCHET; BUONANNO; FAYE, 2010; CARRUTHERS; JACOBSON, 2011; GARFINKLE; JACOBSON, 2011; DONNELLY; JACOBSON, 2010). In this theory, in addition to the space–time metric there is a scalar field (named æther) indicating a Universal notion of time. Thus the theory has a preferred frame of reference and as a result, there is a Lorentz violation of the theory (i.e., the theory breaks the invariance under boosts)(KOSTELECKY, 2004; CARROLL; TAM; WEHUS, 2009; MEWES, 2019; CARROLL; LIM, 2004; HEINICKE; BAEKLER; HEHL, 2005; ALMEIDA et al., 2017; NETTO, 2018; HAGHANI et al., 2015; JACOBSON; MATTINGLY, 2004). Further, due to this symmetry breaking, a Higgs mechanism for the gravitation gives a variation at large distance physics and it may be a possible explanation for the recent Supernova data which is well explained by cosmological constant. One may note that Einstein–æther theory goes over to Einstein gravity if it preserves locality as well as covariance formulation (ELING; JACOBSON; MILLER, 2007; BONVIN et al., 2008; KUCUKAKCA; AKBARIEH, 2020; DING; WANG; WANG, 2015; CHAN; SILVA; SATHEESHKUMAR, 2020; LIN; WU, 2017; AKHOURY; GARFINKLE; GUPTA, 2018; COLEY; LEON, 2019). Also, in the classical limit the Horava-Lifshitz (HORAVA, 2009) gravity corresponds to this modified gravity theory but not in reverse way (SOTIRIOU; VISSER; WEINFURTNER, 2009a; SOTIRIOU; VISSER; WEINFURTNER, 2009b). Moreover, this Lorentz violating gravity theory is important from the point of view of quantum gravity theory. In recent past, gravitational models with Lorentz violation are of special interest in the literature.

Scalar filed in cosmology plays an important role for the description of the evolution of the Universe. Such models start with Brans-Dicke theory of gravity. This gravity theory obeys Mach's principle (BRANS; DICKE, 1961a) and is described by the space–time metric with non–minimally coupled scalar field. This scalar field plays an important role to identify the early era of evolution (SEN; SEN, 2001). Further, the higher–order derivatives of the scalar field in the gravitational action integral indicates quantum corrections or modifications of Einstein gravity. In Einstein–æther theory if the scalar field is non–minimally coupled then it may be termed as Einstein–æther scalar tensor theory (BHADRA et al., 2007;

BERTOLAMI; MARTINS, 2000; TSAMPARLIS et al., 2013). There is another well known scalar field model known as dilation filed which is used for description of string cosmology (GASPERINI; VENEZIANO, 1993).

In this context it is worthy to mention that there is monograph on Noether symmetries in theories of Gravity (BAJARDI; CAPOZZIELLO, 2022). In this monograph various alternatives and extensions to Einstein gravity has been examined form symmetry principles to identify physically viable models. Subsequently, the authors constrains the parameters of the viable models by Noether symmetry analysis and compare them with recent observational data.

This chapter is an attempt to study the cosmic evolution in Einstein–æther scalar–tensor theory with background FLRW space–time model both from classical and quantum cosmological view point. Due to homogeneity and isotropy of the space–time, the Æther field is a comoving time–like vector field and the field equations are second order coupled nonlinear differential equations. The well known Noether symmetry has been imposed both for determining the unknown functions parameters in the system and to identify a transformation in the augmented space so that one of the variable becomes cyclic (for application in scalar tensor gravity one may see the works in references (CAPOZZIELLO et al., 1996; CAPOZZIELLO; RITIS, 1993; CAPOZZIELLO; RITIS; SCUDELLARO, 1993; CAPOZZIELLO; RITIS; SCUDELLARO, 1994)). As a results the field equations get simplified and become solvable to have classical solutions. Noether symmetry analysis has an important role in describing quantum cosmology (for application of Noether symmetry in quantum cosmology one may see the works in references (MONIZ, 2022; BAJARDI; CAPOZZIELLO, 2021; CAPOZZIELLO; LAURENTIS; ODINTSOV, 2012; CAPOZZIELLO; LAMBIASE, 2000; CAPOZZIELLO; LAURENTIS, 2014)) Noether symmetry identifies the oscillatory part of the wave function of the Universe by writing the conserved (Noether) charge equation in operator form. As a result the Wheeler-Dewitt (WD) equation becomes simplified and is solvable using separation of variables. The probability measure identifies whether initial singularity can be avoided or not by quantum description. The plan of this chapter is the following:

An overview of this model has been presented in Section **3.2**. In Section **3.3** we have presented a basic concept of Noether symmetry analysis and in Section **3.4** we are able to find the analytical solution of this model using Noether symmetry analysis. Physical metric and its symmetry analysis are presented in Section **3.5**. Section **3.6** deals with the analysis of quantum cosmology of this model and finally, we draw our conclusions in Section **3.7**.

3.2 Einstein–æther scalar tensor gravity: an overview

The Action integral for the present model can be written as

$$S = S_1 + S_2, \tag{3.1}$$

where S_1 stand for the action for the scalar-tensor theory and S_2 represents action for the Einstein–æther theory. In Jordan frame scalar–tensor action takes the form:

$$S_1 = \int d^4x\sqrt{-g}\left(A(\varphi)R + \frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + V(\varphi)\right), \tag{3.2}$$

where $A(\varphi)$ stand for the nonminimal coupling of the scalar field φ with gravity and $V(\varphi)$ is the potential function of the scalar field. If σ^μ denotes the æther field then corresponding action integral takes the form

$$\begin{aligned} S_2 = -\int d^4x\sqrt{-g}\Big[\alpha_1(\varphi) \quad \sigma^{\nu;\mu} \quad \sigma_{\nu;\mu} + \alpha_2(\varphi)(g^{\mu\nu}\sigma_{\mu;\nu})^2 + \alpha_3(\varphi)\sigma^{\nu;\mu}\sigma_{\mu;\nu} \\ + \quad \alpha_4(\varphi)\sigma^\mu\sigma^\nu\sigma_{;\mu}\sigma_\nu - \delta(\sigma^\mu\sigma_\nu + 1)\Big], \end{aligned} \tag{3.3}$$

where α_i's (i = 1,2,3,4) are the coefficient function identifying the coupling between the æther field and the scalar field, te Lagrange multiplier 'δ' indicates the unitarity of the æther field i.e., $\sigma^\mu\sigma_\mu = -1$. Now due to cosmological principle the space–time geometry is described by homogeneous and isotropic FLRW line element. As a result, the explicit form of the action integral takes the form

$$S = \int L dt$$

with the point like Lagrangian (PALIATHANASIS; LEON, 2021b)

$$L = 6\mu(\varphi)a\dot{a}^2 + 6\xi(\varphi)a^2\dot{a}\dot{\varphi} + \frac{1}{2}a^3\dot{\varphi}^2 - a^3V(\varphi). \tag{3.4}$$

Here $\mu(\varphi) = A(\varphi) + \frac{1}{2}\big(\alpha_1(\varphi) + 3\alpha_2(\varphi) + \alpha_3(\varphi)\big)$ and $\xi(\varphi) = \frac{dA}{d\varphi}$ and 'a' is the scale factor of the Universe. It is to be noted that this minisuperspace description is very much suitable for gravitational models. In fact, the field equations corresponding to the above point Lagrangian identify the evolution of a point–like particle while minisuperspace is very much essential for canonical quantization and as a result, it is possible to formulate the WD equation in quantum cosmology. Now varying the above Lagrangian (3.5) with respect to the variable a and φ the field equations are

$$2\dot{H} + 3H^2 + 2\frac{\mu_{,\varphi}}{\mu}H\dot{\varphi} - \frac{\left(\frac{\dot{\varphi}^2}{2} - V(\varphi)\right)}{2} + \frac{\xi}{\mu}\left(\frac{\xi_{,\varphi}}{\xi}\dot{\varphi}^2 + \ddot{\varphi}\right) = 0, \tag{3.5}$$

$$\ddot{\varphi} - \dot{\varphi} + 3H\dot{\varphi} + 6\xi(\varphi)(\dot{H} + 3H^2) - 6\mu(\varphi)_{,\varphi}H^2 + V_{,\varphi} = 0, \tag{3.6}$$

and also we have the scalar constraint equation

$$6\mu(\varphi)\,H^2 + 6\xi(\varphi)\,H\dot{\varphi} + \frac{\dot{\varphi}^2}{2} + V(\varphi) = 0, \tag{3.7}$$

where $H = \frac{\dot{a}}{a}$ is the usual Hubble parameter. Now the field equations (3.5) and (3.6) can be rewritten in the form of the usual Friedmann equations as

$$3H^2 = G_{\text{mod}}\,\rho_{\text{mod}}, \tag{3.8}$$

and

$$2\dot{H} + 3H^2 = G_{\text{mod}}\,p_{\text{mod}}, \tag{3.9}$$

where $G_{\text{mod}} = \frac{1}{-\mu(\varphi)}$ is the modified time varying gravitational constant and the modified energy density and the thermodynamic pressure have the expressions

$$\rho_{\text{mod}} = \frac{1}{2}\left(6\xi(\varphi)\,H\dot{\varphi} + \frac{\dot{\varphi}^2}{2} + V(\varphi)\right), \tag{3.10}$$

$$p_{\text{mod}} = -\left[-2\frac{d\mu}{d\varphi}\,H\dot{\varphi} - \frac{1}{2}\left(\frac{\dot{\varphi}^2}{2} - V(\varphi)\right) + \frac{d\xi}{d\varphi}\dot{\varphi}^2 + \xi(\varphi)\,\ddot{\varphi}\right]. \tag{3.11}$$

Note that gravitational coupling parameter depends on the coupling function $A(\varphi)$ and the æther coefficients $\alpha_i (i = 1,2,3,4)$. For constant α_i's the above model reduces to usual scalar–tensor theory.

3.3 An overview of Noether symmetry analysis and classical solutions

According to Noether, if the Lagrangian of physical system is invariant with respect to the Lie derivative along a vector field in the augmented space (i.e., $\mathcal{L}_{\vec{\chi}}L = \vec{\chi}L = 0$) then the physical system must be associated to some conserved quantities (Noether 1st theorem) (DUTTA; LAKSHMANAN; CHAKRABORTY, 2019; DUTTA; LAKSHMANAN; CHAKRABORTY, 2021; TSAMPARLIS; PALIATHANASIS, 2012; DUTTA; CHAKRABORTY, 2016; BHAUMIK; DUTTA; CHAKRABORTY, 2022c), known as Noether charge. In addition, the symmetry constraints lead the evolution equations to be solvable or the evolution equations in a much simplified form.

In general, for a point-like canonical Lagrangian $L = L[q^\alpha(x^i), \partial_j q^\alpha(x^i)]$, $q^\alpha(x^i)$ being the generalized coordinates, the Euler-Lagrange equations are

$$\partial_j\left(\frac{\partial L}{\partial(\partial_j q^\alpha)}\right) - \frac{\partial L}{\partial q^\alpha} = 0\,,\ \alpha = 1,2,\ldots N \tag{3.12}$$

with N, the dimension of the augmented space. Now contracting the above Euler-Lagrange equations with some unknown function $\mu^{\alpha}(q^{\beta})$ and simplifying algebraically one gets

$$\mathcal{L}_{\overrightarrow{\chi}} L = \overrightarrow{\chi} L = \lambda^{\alpha} \frac{\partial L}{\partial q^{\alpha}} + (\partial_j \lambda^{\alpha}) \left(\frac{\partial L}{\partial(\partial_j q^{\alpha})} \right) = \partial_j \left(\lambda^{\alpha} \frac{\partial L}{\partial(\partial_j q^{\alpha})} \right) \tag{3.13}$$

where

$$\overrightarrow{\chi} = \lambda^{\alpha} \frac{\partial}{\partial q^{\alpha}} + (\partial_j \lambda^{\alpha}) \frac{\partial}{\partial(\partial_j q^{\alpha})} \tag{3.14}$$

is the vector field in the augmented space (i.e., the space consists of the generalized coordinates and their derivatives). Now according to Noether's theorem if $\mathcal{L}_{\overrightarrow{\chi}} L = 0$, then $\overrightarrow{\chi}$ (in equation (3.14)) is termed as Noether symmetry vector field or infinitesimal generator of the Noether symmetry and the associated conserved Noether current has the expression

$$I^{i} = \lambda^{\alpha} \frac{\partial L}{\partial(\partial_i q^{\alpha})} \tag{3.15}$$

with $\partial_i I^i = 0$.

In the present problem $N = 3$ i.e., the 3D augmented space consists of the variables (t, a, φ). Further, due to explicit time independence of the Lagrangian the Hamiltonian i.e., the total energy function is a constant of motion for the system i.e.,

$$E = \dot{q}^{\alpha} \frac{\partial L}{\partial \dot{q}^{\alpha}} - L = \text{conserved} \tag{3.16}$$

In addition, if the conserved quantity due to the symmetry has some physical analog then this symmetry approach can identify the reliable model.

On the other hand, Hamiltonian formulation is more useful than Lagrangian description in the context of quantum Cosmology. Then the Noether's theorem can be stated as

$$\mathcal{L}_{\overrightarrow{X}_H} H = 0 \tag{3.17}$$

with $\overrightarrow{X}_H = \dot{q}\frac{\partial}{\partial q} + \ddot{q}\frac{\partial}{\partial \dot{q}}$ as the symmetry vector. Moreover, in quantum cosmology Hamiltonian formulation is more useful than Lagrangian formalism. One may note that the canonically conjugate momenta corresponding to the conserved current due to Noether symmetry is constant i.e.,

$$\pi_{\rho} = \frac{\partial L}{\partial q^{\rho}} = p_{0\rho}\ , \quad \text{a constant} \tag{3.18}$$

with $\rho = 1, 2, ...m$ indicating the no of symmetries. So in quantum formulation by writing the operator version of equation (3.18) one gets

$$-i\partial_{q^{\rho}} |\psi >= p_{0\rho} |\psi > \tag{3.19}$$

where $|\psi>$ is termed as the wave function of the Universe. Thus solving equation (3.19) one gets the oscillatory part of the wave function. So the wave function of the Universe has the explicit form as

$$|\psi>=\sum_{\rho=1}^{m} e^{ip_{0\rho}q^{\rho}}|\varphi(q^{\sigma})>, \ \sigma<N \tag{3.20}$$

Here 'σ' indicates a direction along which there is no symmetry and N is the dimension of the minisuperspace. The oscillatory part of the wave function indicates Noether symmetry along that direction and the corresponding conjugate momenta is conserved and vice-versa. In other words the symmetry vector identifies the cyclic variables in the physical system.

3.4 Noether symmetry analysis and classical solution to the present modified gravity theory

This section shows how the above coupled non-linear field equations (3.5)-(3.7) can be solved as an application of the Noether symmetry analysis. According to this symmetry, $\exists$ a function $G(t,a,\varphi)$ such that the Lagrangian satisfies the equation

$$X^{[1]}L+LD_t\eta(t,a,\varphi)=D_tG(t,a,\varphi), \tag{3.21}$$

with symmetry vector

$$X=\eta(t,a,\varphi)\frac{\partial}{\partial t}+\alpha(t,a,\varphi)\frac{\partial}{\partial a}+\beta(t,a,\varphi)\frac{\partial}{\partial\varphi}. \tag{3.22}$$

Here the total derivative operator D_t is given by

$$D_t\equiv\frac{\partial}{\partial t}+\dot{a}\frac{\partial}{\partial a}+\dot{\varphi}\frac{\partial}{\partial\varphi} \tag{3.23}$$

with $X^{[1]}$, the first prolongation vector as

$$X^{[1]}=X+(D_t\alpha-\dot{a}D_t\eta)\frac{\partial}{\partial\dot{a}}+(D_t\beta-\dot{\varphi}D_t\eta)\frac{\partial}{\partial\dot{\varphi}} \tag{3.24}$$

Then the conserved quantity associated with the vector field X has the expression

$$I=\eta L+(\alpha-\dot{a}\eta)\frac{\partial L}{\partial\dot{a}}+(\beta-\dot{\varphi}\eta)\frac{\partial L}{\partial\dot{\varphi}}-G \tag{3.25}$$

For the Noether symmetry of the Lagrangian(3.4) on the tangent space $(a,\dot{a},\varphi,\dot{\varphi})$, the constituent partial differential equations are

$$\begin{aligned}
\eta_a = \eta_\varphi &= 0\\
-3\alpha a^2 V(\varphi) - a^3\beta V'(\varphi) - a^3\eta_t V(\varphi) &= G_t\\
12\mu(\varphi)a\alpha_t + 6\xi(\varphi)a^2\beta_t &= G_a\\
6\xi(\varphi)a^2\alpha_t + a^3\beta_t &= G_\varphi\\
6\mu(\varphi)\alpha + 6a\mu'(\varphi)\beta + 12a\mu(\varphi)\alpha_a - 12\mu(\varphi)a\xi_t + 6\xi(\varphi)a^2\beta_a + 6\mu(\varphi)a\eta_t &= 0\\
\frac{3}{2}\alpha a^2 + 6\xi(\varphi)a^2\alpha_\varphi - a^3\eta_t + a^3\beta\varphi + \frac{a^3\eta_t}{2} &= 0\\
12\xi(\varphi)a\alpha + 6\xi'(\varphi)a^2\beta + 6\xi(\varphi)a^2\alpha_a + 12\mu(\varphi)a\alpha_\varphi -&\\
6\xi(\varphi)a^2\eta_t + a^3\beta_a + 6\xi(\varphi)a^2\beta_\varphi &= 0 \quad (3.26)
\end{aligned}$$

Thus due to Noether symmetry the coefficients (i.e., η,α,β) of the infinitesimal generator satisfy an overdetermined set of partial differential equations which are solved using the method of separation of variables as

$$\alpha = \alpha_0 a^{p_1}\varphi^{q_1}\ ,\ \beta = \beta_0 a^{p_2}\varphi^{q_2} \quad (3.27)$$

Therefore we get the solution of above set of differential equation is given below

$$\begin{aligned}
\alpha = \alpha_0 a^{p_1}\varphi^{q_1},\ \beta = k\alpha_0 a^{p_1-1}\varphi^{q_1+1},\ V(\varphi) = V_0\varphi^{-\frac{3}{k}},\\
\mu(\varphi) = \frac{B_0}{2}\varphi^2,\ \xi(\varphi) = B_0\varphi. \quad (3.28)
\end{aligned}$$

where α_0,B_0,V_0,k, being any constant. Hence the infinitesimal generator on the tangent space can be represented as

$$X^{[3]} = (\alpha = \alpha_0 a^{p_1}\varphi^{q_1})\partial_a + (\beta = k\alpha_0 a^{p_1-1}\varphi^{q_1+1})\partial_\varphi. \quad (3.29)$$

Thus symmetry analysis not only gives the symmetry vector but also determine the potential function and the coupling function. Further, the explicit form of the conserved current and conserved energy are gives as

$$I^\alpha = \alpha\frac{\partial L}{\partial \dot{a}} + \beta\frac{\partial L}{\partial \dot{\varphi}} \quad (3.30)$$

$$\text{and } E = \dot{a}\frac{\partial L}{\partial \dot{a}} + \dot{\varphi}\frac{\partial L}{\partial \dot{\varphi}} - L. \quad (3.31)$$

Now, in order to identify a point transformation $(a,\varphi) \to (u,v)$ so that one of the variables becomes cyclic we impose the restriction (CAPOZZIELLO; STABILE; TROISI, 2007)

$$i_X du = 1 \text{ and } i_X dv = 0. \quad (3.32)$$

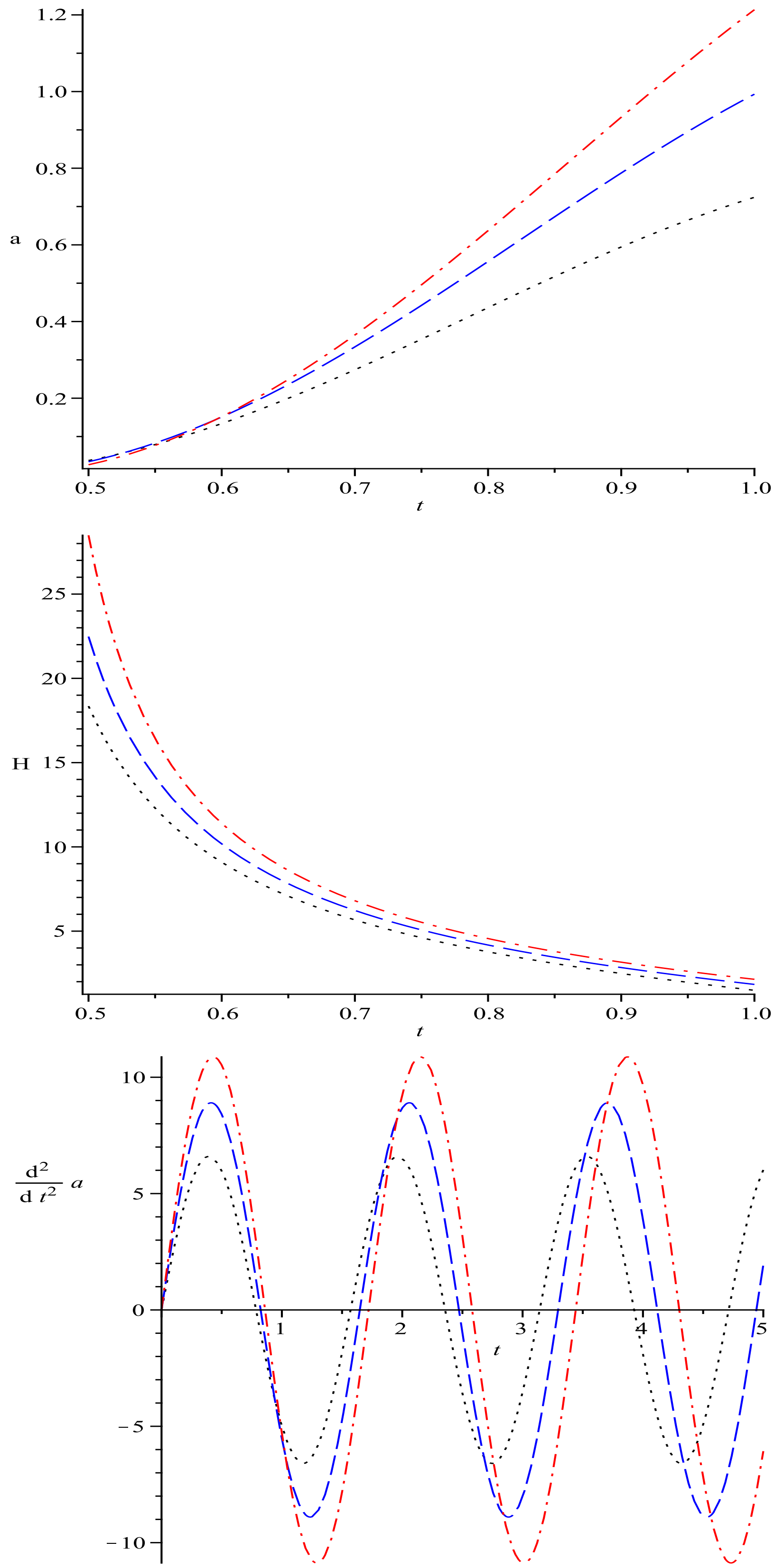

Figure 3.1 – The graphical representation of scale factor $a(t)$ (top left), Hubble parameter $H(t)$ (top right) and acceleration parameter $\ddot{a}(t)$ (bottom) with respect to cosmic time t.

Then the transformed infinitesimal generator

$$\hat{X} = (i_X du)\frac{\partial}{\partial u} + (i_X dv)\frac{\partial}{\partial v} + \left(\frac{d}{dt}(i_X du)\right)\frac{d}{d\dot{u}} + \left(\frac{d}{dt}(i_X dv)\right)\frac{d}{d\dot{v}}. \tag{3.33}$$

simplifies to $\hat{X} = \frac{\partial}{\partial u}$ with $\frac{\partial L}{\partial u} = 0$ i.e., u becomes a cyclic variable. Now solving the equation (3.32) one can get the relation between the old and new variable as

$$u = -\frac{1}{3}\ln\left(a^{-\frac{3}{2}}\varphi\right), \tag{3.34}$$

$$v = \ln\left(\frac{a^{\frac{3}{2}}}{\varphi}\right). \tag{3.35}$$

As a consequence the transformed Lagrangian takes the form

$$L = e^{-2v}[A_1\dot{u}^2 + B_1\dot{v}^2 + C_1\dot{u}\dot{v} - V_0] \tag{3.36}$$

with the conserved energy given by

$$E = e^{-2v}[A_1\dot{u}^2 + B_1\dot{v}^2 + C_1\dot{u}\dot{v} + V_0]. \tag{3.37}$$

Then by forming the Euler-Lagrange equation from equation (3.36), we get the new variable as

$$u(t) = -\frac{C_1}{4A_1}\ln\left[\frac{m}{2V_0}\left(C_0\sin\left(\sqrt{\frac{2V_0}{m}}t\right) - C_2\cos\left(\sqrt{\frac{2V_0}{m}}t\right)\right)^2\right] + D, \tag{3.38}$$

$$v(t) = -\frac{1}{2}\ln\left[\frac{m}{2V_0}\left(C_0\sin\left(\sqrt{\frac{2V_0}{m}}t\right) - C_2\cos\left(\sqrt{\frac{2V_0}{m}}t\right)\right)^2\right]. \tag{3.39}$$

Thus the classical cosmological Solution of this Einstein-æther field model are

$$a(t) = e^{-\left(\frac{2A_1+C_1}{4A_1}\right)\ln\left\{\frac{m}{2V_0}\left(C_0\sin\left(\sqrt{\frac{2V_0}{m}}t\right) - C_2\cos\left(\sqrt{\frac{2V_0}{m}}t\right)\right)^2\right\}+D}, \tag{3.40}$$

$$\varphi(t) = e^{-\left(\frac{2A_1-C_1}{4A_1}\right)\ln\left\{\frac{m}{2V_0}\left(C_0\sin\left(\sqrt{\frac{2V_0}{m}}t\right) - C_2\cos\left(\sqrt{\frac{2V_0}{m}}t\right)\right)^2\right\}-\frac{D}{2}}. \tag{3.41}$$

The above cosmological solution has been presented graphically in FIG.3.1. The scale factor, Hubble parameter and the acceleration parameters has been plotted over cosmic time for different sets of parameter involved. From the figure one may conclude that the present model describes an expanding model of the Universe with Hubble parameter gradually decreases with the evolution. Also the Universe experiences an accelerated expansion in early era and subsequently there is a phase of decelerated expansion and finally, again the Universe is experiencing an accelerated expansion era of evolution. The nature of the cosmic evolution of the present model agrees with the observational evidences. Finally, as the cosmic parameters do not deviate significantly due to small changes in the parameter values so it is reasonable to consider the above cosmic solution to be stable in nature.

3.5 Physical metric and Symmetry analysis

It has been shown by Tsamparlis et.al (TSAMPARLIS et al., 2013)that if the Lagrangian of a physical system can be written in the form of $L = T - V$ then the Noether point symmetries are identified by the elements of the homothetic group of the kinetic metric. As the Lagrangian of the present problem (given by equation (3.4)) can be expressed in the form of point particle so homothetic group of the kinetic metric will be very much relevant to identify the Noether point symmetries. The kinetic metric (obtained from the kinetic part of the Lagrangian) is given by

$$dS_k^2 = 6\mu(\varphi)ada^2 + 6\xi(\varphi)a^2dad\varphi + \frac{1}{2}a^3(d\varphi)^2 \tag{3.42}$$

with effective potential : $V_{\text{eff}}(a,\varphi) = a^3V(\varphi)$.

Substituting the values of $\mu(\varphi)$ and $\xi(\varphi)$ form equation (3.28) and after a simple algebra one gets

$$\begin{aligned} dS_k^2 &= \varphi^2a^3\left[3B_0\left(\frac{da}{a}\right)^2 + 6B_0\frac{da}{a}\frac{d\varphi}{\varphi} + \left(\frac{d\varphi}{\varphi}\right)^2\right] \\ &= e^{3\alpha+2u}\left[3B_0d\alpha^2 + 6B_0d\alpha du + \frac{1}{2}du^2\right] \end{aligned} \tag{3.43}$$

with $e^\alpha = a, e^u = \varphi$.

