# Study of Some Aspects of Clean Semirings

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# CERTIFICATE FROM THE SUPERVISOR(S)

This is to certify that the thesis entitled "**Study of some aspects of clean semirings**" submitted by Sri **Debapriya Das** who got his name registered on $26^{th}$ **March, 2021** for the award of Ph. D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Sukhendu Kar**, Dept. of Mathematics, Jadavpur University and that neither this thesis nor any part of it has been submitted for either any degree/ diploma or any other academic award anywhere before.

28/6/2024

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**(Signature of the Supervisor(s) date with official seal)**

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*Dedicated to*

*my parents*

*Dilip Kumar Das*

*and*

*Manjula Das*

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# Abstract

Rings in which every element is the sum of an idempotent and a unit are said to be clean rings and this notion was introduced by W.K. Nicholson in the study of exchange rings. Since then various generalizations of clean rings have been obtained by many authors. The algebraic theory of semirings has experienced remarkable growth in recent years. A semiring, which extends the concepts of a ring and a distributive lattice, has seen significant development. In this thesis we have introduced the concept of clean semiring and exchange semiring as a generalization of clean ring and exchange ring. We also aim to shed light on different generalizations of clean semiring. Here some characterizations of certain classes of clean semirings are studied.

First, the concept of clean semiring and exchange semiring is discussed. The concept of clean ring was introduced by W.K. Nicholson in his study of exchange rings. In this thesis we have introduced the concept of clean semiring as a generalisation of clean ring. A semiring is said to be clean if its every nonzero element can be written as the sum of an idempotent and a unit. We have studied the notion of clean semiring and obtained some important characterizations of clean semiring. We have also studied the notion of exchange semiring and found out the connection between clean semiring and exchange semiring.

In this thesis, we have introduced and studied the concept of strongly clean semiring. Let S be a semiring. An element $a \in S$ is called strongly clean if a = e+u with e an idempotent in S and u a unit in S such that eu = ue. A semiring S is said to be strongly clean if every nonzero element of S is strongly clean. We have mainly studied the notion of strongly clean semiring and obtained some important characterizations of strongly clean semiring in connection with exchange semiring, antisimple semiring

and additively regular semiring.

We have also introduced the concept of $k$-unit clean semiring which generalizes the notion of clean ring as well as clean semiring. Let $S$ be a semiring with identity 1. An element $a \in S \setminus \{0\}$ is said to be a $k$-unit if there exist $r_1, r_2 \in S$ such that $1 + r_1 a = r_2 a$ and $1 + a r_1 = a r_2$. Basically, $k$-units are the generalization of units in a semiring with zero element. An element $a \in S$ is called $k$-unit clean if $a = e + u$, where $e$ is an idempotent and $u$ is a $k$-unit of $S$. A semiring $S$ is said to be $k$-unit clean if every nonzero element can be written as the sum of an idempotent and a $k$-unit. We have obtained some important characterizations of $k$-unit clean semiring in connection with exchange semiring, antisimple semiring and inverse semiring.

Let $S$ be a semiring. An element $r \in S$ is said to be $k$-regular element if there exist $r_1, r_2 \in S$ such that $r + r r_1 r = r r_2 r$. An element $a \in S$ is called $k$-regular clean if $a = e + r$, where $e$ is an idempotent and $r$ is a $k$-regular element of $S$. A semiring $S$ is said to be a $k$-regular clean semiring if every element can be written as the sum of an idempotent and a $k$-regular element of $S$. In this thesis, we have introduced the concept of $k$-regular clean semiring which generalizes the notion of clean ring, clean semiring and $k$-unit clean semiring. We have obtained some important characterizations of $k$-regular clean semiring in connection with antisimple semiring, additively inverse semiring and zeroic semiring.

Motivated by the work of $k$-unit clean semiring and clean index of a ring, in this thesis, we have introduced the concept of $k$-unit clean index of a semiring $S$. If $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ be the formal triangular matrix semiring and $k$-unit clean index of $S$ is finite then, in this thesis, we have determined $k$-unit clean index of $A$ and $B$. Finally, we have characterized the semirings of $k$-unit clean indices 1 and 2, with the help of some other class of semirings.

We have introduced the concept of nil clean semiring. A semiring is said to be nil clean semiring if every element can be written as the sum of an idempotent and a nilpotent element. We have obtained some important results of nil clean semiring in connection with duo semiring, $k$-duo semiring, exchange semiring, $k$-semipotent semiring.

We have introduced the concept of nil clean index of a semiring $S$. If $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ be the formal triangular matrix semiring and nil clean index of $S$ is finite then we have determined nil clean index of semirings $A$ and $B$. We have also determined the relation between nil clean index and $k$-unit clean index of a semiring $S$. Finally, we have characterized the semirings of nil clean indices 1 and 2, with the help of some other class of semirings.

# Contents

# Chapter 1

# Introduction

# Chapter 1

# Introduction

## 1.1 Introduction

Ring theory plays a role in advanced algebra, both in theoretical exploration and practical application. Advanced algebra is characterized by its emphasis on the systematic investigation of abstract algebraic structures. Several works have been done on different generalizations of ring theory. Among them, the study of semirings has become a great interest of the recent researchers. The algebraic theory of semirings has experienced remarkable growth in recent years. A semiring, which extends the concept of a ring and a distributive lattice, has seen significant development. The concept of semirings was first introduced by Vandiver in 1934. Semirings are algebraic structures that are found in various branches of mathematics and computer science. The study of semirings has evolved since the 1950s and they play a crucial role in different areas. The set of natural numbers, along with the usual addition and multiplication operations forms a semiring. This simple example illustrates how semirings naturally arise in elementary mathematical structures. On the other hand semirings arise naturally in such diverse areas of mathematics as formal languages and automata theory, topology, graph theory, coding theory, linear algebra and matrix theory, commutative ring theory.

The units and idempotents of a ring are key elements determining the structure of the ring. A ring $R$ is said to be regular in the sense of von Neumann, if for every element $r \in R$ there exists an element $x \in R$ such that $r = rxr$. A ring $R$ is unit regular, if for any $a \in R$, $a = aua$ for some unit $u \in R$. Note that if $a = aua$ then $a = eu^{-1}$ where $e = au$; conversely if $a = eu$ then $a = au^{-1}a$. Thus for a unit regular ring $R$, $a = eu$ for some idempotent $e$ and unit $u$ in $R$. In ring theory, there is another notion of regularity which is known as strongly regular ring. The class of strongly regular rings is stronger than the class of unit regular rings. An element $a \in R$ is said to be strongly regular if there exists an element $x \in R$ such that $a = a^2x$ with $ax = xa$. A ring $R$ is called a strongly regular ring if every element of $R$ is strongly regular. Equivalently we can say that for a strongly regular ring $R$, $a = ue = eu$ for some $e^2 = e$ and $u \in U(R)$ where $U(R)$ is the set of all elements in $R$. Clean rings are additive analogs of unit regular rings whereas strongly clean rings are the additive analogs of strongly regular rings. Hence a ring $R$ is said to be a clean ring if for each $a \in R$ there exist an idempotent $e$ and a unit $u$ in $R$ such that $a = e + u$ and $R$ is said to be a strongly clean ring if for each for each $a \in R$ there exist an idempotent $e$ and a unit $u$ in $R$ such that $a = e + u$ with $eu = ue$.

In 1972, R. B. Warfield [4] introduced the concept of exchange ring. Later in 1977, W. K. Nicholson [7] introduced the concept of clean ring as a subclass of exchange ring. From the definition of clean ring it is clear that every Boolean ring, division ring and local ring is a clean ring. But the converse does not hold. After the introduction of clean ring, in 1999, Nicholson [18] introduced another class of clean ring, named as a strongly clean ring. He proved that every strongly clean ring is also an exchange ring.

In this thesis, we have studied the notion of clean rings in a semiring setting. In [62], we have introduced the notion of clean semiring and exchange semiring as a generalization of clean ring and exchange ring and in [63], we have introduced the notion of strongly clean semiring as a generalization of strongly clean ring.

In [11], Adhikari and Sen defined that a semiring $S$ is said to satisfy the condition $(C)$ if for all $a \in S\backslash\{0\}$ and for all $s \in S$, there are $s_1, s_2 \in S$ such that $s+s_1a = s_2a$. If $S$ has an identity 1 then condition $(C')$ is equivalent to the condition $(C)$, $1+s_1a = s_2a$

that holds for each $a \in S \setminus \{0\}$ and suitable $s_1, s_2 \in S$. In [20], Adhikari and Sen defined a nonzero element, satisfying condition $(C)$, as the left semi-invertible element. They defined the right semi-invertible element similarly. According to Adhikari and Sen a nonzero element is semi-invertible if it is both right and left semi-invertible. Throughout this thesis, we have renamed the semi-invertible (left semi-invertible, right semi-invertible) element as the $k$-unit (left $k$-unit, right $k$-unit) element. If $S$ becomes a ring then the $k$-unit element becomes a unit element. But in semiring theory, these two notions are not the same. In this thesis, we have introduced the concept of $k$-unit clean semiring as a generalization of clean semiring.

Following the concept of the clean ring, different works have been done by many authors on the generalization of the clean ring. In 2013, N. Ashrafi and E. Nasibi [43] introduced the notion of $r$-clean ring as a generalization of clean ring where they defined that an element in a ring $R$ is said to be $r$-clean if it can be written as the sum of an idempotent and a regular element and $R$ is called $r$-clean ring if every element of $R$ is $r$-clean. In 1951, S. Bourne [1] first introduced the concept of regularity in semiring theory. According to Bourne, a semiring $S$ is considered regular if, for every element $s \in S$ there exist elements $x', x'' \in S$ such that the equation $s + sx''s = sx's$ holds. But in 1970, H. Subramnian [3] defined that a semiring $S$ is Von Neumann regular if for each $x \in S$ there exists $y \in S$ such that $x = x^2y$. In a ring, Von Neumann regularity and Bourne regularity are the same concept but in a semiring with absorbing zero element, Von Neumann regularity implies Bourne regularity but the converse does not always hold. In 1996, Adhikari, sen and Weinert [14] renamed the notion Bourne regularity of a semiring as $k$-regularity. In this thesis, we have introduced the notion of $k$-regular clean semiring as a generalization of $r$-clean ring.

After the introduction of clean ring by Nicholson [7], in 2012, T.K. Lee and Y. Zhou [41] introduced the concept of clean index of rings. They defined for an element $a$ in a ring $R$, $\mathscr{E}(a) = \{e \in R : e^2 = e, a - e \in U(R)\}$. The clean index of $R$, denoted by $in(R)$, was defined by $in(R) = sup\{|\mathscr{E}(a)| : a \in R\}$. Taking the concept of clean index of rings in mind and using the concept of $k$-unit clean semiring, we have introduced the concept of $k$-unit clean index of a semiring in this thesis.

Among many investigations concerning variants of the clean and strongly clean properties of a ring, two interesting variants that have grabbed the attention of the researchers are nil clean and strongly nil clean properties. In 2013, A. J. Diesl [45], first introduced the notion of nil clean ring and strongly nil clean ring. If $a$ is a nilpotent element in a ring $R$ then $(1 - a)$ is an invertible element in $R$. By using this property, in[45] Diesl proved that every (strongly) nil clean ring is (strongly) clean. Basically clean ring is a generalization of nil clean ring. Following Diesl, many authors ([44], [58], [51]) have worked on nil clean and strongly nil clean rings. In this thesis, we have introduced the notion of nil clean semiring as a generalization of nil clean ring. In semiring theory, every nil clean semiring is not clean. We have also provided an example to support this statement.

After the introduction of nil clean ring by Diesl, in 2014, K. D. Basnet and J. Bhattacharyya introduced the concept of nil clean index of rings. For any element $a$ in a ring $R$, they defined $\eta(a) = \{e \in R : e^2 = e, a - e \in nil(R)\}$ where $nil(R)$ is the set of all nilpotent elements in $R$. The nil clean index of $R$, denoted by $Nin(R) = sup\{|\eta(a)| : a \in R\}$. Taking the concept of nil clean index of rings in mind and using the concept of nil clean semiring, we have introduced the concept of nil clean index of a semiring in this thesis.

The present thesis entitled "Study of Some Aspects of Clean Semirings" has been carried out to study different types of clean semirings.

## 1.2 Overview of the Thesis:

The entire thesis comprises eight chapters mentioned below. Unless otherwise stated, all results of Chapters 2, 3, 4, 5, 6 , 7 and 8 have actually been contributed by the author of the thesis himself under his Ph.D. supervisor, some of which may be the upgradation of existing results for semirings.

**Chapter 1 : Introduction & Preliminaries**

In the first chapter we discuss the background and motivation of this study and also present some preliminary definitions and important results which are relevant for this thesis. We also discuss a brief introductory ideas about the thesis.

**Chapter 2 : Clean Semiring**

In this chapter, the concept of clean semiring and exchange semiring is discussed. The concept of clean ring was introduced by W.K. Nicholson in his study of exchange rings. In this thesis we have introduced the concept of clean semiring as a generalization of clean ring. A semiring is said to be clean if its every nonzero element can be written as the sum of an idempotent and a unit. We study the notion of clean semiring and obtain some important characterizations of clean semiring. We also study the notion of exchange semiring and find out the connection between clean semiring and exchange semiring.

**Chapter 3 : Strongly Clean Semiring**

In this chapter, we have introduced and studied the concept of strongly clean semiring. Let S be a semiring. An element $a \in S$ is called strongly clean if a = e+u with e an idempotent in S and u a unit in S such that eu = ue. A semiring S is said to be strongly clean if every nonzero element of S is strongly clean. We mainly study the notion of strongly clean semiring and obtain some important characterizations of strongly clean semiring in connection with exchange semiring, antisimple semiring and inverse semiring.

**Chapter 4 : On $k$-Unit Clean Semiring**

In chapter 4, we have introduced the concept of $k$-unit clean semiring which generalizes the notion of clean ring as well as clean semiring. Let $S$ be a semiring with identity 1. An element $a \in S \setminus \{0\}$ is said to be a $k$-unit if there exist $r_1, r_2 \in S$ such that $1 + r_1a = r_2a$ and $1 + ar_1 = ar_2$. Basically, $k$-units are the generalization of units in a semiring with zero element. An element $a \in S$ is called $k$-unit clean if $a = e + u$, where $e$ is an idempotent and $u$ is a $k$-unit of $S$. A semiring $S$ is said to be $k$-unit clean if every nonzero element can be written as the sum of an idempotent and a $k$-unit. We obtain some important characterizations of $k$-unit clean semiring in connection with exchange semiring, antisimple semiring and inverse semiring.

**Chapter 5 : On $k$-Regular Clean Semiring**

In this chapter, we introduce the concept of $k$-regular clean semiring which generalizes the notion of clean ring, clean semiring and $k$-unit clean semiring. An element $r$ in a semiring $S$ is said to be $k$-regular element if there exist $r_1, r_2 \in S$ such that $r + rr_1r = rr_2r$. An element $a \in S$ is called $k$-regular clean if $a = e + r$, where $e$ is an idempotent and $r$ is a $k$-regular element of $S$. A semiring $S$ is said to be a $k$-regular clean semiring if every element can be written as the sum of an idempotent and a $k$-regular element of $S$. We obtain some important characterizations of $k$-regular clean semiring in connection with antisimple semiring, inverse semiring and zeroic semiring.

**Chapter 6 : On $k$-Unit Clean Index of Semirings**

A semiring is called $k$-unit clean (resp. uniquely $k$-unit clean) semiring if every nonzero element can be (resp. uniquely) expressed as the sum of an idempotent and a $k$-unit. Motivated by the work on $k$-unit clean semiring, we introduce the concept of $k$-unit clean index of a semiring $S$. For $a \in S$, let $\xi(a) = \{e \in S : e^2 = e,\ a = e+u$, for some $u \in U_k(S)\}$, where $U_k(S)$ is the set of all $k$-unit elements in $S$. We denote the $k$-unit clean index of $S$ by $ind_k(S)$ and we define this by $ind_k(S) = sup\{|\xi(a)| : a \in S\}$. If $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ be the formal triangular matrix semiring then we show that $ind_k(A) < ind_k(S)$ and $ind_k(B) < ind_k(S)$. Finally, we characterize semirings of $k$-unit clean indices 1 and 2.

**Chapter 7 : Nil Clean Semiring**

In this chapter, we have introduced the concept of nil clean semiring. The concept of nil clean ring was introduced by A.J. Diesl in 2013. A semiring is said to be nil clean semiring if every element can be written as the sum of an idempotent and a nilpotent element. We obtain some important results of nil clean semiring in connection with duo semiring, $k$-duo semiring, exchange semiring, $k$-semipotent semiring. We also define nil clean index, $nin(S)$, of a semiring $S$.

**Chapter 8 : On Nil Clean Index of Semirings**

Motivated by the work of nil clean semiring, in this chapter we have introduced the concept of nil clean index of semiring. We have obtained some results related to nil clean index of a semiring and established a connection between $k$-unit clean index and nil clean index of a semiring. Finally, we have characterized the semirings of nil clean indices 1 and 2, with the help of some other class of semirings.

## 1.3 Preliminaries & Prerequisites

In this section, we recall some definitions and results of semirings which we use in this thesis.

**Definition 1.3.1.** *[17] A non-empty set $S$ with two binary operations '$+$' and '$.$' is said to be a semiring if*

*(i) $(S, +)$ is a commutative semigroup,*

*(ii) $(S, .)$ is a semigroup,*

*(iii) $a.(b + c) = a.b + a.c$ and*

*(iv) $(b + c).a = b.a + c.a$ for all $a, b, c \in S$.*

A semiring $S$ is called a semiring with zero element '0' if $a + 0 = 0 + a = a$ and $0.a = a.0 = 0$ for all $a \in S$.

A semiring $S$ is called a semiring with identity '1' if $1.a = a.1 = a$ for all $a \in S$.

A semiring may or may not have a zero and an identity element.

Unless otherwise stated, a semiring $(S, +, .)$ will be denoted simply by $S$ and multiplication '.' will be denoted by juxtaposition.

By the product $AB$ of two non-empty subsets $A$ and $B$ of a semiring $S$, we mean the set $\{\sum_{i=1}^{n} a_i b_i : a_i \in A, b_i \in B \, n \in \mathbb{N}\}$.

**Definition 1.3.2.** *[19] A semiring $(S, +, \cdot)$ is said to be commutative if $(S, \cdot)$ is commutative.*

**Definition 1.3.3.** *Let $(S, +, .)$ be a semiring with zero element '0'. A subset $T$ of $S$ is a subsemiring of $S$ if (i) $0 \in S$, (ii) for any $t_1, t_2 \in S \Longrightarrow t_1 + t_2 \in S$ and $t_1 t_2 \in S$.*

**Definition 1.3.4.** *[16, 24] Let $S$ be a semiring and $I$ be a nonempty subset of $S$. Then $I$ is said to be a left ideal (resp. right ideal) of $S$ if (i) $a_1, a_2 \in I \Longrightarrow a_1 + a_2 \in I$, (ii) $sa \in I$ (resp. $as \in I$) for all $s \in S$ and for all $a \in I$. On the other hand, $I$ is said to be an ideal of $S$ if it is both a left ideal and a right ideal of $S$.*

*If $S$ is a semiring with zero element '0' and $I$ is a left ideal or a right ideal or an ideal of $S$, then $0 \in I$.*

*Let $a$ be an element in semiring $S$ and $0 \in S$. Then $Sa = \{sa : s \in S\}$ ($aS = \{as : s \in S\}$) becomes a left ideal (resp. right ideal) of $S$. The left ideal (resp. right ideal) generated by $a$ is defined as $(a)_l = \{sa : s \in S\} \bigcup \{na : n \in \mathbb{N}\} \bigcup \{sa + na : s \in S, n \in \mathbb{N}\}$ where $na = \sum_{i=1}^{n} a$ ($(a)_r = \{as : s \in S\} \bigcup \{na : n \in \mathbb{N}\} \bigcup \{as + na : s \in S, n \in \mathbb{N}\}$).*

**Definition 1.3.5.** *[17] For each left ideal (resp. right ideal, ideal) $I$ of a semiring $S$, the $k$-closure $\overline{I}$ of $I$ is defined by*

$$\overline{I} = \{a \in S : a + a_1 = a_2 \text{ for some } a_1, a_2 \in I\}.$$

*A left ideal (resp. right ideal, ideal) $I$ of a semiring $S$ is said to be a left $k$-ideal (resp. right $k$-ideal, $k$-ideal) of $S$ if $I = \overline{I}$.*

**Definition 1.3.6.** *If $S$ is a semiring with identity element 1 and $a \in S$, then it can be easily verified that $(a)_l = Sa = \{sa : s \in S\}$. We define this ideal as a principal left ideal generated by $a$. Similarly we define principal right ideal generated by $a$. If $S$ is a commutative semiring then $(a) = (a)_l = (a)_r$. We define this ideal as principal ideal generated by $a$*

**Definition 1.3.7.** *Let $S$ be a semiring with identity '1' and $a$ be an element in $S$. Define the principal left $k$-ideal generated by $a$ as*

$$\overline{(a)_l} = \overline{Sa} = \{b \in S : b + s_1 a = s_2 a \text{ for some } s_1, s_2 \in S\}$$

*Similarly, principal right $k$-ideal generated by $a$ is defined as*

$$\overline{(a)_r} = \overline{aS} = \{b \in S : b + a s_1 = a s_2 \text{ for some } s_1, s_2 \in S\}$$

*If $S$ is a commutative semiring then*

$$\overline{(a)} = \overline{(a)_l} = \overline{(a)_r}$$

. We define this ideal as principal $k$-ideal generated by $a$.

Unless otherwise stated we consider a semiring $(S, +, .)$ with zero element '0' and identity element '1' throughout this thesis.

**Lemma 1.3.8.** *[19] If $I$ and $J$ are two left $k$-ideals (resp. right $k$-ideals, $k$-ideals) of $S$ then $I \cap J$ is also a left $k$-ideal (resp. right $k$-ideal, $k$-ideal) of $S$.*

**Definition 1.3.9.** *[17] Let $A$ be a non-empty subset of a semiring $S$. Then the $k$-closure of $A$, denoted by $\overline{A}$, is defined as : $\overline{A} = \{a \in S : a + b = c$ for some $b, c \in A\}$.*

**Definition 1.3.10.** *[35] A semiring $S$ is said to be left subtractive if every left ideal of $S$ is $k$-ideal i.e. if $I$ is a left ideal of $S$ such that for $x, y \in S$; $x + y \in I$ and $x \in I$ then $y \in I$.*

*Similarly, we define right subtractive semiring. A semiring $S$ is said to be subtractive if it is both left and right subtractive.*

**Lemma 1.3.11.** *[17] Let $S$ be a semiring. Then for any two non-empty subsets $A, B$ of $S$, we have the following :*
*(i) $A \subseteq \overline{A}$, (ii) $A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$, (iii) $\overline{\overline{A}} = \overline{A}$ and (iv) $\overline{AB} = \overline{\overline{A}\,\overline{B}}$.*

**Definition 1.3.12.** *[11] A proper (left, right) $k$-ideal $I$ of a semiring $S$ is called a maximal (left, right) $k$-ideal of $S$ if there is no proper (left, right) $k$-ideal $J$ of $S$ satisfying $I \subset J \subset S$.*

*We denote Jacobson radical of a semiring $S$ by $J_l(S)$ and it is defined as the intersection of all maximal left $k$-ideals of $S$*

**Definition 1.3.13.** *[19] Let $S$ be a commutative semiring. A proper ideal $P$ of $S$ is said to be a prime ideal of $S$ if $ab \in P$ implies that either $a \in P$ or $b \in P$.*

*An ideal $P$ is said to be a prime $k$-ideal of $S$ if $P$ is a prime ideal of $S$ as well as $P = \overline{P}$.*

**Definition 1.3.14.** *[35] A semiring $S$ is said to be left artinian (left $k$-artinian) semiring if for every descending chain of $I_1 \supseteq I_2 \supseteq I_3 \supseteq$ ......... left ideals (left $k$-ideals) there exists a positive integer $n$ such that $I_i = I_n$ for all $i \geq n$. Similarly, we define right artinian (right $k$-artinian) semiring analogously.*

*A semiring $S$ is said to be artinian ($k$-artinian) semiring if it is both left artinian (left $k$-artinian) and right artinian (right $k$-artinian).*

**Definition 1.3.15.** *Let $(S,+,.)$ and $(S',+,.)$ be two semirings with zero elements $0_S$, $0'_S$ and identity elements $1_S$, $1'_S$ respectively. Then a mapping $f : S \longrightarrow S'$ is said to be a semiring homomorphism if $f(x+y) = f(x)+f(y)$, $f(xy) = f(x)f(y)$ for all $x, y \in S$, $f(0_S) = 0'_S$ and $f(1_S) = 1'_S$.*

*A homomorphism $f : S \longrightarrow S'$ is said to be an epimorphism if 'f' is surjective.*

*A homomorphism $f : S \longrightarrow S'$ is said to be a monomorphism if 'f' is injective.*

*A homomorphism $f : S \longrightarrow S'$ is said to be an isomorphism if 'f' is both injective and surjective.*

**Definition 1.3.16.** Let $\{S_i\}_{i\in I}$ (where $I$ is an index set) be a family of semirings. Then the direct product $S = \prod_{i\in I} S_i$ is a semiring with respect to two binary operations $(x_i)_i + (y_i)_i = (x_i + y_i)_i$ and $(x_i)_i(y_i)_i = (x_iy_i)_i$, where $(x_i)_i = (x_1, x_2, ..., x_i, ...)$ and $(y_i)_i = (y_1, y_2, ..., y_i, ...)$ for all $(x_i)_i, (y_i)_i \in S$.

**Definition 1.3.17.** *[19] A semiring $S$ is called an additively cancellative semiring if $a + b = a + c \Longrightarrow b = c$ for all $a, b, c \in S$.*

*An element $a$ of a semiring $S$ is left multiplicatively cancellable if and only if $ab = ac$ only when $b = c$ for any $b, c \in S$. Right multiplicatively cancellable elements are similarly defined. An element of $S$ is multiplicatively cancellable if it is both left and right multiplicatively cancellable. A semiring $S$ is said to be [left, right] multiplicative cancellative if every nonzero element of $S$ is [left, right] multiplicatively cancellable.*

**Definition 1.3.18.** *[19] A non-zero element "a" of a semiring $S$ is said to be a zero divisor if there exists $0_S \neq b \in S$ such that $ab = 0_S$.*

**Definition 1.3.19.** *[20] A semiring $S$ is said to be a semidomain if for any $a, b \in S$, $ab = 0_S$ implies that either $a = 0_S$ or $b = 0_S$.*

*In other words, a semiring is said to be a semidomain if it does not contain any zero divisor.*

**Definition 1.3.20.** *[19] An element $a$ in $S\backslash\{0\}$ is called a unit element if there exists an element $b \in S$ such that $ab = ba = 1$. We denote the set of all unit elements of a* *semiring $S$ by $U(S)$. A semiring $S$ is said to be a division semiring if every non-zero element $a \in S$ is a unit. A commutative division semiring is called a semifield.*

**Definition 1.3.21.** *[20] Let $S$ be a semiring. An element $a \in S \setminus \{0\}$ is said to be a left k-unit (right k-unit) if there exist $r_1, r_2 \in S$ ($s_1, s_2 \in S$) such that $1 + r_1 a = r_2 a$ ($1 + as_1 = as_2$). An element $a \in S \setminus \{0\}$ is said to be a k-unit if it is both left and right k-unit. We denote the set of all k-unit elements of a semiring $S$ by $U_k(S)$. Let $S$ be a commutative semiring. Then $S$ is called a k-semifield if every non-zero element of $S$ is a k-unit.*

**Theorem 1.3.22.** *[20] If $a \in S \setminus \{0\}$, $1 + r_1 a = r_2 a$ and $1 + as_1 = as_2$ hold for suitable $r_i$ and $s_i \in S$, then there exist $r, s \in S$ such that $1 + ra = sa$ and $1 + ar = as$ hold.*

**Example 1.3.23.** *[20] Let $S = \mathbb{N}_0^+$ be the set of all non-negative integers. Define two operations $+$ and $.$ on $S$ by $a + b = max\{a, b\}$ and $a.b$= usual multiplication in $S$ for all $a, b \in S$. Then $(S, +, .)$ is a k-semifield since its every nonzero element is k-unit.*

**Theorem 1.3.24.** *[20] If $a, b \in S \setminus \{0\}$ satisfy the relations*

$(1) 1 + r_1 a = r_2 a, 1 + ar_1 = ar_2$;

$(2) 1 + s_1 b = s_2 b, 1 + bs_1 = bs_2$; *where* $r_1, r_2, s_1, s_2 \in S$

*then* $1 + xab = yab$ *and* $1 + abx = aby$ *hold for suitable* $x, y \in S$.

**Definition 1.3.25.** *[10]* Let $S$ be a semiring and $E^+(S)$ denotes the set of all additive idempotent elements in $S$ i.e. $E^+(S) = \{a \in S : a + a = a\}$.

A (left, right) $k$-ideal $I$ of $S$ is said to be a full (left, right) $k$-ideal of $S$ if $E^+(S) \subseteq I$.

**Definition 1.3.26.** *[19]* Let $S$ be a semiring. Then the center of $S$, denoted by $Z(S)$, is defined by $Z(S) = \{x \in S : xy = yx \text{ for all } y \in S\}$.

Now we define different class of semirings.

**Definition 1.3.27.** *[3] An element '$a$' of a semiring $S$ is said to be regular if there exists an element $x \in S$ such that $a = axa$. A semiring $S$ is said to be regular if every element of $S$ is regular.*

**Definition 1.3.28.** *A semiring $S$ is called strongly regular if for every element $r \in S$ there exist $x, y \in S$ such that $r^2x = r$ and $yr^2 = r$*

**Proposition 1.3.29.** *A semiring $S$ is strongly regular if and only if for each element $r \in S$ there exists some element $z \in S$ such that $rzr = r$ with $rz = zr$.*

*Proof.* Let $S$ be a strongly regular semiring and $r \in S$. Then there exist $x, y \in S$ such that $r^2x = r$ and $yr^2 = r$. Thus $yr^2x = yr$ and $yr^2x = rx$ i.e. $rx = yr$. This implies that $rxr = ryr = r$. Let $z = rx^2$. Then $rzr = r(rx^2)r = r^2x^2r = r^2xxr = rxr = r$. Now $rz = r(rx^2) = r^2x^2 = (r^2x)x = rx$ and hence $zr = (rx^2)r = (rx)(xr) = (yr)(xr) = y(rxr) = yr = rx = rz$.

Converse follows from definition. □

**Definition 1.3.30.** *Let $S$ be a semiring. An element $e \in S$ is said to be an idempotent element if $e^2 = e$.*

*An idempotent $e \in S$ is said to have a complement in $S$ if there exists an idempotent $e_1$ in $S$ such that $e + e_1 = 1$.*

*An idempotent $e \in S$ has an absorbing complement in $S$ if there exists an idempotent $e_1 \in S$ such that $e + e_1 = 1$, $e + see_1 = e$ and $e_1 + e_1es = e_1$ for all $s \in S$.*

*An idempotent $e \in S$ has a strong absorbing complement in $S$ if there exists an idempotent $e_1 \in S$ such that $e_1$ is an absorbing complement of $e$ and $ee_1 = e_1e$.*

*An idempotent $e \in S$ has an orthogonal complement in $S$ if there exists an idempotent $e_1$ in $S$ such that $e + e_1 = 1$ and $ee_1 = e_1e = 0$.*

*We denote the set of all idempotent elements in a semiring $S$ by $Id(S)$.*

*A semiring $S$ is said to be a Boolean semiring if $x^2 = x$ for all $x \in S$.*

**Definition 1.3.31.** *[5]* A semiring $S$ is called an inverse semiring if $(S, +)$ is an inverse semigroup, i.e. for each $a \in S$ there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$.

There are some properties for an inverse semiring $S$ which are as follows :

$(x + y)' = x' + y' = y' + x'$, $(x')' = x$ and $xy' = (xy)' = x'y$ for all $x, y \in S$.

A commutator $[x, y]$ of an inverse semiring $S$ is defined by $[x, y] = xy + y'x = xy + yx'$ for $x, y \in S$.

An inverse semiring $S$ is called centroid semiring if for every $a \in S$, $(a+a')$ is in the center of $S$ i.e. $(a+a')b = b(a+a')$ for all $b \in S$.

An inverse semiring $S$ is called additively absorbing semiring if $a + (a+a')b = a$ for all $a, b \in S$.

An inverse semiring $S$ is said to be an additively absorbing centroid semiring if it is a centroid semiring as well as additively absorbing semiring.

**Definition 1.3.32.** *[24] Let $S$ be a semiring. A left $S$-semimodule is a commutative monoid $(M,+)$ with additive identity $0$ for which we have a function $S \times M \to M$, denoted by $(\lambda, \alpha) \mapsto \lambda\alpha$ and called scalar multiplication, which satisfies the following conditions for all $s_1, s_2 \in S$ and $m_1, m_2 \in M$ :*

*(i) $(s_1s_2)m_1 = s_1(s_2m_1)$, (ii) $s_1(m_1+m_2) = s_1m_1 + s_1m_2$, (iii) $(s_1+s_2)m_1 = s_1m_1 + s_2m_1$, (iv) $1m_1 = m_1$, (v) $s_10 = 0_M = 0m_1$.*

Right $S$-semimodules are defined in an analogous manner. Unless otherwise stated in this thesis, $S$-semimodules will always mean left $S$-semimodules. If $R$ and $S$ are semirings then an $(R,S)$-bisemimodule $(M,+)$ is both a left $R$-semimodule and right $S$-semimodule satisfying the additional that $(rm)s = r(ms)$ for all $m \in M$, $r \in R$ and $s \in S$. If $M$ is a $S$-semimodule then it is in fact an $(S, Z(S))$-bisemimodule with scaler multiplication defined by $m.s = sm$. In particular, if $S$ is commutative then any $S$-semimodule is an $(S,S)$-bisemimodule.

A $S$-semimodule $M$ satisfying the condition that for every element $m \in M$ there exists an element $m' \in M$ satisfying $m + m' = 0_M$ is a left $S$-module.

**Definition 1.3.33.** *[19] A nonempty subset $N$ of a $S$-semimodule $M$ is a subsemimodule of $M$ if and only if $N$ is closed under addition and scalar multiplication. This implies $0_M \in N$. Subsemimodules of right $S$-semimodules and subbisemimodules are defined analogously. A subsemimodule which is an $S$-module is a submodule.*

Let $T$ be a nonempty subset of an $S$-semimodule $M$. Then the intersection of all subsemimodules of $M$ containing $T$ is a subsemimodule of $M$ called the subsemimodule generated by $T$ and it is denoted by $ST$, where $ST = \left\{ \sum_{i=1}^{k} \lambda_i t_i : \lambda_i \in S, t_i \in T, k \in \mathbb{N} \right\}$.

If $T = \{t_1, t_2, ..., t_m\}$ then $ST = \left\{ \sum_{i=1}^{m} \lambda_i t_i : \lambda_i \in S, t_i \in T \right\}$. If $M = ST$ then $M$ is called finitely generated $S$-semimodule.

**Definition 1.3.34.** *[19] A $S$-semimodule $M$ is the direct sum of subsemimodules $N$ and $N'$ of $M$ if every element $m$ of $M$ can be uniquely written as $m = n + n'$, where $n \in N$ and $n' \in N'$ and it is denoted by $M = N \bigoplus N'$. In this case we also say that $N$ is a direct summand of $M$.*

**Definition 1.3.35.** *[19] A nonzero $S$-semimodule $M$ is indecomposable if and only if there do not exist nonzero subsemimodules $N$ and $N'$ of $M$ satisfying $M = N \bigoplus N'$. An $S$-semimodule which is not indecomposable is decomposable.*

**Definition 1.3.36.** *[17] Let $M$ be a $S$-semimodule. Let $U \neq \emptyset$ a subset of $M$. Then $U$ is called linearly independent (over $S$) if $\sum_{u \in F} \alpha_u u = \sum_{u \in F} \beta_u u$ for $\alpha_u, \beta_u \in S$ and any finite subset $F \neq \emptyset$ of $U$ implies $\alpha_u = \beta_u$ for all $u \in F$. The subset $U$ of $M$ is called a basis of $M$ if $U$ is linearly independent and generates $M$. Moreover $M$ is called free $S$-semimodule if $M$ has a basis.*

**Definition 1.3.37.** *[35] A subsemimodule ${}_SK$ of a semimodule ${}_SM$ is said to be a subtractive semimodule if for all $m, m' \in_S M$, that $m + m', m \in {}_SK$ implies $m' \in {}_SK$; ${}_SM$ is said to be subtractive semimodule if it has only subtractive subsemimodules.*

**Definition 1.3.38.** *[19, 24] Let $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$, where $A$ and $B$ be two semirings and $M$ be the ${}_AM_B$ bi-semimodule. Then $S$ becomes a semiring with respect to usual matrix addition and multiplication defined by $\begin{bmatrix} a_1 & m_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & m_2 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & a_1 m_2 + m_1 b_2 \\ 0 & b_1 b_2 \end{bmatrix}$ for all $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $m_1, m_2 \in M$. The semiring $S$ is called formal triangular matrix semiring.*

**Definition 1.3.39.** *(*[14], [54]*) A semiring $S$ is called a $k$-regular semiring if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$.*

**Example 1.3.40.** *The semiring defined in Example 1.3.23, is a $k$-regular semiring.*

**Remark 1.3.41.** *A regular semiring $S$ is also a $k$-regular semiring but the converse is not true, in general. Suppose a semiring $S$ is regular and $a \in S$. Then there exists $x \in S$ such that $a = axa$. This shows that $a + aya = axa + aya$ for any $y \in S$. So $a + aya = a(x + y)a$ that is $a + aya = aza$, where $z = x + y$. This shows that $S$ is $k$-regular. But if we consider Example 1.3.23 , we can check that $(\mathbb{N}_0^+, +, .)$ is $k$-regular but not regular.*

**Definition 1.3.42.** [50] *If $S$ is a commutative semiring with identity element, then $S$ is called a semidomain if $ab = 0$, $a, b \in S$ implies $a = 0$ or $b = 0$.*

**Definition 1.3.43.** *[54] An element $e \in S$ is said to be $k$-idempotent (or almost idempotent) if $e^2 + e = e^2$. If every element of a semiring $S$ is $k$-idempotent or almost idempotent then $S$ is called $k$-idempotent or almost idempotent semiring.*

**Definition 1.3.44.** *[38] Let $S$ be a semiring. An element $a \in S$ is said to be nilpotent if $a^k = 0$ for some positive integer $k$. The least positive integer $k$ satisfying $a^k = 0$ is called the nilpotent index of $a$. A semiring is called nil if every element of the semiring is nilpotent. We denote the set of all nilpotent elements by $N(S)$.*

*A semiring $S$ is called a reduced semiring if $S$ has no nonzero nilpotent element.*

**Definition 1.3.45.** *An element $a$ in a semiring $S$ is right $\pi$-regular if there exists $x \in S$ such that $a^n = a^{n+1}x$ for some integer $n \geq 1$. The left $\pi$-regular is defined analogously. An element $a \in S$ is called strongly $\pi$-regular if it is both left and right $\pi$-regular. A semiring $S$ is called strongly $\pi$-regular semiring if every element of $S$ is strongly $\pi$-regular.*

**Definition 1.3.46.** [19] *A semiring $S$ is called a simple semiring if $1 + a = 1$ for all $a \in S$.*

**Definition 1.3.47.** *[17]* A semiring $S$ is called a mono-semiring if $a + b = ab$ for all $a, b \in S$.

**Definition 1.3.48.** *[19]* Let $S$ be a semiring. Let us define $P(S) = \{0\} \cup \{r + 1 : r \in S\}$. Then $P(S)$ is a subsemiring of the semiring $S$. A semiring $S$ is said to be an antisimple semiring if $S = P(S)$. *A semiring $S$ is said to be semi-antisimple if every nonzero non k-unit element $s$ can be written as $s = s' + 1$ for some $s' \in S$. If $S$ has no nonzero non k-unit element then $S$ is also called semi-antisimple (vacuously).*

**Definition 1.3.49.** *Let $S$ be a semiring and $I$ be a k-ideal of $S$. Now define a relation $k_I$ on $S$ such that $ak_Ib \iff a + i_1 = b + i_2$ for some $i_1, i_2 \in I$. Then $k_I$ becomes a congruence relation on $S$ and it is known as Bourne congruence relation. Let $S/I$ denotes the set of all congruence classes with respect to $k_I$ and we denote $[a]_{k_I} = (a+I)$ as the congruence class of $a$ for any $a \in S$. Hence $S/I = \{(a + I)|a \in S\}$. Now $S/I$ becomes a semiring with respect to the operations $(a + I) + (b + I) = (a + b) + I$ and $(a + I)(b + I) = ab + I$ for all $a, b \in S$. This semiring is called quotient semiring.*

Now we introduce some basic notes about lattices, there is a relation between lattices and semirings as we see in future, e.g. lattices help us to provide a good example for semirings which satisfies some conditions. All of the definitions and facts in this section can be found in [9].

**Definition 1.3.50.** *A nonempty set $L$ together with two binary operations $\vee$ and $\wedge$ (read "join" and "meet" respectively) on $L$ is called a lattice if it satisfies the following identities: L1: (a) $x \vee y = y \vee x$ (b) $x \wedge y = y \wedge x$ (commutative laws)*

*L2: (a) $x \vee (y \vee z) = (x \vee y) \vee z$ (b) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (associative laws)*

*L3: (a) $x \vee x = x$ (b) $x \wedge x = x$ (idempotent laws)*

*L4: (a) $x \vee (x \wedge y) = x$ (b) $x \wedge (x \vee y) = x$ (absorption laws) for all $x, y, z \in L$.*

**Definition 1.3.51.** *A binary relation $\leq$ defined on a set $A$ is a partial order on the set $A$ if the following conditions hold identically in $A$:*

*(i) $a \leq a$ (ii) $a \leq a$ and $b \leq a$ imply $a = b$. (iii) $a \leq b$ and $b \leq c$ imply $a = c$*

If in addition for every $a, b \in A$ $(iv)$ $a \leq b$ or $b \leq a$ then we say that $\leq$ is a totally order relation $A$.

**Definition 1.3.52.** *A nonempty set $P$ with a partial order relation $\leq$ is called a partial order set or poset, denoted by $(P, \leq)$.*

If $a, b$ in a poset$(P, \leq)$ such that $a \leq b$, we say $a$ and $b$ are comparable. Two elements of a poset may not be comparable.

**Example 1.3.53.** *The set of natural numbers* $\mathbb{N}$*, under less than or equal "* $\leq$ *" is a poset.*

**Example 1.3.54.** *The set of natural numbers* $\mathbb{N}$*, under divisibility is a poset.*

**Example 1.3.55.** *Let* $A$ *be any nonempty set and* $P(A)$ *denotes the power set of* $A$*, i.e. the set of all subsets of* $A$*. Then "*$\subseteq$*" is a partial order relation on* $A$*. Hence* $(P(A), \subseteq)$ *is a poset.*

**Example 1.3.56.** *The set of real numbers* $\mathbb{R}$ *with usual ordering "*$\leq$*" is a totally ordered set.*

**Definition 1.3.57.** *Let* $A$ *be a subset of a poset* $P$*,* $a \in P$ *is an upper bound of* $A$*, if* $x \leq a$ *for all* $x \in A$*. An element* $p \in P$ *is the least upper bound of* $A$ *or supremum of* $A$ $(supA)$ *if* $p$ *is an upper bound of* $A$ *and* $a \leq b$ *for every* $a \in A$ *implies that* $p \leq b$*. similarly, we can define what it means for* $p$ *to be a lower bound of* $A$ *and for* $p$ *to be the greatest lower bound of* $A$ *also called the* $infimum$ *of* $A$ $(infA)$*.*

**Definition 1.3.58.** *A poset* $L$ *is a lattice if and only if for every* $a, b \in L$ *both* $sup\{a, b\}$ *and* $inf\{a, b\}$ *exist in* $L$*.*

The two definitions 1.3.50 and 1.3.58 of a lattice are equivalent in the following sense: (1) If $L$ is a lattice by the first definition, then define $\leq$ on $L$ by $a \leq b$ if and only if $a = a \wedge b$. (2) If $L$ is a lattice by the second definition, then define the operations $\vee$ and $\wedge$ by $a \vee b = sup\{a, b\}$ and $a \wedge b = inf\{a, b\}$.

The details of the proof has been given in [9]

**Example 1.3.59.** *Let us consider the subset of natural numbers* $L = \{1, 2, 3, 6\}$*. Define two binary operations* $\vee$ *and* $\wedge$ *on* $L$ *by* $a \vee b = lcm(a, b)$ *and* $a \wedge b = gcd(a, b)$*. Then* $(L, \vee, \wedge)$ *forms a lattice as it satisfies the identities: L1-L4.*

From Definition 1.3.58 the next Theorem follows

**Theorem 1.3.60.** *Let $L$ be a poset, then $L$ is a lattice if and only if every nonempty finite subset of $L$ has sup and inf.*

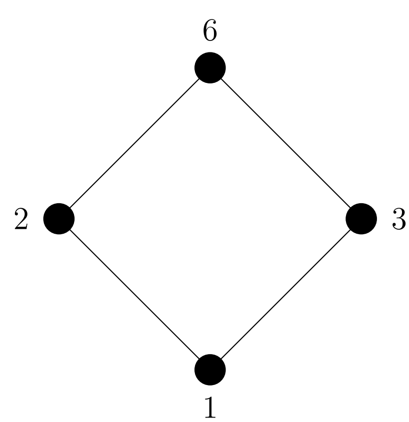
**Theorem 1.3.61.** *[9] Let $L$ be a lattice, then for any $x, y, z \in L$, we have*
*(i)* $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$.
*(ii)* $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

**Definition 1.3.62.** *A distributive lattice is a lattice which satisfies either of the distributive laws: D1:* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
*D2:* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, *for any* $x, y, z \in L$.

**Theorem 1.3.63.** *[9] A lattice $L$ satisfies D1 if and only if it satisfies D2.*

**Definition 1.3.64.** *A lattice* $(L, \vee, \wedge)$ *is a bounded lattice if there exist constants* $0, 1 \in L$ *satisfying the following:*
(1) $x \wedge 1 = x$ *and* $x \vee 1 = 1$ *for all* $x \in L$,
(2) $x \wedge 0 = 0$ *and* $x \vee 0 = x$ *for all* $x \in L$.

Example 1.3.59 is a bounded lattice since 6 is the greatest element and 1 is the least element of $L$. Moreover, $L$ is a bounded distributive lattice.



**Hasse diagram of lattice $L$**

**Definition 1.3.65.** *[9] Let $L$ be a bounded lattice and $x \in L$. An element $y \in L$ is called a complement of $x$ if $x \wedge y = 0$ and $x \vee y = 1$. $L$ is called complemented lattice if every element has a complement.*

**Example 1.3.66.** *Example 1.3.59 is a complemented lattice, since it is a bounded lattice and every element has a complement.*

In the follwing section we introduce some basic concepts of cyclic monoids. All of the definitions and facts in this section can be found in [39, 13].

**Definition 1.3.67** ([13, 39])**.** *A monoid $M$, which can be generated by a single element is called cyclic monoid i.e. $(M, +)$ be a cyclic monoid if for $a \in M$, $M = \{na | n \in \mathbb{N}_0^+\} =< a >$, where $0a = 0$, the identity of $M$. There are two types of cyclic monoids, finite and infinite cyclic monoid. If there is no repetitions in the list $\{0, a, 2a, ...., na, ...\}$ then $(M, +)$ becomes infinite cyclic monoid and isomorphic to $(\mathbb{N}_0^+, +)$. Let $(M, +)$ be a finite cyclic monoid. Now there are two cases:*

**Case-I** *$ma = 0 = 0a$ for some natural number $m$. Then $M$ becomes a cyclic group.*

**Case-II** *$na \neq 0$ for any natural number $n$. Then the set $\{x \in \mathbb{N} : (\exists\ y \in \mathbb{N})\ xa = ya, x \neq y\}$ is non-empty and so has a least element. Let us denote the least element by $m$ and call it index of the element $a$. Now the set $\{x \in \mathbb{N} : (m+x)a = ma\}$ is non-empty and so it too has a least element $r$ which we call the period of $a$. Thus $(m+r)a = ma$. Hence $ma = (m+r)a = ma+ra = (m+r)a+ra = (m+2r)a$ which implies that $ma = (m + qr)a$ for all $q \in \mathbb{N}_0^+$ Let $m + r = l$ where $r \in \mathbb{N}$. Hence $ma = (m + r)a$. By the minimality of $m$ and $r$ we may deduce that $0, a, 2a, ....., ma, ..., (m + r - 1)a$ are all distinct. Let $s \geq m$. Either $s - m \geq r$ or $s - m < r$. If $s - m < r$ then $s = m, m + 1, ....., m + r - 1$. If $s - m \geq r$ then by division algorithm, it follows that $s = m + qr + u$, $q > 0$ and $0 \leq u \leq (r - 1)$. Hence in this case $sa = (m + qr + u)a = (m + qr)a + ua = ma + ua = (m + u)a$. Thus $M =< a >= \{0, a, 2a, ..., ma, ...(m+r-1)a\}$. Hence $| < a > | = m+r$ which is also the order of $a$. It is also proved in [13] that the subset $K_a = \{ma, (m+1)a, (m+2)a, ......., (m+r-1)a\}$ of $M$ is formed a cyclic group.*

*An example of a cyclic monoid with generator $c$ is the monoid defined on the set $\{ic | i \in \{0, 1, 2, ....k + n - 1\}\}$ by means of the following addition rules:*

- *$ic + jc = (i + j)c$ if $i + j < k + n$;*
- *$ic + jc = [k + rem(i + j - k, n)]c$ if $i + j \geq k + n$;*
- *$0c = 0$ is the identity.*

*We refer to this monoid as $C_{k,n}$. Clearly $C_{k,n}$ is cyclic monoid with generator $c$.*

**Theorem 1.3.68.** *(Characterization of cyclic monoids) [39]: Every cyclic monoid is isomorphic with either $(\mathbb{N}_0^+, +)$ or $C_{k,n}$ for some $k, n \in \mathbb{N}_0^+$, $n > 0$.*

**Remark 1.3.69.** *If we think of $C_{k,n}$ in the following way, the reason for the name cyclic becomes clear. First there is the beginning piece of the monoid consisting of $0, c, 2c, ....., kc$. Then comes the cyclic part consisting of $kc, (k+1)c, (k+2)c, ......, (k+n-1)c, (k+n)c = kc$. At the end of the list we are back to the element $kc$. After that the cyclic part repeats itself : $(k + n + 1)c = (k + 1)c, (k + n + 2)c = (k + 2)c, ......$*

We list some properties of the cyclic monoids $C_{k,n}$

- $|C_{k,n}| = k + n$.
- For every $m \in \mathbb{N}_0^+$ with $m > 0$, there are precisely $m$ nonisomorphic cyclic monoids with $m$ elements, viz, $C_{m-k,k}$ for $k = 1, 2, 3, ..., m$.
- If $k > 0$, then no element of $C_{k,n}$ but 0 is invertible.
- In $C_{0,n}$ every element is invertible (in other words, $C_{0,n}$ is a cyclic group of order $n$).

**Remark 1.3.70.** *Let $(M, +)$ be a finite cyclic monoid and $E^+(M)$ be the set of all idempotent elements in $M$. Then from Theorem 1.3.68 it follows that $|E^+(M)| = |E^+(C_{k,n})|$ for some $k, n \in \mathbb{N}_0^+$, where $n > 0$. Hence $|E^+(M)| = 1$ when $k$ in the monoid, $C_{k,n}$ equals to 0 and $|E^+(M)| = 2$ when $k > 0$.*

# Chapter 2

# Clean Semiring

# Chapter 2

# Clean semiring

## 2.1 Introduction

In the last several years, there has been a growing interest in developing the algebraic theory of semirings and their applications in different branches of mathematics, computer science, quantum physics and many other areas of science. Semirings are certainly the most natural generalization of rings and bounded distributive lattices. Investigating semirings and their representations, researchers have to use the methods and techniques of both ring and lattice theory as well as diverse techniques and methods of categorical and universal algebra.

Nowadays, there is a large literature of clean ring theory since many people are interested to study about clean ring and related areas. Certainly the idempotents and units of a ring play very important roll for determining the structure of the clean ring. An element of a ring is called clean if it is the sum of an idempotent and a unit of that ring. A ring $R$ is called clean if every element of $R$ is clean. On the other hand, a ring $R$ is called unit regular if for any $a \in R$, $a = aua$ for some unit $u$ in $R$. Equivalently, one can say a ring $R$ is unit regular if $a = eu$ for some idempotent $e$ and unit $u$ in $R$. Hence a ring $R$ is unit regular if each element of $R$ is the product of an idempotent and a unit. Surprisingly, every unit regular ring is a clean ring i.e. every element of a unit regular ring can also be written as the sum of a unit and an idempotent. Thus the "sum" analog of the unit regular condition is the notion of a clean ring. R. B. Warfield [4] defined a ring $R$ to be an exchange ring if the left $R$-module $R$ has finite exchange property. W. K. Nicholson [7] proved that a ring $R$ is an exchange ring if and only if for each $a \in R$, there exists an idempotent $e$ in $R$ such that $e \in Ra$ and $(1-e) \in R(1-a)$. The notion of clean ring was introduced by W. K. Nicholson [7] in 1977 in a study of exchange rings. He also proved that clean rings are exchange rings but the converse does not always hold. A ring with central idempotents is clean if and only if it is an exchange ring. Since then various results of different types of clean rings have been obtained by many authors ([28], [43], [34], [52], [30], [26], [18], [59], [60], [32]).

In this chapter, we introduce the notion of clean semiring and obtain some important results about clean semiring. Finally, we study the notion of exchange semiring and find out the connenction between these two classes of semirings.

## 2.2 Definition, Examples & Some Basic Results

**Definition 2.2.1.** An element of a semiring $S$ is said to be clean if it can be written as the sum of an idempotent and a unit in $S$. A semiring $S$ is called a clean semiring if every nonzero element of $S$ is clean.

**Remark 2.2.2.** *Let $S$ be a semiring and $a$ be an element in $S$. Note that an element $b$ of $S$ is an additive inverse of $a$ if and only if $a + b = 0$ ([19]). Denote the set of all elements of $S$ having additive inverse, by $V(S)$. The set $V(S)$ is non-empty as $0 \in V(S)$.*

*It can be easily seen that a semiring $S$ is a ring if and only if $V(S) = S$.*

*Now suppose that in a semiring $S$; $0 = e + u$, where $e^2 = e$ and $u \in U(S)$. Since $u \in U(S)$, there exists $v \in S$ such that $uv = vu = 1$. So we have $0 = 0v = (e + u)v = ev + uv = ev + 1$. Let $b = ev$. Then $1 + b = 0$. Thus $x + xb = x(1 + b) = x0 = 0$ for all $x \in S$. So $x \in V(S)$ and hence $S = V(S)$. This shows that $S$ becomes a ring.*

*To avoid this situation, we take a semiring $S$ is clean if every nonzero element of $S$ can be written as the sum of an idempotent and a unit in $S$.*

**Example 2.2.3.** *Every clean ring is an example of clean semiring. But we give some examples of clean semirings, which are not clean rings.*

*(i)* *Every semifield (division semiring) is an example of a clean semiring as its every nonzero element is a unit.*

*Let* $\mathbb{R}_0^+$ *be the set of all non-negative real numbers and* $\mathbb{Q}_0^+$ *be the set of all non-negative rational numbers. Then* $\mathbb{R}_0^+$ *and* $\mathbb{Q}_0^+$ *are semifields with respect to usual addition and multiplication. Thus in particular, both* $\mathbb{R}_0^+$ *and* $\mathbb{Q}_0^+$ *are clean semirings.*

*(ii)* *Let* $\mathbb{Z}$ *be the set of integers. We consider the sets* $S = Z\bigcup\{+\infty\}$ *and* $S' = Z\bigcup\{-\infty\}$. *We define "*$\oplus$*" and "*$\odot$*" on* $S$ *and* $S'$ *respectively by* $a \oplus b = min(a, b)$ *and* $a \odot b = a + b$, $a \oplus b = max(a, b)$ *and* $a \odot b = a + b$ *for all* $a, b \in Z$. *The operations "*$\oplus$*" and "*$\odot$*" are referred to as tropical addition and tropical multiplication respectively.*

*Then* $(S, \oplus, \odot)$ *[*$(S', \oplus, \odot)$*] forms a min tropical semiring (or min-plus semiring) [resp. max tropical semiring (or max-plus semiring )] with zero element "*$+\infty$*" and identity element "0" [resp. with zero element "*$-\infty$*" and identity element "0"]. Hence* $(S, \oplus, \odot)$ *and* $(S', \oplus, \odot)$ *are clean semirings as both these are semifields.*

*(iii)* *Let* $M_2(\mathbb{R}_0^+) = \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R}^+\} \bigcup \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$.

