A Study of Some Aspects of Ideals in Semirings, Ternary Semirings and Ternary Hypersemirings

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CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled "**A Study of Some Aspects of Ideals in Semirings, Ternary Semirings and Ternary Hypersemirings**" submitted by Sri **Sampad Das** who got his name registered on **9th October, 2020 (Index No.: 26/20/Maths./27)** for the award of **Ph.D.** (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Dr. Manasi Mandal** and **Dr. Nita Tamang** that neither this thesis nor any part of it has been submitted for either any degree/ diploma or any other academic award anywhere before.

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Dedicated to
my mother
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Abstract

In this thesis, we have enlighted some aspects of ideals in semirings, ternary semirings and ternary hypersemirings.

Firstly, the concepts of 2-prime and n-weakly 2-prime (resp. weakly 2-prime) ideals in a commutative semiring have been introduced and studied. Characterization of valuation semirings in terms of 2-prime ideals has been obtained. Semirings where 2-prime ideals are prime and semirings where every proper ideal is n-weakly 2-prime (resp. weakly 2-prime) ideals have also been characterized.

Secondly, the concepts of 1-absorbing prime ideals and weakly 1-absorbing prime ideals of commutative semirings have been introduced. Some important properties, results, and various characterizations of 1-absorbing prime (resp. weakly 1-absorbing prime) ideals have been established. The relationships among 1-absorbing prime ideals, prime ideals, 2-prime ideals and 2-absorbing ideals have also been investigated.

After that, the notion of n-ideals has been introduced and studied. Semirings in which every proper ideal is n-ideal have been characterized. The characterization of entire semirings in terms of n-ideals has been established. Moreover, n-ideals under various contexts of constructions such as homomorphic images, localizations, direct products, and expectation semirings have been studied.

Then the notions of left bi-quasi ideals and bi-quasi ideals of a ternary semiring have been introduced, which is an extension of the concept of bi-ideals in a ternary semiring. The concepts of minimal left bi-quasi ideals

and left bi-quasi simple ternary semiring have also been studied. Characterization of regular ternary semirings in terms of left bi-quasi ideals have been obtained.

Next, the concept of 3-prime ideals in a ternary semiring has been introduced. Further, the concept of quasi 3-primary ideals as a generalization of 3-prime ideals and primary ideals in a commutative ternary semiring has been introduced. The relationships among prime ideals, 3-prime ideals, primary ideals, quasi primary and quasi 3-primary ideals have been established. Various results and examples concerning 3-prime ideals and quasi 3-primary ideals have been studied. Analogous theorems to the primary avoidance theorem for quasi 3-primary ideals have been established.

After that, the smallest strongly regular relation δ^* has been studied on a ternary hypersemiring S so that the quotient structure is a ternary semiring. The notion of a fundamental ternary semiring with respect to the fundamental relation δ^* on a ternary hypersemiring S has been introduced. The connection between ternary hypersemirings and ternary semirings via the fundamental relation has been established.

Finally, the notions of prime hyperideals, primary hyperideals, and maximal hyperideals in ternary hypersemirings have been introduced and some of their important properties analogous to the properties in hypersemirings have also been studied. A ternary semiring $P_\circ(S)$ has been constructed from strongly distributive ternary hypersemiring S. Then an inclusion preserving bijection between the set of all prime hyperideals of S and the collections of all prime total subtractive ideals of $P_\circ(S)$ has been established. The concepts of prime and primary avoidance theorems in ternary hypersemirings for C-ternary hyperideals have also been generalized.

Contents

Introduction

At the outset we discuss some relevant history on the development of semirings. The early examples of semirings appeared in the works of Dedekind [24] in 1894, when he worked on the algebra of the ideals of commutative rings. Some of the implicit works in the development of the idea of semiring theory were started by Macaulay [70], and Krull [63]. In 1934, H. S. Vandiver [98] introduced the term semiring in the publication titled "Note on a straightforward form of algebra where the cancellation law of addition is not applicable". In this paper, he defined semiring as an algebraic structure with two associative operations of addition and multiplication such that multiplication distributes over addition, while cancellation law of addition does not hold.

In 1951, Bourne [18] introduced the concept of an ideal in an arbitrary semiring, while developing the concept of Jacobson radical for semirings. Actually, in the last two decades of the last century there had been a tremendous growth of semiring theory in terms of theory itself and in terms of applications to computer science resulting in various research papers. Semirings are not only extensively applied across various domains such as automata theory within theoretical computer science, optimization theory (including combinatorial optimization), and generalized fuzzy computation, but they also represent intriguing extensions of two well-studied algebraic structures: rings and bounded distributive lattices.

The sustained research interest in the theory of semirings is evident through various monographs, as indicated in references [28], [29], [42], [43], [44], [45], [46], [49], [64], [79], [88], [94]. Semiring theory being a generalization of ring theory one aspect of the study of semirings involves the investigation of the validity of the ring theoretic analogues in the semirings.

The literature of semirings is widely scattered and there is no single convention followed for the terminologies appearing in them. Even the very definition of a semiring differs in the two books published so far. Golan assumed that a semiring is additively commutative with an "absorbing zero" and a "unity" whereas Hebisch and Weinert assumed only the commutativity in addition. Since different authors have used the term “semiring” with different meanings, it is essential, from the beginning, to clarify what we mean by semiring.

A semiring S is an algebraic structure $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations $+$ and $\cdot$ on S, called addition and multiplication such that $(S, +)$ is a commutative semigroup and $(S, \cdot)$ is a semigroup, connected by the distributive laws namely, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all a, b, $c \in S$.

We shall write simply ab instead of writing $a \cdot b$ for all a, $b \in S$.

By a commutative semiring $(S, +, \cdot)$, we mean S is commutative with respect to multiplication. A semiring S is called a semiring with zero element ‘0’ if $a+0 = 0+a = a$ and $0 \cdot a = a \cdot 0 = 0$ for all $a \in S$. A semiring S is called a semiring with identity element 1 if $1 \cdot a = a \cdot 1 = a$ for all $a \in S$. We assume $1 \neq 0$.

Throughout this thesis, unless otherwise stated, semirings are assumed to be commutative with zero and identity elements.

In **Chapter 0**, some important preliminaries are given for their frequent use in the sequel.

Prime ideals play a central role in commutative ring theory and so this notion has been generalized and studied in several directions. The importance of some of these generalizations is same as the prime ideals, say primary ideals. The notion of 2-prime (resp. weakly 2-prime) ideals as a generalization of prime (resp. weakly prime) ideals in a commutative ring was introduced and studied by Beddani and Messirdi in [16] (resp. by Suat Koç [60]) and in a commutative semigroup by Khanra and Mandal [57]. Moreover, rings in which concepts of 2-prime, primary ideals coincide and rings in which 2-prime ideals are prime have been studied by Nikandish et al.[80]. These observations tempted us to study 2-prime (resp. weakly 2-prime) ideals in a commutative semiring in *Chapter 1*.

In **Chapter 1**, the notions of 2-prime and n-weakly 2-prime (resp. weakly 2-prime) ideals in a commutative semiring have been introduced and studied. A characterization of valuation semiring has been obtained in terms of 2-prime ideals. Relationships among primary, quasi-primary, and 2-prime ideals have also been established. Semirings where

2-prime ideals are prime and semirings where every proper ideal is n-weakly 2-prime (resp. weakly 2-prime) have also been characterized.

A. Badawi [12], [13] introduced 2-absorbing (resp. weakly 2-absorbing) ideals in commutative rings as a generalization of prime ideals. The concept of 2-absorbing ideals in a commutative semiring was introduced by A. Y. Darani [19]. A proper ideal I of a semiring S is called a 2-absorbing ideal if for all a, b, $c \in S$ such that $abc \in I$ implies $ab \in I$ or $bc \in I$ or $ca \in I$. The concept of 1-absorbing prime (resp. weakly 1-absorbing prime) ideals which is an another extension of prime (resp. weakly prime) ideals was introduced in [58], [97]. Also, the notions of 1-absorbing primary and weakly 1-absorbing primary ideals were investigated in [14, 15, 79]. A proper ideal I of a ring R is called 1-absorbing prime if for all non-unit elements $a, b, c \in R$ such that $abc \in I$, then either $ab \in I$ or $c \in I$. We generalize this notion of 1-absorbing prime ideals in a commutative semiring in *Chapter 2*.

In **Chapter 2**, the notions of 1-absorbing prime ideals and weakly 1-absorbing prime ideals of commutative semirings have been introduced and studied. The relationships among 1-absorbing prime ideals, prime ideals, 2-prime ideals, and 2-absorbing ideals have also been investigated. The concept of 1-absorbing prime ideals of subtractive valuation domain has been studied. Some of the important properties, results, and characterizations of 1-absorbing prime (resp. weakly 1-absorbing prime) ideals have been investigated.

In 2017, Tekir et al., [96] introduced the concept of n-ideals in commutative rings. A proper ideal I of a commutative ring R is called an n-ideal if whenever a, $b \in R$ and $ab \in I$ such that $a \notin \sqrt{0}$, then $b \in I$. Set of n-ideals is a subclass of primary ideals. Note that the set of primary ideals (in any ring) is never empty since every prime ideal is primary. However, by [[96], Theorem 2.12], n-ideals exist only when $\sqrt{0}$ is prime. In the next chapter, we generalize this notion of n-ideals in a commutative semiring.

In **Chapter 3**, the notion of n-ideals has been introduced in commutative semirings and various important properties have been studied. Characterization of those semirings in which every proper ideal is n-ideal have been obtained. Characterization of entire semirings in terms of n-ideals has also been established. Several examples are discussed of such a class of ideals. Moreover, n-ideals under various contexts of constructions such as homomorphic images, localizations, direct products, and expectation semirings have been studied.

On the other hand, there is a large literature dealing with ternary algebra. The notion of ternary algebraic system was first introduced by H. Prufer [82] by the name

'Schar'. The notion of ternary semigroup was introduced by S. Banach. A non-empty set S endowed with a ternary operation, called ternary multiplication satisfying the ternary associative law: $(abc)de = a(bcd)e = ab(cde)$ for all a, b, c, d, $e \in S$, is called a ternary semigroup. After the introduction of ternary semigroup, many mathematicians have taken interest to study ternary semigroup [27], [55], [56], [91], [92]. In the year 1932, D. H. Lehmer [68] investigated certain ternary algebraic systems called triplexes which turn out to be ternary groups. According to Lehmer, a ternary semigroup S is called a ternary group if for a, b, $c \in S$, the equations $abx = c$, $axb = c$ and $xab = c$ have unique solutions in S. In 1971, Lister [69] introduced ternary rings, alongside presenting various representations for this mathematical construct. According to Lister [69], a ternary ring is an algebraic system consisting of a non-empty set S together with a binary operation, called addition and a ternary multiplication, which forms a commutative group relative to addition, a ternary semigroup relative to multiplication and left, right, lateral distributive laws hold. In 2003, Dutta and Kar [32] introduced the notion of a ternary semiring which generalizes the notion of a ternary ring. A ternary semiring is an algebraic system consisting of a set S together with a binary operation '+', called addition, and a ternary multiplication, denoted by juxtaposition, which forms a commutative semigroup relative to addition, a ternary semigroup relative to multiplication and the left, right, lateral distributive laws hold, i.e., for all $a, b, c, d \in S$, $(a+b)cd = acd + bcd$, $a(b+c)d = abd + acd$, $ab(c+d) = abc + abd$. Series of studies on ternary semirings have been done by T.K. Dutta et al. [32], [33], [34], [35], [36], [54].

The notion of quasi ideal was introduced by Otto Steinfeld both in semigroups and rings [93]. Iseki introduced and studied the concept of quasi ideal for semiring [51], [50], [52]. The notion of bi-ideals in semigroups introduced by Lajos [65] and the notion of bi-ideal in semirings is a special case of (m, n)-ideal also introduced by S. Lajos. Dixit and Dewan [26] studied about the quasi-ideals and bi-ideals in ternary semigroups. S. Kar [53] generalized the concept of quasi-ideals and bi-ideals in ternary semirings. M. M. K. Rao introduced the notion of bi-quasi ideals in semirings [84]. Keeping this view in mind we study left bi-quasi ideals of ternary semirings in *Chapter 4*.

In **Chapter 4**, the notions of left bi-quasi ideals and bi-quasi ideals of a ternary semiring have been introduced, which are extensions of the concept of bi-ideals in a ternary semiring. The concepts of minimal left bi-quasi ideals and left bi-quasi simple ternary semirings have also been studied. Characterization of regular ternary semirings in terms of left bi-quasi ideals has been obtained.

The notions of prime ideal and its generalization have an important place in commutative algebra, for their applications in many areas such as graph theory, coding theory, information science, algebraic geometry, topological spaces, etc. In 2016, C. Beddani and W. Messirdi [16] introduced the concept of 2-prime ideals as a generalization of prime ideals in a ring. In [59], S. Koc, U. Tekir, and G. Ulucak introduced a new class of ideals, an intermediate class between the class of primary ideals and the class of quasi-primary ideals in a ring and is called the class of strongly quasi primary ideals. A proper ideal P in a commutative ring R is said to be strongly quasi primary if $ab \in P$ for some a, $b \in R$ implies either $a^2 \in P$ or $b^n \in P$ for some positive integer n. We generalize these notions of rings to ternary semirings in *Chapter 5*.

In **Chapter 5**, we introduce the concept of 3-prime ideals in a ternary semiring as a generalization of prime ideals. The concepts of 3-prime ideals and primary ideals have been generalized, namely as quasi 3-primary ideals in a commutative ternary semiring with zero. The relationships among prime ideals, 3-prime ideals, primary ideals, quasi primary ideals, and quasi 3-primary ideals have been investigated. Various results and examples concerning 3-prime ideals and quasi 3-primary ideals have also been given. Analogous theorems to the primary avoidance theorem for quasi 3-primary ideals have been established.

The field of algebraic hyperstructures is a firmly established branch within classical algebraic theory. It was originally introduced by the French mathematician F. Marty [72] in 1934. In a classical algebraic structure, the composition of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the composition of two elements is a non-empty subset of the same set. A mapping $\circ : A \times A \to P^*(A)$ is called a hyperoperation, where $P^*(A)$ is the set of all non-empty subsets of A. An algebraic system $(A, \circ)$ is called a hypergroupoid. A hyperoperation '$\circ$' is called associative if $a \circ (b \circ c) = (a \circ b) \circ c$ for all a, b, $c \in A$, which means that $\cup_{u \in b \circ c} a \circ u = \cup_{v \in a \circ b} v \circ c$. A hypergroupoid with the associative hyperoperation is called a semihypergroup. Hyperrings, a subclass of hyperstructures, were introduced through various approaches by different researchers. M. Krasner [62] introduced hyperrings, characterized by addition as a hyperoperation and multiplication as a binary operation. On the other hand, M. D. Salvo [87] introduced hyperrings where both addition and multiplication are hyperoperations. Rota [85] delved into the study of hyperrings, where addition is considered a binary operation and multiplication is treated as a hyperoperation. These particular hyperrings are referred to as multiplicative hyperrings.

R. Procesi Ciampi and R. Rota [81] introduced hypersemiring in the year 1987. A hypersemiring is a non-empty set S together with hyperoperations $+$ and $\circ$ such that (i) $(S,+)$ is a commutative semihypergroup (ii) $(S,\circ)$ is a semihypergroup (iii) $(x+y)\circ z = x\circ z + y\circ z$, $x\circ(y+z) = x\circ y + x\circ z$, where for any subsets A, B of S, $A+B = \{a+b : a\in A \text{ and } b\in B\}$, (iv) there exists $0\in S$ such that $0+x=\{x\}$ and $0\circ x = \{0\} = x\circ 0$, for all $x\in S$, (v) there exists $1\in S$ with $1\neq 0$ such that $1\circ x = \{x\} = x\circ 1$. The notion of semihyperring $(S,+,\cdot)$, where $(S,+)$ is a semihypergroup, $(S,\cdot)$ is a semigroup and the operation $\cdot$ is both left and right distributive across the hyperopreration $+$, was introduced by S. Chaopraknoi and Y. Kemprasit. The multiplicative hypersemiring was introduced by M. K. Sen and U. Dasgupta [90] in 2008. The notion of multiplicative hypersemiring is a generalization of the idea of semiring.

Another class of hyperstructures, known as multiplicative ternary hyperrings, was introduced by T.K. Dutta et al. [86] in 2015. In this context, addition serves as a binary operation, while multiplication is defined as a ternary hyperoperation. A multiplicative ternary hyperring $(R,+,\circ)$ is an additive commutative group $(R,+)$ endowed with a ternary hyperoperation '$\circ$' such that the following conditions hold : (i) $(a\circ b\circ c)\circ d\circ e = a\circ(b\circ c\circ d)\circ e = a\circ b\circ(c\circ d\circ e)$; (ii) $(a+b)\circ c\circ d \subseteq a\circ c\circ d + b\circ c\circ d$; $a\circ(b+c)\circ d \subseteq a\circ b\circ d + a\circ c\circ d$; $a\circ b\circ(c+d) \subseteq a\circ b\circ c + a\circ b\circ d$; (iii) $(-a)\circ b\circ c = a\circ(-b)\circ c = a\circ b\circ(-c) = -(a\circ b\circ c)$ for all a, b, $c\in R$; (iv) $0_R\circ x\circ y = x\circ 0_R\circ y = x\circ y\circ 0_R = \{0_R\}$ for all x, $y\in R$ for all a, b, c, d, $e\in S$. We remark here that, if the inclusions in (ii) are replaced by equalities, then the multiplicative ternary hyperring is called a strongly distributive multiplicative ternary hyperring.

Subsequently, in 2018, N. Tamang and M. Mandal [95] defined and conducted a comprehensive study of ternary hypersemirings. These structures represent a generalization encompassing both multiplicative ternary hyperrings and ternary semirings.

When we have a ternary semiring denoted as $(S,+,\circ)$, we can regard $(S,+,\circ)$ as a strongly distributive ternary hypersemiring by defining $a\circ b\circ c = \{abc\}$ for all a, b, $c\in S$. This particular ternary hypersemiring is referred to as "trivial ternary hypersemiring". For more details about ternary hypersemiring, we refer to [71], [95].

Fundamental relations represent a pivotal and intriguing concept within the realm of algebraic hyperstructures. They play a vital role in the derivation of ordinary algebraic structures from these hyperstructures. In the domain of hypergroups, the inaugural fundamental relation in hypergroups known as the β^*-relation, was introduced by Koskas [61] in 1970. Building upon this foundation, Freni [41] later presented the γ-relation for hypergroups, as a generalization of the initial β relation. Within the class of hyperrings, a variety of fundamental relations have been defined over time, each playing a crucial role in transforming a given hyperstructure into an equivalent structure. The primary among these is the γ^*-relation, innovatively introduced by Vougiouklis [100] on a hyperring R, where both addition and multiplication are hyperoperations. This relation leads to a quotient structure that aligns with the classical ring framework. Subsequently, fundamental relations have garnered attention from a multitude of researchers, including B. Davvaz [23] and V. Leoreanu [22], R. Ameri et al. [3], S. Mirvakili and B. Davvaz [73], and Leoreanu-Fotea [21] and many others.

In **Chapter 6**, an equivalence relation δ^* on a ternary hypersemiring S has been introduced. It has been observed that δ^* is the smallest strongly regular relation on S so that the quotient structure is a ternary semiring. The notion of a fundamental ternary semiring with respect to the fundamental relation δ^* on a ternary hypersemiring S has been introduced. Every ternary semiring with unital element is a fundamental ternary semiring has been established. After that, the fundamental relation δ^* in terms of a typical kind of subsets, called ternary strong $\mathcal{C}$-set of ternary hypersemiring S has been studied.

In **Chapter 7**, the notion of prime hyperideals, radical of hyperideals, primary hyperideals and maximal hyperideals in ternary hypersemirings have been introduced and some of their properties analogs to the properties in hypersemirings have been studied. Then corresponding to a strongly distributive ternary hypersemiring S, a ternary semiring $P_\circ(S)$ has been constructed and various results have been obtained among them. An inclusion preserving bijection between the set of all prime hyperideals of S and collection of all prime total subtractive ideals of $P_\circ(S)$ has been established. Finally, the concept of prime and primary avoidance theorems in ternary hypersemirings for $\mathcal{C}$-ternary hyperideals have also been obtained.

List of Abbreviations and Notations

The notations and abbreviations used throughout the thesis is explained as and when it is introduced. In spite of this we give below, for convenience of the readers, a list of notations and abbreviations used frequently in the thesis.

$\mathbb{N}$	The set of all non-negative integers
$\mathbb{Z}$	The set of all integers
$\mathbb{Z}^-$	The set of all negative integers
$\mathbb{Z}_0^-$	The set of all negative integers with zero
$\mathbb{Z}_0^+$	The set of all positive integers with zero
$\mathbb{Z}_n$	Congruent classes of integers modulo n
$< a >$	The ideal generated by a
$U(S)$	The set of all units of semiring S
$Nil(S)$	The set of all nilpotent elements of semiring S
$Z(S)$	The collection of all zero-divisors of S
$Z_I(S)$	$= \{s \in S : rs \in I$ for some $r \in S \setminus I\}$
$\sqrt[2]{I}$	$= \{x \in S : x^2 \in I\}$
I_n	$= (\{x^n : x \in I\})$, an ideal generated by nth powers of elements of an ideal I
$Id(S)$	The collection of all ideals of semiring S
$(I : x)$	The ideal $\{r \in S : rx \in I\}$
(S, M)	The semiring S with unique maximal ideal M
$Spec(S)$	The set of all prime ideals of semiring S

CHAPTER 0

PRELIMINARY IDEAS

0

Preliminary Ideas

In this chapter, some basic notions and results of semirings, ternary semirings, hypersemirings, and ternary hypersemirings have been recalled in order to use them in the sequel.

0.1 Semirings

We recall the following preliminary notions of semiring theory from [1], [5], [7], [9], [10], [19], [31], [42], [44], [47], [49], [74], [89].

Definition 0.1.1. [49] A non-empty set S together with two binary operations addition '+' and multiplication '·' is called semiring if:

(1) $(S, +)$ is a commutative semigroup.

(2) $(S, \cdot)$ is a semigroup.

(3) The multiplication both from left and right is distributes over addition.

A semiring S is called a semiring with zero element '0' if $a + 0 = 0 + a = a$ and $0 \cdot a = a \cdot 0 = 0$ for all $a \in S$.

A semiring S is called a semiring with identity element '1' if $1 \cdot a = a \cdot 1 = a$ for all $a \in S$.

Definition 0.1.2. [42] A semiring S is commutative if $ab = ba$ for all $a, b \in S$.

Throughout this thesis, unless otherwise stated, semirings are assumed to be commutative with zero element '0' and identity element $1(\neq 0)$.

Definition 0.1.3. A non-empty subset I of a semiring S is called an ideal of S if for a, $b \in I$ and $r \in S$, $a + b \in I$ and $ra \in I$.

Definition 0.1.4. An ideal I of a semiring S is called a proper ideal of S if $I \neq S$.

Definition 0.1.5. Let I and J be two ideals of a semiring S. Then the sum of ideals I and J is defined as $I + J = \{i + j : i \in I, j \in J\}$.

Definition 0.1.6. Let I and J be two ideals of a semiring S. Then for the ideals I and J, IJ is defined as $IJ = \{\sum_{i=1}^{n} a_i b_i : a_i \in I, b_i \in J, n \in \mathbb{N}\}$.

Definition 0.1.7. [42] An ideal I of a semiring S is called subtractive ideal (or k-ideal) if a, $a + b \in I$ and $b \in S$, implies $b \in I$.

Definition 0.1.8. [42] A semiring S is called subtractive if every ideal of S is a subtractive ideal.

Definition 0.1.9. [42] Annihilator of an element a in a semiring S is defined as $\text{Ann}(a) = \{x \in S : ax = 0\}$.

Definition 0.1.10. For an ideal I of S and $a \in S$ define $(I : a) = \{x \in S : ax \in I\}$.

It is easy to see that if I is a subtractive ideal of S, then $(I : a)$ is a subtractive ideal of S.

Definition 0.1.11. [102] An element a of S is called regular if there exists an element $b \in S$ such that $aba = a$.
A semiring S is regular semiring if every element of S is regular.

Theorem 0.1.12. *[50] A semiring S is regular if and only if $R \cap L = RL$ for every right ideal R and every left ideal L of S.*

Definition 0.1.13. [5] A proper ideal I of a semiring S is said to be a strong ideal if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$.

Definition 0.1.14. [75] Radical of an ideal I is defined as $\sqrt{I} = \{a \in S : a^n \in I$ for some positive integer $n\}$.

Proposition 0.1.15. *[75] Let S be a semiring and I, J be ideals of S. Then the following statements hold:*

1. *$I \subseteq \sqrt{I}$ and $\sqrt{I} = \sqrt{\sqrt{I}}$.*

2. $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

3. $\sqrt{I} = S$ *if and only if* $I = S$.

4. $\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}}$.

Theorem 0.1.16. *[75] Let S be a semiring and I an ideal of S. Then the following statements hold:*

1. $\sqrt{I} = \bigcap_{P \in V(I)} P$, *where* $V(I) = \{P \in \text{ Spec}(S) : P \supseteq I\}$.

2. $\sqrt{I}$ *is an ideal of* S.

Definition 0.1.17. [42] An element $u \in S \setminus \{0\}$ is called a unit if there exists an element u' such that $uu' = 1$.

Definition 0.1.18. [9] A non-zero element $a \in S$ is said to be a semi-unit in S if there exists $r, s \in S$ such that $1 + ra = sa$.

Definition 0.1.19. [42] An element a of S is called a zero-divisor of S if there exists $0 \neq b \in S$ such that $ab = 0$.

Definition 0.1.20. [42] An element a in a semiring S is said to be nilpotent if there exists a positive integer n (depending on a) such that $a^n = 0$.

Definition 0.1.21. [48] A semiring S is called a reduced semiring if it has no non-zero nilpotent element.

Definition 0.1.22. [42] An ideal I of a semiring S is irreducible if and only if, for any two ideals A, B of S, $A \cap B = I$ implies $A = I$ or $B = I$.

Definition 0.1.23. [42] An ideal I is strongly irreducible if and only if, for ideals A and B of S, we have $A \cap B \subseteq I$ only when $A \subseteq I$ or $B \subseteq I$.

Definition 0.1.24. [42] Let $(S_1, +, \cdot)$ and $(S_2, +, \cdot)$ be two semirings with zero elements 0_{S_1}, 0_{S_2} and identity elements 1_{S_1}, 1_{S_2} respectively, a mapping $f : S_1 \longrightarrow S_2$ is said to be a semiring homomorphism if $f(a+b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$, $f(0_{S_1}) = 0_{S_2}$ and $f(1_{S_1}) = 1_{S_2}$ for all $a, b \in S_1$.

A homomorphism f is called a monomorphism (respectively, epimorphism) if f is injective (respectively, surjective). A homomorphism f is called isomorphism if f is both injective and subjective.

Definition 0.1.25. [42] Let S_1 and S_2 be semirings and $f : S_1 \longrightarrow S_2$ be a homomorphism. Then its kernel is defined as $\ker f = \{a \in S_1 : f(a) = 0\}$ and its image is defined as $\mathrm{im} f = \{r \in S_2 : r = f(a)$ for some $a \in S_1\}$.

Definition 0.1.26. [67] Let I be a proper ideal of a semiring S. Then the congruence on S, denoted by ρ_I and defined by $s_1 \rho_I s_2$ if and only if $s_1 + a_1 = s_2 + a_2$ for some a_1, $a_2 \in I$, is called the Bourne congruence on S defined by the ideal I.

We denote the Bourne congruence ρ_I class of an element r of S by r/I and denote the set of all such congruence classes by S/I.

Definition 0.1.27. [42] Let I be a proper ideal of S. The Bourne factor semiring or simply the factor semiring is defined under the following addition and multiplication on S/I by $p/I + q/I = (p + q)/I$ and $(p/I)(q/I) = pq/I$ for all p, $q \in S$.

Definition 0.1.28. [75] Let U be a multiplicatively closed subset of a semiring S. The relation is defined on the set $S \times U$ by $(s, u) \sim (t, b) \iff xsb = xat$ for some $x \in U$ is an equivalence relation and the equivalence class of $(s, a) \in S \times U$ denoted by s/a. The set of all equivalence classes of $S \times U$ under "$\sim$" is denoted by S_U. S_U forms a semiring, where addition and multiplication are defined by $s/a + t/b = (sb + ta)/ab$ and $(s/a)(t/b) = st/ab$.
Let I be an ideal of S. Then the set $I_U = \{a/b : a \in I, b \in U\}$ is an ideal of S_U. The set I_U is called the localization of the ideal I at U.

Definition 0.1.29. [42] Consider $S = S_1 \times S_2$ where each S_i, $i = 1, 2$ is a semiring and $(a_1, a_2) + (b_1, b_2) = (a_1 + a_2, b_1 + b_2)$, $(a_1, a_2) \cdot (b_1, b_2) = (a_1b_1, a_2b_2)$ for all a_1, $b_1 \in S_1$ and a_2, $b_2 \in S_2$. Then $S_1 \times S_2$ forms a semiring with the operations of addition and multiplication defined, which is called direct product semiring.

Definition 0.1.30. [9], An ideal I of a semiring S is called a partitioning ideal (= Q-ideal) if there exists a subset Q of S such that $S = \cup\{q + I : q \in Q\}$ and if q_1, $q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq \phi$ if and only if $q_1 = q_2$. Let I be a Q-ideal of a semiring S and let $S/I_Q = \{q + I : q \in Q\}$. Then S/I_Q forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$, and $(q_1 + I) \odot (q_2 + I) = q_4 + I$ where $q_4 \in Q$ is the unique element such that $q_1q_2 + I \subseteq q_4 + I$. This semiring S/I_Q is called the quotient semiring of S by I. By definition of Q-ideal, there exists a unique $q' \in Q$ such that $0 + I \subseteq q' + I$. Then $q' + I$ is a zero element of S/I_Q. Clearly, if S is commutative, then so is S/I_Q.

Definition 0.1.31. [74] A semiring S is a semidomain if it is multiplicatively cancellative, i.e., if $ab = ac$ and $a \neq 0$, then $b = c$ for any $a, b, c \in S$.

Definition 0.1.32. [76] A semiring S is a valuation semiring if it is a semidomain and the set of ideals are totally ordered by inclusion.

Definition 0.1.33. [42] A semiring S is said to be an entire semiring if $ab = 0$ implies either $a = 0$ or $b = 0$ for any $a, b \in S$.

Definition 0.1.34. [42] An ideal I of a semiring S is called principal if $I = \{sa : s \in S\}$ for some $a \in S$.

Definition 0.1.35. [74] A semiring S is called a principal ideal semidomain if S is a semidomain and each ideal of S is principal.

Proposition 0.1.36. *[74] Each nonzero prime ideal of a principal ideal semidomain is maximal.*

Definition 0.1.37. [17] A prime ideal P of S is said to be a divided prime ideal of S if $P \subset \langle x \rangle$ for every $x \in S \setminus P$. If each prime ideal of S is divided, then S is called divided semiring.

Proposition 0.1.38. *[17] Any valuation semiring is a divided semiring.*

Definition 0.1.39. [42] A semifield is a commutative semiring in which the non-zero elements form a group under multiplication.

Definition 0.1.40. [42] A semiring S is called Noetherian if it satisfies the ascending chain conditions on its ideals.

Definition 0.1.41. [42] A proper ideal I of S is called a prime if for any a, $b \in S$, $ab \in I$ implies either $a \in I$ or $b \in I$.

The equivalent definition of prime ideal of a commutative semiring S is as follows:

Definition 0.1.42. [50] A proper ideal P of a semiring S is called a prime ideal if for any ideals I, $J \subseteq S$, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Definition 0.1.43. [6] A proper ideal I of S is called a weakly prime if for any a, $b \in S$, $0 \neq ab \in I$ implies either $a \in I$ or $b \in I$.

The equivalent definition of weakly prime ideal of a commutative semiring S is as follows:

Definition 0.1.44. [31] A proper ideal P of a semiring S is called a weakly prime ideal if for any ideals $I,\ J \subseteq S$, $\{0\} \neq IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Definition 0.1.45. [2] A prime ideal P of a semiring S is called a minimal prime ideal of S, if for every prime ideal I, $I \subseteq P$ implies either $I = \langle 0 \rangle$ or $I = P$.

Theorem 0.1.46. *[77] Let $P \supseteq I$ be ideals of a semiring S, where P is prime. Then P is a minimal prime ideal of I if and only if for each $x \in P$, there is some $y \in S \setminus P$ and a nonnegative integer i such that $yx^i \in I$.*

Definition 0.1.47. [42] A proper ideal I of a semiring S is primary if for all $x, y \in S$, we have that $xy \in I$ and $x \notin I$ implies $y^n \in I$ for some positive integer n.

If I is primary, then $\sqrt{I} = P$ is a prime ideal of S [[2], Theorem 38]. In this case, we also say that I is a P-primary ideal of S.

Definition 0.1.48. [19] A proper ideal I of a semiring S is called a 2-absorbing ideal (resp. weakly 2-absorbing ideal) if for any $a, b, c \in S$, $abc \in I$ (resp. $0 \neq abc \in I$) implies $ab \in I$ or $bc \in I$ or $ca \in I$.

Definition 0.1.49. [42] A proper ideal M is maximal (resp. k-maximal) if $M \subseteq M' \subseteq S$ implies either $M = M'$ or $M' = S$ for any ideal (resp. k-ideal) M' of S.

Definition 0.1.50. An ideal I of S is called quasi-primary if $\sqrt{I}$ is a prime ideal.

Theorem 0.1.51. *[2] Let S be a commutative semiring with an identity. If M is a maximal ideal in S, then M is prime.*

The following example shows that a maximal ideal in a semiring without an identity may not be prime.

Example 0.1.52. [2] Let S denote the semiring of positive even integers with the usual addition and multiplication. If $M = \{x \in S : x > 2\}$, then M is a maximal ideal in S. Since $2 \notin M$ and $2 \cdot 2 = 4 \in M$, it follows that M is not prime.

Theorem 0.1.53. *[75] Let S be a Noetherian semiring and I a subtractive ideal of S. If I is irreducible, then it is primary.*

Definition 0.1.54. [75] A semiring is said to be a local semiring if it has only one maximal ideal.

Theorem 0.1.55. *[75] A semiring S is a local semiring if and only if $S \setminus U(S)$ is an ideal of S.*

Definition 0.1.56. [9] The Jacobson radical $Jac(S)$ of a semiring S is defined as the intersection of all maximal k-ideals of S.

Definition 0.1.57. [42] Let $(M, +, 0)$ be a commutative additive monoid. The monoid M is said to be an S-semimodule if S is a semiring and there is a function, called scalar product, $\lambda : S \times M \longrightarrow M$, defined by $\lambda(s, m) = sm$ such that the conditions are satisfied: $s(m + n) = sm + sn$ for all $s \in S$ and $m, n \in M$; $(s + t)m = sm + tm$ and $(st)m = s(tm)$ for all $s, t \in S$ and $m \in M$; $s \cdot 0 = 0$ for all $s \in S$ and $0 \cdot m = 0$ and $1 \cdot m = m$ for all $m \in M$.

Definition 0.1.58. [42] A subset N of the S-semimodule M is called a subsemimodule of M if $a, b \in N$ and $r \in S$ implies $a + b \in N$ and $ra \in N$.

Definition 0.1.59. [42] A subtractive subsemimodule (= k-subsemimodule) N is a subsemimodule of M such that if $x, x + y \in N$, then $y \in N$.

Definition 0.1.60. [42] An S-subsemimodule M is called subtractive if each subsemimodule of M is subtractive.

If N is a proper subsemimodule of an S-semimodule M, then we denote $(N : M) = \{s \in S : sM \subseteq N\}$ and $\sqrt{(N : M)} = \{s \in S : s^n M \subseteq N \text{ for some n} \in \mathbb{N}\}$. Clearly, $(N : M)$ and $\sqrt{(N : M)}$ are ideals of S.

Definition 0.1.61. [25] A proper subsemimodule N of M is said to be a prime subsemimodule of M, if $sm \in N$, $s \in S$ and $m \in M$ then either $m \in N$ or $s \in (N : M)$.

Definition 0.1.62. [25] A subsemimodule N of an S-semimodule M is said to be primary if $rm \in N$, $r \in S$, $m \in M$, then either $r \in \sqrt{(N : M)}$ or $m \in N$.

Definition 0.1.63. [78] Let S be a semiring and M be an S-semimodule. Then $S \times M$ equipped with two operations addition and multiplications as follows:
(1) $(s_1, m_1) + (s_2, m_2) = (s_1 + s_2, m_1 + m_2)$,
(2) $(s_1, m_1)(s_2, m_2) = (s_1 s_2, s_1 m_2 + s_2 m_1)$
forms a semiring, which is denoted by $S \tilde{\oplus} M$, is called the expectation semiring of the S-semimodule M.

Note that in general, if $T \subseteq S$ and $N \subseteq M$, then by $T \tilde{\oplus} N$, we mean that the set of all ordered pairs (t, n) such that $t \in T$ and $n \in N$.

0.2 Ternary Semirings

Definition 0.2.1. [32] A non-empty set S together with a binary operation called addition and a ternary multiplication, denoted by juxtaposition is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:

(1) $(abc)de = a(bcd)e = ab(cde)$,

(2) $(a + b)cd = acd + bcd$,

(3) $a(b + c)d = abd + acd$,

(4) $ab(c + d) = abc + abd$ for all a, b, c, d, $e \in S$.

Example 0.2.2. [33] Let S be a set of continuous functions $f : X \to \mathbb{R}^-$, where X is a topological space and $\mathbb{R}^-$ is the set of all negative real numbers. Define a binary addition and a ternary multiplication on S as follows: For f, g, $h \in S$ and $x \in X$,

(1) $(f + g)(x) = f(x) + g(x)$,

(2) $(fgh)(x) = f(x)g(x)h(x)$.

Then with respect to the binary addition and ternary multiplication, S forms a ternary semiring.

Let A, B and C be three subsets of S. By ABC, we mean the set of all finite sums of the form $\sum a_i b_i c_i$ with $a_i \in A$, $b_i \in B$ and $c_i \in C$.

Definition 0.2.3. [32] A ternary semiring S is called a commutative ternary semiring if $abc = bac = bca$ for all a, b, $c \in S$.

Definition 0.2.4. [32] An additive subsemigroup T of S is called a ternary subsemiring if $t_1t_2t_3 \in T$ for all t_1, t_2, $t_3 \in T$.

Definition 0.2.5. [32] An additive subsemigroup I of S is called a left (resp. right, lateral) ideal of S if s_1s_2i (resp. is_1s_2, s_1is_2) $\in I$, for all s_1, $s_2 \in S$ and $i \in I$. If I is both a left and a right ideal of S, then I is called a two-sided ideal of S. If I is a left, a right, and a lateral ideal of S, then I is called an ideal of S.

Definition 0.2.6. [33] An ideal I of a ternary semiring S is said to be a subtractive ideal (k-ideal) if for x, $y \in S$, $x + y \in I$ and $y \in I$ implies $x \in I$.

Definition 0.2.7. [32] An element a in a ternary semiring S is called regular if there exists an element x in S such that $axa = a$.

Definition 0.2.8. [32] A ternary semiring is called regular if all of its elements are regular.

Definition 0.2.9. [37] A proper ideal P of a ternary semiring S is called a prime ideal if for any three ideals A, B and C of S, $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

Corollary 0.2.10. *[37] A proper ideal P of a commutative ternary semiring S is prime if and only if $abc \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$ for all elements a, b, $c \in S$.*

Definition 0.2.11. [38] A proper ideal Q of a ternary semiring S is called a semiprime ideal of S if $I^3 \subseteq Q$ implies $I \subseteq Q$ for any ideal I of S.

Corollary 0.2.12. *[38] A proper ideal Q of a commutative ternary semiring S is semiprime if and only if $x^3 \in Q$ implies that $x \in Q$ for any element x of S.*

Definition 0.2.13. [38] Let S be a ternary semiring and A be an ideal of S. The radical of A, denoted by $Rad(A)$, is defined to be the intersection of all the prime ideals of S each of which contains A. In a commutative ternary semiring S, $Rad(A) = \{a \in S : a^{2n+1} \in A$ for some positive integer n$\}$.

Definition 0.2.14. [38] A proper ideal I of a ternary semiring S is called a strongly irreducible if for any two ideals, H and K of S, $H \cap K \subseteq I$ implies $H \subseteq I$ or $K \subseteq I$.

Lemma 0.2.15. *[99] Let S be a commutative ternary semiring and I be an ideal of S. Then $(I : a : b)$ is an ideal in S, where $(I : a : b) = \{c \in S : abc \in I\}$.*

Definition 0.2.16. [83] A proper ideal P of a commutative ternary semiring S is called primary if for any $a, b, c \in S$, $abc \in P$ implies $a \in P$ or $b \in P$ or $c^{2n+1} \in P$ for some positive integer n.

Definition 0.2.17. [83] An ideal I of a commutative ternary semiring S is called quasi primary if $Rad(I)$ is prime.

Definition 0.2.18. [53] An additive subsemigroup Q of a ternary semiring S is called a quasi-ideal of S if $QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$.

Definition 0.2.19. [53] A ternary subsemiring B of a ternary semiring S is called a bi-ideal of S if $BSBSB \subseteq B$.

Definition 0.2.20. [30] Let P be an ideal of S. Then P is said to be strongly nilpotent if there exists a positive integer n such that $(PS)^{n-1}P = 0$.

0.3 Multiplcative Hypersemiring

Definition 0.3.1. [90] A multiplicative hypersemiring is an additive commutative semigroup $(S, +)$ endowed with a hyperoperation '$\circ$' such that

(i) $(S, \circ)$ is a semihypergroup,

(ii) $(x + y) \circ z \subseteq x \circ z + y \circ z$, and $x \circ (y + z) \subseteq x \circ y + x \circ z$, $\forall x,\, y,\, z \in S$

(where for any $A,\, B \in P(S)$, $A + B = \{a + b : a \in A \,\text{and}\, b \in B\}$).

If inclusions in (ii) are replaced by equalities, the multiplicative hypersemiring is then referred to as strongly distributive.

Definition 0.3.2. [90] A multiplicative hypersemiring $(S, +, \circ)$ is commutative (resp. weakly commutative) if $x \circ y = y \circ x$ (resp. $x \circ y \cap y \circ x \neq \phi$), for all $x, y \in S$.

Definition 0.3.3. [90] A multiplicative hypersemiring $(S, +, \circ)$ is said to have an absorbing (resp. a strongly absorbing) zero $0 \in S$ if (i) $a + 0 = 0 + a = a$ and (ii) $0 \in a \circ 0 = 0 \circ a$ (resp. $a \circ 0 = 0 \circ a = \{0\}$), for all $a \in S$.

Definition 0.3.4. [90] An element $e \in S \setminus (S \circ 0 \cup 0 \circ S)$ is called a hyperidentity of the multiplicative hypersemiring $(S, +, \circ)$ if for all $x \in S$, $x \in e \circ x = x \circ e$.

Definition 0.3.5. [20] Let $(S, +, \circ)$ be a multiplicative hypersemiring. A non-empty subset I of S is called a left (resp. right) hyperideal of S if

(i) $a + b \in I$ for all $a,\, b \in I$,

(ii) $x \circ a \subseteq I$ (resp. $a \circ x \subseteq I$), for all $x \in S$ and $a \in I$.

A non-empty subset I of S is called a hyperideal of the multiplicative hypersemiring $(S, +, \circ)$ if I is both left hyperideal and right hyperideal of S.

Definition 0.3.6. [20] A hyperideal I of a multiplicative hypersemiring $(S, +, \circ)$ is called a prime hyperideal if for any hyperideals $J,\, K$ of S, $JK \subseteq I \implies J \subseteq I$ or $K \subseteq I$.

Proposition 0.3.7. *[20] Let $(S, +, \circ)$ be a multiplicative hypersemiring with hyperidentity and I be a hyperideal of S. Then the following conditions are equivalent:*

(i) I is a prime hyperideal,

(ii) For any $a,\, b \in S$, $\cup\{a \circ x \circ b : x \in S\} \subseteq I \iff a \in I$ or $b \in I$,

(iii) For any $a,\, b \in S$, $< a >< b > \subseteq I \implies a \in I$ or $b \in I$.

0.4 Ternary Hypersemirings

Definition 0.4.1. [22] Ternary hyperoperation on a set A is a map $\circ : A \times A \times A \to P^*(A)$, where $P^*(A)$ is the collection of all subsets of A.

Definition 0.4.2. [95] A ternary hypersemiring $(S, +, \circ)$ is an additive commutative semigroup $(S, +)$, endowed with a ternary hyperoperation '$\circ$' such that the following condition holds:

(i) $(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b \circ (c \circ d \circ e)$;

(ii) $(a + b) \circ c \circ d \subseteq a \circ c \circ d + b \circ c \circ d$;

(iii) $a \circ (b + c) \circ d \subseteq a \circ b \circ d + a \circ c \circ d$;

(iv) $a \circ b \circ (c + d) \subseteq a \circ b \circ c + a \circ b \circ d$; for all $a, b, c, d \in S$.

Definition 0.4.3. [95] A ternary hypersemiring $(S, +, \circ)$ is said to be commutative if for all a_1, a_2, $a_3 \in S$, $a_1 \circ a_2 \circ a_3 = a_{\sigma(1)} \circ a_{\sigma(2)} \circ a_{\sigma(3)}$ where σ is a permutation of $\{1, 2, 3\}$.

If the inclusions in the Definition 0.4.2 (ii), (iii), (iv) are replaced by equalities, then the ternary hypersemiring is called a strongly distributive ternary hypersemiring.

