Study on Some Submanifolds of Differentiable Manifolds

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Study on Some Submanifolds of Differentiable Manifolds**" submitted by **Smt. Payel Karmakar** who got her name registered on 26th March, 2021 (INDEX NO : 58/21/Maths./27) for the award of Ph.D.(Science) degree of Jadavpur University, is absolutely based upon her own work under the supervision of **Prof. Arindam Bhattacharyya** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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Dedicated to

my loving parents

Mrs. Manju Karmakar

and

Mr. Sibu Karmakar

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PREFACE

The present doctoral thesis is a collection of research works done on the studies of various types of submanifolds of some differentiable manifolds. Most of those works are already published in various reputed national and international journals at different times. This thesis consists of seven chapters.

In the **Introduction** chapter, we have explained the background of the works and the main results at which we have arrived in the following sections.

In the **second chapter**, we have discussed about anti-invariant submanifolds of indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold, indefinite LP-Sasakian manifold and have obtained some results regarding the relation between the structure vector field of a manifold and the anti-invariance of the submanifold. Also we have obtained some results on totally umbilical, totally geodesic submanifolds.

In the **third chapter**, we have discussed $*$-η-Ricci-Yamabe solitons on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection and then we have studied contact CR-submanifolds of trans-Sasakian manifolds with respect to a quarter symmetric non-metric connection. First, We have obtained some results regarding a Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton. Next, we have proved some curvature properties of anti-invariant submanifolds of Kenmotsu manifold admitting a quarter symmetric metric connection. Then, we have obtained a result regarding anti-invariant submanifolds of Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton with respect to a quarter symmetric metric connection. Further, we have studied the nature of a $*$-η-Ricci-Yamabe soliton and solitons appeared as its particular cases on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection when the vector field becomes a conformal Killing vector field. Finally, we have given an example of a 3-dimensional Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton to verify a relation. In this chapter, we have also dealt with the study of trans-Sasakian manifolds with respect to a quarter symmetric non-metric connection. We have stated and proved some results regarding a contact CR-submanifold of a trans-Sasakian manifold with respect to a quarter symmetric non-metric connection. We have investigated totally geodesic leaves and integrability of the distributions. Moreover, we have studied totally umbilical contact CR-submanifolds of trans-Sasakian manifolds. At last, we have provided an example of a 3-dimensional trans-Sasakian manifold admitting a quarter symmetric non-metric connection to verify a relation.

In the **fourth chapter**, we have analysed briefly some properties of a hemi-slant submanifold of an $(LCS)_n$-manifold. We have discussed about some necessary and sufficient conditions for distributions to be integrable and obtain some results in this direction. We have also studied the geometry of leaves. At last, we have constructed a suitable example.

In the **fifth chapter**, we have discussed quasi hemi-slant (QHS) submanifolds of trans-Sasakian manifolds and then, we have introduced the general notion of such submanifolds in metallic Riemannian manifolds. At first, we have obtained various results satisfied by QHS submanifolds of trans-Sasakian manifold. Further, we have obtained necessary and sufficient conditions for integrability of the distributions related to these submanifolds, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. Moreover, we have concluded the necessary and sufficient condition for a QHS submanifold of a trans-Sasakian manifold to be a locally product Riemannian manifold. Then, we have constructed an example of a QHS submanifold of a trans-Sasakian manifold. Next, we have studied some properties of submanifolds, specially QHS submanifolds of metallic and golden Riemannian manifolds. We have found out a necessary and sufficient condition for a submanifold to be QHS in metallic and golden Riemannian manifolds and also have obtained the integrability conditions for the distributions. At last, we have constructed an example of a QHS submanifold of a metallic Riemannian manifold.

In the **sixth chapter**, we have studied screen-slant lightlike submanifolds, totally contact umbilical screen-slant lightlike submanifolds, totally contact umbilical radical screen-transversal lightlike submanifolds, contact screen generic lightlike (CSGL) submanifolds, totally umbilical CSGL submanifolds and minimal CSGL submanifolds of indefinite Kenmotsu manifold. We have proved a characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. We further have proved some results on a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold, such as the necessary and sufficient conditions for the screen distribution to be integrable and for the induced connection on the submanifold to be a metric connection. Next, we have studied CSGL submanifolds of indefinite Kenmotsu manifolds. We have investigated the necessary and sufficient conditions for the induced connection on a CSGL submanifold to be a metric connection, for integrability & parallelism of some associated distributions, and for some distributions to be totally geodesic foliations. We have also discussed about non-parallel distributions and more than one necessary and sufficient conditions for a CSGL submanifold to be mixed geodesic. We have further studied some properties satisfied by proper totally umbilical CSGL submanifolds and the necessary and sufficient conditions for minimality of an associated distribution & also of a CSGL submanifold. At last, we have constructed an example of a CSGL submanifold of an indefinite Kenmotsu manifold.

In the **seventh chapter**, we have dealt with the study of some properties of anti-invariant submanifolds of trans-Sasakian manifold with respect to a new affine connection called Zamkovoy connection. We have discussed the nature of Ricci flat, concircularly flat, ξ-projectively flat, M-projectively flat, ξ-M-projectively flat, pseudo projectively flat and ξ-pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection. Moreover, we have studied Ricci soliton along with η-Ricci-Yamabe soliton and two more solitons arose as its particular cases on Ricci flat, concircularly flat, M-projectively flat and pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold admitting the aforesaid connection. At last, we have made some conclusions after observing all the results and have provided an example of an anti-invariant submanifold of a trans-Sasakian manifold in which all the results of this chapter can be verified very easily.

CHAPTER WISE PUBLICATIONS SUMMARY

Serial No.	Authors	Title of the Paper and Journal Information	Chapter
1.	Payel Karmakar and Arindam Bhattacharyya	Anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds, *Bull. Calcutta Math. Soc.*, Vol. 112, Issue 2, 2020, pp. 95-108	2
2.	Payel Karmakar and Arindam Bhattacharyya	$*$-η-Ricci-Yamabe solitons on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection, *Bull. Calcutta Math. Soc.*, Vol. 114, Issue 3, 2022, pp. 281-294	3
3.	Payel Karmakar and Arindam Bhattacharyya	Contact CR-submanifolds of trans-Sasakian manifolds with respect to quarter symmetric non-metric connection, *Gulf J. Math.*, Vol. 12, Issue 2, 2022, pp. 73-85	3
4.	Payel Karmakar and Arindam Bhattacharyya	Hemi-slant submanifold of $(LCS)_n$-manifold, *J. Indones. Math. Soc.*, Vol. 28, Issue 01, 2022, pp. 75-83	4
5.	Payel Karmakar and Arindam Bhattacharyya	Quasi hemi-slant submanifolds of trans-Sasakian manifold, *Palest. J. Math.*, Vol. 12, Issue 1, 2023, pp. 745-756	5
6.	Payel Karmakar and Arindam Bhattacharyya	Quasi hemi-slant submanifolds of metallic Riemannian manifolds, *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, Vol. 94, Issue 1, 2024, pp. 75-82	5
7.	Payel Karmakar	Totally contact umbilical screen-slant and screen-transversal lightlike submanifolds of indefinite Kenmotsu manifold, *Math. Bohem.*, Published online on April 17, 2024	6
8.	Payel Karmakar	Contact screen generic lightlike submanifolds of indefinite Kenmotsu manifold, *Under Review*	6
9.	Payel Karmakar	Curvature tensors and Ricci solitons with respect to Zamkovoy connection in anti-invariant submanifolds of trans-Sasakian manifold, *Math. Bohem.*, Vol. 147, Issue 3, 2022, pp. 419-434	7
10.	Payel Karmakar	η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection, *Balkan J. Geom. Appl.*, Vol. 27, Issue 2, 2022, pp. 50-65	7

CONTENTS

1

INTRODUCTION

The theory of submanifolds has the origin in the study of geometry of plane curves initiated by Fermat. Nowadays, it has gained prominence in computer design, image processing, economic modeling, mathematical physics and mechanics except modern differential geometry. In Mathematics, more specifically in Differential Geometry, a submanifold of a manifold is a subset which is itself structure wise a manifold such that the inclusion between the subset and the manifold becomes an embedding. The concept of submanifolds has a vast variety of classifications depending upon the decompositions and properties of the tangent and normal bundles of the concerned submanifolds. That's why, more we study about the theory of submanifolds, more we find that there still remain miles to go and undoubtedly, this is perhaps the reason lying behind the beauty of this theory.

History of "Theory of Submanifolds" has been evaluated slowly but fruitfully over time. In the beginning, simplest forms of the submanifolds like invariant and anti-invariant submanifolds were studied. Later, the generalisation process' took place and gradually more and more classifications evolved. Slant, hemi-slant, CR-submanifolds were introduced in this way. Here we have also discussed about such submanifolds in the second, third, fourth and seventh chapters of this thesis. Moreover, the most interesting fact is that, this generalisation process is still ongoing and the recent discovery of quasi hemi-slant (QHS) submanifolds is its proof. Now, we would like to humbly mention that, we have also indulged ourselves in this never ending process by introducing the general notion of QHS submanifolds for metallic Riemannian manifolds which is given in the fifth chapter here.

Although the study of submanifolds of Riemannian manifolds went too far, in case of semi-Riemannian manifolds, the structures of the submanifolds take a new turn such as for lightlike submanifolds due to the degeneracy of the functioning metric, and thus, the study of submanifolds of semi-Riemannian manifolds created much enthusiasm among the geometers. Here, we have also analysed some types of lightlike submanifolds and tried to find some new results as far as possible in the sixth chapter.

We now enlighten about the basic concepts which we have instituted in finding all the results of the subsequent chapters in this thesis.

■ We start with the following definition of almost contact metric manifolds given by D. E. Blair in [23].

Definition 1.1. Let $\tilde{M}$ be an odd dimensional differentiable manifold equipped with a metric structure $(\phi, \xi, \eta, \tilde{g})$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric $\tilde{g}$ satisfying the following relations−

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0,\ \phi\xi = 0, \tag{1.1}$$
$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y), \tag{1.2}$$
$$\tilde{g}(\phi X, Y) = -\tilde{g}(X, \phi Y),\ \eta(X) = \tilde{g}(X, \xi)\ \ \forall X, Y \in \chi(\tilde{M}), \tag{1.3}$$

then $\tilde{M}$ is called *almost contact metric manifold*.

■ Next, Kenmotsu manifold is named after K. Kenmotsu who introduced its notion in 1972 [84] as a particular classification of almost contact manifolds with the following definition.

Definition 1.2. An almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is called *Kenmotsu manifold* if $\forall X, Y \in \chi(\tilde{M})$,

$$(\tilde{\nabla}_X \phi)Y = \tilde{g}(\phi X, Y)\xi - \eta(Y)\phi X, \tag{1.4}$$
$$\tilde{\nabla}_X \xi = X - \eta(X)\xi, \tag{1.5}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$ for the Riemannian metric $\tilde{g}$.

■ Now, we define trans-Sasakian manifold, which was introduced by J. A. Oubina in 1985 [108]. J. C. Marrero characterised the local structure of trans-Sasakian manifolds of dimension $n \geq 5$ in [100] and there he proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu. Trans-Sasakian structures of type $(0,0)$, $(\alpha, 0)$ and $(0, \beta)$ are cosymplectic, α-Sasakian and β-Kenmotsu respectively. The definition is as follows:

Definition 1.3. An odd dimensional manifold $\tilde{M}$ with the almost contact metric structure (ϕ, ξ, η, g) is called a *trans-Sasakian manifold* of type (α, β) if the following equation holds

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{1.6}$$

for smooth functions α and β on $\tilde{M}$ and $\forall X, Y \in \Gamma(T\tilde{M})$. Hence, a trans-Sasakian manifold of type $(0,0)$, $(1,0)$ and $(0,1)$ is called *cosymplectic manifold* [118], *Sasakian manifold* [101] and *Kenmotsu manifold* [4] respectively.

On a trans-Sasakian manifold $\tilde{M}$, authors in [100] and [108] have obtained

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi]. \tag{1.7}$$

■ We now turn our attention to paracontact structures leading to Lorentzian concircular structure which is the context in the fourth chapter of our discussion on hemi-slant submanifolds.

Definition 1.4. An n-dimensional *Lorentzian manifold* $\tilde{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $\tilde{g}$ i.e., $\tilde{M}$ admits a smooth symmetric tensor field $\tilde{g}$ of type $(0,2)$ such that, for each point p, the tensor $\tilde{g}_p : T_p\tilde{M} \times T_p\tilde{M} \to \mathbb{R}$ is a non-degenerate inner-product of signature $(-,+,...,+)$, $T_p\tilde{M}$ denotes the tangent vector space of $\tilde{M}$ at p and $\mathbb{R}$ is the real number space.

Note. A non-zero vector $X_p \in T_p\tilde{M}$ is known to be spacelike, null or light-like, or timelike according as $\tilde{g}_p(X_p, X_p) > 0, = 0$ or < 0 respectively.

Definition 1.5. If $\tilde{M}$ is a differentiable manifold of dimension n and there exists a (ϕ, ξ, η) structure (ϕ is a tensor of type $(1,1)$, ξ is a vector field and η is a 1-form) satisfying

$$\phi^2 = I + \eta \otimes \xi, \ \eta(\xi) = -1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0, \tag{1.8}$$

then M is called an *almost paracontact manifold*.

Definition 1.6. If a differentiable manifold $\tilde{M}$ with an almost paracontact structure (ϕ, ξ, η) admits a Lorentzian metric $\tilde{g}$ satisfying

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) + \eta(X)\eta(Y), \ \tilde{g}(X, \xi) = \eta(X) \tag{1.9}$$

for all vector fields X, Y on $\tilde{M}$, then $\tilde{M}$ is called a *Lorentzian almost paracontact manifold* and we have,

$$\begin{aligned} &\tilde{g}(X, \phi Y) = \tilde{g}(\phi X, Y), \\ &2g(\phi X, Y) = (\tilde{\nabla}_X \eta)Y + (\tilde{\nabla}_Y \eta)X. \end{aligned}$$

Definition 1.7. In a Lorentzian manifold $(\tilde{M}, \tilde{g})$, a vector field P defined by $\tilde{g}(X, P) = A(X)$ for any $X \in \Gamma(T\tilde{M})$, is called *concircular* if

$$(\tilde{\nabla}_X A)(Y) = \alpha\{\tilde{g}(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form and $\tilde{\nabla}$ denotes the operator of covariant differentiation of $\tilde{M}$ with respect to $\tilde{g}$.

Let $\tilde{M}$ admit a unit timelike concircular vector field ξ, called the structure vector field of the manifold, then $\tilde{g}(\xi, \xi) = -1$. Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that $\tilde{g}(X, \xi) = \eta(X)$. The following equations hold−

$$\begin{aligned} &(\tilde{\nabla}_X \eta)Y = \alpha[\tilde{g}(X, Y) + \eta(X)\eta(Y)], \ \alpha \neq 0, \\ &\tilde{\nabla}_X \alpha = X\alpha = d\alpha(X) = \rho\eta(X) \end{aligned}$$

for all vector fields X, Y on $\tilde{M}$ and α is a non-zero scalar function related to ρ, by $\rho = -(\xi\alpha)$.

Definition 1.8. Let $\phi X = \frac{1}{\alpha}\tilde{\nabla}_X \xi$, from which it follows that ϕ is a symmetric $(1,1)$ tensor and we call it the structure tensor on the manifold. Thus, an n-dimensional Lorentzian manifold $\tilde{M}$ together with a unit timelike concircular vector field ξ, its associated 1-form η and a $(1,1)$ tensor field ϕ is called a *Lorentzian Concircular Structure manifold* i.e. $(LCS)_n$*-manifold*. Specially, if $\alpha = 1$, then we obtain *Lorentzian para-Sasakian* structure i.e. *LP-Sasakian structure* of Matsumoto [87].

In an $(LCS)_n$-manifold ($n > 2$), the following relations hold−

$$\phi^2 = I + \eta \otimes \xi,\ \eta(\xi) = -1,$$

where I denotes the identity transformation of the tangent space $T\tilde{M}$,

$$\phi\xi = 0,\ \eta \circ \phi = 0,\ \tilde{g}(X, \phi Y) = \tilde{g}(\phi X, Y),\ rank(\phi) = 2n, \tag{1.10}$$

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) + \eta(X)\eta(Y), \tilde{g}(X, \xi) = \eta(X), \tag{1.11}$$

$$\tilde{R}(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y] \tag{1.12}$$

$\forall X, Y \in \Gamma(T\tilde{M})$.

Also an $(LCS)_n$-manifold satisfies−

$$(\tilde{\nabla}_X \phi)Y = \alpha[\tilde{g}(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{1.13}$$

$$\tilde{\nabla}_X \xi = \alpha\phi X. \tag{1.14}$$

■ The difference between almost contact and indefinite almost contact manifolds lies in the nature of the associated metric and hence one can notice the simple difference in their definitions which is none other than the addition of the symbol ϵ to which assigned the value $\tilde{g}(\xi, \xi) = \pm 1$. Then, let's look upon the definition:

Definition 1.9. A $(2n+1)$-dimensional semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is called an *indefinite almost contact metric manifold* if it admits an indefinite almost contact structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field and η is a 1-form, satisfying for all vector fields Z, W on $\tilde{M}$ [23],

$$\phi^2 Z = -Z + \eta(Z)\xi, \eta \circ \phi = 0, \phi\xi = 0, \eta(\xi) = 1, \tag{1.15}$$

$$\tilde{g}(\phi Z, \phi W) = \tilde{g}(Z, W) - \epsilon\eta(Z)\eta(W), \tag{1.16}$$

$$\tilde{g}(Z, \xi) = \epsilon\eta(Z), \tilde{g}(\phi Z, W) = -\tilde{g}(Z, \phi W). \tag{1.17}$$

Here $\epsilon = \tilde{g}(\xi, \xi) = \pm 1$.

■ Based on the structure equations, an indefinite almost contact metric manifold

$\tilde{M}$ is classified as follows [23]:-

Definition 1.10. An indefinite almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is called an *indefinite Sasakian manifold* if for all vector fields Z, W on $\tilde{M}$,

$$(\tilde{\nabla}_Z\phi)W = \tilde{g}(Z, W)\xi - \epsilon\eta(W)Z, \tag{1.18}$$

$$\tilde{\nabla}_Z\xi = -\epsilon\phi Z, \tag{1.19}$$

where $\tilde{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\tilde{g}$.

■ In [41], U. C. De and A. Sarkar introduced and studied the notion of ϵ-Kenmotsu manifolds with indefinite metric by giving an example of a trans-Sasakian manifold with $\alpha = 0,\ \beta = 1$. Thus indefinite Kenmotsu manifolds comes into the picture and we have thoroughly studied some semi-Riemannian submanifolds in its context in the sixth chapter of this thesis. The definition is given by:

Definition 1.11. An indefinite almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ becomes an *indefinite Kenmotsu manifold* with the structure equations−

$$(\tilde{\nabla}_Z\phi)W = \tilde{g}(\phi Z, W)\xi - \epsilon\eta(W)\phi Z, \tag{1.20}$$

$$\tilde{\nabla}_Z\xi = \epsilon Z - \epsilon\eta(Z)\xi. \tag{1.21}$$

for all vector fields Z, W on $\tilde{M}$.

Definition 1.12. An indefinite almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is called an *indefinite trans-Sasakian manifold* of type (α, β) if

$$(\tilde{\nabla}_Z\phi)W = \alpha[\tilde{g}(Z, W)\xi - \epsilon\eta(W)Z] + \beta[\tilde{g}(\phi Z, W)\xi - \epsilon\eta(W)\phi Z], \tag{1.22}$$

$$\tilde{\nabla}_Z\xi = -\epsilon\alpha\phi Z + \epsilon\beta[Z - \eta(Z)\xi] \tag{1.23}$$

for smooth functions α, β on $\tilde{M}$ and for all vector fields Z, W on $\tilde{M}$.

Definition 1.13. A differentiable manifold $\tilde{M}$ of dimension n is called *indefinite Lorentzian para-Sasakian manifold* (briefly, *indefinite LP-Sasakian manifold*) if it admits an almost paracontact structure (ϕ, ξ, η) along with an indefinite Lorentzian metric $\tilde{g}$ which satisfy for all vector fields Z, W on $\tilde{M}$,

$$\phi^2 Z = Z + \eta(Z)\xi, \eta(\xi) = -1,$$
$$\tilde{g}(\xi, \xi) = -\epsilon, \eta(Z) = \epsilon\tilde{g}(Z, \xi), \phi\xi = 0, \eta(\phi Z) = 0,$$
$$\tilde{g}(\phi Z, \phi W) = \tilde{g}(Z, W) - \epsilon\eta(Z)\eta(W).$$

In this manifold, we also have,

$$(\tilde{\nabla}_Z\phi)W = \tilde{g}(Z, W)\xi + \epsilon\eta(W)Z + 2\epsilon\eta(Z)\eta(W)\xi, \tag{1.24}$$

$$\tilde{\nabla}_Z\xi = \epsilon\phi Z. \tag{1.25}$$

■ Now, we are finally heading to our soul concept i.e. the concept of submanifolds which has opened a door to the researchers for experimenting many beautiful structures in addition to those of the main manifolds and thus it has curved the paths of research of many researchers like me.

Let φ be a differentiable map from a manifold M into a manifold $\tilde{M}$ and let the dimensions of $M, \tilde{M}$ be n, m $(n < m)$ respectively. If at each point p of M, $(\varphi_*)_p$ is a 1-1 map i.e., if rank$(\varphi) = n$, then φ is called an *immersion* of M into $\tilde{M}$.

If an immersion φ is one-one i.e., if $\varphi(p) \neq \varphi(q)$ for $p \neq q$, then φ is called an *imbedding* of M into $\tilde{M}$.

Definition 1.14. If the manifolds $M, \tilde{M}$ satisfy the following two conditions, then M is called a *submanifold* of $\tilde{M}-$
(i) $M \subset \tilde{M}$,
(ii) the inclusion map i from M into $\tilde{M}$ is an imbedding of M into $\tilde{M}$.

Let $\varphi : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M, g) into m-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$.

We denote the Levi-Civita connections on M^n and $\tilde{M}^m$ by ∇ and $\tilde{\nabla}$ respectively. Then the *formulae of Gauss and Weingarten* are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.26}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{1.27}$$

for $X, Y \in T_x(M)$ and $V \in T^{\perp}_x M$, where $h, A, \nabla^{\perp}$ are the *second fundamental form*, the *shape operator*, the *normal connection* respectively.

Moreover, the shape operator A and the second fundamental form h are related by

$$g(h(X, Y), V) = g(A_V X, Y) \tag{1.28}$$

■ As we have said before, there exists a vast variety of submanifolds, some of those, which have made the pillars of our study, are given in the following manner.

Definition 1.15. A submanifold M of a manifold $\tilde{M}$ is called *totally geodesic* if $h = 0$ or equivalently $A_V = 0 \ \ \forall V \in T^{\perp}_x M$.

Definition 1.16. A submanifold M of a manifold $\tilde{M}$ is called *minimal* if the *mean curvature vector* H satisfies $H = \frac{trace(h)}{dim(M)} = 0$.

Definition 1.17. A submanifold M of a manifold $\tilde{M}$ is called *totally umbilical* if

$$h(X, Y) = g(X, Y)H. \tag{1.29}$$

Definition 1.18. A submanifold M of a manifold $\tilde{M}$ is called *invariant* if $\forall X \in T_xM, \ \phi X \in \ T_xM \ \ \forall x \in M$.

Definition 1.19. [151] A submanifold M of a manifold $\tilde{M}$ is called *anti-invariant* if $X \in T_x(M) \Rightarrow \phi X \in T_x^{\perp}M \ \ \forall x \in M$ i.e.

$$g(X, \phi Y) = 0 \ \ \forall X, Y \in \chi(M). \tag{1.30}$$

Definition 1.20. Let M be a submanifold isometrically immersed in a differentiable manifold $\tilde{M}$ and ξ be tangent to M. Then the tangent bundle TM decomposes as $TM = D \oplus < \xi >$, where D is the orthogonal distribution to ξ. Now, for each non-zero vector X tangent to M at x such that X is not proportional to ξ_x, we denote the angle between ϕX and D_x by $\theta(X)$. M is called *slant submanifold* if the angle $\theta(X)$ is constant, which is independent of the choice of $x \in M$ and $X \in T_x(M) - < \xi_x >$. The constant angle $\theta \in [0, \frac{\pi}{2}]$ is then called the slant angle of M in $\tilde{M}$. If $\theta = 0$, then the submanifold is invariant, if $\theta = \frac{\pi}{2}$, then the submanifold is anti-invariant and if $\theta \neq 0, \frac{\pi}{2}$, then the submanifold is *proper slant*.

According to A. Lotta [95], when M is a proper slant submanifold of $\tilde{M}$ with slant angle θ, then $\forall X \in \Gamma(TM)$,

$$T^2(X) = -\cos^2\theta(X - \eta(X)\xi). \tag{1.31}$$

■ A. Carriazo [26] introduced hemi-slant submanifolds as a special case of bi-slant submanifolds and he called them pseudo-slant submanifolds.

Definition 1.21. A submanifold M of a differentiable manifold $\tilde{M}$ is called *hemi-slant* if there exist two orthogonal distributions D_θ and $D^{\perp}$ satisfying [91]−
(i) $TM = D_\theta \oplus D^{\perp} \oplus < \xi >$,
(ii) D_θ is a slant distribution with slant angle $\theta \neq \frac{\pi}{2}$,
(iii) $D^{\perp}$ is totally real i.e., $\phi D^{\perp} \subseteq T^{\perp}M$.

A hemi-slant submanifold is called *proper* if $\theta \neq 0, \frac{\pi}{2}$.

If we denote the dimensions of the distributions $D^{\perp}$ and D_θ by m_1 and m_2 respectively, then we have−
(i) if $m_2 = 0$, then M is anti-invariant,
(ii) if $m_1 = 0, \ \theta = 0$, then M is invariant,
(iii) if $m_1 = 0, \ \theta \neq 0$, then M is proper-slant with slant angle θ,
(iv) if $m_1 m_2 \neq 0, \ \theta \in (0, \frac{\pi}{2})$, then M is proper hemi-slant.

■ R. Prasad et al. introduced the notion of quasi hemi-slant submanifold in Sasakian manifolds [121] as a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant and semi-slant submanifolds.

Definition 1.22. [121] *Quasi hemi-slant submanifold* M of an almost contact manifold $\tilde{M}$ is a submanifold that admits three orthogonal complementary distributions $D,\ D_\theta,\ D^\perp$ such that

(i) TM admits the orthogonal direct decomposition

$$TM = D \oplus D_\theta \oplus D^\perp \oplus <\xi>, \tag{1.32}$$

(ii) the distribution D is invariant i.e., $\phi D = D$,
(iii) the distribution D_θ is slant with constant angle θ and hence, θ is called *slant angle*,
(iv) the distribution $D^\perp$ is ϕ anti-invariant i.e., $\phi D^\perp \subseteq T^\perp M$.

In the above case, θ is called the *quasi hemi-slant angle* of M, and M is called *proper* [121] if $D \neq \{0\},\ D_\theta \neq \{0\},\ D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Let the dimensions of the distributions $D,\ D_\theta,\ D^\perp$ be $n_1,\ n_2,\ n_3$ respectively, then we obtain the following particular cases [121]−
(i) if $n_1 = 0$, then M is a hemi-slant submanifold,
(ii) if $n_2 = 0$, then M is a semi-invariant submanifold,
(iii) if $n_3 = 0$, then M is a semi-slant submanifold.

Now, it can be concluded from the definitions of invariant [78], anti-invariant [131], semi-invariant [13], slant [127], hemi-slant [126] and semi-slant [89] submanifolds that, quasi hemi-slant submanifold is a generalization of all these kinds of submanifolds [121].

From the definition of quasi hemi-slant submanifold given above, it is clear that if $D \neq \{0\},\ D_\theta \neq \{0\},\ D^\perp \neq \{0\}$, then $\dim(D) \geq 2$, $\dim(D_\theta) \geq 2$ and $\dim(D^\perp) \geq 1$. Thus, we have the following remark [121]−

Remark 1.1. *For a proper quasi hemi-slant submanifold M, dim$(M) \geq 6$.*

■ A. Bejancu first gave the definition of CR-submanifolds in 1986 [11], which is given below.

Definition 1.23. A submanifold M of an almost contact manifold $\tilde{M}$ is called *contact CR-submanifold* if ξ is tangent to M and there is a differential distribution D and its orthogonal complementary distribution $D^\perp$ such that
(i) D is invariant under ϕ i.e. $\phi(D) \subseteq D$ and
(ii) $D^\perp$ is anti-invariant under ϕ i.e. $\phi(D^\perp) \subseteq T^\perp M$,
where D (respectively $D^\perp$) is called horizontal (respectively vertical) distribution.

M is called *ξ-horizontal* (respectively *ξ-vertical*) if $\xi \in D$ (respectively $\xi \in D^\perp$).

Now,

$$TM = D \oplus D^{\perp} \; and \; T^{\perp}M = \phi(D^{\perp}) \oplus \mu, \tag{1.33}$$

where μ is a normal sub-bundle invariant to ϕ. For $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ we write

$$X = PX + QX \tag{1.34}$$

and

$$\phi V = BV + CV, \tag{1.35}$$

where $PX \in D,\ QX \in D^{\perp},\ BV = tan(\phi V)$ and $CV = nor(\phi V)$.

Definition 1.24. [101] A contact CR-submanifold M of an almost contact manifold $\tilde{M}$ is called a *contact CR-product* if M is locally a Riemannian product of M^T and $M^{\perp}$, where M^T and $M^{\perp}$ denote the leaves of D and $D^{\perp}$ respectively.

■ On a smooth manifold M, a *distribution* D is the assignment to each point $x \in M$ of a subspace D_x of the tangent space $T_x(M)$. A distribution D is *smooth* at a point x if any tangent vector $X_{(x)} \in D_x$ can be locally extended to a smooth vector field X on some open set $U \in M$ such that $X_{(y)} \in D_y$ for every $y \in U$.

Definition 1.25. A connected submanifold N of M is called an *integral manifold* of the distribution D if $(\varphi_*)_x N = D_x \ \forall x \in N$, φ being the imbedding of N into M. If there is no other integral manifold of D which contains N, then N is called a *maximal integral manifold* of D. Thus, $D = \cup_{x \in M} D_x$ is a distribution on M.

Definition 1.26. We say that a distribution D is *involute* if for every two sections X, Y of D, the commutator $[X, Y]$ is a section of D.

Definition 1.27. Let D denote an integrable distribution on M, then the collection of integrable manifolds of D is called a *foliation*.

Definition 1.28. A maximal connected integral manifold of D is called a *leaf of the foliation*.

■ The concepts of Ricci flow and Yamabe flow were introduced simultaneously by R. S. Hamilton ([70], [71]). He observed that the Ricci flow can be used well in simplifying the structure of a manifold. He developed the concept to answer Thurston's geometric conjecture stating that each closed 3-manifold admits a geometric decomposition. Ricci soliton emerged as a self-similar solution to the Ricci flow [71] and as did Yamabe soliton. These solitons are equivalent in dimension 2 but in greater dimensions, these two do not agree since Yamabe soliton preserves the conformal class of the metric but Ricci soliton does not in general. An interpolation soliton, called Ricci-Bourguignon soliton corresponding to Ricci-Bourguignon flow, was considered and further studied by G. Catino and L. Mazzieri ([27], [28]), but this soliton depends on a single scalar. In 2019, S.

Guler and M. Crasmareanu [62] introduced a new geometric flow called Ricci-Yamabe flow as a scalar combination of Ricci flow and Yamabe flow.

Definition 1.29. *Ricci-Yamabe flow* of type (p, q) is an evolution for the metrics on Riemannian or semi-Riemannian manifolds defined as [62]

$$\frac{\partial}{\partial t}g(t) = -2pRic(t) + qr(t)g(t), \; g(0) = g_0,$$

where p, q are scalars. Due to the signs of p, q, this flow can also be a Riemannian flow or semi-Riemannian flow or singular Riemannian flow. A Ricci-Yamabe flow of type (p, q) is called [62]−
(i) Ricci flow if $p = 1, \; q = 0$;
(ii) Yamabe flow if $p = 0, \; q = 1$;
(iii) Einstein flow [28] if $p = 1, \; q = -1$.

Definition 1.30. Naturally, a soliton to the Ricci-Yamabe flow is called Ricci-Yamabe soliton. *Ricci-Yamabe soliton* of type (p, q) on a Riemannian complex (M, g) is represented by the quintuplet (g, V, λ, p, q) satisfying the following equation−

$$L_V g + 2pS + (2\lambda - qr)g = 0,$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, r is the scalar curvature, S is the Ricci tensor and λ, p, q are scalars. This soliton is called shrinking, steady or expanding according as $\lambda < 0, \; \lambda = 0$ or $\lambda > 0$ respectively. Ricci-Yamabe soliton of type $(0, q)$ and $(p, 0)$ are called *q-Yamabe soliton* and *p-Ricci soliton* respectively. These solitons are studied by many geometers ([18], [19], [22], [35], [38], [61], [136], [137]).

Therefore, A Ricci-Yamabe soliton of type (p, q) is called−

(i) Ricci soliton if $p = 1, \; q = 0$ given by

$$L_V g + 2S + 2\lambda g = 0, \tag{1.36}$$

(ii) Yamabe soliton if $p = 0, \; q = 1$ given by

$$L_V g + (2\lambda - r)g = 0,$$

(iii) Einstein soliton if $p = 1, \; q = -1$ given by

$$L_V g + 2S + (2\lambda + r)g = 0,$$

■ J. T. Cho and M. Kimura introduced the notion of η-Ricci soliton as an advance extension of Ricci soliton in 2009 [37]. Analogously in 2020 [138], M. D. Siddiqi and M. A. Akyol introduced the concept of η-Ricci-Yamabe soliton as a generalization of Ricci-Yamabe soliton.