*Define "+" and "." in* $M_2(\mathbb{R}_0^+)$ *by :*

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} max(a,c) & 0 \\ 0 & max(b,d) \end{bmatrix} \text{ and } \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} . \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}.$$

*Then* $(M_2(\mathbb{R}_0^+), +, .)$ *is a semifield and hence it is a clean semiring.*

*(iv)* *Consider* $S = \{0, 1\}$. *Define the two operations "+" and "." in* $S$ *as follows :*

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| . | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

*Then* $(S, +, .)$ *forms a Boolean semiring which also a semifield and hence it is a clean semiring.*

*(v) Consider* $S = \{0, 1, 2, 3\}$ *. Define the operations "+" and "." in* $S$ *as follows :*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 3 |
| 2 | 2 | 3 | 3 | 3 |
| 3 | 3 | 3 | 3 | 3 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 3 | 3 |
| 3 | 0 | 3 | 3 | 3 |

*Here* 0 *is the zero element and* 1 *is the identity of* $S$*. Also* $1 = 0 + 1$*,* $2 = 1 + 1$*,* $3 = 3 + 1$*. Hence* $(S, +, .)$ *is a clean semiring.*

*(vi) Consider* $S = \{0, 1, 2, 3\}$ *. Define the operations "+" and "." as follows :*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 1 |
| 2 | 2 | 3 | 1 | 2 |
| 3 | 3 | 1 | 2 | 3 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 1 | 3 |
| 3 | 0 | 3 | 3 | 3 |

*Then* $(S, +, .)$ *is a clean semiring, since* $1 = 0 + 1$*,* $2 = 1 + 1$ *and* $3 = 1 + 2$*.*

*(vii) Consider the set* $S = \{1, 2, 3\}$*. Define the operations "+" and "." as follows :*

| + | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 |

| . | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 |

*Let* $\mathbb{R}$ *be the field of real numbers with respect to the usual addition and multiplication.*

*Let us consider the direct product* $S' = \mathbb{R} \times S$*. Then* $S'$ *is a clean semiring with respect to componentwise addition and multiplication which is not a semifield.*

**Proposition 2.2.4.** *Let $S$ be a multiplicative cancellative artinian semiring with identity element 1. Then $S$ is a division semiring hence a clean semiring.*

*Proof.* Let $S$ be a multiplicatively cancellative artinian semiring with identity 1. So $S$ is left artinian as well as right artinian semiring. To show that $S$ is division semiring we have to prove that every nonzero element of S is a unit. Let $a \neq 0 \in S$. Let us consider the principal left ideal generated by $a$, $(a)_l = \{sa : s \in S\}$. Let $y \in (a^2)_l$ then $y = s'a^2 = (s'a)a$ for some $s' \in S$, which belongs to $(a)_l$. So $(a^2)_l \subseteq (a)_l$. Similarly we can show that $(a^3)_l \subseteq (a^2)_l$, $(a^4)_l \subseteq (a^3)_l$ and so on ... So we have $(a)_l \supseteq (a^2)_l \supseteq (a^3)_l \supseteq (a^4)_l \ldots$, which is the decreasing chain of left ideals of $S$. Since $S$ is left artinian, there exists $m \in \mathbb{N}$ such that $(a^i)_l = (a^m)_l$ for all $i \geq m$. Now $a^m \in (a^m)_l = (a^{m+1})_l$. So there exists $s'' \in S$ such that $a^m = s''a^{m+1}$. So we have $a^m = (s''a)a^m$. Since S is multiplicatively cancellative and $a \neq 0$, $a^m \neq 0$ , so we have $1 = s''a \ldots (1)$ . $S$ is also a right artinian semiring. Hence in a similar manner, there exists $k \in \mathbb{N}$ such that $(a^k)_r = (a^i)_r$ for all $i \geq k$. So we have $a^k = a^{k+1}s'''$, where $s''' \in S$. since S is multiplicatively cancellative and $a \neq 0$, $a^k \neq 0$. Therefore,$1 = as''' \ldots (2)$. From equations (1) and (2), we have $as''a = as'''a$. Since $S$ is multiplicatively cancellative and $a \neq 0$, $s'' = s''' = s$. Thus $sa = as = 1$ and $s \neq 0$, since $s'', s''' \neq 0$. Hence $a$ is a unit element of $S$. So every nonzero element is a unit of $S$ . Hence $S$ is a division semiring . Thus $S$ is a clean semiring from Example 2.2.3(i). □

Now we prove that homomorphic image and direct product of clean semiring(s) is clean which are two basic results of clean semirings.

**Theorem 2.2.5.** *Every homomorphic image of a clean semiring is clean.*

*Proof.* Let $S$ be a clean semiring with identity $1_S$ and $S'$ be a semiring with identity $1_{S'}$. Let $f : S \longrightarrow S'$ be an onto homomorphism. Then $S'$ is the homomorphic image of the clean semiring $S$. We have to show that $S'$ is a clean semiring. Since $f$ is an onto homomorphism, so $S' = \{f(a) : a \in S\}$. Now $f(1_S) = 1_{S'}$ is the identity element of $S'$. Let $s' \neq 0 \in S'$. Then there exists $a(\neq 0) \in S$ such that $f(a) = s'$. Since $S$ is clean so we can write $a = e + u$, where $e^2 = e \in S$ and $u$ is a unit in $S$. Thus there exists $v \in S$ such that $uv = vu = 1_S$. So we have $s' = f(a) = f(e+u) = f(e) + f(u)$, since f is a homomorphism. Now $[f(e)]^2 = f(e)f(e) = f(e^2) = f(e)$. Hence $f(e)$ is an idempotent in $S'$. Now $f(u)f(v) = f(uv) = f(1_S) = 1_{S'} = f(1_S) = f(vu) = f(v)f(u)$. This shows that $f(u)$ is a unit in $S'$. So $s'$ is the sum of an idempotent $f(e)$ and a unit $f(u)$ of $S'$. Since $s'$ is an arbitrary nonzero element of $S'$, it follows that $S'$ is a clean semiring. □

**Theorem 2.2.6.** *Let $\{S_i : i = 1, 2, \ldots, n\}$ be a finite family of semirings such that each semiring $S_i$ has no zero element. Then the direct product of semirings $S = \prod_{i=1}^{n} S_i$ is clean if and only if each semiring $S_i$ is clean.*

*Proof.* Suppose that each semiring $S_i$ of the family $\{S_i : i = 1, 2, \ldots, n\}$ be clean. Since no $S_i$ has zero and each $S_i$ is clean, so every element of $S_i$ is written as the sum of an idempotent and a unit for all $i = 1, 2, \ldots, n$. Let $(x_1, x_2, \ldots, x_n) \in S$, where each $x_i \in S_i$. So we have $x_i = e_i + u_i$, where $e_i$ is an idempotent of $S_i$ and $u_i$ is a unit of $S_i$ for all $i = 1, 2, \ldots, n$. Thus it follows that $(x_1, x_2, \ldots, x_n) = (e_1 + u_1, e_2 + u_2, \ldots, e_n + u_n) = (e_1, e_2, \ldots, e_n) + (u_1, u_2, \ldots, u_n)$. Hence every element of $S$ is written as the sum of an idempotent and a unit of $S$. Consequently, $S$ is a clean semiring.

Conversely, suppose that $S = \prod_{i=1}^{n} S_i$ is clean. We have to show that each $S_i$ is clean. Let us consider the mapping $\pi_i : S \longrightarrow S_i$ defined by $\pi_i((x_1, x_2, \ldots, x_n)) = x_i$ for all $(x_1, x_2, \ldots, x_n) \in S$. Then $\pi_i$ is an onto homomorphism from $S = \prod_{i=1}^{n} S_i$ to $S_i$. Thus by Theorem 2.2.5, $S_i$ is clean for all $i = 1, 2, \ldots, n$ and hence the proof. □

## 2.3 Connection With Some Classes of Semirings

**Definition 2.3.1.** A semiring $S$ is said to be strongly clean if every nonzero element $a$ of $S$ can be written as $a = e + u$ for some $e^2 = e$ and $u \in U(S)$ with $eu = ue$.

**Definition 2.3.2.** For a $k$-ideal $I$ of a semiring $S$, we say that idempotents lift modulo $I$ if for each element $x \in S$ such that $x + I = x^2 + I$ there is some idempotent $e \in S$ with $x + I = e + I$.

**Theorem 2.3.3.** *Let $S$ be an additively cancellative strongly clean semiring such that every idempotent of $S$ has a complement in $S$. Then idempotents lift modulo every $k$-ideal of $S$.*

*Proof.* Let $S$ be a strongly clean semiring and $I$ be a $k$-ideal of $S$. Let $x \in S$. If $x \in I$ then clearly the result follows. Let $x$ does not belong to $I$. Since $S$ is strongly clean, $x = f + u$, where $f$ is an idempotent and $u$ is a unit in $S$ with $uf = fu$. Now $f$ has a complement, say $e_1$ in $S$. So we have $e_1 + f = 1$. Also there exists $v \in S$ such that $uv = vu = 1$, since $u$ is a unit in $S$. Let $x + I = x^2 + I$. Then $(f + u) + I = (f + u)^2 + I \Longrightarrow (f + u) + I = (f + fu + uf + u^2) + I \Longrightarrow (u + f) + i_1 = (ux + fu + f + i_2)$, where $i_1, i_2 \in I$. So $u + i_1 = ux + fu + i_2$, since $S$ is additively cancellative. Now multiplying from right side by $v$, we have $uv+i_1' = uxv+fuv+i_2' \Longrightarrow 1+i_1' = uxv+f+i_2'$. Again by multiplying from left side by $v$, we have $v+i_1'' = xv+vf+i_2''$. Now multiplying from right side by $f$, it follows that $vf+i_1''' = xvf+vf+i_2'''$. Since $S$ is additively cancellative, we find that $i_1''' = xvf+i_2'''$. So $xvf \in I$, since $I$ is a $k$-ideal of $S$. Also since $uf = fu$, multiplying from left side by $v$, we have $vuf = vfu \Longrightarrow f = vfu$. Now multiplying from right side by $v$, it follows that $fv = vf$. Thus $xfv \in I$ implies that $xf \in I$. Now $xf = (u + f)f \Longrightarrow xf = uf + f \Longrightarrow xf = fu + f \Longrightarrow xf = f + fu \Longrightarrow xf = f(f + u) \Longrightarrow xf = fx$. So $fx \in I$. Again since $u + i_1 = ux + fu + i_2$, multiplying from left side by $v$, we have $1 + i_3 = x + vfu + i_4 \Longrightarrow (e_1 + f) + i_3 = x + vfu + i_4$. Now multiplying left side by $u$ and right side by $v$, it follows that $ue_1v + ufv + i_5 = uxv + f + i_6$. So adding both side by $fv$, we have $ue_1v + ufv + fv + i_5 = uxv + f + fv + i_6 \Longrightarrow ue_1v + (u + f)fv + i_5 = uxv + f + fv + i_6 \Longrightarrow ue_1v + xfv + i_5 = uxv + f + fv + i_6$. Since $xf \in I$, so $xfv \in I$. Hence $ue_1v + i_7 = uxv + f + fv + i_6$. Also $fx \in I \Longrightarrow f(u+f) \in I \Longrightarrow fu+f \in I$. So $fuv + fv \in I$ which implies that $f + fv \in I$. Thus it follows that $ue_1v + i_7 = uxv + i_8$. Consequently, $ue_1v + I = uxv + I$. Now multiplying from left side by $v + I$, we have $(v+I)(ue_1v+I) = (v+I)(uxv+I) \Longrightarrow (e_1v+I) = (xv+I)$. Again multiplying from right side by $u+I$, it follows that $(e_1v+I)(u+I) = (xv+I)(u+I) \Longrightarrow (e_1+I) = (x+I)$, where $e_1$ is an idempotent. Thus idempotents lift modulo $k$-ideal $I$. Hence the proof. □

**Note 2.3.4.** *Let $S$ be an antisimple reduced semiring. If $I$ be a nonzero proper $k$-ideal of $S$ then there exists an element $x(\neq 0) \in S$ such that $x \in I$. Since $S$ is antisimple, there exists an element $x' \in S$ such that $x = 1 + x'$, where $x' \notin I$. Now we have $x + I = I$. Hence $(1 + x') + I = I$ which implies that $I = (1 + I) + (x' + I)$.*

**Definition 2.3.5.** Let $S$ be a semiring such that every idempotent of $S$ has absorbing complement in $S$. Then $S$ is said to be exchange semiring if for each $x(\neq 0) \in S$, there exists an idempotent $e \in S$ such that $e \in \overline{S(1+x)}$ and $e_1 \in \overline{Sx}$, where $e_1$ is an absorbing complement of $e$ in $S$.

**Theorem 2.3.6.** *Let $S$ be a commutative antisimple reduced semiring such that the $k$-closure of every proper ideal is proper and every idempotent of $S$ has absorbing complement in $S$. Then $S$ is an exchange semiring if and only if $S$ is a clean semiring.*

*Proof.* Let $S$ be a clean semiring. If $I = 0$ then $a + I = a^2 + I$ implies that $a = a^2$, which gives $a$ is an idempotent in $S$. Hence idempotents lift modulo the zero ideal. Let $I$ be a nonzero $k$-ideal of $S$. Since $S$ is a commutative antisimple reduced semiring, from Note 2.3.4 we can say that $S/I$ is additively cancellative. Since every idempotent has absorbing complement in $S$ and $S$ a is clean semiring, Theorem 2.3.3 implies that idempotents lift modulo $I$. Hence idempotents lift modulo every $k$-ideal of $S$. Let $x \neq 0 \in S$. If $(x^2 + x)$ is a unit in $S$ then $x$ and $(1 + x)$ become unit which implies $\overline{(x)} = S$ and $\overline{(1+x)} = S$, so the proof is done. Let $(x^2 + x)$ be non unit in $S$. Now $(x + 1) + 2x(x + 1) = x + 1 + 2x^2 + 2x = 2x^2 + 3x + 1 = x^2 + x + x + 1 + x^2 + x = (x + 1)^2 + x(x + 1)$. So $(x + 1) + \overline{(x^2 + x)} = (x + 1)^2 + \overline{(x^2 + x)}$. Since idempotents lift modulo every $k$-ideal of $S$, there exists an idempotent $e^2 = e \in S$ such that $(x + 1) + \overline{(x^2 + x)} = e + \overline{(x^2 + x)}$. Let $e_1$ be an absorbing complement of $e$. Then $(e_1 + x) + \overline{(x^2 + x)} + [1 + \overline{(x^2 + x)}] = 1 + \overline{(x^2 + x)}$. Let $(x^2 + x) = 0$. Since $S$ is antisimple and $x \neq 0$, $(x^2 + x) = 0$ implies that 1 has an additive inverse in $S$. So we have $e \in \overline{(1 + x)}$ and $e_1 \in \overline{(x)}$. Let $(x^2 + x) \neq 0$. Then $\overline{(x^2 + x)}$ is a nonzero proper $k$-ideal of $S$, otherwise $(x^2 + x)$ becomes unit, since the $k$-closure of every proper ideal is proper. Hence from Note 2.3.4, we have $(e_1 + x) + \overline{(x^2 + x)} = \overline{(x^2 + x)}$ which implies that $(e_1 + x) \in \overline{(x^2 + x)}$. So $e_1 + x \in \overline{(x)}$ which implies that $e_1 \in \overline{(x)}$. Now

$(e_1+x) \in \overline{(1+x)}$. Thus there exist $a, b \in S$ such that $e_1+x+a(1+x) = b(1+x) \Longrightarrow e+e_1+x+a(1+x) = b(1+x)+e \Longrightarrow (1+x)+a(1+x) = b(1+x)+e$. So $e \in \overline{(1+x)}$. Since $x$ is arbitrary nonzero element of $S$, it follows that $S$ is an exchange semiring.

Conversely, suppose that $S$ is an exchange semiring. Let $x \neq 0 \in S$. If $x$ is a unit then $x$ can be written as the sum of an idempotent and a unit. Let $x$ be not a unit element in $S$. Since $S$ is antisimple, we have $x = x' + 1$, where $x' \neq 0 \in S$ otherwise $x$ becomes a unit in $S$. Since $S$ is a exchange semiring and $x' \neq 0$, there exists an idempotent $e^2 = e$ such that $e \in \overline{(1+x')}$ and $e_1 \in \overline{(x')}$, where $e_1$ is an absorbing complement of $e$. This implies that $e^2 = e \in \overline{(x)}$ and $e_1^2 = e_1 \in \overline{(x')}$. Thus there exist $a, b, a_1, b_1 \in S$ such that $e + ax = bx$, $ea = a = ae$, $eb = b = be$ and $e_1 + a_1x' = b_1x'$, $e_1a_1 = a_1 = a_1e_1$, $e_1b_1 = b_1 = b_1e_1$. Since $e_1$ is an absorbing complement of $e$, it follows that $e+e_1 = 1$, $e+see_1 = e$ and $e_1+e_1es = e_1$ for all $s \in S$. So $a_1e + a_1 = a_1$ which implies that $a_1e + a_1e = a_1e$. Now $(a + b + a_1 + b_1)(x' + e) = ax'+bx'+a_1x'+b_1x'+ae+be+a_1e+b_1e = ax'+a+bx'+b+a_1x'+b_1x'+a_1e+b_1ee_1 = ax+bx+a_1x'+b_1x'+a_1e+b_1ee_1 = ax+e+ax+a_1x'+e_1+a_1x'+a_1e+a_1e+b_1ee_1 = 1 + ax' + ae + ax' + ae + a_1x' + a_1e + a_1x' + a_1e = 1 + 2a(x' + e) + 2a_1(x' + e) = 1+(2a+2a_1)(x'+e)$. Since $S$ is commutative, we find that $(x'+e)$ is semi-invertible. Since the $k$-closure of every proper ideal is proper, $(x' + e)$ is an unit of $S$. Thus $x = x' + 1 = (x' + e) + e_1$=unit+idempotent. So $x$ is a clean element in $S$. Since $x$ is arbitrary nonzero element of $S$, it follows that $S$ is a clean semiring. □

**Theorem 2.3.7.** *A Boolean semiring $S$, is clean iff it is antisimple.*

*Proof.* Let $S$ be an antisimple Boolean semiring. We have to show that $S$ is a clean semiring. Let $x \neq 0 \in S$. Since $S$ is antisimple, we can write $x = x' + 1$ for some $x' \in S$. Since $S$ is Boolean, $x'$ is an idempotent of $S$ and 1 is a unit element of $S$. So $x$ is clean and hence $S$ is a clean semiring.

Conversely, let $S$ be a clean semiring. Let $x \neq 0 \in S$. Then $x$ can be written as the sum of an idempotent and a unit in $S$. Since 1 is the only unit in $S$, we can write $x = e + 1$ for some $e^2 = e$. Thus $S$ is an antisimple semiring. □

Now we give an example of an antisimple Boolean semiring i.e. a clean semiring.

**Example 2.3.8.** *Consider the set $S = \{0, 1, 2\}$. Define the operations "+" and "." on $S$ as follows :*

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 1 |
| 2 | 2 | 1 | 2 |

| . | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 2 |

*Then $(S, +, .)$ is an antisimple Boolean semiring and hence it is a clean semiring.*

**Definition 2.3.9.** A semiring $S$ is called strongly regular if for every element $r \in S$ there exist $x, y \in S$ such that $r^2x = r$ and $yr^2 = r$.

**Theorem 2.3.10.** *Let $S$ be an antisimple and strongly regular semiring such that every idempotent of $S$ has an absorbing complement. Then $S$ is a clean semiring.*

*Proof.* To show that $S$ is a clean semiring we have to show that every nonzero element $r \in S$ is the sum of an idempotent and a unit in $S$. If $r$ is a unit in $S$ then we are done. So let us consider $r \neq 0 \in S$ and $r$ is not a unit in $S$. Since $S$ is an antisimple semiring, so there exists $r' \in S$ such that $r = r' + 1$, where $r' \neq 0$. Since $S$ is a strongly regular semiring, for $r' \in S$ there exists $z \in S$ such that $r'zr' = r'$ with $r'z = zr'$, by Proposition 1.3.29. Let us consider $e = r'z$. Then $e^2 = e$ and hence $e$ is an idempotent of the semiring $S$. Also $e$ must be non zero otherwise $r' = 0$. If $e = 1$ then $r'z = zr' = 1$. This shows that $r'$ is an unit and hence $r$ is clean. So let $e \neq 1$. Since every idempotent of $S$ has an absorbing complement, so there exists an idempotent $e' \in S$ such that $e + e' = 1$, $e + ree' = e$ and $e' + e'er = e'$ for all $r \in S$. Here $e' \neq 0$ otherwise $e = 1$. Let $u = r' + e'$ and $v = ze + e'$. Now $uv = (r' + e')(ze + e') = r'ze + r'e' + e'ze + e' = ee + r'zr'e' + e'zr'z + e' = e + r'r'ze' + e'r'zz + e' = e + r'ee' + e'ez + e' = e + e' = 1$. Similarly, $vu = 1$. So $uv = vu = 1$ and hence $u$ is a unit in $S$. Now we have $u + e = (r' + e') + e = r' + (e' + e) = r' + 1 = r$. So $r$ can be written as the sum of an idempotent and a unit in $S$ i.e. $r$ is a clean element in $S$. Since $r$ is an arbitrary nonzero element of $S$, it follows that $S$ is a clean semiring. □

**Example 2.3.11.** *In Example 2.2.3 (vi), we observe that $S$ is an antisimple and strongly regular semiring in which every idempotent of $S$ has absorbing complement. So $S$ is a clean semiring.*

The converse of Theorem 2.3.10 is not true. For this we consider the following example.

**Example 2.3.12.** *Take $S = \mathbb{N}_0^+$, the semiring of all non-negative integers with respect to the usual addition and multiplication. Let us consider $I = 4\mathbb{N}_0^+$. Then $I$ is a $k$-ideal of $S$. Now for all $a, b \in S$, $ak_Ib \Longleftrightarrow a + 4i_1 = b + 4i_2$ for some $i_1, i_2 \in S$, defines a congruence $k_I$ of $S$, where $k_I$ denotes the congruence defined by $I$. Let us consider the set of congruence classes $S' = S/k_I$ with respect to $k_I$. Then $S' = \{[0], [1], [2], [3]\}$ becomes a ring with respect to the operations $[a] + [b] = [a + b]$ and $[a][b] = [ab]$ for all $[a], [b] \in S'$. Since every ring is a semiring, $S'$ is an antisimple clean semiring and every idempotent of $S'$ has absorbing complement. But $S'$ is not strongly regular as $[2]$ can not be written as $[2] = [2]^2[a]$ for any $[a] \in S'$.*

From above Theorem 2.3.10, we have the following result :

**Corollary 2.3.13.** *Let $S$ be a commutative antisimple reduced regular semiring such that every idempotent of $S$ has an orthogonal complement. Then $S$ is a clean semiring.*

## 2.4 Quotient Semiring of Clean Semiring

**Theorem 2.4.1.** *[11, 16] Let $S$ be a semiring with identity 1. Then each proper $k$-ideal (left $k$-ideal) of $S$ is contained in a maximal $k$-ideal (maximal left $k$-ideal) of $S$.*

Theorem 2.2.5 implies that if $S$ is clean semiring then $S/I$ is also clean, but the converse is not true in general, which follows from Example 2.3.12. In this example, $S' = S/k_I = S/I$ is clean semiring since, $[1] = [0] + [1]$, $[2] = [1] + [1]$, $[3] = [0] + [3]$, but $S = \mathbb{N}_0^+$ is not clean. The converse holds for some class of semirings.

Finally, we have the following characterization of clean semiring.

**Theorem 2.4.2.** *Let $S$ be a commutative antisimple reduced semiring such that the $k$-closure of every proper ideal of $S$ is proper and every idempotent has a complement in $S$. Let $I$ be a nonzero $k$-ideal of $S$ and $J_l(S)$ be the Jacobson radical of $S$ such that $I \subseteq J_l(S)$. Then $S$ is clean if and only if $S/I$ is clean and idempotents lift modulo $I$.*

*Proof.* Suppose that $S/I$ is clean and idempotents lift modulo $I$. To show that $S$ is clean we have to show that every nonzero element of $S$ is clean. Let $r \in S$ and $r \neq 0$. Then either $r$ is a unit or a nonzero non unit element of $S$. If $r$ is a unit then $r$ is a clean element in $S$. Let $r \neq 0$ and $r$ be a non unit. Now there are two possibilities : either $r \in I$ or $r \notin I$. If $r \in I$ then $r + I = I$ which is the zero element of $S/I$. Now since $S$ is antisimple and $r \neq 0$, so $r = r' + 1$, where $r' \neq 0$ otherwise $r$ will be a unit element in $S$. Now if $r'$ is non unit then $(r')$ is a proper ideal of $S$. We assume that $k$-closure of every proper ideal is proper, hence $\overline{(r')}$ is a proper $k$-ideal of $S$. since every proper $k$-ideal of $S$ is contained in some maximal $k$-ideal by Theorem 2.4.1, $r' \in M'$, where $M'$ is some maximal $k$-ideal of $S$. Since $S$ is commutative semiring, $J_l(S)$ is the intersection of all maximal $k$-ideals of $S$. Now $r \in I \subseteq J_l(S)$. So we have $r \in M'$ and hence $1 \in M'$ which contradicts that $M'$ is a maximal $k$-ideal of $S$. Thus $r'$ must be a unit of $S$. Hence $r$ is clean. Let $r \notin I$. Then $r + I$ is a nonzero element of $S/I$. Since $S/I$ is clean, so $r+I = (e+I)+(x+I)$, where $e+I$ is an idempotent and $x+I$ is a unit of $S/I$. Now $(e+I)^2 = e^2+I = e+I$. Since idempotents lift modulo $I$, there exist an idempotent $e_1 \in S$ such that $e + I = e_1 + I$. So we have $(r + I) = (e_1 + I) + (x + I)$, where $e_1^2 = e_1 \in S$. Now $I \subseteq J_l(S)$ and $x + I$ is a unit of $S/I$. We have to show that $x$ is a unit of $S$. Since $x + I$ is a unit of $S/I$, so there exists $y + I \in S/I$ such that $(x + I)(y + I) = 1 + I \Longrightarrow xy + I = 1 + I \Longrightarrow (xy + x') + I = I$, by Note 2.3.4. So $(xy + x') \in I \subseteq J_l(S)$, since $I$ is a $k$-ideal. Now if $x$ is non unit, $(x)$ is a proper ideal of $S$. Thus $\overline{(x)}$ is a proper $k$-ideal of $S$ according to our assumption. Since each proper $k$-ideal of $S$ is contained in a maximal $k$-ideal of $S$ by Theorem 2.4.1, $x \in M$, where $M$ is a maximal $k$-ideal of $S$. Therefore, $xy \in M$. So we have $x' \in M$. Now $1 + x' \in J_l(S)$ and $x' \in M$, hence $1 \in M$, which contradicts that $M$ is proper. Hence $x$ is a unit of $S$. Now $r+I = (e_1+I)+(x+I)$, where $e_1 \notin I$, otherwise $r+I$ will be a unit of $S/I$ which implies that $r$ is a unit of $S$. But we take $r$ is a non unit of $S$. Now

$e_1$ is an idempotent element and $x$ is a unit element of $S$. Thus either $e_1 = 1$ or $e_1$ is an idempotent except 1. Suppose that $e_1 \neq 1$. From given condition $e_1$ has a complement in $S$, so there exists an idempotent $e_1'' \in S$ such that $e_1'' + e_1 = 1$ and $e_1'' \neq 0$ otherwise $e_1 = 1$. Now $r + I = (e_1 + I) + (x + I) \Longrightarrow (r + I) + (e_1'' + I) = (x + I) + (1 + I)$. Since $r \notin I$, so $r \neq 0$. Since $S$ is antisimple, there exists $r'' \in S$ such that $r = r'' + 1$, where $r'' \notin I$ otherwise $r$ becomes a unit in $S$. Now $((r'' + e_1'') + I) + (1 + I) = (x + I) + (1 + I)$. From Note 2.3.4, we have $(r'' + e_1'') + I = x + I$. Since $x + I$ is a unit of $S/I$, we have $(r'' + e_1'') + I$ is a unit of $S/I$ and since $I \subseteq J_l(S)$, we have $r'' + e_1''$ is a unit of $S$. Thus we can write $r = r'' + 1 = r'' + (e_1 + e_1'') = e_1 + (r'' + e_1'')$, where $e_1$ is an idempotent and $r'' + e_1''$ is a unit in $S$. So $r$ can be written as the sum of an idempotent and a unit of $S$. Now if $e_1 = 1$, then we have $r + I = (x + I) + (1 + I)$. Now $r = r'' + 1$. So $r'' + I = x + I$. Since $x + I$ is a unit of $S/I$, so $r'' + I$ is a unit of $S/I$. Similarly, we can say that $r''$ is a unit of $S$. Hence we can write $r = 1 + r''$. So every nonzero element of $S$ is written as the sum of an idempotent and a unit in $S$. Hence $S$ is a clean semiring.

Conversely, suppose that $S$ is a clean semiring. We have to show that $S/I$ is a clean semiring. Let us define a mapping $\phi : S \longrightarrow S/I$ by $\phi(x) = x + I$ for all $x \in S$. Then $S/I$ is a homomorphic image of the clean semiring $S$. Hence $S/I$ is clean, by Theorem 2.2.5. Now since $I = (1 + I) + (x' + I)$ and every idempotent of $S$ has a complement in $S$, so we get idempotents lift modulo $I$ by Theorem 2.3.3. $\square$

# Chapter 3

# Strongly Clean Semiring

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# Strongly Clean Semiring

## 3.1 Introduction

There is a large literature about clean ring, strongly clean ring and related areas in ring theory. Idempotents and units play a very crucial role for determining the structure of strongly clean rings. An element of a ring is called clean if it is the sum of an idempotent and a unit where as an element of a ring is called strongly clean if it is the sum of an idempotent and a unit which commute in that ring. A ring $R$ is called clean ring (resp. strongly clean) if every element of $R$ is clean (resp. strongly clean). Now a ring $R$ is called a unit regular ring if for any $a \in R$, $a = aua$ for some unit $u$ in $R$. Equivalently, one can say that a ring $R$ is unit regular if $a = eu$ for some idempotent $e$ and unit $u$ in $R$. In ring theory there is another notion of regularity which is known as strongly regular ring. The class of strongly regular ring is stronger than the class of unit regular rings. An element $a \in R$ is said to be strongly regular if there exists an element $x \in R$ such that $a = a^2x$ with $ax = xa$ and this is equivalent to say that $a = ue = eu$ for some $e^2 = e \in R$ and $u \in U(R)$. A ring $R$ is called a strongly regular ring if every element of $R$ is strongly regular. Clean rings are additive analog of unit regular rings where as strongly clean rings are additive analog of strongly regular rings. Surprisingly, every strongly regular ring is a strongly clean ring. In general, clean rings are not strongly clean. But in the class of commutative rings and also in the class of rings with central idempotents, both the notions clean and strongly clean coincide. R. B. Warfield [4] defined a ring $R$ to be an exchange ring if the left $R$-module $R$ has finite exchange property. W. K. Nicholson [7] proved that a ring $R$ is an exchange ring if and only if for each $a \in R$, there exists an idempotent $e$ in $R$ such that $e \in Ra$ and $(1-e) \in R(1-a)$. A ring with central idempotents is clean if and only if it is an exchange ring. In 1999, Nicholson [26] introduced the notion of strongly clean ring. Since then various results of strongly clean rings have been obtained by some authors ([26], [34], [29], [32], [36]).

There has been a remarkable growth of the theory of semirings and their applications in several branches of mathematics. We have recently introduced the notion of clean semiring in [62] as a generalization of clean ring. The main motivation of this article is to give some characterizations of strongly clean semiring and obtain some results related to strongly clean semiring. We also study the notion of exchange semiring and find out the connection between strongly clean semirings and exchange semirings with the help of some other class of semirings.

## 3.2 Definition & Examples

In the previous chapter we give the definition of strongly clean semiring (Definition 2.3.1). Clearly, every commutative clean semiring is a strongly clean semiring.

**Remark 3.2.1.** *Let $S$ be a semiring with the zero element $0$ and identity element $1$ and $a \in S$. Then an element $b$ of $S$ is an additive inverse of $a$ if and only if $a+b=0$. Denote the set of all elements of $S$ having additive inverses by $V(S)$. The set $V(S)$ is non-empty as $0 \in V(S)$.*

*It can be easily seen that a semiring $S$ is a ring if and only if $V(S)=S$.*

*Now suppose that in a semiring $S$; $0 = e+u$, where $e^2 = e$ and $u \in U(S)$. Since $u \in U(S)$, there exists $v \in S$ such that $uv = vu = 1$. So we have $0 = 0v = (e+u)v = ev + uv = ev + 1$. Let $b = ev$. Then $1 + b = 0$. Thus $x + xb = x(1+b) = x0 = 0$ for all $x \in S$. So $x \in V(S)$ and hence $S = V(S)$. This shows that $S$ becomes a ring.*

*To avoid this situation, we take a semiring $S$ is strongly clean if every nonzero element of $S$ can be written as the sum of an idempotent and a unit in $S$ that commute* *with each other.*

**Example 3.2.2.** *Every strongly clean ring is an example of a strongly clean semiring.*

*But we give some examples of strongly clean semirings that are not strongly clean rings.*

1. *Every semifield (division semiring) is an example of a strongly clean semiring as its every nonzero element is a unit.*

2. *Consider $S = \{0, 1, 2, 3\}$. Define the operations "+" and "." on $S$ as follows :*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 1 |
| 2 | 2 | 3 | 1 | 2 |
| 3 | 3 | 1 | 2 | 3 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 1 | 3 |
| 3 | 0 | 3 | 3 | 3 |

*Take $S_1 = \mathbb{N}_0^+$, the semiring of all non-negative integers with respect to the usual addition and multiplication. Let us consider $I = 4\mathbb{N}_0^+$. Then $I$ is a $k$-ideal of $S_1$. Now for all $a, b \in S_1$, $ak_Ib \iff a + 4i_1 = b + 4i_2$ for some $i_1, i_2 \in S_1$, defines a congruence $k_I$ on $S_1$, where $k_I$ denotes the congruence defined by $I$. Let us consider the set of congruence classes $S' = S_1/k_I$ with respect to $k_I$. Then $S' = \{[0], [1], [2], [3]\}$ becomes a semiring with respect to the operations $[a] + [b] = [a + b]$ and $[a][b] = [ab]$ for all $[a], [b] \in S'$. Now consider $S'' = (S \times S') \setminus \{(0, [1]), (0, [2]), (0, [3])\}$. Then $S''$ is a strongly clean semiring with respect to component wise addition and multiplication.*

3. *Consider $S = \{0, 1, 2\}$. Define the operations "+" and "." on $S$ as follows :*

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 1 |
| 2 | 2 | 1 | 2 |

| . | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 2 |

*Let $(D,+,.)$ be a division ring. Now consider $S_1 = [D \times (S \setminus \{0\})] \bigcup \{(0_D, 0)\}$. Then $S_1$ is a strongly clean semiring with respect to component wise addition and multiplication.*

## 3.3 Elementary Results

**Theorem 3.3.1.** *Let $S$ be an additively absorbing semiring with identity $1$. Suppose that $1+1=2$. If $2 \in U(S)$, then $S$ is strongly clean if and only if every nonzero element belongs to the set $S_1 = \{a \in S \setminus \{0\} : a = (z+1)+u, z^2 = 1, uz = zu, u \in U(S)\}$.*

*Proof.* Let $S$ be a strongly clean semiring and $a \neq 0$. Then $2^{-1}a \neq 0$, since $2 \in U(S)$. Since $S$ is strongly clean, $2^{-1}a = e + u$ and $eu = ue$ for some idempotent $e$ and unit $u$ in $S$. This implies that $a = 2e + 2u$. Since $u$ is a unit and $S$ is an additively absorbing semiring, $u + 1 + 1' = u[1 + u^{-1} + (u^{-1})'] = u1 = u$. Then $a = 2e + 2u = 2e + (2u + 1 + 1') = [(2e + 1') + 1] + 2u$. Now $(2e + 1')(2e + 1') = 2e2e + 2e' + 2e' + 1 = 2e + 2e + 2e' + 2e' + 1 = 1$. $(2e + 1')2u = 2e2u + 1'2u = 22ue + 2u1' = 2u2e + 2u1' = 2u(2e + 1')$. Hence $a \in S_1$.

Conversely, suppose that $a \neq 0$. Then $2a \neq 0$ and $2a \in S_1$. So $2a = (z+1)+u, z^2 = 1, uz = zu$ for some $u \in U(S)$. This implies that $a = 2^{-1}(z+1) + 2^{-1}u$. Now $2^{-1}(z+1)2^{-1}(z+1) = 2^{-1}2^{-1}(z^2+z+z+1) = 2^{-1}2^{-1}2(z+1) = 2^{-1}(z+1)$ and $2^{-1}u$ is a unit, since $2 \in U(S)$. Also $2^{-1}(z+1)2^{-1}u = 2^{-1}2^{-1}(zu+u) = 2^{-1}2^{-1}(uz+u) = 2^{-1}u2^{-1}(z+1)$, since $2 \in U(S)$ and $2s = s2$ for all $s \in S$. So $2^{-1}s = s2^{-1}$ for all $s \in S$. Thus $a$ is a strongly clean element in $S$ and hence $S$ is a strongly clean semiring. □

**Proposition 3.3.2.** *Let $S$ be an additively absorbing semiring and $e^2 = e(\neq 0) \in S$ such that $(e)_l$ be a full principal left ideal of $S$. If $a(\neq 0) \in S$ is strongly clean in $eSe$, then $a$ is strongly clean in $S$.*

*Proof.* Suppose that $a = f + v$ and $fv = vf$, where $f^2 = f \in eSe$, $v \in eSe$ is a unit. Clearly, $e$ is the identity of $eSe$. Thus there exists $w \in eSe$ such that $vw = wv = e$. Then $u = v + (1' + e)$ is a unit in $S$ (with $u^{-1} = w + (1' + e)$) and $e_1 = f + (1 + e')$

is an idempotent because $(1+e')$ is an absorbing complement of the idempotent $e$. Now $e_1+u = f+(1+e')+v+(1'+e) = f+v+e'+e+1+1'$. Again $E^+(S) \subseteq (e)_l$. Since $1+1' \in E^+(S)$, we have $e+1+1' = se$ for some $s \in S$. Thus $e'+1+1' = s'e$. Hence $e+e'+1+1' = se+s'e$. So $e_1+u = f+v+se+s'e = f+[e+(se+s'e)w]v = f+[e+sw+s'w]v = f+1v = f+v = a$, since $S$ is an additively absorbing semiring. Now $e_1u = [f+(1+e')][v+(1'+e)] = fv+f'+fe+v+1'+e+e'v+e+e' = fv+f'+f+v+1'+v'+e = fv+1'+e$. Similarly, $ue_1 = fv+1'+e$, since $vf = fv$. Thus $e_1+u = a$ and $e_1u = ue_1$. Hence $a$ is a strongly clean element in $S$. $\square$

pro

**Definition 3.3.3.** The left annihilator of an element $a$ of a semiring $S$ is $l(a) = \{s \in S : a+sa = a\}$ and the right annihilator of $a$ is $r(a) = \{s \in S : a+as = a\}$.

**Theorem 3.3.4.** *Let $S$ be a strongly clean centroid semiring such that every idempotent of $S$ has an orthogonal complement in $S$. Then $eSe$ is strongly clean semiring for all $e^2 = e(\neq 0) \in S$.*

*Proof.* Let $a(\neq 0) \in eSe$. Then $a \in S$. Since $S$ is strongly clean, $a = g+u$, where $g^2 = g \in S$ and $u$ is a unit such that $gu = ug$. As $g$ has a orthogonal complement in $S$, there exists $g_1^2 = g_1 \in S$ such that $g+g_1 = 1$ and $gg_1 = g_1g = 0$. Let $x \in r(a)$. Then $a+ax = a$ i.e. $(g+u)+(g+u)x = g+u$ i.e. $g(1+x)+u(1+x) = g+u$ $\ldots$ $(i)$. By multiplying on both sides by $g_1$ from left, we have $g_1u(1+x) = g_1u$. Now $g+g_1 = 1 \Longrightarrow gu+g_1u = gu+ug_1$, since $gu = ug$. Hence $g_1u = g_1ug_1$. Similarly, $ug_1 = g_1ug_1$. So $g_1u = ug_1$. Thus it follows that $ug_1(1+x) = ug_1 \Longrightarrow g_1(1+x) = g_1$ $\ldots(ii)$, since $u$ is a unit in $S$. Hence $x \in r(g_1)$. So $r(a) \subseteq r(g_1)$. Now adding equations $(i)$ and $(ii)$, we find that $1+x+u+ux = 1+u \Longrightarrow (1+u)+(1+u)x = (1+u)$. Hence $x \in r(1+u)$. So $r(a) \subseteq r(1+u)$. Similarly, $l(a) \subseteq l(g_1)$ and $l(a) \subseteq l(1+u)$. Now $e$ has an orthogonal complement $e_1$ in $S$, so $e+e_1 = 1$ and $ee_1 = e_1e = 0$. Hence $ae_1 = 0 \Longrightarrow a+ae_1 = a$. So $e_1 \in r(a) \subseteq r(1+u)$. Thus it follows that $(1+u)+(1+u)e_1 = (1+u) \Longrightarrow 1+u+e_1+ue_1 = 1+u$. Similarly, $l(a) \subseteq l(1+u)$ implies that $1+u+e_1+e_1u = 1+u$. Hence $1+u+e_1+ue_1 = 1+u+e_1+e_1u$. Multiply from left on both side by $e$ and from right on both side by $e_1$ and apply $ee_1 = e_1e = 0$, it gives $eue_1 + eue_1 = eue_1$. Now $eue_1 + eu'e_1 = e(u + u')e_1 = ee_1(u + u') = 0$. Hence $eue_1 = 0 \Longrightarrow eu = eue$, adding both side by $eue$ and applying $e + e_1 = 1$. Similarly, $ue = eue$ which implies that $eu = ue$. Now $r(a) \subseteq r(g_1)$ and $l(a) \subseteq l(g_1)$. Hence $e_1 \in r(g_1)$ and $e_1 \in l(g_1)$. So $g_1 + g_1e_1 = g_1$ and $g_1 + e_1g_1 = g_1$. By using $gg_1 = g_1g = 0$, we have $g_1e_1g = 0 \dots (iii)$ and $ge_1g_1 = 0 \dots (iv)$. Adding $ge_1g$ on both side of $(iii)$, $(g + g_1)e_1g = ge_1g \Longrightarrow e_1g = ge_1g$. Similarly, $ge_1 = ge_1g$. Hence $ge_1 = e_1g$. Now $e + e_1 = 1 \Longrightarrow eg + e_1g = ge + e_1g \Longrightarrow eg = ege$, multiplying from left on both side by $e$. Similarly, $ge = ege$. Hence $eg = ge$. Now $(ege)(ege) = egege = eggee = ege$. So $ege$ is an idempotent of $eSe$. Now $eu = ue$ implies that $(eue)(eu^{-1}e) = eueu^{-1}e = euu^{-1}e = e = e1e = eu^{-1}ue = eu^{-1}uee = eu^{-1}eue = (eu^{-1}e)(eue)$. So $eue$ is a unit in $eSe$. Now $a = g + u$ and $a = ese$ for some $s \in S$, $e^2 = e$. So $eae = ese = a$. Therefore, $a = eae = e(g + u)e = ege + eue$. Finally, $(ege)(eue) = egeue = eegue = egue = euge = eugee = euege = (eue)(ege)$. Hence $a$ is a strongly clean element in $eSe$. Consequently, $eSe$ is a strongly clean semiring. □

Since semiring is a generalization of rings and bounded distributive lattices, we have the following remark:

**Remark 3.3.5.** *In Theorem 3.3.4 if we consider* $(S, +, .)$ *as a ring with identity* $1$ *then for each element* $a \in S$ *there exists an unique element* $(-a)$ *such that* $a + (-a) = 0$. *Hence* $a' = (-a)$. *For any* $a, b \in S$, *it also implies that* $a(b - b) = (b - b)a$. *Let* $e$ *be any idempotent of* $S$. *Then* $(1 - e)$ *is an orthogonal complement of* $e$ *as* $(1 - e)^2 = (1 - e)$, $e + (1 - e) = 1$ *and* $e(1 - e) = (1 - e)e = 0$. *Thus* $S$ *becomes a strongly clean centroid ring such that every idempotent has an orthogonal complement in* $S$. *Hence* $eSe$ *is strongly clean ring for all* $e^2 = e(\neq 0) \in S$ *by Theorem 3.3.4, since every ring is a semiring.*

*Let* $(S, \vee, \wedge)$ *be a bounded distributive lattice in Theorem 3.3.4. Let* $1$ *is the greatest element and* $0$ *is the least element of* $S$. *Hence* $s \vee 0 = s$, $s \wedge 0 = 0$ *and* $s \vee 1 = 1$, $s \wedge 1 = s$ *for all* $s \in S$. *Hence* $1$ *is the identity and* $0$ *is the absorbing zero element of* $S$. *Let there exist two elements* $u, v \in S$ *such that* $u \wedge v = 1$. *Then* $u = u \wedge 1 = u \wedge (u \wedge v) = (u \wedge u) \wedge v = u \wedge v$, *since* $s \wedge s = s$ *for all* $s \in S$. *Hence* $u = 1$. *Thus*

1 *is the only unit element of* $S$*. If* $a$ *is the clean element of* $S$ *then* $a = e \vee 1 = 1$*. If* $S$ *is strongly clean bounded distributive lattice then* $S = \{0, 1\}$*. Thus* $(S, \vee, \wedge)$ *is centroid semiring such that every idempotent of* $S$ *has an orthogonal complement in* $S$*. Hence* $eSe$ *is strongly clean semiring for all* $e \neq 0 \in S$ *by Theorem 3.3.4.*

Note that for a nonzero idempotent $e$ in $S$, $eSe$ is strongly clean does not always imply that $S$ is strongly clean which follows from the following example :

**Example 3.3.6.** *Consider* $S = \{1, 2, 3, 6\}$*, the subset of natural numbers* $\mathbb{N}$*. Now define two binary operations* $\wedge$ *and* $\vee$ *on* $S$ *by* $a \wedge b = gcd(a, b)$ *and* $a \vee b = lcm(a, b)$ *for all* $a, b \in S$*. Then* $(S, \vee, \wedge)$ *is a bounded distributive lattice. Hence* $S$ *is a commutative semiring. Let* $2 \in S$*. Then* $2 \wedge S \wedge 2 = \{1, 2\}$ *which is strongly clean semiring, since* $2 = 1 \vee 2 =$ *idempotent+unit. But* $S$ *is not strongly clean semiring.*

## 3.4 Connection With Some Classes of Semirings

**Theorem 3.4.1.** *Let* $S$ *be a strongly clean inverse semiring such that every idempotent has an absorbing complement in* $S$*. Then* $S$ *is an exchange semiring.*

*Proof.* Let $y \neq 0$. Then $y' \neq 0$. Since $y'$ is strongly clean, we can write $y' = e + u$, $eu = ue$, where $e^2 = e \in S$ and $u$ is a unit of $S$. So $ey' = e + eu \Longrightarrow ey + ey' = e + eu + ey \Longrightarrow e(y + y') = e(1 + y) + eu$. Now $y + y' + y = y \Longrightarrow (1 + y) + (y + y') = (1+y) \Longrightarrow (y+y') \in \overline{S(1 + y)}$. So $eu = ue \in \overline{S(1 + y)} \Longrightarrow e \in \overline{S(1 + y)}$. Now $e$ has an absorbing complement $e_1^2 = e_1 \in S$. Thus $e_1 + e = 1 \Longrightarrow e + e_1 e = e \Longrightarrow e + u + e_1 e = e + u \Longrightarrow y' + e_1 e = y' \Longrightarrow e_1 e \in \overline{Sy'}$. Again $e_1 y' = e_1 e + e_1 u \Longrightarrow e_1 u \in \overline{Sy'}$. Since $eu = ue$, $uy' = y'u$ which implies $u^{-1}y' = y'u^{-1}$. Hence $e_1 \in \overline{Sy'} = \overline{Sy}$, since $y + y' + y = y$ and $(S, +)$ is commutative. Hence $S$ is an exchange semiring. □

Since semiring is a generalization of rings and bounded distributive lattices, we have the following remark:

**Remark 3.4.2.** *If* $(S, +, .)$ *is a strongly clean ring in Theorem 3.4.1 then similarly by Remark 3.3.5, it follows that* $S$ *is an inverse semiring such that every idempotent has* *an absorbing complement in $S$. Hence Theorem 3.4.1 implies that $S$ is an exchange semiring.*

*If $(S, \vee, \wedge)$ be a strongly clean bounded distributive lattice in Theorem 3.4.1 then $S = \{0, 1\}$ by Remark 3.3.5, where $0$ is the least element and $1$ is the greatest element of $S$. This implies $(S, \vee, \wedge)$ is an inverse semiring such that every idempotent of $S$ has an absorbing complement in $S$. Hence $S$ is an exchange semiring by Theorem 3.4.1*

The converse of the Theorem 3.4.1 is not true which follows from the following example :

**Example 3.4.3.** *Consider $S = \{1, 2, 3, 6\}$, the subset of natural numbers $\mathbb{N}$. Now define two binary operations $\wedge$ and $\vee$ on $S$ by $a \wedge b = gcd(a, b)$ and $a \vee b = lcm(a, b)$ for all $a, b \in S$. Then $(S, \vee, \wedge)$ is a bounded distributive lattice and hence a semiring. Now $S$ is an inverse exchange semiring such that every idempotent has absorbing complement in $S$, but $S$ is not strongly clean semiring.*

**Definition 3.4.4.** A semiring $S$ is said to be $k$-semipotent if every nonzero left $k$-ideal $I$ of $S$ that is not contained in $J_l(S)$, contains a nonzero idempotent.

**Proposition 3.4.5.** *Let $S$ be an inverse exchange semiring such that the set $[S \setminus U'(S)] \bigcap E^+(S)$ is contained in every nonzero principal left $k$-ideal of $S$. Then $S$ is $k$-semipotent.*

*Proof.* Let $S$ be an inverse exchange semiring. Let $\overline{(x)_l}$ be a nonzero proper left $k$-ideal of $S$ such that $x \notin J_l(S)$. Then to show that $S$ is $k$-semipotent we have to show that there exists $e^2 = e \neq 0$ such that $e \in \overline{(x)_l}$. Suppose $e^2 = e \in \overline{(x)_l}$ implies that $e = 0$. Now if $a \in S$ such that $ax = 0$ then $1 + ax = 1$ is left $k$-unit. Let $a \in S$ such that $ax \neq 0$. Since $S$ is exchange, there exists $e^2 = e \in S$ such that $e \in \overline{(1 + ax)_l}$ and $e_1 \in \overline{(ax)_l}$, where $e_1$ is an absorbing complement of $e$ in $S$. Now $e_1 \in \overline{(ax)_l}$ implies that $e_1 \in \overline{(x)_l}$. So according to the assumption, $e_1 = 0$. Hence $1 \in \overline{(1 + ax)_l}$. This implies that $(1 + ax)$ is left $k$-unit for all $a \in S$. The claim is that $x \in J_l(S)$. If $x \notin J_l(S)$ then there exists a maximal left $k$-ideal $M$ such that $x \notin M$. Here $M$ must be a nonzero maximal left $k$-ideal otherwise $1 \in \overline{(x)_l}$ which contradicts that $\overline{(x)_l}$ is proper. Let $M_1 = \overline{M + (x)_l}$. Then $M_1$ is the left $k$-ideal containing $M$. Hence $1 \in M_1$. So we have $1 + (m_1 + s_1x) = (m_2 + s_2x)$, where $m_1, m_2 \in M$ and $s_1, s_2 \in S$. Since $x \notin U'(S)$, $(s_2x)+(s_2x)' \in [S \setminus U'(S)] \bigcap E^+(S)$. So $1+(m_1+s_1x)+(s_2x)' = m_2+s_2x+(s_2x)' \Longrightarrow 1+(s_1+s_2')x+m_1 = m_2+(s_2x)+(s_2x)'$. Now $b = 1 + (s_1 + s_2')x$ is left $k$-unit. Since $M$ is nonzero maximal left $k$-ideal, $(s_2x) + (s_2x)' \in M$ according to the condition. Hence $b \in M$ which contradicts that $M$ is maximal. Hence $x \in J_l(S)$. So our assumption is wrong. Thus there exists $e^2 = e \neq 0$ such that $e \in \overline{(x)_l}$. Hence $S$ is a $k$-semipotent semiring. □

**Definition 3.4.6.** A nonzero idempotent $e$ of a semiring $S$ is a primitive idempotent if for any nonzero idempotent $f$ of $S$ such that $f \in \overline{Se}$ implies that $\overline{Sf} = \overline{Se}$.

**Definition 3.4.7.** A semiring $S$ is said to be $k$-local semiring if it has unique maximal left $k$-ideal.

An idempotent $e$ of $S$ is said to be $k$-local idempotent if the semiring $eSe$ is $k$-local.