It is evident that the above metric is conformal to the metric

$$dS_c^2 = -d\alpha^2 + d\beta^2 \tag{3.44}$$

with $\beta = \frac{u+2\alpha}{\sqrt{2}}$ and $B_0 = \frac{1}{3}$.

Thus the given Lagrangian with the above transformation of variables simplifies to

$$L = e^{3\alpha+2u}\left[-\left(\frac{\dot{a}}{a}\right) + \frac{1}{2}\dot{\beta}^2 + V(a,\beta)\right] \tag{3.45}$$

Hence the present cosmological model can be consider to be equivalent to FLRW model (of scale factor 'a') with a scalar field having coupled potential.

Due to $2D$ con-formality of the kinematic metric, there exist the gradient Homothetic vector(HV) : $H_\gamma = a\partial_a$ with $e^\beta H_v = 1$. Note that this HV does not generate a Noether symmetry of the Lagrangian. Further, there exists the Killing vectors of the $2D$ equation (3.3) and they span the $E2$ group. In addition, there are $4D$ homothetic Lie algebra with explicit form :

(i) Two gradient translation Killing vectors :

$$\overrightarrow{X}^{(1)} = -\partial_\alpha \ , \overrightarrow{X}^{(2)} = \frac{1}{\alpha}\partial_\beta$$

having gradient Killing functions : $f_k^{(1)} = e^\beta$, $f_k^{(2)} = 0$.

(ii) one non-gradient Killing vector (indicating rotation) ;

$$\overrightarrow{X}^{(3)} = \partial_\beta$$

(iii) The gradient homothetic vector ;

$$\overrightarrow{X}^4 = \alpha\partial_\alpha$$

3.6 Quantum Cosmology in Einstein-æther scalar-tensor gravity

In this present cosmological model, conjugate momenta associated to the 2D configuration space $\{a,\varphi\}$ can be written as

$$\begin{aligned} p_a = \frac{\partial L}{\partial \dot{a}} &= 6B_0 a\varphi(\varphi\dot{a} + a\dot{\varphi}), \\ p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} &= a^2(6B_0\varphi\dot{a} + a\dot{\varphi}). \end{aligned} \tag{3.46}$$

Then the Hamiltonian of the system takes the form as

$$H = \left[\frac{A_2 p_a^2}{a\varphi} + \frac{B_2\varphi p_\varphi^2}{a^3} + \frac{C_2 p_a p_\varphi}{a^2} - V_2 a^3 \varphi^{1-\frac{3}{k}}\right] \tag{3.47}$$

with equivalent Hamilton's equation of motion

$$\begin{aligned} \dot{a} &= \left(\frac{6B_0\varphi p_\varphi - a p_a}{30B_0 a^2\varphi^2}\right), \\ \dot{\varphi} &= \left(\frac{a p_a - \varphi p_\varphi}{a^3\varphi(6B_0 - 1)}\right), \\ \dot{p}_a &= \left[\frac{A_2 p_a^2}{a^2\varphi} + \frac{3B_2\varphi p_\varphi^2}{a^4} + \frac{2C_2 p_a p_\varphi}{a^3} + 3V_2 a^2 \varphi^{1-\frac{3}{k}}\right], \\ \dot{p}_\varphi &= \left[\frac{A_2 p_a^2}{a\varphi^2} - \frac{B_2 p_\varphi^2}{a^3} + V_2\left(1 - \frac{3}{k}\right) a^3 \varphi^{\frac{3}{k}}\right]. \end{aligned} \tag{3.48}$$

In the context of Quantum cosmology, one has to construct the Wheeler Dewitt (WD) equation which is given by

$$\hat{H}\psi(a,\varphi) = 0 \tag{3.49}$$

where $\hat{H}$ is the operator version of the Hamiltonian and $\psi(a,\varphi)$ is the wave function of the Universe. In course of transformation to the operator version, there is a problem which is known as operator ordering problem. One has to consider the ordering consideration:

$$p_a \longrightarrow -i\frac{\partial}{\partial a}$$

and

$$p_\varphi \longrightarrow -i\frac{\partial}{\partial \varphi}.$$

As a result one can get a two parameter family of WD equation as

$$\Big[\frac{A_2}{\varphi}\frac{1}{a^l}\frac{\partial}{\partial a}\frac{1}{a^m}\frac{\partial}{\partial a}\frac{1}{a^n} + \frac{B_2\varphi}{a^3}\frac{\partial^2}{\partial \varphi^2} + C_2\frac{1}{a^{2r}}\frac{\partial}{\partial a}\frac{1}{a^{2s}}\frac{\partial}{\partial \varphi} - V_2a^3\varphi^{1-\frac{3}{k}}\Big]\psi(a,\varphi) = 0. \tag{3.50}$$

Here (l,m,n), the triplet of real numbers, satisfy the condition $l + m + n = 1$. But there are infinitely many possible choices for this triplet. So one can get infinite number of possible ordering. The commonly used choices are discussed below:

- D'Alembert Operator Ordering: $l = 2$, $m = -1$, $n = 0$.
- Vilenkin Operator Ordering: $l = 0$, $m = 1$, $n = 0$.
- No Ordering: $l = 1$, $m = 0$, $n = 0$.

The behaviour of the wave function can be affected by the factor ordering but the semi classical results remain unchanged due to the ordering problem. Now choosing the third option (i.e., no ordering), one can write the explicit form of WD equation as

$$\Big[\frac{A_2}{a\varphi}\frac{\partial^2}{\partial a^2} + \frac{B_2\varphi}{a^3}\frac{\partial^2}{\partial \varphi^2} + \frac{C_2}{a^2}\frac{\partial^2}{\partial a\partial \varphi} - V_2a^3\varphi^{1-\frac{3}{k}}\Big]\psi(a,\varphi) = 0 \tag{3.51}$$

Wave function of the Universe is nothing but the general solution of the above 2nd order hyperbolic partial differential equation. One can construct this solution by separating the Eigen functions of the WD operator as (CARTER; HARTLE, 1987)

$$\psi(a,\varphi) = \int W(Q)\psi(a,\varphi)dQ \tag{3.52}$$

where Q is conserved charge, $W(Q)$ is weight function and ψ is an eigen function of the WD operator. In WD operator, the minisuperspace variables $\{a,\varphi\}$ are highly coupled so it is impossible to have any explicit solution of WD equation even with the separation of variable method. Thus in the context of quantum cosmology, one may analyze this model using the new variables (u,v) in the augmented space.

The canonically conjugate momenta can be written as

$$\begin{aligned} p_u = \frac{\partial L}{\partial \dot{u}} &= e^{-2v}(2A_1\dot{u} + C_1\dot{v}) = \text{Conseved} \\ p_v = \frac{\partial L}{\partial \dot{v}} &= e^{-2v}(2B_1\dot{v} + C_1\dot{u}) \end{aligned} \tag{3.53}$$

Since in the transformed Lagrangian u is cyclic then p_u will be conserved. And hence the Hamiltonian of the system takes the form

$$H = e^{2v}(A_3 p_u^2 + B_3 p_v^2 + C_3 p_u p_v) - V_3 e^{-2v}. \tag{3.54}$$

Here A_3, B_3, C_3, V_3 are arbitrary constants.

Hence WD equation can be written as

$$\left[e^{2v}\left(A_3\frac{\partial^2}{\partial u^2} + B_3\frac{\partial^2}{\partial v^2} + C_3\frac{\partial^2}{\partial u \partial v}\right) + V_3 e^{-2v}\right]\psi(u,v) = 0. \tag{3.55}$$

The operator version of the conserved momentum takes the form

$$i\frac{\partial_\xi(u,v)}{\partial u} = \Sigma_0 \psi(u,v). \tag{3.56}$$

Now we choose, $\psi(u,v) = A(u)B(v)$. Then from (3.56), we will get,

$$\begin{aligned} i\frac{dA}{du} &= \Sigma_0 A \\ \implies A(u) &= A_4 e^{-i\Sigma_0 u} \end{aligned} \tag{3.57}$$

where A_4 is an integration constant.

Now putting the expression of $A(u)$ in the WD equation (3.55), we will get a second order ordinary differential equation which is given by

$$B_3 B''(v) - i\Sigma_0 C_3 B'(v) + (V_3 e^{-4v} - A_3\Sigma_0)B(v) = 0. \tag{3.58}$$

Solution of equation (3.58) is nothing but the non-oscillatory part of the wave function of the Universe for this model. The solution takes the form as

$$B(v) = e^{ivA_5}\left\{EJ_{(-B_4)}(C_4 e^{-2v}) + FJ_{(B_4)}(C_4 e^{-2v})\right\} \tag{3.59}$$

with J as a Bessel function.

Therefore, the wave function of the Universe can be written as

$$\psi(u,v) = e^{i(vA_5 - u\Sigma_0)}\left\{EJ_{(-B_4)}(C_4 e^{-2v}) + FJ_{(B_4)}(C_4 e^{-2v})\right\}. \tag{3.60}$$

The 3D figure in FIG. (3.2) shows the variation of the probability density (i.e., $|\psi|^2$) against the variables 'a' and 'φ'. From the figure, it is clear that there is finite non–zero probability to have zero volume of the Universe.

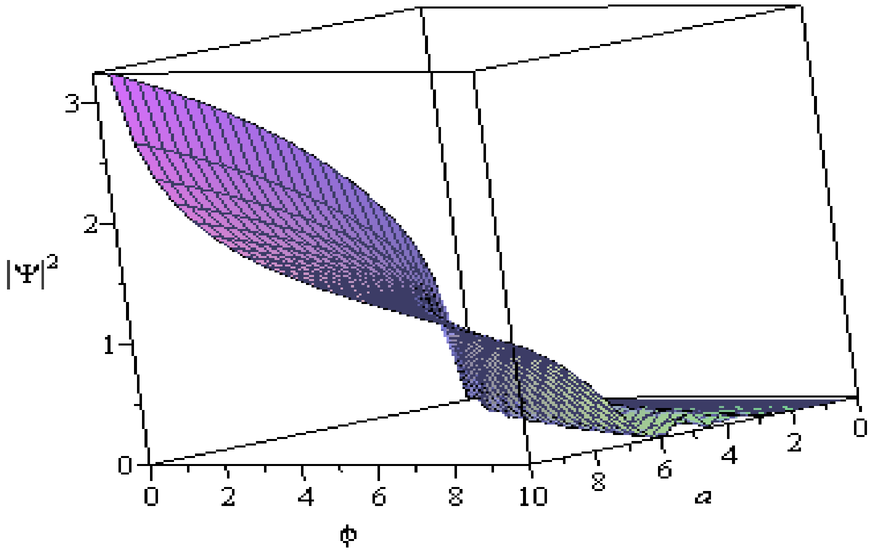

Figure 3.2 – Wave function of the Universe

3.7 Brief discussion and Concluding remarks

This chapter investigates the homogeneous and isotropic Einstein-æther scalar-tensor gravity model both for classical and quantum cosmology using Noether symmetry analysis. The Noether symmetry analysis is a powerful method which in some cases allows (i) to obtain exact dynamics and (ii) determining conserved Noether charge to be linked with some observable quantity. In this chapter, the above symmetry analysis shows two conserved quantity namely Noether charge and energy. Due to symmetry analysis it is possible to identify a cyclic co-ordinate (along the symmetry vector) so that the field equations become in a much simpler form and solutions are obtained. The cosmological solution of the present model shows the evolution of the Universe from the early accelerated expansion to the present era of expansion as reflected in FIG. (3.1). In the subsequent section (i.e., Section-**3.6**) quantum cosmology has been studied by forming the WD equation. The oscillatory part of wave function has been determined from the operator version of the conserved charge due to Noether symmetry. As a consequence, it is possible to have an explicit form of the wave function by solving the WD equation. The graph of probability density in FIG. (3.2) shows that quantum description can not avoid the initial singularity in the present cosmological model.

4 A description of classical and quantum cosmology for a single scalar field torsion gravity

4.1 Prelude

Duality symmetry has a very important role in the development of conformal field theory. To address various issues in the early Universe this symmetry technique has been used in string theory by Veneziano (VENEZIANO, 1991) and it is commonly known today as string cosmology. A dilaton field in the form of a scalar field $\varphi(x^k)$ is coupled to gravity in string cosmology with action integral in the form (GASPERINI; VENEZIANO, 1993; BUSCHER, 1987; BUSCHER, 1988; PALIATHANASIS, 2021):

$$S_{dil} = \int d^D x\sqrt{-g}e^{-2\varphi}\left[R - 4g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + \Lambda\right] \quad (4.1)$$

with Λ the usual cosmological constant. It is to be noted that the above action has some similarities with the BD-theory (BRANS; DICKE, 1961a) and more specifically, one may get back to the Brans-Dicke theory (PALIATHANASIS; CAPOZZIELLO, 2016) with constant BD parameter by the change of variable $\varphi(x^k) \rightarrow -\frac{1}{2}\ln\psi(x^k)$. In the context of cosmology, string cosmology (VENEZIANO, 1997) may describe the inflationary scenario in a natural way without imposing any fine-tune potential. From perturbative view point, galactic magnetic fields corresponds to electro-magnetic perturbations while smallness of matter perturbation agrees with the homogeneity of the Universe (MCALLISTER; SILVERSTEIN, 2008).

In the above action for the D–dimensional spatially flat and homogeneous background space-time, the Lagrangian depends on the scale factor as well as on the scalar field and interestingly there is a duality transformation (VENEZIANO, 1991):

$$a(t) \rightarrow a^{-1}(t)\,,\, \varphi(t) \rightarrow \varphi(t) - (D-1)\ln a \quad (4.2)$$

which leaves the above action integral to be invariant. Subsequently, the above duality transformation has been extended to anisotropic and inhomogeneous space-time models known as Gasperini-Veneziano duality property (GASPERINI; VENEZIANO, 1993; BUSCHER, 1987; BUSCHER, 1988) and is specifically O(d,d) symmetry. Specifically, this duality symmetry is a discrete transformation and there exist an isometry in the background space-time. Subsequently, due to this discrete transformation one gets

$$H \rightarrow -H\,,\, t \rightarrow -t \text{ and } \dot{H}(t) \rightarrow \dot{H}(-t),$$

for FLRW model and hence one obtains the string-driven pre-big-bang cosmology. Further, the above scale factor duality transformation corresponds to a local transformation which leaves the action integral to be invariant (a variational symmetry (PALIATHANASIS; CAPOZZIELLO, 2016)). Thus the above action integral (4.1) can also describe the pre–big bang scenario.

In this chapter, the above scale factor duality transformation has been used for teleparallel dark energy model. The introduction of modified gravity theory as well as the exotic matter is very much relevant in the context of cosmological observations for the last two decays or more. Further, teleparallel gravity has been attained special attention for the years due to a systematic geometric description to explain the cosmological observations. In this modified gravity theory, one gets the GR equivalence by considering the curvatureless Weitzenbӧck connection instead of the torsionless Levi-Civita connection. As a consequence the four linearly independent vierbeins are treated as dynamical fields and gravitational fields are defined by the Weitzenbӧck tensor and its scalar T. There is an analogus scalar-tensor theory with introduction of a scalar field in the teleparallel action integral, having interaction with the scalar T for the Weitzenbӧck tensor. This modified gravity theory is also termed as scalar-torsion theory (SKUGOREVA; SARIDAKIS; TOPORENSKY, 2015; HOHMANN, 2018b; HOHMANN; PFEIFER, 2018; HOHMANN, 2018c). The evolution of cosmological dynamics in teleparallel dark energy model has been investigated in (WEI, 2012; HU; LEON, 2012; OTALORA, 2013) and with an analysis from observational view point in (GENG; SARIDAKIS, 2012; D'AGOSTINO; LUONGO, 2018). The aim of the present chapter is to consider a teleparallel DE model having a discrete transformation from the point of view of Noether symmetry analysis. The plan of the chapter is as follows: Overview of teleparallel DE model is presented in Section-**4.2** and classical and quantum aspects of Noether symmetry is presented in Section-**4.3**. Section-**4.4** deals with classical cosmology of teleparallel DE model, physical metric and symmetry analysis of this model is described in Section-**4.5**. Section-**4.6** presents quantum cosmology and Noether symmetry analysis. The chapter ends with a brief summary.

4.2 An overview of Teleparallel gravity model

In the teleparallel equivalent of general relativity (or teleparallel gravity in brief) instead of torsion, curvature is assumed to vanish and the underlying space-time is known as weitzenbӧck space-time. Though there is fundamental difference, yet the two theories are found to yield equivalent descriptions of the gravitational interaction i.e., both curvature and torsion provide an equivalent description of the gravitational interaction. In teleparallel gravity, the orthogonal tetrad components (i.e., vierbein field) $e_A(x^\mu)$ form an orthonormal basis for the tangent space at each point (x^μ) of the manifold i.e.,

$$e_A e_B = \eta_{AB} = \text{diag}(+1, -1, -1, -1) \tag{4.3}$$

are considered as dynamical variables. So in a co-ordinate basis, the tetrad components can be expressed as

$$e_A = e_A^\mu \partial_\mu \tag{4.4}$$

where e_A^μ are the components of e_A, with $\mu = 0, 1, 2, 3$ and $A = 0, 1, 2, 3$. According to convention, the capital letters refer to the tangent space and co-ordiantes on the manifold are labeled by Greek indices. Thus considering the dual vierbein, one may write the metric tensor as a function of coordinates as

$$g_{\mu\nu}(x) = \eta_{AB} e_\mu^A(x) e_\nu^B(x) \tag{4.5}$$

where $e^A(x^l) = h_\mu^A(x^l) dx^\mu$ is the dual basis with $e^A(e_B) = \delta_B^A$. Here the original 16 degrees of freedom of the tetrad are constrained by the above 10 equations in equation(4.5) and one has 6 new independent degrees of freedom. As the metric tensor has 10 degrees of freedom, so there are 4 new degrees of freedom to completely describe the theory. These are termed as the spin connection coefficients and are given by

$$w_\mu{}^{AB} = e_\nu{}^A \, \Gamma^\nu{}_{\sigma\mu} \, e^{\sigma B} + e_\nu{}^A \, \partial_\mu \, e^{\nu B}.$$

From physical point of view, in canonical formulation of general relativity, a spin connection is defined on spatial slices and can be considered as the guage field generated by the local rotations.

In teleparallel gravity, the fundamental geometric object is the torsion tensor which is described by the antisymmetric part of the affine connection as

$$T_{\mu\gamma}^\sigma = \Gamma_{[\mu\gamma]}^\sigma \tag{4.6}$$

Since the metric tensor is covariantly constant (i.e., $\nabla_c g_{ab} = 0$), a generalized connection can be decomposed into a symmetric and an antisymmetric part as

$$\Gamma_{\beta\gamma}^\alpha = \bar{\Gamma}_{\beta\gamma}^\alpha + k_{\beta\gamma}^\alpha \tag{4.7}$$

where the symmetric part $\bar{\Gamma}_{\beta\gamma}^\alpha$ is the usual christoffel symbols while the antisymmetric part $k_{\beta\gamma}^\alpha$ is known as contortion tensor with $k_{\alpha\beta\gamma} = k_{[\alpha\beta]\gamma}$. The interrelation between this contortion tensor and the above torsion tensor is

$$k_{\alpha\beta\gamma} = T_{\alpha\beta\gamma} + 2T_{(\beta\gamma)\alpha} \tag{4.8}$$

So torsion tensor can be consider as a connecting tool between the intrinsic angular momentum (spin) of the matter and the geometry of the space-time.

If the above antisymmetric connection for the teleparallel gravity is replaced by the Weitzenböck connection $\bar{\Gamma}^{\sigma}_{\mu\nu} = e^{\sigma}_{a}\, \partial_{\mu}\, e^{a}_{\nu}$ then the spin connection $w_{\mu}{}^{AB}$ vanishes identically. As a result one may define torsion vector form the above torsion tensor due to its antisymmetric nature as

$$T_{\alpha} = T^{\beta}{}_{\alpha\beta} = -T^{\beta}{}_{\beta\alpha} \tag{4.9}$$

Hence one may expressed the contracted contortion tensor as

$$k^{\beta}_{\alpha\beta} = 2T_{\alpha} = -k_{\alpha\beta}{}^{\beta} \tag{4.10}$$

Further, the torsion scalar T can be define as

$$T = T^{\sigma}{}_{\alpha\beta} S^{\alpha\beta}_{\sigma} \tag{4.11}$$

where $S^{\alpha\beta}_{\sigma} = \frac{1}{2}\left(k^{\alpha\beta}{}_{\sigma} + \delta^{\alpha}{}_{\sigma} T^{\mu\beta}{}_{\mu} - \delta^{\beta}{}_{\sigma} T^{\mu\beta}{}_{\mu}\right)$ be the super potential.

In this chapter, one considers the homogeneous and isotropic flat FLRW space-time having line-element

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\Omega_2^2\right] \tag{4.12}$$

where $a(t)$ is the scale factor and $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on unit 2-sphere.

So the vierbein field in the diagonal form has the expression

$$h^{A}_{\mu} = \mathrm{diag}(1, a(t), a(t), a(t)) \tag{4.13}$$

with $T = 6H^2$.

Now the action integral in teleparallel dark energy model can be written as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[F(\varphi)\left(T + \frac{\omega}{2}\varphi_{,\mu}\varphi^{,\mu} + V(\varphi)\right)\right] \tag{4.14}$$

where $F(\varphi)$ is a coupling function, ω, a non-zero parameter similar to the Brans-Dicke parameter and $V(\varphi)$ is the potential for the scalar field. Thus for the homogeneous and isotropic FLRW model the Lagrangian has the explicit form (PALIATHANASIS, 2021)

$$L(a,\dot{a},\varphi,\dot{\varphi}) = F(\varphi)\left[\frac{1}{N}\left(6a\dot{a}^2 - \frac{\omega}{2}a^3\dot{\varphi}^2\right) + Na^3V(\varphi)\right] \tag{4.15}$$

having field equations

$$F(\varphi)\left(6H^2 - \frac{\omega}{2N^2}\dot{\varphi}^2 - V(\varphi)\right) = 0, \tag{4.16}$$

$$\left(\frac{2}{N}\dot{H} + 3H^2\right) + \frac{1}{2}\left(\frac{\omega}{2N^2}\dot{\varphi}^2 - V(\varphi) + 2(\ln F(\varphi))_{,\varphi} H\frac{\dot{\varphi}}{N}\right) = 0, \tag{4.17}$$

$$\omega\left(\frac{\ddot{\varphi}}{N^2} + \frac{3}{N}H\dot{\varphi} - \frac{\dot{N}}{N^3}\dot{\varphi}+\right) + \left(\frac{\omega}{2N^2}\dot{\varphi}^2 + V(\varphi)\right) + V_{,\varphi}(\varphi) = 0, \tag{4.18}$$

where N is the lapse function.

The basic difference of this chapter with (DIALEKTOPOULOS; LEON; PALIATHANASIS, 2023) is that multiscalarfield cosmology is considered in ref.(DIALEKTOPOULOS; LEON; PALIATHANASIS, 2023) while the chapter deals with a self interacting scalar field (chosen as DE) in the formulation of teleparallel gravity. The previous work in ref. (DIALEKTOPOULOS; LEON; PALIATHANASIS, 2023) can be reduced to the present chapter if $\beta = 0$ and $\hat{V} = 0$. The above field equations are highly coupled and non-linear in form. So it is very hard to find an analytic solution by solving the above set of differential equations. However, symmetry analysis will be used in the following sections not only to determine the cosmological solution but also to have a description of quantum cosmology.

4.3 Classical and Quantum Description using Noether Symmetry Analysis:a Brief Review

According to Noether's first theorem, if the Lagrangian of a physical system remains invariant with respect to the Lie derivative along an appropriate vector field (CAPOZZIELLO; STABILE; TROISI, 2007) then there exist some conserved quantities associated with the physical system. The Euler-Lagrange equations take the form

$$\partial_j \left(\frac{\partial L}{\partial \partial_j q^\alpha} \right) = \frac{\partial L}{\partial q^\alpha} \tag{4.19}$$

for a point-like canonical Lagrangian $L[q^\alpha(x^i), \dot{q}^\alpha(x^i)]$.

After contracting the equation (4.19) with some unknown function $\eta^\alpha(q^\beta)$, one can get

$$\eta^\alpha \left[\partial_j \left(\frac{\partial L}{\partial \partial_j q^\alpha} \right) - \frac{\partial L}{\partial q^\alpha} \right] = 0 \tag{4.20}$$

i.e.,

$$\eta^\alpha \frac{\partial L}{\partial q^\alpha} + (\partial_j \eta^\alpha) \left(\frac{\partial L}{\partial \partial_j q^\alpha} \right) = \partial_j \left(\eta^\alpha \frac{\partial L}{\partial \partial_j q^\alpha} \right). \tag{4.21}$$

Thus, the Lie derivative of the Lagrangian takes the form as

$$\mathcal{L}_{\overrightarrow{X}} L = \eta^\alpha \frac{\partial L}{\partial q^\alpha} + (\partial_j \eta^\alpha) \frac{\partial L}{\partial \left(\partial_j q^\alpha \right)} = \partial_j \left(\eta^\alpha \frac{\partial L}{\partial \partial_j q^\alpha} \right). \tag{4.22}$$

This $\overrightarrow{X}$ is known as the infinitesimal generator of the Noether symmetry and it is defined by

$$\overrightarrow{X} = \eta^\alpha \frac{\partial}{\partial q^\alpha} + \left(\partial_j \eta^\alpha \right) \frac{\partial}{\partial \left(\partial_j q^\alpha \right)}. \tag{4.23}$$

According to Noether's theorem, $\mathcal{L}_{\overrightarrow{X}} L = 0$ which indicates the physical system to be invariant along the symmetry vector $\overrightarrow{X}$. It is to be noted that the Lagrangian and the symmetry vector are defined on the tangent space of the configuration $TQ\ \{q^{\alpha}, \dot{q}^{\alpha}\}$. Moreover, Noether symmetry analysis has a significant role to identify the conserved quantities of a physical system. From equation (4.22), one can say that there is a constant of motion of the system associated to this symmetry criteria. This conserved quantity is known as Noether current or conserved current which is defined as

$$Q^{i} = \eta^{\alpha} \frac{\partial L}{\partial \left(\partial_{i} q^{\alpha}\right)}. \tag{4.24}$$

Also, this Noether current satisfies the condition

$$\partial_{i} Q^{i} = 0. \tag{4.25}$$

The energy function which is defined by

$$E = \dot{q}^{\alpha} \frac{\partial L}{\partial \dot{q}^{\alpha}} - L \tag{4.26}$$

is a constant of motion if the Lagrangian of the system does not depend on time explicitly. Imposing these symmetry constraints to any physical system, the evolution equations of the physical system become solvable or simpler.

In the context of quantum cosmology, Hamiltonian formulation is very much helpful and one can rewritten the Noether symmetry condition as

$$\mathcal{L}_{\overrightarrow{Y}} H = 0 \tag{4.27}$$

where $\overrightarrow{Y} = \dot{q} \frac{\partial}{\partial q} + \ddot{q} \frac{\partial}{\partial \dot{q}}$.

Due to Noether symmetry, the canonically conjugate momenta which is conserved in nature can be written as

$$p_{l} = \frac{\partial L}{\partial q^{l}} = \Sigma_{l}, \quad l = 1,2,...,m. \tag{4.28}$$

Here m is the number of symmetries. Also the operator version of equation (4.28) can be written as

$$-i \partial_{q^{l}} \left|\psi\right\rangle = \Sigma_{l} \left|\psi\right\rangle. \tag{4.29}$$

This equation (4.29) has oscillatory solution which is given by

$$\left|\psi\right\rangle = \sum_{l=1}^{m} e^{i \Sigma_{l} q^{l}} \left|\varphi(q^{k})\right\rangle, k < n. \tag{4.30}$$

Here, n is the dimension of the minisuperspace and k is the direction where symmetry does not exist. The oscillatory part of the wave function indicates the existence of Noether symmetry and the conjugate momenta should be conserved along the symmetry vector and vice versa. This conserved momenta due to Noether symmetry is useful to solve the Wheeler–DeWitt equation. In fact, using the separation of variables method it is possible to determine the non-oscillatory part of the wave function of the Universe and one may examine whether the initial big–bang singularity may be avoided or not by quantum description.