Definition 1.31. *η-Ricci-Yamabe soliton* of type (p, q) is represented by the sextuplet $(g, V, \lambda, \mu, p, q)$ on a Riemannian manifold M satisfying the following equation−

$$L_V g + 2pS + (2\lambda - qr)g + 2\mu\eta \otimes \eta = 0, \tag{1.37}$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, r is the scalar curvature, S is the Ricci tensor, $\eta \otimes \eta$ is a $(0, 2)$-tensor field and λ, μ, p, q are scalars. The soliton is called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. η-Ricci-Yamabe soliton of type $(0, q)$ and $(p, 0)$ are called *q-η-Yamabe soliton* and *p-η-Ricci soliton* respectively. Recently, in 2021, G. Somashekhara et al. studied η-Ricci-Yamabe solitons on submanifolds of some indefinite almost contact manifolds concerning Riemannian and quarter symmetric metric connection [140].

Definition 1.32. In 2020, S. Dey and S. Roy [46] defined *$*$-η-Ricci soliton* as a generalizalion of η-Ricci soliton as follows−

$$L_\xi g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where S^* is the $*$-Ricci tensor which was first introduced by S. Tachibana [141].

Definition 1.33. Recently, S. Roy et al. [123] introduced the notion of *$*$-η-Ricci-Yamabe soliton* as a generalizalion of η-Ricci-Yamabe soliton and defined it as follows−

$$L_\xi g + 2pS^* + (2\lambda - qr^*)g + 2\mu\eta \otimes \eta = 0, \tag{1.38}$$

where $r^* = Tr(S^*)$ is the $*$-scalar curvature. A $*$-η-Ricci-Yamabe soliton is called−
(i) *$*$-η-Ricci soliton* if $p = 1,\ q = 0$;
(ii) *$*$-η-Yamabe soliton* if $p = 0,\ q = 1$;
(iii) *$*$-η-Einstein soliton* if $p = 1,\ q = -1$.

■ K. L. Duggal and A. Bejancu initiated the study of lightlike submanifolds in 1996 [48]. The fact about these submanifolds that their tangent and normal bundles have non-null intersection unlike the Riemannian submanifolds makes them even more interesting to analyse in detail.

Definition 1.34. A submanifold (M^m, g) immersed in a proper semi-Riemannian manifold $(\tilde{M}^{m+n}, \tilde{g})$ is called a *lightlike submanifold* [48] if the metric g induced from $\tilde{g}$ is degenerate and the radical distribution $Rad(TM) = TM \cap T^\perp M$ is of rank r such that $1 \leq r \leq m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, i.e.

$$TM = Rad(TM) \oplus_{orth} S(TM).$$

Let us consider a screen transversal vector bundle $S(T^\perp M)$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $T^\perp M$, i.e.

$$T^\perp M = Rad(TM) \oplus_{orth} S(T^\perp M).$$

Since for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(T^{\perp}M)$ in $S(TM)^{\perp}$ such that $\tilde{g}(\xi_i, N_j) = \delta_{ij}$ and $\tilde{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be the complementary (not orthogonal) vector bundle to TM in $T\tilde{M}$. Now we have the following decompositions [48]−

$$T\tilde{M}|_M = TM \oplus tr(TM),$$
$$tr(TM) = S(T^{\perp}M) \oplus_{orth} ltr(TM),$$
$$T\tilde{M}|_M = S(TM) \oplus_{orth} [Rad(TM) \oplus ltr(TM)] \oplus_{orth} S(T^{\perp}M).$$

A submanifold $(M, g, S(TM), S(T^{\perp}M))$ of $\tilde{M}$ is called
(i) *r-lightlike* if $r < min\{m, n\}$,
(ii) *co-isotropic* if $r = n < m,\ S(T^{\perp}M) = \{0\}$,
(iii) *isotropic* if $r = m < n,\ S(TM) = \{0\}$,
(iv) *totally lightlike* if $r = m = n,\ S(TM) = \{0\} = S(T^{\perp}M)$.

■ Zamkovoy connection is named after S. Zamkovoy who introduced its notion in 2009 [154].

Definition 1.35. For an n-dimensional almost contact metric manifold $M(\phi, \xi, \eta, g)$ consisting of a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g with the Riemannian connection ∇, *Zamkovoy connection* ∇^* is defined as [154]−

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \tag{1.39}$$

■ Curvature is the central subject in Riemannian geometry. It measures distance between a manifold and a Euclidean space.

Definition 1.36. K. Yano introduced the notion of *concircular curvature tensor* C of type $(1,3)$ on an n-dimensional Riemannian manifold M as [148]−

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type $(1,3)$ and r is the scalar curvature.

Hence, if we consider C^* as the concircular curvature tensor with respect to Zamkovoy connection, then for a $(2n+1)$-dimensional manifold we have

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.40}$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where R^* is the curvature tensor and r^* is the scalar curvature with respect to Zamkovoy connection.

Definition 1.37. K. Yano and S. Bochner introduced the notion of *projective curvature tensor* P of type $(1,3)$ for an n-dimensional Riemannian manifold M as [150]−

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y],$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where S is the Ricci tensor of type $(0,2)$.

Thus, for dimension $(2n+1)$ we have

$$P^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{2n}[S^*(Y,Z)X - S^*(X,Z)Y], \tag{1.41}$$

where we consider P^*, S^* respectively as the projective curvature tensor and the Ricci tensor with respect to Zamkovoy connection.

Both of the above curvature tensors represent the deviation of a manifold from being a manifold of constant curvature ([150], [148]).

■ G. P. Pokhariyal and R. S. Mishra introduced the notion of M-projective curvature tensor on a Riemannian manifold in 1971 [114]. Later R. H. Ojha studied its properties ([105], [106], [107]). This curvature tensor was further discussed by many researchers ([29], [30], [96], [116], [139]).

Definition 1.38. The *M-projective curvature tensor* $\bar{M}$ of rank 3 on an n-dimensional Riemannian manifold M is given by−

$$\begin{aligned}\bar{M}(X,Y)Z =& R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y] \\ & - \frac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY]\end{aligned}$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where Q is the Ricci operator.

Thus, for a $(2n+1)$-dimensional manifold, considering $\bar{M}^*$ as the M-projective curvature tensor with respect to Zamkovoy connection we get

$$\begin{aligned}\bar{M}^*(X,Y)Z = & R^*(X,Y)Z - \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] \\ & - \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y],\end{aligned} \tag{1.42}$$

where Q^* is the Ricci operator with respect to Zamkovoy connection.

■ B. Prasad introduced the notion of pseudo projective curvature tensor in a Riemannian manifold of dimension $n > 2$ in 2002 [117]. Its properties were further studied by many researchers on various manifolds ([98], [103], [104], [132], [145]).

Definition 1.39. The *pseudo projective curvature tensor* $\bar{P}$ of rank 3 on an n-dimensional Riemannian manifold M is given by−

$$\bar{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] + cr[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where $a,\ b,\ c$ are non-zero constants related as $c = -\frac{1}{n}\left(\frac{a}{n-1} + b\right)$.

Thus, for a $(2n+1)$-dimensional manifold, considering $\bar{P}^*$ as the pseudo projective curvature tensor with respect to Zamkovoy connection we get

$$\bar{P}^*(X,Y)Z = aR^*(X,Y)Z + b[S^*(Y,Z)X - S^*(X,Z)Y] + cr^*[g(Y,Z)X - g(X,Z)Y], \tag{1.43}$$

where $a,\ b,\ c$ are non-zero constants related as−

$$c = -\frac{1}{2n+1}\left(\frac{a}{2n} + b\right). \tag{1.44}$$

■ The metallic structure J, a polynomial structure defined by S. I. Goldberg and K. Yano [60], is inspired by the metallic number $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$, which is the positive solution of the equation $x^2 - px - q = 0$ for $p, q \in \mathbb{N}$ [59]. These $\sigma_{p,q}$ numbers are members of the metallic means family or metallic proportions (as generalizations of the golden number $\phi = \frac{1+\sqrt{5}}{2} = 1.618...$) which was introduced by V. W. de Spinadel [45]. Golden mean, silver mean, bronze mean, copper mean, nickel mean etc. are examples of the members of the metallic means family.

M. Crasmareanu and C. E. Hretcanu first instituted the golden structure on a Riemannian manifold in 2008 [39] and later they generalized these structures as metallic structures on Riemannian manifolds in 2013 [77]. Thus we got the notions of metallic and golden Riemannian manifolds.

Definition 1.40. Let $\tilde{M}$ be an n-dimensional manifold endowed with a (1,1) tensor field J. This structure J is called *metallic structure* if it satisfies the following relation−

$$J^2 = pJ + qI \tag{1.45}$$

for $p, q \in \mathbb{N}$, where I is the identity operator on $\Gamma(T\tilde{M})$. Then the pair $(\tilde{M}, J)$ is called *metallic manifold* [77]. In particular, if $p = q = 1$, then this manifold is called *golden manifold* [39].

Definition 1.41. Moreover, if a metallic (or golden) manifold $(\tilde{M}, J)$ is endowed with a Riemannian metric $\tilde{g}$ such that $\tilde{g}$ is J-compatible i.e.

$$\tilde{g}(JX, Y) = \tilde{g}(X, JY) \tag{1.46}$$

$\forall X, Y \in \Gamma(T\tilde{M})$, then the triplet $(\tilde{M}, \tilde{g}, J)$ is called *metallic (or golden) Riemannian manifold* [77]. Then we have

$$\tilde{g}(JX, JY) = \tilde{g}(J^2X, Y) = p\tilde{g}(JX, Y) + q\tilde{g}(X, Y). \tag{1.47}$$

After devoting the **1st Chapter** to introduction, we have dedicated the **2nd Chapter** to the study of anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds. This chapter has been divided into four sections consisting of the discussions on anti-invariant submanifolds of indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold and indefinite LP-Sasakian manifold respectively. We have proved that, when the structure vector field is normal, a submanifold of an indefinite Sasakian manifold becomes totally geodesic and consequently anti-invariant. We have also deduced that, for an anti-invariant submanifold of an indefinite Sasakian manifold or indefinite Kenmotsu manifold or indefinite trans-Sasakian manifold or indefinite LP-Sasakian manifold, with the structure vector field tangent, if the shape operator vanishes for a normal vector field, then the tangent space, with the structure tensor acted upon it, becomes parallel with respect to the normal connection. Again, we have found the result that, for a submanifold of an indefinite LP-Sasakian manifold with the structure vector field tangent, the structure vector field becomes parallel with respect to the induced connection if and only if the submanifold is anti-invariant. In addition, we have provided some examples.

The **3rd Chapter** consists of six sections. After the first and second sections namely Introduction and Preliminaries respectively, in the third section, we have found out the relations between the soliton scalars in a Kenmotsu manifold admitting $*$-η-Ricci-Yamabe soliton and also the relations between the curvature tensors, Ricci tensors, scalar curvatures of an anti-invariant submanifold of a Kenmotsu manifold with respect to the Levi-Civita connection and quarter symmetric connection. We have also shown that the Lie derivatives along the structure vector field with respect to the Levi-Civita connection and quarter symmetric connection are equal in an anti-invariant submanifold of a Kenmotsu manifold admitting $*$-η-Ricci-Yamabe soliton. Moreover, we have studied the nature of a conformal Killing vector field on an anti-invariant submanifold of a Kenmotsu manifold admitting $*$-η-Ricci-Yamabe soliton with respect to quarter symmetric connection. In addition, we have given an example of a Kenmotsu manifold admitting $*$-η-Ricci-Yamabe soliton verifying a relation in the fourth section. The fifth section has three subsections. After establishing the relations between the curvature tensors, Ricci tensors, scalar curvatures of a trans-Sasakian manifold with respect to the Levi-Civita connection and quarter symmetric non-metric connection, in the three subsections, we have studied contact CR-submanifolds of a trans-Sasakian manifold with respect to quarter symmetric non-metric connection, we have investigated totally geodesic leaves and integrability of the distributions and also, we have discussed totally umbilical contact CR-submanifolds of trans-Sasakian manifolds. At last, in the sixth section, we have given an example of a trans-Sasakian manifold admitting quarter symmetric non-metric connection verifying a relation.

We have investigated some properties of a hemi-slant submanifold of an $(LCS)_n$-manifold in the **4th Chapter** dividing it into five sections. After giving introduction in the first section and discussing preliminaries in the second section, we have checked the conditions of integrability of the distributions in the third section and studied the geometry of leaves in the fourth section. In the fifth section, we have constructed an example of a hemi-slant submanifold of an $(LCS)_n$-manifold.

The topic of the **5th Chapter** is quasi hemi-slant (QHS) submanifolds of some differentiable manifolds like trans-Sasakian manifolds and metallic Riemannian manifolds. After Introduction and Preliminaries sections, the third section deals with the study of some properties of a QHS submanifold of a trans-Sasakian manifold. In the fourth section, we have obtained the necessary and sufficient conditions of integrability of some distributions and also for some distributions to define totally geodesic foliations. We have also made the conclusions that both the invariant and slant distributions are not integrable and do not define totally geodesic foliations as well. In the fifth section, we have constructed an example of a QHS submanifold of a trans-Sasakian manifold. The sixth section deals with the study of QHS submanifolds of metallic Riemannian manifold, which notion has been already introduced by us in the Preliminaries section of this chapter. In this section, we have established some properties of a QHS submanifold of a metallic (or golden) Riemannian manifold. Moreover, we have obtained the necessary and sufficient conditions for a submanifold to be QHS in metallic and golden Riemannian manifolds, for integrability of the associated distributions, and also for parallelism of the normal connection on the tangent space of a QHS submanifold of a locally metallic (or locally golden) Riemannian manifold. Finally, in the seventh section, we have constructed an example of a QHS submanifold of a metallic Riemannian manifold.

6th Chapter has been dedicated to the study of various types of lightlike submanifolds of indefinite Kenmotsu manifolds. This chapter has seven chapters except the Introduction and Preliminaries sections summing up to a total of nine sections. In the third and fourth sections, we have stated and proved some results satisfied by screen-slant and totally contact umbilical screen-slant submanifolds of indefinite Kenmotsu manifolds respectively. In the fifth section, we have derived four results on totally contact umbilical radical screen-transversal lightlike submanifolds among which first three produce the necessary and sufficient conditions for the screen distribution to be integrable, for the local second fundamental form on the screen distribution to vanish and for the induced connection to be a metric connection. From the sixth section onwards, we have discussed contact screen generic lightlike (CSGL) submanifolds of indefinite Kenmotsu manifolds, where we have imposed totally umbilicality in the seventh section and minimality in the eighth section. In the sixth section, we have deduced the necessary and sufficient conditions for the induced connection to be a metric connection, for integrability and parallelism of some associated distributions, for some distributions to define totally geodesic foliations and for the submanifold to be mixed geodesic. Moreover, we have also found out the non-parallel distributions. In the seventh section, we have obtained some properties of a proper totally umbilical CSGL submanifold of an indefinite Kenmotsu manifold. In the eighth section, minimality of a distribution and the submanifold itself have been analysed. At last, in the ninth section, an example of a CSGL submanifold of an indefinite Kenmotsu manifold has been constructed.

We have covered the discussion on anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection in the **7th Chapter**, which contains total nine sections, where the first two are devoted to the introduction and preliminaries required for this chapter. Going through all the results of third to seventh sections, at the end of the seventh section, we have made three conclusions on an anti-invariant submanifold M of a trans-Sasakian manifold along with a condition satisfied such as−if M admits Zamkovoy connection and M is Ricci flat, concircularly flat, M-projectively flat or pseudo-projectively flat, then M becomes η-Einstein; if M is concircularly flat, M-projectively flat or pseudo-projectively flat with respect to Zamkovoy connection, then a Ricci soliton becomes shrinking, steady or expanding depending upon three conditions; for horizontal vector fields, M becomes ξ-projectively flat, ξ-M-projectively flat and ξ-psudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection. In the eighth section, we have proved that if M admits an η-Ricci-Yamabe soliton with respect to Zamkovoy connection, then M is η-Einstein with respect to Zamkovoy connection and with respect to Riemannian connection under a condition as well. Observing all the results obtained in the four subsections of the eighth section, we have deduced the conditions under which an η-Ricci-Yamabe soliton, a q-η-Yamabe soliton and a p-η-Ricci soliton are shrinking, steady or expanding when M is concircularly flat, M-projectively flat or pseudo projectively flat, next Ricci flat, concircularly flat, M-projectively flat or pseudo projectively flat, and then, concircularly flat, M-projectively flat or pseudo projectively flat with respect to Zamkovoy connection respectively. Finally, in the ninth section, we have given an example of an anti-invariant submanifold of a trans-Sasakian manifold.

2

ANTI-INVARIANT SUBMANIFOLDS OF SOME INDEFINITE ALMOST CONTACT AND PARACONTACT MANIFOLDS

2.1 Introduction

The study of geometry of anti-invariant submanifolds has been carried out by many researchers ([5], [25], [36], [81], [129], [149]) on various manifolds. In 1977, anti-invariant submanifolds of Sasakian space forms [151] were discussed by K. Yano and M. Kon. In 1985, H. B. Pandey and A. Kumar investigated about anti-invariant submanifolds of almost paracontact manifolds [110]. In 2018, C. S. Bagewadi and S. Venkatesha studied extensively anti-invariant submanifolds of S-manifold and (ϵ)-Sasakian manifold ([6], [7]). Motivated from their works, in this chapter, we have established some new results on anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds.

This chapter consists of six sections. The four sections following the Introduction and Preliminaries sections are devoted to the study of anti-invariant submanifolds of indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold and indefinite LP-Sasakian manifold respectively.

2.2 Anti-invariant submanifolds of an indefinite Sasakian manifold

In this section, we obtain some results on anti-invariant submanifolds of indefinite Sasakian manifold. Those are as follows−

Theorem 2.2.1. *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If M is totally umbilical, then M is totally geodesic.*

Proof. Since ξ is tangent to M, from (1.26) we have,

$$\tilde{\nabla}_X\xi = \nabla_X\xi + h(X,\xi),$$

where the vector fields X and Y are tangent to M.

Using (1.19) we get

$$-\epsilon\phi X = \nabla_X \xi + h(X, \xi).$$

Equating tangential and normal components we obtain

$$\epsilon(\phi X)^T = -\nabla_X \xi,$$
$$\epsilon(\phi X)^\perp = -h(X, \xi).$$

Putting $X = \xi$ in the second equation (as $\phi\xi = 0$) we get $h(\xi, \xi) = 0$.

Let us assume that M is totally umbilical, then $h(X, Y) = g(X, Y)H$. Putting $X = Y = \xi$ we get

$$g(\xi, \xi)H = \pm H = h(\xi, \xi) = 0$$
$$\Rightarrow H = 0$$

and hence, $h = 0$. Thus, M is totally geodesic.

Theorem 2.2.2. *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. Then ξ is parallel with respect to the induced connection on M if and only if M is an anti-invariant submanifold in $\tilde{M}$.*

Proof. Let the structure vector field ξ be tangent to M, then from (1.19) and (1.26) we have,

$$-\epsilon\phi X = \tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{2.2.1}$$

Now, let ξ be parallel with respect to the induced connection on M, then we have $\nabla_X \xi = 0$ and $h(X, \xi) \in T_x^\perp M$.

From (2.2.1) we have,

$$-\epsilon\phi X = h(X, \xi)$$
$$\Rightarrow \phi X = -\epsilon h(X, \xi).$$

Hence, ϕX is normal to M i.e., $\phi X \in T_x^\perp M$. Thus, M is anti-invariant.

Conversely, let M be anti-invariant, then for $X \in T_x(M)$, $\phi X \in T_x^\perp M$, so from (2.2.1) and as $h(X, \xi) \in T_x^\perp M$, we have,

$$-\epsilon\phi X = h(X, \xi)$$
$$\Rightarrow \nabla_X \xi = 0 \quad (by\ (2.2.1))$$

$\Rightarrow \xi$ is parallel with respect to the induced connection on M.

Theorem 2.2.3. *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$. If ξ is normal to M, then M is totally geodesic and consequently M is anti-invariant.*

Proof. Let ξ be normal to M, then (1.27) implies

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi. \tag{2.2.2}$$

Using (1.19), (2.2.2) and taking inner product with Y, we obtain for any X, Y tangent to M,

$$\epsilon g(\phi X, Y) = g(A_\xi X, Y). \tag{2.2.3}$$

Interchanging X, Y in (2.2.3) and then adding we get

$$\begin{aligned} &g(A_\xi X, Y) + g(A_\xi Y, X) = \epsilon[g(\phi X, Y) + g(\phi Y, X)] = 0 \ \ (by\ (1.3)) \\ \Rightarrow &g(A_\xi X, Y) = 0 \ \ (since\ A_\xi\ is\ symmetric) \\ \Rightarrow &g(h(X, Y), \xi) = 0 \ \ (by\ (1.28)) \\ \Rightarrow &h = 0 \\ \Rightarrow &A_\xi X = 0. \end{aligned}$$

Hence, M is totally geodesic.

Thus, by (2.2.2),

$$\begin{aligned} &\nabla^\perp_X \xi = \tilde{\nabla}_X \xi = -\epsilon \phi X \ \ (by\ (1.19)) \\ \Rightarrow &\phi X \in T^\perp_x M. \end{aligned}$$

Hence, M is anti-invariant.

Theorem 2.2.4. *Let M be an anti-invariant submanifold of an indefinite Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If $A_V X = 0$ for any $V \in T^\perp_x M$, then $\phi(T_x(M))$ is parallel with respect to the normal connection.*

Proof. In order to show that $\phi(T_x(M))$ is parallel with respect to the normal connection $\nabla^\perp$, we have to show that for every local section ϕY in $\phi(T_x(M)) \subset T^\perp_x M$, $\nabla^\perp_X(\phi Y)$ is also a local section in $\phi(T_x(M))$.

Now, $(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y)$

$\Rightarrow \tilde{g}(X, Y)\xi - \epsilon\eta(Y)X = -A_{\phi Y} X + \nabla^\perp_X \phi Y - \phi(\nabla_X Y + h(X, Y))$ (by (1.18), (1.26), (1.27))

$\Rightarrow \nabla^\perp_X \phi Y = \tilde{g}(X, Y)\xi - \epsilon\eta(Y)X + A_{\phi Y} X + \phi(\nabla_X Y + h(X, Y))$.

Taking inner product with $V \in T^\perp_x M$, using (1.28) and as $A_V X = 0$ for any $V \in T^\perp_x M$, we have,

$$\begin{aligned} g(\nabla^\perp_X \phi Y, V) &= \tilde{g}(X, Y) g(\xi, V) - \epsilon\eta(Y) g(X, V) + g(A_{\phi Y} X, V) + g(\phi(\nabla_X Y), V) \\ &\quad + g(\phi h(X, Y), V) \\ &= g(\phi(\nabla_X Y), V) \\ \Rightarrow \ \nabla^\perp_X \phi Y &= \phi(\nabla_X Y). \end{aligned}$$

Hence the proof.

Theorem 2.2.5. *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$, then M is anti-invariant if and only if D is integrable, where D is the orthogonal subspace of TM to ξ so that $TM = D\oplus <\xi>$.*

Proof. Let $X, Y \in D$, then $X, Y \in \Gamma(T\tilde{M})$,

$$\begin{aligned} g([X,Y],\xi) &= g(\tilde{\nabla}_X Y, \xi) - g(\tilde{\nabla}_Y X, \xi) \\ &= Xg(Y,\xi) - g(Y, \tilde{\nabla}_X \xi) - Yg(X,\xi) + g(X, \tilde{\nabla}_Y \xi) \\ &= -g(Y, -\epsilon\phi X) + g(X, -\epsilon\phi Y) \quad (by\ (1.19)) \\ &= \epsilon g(\phi X, Y) - \epsilon g(X, \phi Y) \\ &= \epsilon g(\phi X, Y) + \epsilon g(\phi X, Y) \quad (by\ (1.17)) \\ &= 2\epsilon g(\phi X, Y). \end{aligned}$$

Hence, $[X, Y] \in D$ if and only if ϕX is normal to Y i.e., $\phi X \in T_x^{\perp} M$ i.e., M is anti-invariant. Therefore, D is integrable if and only if M is anti-invariant.

2.3 Anti-invariant submanifold of an indefinite Kenmotsu manifold

Here we consider an anti-invariant submanifold of an indefinite Kenmotsu manifold and using its structure equations we state and prove the following theorem:

Theorem 2.3.1. *Let M be an anti-invariant submanifold of an indefinite Kenmotsu manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If $A_V X = 0$ for any $V \in T_x^{\perp} M$, then $\phi(T_x(M))$ is parallel with respect to the normal connection.*

Proof. In order to show that $\phi(T_x(M))$ is parallel with respect to the normal connection $\nabla^{\perp}$, we have to show that for every local section ϕY in $\phi(T_x(M)) \subset T_x^{\perp} M$, $\nabla^{\perp}_X(\phi Y)$ is also a local section in $\phi(T_x(M))$.

Now, $(\tilde{\nabla}_X \phi) Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y)$

$\Rightarrow \tilde{g}(\phi X, Y)\xi - \epsilon\eta(Y)\phi X = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \phi(\nabla_X Y + h(X, Y))$ (by (1.20), (1.26), (1.27))

$\Rightarrow \nabla^{\perp}_X \phi Y = A_{\phi Y} X + \tilde{g}(\phi X, Y)\xi - \epsilon\eta(Y)\phi X + \phi(\nabla_X Y) + \phi(h(X, Y))$.

Taking inner product with $V \in T_x^{\perp}M$, using (1.28) and as $A_V X = 0$ for any $V \in T_x^{\perp}M$, we have,

$$\begin{aligned}
g(\nabla^{\perp}_X \phi Y, V) &= g(A_{\phi Y}X, V) + \tilde{g}(\phi X, Y)g(\xi, V) - \epsilon\eta(Y)g(\phi X, V) \\
&\quad + g(\phi(\nabla_X Y), V) + g(\phi h(X,Y), V) \\
&= -\epsilon\eta(Y)g(\phi X, V) + g(\phi(\nabla_X Y), V) \\
\Rightarrow \nabla^{\perp}_X \phi Y &= \phi(\nabla_X Y - \epsilon\eta(Y)X).
\end{aligned}$$

Hence the result.

■ **Example of an indefinite Kenmotsu manifold.**

We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$, where (x,y,z) are standard coordinates of $\mathbb{R}^3$.

The vector fields $e_1 = \epsilon z\frac{\partial}{\partial x}$, $e_2 = \epsilon z\frac{\partial}{\partial y}$, $e_3 = -\epsilon z\frac{\partial}{\partial z}$ are linearly independent at each point of M.

Indefinite metric $g = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$.

$\phi e_1 = -e_2,\ \phi e_2 = e_1,\ \phi e_3 = 0.$

$\phi^2 Z = -Z + \eta(Z)\xi,\ \eta(Z) = \epsilon g(Z, e_3),\ \eta(\xi) = 1.$

$g(\phi Z, \phi W) = g(Z, W) - \epsilon\eta(Z)\eta(W);\ Z, W \in \Gamma(TM).$

$[e_1, e_3] = \epsilon e_1,\ [e_2, e_3] = \epsilon e_2,\ [e_1, e_2] = 0.$

By Koszul's formula

$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + $
$+ g(Z,[X,Y]),$

we calculate−

$$\begin{aligned}
&\nabla_{e_1} e_3 = \epsilon e_1,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_1 = -\epsilon e_3, \\
&\nabla_{e_2} e_3 = \epsilon e_2,\ \nabla_{e_2} e_2 = -\epsilon e_3,\ \nabla_{e_2} e_1 = 0, \\
&\nabla_{e_3} e_3 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_1 = 0.
\end{aligned}$$

$\nabla_X \xi = \epsilon(X - \eta(X)\xi)$ for $\xi = e_3$.

(ϕ, ξ, η, g) forms an indefinite Kenmotsu structure in $\mathbb{R}^3$.

2.4 Anti-invariant submanifolds of an indefinite trans-Sasakian manifold

In this section, we state and prove two theorems related to submanifolds and anti-invariant submanifolds of an indefinite trans-Sasakian manifold respectively.

Theorem 2.4.1. *Let M be a submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If M is totally umbilical, then M is totally geodesic.*

Proof. Since ξ is tangent to M,

$$\tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

Using (1.23) we get

$$-\epsilon\alpha\phi X + \epsilon\beta[X - \eta(X)\xi] = \nabla_X \xi + h(X, \xi).$$

Equating tangential and normal components we obtain

$$\begin{aligned}
&\nabla_X \xi = -\epsilon\alpha(\phi X)^T + \epsilon\beta X - \epsilon\beta\eta(X)\xi,\\
&h(X, \xi) = -\epsilon\alpha(\phi X)^{\perp}\\
\Rightarrow &h(\xi, \xi) = 0 \ \ (since\ \phi\xi = 0)\\
\Rightarrow &g(\xi, \xi)H = \pm H = 0\\
\Rightarrow &H = 0.
\end{aligned}$$

Hence, $h(X, Y) = g(X, Y)H = 0$ and so, $h = 0 \Rightarrow M$ is totally geodesic.

Theorem 2.4.2. *Let M be an anti-invariant submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If $A_V X = 0$ for any $V \in T_x^{\perp} M$, then $\phi(T_x(M))$ is parallel with respect to the normal connection.*

Proof. In order to show that $\phi(T_x(M))$ is parallel with respect to the normal connection $\nabla^{\perp}$, we have to show that for every local section ϕY in $\phi(T_x(M)) \subset T_x^{\perp} M$, $\nabla_X^{\perp}(\phi Y)$ is also a local section in $\phi(T_x(M))$.

Using (1.22), (1.26), (1.27) we get

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y)$$

$$\Rightarrow -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \phi(\nabla_X Y) - \phi(h(X, Y)) = \alpha[\tilde{g}(X, Y)\xi - \epsilon\eta(Y)X] + \beta[\tilde{g}(\phi X, Y)\xi - \epsilon\eta(Y)\phi X]$$

$$\Rightarrow \nabla_X^{\perp} \phi Y = A_{\phi Y} X + \phi(\nabla_X Y) + \phi(h(X, Y)) + \alpha[\tilde{g}(X, Y)\xi - \epsilon\eta(Y)X] + \beta[\tilde{g}(\phi X, Y)\xi - \epsilon\eta(Y)\phi X].$$

Since $A_V X = 0$ for any $V \in T_x^{\perp}M$, using (1.28) we have,

$$\begin{aligned} g(\nabla^{\perp}_X \phi Y, V) &= g(A_{\phi Y}X, V) + g(\phi(\nabla_X Y), V) + g(\phi h(X,Y), V) \\ &\quad + \alpha[\tilde{g}(X,Y)g(\xi, V) - \epsilon\eta(Y)g(X,V)] \\ &\quad + \beta[\tilde{g}(\phi X, Y)g(\xi, V) - \epsilon\eta(Y)g(\phi X, V)] \\ &= g(\phi(\nabla_X Y), V) - \beta\epsilon\eta(Y)g(\phi X, V) \\ \Rightarrow \nabla^{\perp}_X \phi Y &= \phi(\nabla_X Y - \beta\epsilon\eta(Y)X). \end{aligned}$$

Hence the proof.

■ **Example of an indefinite trans-Sasakian manifold.**

Let $\mathbb{R}^3$ be a 3-dimensional Euclidean space with regular coordinates (x, y, z). In $\mathbb{R}^3$ we define,

$\eta = dz - ydx, \ \xi = \frac{\partial}{\partial z},$

$\phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}, \ \phi(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \ \phi(\frac{\partial}{\partial z}) = 0.$

The semi-Riemannian metric g is defined by the matrix:

$$g = \begin{pmatrix} \epsilon y^2 & 0 & -\epsilon y \\ 0 & 0 & 0 \\ -\epsilon y & 0 & \epsilon \end{pmatrix}.$$

(ϕ, ξ, η, g) forms an indefinite trans-Sasakian structure in $\mathbb{R}^3$.

2.5 Anti-invariant submanifolds of an indefinite LP-Sasakian manifold

In this section, first we state and prove two theorems regarding submanifolds of an indefinite LP-Sasakian manifold and then in the last theorem, we get an important result concerning anti-invariant submanifolds of an indefinite LP-Sasakian manifold.

Theorem 2.5.1. *Let M be a submanifold of an indefinite LP-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If M is totally umbilical, then M is totally geodesic.*

Proof. Since ξ is tangent to M,

$$\tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

Using (1.25) we get

$$\epsilon\phi X = \nabla_X \xi + h(X, \xi).$$

Equating tangential and normal components we obtain

$$\epsilon(\phi X)^T = \nabla_X \xi,$$
$$\epsilon(\phi X)^\perp = h(X, \xi).$$

Putting $X = \xi$ in the second equation we get

$$h(\xi, \xi) = 0 \ \ (as\ \phi\xi = 0).$$

Let us assume that M is totally umbilical, then $h(X, Y) = g(X, Y)H$.

Putting $X = Y = \xi$ we get

$$h(\xi, \xi) = g(\xi, \xi)H \Rightarrow 0 = H \text{ and hence } h(X, Y) = 0.$$

Hence, M is totally geodesic.

Theorem 2.5.2. *Let M be a submanifold of an indefinite LP-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. Then ξ is parallel with respect to the induced connection on M if and only if M is an anti-invariant submanifold in $\tilde{M}$.*

Proof. Using (1.25), (1.26) we have,

$$\epsilon\phi X = \tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{2.5.1}$$

Now let ξ be parallel with respect to the induced connection on M, then we have,

$$\nabla_X \xi = 0.$$

From (2.5.1) we get

$$\epsilon\phi X = h(X, \xi)$$
$$\Rightarrow \phi X = \epsilon h(X, \xi).$$

Hence, ϕX is normal to M, $\phi X \in T_x^\perp M$. Thus, M is anti-invariant.

Conversely, let M be anti-invariant, then for $X \in T_x(M)$, $\phi X \in T_x^\perp M$, and so from (2.5.1) we get as $h(X, \xi) \in T_x^\perp M$,

$$\epsilon\phi X = h(X, \xi)$$

and so from (2.5.1) we have, $\nabla_X \xi = 0 \Rightarrow \xi$ is parallel with respect to the induced connection on M.