**Proposition 3.4.8.** *Let $S$ be a $k$-semipotent inverse semiring such that the set $[S \setminus U'(S)] \bigcap E^+(S)$ is contained in every nonzero principle left $k$-ideal of $S$. Then every primitive idempotent is $k$-local.*

*Proof.* Let $e$ be the primitive idempotent of $S$ and $J_l(S)$ is the intersection of all maximal left $k$-ideals of $S$. Let $a \notin J_l(eSe)$. If $\{0\}$ is the maximal left $k$-ideal of $eSe$ then due to the maximality of $\{0\}$, $a$ is left $k$-unit element of $eSe$. Suppose $\{0\}$ is not the maximal left $k$-ideal of $eSe$. Let $x \in J_l(S) \bigcap eSe$. So $x \in J_l(S)$ and $x \in eSe$. Consider an arbitrary element $y \in S$. Then $(ey)x \in J_l(S)$. Thus $1 + (ey)x$ is left $k$-unit. So there exist $s_1$, $s_2 \in S$ such that $1 + s_1(1 + eyx) = s_2(1+eyx) \Longrightarrow e+es_1(1+eyx) = es_2(1+eyx) \Longrightarrow e+es_1(1+eyx)e = es_2(1+eyx)e \Longrightarrow e + es_1(e + eyx) = es_2(e + eyx) \Longrightarrow e + es_1e(e + (eye)(exe)) = es_2e(e + (eye)(exe))$. Thus $e + (eye)x$ is left $k$-unit in $eSe$ for any $y \in S$. Our claim is that $x \in J_l(eSe)$. Suppose $x \notin J_l(eSe)$. So there exists a maximal left $k$-ideal $M$ of $eSe$ and $M$ is nonzero such that $x \notin M$. So $e \in \overline{M + (x)_l}$ which implies that $e + m_1 + (es_3e)x = m_2 + (es_4e)x$ for $s_3, s_4 \in S$ and $m_1, m_2 \in M$. Now $[e + es_3x + e(s_4'x)] + m_1 =$

$m_2 + e(s_4x + s_4'x) \Longrightarrow e + e(s_3 + s_4')x + m_1 = m_2 + e(s_4 + s_4')ex$. Since $x \in J_l(S)$, so $x \notin U'(S)$. Hence $(s_4x) + (s_4x)' \in [S \setminus U'(S)] \bigcap E^+(S)$. Now $M \neq \{0\}$. So there exists $eme \neq 0 \in M$. According to the condition, $s_4x + s_4'x \in \overline{S(eme)}$. Thus there exist $a, b \in S$ such that $(s_4x + s_4'x) + a(eme) = b(eme) \Longrightarrow e(s_4x + s_4'x) + (ea)(eme) = (eb)(eme) \Longrightarrow e(s_4x + s_4'x) + (eae)(eme) = (ebe)(eme)$. Since $M$ is left $k$-ideal of $(eSe)$, so $e(s_4x + s_4'x) \in M$. Hence $e + e(s_3 + s_4')x \in M$. But $e + e(s_3 + s_4')x$ is left $k$-unit element of $eSe$. This contradicts that $M$ is proper. Hence $x \in J_l(eSe)$. So if $\{0\}$ is not the maximal left $k$-ideal of $eSe$, $J_l(S) \bigcap eSe \subseteq J_l(eSe)$. Now $a$ is an element of $eSe$ such that $a \notin J_l(eSe)$. So $a \notin J_l(S)$. Since $S$ is $k$-semipotent, there exists a nonzero idempotent $f \in S$ such that $f \in \overline{Sa} \subseteq \overline{Se}$. Hence $\overline{Sf} \subseteq \overline{Sa} \subseteq \overline{Se}$. Since $e$ is primitive idempotent, $\overline{Sf} = \overline{Se}$. So $\overline{Sa} = \overline{Se}$. Hence $e \in \overline{Sa}$. Therefore, there exist $c, d \in S$ such that $e + ca = da \Longrightarrow e + eca = eda \Longrightarrow e + (ece)a = (ede)a$. Since $a \notin J_l(eSe)$, so we find that $a$ is a left $k$-unit element of $eSe$. So $eSe$ is a $k$-local semiring. Hence every primitive idempotent is $k$-local. □

From Theorem 3.4.1, Proposition 3.4.5 and Proposition 3.4.8, we have the following result :

**Theorem 3.4.9.** *Let $S$ be a strongly clean inverse semiring such that every idempotent has absorbing complement in $S$ and the set $[S \setminus U'(S)] \bigcap E^+(S)$ is contained in every nonzero principle left $k$-ideal of $S$. Then every primitive idempotent is $k$-local.*

**Example 3.4.10.** *In a semiring $S$, every primitive idempotent is $k$-local does not mean that $S$ is a strongly clean semiring. Let $S = \mathbb{N}_0^+$ be the set of all non-negative integers. Define two operations $+$ and $.$ on $S$ by $a + b = max\{a, b\}$ and $a.b$= usual multiplication in $S$ for all $a, b \in S$. Then $(S, +, .)$ is an additively idempotent $k$-local semiring with idempotents $0$ and $1$. Note that $1$ is the only primitive idempotent of $S$ and $1S1 = S$ is $k$-local. But $S$ is not strongly clean semiring.*

**Lemma 3.4.11.** *Let $S$ be an additively absorbing centroid semiring and $a \in S$ such that $ea = ae$ for some idempotent $e$ in $S$. Then the following conditions are equivalent :*

$(i)$ *$ae$ has a multiplicative inverse in $eSe$.*

(*ii*) $e \in \overline{Sa}$ *and* $l(a) \subseteq l(e)$.

(*iii*) $e \in \overline{aS}$ *and* $r(a) \subseteq r(e)$.

*Proof.* First we prove that $(i) \Longleftrightarrow (ii)$. Suppose that $(i)$ holds. Let $xea = e = eax = aex$, where $x \in eSe$. Then $e \in Sa \subseteq \overline{Sa}$. Let $s \in l(a)$. Then $a + sa = a$. Now $e + se = aex + saex = (a + sa)ex = aex = e$. So $s \in l(e)$ and hence $l(a) \subseteq l(e)$. Now assume that $(ii)$ holds i.e. $e \in \overline{Sa}$ and $l(a) \subseteq l(e)$. Then there exist $b_1, b_2 \in S$ with $eb_1 = b_1$, $eb_2 = b_2$ such that $e + b_1a = b_2a$. Let $b_1e = c_1 = c_1e, b_2e = c_2 = c_2e$ for some $c_1, c_2 \in S$ and $x = e(b_1+b_2)e$. Then $xa = e(b_1+b_2)ea = (eb_1+eb_2)ea = (b_1+b_2)ea$ and $e+xa = e+(b_1+b_2)ea = e+b_1ae+b_2ae = b_2ae+b_2ae = (b_2e+b_2e)a = (c_2+c_2)a = ca$, where $c = 2c_2 \in S$. So $ce = c$. Now $e + xa + x'a = ca + x'a \Longrightarrow e + xa + xa' = (c + x')a \Longrightarrow e + x(a + a') = Xa$, where $X = x' + c$. Since $S$ is additively absorbing semiring and $x = e(b_1+b_2)e$, so $e = Xa = (eXe)(ae)$. Now $(e+aX')a = ea+a(X'a) = ea + a(Xa)' = ea + ae' = ae + ae' = a(e + e')$. So $a + (e + aX')a = a + a(e + e') = a$. Hence $(e + aX') \in l(a) \subseteq l(e)$. Thus $e + (e + aX')e = e \Longrightarrow e + e + a(X'e) = e \Longrightarrow e + e + a(Xe)' = e \Longrightarrow e + e + aX' = e \Longrightarrow e + aX' = e + e' \Longrightarrow e + aX' + aX = e + e' + aX \Longrightarrow e + X(a + a') = e + e' + aX \Longrightarrow Xa + X(a + a') = Xa + (Xa)' + aX$. So $Xa = a(X + X') + aX \Longrightarrow e = aX \Longrightarrow e = (ae)(eXe)$. Hence $ae = ea$ is a unit element in $eSe$. The proof of $(i) \Leftrightarrow (iii)$ is analogous. □

**Definition 3.4.12.** Let $S$ be a semiring such that every idempotent has absorbing complement in $S$. Then $S$ is said to be strongly exchange semiring if for each $a(\neq 0)$ in $S$, the following holds : $E^+(S) \subseteq \overline{(a)_l}$ and there exists an idempotent $e_1 \in S$ with absorbing complement $e$ such that $(i)$ $e_1a = ae_1$, $(ii)$ $e_1 \in \overline{S(1 + a)}$ and $e \in \overline{Sa}$, $(iii)$ $l(1 + a) \subseteq \overline{Se}$ and $l(a) \subseteq \overline{Se_1}$.

**Theorem 3.4.13.** *Let $S$ be an additively absorbing centroid semiring. Then $S$ is strongly clean if and only if $S$ is strongly exchange semiring.*

*Proof.* Let $S$ be strongly exchange and $a(\neq 0) \in S$. Then $a' \neq 0$. Thus there exists $e_1^2 = e_1$ such that $e_1a' = a'e_1$, $e_1 \in \overline{S(1 + a')}$, $e \in \overline{Sa}$, since $\overline{Sa} = \overline{Sa'}$ and $l(1+a') \subseteq \overline{Se}$, $l(a) \subseteq \overline{Se_1}$, since $l(a) \subseteq l(a')$, where $e^2 = e$ is an absorbing complement of $e_1$. Let $y \in \overline{Se}$. Then there exist $A, B \in S$ such that $y + Ae = Be \Longrightarrow y + Ae + (A+$

$B)(e_1+e_1') = Be+(A+B)(e_1+e_1')$. Now $e+e_1 = 1$. So $y+A(1+e_1')+B(e_1+e_1') = B(1+e_1')+A(e_1+e_1') \Longrightarrow y+A(1+e_1')+(A+B) = B(1+e_1')+(A+B) \Longrightarrow y+(1+e_1')+A(1+e_1') = B(1+e_1')+(1+e_1') \Longrightarrow y+(1+A)(1+e_1') = (1+B)(1+e_1') \Longrightarrow y \in \overline{S(1+e_1')}$. Thus $\overline{Se} \subseteq \overline{S(1+e_1')}$. Hence $l(1+a') \subseteq \overline{S(1+e_1')}$. Let $y_1 \in \overline{S(1+e_1')}$. Then there exist $a_1, b_1 \in S$ such that $y_1+a_1(1+e_1') = b_1(1+e_1') \Longrightarrow y_1e_1+a_1(e_1+e_1') = b_1(e_1+e_1') \Longrightarrow y_1e_1+(e_1+e_1')a_1 = (e_1+e_1')b_1 \Longrightarrow y_1e_1+(e_1a_1+e_1a_1') = (e_1b_1+e_1b_1') \Longrightarrow y_1e_1+e_1+e_1(a_1+a_1') = e_1+e_1(b_1+b_1') \Longrightarrow e_1+y_1e_1 = e_1$. Hence $y_1 \in l(e_1)$. So $l(1+a') \subseteq l(e_1)$. From $e_1a' = a'e_1$, we have $e_1(1+a') = (1+a')e_1$. Thus from Lemma 3.4.11, it follows that $e_1(1+a') = (1+a')e_1$ is a unit in $e_1Se_1$. So $[e_1(1+a')]' = e_1(1'+a) = e_1'+e_1a$ is a unit in $e_1Se_1$. Again $E^+(S) \subseteq \overline{Sa}$. Hence $(e_1+e_1') \in \overline{Sa}$. So $(e+e_1+e_1') \in \overline{Sa}$ which implies that $(1+e_1') \in \overline{Sa}$. Now let $y_2 \in \overline{Se_1}$. Then there exist $a_2, b_2 \in S$ such that $y_2+a_2e_1 = b_2e_1 \Longrightarrow y_2e_1'+a_2e_1' = b_2e_1'$. Now adding these two equations, we find that $y_2(1+e_1')+(a_2e_1+a_2e_1') = (b_2e_1+b_2e_1') \Longrightarrow y_2(1+e_1')+e_1'(a_2+a_2') = e_1'(b_2+b_2') \Longrightarrow (1+e_1')+y_2(1+e_1') = (1+e_1')$. Hence $y_2 \in l(1+e_1')$. So $\overline{Se_1} \subseteq l(1+e_1')$. Since $e_1a' = a'e_1$, so $a(1+e_1') = (1+e_1')a$. Similarly, from Lemma 3.4.11, it follows that $a(1+e_1') = (1+e_1')a$ is a unit in $(1+e_1')S(1+e_1')$. Now $u_1 = e_1'+e_1a$ is a unit in $e_1Se_1$ with inverse $v_1$(say) and $u_2 = a+e_1'a$ is a unit in $(1+e_1')S(1+e_1')$ with inverse $v_2$ (say). Since $S$ is an additively absorbing centroid semiring, it follows that $u_1+u_2$ is a unit in $S$ with inverse $v_1+v_2$. So $(a+e_1'a+e_1'+e_1a) = a+e_1'(1+a+a') = a+e_1'$ is a unit in $S$ with inverse $v_1+v_2$. Now $e_1 \in \overline{S(1+a')}$. Thus there exist $a_3, b_3 \in S$ such that $e_1+a_3(1+a') = b_3(1+a') \Longrightarrow e_1+(a_3+a_3a') = (b_3+b_3a') \Longrightarrow e_1+a_3+a_3'a = b_3+b_3'a$. $e_1+a_3+a_3'a+e_1'+a_3'+a_3a = b_3+b_3'a+b_3'+b_3a \Longrightarrow (e_1+e_1')+(a_3+a_3')+(a_3+a_3')a = (b_3+b_3')+(b_3+b_3')a \ldots (i)$. Since $E^+(S) \subseteq \overline{Sa}$, $(a_3+a_3')$, $(b_3+b_3') \in \overline{Sa}$, so there exist $t_1, t_2, t_3, t_4 \in S$ such that $(a_3+a_3')+t_1a = t_2a \ldots (ii)$ and $(b_3+b_3')+t_3a = t_4a \ldots (iii)$. Let $t = t_2+t_3$ and $s = t_1+t_4$. Then $(a_3+a_3')+sa = (b_3+b_3')+ta \ldots (iv)$. Adding both side of equation $(i)$ by $(s+t_3)a$ and using equations $(iii)$ and $(iv)$, we have $(e_1+e_1')+pa+(a_3+a_3')a = (b_3+b_3')a+qa$, where $p = (t+t_4)$ and $q = (s+t_4)$. Thus $(e_1+e_1')+pa+(a_3+a_3')a+(e_1'+e_1)+p'a+(a_3'+a_3)a = qa+(b_3+b_3')a+q'a+(b_3'+b_3)a \Longrightarrow (e_1+e_1')+(p+p')a+(a_3+a_3')a = (q+q')a+(b_3+b_3')a \Longrightarrow a+(e_1+e_1') = a$, by adding both side by a and using the fact that $S$ is an additively absorbing centroid semiring. Hence $a = (a + e'_1) + e_1 =$ unit + idempotent. Now $(a + e'_1)e_1 = ae_1 + e'_1 = e_1 a + e'_1 = e_1(a + e'_1)$, since $a'e_1 = e_1 a'$. Since $a$ is any nonzero element of $S$, so $S$ is a strongly clean semiring.

Conversely, suppose that $S$ is a strongly clean semiring. Let $a \neq 0 \in S$. Then $a' \neq 0$. Let $x \in E^+(S)$. Then $x + x = x$ i.e. $x' = x$. If $u$ is any unit in $S$, then $u + x = u + x + x = u[1 + u^{-1}x + u^{-1}x] = u1 = u$ for any $u \in U(S)$. Since $S$ is strongly clean, so $a' = e_1 + u_1$, $e_1 u_1 = u_1 e_1$, where $e_1^2 = e_1$ and $u_1 \in U(S)$. Hence $a' + x = a' + x + x = e_1 + u_1 + x + x = e_1 + u_1 = a'$. So $x \in \overline{Sa'} = \overline{Sa}$. Thus $E^+(S) \subseteq \overline{Sa}$ for each $a \neq 0 \in S$. Now $a' = e_1 + u_1$, $e_1 u_1 = u_1 e_1$. Let $e_1 \neq 0$. Then $a'e_1 = e_1 + u_1 e_1 = e_1 + e_1 u_1 = e_1(e_1 + u_1) = e_1 a'$. Hence $(e_1 a')' = (a'e_1)' \Longrightarrow e_1 a = ae_1$. Since $S$ is a strongly clean additively absorbing centroid semiring, according to Theorem 3.4.1, we have $e_1 \in \overline{S(1 + a)}$ and $(1 + e'_1) \in \overline{Sa'} = \overline{Sa}$, where $(1 + e'_1)$ is an absorbing complement of idempotent $e_1$. Let $y \in l(1 + a)$. Then $(1 + a) + y(1 + a) = (1 + a) \Longrightarrow 1 + y(1 + a) = 1 \Longrightarrow 1 + y[(1 + e'_1) + u'_1] = 1 \Longrightarrow (1 + e'_1) + y(1 + e'_1) + yu'_1 = (1 + e'_1) \Longrightarrow u'^{-1}_1(1 + e'_1) + yu'^{-1}_1(1 + e'_1) + y = u'^{-1}_1(1 + e'_1)$, since $e_1 u_1 = u_1 e_1$ so $e'_1 u'^{-1}_1 = u'^{-1}_1 e'_1$. Hence $y \in \overline{S(1 + e'_1)} \Longrightarrow l(1 + a) \subseteq \overline{S(1 + e'_1)}$. Let $y_1 \in l(a)$. Then $a + y_1 a = a \Longrightarrow a' + y_1 a' = a' \Longrightarrow e_1 + u_1 + y_1(e_1 + u_1) = e_1 + u_1$, since $e_1 (\neq 0)$ and $e_1$ is strongly clean element. Hence $e_1 + b + b' = e_1$ for any $b \in S$. Therefore, $e_1 + y_1(e_1 + u_1) = e_1 \Longrightarrow e_1 + y_1 e_1 + y_1 u_1 = e_1 \Longrightarrow u_1^{-1} e_1 + y_1 u_1^{-1} e_1 + y_1 = u_1^{-1} e_1$, since $e_1 u_1 = u_1 e_1$. So $y_1 \in \overline{Se_1}$ which implies $l(a) \subseteq \overline{Se_1}$. Hence $S$ is strongly exchange semiring if $e_1 \neq 0$. If $e_1 = 0$ then $a' = u_1$. So $a = u'_1 = u'_1 + 1 + 1'$, where $1 + 1'$ is an idempotent of $S$. Then similarly we can show that $S$ is a strongly exchange semiring. □

Now the focus of our investigation is another question namely whether the nonzero center of a strongly clean additively absorbing centroid semiring is itself strongly clean. But this question has negative answer which follows from following example :

**Example 3.4.14.** *Let $R$ be a strongly clean ring with nonzero center $Z(R)$ such that $Z(R)$ is not strongly clean ([56]). Now consider $S = \{0, 1, 2, 3\}$. Define two* *operations "+" and "." on S as follows :*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 1 |
| 2 | 2 | 3 | 1 | 2 |
| 3 | 3 | 1 | 2 | 3 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 1 | 3 |
| 3 | 0 | 3 | 3 | 3 |

*Now consider* $S_1 = [R \times (S \setminus \{0\})] \bigcup \{(0_R, 0)\}$. *Then* $S_1$ *is a strongly clean additively absorbing centroid semiring with respect to component wise addition and multiplication but its center* $Z(S_1)$ *is not strongly clean.*

**Definition 3.4.15.** *An element* $a$ *of an inverse semiring* $S$ *is optimally clean if it admits a strongly clean decomposition* $a = e + u$, $eu = ue$ *such that for every* $b \in S$, *there exists* $x \in S$ *with* $[a, x] = [e, b]$, *where* $[x, y]$ *denotes the commutator of* $x, y$ *in* $S$.

*An inverse semiring is optimally clean if every nonzero element is optimally clean.*

**Theorem 3.4.16.** *Let* $S$ *be an optimally clean additively absorbing centroid semiring with nonzero center* $Z(S)$. *Then* $Z(S)$ *is a strongly clean semiring.*

*Proof.* Let $a(\neq 0) \in Z(S)$. Then $a = e + u$, $eu = ue$ for some idempotent $e \in S$ and $u \in U(S)$, which in addition, has the property that for every $b \in S$, there exists $x \in S$ with $[a, x] = [e, b]$. This implies that $ax + x'a = eb + b'e \Longrightarrow xa + x'a = eb + b'e$. Adding both side by 1, it implies that $1 = 1 + eb + b'e \Longrightarrow 1 + be = 1 + eb$ for all $b \in S$. For being an additively absorbing centroid semiring $S$, it satisfies $1 + x + x' = 1$ for all $x \in S$. If $u_1, u_2 \in U(S)$ then $U = u_1 u_2 \in U(S)$. Now $U + 1 + 1' = U[1 + U^{-1} + (U^{-1})'] = U$. Since $a \in Z(S)$, $ax = xa$ for all $x \neq 0$. So $ex + ux = xe + xu \Longrightarrow 1 + ex + ux = 1 + ex + xu \Longrightarrow 1 + ux = 1 + xu \Longrightarrow 1 + 1' + ux = 1 + 1' + xu$ for all $x \neq 0$ in $S$. Since $x(\neq 0) \in S$ and $S$ is strongly clean, $x = e_1 + u_1$ and $e_1 u_1 = u_1 e_1$ for some $e_1^2 = e_1$ and $u_1 \in U(S)$. Thus $1 + 1' + ue_1 + uu_1 = 1 + 1' + e_1 u + u_1 u \Longrightarrow ue_1 + uu_1 = e_1 u + u_1 u$. Hence $ux = xu$ for all $x \neq 0$. So $u \in Z(S)$. Again, for all $b \in S$, $eb + ub = be + bu \Longrightarrow eb + ub = be + ub \Longrightarrow ebe + ube = be + ube$, multiplying from right on both side by $e$. Adding both side by

$u'be$, it follows that $ebe + ube + u'be = (1 + u + u')be = be$. Since $u'be = ube'$, we have $ebe + ub(e + e') = be$. Since $S$ is a centroid semiring, $ub(e + e') = (e + e')ub$. So $ebe+eub+e'ub = be \Longrightarrow ebe+eube+e'ube = be \Longrightarrow [e+(e+e')u]be = be \Longrightarrow ebe = be$. Similarly, it can be proved that $eb = ebe$. Hence $eb = be$ for all $b \in S$. So $e \in Z(S)$. Hence $Z(S)$ is a strongly clean semiring. □

**Definition 3.4.17** ([19])**.** *Let $S$ be a semiring with identity. An $S$-semimodule $P$ is projective if $\phi : M \to N$ is surjective $S$-homomorphism of $S$-semimodules $M$ , $N$ and if $\alpha : P \longrightarrow N$ is an $S$-homomorphism then there exists an $S$-homomorphism $\beta : P \longrightarrow M$ satisfying $\phi o \beta = \alpha$.*

**Proposition 3.4.18.** *[19] Every free $S$-semimodule is projective.*

**Definition 3.4.19** ([19])**.** *An $S$-semimodule $N$ is a retract of an $S$-semimodule $M$ if there exists a surjective $S$-homomorphism $\theta : M \longrightarrow N$ and an $S$-homomorphism $\psi : N \longrightarrow M$ satisfying the condition that $\theta o \psi$ is the identity map on $N$.*

**Proposition 3.4.20.** *[19] Any retract of a projective $S$-semimodule is projective.*

**Definition 3.4.21.** *Let $M$ be a finitely generated free $S$-semimodule. Then any basis of $M$ has finite number of elements. A semiring $S$ is called projective-free if every finitely generated projective $S$-semimodule $M$ is free and every basis of $M$ has same number of elements.*

**Proposition 3.4.22.** *Let $S$ be a projective-free semiring and every idempotent of $S$ has an orthogonal complement in $S$. Then the only idempotents of $S$ are $0$ and $1$.*

*Proof.* Let $e_1$ be a nonzero idempotent of $S$ and $e_2$ be its orthogonal complement. Then $e_1 + e_2 = 1$ and $e_1e_2 = e_2e_1 = 0$. Now $e_1 \neq e_2$ otherwise $e_1 = 1 = 0$ which implies $S$ is trivial. Since $e_1 + e_2 = 1$, $\{e_1, e_2\}$ generates $S$-semimodule $S$. Now $S$ is free $S$-semimodule as 1 is a basis. $Se_1$ is the retract of $S$-semimodule $S$, since $\theta : S \to Se_1$ defined by $\theta(s) = se_1$ for all $s \in S$, is a surjective $S$- homomorphism and $\psi : Se_1 \to S$ defined by $\psi(se_1) = se_1$ for all $s \in S$, is an $S$-homomorphism such that their composition $\theta o \psi$ is an identity map on $Se_1$. Now $S$ is projective $S$-semimodule, by Proposition 3.4.18. Hence $Se_1$ is finitely generated projective $S$-semimodule, by Proposition 3.4.20 and so is free. Similarly, $Se_2$ is free $S$-semimodule. Let $A$ and $B$ be bases of $S$-semimodules $Se_1$ and $Se_2$ respectively. Since $e_1e_2 = 0$, $Se_1 \bigcap Se_2 = \{0\}$. It can be easily seen that $A \bigcup B$ is a basis of $S$. Since $S$ is projective-free semiring, $A \bigcup B = \{a\}$ for some $a \neq 0 \in S$. Since $e_1 \neq 0$, so $A = \{a\}$ and $B = \phi$. Since $\phi$ spans only the 0 $S$-semimodule, $Se_2 = 0$. Hence $e_2 = 0$ which implies $e_1 = 1$. Hence the proof. □

**Theorem 3.4.23.** *Let $S$ be a projective-free semiring such that every idempotent has orthogonal complement in $S$. Let $x + x \in J_l(S)$ for all $x \in S \setminus U'(S)$, where $U'(S)$ is the set of all left $k$-unit elements in $S$. Now the following are equivalent :*

*(i) $S$ is a $k$-local semiring,*

*(ii) $S$ is an exchange semiring,*

*(iii) $S$ is a $k$-semipotent semiring.*

*Proof.* $(i) \implies (ii)$. Suppose that $S$ has a unique maximal left $k$-ideal $M$ and $x \neq 0$ in $S$. Let $\overline{(x)_l}$ and $\overline{(1+x)_l}$ be two proper left $k$-ideals of $S$. Since every proper left $k$-ideal is contained in a maximal left $k$-ideal, so $x \in M$ and $(1 + x) \in M$ which implies that $1 \in M$, a contradiction. Hence either $1 \in \overline{(x)_l}$ or $1 \in \overline{(1+x)_l}$. Hence $S$ is an exchange semiring.

$(ii) \implies (iii)$. Let $\overline{(x)_l}$ be a nonzero principal left $k$-ideal of $S$ such that $x \notin J_l(S)$. Suppose $1 \notin \overline{(x)_l}$. Let $s \in S$. If $sx = 0$ then $(1 + sx) = 1$ is left $k$-unit. Suppose $sx \neq 0$. Since $S$ is exchange, either $1 \in \overline{(sx)_l}$ or $1 \in \overline{(1+sx)_l}$. But $1 \notin \overline{(sx)_l}$ as our assumption was $1 \notin \overline{(x)_l}$. Hence $(1+sx)$ is left $k$-unit for all $s \in S$. The claim is that $x \in J_l(S)$. If it is not, then there exists a maximal left $k$-ideal $M$ such that $x \notin M$. So by the maximality of $M$, it follows that $1 + (m_1 + s_1x) = (m_2 + s_2x)$ for some $m_1, m_2 \in M$ and $s_1, s_2 \in S$. Now $(1 + sx) + m_1 = m_2 + (s_2x + s_2x) \ldots (3)$, where $s = (s_1 + s_2)$. Since $1 \notin \overline{(x)_l}$, $s_2x \notin U'(S)$ so $(s_2x + s_2x) \in J_l(S)$ according to the condition. Hence form equation (3), we have $(1+sx) \in M$ which is a contradiction as $(1 + sx)$ is left $k$-unit. Hence $x \in J_l(S)$. So our assumption is wrong. Thus $1 \in \overline{(x)_l}$. Hence $S$ is a $k$-semipotent semiring.

$(iii) \Longrightarrow (i)$. Let $S$ be a $k$-semipotent semiring and $x \neq 0 \in S$. If $x \in J_l(S)$ then we have nothing to proof. Let $x \notin J_l(S)$. Then $\overline{(x)_l} \nsubseteq J_l(S)$. Since $S$ has only nonzero idempotent 1, follows from Proposition 3.4.22 and $S$ is $k$-semipotent, $1 \in \overline{(x)_l}$. Hence $x$ is a left $k$-unit element. Thus $S$ is a $k$-local semiring. □

From Proposition 3.4.22, we have the following result :

**Theorem 3.4.24.** *Let $S$ be a projective-free semiring such that every idempotent has orthogonal complement in $S$. Let $x + x \in J_l(S)$ for all $x \in S \setminus U'(S)$, where $U'(S)$ is the set of all left $k$-unit elements in $S$. Then $S$ is a clean semiring if and only if $S$ is a strongly clean semiring.*

**Theorem 3.4.25.** *Let $S$ be a projective-free semiring such that every idempotent has orthogonal complement in $S$. Let $x + x \in J_l(S)$ for all $x \in S \setminus U'(S)$, where $U'(S)$ is the set of all left $k$-unit elements in $S$. Then $S$ is local, exchange and $k$-semipotent semiring if $S$ is a strongly clean semiring.*

*Proof.* From Proposition 3.4.22, it follows that $S$ has only two idempotents 0 and 1. Let $M$ be the maximal left $k$-ideal and $L$ be a proper left $k$-ideal such that $L \nsubseteq M$. Then there exists $x \in L$ such that $x \notin M$. Let $M_1 = \overline{M + (x)_l}$. Then $M_1$ is the left $k$-ideal containing properly $M$. So $M_1 = S$. Thus there exist $m_1, m_2 \in M$ and $s_1, s_2 \in S$ such that $1 + (m_1 + s_1x) = (m_2 + s_2x) \Longrightarrow 1 + m_1 + (s_1x + s_1x) = m_2 + sx \ldots (1)$, where $s = (s_1 + s_2)$. Since $x \in L$ and $L$ is proper left $k$-ideal, so $s_1x, sx \notin U'(S)$. Thus $(s_1x + s_1x), (sx + sx) \in J_l(S)$. If $sx = 0$ then equation (1) implies that $1 \in M$ which is a contradiction as $M$ is a maximal left $k$-ideal of $S$. Let $sx \neq 0$. Then $sx = 1 + u$ for some unit $u \in S$. Thus $(1 + u) + (1 + u) \in J_l(S) \Longrightarrow (1 + 1)(1 + u) \in J_l(S)$. From equation (1), $1 + m_1 + (s_1x + s_1x) = m_2 + 1 + u \ldots (2)$. If $(1+1) \in U'(S)$, then there exist $c_1, c_2 \in S$ such that $1 + c_1(1+1) = c_2(1+1) \Longrightarrow (1+u) + c_1(1+1)(1+u) = c_2(1+1)(1+u)$ which implies that $(1 + u) \in J_l(S)$. Hence from equation (2), we have $1 \in M$ – a contradiction. Let $(1 + 1) \notin U'(S)$. Then $(1 + 1) + (1 + 1) \in J_l(S)$. Now equation (2) implies that $u \in M$ which is again a contradiction. So $L \subseteq M$. Hence $S$ is a $k$-local semiring. Since $S$ is a semiring as stated in Theorem 3.4.23, the three notions local, exchange and $k$-semipotent are equivalent for $S$. Hence the theorem follows. □

But the converse of Theorem 3.4.25 is not true which follows from the following example.

**Example 3.4.26.** *Let $S = \mathbb{N}_0^+$ be the semiring of all non-negative integers with respect to usual addition and multiplication. Let $P = 2\mathbb{N}_0^+$. Then $P$ is a prime $k$-ideal of $S$. Consider the localization of $S$ at $P$ which is defined as $S_P = \{[m/n] : m \in S, n \in S \backslash P\}$. Then $S_P$ is a commutative semiring with respect to following binary operations : $[m_1/n_1] \bigoplus [m_2/n_2] = [m_1n_2 + m_2n_1/n_1n_2]$ and $[m_1/n_1] \bigodot [m_2/n_2] = [m_1m_2/n_1n_2]$ for all $m_1, m_2 \in S$, $n_1, n_2 \in S \setminus P$. Then $(S_P, \bigoplus, \bigodot)$ is a projective-free $k$-local semiring such that $[1/1] \bigoplus [1/1] = [2/1] \in J_l(S_P)$ but $S_P$ is not a clean semiring. Here $[m/n]$ denotes the equivalence class of $m/n$ such that $a/b \in [m/n]$ if and only if $na = mb$.*

Hence if $S$ is a projective-free semiring such that every idempotent has orthogonal complement in $S$ and $x + x \in J_l(S)$ for all $x \in S \setminus U'(S)$, where $U'(S)$ is the set of all left $k$-unit elements in $S$ then the implications below should now be clear.

Clean Semiring $\Longleftrightarrow$ Strongly Clean Semiring $\not\Longleftrightarrow$ $k$-Local Semiring $\Longleftrightarrow$ Exchange Semiring $\Longleftrightarrow$ $k$-Semipotent Semiring.

# Chapter 4

# On $k$-unit Clean Semiring

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# On $k$-unit Clean Semiring

## 4.1 Introduction

Rings in which every element is the sum of an idempotent and a unit are said to be clean rings and this notion was introduced by W.K. Nicholson [7] in 1977 in the study of exchange rings. In this paper [7], he also proved that a ring with central idempotents is clean if and only if it is exchange. A ring $R$ is unit regular if for any $a \in R$ there exists a unit $u \in R$ such that $a = aua$. Equivalently, a ring $R$ is unit regular if each element $a$ of $R$ is the product of an idempotent and a unit i.e. $a = eu$ for some idempotent $e$ and unit $u$ in $R$. Basically, clean rings are the additive analog of unit regular rings. Since then various generalizations of clean rings have been obtained by many authors ([28], [30], [42], [59]). There has been a remarkable growth of the algebraic theory of semirings over last several years. Semiring is a generalization of ring and distributive lattice. In [62], we have introduced the notion of clean semiring and exchange semiring. Units and idempotents play very important roll for determining the structure of clean ring as well as clean semiring. But in a semiring with absorbing zero element '0' and identity element '1', there is a generalization of units which are called $k$-units. It is quite natural to ask what happens if we replace unit element by $k$-unit element in the definition of clean semiring. Considering this fact in mind, in this paper we introduce the concept of $k$-unit clean semiring which generalizes both clean ring and clean semiring. A semiring is said to be $k$-unit clean semiring if each of its nonzero element can be written as the sum of a $k$-unit and an idempotent. In this paper, we obtain some important results about $k$-unit clean semiring and establish a connection between $k$-unit clean semiring and exchange semiring for some class of semirings.

## 4.2 Definition & Examples of $k$-Unit Clean Semirings

**Definition 4.2.1.** An element of a semiring $S$ is said to be $k$-unit clean if it can be written as the sum of an idempotent and a $k$-unit in $S$. A semiring $S$ is called a $k$-unit clean semiring if every nonzero element of $S$ is $k$-unit clean.

**Definition 4.2.2.** A semiring $S$ is said to be strongly $k$-unit clean semiring if every nonzero element $a$ of $S$ can be written as $a = e + u$ for some $k$-unit $u$ and idempotent $e$ with $eu = ue$.

**Remark 4.2.3.** *Let $S$ be a semiring. Suppose that $a \in S$. Note that an element $b$ of $S$ is an additive inverse of $a$ if and only if $a + b = 0$ ([19]). Denote the set of all elements of $S$ having additive inverses by $V(S)$. The set $V(S)$ is non-empty as $0 \in V(S)$.*

*It can be easily seen that a semiring $S$ is a ring if and only if $V(S) = S$.*

*Now suppose that in a semiring $S$; $0 = e + u$, where $e^2 = e$ and $u$ is a $k$-unit. Since $u$ is a $k$-unit, there exist $s_1, s_2 \in S$ such that $1 + s_1u = s_2u$ and $1 + us_1 = us_2$. So we have $0 = s_2e + s_2u = s_2e + 1 + s_1u = 1 + (s_1u + s_2e)$. Let $b = (s_1u + s_2e)$. Then $1 + b = 0$. Thus $x + xb = x(1 + b) = x0 = 0$ for all $x \in S$. So $x \in V(S)$ and hence $S = V(S)$. This shows that $S$ becomes a ring.*

*To avoid the above situation, we take a semiring $S$ is $k$-unit clean if every nonzero element of $S$ can be written as the sum of an idempotent and a $k$-unit in $S$.*

**Example 4.2.4.** *(i) Every clean semiring and clean ring are examples of $k$-unit clean semiring.*

(ii) *Let $S = \mathbb{N}_0^+$ be the set of all non-negative integers. Define two operations $+$ and $.$ on $S$ by $a + b = max\{a, b\}$ and $a.b =$ usual multiplication in $\mathbb{N}_0^+$. Then $(\mathbb{N}_0^+, +, .)$ forms a semiring with additive identity $0$ and multiplicative identity $1$. In fact, for all nonzero integers $a$, $1+a = a$ holds. Thus $(S, +, .)$ is a $k$-semifield hence $k$-unit clean semiring.*

(iii) *Consider $S = \{0, 1, 2\}$. Define the operations "+" and "." on $S$ as follows :*

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 1 |
| 2 | 2 | 1 | 2 |

| . | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 2 |

*Then $(S, +, .)$ forms a semiring. Let $\mathbb{N}_0^+$ be the set of all non-negative integers. Define two operations $+$ and $.$ on $\mathbb{N}_0^+$ such that $a + b = max\{a, b\}$ and $a.b =$ usual multiplication in $\mathbb{N}_0^+$. Then $(\mathbb{N}_0^+, +, .)$ forms a semiring with multiplicative identity $1$ and additive identity $0$. Let us consider $S_1 = (0, 0) \cup [\mathbb{N} \times (S \setminus \{0\})]$. Then $S_1$ forms a semiring with respect to component wise addition and multiplication. This $S_1$ is an example of $k$-unit clean semiring but it is neither a $k$-semifield nor a clean semiring.*

## 4.3 Elementary Results of $k$-Unit Clean Semiring

**Theorem 4.3.1.** *Homomorphic image of a $k$-unit clean semiring is a $k$-unit clean semiring.*

*Proof.* Let $S$ be a $k$-unit clean semiring with identity $1_S$ and $S'$ be a semiring with identity $1_{S'}$. Let $f : S \longrightarrow S'$ be an onto homomorphism. Then $S'$ is the homomorphic image of the $k$-unit clean semiring $S$. We have to show that $S'$ is a $k$-unit clean semiring. Since $f$ is an onto homomorphism, so $S' = \{f(a) : a \in S\}$. Now $f(1_S) = 1_{S'}$ is the identity element of $S'$. Let $s' \neq 0 \in S'$. Then there exists $a(\neq 0) \in S$ such that $f(a) = s'$. Since $S$ is $k$-unit clean so we can write $a = e + u$, where $e^2 = e \in S$ and $u$

is a $k$-unit in $S$. Thus there exist $s_1, s_2 \in S$ such that $1_S + s_1u = s_2u$ and $1_S + us_1 = us_2$. So we have $s' = f(a) = f(e+u) = f(e) + f(u)$, since $f$ is a homomorphism. Now $[f(e)]^2 = f(e)f(e) = f(e^2) = f(e)$. Hence $f(e)$ is an idempotent in $S'$. Now $f(1_S + s_1u) = f(s_2u) \Longrightarrow f(1_S) + f(s_1u) = f(s_2u) \Longrightarrow 1_{S'} + f(s_1)f(u) = f(s_2)f(u)$, since $f$ is a homomorphism. Similarly we can show that $1'_S + f(u)f(s_1) = f(u)f(s_2)$. Hence $f(u)$ is a $k$-unit in $S'$. So $s'$ is the sum of an idempotent $f(e)$ and a $k$-unit $f(u)$ of $S'$. Since $s'$ is an arbitrary nonzero element of $S'$, it follows that $S'$ is a $k$-unit clean semiring. □

**Theorem 4.3.2.** *Let $S$ be an additively cancellative strongly $k$-unit clean semiring such that every idempotent of $S$ has a complement. Then idempotents lift modulo every $k$-ideal of $S$.*

*Proof.* Let $S$ be an additively cancellative strongly $k$-unit clean semiring such that every idempotent of $S$ has a complement in $S$ and $I$ be a $k$-ideal of $S$. Let $x \in S$. If $x \in I$ then clearly the result follows. Let $x$ does not belong to $I$. Since $S$ is strongly $k$-unit clean, $x = e + u$, where $e$ is an idempotent and $u$ is a $k$-unit in $S$ with $eu = ue$. Now $e$ has a complement, say $e_1$ in $S$. So we have $e + e_1 = 1$. Since $u$ is a $k$-unit in $S$, there exist $s_1, s_2 \in S$ such that $1 + s_1u = s_2u \ \ldots \ (i)$ and $1 + us_1 = us_2 \ \ldots \ (ii)$. Let $x + I = x^2 + I$. Then $(e+u) + I = (e+u)^2 + I \Longrightarrow (e+u) + I = (e^2 + eu + ue + u^2) + I \Longrightarrow (e+u) + i_1 = (e + eu + ux) + i_2$, where $i_1, i_2 \in I$. So $u + i_1 = ux + eu + i_2 \ \ldots \ (iii)$, since $S$ is additively cancellative. Now multiplying from left side by $e$, we have $eu + i'_1 = eux + eu + i'_2 \Longrightarrow i'_1 = eux + i'_2$, where $i'_1 = ei_1 \in I$ and $i'_2 = ei_2 \in I$. Hence $eux \in I$, since $I$ is a $k$-ideal of $S$. Since $eu = ue$ and $xe = ex$, we have $uxe \in I$. Since $I$ is a $k$-ideal of $S$, we have $xe = ex \in I \Longrightarrow e + eu \in I$. Now adding both side of equation $(iii)$ by $e$, we have $(e+u) + i_1 = ux + (e + eu) + i_2$. Since $x = e + u$ and $e + eu \in I$, we have $x + i_1 = ux + i'_2 = xu + i'_2 \Longrightarrow xs_1 + i'_1 = xus_1 + i''_2 \ldots (iv)$, where $i'_1, i''_2 \in I$. Now multiplying equation $(iii)$ from right side by $s_2$, we have $us_2 + i''_1 = uxs_2 + eus_2 + i'''_2$. By using equation $(ii)$, we have $(1 + us_1) + i''_1 = x + xus_1 + e + eus_1 + i'''_2$. Now adding both side by $e_1$, it follows that $e_1 + (1 + us_1) + i''_1 = x + xus_1 + 1 + eus_1 + i'''_2$. Since

$S$ is additively cancellative, it follows that $e_1 + us_1 + i_1'' = x + xus_1 + eus_1 + i_2''''$. Now adding both side by $es_1$, we have $e_1 + xs_1 + i_1'' = x + xus_1 + i_2'''''$. Thus adding both side by $i_1' + i_2''$ and using equation $(iv)$, we get $e_1 + i_3 = x + i_4$, where $i_3, i_4 \in I$. Hence $e_1 + I = x + I$, where $e_1$ is an idempotent. Thus idempotents lift modulo $k$-ideal $I$. □

Now we have the following characterization of $k$-unit clean semiring.

**Theorem 4.3.3.** *Let $S$ be a commutative antisimple semiring such that every idempotent has a complement. Let $I$ be a nonzero $k$-ideal of $S$ and $J_l(S)$ be the Jacobson radical of $S$ such that $I \subseteq J_l(S)$. Then $S$ is $k$-unit clean if and only if $S/I$ is $k$-unit clean and idempotents lift modulo $I$.*

*Proof.* Suppose that $S/I$ is $k$-unit clean and idempotents lift modulo $I$. To show that $S$ is $k$-unit clean, we have to show that every nonzero element of $S$ is $k$-unit clean. Let $r \neq 0 \in S$. Then either $r$ is a $k$-unit or a non $k$-unit element in $S$. If $r$ is a $k$-unit then $r$ is a $k$-unit clean element in $S$. Let $r$ be a non $k$-unit. Then there are two possibilities : either $r \in I$ or $r \notin I$. If $r \in I$ then $r + I = I$. Since $S$ is antisimple and $r \neq 0$, so $r = r' + 1$, where $r' \neq 0$ otherwise $r$ becomes a $k$-unit because every unit is a $k$-unit. Now $r'$ must be $k$-unit otherwise $\overline{(r')}$ is a proper $k$-ideal of $S$. Hence $r' \in M$, where $M$ is a maximal $k$-ideal of $S$. Thus it follows that $1 \in M$ as $r \in I \subseteq J_l(S)$. This is a contradiction. Hence $r$ is $k$-unit clean. Let $r \notin I$. Then $r + I$ is a nonzero element of $S/I$. Since $S/I$ is $k$-unit clean, $r + I = (e + I) + (x + I)$, where $(e + I)$ is an idempotent and $x + I$ is a $k$-unit of $S/I$. Now $(e + I)^2 = e^2 + I = e + I$. Since idempotents lift modulo $I$, there exists an idempotent $e_1 \in S$ such that $e + I = e_1 + I$. So we have $r + I = (e_1 + I) + (x + I)$. Now $I \subseteq J_l(S)$ and $x + I$ is a $k$-unit of $S/I$, so there exist $s_1 + I, s_2 + I \in S/I$ such that $(1 + I) + (s_1 + I)(x + I) = (s_2 + I)(x + I) \Longrightarrow (1 + s_1x) + I = (s_2x + I) \Longrightarrow (1 + s_1x) + i_1 = (s_2x + i_2)$ for some $i_1, i_2 \in I$. Now if $x$ is not a $k$-unit then $\overline{(x)}$ is a proper $k$-ideal of $S$. Since each proper $k$-ideal is contained in a maximal $k$-ideal, so $x \in M_1$, where $M_1$ is a maximal $k$-ideal of $S$. Hence $s_1x, s_2x \in M_1$. Since $I \subseteq J_l(S)$, $i_1, i_2 \in J_l(S)$. Hence $1 \in M_1$, which contradicts that $M_1$ is proper. Thus $x$ is a $k$-unit of $S$. Now $r + I = (e_1 + I) + (x + I)$, where $e_1 \notin I$, otherwise $r + I$ is a $k$-unit of $S/I$ which implies that $r$ is a $k$-unit of $S$. But we take $r$ is a non $k$-unit element. Now $e_1$ is an idempotent and $x$ is a $k$-unit of $S$. Thus $e_1 = 1$ or $e_1$ is an idempotent except 1. Suppose that $e_1 \neq 1$. From given condition $e_1$ has a complement in $S$, so there exists an idempotent $e_2 \in S$ such that $e_2 + e_1 = 1$ and $e_2 \neq 0$ otherwise $e_1 = 1$. Now $r + I = (e_1 + I) + (x + I) \Longrightarrow (r + e_2) + I = (1 + I) + (x + I)\ \ldots\ (i)$. Since $r(\neq 0) \notin I$ and $S$ is antisimple, there exists $r_1 \in S$ such that $r = r_1 + 1$. From equation $(i)$, we have $(r_1 + e_2) + (1 + I) = (1 + I) + (x + I)\ \ldots\ (ii)$. Since $S$ is antisimple and $I$ is a nonzero $k$-ideal of $S$, $S/I$ is additively cancellative from Note 2.3.4 . Hence from equation $(ii)$, we have $(r_1 + e_2) + I = (x + I)$. Thus $(r_1 + e_2) + I$ is a $k$-unit of $S/I$ which implies that $(r_1 + e_2)$ is a $k$-unit of $S$. Thus we can write $r = r_1 + 1 = r_1 + (e_1 + e_2) = e_1 + (r_1 + e_2)$, where $e_1$ is an idempotent and $(r_1 + e_2)$ is a $k$-unit of $S$. Hence $r$ is a $k$-unit clean element of $S$. Now if $e_1 = 1$, then we have $r + I = (x + I) + (1 + I)\ \ldots\ (iii)$. Now $r = r_1 + 1$. Hence from equation $(iii)$, we have $(r_1 + I) = (x + I)$ is a $k$-unit of $S/I$ which implies that $r_1$ is a $k$-unit of $S$. Hence $r$ is $k$-unit clean. Since $r$ is arbitrary nonzero element of $S$, it follows that $S$ is $k$-unit clean.

Conversely, suppose that $S$ is a $k$-unit clean semiring. Let us define a mapping $\phi : S \to S/I$ by $\phi(x) = x + I$ for all $x \in S$. Then $S/I$ is a homomorphic image of the $k$-unit clean semiring $S$. Hence $S/I$ is $k$-unit clean, by Theorem 4.3.1. Since $S/I$ is additively cancellative and every idempotent of $S$ has a complement, we get idempotents lift modulo $I$, by Theorem 4.3.2. □

**Theorem 4.3.4.** *Let $S$ be an antisimple inverse semiring. Then all nonzero idempotent and $k$-idempotent elements of $S$ are $k$-unit clean.*

*Proof.* Let $e$ be a nonzero idempotent element in $S$. Since $S$ is an antisimple inverse semiring, $e$ can be written as $e = e + 1 + 1'$. Now $(2e + 1')(2e + 1') = (2e)^2 + 2e' + 2e' + 1 = 2(2e) + 2e' + 2e' + 1 = 2e + 2e + 2e' + 2e' + 1 = 1 + 2e + 2e'$. Hence $1' + (2e + 1')(2e + 1') = 1' + 2e + 2e' + 1 = (2e + 1') + 1'(2e + 1') = (2e + 1')(1 + 1')$. Thus $1 + (2e + 1')(2e' + 1) = (2e + 1')(1 + 1')$. This implies that $(2e + 1')$ is right

$k$-unit. Similarly, it can be shown that $(2e + 1')$ is left $k$-unit. Hence $(2e + 1')$ is a $k$-unit element of $S$. Now $(1 + e')^2 = 1 + e' + e' + e = 1 + e'$. Hence $(1 + e')$ is an idempotent of $S$. Now $(2e + 1') + (1 + e') = e + e' + e + 1 + 1' = e + 1 + 1' = e$. Consequently, $e$ is a $k$-unit clean element of $S$.

Let $e$ be a nonzero $k$-idempotent in $S$. Then $e^2 + e = e^2 \Longrightarrow ee' + e' = ee' \ \ldots \ (i)$. Now $e = e + 1 + 1'$, since $S$ is an antisimple inverse semiring. By using equation $(i)$, it follows that $(e + 1')(e' + 1') = ee' + e' + e + 1 = ee' + e + 1 = 1 + (e + 1')e'$. Similarly, it can be shown that $1 + e'(e + 1') = (e' + 1')(e + 1')$. So $(e + 1')$ is a $k$-unit element of $S$. Thus $e = e + 1 + 1' = (e + 1') + 1$. Hence $e$ is a $k$-unit clean element of $S$. □

**Theorem 4.3.5.** *Let $S$ be an antisimple strongly regular semiring such that every idempotent has a complement in $S$. Then $S$ is a $k$-unit clean semiring.*

*Proof.* Let $y(\neq 0) \in S$. Then there exists $x \in S$ such that $y = x + 1$, since $S$ is antisimple semiring. Again since $S$ is a strongly regular semiring, there exists $z \in S$ such that $x = xzx$ and $xz = zx$. Let $e = xz$. Then $e$ is an idempotent of $S$. According to the given condition, $e$ has a complement in $S$. Thus there exists $e_1^2 = e_1 \in S$ such that $e + e_1 = 1 \ \ldots \ (i)$. Now $ez = zxz = ze$. From equation $(i)$, we have $eze + e_1ze = ze \Longrightarrow e_1eze + e_1^2ze = e_1ze$. Using $ez = ze$ and $e_1^2 = e_1$, we get $e_1ze + e_1ze = e_1ze \ \ldots \ (ii)$. Now $(x + e_1)(2ze + e_1) = x2ze + xe_1 + e_12ze + e_1 = xze + xze + xe_1 + e_1ze + e_1ze + e_1$. Using equations $(ii)$ and $e = xz$, we have $(x + e_1)(2ze + e_1) = e + e_1 + xe_1 + e_1ze + xze$. Multiplying on both side of equation $(i)$ by $e_1$ and using $e_1^2 = e_1$, we have $e_1ee_1 + e_1 = e_1$. Thus $(x + e_1)(2ze + e_1) = 1 + xee_1 + e_1ee_1 + e_1ze + xze = 1 + (x + e_1)ee_1 + (x + e_1)ze = 1 + (x + e_1)(ze + ee_1)$. Similarly, we can show that $(2ze + e_1)(x + e_1) = 1 + (ze + e_1e)(x + e_1)$. Thus $(x + e_1)$ is a $k$-unit of $S$. Hence $y = x + 1 = x + (e + e_1) = (x + e_1) + e$. So $y$ is a $k$-unit clean element of $S$. Consequently, $S$ is a $k$-unit clean semiring. □

But the converse of Theorem 4.3.5 does not true. In Example 4.2.4(ii), $S$ is an antisimple $k$-unit clean semiring such that every idempotent has a complement in $S$ but there are no strongly regular elements in $S$ except 0 and 1.