4.4 Classical Cosmology of teleparallel DE Model

To solving the above field equations (4.16)-(4.18) we use Noether symmetry analysis. For the point-like Lagrangian (4.15) of the given cosmological model the infinitesimal generator for the Noether symmetry takes the form (TSAMPARLIS; PALIATHANASIS, 2010; PALIATHANASIS et al., 2015; PALIATHANASIS; TSAMPARLIS; BASILAKOS, 2014; PALIATHANASIS; LEON, 2021a; PALIATHANASIS; LEON, 2022; TSAMPARLIS; PALIATHANASIS, 2011)

$$\overrightarrow{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial \varphi} + \dot{\alpha}\frac{\partial}{\partial \dot{a}} + \dot{\beta}\frac{\partial}{\partial \dot{\varphi}} \tag{4.31}$$

where $\alpha = \alpha(a,\varphi)$, $\beta = \beta(a,\varphi)$.

Now by imposing Noether symmetry to this Lagrangian (4.15), we will get a system of partial differential equations :

$$F(\varphi)\alpha + a\beta F'(\varphi) + 2aF(\varphi)\frac{\partial\alpha}{\partial a} = 0, \tag{4.32}$$

$$3F(\varphi)\alpha + a\beta F'(\varphi) + 2aF(\varphi)\frac{\partial\beta}{\partial\varphi} = 0, \tag{4.33}$$

$$12\frac{\partial\alpha}{\partial\varphi} - \omega a^2\frac{\partial\beta}{\partial a} = 0, \tag{4.34}$$

$$V'(\varphi) + V(\varphi)\left(\frac{3\alpha}{a\beta} + \frac{F'(\varphi)}{F(\varphi)}\right) = 0. \tag{4.35}$$

The above set of partial differential equations can be solved by using the method of separation of variables and as a result the symmetry can have explicit form. So α, β take the form

$$\begin{aligned}\alpha \equiv \alpha(a,\varphi) &= \alpha_1(a)\alpha_2(\varphi),\\ \beta \equiv \beta(a,\varphi) &= \beta_1(a)\beta_2(\varphi).\end{aligned} \tag{4.36}$$

As a result the explicit solutions are as follow

$$\alpha = \alpha_0 a, \beta = \beta_0, F(\varphi) = F_0 e^{-\frac{3\alpha_0}{\beta_0}\varphi}, V(\varphi) = V_0 \tag{4.37}$$

with α_0, β_0, F_0, and V_0 as arbitrary constants. Thus symmetry analysis not only determine the symmetry vector but also the potential function and the coupling function.

Further, the symmetry vector helps us to identify the cyclic variable in the augmented space so that the Lagrangian as well as the field equations simplified to a great extend. So a transformation in the augmented space : $(a,\varphi) \to (u,v)$ is chosen such that

$$i_{\overrightarrow{X}} du = 1, \, i_{\overrightarrow{X}} dv = 0 \tag{4.38}$$

where the left hand side of the above equations indicate the inner product between the vector field $\overrightarrow{X}$ and the one form du or dv. As a result the transformed symmetry vector field takes the form

$$\overrightarrow{X}_T = \frac{d}{du}$$

and the conserved current (i.e., conserved charge in the present case) can be written in compact form as

$$Q = i_{\overrightarrow{X}} \theta_L \tag{4.39}$$

with $\theta_L = \frac{\partial L}{\partial \dot{a}} da + \frac{\partial L}{\partial \dot{\varphi}} \varphi$, the Cartan one-form. Thus from the transformation equation (4.38) one has the explicit transformation in the augmented space as

$$u = \frac{\varphi}{\beta_0}, \, v = \ln a - \frac{\alpha_0}{\beta_0}\varphi. \tag{4.40}$$

So the transformed Lagrangian in the new variables takes the form

$$L = F_0 e^{3v} \left\{ \left(6\alpha_0^2 - \frac{\omega}{2}\beta_0^2\right) \dot{u}^2 + 6\dot{v}^2 + 12\alpha_0 \dot{u}\dot{v} + V_0 \right\} \tag{4.41}$$

where u is the cyclic coordinate. The corresponding Euler-Lagrange equations are

$$(12\alpha_0^2 - \omega\beta_0^2)\dot{u} + 12\alpha_0\dot{v} = Ae^{-3v} \tag{4.42}$$

$$\text{and } \; 2\ddot{v} + 3\dot{v}^2 + A_1 e^{-6v} + V_1 = 0 \tag{4.43}$$

where A_1, V_1 are connected to the relation $A_1 = \frac{A^2}{8\omega\beta_0^2}$, $V_1 = \frac{V_0(12\alpha_0^2 - \omega\beta_0^2)}{4\omega\beta_0^2}$. Now we take $A = 0$ to get a solution of above two equation. So after solving these two equations we get

$$u(t) = -\frac{8\alpha_0}{(12\alpha_0^2 - \omega\beta_0^2)} \ln\left\{\cos\left(\frac{\sqrt{3V_1}}{2}(t - 2c_1)\right)\right\} + c_3, \tag{4.44}$$

$$v(t) = \frac{2}{3} \ln\left\{\cos\left(\frac{\sqrt{3V_1}}{2}(t - 2c_1)\right)\right\} + c_2 \tag{4.45}$$

where c_1, c_2, c_3 are arbitrary constant. Now using previous relation (4.40) we get the old coordinates

$$a(t) = c_4 \left\{\cos\left(\frac{\sqrt{3V_1}}{2}(t - 2c_1)\right)\right\}^{c_5},$$

$$\varphi(t) = -\frac{8\alpha_0\beta_0}{(12\alpha_0^2 - \omega\beta_0^2)} \ln\left\{\cos\left(\frac{\sqrt{3V_1}}{2}(t - 2c_1)\right)\right\} + c_3\beta_0 \tag{4.46}$$

where $c_4 = e^{c_2 + \alpha_0 c_3}$, $c_5 = \frac{2}{3} - \frac{8\alpha_0^2}{(12\alpha_0^2 - \omega\beta_0^2)}$ i.e., c_4 is an arbitrary constant and c_5 is a constant. From the above classical solution the cosmological parameters namely the scale factor,

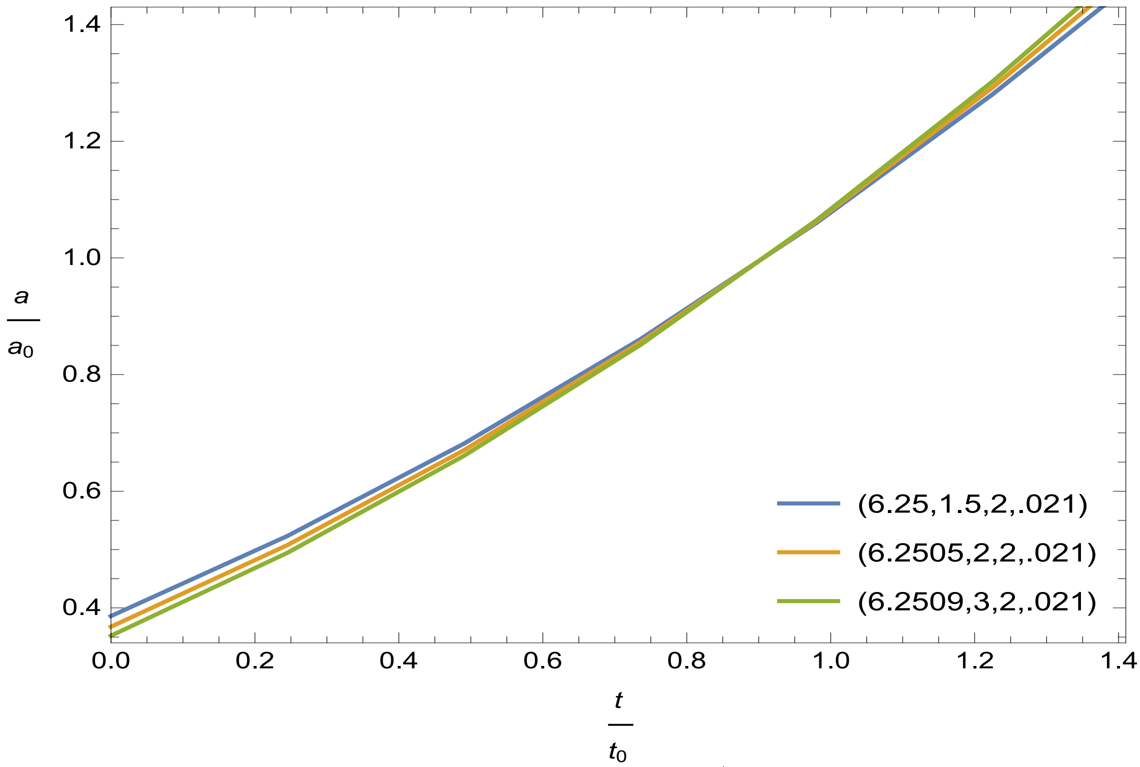

Figure 4.1 – Graphical representation of $\frac{a}{a_0}$ with respect to $\frac{t}{t_0}$ for different values of (c_1,c_4,c_5,V_1) parameter spaces with $t_0 = .011$.

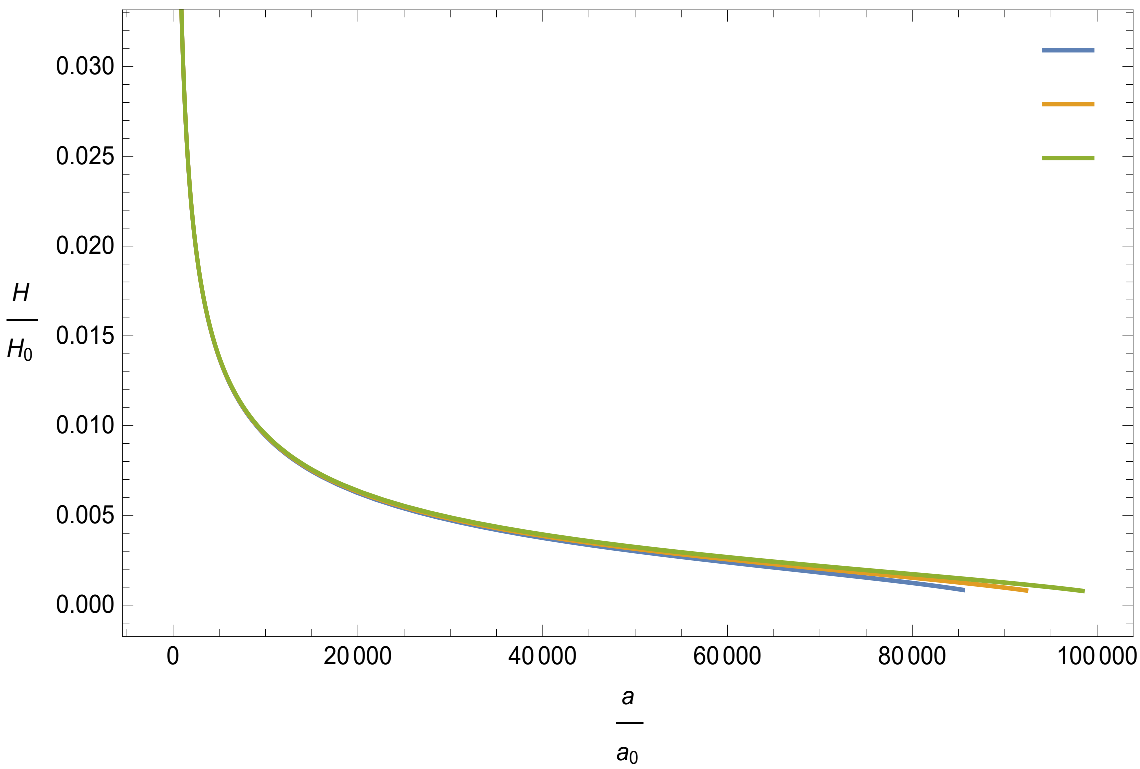

Figure 4.2 – The graphical representation of $\frac{H}{H_0}$ against $\frac{a}{a_0}$ where $H_0 = 72.9798$.

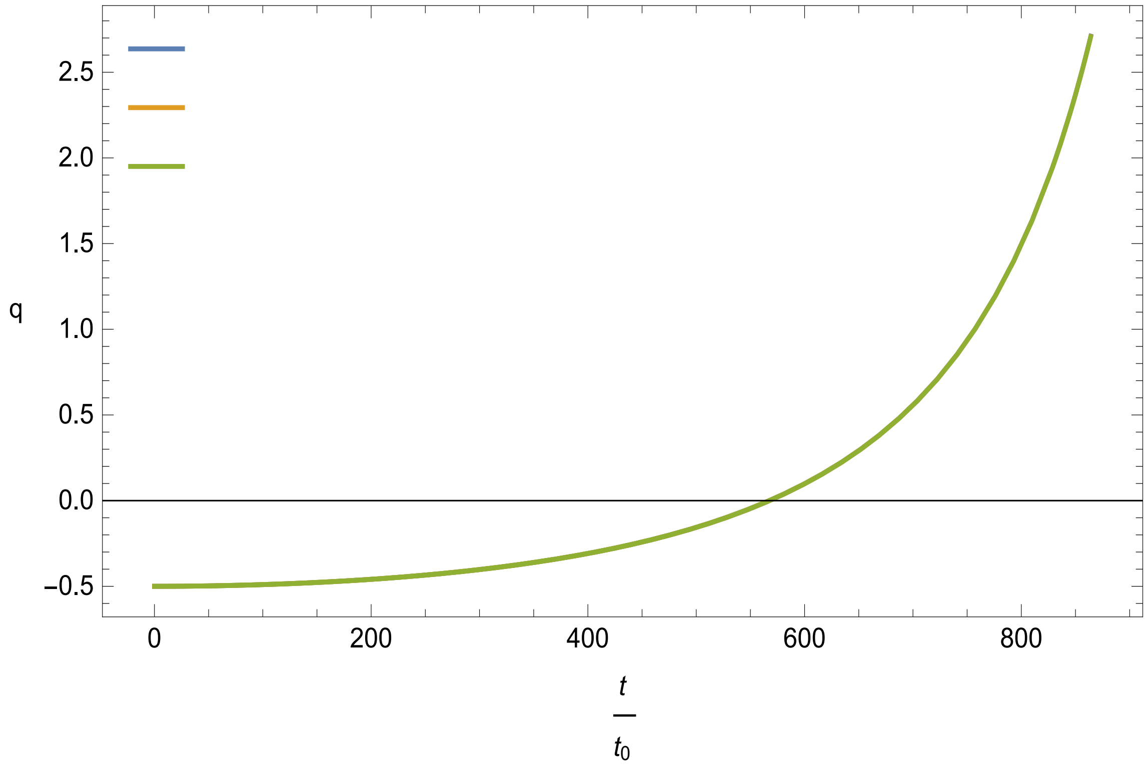

Figure 4.3 – The graphical representation of deceleration parameter $q = -1 - \frac{\dot{H}}{H^2}$ with respect to $\frac{t}{t_0}$.

Hubble parameter and the deceleration parameter have been shown graphically in figures (4.1)-(4.3) for the various choices of the parameters involved.

In FIG. (4.1) the dimensionless scale factor $\frac{a}{a_0}$ has been plotted against the cosmic time $\frac{t}{t_0}$ and the figure shows that the Universe is in an expanding phase throughout the evolution. The graphical representation of the dimensionless Hubble parameter $\frac{H}{H_0}$ has been presented in FIG. (4.2). The nature of the graph shows that the Hubble parameter is positive throughout but decreasing in nature. So, the Universe is in expanding phase but the rate of expansion gradually decreases. The deceleration parameter has been plotted in FIG. (4.3). The graph shows that the present model describes the early inflationary era to the matter dominated era. So, this model is not able to describe the present accelerated expansion of the Universe. Further, it is to be noted that the nature of the scale factor and the Hubble parameter shown graphically (in FIG. (4.1) and (4.2)) agree with the observational evidences since the very early era (RIESS et al., 2022; AGHANIM et al., 2020).

4.5 Physical metric and symmetry analysis

The Noether point symmetries of Lagrange equations having first order Lagrangian are shown to be generated by the elements of the homothetic group of the kinetic metric. In general if the field equations do not have Noether point symmetries then they are not

Noether integrable. However, it is possible to have extra Noether point symmetries by the above homothetic algebra.

For the present physical problem described by the Lagrangian (4.15) one may define the kinetic space as a 2D space having line element

$$dS_2^{(k)^2} = 6aF(\varphi)da^2 - \frac{\omega}{2}a^3F(\varphi)d\varphi^2 \tag{4.47}$$

having effective potential $V_{eff} = a^3F(\varphi)V(\varphi)$.

As the present Lagrangian (in equation (4.15)) is in the form of a point Lagrangian : $L = T - V_{eff}$, where T can be considered as the K.E. of a point particle so the Noether point symmetries are generated by the elements of the homothetic group of the kinetic metric

$$dS_2^{(k)^2} = a^3F(\varphi)\left[\frac{6}{a^2}da^2 - \frac{\omega}{2}d\varphi^2\right]. \tag{4.48}$$

Thus we have associated conformal (1+1) decomposable metric as

$$dS_2^{(c)^2} = \frac{6}{a^2}da^2 - \frac{\omega}{2}d\varphi^2 = 6du^2 - \frac{\omega}{2}d\varphi^2, u = \ln a. \tag{4.49}$$

Further, the above kinetic line-element has the usual gradient homothetic vector (HV) $H_V = \frac{2}{3}a\partial_a$ with $\psi H_\psi = 1$, but it does not generate a Noether point symmetry of the present Lagrangian. Moreover, the $2D$ metric in (u,φ)-plane is a space of constant curvature having 2 Killing vectors which generate the So(2) group. In addition, the above conformally flat metric (4.49) has 4 $\left(\frac{n(n+1)}{2} + 1 \text{ for } n = 2\right)$ dimensional homothetic Lie algebra as

(i) $\quad \overrightarrow{H}_V = u\frac{\overrightarrow{\partial}}{\partial u}$ with $\psi.H_V = 1$ (gradientHV)

(ii) $\quad \overrightarrow{K}^{(1)} = \cosh\varphi\partial_u - \frac{1}{u}\sinh\varphi\partial_\varphi$ (gradient Killing vector field)

(iii) $\quad \overrightarrow{K}^{(2)} = \sinh\varphi\partial_u - \frac{1}{u}\cosh\varphi\partial_\varphi$ (gradient Killing vector field)

(iv) $\quad \overrightarrow{K}^{(4)} = \frac{\overrightarrow{\partial}}{\partial\varphi}$ (non-gradient (rotational) Killing vector, generate the $So(2)$ algebra)

The gradient Killing functions corresponding to $\overrightarrow{K}^{(1)}$ and $\overrightarrow{K}^{(2)}$ are

$$g_1 = u, g_2 = u.$$

The above analysis shows that the determination of Noether point symmetries reduce to a problem of differential geometry.

4.6 Quantum cosmology and Noether symmetry analysis

The canonically conjugate momenta (for the transformed Lagrangian) due to Noether symmetry can be written as

$$p_u = \frac{\partial L}{\partial \dot{u}} = F_0 e^{3v}\left\{(12\alpha_0^2 - \omega\beta_0^2)\dot{u} + 12\alpha_0\dot{v}\right\}, \quad (4.50)$$

$$p_v = \frac{\partial L}{\partial \dot{v}} = F_0 e^{3v}\left\{12\dot{v} + 12\alpha_0\dot{u}\right\}. \quad (4.51)$$

Then the Hamiltonian of the system can be written as

$$H = A_2 e^{-3v} p_u^2 + A_3 e^{-3v} p_v^2 - A_4 e^{-3v} p_u p_v - V_2 e^{3v} \quad (4.52)$$

where A_2, A_3, A_4 and V_2 are arbitrary constants.

It is to be noted that as 'u' is cyclic coordinate so the corresponding momentum p_u is conserved i.e.,

$$F_0 e^{3v}\left\{(12\alpha_0^2 - \omega\beta_0^2)\dot{u} + 12\alpha_0\dot{v}\right\} = \text{conserved} = \sigma. \quad (4.53)$$

In quantum cosmology, the wave function of the Universe is a solution of the Wheeler-DeWitt(WD) equation, a second order hyperbolic partial differential equation. In fact, the WD equation is the operator version of the Hamiltonian constraint i.e., $\hat{H}\psi(u,v) = 0$. But there is a problem of operator ordering in course of conversion to the operator i.e, $p_u \to -i\frac{\partial}{\partial u}$ and $p_v \to -i\frac{\partial}{\partial v}$. So the explicit form of the WD equation can be written as

$$\left[A_2 e^{-3v}\frac{\partial^2}{\partial u^2} + A_3 e^{-3l_1 v}\frac{\partial}{\partial v}e^{-3l_2 v}\frac{\partial}{\partial v}e^{-3l_3 v} - A_4\frac{\partial}{\partial u}e^{-3m_1 v}\frac{\partial}{\partial v}e^{-3m_2 v} + V_2 e^{3v}\right]\psi(u,v) = 0 \quad (4.54)$$

where the number triplet (l_1, l_2, l_3) and the doublet (m_1, m_2) are arbitrary except the restrictions $l_1 + l_2 + l_3 = 0$ and $m_1 + m_2 = 0$. Note that though there are infinite possible choices for the above triplet and doublet, still the following are the preferred choices namely

(i) $l_1 = 2, l_2 = -1, l_3 = 0, m_1 = 2, m_2 = -1$ (D'Alembert operator ordering),

(ii) $l_1 = l_3 = 0, l_2 = 1, m_1 = 0, m_2 = 1$ (Vilenkin ordering),

(iii) $l_1 = 1, l_2 = l_3 = 0, m_1 = 1, m_2 = 0$(No ordering).

It is to be noted that the issue of factor ordering influences the behaviour of the wave function, still at the semi classical level factor ordering has no effect(TAVAKOLI; VAKILI, 2019). Now due to simplicit of the form of the WD equation we shall restrict to the above third choice i.e, no ordering and as a result the above WD equation simplifies to

$$\left[A_2 e^{-3v}\frac{\partial^2}{\partial u^2} + A_3 e^{-3v}\frac{\partial^2}{\partial v^2} - A_4 e^{-3v}\frac{\partial^2}{\partial u \partial v} + V_2 e^{3v}\right]\psi(u,v) = 0 \quad (4.55)$$

As the wave function of the Universe is the general solution of the above WD equation so it can be expressed as a superposition of the eigenfunction of the WD operator i.e.,

$$\Psi(u,v) = \int W(\sigma)\psi(u,v,\sigma)d\sigma \tag{4.56}$$

where ψ is an eigen function of the WD operator and $W(\sigma)$ represents the weight function which by proper choice gives us the desire wave packet. However, in analogy with classical description it is desirable to construct a coherent wave packet with good asymptotic behaviour in the minisuperspace, peaked around the classical trajectory.

Moreover, it is nice to examine whether the wave function can predict the evolution of the dynamical variables. For a consistent quantum cosmological model the classical solution can be predicted at late time only from the quantum description but quantum behaviour at early epochs is distinct from classical solution and is singularity free.

The operator version of conserved momentum which is given by equation (4.53) takes the form

$$-i\frac{\partial\psi(u,v)}{\partial u} = \sigma\psi(u,v). \tag{4.57}$$

Solution of equation (4.57) is oscillatory in nature and it indicates the existence of Noether symmetry.

From (4.57), we get

$$\psi_1(u) = \sigma_0 e^{i\sigma u} \tag{4.58}$$

where σ_0 is an arbitrary constant.

Putting the expression of $\psi_1(u)$ in the WD equation (4.55), one can get a second order differential equation as

$$A_3\frac{d^2\psi_2(v)}{dv^2} - iA_4\sigma\frac{d\psi_2(v)}{dv} + V_2e^{6v}\psi_2(v) - A_2\sigma^2\psi_2(v) = 0. \tag{4.59}$$

The solution of the above equation (4.59) gives the non-oscillatory part of the wave function which is given bellow:

$$\psi_2(v) = e^{iA_5v}\left\{\mu_1\Gamma(1-A_6)J_{(-A_6)}(A_7e^{3v}) + \mu_2\Gamma(1+A_6)J_{(-A_6)}(A_7e^{3v})\right\} \tag{4.60}$$

where μ_1, μ_2, A_5, A_6, A_7 are arbitrary constants Γ is the usual incomplete Gamma function. Therefore the wave function of the Universe for this model can be written as

$$\psi(u,v) = \sigma_0e^{i(\sigma u + A_5v)}\left\{\mu_1\Gamma(1-A_6)J_{(-A_6)}(A_7e^{3v}) + \mu_2\Gamma(1+A_6)J_{(-A_6)}(A_7e^{3v})\right\}. \tag{4.61}$$

Also in (a,φ) coordinate system, the above wave function takes the form:

$$\begin{aligned}\psi(a,\varphi) &= \sigma_0e^{i\left(\sigma\frac{\varphi}{\beta_0}+A_5\left(\ln a-\frac{\alpha_0}{\beta_0}\varphi\right)\right)} \\ &\left\{\mu_1\Gamma(1-A_6)J_{(-A_6)}(A_7a^3e^{-\frac{3\alpha_0}{\beta_0}\varphi}) + \mu_2\Gamma(1+A_6)J_{(-A_6)}(A_7a^3e^{-\frac{3\alpha_0}{\beta_0}\varphi})\right\}.\end{aligned} \tag{4.62}$$

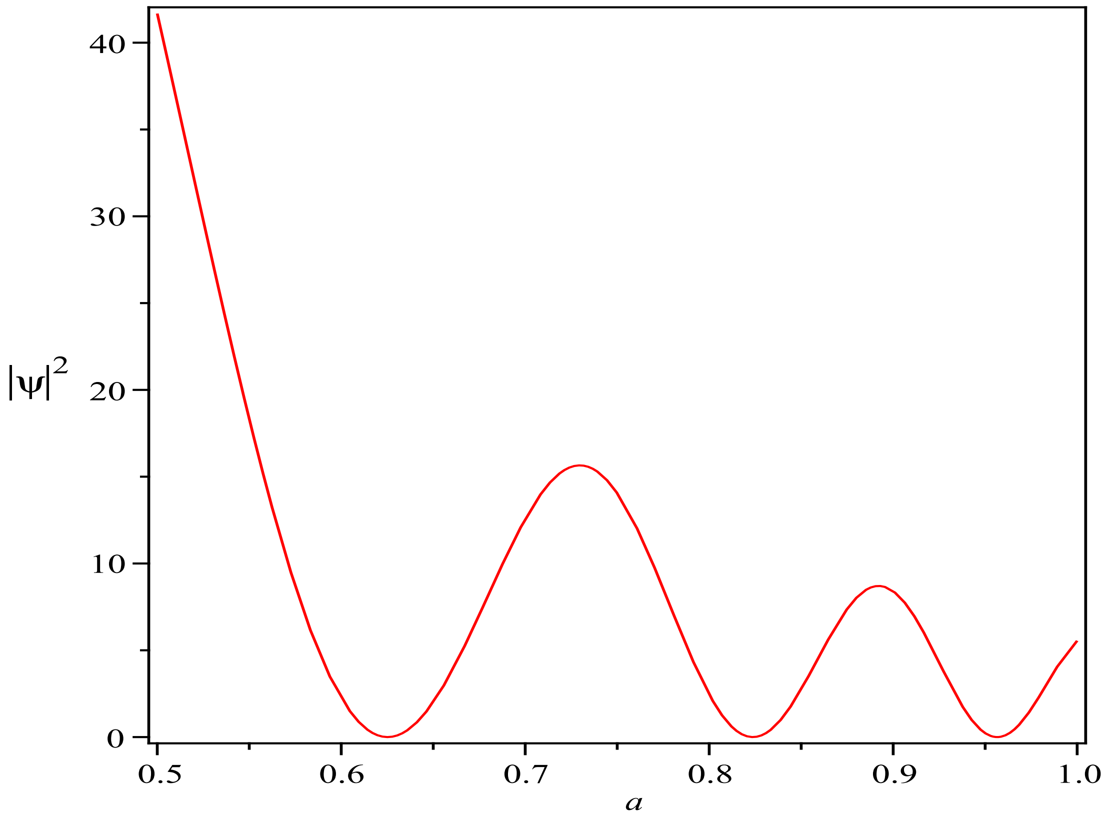

Figure 4.4 – Wave function of the Universe

Here we have plotted $|\psi|^2$ with respect to the scale factor a (FIG. (4.4)). From the graph it is clear that $|\psi|^2$ takes a non-zero finite value at zero volume. So this cosmological model may not overcome the big-bang singularity using quantum cosmology.