Theorem 2.5.3. *Let M be an anti-invariant submanifold of an indefinite LP-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If $A_V X = 0$ for any $V \in T_x^\perp M$, then $\phi(T_x(M))$ is parallel with respect to the normal connection.*

Proof. To show that $\phi(T_x(M))$ is parallel with respect to the normal connection $\nabla^{\perp}$, we have to show that for every local section ϕY in $\phi(T_x(M))) \subset T_x^{\perp}M$, $\nabla_X^{\perp}(\phi Y)$ is also a local section in $\phi(T_x(M))$.

Using (1.24), (1.26), (1.27) we obtain

$$\begin{aligned}
&\tilde{g}(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi = (\tilde{\nabla}_X\phi)Y = \tilde{\nabla}_X\phi Y - \phi(\tilde{\nabla}_X Y)\\
&= -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \phi(\nabla_X Y + h(X,Y))\\
\Rightarrow &\nabla_X^{\perp}\phi Y = \tilde{g}(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi + A_{\phi Y}X + \phi(\nabla_X Y) + \phi(h(X,Y)).
\end{aligned}$$

Since $A_V X = 0$ for any $V \in T_x^{\perp}M$, using (1.28) we have

$$\begin{aligned}
g(\nabla_X^{\perp}\phi Y, V) &= g(X,Y)g(\xi,V) + \epsilon\eta(Y)g(X,V) + 2\epsilon\eta(X)\eta(Y)g(\xi,V)\\
&\quad + g(A_{\phi Y}X, V) + g(\phi(\nabla_X Y), V) + g(\phi h(X,Y), V)\\
&= g(\phi(\nabla_X Y), V)\\
\Rightarrow \nabla_X^{\perp}\phi Y &= \phi(\nabla_X Y).
\end{aligned}$$

Hence the proof.

■ **Example of an indefinite LP-Sasakian manifold.**

Let $\mathbb{R}^3$ be a 3-dimensional Euclidean space with regular coordinates (x,y,z). In $\mathbb{R}^3$ we define,

$\eta = -dz - ydx$, $\xi = \frac{\partial}{\partial z}$ and $\phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$, $\phi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$, $\phi(\frac{\partial}{\partial z}) = 0$.

The Lorentzian metric g is defined by the matrix: $g = \begin{pmatrix} -\epsilon y^2 & 0 & \epsilon y \\ 0 & 0 & 0 \\ \epsilon y & 0 & -\epsilon \end{pmatrix}$.

(ϕ, ξ, η, g) forms an indefinite Lorentzian Para-Sasakian structure in $\mathbb{R}^3$.

3

SOME SUBMANIFOLDS OF SOME DIFFERENTIABLE MANIFOLDS WITH RESPECT TO QUARTER SYMMETRIC CONNECTIONS AND ADMITTING $*$-η-RICCI-YAMABE SOLITON

3.1 Introduction

CR-submanifold was introduced by A. Bejancu ([9], [10], [11]) and several research papers on the geometry of CR-submanifolds such as [90], [92], [93], [101] have been written till date. Some research papers on CR-submanifolds of trans-Sasakian manifolds are given by [85], [102], [128], [130]. M. Ahmad studied CR-submanifolds of Lorentzian para-Sasakian manifolds endowed with a quarter symmetric metric connection [2] and T. Pal et al. discussed CR-submanifolds of $(LCS)_n$-manifolds with respect to a quarter symmetric non-metric connection [109] in detail. Motivated from their works, in this chapter, we have studied some submanifolds like anti-invariant submanifolds and CR-submanifolds of some differentiable manifolds like Kenmotsu manifolds and trans-Sasakian manifolds with respect to quarter symmetric metric connection and non-metric connection respectively.

This chapter consists of six sections. After Introduction and Preliminaries sections, in the third section, we have discussed $*$-η-Ricci-Yamabe solitons on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection. First, we have obtained some results regarding a Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton. Next, we have proved some curvature properties of anti-invariant submanifolds of Kenmotsu manifold admitting a quarter symmetric metric connection. Then, we have obtained a result regarding anti-invariant submanifolds of Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton with respect to a quarter symmetric metric connection. Further, we have studied the nature of a $*$-η-Ricci-Yamabe soliton and solitons appeared as its particular cases on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection when the vector field becomes a conformal Killing vector field. Finally, in the fourth section, we have given an example of a 3-dimensional Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton to verify a relation. In the fifth section, we have dealt with the study of trans-Sasakian manifolds with respect to a quarter symmetric non-metric connection. We have stated and proved some results regarding a contact CR-submanifold of a trans-Sasakian manifold with respect to a quarter symmetric non-metric connection. We have investigated totally geodesic leaves and integrability of the distributions. Moreover, we have studied the totally umbilical contact CR-submanifolds of trans-Sasakian manifolds. At last, in the sixth section, we have provided an example of a 3-dimensional trans-Sasakian manifold admitting a quarter symmetric non-metric connection to verify a relation.

3.2 Preliminaries

On an n-dimensional (n is odd) Kenmotsu manifold $M(\phi, \xi, \eta, g)$, the following relations hold [84] $\forall X, Y \in \chi(M)$,

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.2.1}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{3.2.2}$$

$$R(X,\xi)\xi = \eta(X)\xi - X, \tag{3.2.3}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{3.2.4}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{3.2.5}$$

$$Q\xi = -(n-1)\xi, \tag{3.2.6}$$

where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator of M.

Now from [72] we have that the $*$-Ricci tensor S^* on an n-dimensional α-cosymplectic manifold M is given by

$$S^*(X,Y) = S(X,Y) + \alpha^2(n-2)g(X,Y) + \alpha^2\eta(X)\eta(Y) \ \ \forall X, Y \in \chi(M).$$

Hence, putting $\alpha = 1$, we obtain that, on an n-dimensional Kenmotsu manifold M, S^* is given by $\forall X, Y \in \chi(M)$,

$$S^*(X,Y) = S(X,Y) + (n-2)g(X,Y) + \eta(X)\eta(Y), \tag{3.2.7}$$

and hence, we get the $*$-scalar curvature r^* is given by

$$r^* = r + (n-1)^2, \tag{3.2.8}$$

where r is the scalar curvature of M.

We now write down some definitions.

Definition 3.2.1. *Quarter symmetric linear connection* on a smooth manifold $\tilde{M}$ introduced by S. Golab [58], is a linear connection $\bar{\nabla}$ such that its torsion tensor T defined by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] \ \ \forall X, Y \in \chi(M)$$

satisfies the following equation−

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{3.2.9}$$

If $\phi X = X$ in particular, then it reduces to *semi-symmetric connection* introduced by A. Friedmann and J. A. Shouten [56]. Further, if $(\bar{\nabla}_X g)(Y,Z) = 0$ (or $\neq 0$) $\forall X,Y,Z \in \chi(\tilde{M})$, then $\bar{\nabla}$ is called a *quarter symmetric metric (or non-metric) connection* [43].

Let us consider a linear connection $\bar{\bar{\tilde{\nabla}}}$ on a trans-Sasakian manifold $\tilde{M}$ given by [109]

$$\bar{\bar{\tilde{\nabla}}}_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X + a(X)\phi Y, \tag{3.2.10}$$

where a is a 1-form associated to a vector field A on $\tilde{M}$ by

$$g(X,A) = a(X) \tag{3.2.11}$$

$\forall X,Y \in \chi(\tilde{M})$. If $\bar{\bar{\tilde{T}}}$ be the torsion tensor of $\tilde{M}$ with respect to $\bar{\bar{\tilde{\nabla}}}$, then from (3.2.10) we find,

$$\bar{\bar{\tilde{T}}}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y + a(X)\phi Y - a(Y)\phi X. \tag{3.2.12}$$

Furthermore $\forall X,Y,Z \in \chi(\tilde{M})$,

$$(\bar{\bar{\tilde{\nabla}}}_X g)(Y,Z) = -\eta(Y)g(\phi X,Z) - \eta(Z)g(\phi X,Y). \tag{3.2.13}$$

Thus, $\bar{\bar{\tilde{\nabla}}}$ given in (3.2.10) satisfying (3.2.12) and (3.2.13) is a quarter symmetric non-metric connection.

Now, for $\tilde{M}$ with respect to $\bar{\bar{\tilde{\nabla}}}$ we get

$$(\bar{\bar{\tilde{\nabla}}}_X \phi)Y = \alpha g(X,Y)\xi + \beta g(\phi X,Y)\xi + (1-\alpha)\eta(Y)X - \beta\eta(Y)\phi X - \eta(X)\eta(Y)\xi \tag{3.2.14}$$

and

$$\bar{\bar{\tilde{\nabla}}}_X \xi = (1-\alpha)\phi X + \beta[X - \eta(X)\xi]. \tag{3.2.15}$$

Next, let ∇ and $\tilde{\nabla}$ be the Levi-Civita connection and a linear connection on an almost contact metric manifold M respectively such that $\forall X,Y \in \chi(M)$,

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{3.2.16}$$

where H is a $(1,1)$ tensor field. For $\tilde{\nabla}$ to be a quarter symmetric metric connection on M we have [43],

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{3.2.17}$$

where $\forall X, Y, Z \in \chi(M)$,

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{3.2.18}$$

Now, using (3.2.9) in (3.2.18) we get

$$T'(X,Y) = g(X,\phi Y)\xi - \eta(X)\phi Y. \tag{3.2.19}$$

Then, applying (3.2.9) and (3.2.19) on (3.2.17) we obtain

$$H(X,Y) = -\eta(X)\phi Y. \tag{3.2.20}$$

Hence, from (3.2.16) we have, a quarter symmetric metric connection $\tilde{\nabla}$ on a Kenmotsu manifold M is given by $\forall X, Y \in \chi(M)$,

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.2.21}$$

Definition 3.2.2. A vector field V is called *conformal Killing vector field* if and only if the following relation holds−

$$L_V g = 2\kappa g, \tag{3.2.22}$$

where κ is a function of the co-ordinates. If κ is not a constant, then V is called proper. Also, if κ is a constant, then V is called homothetic vector field and if κ is non-zero, then V is called proper homothetic vector field. If $\kappa = 0$, then V becomes Killing vector field.

3.3 $*$-η-Ricci-Yamabe solitons on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection

Let $\tilde{M}(\phi, \xi, \eta, g)$ be an n-dimensional Kenmotsu manifold (n is odd) admitting a $*$-η-Ricci-Yamabe soliton given by (1.38). Applying (1.5) on the following equation−

$$(L_\xi g)(Y,Z) = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) \ \ \forall Y, Z \in \chi(\tilde{M}),$$

we get

$$(L_\xi g)(Y,Z) = 2[g(Y,Z) - \eta(Y)\eta(Z)]. \tag{3.3.1}$$

Now, from (1.38) we have

$$(L_\xi g)(Y,Z) + 2pS^*(Y,Z) + (2\lambda - qr^*)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0. \tag{3.3.2}$$

Applying (3.2.7), (3.2.8) on (3.3.2) we get

$$(L_\xi g)(Y,Z) + 2pS(Y,Z) + [2\lambda - q\{r + (n-1)^2\} + 2p(n-2)]g(Y,Z)$$
$$+2(p+\mu)\eta(Y)\eta(Z) = 0. \tag{3.3.3}$$

Using (3.3.1) in (3.3.3) we get

$$2pS(Y,Z) + [2(\lambda+1) - q\{r + (n-1)^2\} + 2p(n-2)]g(Y,Z)$$
$$+2(p+\mu-1)\eta(Y)\eta(Z) = 0. \tag{3.3.4}$$

Putting $Y = Z = \xi$ in (3.3.4) and then using (3.2.5) we obtain

$$\lambda + \mu = \frac{q}{2}\{r + (n-1)^2\}. \tag{3.3.5}$$

Hence, we state that−

Theorem 3.3.1. *If an n-dimensional Kenmotsu manifold admits a ∗-η-Ricci-Yamabe soliton, then the soliton scalars λ, μ, q are related by the following equation−*

$$\lambda + \mu = \frac{q}{2}\{r + (n-1)^2\}.$$

Let $\{e_i\}_{i=1}^n$ be an orthonormal frame of $T\tilde{M}$, then putting $Y = Z = e_i$ and replacing the value of μ from (3.3.5) in (3.3.3) we get

$$2div(\xi) + 2pr + [2\lambda - q\{r + (n-1)^2\} + 2p(n-2)]n + 2p + q\{r + (n-1)^2\} - 2\lambda = 0$$

$$\Rightarrow \lambda = -\frac{div(\xi)}{n-1} + \left(\frac{q}{2} - \frac{p}{n-1}\right)\{r + (n-1)^2\}, \tag{3.3.6}$$

where $div(\xi)$ is the divergence of ξ.

Replacing the value of λ from (3.3.6) in (3.3.5) we obtain

$$\mu = \frac{1}{n-1}[div(\xi) + p\{r + (n-1)^2\}]. \tag{3.3.7}$$

Thus, we have the following corollary−

Corollary 3.3.1. *If an n-dimensional Kenmotsu manifold admits a ∗-η-Ricci-Yamabe soliton, then the soliton scalars λ, μ, p, q are related by the equations (3.3.6) and (3.3.7).*

Now, let us consider an m-dimensional anti-invariant submanifold M of an n-dimensional ($n > m$) Kenmotsu manifold $\tilde{M}$ admitting a quarter symmetric metric connection $\tilde{\nabla}$ given by (3.2.21). Let $R,\ S,\ r$ be the Riemannian curvature tensor, Ricci tensor, scalar curvature of M with respect to the Levi-Civita connection ∇ respectively and $\tilde{R},\ \tilde{S},\ \tilde{r}$ be the curvature tensor, Ricci tensor, scalar curvature of M with respect to $\tilde{\nabla}$ respectively.

Using (1.4), (1.5), (1.30), (3.2.21) in

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z \ \ \forall X,Y,Z \in \chi(M)$$

we obtain

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X. \tag{3.3.8}$$

Taking inner product of (3.3.8) with $W \in \chi(M)$ and applying (1.30) in the resultant equation we get

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W). \tag{3.3.9}$$

Contracting (3.3.9) over X and W we obtain

$$\tilde{S}(Y,Z) = S(Y,Z), \tag{3.3.10}$$

and again contracting (3.3.10) over Y and Z we obtain

$$\tilde{r} = r. \tag{3.3.11}$$

Hence, we state that−

Theorem 3.3.2. *For an anti-invariant submanifold M of a Kenmotsu manifold admitting the quarter symmetric metric connection $\tilde{\nabla}$,*

(i) the curvature tensor $\tilde{R}$ of M with respect to $\tilde{\nabla}$ is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X \ \ \forall X,Y,Z \in \chi(M),$$

(ii) $\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) \ \ \forall X,Y,Z,W \in \chi(M)$,
(iii) the Ricci tensors S and $\tilde{S}$ of M with respect to ∇ and $\tilde{\nabla}$ respectively are equal,
(iv) the scalar curvatures r and $\tilde{r}$ of M with respect to ∇ and $\tilde{\nabla}$ respectively are equal.

Now, let M admit a $*$-η-Ricci-Yamabe soliton with respect to $\tilde{\nabla}$. Replacing n by m in (3.3.3) we get $\forall Y,Z \in \chi(M)$,

$$(L_\xi g)(Y,Z) + 2pS(Y,Z) + [2\lambda - q\{r + (m-1)^2\} + 2p(m-2)]g(Y,Z)$$
$$+2(p+\mu)\eta(Y)\eta(Z) = 0. \tag{3.3.12}$$

Similarly, with respect to $\tilde{\nabla}$ we have

$$(\tilde{L}_\xi g)(Y,Z) + 2p\tilde{S}(Y,Z) + [2\lambda - q\{\tilde{r} + (m-1)^2\} + 2p(m-2)]g(Y,Z)$$
$$+2(p+\mu)\eta(Y)\eta(Z) = 0, \tag{3.3.13}$$

where $\tilde{L}_\xi$ is the Lie derivative along ξ with respect to $\tilde{\nabla}$.

Using (3.3.10), (3.3.11) in (3.3.13) and comparing the resultant equation with (3.3.12) we obtain

$$(\tilde{L}_\xi g)(Y,Z) = (L_\xi g)(Y,Z). \tag{3.3.14}$$

Thus, we have the following theorem−

Theorem 3.3.3. *For an anti-invariant submanifold of a Kenmotsu manifold admitting the $*$-η-Ricci-Yamabe soliton with respect to the quarter symmetric metric connection $\tilde{\nabla}$, the Lie derivatives along ξ with respect to ∇ and $\tilde{\nabla}$ are equal.*

Next, let M be an m-dimensional anti-invariant submanifold of an n-dimensional ($n > m$) Kenmotsu manifold $\tilde{M}$ admitting a $*$-η-Ricci-Yamabe soliton $(g, V, \lambda, \mu, p, q)$ with respect to a quarter symmetric metric connection $\tilde{\nabla}$ given by (3.2.21) such that V is a conformal Killing vector field given by (3.2.22). Then, replacing ξ by V in (3.3.13) we get $\forall Y, Z \in \chi(M)$,

$$(\tilde{L}_V g)(Y,Z) + 2p\tilde{S}(Y,Z) + [2\lambda - q\{\tilde{r} + (m-1)^2\} + 2p(m-2)]g(Y,Z)$$
$$+2(p+\mu)\eta(Y)\eta(Z) = 0. \tag{3.3.15}$$

Applying (3.3.14) (by replacing ξ by V), (3.2.22), (3.3.10), (3.3.11) on (3.3.15) we obtain

$$[2\kappa + 2\lambda - q\{r + (m-1)^2\} + 2p(m-2)]g(Y,Z) + 2pS(Y,Z) + 2(p+\mu)\eta(Y)\eta(Z) = 0. \tag{3.3.16}$$

Taking $Y = Z = \xi$ in (3.3.16) and using (3.2.5) (by replacing n by m) we get

$$[2\kappa + 2\lambda - q\{r + (m-1)^2\} + 2p(m-2)] - 2p(m-1) + 2(p+\mu) = 0$$
$$\Rightarrow \kappa = -(\lambda+\mu) + \frac{q}{2}\{r + (m-1)^2\}. \tag{3.3.17}$$

Therefore, we have the following theorem−

Theorem 3.3.4. *Let an m-dimensional anti-invariant submanifold of a Kenmotsu manifold admit a $*$-η-Ricci-Yamabe soliton $(g, V, \lambda, \mu, p, q)$ with respect to the quarter symmetric metric connection $\tilde{\nabla}$. If V is a conformal Killing vector field given by (3.2.22), then*

$$\kappa = -(\lambda+\mu) + \frac{q}{2}\{r + (m-1)^2\}.$$

Now, putting $q = 0$ in (3.3.17) we get

$$\kappa = -(\lambda+\mu) \tag{3.3.18}$$

and hence, we state that−

Corollary 3.3.2. *Let an m-dimensional anti-invariant submanifold of a Kenmotsu manifold admit a $*$-η-Ricci soliton with respect to the quarter symmetric metric connection $\tilde{\nabla}$. If V is a conformal Killing vector field, then (3.3.18) holds.*

Again, putting $q = 1$ in (3.3.17) we have

$$\kappa = -(\lambda + \mu) + \frac{1}{2}\{r + (m-1)^2\} \tag{3.3.19}$$

and thus, we have the following corollary−

Corollary 3.3.3. *Let an m-dimensional anti-invariant submanifold of a Kenmotsu manifold admit a $*$-η-Yamabe soliton with respect to the quarter symmetric metric connection $\tilde{\nabla}$. If V is a conformal Killing vector field, then (3.3.19) holds.*

Finally, putting $q = -1$ in (3.3.17) we obtain

$$\kappa = -(\lambda + \mu) - \frac{1}{2}\{r + (m-1)^2\} \tag{3.3.20}$$

and hence, we have another corollary−

Corollary 3.3.4. *Let an m-dimensional anti-invariant submanifold of a Kenmotsu manifold admit a $*$-η-Einstein soliton with respect to the quarter symmetric metric connection $\tilde{\nabla}$. If V is a conformal Killing vector field, then (3.3.20) holds.*

3.4 Example of a Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton

Now, we give the following example of a 3-dimensional Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton and verify the relation (3.3.5) on it.

Let us consider the 3-dimensional manifold $\tilde{M}$ =$\{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are standard co-ordinates of $\mathbb{R}^3$.

The vector fields $e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$, $e_3 = -z\frac{\partial}{\partial z}$ form a linearly independent frame on $\tilde{M}$.

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = 1 \; if \; i = j \; and \; g(e_i, e_j) = 0 \; if \; i \neq j, \; i, j = 1, 2, 3.$$

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi e_1 = -e_2, \; \phi e_2 = e_1, \; \phi e_3 = 0.$$

Hence, we have $\phi^2 Z = -Z + \eta(Z)\xi$ and $g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$ $\forall Z, W \in \chi(\tilde{M})$, where η is the 1-form defined by $\eta(Z) = g(Z, e_3)$ $\forall Z \in \chi(\tilde{M})$ so that $\xi = e_3$ satisfying $\eta(\xi) = 1$.

Let ∇ be the Levi-Civita connection with respect to g, then we have

$$[e_1, e_3] = e_1,\ [e_2, e_3] = e_2 \ and\ [e_i, e_j] = 0 \ for\ all\ other\ values\ of\ i, j.$$

By Koszul's formula given by $\forall X, Y, Z \in \chi(\tilde{M})$,

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z])$$

we calculate—

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = e_1, \\ &\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -e_3,\ \nabla_{e_2} e_3 = e_2, \\ &\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0. \end{aligned}$$

Hence, it can be easily verified that $\nabla_X \xi = X - \eta(X)\xi$ and $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ $\forall X, Y \in \chi(\tilde{M})$ for $\xi = e_3$.

Therefore, $\tilde{M}(\phi, \xi, \eta, g)$ is a Kenmotsu manifold.

Now, let $\tilde{M}$ admit a $*$-η-Ricci-Yamabe soliton given by (1.38). We now verify the relation (3.3.5).

Let us take $Y = Z = e_3$. Then, from (3.3.1) we have

$$(L_\xi g)(e_3, e_3) = 0. \tag{3.4.1}$$

The Riemannian curvature tensor R is given by $\forall X, Y, Z \in \chi(\tilde{M})$,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Therefore, we calculate—

$$\begin{aligned} &R(e_1, e_1)e_1 = 0,\ R(e_2, e_1)e_1 = -e_2,\ R(e_3, e_1)e_1 = -e_3, \\ &R(e_1, e_2)e_2 = -e_1,\ R(e_2, e_2)e_2 = 0,\ R(e_3, e_2)e_2 = -e_3, \\ &R(e_1, e_3)e_3 = -e_1,\ R(e_2, e_3)e_3 = -e_2,\ R(e_3, e_3)e_3 = 0. \end{aligned}$$

From the above relations we obtain

$$S(e_1, e_1) = -2,\ S(e_2, e_2) = -2,\ S(e_3, e_3) = -2. \tag{3.4.2}$$

Hence, the scalar curvature is given by

$$r = \sum_{i=1}^{3} S(e_i, e_i) = -6. \tag{3.4.3}$$

Now, putting $n = 3$ and $Y = Z = e_3$ in (3.3.3) we get

$$(L_\xi g)(e_3, e_3) + 2pS(e_3, e_3) + [2\lambda - q\{r + 2^2\} + 2p]g(e_3, e_3) + 2(p + \mu) = 0.$$

Using (3.4.1), (3.4.2), (3.4.3) in the above equation we obtain

$$\lambda + \mu = -q = \frac{q}{2}\{-6 + 2^2\} = \frac{q}{2}\{r + (n-1)^2\}$$

, which shows that λ, μ, q satisfy the relation (3.3.5).

3.5 Trans-Sasakian manifolds with respect to a quarter symmetric non-metric connection

This section consists of three subsections but before proving the results of those subsections, we now first compute the curvature tensor, Ricci tensor and scalar curvature of a trans-Sasakian manifold with respect to the quarter symmetric non-metric connection $\bar{\tilde{\nabla}}$ given in (3.2.10).

Let with respect to $\bar{\tilde{\nabla}}$ and $\tilde{\nabla}$, the curvature tensors of an n-dimensional trans-Sasakian manifold $\tilde{M}$ (n is odd) of type (α, β) be $\bar{\tilde{R}}$ and $\tilde{R}$ respectively, the Ricci tensors of $\tilde{M}$ be $\bar{\tilde{S}}$ and $\tilde{S}$ respectively and the scalar curvatures of $\tilde{M}$ be $\bar{\tilde{r}}$ and $\tilde{r}$ respectively. Then we obtain $\forall X, Y, Z \in \chi(\tilde{M})$,

$$\begin{aligned}\bar{\tilde{R}}(X,Y)Z = \tilde{R}(X,Y)Z + da(X,Y)\phi Z + (1-\alpha)[a(Y)\eta(Z)X - a(X)\eta(Z)Y] \\ +\alpha[-\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - a(X)g(Y,Z)\xi + a(Y)g(X,Z)\xi + g(\phi Y,Z)\phi X \\ -g(\phi X,Z)\phi Y] + \beta[2\eta(Z)g(\phi X,Y)\xi - a(X)g(\phi Y,Z)\xi + a(Y)g(\phi X,Z)\xi \\ +a(X)\eta(Z)\phi Y - a(Y)\eta(Z)\phi X - g(Y,Z)\phi X + g(X,Z)\phi Y] + a(X)\eta(Y)\eta(Z)\xi \\ -a(Y)\eta(X)\eta(Z)\xi,\end{aligned} \tag{3.5.1}$$

after contraction we obtain

$$\begin{aligned}\bar{\tilde{S}}(Y,Z) = \tilde{S}(Y,Z) + da(Y,\phi Z) + [(n-1)(1-\alpha) + \alpha - \lambda\beta - 1]a(Y)\eta(Z) \\ +[-\alpha n + a(\xi)]\eta(Y)\eta(Z) + [\alpha\{1 - a(\xi)\} - \lambda\beta]g(Y,Z) \\ +[-\alpha\lambda + \beta\{1 - a(\xi)\}]g(\phi Y,Z) + \beta a(\phi Y)\eta(Z)\end{aligned} \tag{3.5.2}$$

and

$$\bar{\bar{r}} = \tilde{r} + \mu + [(n-1)(1-2\alpha) - 2\lambda\beta]a(\xi) - \lambda\beta(n-1) - \alpha\lambda^2, \qquad (3.5.3)$$

where $\lambda = trace(\phi)$ and $\mu = trace(da)$. Thus, we have the following:

Theorem 3.5.1. *$\bar{\bar{R}}$, $\bar{\bar{S}}$ and $\bar{\bar{r}}$ of an n-dimensional trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$ are given in (3.5.1), (3.5.2) and (3.5.3) respectively.*

3.5.1 Contact CR-submanifolds of a trans-Sasakian manifold with respect to a quarter symmetric non-metric connection

In this subsection, we state and prove some results regarding a contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ with respect to the quarter symmetric non-metric connection given in (3.2.10).

Let ∇ be the induced connection on M from the connection $\tilde{\nabla}$ and $\bar{\nabla}$ be the induced connection on M from the connection $\bar{\bar{\nabla}}$. Let h and $\bar{h}$ be second fundamental forms with respect to ∇ and $\bar{\nabla}$ respectively. Then we have for $X, Y \in \Gamma(TM)$,

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X, Y). \qquad (3.5.4)$$

From (1.26), (3.2.10) and (3.5.4) we get

$$\bar{\nabla}_X Y + \bar{h}(X, Y) = \nabla_X Y + h(X, Y) + \eta(Y)\phi X + a(X)\phi Y. \qquad (3.5.5)$$

Using (1.34) in (3.5.5) we get

$$P\bar{\nabla}_X Y + Q\bar{\nabla}_X Y + \bar{h}(X, Y) = P\nabla_X Y + Q\nabla_X Y + h(X, Y) + \eta(Y)\phi PX + \eta(Y)\phi QX$$
$$+ a(X)\phi PY + a(X)\phi QY. \qquad (3.5.6)$$

Comparing horizontal, vertical and normal parts from both sides of (3.5.6) we get

$$P\bar{\nabla}_X Y = P\nabla_X Y + \eta(Y)\phi PX + a(X)\phi PY, \qquad (3.5.7)$$
$$Q\bar{\nabla}_X Y = Q\nabla_X Y, \qquad (3.5.8)$$
$$\bar{h}(X, Y) = h(X, Y) + \eta(Y)\phi QX + a(X)\phi QY. \qquad (3.5.9)$$

Now, for $X, Y \in D$ from (3.5.5) we get

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X + a(X)\phi Y \qquad (3.5.10)$$

and

$$\bar{h}(X, Y) = h(X, Y). \qquad (3.5.11)$$

Hence, we have the following:

Theorem 3.5.1.1. *If M is a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ admitting the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$, then*
(i) the induced connection $\bar{\nabla}$ on M is also a quarter symmetric non-metric connection,
(ii) the second fundamental forms h and $\bar{h}$ are equal.

Again for $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ from Weingarten formula for quarter symmetric non-metric connection $\bar{\bar{\nabla}}$ we have

$$\bar{\bar{\nabla}}_X V = -\bar{A}_V X + \bar{\nabla}^{\perp}_X V. \tag{3.5.12}$$

Also from (1.27) and (3.2.10) we get

$$\bar{\bar{\nabla}}_X V = -A_V X + \nabla^{\perp}_X V + a(X)\phi V. \tag{3.5.13}$$

From (3.5.12) and (3.5.13) we obtain

$$-\bar{A}_V X + \bar{\nabla}^{\perp}_X V = -A_V X + \nabla^{\perp}_X V + a(X)\phi V. \tag{3.5.14}$$

Now, for $Z \in \Gamma(D^{\perp})$ we have $\phi Z \in \Gamma(T^{\perp}M)$ and hence, from (3.5.14) we obtain

$$-\bar{A}_{\phi Z} X + \bar{\nabla}^{\perp}_X \phi Z = -A_{\phi Z} X + \nabla^{\perp}_X \phi Z + a(X)[-Z + \eta(Z)\xi],$$

from which we get

$$\bar{A}_{\phi Z} X = A_{\phi Z} X - a(X)[-Z + \eta(Z)\xi] \tag{3.5.15}$$

and

$$\bar{\nabla}^{\perp}_X \phi Z = \nabla^{\perp}_X \phi Z. \tag{3.5.16}$$

Theorem 3.5.1.2. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$, then $\forall X, Y \in \Gamma(TM)$,*

$$P\bar{\nabla}_X \phi PY - P\bar{A}_{\phi QY} X = \phi P(\bar{\nabla}_X Y) + \alpha g(X,Y)P\xi + \beta g(\phi X, Y)P\xi$$
$$+(1-\alpha)\eta(Y)PX - \beta\eta(Y)\phi PX - \eta(X)\eta(Y)P\xi, \tag{3.5.17}$$

$$Q\bar{\nabla}_X \phi PY - Q\bar{A}_{\phi QY} X = B\bar{h}(X,Y) + \alpha g(X,Y)Q\xi + (1-\alpha)\eta(Y)QX$$
$$+\beta g(\phi X, Y)Q\xi - \eta(X)\eta(Y)Q\xi, \tag{3.5.18}$$

$$\bar{h}(X, \phi PY) + \bar{\nabla}^{\perp}_X \phi QY = \phi Q(\bar{\nabla}_X Y) + C\bar{h}(X,Y) - \beta\eta(Y)\phi QX. \tag{3.5.19}$$

Proof. From (3.2.14) we have $\forall X, Y \in \Gamma(TM)$,

$$\bar{\bar{\nabla}}_X \phi Y - \phi(\bar{\bar{\nabla}}_X Y)$$

$$= \alpha g(X,Y)\xi + \beta g(\phi X,Y)\xi + (1-\alpha)\eta(Y)X - \beta\eta(Y)\phi X - \eta(X)\eta(Y)\xi. \quad (3.5.20)$$

Applying (1.34), (1.35), (3.5.4) and (3.5.12) on (3.5.20) we get

$$P\bar{\nabla}_X\phi PY + Q\bar{\nabla}_X\phi PY + \bar{h}(X,\phi PY) - P\bar{A}_{\phi QY}X - Q\bar{A}_{\phi QY}X + \bar{\nabla}^{\perp}_X\phi QY - \phi P(\bar{\nabla}_X Y)$$

$$-\phi Q(\bar{\nabla}_X Y) - B\bar{h}(X,Y) - C\bar{h}(X,Y) = \alpha g(X,Y)P\xi + \alpha g(X,Y)Q\xi + \beta g(\phi X,Y)P\xi$$

$$+\beta g(\phi X,Y)Q\xi + (1-\alpha)\eta(Y)PX + (1-\alpha)\eta(Y)QX - \beta\eta(Y)\phi PX - \beta\eta(Y)\phi QX$$

$$-\eta(X)\eta(Y)P\xi - \eta(X)\eta(Y)Q\xi. \quad (3.5.21)$$

Equating horizontal, vertical and normal parts from both sides of (3.5.21) we get (3.5.17), (3.5.18) and (3.5.19) respectively.

3.5.2 Totally geodesic leaves and integrability of the distributions

In this subsection, we obtain the necessary and sufficient conditions of integrability of the distributions D and $D^{\perp}$ of a contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ in different cases. We also discuss the cases where the leaves of D and $D^{\perp}$ are totally geodesic.

Lemma 3.5.2.1. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. Then $\forall Z,W \in D^{\perp}$,*

$$\begin{aligned}\phi P[W,Z] = & A_{\phi W}Z - A_{\phi Z}W - a(Z)[-W + \eta(W)\xi] + a(W)[-Z + \eta(Z)\xi] \\ & + (\alpha-1)[\eta(Z)W - \eta(W)Z].\end{aligned}$$

Proof. $\forall Z,W \in D^{\perp}$, $\bar{\bar{\nabla}}_Z\phi W = (\bar{\bar{\nabla}}_Z\phi)W + \phi(\bar{\bar{\nabla}}_Z W)$.