**Definition 4.3.6.** *A semiring $S$ is called strongly $p$-regular if for each $a \in S$, there exists some $b \in S$ such that $na + aba = (n+1)a$ with $ab = ba$, for some $n \in \mathbb{N}$.*

**Definition 4.3.7.** *An element $e$ in a semiring $S$ is called $p$-idempotent if $ne + e^2 = (n+1)e$ for some $n \in \mathbb{N}$.*

**Theorem 4.3.8.** *Let $S$ be an antisimple strongly $p$-regular semiring such that for every $p$-idempotent $e \in S$ there exists an idempotent $e_1 \in S$ such that $e + e_1 = 1$. Then $S$ is a $k$-unit clean semiring.*

*Proof.* Let $S$ be an antisimple strongly $p$-regular semiring satisfying above condition. Let $x(\neq 0) \in S$. Since $S$ is antisimple, there exists $a \in S$ such that $x = a + 1$. Now for $a \in S$ there exists $b \in S$ such that $na + aba = (n+1)a \ \ldots \ (i)$ with $ab = ba$, for some $n \in \mathbb{N}$. Multiplying from right on both side of equation $(i)$ by $b$, we find that $nab + (ab)^2 = (n+1)ab$. Let $e = ab = ba$. Then $e$ is a $p$-idempotent element in $S$. According to the given condition, there exists an idempotent $e_1 \in S$ such that $e + e_1 = 1 \ \ldots \ (ii)$. Now $(a + e_1)((n+1)be + e_1 + n) = a(n+1)be + ae_1 + na + e_1(n+1)be + e_1 + ne_1 \ \ldots \ (iii)$. Since $ab = ba$, $eb = be$. Multiplying equation $(ii)$ by $be$, we have $e_1be + ebe = be \Longrightarrow e_1be + be^2 = be \Longrightarrow e_1be + be^2 + nbe = be + nbe \Longrightarrow e_1be+(n+1)be = (n+1)be$, since $e^2+ne = (n+1)e$. Hence from equation $(iii)$, we have $(a+e_1)((n+1)be+e_1+n) = a[e_1be+(n+1)be]+ae_1+na+e_1(n+1)be+e_1+ne_1 \ \ldots \ (iv)$. Equation $(i)$ can be written as $na + ae = (n+1)a \Longrightarrow nae_1 + aee_1 = (n+1)ae_1 \Longrightarrow ae_1 + ae_1 + \ldots + ae_1$ ($n$-times) $+aee_1 = (n+1)ae_1 \ \ldots \ (v)$. From equation $(ii)$, it follows that $e_1 + ee_1 = e_1 \Longrightarrow ae_1 + aee_1 = ae_1$. Hence equation $(v)$ can be written as $nae_1 = (n+1)ae_1 \Longrightarrow na = na + ae_1$, by using equation $(ii)$. Now equation $(iv)$ can be written as $(a + e_1)((n+1)be + e_1 + n) = ae_1be + a(n+1)be + na + e_1(n+1)be + e_1 + ee_1 + ne_1 = e^2 + ee_1 + e_1 + (a + e_1)e_1be + n(a + e_1)be + n(a + e_1) = 1 + (a + e_1)(e_1be + nbe + n)$. Thus $(a + e_1)$ is a right $k$-unit element of $S$. Similarly, we can show that $((n+1)be + e_1 + n)(a + e_1) = 1 + (bee_1 + nbe + n)(a + e_1)$. Thus $(a + e_1)$ becomes a left $k$-unit element of $S$. So $(a + e_1)$ is a $k$-unit element of $S$. Since every idempotent element of $S$ is a $p$-idempotent of $S$, $e_1$ is also a $p$-idempotent of $S$. Thus there exists an idempotent $e_2 \in S$ such that $e_1 + e_2 = 1$. Therefore,

$x = a + 1 = a + (e_1 + e_2) = (a + e_1) + e_2$. Hence $x$ is a $k$-unit clean element. This implies that $S$ is a $k$-unit clean semiring. □

## 4.4 Connection With Some Classes of Semirings

**Definition 4.4.1.** *An element $a$ in a semiring $S$ is right $\pi$-regular if there exists $x \in S$ such that $a^n = a^{n+1}x$ for some integer $n \geq 1$. The left $\pi$-regular is defined analogously. An element $a \in S$ is called strongly $\pi$-regular if it is both left and right $\pi$-regular. A semiring $S$ is called strongly $\pi$-regular semiring if every element of $S$ is strongly $\pi$-regular.*

**Proposition 4.4.2.** *If $a$ is strongly $\pi$-regular element in a semiring $S$ then there exist $x, y \in S$ such that $a^{n+1}x = a^n$ and $ya^{n+1} = a^n$ for some integer $n \geq 1$.*

*Proof.* Since $a$ is strongly $\pi$-regular, there exist $x, y \in S$ such that $a^{n+1}x = a^n$ and $ya^{m+1} = a^m$ for some integers $m, n \geq 1$. If $m \leq n$, $ya^{m+1} = a^m \implies ya^{m+1}a^{n-m} = a^m a^{n-m} \implies ya^{n+1} = a^n$. If $m > n$, then $a^n = a^{n+1}x = a^{n+2}x^2 = a^{n+3}x^3 = \ldots = a^{n+m-n}x^{m-n} = a^m x^{m-n}$. Now $ya^{n+1} = ya^{m+1}x^{m-n} = a^m x^{m-n} = a^n$. Hence the proof. □

**Proposition 4.4.3.** *If $a$ is a strongly $\pi$-regular element in a semiring $S$ then $a^n$ is strongly regular for some integer $n \geq 1$ and there exists $b \in S$ such that $ab = ba$, $a^n = a^{n+1}b$.*

*Proof.* The proof of $a^n$ is strongly regular, follows from Proposition 4.4.2. If $r \in S$ is strongly regular then there exists $z \in S$ such that $r^2z = r$, $rz = zr$, $rz^2 = z$ follows from [62]. Let $c \in S$ such that $cr = rc$. Then $zrc = zcr = zcr^2z = zr^2cz = rcz = crz$, i.e. $c$ commutes with $rz = zr$. Now $zc = z^2rc = z.zrc = zczr = zrcz = czrz = cz$. Hence $z$ commutes with every element $c$ which commutes with $r$. Here $a^n$ is strongly regular for some integer $n \geq 1$. Therefore, there exists $z \in S$ such that $a^{2n}z = a^n$ and $az = za$, since $a$ commutes with $a^n$. Thus $a^n = a^{n+1}a^{n-1}z = a^{n+1}b$, where $b = a^{n-1}z$ and $ab = a^n z = a^{n-1}za = ba$. □

**Theorem 4.4.4.** *Let $S$ be an antisimple strongly $\pi$-regular semiring such that every idempotent has strong absorbing complement in $S$. Then $S$ is a $k$-unit clean semiring.*

*Proof.* Let $a(\neq 0) \in S$. Since $S$ is antisimple and $a \neq 0$, $a = a_1 + 1$ for some $a_1 \in S$. Since $S$ is strongly $\pi$-regular semiring, there exists $b_1 \in S$ such that $a_1^n = a_1^{n+1}b_1$ for some integer $n \geq 1$ and $a_1b_1 = b_1a_1$, by Proposition 4.4.3. Hence $a_1^n = a_1^{n+1}b_1 = a_1^{n+2}b_1^2 = a_1^{n+3}b_1^3 = \ldots = a_1^{n+n}b_1^n = a_1^{2n}b_1^n = a_1^nb_1^na_1^n$, since $a_1b_1 = b_1a_1$. Let $f = a_1^nb_1^n$. Then $f$ is an idempotent in $S$. Then $f^2 = f$. According to given condition, there exists $f_1^2 = f_1$ such that $f + f_1 = 1 \ldots (i)$, $ff_1 = f_1f$, $f + sff_1 = f$ and $f_1 + f_1fs = f_1$ for all $s \in S$. Since $a_1b_1 = b_1a_1$, $a_1f = fa_1$. From equation $(i)$, we can write $a_1f + a_1f_1 = fa_1 + f_1a_1 \Longrightarrow a_1f + a_1f_1 = a_1f + f_1a_1 \Longrightarrow a_1ff_1 + a_1f_1 = a_1ff_1 + f_1a_1f_1$. Adding both side by $f$, it follows that $f + a_1f_1 = f + f_1a_1f_1$. Now adding both side by $f_1$, we get $1 + a_1f_1 = 1 + f_1a_1f_1 \ldots (ii)$. Similarly, like equation $(ii)$, we have $1 + f_1a_1 = 1 + f_1a_1f_1 \ldots (iii)$. Thus from $(ii)$ and $(iii)$, it follows that $1 + a_1f_1 = 1 + f_1a_1 \ldots (iv)$. Multiplying equation $(iv)$ from left and right side by $f_1$ we have $f_1 + f_1a_1 = f_1 + f_1a_1f_1 = f_1 + a_1f_1 \ldots (v)$. Similarly, for all integers $n \geq 1$, we can show that $1 + a_1^nf_1 = 1 + f_1a_1^n \ldots (vi)$ and $f_1 + f_1a_1^n = f_1 + a_1^nf_1 \ldots (vii)$. Now $a_1^nf = a_1^n$. Let $w = (a_1^n + f_1)$ and $w_1 = (b_1^nf + f_1)$. Then $ww_1 = (a_1^n + f_1)(b_1^nf + f_1) = a_1^nb_1^nf + a_1^nf_1 + f_1b_1^nf + f_1 = f + a_1^nff_1 + f_1b_1^nf + f_1 = f + a_1^nff_1 + f_1fb_1^n + f_1 = f + f_1 = 1$, where $b_1^nf = fb_1^n$, since $a_1b_1 = b_1a_1$. Similarly, we can show that $w_1w = 1$. Hence $w$ is a unit in $S$. Now $fw = f(a_1^n + f_1) = fa_1^n + ff_1 = a_1^nb_1^na_1^n + ff_1 = a_1^nf + a_1^nb_1^nff_1 = a_1^n(f + b_1^nff_1) = a_1^nf = a_1^n$ and $wf = (a_1^n + f_1)f = a_1^n + f_1f = a_1^n + ff_1 = f(a_1^n + f_1) = fw$. Hence $wf = fw = a_1^n$. Let $n$ be an even integer such that $n \geq 2$. Now $[fa_1 + f_1(1 + a_1)][a_1^{n-1}w^{-1}f + (2a_1 + 2a_1^3 + \ldots + 2a_1^{n-1})f + f_1(1 + a_1 + a_1^2 + a_1^3 + \ldots + a_1^{n-1})] = f + (2a_1 + 2a_1^3 + \ldots + 2a_1^{n-1})fa_1 + fa_1f_1(1 + a_1 + a_1^2 + \ldots + a_1^{n-1}) + f_1(1 + a_1)a_1^{n-1}w^{-1}f + f_1(1 + a_1)(2a_1 + 2a_1^3 + \ldots + 2a_1^{n-1})f + f_1(1 + a_1)f_1(1 + a_1 + a_1^2 + \ldots + a_1^{n-1}) \ldots (viii)$. So we have $f_1(1 + a_1)f_1(1 + a_1 + a_1^2 + \ldots + a_1^{n-1}) = f_1 + f_1a_1 + f_1a_1^2 + \ldots + f_1a_1^{n-1} + f_1a_1 + f_1a_1^2 + f_1a_1^3 + \ldots + f_1a_1^n \ldots (ix)$. Now from equation $(i)$, we have $fw + f_1fw = fw \Longrightarrow f_1fw + f_1fw = f_1fw \Longrightarrow f_1a_1^n + f_1a_1^n = f_1a_1^n$. Hence from equation $(ix)$, we get $f_1(1 + a_1)f_1(1 + a_1 + a_1 + a_1^2 + \ldots + a_1^{n-1}) = f_1 + (f_1a_1 + f_1a_1^2) + (f_1a_1 + f_1a_1^2) + (f_1a_1^3 + f_1a_1^4) + (f_1a_1^3 + f_1a_1^4) + \ldots + (f_1a_1^{n-1} + f_1a_1^n) + (f_1a_1^{n-1} + f_1a_1^n) =$

$f_1 + f_1(1+a_1)(a_1+a_1) + f_1(1+a_1)(a_1^3+a_1^3) + \ldots + f_1(1+a_1)a_1^{n-1} + f_1(1+a_1)a_1^{n-1} = f_1+f_1(1+a_1)(2a_1+2a_1^3+\ldots+2a_1^{n-1})$. Now using equation $(vii)$ and using the equation $a_1f = fa_1$, we can write the equation $(viii)$ as $[fa_1+f_1(1+a_1)][a_1^{n-1}w^{-1}f+(2a_1+2a_1^3+\ldots+2a_1^{n-1})f+f_1(1+a_1+a_1^2+a_1^3+\ldots+a_1^{n-1})] = f+fa_1(2a_1+2a_1^3+\ldots+2a_1^{n-1})+a_1(1+a_1+a_1^2+\ldots+a_1^{n-1})ff_1+f_1f(1+a_1)a_1^{n-1}w^{-1}+f_1f(1+a_1)(2a_1+2a_1^3+\ldots+2a_1^{n-1})+f_1+f_1(1+a_1)(2a_1+2a_1^3+\ldots+2a_1^{n-1}) = 1+[fa_1+f_1(1+a_1)][2a_1+2a_1^3+\ldots+2a_1^{n-1}]$. Hence $[fa_1+f_1(1+a_1)] = (a_1+f_1)$ is a right $k$-unit element of $S$. Similarly, it can be shown that $[fw^{-1}a_1^{n-1}+(2a_1+2a_1^3+\ldots+2a_1^{n-1})f+f_1(1+a_1+\ldots+a_1^{n-1})][fa_1+f_1(1+a_1)] = 1+[2a_1+2a_1^3+\ldots+2a_1^{n-1}][fa_1+f_1(1+a_1)]$. Hence $fa_1+f_1(1+a_1) = (a_1+f_1)$ is left $k$-unit of $S$. This implies that $(a_1+f_1)$ is a $k$-unit element of $S$ if $n \geq 2$ is an even integer. Let $n$ be an odd positive integer. If $n = 1$ then $a_1$ is a strongly regular element and hence by Theorem 4.3.5, it follows that $(a_1+f_1)$ is a $k$-unit of $S$. Let $n \geq 3$ be an odd integer. Now $[fa_1+f_1(1+a_1)][a_1^{n-1}w^{-1}f+f(2a_1+2a_1^3+\ldots+2a_1^{n-2})+f_1(1+a_1+a_1^2+\ldots+a_1^{n-1})] = f+fa_1(2a_1+2a_1^3+\ldots+2a_1^{n-2})+a_1ff_1(1+a_1+a_1^2+\ldots+a_1^{n-1})+f_1(1+a_1)a_1^{n-1}w^{-1}f+f_1f(1+a_1)(2a_1+2a_1^3+\ldots+2a_1^{n-2})+f_1(1+a_1)f_1(1+a_1+a_1^2+\ldots+a_1^{n-1})\ldots(x)$. Since $a_1^n = fw$, we can write the equation $(ix)$ as $f_1(1+a_1)f_1(1+a_1+a_1^2+\ldots+a_1^{n-1}) = f_1 + f_1a_1 + f_1a_1^2 + \ldots + f_1a_1^{n-1} + f_1a_1 + f_1a_1^2 + f_1a_1^3 + \ldots + f_1a_1^n = f_1 + f_1fw + (f_1a_1+f_1a_1^2)+(f_1a_1+f_1a_1^2)+(f_1a_1^3+f_1a_1^4)+(f_1a_1^3+f_1a_1^4)+\ldots+(f_1a_1^{n-2}+f_1a_1^{n-1})+(f_1a_1^{n-2}+f_1a_1^{n-1}) = f_1 + f_1(1+a_1)2a_1 + f_1(1+a_1)2a_1^3 + \ldots + f_1(1+a_1)2a_1^{n-2}) = f_1+f_1(1+a_1)[2a_1+2a_3+\ldots+2a^{n-2}]$. Put the value of this in equation $(x)$ and using $a_1f = fa_1$, the equation $(x)$ becomes $[fa_1+f_1(1+a_1)][a_1^{n-1}w^{-1}f+f(2a_1+2a_1^3+\ldots+2a_1^{n-2})+f_1(1+a_1+a_1^2+\ldots+a_1^{n-1})] = f+fa_1(2a_1+2a_1^3+\ldots+2a_1^{n-2})+f_1+f_1(1+a_1)(2a_1+2a_1^3+\ldots+2a_1^{n-2}) = 1+[fa_1+f_1(1+a_1)][2a_1+2a_1^3+\ldots+2a_1^{n-2}]$. Hence $[fa_1+f_1(1+a_1)] = (a_1+f_1)$ is right $k$-unit of $S$. Similarly, it can be shown that $[fw^{-1}a_1^{n-1}+(2a_1+2a_1^3+\ldots+2a_1^{n-2})f+f_1(1+a_1+\ldots+a_1^{n-1})][fa_1+f_1(1+a_1)] = 1+[2a_1+2a_1^3+\ldots+2a_1^{n-2}][fa_1+f_1(1+a_1)]$. This implies that $(a_1+f_1)$ is left $k$-unit and hence a $k$-unit in $S$ for an odd integer $n \geq 3$. Thus $(a_1+f_1)$ is $k$-unit in all possible cases of $n$. Hence $a = a_1+1 = (a_1+f_1)+f$. So $S$ is a $k$-unit clean semiring. □

**Corollary 4.4.5.** *Let $S$ be an antisimple artinian semiring such that every idempotent* *has strong absorbing complement in $S$. Then $S$ is a $k$-unit clean semiring.*

*Proof.* Since $S$ is artinian, it is left and right artinian. Let $a \in S$. Then there exist some integers $m, n \geq 1$ such that $(a)_l \supseteq (a^2)_l \supseteq (a^3)_l \supseteq \ldots \supseteq (a^m)_l = (a^{m+1})_l = \ldots$ and $(a)_r \supseteq (a^2)_r \supseteq (a^3)_r \supseteq \ldots \supseteq (a^n)_r = (a^{n+1})_r = \ldots$, where $(a)_l$ and $(a)_r$ denote the principal left and right ideal generated by $a$ respectively. Thus there exist $x, y \in S$ such that $a^m = xa^{m+1}$ and $a^n = a^{n+1}y$. This implies that $a$ is strongly $\pi$-regular. The rest of the proof follows from Theorem 4.4.4. □

**Corollary 4.4.6.** *Let $S$ be an antisimple finite semiring such that every idempotent has strong absorbing complement in $S$. Then $S$ is a $k$-unit clean semiring.*

**Theorem 4.4.7.** *Let $S$ be an inverse strongly $k$-unit clean semiring such that every idempotent has an absorbing complement in $S$. Then $S$ is an exchange semiring.*

*Proof.* Let $y \neq 0$ in $S$. Then $y' \neq 0$ in $S$. Since $y'$ is strongly $k$-unit clean, we can write $y' = e + u$, $eu = ue$, where $e^2 = e \in S$ and $u$ is a $k$-unit of $S$. So $ey' = e + eu \implies ey + ey' = e + eu + ey \implies e(y + y') = e(1 + y) + eu$. Now $y + y' + y = y \implies (1 + y) + (y + y') = (1 + y) \implies (y + y') \in \overline{S(1 + y)}$. Thus $eu = ue \in \overline{S(1 + y)}$ . Since $u$ is a $k$-unit of $S$, so there exist $s_1, s_2 \in S$ such that $1 + s_1u = s_2u \ \ldots \ (i)$ and $1 + us_1 = us_2 \ \ldots \ (ii)$. Multiplying from right on both side of equation $(i)$ by $e$, we have $e + s_1ue = s_2ue$. Hence $e \in \overline{S(1 + y)}$. According to given condition, $e$ has an absorbing complement $e_1^2 = e_1 \in S$. Now $e_1 + e = 1 \implies e + e_1e = e \implies e + u + e_1e = e + u \implies y' + e_1e = y' \implies e_1e \in \overline{Sy'}$. Again $e_1y' = e_1e + e_1u \implies e_1u \in \overline{Sy'}$. Now $e + e_1 = 1 \implies eu + e_1u = eu + ue_1 \ \ldots \ (iii)$, since $eu = ue$. Multiplying from right on both side of equation $(i)$ by $e_1$, we have $e_1 + s_1ue_1 = s_2ue_1 \implies e_1 + (s_1 + s_2)eu + s_1ue_1 = (s_1 + s_2)eu + s_2ue_1$. Using equation $(iii)$, it implies that $e_1 + (s_1 + s_2)eu + s_1e_1u = (s_1 + s_2)eu + s_2e_1u \implies e_1 + seu + s_1e_1u = seu + s_2e_1u$, where $s = (s_1 + s_2)$. Hence $e_1 + se + seu + s_1e_1u = se + seu + s_2e_1u \implies e_1 + sey' + s_1e_1u = sey' + s_2e_1u$, since $ey' = e + eu$. Now $e_1u \in \overline{Sy'}$ implies that $e_1 \in \overline{Sy'} = \overline{Sy}$. □

The converse of Theorem 4.4.7 is not true. This follows from the following example

:

**Example 4.4.8.** *Consider the semiring $S$ which is defined in Example 3.4.3. Then $S$ is an inverse exchange semiring such that every idempotent has absorbing complement in $S$, but $S$ is not strongly $k$-unit clean semiring.*

**Definition 4.4.9.** *Let $S$ be a commutative semiring and $J_l(S)$ be the Jacobson radical of $S$. Then $S$ is said to be semi-boolean semiring if $S/J_l(S)$ is Boolean and idempotents lift modulo $J_l(S)$.*

**Theorem 4.4.10.** *Let $S$ be an antisimple commutative semi-Boolean semiring such that every idempotent has a complement in $S$. Then $S$ is a $k$-unit clean semiring and each $k$-unit $u$ of $S$ can be expressed as $u = v + f$ with $v \in 1 + J_l(S)$ and $f$ is an idempotent.*

*Proof.* If $J_l(S) = \{0\}$, then $S$ becomes an antisimple Boolean semiring. Thus from Theorem 2.3.7, it follows that $S$ is a clean semiring and hence $S$ is a $k$-unit clean semirng as every unit of $S$ is a $k$-unit of $S$. Let $u$ be a $k$-unit in $S$. Then $u \neq 0$. Since $S$ is antisimple, there exists $u_1 \in S$ such that $u = u_1 + 1$. Now $u_1^2 = u_1$, since $S$ is Boolean. Hence $u = u_1 + 1 = (1 + 0) + u_1$, where $(1 + 0) \in 1 + J_l(S)$.

Let $J_l(S) \neq \{0\}$. Let $x \neq 0$. Then there exists $x_1 \in S$ such that $x = x_1 + 1$. Now $(x + J_l(S))^2 = x^2 + J_l(S) = x + J_l(S)$. Since idempotents lift modulo $J_l(S)$, $x + J_l(S) = e + J_l(S) \ \ldots \ (i)$, where $e^2 = e \in S$. Now $e$ has a complement in $S$. This implies that $e + e_1 = 1$, for some idempotent $e_1 \in S$. Hence from equation $(i)$, we have $(x + e_1) + J_l(S) = 1 + J_l(S) \Longrightarrow (x_1 + 1 + e_1) + J_l(S) = 1 + J_l(S) \ \ldots \ (ii)$. Since $S$ is antisimple and $J_l(S) \neq \{0\}$, from Note 2.3.4, we can write equation $(ii)$ as $(x_1 + e_1) + J_l(S) = J_l(S) \Longrightarrow (x_1 + e_1) \in J_l(S)$. Hence $x = (x_1 + e_1) + e$. This implies that if $J_l(S) \neq \{0\}$, every element of $S$ can be written as a sum of an idempotent and an element of its Jacobson radical $J_l(S)$. Now let $a \neq 0$. Then there exists $b \in S$ such that $a = b + 1 = (e + b_1) + 1 = e + (1 + b_1)$, where $b_1 \in J_l(S)$. Since $b_1 \in J_l(S)$, so $(1 + b_1)$ is a $k$-unit of $S$. Hence $S$ is a $k$-unit clean semiring. If $u$ is a $k$-unit, then by similar argument, we get $u = f + c$, where $c \in 1 + J_l(S)$ and $f$ is an idempotent. $\square$

The converse of the above result is not true, in general.

**Example 4.4.11.** *Consider the semiring $S$ defined in Example 2.2.3(vi). Then $S$ is a clean semiring and hence a $k$-unit clean semiring. Every $k$-unit element $u$ can be written as $u = v + f$, where $v \in 1 + J_l(S)$ and $f^2 = f$. But $S$ is not semi-boolean semiring because there exists no idempotent element $e \in S$ such that $2 + J_l(S) = e + J_l(S)$ holds.*

But in particular, if the set of idempotents of $S$ i.e. $Id(S)$ forms a subsemiring of $S$ then the converse of the above result also holds.

Let $x \notin J_l(S)$. Then $x \neq 0$. So $x + J_l(S) = (e + u) + J_l(S)$, where $u$ is a $k$-unit and $e$ is an idempotent of $S$. According to the given condition, $u = v + f$, where $v \in 1 + J_l(S)$ and $f \in Id(S)$. Thus $v = 1 + b$, where $b \in J_l(S)$. Hence $x + J_l(S) = (e + v + f) + J_l(S) = (e + 1 + b + f) + J_l(S) = (e + f + 1) + J_l(S)$. Since $Id(S)$ is a subsemiring of $S$, $E = e + f + 1$ is an idempotent of $S$ which implies $x + J_l(S) = E + J_l(S)$. Hence $S$ is a semi-boolean semiring.

**Theorem 4.4.12.** *Let $S$ be a commutative inverse semiring with idempotents* $0$ *and* $1$*. Then $S$ a $k$-unit clean semiring if and only if $S$ is a $k$-local semiring and semi-antisimple semiring.*

*Proof.* Let $S$ be a $k$-local semiring and semi-antisimple semiring with unique maximal $k$-ideal $M$. Let $x \neq 0$. If $x$ is a $k$-unit of $S$ then we have nothing to proof. Let $x$ be not a $k$-unit element of $S$. Then $x \in M$, since $S$ is a $k$-local semiring. Since $S$ is semi-antisimple, there exists $x_1 \in S$ such that $x = x_1 + 1$. If $x_1$ is not a $k$-unit then $x_1 \in M$. This implies that $1 \in M$, which is a contradiction. Hence $x_1$ is a $k$-unit element of $S$. Thus $x$ is a $k$-unit clean element of $S$.

Conversely, let $S$ be a $k$-unit clean semiring. Let $x \neq 0$. Let $S$ be an additive idempotent semiring. Then either $x = u_p$ or $x = u_q + 1$ where $u_p, u_q \in U_k(S)$. Hence either $x = u_p$ or $x = x + 1$. In both cases $x$ is a $k$-unit element of $S$. Hence every nonzero element is a $k$-unit which implies $S$ is $k$-local and semi-antisimple(vacuously). Let $S$ be not an additive idempotent semiring. Let $x \neq 0$ be not a $k$-unit of $S$. Then $x = u + 1$ for some $k$-unit $u \in S$. Hence $S$ is semi-antisimple. Let $I$ be any nonzero proper $k$-ideal in $S$. Then there exists $a(\neq)0 \in I$ such that $a \notin U_k(S)$. Hence

$a = u_1 + 1$ for some $u_1 \in U_k(S)$. Thus $a + 1 + 1' = u_1 + 1 + 1' + 1 = u_1 + 1 = a \Longrightarrow (1 + 1') \in I$. So $E^+(S) \subseteq I$. Thus every nonzero proper $k$-ideal in $S$ is full $k$-ideal. Since $x \notin U_k(S)$, $(sx') \notin U_k(S)$ for any $s \in S$. Thus if $sx' \neq 0$ then $sx' = 1 + u_2$ for some $u_2 \in U_k(S)$. Hence $sx' + sx = 1 + sx + u_2 \Longrightarrow s(x + x') = (1 + sx) + u_2$. Here $(1 + sx) \neq 0$, otherwise $x$ becomes a $k$-unit of $S$. If $(1 + sx)$ is not a $k$-unit then $(1 + sx) \in J$, where $J$ is a nonzero proper $k$-ideal of $S$. By previous argument, $E^+(S) \subseteq J$ which implies that $s(x + x') \in J$. Hence $u_2 \in J$ which is a contradiction. So $(1 + sx)$ is a $k$-unit of $S$. Thus $(1 + sx) \in U_k(S)$ for any $s \in S$. Let $x \notin J_l(S)$. Then there exists a maximal $k$-ideal $M_1$ such that $x \notin M_1$. Since $x$ is not a $k$-unit, $M_1 \neq 0$. Now there exist $m_1, m_2 \in M_1$ and $s_1, s_2 \in S$ such that $1 + (m_1 + s_1x) = (m_2 + s_2x) \Longrightarrow 1 + m_1 + (s_1 + s_2')x = m_2 + s_2(x + x')$. Since $E^+(S) \subseteq M_1$, $b = 1 + (s_1 + s_2')x \in M_1$ which is a contradiction because $b \in U_k(S)$. Hence $x \in J_l(S)$ if $x$ is nonzero and not a $k$-unit of $S$. Therefore, $S$ is a $k$-local semiring. □

## 4.5 Extensions of $k$-Unit Clean Semiring

Let $S$ be a semiring and $x$ be an indeterminate. Suppose that $S[[x]]$ denotes the set of all expressions of the form $f = \sum_{i=0}^{\infty} a_i x^i$, $a_i \in S$. If $g = \sum_{i=0}^{\infty} b_i x^i$ is also an element of $S[[x]]$, we define addition and multiplication in $S[[x]]$ as follows :

$$f + g = \sum_{i=0}^{\infty}(a_i + b_i)x^i \text{ and } fg = \sum_{i=0}^{\infty} d_i x^i, \text{ where } d_i = \sum_{r+s=i} a_r b_s,\ i = 0, 1, 2, \ldots.$$

Then $S[[x]]$ is called the semiring of formal power series (in the indeterminate $x$) over $S$. We say that $f = g$ if and only if $a_i = b_i$ for all non-negative integers $i$ and $a_0$ is called the constant term of $f$. If $a_i = 0$ for $i = 0, 1, 2, \ldots, r - 1$ but $a_r \neq 0$, then $f$ is said to be of order $r$. If $a_0 \neq 0$ then $f$ is said to be of order 0.

**Proposition 4.5.1.** *[20] An element $f = \sum_{i=0}^{\infty} a_i x^i$ is $k$-unit in $S[[x]]$ if and only if $a_0$ is $k$-unit in $S$.*

**Theorem 4.5.2.** *Let $S$ be a semiring. Then $S$ is $k$-unit clean if and only if every nonzero element of $S[[x]]$ of order $r$ ($r \geq 0$) is of the form $x^r u$ such that $u$ is a $k$-unit* *clean element of $S[[x]]$.*

*Proof.* Let $S$ be a $k$-unit clean semiring and $f = \sum_{i=0}^{\infty} a_i x^i$ be a nonzero element of order $r$. Then $a_r (\neq 0)$. Since $S$ is $k$-unit clean, $a_r = e_r + u_r$, where $e_r$ is an idempotent and $u_r$ is a $k$-unit in $S$. Now $f = x^r(a_r + a_{r+1}x + a_{r+2}x^2 + \ldots) = x^r u$, where $u = a_r + a_{r+1}x + a_{r+1}x^2 + \ldots \in S[[x]]$. Then $u = e_r + u_r + a_{r+1}x + a_{r+2}x^2 + \ldots = e_r + g$. By Proposition 4.5.1, $g$ is a $k$-unit element in $S[[x]]$. Since $S$ is a subsemiring of $S[[x]]$, $e_r$ is an idempotent of $S[[x]]$. Hence $u$ is a $k$-unit clean element of $S[[x]]$.

Conversely, suppose that $a(\neq 0) \in S$ and every nonzero element of $S[[x]]$ of order $r$ $(r \geq 0)$ is of the form $x^r u$ such that $u$ is a $k$-unit clean element of $S[[x]]$. Then $a$ is a nonzero element of $S[[x]]$ of order $r = 0$. Then $a = e + u$, where $e = e_0 + e_1 x + e_2 x^2 + \ldots$ is an idempotent of $S[[x]]$ and $u = u_0 + u_1 x + u_2 x^2 + \ldots$ is a $k$-unit of $S[[x]]$. Hence $a = e_0 + u_0$, where $e_0$ is an idempotent of $S$ and $u_0$ is a $k$-unit of $S$, by Proposition 4.5.1. Hence $a$ is a $k$-unit clean element of $S$. This implies that $S$ is a $k$-unit clean semiring. □

But the Theorem 4.5.2 does not hold for polynomial semiring $S[x]$ which follows from the following example :

**Example 4.5.3.** *Consider $S = \{0, 1, 2, 3\}$. Define the operations "+" and "." as follows :*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 1 |
| 2 | 2 | 3 | 1 | 2 |
| 3 | 3 | 1 | 2 | 3 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 1 | 3 |
| 3 | 0 | 3 | 3 | 3 |

*Now $S$ is a clean semiring and hence a $k$-unit clean semiring.*

*Let $S$ be a semiring which is defined above. Consider the polynomial $f(x) = 2x + x^2 = x(2 + x)$ in $S[x]$. Here $(2 + x)$ is not a $k$-unit clean element in $S[x]$ because neither $(2 + x)$ nor $(1 + x)$ is a $k$-unit element in $S[x]$.*

**Theorem 4.5.4.** *Let $A$ and $B$ be two semirings without zero element and with multiplicative identity $1_A$ and $1_B$ respectively. Let $(M,+)$ and $(N,+)$ be two additive abelian monoids with additive identity $0_M$ and $0_N$ are $(B,A)$ bi-semimodule and $(A,B)$ bi-semimodule respectively such that $b0_M = 0_M = 0_M a$ and $a0_N = 0_N = 0_N b$ for all $a \in A$ and $b \in B$. Let us consider the set of all matrices of the form $T = \left\{ \begin{bmatrix} a & n \\ m & b \end{bmatrix} : a \in A, m \in M, n \in N, b \in B \right\}$. Then $T$ forms a semiring in which addition is defined componentwise and multiplication is defined by $\begin{bmatrix} a_1 & n_1 \\ m_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & n_2 \\ m_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_1n_2 + n_1b_2 \\ m_1a_2 + b_1m_2 & b_1b_2 \end{bmatrix}$. Then $T$ is a $k$-unit clean semiring if and only if $A$ and $B$ are $k$-unit clean semirings.*

*Proof.* Let $A$ and $B$ be two $k$-unit clean semirings without zero element and with multiplicative identity $1_A$ and $1_B$ respectively. Let $\begin{bmatrix} a & n \\ m & b \end{bmatrix} \in T$, where $a \in A$, $b \in B$, $m \in M$ and $n \in N$. Then there exist idempotents $e_1 \in A$, $e_2 \in B$ and $k$-units $u_1 \in A$, $u_2 \in B$ such that $a = e_1 + u_1$ and $b = e_2 + u_2$. Now $\begin{bmatrix} a & n \\ m & b \end{bmatrix} = \begin{bmatrix} e_1 & 0_N \\ 0_M & e_2 \end{bmatrix} + \begin{bmatrix} u_1 & n \\ m & u_2 \end{bmatrix}$. Since $u_1 \in A$ and $u_2 \in B$ are two $k$-units, there exist $s_1, s_2 \in A$ and $s_3, s_4 \in B$ such that $1_A + s_1u_1 = s_2u_1$, $1_A + u_1s_1 = u_1s_2$ and $1_B + s_3u_2 = s_4u_2$, $1_B + u_2s_3 = u_2s_4$.

Now $\begin{bmatrix} 1_A & 0_N \\ 0_M & 1_B \end{bmatrix} + \begin{bmatrix} u_1 & n \\ m & u_2 \end{bmatrix} \begin{bmatrix} s_1 & s_1ns_3 + s_2ns_4 \\ s_3ms_1 + s_4ms_2 & s_3 \end{bmatrix} = \begin{bmatrix} 1_A & 0_N \\ 0_M & 1_B \end{bmatrix}$

$$+ \begin{bmatrix} u_1s_1 & u_1s_1ns_3 + u_1s_2ns_4 + ns_3 \\ ms_1 + u_2s_3ms_1 + u_2s_4ms_2 & u_2s_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1_A + u_1s_1 & u_1s_1ns_3 + u_1s_2ns_4 + ns_3 \\ ms_1 + u_2s_3ms_1 + u_2s_4ms_2 & 1_B + u_2s_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1s_2 & u_1s_1ns_3 + u_1s_2ns_4 + ns_3 \\ ms_1 + u_2s_3ms_1 + u_2s_4ms_2 & u_2s_4 \end{bmatrix}$$

$$= \begin{bmatrix} u_1s_2 & (1_A + u_1s_1)ns_3 + u_1s_2ns_4 \\ (1_B + u_2s_3)ms_1 + u_2s_4ms_2 & u_2s_4 \end{bmatrix}$$

$= \begin{bmatrix} u_1s_2 & u_1s_2n(s_3+s_4) \\ u_2s_4m(s_1+s_2) & u_2s_4 \end{bmatrix}.$

Now $\begin{bmatrix} u_1 & n \\ m & u_2 \end{bmatrix} \begin{bmatrix} s_2 & s_2ns_3+s_1ns_4 \\ s_4ms_1+s_3ms_2 & s_4 \end{bmatrix}$

$= \begin{bmatrix} u_1s_2 & u_1s_2ns_3+u_1s_1ns_4+ns_4 \\ ms_2+u_2s_4ms_1+u_2s_3ms_2 & u_2s_4 \end{bmatrix}$

$= \begin{bmatrix} u_1s_2 & u_1s_2ns_3+(u_1s_1+1_A)ns_4 \\ (1_B+u_2s_3)ms_2+u_2s_4ms_1 & u_2s_4 \end{bmatrix}$

$= \begin{bmatrix} u_1s_2 & u_1s_2n(s_3+s_4) \\ u_2s_4m(s_1+s_2) & u_2s_4 \end{bmatrix}.$

Hence $\begin{bmatrix} u_1 & n \\ m & u_2 \end{bmatrix}$ is a right $k$-unit element of $T$. Similarly, it can be shown that $\begin{bmatrix} 1_A & 0_N \\ 0_M & 1_B \end{bmatrix} + \begin{bmatrix} s_1 & s_1ns_3+s_2ns_4 \\ s_3ms_1+s_4ms_2 & s_3 \end{bmatrix} \begin{bmatrix} u_1 & n \\ m & u_2 \end{bmatrix} = \begin{bmatrix} s_2 & s_2ns_3+s_1ns_4 \\ s_4ms_1+s_3ms_2 & s_4 \end{bmatrix} \begin{bmatrix} u_1 & n \\ m & u_2 \end{bmatrix}$. This implies that $\begin{bmatrix} u_1 & n \\ m & u_2 \end{bmatrix}$ is a left $k$-unit element of $T$ and hence a $k$-unit element of $T$. Since $e_1$ and $e_2$ are idempotent elements of $A$ and $B$ respectively, $\begin{bmatrix} e_1 & 0_N \\ 0_M & e_2 \end{bmatrix}$ is an idempotent element of $T$. Hence $\begin{bmatrix} a & n \\ m & b \end{bmatrix}$ is a $k$-unit clean element of $T$. Thus it follows that $T$ is a $k$-unit clean semiring.

Conversely, let $T$ be a $k$-unit clean semiring. Let $a \in A$ and $b \in B$. Then $\begin{bmatrix} a & 0_N \\ 0_M & b \end{bmatrix} \in T$ and hence $k$-unit clean element of $T$. Thus $\begin{bmatrix} a & 0_N \\ 0_M & b \end{bmatrix} = \begin{bmatrix} e_1 & n_1 \\ m_1 & e_2 \end{bmatrix} + \begin{bmatrix} u_1 & n_2 \\ m_2 & u_2 \end{bmatrix} = E + U$, where $E$ is an idempotent and $U$ is a $k$-unit element of $T$. Thus $a = e_1 + u_1$ and $b = e_2 + u_2$. It can be easily proved that $e_1$ and $e_2$ are idempotents and $u_1$ and $u_2$ are $k$-units of $A$ and $B$ respectively. Hence $A$ and $B$ are $k$-unit clean semirings. □

From Theorem 4.5.4, by taking $N = O$ which is $(A, B)$ bi-semimodule, we have the following result :

**Corollary 4.5.5.** *Let $A$ and $B$ be two semirings without zero element and with multiplicative identity $1_A$ and $1_B$ respectively. Let $(M, +)$ be an additive abelian monoid with additive identity $0_M$ which is $(B, A)$ bi-semimodule such that $b0_M = 0_M = 0_M a$ for all $a \in A$ and $b \in B$. Let us consider the set of all matrices of the form $T = \left\{ \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} : a \in A, m \in M, b \in B \right\}$. Then $T$ forms a semiring in which addition is defined componentwise and multiplication is defined by $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ m_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 \\ m_1a_2 + b_1m_2 & b_1b_2 \end{bmatrix}$.*

*Then $T$ is a $k$-unit clean semiring if and only if $A$ and $B$ are $k$-unit clean semirings.*

# Chapter 5

# On $k$-regular Clean Semiring

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# On $k$-regular Clean Semiring

## 5.1 Introduction

Rings in which every element is the sum of an idempotent and a unit are said to be clean rings. The notion of clean ring was introduced by W. K. Nicholson [7] in 1977 in the study of exchange rings. A ring $R$ is called Von Neumann regular if for any $a \in R$ there exists an element $x \in R$ such that $a = axa$. Also a ring $R$ is said to be unit regular if for any $a \in R$, there exists a unit $u \in R$ such that $a = aua$. Basically, clean rings are the additive analog of unit regular rings. Since then various generalizations of clean rings have been obtained by many authors ([28], [30], [42], [59]). There has been a remarkable growth of the algebraic theory of semirings over last several years. Semiring is a generalization of ring and distributive lattice. In [1], S. Bourne introduced the notion of regularity in semiring as a generalization of regular ring. According to Bourne, a semiring $S$ is regular if for every element $s \in S$ there exist elements $x', x'' \in S$ such that $s + sx''s = sx's$. If $S$ becomes a ring then this definition coincides with Von Neumann regularity of a ring. But Von Neumann regularity and Bourne regularity are not equivalent in semiring. To distinguish the notion of Bourne regularity of a semiring from the notion of Von Neumann regularity it was renamed as $k$-regularity in [14] by Adhikari, Sen and Weinert. In [62], we have introduced the notion of clean semiring and we have also worked on the concept of $k$-unit clean semiring as a generalization of clean semiring in chapter 4. We notice that every $k$-unit element in a semiring $S$ is $k$-regular but the converse is not true. Hence $k$-regular elements are the generalization of $k$-unit elements in a semiring $S$. In this chapter, we introduce the concept of $k$-regular clean semiring which generalizes the notion of $k$-unit clean semiring. A semiring $S$ is said to be $k$-regular clean semiring if every element of $S$ can be written as the sum of an idempotent and a $k$-regular element of $S$. In this chapter, we obtain some important characterizations about $k$-regular clean semiring.

## 5.2 Definition & Example

**Definition 5.2.1.** An element of a semiring $S$ is said to be $k$-regular clean if it can be written as the sum of an idempotent and a $k$-regular element of $S$. A semiring $S$ is called a $k$-regular clean semiring if every element of $S$ is $k$-regular clean.

**Example 5.2.2.** *(i) Every clean ring, clean semiring, $k$-unit clean semiring are $k$-regular clean semirings.*

*(ii) Every $k$-regular semiring is a $k$-regular clean semiring.*

*(iii) Every Boolean semiring is a $k$-regular clean semiring.*

*(iv) Let $S = \mathbb{N}_0^+$ be the set of all non-negative integers. Define two binary operations '$+$' and '$.$' on $S$ by $a + b = max\{a, b\}$ and $a.b = ab$. Then $(\mathbb{N}_0^+, +, .)$ forms a semiring with additive identity $0$ and multiplicative identity $1$. Let $M_2(\mathbb{N}_0^+) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{N}_0^+ \right\}$. Since $S$ is additive idempotent $k$-regular semiring with zero element, $M_2(\mathbb{N}_0^+)$ is $k$-regular semiring with zero element. Now $S = \mathbb{N}_0^+$ is also the semiring of all non-negative integers with respect to the usual addition and multiplication. Let $I = 4\mathbb{N}_0^+$. Then $I$ is a $k$-ideal of $S$. Let $k_I$ denotes the bourne congruence defined by $I$. Let $S' = S/k_I$ denotes the set of all congruence classes with respect to $k_I$. Then $S' = \{[0], [1], [2], [3]\}$ becomes a semiring with respect to the operations $[a_1] + [b_1] = [a_1 + b_1]$ and*

*$[a_1][b_1] = [a_1b_1]$ for all $[a_1], [b_1] \in S'$. Let $E = M_2(\mathbb{N}_0^+) \times S'$. Then $E$ is a $k$-regular clean semiring with respect to componentwise addition and multiplication but $E$ is not clean, $k$-unit clean and $k$-regular semiring.*

## 5.3 Elementary Results

**Proposition 5.3.1.** *Let $a \in S$ and $a + axa$ be $k$-regular element for some $x \in S$. Then $a$ is a $k$-regular element in $S$.*

*Proof.* Since $a + axa$ is $k$-regular element for some $x \in S$, there exist $b, c \in S$ such that $(a + axa) + (a + axa)b(a + axa) = (a + axa)c(a + axa)$. This implies that $a + a(x + b + bax + xab + xabax)a = a(c + cax + xac + xacax)a$. This shows that $a$ is a $k$-regular element in $S$. □

**Proposition 5.3.2.** *Homomorphic image of a $k$-regular clean semiring is $k$-regular clean.*

*Proof.* Let $(S', +, .)$ be the homomorphic image of the semiring $(S, +, .)$. Then $f : S \longrightarrow S'$ be an onto homomorphism. Let $s' \in S'$. Since $f$ is onto, there exists $s \in S$ such that $f(s) = s'$. Since $S$ is $k$-regular clean semiring, $s = e + r$, where $e^2 = e \in S$ and $r$ is a $k$-regular element in $S$. $s' = f(s) = f(e+r) = f(e)+f(r)$. Now $f(e)f(e) = f(e^2) = f(e)$, since $f$ is a homomorphism. Since $r$ is a $k$-regular element in $S$, there exist $x, y \in S$ such that $r + rxr = ryr$. Hence $f(r+rxr) = f(ryr)$, since $f$ is well-defined. So $f(r) + f(r)f(x)f(r) = f(r)f(y)f(r)$, since $f$ is a homomorphism. Thus $f(e)$ is an idempotent and $f(r)$ is a $k$-regular element in $S'$. Hence $s'$ is a $k$-regular clean element in $S'$ which implies $S'$ is $k$-regular clean semiring. □

**Proposition 5.3.3.** *Arbitrary direct product of semirings $S = \prod_{i \in I} S_i$ ($I$ is an index set) is $k$-regular clean semiring if and only if each $S_i$ is $k$-regular clean.*

*Proof.* Let $S_i$ be $k$-regular clean semiring for each $i \in I$. Let $(a_i)_i = (a_1, a_2, a_3, \ldots, a_i, \ldots) \in S$. Now $a_i = e_i + r_i$ where $e_i$ is idempotent of $S_i$ and $r_i$ is $k$-regular element of $S_i$ for each $i \in I$. Thus there exist $x_i, y_i \in S_i$ such that $r_i + r_ix_ir_i = r_iy_ir_i$ for each $i \in I$.

Hence $(a_i)_i = (e_i)_i + (r_i)_i$. Now it can be easily proved that $(e_i)_i$ is an idempotent of $S$ and $(r_i)_i + (r_i)_i(x_i)_i(r_i)_i = (r_i)_i(y_i)_i(r_i)_i$ which implies that $(r_i)_i$ is a $k$-regular element in $S$. Hence $(a_i)_i$ is a $k$-regular clean element in $S$. So $S$ is $k$-regular clean semiring.

Conversely, let $S$ be a $k$-regular clean semiring. Let us define a mapping $\pi_i : S \to S_i$, by $\pi_i((s_i)_i) = s_i$ for all $(s_i)_i \in S$. Clearly $\pi_i$ is an onto homomorphism from $S$ to $S_i$. Hence $S_i$ is a homomorphic image of $S$ for each $i \in I$. Thus by Proposition 5.3.2, it follows that $S_i$ is $k$-regular clean for each $i \in I$. □

Let $S$ be a semiring and $I$ be a $k$-ideal of $S$. Now define a relation $k_I$ on $S$ such that $ak_Ib \iff a + i_1 = b + i_2$ for some $i_1, i_2 \in I$. Then $k_I$ becomes a congruence relation on $S$ and it is known as Bourne congruence relation. Let $S/I$ denotes the set of all congruence classes with respect to $k_I$ and we denote $[a]_{k_I} = (a + I)$ as the congruence class of $a$ for any $a \in S$. Hence $S/I = \{(a + I) \mid a \in S\}$. Now $S/I$ becomes a semiring with respect to the operations $(a+I)+(b+I) = (a+b)+I$ and $(a + I)(b + I) = ab + I$ for all $a, b \in S$.

**Proposition 5.3.4.** *Let $S$ be a $k$-regular clean semiring and $I$ be a $k$-ideal of $S$. Then $S/I$ is a $k$-regular clean semiring.*

*Proof.* Define a mapping $\phi : S \longrightarrow S/I$ by $\phi(a) = [a]_{k_I} = (a + I)$ for all $a \in S$. Then $\phi$ is an onto homomorphism, where $\phi(0_S) = I = 0_{S/I}$ and $\phi(1_S) = (1 + I) = 1_{S/I}$. Hence the proof follows from proposition 5.3.2. □

**Theorem 5.3.5.** *Let $S$ be an antisimple inverse semiring such that every idempotent of $S$ has a complement. Let $I$ be a nonzero $k$-regular $k$-ideal in $S$ such that idempotents can be lifted modulo $I$. Then $S$ is a $k$-regular clean semiring if and only if $S/I$ is a $k$-regular clean semiring.*

*Proof.* Let $S$ be a $k$-regular clean semiring. Then $S/I$ is also a $k$-regular clean semiring, by Proposition 5.3.4.

Conversely, let $S/I$ be a $k$-regular clean semiring. Since $I$ is a nonzero $k$-ideal of $S$, so there exists $x(\neq 0) \in I$. Again $x = x_1 + 1$ for some $x_1 \in S$, since $S$ is antisimple. Again since $S$ is an inverse semiring, so there exists $1' \in S$ such that $1 + 1' + 1 = 1$. Thus $x + 1 + 1' = x_1 + 1 + 1' + 1 = x_1 + 1 = x$. Since $I$ is a $k$-ideal of $S$, $1 + 1' \in I$ which implies $E^+(S) \subseteq I$. Hence $I$ is a full $k$-ideal of $S$. Let $a \in I$. Then $a$ is a $k$-regular element in $I$ and hence in $S$, since $I$ is a $k$-regular ideal of $S$. Thus $a = 0 + a$ is a $k$-regular clean element of $S$. Let $a \notin I$. Then there exists $a_1 \in S$ such that $a = a_1 + 1$, since $S$ is antisimple. Thus $(a + I) \in S/I$. So $(a + I) = (e + I) + (r + I) \ldots (i)$, where $(e + I)$ is an idempotent of $S/I$ and $(r + I)$ is a $k$-regular element of $S/I$. Now $(e + I)^2 = e^2 + I = e + I$. Since idempotents lift modulo $I$, there exists $e_1^2 = e_1 \in S$ such that $e + I = e_1 + I$. Since $(r + I)$ is a $k$-regular element of $S/I$, so there exist $(x + I), (y + I) \in S/I$ such that $(r + I) + (r + I)(x + I)(r + I) = (r + I)(y + I)(r + I) \Longrightarrow (r + I) + (rxr + I) = (ryr + I) \Longrightarrow (r + I) + (rxr + ry'r + I) = r(y + y')r + I$. Since $E^+(S) \subseteq I$, so $y + y' \in I$. Thus $(r + r(x + y')r) + I = I$. Therefore, $r + r(x + y')r \in I$. Since $I$ is a $k$-regular $k$-ideal of $S$, $r + r(x + y')r$ is a $k$-regular element of $I \subseteq S$. Thus by Proposition 5.3.1, it follows that $r$ is a $k$-regular element of $S$. Since every idempotent has a complement in $S$, there exists $e_2^2 = e_2$ such that $e_1 + e_2 = 1$. Hence from equation $(i)$, we have $(a + e_2) + I = (1 + I) + (r + I) \Longrightarrow [(a_1 + e_2) + I] + (1 + I) = (1 + I) + (r + I) \ldots (ii)$. Since $S$ is antisimple and $I$ be a nonzero $k$-ideal of $S$, $S/I$ is additively cancellative from Note 2.3.4. Hence from equation $(ii)$, we have $(a_1 + e_2) + I = (r + I)$. Thus $(a_1 + e_2)$ is a $k$-regular element of $S$. Therefore, $a = (a_1 + 1) = (a_1 + e_2) + e_1$. Hence $a$ is a $k$-regular clean element of $S$ if $a \notin I$. Considering all cases, we find that $S$ is a $k$-regular clean semiring. □

**Theorem 5.3.6.** *Let $S$ be an antisimple inverse semiring in which $1+1 = 2$ is a unit element and every idempotent of $S$ has a complement. Then $S$ is a $k$-regular clean semiring if and only if every nonzero element of $S$ is the sum of a $k$-regular element and a square root of $1$.*

*Proof.* Suppose that $S$ is a $k$-regular clean semiring. Let $x \neq 0$. Now $(x + 1)/2 \in S$. So $(x + 1)/2 = r + e \ldots \quad (i)$, where $r$ is a $k$-regular element and $e$ is an idempotent element of $S$. Since $S$ is antisimple, so $x = x_1 + 1$ for some $x_1 \in S$. Hence $x + 1 + 1' =$

$x_1+1+1'+1 = x_1+1 = x$. Thus $x+1 = 2r+2e \Longrightarrow x+1+1' = (2e+1')+2r \Longrightarrow x = (2e+1')+2r$. Since $e$ has a complement in $S$, there exists $e_1^2 = e_1$ such that $e+e_1 = 1$. Hence $1+e+e' = e_1+e+e+e' = e_1+e = 1$. Now $(2e+1')^2 = (2e+1')(2e+1') = 2e2e+2e'+2e'+1 = 2(2e)+2(2e')+1 = e+e+e+e+e'+e'+e'+e'+1 = e+e'+1$, since $e+e' \in E^+(S)$. Hence $(2e+1')^2 = 1$. Since $r$ is a $k$-regular element in $S$, there exist $r_1, r_2 \in S$ such that $r + rr_1r = rr_2r \Longrightarrow 2r + (2r)(r_1/2)(2r) = (2r)(r_2/2)(2r)$. This imples that $2r$ is a $k$-regular element of $S$. Hence $x =$ sum of $k$-regular element of $S$ and an element of square root of 1.

Conversely, let $x \neq 0$. Then $2x \neq 0$, since $2 \in U(S)$. If $x$ is a unit element then $x$ is a $k$-regular clean element, since $x = 0+x$ and every unit element is $k$-regular. Assume $x$ is not a unit in $S$. Since $S$ is antisimple, so $2x+1+1' = 2x$. Hence $2x+1' \neq 0$ otherwise $x$ becomes a unit. According to the condition, $2x+1' = t+r$, where $t \in S$ such that $t^2 = 1$ and $r$ is a $k$-regular element in $S$. So $2x = t+1+r \Longrightarrow x = (t+1)/2+r/2$. Now $[(t+1)/2][(t+1)/2] = (t^2+t+t+1)/2.2 = (1+1+t+t)/2.2 = (2+2t)/2.2 = (t+1)/2$, since $2 \in U(S)$. Since $r$ is a $k$-regular element and $2 \in U(S)$, $r/2$ is also a $k$-regular element of $S$. Hence $x$ is a $k$-regular clean element in $S$. Consequently, $S$ is a $k$-regular clean semiring. □

**Definition 5.3.7.** *[19] A semiring $S$ is said to be zeroic semiring if for each $a \in S$, there exists $x \in S$ such that $a+x = x$.*

**Proposition 5.3.8.** *Let $S$ be a zeroic semiring. Then $S$ is a $k$-regular clean semiring if and only if $S$ is a $k$-regular semiring.*

*Proof.* Let $S$ be a $k$-regular clean semiring. Let $a \in S$. Then $a = e+r$, where $e^2 = e$ and $r$ is a $k$-regular element of $S$. Since $S$ is zeroic and $r$ is $k$-regular, there exists $z \in S$ such that $r+rzr = rzr$. Similarly, there exists $z_1 \in S$ such that $e+ez_1e = ez_1e$. So $a+a(z+z_1)a = (e+r)+(e+r)(z+z_1)(e+r) = (e+r)+(ez+ez_1+rz+rz_1)(e+r) = e+r+eze+ezr+ez_1e+ez_1r+rze+rzr+rz_1e+rz_1r = e+ez_1e+r+rzr+eze+ezr+ez_1r+rze+rz_1e+rz_1r = ez_1e+rzr+eze+ezr+ez_1r+rze+rz_1e+rz_1r = a(z+z_1)a$. Thus $a$ is a $k$-regular element of $S$. Hence $S$ is a $k$-regular semiring.

The converse part clearly follows because every $k$-regular semiring is a $k$-regular clean semiring. □

## 5.4 Ideals & $k$-Ideals of A Zeroic $k$-Regular Clean Semiring

**Proposition 5.4.1.** *Let $S$ be a $k$-regular semiring. Then every ideal of $S$ is $k$-regular.*

*Proof.* Let $I$ be an ideal of a $k$-regular semiring $S$. Let $r \in I \subseteq S$. Since $S$ is $k$-regular semiring, there exist $x, y \in S$ such that $r + rxr = ryr$. Hence $rxr + rxrxr = ryrxr$ and $ryr + rxryr = ryryr$ which implies that $r + rxr + rxryr = ryryr$. Hence $r+rxr+rxryr+rxrxr = ryryr+rxrxr \Longrightarrow r+rxryr+ryrxr = ryryr+rxrxr \Longrightarrow r + r(xry + yrx)r = r(yry + xrx)r$. Let $a = (xry + yrx)$ and $b = (xrx + yry)$. Then $a, b \in I$ since $r \in I$ and $I$ is an ideal of $S$. Thus $r + rar = rbr$. Hence $r$ is a $k$-regular element in $I$ which implies that $I$ is $k$-regular ideal of $S$. □

**Proposition 5.4.2.** *Let $S$ be a zeroic semiring. Then every ideal of $S$ is $k$-regular clean if $S$ is $k$-regular clean semiring.*

*Proof.* Let $I$ be an ideal of a semiring $S$. Let $a \in I \subseteq S$. Since $S$ is zeroic $k$-regular clean semiring by Proposition 5.3.8 it follows that $a$ is a $k$-regular element in $S$. Again from Proposition 5.4.1 it follows that $a$ is a $k$-regular element in $I$. Thus $a = 0 + a$ is the $k$-regular clean expression of $a$ in ideal $I$. Since $a$ is an arbitrary element of $I$, $I$ is $k$-regular clean semiring. □

**Proposition 5.4.3.** *[49] Let $S$ be a $k$-regular semiring. If $R$ be a right $k$-ideal of $S$ and $L$ be a left $k$-ideal of $S$ then $R \cap L = \overline{RL}$.*

**Proposition 5.4.4.** *[15] Let $S$ be a semiring. Then $\overline{\overline{<a>_r}\ \overline{<a>_l}} \subseteq \overline{aSa}$.*

**Definition 5.4.5.** *A proper $k$-ideal $Q$ of a semiring $S$ is called a $k$-semiprime ideal of $S$ if $I^2 \subseteq Q$ implies that $I \subseteq Q$ for any $k$-ideal $I$ of $S$.*

**Theorem 5.4.6.** *Let $S$ be a commutative zeroic semiring. Then $S$ is a $k$-regular clean semiring if and only if every $k$-ideal of $S$ is $k$-semiprime.*

*Proof.* Let $I$ be a $k$-ideal of $S$ such that $A^2 \subseteq I$ for any $k$-ideal $A$ of $S$. Since $S$ is a zeroic $k$-regular clean semiring, by Proposition 5.3.8, it follows that $S$ is a $k$-regular semiring. Thus by Proposition 5.4.3, it follows that $A = A \cap A = \overline{AA} = \overline{A^2} \subseteq I$, since $A^2 \subseteq I$ and $\overline{I} = I$. Thus $I$ is a $k$-semiprime ideal of $S$.