4.7 Brief Summary

This chapter gives an illustration how symmetry analysis helps us to study both classical and quantum cosmology of a model Universe. Here Noether symmetry analysis explores both classical and quantum cosmological description of teleparallel dark energy model. The Noether symmetry vector in the augmented space identifies a cyclic variable and as a result the field equations become solvable. The classical cosmological solution has been presented analytically in equation (4.46) while the relevant cosmological parameters namely the scale factor, Hubble parameter and the deceleration parameter has been shown graphically in FIG ((4.1)-(4.3)). The graphical description shows that the Universe expands throughout the evolution while the rate of expansion gradually decreases as predicted by the recent observational evidences. However, the graphical representation of the deceleration parameter relates that the present cosmological model describes the early evolution of the Universe to the matter dominated era. In quantum cosmology WD equation (a second order hyperbolic partial differential equation) has been constructed and the operator ordering issue has been discussed. The conserved momentum associated with the cyclic variable gives us a first order linear differential equation due to operator conversion and this first order differential equation gives us an oscillatory solution. Subsequently, it is possible to have solution of the WD equation with the help of the above periodic solution. The probability amplitude has been shown graphically in FIG. (4.4) and it is found that classical singularity can not be avoided by quantum description.

For future work, we can extend this chapter by choosing non–minimally coupled scalar field or by considering multi–scalar field model. Also it will be interesting to perform cosmological perturbations about the initial cosmic time.

5 Noether Symmetry Analysis in Scalar Tensor Cosmology : A Study of Classical and Quantum Cosmology

5.1 Prelude

Since the end of the last century, standard cosmology is facing a great challenge, to explain the observational evidences which indicate that our Universe is going through an accelerated phase. To accommodate this fact in cosmological context, the cosmologist are sharing two possible modifications of standard cosmology. One of the groups introduced an extra term in Einstein-Hilbert action (WANDS, 1994) (i.e., modification of gravity theory) while the other group prefers an exotic matter within the framework of Einstein gravity. This exotic matter is known as dark energy having large negative pressure. A hypothetical scalar field (known as inflaton (RUBIO, 2019),(GRANDA; JIMENEZ, 2019),(PARSONS; BARROW, 1995),(BARROW; PALIATHANASIS, 2018)) is responsible for the early accelerated era of evolution i.e., inflationary era. In analogy, the scalar fields describing the late time acceleration (known as dark energy (LINDER, 2004),(BARSHAY; KREYERHOFF, 1998),(MATOS et al., 2009),(LIDDLE; PAHUD; URENA-LOPEZ, 2008),(DEFFAYET; DESER; ESPOSITO-FARESE, 2009),(CHIMENTO; ZUCCALA; MENDEZ, 1999),(BERTACCA; BARTOLO; MATARRESE, 2010)) must have large -ve pressure.

Multiscalar field cosmology has a significant role in studying hybrid inflation, double inflation, α-attractors. Quintom model (HU, 2005), Chiral model are two well known examples of multiscalar field cosmological models. The quintom model (where one is a quintessence field while the other one is a phantom field) is a DE model while the Chiral model leads to hyperbolic inflation (WANDS, 2008). For introducing a multiscalar field model, one can consider the existence of a complex scalar field (SCIALOM, 1996),(ROSEN, 2010),(ARIK; CALIK; KATIRCI, 2011),(SHEN; GE, 2006),(FOIDL; RINDLER-DALLER, 2022),(BEVILACQUA; KOWALSKI-GLIKMAN; WISLICKI, 2022),(LI; SHAPIRO; RINDLER-DALLER, 2016),(GODUNOV et al., 2016) whose real part and imaginary part give the equivalent of a two scalar-field theory (KHALATNIKOV; MEZHLUMIAN, 1992),(KHALATNIKOV, 1995). In this chapter, scalar-tensor theory is studied with a complex scalar field (FARAONI, 2004). The scalar field in scalar tensor theory is minimally coupled to gravity and it interacts with the gravitational action integral of Einstein's general relativity. The scalar tensor theory is usually defined in Jordan frame (PALIATHANASIS, 2022c) with Mach Principle (BRANS;

DICKE, 1961a). The scalar tensor theory with teleparallel gravity describes the scalar torsion theory.

Since the last century, symmetry analysis has an important role in studying the internal symmetries of the space-time, global continuous symmetries and permutation symmetries in quantum field theory (CAPOZZIELLO et al., 2007),(HALDER; PALIATHANASIS; LEACH, 2018). In particular, Noether symmetry has a great role to identify the conserved quantities associated with a physical system. Also Noether integral can simplify a system of differential equations to a great extent (DUTTA; CHAKRABORTY, 2016),(DUTTA; PANJA; CHAKRABORTY, 2016a),(DUTTA; PANJA; CHAKRABORTY, 2016b),(DUTTA; LAKSHMANAN; CHAKRABORTY, 2016),(CAMCI; KUCUKAKCA, 2007),(KUCUKAKCA; CAMCI; SEMIZ, 2012). In addition, using Noether symmetry, any arbitrary function associated in the action integral of a physical system can be obtained uniquely.

This chapter is an example of using Noether symmetry analysis to a cosmological model having scalar-tensor theory with a complex scalar field. Classical cosmological solutions for this model are obtained using Noether symmetry analysis. Conserved quantities associated to this system are also obtained. In the context of quantum cosmology, formulating the Wheeler DeWitt (WD) equation, the wave function of the Universe is obtained by identifying the periodic part of the solution from the quantum version of the conserved charge. The plan of this chapter is as follows: section-**5.2** presents the basic equations of scalar-tensor cosmological model while in section-**5.3** Noether symmetry approach has been used for finding the analytic solutions of the present model. In section-**5.4**, the formation of WD equation in the present cosmological model and its possible solution using Noether symmetry approach have been discussed and the chapter ends with a brief review in section-**5.5**.

In Noether's theorem, the invariance of the functional of the calculus of variations or in mechanics the invariance of the action integral is examined under an infinitesimal transformation. In general such transformations are generated by a differential operator termed as Noether symmetry vector. However, in this chapter we are confined ourselves only to point transformation. For a complete classification one may refer to (TSAMPARLIS; PALIATHANASIS, 2018).

5.2 Basic equations of scalar-tensor cosmology

The scalar tensor and the scalar torsion theories are considered in this chapter where the scalar field is complex in nature and minimally coupled to gravity. Considering ξ as the complex scalar field in scalar-tensor theory, the action integral takes the form (PALIATHANA-

SIS, 2022b)

$$A = \int d^4x\sqrt{-g}\left[F(|\xi|)R + \frac{1}{2}g^{mn}\xi_{,m}\xi^*_{,n} - V(|\xi|)\right]. \tag{5.1}$$

Here R is the usual Ricci scalar which is related to the Levi-Civita connection for the metric tensor g_{mn}. Also ξ is the complex scalar field and $|\xi|$ determines its norm i.e., $|\xi|^2 = \xi\xi^*$. $F(|\xi|)$ denotes the coupling function between the gravitational and the scalar field and the potential function $V(|\xi|)$ drives the dynamics.

If the coupling function $F(|\xi|)$ takes the value $F_0|\xi|^2$ where F_0 is a constant, then one can get the Brans-Dicke theory with a complex scalar field and the action integral (5.1) transforms as (PALIATHANASIS, 2022b)

$$A_{BD} = \int d^4x\sqrt{-g}\left[F_0|\xi|^2R + \frac{1}{2}g^{mn}\xi_{,m}\xi^*_{,n} - V(|\xi|)\right]. \tag{5.2}$$

The line element for a spatially flat FLRW Universe can be written as

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{5.3}$$

where $a(t)$ is the scale factor and $N(t)$ denotes the lapse function.

Also the Ricci scalar takes the expression as

$$R = 6\left(\frac{1}{N}\dot{H} + 2H^2\right), \tag{5.4}$$

where H denotes the usual Hubble parameter defined by $H = \frac{1}{N}\frac{\dot{a}}{a}$ and the overdot indicates the differentiation with respect to the cosmic time t.

Integrating equation (5.1) by parts after using the expression of Ricci scalar (from equation (5.4)) one gets the point-like Lagrangian as

$$L(N,a,\dot{a},\xi,\dot{\xi}) = \frac{1}{N}\left[6F(|\xi|)a\dot{a}^2 + 6\dot{F}(|\xi|)a^2\dot{a} + \frac{1}{2}a^3\left(\dot{\xi}\dot{\xi}^*\right)\right] - a^3NV(|\xi|). \tag{5.5}$$

Similarly, integrating equation (5.2) by parts and using the expression (5.4) one can get the point-like Lagrangian for Brans-Dicke theory as

$$L_{\mathrm{BD}}(N,a,\dot{a},\xi,\dot{\xi}) = \frac{1}{N}\left[6F_0|\xi|^2a\dot{a}^2 + 6F_0(|\xi|^2)a^2\dot{a} + \frac{1}{2}a^3\left(\dot{\xi}\dot{\xi}^*\right)\right] - a^3NV(|\xi|). \tag{5.6}$$

Using the polar form, the complex scalar field ξ can be written as

$$\xi(t) = \varphi(t)e^{i\theta(t)}. \tag{5.7}$$

Using equation (5.7), the Lagrangian (5.6) transforms as (PALIATHANASIS, 2022b)

$$L_{\mathrm{BD}}(N,a,\dot{a},\varphi,\dot{\varphi},\theta,\dot{\theta}) = \frac{1}{N}\left[6F_0a\varphi^2\dot{a}^2 + 12F_0a^2\varphi\dot{a}\dot{\varphi} + \frac{1}{2}a^3\left(\dot{\varphi}^2 + \varphi^2\dot{\theta}^2\right)\right] - a^3NV(\varphi) \tag{5.8}$$

Clearly, the above Lagrangian represents a multiscalar field cosmological model where φ is the Brans-Dicke field and the second scalar field θ is minimally coupled to gravity and also it is minimally coupled to φ. Further, the present cosmological model with $V(\varphi) = \varphi^{-\frac{3\alpha_0+\delta_0}{\beta_0}}$ represents the usual Brans-Dicke field in FLRW model with a cosmological constant and an uncoupled scalar field θ. Such cosmological model has been widely used in the literature (HRYCYNA; SZYDŁOWSKI, 2013),(PAPAGIANNOPOULOS et al., 2017).

The field equations for Brans Dicke cosmological model corresponding to the Lagrangian (5.8) can be written as

$$\frac{1}{N}\left(6F_0\varphi^2H^2 + 12F_0\varphi H\dot{\varphi} + \frac{1}{2}\left(\dot{\varphi}^2 + \varphi^2\dot{\theta}^2\right)\right) + NV(\varphi) = 0 \quad (5.9)$$

$$\frac{1}{N}\left(2F_0\varphi^2\left(2\dot{H} + 3H^2\right) + 8F_0H\varphi\dot{\varphi} - \frac{1}{2}\dot{\varphi}^2 + 4F_0\dot{\varphi}^2 - \frac{1}{2}\varphi^2\dot{\theta}^2 + 4F_0\varphi\ddot{\varphi}\right) + NV(\varphi) = 0 \quad (5.10)$$

$$\frac{1}{N}\left(\ddot{\varphi} + \varphi\left(12F_0\dot{H} - \dot{\theta}^2\right) + 3\dot{\varphi}\dot{H} + 12F_0\varphi H^2\right) + NV_{,\varphi} = 0 \quad (5.11)$$

$$\frac{1}{N}\left(\varphi\ddot{\theta} + (2\dot{\varphi} + 3H\varphi)\,\dot{\theta}\right) = 0 \quad (5.12)$$

5.3 Analytic solution using Noether symmetry approach

If the Lagrangian of a physical system remains invariant with respect to the Lie derivative (TSAMPARLIS; PALIATHANASIS, 2012) along an appropriate vector field then the corresponding physical system is associated with some conserved quantities (Noether's first theorem (BHAUMIK; DUTTA; CHAKRABORTY, 2022a)).

If $L(q^\alpha(x^i),\dot{q}^\alpha(x^i))$ is the point-like canonical Lagrangian then,

$$\partial_j\left(\frac{\partial L}{\partial\partial_j q^\alpha}\right) = \frac{\partial L}{\partial q^\alpha} \quad (5.13)$$

are the corresponding Euler-Lagrange equations.

After contracting the equation (5.13) with $\mu^\alpha\left(q^\beta\right)$ (some unknown functions), one get the following result

$$\mu^\alpha\frac{\partial L}{\partial q^\alpha} + (\partial_j\mu^\alpha)\left(\frac{\partial L}{\partial\partial_j q^\alpha}\right) = \partial_j\left(\mu^\alpha\frac{\partial L}{\partial\partial_j q^\alpha}\right). \quad (5.14)$$

Thus, the Lie derivative of the Lagrangian takes the form

$$\mathcal{L}_{\overrightarrow{X}}L = \mu^\alpha\frac{\partial L}{\partial q^\alpha} + (\partial_j\mu^\alpha)\left(\frac{\partial L}{\partial\partial_j q^\alpha}\right) = \partial_j\left(\mu^\alpha\frac{\partial L}{\partial\partial_j q^\alpha}\right) \quad (5.15)$$

This vector field $\overrightarrow{X}$ defined by (DIALEKTOPOULOS; KOIVISTO; CAPOZZIELLO, 2019), (CAPOZZIELLO et al., 2009)

$$\overrightarrow{X} = \mu^\alpha\frac{\partial}{\partial q^\alpha} + (\partial_j\mu^\alpha)\frac{\partial}{\partial\left(\partial_j q^\alpha\right)}. \quad (5.16)$$

is known as infinitesimal generator of the symmetry. Now, according to Noether's first theorem if $\mathcal{L}_{\overrightarrow{X}}L = 0$ then the physical system will be invariant with respect to the vector field $\overrightarrow{X}$.

Noether symmetry approach is very much useful to identify the conserved quantities of a physical system. The symmetry condition is associated with a constant of motion for the Lagrangian having conserved phase flux along the infinitesimal generator $\overrightarrow{X}$. Furthermore, from equation (5.15) one can conclude that associated to this symmetry criteria there is a constant of motion of the system which is known as Noether current or conserved current Q^i. It is defined by (DUTTA; CHAKRABORTY, 2016) (DUTTA; LAKSHMANAN; CHAKRABORTY, 2021),(DUTTA; LAKSHMANAN; CHAKRABORTY, 2019)

$$Q^i = \mu^\alpha \frac{\partial L}{\partial\left(\partial_i q^\alpha\right)}. \tag{5.17}$$

Also Q^i satisfy the condition

$$\partial_i Q^i = 0. \tag{5.18}$$

The energy function associated to this system can be written as (DUTTA; LAKSHMANAN; CHAKRABORTY, 2016),(DUTTA; LAKSHMANAN; CHAKRABORTY, 2018)

$$E = \dot{q}^\alpha \frac{\partial L}{\partial \dot{q}^\alpha} - L. \tag{5.19}$$

If the Lagrangian does not contain time explicitly, then this energy function which is also known as the Hamiltonian of the system, is a constant of motion. Using these Symmetry constraints, the evolution equations of a physical system can either be solvable or simplified to a great extent.

For the present model the configuration space is a 3D space (a,φ,θ). Also from the Lagrangian (5.8), one can see that θ is a cyclic variable and the infinitesimal generator takes the form

$$\overrightarrow{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial\varphi} + \delta\frac{\partial}{\partial N} + \dot{\alpha}\frac{\partial}{\partial\dot{a}} + \dot{\beta}\frac{\partial}{\partial\dot{\varphi}} + \dot{\gamma}\frac{\partial}{\partial\dot{\theta}}. \tag{5.20}$$

Here, $\alpha = \alpha(a,\varphi)$, $\beta = \beta(a,\varphi)$, $\gamma = \gamma(\theta)$ and $\delta = \delta(N)$ are the coefficients of the infinitesimal generator and

$$\begin{aligned}
\dot{\alpha} &= \frac{\partial\alpha}{\partial a}\dot{a} + \frac{\partial\alpha}{\partial\varphi}\dot{\varphi} \\
\dot{\beta} &= \frac{\partial\beta}{\partial a}\dot{a} + \frac{\partial\beta}{\partial\varphi}\dot{\varphi} \\
\dot{\gamma} &= \frac{\partial\gamma}{\partial\theta}\dot{\theta}.
\end{aligned} \tag{5.21}$$

Now, imposing the Noether's first theorem to the Lagrangian (5.8) one gets

$$\mathcal{L}_{\overrightarrow{X}}L_{\mathrm{BD}} = 0 \tag{5.22}$$

The explicit form of the equation (5.22) gives a system of partial differential equations as follows:

$$\varphi\alpha + 2a\beta + 2a\varphi\frac{\partial\alpha}{\partial a} + 2a^2\frac{\partial\beta}{\partial a} - \frac{a\varphi}{N}\delta = 0 \quad (5.23)$$

$$3\alpha + 24F_0\varphi\frac{\partial\alpha}{\partial\varphi} + 2a\frac{\partial\beta}{\partial\varphi} - \frac{a}{N}\delta = 0 \quad (5.24)$$

$$3\varphi\alpha + 2a\beta + 2a\varphi\frac{\partial\gamma}{\partial\theta} - \frac{a\varphi}{N}\delta = 0 \quad (5.25)$$

$$24F_0\varphi\alpha + 12F_0a\beta + 12F_0\varphi^2\frac{\partial\alpha}{\partial\varphi} + 12F_0a\varphi\frac{\partial\alpha}{\partial a} + a^2\frac{\partial\beta}{\partial a} + 12F_0a\varphi\frac{\partial\beta}{\partial\varphi} - 12F_0a\varphi\frac{\delta}{N} = 0 \quad (5.26)$$

$$3\alpha V(\varphi) + a\beta V'(\varphi) + aV(\varphi)\frac{\delta}{N} = 0 \quad (5.27)$$

For solving the above set of partial differential equations one can use the method of separation of variable i.e., $\alpha(a,\varphi) = \alpha_1(a)\alpha_2(\varphi)$, $\beta(a,\varphi) = \beta_1(a)\beta_2(\varphi)$, $\gamma = \gamma(\theta)$ and $\delta = \delta(N)$.

Solving the equations (5.23-5.26) one can get

$$\begin{aligned} \alpha &= \alpha_0 a \\ \beta &= \beta_0\varphi \\ \gamma &= \gamma_0 \\ \delta &= \delta_0 N \end{aligned} \quad (5.28)$$

where α_0, β_0, γ_0 and δ_0 are integration constants with the relation $3\alpha_0 + 2\beta_0 - \delta_0 = 0$.

Putting the values of α, β and δ from (5.28) into equation (5.27) one gets on integration

$$V(\varphi) = V_0\varphi^{-\frac{3\alpha_0+\delta_0}{\beta_0}} \quad (5.29)$$

where V_0, an integration constant is strictly positive.

Thus imposing the symmetry condition on the Lagrangian one can find out the infinitesimal generator of Noether symmetry. Also the poential function $V(\varphi)$ is determined using the symmetry criteria rather than choosing phenomenologically.

Another important feature of Noether Symmetry is that there are some conserved quantities associated with it. There is no well defined notion of energy for a field theory in a curved space. But when there exists a time like killing vector in the system, then an associated conserved energy exists. It is well known that there is no time like killing vector in FLRW space-time. But the Lagrangian density is explicit time independent. Hence, for a point-like Lagrangian, one can define a conserved energy. So associated to this symmetry criteria there are two conserved quantities, namely, conserved charge (defined in equation (5.17)) and conserved energy (defined in equation (5.19)) which have the explicit form as follows:

$$Q = \frac{1}{N}\left(12F_0(\alpha_0+\beta_0)a^2\varphi^2\dot{a} + (12F_0\alpha_0+\beta_0)a^3\varphi\dot{\varphi} + \gamma_0 a^3\varphi^2\dot{\theta}\right), \tag{5.30}$$

$$E = \frac{1}{N}\left(6F_0 a\varphi^2\dot{a}^2 + 12F_0 a^2\varphi\dot{a}\dot{\varphi} + \frac{1}{2}a^3\left(\dot{\varphi}^2 + a^2\dot{\theta}^2\right)\right) + NV_0 a^3\varphi^{-\frac{3\alpha_0+\delta_0}{\beta_0}}. \tag{5.31}$$

Usually equation (5.17) gives the conserved current associated to this symmetry criteria. Integrating the time component of the conserved current over the spatial volume one can get conserved charge. But all the variables in the present model are time dependent. So equation (5.30) gives the conserved charge associated to this symmetry. Moreover, this conserved charge also can be expressed as the inner product of the infinitesimal generator with Cartan one form as (CAPOZZIELLO; STABILE; TROISI, 2007)

$$Q = i_{\vec{X}}\Theta_L \tag{5.32}$$

where Cartan one form Θ_L is defined as

$$\Theta_L = \frac{\partial L}{\partial \dot{a}}da + \frac{\partial L}{\partial \dot{\varphi}}d\varphi + \frac{\partial L}{\partial \dot{\theta}}d\theta \tag{5.33}$$

and $i_{\vec{X}}$ denotes the inner product with the vector field $\vec{X}$.

Now, we want to make a point transformation $(a,\varphi,\theta,N) \longrightarrow (u,v,\theta,w)$ in such a way that u becomes cyclic because cyclic variable is very much useful for solving non-linear coupled evolution equations. For the above transformation, the transformed infinitesimal generator takes the form as

$$\begin{aligned}\vec{X}_T &= \left(i_{\vec{X}}du\right)\frac{\partial}{\partial u} + \left(i_{\vec{X}}dv\right)\frac{\partial}{\partial v} + \left(i_{\vec{X}}d\theta\right)\frac{\partial}{\partial \theta} + \left(i_{\vec{X}}dw\right)\frac{\partial}{\partial w} + \left\{\frac{d}{dt}\left(i_{\vec{X}}du\right)\right\}\frac{d}{d\dot{u}} \\ &+ \left\{\frac{d}{dt}\left(i_{\vec{X}}dv\right)\right\}\frac{d}{d\dot{v}} + \left\{\frac{d}{dt}\left(i_{\vec{X}}d\theta\right)\right\}\frac{d}{d\dot{\theta}}\end{aligned} \tag{5.34}$$

For making u cyclic, one can restrict the transformation as

$$i_{\vec{X}}du = 1,\ i_{\vec{X}}dv = 0 \text{ and } i_{\vec{X}}dw = 0. \tag{5.35}$$

The explicit form of equation (5.35) can be written as

$$\begin{aligned} u &= \frac{1}{\alpha_0}\ln a, \\ v &= \ln\left(\frac{a^{\beta_0}}{\varphi^{\alpha_0}}\right), \\ w &= \ln\left(\frac{a^3\varphi^2}{N}\right). \end{aligned} \tag{5.36}$$

Equation (5.36) actually describes the relation between the old co-ordinates and new co-ordinates. Then using equation (5.36), the Lagrangian (5.8) transforms as

$$L_T = e^w\left\{A\dot{u}^2 + C\dot{v}^2 - B\dot{u}\dot{v} + \frac{1}{2}\dot{\theta}^2 - V_0 e^{-2w}e^{\frac{6v}{\beta_0}}\right\}. \tag{5.37}$$

Here A, B and C are arbitrary constants.

Now, Euler-Lagrange equations for the transformed Lagrangian can be written as

$$
\begin{aligned}
&2A\dot{u} - B\dot{v} = R \ \ \text{(constant)},\\
&\dot{\theta} = c_1 \ \ \text{(constant)},\\
&e^{w}(-B\ddot{u} + 2C\ddot{v}) + \frac{6V_0}{\beta_0}e^{-w}e^{\frac{6v}{\beta_0}} = 0,\\
&e^{w}\left(A\dot{u}^2 + C\dot{v}^2 - B\dot{u}\dot{v} + \frac{1}{2}\dot{\theta}^2 + V_0 e^{-2w}e^{\frac{6v}{\beta_0}}\right) = 0.
\end{aligned} \tag{5.38}
$$

Solving the set of equations (5.38) one can write the new variables as

$$u(t) = \frac{B\beta_0}{12A}\ln\left[\frac{c_1}{2N_1}\left\{\tanh^2\left(\frac{1}{2M}\sqrt{\frac{6Mc_1}{\beta_0}}(t+c_2)\right)-1\right\}\right] + \frac{Rt}{2A} + \frac{l}{2A}, \tag{5.39}$$

$$v(t) = \frac{\beta_0}{6}\ln\left[\frac{c_1}{2N_1}\left\{\tanh^2\left(\frac{1}{2M}\sqrt{\frac{6Mc_1}{\beta_0}}(t+c_2)\right)-1\right\}\right], \tag{5.40}$$

$$\theta(t) = c_1 t + c_3. \tag{5.41}$$

Here, $c_1, c_2, c_3, M, N_1, R, l$ are arbitrary constants. Also we may get "w" by using the relation (5.38). The consequence of this value we may determine the Lapse function.

Using the relation (5.36) and the solutions (5.39)-(5.41) one can find out the solutions of the evolution equations of Brans-Dicke cosmological model for the old variables as:

$$a(t) = Ee^{\frac{\alpha_0 Rt}{12A}}\left\{\tanh^2\left(\frac{1}{2M}\sqrt{\frac{6Mc_1}{\beta_0}}(t+c_2)\right)-1\right\}^{\frac{\alpha_0\beta_0 B}{12A}}, \tag{5.42}$$

$$\varphi(t) = Fe^{\frac{\beta_0 Rt}{12A}}\left\{\tanh^2\left(\frac{1}{2M}\sqrt{\frac{6Mc_1}{\beta_0}}(t+c_2)\right)-1\right\}^{\frac{B\beta_0^2}{12A}-\frac{\beta_0}{6\alpha_0}}, \tag{5.43}$$

$$\theta(t) = c_1 t + c_3. \tag{5.44}$$

The variation of the dimensionless scale factor $\frac{a(t)}{a_0}$ has been presented with respect to the dimensionless cosmic time $\frac{t}{t_0}$ in Figure (5.1) (here a_0 is the value of the scale factor and H_0 is the value of the Hubble parameter at the present cosmic time t_0). Similarly the dimensionless hubble parameter $\frac{H}{H_0}$ and the deceleration parameter $q(t)$ has been presented graphically with the variation of the dimensionless scale factor $\frac{a(t)}{a_0}$ in Figures (5.2)-(5.3). Figure (5.1) shows that in the present model the Universe is an expanding model with rate of expansion gradually diminishes as reflected in Figure (5.2). The graphical representation of the deceleration parameter shows that initially the Universe was in an accelerating phase, subsequently there was an era of deceleration and then again at present the Universe has entered an acceleration epoch (from Figure (5.3)). Lastly it is to be mentioned that for proper choice of the parametric symbol the present value of the deceleration parameter for the model with the observed value " – 0.55615" (AGHANIM et al., 2020).

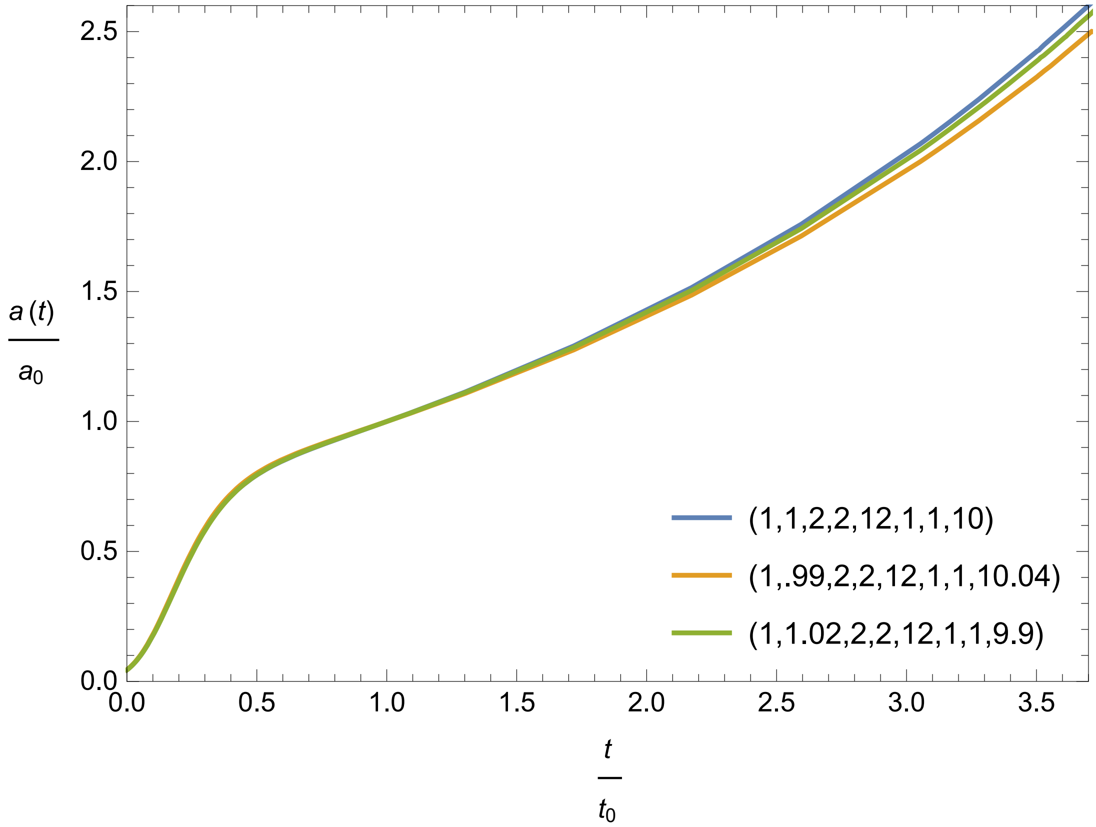

Figure 5.1 – Graphical representation of the scale factor with respect to cosmic time t for different values of $(E,M,c_1,\beta_0,\alpha_0,B,A,R)$ parameter spaces.