Using (1.34), (1.35), (3.2.14), (3.5.4) and (3.5.12) in the above equation we get

$$\bar{\bar{\nabla}}^{\perp}_Z\phi W = \bar{A}_{\phi W}Z + \phi P(\bar{\nabla}_Z W) + \phi Q(\bar{\nabla}_Z W) + B\bar{h}(W,Z) + C\bar{h}(W,Z)$$

$$+\alpha g(W,Z)\xi + (1-\alpha)\eta(W)Z - \beta\eta(W)\phi Z - \eta(Z)\eta(W)\xi. \quad (3.5.22)$$

Also from (3.5.19) we get

$$\bar{\bar{\nabla}}^{\perp}_Z\phi W = \phi Q(\bar{\nabla}_Z W) + C\bar{h}(W,Z) - \beta\eta(W)\phi Z. \quad (3.5.23)$$

Using (3.5.23) in (3.5.22) we obtain

$$\phi P(\bar{\nabla}_Z W) = -\bar{A}_{\phi W}Z - B\bar{h}(W,Z) - \alpha g(W,Z)\xi + (\alpha-1)\eta(W)Z + \eta(Z)\eta(W)\xi. \quad (3.5.24)$$

Interchanging Z and W in (3.5.24) and subtracting (3.5.24) from the resultant equation we get

$$\phi P[W,Z] = \bar{A}_{\phi W}Z - \bar{A}_{\phi Z}W + (\alpha - 1)[\eta(Z)W - \eta(W)Z].$$

Using (3.5.15) in the above equation we obtain

$$\begin{aligned}\phi P[W,Z] =& A_{\phi W}Z - A_{\phi Z}W - a(Z)[-W + \eta(W)\xi] + a(W)[-Z + \eta(Z)\xi] \\ &+ (\alpha - 1)[\eta(Z)W - \eta(W)Z].\end{aligned}$$

Theorem 3.5.2.1. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*

$$\begin{aligned}A_{\phi W}Z - A_{\phi Z}W =& a(Z)[-W + \eta(W)\xi] - a(W)[-Z + \eta(Z)\xi] \\ &+ (\alpha - 1)[\eta(W)Z - \eta(Z)W].\end{aligned}$$

Proof. It is obvious from Lemma 3.5.2.1.

Corollary 3.5.2.1. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. Then the distribution $D^{\perp}$ is integrable if and only if*

$$\forall Z, W \in D^{\perp},\ A_{\phi W}Z - A_{\phi Z}W = a(W)Z - a(Z)W.$$

Remark 3.5.2.1. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\tilde{\nabla}$. Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*

$$A_{\phi W}Z - A_{\phi Z}W = \alpha[\eta(W)Z - \eta(Z)W].$$

Remark 3.5.2.2. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\tilde{\nabla}$. Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*

$$A_{\phi W}Z = A_{\phi Z}W.$$

Theorem 3.5.2.2. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. Then the distribution D is integrable if and only if $\forall X, Y \in D$,*

$$h(X, \phi Y) = h(\phi X, Y).$$

Proof. From (3.5.11) and (3.5.19) we have $\forall X, Y \in D$,

$$\phi Q(\bar{\nabla}_X Y) = h(X, \phi Y) - Ch(X, Y). \tag{3.5.25}$$

Interchanging X and Y in (3.5.25) and subtracting the resultant equation from (3.5.25) we get

$$\phi Q[X, Y] = h(X, \phi Y) - h(Y, \phi X).$$

Hence, the distribution D is integrable if and only if $\forall X, Y \in D$, $h(X, \phi Y) = h(\phi X, Y)$.

Remark 3.5.2.3. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\tilde{\nabla}$. Then the distribution D is integrable if and only if $\forall X, Y \in D$, $h(X, \phi Y) = h(\phi X, Y)$.*

Theorem 3.5.2.3. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. If the leaf of D is totally geodesic in M, then $\forall X, Y \in D, Z \in D^{\perp}$,*

$$-g(h(X,Y), \phi Z) + (\alpha - 1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X, Y) + \eta(X)\eta(Y)\eta(Z) = 0.$$

Proof. As the leaf of D is totally geodesic in M, $\bar{\nabla}_X \phi Y \in D \quad \forall X, Y \in D$ (since $\phi Y \in D$). Now, $\forall Z \in D^{\perp}$ from (3.5.21) we have

$$\begin{aligned} \phi P(\bar{\nabla}_X Z) = &-\bar{A}_{\phi Z} X + \nabla^{\perp}_X \phi Z - \phi Q(\bar{\nabla}_X Z) - \phi \bar{h}(X, Z) \\ &+(\alpha - 1)\eta(Z)X + \beta\eta(Z)\phi X + \eta(X)\eta(Z)\xi. \end{aligned} \tag{3.5.26}$$

Using (1.3), (1.28), (1.34), (1.35) and (3.5.26) we get

$$\begin{aligned} 0 &= g(\bar{\nabla}_X \phi Y, Z) = -g(\phi Y, \bar{\nabla}_X Z) = -g(\phi Y, P(\bar{\nabla}_X Z)) = g(Y, \phi P(\bar{\nabla}_X Z)) \\ &= -g(\bar{A}_{\phi Z} X, Y) + (\alpha - 1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X, Y) + \eta(X)\eta(Z)\eta(Y) \\ &= -g(h(X,Y), \phi Z) + (\alpha - 1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X, Y) + \eta(X)\eta(Y)\eta(Z). \end{aligned}$$

Corollary 3.5.2.2. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$, then the leaf of D is totally geodesic in M if and only if $\forall X, Y \in D, Z \in D^{\perp}$,*

$$g(h(X,Y), \phi Z) = 0.$$

Proof. The direct part follows from Theorem 3.5.2.3.

Conversely, $\forall X, Y \in D, Z \in D^{\perp}$ (since $\phi Y \in D$),

$$0 = g(h(X, \phi Y), \phi Z) = g(\bar{\bar{\nabla}}_X \phi Y, \phi Z) = g(\phi \bar{\bar{\nabla}}_X Y, \phi Z) = g(\bar{\bar{\nabla}}_X Y, Z) = g(\bar{\nabla}_X Y, Z)$$

which implies that $\forall X, Y \in D$, $\bar{\nabla}_X Y \in D$. Hence, the leaf of D is totally geodesic in M.

Theorem 3.5.2.4. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. If the leaf of $D^{\perp}$ is totally geodesic in M, then $\forall Z, W \in D^{\perp}, X \in D$,*

$$g(h(X,Z), \phi W) + a(X)g(Z,W) + \alpha\eta(X)g(Z,W) - a(X)\eta(Z)\eta(W) - \eta(X)\eta(Z)\eta(W) = 0.$$

Proof. As the leaf of $D^\perp$ is totally geodesic in M, $\forall Z, W \in D^\perp$, $\bar{\nabla}_Z W \in D^\perp$. Now from (3.5.21) we have

$$\phi P(\bar{\nabla}_Z W) = -\bar{A}_{\phi W} Z + \bar{\nabla}^\perp_Z \phi W - \phi Q(\bar{\nabla}_Z W) - \phi \bar{h}(Z, W)$$
$$+(\alpha - 1)\eta(W)Z + \beta\eta(W)\phi Z + \eta(Z)\eta(W)\xi - \alpha g(Z, W)\xi. \quad (3.5.27)$$

Taking inner product of (3.5.27) with $X \in D$ we get (since $\bar{\nabla}_Z W \in D^\perp$)

$$0 = g(\phi P(\bar{\nabla}_Z W), X) = -g(\bar{A}_{\phi W} Z, X) - \alpha g(Z, W)\eta(X) + \eta(Z)\eta(W)\eta(X). \quad (3.5.28)$$

Using (1.3), (1.28) and $\bar{h}(X, Z) = h(X, Z) + a(X)\phi Z$ (obtained from (3.5.9)) in (3.5.28) we get

$$0 = -\left[g(h(X, Z), \phi W) + a(X)g(Z, W) - a(X)\eta(Z)\eta(W)\right] - \alpha\eta(X)g(Z, W)$$
$$+ \eta(X)\eta(Z)\eta(W).$$

Corollary 3.5.2.3. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. If the leaf of $D^\perp$ is totally geodesic in M, then $\forall Z, W \in D^\perp, X \in D$,*

$$g(h(X, Z), \phi W) + a(X)g(Z, W) + \alpha\eta(X)g(Z, W) = 0.$$

Corollary 3.5.2.4. *Let M be a ξ-vertical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. If the leaf of $D^\perp$ is totally geodesic in M, then $\forall Z, W \in D^\perp, X \in D$,*

$$g(h(X, Z), \phi W) + a(X)g(Z, W) - a(X)\eta(Z)\eta(W) = 0.$$

Theorem 3.5.2.5. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the quarter symmetric non-metric connection $\bar{\bar{\nabla}}$. If M is a contact CR-product, then $\forall X \in D, W \in D^\perp$,*

$$A_{\phi W} X + a(X)W + \alpha\eta(X)W = 0.$$

Proof. As the leaf of $D^\perp$ is totally geodesic in M, from Corollary 3.5.2.3 we have $\forall Z, W \in D^\perp, X \in D$,

$$g(A_{\phi W} X + a(X)W + \alpha\eta(X)W, Z) = 0,$$

which implies that

$$A_{\phi W} X + a(X)W + \alpha\eta(X)W \in D. \quad (3.5.29)$$

Again as the leaf of D is totally geodesic in M and $\forall Y \in D$, $\phi Y \in D$, we have $\bar{\nabla}_X \phi Y \in D$. Hence, we get

$$g(A_{\phi W} X + a(X)W + \alpha\eta(X)W, Y) = g(A_{\phi W} X, Y) = g(h(X, Y), \phi W)$$
$$= -g(\phi h(X, Y), W) = -g(\phi(\bar{\bar{\nabla}}_X Y - \bar{\nabla}_X Y), W) = -g(\phi(\bar{\bar{\nabla}}_X Y), W)$$
$$= -g(\bar{\bar{\nabla}}_X \phi Y, W) = -g(\bar{\nabla}_X \phi Y, W) = 0 \ \ (using\ (1.3), (1.28), (3.2.14), (3.5.4), (3.5.11)),$$

which implies that

$$A_{\phi W} X + a(X)W + \alpha\eta(X)W \in D^\perp. \quad (3.5.30)$$

From (3.5.29) and (3.5.30) we obtain $A_{\phi W} X + a(X)W + \alpha\eta(X)W = 0$.

3.5.3 Totally umbilical contact CR-submanifolds

In this subsection, we study totally umbilical contact CR-submanifolds of a trans-Sasakian manifold with respect to $\tilde{\nabla}$.

Let M be a totally umbilical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\tilde{\nabla}$, then from (1.6) we have $\forall Z, W \in D^{\perp}$,

$$\tilde{\nabla}_Z \phi W - \phi(\tilde{\nabla}_Z W) = \alpha[g(Z,W)\xi - \eta(W)Z] - \beta\eta(W)\phi Z. \tag{3.5.31}$$

Using (1.26), (1.27) and (1.34) in (3.5.31) we get

$$\begin{aligned} -A_{\phi W}Z + \nabla^{\perp}_Z \phi W &= \phi P(\nabla_Z W) + \phi Q(\nabla_Z W) + \phi h(Z,W) \\ &+\alpha[g(Z,W)\xi - \eta(W)Z] - \beta\eta(W)\phi Z. \end{aligned} \tag{3.5.32}$$

Taking inner product of (3.5.32) with Z and using (1.28) we get

$$-g(h(Z,Z),\phi W) = g(\phi h(Z,W),Z) + \alpha[g(Z,W)\eta(Z) - \eta(W)\|Z\|^2]. \tag{3.5.33}$$

Using (1.29) in (3.5.33) we get

$$g(H,\phi W) = -\frac{1}{\|Z\|^2}[g(Z,W)g(\phi H,Z) + \alpha\{g(Z,W)\eta(Z) - \eta(W)\|Z\|^2\}]. \tag{3.5.34}$$

Interchanging Z and W in (3.5.34) and using (1.3) we get

$$g(\phi H,Z) = \frac{1}{\|W\|^2}[-g(Z,W)g(H,\phi W) + \alpha\{g(Z,W)\eta(W) - \eta(Z)\|W\|^2\}]. \tag{3.5.35}$$

Substituting (3.5.35) in (3.5.34) we obtain

$$\begin{aligned} g(H,\phi W) &= -\frac{g(Z,W)}{\|Z\|^2}\Big[-\frac{1}{\|W\|^2}g(Z,W)g(H,\phi W) \\ &+\frac{\alpha}{\|W\|^2}\{g(Z,W)\eta(W) - \eta(Z)\|W\|^2\}\Big] - \frac{\alpha}{\|Z\|^2}\{g(Z,W)\eta(Z) - \eta(W)\|Z\|^2\} \\ \Rightarrow \Big[1 - \frac{g(Z,W)^2}{\|Z\|^2\|W\|^2}\Big]&g(H,\phi W) + \alpha\frac{g(Z,W)}{\|Z\|^2}\Big[\frac{\eta(W)g(Z,W)}{\|W\|^2} - \eta(Z)\Big] \\ &+\alpha\Big[\frac{\eta(Z)g(Z,W)}{\|Z\|^2} - \eta(W)\Big] = 0. \end{aligned} \tag{3.5.36}$$

Hence, from (3.5.36) we get the following theorems:

Theorem 3.5.3.1. *Let M be a ξ-vertical totally umbilical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\tilde{\nabla}$, then dim($D^{\perp}$)=1.*

Theorem 3.5.3.2. *Let M be a ξ-horizontal totally umbilical contact CR-submanifold of a*

trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\tilde{\nabla}$, then
(i) M is minimal in $\tilde{M}$
or (ii) dim($D^{\perp}$)=1
or (iii) $H \in \Gamma(\mu)$.

Remark 3.5.3.1. *Theorem 3.5.3.2 also holds good in case of considering $\tilde{M}$ with respect to the quarter symmetric non-metric connection $\bar{\tilde{\nabla}}$.*

3.6 Example of a trans-Sasakian manifold admitting a quarter symmetric non-metric connection

Here we give an example of a 3-dimensional trans-Sasakian manifold from [115] and verify the relation (3.5.1) on it.

Let us consider a 3-dimensional manifold $\tilde{M} = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are the standard co-ordinates of $\mathbb{R}^3$. Let the vector fields

$$E_1 = e^{-2z}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \; E_2 = -e^{-2z}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right), \; E_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of $\tilde{M}$. Let g be the Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 0, & for\ i \neq j \\ 1, & for\ i = j \end{cases}$$

and η be the 1-form defined by $\eta(X) = g(X, E_3)\ \forall X \in \chi(\tilde{M})$. Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = E_2,\ \phi E_2 = -E_1,\ \phi E_3 = 0$. Then we have $\eta(E_3) = 1,\ \phi^2(X) = -X + \eta(X)E_3,\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\ \forall X, Y \in \chi(\tilde{M})$. Let $\tilde{\nabla}$ be the Riemannian connection on $\tilde{M}$ with respect to the metric g. Then we obtain $[E_1, E_2] = 0,\ [E_1, E_3] = 2E_1,\ [E_2, E_3] = 2E_2$. Now, using Koszul's formula for g, it can be calculated that

$$\begin{aligned}
&\tilde{\nabla}_{E_1}E_1 = -2E_3,\ \tilde{\nabla}_{E_1}E_2 = 0,\ \tilde{\nabla}_{E_1}E_3 = 2E_1,\\
&\tilde{\nabla}_{E_2}E_1 = 0,\ \tilde{\nabla}_{E_2}E_2 = -2E_3,\ \tilde{\nabla}_{E_2}E_3 = 2E_2,\\
&\tilde{\nabla}_{E_3}E_1 = 0,\ \tilde{\nabla}_{E_3}E_2 = 0,\ \tilde{\nabla}_{E_3}E_3 = 0.
\end{aligned}$$

Since $\{E_1, E_2, E_3\}$ forms a basis for $\tilde{M}$, then any vector field $X, Y \in \chi(\tilde{M})$ can be written as

$$X = x_1E_1 + x_2E_2 + x_3E_3,\ Y = y_1E_1 + y_2E_2 + y_3E_3,$$

where $x_i, y_i \in \mathbb{R},\ i = 1, 2, 3$. Hence, $g(X, Y) = x_1y_1 + x_2y_2 + x_3y_3$.

Thus

$$\tilde{\nabla}_X Y = 2x_1y_3E_1 + 2x_2y_3E_2 - 2(x_1y_1 + x_2y_2)E_3. \tag{3.6.1}$$

Therefore $\tilde{\nabla}_X\xi = -\alpha\phi X + \beta[X - \eta(X)\xi]$ $\forall X \in \chi(\tilde{M})$ holds for $\alpha = 0,\ \beta = 2$ and $\xi = E_3$. Thus, $(\tilde{M}, g)$ is a 3-dimensional trans-Sasakian manifold of type $(0, 2)$.

We set $A = E_1$. Then $a(X) = g(X, E_1) = x_1$ $\forall X = x_1E_1 + x_2E_2 + x_3E_3 \in \chi(\tilde{M})$. Hence, using (3.6.1) in (3.2.10) we get

$$\tilde{\bar{\bar{\nabla}}}_X Y = (2x_1y_3 - x_2y_3 - x_1y_2)E_1 + (2x_2y_3 + x_1y_3 + x_1y_1)E_2 - 2(x_1y_1 + x_2y_2)E_3. \tag{3.6.2}$$

Also for $Z = z_1E_1 + z_2E_2 + z_3E_3 \in \chi(\tilde{M})$ we have

$$(\tilde{\bar{\bar{\nabla}}}_X g)(Y, Z) = x_2y_3z_1 - x_1y_3z_2 + (x_2y_1 - x_1y_2)z_3 \neq 0.$$

Hence, $\tilde{\bar{\bar{\nabla}}}$(given by (3.6.2)) is a quarter symmetric non-metric connection on $\tilde{M}$.

Now, we will verify the relation (3.5.1) for $X = E_1, Y = E_2, Z = E_2$.

Using the values of $\tilde{\nabla}_{E_i}E_j$ $(i, j = 1, 2)$ given above, we obtain

$$\tilde{R}(E_1, E_2)E_2 = -4E_1, \tag{3.6.3}$$

and using (3.6.2) we obtain

$$\tilde{\bar{\bar{R}}}(E_1, E_2)E_2 = -4E_1 - 2E_2. \tag{3.6.4}$$

Now, from (3.5.1) we get $\tilde{\bar{\bar{R}}}(E_1, E_2)E_2 = \tilde{R}(E_1, E_2)E_2 - 2E_2$ which is satisfied by (3.6.3) and (3.6.4). Hence, the relation (3.5.1) holds for $X = E_1, Y = E_2, Z = E_2$. Similarly we can prove that the relation (3.5.1) holds for other values of $X, Y, Z \in \chi(\tilde{M})$.

4

HEMI-SLANT SUBMANIFOLD OF $(LCS)_n$- MANIFOLD

4.1 Introduction

There are various types of works done on hemi-slant submanifolds. H. I. Abutuqayqah worked on geometry of hemi-slant submanifolds of almost contact manifolds [1]. M. A. Khan et al. discussed about totally umbilical hemi-slant submanifolds of K*ä*hler [3] and cosymplectic manifolds [86], and they also discussed about a classification on totally umbilical proper slant and hemi-slant submanifolds of nearly trans-Sasakian manifold [88]. B. Laha et al. studied totally umbilical hemi-slant submanifolds of LP-Sasakian manifold [91] and hemi-slant submanifolds of Kenmotsu manifold [112]. H. M. Tastan et al. discussed about hemi-slant submanifolds of a locally product Riemannian manifold [144] and a locally conformal K*ä*hler manifold [143]. Another important works on hemi-slant submanifolds were done by A. Lotta in 1996 [95], by M. A. Lone et al. in 2016 [94] and by M. S. Siddesha et al. in 2018 [134]. Also, S. K. Hui et al. studied totally real and contact CR-submanifolds of $(LCS)_n$-manifolds in 2018 [79] and in 2021 [80] respectively. Motivated from these works, in this chapter, we have analysed some properties regarding distributions and leaves of hemi-slant submanifold of $(LCS)_n$-manifold.

This chapter consists of five sections. After Introduction and Preliminaries sections, in the third and fourth sections, we have stated and proved all the main results. At last, in the fifth section, we have constructed a suitable example.

4.2 Preliminaries

Let M be a submanifold of an $(LCS)_n$-manifold $\tilde{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g), and let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$, ∇ be the induced connection on the tangent bundle TM and $\nabla^{\perp}$ be the induced connection on the normal bundle $T^{\perp}M$ of M.

For any $X \in \Gamma(TM)$,

$$\phi X = TX + FX, \qquad (4.2.1)$$

where TX is the tangential component and FX is the normal component of ϕX.

Similarly, for any $V \in \Gamma(T^{\perp}M)$,

$$\phi V = tV + fV, \tag{4.2.2}$$

where tV and fV are the tangential component and the normal component of ϕV respectively.

The covariant derivatives of the tensor fields T, F, t, f are defined as−

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{4.2.3}$$
$$(\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{4.2.4}$$
$$(\tilde{\nabla}_X t)V = \nabla_X tV - t\nabla^{\perp}_X V, \tag{4.2.5}$$
$$(\tilde{\nabla}_X f)V = \nabla^{\perp}_X fV - f\nabla^{\perp}_X V \tag{4.2.6}$$

$\forall X, Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$.

Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$, then for any $X \in \Gamma(TM)$,

$$X = P_1X + P_2X + \eta(X)\xi, \tag{4.2.7}$$

where P_1, P_2 are projection maps on the distributions $D^{\perp}, D_{\theta}$ respectively. Now, operating ϕ on (4.2.7), we get

$$\phi X = \phi P_1 X + \phi P_2 X + \eta(X)\phi\xi.$$

Using (1.8) and (4.2.1), we obtain

$$TX + FX = FP_1X + TP_2X + FP_2X.$$

On comparing the tangential and normal components, we get

$$TX = TP_2X,$$
$$FX = FP_1X + FP_2X.$$

If we denote the orthogonal complement of $\phi(TM)$ in $T^{\perp}M$ by μ, then the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = F(D^{\perp}) \oplus F(D_{\theta}) \oplus \mu. \tag{4.2.8}$$

Since $D^{\perp}$ and D_{θ} are orthogonal distributions, $g(X, Y) = 0$ for each $X \in D^{\perp}$ and $Y \in D_{\theta}$. Hence, by (1.11) and (4.2.1), we have

$$\forall Z \in D^{\perp}, W \in D_{\theta},\ g(FZ, FW) = g(\phi Z, \phi W) = g(Z, W) = 0,$$

which shows that $F(D^{\perp}), F(D_{\theta})$ are mutually perpendicular. So, (4.2.8) is an orthogonal direct decomposition.

4.3 Conditions of integrability of the distributions of hemi-slant submanifolds of $(LCS)_n$-manifold

In this section, we discuss about some necessary and sufficient conditions for distributions to be integrable and obtain some results in this direction.

Theorem 4.3.1. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$, then $\forall Z, W \in D^{\perp}$, $A_{\phi W}Z = A_{\phi Z}W - \alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi$.*

Proof. On using (1.28), we have

$$\begin{aligned} g(A_{\phi W}Z, X) = g(h(Z,X), \phi W) = g(\phi h(Z,X), W) &= g(\phi\tilde{\nabla}_X Z, W) - g(\phi\nabla_X Z, W) \\ &= g(\phi\tilde{\nabla}_X Z, W) = g(\tilde{\nabla}_X\phi Z, W) - g((\tilde{\nabla}_X\phi)Z, W). \end{aligned}$$

Again using (1.13) and (1.27), we get

$$\begin{aligned} g(A_{\phi W}Z, X) &= g(A_{\phi Z}X + \nabla^{\perp}_X\phi Z, W) - \alpha g(g(X,Z)\xi + 2\eta(X)\eta(Z)\xi + \eta(Z)X, W) \\ &= g(A_{\phi Z}X, W) - \alpha g(X,Z)\eta(W) - 2\alpha\eta(X)\eta(Z)\eta(W) - \alpha\eta(Z)g(X,W) \\ &= g(h(W,X), \phi Z) - \alpha g(X,Z)\eta(W) - \alpha\eta(Z)g(X,W) - 2\alpha\eta(X)\eta(Z)\eta(W) \\ &= g(A_{\phi Z}W - \alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi, X) \\ &\Rightarrow A_{\phi W}Z = A_{\phi Z}W - \alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi. \end{aligned}$$

Theorem 4.3.2. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. Then the distribution $D_\theta \oplus D^{\perp}$ is integrable.*

Proof. For $X, Y \in D_\theta \oplus D^{\perp}$,

$$\begin{aligned} g([X,Y],\xi) &= g(\tilde{\nabla}_X Y, \xi) - g(\tilde{\nabla}_Y X, \xi) \\ &= -g(\tilde{\nabla}_X\xi, Y) + g(\tilde{\nabla}_Y\xi, X) \\ &= -g(\alpha\phi X, Y) + g(\alpha\phi Y, X) \\ &= 0. \quad (by\ (1.10)) \end{aligned}$$

Since $TM = D_\theta \oplus D^{\perp} \oplus <\xi>$, therefore $[X,Y] \in D_\theta \oplus D^{\perp}$. So, $D_\theta \oplus D^{\perp}$ is integrable.

Theorem 4.3.3. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if $\forall W \in D^{\perp}$, W is a scalar multiple of ξ.*

Proof. For $Z, W \in D^{\perp}$, from (1.13), we have

$$(\tilde{\nabla}_Z\phi)W = \alpha[g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z]. \tag{4.3.1}$$

After some calculations and using (4.2.1), (4.2.2), we get

$$-A_{FW}Z + \nabla^{\perp}_{Z}FW - T\nabla_{Z}W - F\nabla_{Z}W - th(Z,W) - fh(Z,W)$$
$$= \alpha[g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z]. \quad (4.3.2)$$

Comparing tangential components, we have

$$-A_{FW}Z - T\nabla_{Z}W - th(Z,W) = \alpha[g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z]. \quad (4.3.3)$$

Interchanging Z, W, we obtain

$$-A_{FZ}W - T\nabla_{W}Z - th(W,Z) = \alpha[g(W,Z)\xi + 2\eta(W)\eta(Z)\xi + \eta(Z)W]. \quad (4.3.4)$$

Subtracting (4.3.3) from (4.3.4) and using the fact that h is symmetric, we have

$$A_{FW}Z - A_{FZ}W + T(\nabla_{Z}W - \nabla_{W}Z) = \alpha[\eta(Z)W - \eta(W)Z]. \quad (4.3.5)$$

From (4.3.5), we have

$$A_{FW}Z - A_{FZ}W + T([Z,W]) = \alpha[\eta(Z)W - \eta(W)Z]. \quad (4.3.6)$$

Now $D^{\perp}$ is integrable if and only if $[Z,W] \in D^{\perp}$ and as $D^{\perp}$ is anti-invariant, $\phi D^{\perp} \subseteq T^{\perp}M$ and so, $T[Z,W] = 0$.

Hence, from (4.3.6), $D^{\perp}$ is integrable if and only if $A_{FW}Z - A_{FZ}W = \alpha[\eta(Z)W - \eta(W)Z]$.

From Theorem 4.3.1, we have as $TW = 0 = TZ$,

$$A_{FW}Z - A_{FZ}W = A_{\phi W}Z - A_{\phi Z}W = -\alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi$$
$$\Rightarrow \alpha[\eta(Z)W - \eta(W)Z] = -\alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi$$
$$\Rightarrow 2\alpha\eta(Z)W + 2\alpha\eta(Z)\eta(W)\xi = 0$$
$$\Rightarrow \eta(Z)W + \eta(Z)\eta(W)\xi = 0$$
$$\Rightarrow W + \eta(W)\xi = 0.$$

Hence the result is proved.

Theorem 4.3.4. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. Then the slant distribution D_θ is integrable if and only if $\forall X, Y \in D_\theta$,*

$$P_1(\nabla_X TY - \nabla_Y TX - A_{FY}X + A_{FX}Y) = 0. \quad (4.3.7)$$

Proof. We denote by P_1, P_2 the projections on $D^{\perp}, D_\theta$ respectively. $\forall X, Y \in D_\theta$, we have from (1.13),

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]. \quad (4.3.8)$$

On applying (1.26), (1.27), (4.2.1), (4.2.2), we have

$$(\tilde{\nabla}_X \phi)Y = \nabla_X TY + h(X, TY) - A_{FY}X + \nabla^{\perp}_X FY$$
$$-(T\nabla_X Y + F\nabla_X Y) - (th(X,Y) + fh(X,Y))$$
$$= \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]. \qquad (4.3.9)$$

Comparing tangential components, we get

$$\nabla_X TY - A_{FY}X - T\nabla_X Y - th(X,Y) = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]. \qquad (4.3.10)$$

Interchanging X, Y in (4.3.10) and subtracting the resultant from (4.3.10), we obtain

$$\nabla_X TY - \nabla_Y TX - A_{FY}X + A_{FX}Y - T\nabla_X Y + T\nabla_Y X = \alpha[\eta(Y)X - \eta(X)Y]$$

$$\Rightarrow T[X,Y] = \nabla_X TY - \nabla_Y TX - A_{FY}X + A_{FX}Y - \alpha[\eta(Y)X - \eta(X)Y]. \qquad (4.3.11)$$

Since $X, Y \in D_\theta,\ P_1X = 0 = P_1Y$. Applying P_1 to both sides of (4.3.11), we have

$$P_1T[X,Y] = P_1(\nabla_X TY - \nabla_Y TX - A_{FY}X + A_{FX}Y),$$

which implies (4.3.7).

4.4 Geometry of leaves of hemi-slant submanifold of $(LCS)_n$-manifold

In this section, we derive a result regarding the totally geodesic leaves of an associated distribution.

Theorem 4.4.1. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. If the leaves of $D^\perp$ are totally geodesic in M, then $\forall X \in D_\theta$ and $Z, W \in D^\perp$,*

$$g(h(Z,X), FW) + g(th(Z,W), X) = 0. \qquad (4.4.1)$$

Proof. From (1.13), (1.26), (1.27), (4.2.1) we have

$$-A_{FW}Z + \nabla^{\perp}_Z FW - \phi\nabla_Z W - \phi h(Z,W) = \alpha[g(Z,W)\xi + 2\eta(W)\eta(Z)\xi + \eta(W)Z].$$

Comparing tangential components and on taking inner product with $X \in D_\theta$, we obtain

$$-g(A_{FW}Z, X) - g(T\nabla_Z W, X) - g(th(Z,W), X) = 0.$$

The leaves of $D^\perp$ are totally geodesic in M if for $Z, W \in D^\perp, \nabla_Z W \in D^\perp$. So, $T\nabla_Z W = 0$.

Thus, $g(A_{FW}Z, X) + g(th(Z,W), X) = g(h(Z,X), FW) + g(th(Z,W), X) = 0.$

4.5 Example of a hemi-slant submanifold of an $(LCS)_n$-manifold

Let $\tilde{M}(\mathbb{R}^9, \phi, \xi, \eta, g)$ denote the manifold $\mathbb{R}^9$ with the (LCS)-structure given by−

$$\xi = 3\frac{\partial}{\partial z},\ \eta = \frac{1}{3}(-dz + \sum_{i=1}^{4} b^i da^i),$$
$$g = \frac{1}{9}\sum_{i=1}^{4}(da^i \otimes da^i \oplus db^i \otimes db^i) - \eta \otimes \eta,$$
$$\phi(\frac{\partial}{\partial z}) = 0,\ \phi(\frac{\partial}{\partial a^i}) = \frac{\partial}{\partial b^i},\ i = 1, 2, 3, 4,\ and$$
$$\phi(\frac{\partial}{\partial b^i}) = \frac{\partial}{\partial a^i}\ for\ i = 1, 2\ and\ \phi(\frac{\partial}{\partial b^i}) = -\frac{\partial}{\partial a^i}\ for\ i = 3, 4,$$

where $(a^1, a^2, a^3, a^4, b^1, b^2, b^3, b^4, z) \in \mathbb{R}^9$.

Let us consider a 5-dimensional submanifold M of $\tilde{M}$ defined by

$$(a^1, a^2, a^3, a^4, b^1, b^2, b^3, b^4, z) \mapsto (\cos\alpha a^1 + \sin\alpha a^2, \cos\beta b^1 + \sin\beta b^2, \frac{a^3 - b^3}{\sqrt{3}}, \frac{a^4 - b^4}{\sqrt{3}}, 3z).$$

Then it can be easily proved that M is a hemi-slant submanifold of $\tilde{M}$ by choosing the slant distribution $D_\theta =< e_1, e_2 >$ with slant angle $|\alpha - \beta|$ and the totally real distribution $D^\perp =< e_3, e_4 >$, where $e_1 = \sin\alpha\frac{\partial}{\partial a^1} - \cos\alpha\frac{\partial}{\partial a^2}$, $e_2 = \sin\beta\frac{\partial}{\partial b^1} - \cos\beta\frac{\partial}{\partial b^2}$, $e_3 = \frac{\partial}{\partial a^3} + \frac{\partial}{\partial b^3}$, $e_4 = \frac{\partial}{\partial a^4} + \frac{\partial}{\partial b^4}$ such that $\{e_1, e_2, e_3, e_4, \xi\}$ forms an orthogonal frame on TM so that $TM = D_\theta \oplus D^\perp \oplus < \xi >$.

5

QUASI HEMI-SLANT SUBMANIFOLDS OF SOME DIFFERENTIABLE MANIFOLDS

5.1 Introduction

The notion of slant submanifold was introduced by B. Y. Chen in 1990 [33] as a generalization of holomorphic and totally real immersions. Later he collected many consequent results in his book [34]. Further slant submanifold was generalized as semi-slant, pseudo-slant, bi-slant and hemi-slant submanifolds etc. in different types of differentiable manifolds. Many geometers studied invariant [78], anti-invariant [131], semi-invariant [147], slant [127], semi-slant [89], pseudo-slant [42] and bi-slant [135] submanifolds of trans-Sasakian manifolds in different times. On the other hand, the concept of quasi hemi-slant submanifold was introduced recently by R. Prasad et al. in 2020 [121] as a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant and semi-slant submanifolds. Later in 2020-2021, R. Prasad along with some other researchers discussed this submanifold in various types of manifolds ([119], [120], [122]). However, the general notion of quasi hemi-slant submanifolds of metallic Riemannian manifolds has not been introduced yet.