Conversely, suppose that every $k$-ideal of $S$ is $k$-semiprime. Since $S$ is commutative, $\overline{aSa}$ is a $k$-ideal of $S$ and hence $k$-semiprime. Now from Proposition 5.4.4, we find that $\overline{<a>}\ \overline{<a>} \subseteq \overline{\overline{<a>}\ \overline{<a>}} \subseteq \overline{aSa} \Longrightarrow \overline{<a>} \subseteq \overline{aSa}$ because $\overline{aSa}$ is $k$-semiprime. Since $a \in \overline{<a>}$, so $a \in \overline{aSa}$. Thus $a$ is a $k$-regular element of $S$. Hence by Proposition 5.3.8, it follows that $S$ is a $k$-regular clean semiring. □

## 5.5 Extensions of $k$-Regular Clean Semiring

The formal power series of a $k$-regular clean semiring is not $k$-regular clean which follows from the following example :

**Example 5.5.1.** *Let $S = (\mathbb{N}_0, +, .)$, where $a+b = min\{a, b\}$ and $a.b = max\{a, b\}$ for all $a, b \in \mathbb{N}$ and $a+0 = a = 0+a$ and $a.0 = 0 = 0.a$ for all $a \in \mathbb{N}_0$. Then $S$ forms a $k$-regular clean semiring. Since $S$ is an additive idempotent semiring as well as a zeroic semiring, $S[[x]]$ is also an additive idempotent semiring as well as a zeroic semiring. If possible, let $S[[x]]$ be a $k$-regular clean semiring. Then by Proposition 5.3.8, it follows that $S[[x]]$ is a $k$-regular semiring. Let $f(x) = 2 + x + x^2 + x^3 + \ldots \in S[[x]]$. Then there exists $g(x) = a_0 + a_1x + a_2x^2 + \ldots \in S[[x]]$ such that $f(x) + f(x)g(x)f(x) = f(x)g(x)f(x) \Longrightarrow (2 + x + x^2 + \ldots) + (2 + x + x^2 + \ldots)(a_0 + a_1x + a_2x^2 + \ldots)(2 + x + x^2 + \ldots) = (2 + x + x^2 + \ldots)(a_0 + a_1x + a_2x^2 + \ldots)(2 + x + x^2 + \ldots) \Longrightarrow (2+x+x^2+\ldots)+[2a_0+(2a_0+2a_1+2a_0)x+(2a_0+2a_1+a_0+2a_2+2a_1+2a_0)x^2+\ldots] = [2a_0 + (2a_0 + 2a_1 + 2a_0)x + (2a_0 + 2a_1 + a_0 + 2a_2 + 2a_1 + 2a_0)x^2 + \ldots]$. Comparing the coefficients on both sides we have $2 + 2a_0 = 2a_0 \ldots (i)$, $1 + 2a_1 + 2a_0 = 2a_0 + 2a_1 \Longrightarrow 1 + 2(a_0 + a_1) = 2(a_0 + a_1) \ldots (ii)$ and so on. Equation $(ii)$ implies that $2$ is a $k$-unit element of $S$, which is a contradiction. Hence $f(x)$ is not $k$-regular clean element. Thus $S[[x]]$ is not $k$-regular clean semiring.*

**Proposition 5.5.2.** *Let $S$ be a semiring. Let $f = \sum_{i=0}^{\infty} a_i x^i$ be a formal power series such that $a_i \in \overline{a_0 S a_0}$ for all $i = 1, 2, 3, \ldots$ Then $f$ is a $k$-regular element in $S[[x]]$ if and only if $a_0$ is a $k$-regular element in $S$.*

*Proof.* Let $f = \sum_{i=0}^{\infty} a_i x^i$ be a $k$-regular element in $S[[x]]$. Then there exists $g = \sum_{i=0}^{\infty} b_i x^i \in S[[x]]$ and $h = \sum_{i=0}^{\infty} c_i x^i \in S[[x]]$ such that $f + fgf = fhf$. Thus it follows that $a_0 + a_0 b_0 a_0 = a_0 c_0 a_0$. This implies that $a_0$ is a $k$-regular element in $S$.

Conversely, let $a_0$ be a $k$-regular element in $S$. Then there exist $b_0, c_0 \in S$ such that $a_0 + a_0 b_0 a_0 = a_0 c_0 a_0 \ldots (i)$. We are now looking for $g, h \in S[[x]]$ such that $f + fgf = fhf$ holds. According to the condition $a_i + a_0 x_i a_0 = a_0 y_i a_0$ for all $i = 1, 2, 3, \ldots$, where $x_i, y_i \in S$. Multiplying both sides of equation $(i)$ from right by $y_1 a_0$, we have $a_0 y_1 a_0 + a_0 b_0 a_0 y_1 a_0 = a_0 c_0 a_0 y_1 a_0 \Longrightarrow a_1 + a_0 x_1 a_0 + a_0 b_0 a_1 + a_0 b_0 a_0 x_1 a_0 = a_0 c_0 a_1 + a_0 c_0 a_0 x_1 a_0 \Longrightarrow a_1 + a_0 b_0 a_1 + a_0 c_0 a_0 x_1 a_0 = a_0 c_0 a_1 + a_0 c_0 a_0 x_1 a_0 \ldots (ii)$. Then $a_1 b_0 a_0 + a_0 b_0 a_1 b_0 a_0 + a_0 c_0 a_0 x_1 a_0 b_0 a_0 = a_0 c_0 a_1 b_0 a_0 + a_0 c_0 a_0 x_1 a_0 b_0 a_0 \ldots (iii)$ and $a_1 c_0 a_0 + a_0 b_0 a_1 c_0 a_0 + a_0 c_0 a_0 x_1 a_0 c_0 a_0 = a_0 c_0 a_1 c_0 a_0 + a_0 c_0 a_0 x_1 a_0 c_0 a_0 \ldots (iv)$. Adding equations $(iii)$ and $(iv)$, we have $a_1 b_0 a_0 + a_0[b_0 a_1 b_0 + c_0 a_1 c_0 + c_0 a_0 x_1 a_0 b_0 + c_0 a_0 x_1 a_0 c_0] a_0 = a_1 c_0 a_0 + a_0[b_0 a_1 c_0 + c_0 a_1 b_0 + c_0 a_0 x_1 a_0 c_0 + c_0 a_0 x_1 a_0 b_0] a_0 \ldots (v)$. Adding equations $(ii)$ and $(v)$, we have $a_1 + a_0 b_0 a_1 + a_1 b_0 a_0 + a_0[b_0 a_1 b_0 + c_0 a_1 c_0 + c_0 a_0 x_1 a_0 b_0 + c_0 a_0 x_1 a_0 c_0 + c_0 a_0 x_1] a_0 = a_0 c_0 a_1 + a_1 c_0 a_0 + a_0[b_0 a_1 c_0 + c_0 a_1 b_0 + c_0 a_0 x_1 a_0 c_0 + c_0 a_0 x_1 a_0 b_0 + c_0 a_0 x_1] a_0 \Longrightarrow a_1 + a_0 b_0 a_1 + a_1 b_0 a_0 + a_0 b_1 a_0 = a_0 c_0 a_1 + a_1 c_0 a_0 + a_0 c_1 a_0$, where $b_1 = b_0 a_1 b_0 + c_0 a_1 c_0 + k_1$, $c_1 = b_0 a_1 c_0 + c_0 a_1 b_0 + k_1$ and $k_1 = c_0 a_0 x_1 + c_0 a_0 x_1 a_0 b_0 + c_0 a_0 x_1 a_0 c_0$. Suppose there exist $b_2, b_3, \ldots, b_m$, $c_2, c_3, \ldots, c_m$ and $k_2, k_3, \ldots, k_m$ such that the equation $a_m + a_0 b_0 a_m + \ldots + a_0 b_m a_0 + \ldots + a_m b_0 a_0 = a_0 c_0 a_m + \ldots + a_0 c_m a_0 + \ldots + a_m c_0 a_0$ holds, where $b_m = b_0 a_m b_0 + c_0 a_m c_0 + b_0 a_{m-1} b_1 + c_0 a_{m-1} c_1 + \ldots + b_0 a_1 b_{m-1} + c_0 a_1 c_{m-1} + k_m$ and $c_m = b_0 a_m c_0 + c_0 a_m b_0 + b_0 a_{m-1} c_1 + c_0 a_{m-1} b_1 + \ldots + b_0 a_1 c_{m-1} + c_0 a_1 b_{m-1} + k_m$. Now $a_n \in \overline{a_0 S a_0}$ for $n = 1, 2, 3, \ldots$ By using this condition, similarly as equation $(ii)$, we get $a_n + a_0 b_0 a_n + a_0 c_0 a_0 x_n a_0 = a_0 c_0 a_n + a_0 c_0 a_0 x_n a_0$ for $n = 1, 2, 3, \ldots$ Again similar as equation $(v)$, we get $a_n b_t a_r + a_0 b_0 a_n b_t a_r + a_0 c_0 a_n c_t a_r + a_0 c_0 a_0 x_n a_0 b_t a_r + a_0 c_0 a_0 x_n a_0 c_t a_r = a_n c_t a_r + a_0 b_0 a_n c_t a_r + a_0 c_0 a_n b_t a_r + a_0 c_0 a_0 x_n a_0 c_t a_r + a_0 c_0 a_0 x_n a_0 b_t a_r \ldots (vi)$.

Putting $n = m+1$; $t = 0$ and $r = 0$ in equation $(vi)$, we have $a_{m+1}b_0a_0 + a_0b_0a_{m+1}b_0a_0 + a_0c_0a_{m+1}c_0a_0 + a_0c_0a_0x_{m+1}a_0b_0a_0 + a_0c_0a_0x_{m+1}a_0c_0a_0 = a_{m+1}c_0a_0 + a_0b_0a_{m+1}c_0a_0 + a_0c_0a_{m+1}b_0a_0 + a_0c_0a_0x_{m+1}a_0c_0a_0 + a_0c_0a_0x_{m+1}a_0b_0a_0$.

Putting $n = m$; $t = 0, r = 1$ and $t = 1, r = 0$ in equation $(vi)$, we get $a_mb_0a_1 + a_0b_0a_mb_0a_1 + a_0c_0a_mc_0a_1 + a_0c_0a_0x_ma_0b_0a_1 + a_0c_0a_0x_ma_0c_0a_1 = a_mc_0a_1 + a_0b_0a_mc_0a_1 + a_0c_0a_mb_0a_1 + a_0c_0a_0x_ma_0c_0a_1 + a_0c_0a_0x_ma_0b_0a_1$ and

$a_mb_1a_0 + a_0b_0a_mb_1a_0 + a_0c_0a_mc_1a_0 + a_0c_0a_0x_ma_0c_1a_0 + a_0c_0a_0x_ma_0b_1a_0 = a_mc_1a_0 + a_0b_0a_mc_1a_0 + a_0c_0a_mb_1a_0 + a_0c_0a_0x_ma_0c_1a_0 + a_0c_0a_0x_ma_0b_1a_0$.

Again putting $n = m-1$; $t = 0, r = 2$, $t = 2, r = 0$ and $t = 1, r = 1$ in equation $(vi)$ we have, $a_{m-1}b_0a_2 + a_0b_0a_{m-1}b_0a_2 + a_0c_0a_{m-1}c_0a_2 + a_0c_0a_0x_{m-1}a_0c_0a_2 + a_0c_0a_0x_{m-1}a_0b_0a_2 = a_{m-1}c_0a_2 + a_0b_0a_{m-1}c_0a_2 + a_0c_0a_{m-1}b_0a_2 + a_0c_0a_0x_{m-1}a_0c_0a_2 + a_0c_0a_0x_{m-1}a_0b_0a_2$,

$a_{m-1}b_2a_0 + a_0b_0a_{m-1}b_2a_0 + a_0c_0a_{m-1}c_2a_0 + a_0c_0a_0x_{m-1}a_0c_2a_0 + a_0c_0a_0x_{m-1}a_0b_2a_0 = a_{m-1}c_2a_0 + a_0b_0a_{m-1}c_2a_0 + a_0c_0a_{m-1}b_2a_0 + a_0c_0a_0x_{m-1}a_0c_2a_0 + a_0c_0a_0x_{m-1}a_0b_2a_0$ and

$a_{m-1}b_1a_1 + a_0b_0a_{m-1}b_1a_1 + a_0c_0a_{m-1}c_1a_1 + a_0c_0a_0x_{m-1}a_0c_1a_1 + a_0c_0a_0x_{m-1}a_0b_1a_1 = a_{m-1}c_1a_1 + a_0b_0a_{m-1}c_1a_1 + a_0c_0a_{m-1}b_1a_1 + a_0c_0a_0x_{m-1}a_0c_1a_1 + a_0c_0a_0x_{m-1}a_0b_1a_1$.

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Putting $n = 1$; $t = m, r = 0$, $t = m-1, r = 1$, $t = m-2, r = 2$ ......... $t = 1, r = m-1$ and $t = 0, r = m$.

$a_1b_ma_0 + a_0b_0a_1b_ma_0 + a_0c_0a_1c_ma_0 + a_0c_0a_0x_1a_0c_ma_0 + a_0c_0a_0x_1a_0b_ma_0 = a_1c_ma_0 + a_0b_0a_1c_ma_0 + a_0c_0a_1b_ma_0 + a_0c_0a_0x_1a_0c_ma_0 + a_0c_0a_0x_1a_0b_ma_0$,

$a_1b_{m-1}a_1 + a_0b_0a_1b_{m-1}a_1 + a_0c_0a_1c_{m-1}a_1 + a_0c_0a_0x_1a_0c_{m-1}a_1 + a_0c_0a_0x_1a_0b_{m-1}a_1 = a_1c_{m-1}a_1 + a_0b_0a_1c_{m-1}a_1 + a_0c_0a_1b_{m-1}a_1 + a_0c_0a_0x_1a_0c_{m-1}a_1 + a_0c_0a_0x_1a_0b_{m-1}a_1$,

$a_1b_{m-2}a_2 + a_0b_0a_1b_{m-2}a_2 + a_0c_0a_1c_{m-2}a_2 + a_0c_0a_0x_1a_0c_{m-2}a_2 + a_0c_0a_0x_1a_0b_{m-2}a_2 = a_1c_{m-2}a_2 + a_0b_0a_1c_{m-2}a_2 + a_0c_0a_1b_{m-2}a_2 + a_0c_0a_0x_1a_0c_{m-2}a_2 + a_0c_0a_0x_1a_0b_{m-2}a_2$,

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$a_1b_1a_{m-1} + a_0b_0a_1b_1a_{m-1} + a_0c_0a_1c_1a_{m-1} + a_0c_0a_0x_1a_0c_1a_{m-1} + a_0c_0a_0x_1a_0b_1a_{m-1} = a_1c_1a_{m-1} + a_0b_0a_1c_1a_{m-1} + a_0c_0a_1b_1a_{m-1} + a_0c_0a_0x_1a_0c_1a_{m-1} + a_0c_0a_0x_1a_0b_1a_{m-1}$

and $a_1b_0a_m + a_0b_0a_1b_0a_m + a_0c_0a_1c_0a_m + a_0c_0a_0x_1a_0c_0a_m + a_0c_0a_0x_1a_0b_0a_m = a_1c_0a_m + a_0b_0a_1c_0a_m + a_0c_0a_1b_0a_m + a_0c_0a_0x_1a_0c_0a_m + a_0c_0a_0x_1a_0b_0a_m$.

and also, $a_{m+1} + a_0b_0a_{m+1} + a_0c_0a_0x_{m+1}a_0 = a_0c_0a_{m+1} + a_0c_0a_0x_{m+1}a_0$.

Adding all the above equations we have, $a_{m+1} + a_0b_0a_{m+1} + a_0b_ma_1 + a_0b_{m-1}a_2 + \ldots + a_0b_2a_{m-1} + a_0b_1a_m + a_0b_{m+1}a_0 + a_mb_0a_1 + a_mb_1a_0 + a_{m-1}b_0a_2 + a_{m-1}b_2a_0 + a_{m-1}b_1a_1 + \ldots + a_1b_ma_0 + a_1b_{m-1}a_1 + \ldots + a_1b_0a_m + a_{m+1}b_0a_0 = a_0c_0a_{m+1} + a_0c_ma_1 + a_0c_{m-1}a_2 + \ldots + a_0c_2a_{m-1} + a_0c_1a_m + a_0c_{m+1}a_0 + a_mc_0a_1 + a_mc_1a_0 + a_{m-1}c_0a_2 + a_{m-1}c_2a_0 + a_{m-1}c_1a_1 + \ldots + a_1c_ma_0 + a_1c_{m-1}a_1 + \ldots + a_1c_0a_m + a_{m+1}c_0a_0$, where $b_{m+1} = b_0a_{m+1}b_0 + c_0a_{m+1}c_0 + b_0a_mb_1 + c_0a_mc_1 + \ldots + b_0a_1b_m + c_0a_1c_m + k_{m+1}$, $c_{m+1} = b_0a_{m+1}c_0 + c_0a_{m+1}b_0 + b_0a_mc_1 + c_0a_mb_1 + \ldots + b_0a_1c_m + c_0a_1b_m + k_{m+1}$ and $k_{m+1} = c_0a_0x_{m+1} + \ldots + c_0a_0x_{m+1}a_0c_0 + c_0a_0x_{m+1}a_0b_0$. Thus we find $g, h \in S[[x]]$ such that $f + fgf = fhf$ holds. Hence $f$ is a $k$-regular element in $S[[x]]$. □

From Proposition 5.3.8 and Proposition 5.5.2, we have the following result :

**Corollary 5.5.3.** *Let $S$ be a zeroic semiring. Let $f = \sum_{i=0}^{\infty} a_ix^i$ be a formal power series such that $a_i \in \overline{a_0Sa_0}$ for all $i = 1, 2, 3...$ Then $f$ is a $k$-regular clean element in $S[[x]]$ if and only if $a_0$ is a $k$-regular clean element in $S$.*

**Theorem 5.5.4.** (1) *Let $A$ be a $k$-unit clean semiring without zero element and with multiplcative idendity $1_A$ and $B$ be a $k$-regular clean semiring with zero element $0_B$ and with multiplicative idendity $1_B$. Let $(M, +)$ and $(N, +)$ be two additive abelian monoids with additive identity $0_M$ and $0_N$ which are $(B, A)$ bi-semimodule and $(A, B)$ bi-semimodule respectively such that $b0_M = 0_M = 0_Ma$ and $a0_N = 0_N = 0_Nb$ for all $a \in A$ and $b \in B$. Let us consider the set of all matrices of the form $T = \left\{ \begin{bmatrix} a & n \\ m & b \end{bmatrix} : a \in A, m \in M, n \in N, b \in B \right\}$. Then $T$ forms a semiring in which addition is defined componentwise and multiplication is defined by $\begin{bmatrix} a_1 & n_1 \\ m_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & n_2 \\ m_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_1n_2 + n_1b_2 \\ m_1a_2 + b_1m_2 & b_1b_2 \end{bmatrix}$. Then $T$ is a $k$-regular clean semiring.*

(2) *Let $T$ be a $k$-regular clean semiring. Then Both $A$ and $B$ are $k$-regular clean semirings.*

*Proof.* (1) Let $A$ be a $k$-unit clean semiring without zero element $0_A$ and with multiplicative identity $1_A$ and $B$ be a $k$-regular clean semiring with zero element $0_B$ and with multiplicative idendity $1_B$. Let $\begin{bmatrix} a & n \\ m & b \end{bmatrix} \in T$, where $a \in A$, $b \in B$, $m \in M$ and $n \in N$. Then there exist idempotents $e_1 \in A$, $e_2 \in B$, $k$-unit element $u_1 \in A$ and $k$-regular element $r_2 \in B$ such that $a = e_1 + u_1$ and $b = e_2 + r_2$. Now $\begin{bmatrix} a & n \\ m & b \end{bmatrix} = \begin{bmatrix} e_1 & 0_N \\ 0_M & e_2 \end{bmatrix} + \begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix}$. Since $u_1 \in A$ is $k$-unit, there exist $s_1, s_2 \in A$ such that $1_A + s_1u_1 = s_2u_1$, $1_A + u_1s_1 = u_1s_2$ and since $r_2$ is a $k$-regular element in $B$, there exist $x, y \in B$ such that $r_2 + r_2xr_2 = r_2yr_2$.

Now $\begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix} + \begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix} \begin{bmatrix} s_1 & s_1nx + s_2ny \\ xms_1 + yms_2 & x \end{bmatrix} \begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix}$

$$= \begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix} + \begin{bmatrix} u_1s_1 & u_1s_1nx + u_1s_2ny + nx \\ ms_1 + r_2xms_1 + r_2yms_2 & r_2x \end{bmatrix} \begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix} + \begin{bmatrix} u_1s_1u_1 & u_1s_1n + u_1s_1nxr_2 + u_1s_2nyr_2 + nxr_2 \\ ms_1u_1 + r_2xms_1u_1 + r_2yms_2u_1 + r_2xm & r_2xr_2 \end{bmatrix}.$$

$$\begin{bmatrix} u_1 + u_1s_1u_1 & n + u_1s_1n + u_1s_1nxr_2 + u_1s_2nyr_2 + nxr_2 \\ m + ms_1u_1 + r_2xms_1u_1 + r_2yms_2u_1 + r_2xm & r_2 + r_2xr_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1(1_A + s_1u_1) & (1_A + u_1s_1)n + (u_1s_1 + 1_A)nxr_2 + u_1s_2nyr_2 \\ m(1_A + s_1u_1) + r_2xm(1_A + s_1u_1) + r_2yms_2u_1 & r_2yr_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1s_2u_1 & u_1s_2n + u_1s_2nxr_2 + u_1s_2nyr_2 \\ ms_2u_1 + r_2xms_2u_1 + r_2yms_2u_1 & r_2yr_2 \end{bmatrix} \ldots (1)$$

Also $\begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix} \begin{bmatrix} s_2 & s_2nx + s_1ny \\ xms_2 + yms_1 & y \end{bmatrix} \begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix}$

$$= \begin{bmatrix} u_1s_2 & u_1s_2nx + u_1s_1ny + ny \\ ms_2 + r_2xms_2 + r_2yms_1 & r_2y \end{bmatrix} \begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1s_2u_1 & u_1s_2n + u_1s_2nxr_2 + u_1s_1nyr_2 + nyr_2 \\ ms_2u_1 + r_2xms_2u_1 + r_2yms_1u_1 + r_2ym & r_2yr_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1s_2u_1 & u_1s_2n + u_1s_2nxr_2 + (1_A + u_1s_1)nyr_2 \\ ms_2u_1 + r_2xms_2u_1 + r_2ym(s_1u_1 + 1_A) & r_2yr_2 \end{bmatrix}$$

$$= \begin{bmatrix} u_1s_2u_1 & u_1s_2n + u_1s_2nxr_2 + u_1s_2nyr_2 \\ ms_2u_1 + r_2xms_2u_1 + r_2yms_2u_1 & r_2yr_2 \end{bmatrix} \ldots (2).$$

From (1) and (2), it follows that $\begin{bmatrix} u_1 & n \\ m & r_2 \end{bmatrix}$ is a $k$-regular element of $T$. Since $e_1$ and $e_2$ are idempotent elements of $A$ and $B$ respectively, $\begin{bmatrix} e_1 & 0_N \\ 0_M & e_2 \end{bmatrix}$ is an idempotent element of $T$. Hence $\begin{bmatrix} a & n \\ m & b \end{bmatrix}$ is a $k$-regular clean element of $T$. Consequently, it follows that $T$ is a $k$-regular clean semiring.

(2) let $T$ be a $k$-regular clean semiring. Define a mapping $\phi : T \longrightarrow A$ by $\phi\left(\begin{bmatrix} a & n \\ m & b \end{bmatrix}\right) = a$ for any $\begin{bmatrix} a & n \\ m & b \end{bmatrix} \in T$. Now it can be proved that $\phi$ is an onto homomorphism from semiring $T$ to $A$. Hence $A$ is the homomorphic image of $T$. Thus $A$ is $k$-regular clean semiring from Proposition 5.3.2. Similarly, we can prove that $B$ is $k$-regular clean semiring. □

In Theorem 5.5.4, by taking $N = O$ which is $(A, B)$ bi-semimodule, we have the following result :

**Corollary 5.5.5.** *Let $A$ be a $k$-unit clean semiring without zero element and with multiplcative idendity $1_A$ and $B$ be a $k$-regular clean semiring with zero element $0_B$ and with multiplicative idendity $1_B$. Let $(M, +)$ be an additive abelian monoid with additive identity $0_M$ which is $(B, A)$ bi-semimodule such that $b0_M = 0_M = 0_Ma$ for all $a \in A$ and $b \in B$. Let us consider the set of all matrices of the form $T = \left\{ \begin{bmatrix} a & 0 \\ m & b \end{bmatrix} : a \in A, m \in M, b \in B \right\}$. Then $T$ forms a semiring in which addition is defined componentwise and multiplication is defined by $\begin{bmatrix} a_1 & 0 \\ m_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ m_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 \\ m_1a_2 + b_1m_2 & b_1b_2 \end{bmatrix}$. Then $T$ is a $k$-regular clean semiring.*

*Similarly, if $T$ is a $k$-regular clean semiring then both $A$ and $B$ are $k$-regular clean semirings.*

Let $S$ be a semiring and $M$ be a $S$-semimodule. The trivial semiring extension of $S$ by $M$ is the semiring $S_1 = S \propto M$ whose underlying semiring is $S \times M$ with addition defined by $(a, m) + (a_1, m_1) = (a + a_1, m + m_1)$ and multiplication defined by $(a, m)(a_1, m_1) = (aa_1, am_1 + a_1m)$.

**Theorem 5.5.6.** *Let $S$ be a semiring without zero element and $M$ be a $S$-semimodule. Then the trivial semiring extension of $S$ by $M$, $S_1 = S \propto M$ is a $k$-regular clean semiring if $S$ is a $k$-unit clean semiring.*

*Proof.* Let $S$ be a $k$-unit clean semiring without zero element. Let $(a, m) \in S_1$. Then $a = e + u$, where $e$ is an idempotent and $u$ is a $k$-unit element in $S$. Thus there exist $s_1, s_2 \in S$ such that $1 + s_1u = s_2u$ and $1 + us_1 = us_2$. Now we have $(a, m) = (e+u, m) = (e, 0_M)+(u, m)$. This implies that $(u, m)+(u, m)(s_1, s_1^2m+s_2^2m)(u, m) = (u, m) + (us_1, us_1^2m + us_2^2m + s_1m)(u, m) = (u, m) + (us_1u, us_1m + u^2s_1^2m + u^2s_2^2m + us_1m) = (u + us_1u, m + us_1m + u^2s_1^2m + u^2s_2^2m + us_1m) = (us_2u, us_2m + us_1m + u^2s_1^2m + u^2s_2^2m) = (us_2u, us_2m + u(1 + us_1)s_1m + u^2s_2^2m) = (us_2u, u^2s_2s_1m + u^2s_2^2m + us_2m) = (us_2u, u^2s_2(s_1 + s_2)m + us_2m) \ldots (i)$. Again $(u, m)(s_2, s_1s_2m + s_2s_1m)(u, m) = (us_2, us_1s_2m + us_2s_1m + s_2m)(u, m) = (us_2u, us_2m + u^2s_1s_2m + u^2s_2s_1m + us_2m) = (us_2u, us_2m + (u^2s_1 + u)s_2m + u^2s_2s_1m) = (us_2u, us_2m + u^2s_2^2m + u^2s_2s_1m) = (us_2u, u^2s_2(s_2 + s_1)m + us_2m) \ldots (ii)$. Now from equations $(i)$ and $(ii)$ we can say that $(u, m)$ is a $k$-regular element in $S_1$. Also $(e, 0_M)(e, 0_M) = (e, e0_M + e0_M) = (e, 0_M)$. So we find that $(e, 0_M)$ is an idempotent in $S_1$. Thus $(a, m)$ is a $k$-regular clean element in $S_1$. Consequently, $S_1$ is a $k$-regular clean semiring. □

In the following corollary we consider semimodules over semirings whose addition is idempotent and we call such semimodules as idempotent semimodules.

**Corollary 5.5.7.** *Let $S$ be a semiring with zero element such that it is not a ring, $1 \neq 0$ and $M$ be an idempotent $S$-semimodule. Suppose that $S_1 = S \propto M$ is the trivial extension of $S$ by $M$. Let $T = \{(0, 0_M)\} \bigcup \{(a, m) \in S_1 : a \neq 0\}$. Then $T$ is a $k$-regular clean semiring if $S$ is a $k$-unit clean semiring.*

*Proof.* Suppose $(a, m) \in T$ such that $a \neq 0$. Then $a = e+u$, where $e$ is an idempotent in $S$ and $u$ is a $k$-unit in $S$. Thus there exist $s_1, s_2 \in S$ such that $1 + s_1u = s_2u$ and

$1 + us_1 = us_2$. Now there are two possible cases - either $s_1, s_2 \neq 0$ or $s_1 = 0, s_2 \neq 0$. Note that $s_2$ can not be zero because $S$ is not a ring. If $s_1 = 0$ then $u = us_2u$ and $m = us_2m$. Thus $(u, m) + (u, m)(0, 0_M)(u, m) = (u, m)(s_2, 0_M)(u, m)$, since $m + m = m$ for all $m \in M$. Hence $(a, m) = (e, 0_M) + (u, m)$. So $(a, m)$ is a $k$-regular clean element in $T$. If $s_1, s_2 \neq 0$ then from Theorem 5.5.6, it follows that $(a, m)$ is a $k$-regular clean element in $T$. Hence $T$ is a $k$-regular clean semiring. $\square$

But the converse of the above Corollary 5.5.7 is not true which follows from the following example.

**Example 5.5.8.** *Let $\mathbb{Z}_0^+$ be the set of all non-negative integers. Define two operations $+$ and $.$ on $S$ by $a + b = max\{a, b\}$ and $a.b =$ usual multiplication in $\mathbb{Z}_0^+$. Then $(\mathbb{Z}_0^+, +, .)$ forms a $k$-regular clean semiring with additive identity $0$ and multiplicative identity $1$. Let $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}_0^+ \right\}$. Then $S$ forms a semiring under usual matrix addition and multiplication. Let $M = \{0_S\}$. Then $M$ is an idempotent $S$-semimodule. Let $S_1 = S \propto M$ be the trivial extension of $S$ by $M$. Then $S_1 = \{(O_S, O_S)\} \bigcup \{(A, O_S) : A \neq O_S \in S\}$. If $S$ is a $k$-unit clean semiring, then $S$ becomes a $k$-unit semiring, since $S$ is additive idempotent i.e. every element of $S$ becomes a $k$-unit. But $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not a $k$-unit element of $S$. Hence $S_1$ is a $k$-regular clean semiring but $S$ is not a $k$-unit clean semiring.*

**Proposition 5.5.9.** *Let $S$ be a commutative semiring. Suppose that $R_k(S)$ denotes the set of all $k$-regular elements of $S$. Let $S_1 = S \propto M$ be the trivial extension of $S$ by $M$. Then $R_k(S_1) = \{(a, m) \in S_1 : a \in R_k(S), m \in \overline{aM}\}$.*

*Proof.* Suppose that $(a, m) \in S_1$, where $a \in R_K(S)$ and $m \in \overline{aM}$. Then there exist $m_1, m_2 \in M$ such that $m + am_1 = am_2 \dots (i)$. Since $a \in R_k(S)$, there exist $x, y \in S$ such that $a + axa = aya$. Thus $a + a^2x = a^2y \dots (ii)$. Adding both side of equation $(i)$ by $a^2x(m_1 + m_2)$ and using equation $(ii)$, we have $m + (a + a^2x)m_1 + a^2xm_2 = (a + a^2x)m_2 + a^2xm_1 \implies m + a^2(ym_1 + xm_2) = a^2(ym_2 + xm_1) \implies m + a^2m_3 = a^2m_4 \dots (iii)$, where $m_3 = ym_1 + xm_2$ and $m_4 = ym_2 + xm_1$. Now $(a, m)+$

$(a,m)(0,m_3)(a,m) = (a,m)+(0,am_3)(a,m) = (a,m)+(0,a^2m_3) = (a,m+a^2m_3) = (a,a^2m_4) = (a,am_5)$, by using equation $(iii)$. Now $(a,am_5) + (a,am_5)(x,x^2am_5 + y^2am_5)(a,am_5) = (a,am_5) + (ax,ax^2am_5 + ay^2am_5 + xam_5)(a,am_5) = (a,am_5) + (axa,axam_5 + a^2x^2am_5 + a^2y^2am_5 + axam_5) = (aya,ayam_5 + axam_5 + a^2x^2am_5 + a^2y^2am_5) = (aya,ayam_5 + (a+a^2x)xam_5 + a^2y^2am_5) = (aya,ayam_5 + a^2yxam_5 + a^2y^2am_5) = (aya,ayam_5 + a^2y(x+y)am_5)\ldots(iv)$. Also we have $(a,am_5)(y,xyam_5 + yxam_5)(a,am_5) = (ay,axyam_5+ayxam_5+yam_5)(a,am_5) = (aya,ayam_5+a^2xyam_5+a^2yxam_5 + ayam_5) = (aya,ayam_5 + (a+a^2x)yam_5 + a^2yxam_5) = (aya,ayam_5 + a^2y^2am_5 + a^2yxam_5) = (aya,ayam_5 + a^2y(x+y)am_5)\ldots(v)$. Equations $(iv)$ and $(v)$ are equal, which implies that $(a,am_5)$ is a $k$-regular element in $S_1$. Thus from Proposition 5.3.1, we find that $(a,m)$ is a $k$-regular element in $S_1$. Thus $(a,m) \in R_k(S_1)$. Again if $(a,m) \in R_k(S_1)$, then there exist $(a_1,m_1),(a_2,m_2) \in S_1$ such that $(a,m) + (a,m)(a_1,m_1)(a,m) = (a,m)(a_2,m_2)(a,m)$. Thus $(a,m) + (aa_1, am_1 + a_1m)(a,m) = (aa_2, am_2 + a_2m)(a,m) \Longrightarrow (a,m) + (aa_1a, 2aa_1m + a^2m_1) = (aa_2a, 2aa_2m + a^2m_2)$. Hence $a + aa_1a = aa_2a$ and $m + a(2a_1m + am_1) = a(2a_2m + am_2)$. Thus $a \in R_k(S)$ and $m \in \overline{aM}$. Hence the proof. □

**Theorem 5.5.10.** *Let $S$ be a commutative zeroic semiring not necessarily contain multiplicative identity and $M$ be a $S$-semimodule. Let $S_1 = S \propto M$ and $T = \{(a,m) \in S_1 : m \in \overline{aM}\}$. Then $T$ is a $k$-regular clean subsemiring of $S_1$, consisting of all the $k$-regular clean elements of $S_1$ if and only if $S$ is a $k$-regular clean semiring.*

*Proof.* Let $S$ be a $k$-regular clean semiring. Then $S$ is a $k$-regular semiring which follows from Proposition 5.3.8. Let $(a_1,m_1)$, $(a_2,m_2) \in T$. Then $m_1 \in \overline{a_1M}$ and $m_2 \in \overline{a_2M}$. Then there exist $m_3, m_4, m_5, m_6 \in M$ such that $m_1 + a_1m_3 = a_1m_4\ldots(i)$ and $m_2 + a_2m_5 = a_2m_6\ldots(ii)$. Since $a_1$ is a $k$-regular element and $S$ is a commutative zeroic semiring, there exists $x \in S$ such that $a_1 + a_1^2x = a_1^2x$. Now from equation $(i)$, we have $m_1 + a_1m_3 + a_1^2x(m_3 + m_4) = a_1m_4 + a_1^2x(m_3 + m_4) \Longrightarrow m_1 + a_1^2x(m_3 + m_4) = a_1^2x(m_3 + m_4) \Longrightarrow m_1 + a_1m' = a_1m'\ldots(iii)$, where $m' = a_1x(m_3 + m_4) \in M$. Similarly, from equation $(ii)$, we have $m_2 + a_2m'' = a_2m''\ldots(iv)$, where $m'' = a_2y(m_5 + m_6) \in M$ for some $y \in S$. Equation $(iii)$ implies that $m_1 + a_1m' + a_2m' =$

$a_1m'+a_2m' \Longrightarrow m_1+(a_1+a_2)m' = (a_1+a_2)m' \ldots (v)$. Similarly, equation $(iv)$ implies that $m_2+(a_1+a_2)m'' = (a_1+a_2)m'' \ldots (vi)$. Adding equations $(v)$ and $(vi)$, we have $(m_1+m_2)+(a_1+a_2)(m'+m'') = (a_1+a_2)(m'+m'') \Longrightarrow (m_1+m_2) \in \overline{(a_1+a_2)M}$. Multiplying both side of equations $(i)$ by $a_2$ and $(ii)$ by $a_1$, we have $a_2m_1+a_1a_2m_3 = a_1a_2m_4$, $a_1m_2+a_1a_2m_5 = a_1a_2m_6$. Now adding these two equations, we find that $a_1m_2+a_2m_1+a_1a_2(m_3+m_5) = a_1a_2(m_4+m_6) \Longrightarrow (a_1m_2+a_2m_1) \in \overline{a_1a_2M}$. Hence $T$ becomes a subsemiring of $S_1$. Let $(a,m) \in T$. Then by previous method there exists $x_1 \in S$ and $m_7 \in M$ such that $a+ax_1a = ax_1a$ and $m+a^2x_1m_7 = a^2x_1m_7$. Now $(a,m)+(a,m)(x_1,x_1m_7)(a,m) = (a,m)+(ax_1,ax_1m_7+x_1m)(a,m) = (a,m)+(ax_1a,ax_1m+a^2x_1m_7+ax_1m) = (ax_1a,2ax_1m+a^2x_1m_7) = (a,m)(x_1,x_1m_7)(a,m)$. Hence $(a,m)$ is a $k$-regular clean element in $T$, since $(a,m) = (0,0_M)+(a,m)$, where $(0,0_M)$ is an idempotent element in $T$. Hence $T$ is a $k$-regular clean subsemiring of $S_1$. Now let $(a,m)$ be any $k$-regular clean element of $S_1$. Then $(a,m) = (e_1,m_8)+(r_1,m_9)$, where $(e_1,m_8)^2 = (e_1,m_8)$ and $(r_1,m_9)$ be a $k$-regular element of $S_1$ for $m_8,m_9 \in M$ and $e_1,r_1 \in S$. Hence $a = e_1+r_1$ and $m = m_8+m_9$. By Proposition 5.5.9, we have $r_1$ is a $k$-regular element of $S$ and $m_9 \in \overline{r_1M}$. Thus there exists $m''' \in M$ such that $m_9+r_1m''' = r_1m''' \Longrightarrow m_9+r_1(m'''+m_8) = r_1(m'''+m_8) \ldots (vii)$. Now $(e_1,m_8') = (e_1^2,e_1m_8+e_1m_8) \Longrightarrow e_1^2 = e_1$, $e_1m_8+e_1m_8 = m_8$. Hence $e_1m_8+e_1m_8 = e_1m_8 = m_8$ which implies that $m_8+e_1m_8 = e_1m_8 \Longrightarrow m_8+e_1(m_8+m''') = e_1(m_8+m''') \ldots (viii)$. Adding equations $(vii)$ and $(viii)$, we have $(m_8+m_9)+(e_1+r_1)(m_8+m''') = (e_1+r_1)(m_8+m''') \Longrightarrow m+a(m_8+m''') = a(m_8+m''') \Longrightarrow m \in \overline{aM}$. Hence $T$ consists of all the $k$-regular clean elements of $S_1$.

Conversely, suppose that the condition holds for $T$. Let $a \in S$. Then $(a,0_M) \in T$. Thus $(a,0_M)$ is a $k$-regular clean element of $S_1$. Hence $(a,0_M) = (e,m_{10})+(r,m_{11})$ where $(e,m_{10})$ is an idempotent and $(r,m_{11})$ is a $k$-regular element in $S_1$. Thus $a = e+r$. Now clearly $e^2 = e$ and using the Proposition 5.5.9, we have $r$ is a $k$-regular element of $S$. Thus $a$ is a $k$-regular clean element of $S$. Hence $S$ is a $k$-regular clean semiring. □

## 5.6 Corner Semiring of $k$-Regular Clean Semiring Corresponding To A Pierce Trivial Idempotent

**Definition 5.6.1.** *[57] Let $S$ be a semiring and $e \in Id(S)$. Then $e$ is called the pierce trivial idempotent of $S$ if $e$ has a complement $e_1$ such that $eSe_1Se = 0 = e_1SeSe_1$ and $e_1$ is called the pierce trivial complement of $e$.*

**Proposition 5.6.2.** *Let $S$ be a semiring with $e \in Id(S)$. Let $e_1$ be the Pierce trivial complement of $e$. If $S$ is a $k$-regular clean semiring then both $eSe$ and $e_1Se_1$ are $k$-regular clean semirings.*

*Proof.* Let $S$ be a $k$-regular clean semiring. Let $exe \in eSe$ for $x \in S$. Since $S$ is a $k$-regular clean semiring, $x = e_2 + r_1$, where $e_2 \in Id(S)$ and $r_1$ is a $k$-regular element in $S$. Thus there exist $x_1, y_1 \in S$ such that $r_1 + r_1x_1r_1 = r_1y_1r_1$. Hence $exe = ee_2e + er_1e \ldots (i)$ and $er_1e + er_1x_1r_1e = er_1y_1r_1e \ldots (ii)$. Now $(ee_2e)(ee_2e) = ee_2ee_2e + 0 = ee_2ee_2e + ee_2e_1e_2e = ee_2(e+e_1)e_2e = ee_2e$ since $eSe_1Se = 0$. Hence $ee_2e$ is an idempotent of $eSe$. Now $er_1x_1r_1e = er_1(e+e_1)x_1(e+e_1)r_1e = (er_1e+er_1e_1)x_1(er_1e+e_1r_1e) = (er_1ex_1+er_1e_1x_1)(er_1e+e_1r_1e) = er_1ex_1er_1e+er_1ex_1e_1r_1e+er_1e_1x_1er_1e+er_1e_1x_1e_1r_1e$. Since $eSe_1Se = 0$, $er_1x_1r_1e = er_1ex_1er_1e = (er_1e)(ex_1e)(er_1e)$. Similarly, $er_1y_1r_1e = (er_1e)(ey_1e)(er_1e)$. Hence $er_1e$ is $k$-regular element in $eSe$. Thus $eSe$ is a $k$-regular clean semiring. Again since $e_1SeSe_1 = 0$, in a similar way it can be proved that $e_1Se_1$ is a $k$-regular clean semiring. □

But the converse of the Proposition 5.6.2 is not true which follows from the following example :

**Example 5.6.3.**

*Consider $S = \{0, 1, 2, 3\}$ . Define the operations "+" and "." in $S$ as follows :*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 3 |
| 2 | 2 | 3 | 3 | 3 |
| 3 | 3 | 3 | 3 | 3 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 3 | 3 |
| 3 | 0 | 3 | 3 | 3 |

*Now consider* $S' = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in S \right\}$. *Define two binary operations on* $S'$ *such that*
$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \bigoplus \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$ *and*
$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \bigodot \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & d_1d_2 \end{bmatrix}$ *for all* $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in S'$. *Then* $(S', \bigoplus, \bigodot)$ *forms a semiring.*

Let $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $E_1 \bigoplus E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and
$E_1 \bigodot S' \bigodot E_2 \bigodot S' \bigodot E_1 = E_2 \bigodot S' \bigodot E_1 \bigodot S' \bigodot E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Now $E_1 \bigodot S' \bigodot E_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in S \right\}$ and $E_2 \bigodot S' \bigodot E_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} : d \in S \right\}$ both are $k$-regular clean semirings. Also $S'$ is a zeroic semiring and $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ is not a $k$-regular element. Thus from Proposition 5.3.8, we can conclude that $S'$ is not a $k$-regular clean semiring.

**Definition 5.6.4.** *[53] Let $S$ be a semiring and $e \in Id(S)$. Then $e$ is called a right semicentral (left semicentral) idempotent of $S$ if $e$ has a complement $e_1$ such that $eSe_1 = 0$ (resp. $e_1Se = 0$). If $e$ is both right semicentral and left semicentral then $e$ is called central idempotent of $S$.*

**Theorem 5.6.5.** *Let $E = \{e_1, e_2, \ldots, e_m\}$ be pairwise orthogonal right semicentral or left semicentral idempotents of a semiring $S$ such that $e_1 + e_2 + \ldots + e_m = 1$. Then*

*$S$ is a $k$-regular clean semiring if and only if $e_iSe_i$ is a $k$-regular clean semiring for each $i = 1, 2, 3, \ldots, m$.*

*Proof.* Since $E$ is the set of all pairwise orthogonal idempotents, so $e_ie_j = \delta_{ij}e_i$. Let $e_i$ be right semicentral idempotent for each $i = 1, 2, \ldots, m$. Thus there exists $e'_i \in Id(S)$ such that $e_i + e'_i = 1$ and $e_iSe'_i = 0$ for $i = 1, 2, 3, \ldots, m$. Let $S$ be a $k$-regular clean semiring. Now for each $i$, we have $e_iSe'_iSe_i = 0 = e'_iSe_iSe'_i$. Thus $e_i$ is pierce trivial idempotent semiring for each $i$. Hence $e_iSe_i$ is a $k$-regular clean semiring for each $i = 1, 2, \ldots, m$, by Proposition 5.6.2.

Conversely, let $e_iSe_i$ be a $k$-regular clean semiring for each $i = 1, 2, \ldots, m$. Let us define a mapping $\phi : S \longrightarrow \begin{bmatrix} e_1Se_1 & O & O & \cdots & O \\ O & e_2Se_2 & O & \cdots & O \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ O & O & O & \cdots & e_mSe_m \end{bmatrix}$ by

$\phi(s) = \begin{bmatrix} e_1se_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2se_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e_mse_m \end{bmatrix}$ for all $s \in S$. Since $e_iSe'_i = 0$ for each $i = 1, 2, 3, \ldots, m$, we have $e_iSe_i = e_iS$. Thus it can be shown that $\phi$ is a homomorphism. Now let $\phi(s_1) = \phi(s_2) \Longrightarrow e_1s_1e_1 = e_1s_2e_1, e_2s_1e_2 = e_2s_2e_2, \ldots, e_ms_1e_m = e_ms_2e_m \Longrightarrow e_1s_1 = e_1s_2, e_2s_1 = e_2s_2, \ldots, e_ms_1 = e_ms_2 \Longrightarrow (e_1 + e_2 + \ldots + e_m)s_1 = (e_1+e_2+\ldots+e_m)s_2 \Longrightarrow s_1 = s_2$, since $e_1+e_2+\ldots+e_m = 1$. Hence $\phi$ is one-one. Let there exist $x_1, x_2, x_3, \ldots, x_m \in S$ such that $\begin{bmatrix} e_1x_1e_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2x_2e_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e_mx_me_m \end{bmatrix} \in$

$$\begin{bmatrix} e_1Se_1 & O & O & \cdots & O \\ O & e_2Se_2 & O & \cdots & O \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ O & O & O & \cdots & e_mSe_m \end{bmatrix}.$$

Then there exists $s_1 = e_1x_1 + e_2x_2 + \ldots + e_mx_m \in S$ such that

$\phi(s_1) = \begin{bmatrix} e_1x_1e_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2x_2e_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e_mx_me_m \end{bmatrix}$, since $e_ie_j = \delta_{ij}e_i$. Hence $\phi$ is onto. Thus $S \cong \begin{bmatrix} e_1Se_1 & O & O & \cdots & O \\ O & e_2Se_2 & O & \cdots & O \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ O & O & O & \cdots & e_mSe_m \end{bmatrix}$. Since $e_iSe_i$ is a $k$-regular clean semiring for each $i = 1, 2, 3, \ldots, m$; it follows that $\begin{bmatrix} e_1Se_1 & O & O & \cdots & O \\ O & e_2Se_2 & O & \cdots & O \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ O & O & O & \cdots & e_mSe_m \end{bmatrix}$ is a $k$-regular clean semiring. Since every homomorphic image of a $k$-regular clean semiring is a $k$-regular clean semiring [Proposition 5.3.2] , we find that $S$ is a $k$-regular clean semiring.

Similarly, the proof holds if each $e_i$ is a left semicentral idempotent for $i = 1, 2, \ldots, m$. □

**Corollary 5.6.6.** *Let $E = \{e_1, e_2, \ldots, e_m\}$ be pairwise orthogonal central idempotents of a semiring $S$ such that $e_1 + e_2 + \ldots + e_m = 1$. Then $S$ is $k$-regular clean semiring if and only if $e_iSe_i$ is a $k$-regular clean semiring for each $i = 1, 2, 3, \ldots, m$.*

*Proof.* Since $e_i$ is a central idempotent for each $i = 1, 2, 3, \ldots, m$, $e_is = se_i$ for all $s \in S$. Thus $e_i + (e_1 + e_2 + \ldots + e_{i-1} + e_{i+1} + \ldots + e_m) = 1$ and $e_iS(e_1 + e_2 + \ldots + e_{i-1} + e_{i+1} + \ldots + e_m) = 0$. Since $e_ie_j = \delta_{ij}e_i$, $(e_1 + e_2 + \ldots + e_{i-1} + e_{i+1} + \ldots + e_m) \in Id(S)$. Hence $e_i$ is a right semicentral idempotent for each $i = 1, 2, \ldots, m$. Thus the proof follows from Theorem 5.6.5. □

Converse of the Proposition 5.6.2 holds with respect to some conditions, which follows from the following result.

**Theorem 5.6.7.** *Let $S$ be a semiprime semiring with $e \in Id(S)$. Let $e_1$ be the pierce trivial complement of $e$. Then $eSe$ and $e_1Se_1$ both are $k$-regular clean semirings if* *and only if $S$ is a $k$-regular clean semiring.*

*Proof.* Since $e_1$ is the pierce complement of $e$, so $eSe_1Se = 0 = e_1SeSe_1$. Now $SeSe_1S = \left\{ \sum_{i=0}^{n} x_iey_ie_1z_i : x_i, y_i, z_i \in S, n \in \mathbb{N} \right\}$ is an ideal of $S$ and $(SeSe_1S)^2 = 0$. Since $S$ is a semiprime semiring, it follows that $SeSe_1S = 0$. Hence $eSe_1 = 0$. Similarly, $Se_1SeS = 0$ implies that $e_1Se = 0$. Thus $e$, $e_1$ are central idempotents and $e$, $e_1$ are othogonal idempotents such that $e + e_1 = 1$. Hence the proof follows from Corollary 5.6.6. □

# Chapter 6

# On $k$-unit Clean Index Of Semirings

# Chapter 6

# On $k$-unit Clean Index Of Semirings

## 6.1 Introduction

Rings in which every element is the sum of an idempotent and a unit are said to be clean rings and this notion was introduced by W.K. Nicholson [7] in 1977 in the study of exchange rings. Since then, several authors have acquired various generalizations of clean rings ([28], [30], [42], [59]). Lee and Zhou introduced the concept of clean index of a ring in [41] and in [47] they extended their study about the concept of clean index of a ring. In [55], Basnet and Bhattacharyya defined weakly clean index of a ring $R$. The algebraic theory of semirings has experienced remarkable growth in recent years. A semiring, which extends the concepts of a ring and a distributive lattice, has seen significant development. In [62], we have introduced the notion of clean semiring and exchange semiring. Units and idempotents play very important roll for determining the structure of clean ring as well as clean semiring. However, in semirings, there is a broader concept of units, known as $k$-units. Let $S$ be a semiring. An element $a \in S \setminus \{0\}$ is said to be a $k$-unit if there exist $r_1, r_2 \in S$ such that $1 + r_1a = r_2a$ and $1 + ar_1 = ar_2$. We denote the set of all $k$-unit elements of a semiring $S$ by $U_k(S)$. An element $a \in S$ is said to be $k$-unit clean if $a = e + u$ for some $e^2 = e$ and $u \in U_k(S)$. A semiring $S$ is called $k$-unit clean if every nonzero element of $S$ is $k$-unit clean. Inspired by the works done in [55],[41],[47] and the concept of $k$-unit clean semiring, we introduce the notion of $k$-unit clean index of a semiring $S$. For $a \in S$, let $\xi(a) = \{e \in S : e^2 = e,\ a = e + u$ for some $u \in U_k(S)\}$. The $k$-unit clean index of $S$, denoted $ind_k(S)$ and is defined by $ind_k(S) = sup\{|\xi(a)| : a \in S\}$, where $|\xi(a)|$ denotes the cardinality of the set $\xi(a)$. Let $S$ be a semiring and $P(S) = \{0\} \cup \{s = s' + 1$ for some $s' \in S\}$. Then $P(S)$ becomes a subsemiring of $S$. The semiring $S$ is said to be a antisimple semiring if $S = P(S)$. In this paper, we determine the $k$-unit clean index of any subsemiring $T$ of a semiring $S$, where $S$ is an antisimple semiring with finite $k$-unit clean index such that every idempotent of $S$ has a complement. We also give some examples of semirings with finite $k$-unit clean index. If $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ be the formal triangular matrix semiring then we determine that $ind_k(A) < ind_k(S)$ and $ind_k(B) < ind_k(S)$. Finally, we characterize the semirings of $k$-unit clean indices 1 and 2, with the help of some other class of semirings.

## 6.2 Definition and Examples

**Definition 6.2.1.** A semiring $S$ is said to be uniquely $k$-unit clean semiring if every nonzero element $a$ of $S$ can be uniquely written as $a = e + u$ for some $k$-unit $u$ and idempotent $e$.