5.4 Formation of WD equation and wave function of the Universe: a description of quantum cosmology

In the context of quantum cosmology, the Noether symmetry condition can be rewritten as

$$\mathcal{L}_{\overrightarrow{X}_H} H = 0. \tag{5.45}$$

Here H is the Hamiltonian of the system which is very much useful to derive the Wheeler Dewitt (WD) equation and $\overrightarrow{X}_H$ is defined by

$$\overrightarrow{X}_H = \dot{q}\frac{\partial}{\partial q} + \ddot{q}\frac{\partial}{\partial \dot{q}}, \tag{5.46}$$

the symmetry vector in phase space.

In minisuperspace models of quantum cosmology, symmetry analysis can appropriately interpret the wave function of the Universe as follows:

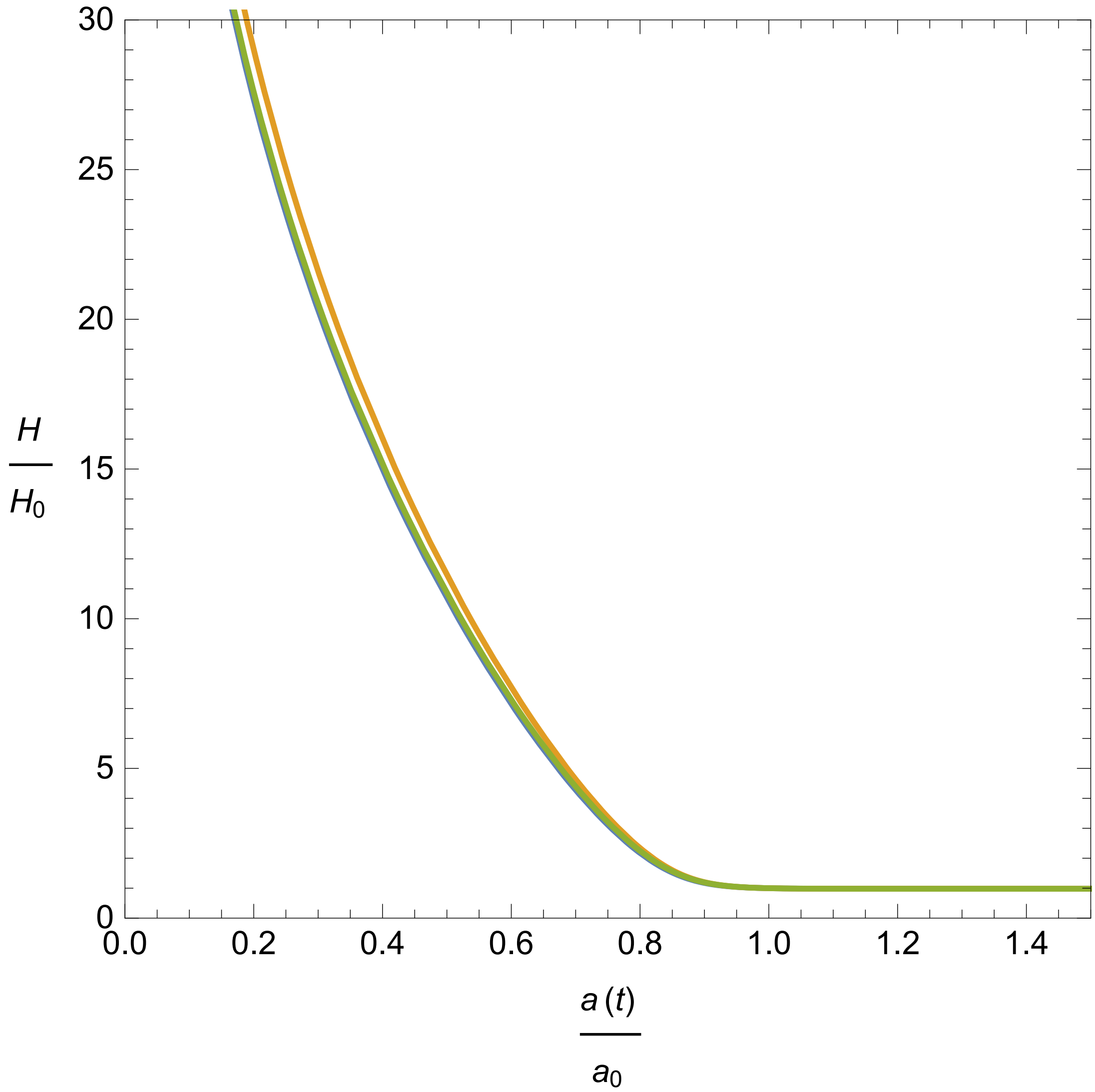

Figure 5.2 – Graphical representation of the dimensionless Hubble parameter with respect to scale factor $a(t)$.

The conserved canonically conjugate momenta can be written as

$$p_l = \frac{\partial L}{\partial \dot{q}^l} = \Sigma_l, \qquad l = 1,2,...,m \tag{5.47}$$

where m is the number of symmetries.

The operator version of equation (5.47) can be written as

$$-i\partial_{q^l} |\psi\rangle = \Sigma_l |\psi\rangle . \tag{5.48}$$

The above equation (5.48) gives an oscillatory solution which is given by

$$|\psi\rangle = \sum_{l=1}^{m} e^{i\Sigma_l q^l} |\varphi(q^k)\rangle , k < n. \tag{5.49}$$

Here k stands for the direction along which there is no symmetry. Thus the oscillatory part of the wave function indicates the existence of Noether symmetry.

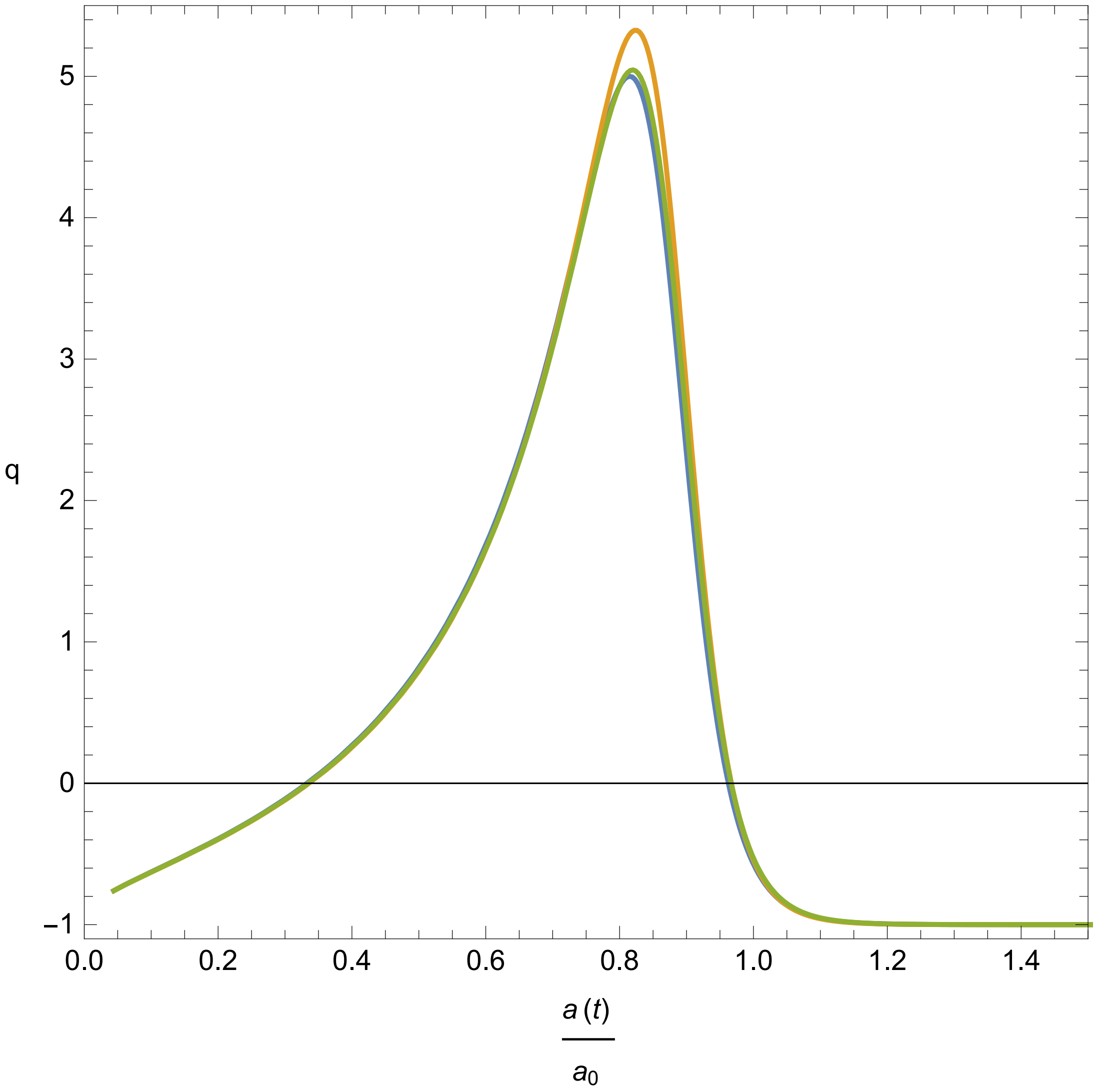

Figure 5.3 – Graphical representation of deceleration parameter $q(t)$ with respect to scale factor $a(t)$.

In 3D configuration space $\{a,\varphi,\theta\}$, the canonically conjugate momenta associated to this model can be written as

$$p_a = \frac{1}{N}(12F_0 a\varphi^2\dot{a} + 12F_0 a^2\varphi\dot{\varphi}), \quad (5.50)$$

$$p_\varphi = \frac{1}{N}(12F_0 a^2\varphi\dot{a} + a^3\dot{\varphi}), \quad (5.51)$$

$$p_\theta = \frac{1}{N}a^3\varphi^2\dot{\theta}. \quad (5.52)$$

Then the Hamiltonian of the system takes the form

$$H = N\left[\left(\frac{l_{11}}{a\varphi^2} - \frac{l_{22}}{a\varphi}\right)p_a^2 + \left(\frac{l_{33}}{a^3} - \frac{l_{22}\varphi}{a^3}\right)p_\varphi^2 + \left(\frac{l_{44}}{a^2} - \frac{l_{55}}{a^2\varphi}\right)p_a p_\varphi + \frac{p_\theta^2}{2a^3\varphi^2} + V_0 a^3 \varphi^{-\frac{3\alpha_0+\delta_0}{\beta_0}}\right] \quad (5.53)$$

where $l_{11}, l_{22}, l_{33}, l_{44}, l_{55}$ are arbitrary constants. In quantum cosmology, the wave function of the Universe is a solution of the Wheeler DeWitt (WD) equation. WD equation is a second order hyperbolic partial differential equation. Actually it is the operator version of the Hamiltonian constraint. WD equation can be written as $\hat{H}\psi(a,\varphi,\theta) = 0$, where $\hat{H}$ is the operator version of the Hamiltonian and $\psi(a,\varphi,\theta)$ is the wave function of the Universe. But in course of conversion to the operator version we have encountered a problem which is known as operator ordering problem. Here $p_a \to -i\frac{\partial}{\partial a}$, $p_\varphi \to -i\frac{\partial}{\partial \varphi}$ and $p_\theta \to -i\frac{\partial}{\partial \theta}$. Then WD equation corresponding the Hamiltonian (5.53) takes the form as

$$\Bigg[-\left(\frac{l_{11}}{a\varphi^2}-\frac{l_{22}}{a\varphi}\right)\frac{\partial^2}{\partial a^2}-\left(\frac{l_{33}}{a^3}-\frac{l_{22}\varphi}{a^3}\right)\frac{\partial^2}{\partial \varphi^2}-\left(\frac{l_{44}}{a^2}-\frac{l_{55}}{a^2\varphi}\right)\frac{\partial^2}{\partial a\partial\varphi} -\frac{1}{2a^3\varphi^2}\frac{\partial^2}{\partial\theta^2}+V_0a^3\varphi^{-\frac{3\alpha_0+\delta_0}{\beta_0}}\Bigg]\psi(a,\varphi,\theta) = 0. \tag{5.54}$$

The solution of this equation can be obtained by separation of eigen functions of the WD operator as follows: (HARTLE, 1986)

$$\psi(a,\varphi,\theta) = \int W(Q)\psi(a,\varphi,\theta,Q)dQ. \tag{5.55}$$

Here ψ is treated as the eigen function of the WD operator, Q is the conserved charge and $W(Q)$ is a weight function. But we can't get any explicit solution from the above WD equation (5.54) using separation of variables because the minisuperspace variables are highly coupled in WD operator. Thus one may analyze this model using the new variables (u,v,θ). So associated to the point-like Lagrangian (5.37), the canonically conjugate momenta can be written as (TAVAKOLI; VAKILI, 2019)

$$p_u = \frac{\partial L}{\partial \dot{u}} = e^w(2A\dot{u}-B\dot{v}) = \sigma \quad (\text{conserved}), \tag{5.56}$$

$$p_v = \frac{\partial L}{\partial \dot{v}} = e^w\,(2C\dot{v}-B\dot{u}), \tag{5.57}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = e^w\dot{\theta} = \Sigma \quad (\text{conserved}). \tag{5.58}$$

Since u and θ are cyclic in the Lagrangian (5.37), p_u and p_θ will be conserved in nature.

Then the Hamiltonian of the system takes the form as follows:

$$H = A'p_u^2 + B'p_v^2 + C'p_up_v + D'p_\theta^2 + V_0e^{-w}e^{\frac{6v}{\beta_0}}. \tag{5.59}$$

Here, A', B', C' and D' are arbitrary constants.

As mentioned earlier the issue of factor ordering in the quantization scheme is nothing but the ordering of a variable and its conjugate momentum. So with the usual operator conversion : $p_u \to -i\frac{\partial}{\partial u}$, $p_v \to -i\frac{\partial}{\partial v}$ and $p_\theta \to -i\frac{\partial}{\partial \theta}$ one gets a six parameter family of Wheeler-DeWitt(WD) equation

$$\left[A'e^{-\frac{6v}{\beta_0}}\frac{\partial^2}{\partial u^2}+B'e^{-l_1\frac{6v}{\beta_0}}\frac{\partial}{\partial v}e^{-l_2\frac{6v}{\beta_0}}\frac{\partial}{\partial v}e^{-l_3\frac{6v}{\beta_0}}+C'\frac{\partial}{\partial u}\left(e^{-m_1\frac{6v}{\beta_0}}\frac{\partial}{\partial v}e^{-m_2\frac{6v}{\beta_0}}\right)+D'\frac{\partial^2}{\partial\theta^2}-V_0e^{-w}\right]\psi(u,v,\theta) = 0$$

with the restriction $l_1 + l_2 + l_3 = 1$ and $m_1 + m_2 = 1$. As there are infinite number of possibilities for the choice of the triplet (l_1,l_2,l_3) and the dublet (m_1,m_2) so it is possible to have infinite possible ordering. In the literature, there are some preffered choices for the above triplet and dublet as follows :

i) Vilenkin operator ordering : $l_1 = l_3 = 0, l_2 = 1;\ m_1 = 0, m_2 = 1.$

ii) D'Alembert operator ordering : $l_1 = 2, l_2 = -1, l_3 = 0;\ m_1 = 2, m_2 = -1.$

iii) no ordering : $l_1 = 1, l_2 = 0 = l_3;\ m_1 = 1, m_2 = 0.$

Through the nature of the wave function depends on the choice of operator ordering, yet the semi-classical description remain unaltered (TAVAKOLI; VAKILI, 2019), (STEIGL; HINTERLEITNER, 2006). For simplicity we shall confine ourselves to the above 3rd choice (i.e., no ordering) and the WD equation takes the form

$$\left[A'\frac{\partial^2}{\partial u^2} + B'\frac{\partial^2}{\partial v^2} + C'\frac{\partial^2}{\partial u \partial v} + D'\frac{\partial^2}{\partial \theta^2} - V_0 e^{-w} e^{\frac{6v}{\beta_0}}\right]\psi(u,v,\theta) = 0. \tag{5.60}$$

For solving equation (5.60) one can use the method of separation of variable as

$$\psi(u,v,\theta) = \psi_1(u)\psi_2(v)\psi_3(\theta). \tag{5.61}$$

The operator version of (5.56) and (5.58) can be written as

$$-i\frac{\partial \psi}{\partial u} = \sigma\psi, \tag{5.62}$$

$$-i\frac{\partial \psi}{\partial \theta} = \Sigma\psi. \tag{5.63}$$

Solving (5.62) and (5.63) using (5.61) one can get

$$\psi_1(u) = \sigma_0 e^{i\sigma u}, \tag{5.64}$$

$$\psi_3(\theta) = \Sigma_0 e^{i\Sigma\theta}. \tag{5.65}$$

where σ_0 and Σ_0 are integration constants. Actually equations (5.64) and (5.65) describes the oscillatory part of the wave function.

Putting the value of $\psi_1(u)$ and $\psi_3(\theta)$ in WD equation (5.60) one gets a second order differential equation as

$$B'\frac{d^2\psi_2(v)}{dv^2} + i\sigma C'\frac{d\psi_2(v)}{dv} - V_0 e^{-w}\psi_2(v) - \left(A'\sigma^2 + D'\Sigma^2\right)\psi_2(v) = 0. \tag{5.66}$$

The solution of equation (5.66) describes the non-oscillatory part of the wave function and is given by

$$\psi_2(v) = e^{-iv\mu + \frac{A_3\pi}{2}}\left[\Gamma(1-A_3)I_{-A_3}\left(A_4 e^{\frac{3v}{\beta_0}}\right) + (-1)^{A_3}\Gamma(1+A_3)I_{A_3}\left(A_4 e^{\frac{3v}{\beta_0}}\right)\right]. \tag{5.67}$$

Here, I is the modified Bessel function of first kind and gamma function is denoted by Γ. A_2, A_3 and A_4 are arbitrary constants. So the wave function of the Universe becomes

$$\begin{aligned}\psi(u,v,\theta) &= \Sigma_0\sigma_0 e^{\frac{A_3\pi}{2}} e^{i(-v\mu+\sigma u+\Sigma\theta)}. \\ &\left[\Gamma(1-A_3)I_{-A_3}\left(A_4 e^{\frac{3v}{\beta_0}}\right)+(-1)^{A_3}\Gamma(1+A_3)I_{A_3}\left(A_4 e^{\frac{3v}{\beta_0}}\right)\right] \\ \text{i.e., } \psi(a,\varphi,\theta) &= \Sigma_0\sigma_0 e^{\frac{A_3\pi}{2}} e^{i\left(-\mu\ln\left(\frac{a^{\beta_0}}{\varphi^{\alpha_0}}\right)+\frac{\sigma}{\alpha_0}\ln a+\Sigma\theta\right)}. \\ &\left[\Gamma(1-A_3)I_{-A_3}\left(\frac{A_4 a^3}{\varphi^{\frac{\alpha_0}{\beta_0}}}\right)+(-1)^{A_3}\Gamma(1+A_3)I_{A_3}\left(\frac{A_4 a^3}{\varphi^{\frac{\alpha_0}{\beta_0}}}\right)\right] \end{aligned} \tag{5.68}$$

It is to be noted that $|\psi|^2$ gives the probability measure on the minisuperspace.

The figures (5.4) and (5.5) shows that the measure of probability depends on the sign of the constant A_3. There is finite non-zero probability at zero volume for $A_3 = 0$ while the probability at zero volume will be zero if $A_3 > 0$ or A_3 is a $-ve$ integer. However, the probability can not be defined for negative non-integral values of A_3. Thus quantum description allow the big-bang singularity for $A_3 = 0$ while quantum formulation overcomes the initial singularity for $A_3 > 0$ or a $-ve$ integer.

For WKB approximation in the semi classical limit one may write

$$\psi = \exp\left(\frac{i}{\hbar}S\right) \tag{5.69}$$

where the classical HJ function S can be expanded as power series in $\hbar$ as

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + ... \tag{5.70}$$

Thus the wave packet

$$\psi = \int S(\overrightarrow{k})\exp\left(\frac{i}{\hbar}S_0\right)d\overrightarrow{k} \tag{5.71}$$

Characterizes the classical solution with $\overrightarrow{k} = (k_1, k_2, k_3)$ as arbitrary parameters (i.e., separation constants). The above semiclassical limit in the WD equation gives the HJ equation (in zeroth order) for so as

$$\left[A'e^{-\frac{6v}{\beta_0}}\left(\frac{\partial S_0}{\partial u}\right)^2 + B'e^{-\frac{6v}{\beta_0}}\left(\frac{\partial S_0}{\partial v}\right)^2 + C'e^{-\frac{6v}{\beta_0}}\frac{\partial^2 S_0}{\partial u \partial v} + D'e^{-\frac{6v}{\beta_0}}\left(\frac{\partial S_0}{\partial \theta}\right)^2 - V_0 e^{-w}\right] = 0. \tag{5.72}$$

For an explicit solution for S_0 the following separation form is suitable

$$S_0(u,v,\theta) = S_u(u) + S_v(v) + S_\theta(\theta) \tag{5.73}$$

with

$$S_u = \frac{1}{4}\left(\frac{l_0}{\sqrt{A'}}u + c_0\right)^2, \tag{5.74}$$

$$S_\theta = \frac{1}{4}\left(\frac{m_0}{\sqrt{D'}}\theta + d_0\right)^2 \tag{5.75}$$

and

$$S_v = \beta_1^2 \left[\left(v' - k^2 e^{-\frac{6v}{\beta_0}} \right)^{\frac{3}{2}} + V_0 \right]^2 . \tag{5.76}$$

Here l_0, m_0 are separation constants and c_0, d_0 and V_0 are constants of integration. Thus the wave packet (5.71) has the explicit form as

$$\psi(u,v,\theta) = \int \int \int \mu(l_0,m_0) \exp\left[\frac{i}{\hbar} S_u(u,l_0) S_v(v,l_0,m_0) S_\theta(\theta,m_0) \right] dl_0 dm_0 \tag{5.77}$$

where μ has the bivariate Gaussian distribution having means $\bar{l}_0 > 0$, $\bar{m}_0 > 0$. For l_0, $m_0 > 0$, the wave function will oscillate rapidly if $u,\theta \to -\infty$ i.e., $a \to 0$. Thus constructive interference is possible provided

$$\left(\frac{\partial S_0}{\partial l_0}\bigg|_{l_0=\bar{l}_0} \right)^2 + \left(\frac{\partial S_0}{\partial m_0}\bigg|_{m_0=\bar{m}_0} \right)^2 = 0. \tag{5.78}$$

This is supported by the classical solutions. Thus as usual the classical limit will be obtained when $a \to 0$.

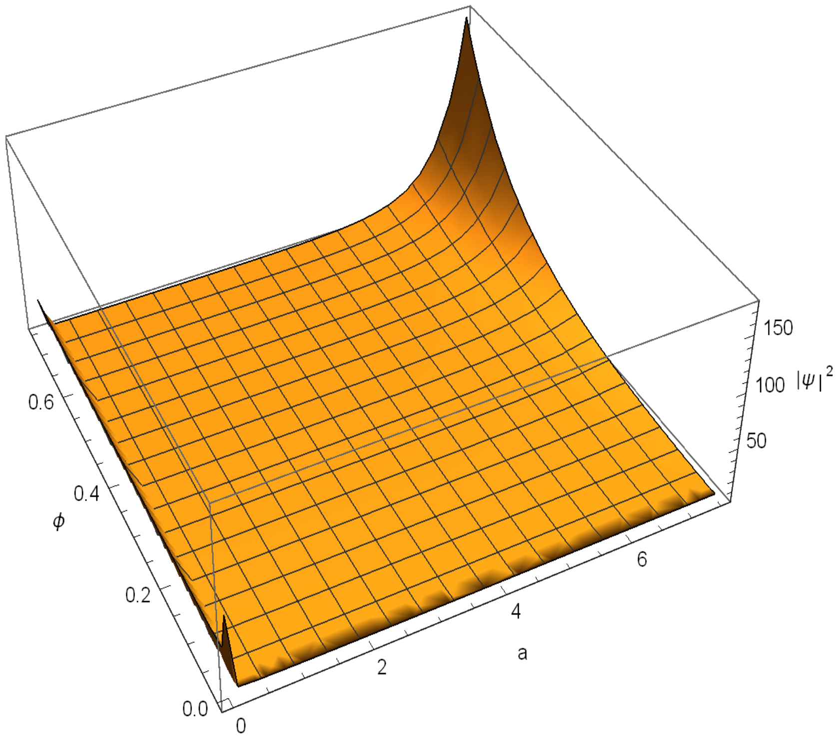

Figure 5.4 – The graphical representation of $|\psi|^2$ when $(\alpha_0,\beta_0,A_3,A_4) = (1.07, -1.4, .035, .01)$.

In metric formulation of Einstein gravity, there are four constraints of which the super momentum constrain (a group of three) or the vector constraint vanish identically for minisuperspaces homogeneous in nature. The only non-trivial constraint equation is known as Hamiltonian constraint or scalar constraint having operator form in quantum version is nothing but the WD equation

$$\hat{H}\left(\hat{q}_\alpha(t), \hat{p}^\alpha(t) \right) \psi(q_\alpha) = 0. \tag{5.79}$$

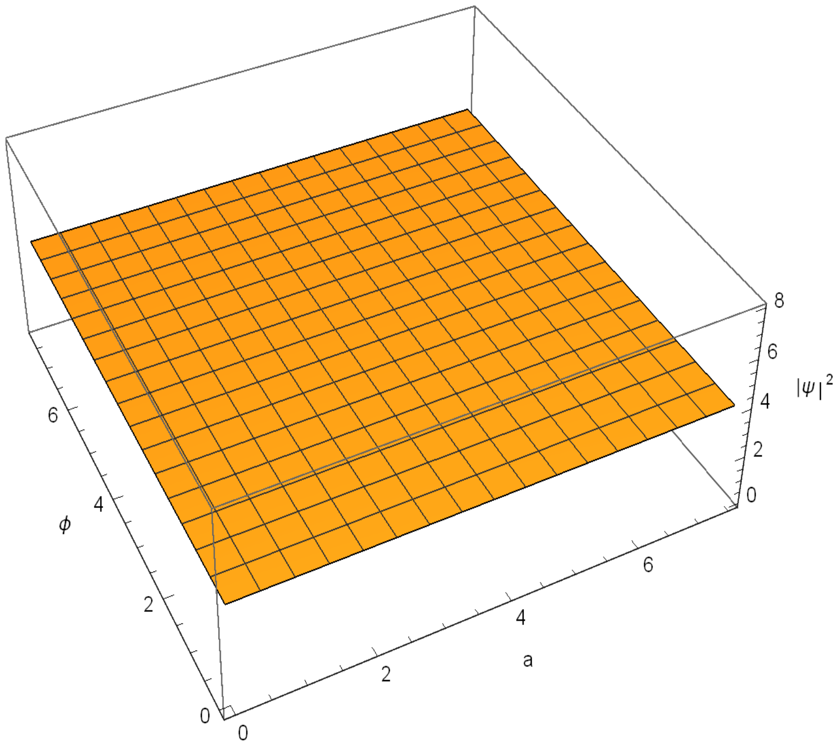

Figure 5.5 – The graphical representation of $|\psi|^2$ when $(\alpha_0,\beta_0,A_3,A_4) = (1.07,-1.4,0,0)$.