The notion of golden structure on a Riemannian manifold was introduced by M. Crasmareanu and C. E. Hretcanu in 2008 [39]. They also investigated the properties of golden structure related to the almost product structure [76] and on some invariant submanifolds in a Riemannian manifold [75]. Later they generalized metallic structures as golden structures on Riemannian manifolds [77]. A. M. Blaga studied the properties of the conjugate connections by a golden structure and expressed their virtual and structural tensor fields and their behaviour on invariant distributions. Also, she studied the impact of the duality between the golden and almost product structures on golden and product conjugate connections [16]. Further, she along with C. E. Hretcanu discussed the properties of the metallic conjugate connections [20] where they expressed the virtual and structural tensor fields and analysed their behaviour on invariant distributions. Recently in 2018, they worked on invariant, anti-invariant and slant submanifolds [21], and also on semi-slant submanifolds [74] in metallic Riemannian manifolds. Some properties regarding the integrability of the golden Riemannian structures were investigated by A. Gezer et al. in 2013 [57]. In 2017, the connection adapted on the almost golden Riemannian structure was studied by F. Etayo et al. [55].

Motivated from the aforesaid research works, in this chapter, we have discussed quasi hemi-slant (QHS) submanifolds of trans-Sasakian manifolds and then, we have introduced the general notion of such submanifolds in metallic Riemannian manifolds besides establishing some results later. This chapter consists of seven sections. After Introduction and Preliminaries sections, the third section deals with some results satisfied by a QHS submanifold of a trans-Sasakian manifold. In the fourth section, we have obtained the necessary and sufficient conditions for integrability of the distributions related to this submanifold, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. At the end of the fourth section, we have concluded the necessary and sufficient condition for a QHS submanifold of a trans-Sasakian manifold to be a locally product Riemannian manifold and also we have made two other conclusions after observing the results. In the fifth section, we have constructed an example of a QHS submanifold of a trans-Sasakian manifold. Next, in the sixth section, we have obtained a necessary and sufficient condition for a submanifold to be QHS in metallic and golden Riemannian manifolds, and also the integrability conditions for the associated distributions along with some properties satisfied by them. At last, in the seventh section, we have constructed an example of a QHS submanifold of a metallic Riemannian manifold.

5.2 Preliminaries

Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ with almost contact metric structure (ϕ, ξ, η, g), and let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$, ∇ be the induced connection on the tangent bundle TM and $\nabla^{\perp}$ be the induced connection on the normal bundle $T^{\perp}M$ of M.

We have $\forall X \in \Gamma(TM)$,

$$\phi X = TX + NX, \tag{5.2.1}$$

where $TX,\ NX$ are the tangential and normal components of ϕX on M respectively.

Similarly, we have $\forall V \in \Gamma(T^{\perp}M)$,

$$\phi V = tV + nV, \tag{5.2.2}$$

where $tV,\ nV$ are the tangential and normal components of ϕV on M respectively.

The covariant derivatives of the tangential and normal components written in the equations (5.2.1), (5.2.2) are given by $\forall X, Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{5.2.3}$$
$$(\tilde{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{5.2.4}$$
$$(\tilde{\nabla}_X t)V = \nabla_X tV - t\nabla^{\perp}_X V, \tag{5.2.5}$$
$$(\tilde{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V. \tag{5.2.6}$$

Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ and the projections of $X \in \Gamma(TM)$ on the distributions $D,\ D_\theta,\ D^{\perp}$ be $PX,\ QX,\ RX$ respectively, then we have $\forall X \in \Gamma(TM)$,

$$X = PX + QX + RX + \eta(X)\xi. \tag{5.2.7}$$

Using (5.2.1) in (5.2.7) we get

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since $\phi D = D,\ \phi D^{\perp} \subseteq T^{\perp}M$, we have $NPX = 0,\ TRX = 0$ and hence, we obtain

$$\phi X = TPX + TQX + NQX + NRX. \tag{5.2.8}$$

Comparing (5.2.8) with (5.2.1) we have

$$TX = TPX + TQX, \tag{5.2.9}$$
$$NX = NQX + NRX. \tag{5.2.10}$$

From (5.2.8) we have the following decomposition$-$

$$\phi(TM) = TD \oplus TD_\theta \oplus ND_\theta \oplus ND^{\perp}. \tag{5.2.11}$$

Again, since $ND_\theta \subseteq \Gamma(T^{\perp}M),\ ND^{\perp} \subseteq \Gamma(T^{\perp}M)$, we have another decomposition$-$

$$T^{\perp}M = ND_\theta \oplus ND^{\perp} \oplus \mu, \tag{5.2.12}$$

where μ is the orthogonal complement of $ND_\theta \oplus ND^{\perp}$ in $\Gamma(T^{\perp}M)$ and it is anti-invariant with respect to ϕ [121].

Next, let M be an m-dimensional submanifold of the n-dimensional metallic (or golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with $n, m \in \mathbb{N}$ and $n > m$. Let $T_xM,\ T_x^{\perp}M$ be the tangent space and normal space of M at $x \in M$ respectively. Then the tangent space $T_x\tilde{M}$ of $\tilde{M}$ can be decomposed as $T_x\tilde{M} = T_xM \oplus T_x^{\perp}M$. Let for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, the tangential parts are $TX, tV \in \Gamma(TM)$ and the normal parts are $NX, nV \in \Gamma(T^{\perp}M)$ of $JX,\ JV$ respectively so that

$$JX = TX + NX, \tag{5.2.13}$$
$$JV = tV + nV. \tag{5.2.14}$$

T, N, t, n satisfy the following relations $\forall X, Y \in \Gamma(TM),\ U, V \in \Gamma(T^{\perp}M)$,

$$\tilde{g}(TX, Y) = \tilde{g}(X, TY), \tag{5.2.15}$$
$$\tilde{g}(nU, V) = \tilde{g}(U, nV), \tag{5.2.16}$$
$$\tilde{g}(NX, U) = \tilde{g}(X, tU). \tag{5.2.17}$$

Let $r = n - m$ be the co-dimension of M in $\tilde{M}$ and $\{N_i\}_{i=1}^{r}$ be a local orthonormal basis of $T_x^{\perp}M$ for $x \in M$. We assume that the indices α, β run over the range $\{1, ..., r\}$. Then JX and JN_α can be decomposed into tangential and normal components as [77]

$$JX = TX + \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha, \tag{5.2.18}$$
$$JN_\alpha = \xi_\alpha + \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta, \tag{5.2.19}$$

where ξ_α are vector fields, u_α are 1-forms and $(a_{\alpha\beta})_r$ is an $r \times r$ matrix of smooth real functions on M.

Using (5.2.13), (5.2.14) in (5.2.18), (5.2.19) we get

$$NX = \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha, \tag{5.2.20}$$
$$tN_\alpha = \xi_\alpha, \tag{5.2.21}$$
$$nN_\alpha = \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta. \tag{5.2.22}$$

Remark 5.2.1. [74] *If $\{N_i\}_{i=1}^{r}$ be a local orthonormal basis of $T_x^{\perp}M$, where r is the co-dimension of M in $\tilde{M}$, and $A_\alpha = A_{N_\alpha}$ for any $\alpha \in \{1, ..., r\}$, then we obtain $\forall X, Y \in \Gamma(TM)$,*

$$\tilde{\nabla}_X N_\alpha = -A_\alpha X + \nabla_X^{\perp} N_\alpha, \tag{5.2.23}$$
$$h_\alpha(X, Y) = \tilde{g}(A_\alpha X, Y). \tag{5.2.24}$$

Remark 5.2.2. [74] *The normal connection $\nabla_X^{\perp} N_\alpha$ has the decomposition $\nabla_X^{\perp} N_\alpha = \sum_{\beta=1}^{r} l_{\alpha\beta}(X) N_\beta$ for $\alpha \in \{1, ..., r\}$ and $\forall X \in \Gamma(TM)$, where $(l_{\alpha\beta})_r$ is an $r \times r$ matrix of 1-forms on M. Moreover [73] $\tilde{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}$ implies that $\tilde{g}(\nabla_X^{\perp} N_\alpha, N_\beta) + \tilde{g}(N_\alpha, \nabla_X^{\perp} N_\beta) = 0$ which is equivalent to $l_{\alpha\beta} = -l_{\beta\alpha}$ for any $\alpha, \beta \in \{1, ..., r\}$ and $\forall X \in \Gamma(TM)$.*

From (1.46) it follows that $\forall X, Y, Z \in \Gamma(TM)$,

$$\tilde{g}((\tilde{\nabla}_X J)Y, Z) = \tilde{g}(Y, (\tilde{\nabla}_X J)Z). \tag{5.2.25}$$

Hence, if M is an isometrically immersed submanifold of the metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then [20] $\forall X, Y, Z \in \Gamma(TM)$,

$$\tilde{g}((\tilde{\nabla}_X T)Y, Z) = \tilde{g}(Y, (\tilde{\nabla}_X T)Z). \tag{5.2.26}$$

Now we can define locally metallic (or locally golden) Riemannian manifold analogously as a locally product manifold [113] in the following manner [73]−

Definition 5.2.1. A metallic (or golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ is called *locally metallic (or locally golden) Riemannian manifold* if J is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$, i.e. $\tilde{\nabla} J = 0$.

We now state some propositions regarding submanifolds of locally metallic (or locally golden) Riemannian manifolds [74].

Proposition 5.2.1. *Let M be a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then $\forall X, Y \in \Gamma(TM)$,*

$$T[X, Y] = \nabla_X TY - \nabla_Y TX - A_{NY}X + A_{NX}Y, \tag{5.2.27}$$
$$N[X, Y] = h(X, TY) - h(TX, Y) + \nabla^{\perp}_X NY - \nabla^{\perp}_Y NX. \tag{5.2.28}$$

Proposition 5.2.2. *Let M be a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then $\forall X, Y \in \Gamma(TM),\ V \in \Gamma(T^{\perp}M)$,*

$$(\nabla_X T)Y = A_{NY}X + th(X, Y), \tag{5.2.29}$$
$$(\tilde{\nabla}_X N)Y = nh(X, Y) - h(X, TY), \tag{5.2.30}$$
$$(\nabla_X t)V = A_{nV}X - TA_V X, \tag{5.2.31}$$
$$(\tilde{\nabla}_X n)V = -h(X, tV) - NA_V X. \tag{5.2.32}$$

Proposition 5.2.3. *Let M be a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with co-dimension r, then the structure $(T,\ g,\ u_\alpha,\ \xi_\alpha,\ (a_{\alpha\beta})_r)$ induced on M by the metallic (or golden) Riemannian structure $(\tilde{g}, J)$ satisfies the following properties* [73] *$\forall X, Y \in \Gamma(TM)$−*

$$(\nabla_X T)Y = \sum_{\alpha=1}^{r} h_\alpha(X, Y)\xi_\alpha + \sum_{\alpha=1}^{r} u_\alpha(Y)A_\alpha X, \tag{5.2.33}$$

$$(\nabla_X u_\alpha)Y = -h_\alpha(X, TY) + \sum_{\beta=1}^{r} [u_\beta(Y) l_{\alpha\beta}(X) + h_\beta(X, Y) a_{\beta\alpha}]. \tag{5.2.34}$$

Proposition 5.2.4. *Let M be a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then $\forall X, Y \in \Gamma(TM)$,*

$$T[X,Y] = \nabla_X TY - \nabla_Y TX - \sum_{\alpha=1}^{r} [u_\alpha(Y) A_\alpha X - u_\alpha(X) A_\alpha Y], \tag{5.2.35}$$

$$N[X,Y] = \sum_{\alpha=1}^{r} [\{(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y\} + \sum_{\beta=1}^{r} \{u_\alpha(X) l_{\alpha\beta}(Y) - u_\alpha(Y) l_{\alpha\beta}(X)\}] N_\alpha. \tag{5.2.36}$$

■ Now, we introduce the following definition−

Definition 5.2.2. *Quasi hemi-slant (QHS) submanifold M* of a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ is a submanifold that admits three orthogonal complementary distributions $D,\ D_\theta,\ D^\perp$ such that

(i) TM admits the orthogonal direct decomposition

$$TM = D \oplus D_\theta \oplus D^\perp, \tag{5.2.37}$$

(ii) the distribution D is invariant i.e., $JD = D$,
(iii) the distribution D_θ is slant with constant angle θ and hence, θ is called *slant angle*,
(iv) the distribution $D^\perp$ is anti-invariant i.e., $JD^\perp \subseteq T^\perp M$.

In the above case, θ is called the *quasi hemi-slant angle* of M, and M is called *proper* if $D \neq \{0\},\ D_\theta \neq \{0\},\ D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Let the dimensions of the distributions $D,\ D_\theta,\ D^\perp$ be $n_1,\ n_2,\ n_3$ respectively, then we obtain the following particular cases−
(i) if $n_1 = 0$, then M is a hemi-slant submanifold,
(ii) if $n_2 = 0$, then M is a semi-invariant submanifold,
(iii) if $n_3 = 0$, then M is a semi-slant submanifold.

Let M be a QHS submanifold of a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ and the projections of $X \in \Gamma(TM)$ on the distributions $D,\ D_\theta,\ D^\perp$ be $PX,\ QX,\ RX$ respectively, then we have $\forall X \in \Gamma(TM)$,

$$X = PX + QX + RX. \tag{5.2.38}$$

Using (5.2.13) in (5.2.38) we get

$$JX = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since $JD = D,\ JD^\perp \subseteq T^\perp M$, we have

$$JPX = TPX,\ NPX = 0,\ TRX = 0, \tag{5.2.39}$$

and hence, we obtain

$$JX = TPX + TQX + NQX + NRX. \tag{5.2.40}$$

Comparing (5.2.40) with (5.2.13) we have

$$TX = TPX + TQX, \tag{5.2.41}$$

$$NX = NQX + NRX. \tag{5.2.42}$$

From (5.2.40) we have the following decomposition−

$$J(TM) = TD \oplus TD_\theta \oplus ND_\theta \oplus ND^\perp. \tag{5.2.43}$$

Again, since $ND_\theta \subseteq \Gamma(T^\perp M)$, $ND^\perp \subseteq \Gamma(T^\perp M)$, we have another decomposition−

$$T^\perp M = ND_\theta \oplus ND^\perp \oplus \mu, \tag{5.2.44}$$

where μ is the orthogonal complement of $ND_\theta \oplus ND^\perp$ in $\Gamma(T^\perp M)$ and it is anti-invariant with respect to J.

Moreover, for any $X \in \Gamma(TM)$ we have

$$\cos\theta(X) = \frac{\tilde{g}(JQX, TQX)}{\|TQX\|.\|JQX\|} = \frac{\|TQX\|}{\|JQX\|}. \tag{5.2.45}$$

5.3 QHS submanifolds of trans-Sasakian manifold

This section deals with some results satisfied by a QHS submanifold of a trans-Sasakian manifold.

Theorem 5.3.1. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then $\forall X, Y \in \Gamma(TM)$,*

$$\nabla_X TY - A_{NY}X - T(\nabla_X Y) - th(X,Y) = \alpha[g(X,Y)\xi - \eta(Y)X]$$
$$+\beta[g(TX,Y)\xi - \eta(Y)TX], \tag{5.3.1}$$

$$h(X, TY) + \nabla^\perp_X NY - N(\nabla_X Y) - nh(X,Y) = -\beta\eta(Y)NX. \tag{5.3.2}$$

Proof. Using (5.2.1) in (1.6) we get

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)(TX + NX)]. \tag{5.3.3}$$

Again, using (1.26), (1.27), (5.2.1) and (5.2.2) in $(\tilde{\nabla}_X\phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y)$ we obtain

$$(\tilde{\nabla}_X \phi)Y = \nabla_X TY + h(X, TY) - A_{NY}X + \nabla^\perp_X NY - th(X,Y) - nh(X,Y)$$

$$-T(\nabla_X Y) - N(\nabla_X Y). \qquad (5.3.4)$$

Equating tangential and normal components of (5.3.3), (5.3.4) we obtain (5.3.1) and (5.3.2) respectively.

Using (5.2.3) and (5.2.4) respectively in (5.3.1) and (5.3.2), we conclude the following−

Corollary 5.3.1. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then $\forall X, Y \in \Gamma(TM)$,*

$$(\tilde{\nabla}_X T)Y = A_{NY}X + th(X,Y) + \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)TX], \qquad (5.3.5)$$

$$(\tilde{\nabla}_X N)Y = -h(X,TY) + nh(X,Y) - \beta\eta(Y)NX. \qquad (5.3.6)$$

Next, we state the following theorem [121]−

Theorem 5.3.2. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then*

$$TD = D,\ TD_\theta = D_\theta,\ TD^\perp = \{0\},\ tND_\theta = D_\theta,\ tND^\perp = D^\perp.$$

Now, using (5.2.1) and (5.2.2) on $\phi^2 = -I + \eta \otimes \xi$ we get the following theorem−

Theorem 5.3.3. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then*
(i) $T^2 + nN = -I + \eta \otimes \xi$ on TM,
(ii) $NT + tN = 0$ on TM,
(iii) $Tt + n^2 = -I$ on $T^\perp M$,
(iv) $Nt + tn = 0$ on $T^\perp M$,
where I is the identity operator.

Next, we have the following theorem [111]−

Theorem 5.3.4. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then $\forall X, Y \in \Gamma(D_\theta)$,*
(i) $T^2X = -(\cos^2\theta)X$,
(ii) $g(TX,TY) = (\cos^2\theta)g(X,Y)$,
(iii) $g(NX,NY) = (\sin^2\theta)g(X,Y)$.

Theorem 5.3.5. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then $\forall X \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,*

$$\nabla_X tV - A_{nV}X + T(A_V X) - t\nabla^{\perp}_X V = \beta g(NX, V)\xi, \tag{5.3.7}$$
$$h(X, tV) + \nabla^{\perp}_X nV + N(A_V X) - n\nabla^{\perp}_X V = 0. \tag{5.3.8}$$

Proof. Using (1.26), (1.27), (5.2.1) and (5.2.2) in $(\tilde{\nabla}_X \phi)V = \tilde{\nabla}_X \phi V - \phi(\tilde{\nabla}_X V)$ we get

$$(\tilde{\nabla}_X \phi)V = \nabla_X tV + h(X, tV) - A_{nV}X + \nabla^{\perp}_X nV + T(A_V X) + N(A_V X) - t\nabla^{\perp}_X V - n\nabla^{\perp}_X V.$$

Again, applying (1.6) and then (5.2.1) in the left hand side of the above equation we obtain

$$\beta[g(NX, V)\xi] = \nabla_X tV + h(X, tV) - A_{nV}X + \nabla^{\perp}_X nV + T(A_V X) + N(A_V X)$$
$$-t(\nabla^{\perp}_X V) - n(\nabla^{\perp}_X V). \tag{5.3.9}$$

Equating tangential and normal components from both sides of (5.3.9) we get (5.3.7) and (5.3.8) respectively.

Now, using (5.2.5) and (5.2.6) in (5.3.7) and (5.3.8) respectively we conclude the following−

Corollary 5.3.2. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then $\forall X \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,*

$$(\tilde{\nabla}_X t)V = A_{nV}X - T(A_V X) + \beta g(NX, V)\xi, \tag{5.3.10}$$
$$(\tilde{\nabla}_X n)V = -h(X, tV) - N(A_V X). \tag{5.3.11}$$

Theorem 5.3.6. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then $\forall X \in \Gamma(TM)$,*

$$\nabla_X \xi = -\alpha TX - \beta T^2 X, \tag{5.3.12}$$
$$h(X, \xi) = -\alpha NX - \beta nNX. \tag{5.3.13}$$

Proof. Using (1.26), (5.2.1) and Theorem 5.3.3.(i) in (1.7) we obtain

$$\nabla_X \xi + h(X, \xi) = -\alpha(TX + NX) + \beta[-T^2 - nN]X.$$

Equating tangential and normal components from both sides of the above equation we get (5.3.12) and (5.3.13) respectively.

Theorem 5.3.7. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then $\forall X, Y \in \Gamma(D^{\perp})$,*

$$A_{\phi X}Y = A_{\phi Y}X \ if \ and \ only \ if \ \phi[X, Y] = 2\beta g(X, \phi Y)\xi. \tag{5.3.14}$$

Proof. Replacing V by ϕY in (1.27) and then applying (1.6), (1.26) and the fact that $Y \in \Gamma(D^{\perp})$ we get

$$\alpha g(X,Y)\xi + \beta g(\phi X,Y)\xi + \phi(\nabla_X Y) + \phi h(X,Y) = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y.$$

Equating tangential components from both sides of the above equation we obtain

$$A_{\phi Y}X = -\alpha g(X,Y)\xi - \beta g(\phi X,Y)\xi - \phi(\nabla_X Y). \tag{5.3.15}$$

Interchanging X, Y in (5.3.15) and then subtracting (5.3.15) from the resultant equation we have

$$A_{\phi X}Y - A_{\phi Y}X = \phi[X,Y] - 2\beta g(X,\phi Y)\xi, \tag{5.3.16}$$

from which we get (5.3.14).

Theorem 5.3.8. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then $\forall X, Y \in \Gamma(D \oplus D_\theta \oplus D^{\perp})$,*

$$g([X,Y],\xi) = 2\alpha g(TX,Y), \tag{5.3.17}$$

$$g(\tilde{\nabla}_X Y,\xi) = \alpha g(TX,Y) - \beta\cos^2\theta g(X,Y). \tag{5.3.18}$$

Proof. Applying (5.3.12) and Theorem 5.3.4.(i) on the following equation

$$g([X,Y],\xi) = g(\nabla_X Y,\xi) - g(\nabla_Y X,\xi) = -g(Y,\nabla_X \xi) + g(X,\nabla_Y \xi)$$

and after simplifying we obtain (5.3.17).

Again, using (1.26) we have

$$g(\tilde{\nabla}_X Y,\xi) = g(\nabla_X Y,\xi) + h(X,Y)\eta(\xi) = -g(Y,\nabla_X \xi) + h(X,Y).$$

Now, applying (5.3.12) and Theorem 5.3.4.(i) on the above equation we get (5.3.18).

Thus the proof is completed.

5.4 Integrability of distributions and decomposition theorems

In this section, we obtain the necessary and sufficient conditions for integrability of the distributions related to the proper QHS submanifolds of a trans-Sasakian manifold, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. At the end, we make three conclusions after observing the results.

Theorem 5.4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the invariant distribution D is not integrable.*

Proof. Let $X, Y \in \Gamma(D)$, then using (1.26), $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$ and then (1.7), $g(\phi X, Y) = -g(X, \phi Y)$ in the following equation

$$g([X,Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi)$$

we get on simplifying,

$$g([X,Y], \xi) = 2\alpha g(\phi X, Y). \tag{5.4.1}$$

Applying (5.2.7), (5.2.8) on (5.4.1) we obtain $g([X,Y], \xi) = 2\alpha g(TPX, PY) \neq 0$. Thus, D is not integrable.

Theorem 5.4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D \oplus < \xi >$ is integrable if and only if $\forall X, Y \in \Gamma(D \oplus < \xi >), Z \in \Gamma(D_\theta \oplus D^\perp)$,*

$$g(T\nabla_X Y - T\nabla_Y X, TQZ) + g(nh(X,Y) - nh(Y,X), NQZ + NRZ) = 0. \tag{5.4.2}$$

Proof. Using (1.2) in $g([X,Y], Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X,Y], Z) = g(\phi\tilde{\nabla}_X Y, \phi Z) - g(\phi\tilde{\nabla}_Y X, \phi Z),$$

on which applying (1.26), (5.2.1), (5.2.2), (5.2.8) and after simplifying we get

$$g([X,Y], Z) = g(T\nabla_X Y - T\nabla_Y X, TQZ) + g(nh(X,Y) - nh(Y,X), NQZ + NRZ).$$

Hence, $g([X,Y], Z) = 0$ if and only if (5.4.2) holds and thus the proof is completed.

Theorem 5.4.3. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the slant distribution D_θ is not integrable.*

Proof. Let $X, Y \in \Gamma(D_\theta)$. Applying (5.2.7) and (5.2.8) in (5.4.1) we have $g([X,Y], \xi) = 2\alpha g(TQX + NQX, QY) \neq 0$ and hence the proof is completed.

Theorem 5.4.4. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D_\theta \oplus < \xi >$ is integrable if and only if $\forall X, Y \in \Gamma(D_\theta \oplus < \xi >), Z \in \Gamma(D \oplus D^\perp)$,*

$$g(n(\nabla^\perp_X Y) - n(\nabla^\perp_Y X), NRZ) + \cos^2\theta g(A_X Y - A_Y X, PZ) = 0. \tag{5.4.3}$$

Proof. Using (1.2) in $g([X,Y], Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X,Y], Z) = g(\phi\tilde{\nabla}_X Y, \phi Z) - g(\phi\tilde{\nabla}_Y X, \phi Z),$$

on which applying (1.27), (5.2.1), (5.2.2), (5.2.8), Theorem 5.3.4.(ii) and after simplifying we get

$$g([X,Y], Z) = \cos^2\theta g(A_X Y - A_Y X, PZ) + g(n(\nabla^\perp_X Y) - n(\nabla^\perp_Y X), NRZ).$$

Therefore, $g([X,Y],Z) = 0$ if and only if (5.4.3) holds and hence the proof is completed.

From the above theorem, using (1.32) and (5.2.12) respectively we conclude the following–

Corollary 5.4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the distribution $D_\theta \oplus <\xi>$ is integrable if $\forall X, Y \in \Gamma(D_\theta \oplus <\xi>)$,*

$$A_X Y - A_Y X \in \Gamma(D_\theta \oplus D^\perp), \tag{5.4.4}$$

$$n(\nabla^\perp_X Y) - n(\nabla^\perp_Y X) \in \Gamma(ND_\theta \oplus \mu). \tag{5.4.5}$$

Theorem 5.4.5. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the anti-invariant distribution $D^\perp$ is integrable if and only if $\forall X, Y \in \Gamma(D^\perp), Z \in \Gamma(D \oplus D_\theta)$*

$$g(\nabla^\perp_X \phi Y - \nabla^\perp_Y \phi X, NQZ) = 0. \tag{5.4.6}$$

Proof. Using (1.2) in $g([X,Y],Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X,Y],Z) = g(\tilde{\nabla}_X \phi Y, \phi Z) - g(\tilde{\nabla}_Y \phi X, \phi Z),$$

on which applying (1.27), (1.28), (5.2.8) and Theorem 5.3.2 we get after simplification,

$$g([X,Y],Z) = g(\nabla^\perp_X \phi Y - \nabla^\perp_Y \phi X, NQZ).$$

Thus, $g([X,Y],Z) = 0$ if and only if (5.4.6) holds and hence the proof is completed.

Using (5.2.12) in the above theorem we conclude the following–

Corollary 5.4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the anti-invariant distribution $D^\perp$ is integrable if $\forall X, Y \in \Gamma(D^\perp)$, $\nabla^\perp_X \phi Y - \nabla^\perp_Y \phi X \in \Gamma(ND^\perp \oplus \mu)$.*

Theorem 5.4.6. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then M is totally geodesic if and only if $\forall X, Y \in \Gamma(TM), V \in \Gamma(T^\perp M)$,*

$$g(\nabla_X TY - A_{NY} X, tV) + g(h(X,TY) + \nabla^\perp_X NY, nV) = 0. \tag{5.4.7}$$

Proof. Applying (1.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$.

Further, using (1.26), (1.27), (5.2.1), (5.2.2) in the above equation we obtain on simplifying,

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TY - A_{NY} X, tV) + g(h(X,TY) + \nabla^\perp_X NY, nV). \tag{5.4.8}$$

Now, M is totally geodesic $\iff h = 0 \iff \forall X, Y \in \Gamma(TM), \tilde{\nabla}_X Y = \nabla_X Y$ (from (1.26)) $\iff g(\tilde{\nabla}_X Y, V) = 0\ \forall V \in \Gamma(T^\perp M)$. Hence, from (5.4.8) we have, M is totally geodesic if and only if (5.4.7) holds. Thus the proof is completed.

Theorem 5.4.7. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the invariant distribution D does not define a totally geodesic foliation on M.*

Proof. Let $X, Y \in \Gamma(D)$. Using (1.7) and the fact that $X \in \Gamma(D)$ in $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$ we get $g(\tilde{\nabla}_X Y, \xi) = -\beta g(X, Y) + \alpha g(Y, \phi X) \neq 0$, and hence the proof is completed.

Theorem 5.4.8. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D \oplus < \xi >$ defines a totally geodesic foliation on M if and only if $\forall X, Y \in \Gamma(D \oplus < \xi >), Z \in \Gamma(D_\theta \oplus D^\perp), V \in \Gamma(T^\perp M)$,*

$$g(\nabla_X TY, TQZ) = -g(h(X, TY), NZ), \tag{5.4.9}$$
$$g(\nabla_X TY, tV) = -g(h(X, TY), nV). \tag{5.4.10}$$

Proof. Applying (1.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (1.26) and (5.2.8) we get

$$g(\tilde{\nabla}_X Y, Z) = g(\nabla_X TY, TQZ) + g(h(X, TY), NZ),$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (5.4.9) holds.

Again, applying (1.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (1.26), (5.2.2) and (5.2.8) we obtain

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TY, tV) + g(h(X, TY), nV).$$

Hence, we have $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (5.4.10) holds.

Thus the proof is completed.

Theorem 5.4.9. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the slant distribution D_θ does not define a totally geodesic foliation on M.*

Proof. Let $X, Y \in \Gamma(D_\theta)$. Applying (1.7) and the fact that $X \in \Gamma(D_\theta)$ on $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$, we get $g(\tilde{\nabla}_X Y, \xi) = -\beta g(X, Y) + \alpha g(\phi X, Y) \neq 0$.

Hence the proof is completed.

Theorem 5.4.10. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$*

of type (α, β), then the distribution $D_\theta \oplus < \xi >$ defines a totally geodesic foliation on M if and only if $\forall X, Y \in \Gamma(D_\theta \oplus < \xi >), Z \in \Gamma(D \oplus D^\perp), V \in \Gamma(T^\perp M)$,

$$g(\nabla_X TQY - A_{NQY}X, TPZ) + g(h(X, TQY) + \nabla^\perp_X NQY, NRZ) = 0, \tag{5.4.11}$$

$$g(\nabla_X TQY - A_{NQY}X, tV) + g(h(X, TQY) + \nabla^\perp_X NQY, nV) = 0. \tag{5.4.12}$$

Proof. Applying (1.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (1.26), (1.27) and (5.2.8) we get

$$g(\tilde{\nabla}_X Y, Z) = g(\nabla_X TQY - A_{NQY}X, TPZ) + g(h(X, TQY) + \nabla^\perp_X NQY, NRZ),$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (5.4.11) holds.

Again, applying (1.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (1.26), (1.27), (5.2.2) and (5.2.8) we obtain

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TQY - A_{NQY}X, tV) + g(h(X, TQY) + \nabla^\perp_X NQY, nV),$$

which implies that $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (5.4.12) holds.

Thus the proof is completed.

Theorem 5.4.11. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the anti-invariant distribution $D^\perp$ defines a totally geodesic foliation on M if and only if $\forall X, Y \in \Gamma(D^\perp), Z \in \Gamma(D \oplus D_\theta), V \in \Gamma(T^\perp M)$,*

$$g(A_{NY}X, TZ) = g(\nabla^\perp_X NY, NQZ), \tag{5.4.13}$$

$$g(A_{NY}X, tV) = g(\nabla^\perp_X NY, nV). \tag{5.4.14}$$

Proof. Applying (1.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (1.27) and (5.2.8) we obtain

$$g(\tilde{\nabla}_X Y, Z) = -g(A_{NY}X, TZ) + g(\nabla^\perp_X NY, NQZ),$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (5.4.13) holds.

Now, applying (1.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (1.27), (5.2.2) and (5.2.8) we get

$$g(\tilde{\nabla}_X Y, V) = -g(A_{NY}X, tV) + g(\nabla^\perp_X NY, nV),$$

which implies that $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (5.4.14) holds.

Thus the proof is completed.

From Theorems 5.4.8, 5.4.10 and 5.4.11, we reach to the following conclusion—

Conclusion 5.4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then M is a locally product Riemannian manifold of the form $M_D \times M_{D_\theta} \times M_{D^\perp}$ if and only if equations (5.4.9)-(5.4.14) hold, where M_D, M_{D_θ}, $M_{D^\perp}$ are leaves of the distributions D, D_θ, $D^\perp$ respectively.*

Next, Theorems 5.4.1 and 5.4.3 give us the following conclusion—

Conclusion 5.4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then both the invariant distribution D and the slant distribution D_θ are not integrable.*

Again, observing Theorems 5.4.7 and 5.4.9 we conclude the following—

Conclusion 5.4.3. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then both the invariant distribution D and the slant distribution D_θ do not define a totally geodesic foliation on M.*

5.5 Example of a QHS submanifold of a trans-Sasakian manifold

Let $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$-dimensional Euclidean space endowed with the almost contact metric structure (ϕ, ξ, η, g) defined by

$$\phi(x^1, x^2, ..., x^{2n}, t) = (-x^{n+1}, -x^{n+2}, ..., -x^{2n}, x^1, x^2, ..., x^n, 0),$$
$$\xi = e^t \frac{\partial}{\partial t}, \ \eta = e^{-t} dt, \ g = e^{-2t} k,$$

where $(x^1, x^2, ..., x^{2n}, t)$ are cartesian coordinates and k is the Euclidean Riemannian metric on $\mathbb{R}^{2n+1}$. Then (ϕ, ξ, η, g) is a trans-Sasakian structure on $\mathbb{R}^{2n+1}$ of type $(0, -e^{-t})$.

For $\theta \in \left(0, \frac{\pi}{2}\right)$, we have, the map given by

$$x(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = (u_1, u_2 \cos\theta, 0, u_2 \sin\theta, u_3, u_4, u_5, u_6, 0, 0, u_7)$$

defines a 7-dimensional submanifold M of $\mathbb{R}^{11}$ with the trans-Sasakian structure described above. Further, let

$$E_1 = e^t \frac{\partial}{\partial x^1}, \ E_2 = e^t \frac{\partial}{\partial x^6},$$
$$E_3 = e^t \left(\cos\theta \frac{\partial}{\partial x^2} + \sin\theta \frac{\partial}{\partial x^4} \right), \ E_4 = e^t \frac{\partial}{\partial x^7},$$
$$E_5 = e^t \frac{\partial}{\partial x^5}, \ E_6 = e^t \frac{\partial}{\partial x^8}, \ E_7 = e^t \frac{\partial}{\partial t} = \xi,$$

then $\{E_i\}_{i=1}^{7}$ is an orthonormal frame of TM.