Definition 0.4.4. [95] Let $(S, +, \circ)$ be a ternary hypersemiring. An element $0 \in S$ is called a zero element or absorbing zero or simply zero of S if $0 \in 0 \circ x \circ y = x \circ 0 \circ y = x \circ y \circ 0$ for all $x, y \in S$ (strongly absorbing zero if $0 \circ x \circ y = x \circ 0 \circ y = x \circ y \circ 0 = \{0\}$).

Definition 0.4.5. [95] An additive subsemigroup T of a ternary hypersemiring $(S, +, \circ)$ is called a ternary subhypersemiring if $t_1 \circ t_2 \circ t_3 \subseteq T$ for all $t_1, t_2, t_3 \in T$.

Definition 0.4.6. [95] Let $(S, +, \circ)$ be a ternary hypersemiring. A finite subset $\epsilon = \{(e_i, f_i); i = 1, 2...., n\}$ of $S \times S$ is called a left (lateral or right) identity set of S if for any $a \in S$, $a \in \Sigma_{i=1}^{n} e_i \circ f_i \circ a$ ($a \in \Sigma_{i=1}^{n} e_i \circ a \circ f_i$ or $a \in \Sigma_{i=1}^{n} a \circ e_i \circ f_i$).

A non-empty finite subset $\epsilon = \{(e_i; f_i); i = 1, 2....n\}$ of $S \times S$, where S is a ternary hypersemiring is called an identity set if it is a left, a lateral and a right identity set of S.

An element $e \in S$ is called a hyperidentity or unital element of S if $a \in (e \circ e \circ a) \cap (e \circ a \circ e) \cap (a \circ e \circ e)$ for all $a \in S$.

Definition 0.4.7. [95] Let $(S, +, \circ)$ be a ternary hypersemiring. An additive subsemigroup I of S is called (i) a left hyperideal of S if $s_1 \circ s_2 \circ i \subseteq I$ for all $s_1,\ s_2 \in S$ and $i \in I$.
(ii) a right hyperideal of S if $i \circ s_1 \circ s_2 \subseteq I$ for all $s_1,\ s_2 \in S$ and $i \in I$.
(iii) a lateral hyperideal of S if $s_1 \circ i \circ s_2 \subseteq I$ for all $s_1,\ s_2 \in S$ and $i \in I$.
(iv) a two-sided hyperideal of S if I is both a left and a right hyperideal of S.
(v) a hyperideal of S if I is a left, a right, and a lateral ideal of S.

Definition 0.4.8. [95] Let $(S, +, \circ)$ be a ternary hypersemiring. If A, B and C are three non-empty subsets of S, then $A \circ B \circ C = \cup\{\sum_{finite} a_i \circ b_i \circ c_i : a_i \in A, b_i \in B, c_i \in C\}$.

Throughout this thesis, we denote $A \circ B \circ C$ by ABC.

Definition 0.4.9. [40] Let $(R, +, \circ)$ and $(S, +, \circ)$ be ternary hypersemirings. A mapping $f : R \to S$ is said to be homomorphism if $f(a+b) = f(a)+f(b)$ and $f(a \circ b \circ c) \subseteq f(a) \circ f(b) \circ f(c)$. In particular, a homomorphism is called a good homomorphism if $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c)$.

Definition 0.4.10. [95] A hyperideal I is said to be k-hyperideal of a ternary hypersemiring S if $x + y \in I, x \in S$ and $y \in I$ imply that $x \in I$.

Definition 0.4.11. [40] Given an equivalence relation δ on a non-empty set S, two relations $\bar{\delta}$ and $\bar{\bar{\delta}}$ on the power set $P^*(S)$ are defined as follows:

(i) $U\bar{\delta}V$ if and only if for each element u in set U, there exists an element v in set V such that $u\delta v$ holds, and for each v' in set V, there exists an u' in set U such that $u'\delta v'$ holds.

(ii) $U\bar{\bar{\delta}}V$ is true if and only if for all $u \in U$ and $v \in V$, $u\delta v$ holds.

Definition 0.4.12. [40] An equivalence relation δ defined on a ternary hypersemiring $(S, +, \circ)$ is classified as follows:

(i) δ is considered regular if it is a congruence on the commutative semigroup $(S, +)$, which implies that for any $x,\ y,\ z \in S$, if $x\delta y$ holds, then $(x + z)\delta(y + z)$, and if $x\delta u,\ y\delta v,\ z\delta w$, then $(x \circ y \circ z)\bar{\delta}(u \circ v \circ w)$ for $x,\ y,\ z,\ u,\ v,\ w \in S$.

(ii) δ is termed strongly regular when it is congruence on the commutative semigroup $(S, +)$, this means that for any $x,\ y,\ z \in S$, if $x\delta y$, then $(x + z)\delta(y + z)$, and if $x\delta u,\ y\delta v,\ z\delta w$, then $(x \circ y \circ z)\bar{\bar{\delta}}(u \circ v \circ w)$ for any $x, y, x, u, v, w \in S$.

The second conditions mentioned in (i) and (ii) of this definition can be equivalently expressed as follows: For all u, v, x, y in S, (i) In the regular case: $u\delta v$ implies $(u \circ x \circ y)\,\bar{\delta}\,(v \circ x \circ y)$, $(x \circ u \circ y)\,\bar{\delta}\,(x \circ v \circ y)$, and $(x \circ y \circ u)\,\bar{\delta}\,(x \circ y \circ v)$ (ii) In the strongly regular case: $u\delta v$ implies $(u \circ x \circ y)\,\bar{\bar{\delta}}\,(v \circ x \circ y)$, $(x \circ u \circ y)\,\bar{\bar{\delta}}\,(x \circ v \circ y)$, and $(x \circ y \circ u)\,\bar{\bar{\delta}}\,(x \circ y \circ v)$.

Theorem 0.4.13. *[95] If A, B, and C are respectively right, lateral and left hyperideals of a ternary hypersemiring S, then $ABC \subseteq A \cap B \cap C$.*

CHAPTER 1

2-PRIME IDEALS OF SEMIRINGS

1

2-prime Ideals of Semirings

In this chapter, the notions of 2-prime and n-weakly 2-prime (resp. weakly 2-prime) ideals in a commutative semiring have been introduced and studied. A characterization of valuation semiring has been obtained in terms of 2-prime ideals. Relationships among primary, quasi-primary and 2-prime ideals have also been obtained. Semirings where 2-prime ideals are prime and semirings where every proper ideal is n-weakly 2-prime (resp. weakly 2-prime) have also been characterized.

This chapter has been organized as follows: In *Section 1*, we first introduce the notion of 2-prime ideals in commutative semirings (*cf.* Definition 1.1.1). We obtain the relationships of 2-prime ideals with prime and quasi-primary ideals (cf. Lemma 1.1.2). Then we prove that every maximal ideal of a semiring without unity is also 2-prime (*cf.* Theorem 1.1.6). We then define valuation ideal in a semiring (*cf.* Definition 1.1.9) and prove that a semidomain is a valuation semiring if and only if every proper ideal of the semidomain is 2-prime (cf. Theorem 1.1.11). Also, we show that in a principal ideal semidomain the concepts of 2-prime ideals, primary ideals and quasi-primary ideals coincide (*cf.* Theorem 1.1.15). We then prove 2-prime ideals are preserved under the semiring homomorphism with certain conditions (*cf.* Theorem 1.1.17). Then we study the structure of 2-prime ideals in product semirings (*cf.* Theorem 1.1.19) and expectation semirings (*cf.* Theorem 1.1.21).

This chapter is mainly based on the works published in the following paper:

- **Sampad Das et al.,** ***A note on 2-prime and n-weakly 2-prime ideal of semiring, Quasigroups and Related Systems*, 30 (2022), 241- 256**.

In *Section 2*, we characterize semirings in which 2-prime ideals are prime, defined as 2-P-semiring (*cf.* Definition 1.2.1, Theorem 1.2.3), and obtain some results of 2-P-semiring (*cf.* Theorems 1.2.6, 1.2.7).

In *Section 3*, we define n-weakly 2-prime (resp. weakly 2-prime) ideals in a semiring (*cf.* Definitions 1.3.1 & 1.3.2). We then characterize semirings in which every proper ideal is weakly 2-prime (resp. n-weakly 2-prime) (cf. Theorem 1.3.5) (resp. cf. Theorem 1.3.6) and also studied some further properties of these ideals.

1.1 2-prime ideals of commutative semirings

Definition 1.1.1. A proper ideal I of a semiring S is said to be a 2-prime ideal if $xy \in I$ for some $x, y \in S$ implies either $x^2 \in I$ or $y^2 \in I$.

The following lemma is obvious, hence we omit the proof.

Lemma 1.1.2. (1) *Every prime ideal of S is a 2-prime ideal of S.*
(2) *Every 2-prime ideal of S is a quasi-primary ideal of S.*

Therefore, if I is a 2-prime ideal of S, then $\sqrt{I} = P$ is a prime ideal of S.

Remark 1.1.3. For a 2-prime ideal I of a semiring S, we refer to the prime ideal $P = \sqrt{I}$ as the associated prime ideal of I and I is referred to as a P-2-prime ideal of S.

The following examples show that the converse of the above lemma may not be true.

Example 1.1.4. (1) Consider the ideal $I = \{m \in \mathbb{N} : m \geq 3\} \cup \{0\}$ in the semiring $S = \{\mathbb{N} \cup \{0\}, +, \cdot\}$. Clearly, I is 2-prime but not a prime ideal of S, since $2 \cdot 2 \in I$ but $2 \notin I$.

Example 1.1.5. Consider the ideal $I = (\{X_n^n\}_{n=1}^{\infty})$ in the semiring $S = \mathbb{Z}_2[\{X_i\}_{i=1}^{\infty}]$. Clearly, I is a quasi-primary ideal of S, since $\sqrt{I}$ is a prime ideal of S. But I is not a 2-prime ideal of S, as $X_6^2 \cdot X_6^4 = X_6^6 \in I$ and neither $(X_6^2)^2 \notin I$ nor $(X_6^4)^2 \notin I$.

If S is a semiring with unity, then every maximal ideal of S is prime (cf. Theorem 0.1.51) and hence 2-prime. If S is a semiring without unity then maximal ideal of S need not be prime for example see (cf. Example 0.1.52) but there is a relation between maximal and 2-prime ideal of S, as follows:

Theorem 1.1.6. *Let S be semiring without unity and assume maximal ideal exists. Then every maximal ideal of S is a 2-prime ideal of S.*

Proof. Let $xy \in M$ with $x^2 \notin M$ for some x, $y \in S$, where M is a maximal ideal of S. If $y^2 \notin M$, then clearly x, $y \in S - M$. Hence $M + (x) = M + (y) = S$. Since $x \in S$, $x^2 = (p + s_1x + n_1x)(q + s_2y + n_2y)$ for some p, $q \in M$, s_1, $s_2 \in S$ and n_1, $n_2 \in \mathbb{Z}_0^+$. This implies $x^2 \in M$, a contradiction. Consequently, $y^2 \in M$. Hence M is a 2-prime ideal of S. □

Proposition 1.1.7. *Let I be an ideal of a semiring S.*

(1) If I is a 2-prime ideal of S, then there is exactly one prime ideal of S that is minimal over I.

(2) If I is a prime ideal of S, then I^2 is a 2-prime ideal of S.

(3) An ideal I of S is prime if and only if it is both 2-prime and semiprime.

(4) If I is a 2-prime ideal of S and J_1, J_2,..., J_n are ideals of S such that $\bigcap J_i \subseteq \sqrt{I}$, then $J_i \subseteq \sqrt{I}$ for some $i \in \{1, 2, ..., n\}$.
In particular, if $\bigcap J_i = \sqrt{I}$, then $J_i = \sqrt{I}$ for some $i \in \{1, 2, ..., n\}$.

(5) If I is a P-2-prime ideal of S, then $(I : a^2)$ is a 2-prime ideal of S, for all $a \in S$ such that $a^2 \notin I$. In particular $(I : a^2)$ is a P-2-prime ideal of S for all $a \in S - \sqrt{I}$.

(6) If I is a 2-prime ideal of S and $(I : a) = (I : a^2)$ for all $a \in S - I$, then $(I : a)$ is a 2-prime ideal of S.

(7) If I is a proper ideal of S and U be a multiplicatively closed subset of S, then the following statements hold:
(i) If I is a 2-prime ideal of S such that $I \cap A = \phi$, then I_U is a 2-prime ideal of S_U.
(ii) If I_U is a 2-prime ideal of S_U with $Z_I(S) \cap S = \phi$, then I is a 2-prime ideal of S.

(8) If I is a P-primary ideal for some prime ideal P of S such that $P^2 \subseteq I$. Then I is a 2-prime ideal of S.

Proof. (1) If possible, let J_1 and J_2 be two distinct prime ideals that are minimal over I. Hence there exist $j_1 \in J_1 - J_2$ and $j_2 \in J_2 - J_1$. By Theorem 0.1.46, there is $a_1 \notin J_1$ and $a_2 \notin J_2$ such that $a_1 j_1^n \in I$ and $a_2 j_2^m \in I$ for some integer m, $n \geq 1$. Since j_1, $j_2 \notin I \subseteq J_1 \cap J_2$ and I is 2-prime , hence $a_1^2 \in I \subseteq J_1 \cap J_2$ and $a_2^2 \in I \subseteq J_1 \cap J_2$. Therefore $a_1^2 \in J_1$. Since J_1 is prime, $a_1 \in J_1$-a contradiction. Similarly, if $a_2^2 \in J_2$, then $a_2 \in J_2$-a contradiction. Hence there is exactly one prime ideal which is minimal over I.

(2) Since $I^2 \subseteq I$ for any ideal I of S, it is clear.

(3) If an ideal I is prime, then clearly it is 2-prime and semiprime.
Conversely, let $ab \in I$ for some a, $b \in S$. Since I is 2-prime we have $a^2 \in I$ or $b^2 \in I$, which implies $a \in I$ or $b \in I$, since I is semprime also. Consequently, I is a prime ideal of S.

(4) Let $J_i \nsubseteq \sqrt{I}$ for all $i \in \{1, 2, ..., n\}$. Then there exists $a_i \in J_i$ but $a_i \notin \sqrt{I}$ for all $i \in \{1, 2, ..., n\}$. Let $x = a_1 a_2 ... a_n$. Then $x \in \bigcap J_i$ but $x \notin \sqrt{I}$, since $\sqrt{I}$ is a prime ideal of S, a contradiction . Hence $J_i \subseteq \sqrt{I}$ for some $i \in \{1, 2, ..., n\}$.
Again if, $\bigcap J_i = \sqrt{I}$, then $\sqrt{I} \subseteq J_i$ for all $i \in \{1, 2, ..., n\}$. Hence $J_i = \sqrt{I}$ for some $i \in \{1, 2, ..., n\}$.

(5) Let $xy \in (I : a^2)$ with $x^2 \notin (I : a^2)$ for x, $y \in S$. Then $xya^2 = (xa)(ya) \in I$. Hence $(ya)^2 = y^2 a^2 \in I$, since I is a 2-prime ideal of S and $x^2 a^2 \notin I$. Consequently, $(I : a^2)$ is a 2-prime ideal of S.

Again let $a \in S - P$ and $x \in (I : a^2)$. Then $a^2 x \in I \subseteq P$. Hence $x^2 \in I$, since $a \notin P$ and I is a 2-prime ideal of S. Thus $I \subseteq (I : a^2) \subseteq P$, which implies $P = \sqrt{I} \subseteq \sqrt{(I : a^2)} \subseteq \sqrt{P} = P$. Consequently $(I : a^2)$ is a P-2-prime ideal of S.

(6) Clearly follows from (5).

(7) (*i*) Let $(a/s)(b/t) \in I_U$ for some a, $b \in S$ and s, $t \in U$. Then there exists $u \in U$ such that $abu \in I$. Then $a^2 \in I$ or $b^2 u^2 \in I$, since I is a 2-prime ideal of S. If $a^2 \in I$, then $(a/s)^2 = (ua^2/us^2) \in I_U$ and if $b^2 u^2 \in I$ then $(b/s)^2 = (b^2 u^2/s^2 u^2) \in I_U$. Therefore I_U is a 2-prime ideal of S_U.

(*ii*) Let $xy \in I$ for some x, $y \in S$. Then $\frac{x}{1}\frac{y}{1} \in I_U$ implies $\frac{x^2}{1} \in I_U$ or $\frac{y^2}{1} \in I_U$.

Hence $ax^2 \in I$ or $by^2 \in I$ for some $a, b \in S$. Since $U \cap Z_I(S) = \phi$, we have either $x^2 \in I$ or $y^2 \in I$, as desired.

(8) Let $ab \in I$ for some $a, b \in S$, where I is a P-primary ideal of S such that $P^2 \subseteq I$. Then either $a \in I$ or $b \in \sqrt{I} = P$. If $a \in I$, then $a^2 \in I^2$ and if $b \in P$, then $b^2 \in P^2 \subseteq I$. Consequently, I is a 2-prime ideal of S.

□

Theorem 1.1.8. *Let P be a proper ideal of a semiring S. Then the following statements are equivalent:*

(1) P is a 2-prime ideal of S,

(2) for any ideals J, K of S with $JK \subseteq P$, either $J_2 \subseteq P$ or $K_2 \subseteq P$,

(3) for every $s \in S$, either $(s) \subseteq (P : s)$ or $(P : s) \subseteq \sqrt[2]{P}$,

(4) for any ideals A and B of S with $AB \subseteq P$, either $A_2 \subseteq P$ or $B \subseteq \sqrt[2]{P}$,

(5) for every $s \in S$, either $s^2 \in P$ or $(P : s)_2 \subseteq P$.

Proof. (1) ⇒ (2) Let P be a 2-prime ideal of a semiring S and $JK \subseteq P$ for some ideal J, K of S with $J_2 \nsubseteq P$. Then there exists an element $p \in J$ such that $p^2 \notin P$. If possible, let $K_2 \nsubseteq P$. Then there exists $k \in K$ such that $k^2 \notin P$. Also we have $pk \in P$. Since P is a 2-prime ideal of S either $p^2 \in P$ or $k^2 \in P$, a contradiction. Therefore, $K_2 \subseteq P$.
(2) ⇒ (1) Let $ab \in P$ for some a,$b \in S$ and $a^2 \notin P$. Let $J = (a)$ and $K = (b)$. Then $JK \subseteq P$ and $J_2 \nsubseteq P$, otherwise $a^2 \in P$. Hence $K_2 \subseteq P$ implies $b^2 \in P$. Consequently, P is a 2-prime ideal of S.
(1) ⇒ (3) Let $s \in S$. If $s^2 \in P$, then $s \in (P : s)$ implies $(s) \subseteq (P : s)$. Let $s^2 \notin P$ and $r \in (P : s)$ for some $r \in S$. Hence $rs \in P$ implies $r^2 \in P$, since P is 2-prime and $s^2 \notin P$. Consequently, $(P : s) \subseteq \sqrt[2]{P}$.
(3) ⇒ (4) Let $AB \subseteq P$ for some ideals A, B of S. Let $B \nsubseteq \sqrt[2]{P}$. Then there exists $b \in B - \sqrt[2]{P}$ and $ab \in P$ for all $a \in A$. Since $b \in (P : a) - \sqrt[2]{P}$, we have $(P : a) \nsubseteq \sqrt[2]{P}$. Hence by hypothesis, $(a) \subseteq (P : a)$ implies $a^2 \in P$. Consequently $A_2 \subseteq P$.
(4) ⇒ (5) Let $s \in S$. If $s^2 \in P$, there is nothing to prove. So let $s^2 \notin P$ and $A = (P : s)$, $B = (s)$. Then $AB = (P : s)(s) \subseteq P$. Since $B \nsubseteq \sqrt[2]{P}$, we have $A_2 = (P : s)_2 \subseteq P$.
(5) ⇒ (1) Let $xy \in P$ with $x^2 \notin P$ for some x, $y \in S$. Then $y \in (P : x)$. Hence by hypothesis, $y^2 \in (P : s)_2 \subseteq P$, as desired. □

The concept of valuation semiring has been defined by P. Nasehpour in [76], here we define valuation ideal of a semiring, as follows:

Definition 1.1.9. Let S be a semidomain and K be its semifield of fractions. Then an ideal I in S is a valuation ideal if I is the intersection of S with an ideal of a valuation semiring S_v containing S. Moreover, if v is the corresponding M-valuation we say I is a valuation ideal associated with the M-valuation v or I is a v-ideal.

Lemma 1.1.10. *Let v be an M-valuation on K and I an ideal of a semidomain S. Then the following are equivalent:*
(1) *I is a valuation ideal,*
(2) *for each $x \in S$, $y \in I$, the inequality $v(x) \geq v(y)$ implies $x \in I$,*
(3) *I is of the form $I = S_v I \cap S$.*

Proof. The proof is similar to ([101], page 340). □

Theorem 1.1.11. *Let S be a semidomain. Then the following are equivalent:*
(1) *Every ideal of S is 2-prime,*
(2) *every principal ideal of S is 2-prime,*
(3) *S is a valuation semiring.*

Proof. (1) ⇒ (2) It is clear.
(2) ⇒ (3) Let $x \in K - \{0\}$, where K is the semifield of fractions of S. Then $x = \frac{a}{b}$ for some $a, b \in S - \{0\}$. Consider the principal ideal $I = (ab)$ of S. Then I is 2-prime and since $ab \in (ab) = I$, we have $a^2 \in I$ or $b^2 \in I$. If $a^2 \in I$, then there exists an element $c \in S$ such that $a^2 = cab$, hence $x = \frac{a}{b} = c \in S$. Similarly, if $b^2 \in I$, we have $x^{-1} \in S$. Consequently, S is a valuation semiring (*cf.* Theorem 2.4, [76]).
(3) ⇒ (1) Let I be a v-ideal on S where v is a valuation on S. Let $xy \in I$ for some $x, y \in S$. If $v(x) \geq v(y)$, we get $v(x^2) \geq v(xy)$ and as I is a v-ideal we have $x^2 \in I$. Similarly, $v(y) \geq v(x)$ implies $y^2 \in I$. Consequently, I is a 2-prime ideal of S. □

The following lemmas are obvious, hence we omit the proof

Lemma 1.1.12. *Let S be a semidomain and $a, b \in S - \{0\}$. Then a and b are associates if and only if $(a) = (b)$.*

Lemma 1.1.13. *Let S be a semidomain and $p \in S - \{0\}$. Then p is an irreducible element of S if and only if (p) is a maximal ideal of S.*

Lemma 1.1.14. *Let I be a P-primary ideal of a semiring S. Then P is the unique minimal prime ideal of I in S.*

Proof. Let Q be another minimal prime of I in S. Then $I \subseteq Q$ implies $P = \sqrt{I} \subseteq \sqrt{Q} = Q$. Hence P is the unique minimal prime ideal of I in S. □

Theorem 1.1.15. *Let I be a proper ideal of a principal ideal semidomain S. Then the following are equivalent:*
(1) *I is a quasi-primary ideal of S,*
(2) *I is a primary ideal of S,*
(3) *I is of the form (p^n), where n is a positive integer and $p = 0$ or an irreducible element of S,*
(4) *I is a 2-prime ideal of S.*

Proof. (1) ⇒ (2) Since every nonzero prime ideal of a principal ideal semidomain S is a maximal ideal ($cf.$ Proposition 0.1.36), it follows clearly from ($cf.$ Theorem 40, [2]).
(2) ⇒ (1) It is obvious.
(2) ⇒ (3) Let I be a nonzero primary ideal of S. Then $I = (a)$ for some nonzero non-unit element $a \in S$. Since every principal ideal semidomain is a unique factorization semidomain ($cf.$ Theorem 3.2, [74]), a can written as a product of irreducible elements of S. If a is divisible by two irreducible elements x and y of S, which are not associates, then by Lemma 1.1.12 and 1.1.13, (x) and (y) would be distinct maximal ideals of S. Therefore, they would both minimal prime ideal of (a), which contradicts Lemma 1.1.14. Hence $I = \{(p^n) : p = 0$ or p is an irreducible elements of S and $n \in \mathbb{N}\}$.
(3) ⇒ (2) Since S is a semidomain, $\{0\}$ is prime and hence primary. Let p be an irreducible element of S and $n \in \mathbb{N}$, then by Lemma 1.1.13, (p^n) is a power of a maximal ideal and hence is a primary ideal of S ($cf.$ Theorem 40, [2]).
(3) ⇔ (4) The proof is similar as that of ($cf.$ Theorem 2.3, [80]). □

Example 1.1.16. Let I be an ideal of a von neuman regular semiring S. Then $I = I^2 = \sqrt{I}$ ($cf.$ Proposition 1, [94]). Hence the concepts of prime, primary, 2-prime and semiprimary ideal coincide in a regular semiring S.

Theorem 1.1.17. *Let $f : S_1 \to S_2$ be a homomorphism of semirings. Then the following statements hold:*
(1) *If J is a 2-prime ideal of S_2, then $f^{-1}(J)$ is a 2-prime ideal of S_1.*
(2) *If f is onto steady homomorphism such that $kerf \subseteq I$ and I is a 2-prime subtractive ideal of S_1, then $f(I)$ is a 2-prime subtractive ideal of S_2.*

Proof. (1) Let $ab \in f^{-1}(J)$ for some $a, b \in S_1$. Then $f(ab) \in J$, hence $f(a^2) \in J$ or $f(b^2) \in J$, since f is a homomorphism and J is a 2-prime of S_2. Therefore $a^2 \in f^{-1}(J)$ or $b^2 \in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is a 2-prime ideal of S_1.
(2) Let $xy \in f(I)$ for some $x, y \in S_2$. Then there exists $a, b \in S_1$ such that $f(a) = x$ and $f(b) = y$. Then $xy = f(a)f(b) = f(ab) \in f(I)$. Hence $f(ab) = f(r)$ for some $r \in I$. So we have $ab + s = r + t$ for some $s, t \in I$, since f is steady. Hence $ab \in I$, since $\ker f \subseteq I$ and I is a subtractive ideal of S_1. Hence either $a^2 \in I$ or $b^2 \in I$, since I is a 2-prime ideal of S_1. Thus either $f(a^2) \in f(I)$ or $f(b^2) \in f(I)$. Consequently, $f(I)$ is a 2-prime subtractive ideal of S_2. □

Corollary 1.1.18. *If $S \subseteq R$ is an extension of semiring and I is a 2-prime ideal of R, then $I \cap S$ is a 2-prime ideal of S.*

Theorem 1.1.19. *Let $S = S_1 \times S_2$ and $I = I_1 \times I_2$, where I_i are ideals of S_i for $i = 1, 2$. Then the following are equivalent:*
(1) I is a 2-prime ideal of S,
(2) $I_1 = S_1$ and I_2 is a 2-prime ideal of S_2 or $I_2 = S_2$ and I_1 is a 2-prime ideal of S_1.

Proof. (1) ⇒ (2) Let I be a 2-prime ideal of S. Then $\sqrt{I} = \sqrt{I_1} \times \sqrt{I_2}$ is a prime ideal of S. Hence either $I_1 = S_1$ or $I_2 = S_2$. Let $I_2 = S_2$ and $ab \in I_1$ for some $a, b \in S_1$. Then $(a, 1)(b, 1) \in I$. Hence $(a, 1)^2 \in I$ or $(b, 1)^2 \in I$, since I is a 2-prime ideal of S. This implies $a^2 \in I_1$ or $b^2 \in I_1$. Consequently, I_1 is a 2-prime of S_1. Similarly, if $I_1 = S_1$, we can show that I_2 is a 2-prime ideal of S_2.
(2) ⇒ (1) Assume $I_1 = S_1$ and I_2 is a 2-prime ideal of S_2. Let $(a, x)(b, y) \in I$ for some $a, b \in S_1$ and $x, y \in S_2$. Then $xy \in I_2$ and this implies $x^2 \in I_2$ or $y^2 \in I_2$. Hence $(a, x)^2 \in I$ or $(b, y)^2 \in I$, as desired. In a similar way, one can prove the other case. □

Corollary 1.1.20. *Let $S = S_1 \times S_2 \times ... \times S_n$ and $I = I_1 \times I_2 \times ... \times I_n$, where I_i are ideals of S_i and $n \in \mathbb{N}$. Then the following are equivalent:*
(1) I is a 2-prime ideals of S,
(2) I_i is a 2-prime ideal of S_i for some $i \in \{1, 2, ..., n\}$ and $I_j = S_j$ for all $j \neq i$.

Proof. By using Theorem 1.1.19 and induction on n, the proof is straightforward. □

Theorem 1.1.21. *Let M be a S-semimodule, I a proper ideal of S and $N \neq M$ an S-subsemimodule of M. Then*
(1) If $I\tilde{\oplus}N$ is a 2-prime ideal of $S\tilde{\oplus}M$, then I is a 2-prime ideal of S.
(2) If the ideal I of S is 2-prime and $\sqrt[2]{I}M \subseteq N$, then $I\tilde{\oplus}N$ is a 2-prime ideal of $S\tilde{\oplus}M$.

Proof. (1) Let $ab \in I$ with $a^2 \notin I$ for some a, $b \in S$. Then $(a, 0)(b, 0) \in I\tilde{\oplus}N$ while $(a, 0)^2 \notin I\tilde{\oplus}N$. Hence $(b, 0)^2 \in I\tilde{\oplus}N$, since $I\tilde{\oplus}N$ is a 2-prime ideal of $S\tilde{\oplus}M$. Consequently, $b^2 \in I$, as desired.
(2) Let $(a, m)(b, n) \in I\tilde{\oplus}N$ for some a, $b \in S$, m, $n \in M$. This implies $ab \in I$ implies $a^2 \in I$ or $b^2 \in I$. If $a^2 \in I$, then $am \in \sqrt[2]{I}M \subseteq N$ and this yields $(a, m)^2 = (a^2, 2am) \in I\tilde{\oplus}N$. Again if $b^2 \in I$, we have $(b, m)^2 \in I\tilde{\oplus}N$. Consequently, $I\tilde{\oplus}N$ is a 2-prime ideal of $S\tilde{\oplus}N$. □

1.2 2-P-semiring

Definition 1.2.1. A semiring S is said to be a 2-P-semiring if 2-prime ideals of S are prime.

Example 1.2.2. Every idempotent semiring is a 2-P-semiring.

Theorem 1.2.3. *A semiring S is 2-P-semiring if and only if one of the following conditions hold:*
(1) 2*-prime ideals are semiprime.*
(2) *Prime ideals are idempotent and every* 2*-prime ideal is of the form A^2, where A is a prime ideal of S.*

Proof. $\Rightarrow$ (1) If S is a 2-P-semiring, clearly 2-prime ideals are semiprime.
Converse follows easily from Proposition 1.1.7 (3).
$\Rightarrow$ (2) Let P be a prime ideal of a 2-P-semiring S. Then P^2 is a prime ideal of S (*cf.* Proposition 1.1.7(2)) and hence $P \subseteq P^2$. Clearly $P^2 \subseteq P$. Therefore prime ideals of S are idempotent. Again, let I be a 2-prime ideal of S. Then I is prime and hence $I = I^2$.
Conversely, let I be a 2-prime ideal of S. Then $I = P^2 = P$ for some prime ideal P of S. Consequently, S is a 2-P semiring. □

Lemma 1.2.4. *Let (S, M) be a local semiring. Then for every prime ideal I of S, IM is a* 2*-prime ideal of S. Furthermore, IM is prime if and only if $IM = I$*

Proof. Let $xy \in IM \subseteq I$. Then either $x \in I$ or $y \in I$, since I is a prime ideal of S. Let $x \in I$ implies $x^2 \in IM$, since $I \subseteq M$. Hence IM is a 2-prime ideal of S. □

Definition 1.2.5. Let I be an ideal of a semiring S. We define a 2-prime ideal P to be a minimal 2-prime ideal over I if there does not exist a 2-prime ideal K of S such that $I \subseteq K \subset P$. We denote the set of minimal 2-prime ideals over I by 2-$\mathrm{Min}_S(I)$.

Theorem 1.2.6. *Let S be a subtractive semiring with unique maximal ideal M such that $(\sqrt{I})^2 \subseteq I$ for every 2-prime ideal I of S. Then the following statements are equivalent:*
(1) *S is a 2-P-semiring,*
(2) *if P is the minimal prime ideal over a 2-prime ideal I, then $IM = P$,*
(3) *for every prime ideal P of S, 2-$Min_S(P^2) = \{P\}$.*

Proof. (1) $\Rightarrow$ (2) Let P be the minimal prime ideal over a 2-prime ideal I of a 2-P-semiring S. Then clearly $IM = P$ ($cf.$ Lemma 1.2.4).
(2) $\Rightarrow$ (1) Let I be a 2-prime ideal of a subtractive semiring S with unique maximal ideal M and P is the minimal prime ideal over I such that $IM = P$. Then $I \subseteq P = IM \subseteq I \cap M = I$ implies $I = P$. Hence S is a 2-P-semiring.
(2) $\Rightarrow$ (3) Let P be a prime ideal of S and I be a 2-prime ideal of S such that $I \in$ 2-$\text{Min}_S(P^2)$. Let J be a prime ideal of S such that $I \subseteq J \subseteq P$. Clearly, $P^2 \subseteq I \subseteq J \subseteq P$. Let $a \in P$ then $a^2 \in P^2$. Therefore $a^2 \in J$ implies $a \in J$, since J is prime. Hence $J = P$. Now by hypothesis, $IM = P$ implies $P = IM \subseteq I \subseteq P$. Consequently, 2-$\text{Min}_S(P^2) = \{P\}$.
(3) $\Rightarrow$ (2) Let P is the minimal prime ideal over a 2-prime ideal I of S. Then $\sqrt{I} = P$. Hence by hypothesis, $P^2 \subseteq I \subseteq P$. Therefore 2-$\text{Min}_S(P^2) = \{P\}$. Clearly $I = P$ implies IM is 2-prime ($cf.$ Lemma 1.2.4). Now $P^2 \subseteq PM \subseteq P$ so $IM = PM = P$. □

Theorem 1.2.7. *Let $S \subseteq R$ be an extension of semiring and spec(S)=spec(R), where spec(S) and spec(R) denote set of all prime ideals of S and R respectively. If S is a 2-P-semiring, then R is 2-P-semiring.*

Proof. Let I be a 2-prime ideal of R. Then $\sqrt{I} = P \in spec(R) = spec(S)$. Clearly $I \subseteq P$. Also $I \cap S$ is a 2-prime ideal of S ($cf.$ Corollary 1.1.18), hence prime, since S is 2-P-semiring. Therefore $I \cap S = \sqrt{I \cap S} = P$ and $P^2 \subseteq I \cap S$. Let $x \in P$. Then $x^2 \in P^2 \subseteq I \cap S \in spec(A)$. Hence $x \in I \cap S \subseteq I$. Consequently, $I = P$, as desired. □

1.3 n-weakly 2-prime ideals

Definition 1.3.1. A proper ideal I of a semiring S is said to be n-weakly 2-prime if for $a, b \in S$, $ab \in I - I^n$ implies that $a^2 \in I$ or $b^2 \in I$.

Definition 1.3.2. A proper ideal I of a semiring S is said to be a weakly 2-prime ideal of S if $0 \neq xy \in I$ for some $x, y \in S$ implies $x^2 \in I$ or $y^2 \in I$.

The following lemma is obvious, hence we omit the proof.

Lemma 1.3.3. (1) *Every* 2*-prime ideal of* S *is a weakly* 2*-prime ideal of* S*.*
(2) *Every weakly prime ideal of* S *is a weakly* 2*-prime ideal of* S*.*
(3) *Every weakly* 2*-prime ideal of* S *is a* n*-weakly* 2*-prime ideal of* S*.*
(4) *Every* n*-weakly* 2*-prime is a* $(n-1)$*-weakly* 2*-prime ideal, for each* $n \geq 3$*.*

Proposition 1.3.4. *Let* I *be a subtractive ideal of a semiring* S*. Then*
(1) *If* I *is weakly* 2*-prime but not a* 2*-prime ideal of* S*, then* (i) $I^2 = 0$*.*
(ii) $\sqrt{I} = \sqrt{0}$*.*
(2) *Let* (S, M) *be a local semiring with* $M^2 = 0$*. Then every proper subtractive ideal of* S *is a weakly prime and hence weakly* 2*-prime ideal of* S*.*
(3) *Let* P *be a weakly prime ideal of* S *and* Q *be an ideal of* S *containg* P*, then* PQ *is a weakly* 2*-prime ideal of* S*. In particular, for every weakly prime ideal* P *of* S*,* P^2 *is a weakly* 2*-prime ideal of* S*.*
(4) $\sqrt{I}$ *is a prime (resp. weakly prime) ideal of* S *if and only if* $\sqrt{I}$ *is a* 2*-prime (resp. weakly* 2*-prime) ideal of* S*.*
(5) *Let* I *be a* n*-weakly* 2*-prime ideal of* S *and* U *be a multiplicatively closed subset of* S *with* $U \cap I = \phi$ *and* $I_U^n \subseteq (I_U)^n$*. Then* I_U *is a* n*-weakly* 2*-prime ideal of* S_U*.*

Proof. (1) (i)We first show that if $ab = 0$ for some a, $b \in S - I$, then we have $aI = bI = 0$. Let $ai \neq 0$ for some $i \in I$. Then $0 \neq a(b+i) \in I$. Since I is a subtractive weakly 2-prime ideal of S, either $a^2 \in I$ or $b^2 \in I$, a contradiction. Therefore $aI = 0$. Similarly, we can show $Ib = 0$. Now let $xy \neq 0$ for some x, $y \in I$ and $ab = 0$ for some a, $b \notin I$. Then we have $(a+x)(b+y) = xy \neq 0$. Since I is subtractive weakly 2-prime ideal of S, either $a^2 \in I$ or $b^2 \in I$, a contradiction. Hence $I^2 = 0$.
(ii) By (i), $I^2 = \{0\}$. So we have $I \subseteq \sqrt{0}$ implies $\sqrt{I} \subseteq \sqrt{0}$. Also we have $\sqrt{0} \subseteq \sqrt{I}$. Therefore $\sqrt{I} = \sqrt{0}$.
(2) Let I be a proper ideal of a local semiring (S, M) such that $M^2 = 0$ and $0 \neq ab \in I$ for some a, $b \in S$. Then either $a \in M$ or $b \in M$ but both a, b does not belongs to M, otherwise $ab \in M^2 = 0$, a contradiction. Hence a or b must be semi-unit, let a be a semi-unit of S. Then there exists p, $q \in S$ such that $1 + pa = qa$ implies $b + pab = qab \in I$. Also $pab \in I$ implies $b \in I$, since I is a subtructive ideal of S. Similarly, if b is a semi-unit then $a \in I$. Consequently, I is a weakly 2-prime ideal of S, as desired.
(3) Let $0 \neq ab \in PQ$ for some a, $b \in I$. Since $PQ \subseteq P$ and P is weakly prime ideal of S, we have either $a \in P \subseteq Q$ or $b \in P \subseteq Q$. Hence either $a^2 \in PQ$ or $b^2 \in PQ$.

Consequently, PQ is a weakly 2-prime ideal of S, in particular, P^2 is a weakly 2-prime ideal of S.
(4) Since $\sqrt{\sqrt{I}} = \sqrt{I}$ for any ideal I of S, it is straightforward.
(5) Let a, $b \in S$ and x, $y \in U$ such that $\frac{a}{x}\frac{b}{y} \in I_U - (I_U)^n$. Then there exists $u \in U$ such that $uab \in I$. Again $vab \notin I^n$ for any $v \in U$ because if $vab \in I^n$, then $\frac{a}{x}\frac{b}{y} \in I_U \subseteq (I_U)^n$, a contradiction. So $abu \in I - I^n$, implies $a^2 \in I$ or $b^2u^2 \in I$, since I is a n-weakly 2-prime ideal of S. Hence $(\frac{a}{x})^2 \in I_U$ or $(\frac{b}{y})^2 \in I_U$. Thus I_U is a n-weakly 2-prime ideal of S_U. □

The following is a characterization of a semiring in which every proper ideal is weakly 2-prime.

Theorem 1.3.5. *Let S be a semiring. Then every proper ideal of S is weakly 2-prime if and only if $(a^2) \subseteq (ab)$ or $(b^2) \subseteq (ab)$ or $ab = 0$, for any a, $b \in S$ such that $(ab) \neq S$.*

Proof. Let every proper ideal of a semiring S is weakly 2-prime and a, $b \in S$ such that $(ab) \neq S$. If $ab \neq 0$, then $0 \neq ab \in (ab)$ and (ab) is weakly 2-prime, hence $a^2 \in (ab)$ or $b^2 \in (ab)$. Consequently, $(a^2) \subseteq (ab)$ or $b^2 \subseteq (ab)$.

Conversely, let I be a proper ideal of a semiring S and $0 \neq ab \in I$ for some a, $b \in S$. Then $0 \neq ab \in (ab) \subseteq I$ implies $a^2 \in (a^2) \subseteq (ab) \subseteq I$ or $b^2 \in (b^2) \subseteq (ab) \subseteq I$. Hence, I is weakly 2-prime ideal of S, as desired. □

Theorem 1.3.6. *Let I be a subtractive ideal of a semiring S with $I^2 \nsubseteq I^n$. Then I is a 2-prime ideal of S if and only if I is a n-weakly 2-prime ideal of S.*

Proof. Let I be a subtractive n-weakly 2-prime ideal of S such that $I^2 \subseteq I^n$ and $ab \in I$ for some a, $b \in S$. If $ab \notin I^n$, then $a^2 \in I$ or $b^2 \in I$, since I is n-weakly 2-prime. So we assume $ab \in I^n$. First, we suppose $aI \nsubseteq I^n$. Then for some $i \in I$, $ai \notin I^n$ implies $a(b+i) \notin I^n$, since I is subtractive and $ab \in I^n$. Hence $a(b+i) \in I - I^n$ implies $a^2 \in I$ or $b^2 \in I$. So we can assume $aI \subseteq I^n$. Similarly, we can assume $Ib \subseteq I^n$. Now since $I^2 \nsubseteq I^n$, there exists a_1, $b_1 \in I$ such that $a_1b_1 \notin I^n$. Hence $(a + a_1)(b + b_1) \in I - I^n$ because, if $(a+a_1)(b+b_1) \in I^n$ then $a_1b_1 = (a+a_1)(b+b_1) = (ab+aa_1+bb_1+a_1b_1) \in I^n$, which contradicts that $a_1b_1 \notin I^n$. Hence $(a + a_1)^2 \in I$ or $(b + b_1)^2 \in I$, since I is n-weakly 2-prime ideal of S. Therefore $a^2 \in I$ or $b^2 \in I$, since I is subtractive ideal of S, as desired. The other part is obvious. □

Proposition 1.3.7. *Let $f : S \to S_1$ be an epimorphism of semirings such that $f(0) = 0$ and I be a subtractive strong ideal of S. Then*

(1) *If I is a weakly 2-prime ideal of S such that $ker f \subseteq I$, then $f(I)$ is a weakly 2-prime ideal of S_1.*
(2) *If I is a 2-prime ideal of S such that $ker f \subseteq I$, then $f(I)$ is a 2-prime ideal of S_1.*

Proof. (1) Let $a_1, b_1 \in S_1$ be such that $0 \neq a_1 b_1 \in f(I)$. So there exists an element $p \in I$ such that $0 \neq a_1 b_1 = f(p)$. Also there exist $a, b \in S$ such that $f(a) = a_1$, $f(b) = b_1$, since f is an epimorphism. Since I is a strong ideal of S and $p \in I$, there exists $q \in I$ such that $p + q = 0$. This implies $f(p+q) = 0$, that is, $f(ab+q) = 0$, implies $ab + q \in ker f \subseteq I$, Hence $0 \neq ab \in I$, as I is a subtractive ideal of S and if $ab = 0$, then $f(p) = 0$, a contradiction. Thus $a^2 \in I$ or $b^2 \in I$, since I is a weakly 2-prime ideal of S. Thus $a_1^2 \in f(I)$ or $b_1^2 \in f(I)$. Hence, $f(I)$ is a weakly 2-prime ideal of S.
(2) It is clear from (1). □

Proposition 1.3.8. *Let S_1 and S_2 be two semirings and I be a proper ideal of S_1. Then the following are equivalent:*
(1) *I is a 2-prime ideal of S_1,*
(2) *$I \times S_2$ is a 2-prime ideals of $S_1 \times S_2$,*
(3) *$I \times S_2$ is a weakly 2-prime ideals of $S_1 \times S_2$.*

Proof. (1) $\Rightarrow$ (2) Let $(a_1, b_1)(c_1, d_1) \in I \times S_2$ for some $(a_1, b_1) \in S_1 \times S_2$ and $(c_1, d_1) \in S_1 \times S_2$. Then $(a_1 c_1, b_1 d_1) \in I \times S_2$ implies $a_1^2 \in I$ or $c_1^2 \in I$, since I is a 2-prime ideal of S_1. Now if $a_1^2 \in I$, then $(a_1, b_1)^2 = (a_1^2, b_1^2) \in I \times S_2$. Similarly if $c_1^2 \in I$, then $(c_1, d_1)^2 = (c_1^2, d_1^2) \in I \times S_2$. Consequently, $I \times S_2$ is a 2-prime ideal of $S_1 \times S_2$.
(2) $\Rightarrow$ (3) It is clear.
(3) $\Rightarrow$ (1) Let $ab \in I$ for some $a, b \in S$. Then $(0,0) \neq (a,1)(b,1) \in I \times S_2$. This implies $(a^2, 1) \in I \times S_2$ or $(b^2, 1) \in I \times S_2$, since $I \times S_2$ is a 2-prime ideal of $S_1 \times S_2$. Hence, $a^2 \in I$ or $b^2 \in I$, as desired. □

CHAPTER 2

1-absorbing Prime Ideals and Weakly 1-absorbing Prime Ideals of Commutative Semirings

2

1-absorbing Prime Ideals and Weakly 1-absorbing Prime Ideals of Commutative Semirings

In this chapter, the notions of 1-absorbing prime ideals and weakly 1-absorbing prime ideals of commutative semirings have been introduced and studied. The relationships among 1-absorbing prime ideals, prime ideals, 2-prime ideals, and 2-absorbing ideals have also been investigated. The concept of 1-absorbing prime ideals of subtractive valuation semiring has been studied. It has been shown that, if a semiring S admits a 1-absorbing prime ideal which is not a prime ideal, then S is a local semiring. Some of the important properties, results, and characterizations of 1-absorbing prime (resp. weakly 1-absorbing prime) ideals have been investigated.