The homogeneous degrees of freedom $q_\alpha(t)$ and $p^\alpha(t)$ can be deduced from the three metric h_{ij} and the conjugate momenta π^{ij}. Now analogous to WKB approximation if the wave function ψ can be written as

$$\psi = A(q)\exp\left[\frac{i}{\hbar}B(q)\right] \tag{5.80}$$

then from the WD equation one gets the quantum modified Hamilton-Jacobi equation as

$$\frac{1}{2}l_{\mu\nu}(q_\alpha)\frac{\partial S}{\partial q_\mu}\frac{\partial S}{\partial q_\nu} + \varphi(q_\mu) + \xi(q_\mu) = 0 \tag{5.81}$$

where $l_{\mu\nu}$ is the reduced supermetric to the given minisuperspace (CHAKRABORTY, 2001a), φ is the particularization of the scalar curvature density $\left(-h^{\frac{1}{2}\,(3)}R\right)$ of the spacelike hypersurfaces and

$$\xi(q_\mu) = -\frac{1}{R}l_{\mu\nu}\frac{\partial^2 R}{\partial q_\mu \partial q_\nu} \tag{5.82}$$

is termed as quantum potential.

Now, due to causal interpretation in quantum cosmology the trajectories $q_\alpha(t)$ should be real and independent of any observations. Such trajectories are identified by the corresponding HJ equation. By identifying

$$p^\alpha = \frac{\partial S}{\partial q_\alpha} \tag{5.83}$$

with the usual momentum-velocity relation

$$p^{\alpha} = l^{\alpha\beta}\frac{1}{N}\frac{\partial q_{\beta}}{\partial t}, \tag{5.84}$$

the first order trajectories namely

$$\frac{\partial S}{\partial q_{\alpha}} = l^{\alpha\beta}\frac{1}{N}\frac{\partial q_{\beta}}{\partial t} \tag{5.85}$$

are termed as Bohmian trajectories (i.e., quantum trajectories). One may note that these quantum trajectories are invariant under time reparametrization (CHAKRABORTY, 2001a).

In the present context the Bohmian trajectories are characterized by (choosing $\omega = 0$)

$$\frac{\partial S}{\partial u} = 2A\dot{u} - B\dot{v}, \frac{\partial S}{\partial v} = 2C\dot{v} - B\dot{u}, \frac{\partial S}{\partial \theta} = \dot{\theta}. \tag{5.86}$$

Also the quantum corrected HJ equation takes the form

$$A'\left(\frac{\partial S}{\partial u}\right)^2 + B'\left(\frac{\partial S}{\partial v}\right)^2 + C'\frac{\partial^2 S_0}{\partial u \partial v} + D'\left(\frac{\partial S}{\partial \theta}\right)^2 - V_0 e^{\frac{6v}{\beta_0}} + \xi = 0 \tag{5.87}$$

with

$$\xi = \frac{1}{A}\left[A'\frac{\partial^2 A}{\partial u^2} + B'\frac{\partial^2 A}{\partial v^2} + C'\frac{\partial^2 A}{\partial u \partial v} + D'\frac{\partial^2 A}{\partial \theta^2}\right] \tag{5.88}$$

as the explicit form of the quantum potential.

5.5 A Brief Summary

The present chapter deals with a multiscalar field cosmological model where one scalar field is non-minimally coupled to both gravity and the other scalar field while the second scalar field is minimally coupled to gravity. Due to highly coupled and non-linear field equations the present model cannot be studied cosmologically in the usual way. However, here it is shown how the symmetry analysis specifically the Noether symmetry helps us to analyze the present cosmological model both classically and quantum mechanically. The identification of a cyclic variable through the symmetry vector simplifies the Lagrangian to a great extend so that the field equations become solvable. From the classical solution, the relevant cosmological parameters are plotted in figures (5.1) – (5.3). Most specifically figure (5.1) shows that the Universe is expanding through out the evolution. The graph of the dimensionless Hubble parameter in figure (5.2) indicates that though the Universe is expanding but the rate of expansion gradually diminishes with the evolution. Figure (5.3) indicates that the present model describes all the three phases of evolution (initially accelerating era, subsequently decelerating phase and lastly the present accelerated expansion) after the big-bang. Also with proper choice of the parameter involved the present theoratical prediction of the deceleration parameter mathces the observed value. So one may say that the present model agrees with observation at least qualitatively. The Noether symmetry takes a crucial role in analyzing quantum cosmology with canonical quantization. The fundamental equation in quantum cosmology namely the WD equation is a 2nd order hyperbolic type p.d.e. The operator version of the conserved charge not only identifies the periodic part of the wave function but also helps to solve the WD equation. As a result, one may examine whether the big-bang singularity may be eliminated by quantum description or not. Here it is found that with proper choice of the parameter A_3, the singularity may be avoided by quantum formulation. Therefore, Noether symmetry analysis plays a crucial role to study any cosmological model.

6 Noether Symmetry Analysis in $f(T, T_G)$ Gravity Theory : A Study of Classical Cosmology

6.1 Prelude

In standard cosmology, the observationally supported cosmic evolution is due to ΛCDM model. According to this model, the evolution of the universe (MISNER; WHEELER, 1932; CLIFTON et al., 2012; AGHANIM et al., 2020) starting form initial big-bang singularity is driven by an inflationary phase, followed by matter dominated decelerated era and subsequently the present state of accelerated expansion, due to dark energy (DE) (RIESS et al., 1998; PERLMUTTER et al., 1999). Though observationally cosmological constant is the best choice for DE, still it has two long standing unresolved issues, namely the cosmological constant problem (due to mismatch of the theoretical and observationally measured value) and the coincidence problem. The other matter component in the model namely the cold dark matter (CDM) is effective on the galactic scale and is responsible for the large scale structure formation of the universe. In spite of several attempts for particle model of CDM, none is observationally supported (BAUDIS, 2016; BERTONE; HOOPER; SILK, 2005). For the last few years the observational evidences show a new challenge for ΛCDM. There is a growing disagreement between the local measurements of the Hubble constant H_0 (which are independent of cosmological theory) (RIESS et al., 2019; WONG et al., 2020) and the predicted value of H_0 using early universe data matching with this model (BERNAL; VERDE; RIESS, 2016; VALENTINO et al., 2021c; VALENTINO et al., 2021a). Not only the H_0 tension appears to be an increasing problem in ΛCDM, there are other tensions in cosmology which are also present in ΛCDM model (RIESS et al., 2022; BROUT et al., 2022; SCOLNIC et al., 2022; ABDALLA et al., 2022; VALENTINO et al., 2021b).

To resolve the above issues there are two ways of modification of the standard cosmology. The first and the obvious one is the modification in the matter part so that the evolution history at different epochs gets alter appropriately. Alternatively one may consider distinct description of gravity in standard cosmological model (CLIFTON et al., 2012; CAPOZZIELLO; LAURENTIS, 2011; AKRAMI et al., 2021; NOJIRI; ODINTSOV, 2011). The simplest way of doing such modification is to alter the Einstein-Hillbert action (CAPOZZIELLO; LAURENTIS, 2011; FARAONI, 2008) so that the gravitational field still interacts with matter due to Levi-Civita connection, the source of curvature in GR (MISNER; WHEELER, 1932;

NAKAHARA, 2003). The torsion can be considered as an alternative connection (known as teleparallel connection) for expressing the gravitational interactions and such a gravity theory is termed as teleparallel gravity (TG) (ALDROVANDI; PEREIRA, 2012; CAI et al., 2016; KRSSAK et al., 2019; BAHAMONDE et al., 2023). In this formulation as the connection is not associated with any form of curvature so all curvature related measures are chosen to be identically zero. (Note that still the usual Ricci scalar is non-zero). Also similar to the Ricci scalar there is associated a scalar T termed as torsion scalar and the difference between these two scalars is a surface term (i.e., boundary term). As a result, the gravity theory constructed due to torsion scalar T is known as teleparallel equivalent of general relativity (TEGR) and the equations are dynamically equivalent to Einstein gravity.

In this context, a natural question arises "is it possible to modify the TEGR in accordance with the modification of GR ?". The answer is yes. In analogy to the simplest modification i.e., $f(R)$ gravity theory there is $f(T)$ gravity theory where the teleparallel Lagrangian T is replaced by an arbitrary function $f(T)$ (BAHAMONDE et al., 2023; HOHMANN, 2023; HOHMANN et al., 2018; CHAKRABORTTY MANAS;SK, 2023). However, the interesting feature is that $f(T)$ is a distinct gravitational modification without an equivalent curvature analog (CAI et al., 2016; Saridakis, 2018). A further extension of TEGR is to include higher order torsion invariant. In curvature formalism an higher order curvature invariant is the Gauss-Bonnet (GB) scalar (CRUZ-DOMBRIZ et al., 2017; CRUZ-DOMBRIZ et al., 2018; KOFINAS; SARIDAKIS, 2014)

$$S_{GB} = \bar{R}^2 - 4\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \bar{R}_{\mu\nu\lambda\rho}\bar{R}^{\mu\nu\lambda\rho} \tag{6.1}$$

where an over bar indicates the curvature term in Levi-civita connection. Note that the Gauss-Bonnet term can act as the source of inflation and also identifies the dynamical dark energy. S_{GB} is only nontrivial in $4 + D$ dimension. In $4D$ it reduces to a topological surface term . Though S_{GB} is quadratic in the Reimann tensor (and Ricci tensor) terms i.e., terms containing more than $2nd$ order partial derivatives of the metric cancel out, making the field equations 2nd order quasi linear partial differential equation in the metric. So there is no additional dynamical degrees of freedom as in $f(R)$-gravity. Also GB gravity has been connected to the classical electrodynamics by means of complete gague invariance w.r.t. Noether theorem. The analogous Gauss-Bonnet term in torsion gravity is the invariant scalar T_G (defined in the next section). The motivation for generalizing TEGR to $f(T)$ and then to the TG nonminimally coupled scalar seems purely algebric in nature.

As the present modified gravity theory is highly coupled and complicated in nature so cosmological investigation with this gravity theory by analyzing classical solution is very much tedious. The mathematical technique of symmetry analysis to differential equations has a crucial role in this context. More specifically, Noether symmetry analysis is very effective in two ways, namely (i) the identification of the symmetry vector results a transformation in the augmented space so that it may be possible to have a cyclic coordinate so that the physical Lagrangian and the field equations get simplified to a great extend and even they may be solvable, (ii) the conserved quantity due to Noether's theory may identify the true physical system among the similar physical systems and also the conserved quantity may have a role in the process of solving the field equations. Further, in the context of quantum cosmology, the conserved momenta due to Noether symmetry identifies the oscillatory part of the wave function and as a result it may me possible to have solution of the Wheeler-DeWitt equation.

In this present chapter both classical and quantum cosmology based on $f(T,T_G)$ gravity theory has been studied. As the classical field equations are highly non-linear and coupled in nature, so for cosmological investigation Noether symmetry analysis has been useful to simplify the Lagrangian as well as the field equations so that they may be solvable for cosmological investigation. The plan of the chapter is as follows: section-**6.2** describes an overview of the Teleparallel gravity while the basic features of Noether symmetry analysis has been presented in section-**6.3**. Section-**6.4** deals with symmetry analysis for $f(T,T_G)$ gravity. As a result the Lagrangian as well as the field equations simplify to a great extend so that the classical solutions are obtained in this section. Finally the quantum cosmology has been studied end section-**6.5**. The chapter ends with a brief discussion of the classical cosmological solutions and results of the quantum cosmology in section-**6.6**.

6.2 The teleparallel gravity : An overview

In teleparallel gravity (TG) theory the curvature term in GR is replaced by torsion term T through the use of the teleparallel connection rather than the Levi-Civita connection (BAHAMONDE et al., 2023) in GR. It is to be noted that though the teleparallel connection satisfies the metricity, but the Riemann tensor for this connection vanishes (the regular Levi-Civita Riemann tensor does not vanish) (ALDROVANDI; PEREIRA, 2012; CAI et al., 2016; KRSSAK et al., 2019). Using the tetrads e^i_μ (with $\bar{e}^\mu_i$ as its inverse) one may write down the metric both on the manifold as well as on the tangent space as

$$g_{\alpha\beta} = e^i{}_\alpha e^j{}_\beta \eta_{ij}, \quad \eta_{ij} = \bar{e}_i{}^\alpha \bar{e}_j{}^\beta g_{\alpha\beta}. \tag{6.2}$$

The orthogonality property of the tetrads are

$$e^i{}_\alpha \bar{e}_j{}^\alpha = \delta^i{}_j, \quad e^i{}_\alpha \bar{e}_i{}^\beta = \delta_\alpha{}^\beta. \tag{6.3}$$

For convenience we have used the latin indices to represent co-ordinates on the tangent space and indices on the general manifold are represented by Greek indices. Note that tetrads may be used in curvature-based gravity but they are usually suppressed due to their non flat nature in these settings while they are considered as flat connection in TG theory. Now the teleparallel connection can be defined in terms of these tetrad as

$$\Gamma^{\alpha}{}_{\beta\delta} = \bar{e}_i{}^{\alpha}\left(\partial_\delta e^i{}_\beta + w^i{}_{j\delta} e^j{}_\beta\right). \tag{6.4}$$

Here the spin connection $w^i{}_{\beta\delta}$ has six degrees of freedom of the local Lorentz invariance. As a result the tetrad-spin connection pair corresponds to equations of motion that uses both the Lorentz and gravitational degrees of freedom. Then one can define the torsion tensor and the contortion tensor as

$$\begin{aligned} T^{\sigma}{}_{\mu\nu} &= 2\Gamma^{\sigma}{}_{[\gamma\mu]}, \\ K^{\sigma}{}_{\mu\nu} &= \Gamma^{\sigma}{}_{\mu\gamma} - \tilde{\Gamma}^{\sigma}_{\mu\gamma} = \frac{1}{2}\left(T_{\mu}{}^{\sigma}{}_{\gamma} + T_{\gamma}{}^{\sigma}{}_{\mu} - T^{\sigma}{}_{\mu\gamma}\right) \end{aligned} \tag{6.5}$$

where $\Gamma^{\sigma}{}_{\mu\gamma}$ is the Levi-Civita connection and the square bracket stands for antisymmetrization. Note that torsion tensor is a measure of the antisymmetry of the connection (ALDROVANDI; PEREIRA, 2012) while the contortion tensor is related to curvature and torsional quantities. Now, contraction of the torsion gives the torsion scalar as

$$T = \frac{1}{4}T^{\alpha}{}_{\mu\gamma}T_{\alpha}{}^{\mu\gamma} + \frac{1}{2}T^{\alpha}{}_{\mu\gamma}T^{\gamma\mu}{}_{\alpha} - T^{\alpha}{}_{\mu\alpha}T^{\beta\mu}{}_{\beta}. \tag{6.6}$$

Thus the action integral with this torsion scalar results the same equations of motion as Einstein-Hilbert action except a total divergence term. The earlier works on Noether symmetries in $f(T, T_G)$ cosmology in ref (KADAM; MISHRA; SAID, 2023; CAPOZZIELLO; LAURENTIS; DIALEKTOPOULOS, 2016) can be briefly describe as follows.

Though the teleparallel connection has zero curvature (i.e., Ricci scalar vanishes), still the Ricci scalar in Levi-Civita connection i.e., $\hat{R} = \hat{R}\left(\hat{\Gamma}^{\sigma}_{\mu\gamma}\right) \neq 0$ is related to the torsion scalar as

$$\hat{R} = B - T \tag{6.7}$$

where B is the total divergence term, defined as

$$B = \frac{2}{e}\partial_\rho\left(eT^{\mu}{}_{\mu}{}^{\rho}\right) \tag{6.8}$$

with $e = det(e^i{}_\mu) = \sqrt{-g}$, the determinant of the tetrad. Note that the relation (6.7) is responsible for identical equations of motion in the classical regime (i.e., dynamical equivalent).

In torsion gravity the above Gauss-Bonnet term has the complicated expression as

$$T_G = \left(K_a{}^i{}_e K_b{}^{ej} K_c{}^k{}_f K_d{}^{fl} - 2K_a{}^{ij} K_b{}^k{}_e K_c{}^e{}_f K_d{}^{fl} + 2K_a{}^{ij} K_b{}^k{}_e K_f{}^{el} K_d{}^f{}_c + 2K_a{}^{ij} K_b{}^k{}_e K_{c,d}{}^{el}\right)\delta^{abcd}_{ijlk} \tag{6.9}$$

with $\delta^{abcd}_{ijkl} = \varepsilon^{abcd}\varepsilon_{ijkl}$, the generalized Kronecker delta function (BAHAMONDE; BÖHMER, 2016). The above Gauss-Bonnet(GB) terms in the two gravity theories are related by the relation

$$G = -T_G + B_G \tag{6.10}$$

with $B_G = \frac{1}{e}\delta^{abcd}_{ijlk}\partial_a\left[K_b{}^{ij}\left(K_c{}^{kl}{}_{,d} + K_d{}^{m}{}_{c}K^{kl}_{m}\right)\right]$, a total divergence term. It is worthy to mention that GB term can be considered as responsible for inflation as well as for identifying dynamical dark energy. Thus it is reasonable to generalized Einstein-Hillbert action by considering both the torsion scalar as well as the GB term in the action integral as

$$S_{f(T,T_G)} = \frac{1}{2\kappa^2}\int d^4xeNf(T,T_G) + \int d^4xeN\mathcal{L}_m \tag{6.11}$$

with $\kappa = 8\pi G$ and $\mathcal{L}_m$, the matter Lagrangian (Jordan frame). Note that the lapse function $N(t)$ is chosen to be unity without any loss of generality. The variation of the action with respect to the lapse function gives as the constraint equation which we have considered in the field equations both for classical cosmology (equation (6.54)), as well as in quantum description (which gives Hamiltonian of the system). In the evolution equations with the above action the second and fourth order contributions to the field equations correspond to torsion scalar and the GB term respectively.

In this chapter this modified gravity theory has been considered for homogeneous and isotropic FLRW model with the choice of the tetrad as

$$e^i{}_\mu = diag(N,a(t),a(t),a(t)) \tag{6.12}$$

having $a(t)$ as the scale factor. The above tetrad is compatible with the so-called Weitzenböck gauge with $w^i{}_{j\alpha} = 0$. Thus the torsion scalar and GB scalar has the explicit form as

$$T = \frac{6H^2}{N^2} \quad \text{and} \quad T_G = \frac{24H^2}{N^4}(\dot{H} + H^2). \tag{6.13}$$

Then the action (6.11) (neglecting the matter term) can be written to a canonical point-like action using Lagrange multiplier as

$$S_{f(T,T_G)} = \frac{1}{2\kappa^2}\int dtN\left[a^3f(T,T_G) - \lambda_1\left\{T_G - \frac{24H^2}{N^4}(\dot{H} + H^2)\right\} - \frac{\lambda_2}{N^2}(T + 6H^2)\right]. \tag{6.14}$$

The Lagrangian multipliers λ_1 and λ_2 are given by

$$\lambda_1 = a^3\frac{\partial f}{\partial T_G}, \lambda_2 = a^3\frac{\partial f}{\partial T},$$

using the variation of the action with respect to T_G and T respectively. As a result, the above action takes the form

$$S_{f(T,T_G)} = \int \frac{dt}{2\kappa^2}a^3N\left[f(T,T_G) - f_{T_G}\left(T_G - \frac{24H^2}{N^4}(\dot{H} + H^2)\right) - f_T\left(T + \frac{6H^2}{N^2}\right)\right]. \tag{6.15}$$

If the total derivative terms are omitted then the Lagrangian takes the point-like canonical form

$$L = a^3N(f - T_G f_{T_G} - T f_T) - \frac{8\dot{a}^3}{N^3}(\dot{T}_G f_{T_G T_G} + \dot{T} f_{T_G T}) - \frac{6 f_T a \dot{a}^2}{N^2}. \tag{6.16}$$

Hence Friedmann equations has the explicit form as (choosing $N = 1$ without loss of generality)

$$f - 12H^2 f_T - T_G f_{T_G} + 24H^3 \dot{f}_{T_G} - 2\kappa^2\rho = 0, \tag{6.17}$$

$$f - 4(\dot{H} + 3H^2) f_T - 4H\dot{f}_T - T_G f_{T_G} + \frac{2}{3H}\dot{T}_G \dot{f}_{T_G} + 8H^2 \ddot{f}_{T_G} + 2\kappa^2 \rho = 0 \tag{6.18}$$

where ρ, p are the energy density and the thermodynamic pressure of the matter and an overdot indicates differentiation with respect to time.

6.3 Basic Features in Noether Symmetry Analysis: Classical and Quantum cosmology

The Noether symmetry analysis to a physical system is based on a result due to Noether (known as Noether 1st theorem) in (DUTTA; LAKSHMANAN; CHAKRABORTY, 2021; DUTTA; LAKSHMANAN; CHAKRABORTY, 2019; LAYA; BHAUMIK; CHAKRABORTY, 2023; LAYA; DUTTA; CHAKRABORTY, 2023; LAYA et al., 2023b; LAYA et al., 2023a). According to Noether, if there exists vector field $\overrightarrow{X}$ in the augmented space (having dimension n) such that the Lagrangian of the physical system remains invariant with respect to Lie derivative along the vector field then the physical system is associated with some conserved quantity. Mathematically, if

$$\mathcal{L}_{\overrightarrow{X}} L = \overrightarrow{X} L = 0, \tag{6.19}$$

then there is a conserved quantity (known as conserved current)

$$Q^i = \eta^\mu \frac{\partial L}{\partial(\partial_i q^\mu)}, \quad \mu = 1,2,3,\ldots,n. \tag{6.20}$$

Here $L = L[q^\mu(x_i), \partial_j(q^\mu(x_i))]$, $q^\mu(x_i)$ are the generalised coordinates, $\eta^\mu(q^\beta)$ are some unknown functions in the augmented space and

$$\overrightarrow{X} = \eta^\mu \frac{\partial}{\partial q^\mu} + (\partial_j \eta^\mu) \frac{\partial}{\partial(\partial_j q^\mu)} \tag{6.21}$$

is the symmetry vector. Note that the conservation of Q^i (*i.e.*, $\partial_j Q^j = 0$) is assured due to Euler-Lagrange equations. Thus Noether symmetry analysis is very much useful to determine the conserved quantities of any physical system.

Further, the symmetry vector can be used to identify the cyclic variable in the augmented space and as a result both the Lagrangian and the field equations get simplified appropriately. For this it is desirable to have a transformation in the augmented space: $\{q^\alpha\} \to \{Q^\alpha\}$ such that

$$i_{\overrightarrow{X}} dQ^1 = 1 \text{ and } i_{\overrightarrow{X}} dQ^i = 0, \quad i = 2,3,...n \tag{6.22}$$

and as result the transformed symmetry vector takes the simple form: $\overrightarrow{X}_T = \frac{\partial}{\partial Q^1}$ while the conserved charge has the compact form

$$Q = i_{\overrightarrow{X}} \theta_L \tag{6.23}$$

where the one Cartan one-form θ_L has the expression

$$\theta_L = \frac{\partial L}{\partial q^\alpha} dq^\alpha \tag{6.24}$$

and $i_{\overrightarrow{X}}$ indicates the inner product between the vector field $\overrightarrow{X}$ and the given one form.

Moreover, if the Lagrangian of a physical system is explicitly time independent then the energy function

$$E = \dot{q}^\mu \frac{\partial L}{\partial \dot{q}^\mu} - L \tag{6.25}$$

is known as the Hamiltonian of the system and is also a constant of motion. Further, if the conserved quantities due to symmetry approach has some physical analogy then the symmetry approach can identify a reliable model. Lastly, by imposing the symmetry constraints to a coupled physical system, either system becomes solvable or the system is simplified to a great extend.

From the point of view of quantum cosmology as the Hamiltonian formulation is very useful so one may write down the symmetry condition as

$$\mathcal{L}_{\overrightarrow{X}_H} H = 0 \tag{6.26}$$

with $\overrightarrow{X}_H = \dot{q}^\mu \frac{\partial}{\partial q^\mu} + \dot{p}^\mu \frac{\partial}{\partial p^\mu}$ as the symmetry vector. Further, in minisuperspace models of quantum cosmology the symmetry analysis has an extra advantage namely identification of the periodic part of the wave function of the universe. The associated conserved quantity due to Noether symmetry is the canonically conjugate momenta in the minisuperspace and one may write

$$\Pi_\mu = \frac{\partial L}{\partial \dot{q}^\mu} = \Sigma_\mu, \text{ a constant} \tag{6.27}$$

with $\mu = 1,2,...,m$, the number of symmetries. Thus in quantization process the operator version of the above conserved momentum takes the form of a 1st order differential equations as

$$-i \frac{\partial}{\partial q^\mu} |\psi\rangle = \Sigma_\mu |\psi\rangle \tag{6.28}$$

where $|\psi\rangle$ is known as the wave function of the Universe. The above equation can be interpreted as a translation along q^{μ} axis due to symmetry. Moreover, the above equation-(6.28) has a simple periodic solution for real Σ_{μ} and the wave function of the universe takes the form

$$|\psi\rangle = \sum_{\mu=1}^{m} e^{i\Sigma_{\mu}q^{\mu}} |\varphi(q^{\alpha})\rangle, \quad \alpha < n \tag{6.29}$$

where k stands for the directions along which there is no symmetry and n is the dimension of the minisuperspace. Thus Noether symmetry analysis identifies the periodic part of the solution of the WD equation. Further, according to Hartle (COMPANY, 1987) the wave function of the universe with conserved momenta (due to symmetry) can be associated with classical trajectories as solution of classical cosmology.

6.4 Classical solution of $f(T,T_G)$ cosmology

This section contains the main results for the application of Noether's theorem to the Gauss-Bonnet generalization. As before the lapse function $N = N(t)$ is chosen to be unity without loss of generality. The configuration space for the present model is a $3D$ space (a,T,T_G). In the background of homogeneous and isotropic flat FLRW spacetime model, the Lagrangian of the above cosmological model is given by

$$\begin{aligned} L(a,T,T_G) &= a^3\left(f - Tf_T - T_G f_{T_G}\right) + 6a\dot{a}^2 f_T - 8\dot{a}^3 \dot{f}_{T_G} \\ &= a^3\left(f - Tf_T - T_G f_{T_G}\right) + 6a\dot{a}^2 f_T - 8\dot{a}^3\left(\dot{T}_G f_{T_G T_G} + \dot{T} f_{TT_G}\right). \end{aligned} \tag{6.30}$$

Let us take $T = \varphi$, $T_G = \psi$ (for simplicity of notation) then the above Lagrangian takes the form

$$L(a,\varphi,\psi) = a^3\left(f - \varphi f_{\varphi} - \psi f_{\psi}\right) + 6a\dot{a}^2 f_{\varphi} - 8\dot{a}^3\left(\dot{\psi} f_{\psi\psi} + \dot{\varphi} f_{\varphi\psi}\right). \tag{6.31}$$

For solving the Euler-Lagrange equations we use Noether symmetry. Now, the infinitesimal generator for Noether symmetry can be written as

$$\vec{X} = \alpha\frac{\partial}{\partial a} + \beta\frac{\partial}{\partial\varphi} + \gamma\frac{\partial}{\partial\psi} + \dot{\alpha}\frac{\partial}{\partial\dot{a}} + \dot{\beta}\frac{\partial}{\partial\dot{\varphi}} + \dot{\gamma}\frac{\partial}{\partial\dot{\psi}} \tag{6.32}$$

where, $\alpha = \alpha(a,\varphi,\psi)$, $\beta = \beta(a,\varphi,\psi)$ and $\gamma = \gamma(a,\varphi,\psi)$ are the coefficient function of the infinitesimal generator also dot means the derivative with respect to cosmic time t i.e.,

$$\begin{aligned} \dot{\alpha} &= \frac{\partial\alpha}{\partial a}\dot{a} + \frac{\partial\alpha}{\partial\varphi}\dot{\varphi} + \frac{\partial\alpha}{\partial\psi}\dot{\psi}, \\ \dot{\beta} &= \frac{\partial\beta}{\partial a}\dot{a} + \frac{\partial\beta}{\partial\varphi}\dot{\varphi} + \frac{\partial\beta}{\partial\psi}\dot{\psi}, \\ \dot{\gamma} &= \frac{\partial\gamma}{\partial a}\dot{a} + \frac{\partial\gamma}{\partial\varphi}\dot{\varphi} + \frac{\partial\gamma}{\partial\psi}\dot{\psi}. \end{aligned} \tag{6.33}$$

Now we impose Noether symmetry to this Lagrangian, i.e., equation (6.19). So α,β and γ will satisfying the set of partial differential equations

$$\begin{aligned}
\alpha f_{\varphi} + a\beta f_{\varphi\varphi} + a\gamma f_{\psi\varphi} + 2af_{\varphi}\alpha_a &= 0, && (6.34)\\
\beta f_{\varphi\varphi\psi} + \gamma f_{\psi\varphi\psi} + 3f_{\varphi\psi}\alpha_a + f_{\varphi\psi}\beta_{\varphi} + f_{\psi\psi}\gamma_{\varphi} &= 0, && (6.35)\\
\beta f_{\varphi\psi\psi} + \gamma f_{\psi\psi\psi} + 3f_{\psi\psi}\alpha_a + f_{\varphi\psi}\beta_{\psi} + f_{\psi\psi}\gamma_{\psi} &= 0, && (6.36)\\
f_{\varphi\psi}\beta_a + f_{\psi\psi}\gamma_a &= 0, && (6.37)\\
f_{\psi\psi}\alpha_{\varphi} + f_{\varphi\psi}\alpha_{\psi} &= 0, && (6.38)\\
f_{\varphi\psi}\alpha_{\varphi} &= 0, && (6.39)\\
f_{\psi\psi}\alpha_{\psi} &= 0, && (6.40)\\
f_{\varphi}\alpha_{\varphi} &= 0, && (6.41)\\
f_{\varphi}\alpha_{\psi} &= 0, && (6.42)\\
3\alpha(f - \varphi f_{\varphi} - \psi f_{\psi}) - a\beta(\varphi f_{\varphi\varphi} + \psi f_{\varphi\psi}) - a\gamma(\varphi f_{\psi\varphi} + \psi f_{\psi\psi}) &= 0. && (6.43)
\end{aligned}$$

To solve the above set of partial differential equations one can use the method of separation of variable. So, α, β, γ takes the form

$$\begin{aligned}
\alpha &= \alpha(a,\varphi,\psi) = \alpha_1(a)\alpha_2(\varphi)\alpha_3(\psi),\\
\beta &= \beta(a,\varphi,\psi) = \beta_1(a)\beta_2(\varphi)\beta_3(\psi),\\
\gamma &= \gamma(a,\varphi,\psi) = \gamma_1(a)\gamma_2(\varphi)\gamma_3(\psi). && (6.44)
\end{aligned}$$

Therefore, the solution of the above set of partial differential equations (6.34)-(6.43) is given by

$$\alpha = \alpha_0,\ \beta = -2\alpha_0\frac{\varphi}{a},\ \gamma = -3\alpha_0\frac{\psi}{a},\ f(\varphi,\psi) = f_0\varphi^{l}\psi^{1-\frac{2l}{3}} \qquad (6.45)$$

where α_0, f_0 and l are arbitrary constants.