**Example 6.2.2.** *(i) Let $S = \mathbb{N}_0^+$ be the set of all non-negative integers. Define two operations $+$ and $.$ on $S$ by $a + b = max\{a, b\}$ and $a.b$ = usual multiplication in $\mathbb{N}_0^+$. Then $(\mathbb{N}_0^+, +, .)$ forms a semiring with additive identity $0$ and multiplicative identity $1$. In fact, for all nonzero integers $a$ in $S$, $1 + a = a$ holds. Thus every nonzero element of $S$ is a $k$-unit in $S$. Now for each $a(\neq 0) \in S$, we have $\xi(a) = \{0, 1\}$. Hence $ind_k(S) = 2$.*

*(ii) Let $n \in \mathbb{N}$ and consider $I_n = \{1, 2, 3, \dots, n\}$. Then $I_n$ forms a semiring with respect to binary operations $a + b = min\{a, b\}$ and $a.b = max\{a, b\}$ for all $a, b \in I_n$. Note that $1$ is the identity and $n$ is the zero element of $I_n$. Now $\xi(1) = \{1, 2, 3, \dots, n\} = I_n$. Hence $ind_k(S) \geq n$. Again each element in $I_n$ is idempotent.*

*So $ind_k(S) \leq n$. Consequently, $ind_k(S) = n$.*

## 6.3 Elementary Results

**Lemma 6.3.1.** *Let $I$ be a nonzero $k$-ideal of a commutative semiring $S$ such that $I \subseteq J_l(S)$. If $(u_1 + I)$ is a $k$-unit of $S/I$ then $u_1$ is a $k$-unit of $S$.*

*Proof.* Let $(u_1 + I)$ be a $k$-unit of $S/I$. Then there exist $(s_1 + I), (s_2 + I) \in S/I$ such that $(1 + I) + (s_1 + I)(u_1 + I) = (s_2 + I)(u_1 + I)$ i. e. $(1 + s_1u_1) + I = (s_2u_1 + I)$. This implies that $(1 + s_1u_1) + i_i = s_2u_1 + i_2 \ldots (1)$, where $i_1, i_2 \in I \subseteq J_l(S)$. If $u_1$ is not a $k$-unit in $S$, $\overline{(u_1)}$ is a proper $k$-ideal of $S$. Since every proper $k$-ideal of $S$ is contained in some maximal $k$-ideal of $S$ by Theorem 2.4.1, it follows that $u_1 \in M$ for some maximal $k$-ideal $M$ of $S$. Since $i_1, i_2 \in J_l(S) \subseteq M$, from equation (1) we have $1 \in M$ which contradicts that $M$ is proper. Hence $u_1$ is a $k$-unit in $S$. □

**Lemma 6.3.2.** *Let $S$ be a commutative antisimple semiring. Suppose $I$ is a nonzero $k$-ideal of $S$ such that $I \subseteq J_l(S)$ and idempotents lift modulo $I$. Then $|\xi(0 + I)| \leq 1$ for the quotient semiring $S/I$, where $0 + I$ is the zero element of $S/I$.*

*Proof.* To prove $|\xi(0 + I)| \leq 1$ it is sufficient to show that there exists at most one idempotent of $S/I$, belongs to $\xi(0 + I)$. If possible, let $0 + I = (e + I) + (u + I)$, where $(e + I) \in Id(S/I)$ and $(u + I) \in U_k(S/I)$. Since idempotents lift modulo $I$, $(e + I) = (e' + I)$, where $e' \in Id(S)$. Also by Lemma 6.3.1, it follows that $u \in U_k(S)$ . Then $0 + I = (e' + u) + I \Longrightarrow (e' + u) \in I \subseteq J_l(S) \ldots (1)$. If $e' \notin U_k(S)$ then $\overline{(e')}$ is a proper $k$-ideal of $S$. Again by Theorem 2.4.1, we find that $e' \in M$ for some maximal $k$-ideal $M$ of $S$. Since $J_l(S) \subseteq M$, from equation (1) we have $u \in M$. This contradicts that $M$ is a proper $k$-ideal of $S$. Hence $e' \in U_k(S)$. Thus there exist $s, t \in S$ such that $1 + se' = te' \Longrightarrow (1 + se') + I = te' + I \Longrightarrow (e' + se') + I = te' + I \Longrightarrow (1 + e' + se') + I = (1 + te') + I \Longrightarrow (e' + te') + I = (1 + te') + I \ldots (2)$. Since $S$ is antisimple and $I$ is a nonzero $k$-ideal of $S$, it follows that $S/I$ is an additively cancellative semiring from Note 2.3.4. Hence from equation (2), it follows that $(e' + I) = (1 + I) = (e + I)$. So if there exists any $(e + I) \in \xi(0 + I)$ then $e + I = 1 + I$. Thus $|\xi(0 + I)| \leq 1$. □

**Lemma 6.3.3.** *Let $S$ be a commutative antisimple semiring such that $ind_k(S) = n < \infty$ and every idempotent of $S$ has a complement in $S$. Suppose $I$ is a nonzero $k$-ideal of $S$ such that $I \subseteq J_l(S)$ and idempotents lift modulo $I$. Then $ind_k(S/I) \leq n$.*

*Proof.* Let us suppose that $ind_k(S/I) > n$. Then from Lemma 6.3.2, and since idempotents lift modulo $I$, there exists $(a + I) \neq (0 + I) \in S/I$ for some $a \in S$ such that $a + I = (f_i + I) + (u_i + I) \ldots (1)$ for $i = 1, 2, 3, \ldots, m$, $m > n$, where $(f_i + I) \in Id(S/I)$ and $(u_i + I) \in U_k(S/I)$ for all $i = 1, 2, \ldots, m$, $m > n$ such that $(f_i + I) \neq (f_j + I)$ for $i \neq j$, $i, j \in \{1, 2, \ldots, m\}$, $m > n$. Since idempotents lift modulo $I$, there exists $e_i \in Id(S)$ such that $(f_i + I) = (e_i + I)$ for each $i = 1, 2, \ldots, m$. So $e_i \neq e_j$ for $i \neq j$; $i, j \in \{1, 2, 3, \ldots, m\}$. Since $(a + I) \neq (0 + I)$, we have $a \neq 0$. Since $S$ is antisimple, $a = a_1 + 1$ for some $a_1 \in S$. Thus from equation (1) we have, $(a_1 + 1) + p_i = (e_i + u_i) + q_i \ldots (2)$ for some $p_i, q_i \in I \subseteq J_l(S)$. Since every idempotent has a complement, there exists $e'_i \in Id(S)$ such that $e_i + e'_i = 1$ for $i = 1, 2, 3, \ldots, m$, $m > n$. Now adding both side of equation (2) by $e'_i$, we have $[(a_1 + e'_i) + 1] + p_i = (1 + u_i) + q_i \Longrightarrow [(a_1 + e'_i) + 1] + I = (1 + u_i) + I$. Since $S$ is antisimple and $I$ is a nonzero $k$-ideal of $S$, $S/I$ is additively cancellative from Note 2.3.4. Hence $(a_1 + e'_i) + I = u_i + I \in U_k(S/I)$. From Lemma 6.3.1, it follows that $(a_1 + e'_i)$ is $k$-unit for each $i$. Hence $a = a_1 + 1 = (a_1 + e'_i) + e_i = e_i + (a_1 + e'_i)$ for $i = 1, 2, 3, \ldots, m$, $m > n$ such that $e_i \neq e_j$ for $i \neq j$ which contradicts that $ind_k(S) = n$. Hence $ind_k(S/I) \leq n$. □

The above Lemma 6.3.3 does not hold for arbitrary semiring which follows from the following example :

**Example 6.3.4.** *Let $S = \mathbb{N}_0^+$ be the semiring of all non-negative integers with respect to usual addition and multiplication. Then $J_l(S) = \{0\}$. Let $I = 3\mathbb{N}_0^+$. Then $I$ is a $k$-ideal in $S$ such that $I \nsubseteq J_l(S)$. Then $S/I = S/k_I = \{[0], [1], [2]\}$, where $k_I$ is the Bourne congruence defined by $I$ on $S$. Now $ind_k(S) = 1$ and $[2] = [1] + [1] = [0] + [2]$. Hence $ind_k(S/I) \geq 2 > 1 = ind_k(S)$.*

**Lemma 6.3.5.** *If $S$ be a commutative semiring such that $ind_k(S) \leq n$, $n \geq 1$, every idempotent has a complement in $S$ and $I$ be a $k$-ideal of $S$ with $I \subseteq J_l(S)$, then every* *idempotent of $S/I$ can be lifted to at most $n$ idempotents of $S$.*

*Proof.* Let $a \in S$ such that $a^2 + I = a + I$. If possible, let $a + I = e + I$, where $e$ is an idempotent in $S$. Suppose that $e$ has a complement $e_1$ in $S$. So $e + e_1 = 1$. Hence $(a + e_1) + I = 1 + I$. Thus from Lemma 6.3.1, it follows that $(a + e_1)$ is a $k$-unit in $S$. Hence $a + 1 = a + (e_1 + e) = e + (a + e_1)$. Thus $e \in \xi(1 + a)$. Since $|\xi(1 + a)| \leq n$, there are atmost $n$ such idempotents $e$. □

In [1], Bourne defined that an element $r$ in a semiring $S$ is said to be right semiregular element of $S$ if there exist $r', r'' \in S$ such that $r + r' + rr' = r'' + rr''$. If $S$ becomes a ring, this definition reduces to one usually given for right quasi-regularity in a ring.

We define right quasi-regular element in a semiring $S$ in following way :

**Definition 6.3.6.** *An element $s$ in a semiring $S$ is said to be right quasi-regular element of $S$ if there exist $s_1, s_2 \in S$ such that $s + s_1 + ss_1 = s_2 + ss_2$. Left quasi-regular element of $S$ is defined analogously. An element of $S$ is said to be quasi-regular if it is both right and left quasi-regular.*

We denote the set of all quasi-regular elements of a semiring $S$ by $Q(S)$.

**Proposition 6.3.7.** *Let $S$ be a semiring. Then $1 + Q(S) \subseteq U_k(S)$.*

*Proof.* Let $q \in Q(S)$. Then $q$ is right and left quasiregular in $S$. Thus there exist $q_1, q_2, q_3, q_4 \in S$ such that $q + q_1 + qq_1 = q_2 + qq_2 \ldots (1)$ and $q + q_3 + q_3q = q_4 + q_4q \ldots (2)$. Adding both side of equation (1) by 1, we have $1 + q + q_1 + qq_1 = 1 + q_2 + qq_2$. This implies that $(1 + q)(1 + q_1) = 1 + (1 + q)q_2$. Hence $(1 + q)$ is a right $k$-unit element of $S$. Similarly, by using equation (2) it can be proved that $(1 + q)$ a is left $k$-unit element of $S$. Thus $(1 + q) \in U_k(S)$. Consequently, $1 + Q(S) \subseteq U_k(S)$. □

The converse of the above Proposition 6.3.7 does not hold. This follows from the following example :

**Example 6.3.8.** *Let us Consider the semiring $S$ which is defined in Example 2.2.3(iv) . Then $S$ is an additive idempotent semiring. Consider the set of all $2 \times 2$ matrices* *over* $S$ *by* $M_2(S) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in S \right\}$. *Then* $M_2(S)$ *becomes a semiring with respect to usual matrix addition and multiplication. Since* $S$ *is an additive idempotent semiring,* $M_2(S)$ *is also an additive idempotent semiring. Let* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be any element in* $M_2(S)$. *Now* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Hence* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Q(M_2(S))$. *Thus* $M_2(S) = Q(M_2(S))$. *Consider the element* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in M_2(S)$. *Then* $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{M_2(S)}$. *Hence* $A$ *is a unit element in* $M_2(S)$ *which implies that* $A \in U_k(M_2(S))$. *But there does not exist any* $B \in M_2(S)$ *such that* $A = I_{M_2(S)} + B$ *holds. Hence* $U_k(M_2(S)) \nsubseteq I_{M_2(S)} + Q(M_2(S))$.

But in particular, we have the following result :

**Proposition 6.3.9.** *Let* $S$ *be a nontrivial semiring such that* $U_k(S) \subseteq P(S)$. *Then* $1 + Q(S) = U_k(S)$.

*Proof.* Let $u \in U_k(S)$. Since $S$ is nontrivial semiring and $U_k(S) \subseteq P(S)$, there exists $u_1 \in S$ such that $u = u_1 + 1$. Since $u \in U_k(S)$, there exist $s_1, s_2 \in S$ such that $1 + s_1u = s_2u \ldots (1)$ and $1 + us_1 = us_2 \ldots (2)$. Using $u = u_1 + 1$ in equation (2), we have $1+(u_1+1)s_1 = (u_1+1)s_2$. Now $1+u_1s_1+s_1 = u_1s_2+s_2 \Longrightarrow u_1+u_1(u_1s_1)+u_1s_1 = u_1(u_1s_2)+u_1s_2$ which implies that $u_1$ is right quasiregular. Similarly, using $u = u_1+1$ in equation (1) we can show that $u_1$ is left quasiregular. Hence $u_1$ is a quasiregular element in $S$. Thus $U_k(S) \subseteq 1 + Q(S)$. The reverse part follows from Proposition 6.3.7. So we find that $1 + Q(S) = U_k(S)$. □

**Lemma 6.3.10.** *Let* $S$ *be an abelian semiring and* $e$ *be a nonzero idempotent such that* $U_k(eS) \subseteq P(eS)$ *and* $e$ *has a complement* $e_1 \in S$. *Let* $ind_k(S) = n < \infty$. *Then* $1 \leq ind_k(eS) \leq ind_k(S)$.

*Proof.* Since $ind_k(S) = n < \infty$, there exists an element $x \in S$ such that $x = e_i + u_i$ for $i = 1, 2, 3, \ldots, n$ such that $e_i \neq e_j$ for $i \neq j$. Again since $e \neq 0$, we have

$ex = ee_i + eu_i$ for $i = 1, 2, 3, \ldots, n$. Since $S$ is abelian, so $ee_i$ are idempotents and $eu_i$ are $k$-units of $eS$ for all $i$. Hence $ind_k(eS) \geq 1$. We prove that $ind_k(eS) \ngtr ind_k(S)$. If possible, let $ind_k(eS) > n$. Then there exists $ey \in eS$ for some $y \in S$ such that $ey = eE_i + eU_i \ldots (1)$ for $i = 1, 2, 3, \ldots, m$ $m > n$ such that $eE_i \neq eE_j$ for $i \neq j$, where $eE_i \in Id(eS)$ and $eU_i \in U_k(eS)$ for $i = 1, 2, 3 \ldots, m$, $m > n$. Thus from Proposition 6.3.9, $U_k(eS) = e + Q(eS)$. So $eU_i = e + es_i$, where $es_i \in Q(eS)$ for $i = 1, 2, \ldots, m$, $m > n$. Hence equation (1) implies that $ey = eE_i + e + es_i \ldots (2)$ for $i = 1, 2, 3, \ldots, m$. Since $e$ has a complement $e_1$ in $S$, so $e_1^2 = e_1$ and $e + e_1 = 1$. Adding both side of equation (2) by $e_1$, it follows that $ey + e_1 = eE_i + 1 + es_i \ldots (3)$ for $i = 1, 2, 3, \ldots, m$, $m > n$. Now $eE_i \in Id(eS) \subseteq Id(S)$ for $i = 1, 2, 3, \ldots, m$, $m > n$ and $eE_i \neq eE_j$ for $i \neq j$, $i, j \in \{1, 2, 3, \ldots, m\}$. Now $Q(eS) \subseteq Q(S)$ and by Proposition 6.3.7, it follows that $1 + Q(S) \subseteq U_k(S)$. Thus from equation (3), we find an element $t = (ey + e_1) \in S$ such that $|\xi(t)| \geq m$, $m > n$ which contradicts that $ind_k(S) = n$. Hence $ind_k(eS) \leq ind_k(S)$. □

**Lemma 6.3.11.** *Let $S$ be an antisimple abelian semiring such that $ind_k(S) < \infty$ and every idempotent of $S$ has a complement in $S$. If $T$ be any nonzero subsemiring of $S$ with identity element $1_T$ such that $S$ and $T$ may or may not share the same identity element, then $ind_k(T) \leq ind_k(S)$.*

*Proof.* Let $t \in U_k(T)$. Then there exist $t_1, t_2 \in T$ such that $1_T + tt_1 = tt_2 \ldots (1)$ and $1_T + t_1 t = t_2 t \ldots (2)$. Since $T$ is a nonzero subsemiring of $S$ and $t$ is a $k$-unit element in $T$, so $t \neq 0$, otherwise $T$ becomes a zero subsemiring of $S$. Since $S$ is antisimple, there exists $s \in S$ such that $t = 1 + s \Longrightarrow t = 1_T + s1_T = 1_T + 1_T s$ for all $s \in S$. From equation (1), it follows that $1_T + (1+s)t_1 = (1+s)t_2 \Longrightarrow 1_T + t_1 + st_1 = t_2 + st_2 \Longrightarrow$ $s1_T + st_1 + s^2 t_1 = st_2 + s^2 t_2 \ldots (3)$. Since $1_T$ is the identity element of $T$, $1_T^2 = 1_T$. Now $1_T s = s1_T$ for all $s \in S$, since $S$ is abelian. Thus from equation (3), it follows that $s1_T + st_1 + (s1_T)(st_1) = st_2 + (s1_T)(st_2)$. Hence $s1_T$ is right quasiregular element in $S$. Similarly, from equation (2), it follows that $s1_T$ is left quasiregular element in $S$. So $s1_T \in Q(S)$ and hence $U_k(T) \subseteq 1_T + Q(S)$. Let $ind_k(S) = n$. If $ind_k(T) > n$ then there exists $t' \in T$ such that $t' = e_i + u_i$ for $i = 1, 2, 3, \ldots, m$, $m > n$. Also

$e_i \neq e_j$ for $i \neq j$, where $e_i \in Id(T)$ and $u_i \in U_k(T)$ for all $i = 1, 2, 3, \ldots, m$. Thus $t' = e_i + 1_T + q_i$, where $q_i \in Q(S)$ for $i = 1, 2, 3, \ldots, m$. Let $e$ be the complement of $1_T$, since $1_T \in Id(S)$. Hence $t' + e = e_i + (1_T + e) + q_i \Longrightarrow t' + e = e_i + 1 + q_i \ldots (4)$. From Proposition 6.3.7, it follows that $1 + Q(S) \subseteq U_k(S)$. Hence $1 + q_i \in U_k(S)$ for $i = 1, 2, 3, \ldots, m$. If $b = t' + e$ then equation (4) implies that $|\xi(b)| \geq m$ and hence $ind_k(S) \geq m > n$ which is a contradiction. Thus $ind_k(T) \leq ind_k(S)$. □

Similar to the proof of Lemma 3 in [41], we have the following result :

**Lemma 6.3.12.** *Let $S = A \times B$ be the direct product of of two semirings $A$ and $B$ such that $ind_k(A) = m$ and $ind_k(B) = n$. Then $ind_k(S) = ind_k(A) ind_k(B) = mn$.*

**Lemma 6.3.13.** *Let $(A, +_1, \cdot_1)$ and $(B, +_2, \cdot_2)$ be two semirings such that $A \cong B$. If $ind_k(A) = n$ for some $n \in \mathbb{N}$, then $ind_k(B) = n$.*

*Proof.* Since $A \cong B$, there exists an isomorphism $\phi : A \to B$ such that $\phi(0_A) = 0_B$ and $\phi(1_A) = 1_B$. Since $ind_k(A) = n$, there exists an element $a \in A$ such that $a = e_i +_1 u_i$ for $(i = 1, 2, 3, \ldots, n)$ and $e_i \neq e_j$ for $i \neq j$, where $e_i^2 = e_i \in A$ and $u_i \in U_k(A)$ for each $i$. Since $\phi$ is a homomorphism from $A$ to $B$, so $\phi(a) = \phi(e_i) +_2 \phi(u_i)$, $\phi(e_i)$ is an idempotent and $\phi(u_i)$ is a $k$-unit in $B$ for each $i$. Now if $\phi(e_i) = \phi(e_j)$ for some $i \neq j$, $i, j \in \{1, 2, 3, \ldots, n\}$, then $e_i = e_j$, since $\phi$ is one-one. This contradicts that $e_i \neq e_j$ for $i \neq j$. Thus $\phi(e_i) \neq \phi(e_j)$ for $i \neq j$. Hence $ind_k(B) \geq n$. Suppose $ind_k(B) > n$. Then there exists an element $b \in B$ such that $|\xi(b)| > n$. Thus there exist $E_1, E_2, E_3, \ldots, E_m \in Id(B)$, $m > n$ such that $b = E_i +_2 U_i$, where $U_i \in U_k(B)$ for each $i = 1, 2, 3, \ldots, m$ and $E_i \neq E_j$ for $i \neq j$. Since $\phi$ is an onto homomorphism from $A$ to $B$, there exist $a, e_i, u_i \in A$ for $i = 1, 2, 3, \ldots, m$ such that $\phi(a) = b$, $\phi(e_i) = E_i$ and $\phi(u_i) = U_i$ for $i = 1, 2, 3, \ldots, m$. Again it can be easily verified that $e_i \in Id(A)$ and $u_i \in U_k(A)$ for $i = 1, 2, 3, \ldots, m$. Thus $\phi(a) = \phi(e_i +_1 u_i) \Longrightarrow a = e_i +_1 u_i$ for $i = 1, 2, 3, \ldots, m$, since $\phi$ is one-one homomorphism. If $e_i = e_j$ for $i \neq j$ then $\phi(e_i) = \phi(e_j) \Longrightarrow E_i = E_j$ which is a contradiction. Hence $|\xi(a)| \geq m > n$ which contradicts that $ind_k(A) = n$. Thus $ind_k(B) \leq n$ and hence $ind_k(B) = n$. □

But the converse of the above Lemma does not hold which follows from the following Example.

**Example 6.3.14.** *Let $A$ be the Boolean semifield, defined in Example 2.2.3 (iv) and $B$ be the semiring, defined in Example 4.2.4 (ii). In both semirings $A$ and $B$ there are only trivial idempotents $\{0,1\}$. Hence $ind_k(A) \leq 2$ and $ind_k(B) \leq 2$. Now in semiring $A$, $1 = 1+1$ and $1 = 0+1$. Hence $|\xi(1)| = 2$. Thus $ind_k(A) = 2$. Again, in the semiring $B$, every nonzero element is a $k$-unit. Hence for each $a \neq 0 \in B$, $|\xi(a)| = 2$. Thus $ind_k(B) = 2$. But two semirings $A$ and $B$ are not isomorphic.*

## 6.4 $k$-Unit Clean Index of Formal Triangular Matrix Semiring

**Lemma 6.4.1.** *Let $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$, where $A$ and $B$ be two semirings with zero element $0_A$ and $0_B$, identity element $1_A$ and $1_B$ respectively, $M$ be the ${}_AM_B$ bi-semimodule. Then $U_k(S) = \left\{ \begin{bmatrix} u_1 & m \\ 0 & u_2 \end{bmatrix} : u_1 \in U_k(A),\ u_2 \in U_k(B),\ m \in M \right\}$.*

*Proof.* Let $T = \begin{bmatrix} u_1 & m \\ 0 & u_2 \end{bmatrix}$, where $u_1 \in U_k(A)$ and $u_2 \in U_k(B)$. Then there exist $u_1, u_2 \in A$ and $u_3, u_4 \in B$ such that $1_A + s_1u_1 = s_2u_1$, $1_A + u_1s_1 = u_1s_2$ and $1_B + s_3u_2 = s_4u_2$, $1_B + u_2s_3 = u_2s_4$. Now $\begin{bmatrix} 1_A & 0 \\ 0 & 1_B \end{bmatrix} + \begin{bmatrix} s_1 & s_1ms_3 + s_2ms_4 \\ 0 & s_3 \end{bmatrix} \begin{bmatrix} u_1 & m \\ 0 & u_2 \end{bmatrix} =$
$\begin{bmatrix} 1_A & 0 \\ 0 & 1_B \end{bmatrix} +$
$\begin{bmatrix} s_1u_1 & s_1m + s_1ms_3u_2 + s_2ms_4u_2 \\ 0 & s_3u_2 \end{bmatrix} = \begin{bmatrix} s_2u_1 & s_1m(1_B + s_3u_2) + s_2ms_4u_2 \\ 0 & s_4u_2 \end{bmatrix} =$
$\begin{bmatrix} s_2u_1 & s_1ms_4u_2 + s_2ms_4u_2 \\ 0 & s_4u_2 \end{bmatrix}$. Again $\begin{bmatrix} s_2 & s_1ms_4 + s_2ms_3 \\ 0 & s_4 \end{bmatrix} \begin{bmatrix} u_1 & m \\ 0 & u_2 \end{bmatrix} =$
$\begin{bmatrix} s_2u_1 & s_2m + s_1ms_4u_2 + s_2ms_3u_2 \\ 0 & s_4u_2 \end{bmatrix} = \begin{bmatrix} s_2u_1 & s_2m(1_B + s_3u_2) + s_1ms_4u_2 \\ 0 & s_4u_2 \end{bmatrix} =$
$\begin{bmatrix} s_2u_1 & s_1ms_4u_2 + s_2ms_4u_2 \\ 0 & s_4u_2 \end{bmatrix}$. Hence $T$ is left $k$-unit in $S$. Similarly it can be

proved that $T$ is right $k$-unit in $S$. Hence $T \in U_k(S)$.

Conversely, let $\begin{bmatrix} u_1 & m \\ 0 & u_2 \end{bmatrix} \in U_k(S)$. Then it is easy to prove that $u_1 \in U_k(A)$ and $u_2 \in U_k(B)$. □

**Lemma 6.4.2.** *Let* $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$, *where* $A$ *and* $B$ *be two antisimple inverse semiring such that every idempotent of* $A$ *and* $B$ *has a complement. Let* $ind_k(A) = m$ *and* $ind_k(B) = n$, $M$ *be the* ${}_AM_B$ *bi-semimodule. Then either* $ind_k(S) \geq n(m-1)+1$ *or* $ind_k(S) \geq 2nm$.

*Proof.* Since $ind_k(A) = m$ and $ind_k(B) = n$, there exist $a \in A$ and $b \in B$ such that $a = e_i + u_i$ for $i = 1, 2, 3, \ldots, m$, $e_i \neq e_k$ for $i \neq k$ and $b = E_j + U_j$, for $j = 1, 2, 3, 4, \ldots, n$, $E_j \neq E_t$ for $j \neq t$, where $e_i \in Id(A)$, $u_i \in U_k(A)$ for $i = 1, 2, 3, \ldots, m$ and $E_j \in Id(B)$, $U_j \in U_k(B)$ for $j = 1, 2, 3, \ldots, n$.

**Case - I :** Let $e_{i_0}M(1_B + E'_{j_0}) + (1_A + e'_{i_0})ME_{j_0} = 0$ for some $i_0 \in \{1, 2, 3, \ldots, m\}$ and $j_0 \in \{1, 2, 3, \ldots, n\}$. Then $e_{i_0}M(1_B + E'_{j_0}) = 0$, $(1_A + e'_{i_0})ME_{j_0} = 0$ and $e_{i_0}w(1_B + E'_{j_0}) = 0$ for all $w \in M$. Thus $e_{i_0}w + e_{i_0}wE'_{j_0} = 0 \Longrightarrow e_{i_0}w + e_{i_0}wE'_{j_0} + e_{i_0}wE_{j_0} = e_{i_0}wE_{j_0} \Longrightarrow e_{i_0}w(1_B + E_{j_0} + E'_{j_0}) = e_{i_0}wE_{j_0} \ldots (1)$. Let $e \in Id(B)$ and $e$ has a complement $e_1 \in B$. Then $e + e_1 = 1_B$. Thus $1_B + e + e' = e_1 + e + e + e' = e_1 + e = 1_B$. Hence from equation (1), it follows that $e_{i_0}w = e_{i_0}wE_{j_0}$ for all $w \in M$. Similarly, from $(1_A + e'_{i_0})ME_{j_0} = 0$, it follows that $wE_{j_0} = e_{i_0}wE_{j_0}$ for all $w \in M$. Hence $e_{i_0}w = wE_{j_0}$ for all $w \in M$. Let $A = \begin{bmatrix} 1_A + a' & w \\ 0 & b \end{bmatrix} \in S$, where $w(\neq 0) \in M$. Then $\begin{bmatrix} 1_A + a' & w \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1_A + e'_i & 0 \\ 0 & E_j \end{bmatrix} + \begin{bmatrix} u'_i & w \\ 0 & U_j \end{bmatrix}$ for $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$. Since $e_i \in Id(A)$ for each $i = 1, 2, 3, \ldots, m$, $(1 + e'_i) \in Id(A)$ for each $i$. Hence $\begin{bmatrix} 1_A + e'_i & 0 \\ 0 & E_j \end{bmatrix} \in Id(S)$ and $\begin{bmatrix} u'_i & w \\ 0 & U_j \end{bmatrix} \in U_k(S)$ follows from Lemma 6.4.1 for each $i = 1, 2, 3, \ldots, m$ and for each $j = 1, 2, 3, \ldots, n$. If $a(\neq 0) \in A$ then $a = a_1 + 1_A$ for some $a_1 \in A$, since $A$ is antisimple. Hence $a + 1_A + 1'_A = a_1 + 1_A + 1_A + 1'_A = a_1 + 1_A = a$. Since $e_i \in \xi(a)$ for $i = 1, 2, 3, \ldots, m$ and $e_i \neq e_k$ for $i \neq k$, $i, k \in \{1, 2, 3, \ldots, m\}$ at least $(m-1)$, $e_i$ are not equal to

0. Let $e_i, e_k(\neq 0) \in \xi(a)$ and $i \neq k$. If possible, let $1_A + e'_i = 1_A + e'_k$. Then $1_A + e'_i + e_i + e_k = 1_A + e'_k + e_i + e_k \Longrightarrow 1_A + e_k = 1_A + e_i \Longrightarrow 1'_A + 1_A + e_k = 1'_A + 1_A + e_i \Longrightarrow e_k = e_i$ which contradicts that $e_i \neq e_k$ for $i \neq k$. Hence $1_A + e'_i \neq 1_A + e'_k$ for $e_i, e_k(\neq 0)$ and $i \neq k$. Again $\begin{bmatrix} 1_A + a' & w \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1_A + e'_{i_0} & w \\ 0 & E_{j_0} \end{bmatrix} + \begin{bmatrix} u'_{i_0} & 0 \\ 0 & U_{j_0} \end{bmatrix}$. Now $\begin{bmatrix} 1_A + e'_{i_0} & w \\ 0 & E_{j_0} \end{bmatrix} \begin{bmatrix} 1_A + e'_{i_0} & w \\ 0 & E_{j_0} \end{bmatrix} = \begin{bmatrix} 1_A + e'_{i_0} & w + e'_{i_0}w + wE_{j_0} \\ 0 & E_{j_0} \end{bmatrix} = \begin{bmatrix} 1_A + e'_{i_0} & w + e'_{i_0}w + e_{i_0}w \\ 0 & E_{j_0} \end{bmatrix} = \begin{bmatrix} 1_A + e'_{i_0} & (1_A + e'_{i_0} + e_{i_0})w \\ 0 & E_{j_0} \end{bmatrix} = \begin{bmatrix} 1_A + e'_{i_0} & w \\ 0 & E_{j_0} \end{bmatrix}$. Hence $\begin{bmatrix} 1_A + e'_{i_0} & w \\ 0 & E_{j_0} \end{bmatrix} \in Id(S)$ Again, from Lemma 6.4.1, it follows that $\begin{bmatrix} u'_{i_0} & 0 \\ 0 & U_{j_0} \end{bmatrix} \in U_k(S)$. Hence $ind_k(S) \geq n(m-1)+1$.

**Case - II :** Let $e_iM(1_B + E'_j) + (1_A + e'_i)ME_j \neq 0$ for all $i, j$. Let $w_{ij} \neq 0 \in e_iM(1_B + E'_j) + (1_A + e'_i)ME_j$ for each pair $(i, j)$. Let $w_{ij} = e_im_1(1_B + E'_j) + (1_A + e'_i)m_2E_j$. Now $\begin{bmatrix} e_i & w_{ij} \\ 0 & E_j \end{bmatrix} \begin{bmatrix} e_i & w_{ij} \\ 0 & E_j \end{bmatrix} = \begin{bmatrix} e_i & e_iw_{ij} + w_{ij}E_j \\ 0 & E_j \end{bmatrix}$. $e_iw_{ij} = e_im_1(1_B + E'_j) + (e_i + e'_i)m_2E_j = e_im_1 + e_im_1E'_j + e_im_2E_j + e'_im_2E_j$ and $w_{ij}E_j = e_im_1(E_j + E'_j) + (1_A + e'_i)m_2E_j = e_im_1E_j + e_im_1E'_j + m_2E_j + e'_im_2E_j$. So $e_iw_{ij} + w_{ij}E_j = e_im_1 + e_im_1E'_j + e_im_2E_j + e'_im_2E_j + e_im_1E_j + e_im_1E'_j + m_2E_j + e'_im_2E_j = e_im_1(1_B + E'_j) + e'_im_2E_j + m_2E_j = e_im_1(1_B + E'_j) + (1_A + e'_i)m_2E_j = w_{ij}$. Thus $\begin{bmatrix} e_i & w_{ij} \\ 0 & E_j \end{bmatrix} \in Id(S)$ for each pair $(i, j)$. Let $k_1 = \{1, 2, \ldots, m\}$ and $k_2 = \{1, 2, \ldots, n\}$. Let $w = \underset{i \in k_1, j \in k_2}{\Sigma} w_{ij}$. Then $w \in M$. Let $C = \begin{bmatrix} a & w \\ 0 & b \end{bmatrix}$. Then $\xi(C) \supseteq \{ \begin{bmatrix} e_i & w_{ij} \\ 0 & E_j \end{bmatrix}, \begin{bmatrix} e_i & 0 \\ 0 & E_j \end{bmatrix}; 1 \leq i \leq m, 1 \leq j \leq n, w_{ij} \neq 0 \in M\}$. Hence $ind_k(S) \geq 2nm$. From Case : I and Case : II it follows that either $ind_k(S) \geq n(m-1)+1$ or $ind_k(S) \geq 2nm$. □

**Theorem 6.4.3.** *Let $A$ and $B$ be two semirings and ${}_AM_B$ be a nontrivial bisemimodule. If $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ is a formal triangular matrix semiring, then $ind_k(A) < ind_k(S)$ and $ind_k(B) < ind_k(S)$.*

*Proof.* Let $ind_k(A) = n$. Then there exists an element $a \in A$ such that $a = e_i + u_i$,

$e_i \neq e_j$ for $i \neq j$, where $e_i \in Id(A)$ and $u_i \in U_k(A)$ for $i = 1, 2, 3, \ldots, n$.

**Case - I :** Let $e_1 M = 0_M$. Let $m \neq 0_M$. Then $e_1 m = 0_M$. Consider the element $A_1 \in S$ such that $A_1 = \begin{bmatrix} a & m \\ 0 & (1_B + 1_B) \end{bmatrix}$. Then $A_1 = \begin{bmatrix} e_i & 0_M \\ 0 & 1_B \end{bmatrix} + \begin{bmatrix} u_i & m \\ 0 & 1_B \end{bmatrix}$ $(i = 1, 2, 3, \ldots, n)$. Also $A_1 = \begin{bmatrix} a & m \\ 0 & (1_B + 1_B) \end{bmatrix} = \begin{bmatrix} e_1 & m \\ 0 & 1_B \end{bmatrix} + \begin{bmatrix} u_1 & 0_M \\ 0 & 1_B \end{bmatrix}$. Since $e_i \in Id(A)$ for $i = 1, 2, 3, \ldots, n$ and $e_1 M = 0_M$, $\begin{bmatrix} e_i & 0_M \\ 0 & 1_B \end{bmatrix}$ for each $i$ and $\begin{bmatrix} e_1 & m \\ 0 & 1_B \end{bmatrix} \in Id(S)$. Again from Lemma 6.4.1, it follows that $\begin{bmatrix} u_i & m \\ 0 & 1_B \end{bmatrix} \in U_k(A)$ for each $i$ and for all $m \in M$. Hence $\xi(A_1) \supseteq \left\{ \begin{bmatrix} e_i & 0_M \\ 0 & 1_B \end{bmatrix} (i = 1, 2, 3, \ldots, n), \begin{bmatrix} e_1 & m \\ 0 & 1_B \end{bmatrix} \right\}$. Hence $ind_k(S) \geq (n+1) > n$.

**Case - II :** Let $e_1 M \neq 0_M$. Then there exists $m_1 \in M$ such that $e_1 m_1 \neq 0_M$. Let $B_1 = \begin{bmatrix} a & e_1 m_1 \\ 0 & 1_B \end{bmatrix} \in S$. Then $B_1 = \begin{bmatrix} e_i & 0_M \\ 0 & 0_B \end{bmatrix} + \begin{bmatrix} u_i & e_1 m_1 \\ 0 & 1_B \end{bmatrix}$ $(i = 1, 2, 3, \ldots, n)$. Also $B_1 = \begin{bmatrix} a & e_1 m_1 \\ 0 & 1_B \end{bmatrix} = \begin{bmatrix} e_1 & e_1 m_1 \\ 0 & 0_B \end{bmatrix} + \begin{bmatrix} u_1 & 0_M \\ 0 & 1_B \end{bmatrix}$. Hence $\xi(B_1) \supseteq \left\{ \begin{bmatrix} e_i & 0_M \\ 0 & 0 \end{bmatrix} (i = 1, 2, 3, \ldots, n), \begin{bmatrix} e_1 & e_1 m_1 \\ 0 & 0_B \end{bmatrix} \right\}$. In this case also $ind_k(S) \geq (n+1) > n$. Similarly, it can be proved that $ind_k(B) < ind_k(S)$. □

**Remark 6.4.4.** *Let $S$ be a commutative $k$-local semiring such that every idempotent has an orthogonal complement. Then $S$ has only trivial idempotents.*

*Proof.* Let $e$ be an idempotent element in $S$ and $e_1$ be its orthogonal complement. Then $e + e_1 = 1$ and $ee_1 = e_1 e = 0$. Since $S$ is a commutative $k$-local semiring, $S$ has unique maximal $k$-ideal $J_l(S)$. If $e \in U_k(S)$, then there exist $s_1, s_2 \in S$ such that $1 + s_1 e = s_2 e \Longrightarrow e_1 + s_1 e e_1 = s_2 e e_1 \Longrightarrow e_1 = 0$. Hence $e = 1$. Let $e \notin U_k(S)$. Then by Theorem 2.4.1, it follows that $e \in J_l(S)$. If $e_1 \in J_l(S)$, then $e + e_1 = 1 \in J_l(S)$ which is a contradiction. Hence $e_1 \in U_k(S)$. Thus $e_1 = 1$ and hence $e = 0$. So $S$ has only trivial idempotents. □

**Remark 6.4.5.** *[37] The cardinality of any set is obviously a cardinal number.*

**Lemma 6.4.6.** *[37] If $K$ is a set of cardinal numbers, then $sup(K)(=\bigcup K)$ is a cardinal number..*

We refer to [37] for rudimentary definitions and results, related to cardinal numbers and their elementary properties.

**Theorem 6.4.7.** *Let $S=\begin{bmatrix} A & M \\ O & B \end{bmatrix}$, where $A$ be a commutative $k$-local semiring such that every idempotent of $A$ has orthogonal complement and $A/J_l(A)\ncong \mathbb{N}_0^+/2\mathbb{N}_0^+$, $B$ is a uniquely $k$-unit clean semiring such that every idempotent has an orthogonal complement in $B$, $(M,+)$ is a nontrivial (i.e $M\neq 0$) cyclic monoid which is not a group and ${}_AM_B$ is bisemimodule. Then either $ind_k(S)=|M|+1$ or $ind_k(S)=|M|+2$.*

*Proof.* Let $1_A+u\in J_l(A)$ for all $u\in U_k(A)$. Then $1_A+1_A=2_A\in J_l(A)$, since $1_A\in U_k(A)$. Now let $x\notin J_l(A)$. Since $A$ is a commutative $k$-local semiring, Theorem 2.4.1 implies that $x\in U_k(A)$. Hence $1_A+x\in J_l(A)$ according to our assumption. Thus $(1_A+x)+J_l(A)=J_l(A)$. Since $2_A\in J_l(A)$, so $1_A+J_l(A)=x+J_l(A)$. Hence $A/J_l(A)\cong \mathbb{N}_0^+/2\mathbb{N}_0^+$ by the mapping $\phi: A/J_l(A)\to \mathbb{N}_0^+/2\mathbb{N}_0^+$ defined by $\phi(0_A+J_l(A))=0+2\mathbb{N}_0^+$ and $\phi(1_A+J_l(A))=1+2\mathbb{N}_0^+$, which contradicts our assumption. Hence there exists $u_0\in U_k(A)$ such that $1+u_0\in U_k(A)$. Now we have following two cases :

**Case - I :** Let $(M,+)$ be a finite cyclic monoid such that it is not a group. Remark 1.3.67 implies that there exists a generator $a_1$ of $M$ with index $m\in\mathbb{N}$ and period $r\in\mathbb{N}$. Hence $(m+r)a_1=ma_1$. So $M=\{0_M,a_1,2a_1,\ldots ma_1,\ldots(m+r-1)a_1\}$. Let $E^+(M)$ denotes the set of all idempotent elements in $M$. From Remark 1.3.67, it follows that $K_{a_1}=\{ma_1,(m+1)a_1,\ldots,(m+r-1)a_1\}$ becomes a cyclic group. Hence $K_{a_1}$ has only one idempotent element. Again due to the minimality of the index and period of $a_1$, $0_M$ is the only idempotent element of $M$ in $\{0_M,a_1,2a_1,\ldots,(m-1)a_1\}$. Hence $|E^+(M)|=2$. Consider the element $\alpha=\begin{bmatrix} 1_A+u_0 & ma_1 \\ 0 & 1_B \end{bmatrix}\in S$. Then $\xi(\alpha)\supseteq\left\{\begin{bmatrix} 0_A & 0_M \\ 0 & 0_B \end{bmatrix},\begin{bmatrix} 1_A & w \\ 0 & 0_B \end{bmatrix}\right\}$ for all $w\in M$. Hence $ind_k(S)\geq |M|+1$. Let

$\alpha_1 = \begin{bmatrix} a & w_0 \\ 0 & b \end{bmatrix} \in S$. Since $A$ is a commutative $k$-local semiring such that every idempotent of $A$ has orthogonal complement, by Remark 6.4.4 it follows that $Id(A) = \{0_A, 1_A\}$. Then $\xi(\alpha_1) \subseteq \left\{ \begin{bmatrix} e & w \\ 0 & f \end{bmatrix} \in S \ : \ e \in \xi(a),\ f \in \xi(b),\ w = ew + wf \right\}$. Since $B$ is uniquely $k$-unit clean semiring, so $f = f_0 \in \xi(b)$. Thus we have $\xi(\alpha_1) \subseteq \left\{ \begin{bmatrix} e & w \\ 0 & f \end{bmatrix} \in S \ : \ e \in \{0, 1_A\},\ f = f_0,\ w = ew + wf_0 \right\} = V_0 \bigcup V_1$ (say), where we have $V_0 = \left\{ \begin{bmatrix} 0_A & w \\ 0 & f_0 \end{bmatrix} \in S \ : \ w = wf_0 \right\}$ and $V_1 = \left\{ \begin{bmatrix} 1_A & w \\ 0 & f_0 \end{bmatrix} \in S \ : \ w = w + wf_0 \right\}$. Now $f_0$ has a complement $f_1 \in B$ which implies that $f_0 + f_1 = 1_B$. Hence $w = w + wf_0 \iff wf_0 + wf_0 = wf_0 \iff wf_0 \in E^+(M)$. Hence $|\xi(\alpha_1)| \leq |V_0 \bigcup V_1| \leq |V_0| + |V_1| \leq |M| + |E^+(M)| \leq |M| + 2$. Since $\alpha_1$ is an arbitrary element of $S$, $ind_k(S)$ is equal to either $|M| + 1$ or $|M| + 2$.

**Case - II :** Let $(M, +)$ be an infinite cyclic monoid. Then $M = \{0, a_1, 2a_1, 3a_1, \ldots\}$, where $na_1 \neq 0$ for all $n \in \mathbb{N}$ and $na_1 = ma_1$ if and only if $n = m$ for $n, m \in \mathbb{N}$. So $(M, +) \cong (\mathbb{N}_0^+, +)$ from Theorem 1.3.68, which implies that $|M| = |\mathbb{N}_0^+| = \aleph_0$. Hence $|E^+(M)| = 1$, since $(\mathbb{N}_0^+, +)$ has only idempotent element 0. For any $n \in \mathbb{N}_0^+$, consider $\alpha_n = \begin{bmatrix} 1_A + u_0 & na_1 \\ 0 & 1_B \end{bmatrix}$. Then $\xi(\alpha_n) \supseteq \left\{ \begin{bmatrix} 1_A & ia_1 \\ 0 & 0_B \end{bmatrix}, \begin{bmatrix} 1_A & 0_M \\ 0 & 0_B \end{bmatrix}, \begin{bmatrix} 0_A & 0_M \\ 0 & 0_B \end{bmatrix} \right\}$ for $i = 1, 2, 3, \ldots n$. Hence $|\xi(\alpha_n)| \geq n + 2$. Now $ind_k(S) = \sup\{|\xi(\alpha)| : \alpha \in S\}$. Thus $ind_k(S) \geq |\xi(\alpha_n)|$ for every $n \in \mathbb{N}_0^+ \implies ind_k(S) \geq (n+2) > n$ for every $n \in \mathbb{N}_0^+$. Remark 6.4.5 implies that $|\xi(\alpha)|$ is a cardinal number for each $\alpha \in S$. Hence Lemma 6.4.6 implies that $ind_k(S)$ is also a cardinal number. Let $ind_k(S) < \aleph_0$. Since $\aleph_0$ is the smallest infinite cardinal number [37], $ind_k(S)$ is a finite cardinal number. Hence there exists a positive integer $m$ such that $ind_k(S) = m$ which contradicts that $ind_k(S) > n$ for every $n \in \mathbb{N}_0^+$. Hence $ind_k(S) \geq \aleph_0 = |M| + 1$. Let $\beta_1 = \begin{bmatrix} a & w_0 \\ 0 & b \end{bmatrix}$ be any element of $S$. Then similarly, like Case - I, we have $|\xi(\beta_1)| \leq |M| + |E^+(M)|$. Since $|E^+(M)| = 1$, $|\xi(\beta_1)| \leq |M| + 1$ and $\beta_1$ is an arbitrary element of $S$, $ind_k(S) = |M| + 1$. □

**Lemma 6.4.8.** *Let $S$ be a semiring and $(M, +)$ be a nontrivial finite cyclic monoid* *which is not a group. Let $M$ be a right $S$-semimodule. Then $(M,+)$ is indecomposable.*

*Proof.* Since $M$ is a nontrivial finite cyclic monoid which is not a group, there exists $\alpha \neq 0_M \in M$ such that $M = \{0_M, \alpha, \dots, m\alpha, \dots, (m+r-1)\alpha\}$, where $m \in \mathbb{N}$ is the index of $\alpha$ and $r \in \mathbb{N}$ is the period of $\alpha$. If possible, let $M$ be not indecomposable. Then there exist nonzero subsemimodules $M_1$ and $M_2$ of $M$ satisfying $M = M_1 \oplus M_2$. Thus we can write $\alpha = m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Our claim is that neither $m_1 \neq 0_M$ nor $m_2 \neq 0_M$. If $m_1 = 0_M$ then $\alpha = m_2 \in M_2$. If $m_1' \in M_1$ then $m_1' = t\alpha$, where $0 \leq t \leq (m+r-1)$. Thus $m_1' = tm_2 \in M_2$. Hence $m_1' = m_1' + 0_M = 0_M + tm_2'$. By the uniqueness of the representation, we have $m_1' = 0_M$ which implies that $M_1 = 0_M$, a contradiction. Thus $m_1 \neq 0_M$. Similarly, it can be shown that $m_2 \neq 0_M$. Hence $m_1 = t_1\alpha$ and $m_2 = t_2\alpha$ for $0 < t_1, t_2 \leq (m+r-1)$. So $(t_1t_2)\alpha = t_1(t_2\alpha) = t_2(t_1\alpha) \in M_1 \bigcap M_2 = \{0_M\}$. Hence $(t_1t_2)\alpha = 0_M = 0\alpha$. This contradicts that $(M,+)$ is not a group. Thus $(M,+)$ is indecomposable. □

**Lemma 6.4.9.** *Let $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$, where $A$ be a semiring such that every idempotent of $A$ has an orthogonal complement and $ind_k(A) = n$, $B$ is a semiring such that $ind_k(B) = m$, ${}_AM_B$ is a nontrivial bisemimodule (i.e $M \neq (0)$). If $(M,+)$ be a finite cyclic monoid that is not a group, then $ind_k(S) \geq n + [n/2)(|M|-1)$, where $[n/2)$ denotes the least integer greater than or equal to $n/2$.*

*Proof.* Since $M$ is nontrivial finite cyclic monoid, there exists $\alpha(\neq 0_M) \in M$ such that $M = \{0_M, \alpha, \dots, m\alpha, \dots, (m+r-1)\alpha\}$, where $m \in \mathbb{N}$ is the index of $\alpha$ and $r \in \mathbb{N}$ is the period of $\alpha$. Let $q = |M| = m + r$ and $a = e_i + u_i$ $(i = 1, 2, 3, \dots, n)$ be distinct $k$-unit clean expressions of $a$ in $A$. Let $e^2 = e \in A$ and $E$ be the orthogonal complement of $e$. Hence $(M,+) = eM \oplus EM$. Remark 6.4.8 implies that $(M,+)$ is indecomposable. Hence either $eM = M$ or $EM = M$. Let us assume that $e_1M = M$, $e_2M = M$, $\dots$, $e_sM = M$ and $E_{s+1}M = M$, $\dots$, $E_nM = M$.

**Case - I :** Let $s \geq n - s$ $(i.e, s \geq [n/2))$. Then for $T = \begin{bmatrix} a & (m+r-1)\alpha \\ 0 & 1_B \end{bmatrix}$, we have $\xi(T) \supseteq \left\{ \begin{bmatrix} e_i & 0_M \\ 0 & 0_B \end{bmatrix}, \begin{bmatrix} e_j & w \\ 0 & 0_B \end{bmatrix}, 1 \leq i \leq n,\ 1 \leq j \leq s,\ w\ (\neq 0_M) \in M \right\}$.

Then $ind_k(S) \geq n + s(q-1) \geq n + [n/2)(|M|-1)$.

**Case - II :** Let $n-s \geq s$. Then $n/2 \geq s$ and hence $n-s \geq n/2$ ($i.e, n-s \geq [n/2)$).
Thus for $T_1 = \begin{bmatrix} a & (m+r-1)\alpha \\ 0 & 1_B + 1_B \end{bmatrix}$, it follows that $\xi(T_1) \supseteq \left\{ \begin{bmatrix} e_i & 0_M \\ 0 & 1_B \end{bmatrix}, \begin{bmatrix} e_j & w \\ 0 & 1_B \end{bmatrix} \right\}$, $1 \leq i \leq n,\ s+1 \leq j \leq n,\ w\ (\neq 0_M) \in M$. So we find that $ind_k(S) \geq n+(n-s)(q-1) \geq n+[n/2)(|M|-1)$. Again, if we take $E_1M = M, E_2M = M, \ldots, E_sM = M$ and $e_{s+1}M = M, \ldots, e_nM = M$ then similarly we can prove that $ind_k(S) \geq n+[n/2)(|M|-1)$. □

## 6.5 Characterization of Semirings of $k$-Unit Clean Indices 1 and 2

**Theorem 6.5.1.** *Let $S$ be an antisimple semiring such that every idempotent of $S$ has an absorbing complement. Then $ind_k(S) = 1$ if and only if $S$ is abelian and for any $0 \neq e^2 = e \in S$, $e + u \neq v$ for any $u, v \in U_k(S)$.*

*Proof.* Let $e^2 = e \in S$ and $e_1$ be the absorbing complement of $e$ in $S$. Then $e+e_1 = 1$, $e + see_1 = e$ and $e_1 + e_1es = e_1$ for all $s \in S$. Now $e0 = 0e = 0$. Let $s\ (\neq 0) \in S$. Since $S$ is antisimple, there exists $s_1 \in S$ such that $s = s_1 + 1$. Now $e + e_1s_1e + 1 = (e + e_1s_1e) + 1 = e + (1 + e_1s_1e)$. Also $(e + e_1s_1e)(e + e_1s_1e) = e + ee_1s_1e + e_1s_1e + e_1s_1ee_1s_1e = e + (e + e_1s_1ee_1)e_1s_1e + e_1s_1e = e + ee_1s_1e + e_1s_1e = e + (e_1 + ee_1)s_1e = e + e_1s_1e$. Hence $(e + e_1s_1e)$ is an idempotent of $S$. Again $1+(1+e_1s_1e)(1+e_1s_1e) = 1+1+e_1s_1e+e_1s_1e+e_1s_1ee_1s_1e = 1+1+ee_1s_1e+e_1s_1e+e_1s_1e+e_1s_1ee_1s_1e = 1+1+(e+e_1s_1ee_1)e_1s_1e+e_1s_1e+e_1s_1e = 2+ee_1s_1e+e_1s_1e+e_1s_1e = 2+e_1s_1e+e_1s_1e = 2(1+e_1s_1e)$. Thus $(1+e_1s_1e)$ is a $k$-unit in $S$. Since $ind_k(S) = 1$, we have $e + e_1s_1e = e \Longrightarrow e + es_1e + e_1s_1e = e + es_1e \Longrightarrow e + s_1e = e + es_1e \Longrightarrow (1 + s_1)e = e(1 + s_1)e \Longrightarrow se = ese$ for all $s \in S$. Similarly, it can be shown that $es = ese$ for all $s \in S$. Thus $es = se$ for all $s \in S$. Hence $S$ is abelian. If $e + u = v$, where $e^2 = e \in S$ and $u, v \in U_k(S)$, then $e+u = v+0$ are two $k$-unit clean expressions of same element in $S$. Consequently, $e = 0$, since $ind_k(S) = 1$.

Conversely, let $a = e_1 + u_1 = e_2 + u_2 \ldots (1)$ be two $k$-unit clean expressions of $a$ in $S$. Now $e_1$ and $e_2$ are idempotents in $S$. Let $e_3^2 = e_3$ be an absorbing complement of $e_1$ and $e_4^2 = e_4$ be an absorbing complement of $e_2$. Since $e_2 + se_2e_4 = e_2$ for all $s \in S$, we have $e_1e_4 + u_1e_4 = e_2e_4 + u_2e_4 \Longrightarrow e_2 + e_1e_4 + u_1e_4 = e_2 + u_2e_4 \Longrightarrow$ $e_1e_4 + (u_1e_4 + e_2) = (u_2e_4 + e_2)$. Since $S$ is abelian, so $(e_1e_4)(e_1e_4) = e_1^2e_4^2 = e_1e_4$. Now $u_1, u_2 \in U_k(S)$. Thus there exist $s_1, s_2 \in S$ such that $1 + s_1u_1 = s_2u_1$ and $1 + u_1s_1 = u_1s_2$. Also $(s_2e_4 + 1)(u_1e_4 + e_2) = s_2u_1e_4 + s_2e_4e_2 + u_1e_4 + e_2 \Longrightarrow$ $e_4 + s_1u_1e_4 + s_2e_2e_4 + u_1e_4 + e_2 = e_2 + e_4 + (s_1 + 1)u_1e_4 = 1 + (s_1 + 1)u_1e_4$. Since $e_4$ is the absorbing complement of $e_2$ and $S$ is abelian, $1 + (s_1 + 1)u_1e_4 = e_4 + e_2 + (s_1 + 1)u_1e_4 + (s_1 + 1)e_4e_2 = 1 + (s_1e_4 + e_4)(u_1e_4 + e_2)$. Thus $(u_1e_4 + e_2)$ is a left $k$-unit of $S$. Again $(u_1e_4 + e_2)(s_2e_4 + 1) = u_1e_4s_2e_4 + u_1e_4 + e_2s_2e_4 + e_2 = u_1s_2e_4 + u_1e_4 + e_2s_2e_4 + e_2 =$ $e_4 + u_1s_1e_4 + u_1e_4 + e_2s_2e_4 + e_2 = e_4 + e_2 + s_2e_2e_4 + u_1s_1e_4 + u_1e_4 = 1 + u_1e_4(s_1 + 1)$, since $S$ is abelian. Now $1 + (u_1e_4 + e_2)e_4(s_1 + 1) = 1 + (u_1e_4 + e_2)(e_4s_1 + e_4) =$ $1 + u_1e_4s_1 + u_1e_4 + e_2e_4s_1 + e_2e_4 = e_2 + e_4 + u_1e_4s_1 + u_1e_4 + e_2e_4 + e_2e_4s_1 = e_2 +$ $e_4 + e_2e_4 + e_4e_2s_1 + u_1e_4 + u_1e_4s_1 = e_2 + e_4 + u_1e_4s_1 + u_1e_4 = 1 + u_1e_4(s_1 + 1)$, since $e_4$ is the absorbing complement of $e_2$. Thus $(u_1e_4 + e_2)$ is a right $k$-unit of $S$. Hence $(u_1e_4 + e_2)$ is a $k$-unit of $S$. Similarly, $(u_2e_4 + e_2)$ is also a $k$-unit of $S$. According to the condition, $e_1e_4 = 0$ which implies $e_1 = e_1e_2$. Again multiplying both side of equation (1) by $e_3$, we have $e_1e_3 + u_1e_3 = e_2e_3 + u_2e_3 \Longrightarrow e_1 + e_1e_3 + u_1e_3 =$ $e_1 + e_2e_3 + u_2e_3 \Longrightarrow e_1 + u_1e_3 = e_2e_3 + (e_1 + u_2e_3)$. Similarly, by previous method we can prove that $(u_1e_3 + e_1)$ and $(u_2e_3 + e_1)$ are $k$-units of $S$. According to the condition $e_2e_3 = 0 \Longrightarrow e_1e_2 = e_2$. Hence $e_1 = e_2$. Thus $ind_k(S) = 1$. □

**Theorem 6.5.2.** *Let $S$ be a semiring such that every idempotent of $S$ has an orthogonal complement. Then $ind_k(S) = 1$ if and only if $S$ is abelian and for any $0 \neq e^2 = e \in S$, $e + u \neq v$ for any $u, v \in U_k(S)$.*

*Proof.* For every idempotent $e \in S$, there exists an idempotent $e_1 \in S$ such that $e + e_1 = 1$ and $ee_1 = e_1e = 0$. Now the rest of the proof is similar to the proof of Theorem 6.5.1. □

**Theorem 6.5.3.** *Let $S$ be a commutative antisimple $k$-semipotent semiring such that $J_l(S) \neq 0$ and every idempotent has an absorbing complement in $S$. Then $ind_k(S) = 1$ if and only if $S/J_l(S)$ is Boolean and $Id(J_l(S)) = \{0\}$.*

*Proof.* Let $S$ be a commutative antisimple $k$-semipotent semiring such that the condition holds. Let $e^2 = e \in S$ such that $e + u = v$ for any $u, v \in U_k(S)$. Hence $(e + u) + J_l(S) = v + J_l(S) \Longrightarrow (e + J_l(S)) + (u + J_l(S)) = (v + J_l(S)) \ldots (1)$. Since $u \in U_k(S)$, there exist $s_1, s_2 \in S$ such that $1 + s_1u = s_2u$. Hence $(1 + s_1u) + J_l(S) = s_2u + J_l(S) \Longrightarrow (1 + J_l(S)) + (s_1u + J_l(S)) = (s_2u + J_l(S))$. Since $S/J_l(S)$ is Boolean, $(u + J_l(S)) + (s_1u + J_l(S)) = (s_2u + J_l(S)) \Longrightarrow (u + J_l(S)) + (s_2u + J_l(S)) = (1 + J_l(S)) + (s_2u + J_l(S))$. Since $S$ is antisimple and $J_l(S) \neq 0$, from Note 2.3.4 it implies that $(u + J_l(S)) = (1 + J_l(S))$. Similarly, it can be proved that $v + J_l(S) = 1 + J_l(S)$. Hence from equation (1) it follows that $(e + J_l(S)) + (1 + J_l(S)) = (1 + J_l(S))$. Hence $(e + J_l(S)) = J_l(S) \Longrightarrow e \in J_l(S)$. Thus $e = 0$ according to the condition. Hence $ind_k(S) = 1$ from Theorem 6.5.1.