If we define the distributions as

$$D =< E_1, E_2 >,\ D_\theta =< E_3, E_4 >,\ D^\perp =< E_5, E_6 >,$$

then it is clear that

$$TM = D \oplus D_\theta \oplus D^\perp \oplus < \xi >$$

and D is an invariant distribution since $\phi E_1 = E_2$ and $\phi E_2 = -E_1$, D_θ is a slant distribution with slant angle $\theta \in \left(0, \frac{\pi}{2}\right)$ since $g(\phi E_3, E_4) = \cos\theta = -g(E_3, \phi E_4)$, $D^\perp$ is an anti-invariant distribution since $\phi E_5 = e^t \frac{\partial}{\partial x^{10}}$ and $\phi E_6 = -e^t \frac{\partial}{\partial x^3}$.

Therefore, M is a QHS submanifold of the trans-Sasakian manifold $(\mathbb{R}^{11}, \phi, \xi, \eta, g)$.

5.6 QHS submanifolds of metallic Riemannian manifolds

In this section, we find out a necessary and sufficient condition for a submanifold to be QHS in metallic and golden Riemannian manifolds and also obtain the integrability conditions for the associated distributions along with some properties satisfied by them. At last, we construct an example of a QHS submanifold of a metallic Riemannian manifold.

Theorem 5.6.1. *If M is a QHS submanifold of a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with the quasi hemi-slant angle θ, then $\forall X, Y \in \Gamma(TM)$,*

$$\tilde{g}(TQX, TQY) = \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)], \tag{5.6.1}$$

$$\tilde{g}(NX, NY) = -\sin^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)] - [p\tilde{g}(JRX, RY) + q\tilde{g}(RX, RY)]. \tag{5.6.2}$$

Proof. Replacing X by $X + Y$ in (5.2.45) and then using (1.47) we get

$$\tilde{g}(TQX, TQY) = \cos^2\theta\tilde{g}(JQX, JQY) = \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)],$$

which gives (5.6.1).

Again from (5.2.40), (5.2.42) we have

$$TQX = JX - TPX - NX,$$

on which applying (5.2.38), (5.2.39) we obtain

$$TQX = JQX + JRX - NX$$

and using it in (5.6.1) we get

$$\tilde{g}(JQX, JQY) + \tilde{g}(JRX, JRY) + \tilde{g}(NX, NY) = \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)].$$

Now applying (1.47) on the above equation we get

$$[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)] + [p\tilde{g}(JRX, RY) + q\tilde{g}(RX, RY)] + \tilde{g}(NX, NY) \\ = \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)],$$

which gives (5.6.2).

Corollary 5.6.1. *If M is a QHS submanifold of a golden Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with the quasi hemi-slant angle θ, then $\forall X, Y \in \Gamma(TM)$,*

$$\tilde{g}(TQX, TQY) = \cos^2\theta[\tilde{g}(JQX, QY) + \tilde{g}(QX, QY)], \tag{5.6.3}$$

$$\tilde{g}(NX, NY) = -\sin^2\theta[\tilde{g}(JQX, QY) + \tilde{g}(QX, QY)] - [\tilde{g}(JRX, RY) + \tilde{g}(RX, RY)]. \tag{5.6.4}$$

Proof. Putting $p = q = 1$ in Theorem 5.6.1, we get the required results.

Theorem 5.6.2. *If M is a QHS submanifold of a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with the quasi hemi-slant angle θ, then*

$$T^2Q = \cos^2\theta[pJQ + qQ]. \tag{5.6.5}$$

Proof. Using (5.2.15) in (5.6.1) we get

$$\tilde{g}(T^2QX, QY) = \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)]$$

which gives (5.6.5).

Corollary 5.6.2. *If M is a QHS submanifold of a golden Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with the quasi hemi-slant angle θ, then*

$$T^2Q = \cos^2\theta(J + I)Q, \tag{5.6.6}$$

where I is the identity mapping on $\Gamma(D_\theta)$.

Proof. Putting $p = q = 1$ in Theorem 5.6.2, we get the required result.

Theorem 5.6.3. *An immersed submanifold M of a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ is QHS if and only if there exists a constant $\lambda \in [0, 1)$ such that $D^* = \{X \in \Gamma(TM) : T^2X = \lambda(pJX + qX)\}$ is a distribution and $D^{*\perp} = D$.*

Proof. If M is QHS, then putting $\cos^2\theta = \lambda \in [0, 1)$ in (5.6.5) we get $\forall X \in \Gamma(TM)$,

$$T^2QX = \lambda[pJQX + qQX],$$

which implies that $QX \in \Gamma(D^*)$ and hence, $D_\theta \subseteq D^*$.

Again, since from (5.2.39) we have $TRX = 0$, $T^2RX = \lambda(pJRX + qRX)$ holds for $\lambda = 0$ and thus, $RX \in \Gamma(D^*)$ which implies that $D^\perp \subseteq D^*$.

Therefore, $D_\theta \oplus D^\perp \subseteq D^*$.

Next, let $X \in \Gamma(D^*)$ be a non-zero vector field, then by (5.2.38) we have

$$X = PX + QX + RX.$$

Now using (5.2.39), (1.45) we have

$$pTPX + qPX = pJPX + qPX = J^2PX = J(JPX) = J(TPX) = T(TPX) = T^2PX.$$

Again, as $X \in \Gamma(D^*)$, from the above equation we get on using (5.2.39),

$$\begin{aligned}
&pTPX + qPX = \lambda(pTPX + qPX) \\
\Rightarrow &(\lambda - 1)(pTPX + qPX) = 0 \\
\Rightarrow &pTPX + qPX = 0 \ \ (since\ \lambda \neq 1) \\
\Rightarrow &TPX = \frac{-q}{p}PX \Rightarrow PX = 0 \ \ (since\ \frac{q^2}{p^2} \neq 0\ for\ p, q \in \mathbb{N}) \\
\Rightarrow &X = QX + RX \in D_\theta \oplus D^\perp \\
\Rightarrow &D^* \subseteq D_\theta \oplus D^\perp.
\end{aligned}$$

Thus, we conclude that $D^* = D_\theta \oplus D^\perp$ and consequently $D^{*\perp} = D$.

Conversely, let there exists a constant $\lambda \in [0, 1)$ such that $D^* = \{X \in \Gamma(TM) : T^2X = \lambda(pJX + qX)\}$ is a distribution and $D^{*\perp} = D$. Then from (5.6.5) we get for $X \in \Gamma(D^*)$,

$$\cos^2\theta(X) = \lambda \Rightarrow \cos\theta(X) = \sqrt{\lambda} \Rightarrow \theta(X) = \cos^{-1}(\sqrt{\lambda}),$$

which does not depend on X.

We can consider the orthogonal decomposition $TM = D \oplus D_\theta \oplus D^\perp$.

Now, for $X \in \Gamma(D^*)$, $Y \in \Gamma(D^{*\perp}) = \Gamma(D)$ we have on applying (5.2.13), (5.2.15),

$$\tilde{g}(X, J^2Y) = \tilde{g}(X, J(JY)) = \tilde{g}(X, TJY) = \tilde{g}(TX, JY) = \tilde{g}(TX, TY) = \tilde{g}(T^2X, Y).$$

Again, since $X \in \Gamma(D^*)$, using (1.47) we get from the above equation

$$\begin{aligned}
& p\tilde{g}(X, JY) + q\tilde{g}(X, Y) = \lambda[p\tilde{g}(JX, Y) + q\tilde{g}(X, Y)] \\
\Rightarrow & \tilde{g}(X, JY) = \lambda\tilde{g}(JX, Y) \\
\Rightarrow & \tilde{g}(X, JY) = \lambda\tilde{g}(X, JY) \ \ (by\ (1.46)) \\
\Rightarrow & \tilde{g}(X, JY) = 0 \ \ (since\ \lambda \neq 1) \\
\Rightarrow & JY \in \Gamma(D^{*\perp}) = \Gamma(D) \\
\Rightarrow & JD \subseteq D \ \ i.e.\ D\ is\ invariant.
\end{aligned}$$

Next, let $X \in \Gamma(D^{\perp}) \subseteq \Gamma(D_\theta \oplus D^{\perp}) = \Gamma(D^*)$ and $Y \in \Gamma(TM) = \Gamma(D \oplus D_\theta \oplus D^{\perp})$. Then as before we get

$$\begin{aligned}
& \tilde{g}(X, JY) = 0 \\
\Rightarrow & \tilde{g}(JX, Y) = 0 \ \ (by\ (1.46)) \\
\Rightarrow & JX \in \Gamma(T^{\perp}M) \\
\Rightarrow & JD^{\perp} \subseteq T^{\perp}M \ \ i.e.\ D^{\perp}\ is\ anti-invariant.
\end{aligned}$$

Therefore M is a QHS submanifold of $(\tilde{M}, \tilde{g}, J)$.

Corollary 5.6.3. *An immersed submanifold M of a golden Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ is QHS if and only if there exists a constant $\lambda \in [0, 1)$ such that $D^* = \{X \in \Gamma(TM) : T^2X = \lambda(JX + X)\}$ is a distribution and $D^{*\perp} = D$.*

Proof. Putting $p = q = 1$ in Theorem 5.6.3, we get the required results.

We now state and prove some results on QHS submanifolds of locally metallic (or locally golden) Riemannian manifolds:−

Theorem 5.6.4. *If M is a QHS submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with co-dimension r, then*

(i) the distribution D is integrable if and only if $\forall X, Y \in \Gamma(D)$,

$$(\nabla_Y u_\alpha)X = (\nabla_X u_\alpha)Y \ \ \forall \alpha \in \{1, ..., r\}, \tag{5.6.7}$$

(ii) the distribution D_θ is integrable if and only if $\forall X, Y \in \Gamma(D_\theta)$,

$$P(\nabla_X TY - \nabla_Y TX) = \sum_{\alpha=1}^{r} [u_\alpha(Y)PA_\alpha X - u_\alpha(X)PA_\alpha Y], \tag{5.6.8}$$

(iii) the distribution $D^{\perp}$ is integrable if and only if $\forall X, Y \in \Gamma(D^{\perp})$,

$$u_\alpha(X)A_\alpha Y = u_\alpha(Y)A_\alpha X \ \ \forall \alpha \in \{1, ..., r\}. \tag{5.6.9}$$

Proof. (i) Let $X, Y \in \Gamma(D)$, then $X = PX,\ Y = PY$.

Now, D is integrable if and only if $[X, Y] \in \Gamma(D)$ and $[X, Y] \in \Gamma(D)$ if and only if $N[X, Y] = 0$. Thus, D is integrable if and only if $N[X, Y] = 0$.

As $JD \subseteq D$, $NX = 0 = NY$. Hence, from (5.2.35) we have $u_\alpha(X)l_{\alpha\beta}(Y) = u_\alpha(Y)l_{\alpha\beta}(X) = 0$.

Thus, from (5.2.36) we get

$$N[X, Y] = 0 \iff \sum_{\alpha=1}^{r} [(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y] = 0$$

$$\Rightarrow D \text{ is integrable if and only if } (\nabla_Y u_\alpha)X = (\nabla_X u_\alpha)Y \ \ \forall \alpha \in \{1, ..., r\}.$$

(ii) Let $X, Y \in \Gamma(D_\theta)$, then $X = QX,\ Y = QY$.

Now, D_θ is integrable if and only if $[X, Y] \in \Gamma(D_\theta)$ and $[X, Y] \in \Gamma(D_\theta)$ if and only if $PT[X, Y] = 0$. Therefore, D_θ is integrable if and only if $PT[X, Y] = 0$.

Again, from (5.2.35) we obtain

$$PT[X, Y] = 0 \iff P(\nabla_X TY - \nabla_Y TX) = \sum_{\alpha=1}^{r} [u_\alpha(Y)PA_\alpha X - u_\alpha(X)PA_\alpha(Y)].$$

(iii) Let $X, Y \in \Gamma(D^\perp)$, then $X = RX,\ Y = RY$.

Now, $D^\perp$ is integrable if and only if $[X, Y] \in \Gamma(D^\perp)$ and $[X, Y] \in \Gamma(D^\perp)$ if and only if $T[X, Y] = 0$. Thus, $D^\perp$ is integrable if and only if $T[X, Y] = 0$.

As $JD^\perp \subseteq T^\perp M$, $TX = 0 = TY$. Hence, from (5.2.35) we have

$$T[X, Y] = 0 \iff u_\alpha(X)A_\alpha Y = u_\alpha(Y)A_\alpha X \ \ \forall \alpha \in \{1, ..., r\}.$$

Theorem 5.6.5. *If M is a QHS submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then*

(i) the distribution D is integrable if and only if $\forall X, Y \in \Gamma(D)$,

$$h(X, TY) = h(TX, Y), \tag{5.6.10}$$

(ii) the distribution D is integrable if and only if $\forall X \in \Gamma(D),\ V \in \Gamma(T^\perp M)$,

$$JA_V X = A_V JX, \tag{5.6.11}$$

(iii) the distribution D_θ is integrable if and only if $\forall X, Y \in \Gamma(D_\theta)$,

$$P(\nabla_X TY - \nabla_Y TX) = P(A_{NY}X - A_{NX}Y), \tag{5.6.12}$$

(iv) the distribution $D^{\perp}$ is integrable if and only if $\forall X, Y \in \Gamma(D^{\perp})$,

$$A_{NX}Y = A_{NY}X. \tag{5.6.13}$$

Proof. (i) D is integrable if and only if $[X, Y] \in \Gamma(D)$ and $[X, Y] \in \Gamma(D)$ if and only if $N[X, Y] = 0$ which implies that D is integrable if and only if $N[X, Y] = 0$.

As $JD \subseteq D$, $NX = 0 = NY$. Hence, from (5.2.28) we have

$$N[X, Y] = 0 \iff h(X, TY) = h(TX, Y).$$

(ii) Again, for $X, Y \in \Gamma(D)$ and $V \in \Gamma(T^{\perp}M)$,

$$\begin{aligned}
0 &= \tilde{g}(h(X, TY) - h(TX, Y), V) \\
&= \tilde{g}(h(X, JY), V) - \tilde{g}(h(JX, Y), V) \quad (\text{since } TX = JX,\ TY = JY) \\
&= \tilde{g}(A_V X, JY) - \tilde{g}(A_V JX, Y) \quad (\text{by } (1.28)) \\
&= \tilde{g}(JA_V X, Y) - \tilde{g}(A_V JX, Y) \quad (\text{by } (1.46)) \\
\Rightarrow & \tilde{g}(h(X, TY) - h(TX, Y), V) = 0 \iff JA_V X = A_V JX.
\end{aligned}$$

Hence, from Theorem 5.6.5.(i) we obtain (5.6.11).

(iii) D_θ is integrable if and only if $[X, Y] \in \Gamma(D_\theta)$ and $[X, Y] \in \Gamma(D_\theta)$ if and only if $PT[X, Y] = 0$ which implies that D_θ is integrable if and only if $PT[X, Y] = 0$.

Hence, from (5.2.27) we obtain

$$PT[X, Y] = 0 \iff P(\nabla_X TY - \nabla_Y TX) = P(A_{NY}X - A_{NX}Y).$$

(iv) $D^{\perp}$ is integrable if and only if $[X, Y] \in \Gamma(D^{\perp})$ and $[X, Y] \in \Gamma(D^{\perp})$ if and only if $T[X, Y] = 0$ which implies that $D^{\perp}$ is integrable if and only if $T[X, Y] = 0$.

As $JD^{\perp} \subseteq T^{\perp}M$, $TX = 0 = TY$. Hence, from (5.2.27) we get

$$T[X, Y] = 0 \iff A_{NX}Y = A_{NY}X.$$

Theorem 5.6.6. *If M is a QHS submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then N is parallel if and only $\forall X \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$,*

$$A_{nV}X = TA_V X = A_V TX. \tag{5.6.14}$$

Proof. Now, for $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$,

$$\tilde{g}(nh(X, Y), V) = \tilde{g}(Jh(X, Y), V) = \tilde{g}(h(X, Y), nV) \quad (\text{by } (1.27)).$$

From (5.2.30) we have

$$\begin{aligned}
\tilde{g}((\tilde{\nabla}_X N)Y, V) &= \tilde{g}(nh(X, Y), V) - \tilde{g}(h(X, TY), V) \\
&= \tilde{g}(h(X, Y), nV) - \tilde{g}(h(X, TY), V) \quad (\text{by } (5.2.16)) \\
&= \tilde{g}(A_{nV}X, Y) - \tilde{g}(A_V X, TY) \quad (\text{by } (1.28)) \\
&= \tilde{g}(A_{nV}X, Y) - \tilde{g}(TA_V X, Y) \quad (\text{by } (5.2.15)) \\
&= \tilde{g}(A_{nV}X - TA_V X, Y) = 0 \\
&\Rightarrow A_{nV}X = TA_V X.
\end{aligned} \tag{5.6.15}$$

Again from (5.2.30) we have

$$\begin{aligned}
\tilde{g}((\tilde{\nabla}_X N)Y, V) &= \tilde{g}(nh(X,Y), V) - \tilde{g}(h(X,TY), V) \\
&= \tilde{g}(h(X,Y), nV) - \tilde{g}(h(X,TY), V) \quad (by\ (5.2.16)) \\
&= \tilde{g}(A_{nV}Y, X) - \tilde{g}(A_V TY, X) \quad (by\ (1.28)) \\
&= \tilde{g}(A_{nV}Y - A_V TY, X) = 0 \\
&\Rightarrow A_{nV}Y = A_V TY. \qquad (5.6.16)
\end{aligned}$$

Combining (5.6.15), (5.6.16) we obtain (5.6.14).

5.7 Example of a QHS submanifold of a metallic Riemannian manifold

Let us consider the Euclidean space $\mathbb{R}^{10}$ with usual Euclidean metric. Let $J : \mathbb{R}^{10} \to \mathbb{R}^{10}$ be the metallic structure defined by

$$\begin{aligned}
&J(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}) \\
&= (\sigma X_1, \sigma X_2, \bar{\sigma} X_3, \bar{\sigma} X_4, \sigma X_5, \sigma X_6, \bar{\sigma} X_7, \bar{\sigma} X_8, \sigma X_9, \sigma X_{10}),
\end{aligned}$$

where $\sigma = \sigma_{pq} = \frac{p+\sqrt{p^2+4q}}{2} > 0$ is a metallic number, $\bar{\sigma} = \frac{p-\sqrt{p^2+4q}}{2} = p - \sigma < 0$, and $p, q \in \mathbb{N}$.

Now, by simple calculations we get $\sigma^2 = p\sigma + q$ and similarly $\bar{\sigma}^2 = p\bar{\sigma} + q$.

Hence, we have

$$\begin{aligned}
&J^2(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10}) \\
&= (pJ + qI)(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10})
\end{aligned}$$

so that $\mathbb{R}^{10}$ forms a metallic Riemannian manifold together with the usual Euclidean metric and the metallic structure J defined above.

Next, let $M = \{(u, \alpha_1, \alpha_2, \alpha_3) : u > 0, \alpha_i \in (0, \frac{\pi}{2})\}$ and $f : M \to \mathbb{R}^{10}$ be the immersion given by

$$f(u, \alpha_1, \alpha_2, \alpha_3) = (u\cos\alpha_1, u\sin\alpha_1, u\cos\alpha_2, u\sin\alpha_2, u\cos\alpha_3, u\sin\alpha_3, u, \alpha_1, \alpha_2, \alpha_3).$$

We consider an orthonormal frame $\{Z_1, Z_2, Z_3, Z_4\}$ on TM such that

$$Z_1 = \frac{1}{2}\Big(\cos\alpha_1\frac{\partial}{\partial x_1} + \sin\alpha_1\frac{\partial}{\partial x_2} + \cos\alpha_2\frac{\partial}{\partial x_3} + \sin\alpha_2\frac{\partial}{\partial x_4} + \cos\alpha_3\frac{\partial}{\partial x_5} + \sin\alpha_3\frac{\partial}{\partial x_6}$$
$$+ \frac{\partial}{\partial x_{10}}\Big),$$
$$Z_2 = \frac{1}{\sqrt{2}}\Big(-\sin\alpha_1\frac{\partial}{\partial x_1} + \cos\alpha_1\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_9}\Big),$$
$$Z_3 = \frac{1}{\sqrt{2}}\Big(-\sin\alpha_2\frac{\partial}{\partial x_3} + \cos\alpha_2\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_8}\Big),$$
$$Z_4 = \frac{1}{\sqrt{q(\sigma-\bar{\sigma})}}\Big(-\bar{\sigma}\sqrt{\sigma}\frac{\partial}{\partial x_5}\sin\alpha_3 + \bar{\sigma}\sqrt{\sigma}\frac{\partial}{\partial x_6}\cos\alpha_3 + \sigma\sqrt{-\bar{\sigma}}\frac{\partial}{\partial x_7}\Big).$$

Thus, we have

$$JZ_1 = \frac{1}{2}\Big(\sigma\cos\alpha_1\frac{\partial}{\partial x_1} + \sigma\sin\alpha_1\frac{\partial}{\partial x_2} + \bar{\sigma}\cos\alpha_2\frac{\partial}{\partial x_3} + \bar{\sigma}\sin\alpha_2\frac{\partial}{\partial x_4} + \sigma\cos\alpha_3\frac{\partial}{\partial x_5}$$
$$+ \sigma\sin\alpha_3\frac{\partial}{\partial x_6} + \sigma\frac{\partial}{\partial x_{10}}\Big),$$
$$JZ_2 = \sigma Z_2,\ JZ_3 = \bar{\sigma}Z_3,$$
$$JZ_4 = \frac{1}{\sqrt{q(\sigma-\bar{\sigma})}}\Big(q\sqrt{\sigma}\frac{\partial}{\partial x_5}\sin\alpha_3 - q\sqrt{\sigma}\frac{\partial}{\partial x_6}\cos\alpha_3 - q\sqrt{-\bar{\sigma}}\frac{\partial}{\partial x_7}\Big).$$

Hence, taking the distributions as

$$D =< Z_2, Z_3 >,\ D_\theta =< Z_1 >,\ D^\perp =< Z_4 >$$

we have that the distribution D is invariant, the distribution $D^\perp$ is anti-invariant and the distribution D_θ is slant with the slant angle θ given by

$$\theta = \cos^{-1}\frac{\tilde{g}(JZ_1, Z_1)}{\|Z_1\|.\|JZ_1\|} = \cos^{-1}\frac{2p + \sqrt{p^2+4q}}{\sqrt{3p\sigma + p\bar{\sigma} + 4q}}.$$

Therefore, $TM = D \oplus D_\theta \oplus D^\perp$ and hence, M is a QHS submanifold of $\mathbb{R}^{10}$.

Note. According to V. W. de Spinadel, besides carrying the name of metals, the metallic means family have common mathematical properties that attach a fundamental importance to them in modern investigations about the search of universal roads to chaos, and the metallic numbers found many applications in researches that analyse the behaviour of non linear dynamical systems when they proceed from a periodic regime to a chaotic one.

Golden mean is known from ancient times as an expression of harmony of many famous constructions (Taj Mahal, the Parthenon in Athens, the Great

Pyramid of Giza, Temple of Zeus, the Great Mosque of Kairouan, the United Nations Building in New York), paintings (the Last Supper Monalisa by Leonardo da Vinci, the Starry Night by Vincent van Gogh) and music (many of Mozart's piano sonatas', Beethoven's fifth symphony, Genesis' Firth of Fifth song). It also appears as an expression of the objects from the natural world (snail shells, pine cones, strawberry seeds, tree branches, flower petals, spiral galaxy, hurricanes, DNA molecules) possessing pentagonal symmetry. Even, the ratios of human forearm to human hand and systolic blood pressure to diastolic blood pressure of people less prone to cardiac arrest are golden ratios.

6

LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KENMOTSU MANIFOLD

6.1 Introduction

K. L. Duggal introduced the geometry of lightlike submanifolds in 1996 along with A. Bejancu [48] and later in 2010, he along with B. Sahin wrote another book on it [54]. In between, B. Sahin characterized lightlike submanifolds in many ways. In 2006, he introduced the notion of transversal lightlike submanifolds and studied some differential geometric properties of those submanifolds [124]. In 2008, he initiated the study of screen transversal lightlike submanifolds [125]. R. S. Gupta introduced the notions of slant and screen slant submanifolds in indefinite Kenmotsu manifolds respectively in 2011 with A. Sharfuddin [66] and in 2010 with A. Upadhyay [67]. R. S. Gupta and A. Sharfuddin also conceptualised screen transversal lightlike submanifolds in the context of indefinite cosymplectic manifolds in 2010 [64] and later in the context of indefinite Kenmotsu manifolds in 2011 [65]. In 2012, S. M. K. Haider et al. studied totally contact umbilical screen transversal lightlike submanifolds of an indefinite Sasakian manifold [69] and recently, in 2021, A. Yadav et al. investigated the existence of totally contact umbilical screen-slant lightlike submanifolds of indefinite Sasakian manifolds [146]. As the tangent and normal bundles have non-trivial intersection in lightlike submanifolds, many researchers used this theory widely in their works such as [12], [47], [49], [50], [51], [53], [83].

K. Yano and M. Kon introduced the notion of generic submanifolds as the generalization of CR-submanifolds in 1980 [152]. Generic submanifold is the most general case of submanifolds because CR-submanifolds include holomorphic, as well as totally real submanifolds as subspaces. Also, screen CR-lightlike submanifold has invariant and anti-invariant lightlike submanifolds as its particular cases. Hence, generic lightlike submanifolds must include CR-lightlike submanifolds. K. L. Duggal and D. H. Jin introduced the concept of generic lightlike submanifolds of an indefinite Sasakian manifold in 2012 [50]. In 2015, D. H. Jin and J. W. Lee further studied generic lightlike submanifolds of an indefinite Kahler manifold [83] but yet, this concept did not contain proper screen CR-lightlike submanifolds. Hence, later in 2019, screen generic lightlike submanifold was introduced by B. Dogan et al. [47]. In 2020, R. S. Gupta modified this concept in the context of contact geometry and introduced a general notion of screen generic lightlike submanifolds of an indefinite Sasakian manifold with the structure vector field tangent to the submanifold [63].

Motivated from the works mentioned above, in this chapter, we have studied various types of lightlike submanifolds of indefinite Kenmotsu manifolds. This chapter is divided into nine sections. After Introduction and Preliminaries sections, in the third section, we have proved some results regarding screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. In the fourth section, we have proved a characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. In the fifth section, we have further proved some results on a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold, such as the necessary and sufficient conditions for the screen distribution $S(TM)$ to be integrable and for the induced connection ∇ to be a metric connection. In the sixth section, we have investigated the necessary and sufficient conditions for the induced connection on a contact screen generic lightlike (CSGL) submanifold of an indefinite Kenmotsu manifold to be a metric connection, for integrability & parallelism of some associated distributions, and for some distributions to be totally geodesic foliations. We have also discussed about non-parallel distributions and more than one necessary and sufficient conditions for a CSGL submanifold to be mixed geodesic. In the seventh and eighth sections respectively, we have further studied some properties satisfied by proper totally umbilical CSGL submanifolds and the necessary and sufficient conditions for minimality of an associated distribution & also of a CSGL submanifold. At last, in the ninth section, we have constructed an example of a CSGL submanifold of an indefinite Kenmotsu manifold.

6.2 Preliminaries

In this section, we write down some definitions related to lightlike submanifolds which we have taken into account while obtaining the results of the sections following this chapter.

We start with the definition of totally contact umbilical lightlike submanifold given below.

Definition 6.2.1. [153] A lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$, with the structure vector field ξ tangent to M, is called a *totally contact umbilical lightlike submanifold* if for a vector field α transversal to M and $\forall Z, W \in \Gamma(TM)$,

$$h(Z, W) = [g(Z, W) - \eta(Z)\eta(W)]\alpha + \eta(Z)h(W, \xi) + \eta(W)h(Z, \xi), \tag{6.2.1}$$

where h is a symmetric bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ known as the *second fundamental form* of M. If $\alpha = 0$, then M is called a *totally contact geodesic lightlike submanifold*.

Now, equating components of (6.2.1) belonging to $ltr(TM)$ and $S(T^{\perp}M)$ respectively we have [52]

$$h^l(Z,W) = [g(Z,W) - \eta(Z)\eta(W)]\alpha_l + \eta(Z)h^l(W,\xi) + \eta(W)h^l(Z,\xi), \quad (6.2.2)$$
$$h^s(Z,W) = [g(Z,W) - \eta(Z)\eta(W)]\alpha_s + \eta(Z)h^s(W,\xi) + \eta(W)h^s(Z,\xi), \quad (6.2.3)$$

where $h^l(Z,W) = L(h(Z,W))$, $h^s(Z,W) = S(h(Z,W))$ (L, S are the projection morphisms of $tr(TM)$ on $ltr(TM)$, $S(T^{\perp}M)$ respectively) and $\alpha_l \in \Gamma(ltr(TM))$, $\alpha_s \in \Gamma(S(T^{\perp}M))$. h^l and h^s are called the *lightlike second fundamental form* and the *screen second fundamental form* of M respectively.

Let M be a lightlike submanifold of an indefinite Kenmotsu manifold $\tilde{M}$ and ∇, $\tilde{\nabla}$ be the Levi-Civita connections on M, $\tilde{M}$ respectively. The Gauss and Weingarten formulae are given by−

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \;\; \forall X,Y \in \Gamma(TM), \quad (6.2.4)$$
$$\tilde{\nabla}_X V = -A_V X + \nabla^t_X V \;\; \forall X \in \Gamma(TM),\; V \in \Gamma(tr(TM)), \quad (6.2.5)$$

where $\nabla_X Y, A_V X \in \Gamma(TM)$ and $h(X,Y), \nabla^t_X V \in \Gamma(tr(TM))$. Here A is a linear operator on TM known as the *shape operator* and ∇^t is a linear connection on $tr(TM)$ known as the *transversal linear connection* on M.

Now, the equations (6.2.4) and (6.2.5) further reduce to−

$$\tilde{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y) \;\; \forall X,Y \in \Gamma(TM), \quad (6.2.6)$$

$$\tilde{\nabla}_X V = -A_V X + D^l(X,V) + D^s(X,V) \;\; \forall X \in \Gamma(TM),\; V \in \Gamma(tr(TM)),$$

where $D^l(X,W) = L(\nabla^t_X W)$, $D^s(X,W) = S(\nabla^t_X W)$.

In particular, we have

$$\tilde{\nabla}_X U = -A_U X + \nabla^l_X U + D^s(X,U) \;\; \forall U \in \Gamma(ltr(TM)), \quad (6.2.7)$$
$$\tilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W) \;\; \forall W \in \Gamma(S(T^{\perp}M)), \quad (6.2.8)$$

where ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(T^{\perp}M)$ called the *lightlike transversal connection* and the *screen transversal connection* on M respectively.

Again, from (6.2.6)-(6.2.8) we get

$$\tilde{g}(h^s(X,Y),W) + \tilde{g}(Y, D^l(X,W)) = g(A_W X, Y), \quad (6.2.9)$$
$$\tilde{g}(D^s(X,U),W) = \tilde{g}(U, A_W X). \quad (6.2.10)$$

Let $\bar{P}$ be the projection morphism of TM on $S(TM)$, then we have $\forall X, Y \in \Gamma(TM)$, $V \in \Gamma(Rad(TM))$,

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{6.2.11}$$
$$\nabla_X V = -A^*_V X + \nabla^{*t}_X V, \tag{6.2.12}$$

where h^* is the local second fundamental form on $S(TM)$ and A^* is the shape operator of $Rad(TM)$, $\nabla^*_X \bar{P}Y, A^*_V X \in \Gamma(S(TM))$ and $h^*(X, \bar{P}Y), \nabla^{*t}_X V \in \Gamma(Rad(TM))$. Here ∇^* and ∇^{*t} are induced connections on $S(TM)$ and $Rad(TM)$ respectively.

From (6.2.11) and (6.2.12) we have

$$\tilde{g}(h^l(X, \bar{P}Y), Z) = g(A^*_Z X, \bar{P}Y), \tag{6.2.13}$$
$$\tilde{g}(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y), \tag{6.2.14}$$
$$\tilde{g}(h^l(X, Z), Z) = 0, \; A^*_Z Z = 0. \tag{6.2.15}$$

Although the induced connection ∇ on M is not a metric connection, ∇^* and ∇^{*t} are metric connections on $S(TM)$ and $Rad(TM)$ respectively. As $\tilde{\nabla}$ is a metric connection on $\tilde{M}$, from (6.2.6) we get $\forall X, Y, Z \in \Gamma(TM)$,

$$(\nabla_X g)(Y, Z) = \tilde{g}(h^l(X, Y), Z) + \tilde{g}(h^l(X, Z), Y). \tag{6.2.16}$$

Definition 6.2.2. [49] A lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$, with the structure vector field ξ tangent to M, is called *totally umbilical* if there exists a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, which is called the *transversal curvature vector field* of M, such that $\forall Z, W \in \Gamma(TM)$,

$$h(Z, W) = g(Z, W)H. \tag{6.2.17}$$

From (6.2.6), (6.2.8) and (6.2.17), we easily conclude that M is totally umbilical if and only if on each coordinate neighbourhood, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$, $H^s \in \Gamma(S(T^\perp M))$, such that $\forall V \in \Gamma(S(T^\perp M))$,

$$h^l(Z, W) = g(Z, W)H^l, \; D^l(Z, V) = 0, \tag{6.2.18}$$
$$h^s(Z, W) = g(Z, W)H^s. \tag{6.2.19}$$

Definition 6.2.3. [8] A lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$, with the structure vector field ξ tangent to M, is called *minimal* if
(i) $h^s = 0$ on $Rad(TM)$,
(ii) $trace(h) = 0$ with respect to g restricted to $S(TM)$.