Thus we can say Noether symmetry analysis not only gives the symmetry vector but also the coupling function can be determined with the help of symmetry criteria. One of the most beautiful features of Noether symmetry is that there are some conserved quantities associated to the system. For a field theory in a augmented space there is no well define notion of energy. The conserved quantity associated to the symmetry is the energy-momentum tensor. It is well known that, FLRW space-time have no time-like killing vector. So, for usually FLRW model there are no conserved energy. But as the Lagrangian for this model has not explicit time dependence, so analogically we can say that it is possible to have some conserved quantity for this point-like Lagrangian. Thus the possible conserved quantities associated to the above point-like Lagrangian, are

$$\begin{aligned}
Q &= \alpha_0 f_0\left[12\dot{a}a\psi^{\frac{1}{3}} - \frac{8}{3}a^2\varphi\psi^{-\frac{2}{3}}\left(3\frac{\dot{\varphi}}{\varphi} - 2\frac{\dot{\psi}}{\psi}\right)\right], && (6.46)\\
E &= f_0\left[6\dot{a}^2a\psi^{\frac{1}{3}} + \frac{1}{3}a^3\varphi\psi^{\frac{1}{3}} - \frac{8}{3}\dot{a}^3\dot{\varphi}\psi^{-\frac{2}{3}} + \frac{16}{9}\dot{a}\dot{\psi}\varphi\psi^{-\frac{5}{3}} - \frac{32}{9}\dot{a}^3\varphi\psi^{-\frac{2}{3}}\left(3\frac{\dot{\varphi}}{\varphi} - 2\frac{\dot{\psi}}{\psi}\right)\right] && (6.47)
\end{aligned}$$

Note that one can get the Noether(conserved) charge from conserved current if one integrates the time component of conserved current over spatial volume. But for present model as all the variables are time dependent, so the conserved charge can be represented as the inner product of the infinitesimal generator with Cartan one form as

$$Q = i_{\vec{X}}\theta_L \tag{6.48}$$

where $\theta_L = \frac{\partial L}{\partial \dot{a}}da + \frac{\partial L}{\partial \dot{\varphi}}d\varphi + \frac{\partial L}{\partial \dot{\psi}}d\psi$, the Carton one form and $i_{\vec{X}}$ stands for the inner product with the vector field $\vec{X}$. This geometric inner product representation is useful to identify the cyclic variable in the augmented space so that the field equations become much simpler. Since cyclic variable is very much useful for solving non-linearly coupled evolution equations, so we are trying to find a transformation in such a way from (a,φ,ψ) to (u,v,w) so that one of the new variables become cyclic. Then the transformed infinitesimal generator can be written as

$$\vec{X}_T = (i_{\vec{X}}du)\frac{\partial}{\partial u}+(i_{\vec{X}}dv)\frac{\partial}{\partial v}+(i_{\vec{X}}dw)\frac{\partial}{\partial w}+\left[\frac{d}{dt}(i_{\vec{X}}du)\right]\frac{d}{d\dot{u}}+\left[\frac{d}{dt}(i_{\vec{X}}dv)\right]\frac{d}{d\dot{v}}+\left[\frac{d}{dt}(i_{\vec{X}}dw)\right]\frac{d}{d\dot{w}}. \tag{6.49}$$

Now, without loss of generality we impose restriction on above transformed infinitesimal generator to get a cyclic variable as

$$i_{\vec{X}}du = 1,\ i_{\vec{X}}dv = 0,\ i_{\vec{X}}dw = 0, \tag{6.50}$$

so that the above $\vec{X}_T$ simplifies to

$$\vec{X}_T = \frac{\partial}{\partial u} \text{ and } \frac{\partial L_T}{\partial u} = 0. \tag{6.51}$$

Thus 'u' becomes cyclic variable in the augmented space. The explicit form of the equation (6.50) for the present problem result to

$$u = a,\ v = a\varphi^{\frac{1}{2}},\ w = a\psi^{\frac{1}{3}}. \tag{6.52}$$

Actually above relations are a bridge between old coordinate and new coordinate. As a consequences, the point-like Lagrangian simplifies to

$$L_T = f_0\left[-\frac{1}{3}v^2w + 6\dot{u}^2w - \frac{16}{3}\frac{v^2\dot{u}^3}{w^2}\left(\frac{\dot{v}}{v} - \frac{\dot{w}}{w}\right)\right] \tag{6.53}$$

for $l = 1$, with u as the cyclic co-ordinate.

Now the Euler-Lagrange equations for the transformed Lagrangian can be written as

$$3\dot{u}w - 4\frac{v^2\dot{u}^2}{w^2}\left(\frac{\dot{v}}{v} - \frac{\dot{w}}{w}\right) - c_1 = 0, \tag{6.54}$$

$$\frac{d}{dt}\left(\frac{v\dot{u}^3}{w^2}\right) - \frac{vw}{8} - \frac{v\dot{u}^3}{w^2}\left(\frac{\dot{v}}{v} - 2\frac{\dot{w}}{w}\right) = 0, \tag{6.55}$$

$$\frac{d}{dt}\left(\frac{v^2\dot{u}^3}{w^3}\right) + \frac{v^2}{16} - \frac{9}{8}\dot{u}^2 + \frac{v^2\dot{u}^3}{w^3}\left(3\frac{\dot{w}}{w} - 2\frac{\dot{v}}{v}\right) = 0. \tag{6.56}$$

From (6.55) and (6.56) after some algebraic simplification we get,

$$\dot{u} = \pm \frac{v}{\sqrt{6}}. \tag{6.57}$$

In order to solve the above evolution equations we assume $w = v^n$ i.e., $\frac{\dot{w}}{w} = n\frac{\dot{v}}{v}$, n being any non-zero real number. Then using (6.57) in (6.54) we obtain

$$\int \frac{2v^{3-2n}}{\sqrt{6}v^{n+1} - 2c_1} dv = \frac{3}{2(1-n)}(t + c_2) \tag{6.58}$$

where c_1, c_2 are arbitrary integration constants. The above integral can be solved analytically both for $c_1 = 0$ and $c_1 \neq 0$ as follows.

Case-I: $c_1 \neq 0$

In this case integrating equation (6.58) we obtain

$$-\frac{v^{4-2n}}{c_1(2-n)} {}_2F_1\left[1, \frac{4-2n}{1+n}, \frac{5-n}{1+n}, \sqrt{\frac{3}{2}}\frac{1}{c_1}v^{n+1}\right] = \frac{3}{2(1-n)}(t + c_2) \tag{6.59}$$

where ${}_2F_1$ is the usual hyper-geometric function. From the above solution it is not possible to obtain an explicit solution for $v(t)$ and hence $u(t)$ and $w(t)$. So we can't infer the cosmological evolution. Thus we choose some specific value of the parameter in following.
Subcase: I $n = -1$
The solution has the explicit form

$$v(t) = A_1(t + c_2)^{\frac{1}{6}}, \tag{6.60}$$
$$u(t) = \frac{1}{7}A_1(t + c_2)^{\frac{7}{6}} + c_3, \tag{6.61}$$
$$w(t) = A_1^{-1}(t + c_2)^{-\frac{1}{6}} \tag{6.62}$$

where the c_3, A_1 are arbitrary constants. Using (6.52), the above classical solution for the old variables have the expression

$$a(t) = u(t) = \frac{1}{7}A_1(t + c_2)^{\frac{7}{6}} + c_3, \tag{6.63}$$
$$T(t) = \varphi(t) = \left[\frac{A_1(t + c_2)^{\frac{1}{6}}}{\frac{1}{7}A_1(t + c_2)^{\frac{7}{6}} + c_3}\right]^2, \tag{6.64}$$
$$T_G(t) = \psi(t) = \left[\frac{A_1^{-1}(t + c_2)^{-\frac{1}{6}}}{\frac{1}{7}A_1(t + c_2)^{\frac{7}{6}} + c_3}\right]^3. \tag{6.65}$$

The cosmological parameters namely the scale factor "a", the Hubble parameter "$H\left(=\frac{\dot{a}}{a}\right)$" and deceleration parameter "$q\left(=-\frac{\ddot{a}a}{\dot{a}^2}\right)$" has been presented graphically in FIG. (6.1). The figure shows that the above solution describe an accelerated expanding era of evolution.

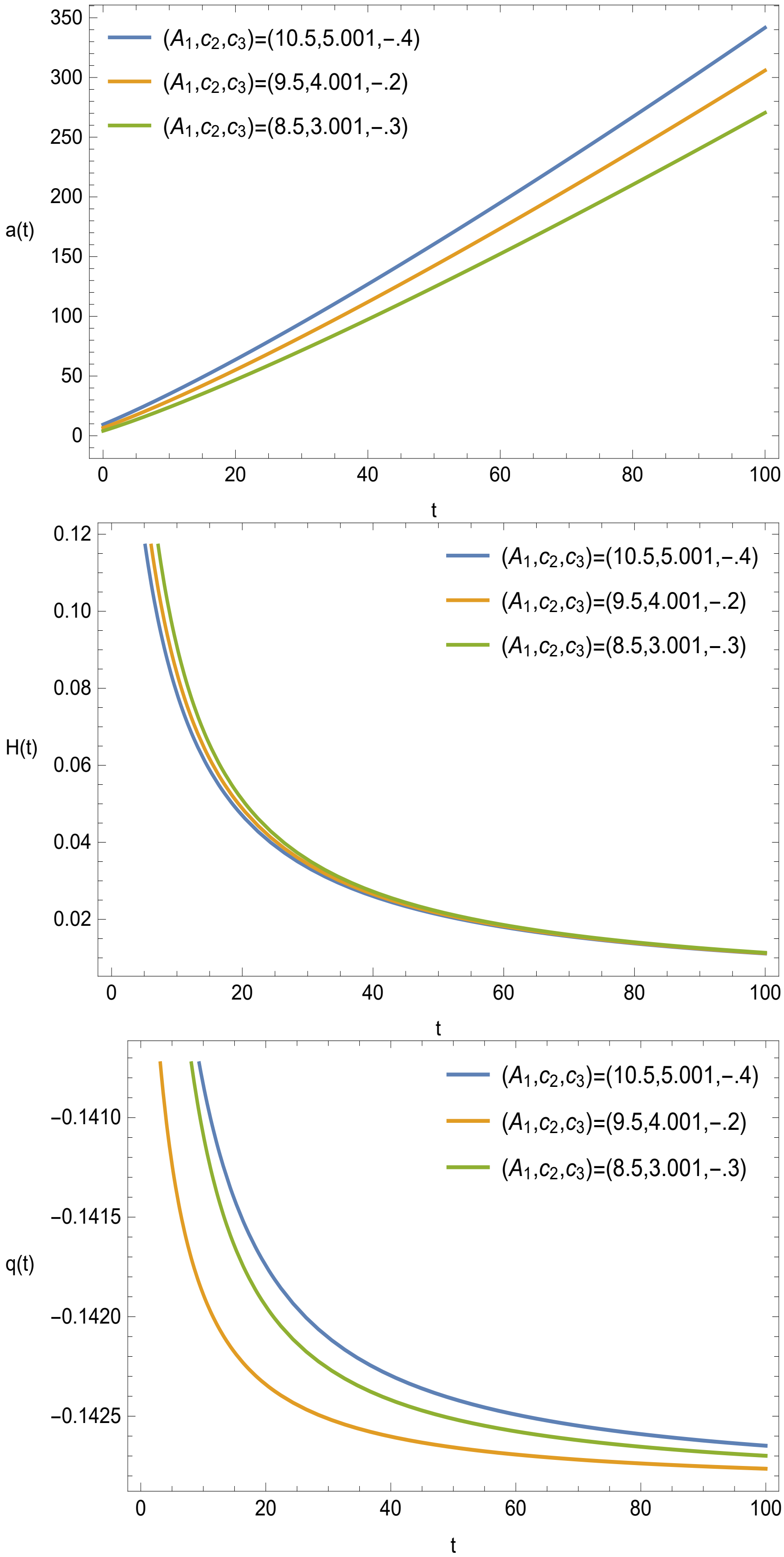

Figure 6.1 – The graphical representation of scale factor $a(t)$ (top left), Hubble parameter $H(t)$ (top right) and deceleration parameter q (bottom) with respect to cosmic time t.

Subcase: II $n = 2$.
For this choice of n the explicit solution is as follows

$$
\begin{aligned}
u(t) &= a(t) = \frac{A_2}{\sqrt{3}} \tan^{-1}\left(\frac{1 + \frac{2e^{\frac{t}{2}}}{\left(z_0 + e^{\frac{3t}{2}}\right)^{\frac{2}{3}}}}{\sqrt{3}} \right) \\
&+ \frac{A_2}{6} \log\left(\frac{\left(z_0 + e^{\frac{3t}{2}}\right)^{\frac{2}{3}} + e^t + e^{\frac{t}{2}}\left(z_0 + e^{\frac{3t}{2}}\right)^{\frac{1}{3}}}{\left(z_0 + e^{\frac{3t}{2}}\right)^{\frac{2}{3}} + e^t - 2e^{\frac{t}{2}}\left(z_0 + e^{\frac{3t}{2}}\right)^{\frac{1}{3}}} \right), \qquad (6.66) \\
v(t) &= \frac{e^{\frac{t}{2}}}{\left(B_1 e^{\frac{3t}{2}} + B_2\right)^{\frac{1}{3}}}, \qquad (6.67) \\
w(t) &= \frac{e^t}{\left(B_1 e^{\frac{3t}{2}} + B_2\right)^{\frac{2}{3}}} \qquad (6.68)
\end{aligned}
$$

where z_0, A_2, B_1, B_2 are arbitrary integration constants.

As in subcase: I we can obtain the old variable using the relation in equation (6.52). Again the graphical representation in this case has been presented in FIG-(6.2). It shows a decelerated era of expansion.

Subcase: III $n = \frac{1}{2}$.
It is to be noted we can integrate equation (6.58) for $n = 0{,}2$ but v can't be obtain explicitly as a function of cosmic time. So we have not presented the solution here.

Case-II: $c_1 = 0$

For this choice after integrating equation (6.58) we obtain the following solution (6.69), (6.70) and (6.71)

$$
\begin{aligned}
u(t) &= a(t) = A_4 \frac{3(1-n)}{4-n}(t - c_2)^{\frac{4-3n}{3(1-n)}} + B_4, \qquad (6.69) \\
v(t) &= A_4 (t - c_2)^{\frac{1}{3(1-n)}}, \qquad (6.70) \\
w(t) &= {A_4}^n (t - c_2)^{\frac{n}{3(1-n)}}. \qquad (6.71)
\end{aligned}
$$

where A_4,B_4,c_2 are arbitrary integrating constants. As before the cosmological parameters have been presented graphically in FIG-(6.3).

From the above three particular solutions we note that 1st and 3rd solutions are power law in cosmic time while the 2nd one is complicated in form. The graphical representation of the cosmological parameters for the 1st two cases are presented in figures (6.1) and (6.2)

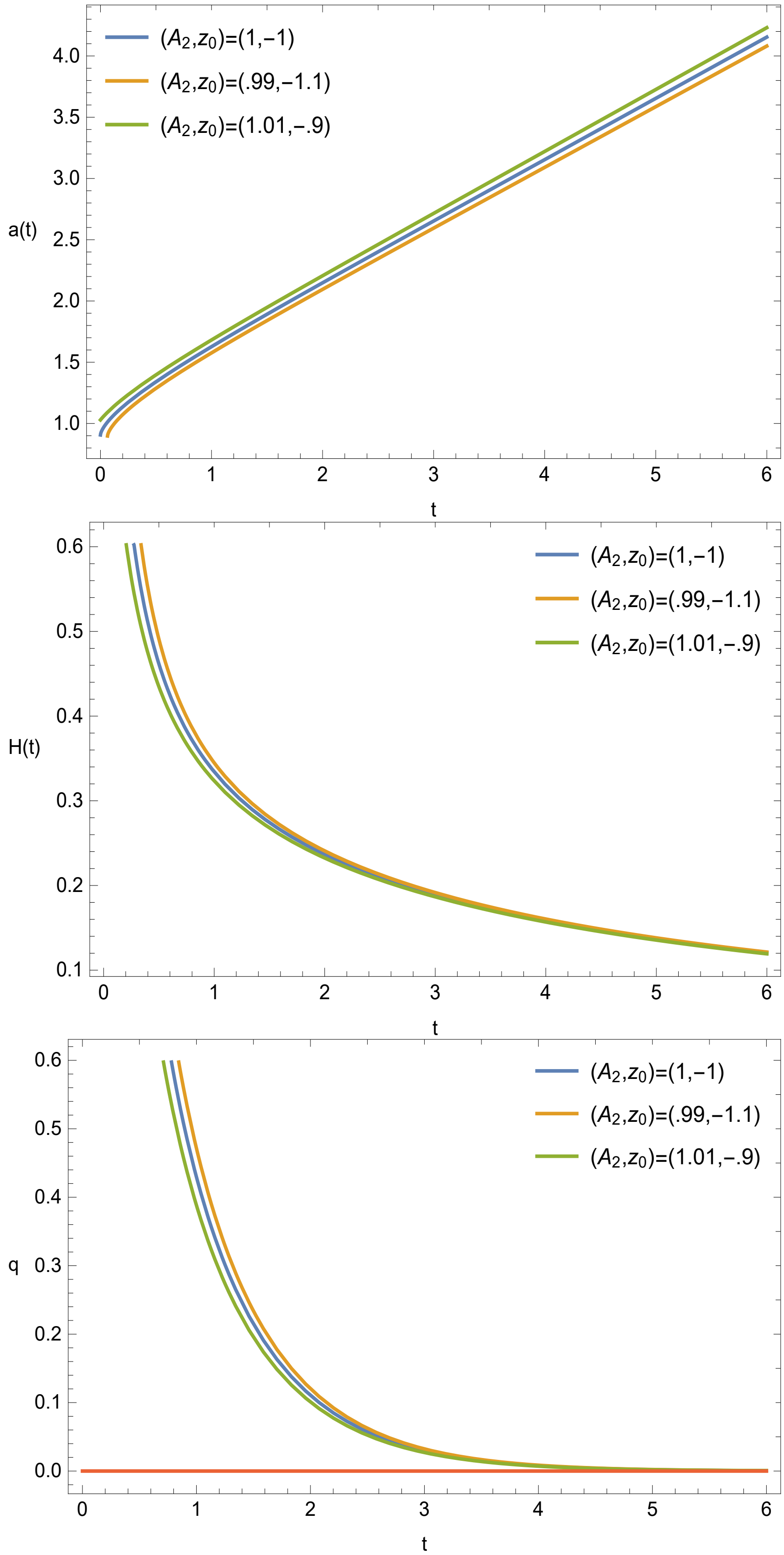

Figure 6.2 – Shows the graphical representation of scale factor $a(t)$ (top left), Hubble parameter $H(t)$ (top right) and q (bottom) with respect to cosmic time t for Subcase II.

respectively and they showed that the Universe is in a decelerating phase for the second case while the universe is in an accelerating era throughout the evolution for the 1st case. But third case is interesting because in this case the power of the cosmic time is arbitrary as the parameter n is arbitrary. The solution in this case can be written in compact form as

$$a = a_0 t^{\mu} + b_0 \tag{6.72}$$

where $a_0 = A_4 \frac{3(1-n)}{4-n}$, $b_0 = B_4$ and $\mu = \frac{4-3n}{3(1-n)}$. So, in this case the deceleration parameter has the expression

$$q = -\left(\frac{\mu-1}{\mu}\right)\left(1+\left(\frac{b_0}{a_0}\right)t^{-\mu}\right). \tag{6.73}$$

This shows that as t goes to infinity $q \longrightarrow -1+\frac{1}{\mu}$, provided $\mu > 0$. Thus if μ is very large then $q \longrightarrow -1$ i.e., the model asymptotically corresponds to ΛCDM model. This is supported by graphical representation of the deceleration parameter in FIG-(6.3). Also in this case the present value in the deceleration parameter is obtained at some finite 't' which may be chosen at the present era of evaluation. Hence this solution is in analogy with the recent observation.

6.5 A description of Quantum Cosmology: derivation of Wheeler-DeWitt Equation

An attempt will be made to study quantum cosmology in the framework of $f(T,T_G)$ gravity theory. Though Noether symmetry analysis in superspace are identified by the metric and matter field, still in quantum cosmology minisuperspaces (which are restricted from the geometrodynamics of the superspace) are considered due to simplicity. The simplest choice of minisuperspace model is chosen as homogeneous and isotropic in nature so that $N = N(t)$ is the lapse function and N^{α}, the shift vector vanishes identically. So, in (3+1)-decomposion, the minisuperspace line element can be explicitly written as

$$dS^2 = -dt^2 + q_{ij}(x,t)dx^i dx^j \tag{6.74}$$

with Einstein-Hillbert action (COLISTETE JR.; FABRIS; PINTO-NETO, 1998)

$$\mathcal{A}(q_{ij}) = \int dt d^3x \sqrt{q}\left[K_{ij}K^{ij} - K^2 + {}^{(3)}R - 2\Lambda\right]. \tag{6.75}$$

Here $K = K_{ij}q^{ij}$ is the trace of the extrinsic curvature K_{ij}, ${}^{(3)}R$ is the usual 3-space curvature, N is chosen to be unity (a rescaling of the time parameter) and Λ stands for the cosmological constant. As the three space is homogeneous so the 3 space metric can be expressed by a finite number of functions $x^{\alpha}(t)$, $\alpha = 0,1,2,...,(n-1)$. As a result the above action takes the form

$$\mathcal{A}(x^{\alpha}(t)) = \int_0^t dt \left[\frac{1}{2} l_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} - V(x)\right] \tag{6.76}$$

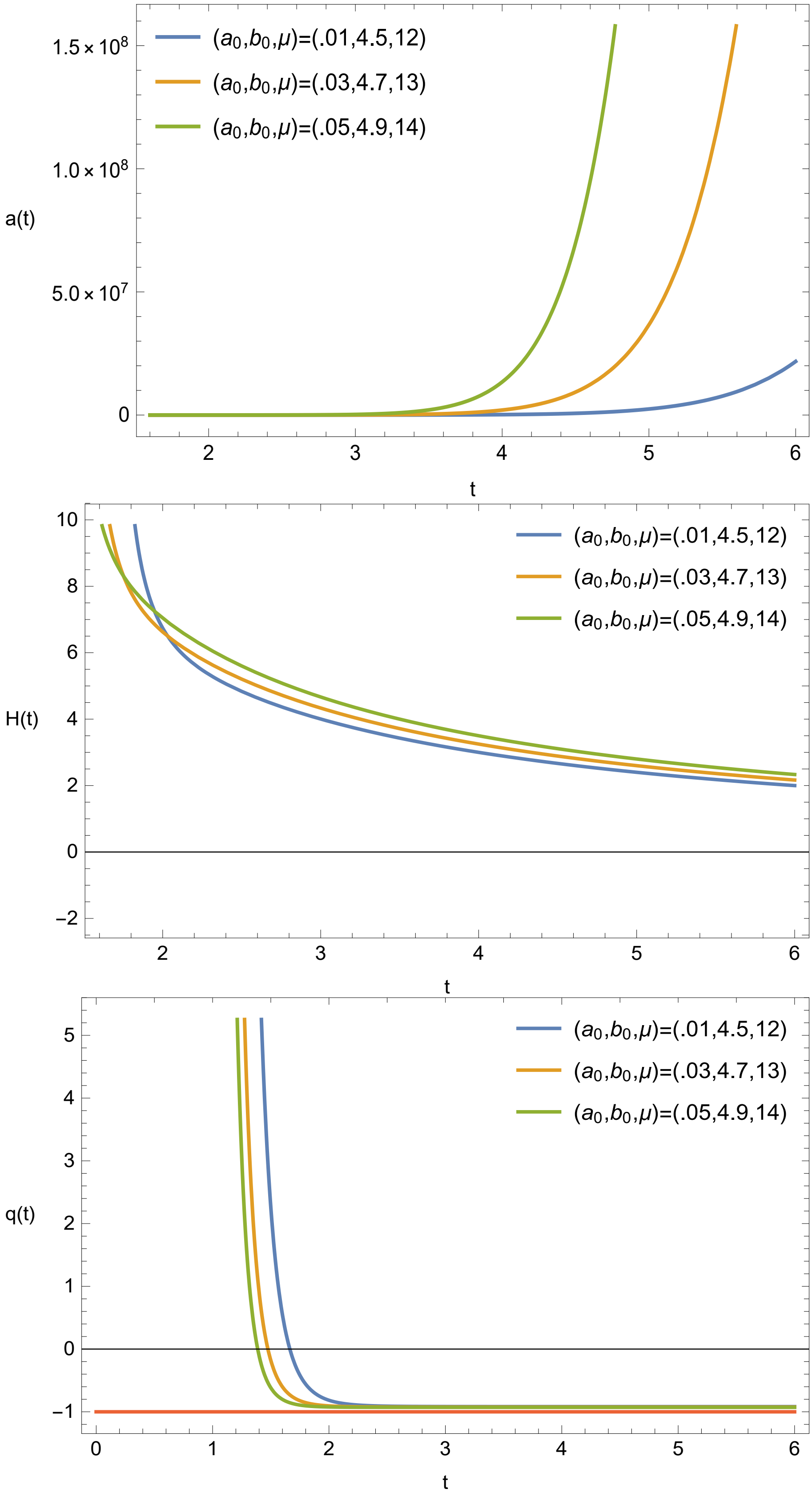

Figure 6.3 – Evolution of Universe's deceleration parameter over cosmic time

with metric on the 3-space as $l_{\alpha\beta}$. This action corresponds to a point Lagrangian describing a relativistic point particle moving in nD with metric $l_{\alpha\beta}$ and $V(x)$ represents the self interacting potential. Thus the equation of motion is described by 2nd order quasi linear differential equation as

$$\frac{d}{dt}\left(\dot{u}^{\alpha}\right) + \Gamma^{\alpha}_{\delta\beta}\dot{x}^{\delta}\dot{x}^{\beta} + l^{\alpha\beta}\frac{\partial V}{\partial x^{\beta}} = 0 \tag{6.77}$$

Also there is a constraint equation related to these equations of motion and it can be obtained by variation of the action with respect to the lapse function as

$$\frac{1}{2}l_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} + V(x) = 0. \tag{6.78}$$

In quantum context, Hamiltonian formulation is desirable and hence one has to determine the canonical momenta corresponding to the dynamical variables as

$$\Pi_{\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\alpha}} = l_{\alpha\beta}\dot{x}^{\beta}. \tag{6.79}$$

So the Hamiltonian of the physical system is given by

$$H = \Pi_{\alpha}\dot{x}^{\alpha} - \mathcal{L} = \left[\frac{1}{2}l^{\alpha\beta}\Pi_{\alpha}\Pi_{\beta} + V(x)\right]. \tag{6.80}$$

It is interesting to note that the above Hamiltonian is nothing but the constraint equation (6.78) in phase space. Now in the standard quantization programme the momenta are replaced by their operator version i.e., $\Pi_{\alpha} \to -i\frac{\partial}{\partial x^{\alpha}}$ so that the above Hamiltonian becomes a second order hyperbolic partial differential equation

$$\left[-\frac{1}{2}l^{\alpha\beta}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}} + V(x)\right]\psi(x) = 0 \tag{6.81}$$

This basic equation in quantum cosmology is termed as Wheeler-DeWitt (WD) equation and $\Psi(x)$ is known as the wave function of the Universe. In the above formulation one has to face an ambiguity related to operator ordering. However, It may be resolve considering the quantization scheme to be of covariant nature i.e., WD equation is invariant under the transformation: $x^{\alpha} \to \bar{x}^{\alpha}(x^{\alpha})$. Further, due to hyperbolic nature of the WD equation, the conserved probability current (to have a probability measure) has the expression

$$\overrightarrow{J} = \frac{i}{2}\left(\psi^{*}\nabla\psi - \psi\nabla\psi^{*}\right) \tag{6.82}$$

with $\nabla\overrightarrow{J} = 0$. So the probability measure can be defined as

$$dp = |\psi(u^{\mu})|^2 dV \tag{6.83}$$

where dV is volume element in the minisuperspace.