This chapter has been organized as follows.

In *Section 1*, we first introduce the notion of 1-absorbing prime ideals (*cf.* Definition 2.1.1) as a generalization of prime ideals in a semiring. We then show that these ideals form an intermediate class of ideals between prime ideals and 2-absorbing ideals (*cf.* Examples 2.1.2, 2.1.3). Then observe that every 1-absorbing prime ideal is a 2-prime ideal (*cf.* Theorem 2.1.4) but the converse is not true which is evident from Example 2.1.5. In a commutative non-local semiring, a proper ideal is a 1-absorbing prime if and

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This chapter is mainly based on the works published in the following paper:

- **Sampad Das et al., *On 1-absorbing Prime and Weakly 1-absorbing Prime Ideals in Semirings* (Communicated)**.

only if it is prime (*cf.* Corollary 2.1.9). Then it is shown that in a semiring S, $\langle 0 \rangle$ is a 1-absorbing ideal of S if and only if S is an entire semiring or (S, M) is a local semiring such that $M^2 = \langle 0 \rangle$ (*cf.* Theorem 2.1.12). We then establish a characterization of 1-absorbing prime ideals (*cf*. Theorem 2.1.13). Then obtain various properties of 1-absorbing prime ideals (*cf.* Proposition 2.1.14, Theorems 2.1.16, 2.1.17, 2.1.18, 2.1.21). We show (*cf*. Theorem 2.1.23) that in a subtractive valuation semiring S, an ideal I is a 1-absorbing prime ideal of S if and only if either $I = P$ or $I = P^2$, where $P = \sqrt{I}$ is a prime ideal of S.

In *Section 2*, we define weakly 1-absorbing prime ideal (*cf.* Definition 2.2.1) as a generalization of weakly prime ideals. Some properties of 1-absorbing prime ideals are studied. We characterize a strong local semiring whose all proper ideals are 1-absorbing prime (*cf*. Theorem 2.2.11). It is shown that if S is a local semiring and I is a subtractive weakly 1-absorbing prime ideal of S, then either I is a 1-absorbing prime ideal of S or $I^3 = 0$. (*cf.* Theorem 2.2.9) At the end, we characterize 1-absorbing prime ideals in a decomposable semiring (*cf*. Theorem 2.2.13).

2.1 1-absorbing Prime Ideals

Definition 2.1.1. A proper ideal I of a commutative semiring S is said to be a 1-absorbing prime ideal if whenever $abc \in I$ for some non-units a, b, $c \in S$, then either $ab \in I$ or $c \in I$.

It is clear that every prime ideal is a 1-absorbing prime ideal and every 1-absorbing prime ideal is a 2-absorbing ideal but the converse may not be true in general which can be seen from the following examples.

Example 2.1.2. In semiring $S = \mathbb{Z}_0^+$, the ideal $I = 3\mathbb{Z}_0^+ \setminus \{3\}$ is a 1-absorbing prime ideal which is not a prime ideal.

Example 2.1.3. Let $S = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and $I = 2\mathbb{Z}_0^+ \times 3\mathbb{Z}_0^+$. Then I is a 2-absorbing ideal, however I is not a 1-absorbing prime ideal of S. Indeed, $(2,5)(4,2)(7,3) \in I$ but neither $(2,5)(4,2) \in I$ nor $(7,3) \in I$.

Theorem 2.1.4. *Every 1-absorbing prime ideal of a semiring S is a 2-prime ideal of S.*

Proof. Suppose that I is a 1-absorbing prime ideal of S and $xy \in I$ for some elements x, $y \in S$. If either x or y is a unit, then I is a 2-prime ideal of S. So assume that x, y

are non-unit elements of S. Then $x^2y \in I$, this implies either $x^2 \in I$ or $y \in I$. Thus either $x^2 \in I$ or $y^2 \in I$ and hence I is a 2-prime ideal of S. □

The converse of Theorem 2.1.4 may not be true, as shown in the following example:

Example 2.1.5. In semiring $S = \mathbb{Z}_0^+$, the ideal $I = 4\mathbb{Z}_0^+$ is a 2-prime ideal but I is not a 1-absorbing prime ideal, since $2 \cdot 3 \cdot 2 \in I$ but neither $2 \cdot 3 \in I$ nor $2 \in I$.

So the class of all 1-absorbing prime ideals of a semiring S is an intermediate class between the class of all prime ideals and the class of 2-prime ideals. Also, it is an intermediate class between the class of prime ideals and that of 2-absorbing ideals of S.

Theorem 2.1.6. *Let I be a 1-absorbing prime ideal of a semiring S. Then $x^2 \in I$ for every $x \in \sqrt{I}$ and $\sqrt{I}$ is a prime ideal of S.*

Proof. Suppose $x \in \sqrt{I}$. Then there exists a smallest positive integer n such that $x^n \in I$. For $n = 1$ or $n = 2$, it is clear. For $n \geq 3$, $x^n = xxx^{n-2} \in I$. That implies either $x^2 \in I$ or $x^{n-2} \in I$. Again, if $x^{n-2} \in I$ and $n - 2 \geq 3$, we get either $x^2 \in I$ or $x^{n-4} \in I$. Continuing this process, it is easy to conclude that $x^2 \in I$. Now consider $ab \in \sqrt{I}$ for some a, $b \in S$. If one of a or b is a unit, then there is nothing to prove. Assume that both a and b are non-units and $ab \in \sqrt{I}$. Then $(ab)^2 = aab^2 \in I$, which implies that either $a^2 \in I$ or $b^2 \in I$. That is either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Therefore, $\sqrt{I}$ is a prime ideal of S. □

Theorem 2.1.7. *Let I be a 2-prime ideal of S. If $(P^2 : x) \subseteq I$ for any $x \in P \setminus I$, where $\sqrt{I} = P$. Then I is a 1-absorbing prime ideal of S.*

Proof. Suppose that $xyz \in I$ and $xy \notin I$ for some non-units $x, y, z \in S$. Since I is a 2-prime ideal and $(xy)z \in I$, we obtain either $(xy)^2 \in I$ or $z^2 \in I$. If $(xy)^2 \in I$, then $xy \in P \setminus I$. But then $xy \in (P^2 : xy) \subseteq I$, which is a contradiction. So $z^2 \in I$, that is $z \in P$. If $z \in I$, then we are done. If $z \in P \setminus I$, then $z \in (P^2 : z) \subseteq I$, a contradiction. Therefore, I is a 1-absorbing prime ideal of S. □

Theorem 2.1.8. *Suppose that a semiring S has a 1-absorbing prime ideal, which is not a prime ideal. Then, S is a local semiring.*

Proof. Assume that I is a 1-absorbing prime ideal, which is not a prime ideal of S. So there exist non-unit elements a, $b \in S$ such that $ab \in I$ but $a \notin I$, $b \notin I$. Now consider the set of all non-units $S \setminus U(S)$. Let $x, y \in S \setminus U(S)$. Then $xab \in I$ and $yab \in I$. Since I is a 1-absorbing prime ideal of S and $b \notin I$, we have $xa \in I$ and $ya \in I$. Hence, $(x+y)a = xa + ya \in I$. If $(x+y)$ is a unit, then $a \in I$, which is a contradiction. Therefore, $x + y \in S \setminus U(S)$. Again for any $s \in S$ and $x \in S \setminus U(S)$, we have $sx \in S \setminus U(S)$ (*cf*. Example 6.1, [42]) and hence $S \setminus U(S)$ is an ideal of S. Therefore, S is a local semiring. □

Corollary 2.1.9. *In a commutative non-local semiring, a proper ideal is a 1-absorbing prime if and only if it is prime.*

Theorem 2.1.10. *Let S be a local semiring with maximal ideal M. A proper ideal I of S is 1-absorbing prime ideal if and only if either I is a prime ideal or $M^2 \subseteq I$.*

Proof. Suppose that I is a 1-absorbing prime ideal of S. If I is not a prime ideal, then there exist non-units a, $b \in S$ such that $ab \in I$ but $a \notin I$ and $b \notin I$. Let x, $y \in M$. Here $xyab \in I$. Since I is a 1-absorbing prime ideal and $b \notin I$, we have $xya \in I$. Again $xy \in I$, since $a \notin I$. Therefore $xy \in I$ for any x, $y \in M$. Hence $M^2 \subseteq I$.

Conversely, if I is a prime ideal of S, then I is a 1-absorbing prime ideal of S. Consider $M^2 \subseteq I$ and $abc \in I$ for some non-units a, b and $c \in S$. Since a, b, c are non-units, so a, $b$and $c \in M$. Thus $ab \in M^2 \subseteq I$ and hence I is a 1-absorbing prime ideal of S. □

Corollary 2.1.11. *In a local semiring S with unique maximal ideal M, if $M^2 = \langle 0 \rangle$, then every proper ideal of S is a 1-absorbing prime ideal.*

Theorem 2.1.12. *Let S be a semiring. Then, $\langle 0 \rangle$ is a 1-absorbing ideal of S if and only if S is an entire semiring or (S, M) is a local semiring such that $M^2 = \langle 0 \rangle$.*

Proof. Suppose that $\langle 0 \rangle$ is a 1-absorbing ideal of S and S is not an entire semiring. So $\langle 0 \rangle$ is a 1-absorbing prime ideal of S which is not a prime ideal. Thus, by Theorem 2.1.8 and Theorem 2.1.10, S is a local semiring with maximal ideal M such that $M^2 = \langle 0 \rangle$.

Conversely, if S is an entire semiring, then $\langle 0 \rangle$ is a prime ideal of S and so $\langle 0 \rangle$ is a 1-absorbing prime ideal of S. If S is a local semiring with maximal ideal M such that $M^2 = \langle 0 \rangle$, then by Corollary 2.1.11, $\langle 0 \rangle$ is a 1-absorbing prime ideal of S. □

Theorem 2.1.13. *Let I be a proper ideal of a semiring S, then the following are equivalent:*

(1) I is a 1-absorbing prime ideal of S,

(2) if $abJ \subseteq I$ for non-units a, $b \in S$, then either $ab \in I$ or $J \subseteq I$,

(3) for any proper ideals K, L and non-unit element a of S, if $aKL \subseteq I$, then either $aK \subseteq I$ or $L \subseteq I$,

(4) if for any ideals J, K and L of S, $JKL \subseteq I$, then either $JK \subseteq I$ or $L \subseteq I$.

Proof. (1) $\implies$ (2) Suppose that $abJ \subseteq I$ and $ab \notin I$. Consider $j \in J$. Then $abj \in abJ \subseteq I$. Since I is a 1-absorbing prime ideal of S, we have $j \in I$. This implies $J \subseteq I$.

(2) $\implies$ (3) Assume that $aKL \subseteq I$ and $aK \nsubseteq I$. Then there exists $k \in K$ such that $ak \notin I$. As $akL \subseteq I$, by (2), $L \subseteq I$. Therefore, either $aK \subseteq I$ or $L \subseteq I$.

(3) $\implies$ (4) Suppose $JKL \subseteq I$ and $JK \nsubseteq I$ for some ideals J, K and L of S. Then there exists an element $j \in J$ such that $jK \nsubseteq I$. By (3), we have $L \subseteq I$. Therefore, either $JK \subseteq I$ or $L \subseteq I$.

(4) $\implies$ (1) Assume that $abc \in I$ for some non-units $a, b, c \in S$. Consider $J = \langle a \rangle$, $K = \langle b \rangle$, $L = \langle c \rangle$. Then $JKL = abcS \subseteq I$. By (4), either $JK \subseteq I$ or $L \subseteq I$. So either $ab \in I$ or $c \in I$. Hence I is a 1-absorbing prime ideal of S. □

Proposition 2.1.14. *Let I be a 1-absorbing prime ideal of a semiring S. Then for every non-unit $c \in S \setminus I$, $(I : c)$ is a prime ideal of S.*

Proof. Suppose $ab \in (I : c)$ for some a, $b \in S$ and non-unit $c \in S \setminus I$. Without loss of generality, we may assume that a and b are non-units. Since I is a 1-absorbing prime ideal of S and $acb \in I$, we have either $ac \in I$ or $b \in I$. This implies either $a \in (I : c)$ or $b \in I \subseteq (I : c)$. Therefore, $(I : c)$ is a prime ideal of S for every non-unit $c \in S \setminus I$. □

Theorem 2.1.15. *If I is a $P-$primary ideal of S such that $(P^2 : x) \subseteq I$ for all $x \in P \setminus I$, then I is a 1-absorbing prime ideal of S.*

Proof. Let $abc \in I$ for some non-units $a, b, c \in S$ and $ab \notin I$. If possible, let $c \notin I$. Since I is a $P-$primary ideal of S and $ab \notin I$, we have $c \in P$. Also $c \notin I$ implies $ab \in P$. So $abc \in P^2$. By the given hypothesis, $(P^2 : c) \subseteq I$. Thus $ab \in (P^2 : c) \subseteq I$, which is a contradiction. Hence $c \in I$ and thus I is a 1-absorbing prime ideal of S. □

Theorem 2.1.16. *Let I be an irreducible subtractive ideal of S and $(I : x) = (I : x^2)$ for every $x \in S \setminus I$, then I is a 1-absorbing prime ideal of S.*

Proof. Let $abc \in I$ and $c \notin I$ for some non-units a, b, $c \in S$. By the assumption, $(I : c) = (I : c^2)$. If possible, let $ab \notin I$. Consider $r \in (I + (ab)) \cap (I + (c))$. Then $r = i_1 + abs_1 = i_2 + cs_2$ for some i_1, $i_2 \in I$ and s_1, $s_2 \in S$. Thus $rc = i_1c + abcs_1 = i_2c + c^2s_2 \in I$. Since I is a subtractive ideal of S, we get $c^2s_2 \in I$. So $cs_2 \in I$ (as $(I : c) = (I : c^2)$). Therefore, $r = i_2 + cs_2 \in I$. This shows that $(I + (ab)) \cap (I + (c)) \subseteq I$, and so $(I + (ab)) \cap (I + (c)) = I$, a contradiction because I is an irreducible ideal. Thus $ab \in I$ and so I is a 1-absorbing prime ideal of S. □

Theorem 2.1.17. *Let I be a 1-absorbing prime subtractive ideal of S with $\sqrt{I} = P$. If $I \neq P$, then $\Omega = \{(I : x) : x \in P \setminus I\}$ under set inclusion is a totally ordered set.*

Proof. If possible, let there exist a, $b \in P \setminus I$ such that neither $(I : a) \subseteq (I : b)$ nor $(I : b) \subseteq (I : a)$. Then there exists x, $y \in S \setminus I$ so that $x \in (I : a) \setminus (I : b)$ and $y \in (I : b) \setminus (I : a)$. Since $P \subseteq (I : c)$ for any $c \in S \setminus I$, $x \in (I : a) \setminus P$ and $y \in (I : b) \setminus P$. Thus $xy \notin P$, since P is a prime ideal of S. Consider $x(a + b)y = xay + xby \in I$. This implies $x(a+b) = xa+xb \in I$. Since I is a subtractive ideal and $xa \in I$, we get $xb \in I$, that is $x \in (I : b)$, which is a contradiction. Hence $\Omega = \{(I : x) : x \in P \setminus I\}$ under set inclusion is a totally ordered set. □

Theorem 2.1.18. *In a regular semiring S, every irreducible ideal of S is a 1-absorbing prime ideal of S.*

Proof. Let S be a regular semiring and I be an irreducible ideal of S. Let $abc \in I$ and $ab \notin I$ for some non-units a, b, $c \in S$. If possible, let $c \notin I$. Now consider the ideals $J = (I + (ab))$ and $K = (I + (c))$, properly containing I. Since I is an irreducible ideal of S, $J \cap K \nsubseteq I$. Thus, there exists $p \in S$ such that $p \in (I + (ab)) \cap (I + (c)) \setminus I$. So $p \in (I + (ab)(I + (c)) \setminus I$, by regularity of S (*cf.* Proposition 6.35, [42]). Then, there are $p_1, p_2 \in I$ and $s_1, s_2 \in S$ such that $p = (p_1 + s_1ab)(p_2 + s_2c) = p_1p_2 + p_1s_2c + s_1abp_2 + s_1s_2abc$. This implies that $p \in I$, which is a contradiction. Hence $c \in I$, and so I is a 1-absorbing prime ideal of S. □

Theorem 2.1.19. *Let S be a commutative semiring and U be a multiplicative closed subset of S. If I is a 1-absorbing prime ideal of S with $U \cap I = \phi$, then I_U is a 1-absorbing prime ideal of S_U.*

Proof. Let $(a/s)(b/t)(c/r) \in I_U$ for some non-units (a/s), (b/t), $(c/r) \in S_U$, where a, b, $c \in S$ and s, t, $r \in U$. Then $(ua)bc \in I$ for some $u \in U$. So either $uab \in I$ or $c \in I$, as I is a 1-absorbing prime ideal of S. Thus either $(a/s)(b/t) = (uab/ust) \in I_U$ or $(c/r) \in I_U$. Therefore, I_U is a 1-absorbing prime ideal of S_U. □

Theorem 2.1.20. *Let P be a nonzero divided prime ideal of a semidomain S. Then, P^2 is a 1-absorbing prime ideal of S.*

Proof. First, we show that P^2 is a $P-$primary ideal of S. Let $xy \in P^2$ and $y \notin P$. Suppose $xy = \sum_{i=1}^{n} x_i y_i$ for some $x_i, y_i \in P$, $i = 1, 2, .., n$. Since P is a divided prime ideal and $y \notin P$, we have $P \subseteq Sy$. Thus $x_i = s_i y$, where $s_i \in S$, for $i = 1, 2, .., n$. Since P is a prime ideal and $y \notin P$, we have $s_i \in P$. Also since S is a semidomain, $xy = \sum_{i=1}^{n} s_i y y_i = (\sum_{i=1}^{n} s_i y_i)y$ implies $x = \sum_{i=1}^{n} s_i y_i$, that is $x \in P^2$. Thus P^2 is a $P-$primary ideal of S. Also, $(P^2 : x) \subseteq P^2$ for every $x \in P \setminus P^2$. So by Theorem 2.1.15, P^2 is a 1-absorbing prime ideal of S. □

Theorem 2.1.21. *In a semiring S, for any 1-absorbing prime ideal I of S, there exists exactly one minimal prime ideal containing I.*

Proof. If possible, let P_1 and P_2 be two minimal prime ideals containing I. Then there exist elements $a, b \in S$ such that $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$. By Theorem 0.1.46, there exist $p_2 \notin P_2$ and $p_1 \notin P_1$ such that $ap_2^n \in I$, $bp_1^n \in I$. Since p_1, $p_2 \notin I$ and I is a 1-absorbing prime ideal, $ap_2^n \in I$ implies $ap_2 \in I$ and $bp_1^n \in I$ implies $bp_1 \in I$. Thus $a^2 \in I$ and $b^2 \in I$, whence $a^2 \in I \subseteq P_2$ implies $a \in P_2$ and $b^2 \in I \subseteq P_1$ implies $b \in P_1$, which is a contradiction. Therefore, there exists one minimal prime ideal of S containing I. □

Theorem 2.1.22. *Let P be a non-zero divided prime ideal of a subtractive semiring S and I be an ideal of S such that $\sqrt{I} = P$. If I is a 1-absorbing prime ideal of S, then I is a $P-$primary ideal of S such that $P^2 \subseteq I$.*

Proof. Let $ab \in I$ for some $a, b \in S$ and $b \notin P$. Since P is a divided prime ideal of S, we have $P \subseteq Sb$. Thus $a = sb$ for some $s \in S$. So, we get $ab = b^2 s \in I$. Since $b^2 \notin I$ and I is a 1-absorbing prime ideal of S, we conclude that $s \in I$. Therefore, $a = sb \in I$ and so I is a $P-$primary ideal of S. Now, let x, $y \in P$. Then by Theorem 2.1.6, x^2, $y^2 \in I$. Thus $x(x+y)y = x^2 y + xy^2 \in I$. Since I is a 1-absorbing prime ideal of S, we have either $x(x+y) = x^2 + xy \in I$ or $y \in I$. Since S is a subtractive semiring, $xy \in I$ and hence $P^2 \subseteq I$. □

Theorem 2.1.23. *Suppose that S is a subtractive valuation semiring and I is an ideal of S. Then I is a 1-absorbing prime ideal of S if and only if either $I = P$ or $I = P^2$, where $P = \sqrt{I}$ is a prime ideal of S.*

Proof. Suppose I is a 1-absorbing prime ideal of S. Since every subtractive valuation semiring is a divided semiring (cf. Proposition 1.4, [17]), so I is a P-primary ideal of S such that $P^2 \subseteq I$, by Theorem 2.1.22. Therefore, either $I = P$ or $I = P^2$ (cf. Theorem 2.13, [17]).

Conversely, suppose that either $I = P$ or $I = P^2$, where $P = \sqrt{I}$ is a prime ideal of S. If $I = P$, then I is a 1-absorbing prime ideal of S. If $I = P^2$, then by Theorem 2.1.20, I is a 1-absorbing prime ideal of S. □

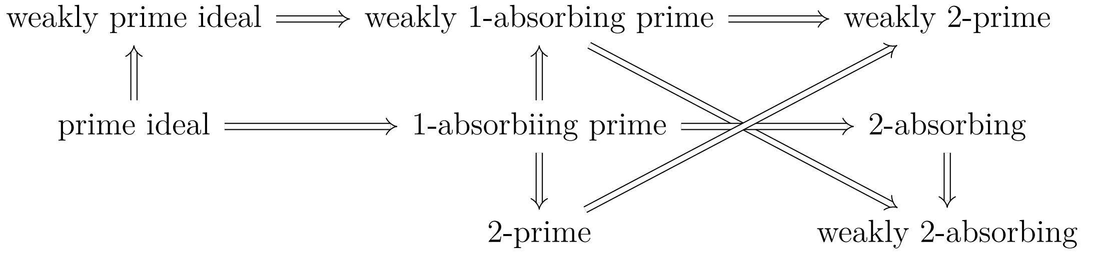
2.2 Weakly 1-absorbing Prime Ideals

Definition 2.2.1. A proper ideal I of a commutative semiring S is said to be a weakly 1-absorbing prime ideal if whenever $0 \neq abc \in I$ for some non-units a, b, $c \in S$, either $ab \in I$ or $c \in I$.

Clearly, every 1-absorbing prime ideal is also a weakly 1-absorbing prime ideal but the converse may not be true in general which can be seen from the following example.

Example 2.2.2. In the semiring $\mathbb{Z}_{15}$, the ideal $I = \{0\}$ is clearly a weakly 1-absorbing prime ideal of S but not 1-absorbing prime, as $0 = \bar{3} \cdot \bar{3} \cdot \bar{5} \in I$ but neither $\bar{3} \cdot \bar{3} \in I$ nor $\bar{5} \in I$.

Then it is easy to check that every weakly prime ideal of S is weakly 1-absorbing prime ideal and every weakly 1-absorbing prime ideal is weakly 2-prime, weakly 2-absorbing prime ideal but the converse inclusions does not hold in general. So we have the interrelation among some generalizations of prime ideals, which is clear from the following diagram:



Theorem 2.2.3. *If I is a weakly 1-absorbing prime of a reduced semiring S, then $\sqrt{I}$ is a weakly prime ideal of S.*

Proof. Suppose that $0 \neq ab \in \sqrt{I}$ for some non-units a, $b \in S$. Then $(ab)^n \in I$ for some positive integer n. Since S is a reduced semiring, so $0 \neq (ab)^n = aa^{n-1}b^n \in I$.

Thus either $aa^{n-1} = a^n \in I$ or $b^n \in I$ which implies either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. So $\sqrt{I}$ is a weakly prime ideal of S. □

Theorem 2.2.4. *If the zero ideal of a semiring S is a 1-absorbing prime, then every weakly 1-absorbing prime ideal is a 1-absorbing prime of S.*

Proof. Suppose that I is a weakly 1-absorbing prime ideal of S and $abc \in I$ for some non-units a, b, $c \in S$. If $0 \neq abc$, then either $ab \in I$ or $c \in I$, that is, I is a 1-absorbing prime ideal. If $abc = 0$ and since zero ideal is 1-absorbing prime, either $ab = 0 \in I$ or $c = 0 \in I$. Therefore, I is a 1-absorbing prime ideal of S. □

Theorem 2.2.5. *Let S_1 and S_2 be two semirings and $f : S_1 \longrightarrow S_2$ be a monomorphism such that $f(a)$ is a non-unit in S_2 for every non-unit element a in S_1. Then the following statements hold:*

(1) If I is a weakly 1-absorbing prime ideal of S_2, then $f^{-1}(I)$ is a weakly 1-absorbing prime ideal of S_1.

(2) If J is a weakly 1-absorbing prime subtractive ideal of S_1 with $kerf \subseteq J$ and f is onto steady homomorphism, then $f(J)$ is a weakly 1-absorbing prime subtractive ideal of S_2.

Proof. (1) Assume that $0 \neq abc \in f^{-1}(I)$ for some non-units a, b, $c \in S_1$. Since f is monomorphism, we have $0 \neq f(abc) = f(a)f(b)f(c) \in I$ for non-units $f(a)$, $f(b)$, $f(c) \in S_2$. As I is a weakly 1-absorbing prime ideal of S_2, so either $f(a)f(b) = f(ab) \in I$ or $f(c) \in I$. That is either $ab \in f^{-1}(I)$ or $c \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is a weakly 1-absorbing prime ideal of S_1.

(2) Clearly, $f(J)$ is a subtractive ideal of S_2 as J is a subtractive ideal of S_1. Now consider, $0 \neq abc \in f(J)$ for some non-units a, b, $c \in S_2$. There exist non-units x, y, $z \in S_1$ such that $f(x) = a$, $f(y) = b$, $f(z) = c$. So $0 \neq abc = f(x)f(y)f(z) = f(xyz) \in f(J)$. Thus $f(xyz) = f(j)$ for some $j \in J$. Since f is steady, we have $yxz + k_1 = j + k_2$ for some k_1, $k_2 \in \ker f$. As J is a subtractive ideal of S_1 and $\ker f \subseteq J$, we obtain $0 \neq xyz \in J$. Since J is a weakly 1-absorbing prime ideal of S_1, either $xy \in J$ or $z \in J$. This implies either $ab = f(xy) \in f(J)$ or $c = f(z) \in f(J)$. Hence, $f(J)$ is a weakly 1-absorbing prime subtractive ideal of S_2.

□

Definition 2.2.6. Let I be a weakly 1-absorbing prime ideal of S and a, b, $c \in S$ be non-units of S. We call (a, b, c) is a 1-triple zero of I if $abc = 0$, $ab \notin I$ and $c \notin I$.

Theorem 2.2.7. *Let I be a subtractive weakly 1-absorbing prime ideal of S and (a, b, c) be a 1-triple zero of I. Then, $abI = 0$ and if $a, b \notin (I : c)$, then $I^3 = 0$.*

Proof. Suppose that I is a subtractive weakly prime ideal of S and (a, b, c) is a 1-triple zero of I. So $abc = 0$ and $ab \notin I$, $c \notin I$. If possible, let $abI \neq 0$. Then there exists $i \in I$ such that $0 \neq abi$. This implies that $0 \neq ab(c + i) = abc + abi \in I$. Also $c + i$ is a non-unit, because $(c + i)$ is an unit implies $ab \in I$. Since I is weakly 1-absorbing prime and $ab \notin I$, so $c + i \in I$. As I is a subtractive ideal of S, we get $c \in I$, which is a contradiction. Therefore, $abI = 0$.

Next we show that if $a, b \notin (I : c)$, then $bcI = caI = aI^2 = bI^2 = cI^2 = 0$. Since $a, b \notin (I : c)$, $ac \notin I$ and $bc \notin I$. Suppose $bcI \neq 0$. Then there exists $i \in I$ such that $bci \neq 0$. Since $abc = 0$, we have $0 \neq bci = (a+i)bc \in I$. Here $(a+i)$ is non-unit, because if $a + i$ is a unit, then $bc \in I$, which is a contradiction. Since I is weakly 1-absorbing prime, either $(a + i)b \in I$ or $c \in I$, that is either $ab \in I$ or $c \in I$, a contradiction, as I is a subtractive ideal of S. Therefore $bcI = 0$. Similarly, $caI = 0$. Now, we will show that $aI^2 = 0$. Suppose that $ai_1i_2 \neq 0$ for some $i_i, i_2 \in I$. Since $abc = 0$ and $abI = acI = 0$, we have $a(b+i_1)(c+i_2) = abc+abi_2+aci_1+ai_1i_2 = ai_1i_2 \neq 0$. If $c+i_2$ is a unit, then $a(b+i_1) \in I$, so $ab \in I$ also similarly, if $b+i_1$ is a unit, then $ac \in I$, both contradict the hypothesis. Thus either $a(b + i_1) \in I$ or $c + i_2 \in I$, since I is a weakly 1-absorbing prime ideal. So $ab \in I$ or $c \in I$, as I is a subtractive ideal of S, which is a contradiction. Therefore $aI^2 = 0$. Similarly, one can show that $bI^2 = 0$ and $cI^2 = 0$. Now, assume that $I^3 \neq 0$. So there exist i_1, i_2, $i_3 \in I$ such that $0 \neq i_1i_2i_3 \in I$. Since $abI = bcI = caI = aI^2 = bI^2 = cI^2 = 0$, we have $0 \neq i_1i_2i_3 = (a+i_1)(b+i_2)(c+i_3) \in I$ and $(a + i_1)$, $(b + i_2)$, $(c + i_3)$ are non-units. So either $(a + i_1)(b + i_2) \in I$ or $c + i_3 \in I$. Since I is a subtractive ideal, we have either $ab \in I$ or $c \in I$, which is a contradiction. Therefore, $I^3 = 0$.

□

Theorem 2.2.8. *Let I be a Q-ideal and J be a subtractive ideal of a semiring S so that $I \subseteq J$. If J is a weakly 1-absorbing prime ideal of S, then $J/I_{(Q\cap J)}$ is a weakly 1-absorbing prime ideal of S/I_Q.*

Let I be a weakly 1-absorbing prime ideal of S such that $u(S/I) = \{x + I : x \in u(S) \cap Q\}$. If $J/I_{(Q\cap J)}$ is a weakly 1-absorbing prime ideal of S/I_Q, then J is a weakly 1-absorbing prime ideal of S.

Proof. Suppose that J is a weakly 1-absorbing prime ideal of S and $0 \neq (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in J/I_{(Q\cap J)}$ for some non-units $q_1 + I$, $q_2 + I$, $q_3 + I \in S/I_Q$ where

q_1, q_2, $q_3 \in Q$. So there exists a unique element $q_4 \in J \cap Q$ such that $q_1q_2q_3 + I \subseteq q_4 + I \in J/I_{(Q\cap J)}$. It follows that q_1, q_2, q_3 are non-units in S and $0 \neq q_1q_2q_3 \in J$. Since J is a weakly 1-absorbing prime ideal of S, we get $q_1q_2 \in J$ or $q_3 \in J$. Therefore if $q_3 \in J$, then $q_3 + I \in J/I_{(Q\cap J)}$. Otherwise if $q_1q_2 \in J$, then $(q_1 + I) \odot (q_2 + I) = q_5 + I$ where q_5 is the unique element in Q and $q_1q_2 + r = q_5 + s$ for some r, $s \in I$. Since J is a subtractive ideal, $q_5 \in J$. So $(q_1 + I) \odot (q_2 + I) \in J/I_{(Q\cap J)}$. Hence $J/I_{(Q\cap J)}$ is a weakly 1-absorbing prime ideal of S/I_Q.

Let $0 \neq abc \in J$ for some non-units a, b, $c \in S$. If $abc \in I$, then either $ab \in I \subseteq J$ or $c \in I \subseteq J$. Assume that $abc \notin I$. Since I is a Q-ideal, there exist q_1, q_2, $q_3 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$ and $c \in q_3 + I$. Thus $a = q_1 + i_1$, $b = q_2 + i_2$, $c = q_3 + i_3$ for some i_1, i_2, $i_3 \in I$. So $abc = (q_1+i_1)(q_2+i_2)(q_3+i_3) = q_1q_2q_3+i_1q_2q_3+i_2q_1q_3+i_3q_1q_2+i_1i_3q_2+i_1i_2q_3+i_2i_3q_1+i_1i_2i_3$. Since J is a subtractive ideal of S, we have $q_1q_2q_3 \in J$. Also $(q_1 + I) \odot (q_2 + I) \odot (q_3 + I) = q_4 + I$ for some unique element $q_4 \in Q$, where $q_1q_2q_3 + I \subseteq q_4 + I$. Thus $q_1q_2q_3 + i_4 = q_4 + i_5 \in J$. Since J is a subtractive ideal, $q_4 \in J\cap Q$ and $q_4+I \in J/I_{(Q\cap J)}$. By hypothesis, q_1+I, q_2+I, q_3+I are non-units elements of S/I_Q and $(q_1+I)\odot(q_2+I)\odot(q_3+I) \in J/I_{(Q\cap J)}$. If there exists a unique element $q \in Q$ such that $q+I$ is a zero element of S/I_Q and $(q_1+I)\odot(q_2+I)\odot(q_3+I) = q+I$. Thus $q_1q_2q_3 + r = q + s$ for some r, $s \in I$, as J is a subtractive ideal, $q_1q_2q_3 \in I$. So $abc \in I$, which is a contradiction. Hence $0 \neq (q_1 + I) \odot (q_2 + I) \odot (q_3 + I) \in J/I_{(Q\cap J)}$. Since $J/I_{(Q\cap J)}$ is a weakly 1-absorbing prime ideal, we conclude that either $q_1q_2+I \in J/I_{(Q\cap J)}$ or $q_3 + I \in J/I_{(Q\cap J)}$. $(q_1 + I) \odot (q_2 + I) \in J/I_{(Q\cap J)}$ implies $ab \in J$ or $q_3 + I \in J/I$ implies $c \in J$. Therefore, J is a weakly 1-absorbing prime ideal of S. □

Theorem 2.2.9. *Let S be a local semiring and I be a subtractive ideal of S. If I is a weakly 1-absorbing prime ideal of S, then either I is a 1-absorbing prime ideal of S or $I^3 = 0$.*

Proof. Suppose that $I^3 \neq 0$. Then there exists x, y, $z \in I$ such that $xyz \neq 0$. If possible, let I is not a 1-absorbing prime ideal of S. Then there exists a 1-triple zero (a, b, c) of I for some non-units a, b, $c \in S$.

Since S is a local semiring, set of non-units forms an ideal. So $(a + x)$, $(b + y)$, $(c + z)$ are non-units. Consider $(a + x)(b + y)(c + z) = abc + bcx + acy + abz + ayz + bxz + cxy + xyz \in I$.

We claim that $abz = bcx = acy = 0$. If $abz \neq 0$, then $0 \neq abz = ab(c+z) \in I$. Since I is a weakly 1-absorbing prime ideal, either $ab \in I$ or $c + z \in I$. As I is a subtractive ideal of S, either $ab \in I$ or $c \in I$, which is a contradiction. Hence we may assume that

$abz = 0$. In a similar way, one can show $bcx = acy = 0$. Also $ayz = bxz = cxy = 0$. If suppose $ayz \neq 0$, then we have $0 \neq ayz = ayz + abz + acy + abc = a(y + b)(z + c)$. Hence either $a(b + y) \in I$ or $c + z \in I$. Since I is a subtractive ideal, so in both cases we arrive at a contradiction. Thus $ayz = 0$. In a similar way, $bxz = cxy = 0$.

Therefore, $0 \neq xyz = (a + x)(b + y)(c + z) \in I$ but $ab \notin I$, $c \notin I$ and I is a subtractive ideal of S, implies neither $(a + x)(b + y) \in I$ nor $c + z \in I$, which contradicts our hypothesis that I is a weakly 1-absorbing prime ideal of S. Therefore, I is a 1-absorbing prime ideal of S. □

Theorem 2.2.10. *Let S be a semiring and $Jac(S)$ be the Jacobson radical of S. If $Jac(S)$ is a strong ideal of S, then for any a, b, $c \in Jac(S)$, the ideal $I = \langle abc\rangle$ is a weakly 1-absorbing prime ideal of S if and only if $abc = 0$.*

Proof. If $abc = 0$ for some $a, b, c \in Jac(S)$, then clearly $I = \langle abc\rangle$ is a weakly 1-absorbing prime ideal of S. Conversely, assume that I is a weakly 1-absorbing prime ideal of S. If possible, let $abc \neq 0$. Since $abc \in I$, we have either $ab \in I$ or $c \in I$, that is, either $ab = abcx$ or $c = abcy$ for some $x, y \in S$. Suppose $ab = abcx$. As $c \in Jac(S)$ and $Jac(S)$ is a strong ideal, so there exists $c' \in Jac(S)$ such that $c + c' = 0$. This implies $ab(1 + c'x) = 0$ and $(1 + c'x)$ is a semiunit of S (*cf.* Lemma 3.4, [11]). So there exist $s, t \in S$ such that $1 + s(1 + c'x) = t(1 + c'x)$. Thus $ab = 0$ and so $abc = 0$, which is a contradiction. Also by similar arguments, if $c \in I$, we get $c = 0$, a contradiction. Therefore $abc = 0$. □

Theorem 2.2.11. *Let S be a local semiring with strong maximal ideal M. Then every proper ideal of S is a weakly 1-absorbing prime ideal if and only if $M^3 = \{0\}$.*

Proof. Suppose that every proper ideal of S is weakly 1-absorbing prime and $a, b, c \in M$. Then the ideal $I = \langle abc\rangle$ is a weakly 1-absorbing prime. By Theorem 2.2.10, $abc = 0$ and so $M^3 = \{0\}$. Conversely, suppose that $M^3 = \{0\}$ and I is a nonzero proper ideal of S. So there do not exist any non-units a, b, $c \in S$ such that $0 \neq abc \in I$. Hence, I is a weakly 1-absorbing prime ideal of S. □

Theorem 2.2.12. *Let $S = S_1 \times S_2$, where S_1 and S_2 be commutative semirings. If I_1 is a proper ideal of S_1. Then the following are equivalent:*

(1) I_1 is a 1-absorbing prime ideal of S_1,

(2) $I_1 \times S_2$ is a 1-absorbing prime ideal of S,

(3) $I_1 \times S_2$ *is a weakly 1-absorbing prime ideal of* S.

Proof. (1) $\implies$ (2) and (2) $\implies$ (3) are trivial.
(3) $\implies$ (1) Suppose that $I_1 \times S_2$ is a weakly 1-absorbing prime ideal of S and $abc \in I_1$ for some non-units a, b, $c \in S_1$. Then $0 \neq (abc, 1) = (a, 1)(b, 1)(c, 1) \in I_1$. and $(a, 1)$, $(b, 1)$, $(c, 1)$ are non-units of S. Thus, either $(a, 1)(b, 1) = (ab, 1) \in I_1 \times S_2$ or $(c, 1) \in I_1 \times S_2$, that is either $ab \in I_1$ or $c \in I_1$. Therefore, I_1 is a 1-absorbing prime ideal of S_1. □

Theorem 2.2.13. *Let* $S = S_1 \times S_2$, *where* S_1 *and* S_2 *be commutative semirings but not semifields. If* I_1 *and* I_2 *are nonzero ideals of* S_1 *and* S_2 *respectively, then the following are equivalent:*

(1) $I_1 \times I_2$ *is a weakly 1-absorbing prime ideal of* S,

(2) Either I_1 *is a prime ideal of* S_1 *and* $I_2 = S_2$ *or* I_2 *is a prime ideal of* S_2 *and* $I_1 = S_1$,

(3) $I_1 \times I_2$ *is a 1-absorbing prime ideal of* S,

(4) $I_1 \times I_2$ *is a prime ideal of* S.

Proof. (1) $\implies$ (2) Suppose that $I_1 \times I_2$ is a weakly 1-absorbing prime ideal of S and $0 \neq (a, 0) \in I_1 \times I_2$ for some non zero $a \in I_1$. Then $0 \neq (a, 0) = (1, 0)(1, 0)(a, 1) \in I_1 \times I_2$. Therefore, either $(1, 0)(1, 0) = (1, 0) \in I_1 \times I_2$ or $(a, 1) \in I_1 \times I_2$. So either $1 \in I_1$ or $1 \in I_2$, that is, either $I_1 = S_1$ or $I_2 = S_2$.

Consider $I_2 = S_2$ and $ab \in I_1$ for some non-units $a, b \in S_1$. Since S_2 is not a semifield, there exists a non-unit $0 \neq z \in S_2$. Consider $0 \neq (ab, z) = (a, 1)(1, z)(b, 1) \in I_1 \times I_2$, then either $(a, z) = (a, 1)(1, z) \in I_1 \times I_2$ or $(b, 1) \in I_1 \times I_2$. Thus either $a \in I_1$ or $b \in I_2$. Therefore, I_1 is a prime ideal of S_1. Similarly, I_2 is a prime ideal of S_2 when $I_1 = S_1$.

(2) $\implies$ (3) follows from Theorem 2.2.12.

(3) $\implies$ (4) is clear from the Corollary 2.1.9.

(4) $\implies$ (1) is trivial. □

CHAPTER 3

On n-ideals of Commutative Semirings

3

On n-ideals of Commutative Semirings

In this chapter, the notion of n-ideals has been introduced in commutative semirings with non-zero identity and various important properties have been investigated. Characterization of those semirings in which every proper ideal is n-ideal have been obtained. Characterization of entire semirings in terms of n-ideals has also been investigated. Several examples are discussed of such a class of ideals. Moreover, n-ideals under various contexts of constructions such as homomorphic images, localizations, direct products, and expectation semirings have been studied.
This chapter has been organized as follows:

In *Section 1*, we first introduce the notion of n-ideals (*cf.* Definition 3.1.1) and provide some examples (*cf.* Examples 3.1.2, 3.1.3) in a semiring. We obtain some characterization of n-ideals (*cf.* Proposition 3.1.7, Theorem 3.1.8). Then characterize semirings in which every proper ideal is n-ideal (cf. Theorem 3.1.12). Next we show that in general n-ideals are not comparable with prime ideals(*cf.* Examples 3.1.9, 3.1.10, Proposition 3.1.11) and observe their relationship with primary ideals of semirings(*cf.* Proposition 3.1.13, Example 3.1.14). Then characterize the entire semiring in terms of n-ideal (cf. Theorem 3.1.24). We then investigate n-ideals under various contexts of constructions such as homomorphic images (*cf.* Theorem 3.1.25), localizations

This chapter is mainly based on the works published in the following paper:

- **Sampad Das et al.**, *A note on n-ideals in Commutative Semirings* (Communicated).

(*cf.* Theorem 3.1.33), direct products (*cf.* Corollary 3.1.35).

In *Section 2*, we introduce the concept of n-subsemimodule of an S-semimodule M (*cf.* Definition 3.2.1). Some results of n-ideals are studied (*cf.* Theorem 3.2.3, 3.2.3). Finally, we investigate the n-ideals of the expectation semiring (cf. Theorem 3.2.9). At the end of this chapter, we obtain a characterization of n-ideals in expectation semiring for subtractive semimodule (cf. Corollary 3.2.10).

3.1 n-ideals in Semirings

In this section, we introduce the class of n-ideals in commutative semiring and investigate some of their properties. Moreover, we clarify their relations with prime and primary ideals.

Definition 3.1.1. A proper ideal I of a semiring S is called n-ideal if whenever $a,\ b \in S$ with $ab \in I$ and $a \notin \sqrt{0}$ implies $b \in I$.

Example 3.1.2. Let S be the set of real numbers a satisfying $0 \leq a \leq 1$ and define $a \vee b = max\{a, b\}$ and $a \cdot b = (a + b - 1) \vee 0$ for all $a, b \in S$. Then $(S, \vee, \cdot)$ is easily checked to be a commutative semiring with the zero element 0 and the identity element 1. Then each real number r such that $0 \leq r < 1$ defines an ideal $A = \{a \in S : a \leq r\}$ of S which is an n-ideal of S. As we can see that for any $b(\neq 1) \in S$, $b^k = (kb-k+1) \vee 0$. Thus $b^k = 0$ if and only if $b \leq 1 - \frac{1}{k}$. So every non-units are nilpotent and $\sqrt{0} = [0,\ 1)$. Therefore, every proper ideal of S is an n-ideal of S.

Example 3.1.3. Let $S = (\mathbb{Z}_0^+, +, \cdot)$, the semiring of all non-negative integers with respect to the usual addition and multiplication. Consider the ideal $I = 8\mathbb{Z}_0^+$. Now define a relation ρ_I on S by for all $a, b \in S$, $a\rho_I b \iff a + 8i_1 = b + 8i_2$ for some $i_1,\ i_2 \in S$. Then ρ_I is a congruence relation on S. Let us consider the set of congruence classes S/ρ_I with respect to ρ_I. Then $S/\rho_I = \{[0],\ [1],\ [2],\ [3],\ [4],\ [5],\ [6],\ [7]\}$ becomes a semiring with respect to the operations $[a] + [b] = [a + b]$ and $[a][b] = [ab]$ for all $[a],\ [b] \in S/\rho_I$. Now, it is easy to verify that the ideal $\{[0],\ [4]\}$ is an n-ideal of S/ρ_I.