Definition 6.2.4. [63] An r-lightlike submanifold $(M, g, S(TM), S(T^\perp M))$ of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$, with the structure vector field ξ

tangent to M, is called *contact screen generic lightlike (CSGL) submanifold* if the following conditions are satisfied−
(i) $Rad(TM)$ is invariant with respect to ϕ i.e.,

$$\phi(Rad(TM)) = Rad(TM), \tag{6.2.20}$$

(ii) there exists a subbundle D_0 of $S(TM)$ such that

$$D_0 = \phi(S(TM)) \cap S(TM), \tag{6.2.21}$$

where D_0 is a non-degenerate distribution on M.

From Definition 6.2.4 we get

$$S(TM) = D_0 \oplus D' \oplus_{orth} < \xi >, \tag{6.2.22}$$

where D' is a complementary non-degenerate distribution to D_0 in $S(TM)$ such that

$$\phi(D') \nsubseteq S(TM), \ \phi(D') \nsubseteq S(T^{\perp}M).$$

Let P_0, P_1 and P' be the projection morphisms on D_0, $Rad(TM)$ and D' respectively, then we have $\forall X \in \Gamma(TM)$,

$$X = P_0X + P_1X + P'X + \eta(X)\xi \tag{6.2.23}$$

$$\Rightarrow X = PX + P'X + \eta(X)\xi, \tag{6.2.24}$$

where

$$D = D_0 \oplus_{orth} Rad(TM), \tag{6.2.25}$$

so that

$$TM = Rad(TM) \oplus_{orth} S(TM) = Rad(TM) \oplus_{orth} [D_0 \oplus D' \oplus_{orth} < \xi >]$$

$$\Rightarrow TM = D \oplus D' \oplus_{orth} < \xi >, \tag{6.2.26}$$

D is invariant i.e. $\phi(D) = D$ and $PX \in \Gamma(D)$, $P'X \in \Gamma(D')$.

From (6.2.20) we have

$$\phi X = TX + \omega X, \tag{6.2.27}$$

where TX and ωX are the tangential and transversal parts of ϕX respectively. Also, it is clear that $\phi(D') \neq D'$.

Again, $\forall Y \in \Gamma(D')$,

$$\phi Y = TY + \omega Y, \tag{6.2.28}$$

where $TY \in \Gamma(D')$ ande $\omega Y \in \Gamma(S(T^{\perp}M))$.

Similarly, $\forall W \in \Gamma(tr(TM))$,

$$\phi W = BW + CW, \tag{6.2.29}$$

where BW and CW are the tangential and transversal parts of ϕW respectively.

Definition 6.2.5. A lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is called *proper CSGL submanifold* if $D_0 \neq \{0\}$, $D' \neq \{0\}$ and then, from Definition 6.2.4 we have−
(A) $dim(Rad(TM)) = 2s \geq 2$ (by condition (i)),
(B) $dim(D_0) = 2a \geq 2$ (by condition (ii)),
(C) $dim(D') = 2p \geq 2$ so that $dim(M) \geq 7$ and $dim(\tilde{M}) \geq 11$,
(D) any proper 7-dimensional CSGL submanifold must be 2-lightlike,
(E) $index(\tilde{M}) \geq 2$ (by condition (i), since $\tilde{M}$ is an indefinite Kenmotsu manifold).

Proposition 6.2.1. [63] *A contact SCR-lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is a CSGL submanifold such that the distribution D' is totally anti-invariant i.e.,*

$$S(T^{\perp}M) = \omega D' \oplus \mu, \tag{6.2.30}$$

where μ is a non-degenerate invariant distribution $(\phi(\mu) = \mu)$.

Definition 6.2.6. [83] An r-lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is called *generic r-lightlike submanifold* if there exists a screen distribution $S(TM)$ of M such that

$$\phi(S(T^{\perp}M)) \subset S(TM). \tag{6.2.31}$$

Proposition 6.2.2. [63] *A generic r-lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is a screen generic lightlike submanifold with $\mu = \{0\}$.*

Proposition 6.2.3. [63] *Any CSGL submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is an invariant lightlike submanifold if $D' = \{0\}$.*

Definition 6.2.7. [63] A CSGL submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is called *D-geodesic* if

$$h(X, Y) = 0 \ \ \forall X, Y \in \Gamma(D), \tag{6.2.32}$$

which implies that, M is D-geodesic if

$$h^l(X, Y) = 0 = h^s(X, Y) \ \ \forall X, Y \in \Gamma(D). \tag{6.2.33}$$

Definition 6.2.8. M is called *mixed geodesic* if

$$h(X, Y) = 0 \ \ \forall X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >). \tag{6.2.34}$$

Definition 6.2.9. [67] Let (M, g) be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ of index $2q$ such that $2q < dim(M)$ with structure vector field ξ tangent to M, then M is called a *screen-slant lightlike submanifold*

of $\tilde{M}$ if the following conditions are satisfied−
(i) $Rad(TM)$ is invariant with respect to ϕ i.e. $\phi(Rad(TM)) \subseteq Rad(TM)$,
(ii) for any non-zero vector field Y tangent to $S(TM) = D \oplus_{orth} < \xi >$ at $y \in M$, the angle $\theta(Y)$ (known as the *slant angle*) between ϕY and $S(TM)$ is constant, where D is the complementary distribution to $< \xi >$ in $S(TM)$ and $Y,\ \xi$ are linearly independent.

M is called *proper* if $D \neq \{0\}$, $\theta \neq 0, \frac{\pi}{2}$ and is called a *screen real lightlike submanifold* if $\theta = \frac{\pi}{2}$.

Definition 6.2.10. [125] An r-lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is called a *screen-transversal lightlike submanifold* if $\phi(Rad(TM)) \subseteq S(T^{\perp}M)$.

Definition 6.2.11. [125] A screen-transversal lightlike submanifold (M, g) of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$ is called a *radical screen-transversal lightlike submanifold* if $S(TM)$ is invariant with respect to ϕ i.e. $\phi(S(TM)) \subseteq S(TM)$.

Note. [82] Here, without loss of generality, the structure vector field ξ of the indefinite Kenmotsu manifold $\tilde{M}$ is assumed to be spacelike i.e. $\tilde{g}(\xi, \xi) = \epsilon = 1$.

6.3 Screen-Slant Lightlike Submanifolds

In this section, we prove some results regarding screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold.

Let (M, g) be a screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $\tilde{M}(\phi, \xi, \eta, \tilde{g})$. Then we have the following decomposition−

$$TM = Rad(TM) \oplus_{orth} D \oplus_{orth} < \xi > .$$

Let $P,\ Q$ be the projection morphisms of TM on $Rad(TM),\ D$ respectively, then for any $X \in \Gamma(TM)$, we have

$$X = PX + QX + \eta(X)\xi, \tag{6.3.1}$$

where $PX \in \Gamma(Rad(TM)),\ QX \in \Gamma(D)$.

Again, for any $X \in \Gamma(TM)$, we have

$$\phi X = TX + \omega X, \tag{6.3.2}$$

where $TX \in \Gamma(TM)$ and $\omega X \in \Gamma(tr(TM))$ are the tangential and transversal components of ϕX respectively.

Now, applying ϕ on (6.3.1) we get

$$\phi X = TPX + TQX + \omega QX. \tag{6.3.3}$$

$S(T^{\perp}M)$ can be decomposed as

$$S(T^{\perp}M) = \omega Q(S(TM)) \oplus_{orth} \mu,$$

where μ is an invariant subspace of $T\tilde{M}$.

Then for any $W \in \Gamma(S(T^{\perp}M))$, we have

$$\phi W = BW + CW, \tag{6.3.4}$$

where $BW \in \Gamma(S(TM)),\ CW \in \Gamma(S(T^{\perp}M))$.

Also, for any $N \in \Gamma(ltr(TM))$,

$$\phi N = CN, \tag{6.3.5}$$

where $CN \in \Gamma(ltr(TM))$.

Now, we state and prove some results:−

Theorem 6.3.1. *Let (M, g) be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with constant index $2q < dim(M)$, then M is a screen-slant lightlike submanifold if and only if there exists a constant $\lambda \in [-1, 0]$ such that $\forall X \in \Gamma(S(TM))$,*

$$(P \circ T)^2 X = \lambda[-X + \eta(X)\xi], \tag{6.3.6}$$

where $\lambda = \cos^2\theta|_{S(TM)}$.

Proof. The proof follows from Theorem 3.1 in [68].

Corollary 6.3.1. *Let (M, g) be a screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$, then $\forall X, Y \in \Gamma(TM)$,*

$$g(TQX, TQY) = \cos^2\theta|_{S(TM)}[g(X, Y) - \eta(X)\eta(Y)], \tag{6.3.7}$$

$$\tilde{g}(\omega QX, \omega QY) = \sin^2\theta|_{S(TM)}[g(X, Y) - \eta(X)\eta(Y)]. \tag{6.3.8}$$

Proof. The proof follows from Corollary 3.2 in [68].

Theorem 6.3.2. *Let (M, g) be a screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$, then $\forall X, Y \in \Gamma(TM)$,*

$$(\nabla_X T)Y = A_{\omega Y}X + Bh^s(X, Y) + \tilde{g}(\phi X, Y)\xi - \eta(Y)TX, \tag{6.3.9}$$

$$(\nabla_X\omega)Y = Ch^s(X,Y) + Ch^l(X,Y) - h^s(X,TY) - h^l(X,TY) - D^l(X,\omega Y) \\ -\eta(Y)\omega X, \quad (6.3.10)$$

where $(\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y)$ *and* $(\nabla_X\omega)Y = \nabla^s_X\omega Y - \omega(\nabla_X Y)$.

Proof. From (1.20) we get

$$\tilde{\nabla}_X\phi Y = \phi\tilde{\nabla}_X Y + \tilde{g}(\phi X, Y)\xi - \eta(Y)\phi X.$$

Applying (6.3.2) on the above equation we obtain

$$\tilde{\nabla}_X(TY + \omega Y) = \phi\tilde{\nabla}_X Y + \tilde{g}(\phi X, Y)\xi - \eta(Y)(TX + \omega X),$$

on which applying (6.2.6), (6.2.8), (6.3.2), (6.3.4), (6.3.5) we get

$$\nabla_X TY + h^l(X,TY) + h^s(X,TY) - A_{\omega Y}X + \nabla^s_X\omega Y + D^l(X,\omega Y) \\ = T\nabla_X Y + \omega\nabla_X Y + Ch^l(X,Y) + Bh^s(X,Y) + Ch^s(X,Y) + \tilde{g}(\phi X, Y)\xi \\ - \eta(Y)(TX + \omega X).$$

Equating tangential and transversal components of the above equation we obtain (6.3.9) and (6.3.10) respectively.

6.4 Totally Contact Umbilical Screen-Slant Lightlike Submanifolds

In this section, we prove the following characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold:−

Theorem 6.4.1. *Let (M, g) be a totally contact umbilical screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$, then at least one of the following statements is true−*
(i) M is a screen real lightlike submanifold,
(ii) $D = \{0\}$,
(iii) if M is a proper screen-slant lightlike submanifold, then $\alpha_s \in \Gamma(\mu)$.

Proof. For any $Y = QY \in \Gamma(D)$, from (6.2.1) we have

$$h(TQY, TQY) = g(TQY, TQY)\alpha,$$

on which applying (1.17), (1.20), (6.2.4), (6.2.6), (6.2.8), (6.3.1), (6.3.2), (6.3.7) we get

$$\phi(\nabla_{TQY}QY + h^l(TQY, QY) + h^s(TQY, QY)) + A_{\omega QY}TQY - \nabla^s_{TQY}\omega QY \\ - D^l(TQY, \omega QY) - \nabla_{TQY}TQY - g(TQY, TQY)\xi \\ = \cos^2\theta g(Y,Y)\alpha,$$

which (by the help of (6.2.2), (6.2.3), (6.3.2)) reduces to−

$$\begin{aligned}&T\nabla_{TQY}QY + \omega\nabla_{TQY}QY + A_{\omega QY}TQY - \nabla^s_{TQY}\omega QY - D^l(TQY, \omega QY)\\&- \nabla_{TQY}TQY - g(TQY, TQY)\xi\\&= \cos^2\theta g(Y, Y)\alpha,\end{aligned}$$

since $g(TQY, QY) = \tilde{g}(\phi Y, Y) = -\tilde{g}(Y, \phi Y) = -g(TQY, QY) \Rightarrow g(TQY, QY) = 0$.

Equating transversal components of the above equation we obtain

$$\omega\nabla_{TQY}QY - \nabla^s_{TQY}\omega QY - D^l(TQY, \omega QY) = \cos^2\theta g(Y, Y)\alpha. \tag{6.4.1}$$

Now, taking covariant derivative of (6.3.8) with respect to TQY we get

$$\tilde{g}(\nabla^s_{TQY}\omega QY, \omega QY) = \sin^2\theta g(\nabla^s_{TQY}Y, Y). \tag{6.4.2}$$

Again, from (6.3.8) we have

$$\tilde{g}(\omega\nabla_{TQY}QY, \omega QY) = \sin^2\theta g(\nabla^s_{TQY}Y, Y). \tag{6.4.3}$$

Now, taking inner product of (6.4.1) with ωQY we obtain

$$\begin{aligned}&\tilde{g}(\omega\nabla_{TQY}QY, \omega QY) - \tilde{g}(\nabla^s_{TQY}\omega QY, \omega QY) = \cos^2\theta g(Y, Y)\tilde{g}(\alpha_s, \omega QY)\\&\Rightarrow \cos^2\theta g(Y, Y)\tilde{g}(\alpha_s, \omega QY) = 0 \;\; (by\ (6.4.2), (6.4.3))\\&\Rightarrow \theta = \frac{\pi}{2} \; or\ Y = 0\ or\ \alpha_s \in \Gamma(\mu),\end{aligned}$$

which implies either M is a screen real lightlike submanifold or $D = \{0\}$ or $\alpha_s \in \Gamma(\mu)$ if M is proper.

This completes the proof.

6.5 Totally Contact Umbilical Radical Screen-Transversal Lightlike Submanifolds

In this section, we prove some results on a totally contact umbilical radical screen-transversal lightlike submanifold M of an indefinite Kenmotsu manifold $\tilde{M}$, such as the necessary and sufficient conditions for the screen distribution $S(TM)$ to be integrable and for the induced connection ∇ to be a metric connection.

Theorem 6.5.1. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$, then $S(TM)$ is integrable if and only if α_s has no component in $\phi(Rad(TM))$.*

Proof. For any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(Rad(TM))$, using (1.16), (1.17), (1.20), (6.2.3), (6.2.6) we get

$$\tilde{g}([X,Y],N) = \tilde{g}(h^s(X,\phi Y) - h^s(Y,\phi X), \phi N) = 2g(X,\phi Y)\tilde{g}(\alpha_s, \phi N),$$

which implies that $[X,Y] \in \Gamma(S(TM))\ \forall X, Y \in \Gamma(S(TM))$ if and only if $\tilde{g}(\alpha_s, \phi N) = 0\ \forall N \in \Gamma(Rad(TM))$.

This completes the proof.

Theorem 6.5.2. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$, then $h^*(X,Y) = 0$ if and only if α_s has no component in $\phi(Rad(TM))$.*

Proof. For any $X, Y \in \Gamma(S(TM))$, using (1.20), (6.2.6) we have

$$\nabla_X \phi Y + h^l(X,\phi Y) + h^s(X,\phi Y) = \tilde{g}(\phi X, Y)\xi - \eta(Y)\phi X + \phi(\nabla_X Y + h^l(X,Y) + h^s(X,Y)).$$

Taking inner product of the above equation with ϕN for any $N \in \Gamma(Rad(TM))$, we obtain

$$\tilde{g}(h^s(X,\phi Y), \phi N) = \tilde{g}(\phi \nabla_X Y, \phi N).$$

Now, using (1.16), (6.2.3), (6.2.11) in the above equation we get

$$\tilde{g}(\alpha_s, \phi N) g(X, \phi Y) = \tilde{g}(h^*(X,Y), N),$$

which implies our assertion.

Theorem 6.5.3. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$, then the induced connection ∇ on M is a metric connection if and only if α_s has no component in $\phi(Rad(TM))$.*

Proof. For any $X \in \Gamma(TM)$ and $N \in \Gamma(Rad(TM))$, using (1.20) we get

$$\tilde{\nabla}_X \phi N - \phi(\tilde{\nabla}_X N) = \tilde{g}(\phi X, N)\xi,$$

on which applying ϕ and then using (1.15) we obtain

$$\tilde{\nabla}_X N = -\phi(\tilde{\nabla}_X \phi N).$$

Using (6.2.6), (6.2.8) in the above equation, taking inner product with $Y \in \Gamma(S(TM))$ and then using (1.17) we get

$$g(\nabla_X N, Y) = -g(A_{\phi N} X, \phi Y) + \tilde{g}(\nabla^s_X \phi N, \phi Y) + \tilde{g}(D^l(X, \phi N), \phi Y),$$

in which using (6.2.3), (6.2.9) we obtain

$$g(\nabla_X N, Y) = -g(X, \phi Y)\tilde{g}(\alpha_s, \phi N).$$

Therefore, ∇ is a metric connection on M if and only if $Rad(TM)$ is parallel if and only if $\nabla_X N \in \Gamma(Rad(TM))\ \ \forall X \in \Gamma(TM),\ N \in \Gamma(Rad(TM))$ if and only if $\tilde{g}(\alpha_s, \phi N) = 0\ \ \forall N \in \Gamma(Rad(TM))$.

This completes the proof.

Theorem 6.5.4. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$, then*
(i) $A_{\phi N} X = [X - \eta(X)\xi]\tilde{g}(\alpha_s, \phi N) + \eta(X)\phi N + D^l(X, \phi N)\ \ \forall X \in \Gamma(S(TM)),$ $N \in \Gamma(ltr(TM))$,
(ii) $A_{\phi N} X = X\tilde{g}(\alpha_s, \phi N) + D^l(X, \phi N)\ \ \forall X \in \Gamma(Rad(TM)),\ N \in \Gamma(ltr(TM))$.

Proof. Replacing W by ϕN in (6.2.9) we have

$$g(A_{\phi N} X, Y) = \tilde{g}(h^s(X, Y), \phi N) + \tilde{g}(Y, D^l(X, \phi N)),$$

on which applying (1.21), (6.2.3), (6.2.6) we get

$$\begin{aligned}
&g(A_{\phi N} X, Y) \\
=&[g(X, Y) - \eta(X)\eta(Y)]\tilde{g}(\alpha_s, \phi N) + \eta(X)\tilde{g}(Y, \phi N) + \eta(Y)\tilde{g}(X, \phi N) \\
&+ \tilde{g}(Y, D^l(X, \phi N)) \\
\Rightarrow& A_{\phi N} X = [X - \eta(X)\xi]\tilde{g}(\alpha_s, \phi N) + \eta(X)\phi N + \tilde{g}(X, \phi N)\xi + D^l(X, \phi N).
\end{aligned}$$

Then (i) and (ii) follow from the above equation restricting X to $S(TM)$ and $Rad(TM)$ respectively.

6.6 CSGL Submanifolds

In this section, we investigate the necessary and sufficient conditions for the induced connection ∇ on a CSGL submanifold M of an indefinite Kenmotsu manifold $\tilde{M}$ to be a metric connection, for integrability & parallelism of some associated distributions, and for some distributions to be totally geodesic foliations. We also discuss about non-parallel distributions and more than one necessary and sufficient conditions for M to be mixed geodesic.

Theorem 6.6.1. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then the induced connection ∇ on M is a metric connection if and only if $\forall X, Y \in \Gamma(Rad(TM))$, $U \in \Gamma(S(TM))$,*

$$\tilde{g}(h^l(X, \phi Y), \omega U) + \tilde{g}(h^s(X, \phi Y), \omega U) = g(X, Y)\eta(U). \tag{6.6.1}$$

Proof. From (1.20) we have $\forall X, Y \in \Gamma(Rad(TM))$,

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y) = \tilde{g}(\phi X, Y)\xi\ \ (as\ \eta(Y) = 0),$$

on which applying ϕ and then using (1.15) we get

$$\tilde{\nabla}_X Y = -\phi(\tilde{\nabla}_X \phi Y) - g(X,Y)\xi. \tag{6.6.2}$$

Using (6.2.6), (6.2.12), (6.2.27), (6.2.29) in (6.6.2) we obtain

$$\nabla_X Y + h(X,Y) = TA^*_{\phi Y}X + \omega A^*_{\phi Y}X - B\nabla^{*t}_X\phi Y - C\nabla^{*t}_X\phi Y$$

$$-Bh^l(X,\phi Y) - Ch^l(X,\phi Y) - Bh^s(X,\phi Y) - Ch^s(X,\phi Y) - g(X,Y)\xi. \tag{6.6.3}$$

Equating the tangential parts from both sides of (6.6.3) we get

$$\nabla_X Y = TA^*_{\phi Y}X - B\nabla^{*t}_X\phi Y - Bh^l(X,\phi Y) - Bh^s(X,\phi Y) - g(X,Y)\xi. \tag{6.6.4}$$

Now, we know that ∇ is a metric connection if and only if $Rad(TM)$ is a parallel distribution i.e., $g(\nabla_X Y, U) = 0 \ \ \forall U \in \Gamma(S(TM))$.

From (6.6.4), on applying (1.17) and (6.2.27), we have $\forall U \in \Gamma(S(TM))$,

$$g(\nabla_X Y, U) = \tilde{g}(h^l(X,\phi Y), \omega U) + \tilde{g}(h^s(X,\phi Y), \omega U) - g(X,Y)\eta(U),$$

which implies that $g(\nabla_X Y, U) = 0$ if and only if (6.6.1) holds. This completes the proof.

Theorem 6.6.2. *Let (M,g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M},\tilde{g})$ with the structure vector field ξ tangent to M, then*

(i) the distribution D_0 is integrable if and only if $\forall X,Y \in \Gamma(D_0)$, $Z \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$, $V \in \Gamma(S(T^{\perp}M))$,

$$g(\nabla^*_X\phi Y - \nabla^*_Y\phi X, TZ) + g(Bh^*(X,\phi Y) - Bh^*(Y,\phi X), TZ) = 0, \tag{6.6.5}$$
$$\tilde{g}(h^*(X,\phi Y) - h^*(Y,\phi X), \phi N) = 0, \tag{6.6.6}$$
$$g(\nabla^*_X\phi Y - \nabla^*_Y\phi X, BV) + \tilde{g}(h^*(X,\phi Y) - h^*(Y,\phi X), CV) = 0; \tag{6.6.7}$$

(ii) the distribution D' is integrable if and only if (6.6.5), (6.6.6), (6.6.7) hold $\forall X,Y \in \Gamma(D')$;

(iii) the distribution D is integrable if and only if $\forall X,Y \in \Gamma(D)$, $V \in \Gamma(S(T^{\perp}M))$,

$$g(\nabla_X TY - \nabla_Y TX, BV) + \tilde{g}(h(X,TY) - h(Y,TX), CV) = 0. \tag{6.6.8}$$

Proof. (i) $\forall X,Y \in \Gamma(D_0)$, using (1.21) in the following equation

$$\tilde{g}([X,Y],\xi) = \tilde{g}(\tilde{\nabla}_X Y,\xi) - \tilde{g}(\tilde{\nabla}_Y X,\xi) = -\tilde{g}(Y,\tilde{\nabla}_X\xi) + \tilde{g}(X,\tilde{\nabla}_Y\xi),$$

we have

$$\tilde{g}([X,Y],\xi) = 0. \tag{6.6.9}$$

Now, $\forall Z \in \Gamma(Rad(TM))$, using (1.16) we have

$$\tilde{g}([X,Y],Z) = \tilde{g}(\phi\tilde{\nabla}_X Y - \phi\tilde{\nabla}_Y X, \phi Z).$$

Applying (1.15), (1.17), (1.20), (6.2.11), (6.2.27) and (6.2.29) on the above equation we obtain

$$\tilde{g}([X,Y],Z) = g(\nabla^*_X\phi Y - \nabla^*_Y\phi X, TZ) + g(Bh^*(X,\phi Y) - Bh^*(Y,\phi X), TZ). \tag{6.6.10}$$

Again, $\forall N \in \Gamma(ltr(TM))$, using (1.16) we have

$$\tilde{g}([X,Y],N) = \tilde{g}(\phi\tilde{\nabla}_X Y - \phi\tilde{\nabla}_Y X, \phi N),$$

in which using (1.15), (1.17), (1.20) and (6.2.11) we get

$$\tilde{g}([X,Y],N) = \tilde{g}(h^*(X,\phi Y) - h^*(Y,\phi X), \phi N). \tag{6.6.11}$$

Also, $\forall V \in \Gamma(S(T^\perp M))$, using (6.2.16) we have

$$\tilde{g}([X,Y],V) = \tilde{g}(\phi\tilde{\nabla}_X Y - \phi\tilde{\nabla}_Y X, \phi V),$$

on which applying (1.15), (1.17), (1.20), (6.2.11) and (6.2.29) we get

$$\tilde{g}([X,Y],V) = g(\nabla^*_X\phi Y - \nabla^*_Y\phi X, BV) + \tilde{g}(h^*(X,\phi Y) - h^*(Y,\phi X), CV). \tag{6.6.12}$$

From (6.6.9)−(6.6.12), we can conclude that $\forall X,Y \in \Gamma(D_0)$, $[X,Y] \in \Gamma(D_0)$ if and only if $\forall Z \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$, $V \in \Gamma(S(T^\perp M))$, equations (6.6.5), (6.6.6) and (6.6.7) hold.

(ii) The proof is similar as of (i).

(iii) $\forall X,Y \in \Gamma(D)$, similarly as (6.6.9) we have

$$\tilde{g}([X,Y],\xi) = 0. \tag{6.6.13}$$

Now, $\forall V \in \Gamma(S(T^\perp M))$, using (1.16) we have

$$\tilde{g}([X,Y],V) = \tilde{g}(\phi\tilde{\nabla}_X Y - \phi\tilde{\nabla}_Y X, \phi V).$$

Using (1.15), (1.20), (6.2.4) and (6.2.29) in the above equation we obtain

$$\tilde{g}([X,Y],V) = g(\nabla_X TY - \nabla_Y TX, BV) + \tilde{g}(h(X,TY) - h(Y,TX), CV). \tag{6.6.14}$$

From (6.6.13) and (6.6.14), we conclude that $\forall X,Y \in \Gamma(D)$, $[X,Y] \in \Gamma(D)$ if and only if $\forall V \in \Gamma(S(T^\perp M))$, equation (6.6.8) holds.

Theorem 6.6.3. *Let (M,g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M},\tilde{g})$ with the structure vector field ξ tangent to M, then* *(i) the distribution D_0 is not parallel,*

(ii) the distribution D' is not parallel,

(iii) the distribution D is not parallel.

Proof. (i) Let $X, Y \in \Gamma(D_0)$, then using (1.21) in the following equation

$$g(\nabla_X Y, \xi) = -\tilde{g}(Y, \tilde{\nabla}_X \xi),$$

we get $g(\nabla_X Y, \xi) = -\tilde{g}(X, Y) \neq 0$ since D_0 is non-degenerate. Hence, D_0 is not parallel.

(ii) The proof is similar as of (i).

(iii) Let $X, Y \in \Gamma(D) = \Gamma(D_0 \oplus_{orth} Rad(TM))$, then using (1.21) in the following equation

$$g(\nabla_X Y, \xi) = -\tilde{g}(Y, \tilde{\nabla}_X \xi),$$

we get $g(\nabla_X Y, \xi) = -\tilde{g}(X, Y) \neq 0$ since D_0 is non-degenerate. Hence, D is not parallel.

Theorem 6.6.4. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then*

(i) the distribution $D_0 \oplus_{orth} < \xi >$ is parallel if and only if $\forall X \in \Gamma(TM)$, $Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$,

$$\nabla^*_X TY - A_{\omega Y} X \in \Gamma(D_0 \oplus_{orth} < \xi >),$$
$$h^*(X, TY) + \nabla^t_X \omega Y = 0;$$

(ii) the distribution $D' \oplus_{orth} < \xi >$ is parallel if and only if $\forall X \in \Gamma(TM)$, $Y \in \Gamma(D' \oplus_{orth} < \xi >)$,

$$\nabla^*_X TY - A_{\omega Y} X \in \Gamma(D' \oplus_{orth} < \xi >),$$
$$h^*(X, TY) + \nabla^t_X \omega Y = 0;$$

(iii) the distribution $D \oplus_{orth} < \xi >$ is parallel if and only if $\forall X \in \Gamma(TM)$, $Y \in \Gamma(D \oplus_{orth} < \xi >)$, $\tilde{\nabla}_X TY$ has no component in $\phi(S(T^\perp M))$.

Proof. (i) Let $X \in \Gamma(TM)$, $Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$.

Now, for $Z \in \Gamma(Rad(TM))$, using (1.16) we have

$$g(\nabla_X Y, Z) = \tilde{g}(\tilde{\nabla}_X Y, \phi Z),$$

which leads to the following equation with the help of (1.15), (1.20), (6.2.5), (6.2.11) and (6.2.27)−

$$g(\nabla_X Y, Z) = g(\nabla^*_X TY - A_{\omega Y}X, TZ)$$
$$\Rightarrow g(\nabla_X Y, Z) = 0 \iff g(\nabla^*_X TY - A_{\omega Y}X, TZ) = 0. \tag{6.6.15}$$

Again, for $N \in \Gamma(ltr(TM))$, using (6.2.20) we have

$$g(\nabla_X Y, N) = \tilde{g}(\tilde{\nabla}_X Y, \phi N).$$

Applying (1.15), (1.20), (6.2.5), (6.2.11), (6.2.27) and (6.2.29) on the above equation we get

$$g(\nabla_X Y, N) = g(\nabla^*_X TY - A_{\omega Y}X, BN) + \tilde{g}(h^*(X, TY) + \nabla^t_X \omega Y, CN)$$
$$\Rightarrow g(\nabla_X Y, N) = 0 \text{ if and only if}$$
$$g(\nabla^*_X TY - A_{\omega Y}X, BN) = 0, \tag{6.6.16}$$
$$\tilde{g}(h^*(X, TY) + \nabla^t_X \omega Y, CN) = 0. \tag{6.6.17}$$

Also, for $V \in \Gamma(S(T^\perp M))$, using (6.2.20) we have

$$g(\nabla_X Y, V) = \tilde{g}(\tilde{\nabla}_X Y, \phi V),$$

in which using (1.15), (1.20), (6.2.5), (6.2.11), (6.2.27) and (6.2.29) we get

$$g(\nabla_X Y, V) = g(\nabla^*_X TY - A_{\omega Y}X, BV) + \tilde{g}(h^*(X, TY) + \nabla^t_X \omega Y, CV)$$
$$\Rightarrow g(\nabla_X Y, V) = 0 \text{ if and only if}$$
$$g(\nabla^*_X TY - A_{\omega Y}X, BV) = 0, \tag{6.6.18}$$
$$\tilde{g}(h^*(X, TY) + \nabla^t_X \omega Y, CV) = 0. \tag{6.6.19}$$

The distribution $D_0 \oplus_{orth} < \xi >$ is parallel if and only if $\forall X \in \Gamma(TM)$, $Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$, $\nabla_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$.

Now, combining (6.6.15), (6.6.16), (6.6.18) and then (6.6.17), (6.6.19) respectively, we have, $\nabla_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$ if and only if

$$g(\nabla^*_X TY - A_{\omega Y}X, \phi U) = 0 \;\; \forall U \in \Gamma([Rad(TM) \oplus ltr(TM)] \oplus_{orth} S(T^\perp M))$$
$$\iff \nabla^*_X TY - A_{\omega Y}X \in \Gamma(D_0 \oplus_{orth} < \xi >), \; and$$
$$\tilde{g}(h^*(X, TY) + \nabla^t_X \omega Y, \phi W) = 0 \;\; \forall W \in \Gamma(ltr(TM) \oplus_{orth} S(T^\perp M)) = \Gamma(tr(TM))$$
$$\iff h^*(X, TY) + \nabla^t_X \omega Y = 0.$$

(ii) The proof is similar as of (i).

(ii) Let $X \in \Gamma(TM)$, $Y \in \Gamma(D \oplus_{orth} < \xi >)$. For $V \in \Gamma(S(T^{\perp}M))$, using (1.16) we have

$$g(\nabla_X Y, V) = \tilde{g}(\tilde{\nabla}_X Y, \phi V),$$

which leads to the following equation by the help of (1.15), (1.20), (6.2.4) and (6.2.27)−

$$g(\nabla_X Y, V) = \tilde{g}(\tilde{\nabla}_X TY, \phi V). \tag{6.6.20}$$

Now, the distribution $D \oplus_{orth} < \xi >$ is parallel if and only if $\forall X \in \Gamma(TM)$, $Y \in \Gamma(D \oplus_{orth} < \xi >)$, $\nabla_X Y \in \Gamma(D \oplus_{orth} < \xi >)$.

Therefore, from (6.6.20), we get, the distribution $D \oplus_{orth} < \xi >$ is parallel if and only if $\tilde{\nabla}_X TY$ has no component in $\phi(S(T^{\perp}M))$.

Theorem 6.6.5. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then the distribution $D_0 \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\tilde{M}$ if and only if M is $D_0 \oplus_{orth} < \xi >$-geodesic and $D_0 \oplus_{orth} < \xi >$ is parallel with respect to ∇ on M.*

Proof. $D_0 \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\tilde{M}$ if and only if $\forall X, Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$, $\tilde{\nabla}_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$ i.e., $\tilde{g}(\tilde{\nabla}_X Y, Z) = \tilde{g}(\tilde{\nabla}_X Y, N) = \tilde{g}(\tilde{\nabla}_X Y, V) = 0 \;\; \forall Z \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$, $V \in \Gamma(S(T^{\perp}M))$.

Now, from (6.2.30), we have $\forall X, Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$, $N \in \Gamma(ltr(TM))$, $V \in \Gamma(S(T^{\perp}M))$,

$$\tilde{g}(\tilde{\nabla}_X Y, N) = \tilde{g}(h^l(X, Y), N),$$
$$\tilde{g}(\tilde{\nabla}_X Y, V) = \tilde{g}(h^s(X, Y), V).$$

Hence, if $D_0 \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\tilde{M}$, then $\tilde{\nabla}_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$ and thus, from the above two equations, we get $h^l(X, Y) = 0 = h^s(X, Y) \Rightarrow M$ is $D_0 \oplus_{orth} < \xi >$-geodesic and from (6.2.6), $\nabla_X Y = \tilde{\nabla}_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >)$ so that $D_0 \oplus_{orth} < \xi >$ is parallel with respect to ∇ on M.