Conversely, let $ind_k(S) = 1$. Let $S/J_l(S)$ is not Boolean. Then there exists $a \in S$ such that $a^2 + J_l(S) \neq a + J_l(S)$ which implies that $a \neq 0$. Since $S$ is antisimple there exists $c \in S$ such that $a = c + 1$. We claim that $c^2 + c \notin J_l(S)$. If $c^2 + c \in J_l(S)$ then $c^2 + c + J_l(S) = J_l(S) \Longrightarrow c^2 + c + c + 1 + J_l(S) = c + 1 + J_l(S) \Longrightarrow (c + 1)^2 + J_l(S) = (c + 1) + J_l(S) \Longrightarrow a^2 + J_l(S) = a + J_l(S)$, this contradicts our assumption. Hence $c^2 + c \notin J_l(S) \Longrightarrow \overline{(c^2 + c)} \nsubseteq J_l(S)$. Since $S$ is $k$-semipotent there exists $e^2 = e \neq 0 \in S$ such that $e \in \overline{(c^2 + c)}$. So $e + s_1(c^2 + c) = s_2(c^2 + c)$ for some $s_1, s_2 \in S$. Hence $e + (es_1e)(e(c^2 + c)e) = (es_2e)(e(c^2 + c)e) \Longrightarrow e + (es_1e)(ece)(e(c + 1)e) = (es_2e)(ece)(e(c + 1)e)$, since $S$ is commutative. Hence $ece, e(c + 1)e$ are $k$-unit elements in $eSe$. Since $eSe$ is a nonzero subsemiring of $S$, by Lemma 6.3.11, it follows that $ind_k(eSe) \leq ind_k(S)$. Hence $ind_k(eSe) = 1$. Now $0 + e(c + 1)e = ece + e \ldots (1)$. Since $in_k(eSe) = 1$, from equation (1) it implies $e = 0$ which is a contradiction. Hence $S/J_l(S)$ is Boolean semiring. Let $e^2 = e \in J_l(S)$ and $b = 1 + e$. $\overline{(b)} = \{s \in S | s + s_1b = s_2b, s_1, s_2 \in S\}$. since $S$ is commutative, $\overline{(b)}$ is a $k$-ideal of $S$. If $\overline{(b)}$ is a proper $k$-ideal of $S$ then from Theorem 2.4.1 it follows that $b \in M$, where $M$ is a maximal $k$-ideal of $S$. Thus $1 + e \in M \Longrightarrow 1 \in M$, since $e \in J_l(S)$. This contradicts that $M$ is maximal $k$-ideal of $S$. Hence $1 \in \overline{(b)}$ which implies that $b \in U_k(S)$. So $b = 1 + e = 0 + (1 + e)$. Since $ind_k(S) = 1$, $e = 0$. Thus $Id(J_l(S)) = \{0\}$. □

**Lemma 6.5.4.** *If* $T = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$, *where* $A$ *and* $B$ *are two semirings such that* $ind_k(A) = ind_k(B) = 1$ *and* ${}_AM_B$ *is a bisemimodule with* $|M| = 2$, *then* $ind_k(T) = 2$.

*Proof.* Let ${}_AM_B$ be a bisemimodule with $|M| = 2$. So $M = \{0_M, m\}$.

**Case - I :** Suppose that $w + w = w$ for any $w \in M$.

Consider $\alpha_0 = \begin{bmatrix} 1_A & m \\ 0 & 2_B \end{bmatrix} \in T$. Then $\xi(\alpha_0) = \left\{ \begin{bmatrix} 0_A & 0_M \\ 0 & 1_B \end{bmatrix}, \begin{bmatrix} 0_A & m \\ 0 & 1_B \end{bmatrix} \right\}$. This implies that $ind_k(T) \geq |\xi(\alpha_0)| \geq |M| = 2$. For any $\alpha_1 = \begin{bmatrix} a & 0_M \\ 0 & b \end{bmatrix} \in T$, We have $\xi(\alpha_1) = \left\{ \begin{bmatrix} e & 0_M \\ 0 & f \end{bmatrix} : e \in \xi(a), f \in \xi(b) \right\}$. Hence $|\xi(\alpha_1)| \leq 1$, since $|\xi(a)| \leq ind_k(A) = 1$ and $|\xi(b)| \leq ind_k(B) = 1$. Consider any $\alpha_2 = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in T$. Now $w + m = m$ for any $w \in M$, since $|M| = 2$ and $m + m = m$. Hence $\xi(\alpha_2) = \left\{ \begin{bmatrix} e & w \\ 0 & f \end{bmatrix} \in T : e \in \xi(a), f \in \xi(b), w = ew + wf \right\}$. Since $|\xi(a)| \leq 1, |\xi(b)| \leq 1$ and $|M| = 2$, it follows that $|\xi(\alpha_2)| \leq 2$. Hence $ind_k(T) = 2$.

**Case - II :** Suppose that $w + w = 0_M$ for any $w \in M$.

Consider $\beta_0 = \begin{bmatrix} 1_A & 0_M \\ 0 & 2_B \end{bmatrix} \in T$. Then $\xi(\beta_0) = \left\{ \begin{bmatrix} 0_A & w \\ 0 & 1_B \end{bmatrix} : w \in M \right\}$. This implies that $ind_k(S) \geq |\xi(\beta_0)| \geq |M| = 2$. For any $\beta = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in T$, where $x$ is any element in $M$, $\xi(\beta) = \left\{ \begin{bmatrix} e & w \\ 0 & f \end{bmatrix} : e \in \xi(a), f \in \xi(b), w = ew + wf \right\}$. Since $|M| = 2$, $|\xi(a)| \leq 1$ and $|\xi(b)| \leq 1$, it follows that $|\xi(\beta)| \leq 2$. Hence $ind_k(T) = 2$. □

Let $S$ be a semiring and $M$ be the $S$-bisemimodule. The trivial extension of semiring $S$ by $M$ is the semiring $S_1 = S \propto M$ whose underlying semiring is $S \times M$ with addition defined by $(a, m) + (a_1, m_1) = (a + a_1, m + m_1)$ and multiplication defined by $(a, m)(a_1, m_1) = (aa_1, am_1 + a_1m)$.

**Lemma 6.5.5.** *Let $S$ be a semiring and $M$ be the $S$-semimodule. Let $S_1 = S \propto M$. Then $(u, m) \in U_k(S_1)$ if and only if $u \in U_k(S)$.*

*Proof.* Since $u \in U_k(S)$ there exist $s_1, s_2 \in S$ such that $1+s_1u = s_2u$ and $1+us_1 = us_2$. Now $(1, 0_M)+(s_1, s_1^2m+s_2^2m)(u, m) = (1, 0_M)+(s_1u, s_1m+us_1^2m+us_2^2m) = (s_2u, (1+us_1)s_1m + us_2^2m) = (s_2u, us_2s_1m + us_2^2m) = (s_2u, us_2(s_1 + s_2)m) \ldots (1)$. Again, $(s_2, s_1s_2m + s_2s_1m)(u, m) = (s_2u, s_2m + us_1s_2m + us_2s_1m) = (s_2u, (1 + us_1)s_2m + us_2s_1m) = (s_2u, us_2^2m+us_2s_1m) = (s_2u, us_2(s_1+s_2)m) \ldots (2)$. Hence from equations (1) and (2) we can say that $(u, m)$ is left $k$-unit in $S'$. Similarly, we can show that $(u, m)$ is right $k$-unit in $S_1$. Thus $(u, m) \in U_k(S')$.

Conversely let $(u, m) \in U_k(S_1)$. Then it can be easily verified that $u \in U_k(S)$ □

**Theorem 6.5.6.** *Let $S$ be a semiring and $M$ be the $S$-semimodule such that $(M, +)$ is not a group. Let $S_1 = S \propto M$. Then $ind_k(S_1) = 2$ if $|M| = 2$ and $ind_k(S) = 1$.*

*Proof.* Let $M = \{0_M, m\}$. Since $(M, +)$ is a monoid which is not a group, $m + m = m$. Hence $(1, m)(1, m) = (1, 1m + 1m) = (1, m + m) = (1, m)$. Thus $(1, m)$ is an idempotent in $S_1$. Take $\alpha = (2, m) \in S_1$. Then $(2, m) = (1, m) + (1, m) = (1, 0_M) + (1, m)$. Now Lemma 6.5.5 implies that $(1, m)$ is a $k$-unit in $S_1$ since 1 is a $k$-unit in $S$. Hence we have $\xi(\alpha) \supseteq \{(1, 0_M), (1, m)\}$. Hence $ind_k(S_1) \geq |\xi(\alpha)| \geq 2$. Consider, $\alpha_1 = (a, 0_M)$. Then $\xi(\alpha_1) = \{(e, 0_M) \in S_1 : e \in \xi(a)\}$ which implies that $|\xi(\alpha_1)| \leq 1$ since $|\xi(a)| \leq 1$. Take any $\alpha_2 = (a, m) \in S_1$. since $w + m = m$ for any $w \in M$, again using the Lemma 6.5.5 we get $\xi(\alpha_2) = \{(e, w) \in S_1 : e \in \xi(a), w = ew + ew\}$. Since $|\xi(a)| \leq 1$ and $|M| = 2$, we have $|\xi(\alpha_2)| \leq 2$. Hence $ind_k(S_1) \leq 2$. Thus $ind_k(S_1) = 2$. □

**Definition 6.5.7.** *A semiring $S$ is called an elemental semiring if the idempotents of $S$ are trivial and $1 + u = v$ for some $u, v \in U_k(S)$.*

**Theorem 6.5.8.** *Let $S$ be an abelian semiring such that every idempotent of $S$ has an orthogonal complement in $S$. Let $U_k(eS) \subseteq P(eS)$ for every idempotent $e^2 = e\ (\neq 0)$. Then $ind_k(S) = 2$ if and only if one of the following holds :*

(1) *$S$ is an elemental semiring.*

(2) *$S \cong A \times B$, where $A$ is an elemental semiring and $ind_k(B) = 1$.*

*Proof.* Let $S$ be an elemental semiring. Since $S$ has only trivial idempotents, we have $|\xi(a)| \leq 2$ for all $a \in S$. So $ind_k(S) \leq 2$. But $1 + u = v$ for some $u, v \in U_k(S)$ implies that $ind_k(S) \geq 2$, by Theorem 6.5.2. So $ind_k(S) = 2$. If (2) holds then $ind_k(S) = 2$ by (1) and Lemma 6.3.12.

Now we show the reverse part. Since $S$ is abelian and $ind_k(S) \neq 1$, there exists $0 \neq e = e^2 \in S$ such that $e + u = v$ for some $u, v \in U_k(S)$, by Theorem 6.5.2. Thus $e + eu = ev \ldots (i)$. Since $u, v \in U_k(S)$ and $S$ is abelian, so $eu, ev \in U_k(eS)$ and hence $ind_k(eS) \geq 2$, by Theorem 6.5.2. But $ind_k(eS) \leq ind_k(S) = 2$, by Lemma 6.3.10. So $ind_k(eS) = 2$. By assumption, let $e_1$ be the orthogonal complement of $e$. Let $A = eS$ and $B = e_1S$. Let us define $\phi : S \to A \times B$ by $\phi(s) = (es, e_1s)$ for all $s \in S$. Since $S$ is abelian and $e_1$ is the orthogonal complement of $e$, it can be proved that $\phi$ is one-one and onto homomorphism from $S$ to $A \times B$ such that $\phi(0) = (0, 0)$ and $\phi(1) = (e, e_1) =$ identity of $A \times B$. Hence $S \cong A \times B$. Thus by Lemma 6.3.13, it follows that $ind_k(S) = ind_k(A \times B)$. Now $e_1 = 0 + e_1$ be the $k$-unit clean expression of $e_1$ in $B$. Hence $ind_k(B) \geq |\xi(e_1)| \geq 1$. If $ind_k(B) > 2$, then there exists an element $b \in B$ such that $b$ has at least 3 $k$-unit clean expressions in $B$. Let $b = e_i + u_i$ for $i = 1, 2, 3$ such that $e_i \neq e_j$ for $i \neq j$, where $e_i \in Id(B)$ and $u_i \in U_k(B)$ for each $i$. Consider the element $(1_A, b) \in A \times B$. Now $(1_A, b) = (0_A, e_i) + (1_A, u_i)$ for each $i = 1, 2, 3$, we have $(0_A, e_i) \neq (0_A, e_j)$ for $i \neq j$. Hence $ind_k(A \times B) \geq |\xi(1_A, b)| \geq 3$ which contradicts that $ind_k(S) = 2$. Hence $ind_k(B) \leq 2 = finite = k$. Thus from Lemma 6.3.12, it follows that $ind_k(S) = ind_k(A)ind_k(B) = 2k$. Hence $2k = 2$ implies that $k = 1$. So $ind_k(B) = 1$. Let $f$ be a nontrivial idempotent in $A = eS$. According to the assumption, $f$ has an orgthogonal complement $f_1$ in $S$. Hence $F_1 = f_1e = ef_1$ is the orthogonal complement of $f$ in $A$. Similarly, $A \cong fA \times F_1A$. Since $f$ is the nontrivial idempotent of $A$, $f, F_1 \neq 0$. Multiplying both side of equation $(i)$ by $f$ and $F_1$, we have $f + f(eu) = f(ev) \ldots (ii)$ and $F_1 + F_1(eu) = F_1(ev) \ldots (iii)$. Since $S$ is abelian, $f(eu), f(ev) \in U_k(fA)$ and $F_1(eu), F_1(ev) \in U_k(F_1A)$. Thus $ind_k(fA) \geq 2$ and $ind_k(F_1A) \geq 2$ from Theorem 6.5.2. Again since $S$ is abelian, $fA = fS$ and $F_1A = F_1S$. So by assumption, $U_k(fA) \subseteq P(fA)$ and $U_k(F_1A) \subseteq P(F_1A)$. Thus by Lemma 6.3.10, we have $ind_k(fA), ind_k(F_1A) \leq ind_k(A) = 2$. So $ind_k(fA) =$

$ind_k(F_1A) = 2$. Thus Lemma 6.3.12 implies that $ind_k(A) = 2 \times 2 = 4$ which is a contradiction. Hence (1) follows if $e = 1$ or (2) follows if $e \neq 1$. □

**Lemma 6.5.9.** *If a semiring $S$ has an idempotent element $e$ and $e_1$ is its orthogonal complement in $S$ then* $S \cong \begin{bmatrix} eSe & eSe_1 \\ e_1Se & e_1Se_1 \end{bmatrix}$

*Proof.* Let us define a mapping $\phi : S \to \begin{bmatrix} eSe & eSe_1 \\ e_1Se & e_1Se_1 \end{bmatrix}$ by $\phi(s) = \begin{bmatrix} ese & ese_1 \\ e_1se & e_1se_1 \end{bmatrix}$ for all $s \in S$. Clearly the mapping is well-defined. Let $s_1, s_2 \in S$. Now $\phi(s_1 + s_2) = \begin{bmatrix} e(s_1+s_2)e & e(s_1+s_2)e_1 \\ e_1(s_1+s_2)e & e_1(s_1+s_2)e_1 \end{bmatrix} = \begin{bmatrix} es_1e + es_2e & es_1e_1 + es_2e_1 \\ e_1s_1e + e_1s_2e & e_1s_1e_1 + e_1s_2e_1 \end{bmatrix} = \begin{bmatrix} es_1e & es_1e_1 \\ e_1s_1e & e_1s_1e_1 \end{bmatrix} + \begin{bmatrix} es_2e & es_2e_1 \\ e_1s_2e & e_1s_2e_1 \end{bmatrix} = \phi(s_1) + \phi(s_2)$. Now $\phi(s_1s_2) = \begin{bmatrix} es_1s_2e & es_1s_2e_1 \\ e_1s_1s_2e & e_1s_1s_2e_1 \end{bmatrix}$. $\phi(s_1)\phi(s_2) = \begin{bmatrix} es_1e & es_1e_1 \\ e_1s_1e & e_1s_1e_1 \end{bmatrix} \begin{bmatrix} es_2e & es_2e_1 \\ e_1s_2e & e_1s_2e_1 \end{bmatrix} =$

$$\begin{bmatrix} es_1es_2e + es_1e_1s_2e & es_1es_2e_1 + es_1e_1s_2e_1 \\ e_1s_1es_2e + e_1s_1e_1s_2e & e_1s_1es_2e_1 + e_1s_1e_1s_2e_1 \end{bmatrix} = \begin{bmatrix} es_1(e+e_1)s_2e & es_1(e+e_1)s_2e_1 \\ e_1s_1(e+e_1)s_2e & e_1s_1(e+e_1)s_2e_1 \end{bmatrix} =$$

$\begin{bmatrix} es_1s_2e & es_1s_2e_1 \\ e_1s_1s_2e & e_1s_1s_2e_1 \end{bmatrix}$ since $e + e_1 = 1$. Hence $\phi(s_1s_2) = \phi(s_1)\phi(s_2)$. Clearly, $\phi(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\phi(1) = \begin{bmatrix} e & 0 \\ 0 & e_1 \end{bmatrix}$. Hence $\phi$ is a homomorphism. Let $\phi(s_1) = \phi(s_2)$ for $s_1, s_2 \in S$. Then $\begin{bmatrix} es_1e & es_1e_1 \\ e_1s_1e & e_1s_1e_1 \end{bmatrix} = \begin{bmatrix} es_2e & es_2e_1 \\ e_1s_2e & e_1s_2e_1 \end{bmatrix}$. Thus $es_1e = es_2e$, $es_1e_1 = es_2e_1$, $e_1s_1e = e_1s_2e$ and $e_1s_1e_1 = e_1s_2e_1$. Hence $es_1 = es_2$ and $e_1s_1 = e_1s_2$ which implies that $s_1 = s_2$. Hence $\phi$ is one-one. Let $A = \begin{bmatrix} ese & es_1e_1 \\ e_1s_2e & e_1s_3e_1 \end{bmatrix} \in \begin{bmatrix} eSe & eSe_1 \\ e_1Se & e_1Se_1 \end{bmatrix}$ for some $s, s_1, s_2, s_3 \in S$. Take $a = ese + es_1e_1 + e_1s_2e + e_1s_3e_1$. Then $a \in S$. Since $e + e_1 = 1$ and $ee_1 = e_1e = 0$, it can be easily verified that $\phi(a) = A$. Hence $\phi$ is onto. Thus $S \cong \begin{bmatrix} eSe & eSe_1 \\ e_1Se & e_1Se_1 \end{bmatrix}$. □

**Theorem 6.5.10.** *Suppose a semiring $S$ has a non-central idempotent which has* *an orthogonal complement in S. Then* $ind_k(S) = 2$ *if and only if* $S \cong \begin{bmatrix} A & M \\ O & B \end{bmatrix}$, *where A and B are two semirings such that* $ind_k(A) = ind_k(B) = 1$ *and* ${}_AM_B$ *is a bisemimodule with* $|M| = 2$.

*Proof.* Let $e$ be a non-central idempotent of $S$ and $e_1$ be its orthogonal complement in $S$. Then $e + e_1 = 1$ and $ee_1 = e_1e = 0$. Let $ind_k(S) = 2$. From Lemma 6.5.9, it follows that $S \cong \begin{bmatrix} eSe & eSe_1 \\ e_1Se & e_1Se_1 \end{bmatrix}$. Now both $eSe_1$ and $e_1Se$ can not be equal to zero otherwise $e$ becomes a central idempotent. Let $eSe_1 \neq 0$ and $e_1Se \neq 0$. Then there exists $exe_1$ $(\neq 0) \in eSe_1$ and there exists $e_1ye$ $(\neq 0) \in e_1Se$ for $x, y \in S$. Suppose there exists $ex_1e_1 \in eSe_1$ for some $x_1 \in S$ such that $ex_1e_1 \neq 0$ and $ex_1e_1 \neq exe_1$. Let $a = e + 1 + exe_1 + ex_1e_1$. Then $1 + (1 + exe_1 + ex_1e_1)(1 + exe_1 + ex_1e_1) = 1 + (1 + exe_1 + ex_1e_1 + exe_1 + ex_1e_1) = 2 + 2exe_1 + 2ex_1e_1 = 2(1 + exe_1 + ex_1e_1)$. Hence $(1 + exe_1 + ex_1e_1)$ is a $k$-unit in $S$. Similarly, $(1 + exe_1), (1 + ex_1e_1) \in U_k(S)$. Then $(e+exe_1)(e+exe_1) = e+exe_1$. Similarly, $(e+ex_1e_1) \in Id(S)$. Then $(e+exe_1)+(ex_1e_1+1) = (e+ex_1e_1)+(exe_1+1) = e+(exe_1+ex_1e_1+1)$ are three distinct $k$-unit clean expressions of $a$ which contradicts that $ind_k(S) = 2$. Hence $eSe_1 = \{0, exe_1\}$. Similarly, we can prove that $e_1Se = \{0, e_1ye\}$. Now we have the following cases :

**Case - I :** Suppose that $exe_1 + exe_1 = exe_1$ and $e_1ye + e_1ye = e_1ye$. Then $(1+exe_1+e_1ye)(1+exe_1+e_1ye) = 1+exe_1+e_1ye+exe_1+exe_1ye+e_1ye+e_1yexe_1 = 1+exe_1+exe_1exe_1+exe_1ye+e_1ye+e_1yexe_1+e_1yee_1ye$, since $ee_1 = e_1e = 0$. Hence $(1+exe_1+e_1ye)(1+exe_1+e_1ye) = 1+exe_1(1+exe_1+e_1ye)+e_1ye(1+exe_1+e_1ye) = 1+(exe_1+e_1ye)(1+exe_1+e_1ye)$. Thus $(1+exe_1+e_1ye)$ is left $k$-unit. Similarly, it can be shown that $(1+exe_1+e_1ye)$ is right $k$-unit. Thus $(1+exe_1+e_1ye) \in U_k(S)$. Now $(1+exe_1)(1+exe_1) = 1+exe_1+exe_1+exe_1exe_1 = 1+exe_1+exe_1exe_1 = 1+exe_1(1+exe_1) = 1+(1+exe_1)exe_1$. Hence $(1+exe_1) \in U_k(S)$. Similarly, $(1+e_1ye) \in U_k(S)$. Also $(e+exe_1), (e+e_1ye)$ are idempotents in $S$. Let $b = e+exe_1+e_1ye+1$. Then $(e+exe_1)+(1+e_1ye) = (e+e_1ye)+(1+exe_1) = e+(1+exe_1+e_1ye)$ are three distinct $k$-unit clean expressions of $b$. This is not possible as $ind_k(S) = 2$.

**Case - II :** Suppose that $exe_1+exe_1 = exe_1$ and $e_1ye+e_1ye = 0$. Then $exe_1ye =$

$(exe_1 + exe_1)ye = ex(e_1ye + e_1ye) = ex(0) = 0$. Similarly, $e_1yexe_1 = 0$. Now $1 + (1 + exe_1 + e_1ye)(1 + exe_1 + e_1ye) = 1 + 1 + exe_1 + e_1ye + exe_1 + exe_1ye + e_1ye + e_1yexe_1 = 1 + 1 + exe_1 + exe_1 + eye_1 + eye_1 = 2(1 + exe_1 + e_1ye)$. Thus $(1 + exe_1 + e_1ye) \in U_k(S)$. Hence $(e + exe_1) + (1 + e_1ye) = (e + e_1ye) + (1 + exe_1) = e + (1 + exe_1 + e_1ye)$ are three distinct $k$-unit clean expressions of $b$ which is not possible, since $ind_k(S) = 2$.

**Case - III :** Suppose that $exe_1 + exe_1 = 0$ and $e_1ye + e_1ye = e_1ye$. Similar to the Case - II, in this case also we have three distinct $k$-unit clean expressions of $b$.

**Case - IV :** Suppose that $exe_1 + exe_1 = 0$ and $e_1ye + e_1ye = 0$. Then $e + 1 = (e + exe_1) + (1 + exe_1) = (e + e_1ye) + (1 + e_1ye)$ are three distinct $k$-unit clean expressions in $S$ which contradicts that $ind_k(S) = 2$.

Now considering all possible cases, we have either $eSe_1 = \{0\}$ or $e_1Se = \{0\}$. Let $e_1Se = \{0\}$. Then $S \cong \begin{bmatrix} eSe & eSe_1 \\ 0 & e_1Se_1 \end{bmatrix}$. The next claim is that $ind_k(eSe) = ind_k(e_1Se_1) = 1$. Since $e = 0 + e$, $ind_k(eSe) \geq 1$. Similarly, $ind_k(e_1Se_1) \geq 1$. Again from Theorem 6.4.3, it folows that $ind_k(eSe) < ind_k\left(\begin{bmatrix} eSe & eSe_1 \\ 0 & e_1Se_1 \end{bmatrix}\right)$ and $ind_k(e_1Se_1) < ind_k\left(\begin{bmatrix} eSe & eSe_1 \\ 0 & e_1Se_1 \end{bmatrix}\right)$. Since $ind_k(S) = 2$, it follows from Lemma 6.3.13 that $ind_k\left(\begin{bmatrix} eSe & eSe_1 \\ 0 & e_1Se_1 \end{bmatrix}\right) = 2$. Hence $ind_k(eSe) = ind_k(e_1Se_1) = 1$.

Conversely, let $S \cong \begin{bmatrix} A & M \\ O & B \end{bmatrix}$, where $A$ and $B$ are two semirings such that $ind_k(A) = ind_k(B) = 1$ and ${}_AM_B$ be a bisemimodule with $|M| = 2$. Then $ind_k(S) = 2$ follows from Lemma 6.5.4. □

# Chapter 7

# Nil Clean Semiring

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## 7.1 Introduction

In the last several years there has been a growing interest in the structure of rings whose elements are sums of certain special elements. In 1954, Zelinsky [2] introduced the concept of 2-good rings where he defined that a ring $R$ is called 2-good if every element is a sum of two units of $R$. A ring $R$ is called (strongly) clean ring if every element is a sum of an idempotent and a unit (that commute with each other). In 1977 Nicholson [7] introduced the concept of clean rings. After that, in 1999 Nicholson [18] introduced the notion of strongly clean rings. Since then various works on different types of clean rings, have been obtained by many authors ([28], [43], [34], [31] [32]). In 2013, Diesl [45] introduced two interesting variants of the clean property of rings where he defined that an element of a ring is called (strongly) nil clean if it is a sum of an idempotent and a nilpotent element (that commute with each other) and a ring is called (strongly) nil clean if every element is (strongly) nil clean. Surprisingly, nil clean and strongly nil clean rings are naturally connected to clean and strongly clean rings. Diesl in [45] (Proposition 3.4), proved that every (strongly) nil clean ring is (strongly) clean ring. "Let $R$ be a nil clean ring and $n$ be a positive integer. Is $M_n(R)$ nil clean ?". To find the answer to this question, in [44], Breaz et al. proved that if $R$ is any commutative nil clean ring then $M_n(R)$ is nil clean. Later, Kosan et al.[51] continued the study of nil clean and strongly nil clean rings with a focus on the structure and construction of strongly nil clean rings and the question of when a matrix ring is nil clean. The algebraic theory of semirings has experienced remarkable growth in recent years. A semiring, which extends the concepts of a ring and a distributive lattice, has seen significant development. Motivated by the works ([7], [18]), we have already introduced the concept of clean semiring [62] and strongly clean semiring [63]. In this chapter, we have introduced the concept of nil clean semiring. The main motivation of this section is to give some important results related to nil clean semiring. We study the notion of exchange semiring and $k$-semipotent semiring to find its connection with nil clean semiring with the help of some other class of semirings.

## 7.2 Definition, Examples & Some Elementary Results

**Definition 7.2.1.** An element of a semiring $S$ is said to be nil clean if it can be written as the sum of an idempotent and a nilpotent element in $S$. A semiring $S$ is called a nil clean semiring if every element of $S$ is nil clean.

**Definition 7.2.2.** A semiring $S$ is said to be strongly nil clean if every element $a$ of $S$ can be written as $a = e + b$ for some idempotent $e \in S$ and $b \in N(S)$ such that $eb = be$.

**Example 7.2.3.**

*Consider $S = \{0, 1\}$. Define the two operations "+" and "." in $S$ as follows :*

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| . | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

*Then $(S, +, .)$ forms a semiring which is also known as a Boolean semifield. Let $T_2(S) = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in S\}$. Then $T_2(S)$ becomes a nil clean semiring with respect to usual matrix addition and multiplication.*

**Proposition 7.2.4.** (1) *Every homomorphic image of a nil clean semiring is nil clean.*

(2) *Let* $\{S_i : i = 1, 2, \ldots, n\}$ *be a finite family of semirings. Then the direct product of semirings* $S = \prod_{i=1}^{n} S_i$ *is nil clean if and only if each semiring* $S_i$ *is nil clean.*

Any arbitrary direct product of nil clean semirings is not nil clean which follows from the following example:

**Example 7.2.5.** *Let* $S = \mathbb{N}_0^+$, *be the set of all non-negative integers. Then* $I = m\mathbb{N}_0^+$ *becomes a* $k$*-ideal of* $S$ *for any* $m \in S$*. Now consider the arbitrary direct product of quotient semirings* $S' = \prod_{m=1}^{\infty} \mathbb{N}_0^+/2^m\mathbb{N}_0^+$*. Then each* $N_0^+/2^m\mathbb{N}_0^+$ *is nil clean semiring but the element* $([0], [2], [2], \ldots) \in S'$ *is not nil clean.*

**Remark 7.2.6.** *If* $\{S_\alpha\}$ *is a family of semirings for which there is a fixed number* $K$ *such that no nilpotent in any* $S_\alpha$ *has nilpotent index larger than* $K$ *then the direct product* $\prod_{\alpha} S_\alpha$ *is nil clean if and only if each* $S_\alpha$ *is nil clean.*

**Remark 7.2.7.** *Bourne in [1] showed that if* $n$ *is a nilpotent element with nilpotent index* $k$*, in a semiring* $S$ *then there exist* $n', n'' \in S$ *such that* $n + n' + nn' = n'' + nn''$ *and* $n + n' + n'n = n'' + n''n$*, where* $n' = \sum_{i=1}^{i=k-1} n^{2i}$ *and* $n'' = \sum_{i=1}^{i=k} n^{2i-1}$*. Hence* $n \in Q(S)$*. Since Proposition 6.3.7 implies that* $1 + Q(S) \subseteq U_k(S)$*, hence* $(1 + n) \in U_k(S)$*.*

**Proposition 7.2.8.** *Every* $k$*-ideal of a nil clean semiring is nil clean semiring.*

*Proof.* Let $I$ be a $k$-ideal of a nil clean semiring $S$. Let $a \in I$. Since $S$ is nil clean, $a = e + b$ where $e \in Id(S)$ and $b$ is a nilpotent element in $S$. For $b$, there exist $b_1, b_2 \in S$ such that $b + b_1 + bb_1 = b_2 + bb_2$ and $b + b_1 + b_1b = b_2 + b_2b$, which follows from Remark 7.2.7. Now $ea = e + eb \ldots (i)$, $eab_1 = eb_1 + ebb_1 \ldots (ii)$ and $eab_2 = eb_2 + ebb_2 \ldots (iii)$. Adding equations $(i)$ and $(ii)$, we have $ea + eab_1 = e + eb + eb_1 + ebb_1 = e + e(b + b_1 + bb_1) = e + eb_2 + ebb_2 = e + eab_2$ (from equation $(iii)$). Since $I$ is a $k$-ideal of $S$ and $a \in I$, it implies that $e \in I$, hence $b \in I$. Thus, $I$ is a nil clean semiring. $\square$

**Proposition 7.2.9.** *Let $I$ be a left or a right $k$-ideal of a semiring $S$ and $a \in I$. If $a = e + b$ with $eb = be$, is the strongly nil clean expression of $a$ in the semiring $S$ then $e, b \in I$.*

*Proof.* Let $I$ be a left $k$-ideal of the semiring $S$ and $a \in I$. Now $a = e + b$, $e^2 = e \in S$, $b \in N(S)$ with $eb = be$. From Remark 7.2.7, it follows that $b \in Q(S)$. Hence there exist $b_1, b_2 \in S$ such that $b + b_1 + b_1 b = b_2 + b_2 b$. Now $ea = e + eb = e + be$, $b_1 ea = b_1 e + b_1 eb = b_1 e + b_1 be$. Similarly, $b_2 ea = b_2 e + b_2 be$. $b_1 ea + ea = e + be + b_1 e + b_1 be = e + (b + b_1 + b_1 b)e = e + (b_2 e + b_2 be) = e + b_2 ea$. Hence, $e \in \overline{Sa} \subseteq I$. Since $I$ is a left $k$-ideal of $S$, we get $b \in I$. Hence the proof.

Similarly, the result is true when $I$ is a right $k$-ideal of $S$. □

**Corollary 7.2.10.** *If $S$ is a strongly nil clean semiring then every left and right $k$-ideal is strongly nil clean.*

*Proof.* The proof follows from the above Proposition 7.2.9 □

**Remark 7.2.11.** *If $A$ is a non-empty subsemiring of a semiring $S$. Then $\overline{A}$ is also a subsemiring of $S$.*

*Proof.* Since $A$ is non-empty and $A \subseteq \overline{A}$, $\overline{A}$ is also non-empty. Let $x, y \in \overline{A}$. Then $x + a_1 = a_2$ and $y + a_3 = a_4$, for some $a_1, a_2, a_3, a_4 \in A$. Thus, $(x + y) + (a_1 + a_3) = (a_2 + a_4)$. Since $A$ is a subsemiring of $S$, $(a_1 + a_3) \in A$ and $(a_2 + a_4) \in A$. Hence $(x+y) \in \overline{A}$. Let $x_1, y_1 \in \overline{A}$. Hence $x_1 + a_5 = a_6$ and $y_1 + a_7 = a_8$, for $a_5, a_6, a_7, a_8 \in A$. Hence $x_1 y_1 + x_1 a_7 + a_5 y_1 + a_5 a_7 = a_6 a_8$. Now $x_1 a_7 + a_5 a_7 = a_6 a_7$ and $a_5 y_1 + a_5 a_7 = a_5 a_8$. Hence $x_1 y_1 + a_6 a_7 + a_5 y_1 + a_5 a_7 = a_6 a_8 + a_5 a_7 \Longrightarrow x_1 y_1 + a_6 a_7 + a_5 a_8 = a_6 a_8 + a_5 a_7$. Since $A$ is a subsemiring of $S$, $a_6 a_7, a_5 a_8, a_6 a_8, a_5 a_7 \in A$ which implies that $x_1 y_1 \in \overline{A}$. So, $\overline{A}$ is a subsemiring of $S$. □

**Proposition 7.2.12.** *Let $S$ be a semiring and $e$ be any idempotent of $S$. Then $a \in \overline{eSe}$ is strongly nil clean in $S$ if and only if it is strongly nil clean in $\overline{eSe}$.*

*Proof.* Let $a \in \overline{eSe}$ is strongly nil clean in $S$. So, $a = e_1 + b$ for some idempotent $e_1$ and nilpotent $b$ of $S$ such that $e_1 b = b e_1$. Now $\overline{Se}$ is the left $k$-ideal of $S$ and

$\overline{eSe} \subseteq \overline{Se}$. Hence $a \in \overline{Se}$. From Proposition 7.2.9, it follows that $e_1, b \in \overline{Se}$. Thus, $a = e_1 + b \dots (1)$ where $e_1$ is an idempotent element in $\overline{Se}$ and $b$ is a nilpotent element in $\overline{Se}$. From Remark 7.2.7 we can say that there exist $b' = \sum_{i=1}^{i=k-1} b^{2i}$ and $b'' = \sum_{i=1}^{i=k} b^{2i-1}$ such that $b + b' + bb' = b'' + b''$ holds. Since $b \in \overline{Se}$ and $\overline{Se}$ is the left $k$-ideal of $S$, $b', b'' \in \overline{Se}$. Now $a = e_1 + b$ with $e_1 b = b e_1$, which implies $ae_1 + ae_1 b' = e_1 + e_1 b + e_1 b' + e_1 bb' = e_1 + e_1 b'' + e_1 bb'' = e_1 + ae_1 b'' \dots (2)$. Since $e_1 \in \overline{Se}$ and $b', b'' \in \overline{Se}$, $e_1 b', e_1 b'' \in \overline{Se}$. Since $\overline{eSe}$ is the right $k$-ideal of $\overline{Se}$, by using equation (1) and (2) we can say that $e_1, b \in \overline{eSe}$. Hence $a$ is strongly nil clean in $\overline{eSe}$. The reverse implication follows immediately. □

**Corollary 7.2.13.** *If $S$ is a strongly nil clean semiring and $e$ is any idempotent of $S$, then the closure of the corner semiring $\overline{eSe}$ is also strongly nil clean semiring.*

*Proof.* The proof follows from the Proposition 7.2.12. □

## 7.3 Connection With Some Classes of Semirings

**Theorem 7.3.1.** *If $S$ is a strongly nil clean semiring such that every idempotent has a strong absorbing complement in $S$ then $S$ is an exchange semiring.*

*Proof.* Let $a \neq 0 \in S$. Since $S$ is strongly nil clean, $a = e + b$, $e^2 = e$, $b \in N(S)$ with $eb = be$. From the Proposition 7.2.9, it follows that $e \in \overline{Sa}$ since $\overline{Sa}$ is the left $k$-ideal of $S$ containing $a$. Let $e_1$ be the absorbing complement of $e$. $e + e_1 = 1 \Longrightarrow e_1 + e + b = 1 + b \Longrightarrow e_1 + a = 1 + b \Longrightarrow e_1 + e_1 a = e_1(1 + b) \Longrightarrow e_1(1 + a) = e_1(1 + b)$. From Remark 7.2.7, it follows that, $b + b_1 + bb_1 = b_2 + bb_2 \dots (i)$ and $b + b_1 + b_1 b = b_2 + b_2 b \dots (ii)$ where $b_1, b_2 \in S$. Now, adding both side of equations $(ii)$ by 1, it implies that, $1 + s_1(1 + b) = s_2(1 + b)$, where $s_1 = b_2$ and $s_2 = (1 + b_1)$. Now, $e_1 + s_1(1+b)e_1 = s_2(1+b)e_1 \Longrightarrow e_1 + s_1(1+b)e_1 + (s_1+s_2)b = s_2(1+b)e_1 + (s_1+s_2)b \Longrightarrow e_1 + s_1((1 + b)e_1 + b) + s_2 b = s_2((1 + b)e_1 + b) + s_1 b \dots (iii)$. Since $e + e_1 = 1$ and $eb = be$, we have $eb + e_1 b = eb + be_1 \Longrightarrow b + e_1 b = b + be_1 \Longrightarrow b + e_1 b + e_1 = b + be_1 + e_1 \Longrightarrow b + e_1(1 + b) = b + (1 + b)e_1$. Hence from equation $(iii)$, we have

$e_1 + s_1(e_1(1+b)+b) + s_2b = s_2(e_1(1+b)+b) + s_1b \Longrightarrow e_1 + s_1e_1(1+b) + (s_1 + s_2)b = s_2e_1(1+b) + (s_1+s_2)b \Longrightarrow e_1 + s_1e_1(1+b) + (s_1+s_2)b + (s_1+s_2)(1+e) = s_2e_1(1+b) + (s_1+s_2)b + (s_1+s_2)(1+e)$. Since $e_1(1+a) = e_1(1+b)$, we have $e_1+s_1e_1(1+a)+(s_1+s_2)(1+a) = s_2e_1(1+a)+(s_1+s_2)(1+a)$. Thus $e_1 \in \overline{S(1+a)}$. Since $e_1$ is the strongly absorbing complement of $e$, $e+e_1 = 1$, $e+see_1 = e$, $e_1+e_1es = e_1$ for all $s \in S$ such that $ee_1 = e_1e$. Now $e+e_1 = 1$ implies that $1+see_1 = 1$ and $1+e_1es = 1$ for all $s \in S$ . By using $ee_1 = e_1e$, we have $e_1+se_1e = e_1$ and $e+ee_1s = e$ for all $s \in S$. Hence $e$ is also the absorbing complement of $e_1$. Thus $S$ is an exchange semiring. □

**Corollary 7.3.2.** *If $S$ is an abelian nil clean semiring such that every idempotent has an absorbing complement in $S$ then $S$ is an exchange semiring.*

*Proof.* Since $S$ is abelian, all idempotents of $S$ are in the center of $S$. Hence $S$ is strongly nil clean semiring. Thus, the proof follows from Theorem 7.3.1. □

The converse of Theorem 7.3.1 is not true in general, which follows from the following Examples

**Example 7.3.3.** (*i*) *Consider $S = \{0,1,2,3\}$ . Define the operations "+" and "." as follows :*

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 1 |
| 2 | 2 | 3 | 1 | 2 |
| 3 | 3 | 1 | 2 | 3 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 1 | 3 |
| 3 | 0 | 3 | 3 | 3 |

*Then $(S,+,.)$ is a commutative exchange semiring such that every idempotent of $S$ has an absorbing complement, but $S$ is not nil clean semiring.*

(*ii*) *Let $S = \mathbb{N}_0^+$ be the set of all non-negative integers. Define two operations + and . on $S$ by $a+b = max\{a,b\}$ and $a.b$= usual multiplication in $S$ for all $a, b \in S$. Then $S$ is a commutative exchange semiring but not nil clean semiring.*

**Definition 7.3.4.** *A semiring $S$ is called left (right) duo semiring if every left (right) ideal of $S$ is two-sided ideal and $S$ is called left (right) $k$-duo semiring if every left (right) $k$-ideal of $S$ is two sided $k$-ideal.*

**Remark 7.3.5.** *If $S$ is a left duo semiring then for each $a \in S$, $aS \subseteq Sa$.*

*Proof.* Let $a \in S$. Then $Sa$ is a left ideal containing $a$. Let $s$ be any element in $S$. Since $S$ is a left duo semiring $Sa$ becomes a two-sided ideal containing $a$. So, $as \in Sa$ for any $s \in S$. Hence $aS \subseteq Sa$. □

**Note 7.3.6.** *Let $S$ be a left duo semiring. If $a$ is a nilpotent element of $S$ then $sa$ becomes a nilpotent element in $S$ for any $s \in S$.*

*Proof.* Let $a^k = 0$ for some $k \in \mathbb{N}$. Now for any $s \in S$, $(sa)^k = (sa)(sa)(sa)\ldots(sa)$ ($k$ times). Since $S$ is a left duo semiring, by repeatedly using the Remark 7.3.5, we can get $(sa)^k = s_1 a^k = 0$ for some $s_1 \in S$. Hence $sa$ is a nilpotent element in $S$ for any $s$. □

**Definition 7.3.7.** *A semiring $S$ is said to be left (right) quasi duo semiring, if every maximal left (right) ideal of $S$ is two-sided ideal and $S$ is called left (right) $k$-quasi duo semiring if every maximal left (right) $k$-ideal of $S$ is two sided $k$-ideal.*

**Remark 7.3.8.** *A left $k$-duo semiring is always left $k$-quasi duo semiring but the converse is not true in general. For a left $k$-quasi duo semiring $S$, the Jacobson radical $J_l(S)$, is a two-sided ideal.*

**Example 7.3.9.** *Let $T_2(S)$ be the upper triangular matrix semiring over $S$, where $S$ is the semiring, defined in Example 7.2.3. Now* $M_1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ *and* $M_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ *are only two maximal left $k$-ideals in the semiring $T_2(S)$. It can be verified that they are also two-sided $k$-ideals in $T_2(S)$. Hence $T_2(S)$ is the example of left $k$-quasi duo semiring, but it is not left $k$-duo semiring since* $I = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ *is the left $k$-ideal but not a right $k$-ideal of $T_2(S)$.*

**Proposition 7.3.10.** *Let $S$ be a left $k$-quasi duo semiring. Then $S$ is $k$-semipotent, if $S$ is strongly nil clean semiring.*

*Proof.* Let $\overline{(x)_l}$ be a proper left $k$-ideal of $S$ such that $x \notin J_l(S)$. To show that $S$ is $k$-semipotent, we have to show that there exists $e^2 = e \neq 0$ such that $e \in \overline{(x)_l}$. Suppose, $\overline{(x)_l}$ contains no nonzero idempotent. Let $y \in \overline{(x)_l}$. Since $S$ is strongly nil clean semiring there exists idempotent $e_1$ and there exists nilpotent $b_1$ in $S$ such that $y = e_1 + b_1$ with $e_1b_1 = b_1e_1$. Now Proposition 7.2.9 implies that $e_1 \in \overline{(y)_l}$. Since $y \in \overline{(x)_l}$, $\overline{(y)_l} \subseteq \overline{(x)_l}$. Hence $e_1 \in \overline{(x)_l}$. According to our assumption, we have $e_1 = 0$. Hence $y = b_1$ is a nilpotent. Since $x \notin J_l(S)$, there exists a maximal left $k$-ideal $M$ of $S$ such that $x \notin M$. Hence, $M \subset \overline{M + (x)_l}$ which implies that $1 \in \overline{M + (x)_l}$. Thus there exist $m_1, m_2 \in M$ and $s_2, s_3 \in S$ such that $1 + m_1 + s_2x = m_2 + s_3x \implies (1 + n_2) + m_1 = m_2 + n_3 \ldots (1)$, where $s_2x = n_2$ and $s_3x = n_3$ are nilpotent elements of $S$. There exist $k_2, k_3 \in \mathbb{N}$ such that $n_2^{k_2} = 0$ and $n_3^{k_3} = 0$. Now $1 + n_2 \in U_k(S)$, follows from Remark 7.2.7. Suppose $1 + n_2 = u_2$. Hence we have $(u_2 + m_1)^{k_3} = (m_2 + n_3)^{k_3} \ldots (2)$. Since $S$ is left $k$-quasi duo semiring, $M$ is a two-sided $k$ ideal of $S$. Again using $n_3^{k_3} = 0$ in equation (2), we get $u_2^{k_3} \in M$. Since $u_2 \in U_k(S)$, $u_2^{k_3} \in U_k(S)$ from Theorem 1.3.24 which contradicts that $M$ is maximal left $k$-ideal of $S$. Thus our assumption is wrong. Hence there exists $e^2 = e \neq 0 \in \overline{(x)_l}$. Hence $S$ is $k$-semipotent. □

**Example 7.3.11.** *Let $T_2(S)$ be the upper triangular matrix semiring over $S$, where $S$ is the semiring, defined in Example 7.2.3. From Example 7.3.9 we can say that $T_2(S)$ is a left $k$-quasi duo semiring. Now $T_2(S)$ is also a strongly nil clean semiring since its each element is either an idempotent or a nilpotent element. Hence from Proposition 7.3.10, it follows that, $T_2(S)$ is an example of a $k$-semipotent semiring.*

**Corollary 7.3.12.** *Let $S$ be a left $k$-duo semiring. Then $S$ is $k$-semipotent, if $S$ is strongly nil clean semiring.*

*Proof.* Since every left $k$-duo semiring is left $k$-quasi duo semiring, the proof follows from Proposition 7.3.10. □

**Corollary 7.3.13.** *If $S$ is a commutative nil clean semiring, then $S$ is $k$-semipotent semiring.*

*Proof.* Since $S$ is commutative, $S$ is a left $k$-quasi duo semiring. Hence the proof follows from Proposition 7.3.10. □

**Definition 7.3.14.** *An element $a$ of a semiring $S$ is called unipotent if it can be written as $a = 1 + b$ for some nilpotent element $b$.*

**Lemma 7.3.15.** *If $S$ is a left duo semiring then all nilpotent elements are in $J_l(S)$.*

*Proof.* Let $x$ be a nilpotent element of $S$ such that $x \notin J_l(S)$. Then there exists a maximal left $k$-ideal $M$ of $S$ such that $x \notin M$. Now, $\overline{M + (x)_l}$ is a left $k$-ideal of $S$ properly containing $M$. Hence $1 \in \overline{M + (x)_l}$. Thus there exist $m_1, m_2 \in M$ and $s_1, s_2 \in S$ such that $1+(m_1+s_1x) = (m_2+s_2x) \Longrightarrow (1+s_1x)+m_1 = m_2+s_2x \ldots (1)$ . Let $k \in \mathbb{N}$ be the nilpotent index of $x$. Since, $S$ is left duo semiring, from Note 7.3.6, it follows that $(s_1x)^k = d_1x^k = 0$ and $(s_2x)^k = d_2x^k = 0$ for some $d_1, d_2 \in S$. Remark 7.2.7 implies that $(1 + s_1x), (1 + s_2x) \in U_k(S)$. Let $u_1 = 1 + s_1x$ and $u_2 = 1 + s_2x$. Thus from equation (1) we have $(u_1+m_1)^k = (m_2+s_2x)^k \ldots (2)$. Since $S$ is a left duo semiring, $M$ is a two-sided $k$-ideal of $S$. Using, the fact that $M$ is a two sided $k$-ideal and using $(s_1x)^k = 0 = (s_2x)^k$, in equation (2), we have $u_1^k \in M$. Since $u_1 \in U_k(S)$, $u_1^k \in U_k(S)$ from Theorem 1.3.24 which contradicts that $M$ is maximal left $k$-ideal of $S$. Thus $x \in J_l(S)$. □

**Proposition 7.3.16.** *Let $S$ be a left duo semiring such that every idempotent of $S$ has an orthogonal complement. Then a $k$-unit $u$ in $S$ is nil clean if and only if it is unipotent.*

*Proof.* Let $u \in U_k(S)$, such that $u = e + b$ for some idempotent $e$ and nilpotent $b$ in $S$. Let $1 \notin \overline{(e)_l}$. Then $\overline{(e)_l}$ is a proper left $k$-ideal. From Theorem 2.4.1 we can say that $e \in M$, where $M$ is a maximal left $k$-ideal of $S$. From Lemma 7.3.15, it follows that $b \in J_l(S) \subseteq M$. Hence $u \in M$, which contradicts that $M$ is maximal left $k$-ideal. Hence $1 \in \overline{(e)_l}$. Thus there exist $b_1, b_2 \in S$ such that $1 + b_1e = b_2e \ldots (1)$. Let $e_1$ be the orthogonal complement of $e$, $ee_1 = e_1e = 0$. Hence from equation (1) we have

$e_1 = 0$ which implies that $e = 1$. So, $u = 1 + b$ is unipotent. The reverse implication follows immediately. □

**Theorem 7.3.17.** *If $S$ is a commutative semiring with only trivial idempotents then $S$ is nil clean semiring if and only if*

(1) *$S$ is $k$-local semiring.*

(2) *Every $k$-unit element is unipotent.*

(3) *$J_l(S)$ is nil.*

*Proof.* Let $S$ be a nil clean semiring. Since every commutative semiring is left duo semiring, by Lemma 7.3.15 we can say that every nilpotent element is in $J_l(S)$. Let $x \notin J_l(S)$. Since $S$ is nil clean and $S$ has only trivial idempotents, $x = 1 + b$ for some nilpotent element $b$. From Remark 7.2.7, it implies that $x \in U_k(S)$. Thus $S$ is $k$-local semiring and if $u \in U_k(S)$, then from Proposition 7.3.16, it follows that $u$ is unipotent. Let $y \in J_l(S)$. Then $y = 0 + b_1 = b_1$ for some nilpotent $b_1 \in S$. Hence $J_l(S)$ is nil.

Conversely, let conditions (1) ,(2) and (3) hold. Let $a \in J_l(S)$. Then $a$ is nilpotent. Hence $a$ is a nil clean element since $a = 0 + a$. Let $b \notin J_l(S)$. Since $S$ is $k$-local semiring, $b \in U_k(S)$. Hence condition (2) implies that $b$ in unipotent. Thus $b = 1+n$ where $n$ is a nilpotent element. This implies that $S$ is nil clean semiring. □

**Theorem 7.3.18.** *Let $S$ be an abelian left $k$-quasi duo semiring such that every idempotent has an orthogonal complement. Then $S$ is a Boolean semiring if and only if $S$ is nil clean semiring and $J_l(S) = 0$.*

*Proof.* Let $S$ be a nil clean semiring and $J_l(S) = 0$. Let $n$ be a nilpotent element of $S$ with nilpotent index $k_1$. Our claim is that $n \in J_l(S)$. Let $n \notin J_l(S)$. Proposition 7.3.10 implies that $S$ is $k$-semipotent. Hence there exists $e^2 = e \neq 0$ such that $e \in \overline{(n)_l}$. So, $e + s_1 n = s_2 n$ for some $s_1, s_2 \in S$. Since $S$ is abelian, we have $e + (es_1e)(ene) = (es_2e)(ene)$. This gives $en^{k-1}e = en^{k-2}e = en^{k-3}e = \ldots = ene = 0$ which implies that $e = 0$ , a contradiction. Hence $n \in J_l(S)$. According to the condition $J_l(S) = 0$. So, $n = 0$. Since $S$ is nil clean semiring and 0 is the only nilpotent element, every element of $S$ is idempotent. Hence $S$ is Boolean semiring.

Conversely, let $S$ be Boolean semiring. Then clearly, $S$ is a nil clean semiring since every element of $S$ is idempotent. Let $x \in J_l(S)$ and $x_1$ be the orthogonal complement of $x$. Then $x + x_1 = 1 \ldots (1)$ and $xx_1 = x_1x = 0 \ldots (2)$. If $1 \notin \overline{(x_1)_l}$, then from Theorem 2.4.1 we get $x_1 \in M_1$, where $M_1$ is a maximal left $k$-ideal of $S$. Since $J_l(S) \subseteq M_1$, from equation (1) it follows that $1 \in M_1$, which contradicts that $M_1$ is a maximal left $k$-ideal of $S$. Hence $1 \in \overline{(x_1)_l}$. So there exist $s_1, s_2 \in S$ such that $1 + s_1x_1 = s_2x_1$. Thus by using equation (2), we have $x = 0$. Hence $J_l(S) = 0$. □

## 7.4 Extensions of Nil Clean Semiring

**Theorem 7.4.1.** *Let $S$ be a semiring and $n$ be a positive integer. Then $S$ is nil clean semiring if and only if $T_n(S)$ is nil clean, where $T_n(S)$ is the set all $n \times n$ upper-triangular matrices over $S$.*

*Proof.* Let $S$ be a nil clean semiring and $A \in T_n(S)$ where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$.