Proposition 3.1.4. *If I is an n-ideal of a semiring S, then $I \subseteq \sqrt{0}$.*

Proof. Suppose that I is an n-ideal but $I \nsubseteq \sqrt{0}$. Then there exists $a \in I$ such that $a \notin \sqrt{0}$. Now, $a \cdot 1 \in I$ and $a \notin \sqrt{0}$ implies $1 \in I$, which is a contradiction. Therefore, $I \subseteq \sqrt{0}$. □

However, we can find an ideal I of S with $I \subseteq \sqrt{0}$ which is not an n-ideal. For example:

Example 3.1.5. Consider $I = 12\mathbb{Z}_0^+$ in Example 3.1.3, then for the ideal $\{[0], [6]\}$ of semiring S/ρ_I, $\{[0]\,[6]\} \subseteq \sqrt{[0]}$ but $\{[0]\,[6]\}$ is not an n-ideal of S/ρ_I, because $[2][3] \in \{[0]\,[6]\}$ but neither $[2] \in \sqrt{[0]}$ nor $[3] \in \{[0]\,[6]\}$.

Proposition 3.1.6. *$\sqrt{0}$ is an n-ideal of S if and only if $\sqrt{0}$ is a prime ideal of S.*

Proof. The proof is straightforward. □

Proposition 3.1.7. *Suppose I is a proper ideal of a semiring S, then the followings are equivalent:*

(1) I is an n-ideal of S,

(2) $(I : a) = I$ for all $a \notin \sqrt{0}$.

Proof. (1) $\implies$ (2) Suppose I is an n-ideal of a semiring S. It is clear that $I \subseteq (I : a)$. Now let, $x \in (I : a)$ for some $a \notin \sqrt{0}$. Since I is an n-ideal, $ax \in I$ implies $x \in I$ and so $(I : a) = I$.
(2) $\implies$ (1) Let $ab \in I$ and $a \notin \sqrt{0}$. Then by (2), $b \in (I : a) = I$ and hence I is an n-ideal of S. □

In the following, we give some characterizations of n-ideals.

Theorem 3.1.8. *For a proper ideal I of semiring S, the following statements are equivalent:*

(1) I is an n-ideal of S,

(2) $(I : a) \subseteq \sqrt{0}$ for all $a \in S \setminus I$,

(3) whenever $a \in S$ and J is an ideal of S with $aJ \subseteq I$, then $a \in \sqrt{0}$ or $J \subseteq I$,

(4) whenever K and J are ideals of S with $KJ \subseteq I$, then $K \subseteq \sqrt{0}$ or $J \subseteq I$.

Proof. (1) $\implies$ (2) Let $x \in (I : a)$ for some $a \in S \setminus I$ and $x \in S$. That is $xa \in I$. Since I is an n-ideal of S, $xa \in I$ and $a \notin I$ implies $x \in \sqrt{0}$. So $(I : a) \subseteq \sqrt{0}$ for all $a \in S \setminus I$.

(2) $\implies$ (3) Assume that $aJ \subseteq I$ and $J \nsubseteq I$. Then there exists an element $j \in J$ such that $j \notin I$. Thus by (2), $a \in (I : j) \subseteq \sqrt{0}$.

(3) $\implies$ (4) Suppose that $KJ \subseteq I$ and $K \nsubseteq \sqrt{0}$. Then there exists an element $a \in K$ such that $a \in S \setminus \sqrt{0}$. Thus $aJ \subseteq I$ and by (3) we have $J \subseteq I$.

(4) $\implies$ (1) Let $ab \in I$ for some $a, b \in S$ and $a \notin \sqrt{0}$. Set $K = \langle a \rangle$ and $J = \langle b \rangle$. Then by hypothesis, $b \in I$. □

A prime ideal in a semiring may not be an n-ideal, which will be clear from the following example:

Example 3.1.9. In Example 3.1.2, any proper ideal of S is an n-ideal but none of them is prime, except the ideal $[0, 1)$. Suppose consider the ideal $A = \{a \in S : a \leq r\}$ for some $0 \leq r < 1$. Since any ideal of S is an n-ideal, A is an n-ideal but not a prime ideal of S. As we have $\frac{1}{3}(2r+1) \cdot \frac{1}{3}(2r+1) = (\frac{1}{3}(2r+1) + \frac{1}{3}(2r+1) - 1) \vee 0 \in A$ for $0 \leq r < 1$ but $\frac{1}{3}(2r+1) \notin A$.

Also in the following example, a prime ideal may not be an n-ideal of S.

Example 3.1.10. Consider the semiring $S = (\mathbb{Z}_0^+, +, \cdot)$, of all non-negative integers with respect to the usual addition and multiplication. Then for any prime number p, the ideal of the form $p\mathbb{Z}_0^+$ is prime but not an n-ideal of S.

Proposition 3.1.11. *If I is a prime ideal of S, I is an n-ideal of S if and only if $I = \sqrt{0}$.*

Proof. Suppose that I is an n-ideal of S. Then by Proposition-3.1.4, $I \subseteq \sqrt{0}$. Again, since I is a prime ideal of S, we have $\sqrt{0} \subseteq I$. Therefore $I = \sqrt{0}$.

Conversely, assume that $I = \sqrt{0}$ and consider $ab \in I$, $a \notin \sqrt{0}$ for some a, $b \in S$. Since I is a prime ideal of S, clearly $b \in I$. Thus I is an n-ideal of S. □

Next, we characterize semirings in which every proper ideal is an n-ideal.

Theorem 3.1.12. *In a semiring S, the following are equivalent:*

(1) Every proper principal ideal is an n-ideal of S,

(2) every proper ideal is an n-ideal of S,

(3) $\sqrt{0}$ is the unique prime ideal of S,

(4) S is a local semiring with maximal ideal $M = \sqrt{0}$.

Proof. (1) $\implies$ (2) Let I be an ideal of S and $ab \in I$, $a \notin \sqrt{0}$ for some $a, b \in S$. Consider $\langle ab \rangle$. Then by (1), $\langle ab \rangle$ is an n-ideal of S and $a \notin \sqrt{0}$. Therefore $b \in \langle ab \rangle \subseteq I$ and thus I be an n-ideal of S.

(2) $\implies$ (3) Let P be any prime ideal of S. Then by hypothesis, P is an n-ideal of S. By Proposition-3.1.11, $P = \sqrt{0}$ and hence $\sqrt{0}$ is the unique prime ideal of S.

(3) $\implies$ (4) Suppose that M is a maximal ideal of S. Then M is also prime ideal of S. By (3), $M = \sqrt{0}$ and hence S is a local semiring with maximal ideal $M = \sqrt{0}$.

(4) $\implies$ (1) Assume that $\sqrt{0}$ is the unique maximal ideal of local semiring S and I is a proper principal ideal of S. Let $ab \in I$ and $a \notin \sqrt{0}$. Thus the element a is a unit in S. Since $ab \in I$, we have $b \in I$. Thus I is an n-ideal of S. □

Proposition 3.1.13. *Every n-ideal of a semiring S is a primary ideal of S.*

Proof. Let I be an n-ideal of a semiring S and $ab \in I$ with $a \notin \sqrt{I}$ for some $a, b \in S$. Since $\sqrt{0} \subseteq \sqrt{I}$ and I is an n-ideal, we get $b \in I$. Thus I is a primary ideal of S. □

The converse of Proposition 3.1.13 may not be true in general, as shown in the following example:

Example 3.1.14. The ideal $8\mathbb{Z}_0^+$ is a primary ideal but not an n-ideal of semiring $\mathbb{Z}_0^+$. Because, if $8\mathbb{Z}_0^+$ is an n-ideal, it will contradict Proposition 3.1.4, since $8\mathbb{Z}_0^+ \nsubseteq \sqrt{0} = \{0\}$. So $8\mathbb{Z}_0^+$ is not an n-ideal of $\mathbb{Z}_0^+$.

The previous example is a special case of the following theorem.

Theorem 3.1.15. *Let I be a primary ideal of S and $I \subseteq \sqrt{0}$. Then I is an n-ideal of S.*

Proof. Suppose I is a primary ideal of S and $I \subseteq \sqrt{0}$. Let $ab \in I$ and $a \notin \sqrt{0}$ for some $a, b \in S$. As $\sqrt{I} = \sqrt{0}$, so $a \notin I$. Since I is a primary ideal, we have $b \in I$ and thus I is an n-ideal of S. □

Corollary 3.1.16. *Let I be an n-ideal and J be a primary ideal of S such that $J \subseteq I$. Then J is an n-ideal of S.*

Proposition 3.1.17. *Let S be a Noetherian semiring and I be an irreducible subtractive ideal of S such that $I \subseteq \sqrt{0}$. Then I is an n-ideal of S.*

Proof. Suppose I is an irreducible subtractive ideal of Noetherian semiring S. Then I is a primary ideal of S (*cf*. Theorem 0.1.53). Since $I \subseteq \sqrt{0}$, by Theorem 3.1.15, I is an n-ideal of S. □

Theorem 3.1.18. *If I is an n-ideal of semiring S, then $\sqrt{I}$ is an n-ideal of S. Moreover, $\sqrt{I}$ is a prime ideal of S and $\sqrt{I} = \sqrt{0}$.*

Proof. Let I be an n-ideal of S and $ab \in \sqrt{I}$ with $a \notin \sqrt{0}$ for some $a, b \in S$. Then there exists a positive integer m such that $a^m b^m = (ab)^m \in I$. Since $a^m \notin \sqrt{0}$ and I is an n-ideal, so $b^m \in I$. Thus $b \in \sqrt{I}$ and hence $\sqrt{I}$ is an n-ideal of S. Again by Proposition 3.1.13, $\sqrt{I}$ is primary and so $\sqrt{I}$ is a prime ideal of S, as $\sqrt{\sqrt{I}} = \sqrt{I}$. Also, $\sqrt{I} = \sqrt{0}$, by Proposition 3.1.11. □

Corollary 3.1.19. *If $\{0\}$ is an n-ideal of S, then $\sqrt{0}$ is a prime ideal of S.*

Proposition 3.1.20. *For a reduced semiring S, S is an entire semiring if and only if $\{0\}$ is an n-ideal of S.*

Proof. Suppose S is an entire semiring and $ab \in \{0\}$ with $a \notin \sqrt{0}$. Since S is a reduced semiring, $a \notin \sqrt{0} = \{0\}$ and $ab = 0$. That implies $b = 0$, and so $\{0\}$ is an n-ideal of S.

Conversely, suppose that $\{0\}$ is an n-ideal of S and $ab = 0$. So $ab \in \{0\}$. If $a \in \sqrt{0}$, then $a = 0$, since S is reduced and we are done. So suppose $a \notin \sqrt{0}$. Since $\{0\}$ is an n-ideal, we have $b = 0$. Therefore, S is an entire semiring. □

Any semiring S need not have an n-ideal. For example:

Example 3.1.21. It can be easily verified that $S = \{0, 1, 2, 3\}$ equipped with two operations $+$ and $\cdot$ defined by

$+$	0	1	2	3
0	0	1	2	3
1	1	2	1	2
2	2	1	2	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	0
3	0	3	0	3

is a commutative semiring with identity, which has no n-ideals.

Proposition 3.1.22. *Any reduced semiring which is not an entire semiring, has no n-ideals.*

Proof. Suppose S is a reduced semiring which is not an entire semiring. Thus $\sqrt{0} = \{0\}$. If I is an n-ideal of S, by Proposition-3.1.4, $I = \{0\}$. If $\{0\}$ is an n-ideal of S, again by Proposition-3.1.20, S is an entire semiring, which is a contradiction. Therefore, S has no n-ideals. □

Theorem 3.1.23. *Let S be a semiring. Then there exists an n-ideal of S if and only if $\sqrt{0}$ is a prime ideal of S.*

Proof. Suppose there exists an n-ideal I of S. Then by Proposition-3.1.4, $I \subseteq \sqrt{0}$ which implies $\sqrt{I} \subseteq \sqrt{0}$. Thus $\sqrt{I} = \sqrt{0}$. By Proposition 3.1.13, I is a primary ideal. Thus we conclude that $\sqrt{I} = \sqrt{0}$ is a prime ideal of S.

Conversely, assume that $\sqrt{0}$ is a prime ideal of S. Then by Proposition 3.1.6, $\sqrt{0}$ is an n-ideal of S.

□

Theorem 3.1.24. *A semiring S is an entire semiring if and only if $\{0\}$ is the only n-ideal of S.*

Proof. Let S be an entire semiring and $ab \in \{0\}$ with $a \notin \sqrt{0}$ for some $a.b \in S$. Since S is entire and $ab = 0$, we have either $\sqrt{0} = \{0\}$. As $a \notin \sqrt{0}$, thus $b = 0$. So $\{0\}$ is an n-ideal of S. Let I be an n-ideal of S. Then $I \subseteq \sqrt{0} = \{0\}$ (*cf.* Proposition 3.1.4). Therefore, $\{0\}$ is the only n-ideal of S.

Conversely, let $\{0\}$ is the only n-ideal of S. Then by Corollary 3.1.19 and Proposition 3.1.6, we get $\sqrt{0} = \{0\}$ is a prime ideal of S and hence S is an entire semiring. □

Theorem 3.1.25. *Let S_1 and S_2 be semirings, $f : S_1 \longrightarrow S_2$ be a homomorphism. Then the following statements hold:*

(1) If I is a subtractive n-ideal of S_1 such that $kerf \subseteq I$ and f is an onto steady homomorphism, then $f(I)$ is an n-ideal of S_2.

(2) If J is an n-ideal of S_2 such that $kerf \subseteq \sqrt{0_{S_1}}$, then $f^{-1}(J)$ is an n-ideal of S_1.

Proof. (1) Let $ab \in f(I)$ and $a \notin \sqrt{0_{S_2}}$ for some $a, b \in S_2$. Since f is an onto homomorphism, there exist $p, q \in S_1$ such that $f(p) = a$, $f(q) = b$. Thus $ab = f(p)f(q) = f(pq) \in f(I)$. So, $f(pq) = f(i)$ for some $i \in I$. Since f is steady, $pq + k_1 = i + k_2$ for some $k_1, k_2 \in ker(f)$. Hence $pq \in I$, as I is a subtractive ideal of S_1 and $\ker f \subseteq I$. Now, if $p \in \sqrt{0_{S_1}}$, then $p^n = 0_{S_1}$ for some positive integer n. That implies $a^n = (f(p))^n = f(p^n) = f(0_{S_1}) = 0_{S_2}$, which contradicts $a \notin \sqrt{0_{S_2}}$. Thus $p \notin \sqrt{0_{S_1}}$, and since I is an n-ideal, $pq \in I$ so we get $q \in I$. This gives $b = f(q) \in f(I)$ and hence $f(I)$ is an n-ideal of S_2.

(2) Let $pq \in f^{-1}(J)$ and $p \notin \sqrt{0_{S_1}}$. Then $f(pq) = f(p)f(q) \in J$. If $f(p) \in \sqrt{0_{S_2}}$, then there exists a positive integer n such that $f(p)^n = f(p^n) = 0_{S_2}$. That implies $p^n \in$

$\ker f \subseteq \sqrt{0_{S_1}}$, which contradicts $p \notin \sqrt{0}$ and so $f(p) \notin \sqrt{0}$. Since J is an n-ideal of S_2, $f(p)f(q) \in J$ with $f(p) \notin \sqrt{0}$, we conclude that $f(q) \in J$. Thus $q \in f^{-1}(J)$ and hence $f^{-1}(J)$ is an n-ideal of S_1. □

Lemma 3.1.26. *Let I be an n-ideal and J be a non-empty subset of S such that $J \nsubseteq I$. Then $(I : J)$ is an n-ideal of S.*

Proof. If $(I : J) = S$, then $1 \in (I : J)$ and so $J \subseteq I$, a contradiction. Therefore, $(I : J)$ is a proper ideal of S. Let $ab \in (I : J)$ with $a \notin \sqrt{0}$. Then $abJ \subseteq I$ and thus $bJ \subseteq I$, because I is an n-ideal of S. Therefore $b \in (I : J)$ and hence $(I : J)$ is an n-ideal of S. □

Definition 3.1.27. An n-ideal I of S is called maximal n-ideal if there is no n-ideal that contains I properly.

Proposition 3.1.28. *Let I be a maximal n-ideal of a semiring S. Then I is a prime ideal. Moreover, $I = \sqrt{0}$.*

Proof. Suppose I is a maximal n-ideal of S. Let $ab \in I$ with $a \notin I$ for some $a, b \in S$. Since I is an n-ideal of S, by Lemma-3.1.26, $(I : a)$ is an n-ideal. Thus $b \in (I : a) = I$, by maximality of I and so I is a prime ideal. Also, by Proposition 3.1.11, $I = \sqrt{0}$. □

Theorem 3.1.29. *Let S be a semiring, J be a Q-ideal of S and I be a subtractive ideal of S with $J \subseteq I$. If I is an n-ideal of S, then $I/J_{Q\cap I}$ is an n-ideal of S/J_Q. The converse is true if, $J \subseteq \sqrt{0_S}$.*

Proof. Suppose that I is an n-ideal of S and $p_1 + J$, $p_2 + J \in S/J_Q$ be such that $(p_1+J) \odot (p_2+J) = p+J \in I/J_{Q\cap I}$ and $p_1+J \notin \sqrt{0_{I/J_{Q\cap I}}}$ where $p \in Q\cap I$ is a unique element such that $p_1p_2 + J \subseteq p+J \in I/J_{Q\cap I}$. So $p_1p_2 + j_1 = p + j_2$ for some $j_1, j_2 \in J$. Since I is a subtractive ideal and $J \subseteq I$ we have $p_1p_2 \in I$. Now, if $p_1 \in \sqrt{0_S}$ then $p_1^k = 0_S$ for some positive integer k. That implies $(p_1 + J)^k = q + J$ where $q \in Q \cap I$ is a unique element such that $p_1^k + J \subseteq q + J$. That is $0 + J \subseteq q + J$, which contradicts $p_1 + J \notin \sqrt{0_{I/J_{Q\cap I}}}$. Thus $p_1p_2 \in I$, $p_1 \notin \sqrt{0_S}$. Since I is an n-ideal of S, we have $p_2 \in I$. Therefore $p_2 + J \in I/J_{Q\cap I}$ and so $I/J_{Q\cap I}$ is an n-ideal of S/J_Q.

Conversely, if $I/J_{Q\cap I}$ is an n-ideal of S/J_Q. Let $ab \in I$ and $a \notin \sqrt{0_S}$ for some $a, b \in S$. Since J is a Q-ideal of S, there exist p_1, p_2, $p \in Q$ such that $a \in p_1 + J$, $b \in p_2 + J$. Now, $ab \in (p_1 + J) \odot (p_2 + J) = p + J$. So, $ab = p + j \in I$ for some $j \in J$. Since I is a subtractive ideal of S and $J \subseteq I$, we have $p \in I$. So, $(p_1+J)\odot(p_2+J) = p+J \in I/J_{Q\cap I}$.

Also, $p_1 + J \notin \sqrt{0_{I/J_{Q\cap I}}}$, otherwise $(p_1 + J)^k = 0 + J$ for some positive integer k, that is $a^k \in (p_1 + J)^k = p_1^k + J = J \subseteq \sqrt{0_S}$, a contradiction. So $p_2 + J \in I/J_{Q\cap I}$, since $I/J_{Q\cap I}$ is an n-ideal of S/J_Q. Thus $b = p_2 + j_2$ for some $j_2 \in J \subseteq I$ and hence $b \in I$. □

Let I and J be ideals of semiring S with $J \subseteq I$. Then $I/J = \{a + J : a \in I\}$ is an ideal of S/J. Moreover, if I is a subtractive ideal of S, then I/J is a subtractive ideal of S/J [8]. In the following, we can use it to show the next result.

Corollary 3.1.30. *Let S be a commutative semiring and J be an ideal of S. If I is a subtractive n-ideal of S with $J \subseteq I$, then I/J is an n-ideal of S/J.*

Theorem 3.1.31. *Let I be a Q-ideal of a semiring S and J be a subtractive ideal of S such that $I \subseteq J$. If I and J/I are n-ideals of S and S/I respectively, then J is an n-ideal of S.*

Proof. Let $ab \in J$ and $a \notin \sqrt{0}$ for some $a, b \in S$. If $ab \in I$, then $b \in I \subseteq J$ since I is an n-ideal of S and $a \notin \sqrt{0}$. So we can assume that $ab \notin I$. Since I is an Q-ideal, there exist q_1, $q_2 \in Q$ such that $a \in q_1 + I$ and $b \in q_2 + I$. Thus $a = q_1 + i_1$ and $b = q_2 + i_2$ for some i_1, $i_2 \in I$. It follows that $ab = (q_1+i_1)(q_2+i_2) = q_1q_2+q_1i_2+q_2i_1+i_1i_2 \in J$. Since J is a subtractive ideal of S we have $q_1q_2 \in J$. Assume that q is the unique element in Q such that $(q_1+I)\odot(q_2+I) = q+I$ where $q_1q_2+I \subseteq q+I$. Then $q_1q_2+i_3 = q+i_4$ for some i_3, $i_4 \in I$ and so $q \in J \cap Q$ and $q + I \in J/I$. Suppose that $q_0 \in Q$ is the unique element such that $q_0 + I$ is the zero element in J/I. If $(q_1 + I) \odot (q_2 + I) = q_0 + I$, then $q_1q_2 + k = q_0 + l$ for some k, $l \in I$. As I is a Q-ideal of S, it is a subtractive ideal by (*cf*. Corollary 2, [66]). Thus $q_1q_2 \in I$ and so $ab \in I$, a contradiction. Also, if $(q_1 + I)^m = q_0 + I$ for some positive integer m, then $p_1^m + i_5 = q_0 + i_6$ for some i_5, $i_6 \in I$. Since I is subtractive ideal, by Proposition 3.1.11 we have $p_1^m \in I \subseteq \sqrt{0}$, that contradicts $a \notin \sqrt{0}$. Hence $(q_1 + I) \odot (q_2 + I) \in J/I$ with $q_1 + I \notin \sqrt{0_{J/I}}$. So we get $q_2 + I \in J/I$ since J/I is an n-ideal. This implies that $b = q_2 + i_2 \in J$. Therefore J is an n-ideal if S.

□

Proposition 3.1.32. *Let I be an n-ideal and J be a primary ideal of S such that $I \nsubseteq J$. Then $I \cap J$ is an n-ideal of S if and only if $J \subseteq \sqrt{0}$.*

Proof. Suppose that $I \cap J$ is an n-ideal of S. Since $I \nsubseteq J$, there exists an element $y \in I$ such that $y \notin J$. If possible, let $x \in J$ but $x \notin \sqrt{0}$. Since $I \cap J$ is an n-ideal and

$xy \in I \cap J$ and $x \notin \sqrt{0}$, we have $y \in I \cap J \subseteq J$, which is a contradiction. Therefore, $J \subseteq \sqrt{0}$.

Conversely, by Theorem 3.1.15, J is an n-ideal of S. Let $ab \in I \cap J$ with $a \notin \sqrt{0}$. As I, J both are n-ideals and $ab \in I$, $ab \in J$ with $a \notin \sqrt{0}$, we get $b \in I \cap J$ and hence $I \cap J$ is an n-ideal of S. □

Theorem 3.1.33. *Let I be an ideal of S and U be a multiplicative closed subset of S. Then the following are true:*

(1) If I is an n-ideal of S, then I_U is an n-ideal of S_U.

(2) If I_U is an n-ideal of S_U with $Z(S) \cap U = Z_I(S) \cap U = \phi$, then I is an n-ideal of S.

Proof. (1) Let $(a/u)(b/v) \in I_U$ and $a/u \notin \sqrt{0_{S_U}}$ for some $a/u, b/v \in S_U$. Then there exists $t \in U$ such that $abt \in I$. Since $a \notin \sqrt{0}$, as $a/u \notin \sqrt{0_{S_U}}$ and I is an n-ideal of S, we conclude that $bt \in I$. Therefore, $b/v = bt/vt \in I_U$. Consequently, I_U is an n-ideal of S_U.

(2) Let $ab \in I$ and $a \notin \sqrt{0}$ for some $a, b \in S$. Then $(a/1)(b/1) \in I_U$. Also $(a/1) \notin \sqrt{0_{S_U}}$, otherwise $a/1 \in \sqrt{0_{S_U}}$ implies $a^m/1 = 0_{S_U}$ for some positive integer m. So there exists $u \in U$ such that $ua^m = 0$. Since $Z(S) \cap U = \phi$, we have $a^m = 0$, that contradicts $a \notin \sqrt{0}$. Thus $(a/1)(b/1) \in I_U$ and $a/1 \notin \sqrt{0_{S_U}}$ implies $b/1 \in I_U$, since I_U is an n-ideal of S_U. So $vb \in I$ for some $v \in U$. Again, since $Z_I(S) \cap S = \phi$, it follows that $b \in I$ and so I is an n-ideal of S. □

Remark 3.1.34. In (2) of Theorem 3.1.33, such a multiplicative closed subset U exists. For example, consider $U = \{[1], [3]\}$ in Example .. and $I = \{[0], [4]\}$. Then $Z(S) = Z_I(S) = \{[0], [2], [4], [6]\}$ and so $Z(S) \cap U = Z_I(S) \cap U = \phi$.

Theorem 3.1.35. *Let S_1 and S_2 be two semirings and I_1, I_2 be two proper ideals of S_1, S_2 respectively. Then,*

(1) $I_1 \times S_2$ is an n-ideal of $S_1 \times S_2$ if and only if I_1 is an n-ideal of S_1.

(2) $S_1 \times I_2$ is an n-ideal of $S_1 \times S_2$ if and only if I_2 is an n-ideal of S_2.

Proof. (1) Suppose that $I_1 \times S_2$ is an n-ideal of $S_1 \times S_2$. Let $ab \in I_1$ and $a \notin \sqrt{0_{S_1}}$. Then $(a, x)(b, y) = (ab, xy) \in I_1 \times S_2$ for any $x, y \in S_2$. Also $(a, x) \notin \sqrt{0_{S_1 \times S_2}}$, as $a \notin \sqrt{0_{S_1}}$ and $\sqrt{0_{S_1 \times S_2}} = \sqrt{0_{S_1}} \times \sqrt{0_{S_2}}$. Thus $(b, y) \in I_1 \times S_2$ and so $b \in I_1$. Therefore, I_1 is an n-ideal of S_1.

Conversely, let I_1 is an n-ideal of S_1 and $(a, x)(b, y) \in I_1 \times S_2$ with $(a, x) \notin \sqrt{0_{S_1 \times S_2}}$ for some $(a, x)(b, y) \in S_1 \times S_2$. Then $ab \in I_1$ and $a \notin \sqrt{0_{S_1}}$. Since I_1 is an n-ideal of S_1, we have $b \in I_1$. Hence, $(b, y) \in I_1 \times S_2$ and so $I_1 \times S_2$ is an n-ideal of $S_1 \times S_2$.

(2) Silimar to (1). □

Remark 3.1.36. Let I_1 and I_2 be two proper ideals of commutative semirings S_1 and S_2 respectively. Then $I_1 \times I_2$ is not an n-ideal of $S_1 \times S_2$. As because $(1,\ 0)(0,\ 1) = (0,\ 0) \in I_1 \times I_2$, but neither $(1, 0) \in \sqrt{(0,\ 0)}$ nor $(0,\ 1) \in I_1 \times I_2$.

Corollary 3.1.37. *Let I_1 and I_2 be ideals of semirings S_1 and S_2 respectively. Then $I_1 \times I_2$ is an n-ideal of $S_1 \times S_2$ if and only if either I_1 is an n-ideal of S_1 and $I_2 = S_2$, or $I_1 = S_1$ and I_2 is an n-ideal of S_2.*

3.2 n-subsemimodule over Commutative Semiring

In this section, we generalize the concept of n-ideals of semirings to n-subsemimodule of S-semimodules and study n-ideals in expectation semirings.

Definition 3.2.1. A proper subsemimodule N of an S-semimodule M is called an n-subsemimodule if whenever $a \in S$, $m \in M$ and $am \in N$ with $a \notin Nil(M)$, then $m \in N$ where $Nil(M) = \sqrt{(0 : M)}$.

Example 3.2.2. Let $M = (\mathbb{Z}_{27}, +)$ and $S = \mathbb{Z}_0^+$. Then subsemimodule $\{\bar{0}, \bar{9}, \bar{18}\}$ is an n-subsemimodule of M.

Theorem 3.2.3. *Let N be a proper subsemimodule of an S-semimodule M. Then N is an n-subsemmimodule if and only if $(N : m) = S$ or $(N : m) \subseteq Nil(M)$ for any $m \in M$.*

Proof. Suppose N is a proper subsemimodule of M and consider an element $m \in M$. If $a \in N$ then we have $(N : a) = S$. Otherwise, for $m \in M \setminus N$, let $a \in (N : m)$, i.e., $am \in N$. Since $m \notin N$ and N is an N-subsemimodule of M, then $y \in Nil(M)$. Thus $(N : m) \subseteq Nil(M)$. Conversely, let $am \in N$ with $a \notin Nil(M)$ for $a \in S$ and $m \in M$. So $a \in (N : m)$ but $a \notin Nil(M)$. Therefore, by assumption $(N : m) = S$ and it follows that $m \in N$. □

Definition 3.2.4. A non-empty subset U of a semiring S such that $S - \sqrt{0} \subseteq U$. Then U is called an n-multiplicatively closed subset of S, if $xy \in U$ for all $x \in S - \sqrt{0}$ and $y \in U$.

Proposition 3.2.5. *Let I be a proper ideal of S. Then I is an n-ideal of S if and only if $S - I$ is an n-multiplicatively closed subset of S.*

Proof. Suppose that I is an n-ideal of S. Then by Proposition 3.1.4, we have $I \subseteq \sqrt{0}$. Thus $S - \sqrt{0} \subseteq S - I$. Let $x \in S - \sqrt{0}$ and $y \in S - I$. If possible, let $xy \in I$. Since $x \notin \sqrt{0}$ and I is an n-ideal of S, we get $y \in I$, which is a contradiction. Therefore $ab \in S - I$ and so $S - I$ is an n-multiplicatively closed subset of S. Conversely, suppose that $S - I$ is an n-multiplicatively closed subset of S and let $ab \in I$ with $a \notin \sqrt{0}$. Then we have $b \in I$, since otherwise we would have $ab \in S - I$, which is a contradiction. Hence, I is an n-ideal of S. □

Theorem 3.2.6. *Let U be an n-multiplicatively closed subset of S and N a subtractive subsemimodule of a finitely generated S-semimodule M such that $(N : M) \cap U = \phi$. Then there exists a subtractive n-subsemimodule P of M containing N such that $(P : M) \cap U = \phi$.*

Proof. Consider $\Lambda = \{T : T$ is a subtractive subsemimodule of M and $(T : M) \cap U = \phi\}$. Clearly, Λ is non-empty because N itself is a member of Λ. Λ is also partially ordered under the usual set inclusion relation. Since M is finitely generated, any chain of such subsemimodule in the partially ordered set Λ has an upper bound which is their union. Hence by Zorn's lemma, Λ possesses a maximal element P such that $(P : M) \cap U = \phi$. Our claim is that P is an n-subsemimodule of M. On the contrary, assume that $sm \in P$ with $s \notin Nil(M)$ and $b \notin P$. Then $m \in (P : s)$ and $P \subsetneq (P : s)$. By maximality of P, $((P : s) : M) \cap U \neq \phi$. Let $x \in ((P : s) : M) \cap U$. Then $xM \subseteq (P : s)$ and thus $sx \in (P : M)$. Since $x \in U$, $s \in S - Nil(M) \subseteq S - \sqrt{0}$ and U is an n-multiplicative subset of S, then we have $sx \in U$. That contradicts $(P : M) \cap U = \phi$ and hence P is an n-ideal of S. □

Theorem 3.2.7. *Let M and L be two semimodules over a semiring S such that $M \subseteq L$ and $Nil(L) = Nil(M)$. If N is a subtractive n-subsemimodule of M, then there exists a subtractive n-subsemimodule K of L such that $K \cap M = N$.*

Proof. Consider the set $\Omega = \{P :$ P is a subtractive subsemimodule of L with $P \cap M = N\}$. Then Ω is partially ordered by set inclusion. Since $N \in \Omega$, we have Ω is non-empty. If $P_1 \subseteq P_2 \subseteq ...$ is any chain of Ω, then $\bigcup_{i=1}^{\infty} P_i$ is a subtractive subsemimodule of L with $(\bigcup_{i=1}^{\infty} P_i) \cap L = N$. Hence, $\bigcup_{i=1}^{\infty} P_i$ is an upper bound of the chain. Thus by Zorn's lemma, we get a maximal element K of Ω. Now, we claim that K is an

n-subsemimodule of L. Suppose not, then there exist elements $s \in S$ and $l \in L$ such that $sl \in K$ but $s \notin Nil(L)$ and $l \notin K$. So $(K + \langle l \rangle) \cap M \neq N$, by maximality of K. Thus there exists $m \in (K + \langle l \rangle) \cap M$ such that $m \notin N$. Then $m = v + tl$ for some $v \in K$ and $t \in S$. It follows that $sm = sv + tsl \in K \cap M = N$ and $s \notin Nil(L) = Nil(M)$, $m \notin N$ that contradicts N is an n-subsemimodule of M. Hence, K is a subtractive n-subsemimodule of L. □

Proposition 3.2.8. *Let S be a semiring and M be an S-semimodule. Then an ideal I of S is an n-ideal if and only if $I \tilde{\oplus} M$ is an n-ideal of $S \tilde{\oplus} M$.*

Proof. Suppose I is an n-ideal of S and $(s,m)(t,n) \in I \tilde{\oplus} M$ while $(s,m) \notin Nil(S \tilde{\oplus} M)$. Since $Nil(S \tilde{\oplus} M) = Nil(S) \tilde{\oplus} M$, we have $s \notin Nil(S)$. Also $st \in I$. Thus $t \in I$ and so $(t,n) \in I \tilde{\oplus} M$. Therefore, $I \tilde{\oplus} M$ is an n-ideal of $S \tilde{\oplus} M$.

Conversely, let $st \in I$ with $s \notin Nil(S)$. Then $(s,0)(t,0) \in I \tilde{\oplus} M$ while $(s,0) \notin Nil(S \tilde{\oplus} M)$. So we get $(t,0) \in I \tilde{\oplus} M$. Thus $t \in I$ which implies I is an n-ideal of S. □

Theorem 3.2.9. *Let I be an ideal of a semiring S and N be a subsemimodule of S-semimodule M. If $I \tilde{\oplus} N$ is an n-ideal of $S \tilde{\oplus} M$, then I is an n-ideal of S, N is an n-subsemimodule of M, $IM \subseteq N$, and $Nil(S) = Nil(M)$.*

The converse of the statement holds if N is a subtractive subsemimodule of M.

Proof. Suppose $I \tilde{\oplus} N$ is an n-ideal of $S \tilde{\oplus} M$ and let $ab \in I$ with $a \notin Nil(S)$ for some $a, b \in S$. Then $(a,0)(b,0) \in I \tilde{\oplus} N$ and $(a,0) \notin Nil(S \tilde{\oplus} M)$, since $Nil(S \tilde{\oplus} M) = Nil(S) \tilde{\oplus} M$ and $I \tilde{\oplus} N$ is an n-ideal of $S \tilde{\oplus} M$. So $(b,0) \in I \tilde{\oplus} N$. Hence $b \in I$ and so I is an n-ideal of S.

Now, let $am \in N$ with $a \notin Nil(M)$ for some $a \in S$ and $m \in M$. Then $(a,0)(0,m) = (0,am) \in I \tilde{\oplus} N$ and $(a,0) \notin Nil(S \tilde{\oplus} M)$. So $(0,m) \in I \tilde{\oplus} N$, and hence $m \in N$. Therefore, N is an n-subsemimodule of M.

Since $I \tilde{\oplus} N$ is an ideal of $S \tilde{\oplus} M$, we have $IM \subseteq N$ ($cf.$ Theorem 1.6, [78]).

Let $x \in Nil(M)$ and $m \in M \backslash N$. Then $x^n M = 0_M$ for some positive integer n. Here $(x^n, 0)(0, m) = (0,0) \in I \tilde{\oplus} N$ and $(0,m) \notin I \tilde{\oplus} N$. Since $I \tilde{\oplus} N$ is an n-ideal, so we get $(x^n, 0) \in Nil(S \tilde{\oplus} M) = Nil(S) \tilde{\oplus} M$. From this, we get $x^n \in Nil(S)$, which means that $x \in Nil(S)$. Also from the definition of $Nil(M)$ it is easy to observe $Nil(S) \subseteq Nil(M)$. Therefore $Nil(M) = Nil(S)$.

Conversely, let N is subtractive and $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1) \in I \tilde{\oplus} N$ and $(r_1, m_1) \notin Nil(S \tilde{\oplus} M)$ for $(r_1, m_1), (r_2, m_2) \in S \tilde{\oplus} M$. Then $r_1 r_2 \in I$ and $r_1 \notin$

$Nil(S)$, as $Nil(S\tilde{\oplus}M) = Nil(S)\tilde{\oplus}M$. Since I is an n-ideal of S, we get $r_2 \in I$. Thus $r_2m_1 \in IM \subseteq N$ and $r_1m_2 + r_2m_1 \in N$, that implies $r_1m_2 \in N$, as N is subtractive subsemimodule of M. Therefore $r_1m_2 \in N$, $r_1 \notin Nil(M) \subseteq Nil(S)$, since N is an n-subsemimodule of M, we get $m_2 \in N$. Thus, $(r_2, m_2) \in I \tilde{\oplus} N$ and so $I \tilde{\oplus} N$ is an n-ideal of $S \tilde{\oplus} M$. □

Corollary 3.2.10. *Let I be an ideal of S and N be a subsemimodule of a subtractive S-semimodule M. Then $I \tilde{\oplus} N$ is an n-ideal of the expectation semiring $S \tilde{\oplus} M$ if and only if I is an n-ideal of S and N is an n-subsemimodule of M, $IM \subseteq N$ and $Nil(M) \subseteq Nil(S)$.*

CHAPTER 4

Left Bi-quasi Ideals of Ternary Semirings

4

Left Bi-quasi Ideals of Ternary Semirings

In this chapter, the notions of left bi-quasi ideals and bi-quasi ideals of a ternary semiring have been introduced, which are extensions of the concept of bi-ideals in a ternary semiring. The concept of minimal left bi-quasi ideals, left bi-quasi simple ternary semiring have also been studied. Characterization of regular ternary semiring in terms of left bi-quasi ideals has been obtained.
This chapter has been organized as follows:

In *Section 1*, we introduce the notion of left (right, lateral) bi-quasi ideals and bi-quasi ideals in ternary semirings (*cf.* Definition 4.1.1, 4.1.2, 4.1.3), (*cf.* Definition 4.1.4). Then we provide an example (*cf.* Example 4.1.5) of a bi-quasi ideal which together with Proposition 4.1.10 shows that the notion of bi-quasi ideals generalizes the notion of bi-ideals in a ternary semiring. We characterize regular ternary semirings in terms of left bi-quasi ideals (*cf.* Theorem 4.1.12). We also obtain some important properties of left bi-quasi ideals in ternary semirings. Finally, we provide a process of construction of bi-quasi ideals from left ideals, lateral ideals, right ideals, and multiplicatively idempotent elements (*cf.* Theorem 4.1.18), (*cf.* Theorem 4.1.19).

In *Section 2*, we introduce the concept of minimal bi-quasi ideals and bi-quasi simple ternary semirings (*cf.* Definition 4.2.1), (*cf.* Definition 4.2.2). We obtain some characterizations (*cf.* Theorem 4.2.3, Theorem 4.2.4) of left bi-quasi simple ternary

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semirings. We deduce a necessary and sufficient condition for a left bi-quasi ideal to be a minimal left bi-quasi ideal in a ternary semiring (*cf.* Theorem 4.2.5). To conclude this chapter we finally obtain a characterization of regular ternary semirings in terms of the left bi-quasi ideals (*cf.* Theorem 4.2.11).

4.1 Left Bi-quasi Ideals

Definition 4.1.1. A ternary subsemiring L of a ternary semiring S is called a left bi-quasi ideal of S if $SSL \cap LSLSL \subseteq L$.

Definition 4.1.2. A ternary subsemiring R of a ternary semiring S is called a right bi-quasi ideal of S if $RSS \cap RSRSR \subseteq R$.

Definition 4.1.3. A ternary subsemiring M of a ternary semiring S is called a lateral bi-quasi ideal of S if $(SMS + SSMSS) \cap MSMSM \subseteq M$.

Definition 4.1.4. Let S be a ternary semiring. A ternary subsemiring B of S is said to be bi-quasi ideal of S if B satisfies all three conditions $SSB \cap BSBSB \subseteq B$, $(SBS + SSBSS) \cap BSBSB \subseteq B$ and $BSS \cap BSBSB \subseteq B$.

Example 4.1.5. Let $S = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} : a, b, c, d, e, f \in \mathbb{Q}_0^-, f \neq 0 \right\}$ be the ternary semiring with respect to matrix multiplication, where $\mathbb{Q}_0^-$ is the set of all non positive rational numbers. Consider $L = \left\{ \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n \end{pmatrix} : m, n \in \mathbb{Q}_0^-, n \neq 0 \right\}$. Then L is not a bi-ideal of the ternary semiring S but L is a left bi-quasi ideal of S. Let, $x \in SSL \cap LSLSL$. Then

$$x = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & 0 & f_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & e_2 \\ 0 & 0 & f_2 \end{pmatrix} \begin{pmatrix} 0 & m_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & m_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_2 \end{pmatrix} \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & d_3 & e_3 \\ 0 & 0 & f_3 \end{pmatrix} \begin{pmatrix} 0 & m_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_3 \end{pmatrix} \begin{pmatrix} a_4 & b_4 & c_4 \\ 0 & d_4 & e_4 \\ 0 & 0 & f_4 \end{pmatrix} \begin{pmatrix} 0 & m_4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & a_1a_2m_1 & a_1c_2n_1 + b_1e_2n_1 + c_1f_2n_1 \\ 0 & 0 & d_1e_2n_1 + e_1f_2n_1 \\ 0 & 0 & f_1f_2n_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & e_3f_4m_2n_3n_4 \\ 0 & 0 & 0 \\ 0 & 0 & f_3f_4n_2n_3n_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & v \end{pmatrix} (say)$$

Then the equations

$$a_1a_2m_1 = 0 = u,$$

$$a_1c_2n_1 + b_1e_2n_1 + c_1f_2n_1 = e_3f_4m_2n_3n_4 = 0,$$

$$d_1e_2n_1 + e_1f_2n_1 = 0,$$

$$f_1f_2n_1 = f_3f_4n_2n_3n_4 = v(\neq 0),$$

have non-trivial solutions. Hence, $x \in L$. Therefore, $SSL \cap LSLSL \subseteq L$ which implies L is left bi-quasi ideal of S but $LSLSL \nsubseteq L$ which is evident from the above computation. Hence, L is not a bi-ideal of S.

Proposition 4.1.6. *Let S be a ternary semiring. Then every left (right) ideal L of S is a bi-quasi ideal of S.*

Proof. Let L be a left ideal of a ternary semiring S, then $SSL \subseteq L$. Now $SSL \cap LSLSL \subseteq SSL \subseteq L$. So L is a left bi-quasi ideal. Also $(SLS + SSLSS) \cap LSLSL \subseteq LSLSL \subseteq SSSSL \subseteq SSL \subseteq L$. So L is a lateral bi-quasi ideal of S. Similarly we can prove L is a right bi-quasi ideal of S and consequently L is a bi-quasi ideal of S. □

Proposition 4.1.7. *Every lateral ideal M of a ternary semiring S is a bi-quasi ideal of S.*

Proof. Let S be a ternary semiring and M be a lateral ideal of S. Then $(SMS + SSMSS) \subseteq M$. Now $SSM \cap MSMSM \subseteq MSMSM \subseteq SSMSS \subseteq (SMS + SSMSS)$ (Since $0 \in SMS$) $\subseteq M$. Similarly, we can prove M is a right bi-quasi ideal of S. Again $(SMS + SSMSS) \cap MSMSM \subseteq SMS + SSMSS \subseteq M$. So M is a lateral bi-quasi ideal of S. Consequently, M is a bi-quasi ideal of S. □

Corollary 4.1.8. *Every ideal of ternary semiring is a bi-quasi ideal.*

Proposition 4.1.9. *Every quasi-ideal Q of a ternary semiring S is a bi-quasi ideal of S.*

Proof. Let Q be a quasi-ideal of ternary semiring S. Then $SSQ \cap (SQS + SSQSS) \cap QSS \subseteq Q$. We have $QSQSQ \subseteq SQS + SSQSS$ (since $QSQSQ \subseteq SSQSS$ and $0 \in$

SQS). Also $QSQSQ \subseteq Q(SSS)S \subseteq QSS$. Thus $QSQSQ \subseteq QSS \cap (SQS + SSQSS)$. So $SSQ \cap QSQSQ \subseteq SSQ \cap (SQS + SSQSS) \cap QSS \subseteq Q$. Hence Q is a left bi-quasi ideal of S. Similarly, we can prove Q is a right bi-quasi ideal of S. Further, $QSQSQ \subseteq SSQ$ and $QSQSQ \subseteq QSS$ implies that $QSQSQ \subseteq SSQ \cap QSS$. Thus $(SQS + SSQSS) \cap QSQSQ \subseteq SSQ \cap (SQS + SSQSS) \cap QSS \subseteq Q$, that proves Q is a lateral bi-quasi ideal of S. Consequently, Q is a bi-quasi ideal of S. □

Proposition 4.1.10. *Every bi-ideal of a ternary semiring S is a bi-quasi ideal of S.*

Proof. Let B be a bi-ideal of a ternary semiring S. Then $BSBSB \subseteq B$. Therefore $SSB \cap BSBSB \subseteq BSBSB \subseteq B$, $BSS \cap BSBSB \subseteq BSBSB \subseteq B$ and $(SBS + SSBSS) \cap BSBSB \subseteq BSBSB \subseteq B$ and hence B is a bi-quasi ideal of S. □

Theorem 4.1.11. (*cf*. Theorem 3.4, [33]) Let S be a ternary semiring. S is regular if and only if $RML = R \cap M \cap L$, for any right ideal R, lateral ideal M and left ideal L.