Moreover, as the as wave function of the Universe is the general solution of the above WD equation (a second order hyperbolic type partial differential equation) can be expressed as a superposition of the eigen function of the WD operator. Thus one may write

$$\Psi(x^{\mu}) = \int w(\sigma)\psi(x^{\mu},\sigma)d\sigma \tag{6.84}$$

with ψ an eigen function of the WD operator and $w(\sigma)$, the weight function. To have a consistency with classical description the weight function is so chosen that one gets a coherent wave packet with good asymptotic behaviour in the minisuperspace and having a peaked around the classical trajectory. Further, for a quantum cosmological description, it is desirable that classical solution should be predicted at late time only from the quantum description while at early epochs they are quite distinct so that it is possible to have singularity-free early epoch in quantum description.

In the context of quantum cosmology, one may discussed this model for the new variables $\{u,v\}$ in the augmented space. So the canonically conjugate momenta associated to the point-like transformed Lagrangian (6.53) are

$$p_u = \frac{\partial L_T}{\partial \dot{u}} = f_0\left(12\dot{u}v^n + 16(n-1)v^{1-2n}\dot{u}^2\dot{v}\right) = \text{conserved } (\sigma_u, \text{ say}), \tag{6.85}$$

$$p_v = \frac{\partial L_T}{\partial \dot{v}} = \frac{16}{3}f_0(n-1)v^{1-2n}\dot{u}^3. \tag{6.86}$$

Since u is cyclic in the transformed Lagrangian (6.53), $p_u (= \sigma_u)$ will be conserved in nature. Thus the Hamiltonian of the system can be written as

$$\begin{aligned} H &= p_u\dot{u} + p_v\dot{v} - L_T \\ &= \frac{f_0}{3}v^{n+2} - 6f_0A_5{}^2p_v{}^{\frac{2}{3}}v^{\frac{7n-2}{3}} + A_5p_up_v{}^{\frac{1}{3}}v^{\frac{2n-1}{3}} + B_5p_v{}^{\frac{1}{3}}v^{\frac{2n-1}{3}}\left(p_u - 12A_5f_0p_v{}^{\frac{1}{3}}v^{\frac{5n-1}{3}}\right) \\ &= \frac{f_0}{3}v^{n+2} - A_6p_v{}^{\frac{2}{3}}v^{\frac{7n-2}{3}} + A_7p_v{}^{\frac{1}{3}}p_uv^{\frac{2n-1}{3}}, \text{(say)} \end{aligned} \tag{6.87}$$

where A_6, A_7 are arbitrary constants. As $H \equiv 0$ so we have

$$A_6p_v{}^{\frac{2}{3}}v^{\frac{7n-2}{3}} - A_7p_v{}^{\frac{1}{3}}p_uv^{\frac{2n-1}{3}} - \frac{f_0}{3}v^{n+2} = 0. \tag{6.88}$$

Due to complicated and fractional power in the momentum variables it is hard to write the quantum version of this Hamiltonian constraint equation. So we shall consider the following two cases:

Case-I: $A_7 = 0$

In this (6.88) can be written as

$$A_6^3p_v{}^2v^{7n-2} = \left(\frac{f_0}{3}\right)^3 v^{3(n+2)}$$

$$i.e., p_v{}^2 = B_0v^{8-4n}, \ B_0 = \frac{f_0^3}{27A_6^3}$$

which in the operator version $\left(i.e., p_v \to -i\frac{\partial}{\partial v}\right)$ takes the form

$$\frac{\partial^2\psi(u,v)}{\partial^2 v} = -B_0v^{8-4n}\psi(u,v). \tag{6.89}$$

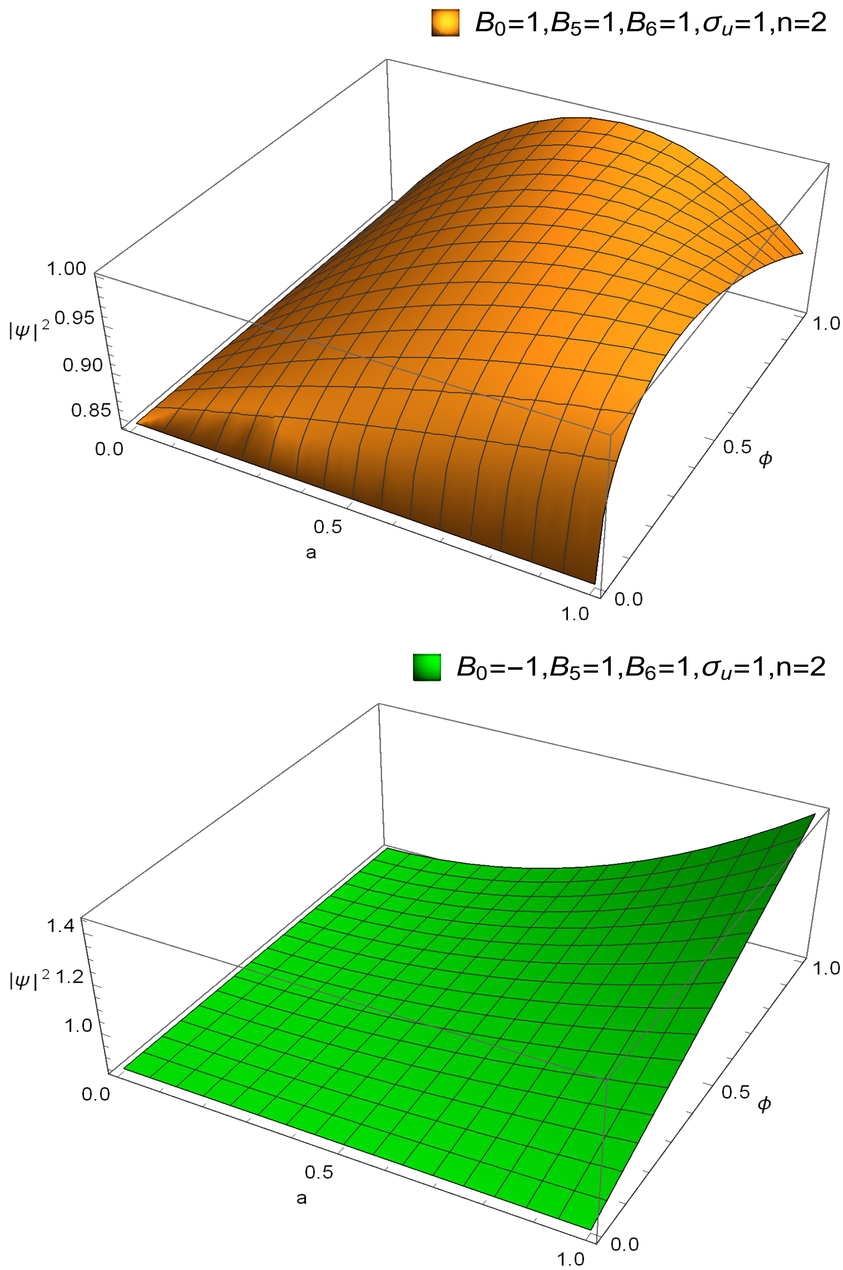

Figure 6.4 – The graphical representation of wave function of the Universe $|\psi|^2$ versus a,φ when $n = 2$.

The Noether symmetry identifies the transformed variable u as a cyclic variable (see the transformed Lagrangian (6.53)) so the corresponding momentum is a conserved quantity as shown in equation (6.85). Thus the operator version of the conserved momenta in equation (6.85) becomes

$$-i\frac{\partial\psi(u,v)}{\partial u} = \sigma_u\psi. \tag{6.90}$$

So for real σ_u the solution can be written as

$$\psi(u,v) = \Phi(v)e^{i\sigma_u u} \tag{6.91}$$

with $\Phi(v)$ an arbitrary function of v. Now using (6.91) in equation (6.89) one has

$$\frac{d^2\Phi(v)}{dv^2} = -B_0 v^{8-4n}\Phi(v) \tag{6.92}$$

when $n = 2$, then $\Phi(v) = B_5 \sin(\sqrt{B_0}v + B_6)$ i.e.,

$$\begin{aligned} \psi(u,v) &= B_5 e^{i\sigma_u u} \sin(\sqrt{B_0}v + B_6), \\ \psi(a,\varphi) &= B_5 e^{i\sigma_u a} \sin(\sqrt{B_0}a\varphi^{\frac{1}{2}} + B_6). \end{aligned} \tag{6.93}$$

When $n = \frac{5}{2}$, then

$$\Phi(v) = \begin{cases} B_7\sqrt{v}\sinh\left(\sqrt{\frac{1-4B_0}{4}}\ln v + B_8\right), & \sqrt{1-4B_0} > 0 \\ B_9\sqrt{v}\sin\left(\sqrt{\frac{1-4B_0}{4}}\ln v + B_{10}\right), & \sqrt{1-4B_0} < 0 \\ (B_{11} + B_{12}\log v)\sqrt{v}, & B_0 = \frac{1}{4} \end{cases} \tag{6.94}$$

Therefore, the equation (6.91) becomes

$$\psi(u,v) = \begin{cases} B_7\sqrt{v}e^{i\sigma_u u}\sinh\left(\sqrt{\frac{1-4B_0}{4}}\ln v + B_8\right), & \sqrt{1-4B_0} > 0 \\ B_9\sqrt{v}e^{i\sigma_u u}\sin\left(\sqrt{\frac{1-4B_0}{4}}\ln v + B_{10}\right), & \sqrt{1-4B_0} < 0 \\ e^{i\sigma_u u}(B_{11} + B_{12}\log v)\sqrt{v}, & B_0 = \frac{1}{4} \end{cases} \tag{6.95}$$

So using (6.52), the wave function of the Universe has an explicit form

$$\psi(a,\varphi) = \begin{cases} B_7\sqrt{a\varphi^{\frac{1}{2}}}e^{i\sigma_u a}\sinh\left(\sqrt{\frac{1-4B_0}{4}}\ln\left(a\varphi^{\frac{1}{2}}\right) + B_8\right), & B_0 < \frac{1}{4} \\ B_9\sqrt{a\varphi^{\frac{1}{2}}}e^{i\sigma_u a}\sin\left(\sqrt{\frac{1-4B_0}{4}}\ln\left(a\varphi^{\frac{1}{2}}\right) + B_{10}\right), & B_0 > \frac{1}{4} \\ e^{i\sigma_u a}\sqrt{a\varphi^{\frac{1}{2}}}\left(B_{11} + B_{12}\log\left(a\varphi^{\frac{1}{2}}\right)\right), & B_0 = \frac{1}{4} \end{cases} \tag{6.96}$$

where $B_7, B_8, B_9, B_{10}, B_{11}, B_{12}$ are arbitrary integrating constants.

In figure (6.5), we have plotted the probability amplitude $|\Psi|^2$ against a,φ for different choices of parameter symbol. From the above graph it is found that with the proper choice of parameter it is possible to have zero probability at the big-bang singularity. So big-bang singularity may be avoided by quantum description in this case.

Case-II: $A_6 = 0$

For this choice equation (6.88) simplifies to

$$p_v {p_u}^3 + c_0 v^{n+7} = 0, \;\; c_0 = \frac{f_0^3}{27A_7^3}$$

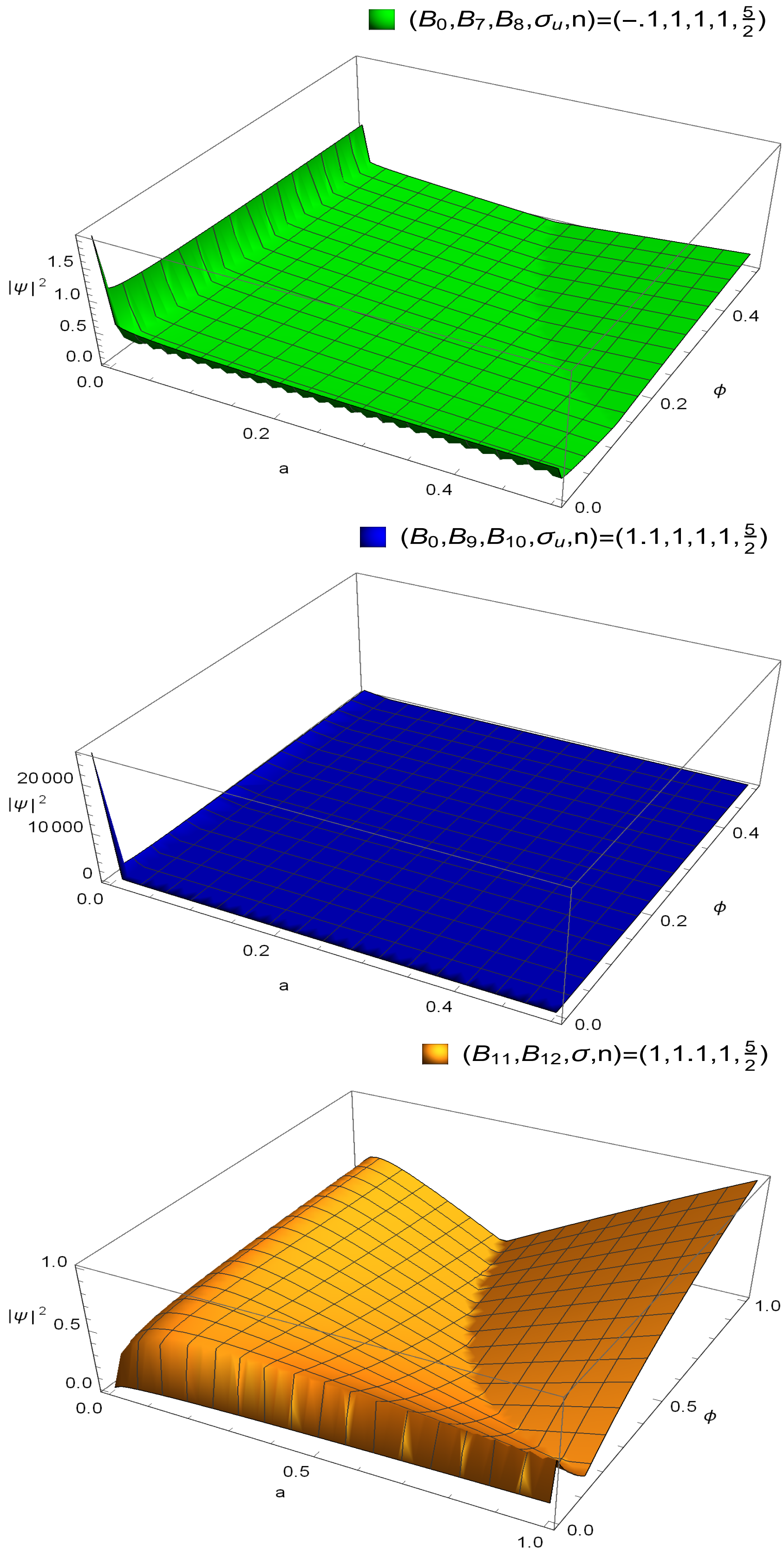

Figure 6.5 – The graphical representation of wave function of the Universe $|\psi|^2$ versus a,φ for various choice of parameters when the value of $n = \frac{5}{2}$.

which in the operator version becomes

$$\frac{\partial^4\psi(u,v)}{\partial v\partial u^3}+c_0v^{n+7}\psi(u,v)=0. \tag{6.97}$$

Now using (6.91) we have

$$\frac{d\Phi(v)}{dv}+ic_0{\sigma_u}^{-3}v^{n+7}\Phi(v)=0 \tag{6.98}$$

then the solution of the above differential equation is

$$\Phi(v)=B_{13}e^{-\frac{ic_0}{\sigma_u^3(n+8)}v^{n+8}} \tag{6.99}$$

therefore from (6.91), we get

$$\begin{aligned}\psi(u,v) &= B_{13}e^{i\sigma_u u}e^{-\frac{ic_0}{\sigma_u^3(n+8)}v^{n+8}}\\ \Longrightarrow \psi(a,\varphi) &= B_{13}e^{i\left(\sigma_u a-\frac{c_0}{\sigma_u^3(n+8)}\left(a\varphi^{\frac{1}{2}}\right)^{n+8}\right)}.\end{aligned} \tag{6.100}$$

In this case the wave function is purely oscillatory in nature and hence it is not possible to remove the big-bang singularity by quantum description.

6.6 Brief summery

The present chapter describes the classical cosmology due to $f(T,T_G)$ gravity theory, in the background of homogeneous and isotropic space-time geometry. Noether symmetry analysis has been imposed to the present physical model and as a result both the Lagrangian as well as the field equations get simplified by a suitable transformation in the augmented space. Thus it is possible to obtain exact analytic solution to the present model. For cosmological analysis the relevant cosmological parameters namely the scale factor, the Hubble parameter and the deceleration parameter have been plotted in figures (6.1)-(6.3). Figure-(6.1) shows that the Universe evolves in an expanding mode with cosmic time. The graph of the Hubble parameter shows that H is decreasing as the scale factor increases and it implies that the rate of expansion gradually decreases with the evolution. The graph of the deceleration parameter exhibits that corresponding to this solution the universe is only in accelerated mode of expansion. The graphical representation of the second solution is presented in Figure-(6.2) and it shows that the universe is fully in decelerated era of expansion. The third solution is quite interesting. Though it describes fully accelerated era of expansion, it approaches the well known ΛCDM model in the asymptotic limit. So for the present model the classical solution obtained due to Noether symmetry analysis, describes decelerated or accelerated mode of expansion and it may have analogy to its ΛCDM era (for the third choice). Also which suitable choice of the value of the parameter it is possible to obtain the present value of the deceleration parameter from the observational point of view.

For quantum cosmology description though the operator version of the Hamiltonian constraint has frictional power but still it is possible to have the wave function of the universe with proper choices for arbitrary parameters involve. In one case the wave function is purely oscillatory in nature, having finite non zero probability at zero volume. So for this choice quantum description can not avoid the classical big-bang singularity. However, for the other choice the wave function has both oscillatory and non oscillatory form and it is found that with proper choice of the parameters involved, one may have zero probability at zero volume (see figures (6.4) and (6.5)), indicating avoidance of singularity by quantum description. The present chapter is an example where a non metricity modified gravity theory has been converted to a much simpler form to obtain a classical solution as well as quantum description is possible. It is interesting to note that the conserved charge has a very crucial role both identifying the classical solution as well as to solve the Wheeler-DeWitt equation. Finally, we conclude that the Noether symmetry analysis is a very powerful mathematical tool in the context of both classical and quantum cosmology.

7 Overall Conclusions and Future Prospect

This chapter contains an overview of the thesis and gives potential future directions for research. The goal is to encapsulate the main points discussed in preceding chapters within distinct paragraphs, providing a concise summary.

7.1 Overall Conclusion

In chapter two it has been shown how to use the Noether symmetry on the coupled Brans-Dicke gravity and discussed the solutions from the cosmological point of view. In classical cosmology, this analysis determines not only the symmetry vector but also the coupling function and provides classical solutions by selecting a suitable transformation in augmented space. Graphical representation of cosmological parameters align qualitatively with observational predictions. The geometric symmetry of physical space shows there is a homothetic vector field, and there is no symmetry along the homothetic vector field direction. In quantum cosmology, the Wheeler-DeWitt (WD) equation presents a highly coupled nonlinear second-order partial differential equation. By identifying the oscillatory part of the wave function, solvability of the WD equation is achieved. The graphical representation of the wave function suggests a finite non-zero probability for the system to have zero volume at the Big Bang singularity, indicating that the quantum version of the model does not eliminate the initial singularities present in classical cosmology. Alternative interpretations of quantum mechanics, such as the de Bohm-Broglie interpretation, may offer insights into this behavior.

Chapter three explores the homogeneous and isotropic Einstein-æther scalar-tensor gravity model in both classical and quantum cosmologies using Noether symmetry analysis. This method enables the determination of exact dynamics and conserved charge, linked with observable quantities. The analysis reveals two conserved quantities: Noether charge and energy. By identifying a transformation in the augmented space, the field equations simplify, leading to obtainable solutions. The cosmological solution illustrates the universe's evolution from early accelerated expansion to the current era, as depicted in (3.1). Also, quantum cosmology is investigated by forming the WD equation. The oscillatory part of the wave function of the Universe is determined from the operator version of the conserved charge, allowing an explicit form of the wave function through WD equation solutions. Probability density graph (3.2) indicates that the quantum description does not overcome the initial singularity.

Chapter four discusses how symmetry analysis aids in examining classical and quantum cosmologies on the teleparallel dark energy model. Noether symmetry analysis explores both classical and quantum cosmological descriptions, revealing a cyclic variable in the augmented space that renders the field equations solvable. The classical cosmological solution is discussed analytically, while key cosmological parameters are graphically represented in the classical section. The graphs indicate that universal expansion with a gradually decreasing rate (positive in nature but never zero), aligning with recent observational evidence. However, the graphical depiction of the deceleration parameter suggests that this model only describes the early evolution of the Universe up to the matter-dominated era. In the quantum section, the WD equation, a second-order hyperbolic partial differential equation, is constructed using the operator ordering issue. Finally, we have discussed the wave function of the Universe from the quantum aspect, and it indicates that classical singularities persist despite the quantum framework.

Fifth chapter investigates a multiscalar field cosmological model characterized by non-minimal coupling between one scalar field and gravity, while the other scalar field is minimally coupled to gravity. Despite the highly coupled and non-linear field equations, Noether symmetry analysis is employed to analyze the model both classically and quantum mechanically. The identification of a cyclic variable simplifies the Lagrangian; as a consequence, the field equations are solvable. Graphical representations of relevant cosmological parameters (a and H) confirm universal expansion throughout evolution, with the rate of expansion gradually diminishing. The model's theoretical predictions qualitatively match observations, particularly in describing the three phases of evolution after the big bang. Noether symmetry analysis plays a crucial role in studying quantum cosmology. By utilizing the operator version of the conserved charge, one can identify the periodic part of the wave function and potentially eliminate the big bang singularity through quantum formulation with proper parameter selection. Overall, Noether symmetry analysis significantly contributes to the study of various cosmological models within quantum frameworks.

Chapter six discusses classical cosmology within the framework of $f(T,T_G)$ gravity theory, focusing on homogeneous and isotropic space-time geometry. Noether symmetry analysis is applied to simplify both the Lagrangian and the field equations through a suitable transformation in augmented space, facilitating exact analytic solutions. Cosmological parameters such as the scale factor, Hubble parameter, and deceleration parameter are analyzed graphically. Results show that the Universe evolves in an expanding mode over cosmic time, with the rate of expansion gradually decreasing. Different solutions depict either accelerated or decelerated modes of expansion, with one solution converging to the well-known ΛCDM model in the asymptotic limit. This classical solution, derived through Noether symmetry analysis, offers insights into the dynamics of cosmic expansion, potentially shedding light on the ΛCDM era. This work explores quantum cosmology within the framework of a non-metricity modified gravity theory, focusing on the operator version of the Hamiltonian constraint. Despite encountering frictional power in this description, it's still feasible to derive the wave function of the universe with appropriate parameter choices. One scenario yields a purely oscillatory wave function, maintaining a finite nonzero probability at zero volume, thus unable to avoid the classical big bang singularity. Conversely, another scenario presents a wave function with both oscillatory and non-oscillatory components, allowing avoidance of singularity with suitable parameter selections. Thi chapter demonstrates the conversion of a non-metricity modified gravity theory into a simpler form to facilitate both classical solutions and quantum descriptions. Notably, the conserved charge plays a pivotal role in identifying classical solutions and solving the Wheeler-DeWitt equation. In conclusion, this chapter highlights the significance of Noether symmetry analysis as a potent mathematical tool in both classical and quantum cosmology contexts.

7.2 Future Prospect

The future potential of applying Noether symmetry analysis in cosmology seems promising for advancing our grasp of the universe's fundamental principles. By utilizing Noether's theorem on cosmological models, scientists can uncover symmetries that simplify complex equations, thereby enabling precise analytical solutions. This method not only offers insights into classical and quantum cosmologies but also has the potential to reveal new perspectives on cosmic evolution, resolving singularities, and understanding the universe's fundamental laws. Moreover, delving into Noether symmetry analysis in cosmology may open up innovative research paths, including investigating geodesic symmetries and their implications for cosmic structures and dynamics. In summary, the future of Noether symmetry analysis in cosmology holds vast potential for enriching our understanding of the cosmos, offering opportunities to delve deeper into quantum and classical cosmologies, potentially unveiling insights into bouncing evolution and singularity-free models in quantum cosmology, and exploring diverse applications in geodesic studies.

The future scope of Noether symmetry analysis in physics promises to continue revolutionizing our understanding of fundamental laws and principles governing the universe. Symmetry approach not only aids in elucidating classical mechanics but also extends to quantum mechanics, field theory, and beyond. Furthermore, the future of Noether symmetry analysis holds potential for advancements in areas such as particle physics, general relativity, and condensed matter physics. Exploring new applications and refining existing techniques in Noether symmetry analysis will likely lead to breakthroughs in our comprehension of the fundamental forces and structures of the universe, paving the way for innovative technologies and furthering our quest for a unified theory of physics. We could broaden our research scope by exploring alternative approaches, such as investigating non-minimally coupled scalar fields or delving into multi-scalar field models. Additionally, conducting cosmological perturbations around the initial cosmic time could provide intriguing insights.

In the future, we aim to demonstrate that certain precise solutions align with various observational datasets concerning the overall expansion of the universe. These datasets include the Pantheon data from Type Ia supernovae (SNeIa), the Hubble diagram for gamma-ray bursts (GRBs), and direct measurements of the Hubble parameter. We plan to conduct an in-depth examination of how interacting dark energy influences the formation and behavior of large-scale structures within the universe.

Noether symmetry analysis holds exciting potential for advancing our comprehension of symmetries across diverse mathematical fields. Beyond its traditional roots in physics, it's increasingly relevant in areas like differential equations, algebraic geometry, and mathematical physics. Researchers are poised to uncover new applications, revealing hidden structures in mathematical systems. This might entail discovering novel symmetries in non-linear equations, delving into geometric aspects of dynamic systems, and exploring the relationship between symmetry and integrability in mathematical models. Additionally, exploring connections with domains like algebraic geometry and number theory could yield further insights. Enhanced computational methods and symbolic algebra will facilitate the exploration of complex mathematical systems through Noether symmetry analysis. Overall, the future of Noether symmetry analysis promises profound insights into fundamental mathematical principles and enriches various branches of mathematical inquiry.

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