Now $a_{ij} = 0$ for $i > j$ and $a_{ij} = e_{ij} + n_{ij}$ for $i \leq j \leq n$, where $i \in \{1, 2, 3, \cdots n\}$. Hence we can write $A = \begin{bmatrix} e_{11} & e_{12} & e_{13} & \cdots & e_{1n} \\ 0 & e_{22} & e_{23} & \cdots & e_{2n} \\ 0 & 0 & e_{33} & \cdots & e_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e_{nn} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} & n_{13} & \cdots & n_{1n} \\ 0 & n_{22} & n_{23} & \cdots & n_{2n} \\ 0 & 0 & n_{33} & \cdots & n_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & n_{nn} \end{bmatrix} =$ $E + N$. Clearly, $E$ is an idempotent of $T_n(S)$. Let $n_{ii}$ be the nilpotent element of $S$ with nilpotent index $k_i$ for each $i = 1, 2, 3, \ldots, n$. Let $k = max\{k_1, k_2, \cdots k_n\}$. Then it can be verified that $N^l = 0$ where $l = nk$. Thus $N$ is a nilpotent element of $T_n(S)$. Hence, $A$ is a nil clean element which implies that $T_n(S)$ is nil clean semiring.

Conversely, let $T_n(S)$ be a nil clean semiring. Since $S$ is the homomomorphic image of $T_n(S)$, from (1) of Proposition 7.2.4, it follows that $S$ is nil clean semiring. □

**Theorem 7.4.2.** *Let* $S = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$, *where* $A$ *and* $B$ *are two semirings,* $M$ *be the* ${}_AM_B$ *bi-semimodule and* $N$ *be the* ${}_BN_A$ *bi-semimodule. Then* $S$ *becomes a semiring with respect to usual matrix addition and multiplication defined by* $\begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & m_2 \\ n_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & a_1m_2 + m_1b_2 \\ n_1a_2 + b_1n_2 & b_1b_2 \end{bmatrix}$ *for all* $a_1, a_2 \in A$, $b_1, b_2 \in B$, $m_1, m_2 \in M$ *and* $n_1, n_2 \in N$. *Then* $S$ *is a nil clean semiring if and only if* $A$ *and* $B$ *are nil clean semirings*

*Proof.* Let $A$ and $B$ be two nil clean semirings. Let $T = \begin{bmatrix} a_1 & m_1 \\ n_1 & b_1 \end{bmatrix} \in S$. Now $a_1 = e_1 + x_1$ and $b_1 = e_2 + x_2$, where $e_1^2 = e_1 \in A$, $x_1$ is a nilpotent element in $A$ and $e_2^2 = e_2 \in B$, $x_2$ is a nilpotent element in $B$. Hence we can write $T = \begin{bmatrix} e_1 & 0_M \\ 0_N & e_2 \end{bmatrix} + \begin{bmatrix} x_1 & m_1 \\ n_1 & x_2 \end{bmatrix} = E + N$. Suppose the nilpotent index of $x_1$ and $x_2$ are $k_1$ and $k_2$ respectively. Let $k = max\{k_1, k_2\}$. Hence $x^k = y^k = 0$. According to the multiplication rule, defined in the semiring $S$, $N^k = \begin{bmatrix} x_1^k & m_2 \\ n_2 & y_1^k \end{bmatrix} = \begin{bmatrix} 0_A & m_2 \\ n_2 & 0_B \end{bmatrix}$ for some $m_2 \in M$ and for some $n_2 \in N$. Hence $N^{2k} = \begin{bmatrix} 0_A & m_2 \\ n_2 & 0_B \end{bmatrix} \begin{bmatrix} 0_A & m_2 \\ n_2 & 0_B \end{bmatrix} = \begin{bmatrix} 0_A & 0_M \\ 0_N & 0_B \end{bmatrix}$. Hence if $l = 2k$, then it can be proved that $N^l = 0$. Thus, $N$ is a nilpotent element in $S$. Clearly, $E$ is an idempotent element in $S$. Hence $S$ is nil clean semiring.

Conversely, let $S$ be a nil clean semiring. Since, $A$ and $B$ are homomorphic images of $S$, from (1) of Proposition 7.2.4, it follows that $A$ and $B$ are two nil clean semirings. □

**Example 7.4.3.** *Consider the semiring* $S$, *where* $S$ *is defined in Example 2.2.3(iv). Then* $S$ *is a commutative nil clean semiring. But the* $2 \times 2$ *matrix semiring over* $S$, $M_2(S)$ *is not nil clean semiring since* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *is not a nil clean element in* $M_2(S)$. *Hence, the matrix semiring of a commutative nil clean semiring is not nil clean.*

**Proposition 7.4.4.** *[24] Let* $S$ *be a semiring and* $n$ *be a positive integer. If* $I$ *is an ideal of* $S$ *then* $M_n(I) = \{[a_{ij}] \in M_n(S) | a_{ij} \in I$ *for all* $1 \leq i, j \leq n\}$ *is an ideal of*

*$M_n(S)$. Moreover every ideal of $M_n(S)$ is of this form, for a unique $I$.*

**Proposition 7.4.5.** *[38] Let $A = (a_{ij}) \in M_n(S)$ where $S$ is a commutative semiring. If every element of $S$ is nilpotent then the matrix $A$ is nilpotent.*

**Theorem 7.4.6.** *Let $S$ be a commutative nil clean semiring such that every idempotent has an orthogonal complement. Let $J(M_n(S))$ be the intersection of all maximal $k$-ideals of $M_n(S)$. Then $J(M_n(S))$ is nil.*

*Proof.* Since $S$ is a commutative nil clean semiring and $J_l(S)$ is the $k$-ideal of $S$, from Proposition 7.2.8, it follows that $J_l(S)$ is also nil clean. Let $a \in J_l(S)$. Then there exists an idempotent $e \in J_l(S)$ and nilpotent element $b \in J_l(S)$ such that $a = e + b$. Let $e_1$ be the orthogonal complement of $e$ in $S$, Then $e + e_1 = 1 \ldots (1)$ and $ee_1 = e_1 e = 0 \ldots (2)$. If $1 \notin \overline{(e_1)}$, then from Theorem 2.4.1, we get $e_1 \in M_1$, where $M_1$ is a maximal $k$-ideal of $S$. Since $J_l(S) \subseteq M_1$, from equation (1) it follows that $1 \in M_1$, which contradicts that $M_1$ is a maximal $k$-ideal of $S$. Hence $1 \in \overline{(e_1)}$. So there exist $s_1, s_2 \in S$ such that $1 + s_1 e_1 = s_2 e_1$. Thus by using equation (2), we have $e = 0$. Hence, $a$ is a nilpotent element in $S$. Thus, $J_l(S)$ is nil. Since $J(M_n(S))$ be the proper $k$-ideal of $M_n(S)$, from Proposition 7.4.4 it follows that $J(M_n(S)) = M_n(I)$ for some ideal $I$. Also, it can be easily shown that $I$ is a proper $k$-ideal of $S$. Our claim is that $I \subseteq J_l(S)$. Suppose, $I \nsubseteq J_l(S)$. Then there exists $x \in I$ such that $x \notin J_l(S)$. Hence there exists a maximal $k$-ideal $M$ such that $x \notin M$. Thus $M \subset \overline{M + (x)}$. So, $1 \in \overline{M + (x)}$ which implies that $1 + m_1 + s_1 x = m_2 + s_2 x \ldots (1)$ where $m_1, m_2 \in M$ and $x_1, x_2 \in S$. Again by using the Proposition 7.4.4, we can say that $M_n(M)$ is an ideal in $M_n(S)$. Since $M$ is a maximal $k$-ideal of $S$, $M_n(M)$ is the proper $k$-ideal of $M_n(S)$. Hence $M_n(M) \subseteq L$ from Theorem 2.4.1, where $L$ is a maximal $k$-ideal of $M_n(S)$. Let $E_{ij}$ denotes the matrix of $M_n(S)$ having the $(i, j)$ entry equals to 1 and all other entries equal to 0 for $1 \leq i, j \leq n$. Let $M_{ij}$ denotes the matrix of $M_n(S)$ whose $(i, i)$ entry is $m_j$ and all other entries equal to 0 and $A_{ij}$ denotes the matrix whose $(i, i)$ entry is $s_j x$ and all other entries equal to 0 for $1 \leq i \leq n$ and $j = 1, 2$. Now, by using equation (1), we can write $E_{ii} + E_{ii} M_{i1} E_{ii} + E_{ii} A_{i1} E_{i1} = E_{ii} M_{i2} E_{ii} + E_{ii} A_{i2} E_{ii}$ for $1 \leq i \leq n$. Since $M_{ij} \in M_n(M) \subseteq L$ and $A_{ij} \in J(M_n(S)) \subseteq L$ for $1 \leq i \leq n$ and for $j = 1, 2$, it follows that $E_{ii} \in L$ for each $i$. Thus $E_{11} + E_{22} + \ldots + E_{nn} = I \in L$ which contradicts that $L$ is a maximal $k$-ideal of $M_n(S)$. Hence $I \subseteq J_l(S)$ which implies that $J(M_n(S)) \subseteq M_n(J_l(S))$. Since $S$ is commutative semiring and $J_l(S)$ is nil, from Proposition 7.4.5, it follows that $J(M_n(S))$ is nil. $\square$

# Chapter 8

# On Nil Clean Index of Semirings

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# On Nil Clean Index of Semirings

## 8.1 Introduction

Lee T. K., Zhou, Y introduced the concept of the clean index of a ring in [41] and [47]. In [55] Basnet et al. defined the weakly clean index of a ring $R$ and in [48] they introduced the concept of nil clean index of a ring. Inspired by these works, in chapter 6 we have already introduced the notion of $k$-unit clean index of a semiring. In this chapter we have introduced the notion of nil clean index of a semiring. For an element $a$ in a semiring $S$, let $\eta(a) = \{e \in S : e^2 = e, a = e + n$ for some nilpotent element $n \in S\}$ and nil clean index of $S$, denoted $nin(S)$, is defined by $nin(S) = sup\{|\eta(a)| : a \in S\}$. We have obtained some results related to nil clean index of a semiring and established a connection between $k$-unit clean index and nil clean index of a semiring. Finally, we have characterized the semirings of nil clean indices 1 and 2, with the help of some other class of semirings.

## 8.2 Elementary Results

**Lemma 8.2.1.** (1) *Let $S$ be a commutative semiring and $n$ be any nilpotent element of $S$. Then $nin(S) \geq 1$ and $|\eta(n)| = 1$.*

(2) *If $f : S \to S$ is a homomorphism, then $e \in \eta(a)$ implies $f(e) \in \eta(f(a))$, and for converse part $f$ must be isomorphism.*

(3) *If a semiring $S$ has atmost $n$ idempotent elements, then $nin(S) \leq n$.*

*Proof.* (1) Since $n$ is a nilpotent element of $S$, $n = 0 + n$. Hence $0 \in \eta(n)$. So, $nin(S) \geq 1$. Let $e$ be another idempotent of $S$ such that $e \in \eta(n)$. Hence, $n = e + n_1$, where $n_1$ is a nilpotent element in $S$. Suppose, there exists a positive integer $k$ such that $n^k = {n_1}^k = 0$. Hence $n^k = (e+n_1)^k \Longrightarrow n^k = e + \binom{k}{1}en_1 + \binom{k}{2}en_1^2 + \binom{k}{3}en_1^3 + \ldots + \binom{k}{k-1}en_1^{k-1} + n_1^k \Longrightarrow 0 = e[1 + \binom{k}{1}n_1 + \binom{k}{2}n_1^2 + \binom{k}{3}n_1^3 + \ldots + \binom{k}{k-1}n_1^{k-1}] = eu$. Since every commutative semiring is a duo semiring, from Lemma 7.3.15, it follows that $n_1 \in J_l(S)$. Thus $u = 1 + b$ for some $b \in J_l(S)$. If $u \notin U_k(S)$ then $\overline{(u)}$ is a proper $k$-ideal of $S$. From Theorem 2.4.1, we can say that $u \in M$, where $M$ is a maximal $k$-ideal of $S$. This implies that $1 \in M$ since $b \in J_l(S) \subseteq M$. This is a contradiction since $M$ is maximal $k$-ideal of $S$. Hence $u$ is a $k$-unit in $S$. Hence there exist $s_1, s_2 \in S$ such that $1 + us_1 = us_2 \Longrightarrow e + eus_1 = eus_2 \Longrightarrow e = 0$. Hence $|\eta(n)| = 1$.

(2) The proof is straightforward and (3) follows from the definition of nil clean index of semiring. □

**Lemma 8.2.2.** *Let $S$ be a commutative additively cancellative semiring such that every idempotent has a complement in $S$ and every $k$-unit is unipotent. If $I$ is a nonzero nil $k$-ideal and $nin(S) \leq n$ then every idempotent of $S/I$ can be lifted to atmost $n$ idempotents of $S$.*

*Proof.* Let $(x + I)$ be an idempotent element in $S/I$ for some $x \in S$. If possible $x + I = e_x + I$, for some idempotent $e_x \in S$. Let $e_1$ be the complement of $e_x$ in $S$. Hence $(x + e_1) + I = 1 + I \ldots (1)$. Since, $I$ is a nil $k$-ideal, every element of $I$ is nilpotent element in $S$. Since $S$ is commutative, Lemma 7.3.15 implies that $I \subseteq J_l(S)$. Now $(x + e_1) + I$ is a $k$-unit in $S/I$ which follows from equation (1). Hence Lemma 6.3.1 implies that $(x+e_1) \in U_k(S)$. Since every $k$-unit is unipotent, $(x+e_1) = (1+b)$ for some nilpotent element $b \in S$. Hence $x + 1 = 1 + (e_x + b)$. Since $S$ is additively cancellative, $x = e_x + b$, which implies $e_x \in \eta(x)$. Since $|\eta(x)| \leq nin(S) \leq n$, there are atmost $n$ such idempotents. □

**Lemma 8.2.3.** *Let $T$ be a subsemiring of a semiring $S$, where $S$ and $T$ may or may not share the same identity, then $nin(T) \leq nin(S)$.*

*Proof.* Since, $T$ is a subsemiring of $S$, all the idempotents and nilpotent elements of $T$ are idempotents and nilpotents of $S$ respectively. If $e^2 = e \in T$ such that $e \in \eta_T(a)$, where $a \in T$, then there exists a nilpotent element $b \in T$ such that $a = e + b$. Also, $e, b \in S$. Hence $e \in \eta_S(a)$. Thus $\eta_T(a) \subseteq \eta_S(a)$. So, $sup_{a\in T}|\eta_T(a)| \leq sup_{a\in T}|\eta_S(a)| \leq sup_{a\in S}|\eta_S(a)|$. Thus, $nin(T) \leq nin(S)$. □

**Lemma 8.2.4.** *If $S = T_1 \times T_2$ be the direct product of two semirings $T_1$ and $T_2$, then $nin(S) = nin(T_1)nin(T_2)$.*

*Proof.* The proof is same as in [48]. □

**Lemma 8.2.5.** *Let $(A, +_1, \cdot_1)$ and $(B, +_2, \cdot_2)$ be two semirings such that $A \cong B$. If $nin(A) = n$ for some $n \in \mathbb{N}$, then $nin(B) = n$.*

*Proof.* The proof of the above Lemma is similar to the proof of Lemma 6.3.13, since definition of $nin(S)$ is similar to that of $ind_k(S)$ where $k$-unit element is replaced by nilpotent element of $S$. □

From Remark 6.4.5 and Lemma 6.4.6 we can say that $nin(S)$ is a cardinal number. Since $\aleph_0$ is also a cardinal number ([37]), it may be possible that there exists a semiring with $nin(S) = \aleph_0$.

**Theorem 8.2.6.** *Let $S$ be a semiring such that $nin(S)$ = any finite natural number or $\aleph_0$. Then $ind_k(S) \geq nin(S)$, where $ind_k(S)$ is the $k$-unit clean index of $S$.*

*Proof.* Definition of $ind_k(S)$ is similar to that of $nin(S)$ where nilpotent element is replaced by $k$-unit. Let $nin(S) = k$ where $k \in \mathbb{N}$. Then there exists an element $x \in S$ such that $x = e_i + n_i$ for $i = 1, 2, 3, \ldots, k$ where $e_i$ is idempotent and $n_i$ is nilpotent for each $i$ and $e_i \neq e_j$ for $i \neq j$. Hence $x + 1 = e_i + (n_i + 1)$. From Remark 7.2.7, it follows that $(n_i + 1) \in U_k(S)$ for each $i$. Hence $e_i \in \xi(x + 1)$ for each $i$. Thus $ind_k(S) \geq k = nin(S)$. Let $nin(S) = \aleph_0$. If $ind_k(S) < \aleph_0$ then $ind_k(S)$ becomes a natural number, since $\aleph_0$ is the smallest infinite cardinal number ([37]). Let $ind_k(S) = n$ where $n \in \mathbb{N}$. Since $nin(S) = \aleph_0$, there exists at least one element $a \in S$ such that $|\eta(a)| > n$. Then similarly by the previous argument we can show that $|\xi(a+1)| > n$ which contradicts that $ind_k(S) = n$. Hence $ind_k(S) \geq \aleph_0 = nin(S)$. □

## 8.3 Nil Clean Index of Formal Triangular Matrix Semiring

**Lemma 8.3.1.** (1) *Let* $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$*, where* $A$*,* $B$ *be two semirings ,* $(M, +)$ *be a nontrivial cyclic monoid which is not a group and* ${}_A M_B$ *bi-semimodule. Let* $nin(A) = n$ *and* $nin(B) = m$*. Then* $nin(S) \geq |M|$*.*

(2) *If* $(M, +)$ *is a finite cyclic monoid which is not a group and every idempotent in* $A$ *has an orthogonal complement then* $nin(S) \geq n + [n/2)(|M| - 1)$*, where* $[n/2)$ *denotes the least integer greater than or equal to* $n/2$*.*

*Proof.* (1) Since $M$ is finite cyclic monoid, there exists $\alpha \neq 0_M \in M$ such that $M = \{0_M, \alpha, \ldots, m\alpha, \ldots, (m + r - 1)\alpha\}$, where $m \in \mathbb{N}_0$ is the index of $\alpha$ and $r \in \mathbb{N}$ is the period of $\alpha$. Let $q = |M| = m + r$. Consider the element $T = \begin{bmatrix} 1_A & (m + r - 1)\alpha \\ 0 & 0_B \end{bmatrix}$. Now for any $w \in M$ there exists $w_1 \in M$ such that $w + w_1 = (m + r - 1)\alpha$. Hence $T = \begin{bmatrix} 1_A & w \\ 0 & 0_B \end{bmatrix} + \begin{bmatrix} 0_A & w_1 \\ 0 & 0_B \end{bmatrix}$ = idempotent + nilpotent in $S$, where $w$ is any element of $M$ and $w_1$ is an element of $M$ corresponding to $w$, such that $w + w_1 = (m + r - 1)\alpha$. Thus $\{\begin{bmatrix} 1_A & w \\ 0 & 0_B \end{bmatrix} | w \in M\} \subseteq \eta(T)$. Hence $nin(S) \geq |\eta(T)| \geq (m + r) = |M|$.

Let $M$ be an infinite cyclic monoid and $a$ be the generator of m. Then $(M, +) \cong (\mathbb{N}_0^+, +)$. Hence $|M| = |\mathbb{N}_0^+| = \aleph_0$. Now $nin(S) = sup\{|\eta(\alpha)| : \alpha \in S\}$. Since the cardinality of any set is obviously a cardinal number, follows from Remark 6.4.5, $|\eta(\alpha)|$ is a cardinal number for each $\alpha \in S$. Hence Lemma 6.4.6 implies that $nin(S)$ is also a cardinal number. Suppose $nin(S) < \aleph_0$. Since $\aleph_0$ is the smallest infinite cardinal number ([37]), $nin(S)$ is the finite cardinal number. Hence there exists $m \in \mathbb{N}$ such that $nin(S) = m$. Let $q \in \mathbb{N}$ such that $q > m$. Consider the element $\alpha_q = \begin{bmatrix} 1_A & qa \\ 0 & 0_B \end{bmatrix} \in S$. Then $\eta(\alpha_q) \supseteq \left\{ \begin{bmatrix} 1_A & ia \\ 0 & 0_B \end{bmatrix} \in S | i = 0, 1, 2, 3, \ldots q \right\} = T$. Since $M$ is infinite cyclic monoid and it is not a group, elements in $T$ are all distinct. Hence $nin(S) \geq |\eta(\alpha_q)| \geq (q + 1) > m$ which contradicts that $nin(S) = m$. So,

$nin(S) \geq \aleph_0 \geq |\mathbb{N}_0^+| = |M|$.

(2) Since $nin(A) = n$, there exists $a \in A$ such that $a = e_i + n_i$ for $(i = 1, 2, 3, \ldots, n)$, where $e_i \neq e_j$ for $i \neq j$. Let $E_i$ be the orthogonal complement of $e_i$ for $(i = 1, 2, 3, \ldots, n)$. Let $e^2 = e \in A$ and $E$ be the orthogonal complement of $e$. Now, $(M, +) = eM \oplus EM$. Lemma 6.4.8 implies that $(M, +)$ is indecomposable. Hence either $eM = M$ or $EM = M$. we can assume $e_1M = M$, $e_2M = M, \ldots, e_sM = M$ and $E_{s+1}M = M$, $E_{s+2}M = M, \ldots, E_nM = M$ where $s \in \{1, 2, 3, \ldots, n\}$.

**Case-1:** $s \geq n - s$ (i.e, $s \geq [n/2)$). Then for $T = \begin{bmatrix} a & (m+r-1)\alpha \\ 0 & 0_B \end{bmatrix}$, we have $\eta(T) \supseteq \{\begin{bmatrix} e_i & 0 \\ 0 & 0_B \end{bmatrix}, \begin{bmatrix} e_j & w \\ 0 & 0_B \end{bmatrix}, 1 \leq i \leq n, 1 \leq j \leq s, w \neq 0_M \in M\}$. Then $nin(S) \geq n + s(q-1) \geq n + [n/2)(|M| - 1)$.

**case-II:** $n - s \geq s \Longrightarrow n/2 \geq s \Longrightarrow n - s \geq n/2$ (i.e, $n - s \geq [n/2)$). Then for $T_1 = \begin{bmatrix} a & (m+r-1)\alpha \\ 0 & 1_B \end{bmatrix}$, we have $\eta(T_1) \supseteq \{\begin{bmatrix} e_i & 0 \\ 0 & 1_B \end{bmatrix}, \begin{bmatrix} e_j & w \\ 0 & 1_B \end{bmatrix}, 1 \leq i \leq n, s+1 \leq j \leq n, w \neq 0_M \in M\}$. Hence $nin(S) \geq n + (n-s)(q-1) \geq n + [n/2)(q-1) \geq n + [n/2)(|M| - 1)$. If we take $E_1M = M$, $E_2M = M, \ldots, E_sM = M$ and $e_{s+1}M = M, \ldots, e_nM = M$ then similarly we can prove that $nin(S) \geq n + [n/2)(|M| - 1)$. □

**Theorem 8.3.2.** *Let $A$ and $B$ be two semirings and ${}_AM_B$ be a nontrivial bi-semimodule. If $S = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ is a formal triangular matrix semiring, then $nin(A) < nin(S)$ and $nin(B) < nin(S)$.*

*Proof.* Let $nin(A) = n$. Hence there exists an element $a \in A$ such that $a = e_i + n_i$, $e_i \neq e_j$ for $i \neq j$, where $e_i \in Id(A)$ and $n_i$ are nilpotent elements of $A$ for $i = 1, 2, 3, \ldots, n$.

**Case - I:** Let $e_1M = 0_M$. Let $m \neq 0_M$. Then $e_1m = 0_M$. Consider the element $A_1 = \begin{bmatrix} a & m \\ 0 & 1_B \end{bmatrix}$. Then $A_1 = \begin{bmatrix} e_i & 0_M \\ 0 & 1_B \end{bmatrix} + \begin{bmatrix} n_i & m \\ 0 & 0_B \end{bmatrix}$ for $(i = 1, 2, 3, \ldots, n)$. Also, $A_1 = \begin{bmatrix} e_1 & m \\ 0 & 1_B \end{bmatrix} + \begin{bmatrix} n_1 & 0_M \\ 0 & 0_B \end{bmatrix}$. Since, $e_i \in Id(A)$ for each $i = 1.2, 3, \ldots, n$ and

$e_1M = 0_M$, $\begin{bmatrix} e_i & 0_M \\ 0 & 1_B \end{bmatrix}$ and $\begin{bmatrix} e_1 & m \\ 0 & 1_B \end{bmatrix} \in Id(S)$ for each $i = 1, 2, 3, \ldots, n$. Again, since $n_i$ are nilpotent elements of $A$ for each $i$, $\begin{bmatrix} n_i & m \\ 0 & 0_B \end{bmatrix}$ are nilpotent elements in $S$ for each $i$ and for all $m \in M$. So, $\eta(A_1) \supseteq \left\{ \begin{bmatrix} e_i & 0_M \\ 0 & 1_B \end{bmatrix} (i = 1, 2, 3, \ldots, n), \begin{bmatrix} e_1 & m \\ 0 & 1_B \end{bmatrix} \right\}$. Hence, $nin(S) \geq |\eta(A_1)| \geq (n+1) > n$.

**Case - II** Let $e_1M \neq 0_M$. Then there exists $m_1 \in M$ such that $e_1m_1 \neq 0_M$. Let $B_1 = \begin{bmatrix} a & e_1m_1 \\ 0 & 0_B \end{bmatrix} = \begin{bmatrix} e_i & 0_M \\ 0 & 0_B \end{bmatrix} + \begin{bmatrix} n_i & e_1m_1 \\ 0 & 1_B \end{bmatrix}$ for $(i = 1, 2, 3, \ldots, n)$. Also $B_1 = \begin{bmatrix} e_1 & e_1m_1 \\ 0 & 0_B \end{bmatrix} + \begin{bmatrix} n_1 & 0_M \\ 0 & 0_B \end{bmatrix}$. Thus $\eta(B_1) \supseteq \left\{ \begin{bmatrix} e_i & 0_M \\ 0 & 0_B \end{bmatrix} (i = 1, 2, 3, \ldots, n), \begin{bmatrix} e_1 & e_1m_1 \\ 0 & 0_B \end{bmatrix} \right\}$ In this case also, $nin(S) \geq |\eta(B_1)| \geq (n+1) > n$. Similarly, it can be proved that $nin(B) < nin(S)$. □

## 8.4 Characterization of Semirings of Nil clean Indices 1 & 2

**Lemma 8.4.1.** *$nin(S) = 1$, if and only if $S$ is abelian and for any $0 \neq e^2 = e \in S$, $e + n_1 \neq n_2$ for any nilpotent $n_1, n_2 \in S$.*

*Proof.* Suppose $nin(S) = 1$. Let $e^2 = e \in S$ and $e_1^2 = e_1$ be the orthogonal complement of $e$. Now for any $s \in S$, $e + ese_1 = (e + ese_1) + 0$ where, $(e + ese_1)(e + ese_1) = e + ese_1 + ese_1e + ese_1ese_1 = e + ese_1$, Since $ee_1 = e_1e = 0$, $(ese_1)^2 = ese_1ese_1 = 0$ and $ese_1e = 0$ . As, $nin(S) = 1$, we have $e = e + ese_1 \implies ee_1 = ee_1 + ese_1 \implies ese_1 = 0 \implies ese + ese_1 = ese \implies es = ese$. If we consider the element $e + e_1se$ then similarly it can be proved that $se = ese$ for all $s \in S$. Hence $es = se$ for all $s \in S$ and for any $e^2 = e \in S$. Hence $S$ is abelian. Let there exists a nonzero idempotent $e$ in $S$ such that $e + n_1 = n_2$ for some nilpotent elements $n_1, n_2 \in S$. Then $e + n_1 = 0 + n_2$. Since $nin(S) = 1$, $e = 0$, which contradicts our assumption.

conversely, let $a = e_1 + n_1 = e_2 + n_2 \ldots (1)$ are two nil clean expressions of $a$ in

$S$. Let $e_3^2 = e_3$ be the orthogonal complement of $e_1$ and $e_4^2 = e_4$ be the orthogonal complement of $e_2$. Thus $e_1e_4+n_1e_4 = e_2e_4+n_2e_4 \Longrightarrow e_1e_4+n_1e_4 = n_2e_4$ since $e_2e_4 = 0$. Since $S$ is abelian, $e_1e_4$ is an idempotent element in $S$. Again since $S$ is abelian and $n_1, n_2$ are nilpotent elements in $S$, $n_1e_4, n_2e_4$ are nilpotent elements in $S$. According to the condition, $e_1e_4 = 0 \Longrightarrow e_1e_4 + e_1e_2 = e_1e_2 \Longrightarrow e_1 = e_1e_2$. Again from equation (1), it follows that $e_1e_3 + n_1e_3 = e_2e_3 + n_2e_3 \Longrightarrow e_2e_3 + n_2e_3 = n_1e_3$ since $e_1e_3 = 0$. Again according to the condition $e_2e_3 = 0 \Longrightarrow e_2e_3+e_2e_1 = e_2e_1 \Longrightarrow e_2 = e_2e_1 = e_1e_2$. Hence $e_1 = e_2$ which implies that $nin(S) = 1$. □

**Theorem 8.4.2.** *If $S$ is a semiring such that every idempotent has an orthogonal complement in $S$, then $S$ is an abelian semiring if and only if $nin(S) = 1$.*

*Proof.* ($\Longrightarrow$) This follows from Lemma 8.4.1.

($\Longleftarrow$) Let $S$ be an abelian semiring and $e$ be a nonzero idempotent of $S$. Suppose, $e + n_1 = n_2$ for nilpotent elements $n_1, n_2 \in S$. Hence $n_1^{k_1} = 0$ and $n_2^{k_2} = 0$ for $k_1, k_2 \in \mathbb{N}$. Let $k = max\{k_1, k_2\}$. Hence $e + en_1 = en_2 \Longrightarrow [e(1 + n_1)]^k = (en_2)^k$. Since $S$ is abelian, we have $e(1 + n_1)^k = en_2^k = 0$. Now Remark 7.2.7 implies that $(1 + n_1) \in U_k(S)$, hence $(1 + n_1)^k \in U_k(S)$ from [20]. Let $u = (1 + n_1)^k$. So, $eu = 0$. Now there exist $s_1, s_2 \in S$ such that $1 + us_1 = us_2 \Longrightarrow e + eus_1 = eus_2 \Longrightarrow e = 0$ which is a contradiction. Hence for any $0 \neq e^2 = e \in S$, $e + n_1 \neq n_2$ for any nilpotent elements $n_1, n_2 \in S$. Thus from Lemma 8.4.1 it follows that $nin(S) = 1$. □

**Lemma 8.4.3.** *Let $S$ be a semiring such that its every proper homomorphic image is nil clean and every idempotent has orthogonal complement. If $nin(S) = 1$, then either $S$ is nil clean or $S$ has only trivial idempotents such that $1 + n_1 \neq n_2$ for any nilpotent $n_1, n_2 \in S$.*

*Proof.* Suppose $S$ is not nil clean. Let $e$ be a non-trivial idempotent of $S$. Since $nin(S) = 1$ and every idempotent of $S$ has orthogonal complement, from Theorem 8.4.2, it follows that $S$ is abelian. Hence $eS$ and $e_1S$ become two ideals of $S$ where $e_1$ is the orthogonal complement of $e$. Let $y \in \overline{eS}$. Then there exist $s_1, s_2 \in S$ such that $y + es_1 = es_2 \ldots (1)$. Multiplying both side of equation (1) by $e_1$, we have $e_1y = 0$. Hence $y = ey \in eS$. Thus $eS$ is the $k$-ideal of $S$. Similarly, $e_1S$ is the $k$-ideal of $S$.

Define a function $f : S \to eS \times e_1S$ by $f(s) = (es, e_1s)$ for all $s \in S$. Clearly, $f$ is well-defined. Let $s, s_1 \in S$. Then $f(s+s_1) = (e(s+s_1), e_1(s+s_1)) = (es+es_1, e_1s+e_1s_1) = (es, e_1s) + (es_1, e_1s_1) = f(s) + f(s_1)$. Now $f(ss_1) = (ess_1, e_1ss_1)$. Since $S$ is abelian, we have $f(ss_1) = (eses_1, e_1se_1s_1) = (es, e_1s)(es_1, e_1s_1) = f(s)f(s_1)$ for all $s, s_1 \in S$. Hence $f$ is a homomorphism. Let $a, b \in S$ such that $f(a) = f(b) \implies (ea, e_1a) = (eb, e_1b) \implies ea = eb, e_1a = e_1b$. Hence, $ea + e_1a = eb + e_1b \implies a = b$ since $e + e_1 = 1$. Thus $f$ is one-one. Let $(ex, e_1y) \in eS \times e_1S$ for some $x, y \in S$. Then $f(ex + e_1y) = (e(ex + e_1y), e_1(ex + e_1y)) = (ex, e_1y)$, since $ee_1 = e_1e = 0$. So, $f$ is onto. Hence $S \cong eS \times e_1S$. Since, $eS, e_1S$ are two homomorphic images of $eS \times e_1S$ and $S \cong eS \times e_1S$, $eS$ and $e_1S$ are two homomorphic images of $S$. Since $e$ is the non-trivial idempotent of $S$ and $e_1$ is the orthogonal complement of $e$, $eS = \overline{eS} \neq S$ and $e_1S = \overline{e_1S} \neq S$. Thus $eS, e_1S$ are two proper homomorphic images of $S$. According to the condition $eS, e_1S$ are nil clean semirings. Thus by (1) and (2) of Proposition 7.2.4, it follows that $S$ is nil clean semiring which contradicts our assumption. Hence $S$ has only trivial idempotents. Again since $nin(S) = 1$, from Lemma 8.4.1, we have $1 + n_1 \neq n_2$ for any nilpotent element $n_1, n_2 \in S$. □

**Lemma 8.4.4.** *Let $S$ be a semiring and $M$ be the $S$-semimiodule. Let $S_1 = S \propto M$. Then $(n, m)$ is a nilpotent in $S_1$ if and only if $n$ is a nilpotent in $S$.*

*Proof.* Let $n$ be a nilpotent element of $S$. Then there exist a positive integer $k$ such that $n^k = 0$. Now $(n, m)^k = (n^k, m_2) = (0, m_2)$ for some $m_2 \in M$. Thus $(n, m)^{2k} = (0, m_2)(0, m_2) = (0, 0m_2 + 0m_2) = (0, 0_M)$. Hence $(n, m)$ is a nilpotent element of $S_1$. □

**Proposition 8.4.5.** *Let $S$ and $M$ be the $S$-semimodule such that $(M, +)$ is not a group. Let $S_1 = S \propto M$. Then $nin(S_1) = 2$ if $|M| = 2$ and $nin(S) = 1$.*

*Proof.* Since $|M| = 2$, $M = \{0_M, m\}$ where $m \neq 0_M$. $(M, +)$ is a monoid which is not a group, hence $m+m = m$. Let $\alpha = (1, m) \in S_1$. Then $(\alpha)^2 = (1, m)(1, m) = (1, m+m) = (1, m) = \alpha$. Hence $\alpha$ is an idempotent of $S_1$. Also $(0, m)(0, m) = (0, 0m+m0) = (0, 0_M)$. Thus $(0, m)$ is a nilpotent element in $S_1$. Now $(1, m) = (1, 0_M) + (0, m) = (1, m) + (0, m) = (1, m) + (0, 0_M)$. Hence we have $\eta(\alpha) \supseteq \{(1, 0_M), (1, m)\}$. Hence

$nin(S_1) \geq 2$. Consider, $\alpha_1 = (a, 0_M)$ for some $a \in S$. Then $\eta(\alpha_1) = \{(e, 0_M) \in S_1 : e \in \xi(a)\}$ which implies that $|\eta(\alpha_1)| \leq 1$ since $|\eta(a)| \leq 1$. Take $\alpha_2 = (a, m) \in S_1$. Since, $w + m = m$, for all $w \in M$, $\eta(\alpha_2) = \{(e, w) \in S_1 : e \in \eta(a), w = ew + ew\}$. Since $|\eta(a)| \leq 1$ and $|M| = 2$, we have $|\eta(\alpha_2)| \leq 2$. Hence $nin(S_1) \leq 2$. So, $nin(S_1) = 2$. □

**Theorem 8.4.6.** *Let $S$ be a semiring such that every idempotent has an orthogonal complement in $S$. Then $nin(S) = 2$ if and only if $S \cong \begin{bmatrix} A & M \\ O & B \end{bmatrix}$, where $A$ and $B$ are two semirings such that $nin(A) = nin(B) = 1$ and ${}_AM_B$ is a bisemimodule with $|M| = 2$.*

*Proof.* Let $nin(S) = 2$. Hence from Theorem 8.4.2 it follows that $S$ is non-abelian semiring. Let $e$ be the non-central idempotent of $S$ and $e_1$ be its othogonal complement. From Lemma 6.5.9 it follows that $S \cong \begin{bmatrix} eSe & eSe_1 \\ e_1Se & e_1Se_1 \end{bmatrix}$. Now both $eSe_1$ and $e_1Se$ can not be equal to zero otherwise $e$ becomes a central idempotent. Let $eSe_1 \neq 0$ and $e_1Se \neq 0$. Then there exist $x, y \in S$ such that $exe_1 \neq 0 \in eSe_1$ and $e_1ye \neq 0 \in e_1Se$. Suppose, $ex_1e_1 \in eSe_1$ for some $x_1 \in S$ such that $ex_1e_1 \neq 0$ and $ex_1e_1 \neq exe_1$. Now $a = e + exe_1 + ex_1e_1 = (e + exe_1) + ex_1e_1 = (e + ex_1e_1) + exe_1 = e + (exe_1 + ex_1e_1)$. Now $(e + exe_1)(e + exe_1) = e + exe_1 + exe_1e + exe_1exe_1 = e + exe_1$, since $e_1$ is the orthogonal complement of $e$, $ee_1 = e_1e = 0$. Hence $e + exe_1$ is an idempotent of $S$. Similarly, $(e + ex_1e_1)$ is an idempotent of $S$. Now $(exe_1)(exe_1) = 0$. Hence $exe_1$ is a nilpotent element of $S$. Similarly $ex_1e_1, (exe_1 + ex_1e_1)$ are nilpotent elements of $S$. Hence the element $a$ has three distinct nil clean expressions which contradicts that $nin(S) = 2$. Hence $eSe_1 = \{0, exe_1\}$. Again, let $e_1y_1e \in e_1Se$ for some $y_1 \in S$ such that $e_1y_1e \neq 0$ and $e_1y_1e \neq e_1ye$. Then in a similar way this contradicts that $nin(S) = 2$. Hence $e_1Se = \{0, e_1ye\}$. Let $p = exe_1$ and $q = e_1ye$. Now there are several cases:

**Case- I:** $p + p = p$ and $q + q = q$. Now $pqp \in \{0, exe_1\}$. Let $pqp = 0 \implies exe_1yexe_1 = 0$. Hence $e + p + q = (e + p) + q = (e + q) + p = e + (p + q)$. previously, we prove that $e + p$, $e + q$ are idempotents of $S$ and $p, q$ are nilpotents of $S$. Now

$(p+q)^4 = (exe_1 + e_1ye)^4 = (exe_1ye + e_1yexe_1)^2 = (exe_1yexe_1ye + e_1yexe_1yexe_1) = 0$, since we take $pqp = exe_1yexe_1 = 0$. Hence $(p+q)$ is a nilpotent element of $S$. If $e = e + p$ then $ee_1 = ee_1 + pe_1$. Since $ee_1 = e_1e = 0$, $pe_1 = 0$. Thus $p = 0$ . Similarly, $q = 0$ if $e = e + q$ which contradicts our assumption. Let $e + p = e + q$. Then also we get $q = 0$ which also contradicts that $q \neq 0$ . Hence there exist three different nil clean representations of $e + p + q$ which is not possible since $nin(S) = 2$. Hence $pqp = 0$ is not possible. Thus $pqp = p$ which implies that $pq$ and $qp$ are idempotents in $S$. Let $f = pq$, $g = qp$ and $A_1 = fSf$, $B_1 = gSg$. Now $A_1$ and $B_1$ are subsemirings of $eSe$ and $e_1Se_1$ respectively. Since $eSe_1 = \{0, p\}$ and $e_1Se = \{0, q\}$, $L = \begin{bmatrix} fSf & eSe_1 \\ e_1Se & gSg \end{bmatrix}$ is a subsemiring of $\begin{bmatrix} eSe & eSe_1 \\ e_1Se & e_1Se_1 \end{bmatrix}$ with respect to usual matrix addition and multiplication. Since $nin(S) = 2$, from Lemma 8.2.5 it follows that $nin\left(\begin{bmatrix} eSe & eSe_1 \\ e_1Se & e_1Se_1 \end{bmatrix}\right) = 2$. Thus Lemma 8.2.3 implies that $nin(L) \leq 2$. Let us consider the element $\alpha = \begin{bmatrix} f & p \\ q & g \end{bmatrix} \in L$. Then $\alpha = \begin{bmatrix} f & p \\ 0 & g \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix} = \begin{bmatrix} f & 0 \\ q & g \end{bmatrix} + \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f & p \\ q & g \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are three distinct nil clean expressions of $\alpha$ which again contradicts that $nin(L) \leq 2$. Hence the **Case - I** is not possible. □

**Case - II** $p + p = p$ and $q + q = 0$. Hence $qp = e_1yexe_1 = e_1y(exe_1 + exe_1) = e_1yexe_1 + e_1yexe_1 = (e_1ye + e_1ye)xe_1 = (0)xe_1 = 0$. Similarly, $pq = 0$. Now $e+p+q = (e+p)+q = (e+q)+p = e+(p+q)$. Clearly, $(e+p)$, $(e+q)$ are idempotents and $p, q$ are nilpotents in $S$. Since $p^2 = 0, q^2 = 0$, $(p+q)^2 = pq + qp = 0$ which implies $(p+q)$ is a nilpotent element in $S$. Hence there exist three different nil clean expressions of $e + p + q$, which contradicts $nin(S) = 2$. Hence this case is not possible.

**Case - III** $p + p = 0$ and $q + q = q$, Similar to the case-II, in this case also we have three different nil clean expressions of $e + p + q$ which is not possible.

**Case - IV** $p + p = 0$ and $q + q = 0$. Then $e = e + 0 = (e + p) + p = (e + q) + q$ are three distinct nil clean expressions of $e$ which contradicts that $nin(S) = 2$.

Considering all possible cases, we have either $eSe_1 = 0$ or $e_1Se = 0$. Without any loss of generality we can assume $e_1Se = 0$. Hence $S \cong \begin{bmatrix} eSe & eSe_1 \\ 0 & e_1Se_1 \end{bmatrix}$ where $|eSe_1| = 2$. Our next claim is that $nin(eSe) = 1$ and $nin(e_1Se_1) = 1$. Now, $e = e + 0$ and $e_1 = e_1 + 0$ are nil clean expression of $e$ and $e_1$. Hence $nin(eSe) \geq 1$ and $nin(e_1Se_1) \geq 1$. From Theorem 8.3.2 it follows that $nin(eSe) < nin(S) = 2$ and $nin(e_1Se_1) < nin(S) = 2$. Hence $nin(eSe) = 1$ and $nin(e_1Se_1) = 1$.

Conversely, let $S \cong \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$, where $A$ and $B$ be two semirings such that $nin(A) = nin(B) = 1$ and ${}_AM_B$ is a bi-semimodule with $|M| = 2$. So $M = \{0_M, m\}$. **Case-I:** $w + w = w$ for any $w \in M$. Consider the element $\alpha_0 = \begin{bmatrix} 1_A & m \\ 0 & 0_B \end{bmatrix}$. Then $\eta(\alpha_0) \supseteq \left\{ \begin{bmatrix} 1_A & 0 \\ 0 & 0_B \end{bmatrix}, \begin{bmatrix} 1_A & m \\ 0 & 0_B \end{bmatrix} \right\}$. It follows that $nin(S) \geq 2$. Consider any $\alpha_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $\eta(\alpha_1) = \left\{ \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} : e \in \eta(a), f \in \eta(b) \right\}$. Hence $|\eta(\alpha_1)| \leq 1$, since $|\eta(a)| \leq 1$ and $|\eta(b)| \leq 1$. Consider any $\alpha_2 = \begin{bmatrix} a & m \\ 0 & b \end{bmatrix}$. Now $w + m = m$ for any $w \in M$, since $|M| = 2$ and $m + m = m$. Hence $\eta(\alpha_2) = \left\{ \begin{bmatrix} e & w \\ 0 & f \end{bmatrix} : e \in \eta(a), f \in \eta(b), w = ew + wf \right\}$. Because $|\eta(a)| \leq 1, |\eta(b)| \leq 1$ and $|M| = 2$ it implies that $|\eta(\alpha_2)| \leq 2$. Hence $nin(S) \leq 2$. So $nin(S) = 2$.

**Case: 2** $w + w = 0$ for any $w \in M$.

Consider the element $\beta_0 = \begin{bmatrix} 1_A & 0 \\ 0 & 0_B \end{bmatrix}$. Then $\eta(\beta_0) \supseteq \left\{ \begin{bmatrix} 1_A & w \\ 0 & 0 \end{bmatrix} : w \in M \right\}$ which implies that $ind_k(S) \geq |\eta(\beta_0)| \geq |M| = 2$. For any $\beta = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in T$, $\eta(\beta) = \left\{ \begin{bmatrix} e & w \\ 0 & f \end{bmatrix} : e \in \eta(a), f \in \eta(b), w = ew + wf \right\}$. Because $|M| = 2$, $|\eta(a)| \leq 1$ and $|\eta(b)| \leq 1$, it follows that $|\eta(\beta)| \leq 2$. Hence $nin(S) = 2$.

# Chapter 9

# Conclusion

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# Conclusion

## 9.1 Introduction

In previous chapters we have defined different types of clean semirings. We have discussed several properties of those semirings. We have also proved several characterization results for those semirings. We conclude this thesis by investigating the inter-relation among those types of semirings.

## 9.2 The relation among different types of clean semirings

In chapter 2 and chapter 3 we have discussed the concept of clean semiring and strongly clean semiring respectively. We have observed that every strongly clean element in a semiring $S$ is clean element but the converse does not always hold. Hence every strongly clean semiring is clean semiring but every clean semiring is not strongly clean semiring. In chapter 4 and 5 we have introduced the concept of $k$-unit clean semiring and $k$-regular clean semiring respectively. Let $u$ be a $k$-unit element in a semiring $S$. Then there exist $s_1, s_2 \in S$ such that $1+s_1u = s_2u \Longrightarrow u+us_1u = us_2u$. So, $u$ is a $k$-regular element in $S$. Hence every $k$-unit element in a semiring is $k$-regular but the reverse implication does not always hold. Hence every $k$-unit clean semiring is $k$-regular clean semiring. If a semiring $S$ has an absorbing zero element then it can be proved that every unit element is $k$-unit element. Hence every clean semiring is $k$-unit clean semiring but the converse is not true always.

Hence we have the following pictorial presentation of different types of clean semirings.

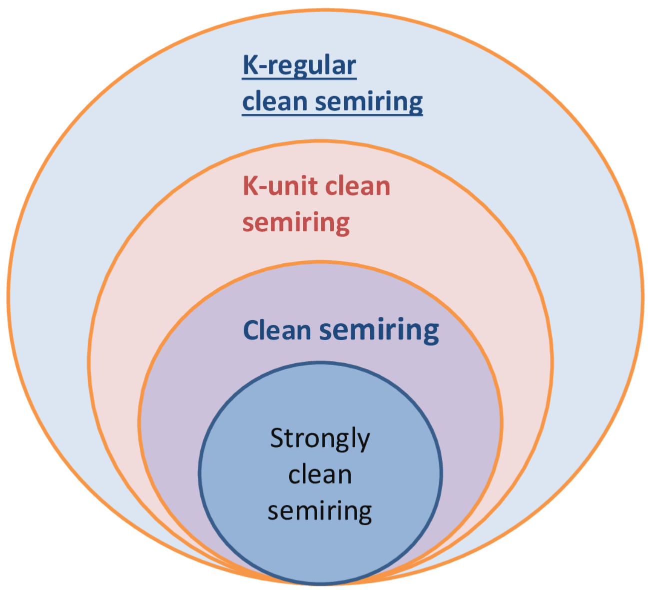


**Figure 1**

we have already stated that the reverse implications in the above diagram does not always hold. Now we provide required examples to support the statement.

**Example 9.2.1.** *Let* $E == M_2(\mathbb{N}_0^+) \times S'$ *be the semiring which is defined in Example 5.2.2(iv). Then* $E$ *is* $k$*-regular clean semiring but not* $k$*-unit clean. Let* $S$ *be the semiring defined in Example 3.4.10. Then* $S$ *is* $k$*-unit clean semiring but not a clean semiring. In [56] J. Šter gave an Example [Example 2.3.] of a clean ring which is* *not strongly clean ring. Since every ring itself is a semiring and every clean ring and strongly clean ring is clean semiring and strongly clean semiring respectively, this is the required example of clean semiring which is not strongly clean semiring.*

In chapter 3 we have already stated that, if $S$ is a commutative semiring or $S$ is a semiring with central idempotents then the two notions clean and strongly clean for the semiring $S$, coincide. In the next Proposition we have tried to prove that when a $k$-unit clean semiring becomes a clean semiring.

**Proposition 9.2.2.** *Let $S$ be an additively absorbing semiring. Then $S$ is a clean semiring if $S$ is $k$-unit clean.*

*Proof.* Let $a \neq 0 \in S$. Since $S$ is $k$-unit clean, $a = e+u$, where $e$ is an idempotent and $u$ is a $k$-unit element in $S$. Hence there exists $s_1, s_2 \in S$ such that $1+s_1u = s_2u \ldots (1)$ and $1 + us_1 = us_2 \ldots (2)$. Since $S$ is additively absorbing centroid semiring we can write from equation (1), $1+s_1u+s'_1u = s_2u+s'_1u \Longrightarrow 1+s_1u+1'(s_1u) = (s_2+s'_1)u \Longrightarrow 1+(1+1')(s_1u) = (s'_1+s_2)u \Longrightarrow (s'_1+s_2)u = 1$. Similarly, from equation (2), we can prove that $u(s'_1+s_2) = 1$. Hence $u$ is a unit elememt in $S$. Thus $a$ is a clean element in $S$. So, $S$ is a clean semiring. □

In the next Proposition we have proved that when a $k$-regular clean semiring is $k$-unit clean.

**Proposition 9.2.3.** *If $S$ is a multiplicatively cancellative $k$-regular clean semiring then $S$ is a $k$-unit clean semiring.*

*Proof.* Let $a \neq 0$. Since $S$ is multiplicatively cancellative semiring, it has only idempotents 0 and 1. Since $S$ is $k$-regular clean, either $a = r$ or $a = 1 + r_1$, where $r$ and $r_1$ are $k$-regular elements in $S$ and $r \neq 0$. If $a = r$ then there exist $x, y \in S$ such that $a + axa = aya$. Since $a \neq 0$ and $S$ is both left and right multiplicatively cancellative, we have $1 + xa = ya$ and $1 + ax = ay$. Hence $a$ is a $k$-unit element in $S$ which implies that $a$ is $k$-unit clean. Let $a = 1 + r_1$. If $r_1 = 0$ then $a = 1$ is $k$-unit clean element. If $r_1 \neq 0$ then similarly we can say that $r_1$ is a $k$-unit. Hence in both cases $a$ is $k$-unit clean. Thus $S$ is $k$-unit clean semiring. □

In [45] A. J. Diesl introduced the concept of nil clean ring and he proved that every nil clean ring is clean. In chapter 7 we have introduced the concept of nil clean semiring. Now we provide an example which implies that a nil clean semiring need not be a clean semiring.

**Example 9.2.4.** *Let $S = \{0, 1\}$ be the Boolean semifield and $T_2(S)$ be the semiring of $2 \times 2$ upper triangular matrices over $S$, which are defined in Example 7.2.3. Then $T_2(S)$ is a zeroic nil clean semiring. If $T_2(S)$ is k-regular clean semiring then $T_2(S)$ becomes a k-regular semiring from Proposition 5.3.8, but $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not a k-regular element in $T_2(S)$. Hence $T_2(S)$ is not a k-regular clean semiring. Thus from Figure 1 of section 9.2, it follows that $T_2(S)$ is not clean and k-unit clean semiring.*

*Conversely, if $S$ is the semiring defined in Example 2.2.3(vi), then $S$ is a clean semiring but not a nil clean semiring since $2$ is not a nil clean element in $S$. If $S$ is the semiring defined in Example 3.4.10, then $S$ is k-unit clean semiring but not a nil clean semiring since $S$ has no nil clean element except $0$ and $1$. Let $M_2(S)$ be the set of all matrices over the Boolean semifield $S = \{0, 1\}$ which is defined in Example 2.2.3(iv). Then $M_2(S)$ becomes a k-regular clean semiring with respect to usual matrix addition and multiplication, but $M_2(S)$ is not a nil clean semiring since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not a nil clean element in $M_2(S)$.*

**Proposition 9.2.5.** *Let $S$ be a nil clean semiring. Then every nonzero element in the subsemiring $P(S)$ of $S$, which is defined in Definition 1.3.48, becomes a k-unit clean element in $S$.*

*Proof.* Let $S$ be a nil clean semiring. Suppose, $a \in P(S)$ and $a \neq 0$. Then there exist an element $a_1 \in S$ such that $a = a_1 + 1$. Since $S$ is nil clean, $a_1$ is a nil clean element of $S$. Let $a_1 = e + b$, where $e$ is an idempotent in $S$ and $b$ is a nilpotent element in $S$. Thus, $a = a_1 + 1 = (e + b) + 1 = e + (1 + b)$. Since $b$ is a nilpotent element in $S$, Remark 7.2.7 implies that $(1 + b) \in U_k(S)$. Hence $a$ is a $k$-unit clean element in $S$. □

**Corollary 9.2.6.** *Let $S$ be a nil clean semiring. Then every element in $P(S)$ becomes a $k$-regular clean element in $S$.*

*Proof.* Since $0 = 0 + 0$ is the $k$-regular clean presentation of 0, 0 is a $k$-regular clean element in $S$. Let $a \neq 0 \in P(S)$. Proposition 9.2.5 implies that $a$ is a $k$-unit clean element in $S$. In section 9.2 we have already proved that every $k$-unit element is $k$-regular element in a semiring, which implies that $a$ is a $k$-regular clean element in $S$. □

Thus If $S$ is an antisimple semiring i.e. if $S = P(S)$ then we have the following pictorial presentation for the semiring $S$:

# List of published and communicated papers based on this thesis

1) Kar, S., Das, D. : *Clean semiring*; Beitr. Algebra Geom. (Springer) (2023) 64: 197-207.

2) Das, D. Kar, S. : *Strongly clean semiring*; Proc. Natl. Acad. Sci. (Springer), India, Sect. A Phys. Sci (April 2024) 94(2): 249-258.

3) Das, D. Kar, S. : *On k-unit clean semiring* (communicated).

4) Das, D. Kar, S. : *On k-regular clean semiring* (communicated).

5) Das, D. Kar, S. : *On k-unit clean index of semiring* (communicated).

6) Das, D. Kar, S. : *Nil clean semiring* (communicated).

7) Das, D. Kar, S. : *On nil clean index of semiring* (communicated).

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# List of symbols

| | |
|---|---|
| $\mathbb{Z}$ | Set of all integers |
| $\mathbb{N}$ | Set of all natural numbers |
| $\mathbb{N}_0^+$ | Set of all non-negative integers |
| $\mathbb{Q}$ | Set of all rationals |
| $\mathbb{Q}_0^+$ | Set of all positive rationals |
| $\mathbb{R}$ | Set of all real numbers |
| $\mathbb{R}_0^+$ | Set of all non-negative real numbers |
| $Id(S)$ | Set of all idempotent elements in $S$ |
| $U(S)$ | Set of all unit elements in semiring $S$ |
| $U_k(S)$ | Set of all $k$-unit elements in semiring $S$ |
| $R_k(S)$ | Set of all $k$-regular elements in semiring $S$ |
| $N(S)$ | Set of all nilpotent elements in semiring $S$ |
| $\|A\|$ | Cardinality of a set $A$ |
| $Z(S)$ | Center of a semiring $S$ |
| $ind_k(S)$ | $k$-Unit clean index of a semiring $S$ |
| $nin(S)$ | Nil clean index of a semiring $S$ |