Theorem 4.1.12. *Let S be a regular ternary semiring. Then a ternary subsemiring L is a left bi-quasi ideal of S if and only if $L = LSL$.*

Proof. Suppose L is a left bi-quasi ideal of a regular ternary semiring S. Let $a \in L$, then there exists $s \in S$ such that $a = asa \in LSL$, that implies $L \subseteq LSL$. Again $LSL \subseteq SSL$ and by regularity, $LSL \subseteq LSLSL$. Thus $LSL \subseteq SSL \cap LSLSL \subseteq L$. Hence $L = LSL$.

Conversely, suppose $L = LSL$. Since S is regular, $L \subseteq LSL \subseteq SSL$. Thus $SSL \cap LSLSL = SSL \cap L \subseteq L$. Consequently, L is a left bi-quasi ideal of S. □

Theorem 4.1.13. *Let S be a regular ternary semiring. Then every left bi-quasi ideal of S is an ideal of S.*

Proof. Let S be a regular ternary semiring and L be a left bi-quasi ideal of S. By Theorem 4.1.11, we have $LSS \cap (SLS + SSLSS) \cap SSL = LSSSLSSSL + LSSSSLSSSSL \subseteq LSLSL + LSSLSSL \subseteq LSLSL + LSLSL$ (*cf.* Theorem 4.1.12) $\subseteq LSLSL$. Also $LSS \cap (SSL + SSLSS) \cap SSL = LSSSLSSSL + LSSSSLSSSSL \subseteq LSLSL + LSSLSSL \subseteq SSSSL + SSSSSSL \subseteq SSL + SSL \subseteq SSL$. Therefore $LSS \cap (SSL + SSLSS) \cap SSL \subseteq SSL \cap LSLSL \subseteq L$. Hence L is a quasi-ideal of S. Since in regular ternary semiring every bi-ideal is quasi-ideal and every quasi ideal is an ideal, therefore in a regular ternary semiring every bi-quasi ideal is an ideal. □

Theorem 4.1.14. *Let S be a regular ternary semiring. Then every left bi-quasi ideal is a bi-ideal of S.*

Proof. Suppose L is a left bi-quasi ideal of S. Clearly, by Theorem 4.1.12, $LSLSL \subseteq L$. Consequently, L is a bi-ideal of S. □

Theorem 4.1.15. *Let S be a ternary semiring and $\{L_i\}_{i\in I}$ be a family of left bi-quasi ideals of S. Then $\bigcap_{i\in I} L_i$ a left bi-quasi ideal of S, provided $\bigcap_{i\in I} L_i \neq \phi$*

Proof. Let $L = \bigcap_{i\in I} L_i$. Since $SSL_i \cap L_iSL_iSL_i \subseteq L_i$ for all $i \in I$, we have $SSL \cap LSLSL \subseteq SSL_i \cap L_iSL_iSL_i \subseteq L_i$ for all $i \in I$. Hence L is left bi-quasi ideal of ternary semiring S. □

We obtain the following proposition by routine verification.

Proposition 4.1.16. *Let L be a left ideal, M be a lateral ideal and R be a right ideal of a ternary semiring S. Then $L \cap M \cap R$ is a left bi-quasi ideal of S.*

Theorem 4.1.17. *If B is a bi-quasi ideal and T is a ternary subsemiring of a ternary semiring S, then $B \cap T$ is a bi-quasi ideal of T.*

Proof. Clearly, $B\cap T$ is a subsemiring of T. Now $TT(B\cap T)\cap((B\cap T)T(B\cap T)T(B\cap T)) \subseteq TTT \cap TTTTT \subseteq T$. Also $TT(B\cap T)\cap((B\cap T)T(B\cap T)T(B\cap T)) \subseteq TTB \cap BTBTB \subseteq B$, which implies that $TT(B\cap T)\cap((B\cap T)T(B\cap T)T(B\cap T)) \subseteq B\cap T$. Consequently, $B \cap T$ is left bi-quasi ideal of T. Similarly, we can prove that $B \cap T$ is a right and lateral bi-quasi ideal of T. Hence $B \cap T$ is a bi-quasi ideal of T. □

Theorem 4.1.18. *Let L be a left ideal, R be a right ideal of a ternary semiring S. If e be a multiplicative idempotent element of S, then RSe and eSL are left bi-quasi ideals of S.*

Proof. To show RSe is a left bi-quasi ideal of S, it is enough if we prove $RSe = R \cap (SeS + SSeSS) \cap SSe$ (*cf.* Proposition 4.1.16). Clearly, $RSe \subseteq R \cap SSe$. Let $a \in R \cap SSe$. Then $a \in R$ and $a \in SSe$. Now $a \in SSe$ implies that $a = \sum_{i=1}^{m} s_it_ie$ for some $s_i, t_i \in S$. Therefore $aee = (\sum_{i=1}^{m} s_it_ie)ee = \sum_{i=1}^{m} s_it_i(eee) = \sum_{i=1}^{m} s_it_ie = a$, this implies $a \in Ree \subseteq RSe$ and hence $RSe = R \cap SSe$. Again $a = aee \in SeS$ and $0 \in SSeSS$ implies that $a + 0 = a \in SeS + SSeSS$. Thus $R \cap SSe \subseteq SeS + SSeSS$. Consequently, $RSe = R \cap (SeS + SSeSS) \cap SSe$.
Similarly, we can show that eSL is a left bi-quasi ideal of S. □

Theorem 4.1.19. *Let e and f be two multiplicative idempotent elements of a ternary semiring S and M be a lateral ideal of S. Then $eSMSf$ is a left bi-quasi ideal of S.*

Proof. Let M be a lateral ideal of ternary semiring S. Then clearly, $eSMSf \subseteq eSS \cap M \cap SSf$. Let $a \in eSS \cap M \cap SSf$. Then $a \in eSS$, $a \in M$ and $a \in SSf$. Now $a \in eSS$ and $a \in SSf$ imply that $a = \sum_{i=1}^{m} es_it_i = \sum_{j=1}^{n} u_jv_jf$ for some $s_i, t_i, u_j, v_j \in S$. Therefore $eeaff = ee(\sum_{i=1}^{m} es_it_i)ff = (\sum_{i=1}^{m} es_it_i)ff = (\sum_{j=1}^{n} u_jv_jf)ff = \sum_{j=1}^{n} u_jv_jf = a$. So $a \in eeMff \subseteq eSMSf$. Thus $eSMSf = eSS \cap M \cap SSf$. Consequently, $eSMSf$ is a left bi-quasi ideal of S. □

4.2 Minimal Left Bi-quasi Ideals

Definition 4.2.1. Let S be a ternary semiring. A left bi-quasi ideal L of S is called a minimal left bi-quasi ideal of S if it does not contain any non-zero left bi-quasi ideal of S.

Definition 4.2.2. A ternary semiring S is called left bi-quasi simple if S is the unique non-zero left bi-quasi ideal of S.

A characterization of left bi-quasi simple ternary semiring is obtained in the following theorem:

Theorem 4.2.3. *A ternary semiring S is left bi-quasi simple if and only if $SSa \cap aSaSa = S$ for all $a \in S$.*

Proof. Let S be a left bi-quasi simple and let $a \in S$. $SSa \cap aSaSa$ being the intersection of two left bi-quasi ideals of S, is a left bi-quasi ideal of S. Therefore, $SSa \cap aSaSa = S$ for all $a \in S$.

Conversely, suppose that $SSa \cap aSaSa = S$ for all $a \in S$ and T is a left bi-quasi ideal of S. Let $b \in T$. Then $S = SSb \cap bSbSb \subseteq SST \cap TSTST \subseteq T \subseteq S$. So $T = S$ and hence S is left bi-quasi simple. □

Theorem 4.2.4. *Let S be a ternary semiring. Then the following statements are equivalent:*

(1) S is left bi-quasi simple,

(2) $SSa = S$ for all $a \in S$,

(3) $\langle a \rangle_{lbq} = S$ for all $a \in S$, where $\langle a \rangle_{lbq}$ be the smallest left bi-quasi ideal of S containing a.

Proof. (1) $\Rightarrow$ (2) Assume that S is left bi-quasi simple. For $a \in S$, SSa is a left ideal of S. So SSa is a left bi-quasi ideal of S. Therefore $SSa = S$ for all $a \in S$.

(2) $\Rightarrow$ (3) Let $\langle a \rangle_{lbq}$ be the smallest left bi-quasi ideal of S containing a. Then $SSa \subseteq \langle a \rangle_{lbq} \Rightarrow S \subseteq \langle a \rangle_{lbq}$. So $S = \langle a \rangle_{lbq}$.

(3) $\Rightarrow$ (1) Suppose A is a left bi-quasi ideal of S and $a \in A$. Now $\langle a \rangle_{lbq} \subseteq A$ (by (3)) $S \subseteq A$ implies $A = S$. Consequently, S is left bi-quasi simple. □

Theorem 4.2.5. *Let S be a ternary semiring and B be a left bi-quasi ideal of S. Then B is minimal if and only if B is the intersection of minimal left ideal and minimal bi-ideal of S.*

Proof. Let B be a minimal left bi-quasi ideal of S. Then $SSB \cap BSBSB \subseteq B$. Let $b \in B$. Then SSb is a left ideal and $bSbSb$ is a bi-ideal of S. So, their intersection $SSb \cap bSbSb$ is a left bi-quasi ideal of S. Further $SSb \cap bSbSb \subseteq SSB \cap BSBSB \subseteq B$. Since B is minimal so $SSb \cap bSbSb = B$. Now it remains to show that SSb and $bSbSb$ are minimal left ideal and minimal bi-ideal of S respectively. If possible, let L be a left ideal of S such that $L \subseteq SSb$. Then $SSL \subseteq L \subseteq SSb$. Thus $SSL \cap bSbSb \subseteq SSb \cap bSbSb = B$. By minimality of B, $SSL \cap bSbSb = B$. This implies that $B \subseteq SSL$. Also $SSb \subseteq SSB \subseteq SS(SSL) \subseteq SSL \subseteq L$. Thus $L = SSb$. Hence SSb is minimal left ideal of S. If possible, let K be a left bi-quasi ideal of S such that $K \subseteq bSbSb$. Then $KSKSK \subseteq K \subseteq bSbSb$. Now $SSb \cap KSKSK \subseteq SSb \cap bSbSb = B$. By minimality of B, $SSL \cap KSKSK = B$. So $B \subseteq KSKSK$. Again $bSbSb \subseteq BSBSB \subseteq (KSKSK)S(KSKSK)S(KSKSK) \subseteq KSSSKSSSKSSSKSSSK \subseteq KSKSKSKSK \subseteq KSSSKSSSK \subseteq KSKSK \subseteq K$. Thus $K = bSbSb$. Hence $bSbSb$ is minimal bi-ideal of S.

Conversely, let L be a minimal left ideal and K be a minimal bi-ideal of S. Then $B = L \cap K$ is a left bi-quasi ideal of S. Let B_1 be a left bi-quasi ideal of S such that $B_1 \subseteq B$. Then $SSB_1 \subseteq SSB \subseteq SSL \subseteq L$. L is minimal left ideal of S implies $L = SSB_1$. Also $B_1SB_1SB_1 \subseteq BSBSB \subseteq KSKSK \subseteq K$. By minimality of K, $K = B_1SB_1SB_1$. Further $B = L \cap K = SSB_1 \cap B_1SB_1SB_1 \subseteq B_1$. Thus $B = B_1$. Hence B is minimal left bi-quasi ideal of S. □

Theorem 4.2.6. (*cf*. Theorem 2.6, [30]) Let B be a minimal bi-ideal of a ternary semiring S with no nonzero strongly nilpotent ideals. Then B can be represented in the form of RML with minimal right ideal R, minimal lateral ideal M and minimal left ideal L of S.

Corollary 4.2.7. *If B is minimal left bi-quasi ideal of ternary semiring S with no non-zero strongly nilpotent ideals then by Theorem 4.2.6, B can be represented in the form $L_1 \cap RML$ with minimal right ideal R, minimal left ideals L, L_1 and minimal lateral ideal M.*

Theorem 4.2.8. *Let B be a left bi-quasi ideal of S. If B is minimal, then any two non-zero elements of B generate the same ideal (left, lateral, right) of S.*

Proof. Let B be a minimal left bi-quasi ideal of S and $0 \neq a \in B$. Then the left ideal $\langle a \rangle_l$, is a left bi-quasi ideal of S. So $B \cap \langle a \rangle_l$ is a left bi-quasi ideal of S. By minimality of B, $B = B \cap < a >_l$. Thus $B \subseteq \langle a \rangle_l$. Now for any non-zero element $b \in B$, $b \in B \subseteq \langle a \rangle_l \Rightarrow \langle b \rangle_l \subseteq \langle a \rangle_l$. Similarly, $\langle a \rangle_l \subseteq \langle b \rangle_l$. Hence $\langle a \rangle_l = \langle b \rangle_l$. □

Lemma 4.2.9. *Let B be a left bi-quasi ideal of a ternary semiring S and T be a ternary subsemiring of S. If T is left bi-quasi simple such that $T \cap B \neq \phi$, then $T \subseteq B$.*

Proof. Let $a \in T \cap B$. Since $TTa \cap aTaTa$ is left bi-quasi ideal of T and T is left bi-quasi simple, so $TTa \cap aTaTa = T$ (*cf.* Theorem 4.2.3). Now $T = TTa \cap aTaTa \subseteq TTB \cap BTBTB \subseteq SSB \cap BSBSB \subseteq B$ (as B is a left bi-quasi ideal of S). Hence $T \subseteq B$. □

In view of the above lemma, we have the following theorem.

Theorem 4.2.10. *Let S be a ternary semiring and L a left bi-quasi ideal of S. Then the following statements are true*

(1) Let L be a left ideal of S and a minimal left bi-quasi ideal of S. Then L is left bi-quasi simple.

(2) Let L be left bi-quasi simple. Then L is a minimal left bi-quasi ideal of S.

Proof. (1) Suppose L is a left ideal of ternary semiring S and minimal left bi-quasi ideal of S. Then $SSL \cap LSLSL \subseteq L$. To show L is left bi-quasi simple, let A be a left bi-quasi ideal of L. Then $LLA \cap ALALA \subseteq A$.

Now define $H = \{h \in A : h \in LLA \cap ALALA\}$. Then $H \subseteq A \subseteq L$. We want to show that H is a left bi-quasi ideal of S. Let $h_1, h_2, h_3 \in H$. Then $h_1 = \sum q_i p_i a_i = \sum b_i r_i c_i s_i d_i$, $h_2 = \sum q_j p_j a_j = \sum b_j r_j c_j s_j d_j$ and $h_3 = \sum q_k p_k a_k = \sum b_k r_k c_k s_k d_k$ for some a_i, a_j, a_k, b_i, b_j, b_k, c_i, c_j, c_k, $d_i, d_j, d_k \in A$ and p_i, p_j, p_k, q_i, q_j, q_k, r_i, r_j, r_k, $s_i, s_j, s_k \in L$. Thus $h_1 + h_2 \in H$.

Then we have, $h_1h_2h_3 = p_iq_ia_ip_jq_ja_jp_kq_ka_k = b_ir_ic_is_id_ib_jr_jc_js_jd_jb_kr_kc_ks_kd_k$. Since L is a left ideal of S, $(p_iq_ia_ip_jq_j)(a_jp_kq_k)a_k \in LLA$ and $b_i(r_ic_is_i)d_i(b_jr_jc_js_jd_jb_kr_kc_ks_k)d_k \in ALALA$. Thus $h_1h_2h_3 \in H$. Therefore H is a subsemiring of S. To show, H is a left bi-quasi ideal of S. Let $h \in SSH \cap HSHSH$, then $h = \sum q_i^{'}p_i^{'}a_i^{'} = \sum b_i^{'}r_i^{'}c_i^{'}s_i^{'}d_i^{'}$ where $q_i^{'}, p_i^{'}, r_i^{'}, s_i^{'} \in S$ and $a_i^{'}, b_i^{'}, c_i^{'}, d_i^{'} \in H$. Now $h = \sum q_i^{'}p_i^{'}a_i^{'} = \sum\sum q_i^{'}p_i^{'}q_{ij}p_{ij}a_{ij} \in LLA$ (since L is a left ideal of S), where $a_i^{'} = \sum q_{ij}p_{ij}a_{ij} = \sum b_{ij}r_{ij}c_{ij}s_{ij}d_{ij}$. Also $h = \sum b_i^{'}r_i^{'}c_i^{'}s_i^{'}d_i^{'} = \sum\sum\sum(b_{ki}r_{ki}c_{ki}s_{ki}d_{ki})r_i^{'}c_i^{'}s_i^{'}(b_{li}r_{li}c_{li}s_{li}d_{li}) = \sum\sum\sum b_{ki}(r_{ki}c_{ki}s_{ki})d_{ki}(r_i^{'}c_i^{'}s_i^{'}b_{li}r_{li}c_{li}s_{li})d_{li} \in ALALA$, (since L is a left ideal) where $b_i^{'} = \sum b_{ki}r_{ki}c_{ki}s_{ki}d_{ki}$ and $d_i^{'} = \sum b_{li}r_{li}c_{li}s_{li}d_{li}$. Consequently, $h \in LLA \cap ALALA \subseteq A$ which implies that $h \in H$. So $SSH \cap HSHSH \subseteq H$. By minimality of L, $H = L$. Thus $L = H \subseteq A \subseteq L$ implies $A = L$. Hence L is simple.

(2) Let L be a left bi-quasi simple ideal of S and L_1 a left bi-quasi ideal of S such that $L_1 \subseteq L$. By Lemma 4.2.9, $L \subseteq L_1$, implies $L = L_1$. Hence L is minimal left bi-quasi ideal of S. □

To conclude this chapter we obtain the following characterization of a regular ternary semiring in terms of left bi-quasi ideals.

Theorem 4.2.11. *Let S be a ternary semiring. Then S is regular ternary semiring if and only if $B = SSB \cap BSBSB$ for every left bi-quasi ideal B of S.*

Proof. Let S be a regular ternary semiring and B a left bi-quasi ideal of S. Then $SSB \cap BSBSB \subseteq B$. Take $a \in B$, then there exist $x \in S$ such that $a = axa$ (as S is regular). Now $a = axa = axaxa \in BSBSB$ also $a = axa \in SSB$. So $a \in SSB \cap BSBSB$. Thus $B \subseteq SSB \cap BSBSB$. Hence $B = SSB \cap BSBSB$ for every left bi-quasi ideal B of S.

Conversely, suppose $B = SSB \cap BSBSB$ for every left bi-quasi ideal B of S. Let R, M, L are right, lateral and left ideals of S respectively. Then $Q = R \cap M \cap L$ is a left bi-quasi ideal of S (*cf.* Proposition 4.1.16). So by the given condition $Q = SSQ \cap QSQSQ$, which implies $R \cap M \cap L = SSQ \cap QSQSQ \subseteq QSQSQ \subseteq RSMSL \subseteq RML$. Also, we have $RML \subseteq R \cap M \cap L$. Hence $RML = R \cap M \cap L$. Consequently, by Theorem 4.1.11, S is regular. □

CHAPTER 5

On 3-Prime and Quasi 3-Primary Ideals of Ternary Semirings

5

On 3-Prime and Quasi 3-Primary Ideals of Ternary Semirings

The purpose of this chapter is to introduce the concept of 3-prime ideals as a generalization of prime ideals. The concepts of 3-prime ideals and primary ideals have been generalized, namely as quasi 3-primary ideals in a commutative ternary semiring with zero. The relationship among prime ideal, 3-prime ideal, primary ideal, quasi primary and quasi 3-primary ideal have been investigated. Various results and examples concerning 3-prime ideals and quasi 3-primary ideals have also been given. Analogous theorems to the primary avoidance theorem for quasi 3-primary ideals have been studied.

This chapter has been organized as follows: In *Section 1*, we introduce the notion of 3-prime ideals as a generalization of prime ideals in a ternary semiring (*cf.* Definition 5.1.1). Various properties and relationships among radical ideals (*cf.* Proposition 5.1.4, Proposition 5.1.6), maximal ideals (*cf.* Proposition 5.1.1), and strongly irreducible ideals in regular ternary semiring (*cf.* Proposition 5.1.16) are studied. We give a characterization of 3-prime ideals in ternary semirings (*cf.* Theorem 5.1.14). Then we study ternary semirings, where every 3-prime ideal is prime (*cf.* Definition 5.1.17, Theorems 5.1.22, 5.1.24).

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- **Sampad Das et al.**, On 3-Prime and Quasi 3-Primary Ideals of Ternary Semirings, *Discussiones Mathematicae-General Algebra and Applications* (Accepted).

In *Section 2*, we define a quasi 3-primary ideal (*cf.* Definition 5.2.1), which is a generalization of 3-prime ideal and is an intermediate class between 3-prime ideals and quasi-primary ideals in a ternary semiring. We show that in regular ternary semirings, the concept of 3-prime ideals, quasi 3-primary ideals and primary ideals are the same. Theorem 5.2.8 is a characterization for quasi 3-primary ideals on a ternary semiring. At the end, we focus on the study of the avoidance theorem for quasi 3-primary ideals by using the techniques of efficient covering (cf. Theorem 5.2.13) and give an extended version of the theorem (cf. Theorem 5.2.14).

5.1 3-Prime Ideals

Throughout the section, unless otherwise stated, S stands for a commutative ternary semiring with zero.

Definition 5.1.1. An ideal I of a ternary semiring S is called a 3-prime ideal if for any x, y, $z \in S$; $xyz \in I$ implies $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$.

Example 5.1.2. In the ternary semiring $\mathbb{Z}_0^-$, the ideal $8\mathbb{Z}_0^-$ is a 3-prime ideal.

It is easy to see that in a ternary semiring, every prime ideal is 3-prime but the converse may not be true. In the above example, $8\mathbb{Z}_0^-$ is 3-prime but not a prime ideal of $\mathbb{Z}_0^-$. If 3-prime ideal is a semiprime ideal, then the converse also holds as is shown in the next result.

Theorem 5.1.3. *If an ideal I of a ternary semiring S is 3-prime as well as semiprime, then I is a prime ideal.*

Proof. Let $xyz \in I$ for some x, y, $z \in S$. Since I is a 3-prime ideal of S, $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$. As I is semiprime, we have $x \in I$ or $y \in I$ or $z \in I$. □

Proposition 5.1.4. *If I is 3-prime, then $Rad(I)$ is prime.*

Proof. Let $xyz \in Rad(I)$ for some x, y, $z \in S$. Then $(xyz)^{2n+1} \in I$ for some $n \in \mathbb{Z}_0^+$. Thus $x^{2n+1}y^{2n+1}z^{2n+1} \in I$, which implies $x^{2n+1} \in I$ or $y^{2n+1} \in I$ or $z^{2n+1} \in I$. So $x \in Rad(I)$ or $y \in Rad(I)$ or $z \in Rad(I)$. □

The converse of the above proposition may not be true, as is shown in the following example:

Example 5.1.5. Consider the ternary subsemiring $\mathbb{Z}_0^- \times 3\mathbb{Z}_0^-$ of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then the ideal $32\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$ is not a 3-prime ideal in $\mathbb{Z}_0^- \times 3\mathbb{Z}_0^-$ but $Rad(32\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-) = 2\mathbb{Z}_0^- \times 3\mathbb{Z}_0^-$ is a prime ideal of $\mathbb{Z}_0^- \times 3\mathbb{Z}_0^-$. This is because $(-4,-3)(-4,-3)(-2,-27) = (-32,-243) \in 32\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$ but $(-4,-3)^3 \notin 32\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$, $(-4,-3)^3 \notin 32\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$ and $(-2,-27)^3 \notin 32\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$.

Proposition 5.1.6. *If $Rad(I)$ is prime and $(Rad(I))^3 \subseteq I$. Then I is 3-prime.*

Proof. Let $Rad(I)$ be prime and $(Rad(I))^3 \subseteq I$. For x, y, $z \in S$, suppose $xyz \in I$. Then $xyz \in Rad(I)$ which implies $x \in Rad(I)$ or $y \in Rad(I)$ or $z \in Rad(I)$. So $x^3 \in (Rad(I))^3 \subseteq I$ or $y^3 \in (Rad(I))^3 \subseteq I$ or $z^3 \in (Rad(I))^3 \subseteq I$. Hence I is 3-prime. □

Theorem 5.1.7. *In a regular ternary semiring, an ideal is prime if and only if it is 3-prime.*

Proof. Clearly, if an ideal I is prime then it is 3-prime.

Conversely, let I be a 3-prime ideal and $xyz \in I$. Then $x^3 \in I$ or $y^3 \in I$ or $z^3 \in I$. Suppose $x^3 \in I$. By regularity, there exist $a, b \in I$ such that $x = xaxbx$, that is, $x = abx^3 \in I$. So I is prime. □

Proposition 5.1.8. *Let S be a ternary semiring. If an ideal I is a 3-prime ideal of S, then $(I : a^3 : b^3)$ is a 3-prime ideal of S, where a, $b \in S \setminus Rad(I)$.*

Proof. Let $xyz \in (I : a^3 : b^3)$ for some x, y, $z \in S$. Then $xyza^3b^3 \in I$. This implies $(xab)(yab)(zab) \in I$. Thus $(xab)^3 = x^3a^3b^3 \in I$ or $(yab)^3 = y^3a^3b^3 \in I$ or $(zab)^3 = z^3a^3b^3 \in I$. Hence $x^3 \in (I : a^3 : b^3)$ or $y^3 \in (I : a^3 : b^3)$ or $z^3 \in (I : a^3 : b^3)$ and so $(I : a^3 : b^3)$ is a 3-prime ideal of S. □

The ternary product of 3-prime ideals may not be 3-prime, as is shown in the next example:

Example 5.1.9. In the ternary semiring $\mathbb{Z}_0^-$, ternary product of the 3-prime ideals $2\mathbb{Z}_0^-$, $3\mathbb{Z}_0^-$ and $5\mathbb{Z}_0^-$ is $30\mathbb{Z}_0^-$, which is not a 3-prime ideal of $\mathbb{Z}_0^-$.

Lemma 5.1.10. *Suppose P be a prime ideal and P', P'' be two ideals with $P \subseteq P'$ and $P \subseteq P''$. Then $PP'P''$ is 3-prime. Moreover, $PP'P''$ is prime if and only if $PP'P'' = P$.*

Proof. Let $abc \in PP'P''$. Then $abc \in PP'P'' \subseteq P$ which implies $a \in P$ or $b \in P$ or $c \in P$. So $a^3 \in P^3 \subseteq PP'P''$ or $b^3 \in P^3 \subseteq PP'P''$ or $c^3 \in P^3 \subseteq PP'P''$.

Now, let $PP'P''$ be a prime ideal of S. Clearly, $PP'P'' \subseteq P$. Consider $a \in P$. It follows that $a^3 \in PP'P''$. As $PP'P''$ is a prime ideal of S, we have $a \in PP'P''$ and so $P \subseteq PP'P''$. Hence $PP'P'' = P$. □

Corollary 5.1.11. *If P is a prime ideal, then P^3 is a 3-prime ideal.*

Proposition 5.1.12. *In a ternary semiring S, every maximal ideal is 3-prime.*

Proof. If S is a ternary semiring with identity, then every maximal ideal is prime and hence 3-prime. Now suppose that S has no identity and M is a maximal ideal of S. Consider $xyz \in M$ and $x^3 \notin M$, $y^3 \notin M$ for some x, y, $z \in S$. If possible, let $z^3 \notin M$. Then clearly x, y, $z \notin M$. Thus we conclude that $M + \langle x\rangle = S$, $M + \langle y\rangle = S$, $M + \langle z\rangle = S$. Now $x^3 = (m_1 + s_1s_2x + n_1x)(m_2 + s_3s_4y + n_2y)(m_3 + s_5s_6z + n_3z)$ for some m_1, m_2, $m_3 \in M$, s_1, s_2, s_3, s_4, s_5, $s_6 \in S$ and $n_1, n_2, n_3 \in \mathbb{Z}_0^+$. This implies $x^3 \in M$. Similarly, $y^3 \in M$. But $x^3 \notin M$ and $y^3 \notin M$, hence $z^3 \in M$ and so M is a 3-prime ideal. □

Lemma 5.1.13. *Let I be a 3-prime ideal of a ternary semiring S. If $abC \subseteq I$ and $a^3 \notin I$, $b^3 \notin I$ for some elements $a, b \in S$ and some ideal C, then $\{c^3 : c \in C\} \subseteq I$.*

Proof. Suppose $abC \subseteq I$ and $a^3 \notin I$, $b^3 \notin I$ for some a, $b \in S$ and some ideal C. Consider any arbitrary element $c \in C$, then $abc \in abC \subseteq I$. Since I is 3-prime, we conclude that $c^3 \in I$. Hence $\{c^3 : c \in C\} \subseteq I$. □

Theorem 5.1.14. *Let I be a proper ideal of a ternary semiring S with identity. Then I is a 3-prime ideal if and only if whenever $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of S, we have $\{a^3 : a \in I_1\} \subseteq I$ or $\{b^3 : b \in I_2\} \subseteq I$ or $\{c^3 : c \in I_3\} \subseteq I$.*

Proof. Suppose that the condition holds and $abc \in I$ for some a, b, c in S. Then $(SSa)(SSb)(SSc) \subseteq I$ and so by the given condition $\{x^3 : x \in SSa\} \subseteq I$ or $\{y^3 : y \in SSb\} \subseteq I$ or $\{z^3 : z \in SSc\} \subseteq I$. Thus $a^3 \in I$ or $b^3 \in I$ or $c^3 \in I$.

Conversely, suppose I is a 3-prime ideal of S and $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, I_3. Also, suppose that $\{a^3 : a \in I_1\} \nsubseteq I$ and $\{b^3 : b \in I_2\} \nsubseteq I$. Then there exist $i_1 \in I_1$, and $i_2 \in I_2$ such that $i_1^3, i_2^3 \notin I$. By Lemma 5.1.13, $\{c^3 : c \in I_3\} \subseteq I$. □

Theorem 5.1.15. *Let $f : S \longrightarrow T$ be a ternary homomorphism of ternary semirings. Then the following statements hold:*

(1) If J is a 3-prime ideal of T, then $f^{-1}(J)$ is a 3-prime ideal of S.

(2) If f is an onto ternary homomorphism and I is a subtractive ideal of S with $\{x \in S :$ for some $a, b \in S$, $x = a + b$ and $f(a) = f(b)\} \subseteq I$, then $f(I)$ is a 3-prime ideal of T if I is a 3-prime ideal of S.

Proof. (1) Let $xyz \in f^{-1}(J)$ for some $x, y, z \in S$. Then $f(xyz) = f(x)f(y)f(z) \in J$, which implies $(f(x))^3 = f(x^3) \in J$ or $(f(y))^3 = f(y^3) \in J$ or $(f(z))^3 = f(z^3) \in J$. Thus $x^3 \in f^{-1}(J)$ or $y^3 \in f^{-1}(J)$ or $z^3 \in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is a 3-prime ideal of S.

(2) Let $xyz \in f(I)$ for some $x, y, z \in S$. Then there exist a, b, $c \in S$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$. So $xyz = f(a)f(b)f(c) = f(abc) \in f(I)$. Then $f(abc) = f(i)$ for some $i \in I$. Thus $abc + i \in I$. Hence $abc \in I$, since I is a subtractive ideal of S and $i \in I$. So $a^3 \in I$ or $b^3 \in I$ or $c^3 \in I$. Therefore, $f(a^3) = (f(a))^3 = x^3 \in f(I)$ or $f(b^3) = (f(b))^3 = x^3 \in f(I)$ or $f(c^3) = (f(c))^3 = x^3 \in f(I)$. Consequently, $f(I)$ is a 3-prime ideal of T. □

Proposition 5.1.16. *If an ideal I is strongly irreducible in a regular ternary semiring S, then I is 3-prime.*

Proof. Assume that S is a regular ternary semiring and I is a strongly irreducible ideal of S. Suppose that $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$ for some $a, b, c \in S$. We have to show that $c^3 \in I$. On the contrary, assume that $c^3 \notin I$. Then I is properly contained in $(I + \langle a^3\rangle) \cap (I + \langle b^3\rangle) \cap (I + \langle c^3\rangle)$. So there exists an element $x \in (I + \langle a^3\rangle) \cap (I + \langle b^3\rangle) \cap (I + \langle c^3\rangle)$ such that $x \notin I$. Since S is regular, we have $x \in (I+\langle a^3\rangle)(I+\langle b^3\rangle)(I+\langle c^3\rangle)=(I+\langle a^3\rangle)\cap(I+\langle b^3\rangle)\cap(I+\langle c^3\rangle)$. Thus for some $i_1, i_2, i_3 \in I$ and $r_1, r_2, s_1, s_2, t_1, t_2 \in S$ $x = (i_1 + r_1r_2a^3)(i_2 + s_1s_2b^3)(i_3 + t_1t_2c^3) \in I$, which is a contradiction. Therefore, I is a 3-prime ideal of S. □

Definition 5.1.17. A ternary semiring S is called a 3-P-ternary semiring if every 3-prime ideal of S is prime.

Example 5.1.18. Every regular ternary semiring is a 3-P-ternary semiring.

Definition 5.1.19. Let A be an ideal of a ternary semiring S. A 3-prime ideal I containing A, is called a minimal 3-prime ideal over A if for any 3-prime ideal Q, $A \subseteq Q \subseteq I$ implies $Q = I$.

Proposition 5.1.20. *A ternary semiring S is a 3-P-ternary semiring if and only if every 3-prime ideal is semiprime.*

Proof. Follows from Theorem 5.1.3. □

Theorem 5.1.21. *A ternary semiring S is a 3-P-ternary semiring if and only if every prime ideal is idempotent and every 3-prime ideal is of the form P^3, for some prime ideal P of S.*

Proof. Let S be a 3-P-ternary semiring and P' be a prime ideal of S. By Corollary 5.1.11, P'^3 is a 3-prime ideal. Thus P'^3 is prime and so $P' \subseteq P'^3$. Also $P'^3 \subseteq P'$ and hence $P'^3 = P'$. Now, consider any 3-prime ideal P'' of S, then P'' is prime. So we have P'' is idempotent as it is needed.

Conversely, let I be a 3-prime ideal of S. Then I is of the form $I = P'^3$ for some idempotent prime ideal P', it follows that $I = P'$, as required. □

Theorem 5.1.22. *Let S be a ternary semiring with unique maximal ideal M. Then for any prime ideal P of S, P^2M is a 3-prime ideal of S. Moreover, P^2M is prime if and only if $P^2M = P$.*

Proof. Since $P \subseteq M$, the proof follows from the Lemma 5.1.10. □

Theorem 5.1.23. *Let S be a ternary semiring with unique maximal ideal M, then S is a 3-P-ternary semiring if and only if for for every 3-prime ideal I, $I^2M = Rad(I)$.*

Proof. Suppose for every 3-prime ideal I, $I^2M = Rad(I)$. Thus $I \subseteq Rad(I) = I^2M \subseteq I$. So $I = Rad(I)$. Hence I is prime. The converse part follows from the Theorem 5.1.22. □

Theorem 5.1.24. *Let S be a ternary semiring with unique maximal ideal M and P be a prime ideal of S. If $(Rad(I))^3 \subseteq I$ for any 3-prime ideal I of S, then the following are equivalent:*

(1) *for every minimal 3-prime ideal I over P^3, if P is minimal prime over I, then $I^2M = P$,*

(2) *for every minimal 3-prime ideal I over P^3 such that $I \subseteq P$, then $I = P$.*

Proof. (1) $\Longrightarrow$ (2) Let I be a minimal 3-prime ideal over P^3 and $I \subseteq P$. We claim that P is a minimal prime ideal over I. If $I \subseteq J \subseteq P$, for some prime ideal J of S. Then for any $x \in P$, $x^3 \in P^3 \subseteq I \subseteq J$. Thus $x \in J$. So $J = P$ and hence P is minimal. By (i), $I^2M = P$. Thus $P = I^2M \subseteq I \subseteq P$ and so $I = P$.

(2) $\Longrightarrow$ (1) Assume that I is a minimal 3-prime ideal over P^3 and P is a minimal prime ideal over I. Since $Rad(I)$ is a prime ideal and $P^3 \subseteq I \subseteq Rad(I)$, it follows that $P = Rad(I)$. By hypothesis, $P^3 \subseteq I \subseteq P$ and so $I = P$. Also $P^3 \subseteq P^2M \subseteq P = I$ and by Theorem 5.1.22, P^2M is 3-prime. Therefore, $P^2M = P^2I = P$, as required. □

5.2 On Quasi 3-Primary Ideals

Throughout the section, unless otherwise stated, S stands for a commutative ternary semiring with zero.

Definition 5.2.1. An ideal I of a ternary semiring S is called a quasi 3-primary ideal if for any a, b, $c \in S$, $abc \in I$ and $a^3 \notin I, b^3 \notin I$ implies there exists an integer $n \in \mathbb{Z}_0^+$ such that $c^{2n+1} \in I$.

It can be easily obtained by the definition that every 3-prime ideal is a quasi 3-primary ideal. The following example shows that the converse may not be true:

Example 5.2.2. Consider the ternary subsemiring $\mathbb{Z}_0^- \times 3\mathbb{Z}_0^-$ of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then the ideal $\{0\} \times 81\mathbb{Z}_0^-$ is strongly quasi primary, but not 3-prime in $\mathbb{Z}_0^- \times 3\mathbb{Z}_0^-$, since $(0,-81) = (-6,-9)(-5,-3)(0,-3) \in \{0\} \times 81\mathbb{Z}_0^-$ and $(-6,-9)^3 \notin \{0\} \times 81\mathbb{Z}_0^-$, $(-5,-3)^3 \notin \{0\} \times 81\mathbb{Z}_0^-$ and $(0,-3)^3 \notin \{0\} \times 81\mathbb{Z}_0^-$ but $(0,-3)^5 \in \{0\} \times 81\mathbb{Z}_0^-$.

Theorem 5.2.3. *Let S be a regular ternary semiring, then an ideal I is quasi 3-primary if and only if I is 3-prime.*

Proof. Let I be a quasi 3-primary ideal of S. Assume that $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$ for some a, b, $c \in S$. Then there exists an integer $n \in \mathbb{Z}_0^+$ such that $c^{2n+1} \in I$. Since S is a regular ternary semiring, there exists $x \in S$ such that $c = xc^{2n+1} \in I$. So $c^3 \in I$ and hence I is a 3-prime ideal of S. □

Theorem 5.2.4. *If I is a quasi 3-primary ideal of ternary semiring S, then I is a quasi primary ideal.*

Proof. Let $abc \in Rad(I)$ for some a, b, $c \in S$ and $a \notin Rad(I)$, $b \notin Rad(I)$. Then there exists an integer $n \in \mathbb{Z}_0^+$ such that $(abc)^{2n+1} = a^{2n+1}b^{2n+1}c^{2n+1} \in I$. Since I is a quasi 3-primary ideal and $a \notin Rad(I)$, $b \notin Rad(I)$, so we have $c^{(2m+1)(2n+1)} \in I$ for some integer $m \in \mathbb{Z}_0^+$. This implies $c \in Rad(I)$ and so I is a quasi primary ideal of S. □

The converse may not be true as is shown in the following example:

Example 5.2.5. Consider the ternary subsemiring $2\mathbb{Z}_0^- \times \mathbb{Z}_0^-$ of the ternary semiring $\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. Then the ideal $16\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$ is quasi primary, since $Rad(16\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-) = 2\mathbb{Z}_0^- \times 3\mathbb{Z}_0^-$ is prime on $2\mathbb{Z}_0^- \times \mathbb{Z}_0^-$. But this ideal is not quasi 3-primary, as $(-2,-27)(-4,-3)(-2,-4) = (-16,-324) \in 16\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$, where $(-2,-27)^3 \notin 16\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$, $(-4,-3)^3 \notin 16\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$ and $(-2,-4)^{2n+1} \notin 16\mathbb{Z}_0^- \times 81\mathbb{Z}_0^-$ for any $n \in \mathbb{Z}_0^+$.

Proposition 5.2.6. *In a ternary semiring S, I is a quasi 3-primary ideal if and only if $Rad(I)$ is a 3-prime ideal.*

Proof. Suppose I is a quasi 3-primary ideal of S. Then $Rad(I)$ is a prime ideal of S, thus $Rad(I)$ is a 3-prime ideal.

Conversely, assume that $Rad(I)$ is a 3-prime ideal of S. Let $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$ for some $a, b, c \in S$. Since $abc \in I \subseteq Rad(I)$, we have $c^3 \in Rad(I)$. Thus there exists an integer $n \in \mathbb{Z}_0^+$ such that $c^{2n+1} \in I$ and hence I is a quasi 3-primary ideal of S. □

In a commutative regular ternary semiring, every non-zero proper ideal is semiprime. Hence it can be easily shown that in a regular ternary semiring the concepts of prime ideal, 3-prime ideal, primary ideal, quasi 3-primary ideal and quasi primary ideal are coincide.

The following example shows that the intersection of quasi 3-primary ideals may not be a quasi 3-primary ideal.

Example 5.2.7. In the ternary semiring $\mathbb{Z}_0^-$, the intersection of quasi 3-primary ideals $3\mathbb{Z}_0^-$, $5\mathbb{Z}_0^-$ and $2\mathbb{Z}_0^-$ is $30\mathbb{Z}_0^-$, which is not a quasi 3-primary ideal.

Theorem 5.2.8. *Let S be a ternary semiring with identity and I be a proper ideal of S, then the following are equivalent:*

(1) *I is a quasi 3-primary ideal of S,*

(2) for any $a, b \in S$, if $\langle a \rangle \nsubseteq (I : a : a)$ and $\langle b \rangle \nsubseteq (I : b : b)$, then $(I : a : b) \subseteq Rad(I)$,

(3) for any three ideals J, K, L of S, $JKL \subseteq I$, $\{a^3 : a \in J\} \nsubseteq I$ and $\{b^3 : b \in K\} \nsubseteq I$ implies $K \subseteq Rad(I)$.

Proof. (1) $\implies$ (2) Suppose I is a quasi 3-primary ideal of S and $\langle a \rangle \nsubseteq (I : a : a)$, $\langle b \rangle \nsubseteq (I : b : b)$. Then $a^3 \notin I$ and $b^3 \notin I$. We have to show $(I : a : b) \subseteq Rad(I)$. Take $c \in (I : a : b)$. Then $abc \in I$. Also $a^3 \notin I$ and $b^3 \notin I$. Thus there exists an integer $n \in \mathbb{Z}_0^+$ such that $c^{2n+1} \in I$ and hence $(I : a : b) \subseteq Rad(I)$.

(2) $\implies$ (3) Consider $JKL \subseteq I$, $\{a^3 : a \in J\} \nsubseteq I$ and $\{b^3 : b \in K\} \nsubseteq I$ for some ideals J, K, L of S. Then $a \in J$ and $b \in K$ such that a^3, $b^3 \notin I$ and so $\langle a \rangle \nsubseteq (I : a : a)$ and $\langle b \rangle \nsubseteq (I : b : b)$. Then by (ii), $(I : a : b) \subseteq Rad(I)$. For any arbitrary element $c \in K$, $abc \in JKL \subseteq I$. So $c \in (I : a : b) \subseteq Rad(I)$. This yields that $K \subseteq Rad(I)$.

(3) $\implies$ (4) Assume that $abc \in I$ and $a^3 \notin I$, $b^3 \notin I$. Then $\{x^3 : x \in \langle a\rangle\} \nsubseteq I$ and $\{y^3 : y \in \langle b\rangle\} \nsubseteq I$. Since $abc \in \langle a\rangle\ \langle b\rangle\ \langle c\rangle \subseteq I$, by (iii) there exists an integer $n \in \mathbb{Z}_0^+$ such that $c^{2n+1} \in I$. So I is a quasi 3-primary ideal of S. □

Proposition 5.2.9. *Let I be a quasi 3-primary ideal of ternary semiring S with identity and $\langle a\rangle = \langle a^3\rangle$ for $a \in S$. If $a \notin (I : a : a)$, then $(I : a : a)$ is a quasi 3-primary ideal of S.*

Proof. Suppose I is a quasi 3-primary ideal of S. Here $\langle a\rangle \nsubseteq (I : a : a)$, since $a \notin (I : a : a)$. So by Theorem 5.2.8, $(I : a : a) \subseteq Rad(I)$. Thus $(I : a : a) = Rad(I)$. Consider $xyz \in (I : a : a)$ and $z^{2n+1} \notin (I : a : a)$ for some $x, y, z \in S$, $n \in \mathbb{Z}_0^+$. Then $(xa^2)yz = xyza^2 \in I$ and $z^{2n+1} \notin I$ implies $(xa^2)^3 \in I$ or $y^3 \in I$. That is $x^3 \in (I : a^3 : a^3) = (I : a : a)$ or $y^3 \in I \subseteq (I : a : a)$. Hence $(I : a : a)$ is a quasi 3-primary ideal of S. □

Proposition 5.2.10. *Suppose that I_1 and I_2 are two ideals of ternary semiring S_1 and S_2 respectively. Consider the ternary semiring $S = S_1 \times S_2$, then the following hold:*

(1) $I_1 \times S_2$ is a quasi 3-primary ideal of S if and only if I_1 is a quasi 3-primary ideal of S_1.

(2) $S_1 \times I_2$ is a quasi 3-primary ideal of S if and only if I_2 is a quasi 3-primary ideal of S_2.

Proof. (1) Suppose that $I_1 \times S_2$ is a quasi 3-primary ideal of S, $abc \in I_1$ for some $a, b, c \in S_1$ and $a^3 \notin I_1$, $b^3 \notin I_1$. Then we have $(abc, 0) = (a, 0)(b, 0)(c, 0) \in I_1 \times S_2$ and $(a, 0)^3 = (a^3, 0) \notin I_1 \times S_2$, $(b, 0)^3 = (b^3, 0) \notin I_1 \times S_2$. So we conclude that there exists an integer $n \in \mathbb{Z}_0^+$ such that $(c, 0)^{2n+1} = (c^{2n+1}, 0) \in I_1 \times S_2$. Thus there exists an integer $n \in \mathbb{Z}_0^+$ such that $c^{2n+1} \in I_1$.

Conversely, assume that I_1 is a quasi 3-primary ideal of S_1. Let $(a, x)(b, y)(c, z) \in I_1 \times S_2$ and $(a, x)^3 \notin I_1 \times S_2$, $(b, y)^3 \notin I_1 \times S_2$. This implies $abc \in I_1$ and $a^3 \notin I_1$, $b^3 \notin I_1$. So there exists an integer $n \in \mathbb{Z}_0^+$ such that $c^{2n+1} \in I_1$. Hence $(c, z)^{2n+1} = (c^{2n+1}, z^{2n+1}) \in I_1 \times S_2$. Therefore, $I_1 \times S_2$ is a quasi 3-primary ideal of S.

(2) The proof is similar to (1). □

Definition 5.2.11. Let $I, I_1, I_2, ..., I_n$ be ideals of a ternary semiring S. The collection $\{I_1, I_2, ..., I_n\}$ is said to be a cover of I if $I \subseteq I_1 \cup I_2 \cup ... \cup I_n$. We call such a cover of I efficient, if I is not contained in the union of any $n - 1$ ideals of $I_1, I_2, ..., I_n$.

Lemma 5.2.12. *Let $\{I_1, I_2, ..., I_n\}$ be an efficient covering of the ideal I of S, where I_1, I_2,..., I_n are subtractive ideals of ternary semiring S and $n > 1$. If $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for each $i \neq j$, then no I_j is a quasi 3-primary ideal of S.*

Proof. We first show that for efficient covering $\{I_1, I_2, ..., I_n\}$ of I, $(\cap_{i \neq k} I_i) \cap I = (\cap_{i=1}^{n} I_i) \cap I$ for all k. Let $x \in (\cap_{i \neq k} I_i) \cap I$. Since the cover is efficient, there exists $x_k \in I_k \cap I$ such that $x_k \notin (\cup_{i \neq k} I_i) \cap I$. Now consider the element $x + x_k$ in I. If $x + x_k \in I_i$ for $i \neq k$, then $x_k \in I_i$ for all $i \neq k$, which is a contradiction. Then $x + x_k \in I_k$ and thus $x \in I_k$. So $(\cap_{i \neq k} I_i) \cap I = (\cap_{i=1}^{n} I_i) \cap I$. If possible, let I_j be a quasi 3-primary ideal of S for some $j = 1, 2, ..., n$. Since $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for each $i \neq j$ we have $I = \cup_{i=1}^{n}(Rad(I_i) \cap I)$. Since $\{Rad(I_i) \cap I : 1 \leq i \leq n\}$ is also an efficient covering of I, there exists an element $x_i \in I \setminus Rad(I_i)$. This yields that $x_i^3 \notin I_i$ for each $i = 1, 2, ..., n$. Also $Rad(I_i) \nsubseteq Rad(I_j)$ for each $i \neq j$. Hence there exist $y_i \in Rad(I_i) \setminus Rad(I_j)$ for every $i \neq j$. Thus $y_i^{2n_i+1} \in I_i$ but $y_i^{2n_i+1} \notin I_j$ for some $n_i \in \mathbb{Z}_0^+$ and $i \neq j$. Consider $y = (y_1)^{n+1} y_2 ... y_{j-1} y_{j+1} ... y_n$. Since $Rad(I_i)$ is prime, we have $y \notin Rad(I_j)$. Assume that $k = max\{2n_1 + 1, 2n_2 + 1, .., 2n_{j-1} + 1, 2n_{j+1} + 1, .., 2n_n + 1\}$, then $y^k \in I_i$ for every $i \neq j$ but $y^k \notin I_j$. Now $y^k x_j x_j \in I \cap I_i$ for every $i \neq j$ but $y^k x_j x_j \notin I \cap I_j$. Since $y^k x_j x_j \in I_j$ and $x_j^3 \notin I_j$, there exists an integer $n \in \mathbb{Z}_0^+$ such that $(y^k)^{(2n+1)} \in I_j$, that is, $y \in Rad(I_j)$, a contradiction. Thus $y^k x_j x_j \in I \cap (\cap_{i \neq j}^{n} I_i)$ but $y^k x_j x_j \notin I \cap I_j$, which also contradicts $(\cap_{i \neq k} I_i) \cap I = (\cap_{i=1}^{n} I_i) \cap I$. Therefore, I_j is not a quasi 3-primary ideal of S. □

By using Lemma 5.2.12, we obtain the following Theorem.

Theorem 5.2.13. *Let I be an arbitrary ideal in a commutative ternary semiring S and I_1, I_2,....,I_n be subtractive ideals of S such that at least $n - 2$ of which are quasi 3-primary ideals. If $\{I_1, I_2, ..., I_n\}$ be a cover of I and $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for each $i \neq j$, then $I \subseteq I_i$ for some i.*

Proof. We may assume that the cover is efficient since the hypothesis remains valid if one reduces the covering to an efficient covering. Then $n \neq 2$. Since $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for each $i \neq j$, by Lemma 5.2.12, we have $n < 2$. Therefore $n = 1$ and hence $I \subseteq I_i$ for some i. □

Theorem 5.2.14. *Let S be a commutative ternary semiring and I_1, I_2,....,I_n be quasi 3-primary subtractive ideals of S such that $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for all $i \neq j$. Let I be an ideal of S such that $aSS + I \nsubseteq \cup_{i=1}^{n} I_i$ for some $a \in S$. Then there exists an element $c \in I$ such that $a + c \notin \cup_{i=1}^{n} I_i$.*

Proof. Assume that a lies in all of I_1, I_2,....,I_k but none of I_{k+1},....,I_n. If $k = 0$, then $a + 0 \notin \cup_{i=1}^{n} I_i$. So consider $k \geq 1$. Now $I \nsubseteq \cup_{i=1}^{k} Rad(I_i)$. If $I \subseteq \cup_{i=1}^{k} Rad(I_i)$, by Theorem 5.2.13, $I \subseteq Rad(I_i)$ for some $1 \leq i \leq k$, which contradicts the hypothesis that $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for all $i \neq j$. So there exists an element $p \in I$ such that $p \notin \cup_{i=1}^{k} Rad(I_i)$. Also, $I_{k+1} \cap .... \cap I_n \nsubseteq Rad(I_1) \cup Rad(I_2) \cup ... \cup Rad(I_k)$. If $I_{k+1} \cap .... \cap I_n \subseteq Rad(I_1) \cup Rad(I_2) \cup ... \cup Rad(I_k)$, then by Theorem 5.2.13, we get $I_{k+1} \cap .... \cap I_n \subseteq Rad(I_j)$ for some $1 \leqslant j \leqslant k$. Thus $(Rad(I_{k+1}))^{n-k} \cap ... \cap Rad(I_n) = Rad((I_{k+1})^{n-k} \cap .... \cap I_n) \subseteq Rad(I_{k+1} \cap .... \cap I_n) \subseteq Rad(I_j)$ and since $Rad(I_j)$ is a prime ideal of S, we conclude that $Rad(I_l) \subseteq Rad(I_j)$ for $k + 1 \leqslant l \leqslant n$, so $I \cap Rad(I_i) \nsubseteq I \cap Rad(I_j)$ for $i \neq j$, which contradicts the hypothesis. Thus there exists $q \in I_{k+1} \cap .... \cap I_n$ such that $q \notin Rad(I_1) \cup Rad(I_2) \cup ... \cup Rad(I_k)$. Consider the element $c = ppq \in I$. Then $c \in I_{k+1} \cap .... \cap I_n$ but $c \notin I_1 \cup I_2 \cup ... \cup I_k$. If $c \in I_1 \cup I_2 \cup ... \cup I_k$, then $c = ppq \in I_i$ for some $1 \leqslant i \leqslant k$. Also $p^3 \notin I_i$. Since I_i is a quasi 3-primary ideal, there exists an integer $n \in \mathbb{Z}_0^+$ such that $q^{2n+1} \in I_i$, a contradiction. Hence $c \in \cup_{j=k+1}^{n} I_j \setminus \cup_{i=1}^{k} I_i$. Again, as $a \in \cup_{i=1}^{k} I_i \setminus \cup_{j=k+1}^{n} I_j$, it follows that $a + c \notin \cup_{i=1}^{n} I_i$. □

CHAPTER 6

Fundamental Relation on Ternary Hypersemirings

6

Fundamental Relation on Ternary Hypersemirings

In this chapter, an equivalence relation δ^* on a ternary hypersemiring S has been introduced. It has been observed that δ^* is the smallest strongly regular relation on S so that the quotient structure is a ternary semiring. The notion of a fundamental ternary semiring with respect to the fundamental relation δ^* on a ternary hypersemiring S has been introduced. Every ternary semiring with unital element is isomorphic to a fundamental ternary semiring has been established. After that, the fundamental relation δ^* in terms of a typical kind of subsets, called ternary strong $\mathcal{C}$-set of ternary hypersemiring S has been studied.

This chapter has been organized as follows:

In *Section 1*, we introduce the smallest strongly regular relation δ^* on ternary hypersemiring S (*cf.* Theorems 6.1.1, 6.1.2), so that quotient S/δ^* is a ternary semiring (*cf.* Corollary 6.1.8) and we define it as fundamental relation on ternary hypersemiring S (*cf.* Definition 6.1.4). Then we introduce the notion of fundamental ternary semiring, that is a ternary semiring which is isomorphic to the ternary semiring of a nontrivial ternary hypersemiring (*cf.* Definition 6.1.7). Then we observe that every ternary semiring with unital element is a fundamental ternary semiring (*cf.* Theorem 6.1.10).

This chapter is based on the work published in the following paper:

- **Sampad Das et al., Fundamental Relation on Ternary Hypersemirings *(Communicated)*.**

In *Section 2*, we introduce the notion of ternary strong $\mathcal{C}$-set (*cf.* Definition 6.2.5). we establish the fundamental relation δ^* in terms of a typical kind of subsets, called ternary strong $\mathcal{C}$-set of ternary hypersemiring S (*cf.* Theorem 6.2.12).

6.1 On Fundamental Relation and Fundamental Ternary Semiring

Let $(S, +, \circ)$ be a ternary hypersemiring. We define a binary relation δ on S as follows: $x\delta y$ if and only if there exist $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2m_i + 1$ such that

$$x = y \text{ or } \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

Clearly, δ is reflexive and symmetric. Consider δ^* be the transitive closure of the relation δ. Then, we have $x\delta^* y$ if and only if there exist elements $a_0, a_1, ..., a_m \in S$ with $x = a_0$ and $y = a_m$ such that $a_i \delta a_{i+1}$ for $i \in 0, 1, ..., m-1$.

Theorem 6.1.1. *The relation δ^* is a strongly regular relation on S.*

Proof. Let $a, b \in S$ such that $a\delta^* b$ holds. If $a = b$, then it is obvious that $(a+c)\, \delta^*\, (b+c)$ and $(a \circ u \circ v)\, \bar{\bar{\delta}}^*\, (b \circ u \circ v)$ for any $c, u, v \in S$. Analogously, $(u \circ a \circ v)\, \bar{\bar{\delta}}^*\, (u \circ b \circ v)$ and $(u \circ v \circ a)\, \bar{\bar{\delta}}^*\, (u \circ v \circ b)$ for any $u, v \in S$. If $a \neq b$, then there exist $n \in \mathbb{N}, m_i \in \mathbb{Z}_0^+, i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2m_i + 1$ such that

$$\{a, b\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

Then, for any $c \in S$, $a + c \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} a_{ij} + c$ and $b + c \in \sum_{i=1}^{n} \prod_{j=1}^{k_i} a_{ij} + c$. Now we set $m_{n+1} = 0$ and $a_{n+1,1} = c$, thus

$$\{a + c, b + c\} \subseteq \sum_{i=1}^{n+1} \prod_{j=1}^{2m_i+1} a_{ij}.$$

So, $(a + c)\, \bar{\bar{\delta}}^*\, (b + c)$ for any $c \in S$.
Again for arbitrary $u, v \in S$, $a \circ u \circ v \subseteq (\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}) \circ u \circ v \subseteq \sum_{i=1}^{n} (\prod_{j=1}^{2m_i+1} a_{ij}) \circ u \circ v$, also $b \circ u \circ v \subseteq \sum_{i=1}^{n} (\prod_{j=1}^{2m_i+1} a_{ij}) \circ u \circ v$. Here, we set $m_i' = m_i + 1$, $a_{i,2m_i+2} = u$, $a_{i,2m_i+3} = v$ for $i = 1, 2, ..., n$. Then

$$\{a \circ u \circ v, b \circ u \circ v\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i'+1} a_{ij}$$

So for any $s \in a \circ u \circ v$ and $t \in b \circ u \circ v$, we have $s\delta t$. Thus $(a \circ u \circ v) \,\bar{\bar{\delta^*}}\, (b \circ u \circ v)$. Similarly, $(u \circ a \circ v) \,\bar{\bar{\delta^*}}\, (u \circ b \circ v)$ and $(u \circ v \circ a) \,\bar{\bar{\delta^*}}\, (u \circ v \circ b)$. Hence δ^* is a strongly regular relation on S.

□

Theorem 6.1.2. *The relation δ^* is the smallest strongly regular relation on ternary hypersemiring S.*

Proof. If possible, let θ be the smallest strongly regular relation on S. We prove that $\theta = \delta^*$. It is clear that, $\theta \subseteq \delta^*$. If $x\delta y$, then there exist $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2m_i + 1$ such that

$$x = y \text{ or } \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

Since θ is a strongly regular relation and $a\theta a$ for $a \in S$, we have $(a \circ b \circ c) \,\bar{\bar{\theta}}\, (a \circ b \circ c)$ for any $b, c \in S$. Also for any $z \in (a \circ b \circ c)$, we have $(z \circ d \circ e) \,\bar{\bar{\theta}}\, (z \circ d \circ e)$ for all $d, e \in S$. That implies $(a \circ b \circ c) \circ d \circ e \,\bar{\bar{\theta}}\, (a \circ b \circ c) \circ d \circ e$. Continuing this process, we get any finite product of elements of S is θ related to itself. Thus, we obtain $(\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}) \,\bar{\bar{\theta}}\, (\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij})$. Hence $x \,\theta\, y$, that implies $\delta \subseteq \theta$. If $x \,\delta^*\, y$, then there exist $x = a_0, a_1, a_2, ..., a_m = y$ such that $a_i \,\delta\, a_{i+1}$ for $i = 0, 1, .., m-1$. Since $\delta \subseteq \theta$ and θ is transitive, we have $\delta^* \subseteq \theta$. Therefore, δ^* is the smallest strongly regular relation on ternary hypersemiring S. □

Corollary 6.1.3. *By Theorem-4.13 [40], the quotient $(S/\delta^*, +, \circ)$ is a ternary semiring.*

Definition 6.1.4. The smallest strongly regular equivalence relation δ^* on ternary hypersemiring S is called the fundamental relation on ternary hypersemiring S.

Proposition 6.1.5. *Let f be a homomorphism from a ternary hypersemiring $(S, +, \circ)$ to a ternary hypersemiring $(T, +', \circ')$. Then, the following are true:*

(1) $x \,\delta^*\, y$ implies $f(x) \,\delta^*\, f(y)$

(2) if f is an one-one homomorphism, then $a\delta b$ implies $x\delta y$ where $f(x) = a$ and $f(y) = b$ for some $x, y \in S$ and $a, b \in T$.

(3) if f is an isomorphism, then $x \,\delta\, y$ if and only if $f(x) \,\delta\, f(y)$

Proof. (1) Let $x\,\delta^*\,y$. Then there exists $a_1, a_2, .., a_n \in S$ with $a_1 = x$, $a_m = y$ such that $a_k\,\delta\,a_{k+1}$ for $1 \leq k \leq m-1$. Thus for $a_k\,\delta\,a_{k+1}$, there exist $n_k \in \mathbb{N}$, $m_{ik} \in \mathbb{Z}_0^+$ for $i = 1, 2, ..., n_k$ and $a_{ijk} \in S$, $1 \leq i \leq n_k$, $1 \leq j \leq 2m_ik + 1$ such that

$$\{a_k, a_{k+1}\} \subseteq \sum_{i=1}^{n_k} \prod_{j=1}^{2m_{ik}+1} a_{ijk}.$$

Since f is a homomorphism, we have $\{f(a_k), f(a_{k+1})\} = f(\{a_k, a_{k+1}\})$ $\subseteq f(\sum_{i=1}^{n_k} \prod_{j=1}^{2m_{ik}+1} a_{ijk}) = \sum_{i=1}^{n_k} f(\prod_{j=1}^{2m_{ik}+1} a_{ijk}) \subseteq \sum_{i=1}^{n_k} \prod_{j=1}^{2m_{ik}+1} f(a_{ijk})$ for all $k = 1, 2, .., m-1$. Therefore, $f(x)\,\delta^*\,f(y)$.

(2) Let $a = f(x)\,\delta f(y) = b$. Then there exist $n \in \mathbb{N}$, $2m_i + 1 \in \mathbb{N}$ where $m_i \in \mathbb{Z}_0^+$ for $i = 1, 2, ..., n$ and $c_{ij} \in T$ such that $\{f(x), f(y)\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} c_{ij}$. Since f is an one-one homomorphism, we have $\{x, y\} = f^{-1}\{f(x), f(y)\} \subseteq f^{-1}(\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} c_{ij}) = \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} f^{-1}(c_{ij})$. So, $x\,\delta\,y$.

(3) By (1) and (2), it is clear.

□

Theorem 6.1.6. *Let $f : S \longrightarrow T$ be an isomomorphism from a ternary hypersemiring $(S, +, \circ)$ to a ternary hypersemiring $(T, +', \circ')$. Then the ternary semirings $(S/\delta^*, +, \cdot)$ and $(T/\delta^*, +', \cdot')$ are isomorphic.*

Proof. We define a mapping $h : S/\delta^* \longrightarrow T/\delta^*$ by $h(\delta^*(a)) = \delta^*(f(a))$ for all $a \in S$. Now $\delta^*(a) = \delta^*(b) \iff a\delta^*b \iff f(a)\,\delta^*\,f(b)$ (by Proposition-6.1.5) $\iff \delta^*(f(a)) = \delta^*(f(b)) \iff h(\delta^*(a)) = h(\delta^*(b))$. Thus h is well-defined and one-one. Also, h is an onto mapping. Now, we have the following equality $h(\delta^*(a) + \delta^*(b)) = h(\delta^*(a + b)) = \delta^*(f(a + b)) = \delta^*(f(a) +' f(b)) = \delta^*(f(a)) +' \delta^*(f(b)) = h(\delta^*(a)) +' h(\delta^*(b))$ and $h(\delta^*(a) \cdot \delta^*(b) \cdot \delta^*(c)) = h(\delta^*(x))$ (for some $x \in a \circ b \circ c$) $= \delta^*(f(x))$ (where $f(x) \in f(a) \circ' f(b) \circ' f(c)$)$=\delta^*(f(a)) \cdot' \delta^*(f(b)) \cdot' \delta^*(f(c)) = h(\delta^*(a)) \cdot' h(\delta^*(b)) \cdot' h(\delta^*(c))$. Hence h is a ternary homomorphism and so S/δ^* and T/δ^* are isomorphic. □

Definition 6.1.7. A ternary semiring $(S, +, \cdot)$ is called a fundamental ternary semiring if there exists a non-trivial ternary hypersemiring, say $(T, +, \circ)$, such that $(T/\delta^*, +', \cdot')$ is isomorphic to $(S, +, \cdot)$.

Theorem 6.1.8. *Let $(T, +, \cdot)$ be a ternary semiring. Then, for any semiring $(S, +', \cdot')$ there exist a binary operation $\oplus$ and a ternary hyperopration $\odot$ such that $(T \times S, \oplus, \odot)$ is a non-trivial ternary hypersemiring.*

Proof. Let $(S, +', \cdot')$ be a semiring. We define a binary operation $\oplus$ and a ternary hyperopration $\odot$ on $T \times S$ by $(r_1, s_1) \oplus (r_2, s_2) = (r_1 + r_2, s_1 +' s_2)$ and $(r_1, s_1) \odot (r_2, s_2) \odot (r_3, s_3) = \{(r_1 \cdot r_2 \cdot r_3, s_1), (r_1 \cdot r_2 \cdot r_3, s_2), (r_1 \cdot r_2 \cdot r_3, s_3), (r_1 \cdot r_2 \cdot r_3, 0)\}$ for (r_1, s_1), (r_2, s_2), $(r_3, s_3) \in T \times S$. Then it is easy to show that $(T \times S, \oplus)$ is a commutative semigroup and '$\odot$' is associative. Now, $((r_1, s_1) \oplus (r_2, s_2)) \odot (r_3, s_3) \odot (r_4, s_4) = \{((r_1 + r_2) \cdot r_3 \cdot r_4, s_1 +' s_2), ((r_1 + r_2) \cdot r_3 \cdot r_4, s_3), ((r_1 + r_2) \cdot r_3 \cdot r_4, s_4), ((r_1 + r_2) \cdot r_3 \cdot r_4, 0)\} \subseteq ((r_1, s_1) \odot (r_3, s_3) \odot (r_4, s_4)) \oplus ((r_2, s_2) \odot (r_3, s_3) \odot (r_4, s_4))$ for (r_1, s_1), (r_2, s_2), (r_3, s_3), (r_4, s_4), $(r_5, s_5) \in T \times S$. In a similar way, the other inclusions hold for ternary hypersemiring. Therefore, $(T \times S, \oplus, \odot)$ is a ternary hypersemiring. □

The ternary hypersemiring $(T \times S, \oplus, \odot)$ is called associated ternary hypersemiring of T via S.

Theorem 6.1.9. *Let $(T, +, \cdot)$ and $(\tilde{T}, \tilde{+}, \tilde{\cdot})$ be two isomorphic ternary semirings. Then, for any semiring $(S, +', \cdot')$, the associated ternary hypersemirings $(T \times S, \oplus, \odot)$ of T via S and $(\tilde{T} \times S, \tilde{\oplus}, \tilde{\odot})$ of $\tilde{T}$ via S are isomorphic.*

Proof. Let $(S, +', \cdot')$ be a semiring and $f : T \longrightarrow \tilde{T}$ be an isomorphism. We define a mapping $h : T \times S \longrightarrow \tilde{T} \times S$ by $h((t, s)) = (f(t), s)$ for any $(t, s) \in T \times S$. Since f is an isomorphism, it is easy to check h is well-defined and a bijection mapping. Now, $h((t_1, s_1) \oplus (t_2, s_2)) = h((t_1 + t_2, s_1 +' s_2)) = (f(t_1 + t_2), s_1 +' s_2) = (f(t_1) \tilde{+} f(t_2), s_1 +' s_2) = (f(t_1), s_1) \tilde{\oplus} (f(t_2), s_2) = h((t_1, s_1)) \tilde{\oplus} h((t_2, s_2))$.
Also, $h((t_1, s_1) \odot (t_2, s_2) \odot (t_3, s_3))$
$= h\left(\left\{(t_1 \cdot t_2 \cdot t_3,\ s_1), (t_1 \cdot t_2 \cdot t_3,\ s_2), (t_1 \cdot t_2 \cdot t_3,\ s_3), (t_1 \cdot t_2 \cdot t_3,\ 0)\right\}\right)$
$= \left\{h((t_1 \cdot t_2 \cdot t_3,\ s_1)), h((t_1 \cdot t_2 \cdot t_3,\ s_2)), h((t_1 \cdot t_2 \cdot t_3,\ s_3)), h((t_1 \cdot t_2 \cdot t_3,\ 0))\right\}$
$= \left\{(f(t_1 \cdot t_2 \cdot t_3),\ s_1), (f(t_1 \cdot t_2 \cdot t_3),\ s_2), (f(t_1 \cdot t_2 \cdot t_3),\ s_3), (f(t_1 \cdot t_2 \cdot t_3),\ 0)\right\}$
$= \left\{(f(t_1) \tilde{\cdot} f(t_2) \tilde{\cdot} f(t_3),\ s_1), (f(t_1) \tilde{\cdot} f(t_2) \tilde{\cdot} f(t_3),\ s_2), (f(t_1) \tilde{\cdot} f(t_2) \tilde{\cdot} f(t_3),\ s_3),\right.$
$\left.(f(t_1) \tilde{\cdot} f(t_2) \tilde{\cdot} f(t_3),\ 0)\right\}$
$= (f(t_1),\ s_1) \tilde{\odot} (f(t_2),\ s_2) \tilde{\odot} (f(t_3),\ s_3) = h((t_1,\ s_1)) \tilde{\odot} h((t_2,\ s_2)) \tilde{\odot} h((t_3,\ s_3))$. Therefore, h is a good homomorphism and so the ternary hypersemirings $(T \times S, \oplus, \odot)$ and $(\tilde{T} \times S, \tilde{\oplus}, \tilde{\odot})$ are isomorphic. □

Theorem 6.1.10. *Every ternary semiring with a unital element is a fundamental ternary semiring.*

Proof. Let $(T, +, \cdot)$ be a ternary semiring with unital element 'e' and $(S, +', \cdot')$ be any semiring. Then, by Theorem 6.1.8, $(T \times S, \oplus, \odot)$ is a non-trivial ternary hypersemiring.

Consequently, by Corollary 6.1.3, $(T \times S)/\delta^*$ is a ternary semiring, where δ^* is the fundamental relation on $T \times S$. Now we show that $\delta^*((t,s)) = \Big\{(t,s') : s' \in S\Big\}$ for $(t,s) \in T \times S$. Since $\Big\{(t,s),(t,s')\Big\} \subseteq (t,s) \odot (e,s') \odot (e,s)$ for any $(t,s') \in T \times S$. Thus we get $(t,s') \in \delta^*((t,s))$. On the other hand, if $(t'',s'')\, \delta^*\, (t,s)$ this implies that $t'' = t$. Therefore, $\delta^*((t,s)) = \Big\{(t,s') : s' \in S\Big\}$. Now we consider a map $f : (T \times S)/\delta^* \longrightarrow T$ by $f(\delta^*((t,s))) = t$ for all $\delta^*((t,s)) \in (T \times S)/\delta^*$. It is easy to check that the map is well-defined and one-one. Also, for any $t \in T$, we have $f(\delta^*((t,0))) = t$. Thus f is an onto map. Now $f(\delta^*(t_1,s_1) + \delta^*(t_2,s_2)) = f(\delta^*((t_1+t_2, s_1+s_2)) = t_1 + t_2 = f(\delta^*(t_1,s_1)) + f(\delta^*((t_2,s_2)))$ and $f(\delta^*((t_1,s_1)) \cdot \delta^*((t_2,s_2)) \cdot \delta^*((t_3,s_3))) = f(\delta^*((t_1t_2t_3, s_1))) = t_1t_2t_3 = f(\delta^*((t_1,s_1)))f(\delta^*((t_1,s_1)))f(\delta^*((t_1,s_1)))$. Thus, f is a homomorphism and so $(T \times S)/\delta^*$ and T are isomorphic. Therefore, $(T,+,\cdot)$ is a fundamental ternary semiring. □

Theorem 6.1.11. *Let $(T,+,\circ)$ be a ternary hypersemiring. Then, there exist a ternary semiring S, binary operation '$\oplus$' and ternary hyperoperation '$\odot$' such that $(T,+,\circ)$ can be embedded in $(T \times S, \oplus, \odot)$.*

Proof. Let $(T,+,\circ)$ be a ternary hypersemiring. By Corollary-6.1.3, $(T/\delta^*, +', \circ')$ is a ternary semiring. We set $S = (T/\delta^*, +', \circ')$. Now, on the set $T \times S$, we define the binary operation $\oplus$ and ternary hyperoperation $\odot$ by $(t_1, \delta^*(s_1)) \oplus (t_2, \delta^*(s_2)) = (t_1+t_2, \delta^*(s_1 + s_2))$ and $(t_1,\delta^*(s_1)) \odot (t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3)) = (t_1 \circ t_2 \circ t_3, \delta^*(s_1 \circ s_2 \circ s_3))$ respectively. Let $(t_1,\delta^*(s_1)) = (t_1', \delta^*(s_1'))$, $(t_2,\delta^*(s_2)) = (t_2',\delta^*(s_2'))$, $(t_3,\delta^*(s_3)) = (t_3',\delta^*(s_3'))$. So $t_1 = t_1'$, $\delta^*(s_1) = \delta^*(s_1')$, $t_2 = t_2'$, $\delta^*(s_2) = \delta^*(s_2')$ and $t_3 = t_3'$, $\delta^*(s_3) = \delta^*(s_3')$. Then $t_1 \circ t_2 = t_1' \circ t_2'$ and $\delta^*(s_1 + s_2) = \delta^*(s_1' + s_2')$ (since δ^* is a congruence on $(T,+)$). Also, $t_1 \circ t_2 \circ t_3 = t_1' \circ t_2' \circ t_3'$ and $\delta^*(s_1 \circ s_2 \circ s_3) = \delta^*(s_1' \circ s_2' \circ s_3')$, since δ^* is a strongly regular relation. Thus the operations $\oplus$ and $\odot$ are well-defined. Next, we will show $(T \times S, \oplus, \odot)$ is a ternary hypersemiring. Clearly, $(T \times S, \oplus)$ is a commutative semigroup. For associativity of '$\odot$', $((t_1,\delta^*(s_1)) \odot (t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3))) \odot (t_4,\delta^*(s_4)) \odot (t_5,\delta^*(s_5)) = ((t_1 \circ t_2 \circ t_3) \circ t_4 \circ t_5, \delta^*((s_1 \circ s_2 \circ s_3) \circ s_4 \circ s_5)) = (t_1 \circ (t_2 \circ t_3 \circ t_4) \circ t_5, \delta^*(s_1 \circ (s_2 \circ s_3 \circ s_4) \circ s_5)) = (t_1,\delta^*(s_1)) \odot ((t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3))) \odot (t_4,\delta^*(s_4))) \odot (t_5,\delta^*(s_5))$. In a similar way, $(t_1,\delta^*(s_1)) \odot ((t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4))) \odot (t_5,\delta^*(s_5)) = (t_1,\delta^*(s_1)) \odot (t_2,\delta^*(s_2)) \odot ((t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4)) \odot (t_5,\delta^*(s_5)))$ for any $(t_i,\delta^*(s_i)) \in T \times S$, $i = 1,2,..,5$. Hence,'$\odot$' is associative. Here, $((t_1,\delta^*(s_1)) \oplus (t_2,\delta^*(s_2))) \odot (t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4)) = ((t_1+t_2) \circ t_3 \circ t_4,\ \delta^*((s_1+s_2) \circ s_3 \circ s_4)) \subseteq ((t_1 \circ t_3 \circ t_4) + (t_2 \circ t_3 \circ t_4), \delta^*((s_1 \circ s_3 \circ s_4) + (s_2 \circ s_3 \circ s_4)) = ((t_1,\delta^*(s_1)) \odot (t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4))) \oplus ((t_2,\delta^*(s_2)) \odot (t_3,\delta^*(s_3)) \odot (t_4,\delta^*(s_4)))$ for any $(t_i,\delta^*(s_i)) \in T \times S$, $i =$

$1, 2, 3, 4$. Silimarly, other inclusions of ternary hypersemiring also hold. So, $(T \times S, \oplus, \odot)$ is a ternary hypersemirng. Now, consider a mapping $\phi : T \longrightarrow T \times S$ defined by $\phi(t) = (t, \delta^*(t))$ for all $t \in T$. Then, $t_1 = t_2 \iff (t_1, \delta^*(t_1)) = (t_2, \delta^*(t_2)) \iff \phi(t_1) = \phi(t_2)$ for $t_1, t_2 \in T$. So ϕ is well-defined and one-one. Let $t_1, t_2 \in T$. Then $\phi(t_1 + t_2) = (t_1 + t_2, \delta^*(t_1 + t_2)) = (t_1, \delta^*(t_1)) \oplus (t_2, \delta^*(t_2)) = \phi(t_1) \oplus \phi(t_2)$. Further, we have $\phi(t_1 \circ t_2 \circ t_3) = (t_1 \circ t_2 \circ t_3, \delta^*(t_1 \circ t_2 \circ t_3)) = (t_1, \delta^*(t_1)) \odot (t_2, \delta^*(t_2)) \odot (t_3, \delta^*(t_3)) = \phi(t_1) \odot \phi(t_2) \odot \phi(t_3)$. This shows that ϕ is a one-one good homomorphism. Therefore, $(T, +, \cdot)$ is embedded in $(T \times S, \oplus, \odot)$. □

6.2 Ternary Strong $\mathcal{C}$-set

In this section, we will first define the concept of ternary $\mathcal{C}$- set and ternary strong $\mathcal{C}$-set, then we will describe the fundamental relation δ^* in terms of strong $\mathcal{C}$-set of ternary hypersemiring S.

Definition 6.2.1. Let $\mathcal{C} = \{\Pi_{i=1}^{2n+1} a_i : a_i \in S, n \in \mathbb{Z}_0^+\}$ be the class of all finite ternary products of elements of a ternary hypersemiring $(S, +, \circ)$. A non-empty subset I is called complete ternary part or ternary $\mathcal{C}$- set if for any $A \in \mathcal{C}$, $I \cap A \neq \phi$ implies $A \subseteq I$.

Definition 6.2.2. Let $(S, +, \circ)$ be a ternary hypersemiring and $\Omega = \{\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} : n \in \mathbb{N}, m_i \in \mathbb{Z}_0^+, a_{ij} \in S \textit{ for } 1 \leq i \leq n, 1 \leq j \leq 2m_i + 1\}$. A non-empty subset I of S is called a ternary strong $\mathcal{C}$-set if for any $A \in \Omega$, $I \cap A \neq \phi$ implies $A \subseteq I$.

Proposition 6.2.3. *For any strongly regular equivalence relation $\wp$ on a ternary hypersemiring S, the equivalence class $\wp(a)$, $a \in S$ is a ternary strong $\mathcal{C}$-set.*

Proof. Let $A \in \Omega = \{\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} : n \in \mathbb{N}, m_i \in \mathbb{Z}_0^+, a_{ij} \in S, \textit{ for } 1 \leq i \leq n, 1 \leq j \leq 2m_i + 1\}$ be such that $A \cap \wp(a) \neq \phi$. Then there exists $x \in S$ such that $x \in A$ and $x \in \wp(a)$. Thus for any $y \in A$, there exist $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $a_{ij} \in S$ for $1 \leq i \leq n$ and $1 \leq j \leq 2m_i + 1$ such that

$$\{x, y\} \subseteq A \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$$

This implies $x\delta^*y$. Since δ^* is the smallest strongly regular equivalence relation and $\wp$ a strongly regular equivalence relation, we have $\delta^* \subseteq \wp$. Thus $x\wp y$, and so $y \in \wp(a)$. Hence $A \subseteq \wp(a)$. Therefore, $\wp(a)$ is a ternary strong $\mathcal{C}$-set of S. □

Theorem 6.2.4. *Let A be a non-empty subset of a ternary hypersemiring S. Then the following are equivalent:*

(1) A is a ternary strong $\mathcal{C}$-set of S,

(2) for $x \in A$, $x\,\delta\,y$ implies $y \in A$,

(3) for $x \in A$, $x\,\delta^*\,y$ implies $y \in A$.

Proof. (1) $\Rightarrow$ (2) Suppose that A is a ternary strong $\mathcal{C}$-set of S and $x \in A$, $x\,\delta\,y$ for some $x, y \in S$. Then there exist $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2m_i + 1$ such that

$$x = y \text{ or } \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}.$$

If $x = y$, then it is clear. Otherwise, $x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap A$. Since A is a ternary strong $\mathcal{C}$-set, we have $\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \subseteq A$ which implies $y \in A$.

(2) $\Rightarrow$ (3) Let $x \in A$ and $x\delta^* y$ for some $x, y \in S$. So there exist $n \in \mathbb{N}$ and $x = y_0, y_1, y_2, ..., y_n = y \in S$ such that $y_i\,\delta\,y_{i+1}$ for all $i \in \{0, 1, ..., n-1\}$. Since $x = y_0 \in A$ we have $y \in A$, by applying (2) n-times.

(3) $\Rightarrow$ (1) Let $(\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}) \cap A \neq \phi$ for some $n \in \mathbb{N}$, $m_i \in \mathbb{Z}_0^+$, $a_{ij} \in S$ for $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Suppose $x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap A$. Now for any $y \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$, we have $x\delta^* y$. So by (3), $y \in A$. Hence $\sum_{i=1}^{n} \prod_{j=1}^{k_i} a_{ij} \subseteq A$. Therefore, A is a ternary strong $\mathcal{C}$-set of S. □

Intersection of any arbitrary collection of ternary strong $\mathcal{C}$-set of a ternary hypersemiring S is again a ternary strong $\mathcal{C}$-set of S.

Definition 6.2.5. Let A be a non-empty subset of a ternary hypersemiring S. The intersection of all the ternary strong $\mathcal{C}$-sets of S containing A is called the ternary strong $\mathcal{C}$-closure of A and denote it by $\mathcal{C}^*(A)$.

Since, S is itself a ternary strong $\mathcal{C}$-set in the ternary hypersemiring S, so $\mathcal{C}^*(A)$ exists for any subset A of S.

Now, for a non-empty subset A of a ternary hypersemiring S, we set

$$K_1(A) = A$$

$$K_{m+1}(A) = \left\{ x \in S : \exists\, a_{ij} \in S; x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \textit{ and } \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap K_m(A) \neq \phi \right\}$$

and $K(A) = \bigcup_{m \geq 1} K_m(A)$

Lemma 6.2.6. *For any non-empty subset A of S, the set $K(A)$ is a ternary strong $\mathcal{C}$-set containing A.*

Proof. Let $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K(A) \neq \phi$ for some $n \in \mathbb{N}$, $k_i \in \mathbb{Z}_0^+$, $a_{ij} \in S$ where $i = 1, 2, ..., n$ and $1 \leq j \leq 2k_i+1$. Then, there exists $m \in \mathbb{N}$ such that $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_m(A) \neq \phi$, which implies that $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \subseteq K_{m+1}(A) \subseteq K(A)$. Also $K_1(A) = A \subseteq K(A)$ and so the set $K(A)$ is a ternary strong $\mathcal{C}$-set containing A. □

Corollary 6.2.7. *$A \subseteq B$ implies $K(A) \subseteq K(B)$, for any two non-empty subsets $A, B \in S$.*

Lemma 6.2.8. *For every $m(\geq 2) \in \mathbb{N}$ and $x \in S$, we have $K_m(K_2(\{x\})) = K_{m+1}(\{x\})$.*

Proof. $K_2(K_2(\{x\})) = \Big\{x \in S : \exists\, a_{ij} \in S; x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$ *and* $\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap K_2(\{x\})) \neq \phi\Big\} = K_3(\{x\})$. So, our assertion is true for $m = 2$. Let the assertion be true for some $l(> 2) \in \mathbb{N}$, i.e., $K_l(K_2(\{x\})) = K_{l+1}(\{x\})$. Let $K_{l+1}(K_2(x)) = \Big\{x \in S : \exists\, a_{ij} \in S; x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$ *and* $\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap K_l(K_2(\{x\})) \neq \phi\Big\} = \Big\{x \in S : \exists\, a_{ij} \in S; x \in \sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij}$ *and* $\sum_{i=1}^{n} \prod_{j=1}^{2m_i+1} a_{ij} \cap K_{l+1}(\{x\}) \neq \phi\Big\} = K_{l+2}(\{x\})$. Hence, by induction $K_m(K_2(\{x\})) = K_{m+1}(\{x\})$ for any $m(\geq 2) \in \mathbb{N}$. □

Theorem 6.2.9. *For any non-empty subset A of a ternary hypersemiring S, $\mathcal{C}^*(A) = K(A)$.*

Proof. By Lemma 6.2.6, we have $\mathcal{C}^*(A) \subseteq K(A)$. Let A' be a ternary strong $\mathcal{C}$-set containing A. Clearly, $K_1(A) \subseteq A'$. Suppose $K_m(A) \subseteq A'$ for some $m \in \mathbb{N}$. Let $x \in K_{m+1}(A)$. Then there exist elements $a_{ij} \in S$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}$ and $\phi \neq (\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}) \cap K_m(A) \subseteq (\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}) \cap A'$. Since A' is a ternary strong $\mathcal{C}$-set, we have $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \subseteq A'$. So $K_{m+1}(A) \subseteq A'$ and thus by induction $K_m \subseteq A'$ for all $m \in \mathbb{N}$. So $K(A) \subseteq A'$. Therefore, $K(A) \subseteq \mathcal{C}^*(A)$, this implies that $\mathcal{C}^*(A) = K(A)$. □

Theorem 6.2.10. *If B is a non-empty subset of a ternary hypersemiring S, then*

$$\bigcup_{b\in B} \mathcal{C}^*(\{b\}) = \mathcal{C}^*(B)$$

.

Proof. Clearly, for every $b \in B$, $\mathcal{C}^*(\{b\}) \subseteq \mathcal{C}^*(B)$, by Theorem 6.2.9 and Corollary 6.2.7. Hence, $\bigcup_{b\in B} \mathcal{C}^*(\{b\}) \subseteq \mathcal{C}^*(B)$.

For the converse part, we first show that $K_m(B) \subseteq \bigcup_{b\in B} K_m(\{b\})$ for any $m \in \mathbb{N}$. For $m = 1$, $K_1(B) = B = \bigcup_{b\in B} K_1(\{b\})$. Now let the assertion be true for some $l \in \mathbb{N}$, that is $K_l(B) \subseteq \bigcup_{b\in B} K_l(\{b\})$. Consider, $x \in K_{l+1}(B)$, then there exist $n \in \mathbb{N}$, $k_i \in \mathbb{Z}_0^+$, $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2k_i + 1$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}$ and $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_l(B) \neq \phi$. Now from induction hypothesis, we have $(\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}) \cap (\bigcup_{b\in B} K_l(\{b\})) \neq \phi$. So, there exists $b_1 \in B$ such that $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_l(\{b_1\}) \neq \phi$. Therefore, $x \in K_{l+1}(\{b_1\})$ and so $K_{l+1}(B) \subseteq \bigcup_{b\in B} K_{l+1}(\{b\})$. Now by Theorem 6.2.9, $y \in \mathcal{C}^*(B)$ implies $y \in K(B) = \bigcup_{m\geq 1} K_m(B) \subseteq \bigcup_{m\geq 1} \bigcup_{b\in B} K_m(\{b\})$. Then for some $b \in B$ and $m \in \mathbb{N}$ we have $y \in K_m(\{b\}) \subseteq K(\{b\}) = \mathcal{C}^*(\{b\}) \subseteq \bigcup_{b\in B} \mathcal{C}^*(\{b\})$. Therefore, $\bigcup_{b\in B} \mathcal{C}^*(\{b\}) = \mathcal{C}^*(B)$. □

We define a binary relation 'μ' on a ternary hypersemiring S as follows: $x \,\mu\, y$ if and only if $x \in K(\{y\})$, for all $x, y \in S$.

Proposition 6.2.11. *The relation μ is an equivalence relation on a ternary hypersemiring S.*

Proof. For any $x \in S$ we have $x \in K(\{x\})$, thus $x\mu x$ holds and so μ is reflexive. To show μ is symmetric, we first prove that $x \in K_m(\{y\}) \iff y \in K_m(\{x\})$ for all $m \in \mathbb{N}$ and we prove it by induction on 'm'. If $m = 2$, there exist $a_{ij} \in S$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}$ and $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap \{y\} \neq \phi$ which implies $x = y$ and so $y \in K_2(\{x\})$. Let the assertion be true for some $l(> 2) \in \mathbb{N}$, i.e., $x \in K_l(\{y\})$ which implies $y \in K_l(\{x\})$. Now for $l + 1 \in \mathbb{N}$, $x \in K_{l+1}(\{y\})$ implies there exist $a_{ij} \in S$ such that $x \in \sum_{i=1}^{n'} \prod_{j=1}^{k_i'} a_{ij}$ and $\sum_{i=1}^{n'} \prod_{j=1}^{k_i'} a_{ij} \cap K_l(\{y\}) \neq \phi$. Suppose $u \in (\sum_{i=1}^{n'} \prod_{j=1}^{k_i'} a_{ij}) \cap K_l(\{y\})$. So $u, x \in \sum_{i=1}^{n'} \prod_{j=1}^{k_i'} a_{ij}$ and $u \in K_l(\{y\})$. Thus $u \in K_2(\{x\})$ and by hypothesis $y \in K_l(\{u\})$. So by Lemma 6.2.8, $y \in K_l(K_2(\{x\})) = K_{l+1}(\{x\})$. Hence by induction, $x \in K_m(\{y\})$, implies $y \in K_m(\{x\})$ for any $m \in \mathbb{N}$. Now let $x \in K(\{y\})$. Then there exists $m \in \mathbb{N}$ such that $x \in K_m(\{y\})$. Thus $y \in K_m(\{x\})$, this means that $y \in K(\{x\})$. Therefore, μ is symmetric. To show μ is transitive, let $x \,\mu\, y$ and $y \,\mu\, z$ hold. Then $x \in K(\{y\}) = \mathcal{C}^*(\{y\})$, $y \in K(\{z\}) = \mathcal{C}^*(\{z\})$. Let A be a ternary strong $\mathcal{C}$-set containing $\{z\}$. Then $y \in A$ and $x \in K(\{y\}) = \mathcal{C}^*(\{y\}) \subseteq A$. So, x in any ternary strong $\mathcal{C}$-set containing $\{z\}$. Thus, $x \in \mathcal{C}^*(\{z\}) = K(\{z\})$. Hence $x \, \mu \, z$. □

Theorem 6.2.12. *The equivalence relation μ on a ternary hypersemiring S coincides with the fundamental relation δ^* on S.*

Proof. Let $x \, \mu \, y$ for some $x, y \in S$. If $x = y$, then $x \, \delta^* \, y$. Consider $x \neq y$, then there exists $m \in \mathbb{N}$ for which $x \in K_m(\{y\})$. So there exist $n \in \mathbb{N}$, $k_i \in \mathbb{Z}_0^+$ for $i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2k_i + 1$ such that $x \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}$ and $\sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_{m-1}(\{y\}) \neq \phi$. Let $a_1 \in \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij} \cap K_{m-1}(\{y\})$. Thus $x \, \delta \, a_1$ and $a_1 \in K_{m-1}(\{y\})$. In similar way, there exists $a_2 \in S$ such that $a_1 \, \delta \, a_2$, continuing this process m-times we get $x = a_0, a_1, a_2, ..., a_m = y$ such that $a_i \, \delta \, a_{i+1}$ for $i \in \{1, 2, ..., m-1\}$. Hence $x \delta^* y$.

Conversely, let $x \, \delta \, y$. Then there exist $n \in \mathbb{N}, k_i \in \mathbb{Z}_0^+, i = 1, 2, ..., n$ and $a_{ij} \in S$ for $1 \leq i \leq n$, $1 \leq j \leq 2k_i + 1$ such that

$$x = y \text{ or } \{x, y\} \subseteq \sum_{i=1}^{n} \prod_{j=1}^{2k_i+1} a_{ij}.$$

Thus $x \in K_2(\{y\})$ and so $x \in K(\{y\})$. Hence, $\delta \subseteq \mu$. Since μ is an equivalence relation, we have $\delta^* \subseteq \mu$ and the proof is complete. □

CHAPTER 7

On Some Properties of Hyperideals in Ternary Hypersemirings

7

On Some Properties of Hyperideals in Ternary Hypersemirings

In this chapter, the notions of prime hyperideals and maximal hyperideals in ternary hypersemirings have been introduced and some of their properties have been studied. Then corresponding to a strongly distributive ternary hypersemiring S, a ternary semiring $P_\circ(S)$ has been constructed and various results have been obtained among them. An inclusion preserving bijection between the set of all prime hyperideals of S and the collection of all prime total subtractive ideals of $P_\circ(S)$ has been established. The notions of radical of hyperideals and primary hyperideals of ternary hypersemiring have also been introduced. Some of their important properties on a particular class of hyperideals called $\mathcal{C}$-ternary hyperideals have been investigated. Finally, we generalize the concepts of prime and primary avoidance theorem in ternary hypersemirings for $\mathcal{C}$-ternary hyperideals.

This chapter has been organized as follows:

In *Section 1*, we introduce the notions of prime hyperideals, maximal hyperideals

This chapter is based on the work published in the following papers:

- **Sampad Das et al., On some properties of hyperideals in ternary hypersemirings, *Asian-European Journal of Mathematics*, 16(1) (2023), 2250230**.
- **Sampad Das et al., On Primary Hyperideals of Ternary Hypersemiring, *Journal of Hyperstructures*, 10(1) (2021), 22-37**.

(cf. Definitions 7.1.1, 7.1.12) and study some of their important properties (*cf.* Theorems 7.1.5, 7.1.11, 7.1.19, 7.1.20).

In *Section 2*, we first provide an example of strongly distributive ternary hypersemiring (*cf.* Example 7.2.1). Then construct a ternary semiring $P_\circ(S)$ from a strongly distributive ternary hypersemiring S and observe that $P_\circ(S)$ forms a ternary semiring (*cf.* Theorem 7.2.3). Further, we introduce the notion of a total ideal of $P_\circ(S)$ (*cf.* Definition 7.2.6). We also establish an inclusion preserving bijection from the set of all hyperideals of strongly distributive ternary hypersemiring S to the set of all total subtractive ideals of $P_\circ(S)$ (cf. Theorem 7.2.17).

In *Section 3*, We introduce the notions of $\mathcal{C}$-ternary hyperideal, radical hyperideal, and primary hyperideal (*cf. Definitions 7.3.1, 7.3.7, 7.3.17 respectively.*) in a ternary hypersemiring and study some of their properties. We also prove the prime avoidance theorem (*cf. Theorem 7.3.25*) for ternary hypersemiring. Lastly, using the technique of efficient covering, we prove the primary avoidance theorem (*cf. Theorem 7.3.28*) and an extended version of the primary avoidance theorem (*cf. Theorem 7.3.29*) for ternary hypersemiring.

7.1 Prime Hyperideals

Throughout the section, unless otherwise stated S stands for a ternary hypersemiring $(S, +, \circ)$ with zero.

Definition 7.1.1. A hyperideal P of a ternary hypersemiring S is called prime hyperideal of $(S, +, \circ)$ if for any three hyperideals A, B and C of S, $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

Example 7.1.2. Consider the commutative semigroup $(\mathbb{Z}_0^-, +)$. Let $A = \{-2, -3\}$. Define a ternary hyperoperation $\circ$ on $\mathbb{Z}_0^-$ by $a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c : x, y \in A\}$. Now $(\mathbb{Z}_0^-, +, \circ)$ is a ternary hypersemiring. Then the hyperideal $5\mathbb{Z}_0^-$ is a prime hyperideal of $\mathbb{Z}_0^-$.

Proposition 7.1.3. *Let P be a prime hyperideal of a ternary hypersemiring S and $I_1, I_2, I_3....., I_{2k+1}$, $k \in \mathbb{Z}_0^+$ be arbitrary hyperideals of S. Then the following are equivalent:*
(1) $I_m \subseteq P$ for some $1 \leq m \leq 2k+1$,

(2) $\cap_{1\leq m\leq 2k+1} I_m \subseteq P$,
(3) $\prod_{1\leq m\leq 2k+1} I_m \subseteq P$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are straightforward.
(2) $\Rightarrow$ (3) Since $\prod_{1\leq m\leq n} I_m \subseteq \cap_{1\leq m\leq n} I_m$ (by Theorem 0.4.13). So $\prod_{1\leq m\leq n} I_m \subseteq P$. □

Lemma 7.1.4. *Let S be a ternary hypersemiring and $a \in S$. Then the hyperideal generated by a is given by $\langle a\rangle = SSa + aSS + SaS + SSaSS + \{na : n \in \mathbb{Z}_0^+\}$.*

Theorem 7.1.5. *Let $(S,+,\circ)$ be a ternary hypersemiring. Then the following conditions are equivalent:*

(1) P is a prime hyperideal of S,

(2) $aSbSc \subseteq P$, $aSSbSSc \subseteq P$, $aSSbScS \subseteq P$ and $SaSbSSc \subseteq P$ implies $a \in P$ or $b \in P$ or $c \in P$,

(3) $\langle a\rangle\langle b\rangle\langle c\rangle \subseteq P \Rightarrow a \in P$ or $b \in P$ or $c \in P$.

Proof. (1) $\Rightarrow$ (2) Suppose P is a prime hyperideal of S and $aSbSc \subseteq P$, $aSSbSSc \subseteq P$, $aSSbScS \subseteq P$ and $SaSbSSc \subseteq P$.
Then

$$\begin{aligned}
&(SaS + SSaSS)(SbS + SSbSS)(ScS + SScSS)\\
&\subseteq (SaS)(SbS)(ScS) + (SSaSS)(SbS)(ScS) + (SaS)(SSbSS)(ScS)+\\
&(SSaSS)(SSbSS)(ScS) + (SaS)(SbS)(SScSS) + (SSaSS)(SbS)(SScSS)+\\
&(SaS)(SSbSS)(SScSS) + (SSaSS)(SSbSS)(SScSS)\\
&\subseteq SPS + SPS + SPS + SSP + SPS + SSPSS + PSS + SSPSS \subseteq P
\end{aligned}$$

Without loss of generality, suppose that $(SaS + SSaSS) \subseteq P$.
Then $\langle a\rangle\langle a\rangle\langle a\rangle = (SSa + aSS + SaS + SSaSS + \{na : n \in \mathbb{Z}_0^+\})(SSa + aSS + SaS + SSaSS + \{na : n \in \mathbb{Z}_0^+\})(SSa + aSS + SaS + SSaSS + \{na : n \in \mathbb{Z}_0^+\}) \subseteq (SaS + SSaSS) \subseteq P$. That implies $\langle a\rangle \subseteq P$, and hence $a \in P$. Similarly, if $(SbS+SSbSS) \subseteq P$ then $b \in P$ and if $(ScS + SScSS) \subseteq P$ then $c \in P$.

(2) $\Rightarrow$ (3) Let $\langle a\rangle\langle b\rangle\langle c\rangle \subseteq P$ for some $a, b, c \in S$. Then $aSbSc = a(SbS)c \subseteq \langle a\rangle\langle b\rangle\langle c\rangle \subseteq P$; $aSSbSSc = (aSS)(bSS)c \subseteq \langle a\rangle\langle b\rangle\langle c\rangle \subseteq P$. Similarly, $aSSbScS \subseteq P$ and $SaSbSSc \subseteq P$, which implies either $a \in P$ or $b \in P$ or $c \in P$.

(3) $\Rightarrow$ (1) Suppose A, B and C be any three hyperideals of S such that $ABC \subseteq P$. Let $A \nsubseteq P$ and $B \nsubseteq P$. Then there exist elements $a \in A$ such that $a \notin P$ and $b \in B$ such that $b \notin P$. Now for any $c \in C$, we have $\langle a\rangle\langle b\rangle\langle c\rangle \subseteq ABC \subseteq P$. By using (3), we get $c \in P$. So $C \subseteq P$ and consequently P is a prime hyperideal of S. □

Definition 7.1.6. A hyperideal P of a ternary hypersemiring S is called completely prime if for any three elements a, b and c of S, $abc \subseteq P$ implies either $a \in P$ or $b \in P$ or $c \in P$.

In a ternary hypersemiring, every completely prime hyperideal is prime but the converse may not be true in general, which will be clear from the next example.

Example 7.1.7. Consider the commutative semigroup $(M_{2\times 3}(\mathbb{Z}_0^-), +)$.

Let $A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Let '$\circ$' be a ternary hyperoperation on $M_{2\times 3}(\mathbb{Z}_0^-)$ defined by $a \circ b \circ c = \{a \cdot i \cdot b \cdot j \cdot c : i, j \in A\}$, where a, b, $c \in M_{2\times 3}(\mathbb{Z}_0^-)$.

Then $(M_{2\times 3}(\mathbb{Z}_0^-), +, \circ)$ is a ternary hypersemiring. In this ternary hypersemiring, the zero hyperideal is prime but not completely prime.

Theorem 7.1.8. *A hyperideal P in a commutative ternary hypersemiring $(S,+,\circ)$ is prime if and only if it is completely prime.*

Proof. Suppose P is a prime hyperideal of commutative ternary hypersemiring S and $abc \subseteq P$ for some $a, b, c \in S$. Then $aSbSc = (abc)SS \subseteq PSS \subseteq P$, $aSSbSSc = (abc)SSSS \subseteq PSSSS \subseteq PSS \subseteq P$. Similarly, $aSSbScS \subseteq P$, $SaSbSSc \subseteq P$. Now by Theorem 7.1.5-(2), either $a \in P$ or $b \in P$ or $c \in P$.

Conversely, let P be a completely prime hyperideal of S. Suppose A, B, C be three hyperideals of S such that $ABC \subseteq P$ and $A \nsubseteq P$, $B \nsubseteq P$. Then there exist $a \in A$, $b \in B$ such that $a \notin P$ and $b \notin P$. Now for any $c \in C$, $abc \subseteq ABC \subseteq P$ implies that $c \in P$. So $C \subseteq P$ and consequently, P is a prime hyperideal of S. □

Proposition 7.1.9. *Let $(S, +, \circ)$ be a ternary hypersemiring and I a prime hyperideal of S. Then for any $a, b, c \in S$, the following conditions are equivalent:*
(1) $a \circ b \circ c \subseteq I \Rightarrow a \in I$ or $b \in I$ or $c \in I$,
(2) $a \circ b \circ c \subseteq I \Rightarrow b \circ a \circ c \subseteq I$ and $a \circ c \circ b \subseteq I$.

Proof. (1) $\Rightarrow$ (2) It is straight forward since I is a hyperideal of S.
(2) $\Rightarrow$ (1) Let $a \circ b \circ c \subseteq I$ where $a, b, c \in S$. Now for any $x, y \in S$, $a \circ b \circ c \circ x \circ y \subseteq I$.

This implies $a \circ (b \circ c \circ x) \circ y \subseteq I \Rightarrow \cup_{t_i \in b \circ c \circ x}(\sum a \circ t_i \circ y) \subseteq I \Rightarrow \cup_{t_i \in b \circ c \circ x}(\sum a \circ y \circ t_i) \subseteq I \Rightarrow a \circ y \circ b \circ c \circ x \subseteq I \Rightarrow \cup_{u_i \in a \circ y \circ b}(\sum u_i \circ c \circ x) \subseteq I \Rightarrow a \circ y \circ b \circ x \circ c \subseteq I$, for all $x, y \in S$. So $aSbSc \subseteq I$. Similarly, we can show that $aSSbSSc \subseteq I$, $aSSbScS \subseteq I$ and $SaSbSSc \subseteq I$. Hence by Theorem 7.1.5-(2), either $a \in I$ or $b \in I$ or $c \in I$. □

Definition 7.1.10. A non-empty subset A of a ternary hypersemiring $(S, +, \circ)$ is called an m-system if for any a, b, $c \in A$, $aSbSc \cap A \neq \emptyset$ or $aSSbSSc \cap A \neq \emptyset$ or $aSSbScS \cap A \neq \emptyset$ or $SaSbSSc \cap A \neq \emptyset$.

Theorem 7.1.11. *A hyperideal I of a ternary hypersemiring S is a prime hyperideal if and only if $I^c = S - I$ is an m-system of S.*

Proof. Let I be a prime hyperideal of S. If possible, let I^c is not an m-system of S. Then there exist $a, b, c \in I^c$ such that $aSbSc \cap I^c = \emptyset$ and $aSSbSSc \cap I^c = \emptyset$ and $aSSbScS \cap I^c = \emptyset$ and $SaSbSSc \cap I^c = \emptyset \Rightarrow aSbSc \subseteq I$, $aSSbSSc \subseteq I$, $aSSbScS \subseteq I$ and $SaSbSSc \subseteq I$. Since I is a prime hyperideal of S, by Theorem 7.1.5-(2), $a \in I$ or $b \in I$ or $c \in I$, which is a contradiction. Hence I^c is an m-system of S.

Conversely, let I^c be an m-system of S. Suppose $aSbSc \subseteq I$, $aSSbSSc \subseteq I$, $aSSbScS \subseteq I$ and $SaSbSSc \subseteq I$ for some a, $b\, c \in S$. If possible, let $a, b, c \in I^c$. Then $aSbSc \cap I^c = \phi$, $aSSbSSc \cap I^c = \phi$, $aSSbScS \cap I^c = \phi$ and $SaSbSSc \cap I^c = \phi$, which contradicts I^c is an m-system. So either $a \in I$ or $b \in I$ or $c \in I$. Hence I is a prime hyperideal of S. □

Definition 7.1.12. A hyperideal M in a ternary hypersemiring S is called maximal if $M \neq S$ and for any hyperideal $N \supseteq M$, either $N = M$ or $N = S$.

In a ternary hypersemiring, a maximal hyperideal may not be prime as shown by the next examples.

Example 7.1.13. Consider the set $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in 2\mathbb{Z}_0^- \right\}$ and define $\circ : M \circ M \circ M \to P^*(M)$ by $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} \circ \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \circ \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix}$

$= \left\{ \begin{pmatrix} a_1 \cdot a_2 \cdot a_3 \cdot x \cdot y & (a_1 \cdot a_2 \cdot b_3 + a_1 \cdot b_2 \cdot c_3 + b_1 \cdot c_2 \cdot c_3) \cdot x \cdot y \\ 0 & c_1 \cdot c_2 \cdot c_3 \cdot x \cdot y \end{pmatrix}, \text{ where} \right.$

$\left. x,\, y \in \{-3\ -5\} \right\}$. Then $(M, +, \circ)$ is a ternary hypersemiring and $K = \left\{ \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} : \right.$

$a^{'} \in 4\mathbb{Z}_0^-, b^{'}, c^{'} \in 2\mathbb{Z}_0^- \Big\}$ is a maximal hyperideal of M. But this is not prime, since for the hyperideal $A = \left\{ \begin{pmatrix} m & s \\ 0 & t \end{pmatrix} : m, s \in 2\mathbb{Z}_0^-, t \in 8\mathbb{Z}_0^- \right\}$, $A^3 \subseteq K$ but $A \nsubseteq K$.

Example 7.1.14. Consider the commutative ternary hypersemiring $(2\mathbb{Z}_0^-, +, \circ)$, where the ternary hyperoperation $\circ : 2\mathbb{Z}_0^- \times 2\mathbb{Z}_0^- \times 2\mathbb{Z}_0^- \to P^*(2\mathbb{Z}_0^-)$ is defined by $a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c : x, y \in 3\mathbb{Z}_0^-\}$. The hyperideal $4\mathbb{Z}_0^-$ is a maximal hyperideal in $(2\mathbb{Z}_0^-, +, \circ)$ but not prime. Since $2 \circ 2 \circ 2 \subseteq 4\mathbb{Z}_0^-$ but $2 \notin 4\mathbb{Z}_0^-$.

Theorem 7.1.15. *Let (S,+,∘) be a commutative ternary hypersemiring with a hyperidentity e, then every maximal hyperideal of S is a prime hyperideal of S.*

Proof. Let M be a maximal hyperideal of S. Consider $abc \subseteq M$ where a, b, $c \in S$. If possible, let $a \notin M$, $b \notin M$, $c \notin M$. Then $\langle M, a\rangle = S$, $\langle M, b\rangle = S$, $\langle M, c\rangle = S$. That implies $e \in \{m_1\} + s_1 \circ t_1 \circ a$, $e \in \{m_2\} + s_2 \circ t_2 \circ b$ and $e \in \{m_3\} + s_3 \circ t_3 \circ c$, where $m_1, m_2, m_3 \in M$ and $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. Then $e \in e \circ e \circ e \subseteq (m_1 + s_1 \circ t_1 \circ a)(m_2 + s_2 \circ t_2 \circ b)(m_3 + s_3 \circ t_3 \circ c) \subseteq M$, since $abc \subseteq M$ and M is a hyperideal of S. This implies $a \in e \circ a \circ e \subseteq M \circ a \circ M \subseteq M$, for all $a \in S$. So $S = M$, which is a contradiction. Hence M is a prime hyperideal of S. □

Example 7.1.16. The zero hyperideal $\langle 0\rangle$, in the ternary hypersemiring $(\mathbb{Z}_0^-, +, \circ)$ is a prime hyperideal but not maximal.

Theorem 7.1.17. *Let I be an m-system of a ternary hypersemiring $(S, +, \circ)$ and N be a hyperideal of S such that $N \cap I = \emptyset$. Then there exists a maximal hyperideal M of S containing N such that $M \cap I = \emptyset$. Moreover, M is also a prime hyperideal of S.*

Proof. Consider the collection of hyperideals $\aleph = \{A : A \supseteq N, A$ is a hyperideal of S such that $A \cap I = \emptyset\}$. Clearly $\aleph$ is non-empty, since $N \in \aleph$. Now under the set inclusion relation, $\aleph$ forms a partial ordered set and any chain of elements in $\aleph$ has an upper bound which is their union. So by Zorn's Lemma, $\aleph$ contains a maximal element M. Therefore from the consideration of $\aleph$, M is the required maximal hyperideal of S containing N such that $M \cap I = \emptyset$.

If possible, let M be not a prime hyperideal of S. So there exist hyperideals J, K, L of S such that $JKL \subseteq M$ but $J \nsubseteq M$, $K \nsubseteq M$ and $L \nsubseteq M$. Now $M \subsetneq M+J$, $M \subsetneq M+K$, $M \subsetneq M+L$. So by the given condition and maximality of M, $(M+J)\cap I \neq \emptyset$, $(M+K)\cap I \neq \emptyset$ and $(M+L)\cap I \neq \emptyset$. Then there exist $i_1, i_2, i_3 \in I$ such that $i_1 = n_1+j$, $i_2 =$

$n_2 + k$, $i_3 = n_3 + l$ for some $n_1, n_2, n_3 \in M$, $j \in J, k \in K, l \in L$. Now $i_1 s_1 s_2 i_2 s_3 s_4 i_3 = (n_1 + j) s_1 s_2 (n_2 + k) s_3 s_4 (n_3 + l) \subseteq n_1 s_1 s_2 n_2 s_3 s_4 n_3 + j s_1 s_2 n_2 s_3 s_4 n_3 + n_1 s_1 s_2 k s_3 s_4 n_3 + j s_1 s_2 k s_3 s_4 n_3 + n_1 s_1 s_2 n_2 s_3 s_4 l + j s_1 s_2 n_2 s_3 s_4 l + n_1 s_1 s_2 k s_3 s_4 l + j s_1 s_2 k s_3 s_4 l \subseteq M$ for all $s_1, s_2, s_3, s_4 \in S$. This implies $i_1 S S i_2 S S i_3 \cap I \subseteq M \cap I = \phi$, which is a contradiction. Hence M is a prime hyperideal of S. □

Theorem 7.1.18. *Let S be a ternary hypersemiring and M a maximal hyperideal of S. Then M is not a prime hyperideal if and only if $S^3 \subseteq M$.*

Proof. Suppose M is not a prime hyperideal of S. Then there exist hyperideals A, B, C of S such that $ABC \subseteq M$ but $A \nsubseteq M$, $B \nsubseteq M$, $C \nsubseteq M$. By maximality of the hyperideal M, $A+M = B+M = C+M = S$. Hence $S^3 = (A+M)(B+M)(C+M) \subseteq ABC + AMC + MBC + MMC + ABM + AMM + MBM + MMM \subseteq M$.

Conversely, let $S^3 \subseteq M$. If M is a prime hyperideal of S then $S \subseteq M$, which is a contradiction. Hence, M is not a prime hyperideal of S. □

Theorem 7.1.19. *Every prime hyperideal I of a ternary hypersemiring S contains a minimal prime hyperideal.*

Proof. Let S be a ternary hypersemiring and I be a prime hyperideal of S. Construct $T = \{H : H \subseteq I,\ H \text{ is prime hyperideal of } S\}$. Since $I \in T$, T is non-empty. Let $\{H_i : i \in \triangle\}$ be a descending chain of prime hyperideals of S. Take $H = \cap_{i \in \triangle} H_i$. Now to prove that H is prime, let $aSbSc \subseteq H$, $aSSbSSc \subseteq H$, $aSSbScS \subseteq H$ and $SaSbSSc \subseteq H$, where $a, b, c \in S$. Suppose $a \notin H$, $b \notin H$, that is, there exists $k \in \triangle$ such that $a, b \notin H_k \Rightarrow c \in H_k$. That implies $c \in H_i$, for all $i \leq k$. If $i > k$ then $H_i \subseteq H_k$ and $a, b \notin H_k \Rightarrow a, b \notin H_i$. Since H_i is prime, $c \in H_i$. So for every $i \in \triangle$, $c \in H$. Therefore, H is a prime hyperideal and a lower bound of $\{H_i : i \in \triangle\}$. Then by Zorn's lemma, T has a minimal element. We claim that H is the minimal prime hyperideal of S. If not, consider there exists a prime hyperideal J such that $J \subsetneq H \subseteq I$. Then $J \in T$. That contradicts the fact that H is the minimal element of T. Therefore, H is the minimal prime hyperideal of S contained in I. □

Theorem 7.1.20. *Let $(S, +, \circ)$ be a strongly distributive commutative ternary hypersemiring with a unital element e. If every hyperideal of S, which is maximal with respect to the property of not having a given m-system I of S is prime then S is regular.*

Proof. Let $(S, +, \circ)$ be a strongly distributive commutative ternary hypersemiring with a unital element e. Take $a \in S$. Now consider the m-system $I = \{a\}$ and $I_a = I \circ S \circ I$.

Then I_a is a hyperideal of S. If $a \in I_a$, then a is a regular element. If possible, let $a \notin I_a$ and take $\Omega = \{B : B$ is a hyperideal of S such that $I_a \subseteq B$ and $a \notin B\}$. Clearly, Ω is non-empty, since $I_a \in \Omega$ and any chain of Ω has upper bound which is their union. Now by Zorn's lemma, Ω contains a maximal element P(say). Then by the hypothesis, P is prime hyperideal of S. We have $a \circ a \circ a \subseteq I_a \subseteq P$, which implies $a \in P$, a contradiction. Therefore, $a \in I_a$. So a is a regular element. Hence, $(S, +, \circ)$ is regular. □

7.2 Strongly distributive ternary hypersemiring S and the ternary semiring $P_\circ(S)$

The notion of strongly distributive ternary hypersemiring has already been defined in Section-2 (cf, Definition- 0.4.2), here we start with an example of strongly distributive ternary hypersemiring.

Example 7.2.1. Let $S = \{n\sqrt{3} : n \in \mathbb{Z}_0^-\}$. Then S is an additive commutative semigroup with respect to usual addition. Suppose that m is an arbitrarily choosen but fixed positive integer. We now define a hyperoperation $\circ$ on S as follows: $a \circ b \circ c = \{abc + mr : r \in S\}$. Clearly, $\circ$ is associative. Let $x \in a \circ b \circ c$ and $y \in a \circ b \circ d$, where $a, b, c, d \in S$. Then $x = abc + mr_1$ and $y = abd + mr_2$ for some $r_1, r_2 \in S$. Thus $x + y = abc + mr_1 + abd + mr_2 = ab(c + d) + m(r_1 + r_2) \in a \circ b \circ (c + d)$. So $a \circ b \circ c + a \circ b \circ d \subseteq a \circ b \circ (c + d)$. Conversely, let $x \in a \circ b \circ (c + d)$. Then $x = ab(c+d)+mr_3$ for some $r_3 \in S$. Now $x = (abc+mr_3)+(abd+0) \in a\circ b\circ c+a\circ b\circ d$. So $a\circ b\circ (c+d) \subseteq a\circ b\circ c+a\circ b\circ d$. Thus $a\circ b\circ (c+d) = a\circ b\circ c+a\circ b\circ d$. Similarly we can show that $a\circ (b+c)\circ d = a\circ b\circ d+a\circ c\circ d$ and $(a+b)\circ c\circ d = a\circ c\circ d+b\circ c\circ d$. Moreover, $0 \in 0 \circ a \circ b = a \circ 0 \circ b = a \circ b \circ 0$ for all $a, b \in S$. So $(S, +, \circ)$ is a strongly distributive ternary hypersemiring with absorbing zero.

Remark 7.2.2. Any ternay semiring $(S, +, .)$ can be regarded as a strongly distributive ternary hypersemiring $(S, +, \circ)$, if we consider the ternary hyperoperation as $a\circ b\circ c = \{a \cdot b \cdot c\}$ for all $a, b, c \in S$.

In this section, first we construct a ternary semiring $P_0(S)$, from a strongly distributive ternary hypersemiring S. For any ternary hypersemiring $(S, +, \circ)$ with absorbing zero, $(P(S)$,+$)$ is a commutative semigroup and $(P(S), \star)$ is a ternary semigroup, where '+' and '$\star$' defined as follows: $A + B = \{a + b : a \in A, b \in B\}$ and $A \star B \star C = \bigcup\{\sum_{finite} a \circ b \circ c; a \in A, b \in B, c \in C\}$, for any $A, B, C \in P(S)$.

Consider the set $P_\circ(S) = \{A \in P(S) : 0 \in A\}$, corresponding to a strongly distributive ternary hypersemiring $(S, +, \circ)$ with absorbing zero '0'. If we take the restriction of the binary operation '+' and the ternary operation '$\star$' over the set $P_\circ(S)$, then we arrive at the following results.

Theorem 7.2.3. *Let $(S, +, \circ)$ be a strongly distributive ternary hypersemiring with absorbing zero. Then $(P_\circ(S), +, \star)$ is a ternary semiring.*

Proof. Let $A, B \in P_\circ(S)$. Then $0 \in A$ and $0 \in B$. Since $0 = 0 + 0 \in A + B$, $A + B \in P_\circ(S)$. Clearly, $P_\circ(S)$ is an additive commutative semigroup.

Since $0 \in 0 \circ 0 \circ 0 \subseteq A \star B \star C$, so $A \star B \star C \in P_\circ(S)$ for any $A, B, C \in P_\circ(S)$. Let $x \in (A \star B \star C) \star D \star E$, for some $A, B, C, D, E \in P_\circ(S)$. Then $x \in \sum_{i=1}^{n} x_i \circ d_i \circ e_i$, where $x_i \in \sum_{j=1}^{m} a_{ij} \circ b_{ij} \circ c_{ij}$ for some $a_{ij} \in A, b_{ij} \in B, c_{ij} \in C, d_i \in D, e_i \in E$.
That implies

$$\begin{aligned}
x &\in \sum_{i=1}^{n}(\sum_{j=1}^{m} a_{ij} \circ b_{ij} \circ c_{ij}) \circ d_i \circ e_i \\
&\subseteq \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} \circ (b_{ij} \circ c_{ij} \circ d_i) \circ e_i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}(\cup_{t_{ij}} a_{ij} \circ t_{ij} \circ e_i) \qquad\qquad \text{where } t_{ij} \in b_{ij} \circ c_{ij} \circ d_i. \\
&\subseteq \bigcup\{\sum_{k=1}^{s} a_k \circ t_k \circ e_k; a_k \in A, t_k \in BCD, e_k \in E\} \\
&= A \star (B \star C \star D) \star E.
\end{aligned}$$

So $(A \star B \star C) \star D \star E \subseteq A \star (B \star C \star D) \star E$. Similarly, one can prove the reverse inclusion. Therefore $(A \star B \star C) \star D \star E = A \star (B \star C \star D) \star E$. In a similar way, it is easy to show that $A \star (B \star C \star D) \star E = (A \star B \star C) \star D \star E$. So $A \star B \star (C \star D \star E) = A \star (B \star C \star D) \star E = (A \star B \star C) \star D \star E$.

Suppose $x \in (A + B) \star C \star D = \bigcup\{\sum_{i=1}^{n}(a_i + b_i) \circ c_i \circ d_i; a_i \in A, b_i \in B, c_i \in C, d_i \in D\} \Rightarrow x \in \sum_{i=1}^{n}(a_i + b_i) \circ c_i \circ d_i$ (for some $a_i \in A, b_i \in B, c_i \in C, d_i \in D$) $\subseteq \sum_{i=1}^{n}(a_i \circ b_i \circ d_i + b_i \circ c_i \circ d_i) \subseteq \sum_{i=1}^{n}(a_i \circ b_i \circ d_i) + \sum_{i=1}^{n}(b_i \circ c_i \circ d_i) \subseteq A \star C \star D + B \star C \star D$. Again $A \star C \star D + B \star C \star D \subseteq (A + \{0\}) \star C \star D + (\{0\} + B) \star C \star D \subseteq (A + B) \star C \star D + (A + B) \star C \star D \subseteq (A + B) \star C \star D$. So $(A + B) \star C \star D = A \star C \star D + B \star C \star D$ and the remaining conditions for ternary semiring will hold similarly. Therefore, $(P_\circ(S), +, \star)$ is a ternary semiring. □

Corollary 7.2.4. *If S is a commutative ternary hypersemiring, then $(P_\circ(S), +, \star)$ is a commutative ternary semiring.*

From now onwards throughout this section, S will stand for strongly distributive ternary hypersemiring, and $P_\circ(S)$ will stand for the corresponding ternary semiring $(P_\circ(S), +, \star)$, also we denote the set $A\star B\star C$ by ABC, unless stated otherwise explicitly.

Theorem 7.2.5. *Let S be a strongly distributive ternary hypersemiring, then the collection of all hyperideals of S forms a ternary subsemiring of $P_\circ(S)$.*

Proof. For any hyperideal I of S, $0 \in I$. So $I \in P_\circ(S)$. Also $I + J \in P_\circ(S)$ and $IJK \in P_\circ(S)$, for any three hyperideals $I, J, K \in P_\circ(S)$. □

For any non-empty subset $\mathcal{A}$ of $P_\circ(S)$, define $\cup\mathcal{A} = \cup\{A : A \in \mathcal{A}\}$.

Definition 7.2.6. Let $(S, +, \circ)$ be a strongly distributive ternary hypersemiring. An ideal $\mathcal{A}$ of ternary semiring $(P_\circ(S), +, \star)$ is called a total ideal if $\cup\mathcal{A} \in \mathcal{A}$.

Theorem 7.2.7. *Let $\mathcal{A}$ be a total ideal of $P_\circ(S)$, corresponding to a strongly distributive ternary hypersemiring S. Then $\cup\mathcal{A} = \cup\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}$ is a total ideal of $P_\circ(S)$, where $\bar{\mathcal{A}}$ is the k-closure of $\mathcal{A}$,.*

Proof. Clearly $\mathcal{A} \subseteq \bar{\mathcal{A}}$, so $\cup\mathcal{A} \subseteq \cup\bar{\mathcal{A}}$. Let $x \in \cup\bar{\mathcal{A}}$, then there exists $B \in \bar{\mathcal{A}}$ such that $x \in B$. Now $B \in \bar{\mathcal{A}}$ implies there exist $A \in \mathcal{A}$ such that $A + B \in \mathcal{A}$, therefore $x = 0 + x \in A + B \in \mathcal{A}$. Thus $x \in \cup\mathcal{A}$. Hence $\cup\mathcal{A} = \cup\bar{\mathcal{A}}$, also $\cup\bar{\mathcal{A}} = \cup\mathcal{A} \in \mathcal{A}$ (since $\mathcal{A}$ is total) $\subseteq \bar{\mathcal{A}}$. Therefore, $\bar{\mathcal{A}}$ is a total ideal of $P_\circ(S)$. □

Lemma 7.2.8. *If $\mathcal{A}$ is an ideal of a ternary semiring $(P_\circ(S), +, \star)$ corresponding to a strongly distributive ternary hypersemiring S, then $\cup\mathcal{A}$ is a hyperideal of S.*

Proof. Since $\mathcal{A} \subseteq P_\circ(S)$, so $0 \in \cup\mathcal{A}$. Let $a, b \in \cup\mathcal{A} = \cup\{A : A \in \mathcal{A}\}$. Then $a \in A$ and $b \in B$ for some $A, B \in \mathcal{A}$. Thus $a + b \in A + B \in \mathcal{A}$ implies $a + b \in \cup\mathcal{A}$. Again, for any $x, y \in S$, $a \circ x \circ y \subseteq a \circ \{0, x\} \circ \{0, y\} \subseteq A \circ \{0, x\} \circ \{0, y\} \in \mathcal{A}$, since $\{0, x\}$,$\{0, y\} \in P_\circ(S)$ and $\mathcal{A}$ is an ideal of $(P_\circ(S)$. Hence $a \circ x \circ y \subseteq \bigcup\mathcal{A}$, similarly $x \circ a \circ y \subseteq \cup\mathcal{A}$ and $x \circ y \circ a \subseteq \cup\mathcal{A}$. Thus $\cup\mathcal{A}$ is a hyperideal of S. □

For any hyperideal I of a strongly distributive ternary hypersemiring S, define $A_I = \{A \in P_\circ(S) : A \subseteq I\}$.

Lemma 7.2.9. *Let I be a hyperideal of a strongly distributive ternary hypersemiring $(S, +, \circ)$. Then A_I is a total ideal as well as subtractive ideal of the ternary semiring $(P_\circ(S), +, \star)$. Moreover, $\cup A_I = I$.*

Proof. Let $A, B \in A_I$. Then $0 \in A$, $0 \in B$ and $A \subseteq I$, $B \subseteq I$. Thus $0 = 0+0 \in A+B$ and $A+B \subseteq I$, since I is a hyperideal of S. So $A+B \in A_I$. Consider $P, Q \in P_\circ(S)$ and $A \in A_I$ be arbitrary. Let $x \in PQA \Rightarrow x \in \sum_{i=1}^{n} p_i \circ q_i \circ a$, for some $p_i \in P$, $q_i \in Q$, $a_i \in A$. Since I is a hyperideal and $a_i \in A \in I$, we have $p_i \circ q_i \circ a_i \subseteq I$ for each i. So $x \in \sum_{i=1}^{n} p_i \circ q_i \circ a \subseteq I$. Hence $PQA \subseteq I$. Also $0 \in 0 \circ 0 \circ 0 \subseteq PQA$. Therefore, $PQA \in A_I$; similarly, $PAQ \subseteq A_I$ and $APQ \subseteq A_I$. Thus A_I is an ideal of the ternary semiring $P_\circ(S)$. Here $A \subseteq I$ for all $A \in A_I$. So $\cup A_I = \cup\{A : A \in A_I\} \subseteq I$. Therefore, $\cup A_I \in A_I$ and hence A_I is a total ideal of the ternary semiring $P_\circ(S)$.

Let $A, B \in P_\circ(S)$ such that $A+B \in A_I$ and $B \in A_I$, which implies $A+B \subseteq I$ and $B \subseteq I$. Thus $A \subseteq A+B \subseteq I$. So $A \in A_I$. Therefore, A_I is a subtractive ideal of the ternary semiring $P_\circ(S)$.

Again, $I \in A_I$ and $A \subseteq I$ for all $A \in A_I$ implies $\cup A_I = I$. □

Definition 7.2.10. Let $(S, +, \circ)$ be a strongly distributive ternary hypersemiring. An ideal of the ternary semiring $P_\circ(S)$ which is total as well as subtractive ideal is called total subtractive ideal of $P_\circ(S)$.

Lemma 7.2.11. *Let S be a strongly distributive ternary hypersemiring and $\mathcal{A}$ be a total subtractive ideal of the ternary semiring $P_\circ(S)$. Then for the hyperideal $I = \cup\mathcal{A}$, $A_I = \mathcal{A}$.*

Proof. By Lemma 7.2.8, $I = \cup\mathcal{A}$ is a hyperideal of S. Then for any $P \in \mathcal{A}$, $0 \in P \subseteq I$ which implies $P \in A_I$, that is $\mathcal{A} \subseteq A_I$. Conversely, let $P \in A_I$. Then $P+I \subseteq I+I \subseteq I$. Now for any $x \in I$ $x = 0 + x \in P + I$, since $0 \in P$. Thus $I \subseteq P + I$. Consequently, $P + I = I$. Since $\mathcal{A}$ is total ideal and $I \in \mathcal{A}$, thus $P + I = I \in \mathcal{A}$ implies $P \in \mathcal{A}$ as $\mathcal{A}$ is a subtractive ideal. So $A_I \subseteq \mathcal{A}$. Hence $A_I = \mathcal{A}$. □

Definition 7.2.12. Let $(P_\circ(S), +, \star)$ be a ternary semiring corresponding to the strongly distributive ternary hypersemiring $(S, +, \circ)$. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are three non empty subsets of the ternary semiring $(P_\circ(S), +, \star)$, then $\mathcal{ABC} = \{\Sigma_{finite} A_i B_i C_i : A_i \in \mathcal{A}, B_i \in \mathcal{B}, C_i \in \mathcal{C}\}$ and $\cup\mathcal{A} = \cup\{A : A \in \mathcal{A}\}$

Lemma 7.2.13. *Let $(S, +, \circ)$ be a strongly distributive ternary hypersemiring and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three ideals of the ternary semiring $(P_\circ(S), +, \star)$. Then $(\cup\mathcal{A})(\cup\mathcal{B})(\cup\mathcal{C}) = \cup\mathcal{ABC}$.*

Proof. Let $x \in (\cup\mathcal{A})(\cup\mathcal{B})(\cup\mathcal{C})$ which implies $x \in \sum_{i=1}^{n} a_i \circ b_i \circ c_i$ for some $a_i \in \cup\mathcal{A}$, $b_i \in \cup\mathcal{B}$, $c_i \in \cup\mathcal{C}$. So there exist $A_i \in \mathcal{A}$, $B_i \in \mathcal{B}$, $C_i \in \mathcal{C}$ such that $a_i \in A_i$, $b_i \in B_i$, $c_i \in C_i$. Hence $x \in \sum_{i=1}^{n} a_i \circ b_i \circ c_i \subseteq \sum_{i=1}^{n} A_i B_i C_i \in \mathcal{ABC}$. Thus $\cup\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \subseteq \cup\mathcal{ABC}$

Conversely, let $x \in \cup\mathcal{ABC}$. Then $x \in \sum_{i=1}^{n} A_iB_iC_i$ for some $A_i \in \mathcal{A}, B_i \in \mathcal{B}, C_i \in \mathcal{C}$. That implies $x \in \sum_{i=1}^{n}(\sum_{j=1}^{m} a_{ij} \circ b_{ij} \circ c_{ij})$ for some $a_{ij} \in A_i$, $b_{ij} \in B_i$, $c_{ij} \in C_i$. Let us assume $A_i' = \{a_{ij} : j = 1, 2, ...m\}$, $B_i' = \{b_{ij} : j = 1, 2, ...m\}$, $C_i' = \{c_{ij} : j = 1, 2, ...m\}$ for each $i(= 1, 2, ...n)$ and consider $A = \cup A_i'$, $B = \cup B_i'$ and $C = C_i'$. Then clearly $A_i' \subseteq A_i \in \mathcal{A}$, $B_i' \subseteq B_i \in \mathcal{B}$ and $C_i' \subseteq C_i \in \mathcal{C}$. So $A \subseteq \cup\mathcal{A}$, $B \subseteq \cup\mathcal{B}$, $C \subseteq \cup\mathcal{C}$. Thus, $x \in \sum_{i=1}^{n}(\sum_{j=1}^{m} a_{ij} \circ b_{ij} \circ c_{ij}) \subseteq ABC \subseteq (\cup\mathcal{A})(\cup\mathcal{B})(\cup\mathcal{C})$. Hence $\cup\mathcal{ABC} \subseteq (\cup\mathcal{A})(\cup\mathcal{B})(\cup\mathcal{C})$. Therefore, $(\cup\mathcal{A})(\cup\mathcal{B})(\cup\mathcal{C}) = \cup\mathcal{ABC}$. □

Definition 7.2.14. Let $(S, +, \circ)$ be a strongly distributive ternary hypersemiring. An ideal of the ternary semiring $P_\circ(S)$ which is prime as well as total subtractive ideal is referred as a prime total subtractive ideal.

Lemma 7.2.15. *Let $(S, +, \circ)$ be a strongly distributive ternary hypersemiring and $\mathcal{A}$ be a prime total subtractive ideal of the ternary semiring $P_\circ(S)$. Then $I = \cup\{A : A \in \mathcal{A}\}$ is a prime hyperideal of S.*

Proof. Let $\mathcal{A}$ be a prime total subtractive ideal of $P_\circ(S)$. By Lemma 7.2.8 and Lemma 7.2.11, $I = \cup\{A : A \in \mathcal{A}\}$ is a hyperideal of S and $A_I = \mathcal{A}$. Consider $JKL \subseteq I$ where J, K, L are three hyperideals of S. Then by Lemma 7.2.9, A_J, A_K, A_L are three ideals of $P_\circ(S)$ and $\cup A_J = J$, $\cup A_K = K$, $\cup A_L = L$. So $JKL = (\cup A_J)(\cup A_K)(\cup A_L) = \cup(A_JA_KA_L) \subseteq I$ (by Lemma 7.2.13). Thus $A_JA_KA_L \subseteq A_I = \mathcal{A}$, since for any $A \in A_JA_KA_L$, $0 \in A \subseteq I$ implies $A \in A_I$. Hence either $A_J \subseteq \mathcal{A}$ or $A_K \subseteq \mathcal{A}$ or $A_L \subseteq \mathcal{A}$ that implies $\cup A_J \subseteq \cup\mathcal{A}$ or $\cup A_K \subseteq \cup\mathcal{A}$ or $\cup A_L \subseteq \cup\mathcal{A}$ i.e., $J \subseteq I$ or $K \subseteq I$ or $L \subseteq I$. Hence I is a prime hyperideal S. □

Theorem 7.2.16. *Let I be a hyperideal of the strongly distributive ternary hypersemiring $(S, +, \circ)$. Then I is a prime hyperideal of S if and only if A_I is a prime ideal of the ternary semiring $P_\circ(S)$.*

Proof. Let I be a prime hyperideal of S. Suppose $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are three ideals of $P_\circ(S)$ such that $\mathscr{A}\mathscr{B}\mathscr{C} \subseteq A_I$. By Lemma 7.2.8, $\cup\mathscr{A} = U$, $\cup\mathscr{B} = V$, $\cup\mathscr{C} = W$ are three hyperideals S, moreover $\mathscr{A} \subseteq A_U$, $\mathscr{B} \subseteq A_V$, $\mathscr{C} \subseteq A_W$. Here $UVW = (\cup\mathscr{A})(\cup\mathscr{B})(\cup\mathscr{C}) = \cup(\mathscr{A}\mathscr{B}\mathscr{C}) \subseteq \cup A_I = I$, by using Lemma 7.2.13 and Lemma 7.2.9. Hence either $U \subseteq I$ or $V \subseteq I$ or $W \subseteq I$. Thus either $A_U \subseteq A_I$ or $A_V \subseteq A_I$ or $A_W \subseteq A_I$. This implies either $\mathscr{A} \subseteq A_I$ or $\mathscr{B} \subseteq A_I$ or $\mathscr{C} \subseteq A_I$. Hence A_I is a prime ideal of $P_\circ(S)$.

Conversely, let A_I be a prime ideal of $P_\circ(S)$. Since I is a hyperideal S, then by Lemma 7.2.9, A_I is a total subtractive ideal of the ternary semiring $P_\circ(S)$ and $\cup A_I = I$. So by Lemma 7.2.15, I is a prime hyperideal of S. □

We conclude the article by proving a bijection from the set of all hyperideals of a strongly distributive ternary hypersemiring $(S, +, \circ)$ to the set of all total subtractive ideals of the ternary semiring $P_\circ(S)$.

Theorem 7.2.17. *There is an inclusion preserving bijection from the set of all hyperideals of a strongly distributive ternary hypersemiring $(S, +, \circ)$ to the set of all total subtractive ideals of the ternary semiring $(P_\circ(S), +, \star)$.*

Proof. Let $\mathcal{S}$ be the set of all hyperideals of S and Ψ_K be the set of all total subtractive ideals of the ternary semiring $P_\circ(S)$. Now define a map $\Phi : \mathcal{S} \to \Psi_K$ by $\Phi(I) = A_I$, where $A_I = \{A \in P_\circ(S) : A \subseteq I\}$. Let I and J be two hyperideals of S and $\Phi(I) = \Phi(J)$. So $A_I = A_J$. Since $I \in A_I \Rightarrow I \in A_J$, that implies $I \subseteq J$. Also $J \in A_J \Rightarrow J \in A_I$, that implies $J \subseteq I$. So $I = J$. Hence Φ is injective. Let $\mathcal{A} \in \Psi_K$. Then by Lemma 7.2.8, $\cup\mathcal{A} = I(say)$ is a hyperideal of S. Now $\Phi(I) = A_I = \mathcal{A}$ (by Lemma 7.2.11). Hence Φ is surjective. Let $I, J \in \mathcal{S}$ such that $I \subseteq J$. Then for any $A \in A_I$. So $A \subseteq I \subseteq J \Rightarrow A \in A_J$. So $A_I \subseteq A_J$. Therefore, $\Phi(I) \subseteq \Phi(J)$. □

Remark 7.2.18. From Theorem 7.2.16 and Theorem 7.2.17, there is an one to one correspondence from the set of all prime hyperideals of S onto the set of all prime total subtractive ideals of $P_\circ(S)$.

7.3 Primary Hyperideals

The aim of this section is to extend some properties of radical and primary ideals of semiring to radical and primary hyperideals of ternary hypersemirings. Throughout the section, unless otherwise stated S stands for a ternary hypersemiring $(S, +, \circ)$ with a hyperidentity.

Definition 7.3.1. Let $\mathcal{C} = \{\Pi_{i=1}^{2n+1} a_i : a_i \in S, n \in \mathbb{Z}_0^+\}$ be the class of all finite ternary products of elements of a ternary hypersemiring $(S, +, \circ)$. A hyperideal I is called complete ternary hyperideal or $\mathcal{C}$-ternary hyperideal if for any $A \in \mathcal{C}$, $I \cap A \neq \phi$ implies $A \subseteq I$.

Example 7.3.2. Consider the ternary hypersemiring $(\mathbb{Z}_0^-, +, \circ)$, where hyperoperation '$\circ$' is defined by $a \circ b \circ c = \{abc + kn : n \in \mathbb{Z}_0^-\}$, k is a fixed positive integer. Then every hyperideal of the form $m\mathbb{Z}_0^+, m \in \mathbb{Z}_0^-$ is a $\mathcal{C}$-ternary hyperideal.

Example 7.3.3. Consider the ternary hypersemiring $([0, 1], +, \circ)$, where binary operation '$+$' and ternary hyperoperation '$\circ$' on S are defined by $a + b = max\{a, b\}$ and

$a \circ b \circ c = [0, x]$, where $x = min\{a, b, c\}$ respectively. In this ternary hypersemiring, the hyperideal $[0, \frac{1}{2}]$ is a $\mathcal{C}$-ternary hyperideal.

Example 7.3.4. Corresponding the set $X = \{2, 3\}$, $(\mathbb{Z}_0^-, +, \circ)$ forms a ternary hypersemiring, where ternary hyperoperation '$\circ$' is defined by $a \circ b \circ c = \{x \cdot a \cdot b \cdot c \cdot y : x, y \in X\}$. In this ternary hypersemiring, the hyperideal $18\mathbb{Z}_0^-$ is not a $\mathcal{C}$-ternary hyperideal. As because, $-18 \in (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1)$ hence, $(-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \cap 18\mathbb{Z}_0^- \neq \phi$ but $-27 \in \{(-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1)$ and $-27 \notin 18\mathbb{Z}_0^-$. So $(-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \nsubseteq 18\mathbb{Z}_0^-$.

Proposition 7.3.5. *Intersection of arbitrary collection of $\mathcal{C}$-ternary hyperideals $\{I_i : i \in \Lambda\}$ of ternary hypersemiring $(S, +, \circ)$ is also a $\mathcal{C}$-ternary hyperideal.*

Proof. Let $A \in \mathcal{C}$ such that $A \cap (\bigcap\limits_{i \in \Lambda} I_i) \neq \phi$, so $A \cap I_i \neq \phi$ for all $i \in \Lambda$. Since $\{I_i : i \in \Lambda\}$ are $\mathcal{C}$-ternary hyperideals of S, $A \subseteq I_i$ for all $i \in \Lambda$. So $A \subseteq (\bigcap\limits_{i \in \Lambda} I_i)$. Hence $(\bigcap\limits_{i \in \Lambda} I_i)$ is a $\mathcal{C}$-ternary hyperideal. □

Proposition 7.3.6. *Let f be a good homomorphism from a ternary hypersemiring S to a ternary hypersemiring T and I, J be k-hyperideals of S and T respectively. Then the following are satisfied:*

(1) If I is a $\mathcal{C}$-ternary hyperideal of S containing the set $\{x \in S :$ there exist $a, b \in S$ such that $x = a + b$ and $f(a) = f(b)\}$ and f is an onto homomorphism, then $f(I)$ is a $\mathcal{C}$-ternary hyperideal of T.

(2) If J is a $\mathcal{C}$-ternary hyperideal of T, then $f^{-1}(J)$ is a $\mathcal{C}$-ternary hyperideal of S.

Proof. (1) Let $\prod_{i=1}^{2n+1} a_i \cap f(I) \neq \phi$ for some $a_1, a_2, ..., a_{2n+1} \in T$. So there exist $s_i \in S$ such that $f(s_i) = a_i$, $1 \leq i \leq 2n+1$. Then $\Pi_{i=1}^{2n+1} f(s_i) \cap f(I) = f(\Pi_{i=1}^{2n+1} s_i) \cap f(I) \neq \phi$, since f is a good homomorphism. So there exists $r \in \Pi_{i=1}^{2n+1} s_i$ such that $f(r) \in f(I)$. Thus $f(r) = f(i)$ for some $i \in I$, that implies $r + i \in I$. Since I is a k-hyperideal, $r \in I$. So $\Pi_{i=1}^{2n+1} s_i \cap I \neq \phi$. Thus $\Pi_{i=1}^{2n+1} s_i \subseteq I$, since I is a $\mathcal{C}$-ternary hyperideal of S. Hence $\Pi_{i=1}^{2n+1} a_i = \Pi_{i=1}^{2n+1} f(s_i) = f(\Pi_{i=1}^{2n+1} s_i) \subseteq f(I)$.

(2) Let $\Pi_{i=1}^{2n+1} s_i \cap f^{-1}(J) \neq \phi$ for some $s_1, s_2, ..., s_{2n+1} \in S$. Suppose $t \in \Pi_{i=1}^{2n+1} s_i \cap f^{-1}(J)$, then $f(t) \in f(\Pi_{i=1}^{2n+1} s_i) \cap J$. It follows that $\Pi_{i=1}^{2n+1} f(s_i) \cap J \neq \phi$. Since J is a $\mathcal{C}$-ternary hyperideal of T, $f(\Pi_{i=1}^{2n+1} s_i) = \Pi_{i=1}^{2n+1} f(s_i) \subseteq J$, which implies $\Pi_{i=1}^{2n+1} s_i \subseteq f^{-1}(J)$. So $f^{-1}(J)$ is a $\mathcal{C}$-ternary hyperideal of S. □

Definition 7.3.7. Let A be a hyperideal of a ternary hypersemiring $(S, +, \circ)$. The intersection of all prime hyperideals of S containing A is called prime radical or simply radical of A, denoted by $Rad(A)$. If the ternary hypersemiring S does not have any prime hyperideal containing I, define $Rad(I) = R$.

Example 7.3.8. For the set $X = \{10, 20\}$, the radicals of the hyperideals $5\mathbb{Z}_0^-$ and $6\mathbb{Z}_0^-$ in the ternary hypersemiring $(\mathbb{Z}_0^-, +, \circ)$ where '$\circ$' is defined by $a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c : x, y \in X\}$ are $\mathbb{Z}_0^-$, $3\mathbb{Z}_0^-$ respectively.

notation 7.3.9. $\Re(A) = \{a \in S : a^{2n+1} \subseteq A$, for some integers $n \geq 0\}$ for any hyperideal A of S.

Theorem 7.3.10. *Let A be a hyperideal of a commutative ternary hypersemiring $(S, +, \circ)$. Then $\Re(A)$ is a hyperideal of S containing A and $\Re(A) \subseteq Rad(A)$.*

Proof. Let $a, b \in \Re(A)$ be arbitrary. Then there exist $m, n \in \mathbb{Z}_0^+$ such that $a^{2m+1} \subseteq A$ and $b^{2n+1} \subseteq A$. If $m = n = 0$, then $\{a + b\} \subseteq A$, so $a + b \in \Re(A)$. If either $m > 0$ or $n > 0$, then $2m + 2n + 1 \geq 3$. Now $(a+b)^{2m+2n+1} \subseteq \sum\limits_{r=0}^{2m+2n+1} \binom{2m+2n+1}{r=0} a^{2m+2n+1-r} b^r$. If $2m + 2n + 1 - r < 2m + 1$, then $r \geq 2n + 1$. Otherwise, $2m + 2n + 1 - r \geq 2m + 1$. So in each case, either $a^{2m+2n+1-r} \subseteq A$ or $b^r \subseteq A$, thus $(a+b)^{2m+2n+1} \subseteq A$. Consequently, $a + b \in \Re(A)$. Again, for any $x, y \in S$ and $a \in \Re(A)$, there exists $n \in \mathbb{Z}_0^+$ such that $a^{2n+1} \subseteq A$. Now for any $t \in x \circ y \circ a$, $t^{2n+1} \subseteq (x \circ y \circ a)^{2n+1} = x^{2n+1} \circ y^{2n+1} \circ a^{2n+1} \subseteq A$, which implies $t \in \Re(A)$. So $x \circ y \circ a \subseteq \Re(A)$. Therefore, $\Re(A)$ is a hyperideal of S. Also for any $a \in A$, $a^1 = \{a\} \subseteq A \Rightarrow a \in \Re(A)$. Hence $A \subseteq \Re(A)$.
Let $a \in \Re(A)$, then $a^{2n+1} \subseteq A$ for some $n \in \mathbb{Z}_0^+$. Therefore for any prime hyperideal P of S containing A, $a^{2n+1} \subseteq P$ implies $a \in P$. So $a \in Rad(A)$ and hence $\Re(A) \subseteq Rad(A)$. □

Theorem 7.3.11. *Let A be a complete ternary k-hyperideal of a commutative ternary hyprsemiring $(S, +, \circ)$. Then $Rad(A) \subseteq \Re(A) = \{a \in S : a^{2n+1} \subseteq A$ for some integers $n \in \mathbb{Z}_0^+\}$.*

Proof. Let $p \notin \Re(A)$. Then $p^{2n+1} \nsubseteq A$ for any $n \in \mathbb{Z}_0^+$. Since A is a complete ternary k-hyperideal, $p^{2n+1} \cap A = \phi$ for all $n \in \mathbb{Z}_0^+$. Now consider $D = \cup\{p^{2n+1} + A$, for any $n \in \mathbb{Z}_0^+\}$. Let $a, b, c \in D$ be arbitrary. Then $a \circ b \circ c \subseteq p^{2m_1+1} \circ p^{2m_2+1} \circ p^{2m_3+1} + A \subseteq p^{2(m_1+m_2+m_3+1)+1} + A \subseteq D$. Since S contains hyperidentity, D is an m-system. Here $D \cap A = \phi$. If not, let $t \in D \cap A$, then $t = x + y$, where $x \in p^{2n+1}$ and $y \in A$. Thus $t \in A$ and $y \in A$. This implies $x \in A$ (since A is a k-hyperideal), which contradicts the fact that $p^{2n+1} \cap A = \phi$ for any $n \in \mathbb{Z}_0^+$. Therefore $D \cap A = \phi$. Hence (*cf.* Theorem 7.1.17) there is a prime hyperideal P containing A and disjoint from D. So $p^{2n+1} \cap P = \phi$ for any $n \in \mathbb{Z}_0^+$. Thus $p \notin P \Rightarrow p \notin Rad(A)$, consequently $Rad(A) \subseteq \Re(A)$. □

Proposition 7.3.12. *Let A be a $\mathcal{C}$-ternary hyperideal of a ternary hypersemiring $(S, +, \circ)$. Then $Rad(A)$ is a $\mathcal{C}$-ternary hyperideal of S.*

Proof. Let $a_1 \circ a_2 \circ a_3 \circ \dots \circ a_{2n+1} \cap Rad(A) \neq \phi$ for some $a_1, a_2, a_3 ..., a_{2n+1} \in S$ and integers $n \in \mathbb{Z}_0^+$. Then there exists $x \in a_1 \circ a_2 \circ a_3 \circ \dots \circ a_{2n+1}$ such that $x^{2m+1} \subseteq A$, where $m \in \mathbb{Z}_0^+$. Also $x^{2m+1} \subseteq (a_1 \circ a_2 \circ a_3 \circ \dots \circ a_{2n+1})^{2m+1}$ which implies $(a_1 \circ a_2 \circ a_3 \circ \dots \circ a_{2n+1})^{2m+1} \cap A \neq \phi$. Since A is a $\mathcal{C}$-ternary hyperideal of S, $(a_1 \circ a_2 \circ a_3 \circ \dots \circ a_{2n+1})^{2m+1} \subseteq A$. Now for any $y \in a_1 \circ a_2 \circ a_3 \circ \dots \circ a_{2n+1}$, $y^{2m+1} \subseteq A$, whence $y \in Rad(A)$, i.e., $a_1 \circ a_2 \circ a_3 \circ \dots \circ a_{2n+1} \subseteq Rad(A)$. Thus $Rad(A)$ is a $\mathcal{C}$-ternary hyperideal of S. □

Proposition 7.3.13. *Let A, B and C be hyperideals of a ternary hypersemiring S. Then*

(1) $A \subseteq Rad(A)$.

(2) $A \subseteq B \Rightarrow Rad(A) \subseteq Rad(B)$.

(3) $Rad(Rad(A)) = Rad(A)$.

(4) $Rad(A) = Rad(A^{2n+1})$ for any $n \in \mathbb{Z}_0^+$.

(5) $Rad(A + B) = Rad(Rad(A) + Rad(B))$.

(6) If S is commutative and A, B and C are complete ternary k-hyperideals of S, then $Rad(ABC) = Rad(A \cap B \cap C) = Rad(A) \cap Rad(B) \cap Rad(C)$.

Proof. (1) Follows immediately from the Definition 7.3.7.

(2) Suppose $A \subseteq B$. Then any prime hyperideal P containing B also contains A. Therefore, $Rad(A) \subseteq Rad(B)$.

(3) By (1) and (2), $A \subseteq Rad(A) \Rightarrow Rad(A) \subseteq Rad(Rad(A))$.
Now let $x \in Rad(Rad(A))$ and $\{P_i\}_{i \in I}$ be the collection of all prime hyperideals containing A, then $Rad(A) \subseteq P_i$ for all $i \in I$. So $x \in Rad(Rad(A)) \subseteq P_i$ for all $i \in I$. Hence $x \in Rad(A)$. Therefore, $Rad(Rad(A)) = Rad(A)$.

(4) Since A is a hyperideal of S, $A^{2n+1} \subseteq A$ for all $n \in \mathbb{Z}_0^+$. By (2), $Rad(A) \supseteq Rad(A^{2n+1})$. Now let, $x \in Rad(A)$. So x is in the set of all prime hyperideals containing A. If possible, let $x \notin Rad(A^{2n+1})$. Then there exists a prime hyperideal P containing A^{2n+1} and $x \notin P$. Here $A^{2n+1} \subseteq P$ which implies $A \subseteq P$. Since P is a prime hyperideal, which contradicts the fact that x is in the set of all prime hyperideals containing A. Hence $Rad(A) = Rad(A^{2n+1})$ for any $n \in \mathbb{Z}_0^+$.

(5) We have $A \subseteq Rad(A)$ and $B \subseteq Rad(B)$. So $A + B \subseteq Rad(A) + Rad(B)$. Thus by (2), $Rad(A + B) \subseteq Rad(Rad(A) + Rad(B))$. Again $A \subseteq A + B$ and $B \subseteq A + B$ implies $Rad(A) \subseteq Rad(A + B)$ and $Rad(B) \subseteq Rad(A + B)$ whence $Rad(A)+Rad(B) \subseteq Rad(A+B)$. Thus by (2) and (3), $Rad(Rad(A)+Rad(B)) \subseteq Rad(Rad(A + B)) = Rad(A + B)$. Therefore, $Rad(A + B) = Rad(Rad(A) + Rad(B))$.

(6) Clearly $ABC \subseteq A \cap B \cap C$. Then by (2) $Rad(ABC) \subseteq Rad(A \cap B \cap C)$. Let $x \in Rad(A \cap B \cap C)$. Then there exists $m \in \mathbb{Z}_0^+$ such that $x^{2m+1} \subseteq A \cap B \cap C$. Then $x^{6m+3} = x^{2m+1} \circ x^{2m+1} \circ x^{2m+1} \subseteq ABC$ which implies $x \in Rad(ABC)$. Hence $Rad(ABC) = Rad(A \cap B \cap C)$. Now let $x \in Rad(A \cap B \cap C)$. Then there exists $n \in \mathbb{Z}_0^+$ such that $x^{2m+1} \subseteq (A \cap B \cap C)$. Therefore $x^{2m+1} \subseteq A$, $x^{2m+1} \subseteq B$, $x^{2m+1} \subseteq C$. This implies $x \in Rad(A)$, $x \in Rad(B)$, $x \in Rad(C)$. So $x \in Rad(A) \cap Rad(B) \cap Rad(C)$. Conversely, let $x \in Rad(A) \cap Rad(B) \cap Rad(C)$. Thus there exist $r, s, t \in \mathbb{Z}_0^+$ such that $x^{2r+1} \subseteq A$, $x^{2s+1} \subseteq B$ $x^{2t+1} \subseteq C$. So $x^{(2r+1)(2s+1)(2t+1)} \subseteq A \cap B \cap C$, that implies $x \in Rad(A \cap B \cap C)$. Consequently, $Rad(A) \cap Rad(B) \cap Rad(C) \subseteq Rad(A \cap B \cap C)$. Hence $Rad(A \cap B \cap C) = Rad(A) \cap Rad(B) \cap Rad(C)$.

□

Proposition 7.3.14. *Let I be a hyperideal in a commutative ternary hypersemiring S, then $Rad(I) = Rad(\Re(I))$.*

Proof. Since $I \subseteq \Re(I)$, Proposition 7.3.13(2) implies the inclusion $Rad(I) \subseteq Rad(\Re(I))$. Now for reverse inclusion, let P be any prime hyperideal containing I. Then it is sufficient to show that $\Re(I) \subseteq P$. Consider $x \in \Re(I)$. Then $x^{2n+1} \subseteq I \subseteq P$ for some integer $n \in \mathbb{Z}_0^+$. So $x \in P$, which implies $\Re(I) \subseteq P$. Thus $Rad(I) = Rad(\Re(I))$. □

Theorem 7.3.15. *Let S_1 and S_2 be commutative ternary hypersemirings, $f : S_1 \to S_2$ be a good homomorphism and I be a k-hyperideal of S_2. Then $f^{-1}(Rad(I)) = Rad(f^{-1}(I))$.*

Proof. Let $x \in f^{-1}(Rad(I))$. Then $f(x) \in Rad(I)$. So there exists an integer $n \in \mathbb{Z}_0^+$ such that $f^{2n+1}(x) = f(x^{2n+1}) \subseteq I$, that implies $x^{2n+1} \subseteq f^{-1}(I)$. Hence $x \in$ Rad $(f^{-1}(I))$.

Conversely, let $x \in Rad(f^{-1}(I))$. Then there exists an integer $n \in \mathbb{Z}_0^+$ such that $x^{2n+1} \in (f^{-1}(I))$. Thus $f^{2n+1}(x) = f(x^{2n+1}) \subseteq I$. So $f(x) \in Rad(I)$ and hence

$x \in f^{-1}(Rad(I))$. Thus $Rad(f^{-1}(I)) \subseteq f^{-1}(Rad(I))$. Therefore, $f^{-1}(Rad(I)) = Rad(f^{-1}(I))$. □

Theorem 7.3.16. *Let S_1 and S_2 be commutative ternary hypersemirings, $f : S_1 \to S_2$ be a good epimorphism and I be a k-hyperideal of S_1 such that $\{x \in S_1$: there exist $a, b \in S_1$ such that $x = a + b$ and $f(a) = f(b)\} \subseteq I$. Then $f(Rad(I)) = Rad(f(I))$.*

Proof. Let $x \in f(Rad(I))$. Then there exists $a \in Rad(I)$ such that $f(a) = x$. So there exists $m \in \mathbb{Z}_0^+$ such that $a^{2m+1} \subseteq I$. Now $x^{2m+1} = (f(a))^{2m+1} = f(a^{2m+1}) \subseteq f(I)$, since $a^{2m+1} \subseteq I$. Thus $x \in Rad(f(I))$. Hence $f(Rad(I)) \subseteq Rad(f(I))$.

For the converse part, let $x \in Rad(f(I))$. So $x^{2n+1} \subseteq f(I)$ for some $n \in \mathbb{Z}_0^+$. Also there exists an element $a \in S$ such that $f(a) = x$. Now $f(a^{2n+1}) = (f(a))^{2n+1} = x^{2n+1} \subseteq f(I)$, then for any element $p \in a^{2n+1}$, there is an element $i \in I$ such that $f(p) = f(i)$. Thus by the given condition $p + i \in I$, and hence $p \in I$, since I is a k-hyperideal. So $a^{2n+1} \subseteq I$, which implies $a \in Rad(I)$. Thus $x = f(a) \in f(Rad(I))$. □

Definition 7.3.17. A hyperideal A of a ternary hypersemiring S is called primary hyperideal of S if for $a, b, c \in S$, $abc \subseteq A$ and $a \notin A, b \notin A$ then there exists an integer $n \in \mathbb{Z}_0^+$ such that $c^{2n+1} \subseteq A$.

Theorem 7.3.18. *Let A be a primary $\mathcal{C}$-ternary hyperideal of a commutative ternary hypersemiring $(S, +, \circ)$, then $Rad(A)$ is a prime hyperideal of S.*

Proof. Let $a \circ b \circ c \subseteq Rad(A)$ and $a \notin Rad(A)$, $b \notin Rad(A)$. Now for any element $x \in a \circ b \circ c$, there exists an integer $n \in \mathbb{Z}_0^+$ such that $x^{2n+1} \subseteq A$. That implies $x^{2n+1} \subseteq (a \circ b \circ c)^{2n+1} = a^{2n+1} \circ b^{2n+1} \circ c^{2n+1}$. So $a^{2n+1} \circ b^{2n+1} \circ c^{2n+1} \cap A \neq \phi$. Since A is a $\mathcal{C}$-ternary hyperideal, $a^{2n+1} \circ b^{2n+1} \circ c^{2n+1} \subseteq A$. Now $a \notin Rad(A)$ and $b \notin Rad(A)$ implies $a^{2n+1} \cap A = \phi$ and $b^{2n+1} \cap A = \phi$. For any $p \in a^{2n+1}$, $q \in b^{2n+1}$ we have $p \notin A$ and $q \notin A$. Here $p \circ q \circ r \subseteq a^{2n+1} \circ b^{2n+1} \circ c^{2n+1} \subseteq A$, where $r \in c^{2n+1}$. Since A is a primary hyperideal, there exists an integer $m \in \mathbb{Z}_0^+$ such that $r^{2m+1} \subseteq A$. Also $r^{2m+1} \subseteq (c^{2n+1})^{2m+1}$. Hence $(c^{2n+1})^{2m+1} \cap A \neq \phi$ implies $(c^{2n+1})^{2m+1} \subseteq A$ whence $c \in Rad(A)$. So $Rad(A)$ is a prime hyperideal of S. □

Theorem 7.3.19. *Let I be a proper hyperideal of a ternary hypersemiring $(S, +, \circ)$ then $Rad(I) = \{s \in S$: every m-system in S which contains s has a non-empty intersection with $I\}$.*

Proof. Consider $\Omega = \{s \in S$: every m-system in S which contains s has a non-empty intersection with $I\}$. Let $x \in Rad(I)$ and $\{P_\lambda : \lambda \in \Lambda\}$ be the collection of all prime hyperideals of S containing I. Then $x \in P_\lambda$ for all $\lambda \in \Lambda$. If possible, let there exists an m-system A which contains x and has empty intersection with I. Then by Theorem 7.1.17, there exists a prime hyperideal P_λ such that $A \cap P_\lambda = \phi$. Since $x \in P_\lambda$, we arrive at a contradiction. So $Rad(I) \subseteq \Omega$.

Conversely, let $x \in \Omega$ and $\{P_\lambda : \lambda \in \Lambda\}$ be the collection of all prime hyperideals of S containing I. If possible, let $x \notin Rad(I)$. Then there exists $\lambda \in \Lambda$ such that $x \notin P_\lambda$. By Theorem 7.1.11, P_λ^c is an m-system of S which contains x and has an empty intersection with I, which is a contradiction. Therefore, $\Omega \subseteq Rad(I)$. □

Definition 7.3.20. Let A be a primary complete ternary k-hyperideal. A is called P-primary complete ternary k-hyperideal whenever $Rad(A) = P$ is a prime hyperideal of a commutative ternary hypersemiring S.

Example 7.3.21. In the ternary hypersemiring $(\mathbb{Z}_0^-, +, \circ)$ where hyperoperation '$\circ$' is defined by $a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c : x, y \in 3\mathbb{Z}_0^-\}$, $P = 2\mathbb{Z}_0^-$ is a prime hyperideal. Here the primary complete ternary k-hyperideal $8\mathbb{Z}_0^-$ is a P-primary complete ternary k-hyperideal, because $Rad(8\mathbb{Z}_0^-) = P$.

Proposition 7.3.22. *If A is a complete ternary k-hyperideal and P be a hyperideal of a commutative ternary hypersemiring $(S, +, \circ)$, then A is a P-primary complete ternary k-hyperideal of S if and only if*

(1) $A \subseteq P \subseteq Rad(A)$ and

(2) $a \circ b \circ c \subseteq A$; $a, b \notin A$ implies $c \in P$.

Proof. If A is a P-primary complete ternary k-hyperideal then clearly the conditions (1) and (2) are satisfied. For the converse part, let $a \circ b \circ c \subseteq A$ and $a, b \notin A$. Then by the given conditions, $c \in P \subseteq Rad(A)$ which implies $c^{2n+1} \subseteq A$ for some integer $n \in \mathbb{Z}_0^+$. So A is a primary hyperideal. To show that $Rad(A) = P$, let $x \in Rad(A)$. Then there exists a least positive integer m such that $x^{2m+1} \subseteq A$. If $m = 0$ then by (1), $x \in P$. If $m \geq 1$, then $x^{2m-1} \nsubseteq A$. Since A is a $\mathcal{C}$-ternary hyperideal, $x^{2m-1} \cap A = \phi$. Now let $y, z \in x^{2m-1}$, then $y \circ z \circ x \subseteq x^{2m-1} \circ x^{2m-1} \circ x \subseteq A$. So by (2), $x \in P$. Hence by (1), $P = Rad(A)$ and thus A is a P-primary complete ternary k-hyperideal of S. □

Proposition 7.3.23. *Let A be a proper hyperideal of ternary hypersemiring S. Then A is a primary hyperideal of S if and only if for any hyperideals I, J, K of S, if $IJK \subseteq A$, $I \nsubseteq A$, $J \nsubseteq A$ implies $K \subseteq \Re(A)$.*

Proof. Let A be a primary hyperideal such that $IJK \subseteq A$, $I \nsubseteq A, J \nsubseteq A$. Then there exist $i \in I, j \in J$ such that $i \notin A$ and $j \notin A$. Take $k \in K$. Now $ijk \subseteq IJK \subseteq A$ implies there exists integer $n \in \mathbb{Z}_0^+$ such that $k^{2n+1} \subseteq A$ i.e., $k \in \Re(A)$. Therefore, $K \subseteq \Re(A)$. Conversely, let $a \circ b \circ c \subseteq A$, $a \notin A$, $b \notin A$. Since $\langle a\rangle \circ \langle b\rangle \circ \langle c\rangle \subseteq \langle a \circ b \circ c\rangle \subseteq A$ and $\langle a\rangle \nsubseteq A, \langle b\rangle \nsubseteq A$. Then $\langle c\rangle \subseteq \Re(A)$. Thus $c^{2n+1} \subseteq A$. So A is primary. □

Proposition 7.3.24. *Let f be a good homomorphism from a ternary hypersemiring S to the ternary hypersemiring T and I, J be k-hyperideals of S and T respectively. Then the following are satisfied:*

(1) If I is a primary hyperideal of S such that $\{x \in S :$ there exist $a, b \in S$ such that $x = a + b$ and $f(a) = f(b)\} \subseteq I$ and f is an epimorphism, then $f(I)$ is a primary hyperideal of T.

(2) If J is a primary hyperideal of T, then $f^{-1}(J)$ is a primary hyperideal of S.

Proof. (1) Let $a \circ b \circ c \subseteq f(I)$ for some $a, b, c \in T$ and $a \notin f(I), b \notin f(I)$. As f is an onto homomorphism, there exist a_1, b_1, $c_1 \in S$ such that $f(a_1) = a, f(b_1) = b, f(c_1) = c$, where $a_1 \notin I$, $b_1 \notin I$. Here $f(a_1 \circ b_1 \circ c_1) = f(a_1)f(b_1)f(c_1) \subseteq f(I)$. Then for any $x \in a_1 \circ b_1 \circ c_1$, there exists $i \in I$ such that $f(x) = f(i)$. Thus $x + i \in I$ and have $x \in I$, since I is s k-hyperideal of S. Therefore, $a_1 \circ b_1 \circ c_1 \subseteq I$ and $a_1 \notin I, b_1 \notin I$ implies $c_1^{2n+1} \subseteq I$ for some $n \in \mathbb{Z}_0^+$. So $c^{2n+1} = f(c_1^{2n+1}) \subseteq f(I)$. Hence $f(I)$ is a primary hyperideal of T.

(2) Let J is a primary hyperideal of T. Consider $a \circ b \circ c \subseteq f^{-1}(J)$ for some a, b, $c \in S$ and $a \notin f^{-1}(J)$, $b \notin f^{-1}(J)$. Now $f(a) \circ f(b) \circ f(c) = f(a \circ b \circ c) \subseteq J$ and $f(a) \notin J$, $f(b) \notin J$. As J is a primary hyperideal of T, $f(c^{2n+1}) = f(c)^{2n+1} \subseteq J$ for some $n \in \mathbb{Z}_0^+$. So $c^{2n+1} \subseteq f^{-1}(J)$. Consequently, $f^{-1}(J)$ is a primary hyperideal of S. □

Theorem 7.3.25 (The Prime Avoidance Theorem)**.** *Let I be an arbitrary hyperideal in a ternary hypersemiring $(S, +, \circ)$ and P_1, P_2, …, P_n be k-hyperideals of S such that at least $n - 2$ of which are $\mathcal{C}$-ternary hyperideals as well as completely prime hyperideals. If $I \subseteq P_1 \cup P_2 \cup ... \cup P_n$, then $I \subseteq P_i$, for some i.*

Proof. The proof is by induction on $n \geq 2$. For $n = 2$ suppose $I \subseteq P_1 \cup P_2$. If $I \nsubseteq P_1$, then there exists $x \in I$ such that $x \notin P_1$. Since $I \subseteq P_1 \cup P_2$, $x \in P_2$. Take $y \in I \cap P_1$. Then $x + y \in I \subseteq P_1 \cup P_2$. If $x + y \in P_1$, then $x \in P_1$(since P_1 is a k-hyperideal), which is a contradiction. Thus $x + y \in P_2$ which implies $y \in P_2$. So $I \cap P_1 \subseteq P_2$. Now $I = (I \cap P_1) \cup (I \cap P_2) \subseteq P_2$. So either $I \subseteq P_1$ or $I \subseteq P_2$.

Assume the result is true for $n-1$, $n \geq 3$. Let $I \subseteq P_1 \cup P_2 \cup ... \cup P_n$, where at least $n-2$ of the P_i are completely prime. Suppose that $I \nsubseteq P_1 \cup P_2 \cup ..P_{i-1} \cup P_{i+1}.. \cup P_n$ for all i. Then there exists $x_i \in I$ such that $x_i \notin P_j$ for all $i \neq j$. So we must have $x_i \in P_i$. Since $n \geq 3$, at least one of the P_i's is completely prime hyperideal. Without loss of generality, let us assume that P_1 is a completely prime hyperideal. Consider the set $x = \{x_1\} + x_2^{n+1} \circ x_3 \circ ... \circ x_n \subseteq I \subseteq P_1 \cup P_2 \cup ... \cup P_n$. Here $x_2^{n+1} \circ x_3 \circ ... \circ x_n \subseteq P_i$, $i \neq 1$ (since P_i is a hyperideal and $x_i \in P_i$). Now for each $y \in x_2^{n+1} \circ x_3 \circ ... \circ x_n$, $x_1 + y \in P_i$ for some i. If for $i \geq 2$, $x_1 + y \in P_i$, then $x_1 \in P_i$, which is a contradiction. Then $x_1 + y \in P_1$ and thus $y \in P_1$. So $(x_2^{n+1} \circ x_3 \circ ... \circ x_n) \cap P_1 \neq \phi$, which implies $(x_2^{n+1} \circ x_3 \circ ... \circ x_n) \subseteq P_1$. Hence $x_k \in P_1$ for some $k = 2, 3, .., n$, which is also a contradiction. Thus $I \subseteq P_1 \cup P_2 \cup ..P_{i-1} \cup P_{i+1}.. \cup P_n$ for some i. By induction assumption, $I \subseteq P_i$, for some i. □

Definition 7.3.26. Let $I, I_1, I_2, ..., I_n$ be hyperideals of a ternary hypersemiring S. The collection $\{I_1, I_2, ..., I_n\}$ is said to be a cover of I if $I \subseteq I_1 \cup I_2 \cup ... \cup I_n$. We call such a cover of I is efficient, if I is not contained in the union of any $n-1$ of the hyperideals $I_1, I_2, ..., I_n$.

Proposition 7.3.27. *Let $(S, +, \circ)$ be a commutative ternary hypersemiring and let $\{Q_1, Q_2, ..., Q_n\}$ be an efficient cover of the hyperideal I, where $Q_1, Q_2, ..., Q_n$ are k-hyperideals of S. If $Rad(Q_i) \nsubseteq Rad(Q_j)$ for each $i \neq j$, then no Q_k is a primary hyperideal of S.*

Proof. We first prove that for efficient covering $I \subseteq Q_1 \cup Q_2 \cup .... \cup Q_n$ of I, $\cap_{i \neq k} Q_i = \cap_{i=1}^{n} Q_i$ for all k. Let $x \in \cap_{i \neq k} Q_i$, since the cover is efficient, there exists $x_k \in Q_k \cap I$ such that $x_k \notin \cup_{i \neq k} Q_i$. Now consider the element $x + x_k$ in I. If $x + x_k \in Q_i, i \neq k$ then $x_k \in Q_i$ for all $i \neq k$, which is a contradiction. Then $x + x_k \in Q_k$ and thus $x \in Q_k$. So $\cap_{i \neq k} Q_i = \cap_{i=1}^{n} Q_i$. Now if possible, let Q_k be a primary hyperideal of S. Here $I \circ I \circ Q_1^{n+1} \circ Q_2 \circ .. \circ Q_{k-1} \circ Q_{k+1} \circ .. \circ Q_n \subseteq Q_i$ for all $i \neq k$. Since $I \cap (\cap_{i=1}^{n} Q_i) = I \cap (\cap_{i \neq k} Q_i) \subseteq I \cap Q_k \subseteq Q_k$ we get $I \circ I \circ Q_1^{n+1} \circ Q_2 \circ .. \circ Q_{k-1} \circ Q_{k+1} \circ .. \circ Q_n \subseteq Q_k$. As $I \nsubseteq Q_k$, by Proposition 7.3.23, $Q_i \subseteq \Re(Q_k)$. Whence $Rad(Q_i) \subseteq Rad(\Re(Q_k)) = Rad(Q_k)$ (by Proposition 7.3.14), which contradicts the hypothesis. □

Now by using Proposition 7.3.27, we obtain the following Theorem.

Theorem 7.3.28 (The Primary Avoidance Theorem). *Let I be an arbitrary hyperideal in a commutative ternary hypersemiring $(S, +, \circ)$ and $Q_1, Q_2, ..., Q_n$ be k-hyperideals* *of S such that at least $n-2$ of which are primary hyperideals. If $I \subseteq Q_1 \cup Q_2 \cup ... \cup Q_n$ and $Rad(Q_i) \nsubseteq Rad(Q_j)$ for each $i \neq j$, then $I \subseteq Q_i$, for some i.*

Proof. We may assume that cover is efficient since the hypothesis remains valid if we convert it to an efficient covering. By Proposition 7.3.27, $n \leqslant 2$. For $n = 2$, $I \subseteq Q_1 \cup Q_2$ implies either $I \subseteq Q_1$ or $I \subseteq Q_2$, which contradicts the cover is efficient. So we have $n = 1$. □

In the next Theorem, we extend the Primary Avoidance Theorem for class of complete ternary hyperideals in a ternary hypesemiring S.

Theorem 7.3.29 (Extended Version of Primary Avoidance Theorem)**.** *Let S be a commutative ternary hypesemiring and P_1, P_2,....,P_n be $\mathcal{C}$-ternary primary k-hyperideals of S such that $Rad(P_i) \nsubseteq Rad(P_j)$ for all $i \neq j$. Let T be a hyperideal of S such that $aSS + T \nsubseteq \cup_{i=1}^{n} P_i$ for some $a \in S$. Then there exists a subset T_1 of T such that $a + T_1 \nsubseteq \cup_{i=1}^{n} P_i$.*

Proof. Assume that a lies in all of P_1, P_2,....,P_k but none of P_{k+1},....,P_n. If $k = 0$, then $a + 0 \notin \cup_{i=1}^{n} P_i$. So consider $k \geq 1$. Now $T \nsubseteq \cup_{i=1}^{k} P_i$. If $T \subseteq \cup_{i=1}^{k} P_i$, by Theorem 7.3.28, $T \subseteq P_i$ for some $1 \leq i \leq k$. Thus $aSS + T \subseteq P_i \subseteq \cup_{i=1}^{n} P_i$, which is a contradiction. So there exists an element $p \in T$ such that $p \notin \cup_{i=1}^{k} P_i$. Also $P_{k+1} \cap .... \cap P_n \nsubseteq P_1 \cup P_2 \cup ... \cup P_k$. If $P_{k+1} \cap .... \cap P_n \subseteq P_1 \cup P_2 \cup ... \cup P_k$, then by Theorem 7.3.28, $P_{k+1} \cap .... \cap P_n \subseteq P_j$ for some $1 \leqslant j \leqslant k$. Thus, $Rad(P_{k+1}) \cap ... \cap Rad(P_n) = Rad(P_{k+1} \cap .... \cap P_n) \subseteq Rad(P_j)$ (*cf.* Proposition 7.3.13). Since $(Rad(P_{k+1}))^{n-k} Rad(P_{k+2}) ... Rad(P_n) \subseteq Rad(P_{k+1} \cap .... \cap P_n) \subseteq Rad(P_j)$ and $Rad(P_j)$ is a prime hyperideal, by Theorem 7.3.18, we have $Rad(P_l) \subseteq Rad(P_j)$ for $k+1 \leqslant l \leqslant n$, which contradicts the hypothesis. Thus, there exists $c \in P_{k+1} \cap .... \cap P_n$ such that $c \notin P_1 \cup P_2 \cup ... \cup P_k$. Now $p \circ c \circ c \subseteq T$ and $p \circ c \circ c \subseteq P_{k+1} \cap .... \cap P_n$ but $p \circ c \circ c \nsubseteq P_1 \cup P_2 \cup ... \cup P_n$. If $p \circ c \circ c \subseteq P_1 \cup P_2 \cup ... \cup P_k$, then $p \circ c \circ c \subseteq P_i$ for some $1 \leqslant i \leqslant k$, since P_1, P_2,....,P_k are $\mathcal{C}$-ternary hyperideals. This implies either $p \in Rad(P_i)$ or $c \in P_i$, which is also a contradiction. Consider $T_1 = p \circ c \circ c$, then $a + T_1 \nsubseteq \cup_{i=1}^{n} P_i$. Since each P_i is a $\mathcal{C}$-ternary primary k-hyperideal of S and $a \in \cup_{i=1}^{k} P_i - \cup_{j=k+1}^{n} P_j$, we have $T_1 \subseteq \cup_{j=k+1}^{n} P_j - \cup_{i=1}^{k} P_i$. □

Scope of Further Study

Here I pose the following problems which raises from this thesis for further studies.

1. In Chapter 1, we introduced and studied the concept of 2-prime and weakly 2-prime ideals in a commutative semiring. So it is interesting to classify semirings where weakly 2-prime ideals are 2-prime.

2. Also we observe that every maximal ideal of any semiring (if exists) is 2-prime but the converse is not true. So it is interesting to consider the problem of classifying semirings where 2-prime ideals are maximal.

3. The concept of 1-absorbing prime ideals in semiring can be generalized in hyperstructure, like multiplicative hypersemiring.

4. The concept of studying n-ideals of a semiring in Chapter 3, may be extended into ternary and hyper ternary semiring.

5. In Chapter 4, we introduced the concept of 3-prime ideals in a ternary semiring as a generalization of 2-prime ideals in a semiring. One can study the concept of 3-prime ideals in a ternary hypersemiring.

6. In Chapter 6, we introduced an equivalence relation δ^* on a ternary hypersemiring S, so that the quotient structure is a ternary semiring. One can try to introduce an equivalence relation λ on S (say), so that the quotient S/λ is a commutative ternary semiring.

7. In Chapter 7, prime hyperideals, radical of hyperideals and primary hyperideals have been introduced and studied. The prime and primary avoidance theorems

for $\mathcal{C}$-ternary hyperideals in ternary hypersemirings have been generalized. There is a huge scope of further study on ternary hypersemiring in terms of prime, primary hyperideals and their generalizations. Moreover, the results obtained in this chapter can be extended to some other algebraic systems like gamma-semirings, partially ordered ternary semiring, etc. also to fuzzy and intuitionistic fuzzy settings.

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List of Publications Based on the Thesis

A list of publications resulted from the work of this thesis has been appended below.

(1) **Sampad Das, Manasi Mandal and Nita Tamang**, *On some properties of hyperideals in ternary hypersemirings*, Asian-European Journal of Mathematics, **16(1)** (2023), 2250230.

(2) **Sampad Das, Manasi Mandal and Nita Tamang**, *On 3-Prime and Quasi 3-Primary Ideals of Ternary Semirings*, Discussiones Mathematicae - General Algebra and Applications (Accepted)

(3) **Sampad Das, Biswaranjan Khanra, Manasi Mandal**, *A note on 2-prime and n-weakly 2-prime ideals of semirings*, Quasigroups and Related Systems, **30** (2022), 241-256.

(4) **Sampad Das, Manasi Mandal and Nita Tamang**, *On Primary Hyperideals of Ternary Hypersemiring*, Journal of Hyperstructure, **10(1)** (2021), 22-37.

(5) **Sampad Das, Manasi Mandal and Nita Tamang**, *Left Bi-quasi and Minimal Left Bi-quasi Ideals of Ternary Semiring*, Southeast Asian Bulletin of Mathematics, **45** (2021), 217-226.

(6) **Sampad Das and Manasi Mandal**, *On 1-absorbing Prime and Weakly 1-absorbing Prime Ideals in Commutative Semiring* (Communicated).

(7) **Sampad Das, Manasi Mandal and Nita Tamang**, *A note on n-ideals in Commutative Semirings* (Communicated).

(8) **Sampad Das and Manasi Mandal**, *Fundamental Relation on Ternary Hypersemirings* (Communicated).

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