Conversely, if M is $D_0 \oplus_{orth} < \xi >$-geodesic, then $h^l(X, Y) = 0 = h^s(X, Y)$ and hence, from (6.2.6), $\tilde{\nabla}_X Y = \nabla_X Y \in \Gamma(TM)$. As $D_0 \oplus_{orth} < \xi >$ is parallel with respect to ∇ on M, $\tilde{\nabla}_X Y = \nabla_X Y \in \Gamma(D_0 \oplus_{orth} < \xi >) \Rightarrow D_0 \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\tilde{M}$.

Theorem 6.6.6. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then the distribution $D' \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\tilde{M}$ if and only if M is $D' \oplus_{orth} < \xi >$-geodesic and $D' \oplus_{orth} < \xi >$ is parallel with respect to ∇ on M.*

Proof. The proof is similar as of Theorem 6.6.5.

Theorem 6.6.7. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then the distribution $D \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\tilde{M}$ if and only if $h^s = 0$ on $D \oplus_{orth} < \xi >$.*

Proof. $D \oplus_{orth} < \xi >$ is a totally geodesic foliation in $\tilde{M}$ if and only if $\forall X, Y \in \Gamma(D \oplus_{orth} < \xi >)$, $\tilde{\nabla}_X Y \in \Gamma(D \oplus_{orth} < \xi >)$ i.e., $\tilde{g}(\tilde{\nabla}_X Y, V) = 0 \quad \forall V \in \Gamma(S(T^{\perp}M))$.

Now, from (6.2.6), we have $\forall X, Y \in \Gamma(D \oplus_{orth} < \xi >)$, $V \in \Gamma(S(T^{\perp}M))$,

$$\begin{aligned} &\tilde{g}(\tilde{\nabla}_X Y, V) = \tilde{g}(\nabla_X Y + h^l(X, Y) + h^s(X, Y), V) = \tilde{g}(h^s(X, Y), V) \\ \Rightarrow &\tilde{g}(\tilde{\nabla}_X Y, V) = 0 \iff h^s(X, Y) = 0. \end{aligned}$$

Hence, the proof is completed.

Theorem 6.6.8. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M. If M is mixed geodesic, then $\forall X \in \Gamma(D)$, $Y \in \Gamma(D' \oplus_{orth} < \xi >)$, $Z \in \Gamma(Rad(TM))$, $V \in \Gamma(S(T^{\perp}M))$,*

$$(i)\ g((\nabla_X T)Y, Z) = g(A_{\omega Y} X - \eta(Y)\phi X, Z),\ D^l(X, \omega Y) = -h^l(X, TY), \tag{6.6.21}$$

$$(ii)\ g(A_{\omega Y} X - \nabla_X TY, BV) = \tilde{g}(\nabla^s_X \omega Y + h^s(X, TY), CV). \tag{6.6.22}$$

Proof. Let M be mixed geodesic, then $\forall X \in \Gamma(D)$, $Y \in \Gamma(D' \oplus_{orth} < \xi >)$,

$$h(X, Y) = 0 \Rightarrow \tilde{g}(h(X, Y), Z) = 0 = \tilde{g}(h(X, Y), V)$$

$$\Rightarrow \tilde{g}(h^l(X, Y), Z) = 0 \ \ \forall Z \in \Gamma(Rad(TM)), \tag{6.6.23}$$

$$\tilde{g}(h^s(X, Y), V) = 0 \ \ \forall V \in \Gamma(S(T^{\perp}M)). \tag{6.6.24}$$

(i) We have, on using (6.2.6) and (6.6.23), $\forall X \in \Gamma(D)$, $Y \in \Gamma(D' \oplus_{orth} < \xi >)$, $Z \in \Gamma(Rad(TM))$,

$$\tilde{g}(\tilde{\nabla}_X Y, Z) = \tilde{g}(\nabla_X Y, Z). \tag{6.6.25}$$

Replacing Z by ϕZ in (6.6.25) we have

$$\tilde{g}(\tilde{\nabla}_X Y, \phi Z) = \tilde{g}(\nabla_X Y, \phi Z),$$

on which applying (1.20), (6.2.6), (6.2.8), (6.2.28) to the left side and (1.17), (6.2.27) to the right side we obtain

$$\begin{aligned} g(\nabla_X TY - A_{\omega Y} X, Z) + \eta(Y) g(\phi X, Z) + \tilde{g}(D^l(X, \omega Y), Z) + \tilde{g}(h^l(X, TY), Z) \\ = g(T(\nabla_X Y), Z). \end{aligned} \tag{6.6.26}$$

Comparing the tangential and transversal parts of (6.6.26) from both sides, we get respectively

$$\begin{aligned} & g(\nabla_X TY - A_{\omega Y}X, Z) + \eta(Y) g(\phi X, Z) = g(T(\nabla_X Y), Z) \\ \Rightarrow & g(\nabla_X TY - A_{\omega Y}X + \eta(Y)\phi X, Z) = g(T(\nabla_X Y), Z) \\ \Rightarrow & g((\nabla_X T)Y, Z) = g(A_{\omega Y}X - \eta(Y)\phi X, Z), \end{aligned}$$

and

$$\begin{aligned} & D^l(X, \omega Y) + h^l(X, TY) = 0 \\ \Rightarrow & D^l(X, \omega Y) = -h^l(X, TY). \end{aligned}$$

(ii) By the help of the equations (6.2.6), (6.2.8), (6.2.28) and (6.2.29) we have

$$\tilde{g}(\tilde{\nabla}_X \phi Y, \phi V) = g(\nabla_X TY - A_{\omega Y}X, BV) + \tilde{g}(\nabla^s_X \omega Y + h^s(X, TY), CV). \quad (6.6.27)$$

Now, using (6.2.6) and (6.6.24) we have

$$\tilde{g}(\tilde{\nabla}_X Y, V) = \tilde{g}(h^s(X, Y), V) = 0. \quad (6.6.28)$$

Again, using (1.16) we have

$$\tilde{g}(\tilde{\nabla}_X Y, V) = \tilde{g}(\phi(\tilde{\nabla}_X Y), \phi V).$$

Using (1.15), (1.16) and (1.20) in the above equation we obtain

$$\tilde{g}(\tilde{\nabla}_X Y, V) = \tilde{g}(\tilde{\nabla}_X \phi Y, \phi V). \quad (6.6.29)$$

Equations (6.6.28) and (6.6.29) imply

$$\tilde{g}(\tilde{\nabla}_X \phi Y, \phi V) = 0. \quad (6.6.30)$$

Equations (6.6.27) and (6.6.30) lead to

$$\begin{aligned} & g(\nabla_X TY - A_{\omega Y}X, BV) + \tilde{g}(\nabla^s_X \omega Y + h^s(X, TY), CV) = 0 \\ \Rightarrow & g(A_{\omega Y}X - \nabla_X TY, BV) = \tilde{g}(\nabla^s_X \omega Y + h^s(X, TY), CV). \end{aligned}$$

Theorem 6.6.9. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then M is mixed geodesic if and only if* $\forall X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >)$,

$$D^l(X, \omega Y) = -h^l(X, TY), \quad (6.6.31)$$

$$\omega(A_{\omega Y}X - \nabla_X TY) = C(h^s(X, TY) + \nabla^s_X \omega Y). \quad (6.6.32)$$

Proof. Let $X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >)$, then from (1.15) we have

$$\begin{aligned} & \phi^2 Y = -Y + \eta(Y)\xi \\ \Rightarrow & \phi(\phi Y) = -Y + \eta(Y)\xi, \end{aligned}$$

on which applying (6.2.28) we get

$$\phi(TY + \omega Y) = -Y + \eta(Y)\xi.$$

Now, differentiating the above equation with respect to X i.e., operating with $\tilde{\nabla}_X$ on both sides we obtain

$$(\tilde{\nabla}_X\phi)\phi Y + \phi(\tilde{\nabla}_X TY) + \phi(\tilde{\nabla}_X \omega Y) = -\tilde{\nabla}_X Y - \tilde{g}(Y, \tilde{\nabla}_X\xi)\xi + \eta(Y)\tilde{\nabla}_X\xi,$$

in which using (1.20), (1.21), (6.2.4), (6.2.6), (6.2.8), (6.2.27) and (6.2.29) we obtain

$$\begin{aligned}
&[T(\nabla_X TY) + \omega(\nabla_X TY) + Bh^l(X, TY) + Ch^l(X, TY) + Bh^s(X, TY) + Ch^s(X, TY)] \\
&\quad +[-TA_{\omega Y}X - \omega A_{\omega Y}X + B\nabla^s_X\omega Y + C\nabla^s_X\omega Y + BD^l(X, \omega Y) + CD^l(X, \omega Y)] \\
&\quad +2\tilde{g}(X, Y)\xi - \eta(X)\eta(Y)\xi \\
&= -\nabla_X Y - h(X, Y) + \eta(Y)X. \qquad (6.6.33)
\end{aligned}$$

Equating the transversal parts from both sides of (6.6.33), we have

$$h(X, Y)$$
$$= [\omega(A_{\omega Y}X - \nabla_X TY) - C(h^s(X, TY) + \nabla^s_X\omega Y)] - C(h^l(X, TY) + D^l(X, \omega Y)). \qquad (6.6.34)$$

Now, M is mixed geodesic if and only if $h(X, Y) = 0 \quad \forall X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >)$. Hence, from (6.6.34), we have, M is mixed geodesic if and only if the equations (6.6.31) and (6.6.32) hold.

Theorem 6.6.10. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M. If M is mixed geodesic, then $\forall X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >)$,*

$$(\nabla_X T)Y = A_{\omega Y}X + g(TX, Y)\xi - \eta(Y)TX, \qquad (6.6.35)$$
$$\omega\nabla_X Y = h^s(X, TY) + \nabla^s_X\omega Y. \qquad (6.6.36)$$

Proof. As M is mixed geodesic, we have, $\forall X \in \Gamma(D),\ Y \in \Gamma(D' \oplus_{orth} < \xi >)$,

$$h(X, Y) = 0. \qquad (6.6.37)$$

From (6.2.28) we have

$$\phi Y = TY + \omega Y,$$

which gives us, on differentiating both sides with respect to X,

$$(\tilde{\nabla}_X\phi)Y + \phi(\tilde{\nabla}_X Y) = \tilde{\nabla}_X TY + \tilde{\nabla}_X\omega Y.$$

Now, using (1.20), (6.2.4), (6.2.6), (6.2.8), (6.2.27), (6.2.29) and (6.6.37) in the above equation we obtain

$$\begin{aligned}
&\tilde{g}(\phi X, Y)\xi - \eta(Y)\phi X + T\nabla_X Y + \omega\nabla_X Y \\
&= \nabla_X TY + h^l(X, TY) + h^s(X, TY) - A_{\omega Y}X + \nabla^s_X\omega Y + D^l(X, \omega Y),
\end{aligned}$$

on which applying (6.6.31) we have

$$\tilde{g}(\phi X, Y)\xi - \eta(Y)\phi X + T\nabla_X Y + \omega\nabla_X Y = \nabla_X TY + h^s(X, TY) - A_{\omega Y}X + \nabla^s_X \omega Y. \tag{6.6.38}$$

Again, comparing the tangential and transversal parts from both sides of (6.6.38), we have respectively

$$\begin{aligned} & g(TX, Y)\xi - \eta(Y)TX = (\nabla_X T)Y - A_{\omega Y}X \quad (\textit{using } (6.2.27)) \\ \Rightarrow & (\nabla_X T)Y = A_{\omega Y}X + g(TX, Y)\xi - \eta(Y)TX, \\ & \omega\nabla_X Y = h^s(X, TY) + \nabla^s_X \omega Y. \end{aligned}$$

6.7 Totally Umbilical CSGL Submanifolds

In this section, we study some properties satisfied by a proper totally umbilical CSGL submanifold M of an indefinite Kenmotsu manifold $\tilde{M}$.

Theorem 6.7.1. *Let (M, g) be a proper totally umbilical CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then $\alpha_s \notin \Gamma(\mu)$.*

Proof. Let $X, Y \in \Gamma(TM)$, then from (6.2.27) we have

$$\phi Y = TY + \omega Y.$$

Now, differentiating the above equation with respect to X, we get

$$(\tilde{\nabla}_X \phi)Y + \phi(\tilde{\nabla}_X Y) = \tilde{\nabla}_X TY + \tilde{\nabla}_X \omega Y.$$

Applying (1.20), (6.2.6), (6.2.8), (6.2.27) and (6.2.29) on the above equation we obtain

$$\begin{aligned} & g(TX, Y)\xi - \eta(Y)TX - \eta(Y)\omega X + T(\nabla_X Y) + \omega(\nabla_X Y) \\ & + Bh^l(X, Y) + Ch^l(X, Y) + Bh^s(X, Y) + Ch^s(X, Y) \\ & = \nabla_X TY + h^l(X, TY) + h^s(X, TY) - A_{\omega Y}X + \nabla^s_X \omega Y + D^l(X, \omega Y). \end{aligned}$$

Comparing the tangential and transversal parts of the above equation, we get respectively

$$g(TX, Y)\xi - \eta(Y)TX + T(\nabla_X Y) + Bh^l(X, Y) + Bh^s(X, Y) = \nabla_X TY - A_{\omega Y}X, \tag{6.7.1}$$

$$\begin{aligned} & -\eta(Y)\omega X + \omega(\nabla_X Y) + Ch^l(X, Y) + Ch^s(X, Y) \\ & = h^l(X, TY) + h^s(X, TY) + \nabla^s_X \omega Y + D^l(X, \omega Y). \end{aligned} \tag{6.7.2}$$

Again, from (6.2.2) and (6.2.3), we have respectively

$$Ch^l(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]C\alpha_l + \eta(X)Ch^l(Y,\xi) + \eta(Y)Ch^l(X,\xi), \quad (6.7.3)$$
$$Ch^s(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]C\alpha_s + \eta(X)Ch^s(Y,\xi) + \eta(Y)Ch^s(X,\xi). \quad (6.7.4)$$

Adding (6.7.3), (6.7.4) and then using (6.7.2) to replace the value obtained in the left hand side of the resultant equation, we get

$$\eta(Y)\omega X - \omega(\nabla_X Y) + h^l(X,TY) + h^s(X,TY) + \nabla^s_X \omega Y + D^l(X,\omega Y)$$
$$= [g(X,Y) - \eta(X)\eta(Y)]C(\alpha_l + \alpha_s) + \eta(X)C[h^l(Y,\xi) + h^s(Y,\xi)]$$
$$+\eta(Y)C[h^l(X,\xi) + h^s(X,\xi)]. \quad (6.7.5)$$

Let $X, Y \in \Gamma(D)$, then $\phi X, \phi Y \in \Gamma(\phi(D)) = \Gamma(D) \Rightarrow \phi X = TX$, $\phi Y = TY$ and $\omega X = 0 = \omega Y$. Also, $\eta(X) = 0 = \eta(Y)$. Hence, from (6.7.5) we obtain

$$-\omega(\nabla_X Y) + h^l(X,TY) + h^s(X,TY) = g(X,Y)C(\alpha_l + \alpha_s). \quad (6.7.6)$$

Equating the $S(T^{\perp}M)$-components from both sides of (6.7.6), we have

$$-\omega(\nabla_X Y) + h^s(X,TY) = g(X,Y)C\alpha_s. \quad (6.7.7)$$

Replacing X by ϕX, Y by ϕY in (6.7.7) and then using (1.16) and $\eta(X) = 0 = \eta(Y)$, we get

$$-\omega(\nabla_{\phi X}\phi Y) + h^s(\phi X, \phi^2 Y) = g(X,Y)C\alpha_s. \quad (6.7.8)$$

Again, from (6.2.3) and with the help of (1.15), (1.16) we have

$$h^s(\phi X, \phi^2 Y) = g(X, \phi Y)\alpha_s. \quad (6.7.9)$$

Now, applying (6.7.9) on (6.7.8) we obtain

$$-\omega(\nabla_{\phi X}\phi Y) + g(X,\phi Y)\alpha_s = g(X,Y)C\alpha_s. \quad (6.7.10)$$

Putting $X = \phi Y$ in (6.7.10) and using the fact that $g(Y,\phi Y) = -g(\phi Y, Y) \Rightarrow g(Y,\phi Y) = 0$, we get

$$g(\phi Y, \phi Y)\alpha_s = \omega(\nabla_{\phi^2 Y}\phi Y),$$

which gives, on replacing ϕY by Y,

$$g(Y,Y)\alpha_s = \omega(\nabla_{\phi Y} Y). \quad (6.7.11)$$

Let $Y \in \Gamma(D_0)$, then (6.7.11) gives $\alpha_s \notin \Gamma(\mu)$ since D_0 is a non-degenerate distribution.

Theorem 6.7.2. *Let (M,g) be a proper totally umbilical CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M},\tilde{g})$ with the structure vector field ξ tangent to M, then the induced connection ∇ is a metric connection on $D \oplus_{orth} < \xi >$.*

Proof. Equating $ltr(TM)$-components from (6.7.6), we have $\forall X, Y \in \Gamma(D)$,

$$h^l(X, TY) = g(X, Y)C\alpha_l. \tag{6.7.12}$$

Replacing X by ϕX, Y by ϕY and then using (1.16), (6.2.2) and $\eta(X) = 0 = \eta(Y)$, we get

$$g(X, \phi Y)\alpha_l = g(X, Y)C\alpha_l. \tag{6.7.13}$$

Now, interchanging X, Y and then applying (1.17), we obtain

$$-g(X, \phi Y) = g(X, Y)C\alpha_l. \tag{6.7.14}$$

Subtracting (6.7.14) from (6.7.13) we get

$$2g(X, \phi Y)\alpha_l = 0. \tag{6.7.15}$$

Putting $X = \phi Y$ in (6.7.15) we have

$$2g(\phi Y, \phi Y)\alpha_l = 0$$

$$\Rightarrow \alpha_l = 0 \tag{6.7.16}$$

since $D = D_0 \oplus_{orth} < \xi >$ and D_0 is non-degenerate.

Again, applying (6.7.16) and $\eta(X) = 0 = \eta(Y)$ on (6.2.45), we get $\forall X, Y \in \Gamma(D)$,

$$h^l(X, Y) = 0. \tag{6.7.17}$$

By (1.21) and (6.2.6) we have

$$\nabla_X \xi + h^l(X, \xi) + h^s(X, \xi) = X - \eta(X)\xi,$$

which gives, on equating the tangential and transversal parts from both sides respectively

$$\nabla_X \xi = X - \eta(X)\xi, \tag{6.7.18}$$
$$h^l(X, \xi) = 0, \tag{6.7.19}$$
$$h^s(X, \xi) = 0. \tag{6.7.20}$$

Combining (6.7.17) and (6.7.19) we get

$$h^l = 0 \text{ on } D \oplus_{orth} < \xi >$$

$$\Rightarrow \nabla g = 0 \text{ on } D \oplus_{orth} < \xi > \text{ (by (6.2.16))}$$

$$\Rightarrow \nabla \text{ is a metric connection on } D \oplus_{orth} < \xi >.$$

6.8 Minimal CSGL Submanifolds

In this section, we find the necessary and sufficient conditions for minimality of the distribution $D_0 \oplus_{orth} < \xi >$ associated to a CSGL submanifold M of an indefinite Kenmotsu manifold $\tilde{M}$ and also of M itself.

Theorem 6.8.1. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then the distribution $D_0 \oplus_{orth} < \xi >$ is minimal if and only if $\nabla_X X + \nabla_{\phi X} \phi X \in \Gamma(D_0 \oplus_{orth} < \xi >)$ $\forall X \in \Gamma(D_0 \oplus_{orth} < \xi >)$.*

Proof. From the description of D_0, it is clear that $D_0 \oplus_{orth} < \xi >$ is minimal if and only if $h^s = 0$ on $D_0 \oplus_{orth} < \xi >$. Now, let $X \in \Gamma(D_0 \oplus_{orth} < \xi >)$.

For $V \in \Gamma(S(T^{\perp}M))$, by the help of the equations (1.17), (1.20), (6.2.6) and (6.2.9), we obtain

$$g(\nabla_X X, \phi V) = -g(A_V X, \phi X) - \tilde{g}(h^s(X, X), \phi V). \tag{6.8.1}$$

Also, using (1.15), (1.16), (1.20), (6.2.6), (6.2.9) and $\eta(X) = 0 = \eta(V)$, we get

$$g(\nabla_{\phi X} \phi X, \phi V) = g(A_V \phi X, X) - \tilde{g}(h^s(\phi X, \phi X), \phi V)$$

$$\Rightarrow g(\nabla_{\phi X} \phi X, \phi V) = g(\phi X, A_V X) - \tilde{g}(h^s(\phi X, \phi X), \phi V) \tag{6.8.2}$$

since A is symmetric on $S(T^{\perp}M)$.

Addition of (6.8.1) and (6.8.2) gives

$$g(\nabla_X X + \nabla_{\phi X} \phi X, \phi V) = -\tilde{g}(h^s(X, X) + h^s(\phi X, \phi X), \phi V),$$

which implies that $h^s = 0$ on $D_0 \oplus_{orth} < \xi > \iff \nabla_X X + \nabla_{\phi X} \phi X \in \Gamma(D_0 \oplus_{orth} < \xi >)$.

Theorem 6.8.2. *Let (M, g) be a CSGL submanifold of an indefinite Kenmotsu manifold $(\tilde{M}, \tilde{g})$ with the structure vector field ξ tangent to M, then M is minimal if and only if $h^s|_{Rad(TM)} = 0$ and $trace(A^*_{Z_k})|_{S(TM)} = 0$, $trace(A_{V_p})|_{S(TM)} = 0$, $Z_k \in \Gamma(Rad(TM))$, $V_p \in \Gamma(S(T^{\perp}M))$.*

Proof. Putting $X = \xi$ in (1.21) and then using (1.15) in the right side and (6.2.4) in the left side, we have

$$\nabla_\xi \xi + h(\xi, \xi) = 0$$

$$\Rightarrow h(\xi, \xi) = 0. \tag{6.8.3}$$

Now, let us consider a quasi orthonormal frame $\{Z_1, ..., Z_{2r}, e_1, ..., e_{m-2r-1}, \xi, N_1, ..., N_{2r}, V_1, ..., V_{n-2r}\}$ such that $\{e_i\}_1^{2a}$ are tangent to D_0 and $\{e_j\}_{2a+1}^{m-2r-1}$ are tangent to D' with signatures $\{\epsilon_i\}_1^{m-2r-1}$, $Z_k \in \Gamma(Rad(TM))$, $N_k \in \Gamma(ltr(TM))$, $V_p \in \Gamma(S(T^{\perp}M))$. Then we have

$$trace(h)|_{S(TM)} = trace(h)|_{D_0} + trace(h)|_{D'} \quad (by\ 6.8.3)$$
$$= \sum_{i=1}^{2a} \epsilon_i [h^l(e_i, e_i) + h^s(e_i, e_i)] + \sum_{j=2a+1}^{m-2r-1} \epsilon_j [h^l(e_j, e_j) + h^s(e_j, e_j)]$$
$$= \sum_{i=1}^{2a} \epsilon_i \left[\frac{1}{2r} \sum_{k=1}^{2r} \tilde{g}(h^l(e_i, e_i), Z_k) N_k + \frac{1}{n-2r} \sum_{p=1}^{n-2r} \tilde{g}(h^s(e_i, e_i), V_p) V_p \right]$$
$$+ \sum_{j=2a+1}^{m-2r-1} \epsilon_j \left[\frac{1}{2r} \sum_{k=1}^{2r} \tilde{g}(h^l(e_j, e_j), Z_k) N_k + \frac{1}{n-2r} \sum_{p=1}^{n-2r} \tilde{g}(h^s(e_j, e_j), V_p) V_p \right]. \quad (6.8.4)$$

Again, from (6.2.9) and (6.2.13), we have respectively

$$\tilde{g}(h^l(e_i, e_i), Z_k) N_k = g(A^*_{Z_k} e_i, e_i) N_k, \quad (6.8.5)$$
$$\tilde{g}(h^s(e_j, e_j), V_p) V_p = g(A_{V_p} e_j, e_j) V_p. \quad (6.8.6)$$

Applying (6.8.5) and (6.8.6) on (6.8.4), we obtain

$$trace(h)|_{S(TM)} = \sum_{k=1}^{2r} trace(A^*_{Z_k})|_{D_0 \oplus D'} + \sum_{p=1}^{n-2r} trace(A_{V_p})|_{D_0 \oplus D'}$$

$$\Rightarrow trace(h)|_{S(TM)} = 0 \iff trace(A^*_{Z_k})|_{S(TM)} = 0 = trace(A_{V_p})|_{S(TM)}, \quad (6.8.7)$$

Using Definition 6.2.3, we conclude that, M is minimal if and only if (6.8.7) holds and $h^s|_{Rad(TM)} = 0$.

6.9 Example of a CSGL submanifold of an indefinite Kenmotsu manifold

Let us consider the 13-dimensional manifold $\tilde{M} = \{(x^1, ..., x^{13}) \in \mathbb{R}^{13} : x^{13} \neq 0\}$, where $(x^1, ..., x^{13})$ are the standard coordinates in $\mathbb{R}^{13}$. Then $\tilde{M}$ forms an indefinite Kenmotsu manifold together with the indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ such that $\tilde{g}$ is the semi-Riemannian metric defined by

$$\tilde{g}(e_i, e_i) = 1 \ for\ i = 1,2,3,4,5,6,13 \ and\ \tilde{g}(e_i, e_i) = -1 \ for\ i = 7,8,9,10,11,12,$$
$$\tilde{g}(e_i, e_j) = 0 \ \ \forall i \neq j,\ i,j = 1,...,13,$$

where $\{e_i\}_{i=1}^{13}$ are linearly independent vector fields at each point of $T\tilde{M}$ given by

$$e_i = x^{13} \frac{\partial}{\partial x^i} \ for\ i = 1,2,3,4,5,6 \ and\ e_i = -x^{13} \frac{\partial}{\partial x^i} \ for\ i = 7,8,9,10,11,12,13;$$
$$\phi e_1 = -e_2,\ \phi e_2 = e_1,\ \phi e_3 = -e_4,\ \phi e_4 = e_3,\ \phi e_5 = -e_6,\ \phi e_6 = e_5,$$
$$\phi e_7 = -e_8,\ \phi e_8 = e_7,\ \phi e_9 = -e_{10},\ \phi e_{10} = e_9,\ \phi e_{11} = -e_{12},\ \phi e_{12} = e_{11},\ \phi e_{13} = 0;$$
$$\xi = e_{13} = -x^{13} \frac{\partial}{\partial x^{13}},\ \eta = -\frac{1}{x^{13}} dx^{13}.$$

Now, the map given by

$$x(u_1, u_2, u_3, u_4, u_5, u_6, u_7) = (u_1, u_2, u_3, u_4, u_5, u_6, 0, 0, u_1, u_2, u_4, u_3, u_7)$$

defines a 7-dimensional submanifold M of $\tilde{M}$, where

$$E_1 = e_1 + e_9,\ E_2 = e_2 + e_{10},\ E_3 = e_5,\ E_4 = e_6,\ E_5 = e_3 + e_{12},\ E_6 = e_4 + e_{11},\ E_7 = \xi$$

form a local orthogonal basis of $TM = Rad(TM) \oplus_{orth} S(TM) = D \oplus D' \oplus_{orth} < \xi >$ such that $D = Rad(TM) \oplus_{orth} D_0$ and $S(TM) = D_0 \oplus D' \oplus_{orth} < \xi >$ with $Rad(TM) = < E_1, E_2 >$, $D_0 = < E_3, E_4 >$, $D' = < E_5, E_6 >$ so that $D = < E_1, E_2, E_3, E_4 >$ and $S(TM) = < E_3, E_4, E_5, E_6, E_7 >$.

Again, $tr(TM) = S(T^{\perp}M) \oplus_{orth} ltr(TM)$ such that $ltr(TM) = < N_1, N_2 >$ and $S(T^{\perp}M) = < V_1, V_2, V_3, V_4 >$, where

$$N_1 = e_1,\ N_2 = e_2$$

so that $\tilde{g}(E_1, N_1) = 1 = \tilde{g}(E_2, N_2)$, $\tilde{g}(E_2, N_1) = 0 = \tilde{g}(E_1, N_2)$, $\tilde{g}(N_1, N_2) = 0$, and

$$V_1 = e_{11},\ V_2 = e_{12},\ V_3 = e_7,\ V_4 = e_8$$

such that $\omega D' = < V_1, V_2 >$ and $\mu = < V_3, V_4 >$ satisfying $\phi(\mu) = \mu$ since $\phi V_3 = -V_4$, $\phi V_4 = V_3$.

Now, $\phi E_1 = -E_2$, $\phi E_2 = E_1$, $\phi E_3 = -E_4$, $\phi E_4 = E_3$, $\phi E_5 = -e_4 + e_{11}$, $\phi E_6 = e_3 - e_{12}$, $\phi E_7 = \phi\xi = 0$ so that $\phi(Rad(TM)) = Rad(TM)$, $D_0 = \phi(S(TM)) \cap S(TM)$, $\phi(D) = D$ and $\phi(D') \not\subseteq S(TM)$, $\phi(D') \not\subseteq S(T^{\perp}M)$.

Therefore, M is a CSGL submanifold of $\tilde{M}$.

Note. The primary difference between the theory of lightlike submanifolds and the classical theory of Riemannian or semi-Riemannian submanifolds arises due to the fact that, in the first case, a part of the normal bundle lies in the tangent bundle of the submanifold such that the intersection of the tangent bundle and the normal bundle is called the radical or lightlike or null distribution, whereas, in the second case, that intersection is null. Hence we can see that, the lightlike or null cone of a semi-Euclidean space is a typical example of lightlike submanifold of a semi-Riemannian manifold. This unique property of lightlike submanifolds has made it an interesting topic for the researchers since its conceptualization and the author is no exception.

Geometry of lightlike submanifolds is used in Mathematical Physics, in particular, in general theory of relativity since lightlike submanifolds produce models of different types of horizons for e.g. event horizons, Cauchy horizons, Kruskal's horizons. Lightlike hypersurfaces are also studied in the theory of electromagnetism, radiation fields, Killing horizons, asymptotically flat spacetimes. Lightlike submanifolds appear as smooth parts of event horizons of the Kruskal and Keer black holes.

7

SOME SOLITONS ON ANTI-INVARIANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLD ADMITTING ZAMKOVOY CONNECTION

7.1 Introduction

The notion of Zamkovoy connection was introduced by S. Zamkovoy in 2009 [154]. Later A. Biswas and K. K. Baishya applied this connection on generalized pseudo Ricci symmetric Sasakian manifolds [15] and on almost pseudo symmetric Sasakian manifolds [14]. This connection was further studied by A. M. Blaga in 2015 [17]. In 2020, A. Mandal and A. Das worked in detail on various curvature tensors of Sasakian and Lorentzian para-Sasakian manifolds admitting this new connection ([96], [97], [98], [40]), and recently in 2021, they discussed LP-Sasakian manifolds equipped with this new connection and conharmonic curvature tensor [99]. Motivated from these research works, in this chapter, we have obtained some results by considering some solitons like Ricci solitons and η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection.

This chapter consists of nine sections. After Introduction and Preliminaries sections, in the third, fourth, sixth and seventh sections, Ricci flat, concircularly flat, M-projectively flat and pseudo projectively flat anti-invariant submanifolds of a trans-Sasakian manifold admitting Zamkovoy connection have been discussed respectively. Ricci solitons on those submanifolds have also been studied. Also we have found out the conditions under which an anti-invariant submanifold of a trans-Sasakian manifold is ξ-projectively, ξ-M-projectively and ξ-pseudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection in the fifth, sixth and seventh sections respectively. At last, three conclusions are made after observing all the results. The eighth section, which concerns the topic η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection, is further subdivided into four subsections dealing with the study of η-Ricci-Yamabe soliton, q-η-Yamabe soliton and p-η-Ricci soliton on an anti-invariant submanifold of a trans-Sasakian manifold, where the submanifold is (i) Ricci flat, (ii) concircularly flat, (iii) M-projectively flat and (iv) pseudo projectively flat respectively with respect to Zamkovoy connection. At last, three conclusions are made after observing all the results of these four subsections. Finally, in the ninth section, an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified very easily.

7.2 Preliminaries

In this section, we first cite some basic definitions from [40], [96] and [97] which we have instituted in the results later on. Those are given as:-

Definition 7.2.1. A $(2n+1)$-dimensional manifold M is called *Ricci flat* with respect to Zamkovoy connection if $S^*(X,Y) = 0 \ \ \forall X, Y \in \chi(M)$.

Definition 7.2.2. [40] A $(2n+1)$-dimensional manifold M is called *concircularly flat* with respect to Zamkovoy connection if $C^*(X,Y)Z = 0 \ \ \forall X, Y, Z \in \chi(M)$.

Definition 7.2.3. [97] A $(2n+1)$-dimensional manifold M is called *projectively flat* with respect to Zamkovoy connection if $P^*(X,Y)Z = 0 \ \ \forall X, Y, Z \in \chi(M)$.

Definition 7.2.4. [97] A $(2n+1)$-dimensional manifold M is called *ξ-projectively flat* with respect to Zamkovoy connection if $P^*(X,Y)\xi = 0 \ \ \forall X, Y \in \chi(M)$.

Definition 7.2.5. [96] A $(2n+1)$-dimensional manifold M is called *M-projectively flat* with respect to Zamkovoy connection if $\bar{M}^*(X,Y)Z = 0 \ \ \forall X, Y, Z \in \chi(M)$.

Definition 7.2.6. [96] A $(2n+1)$-dimensional manifold M is called *ξ-M-projectively flat* with respect to Zamkovoy connection if $\bar{M}^*(X,Y)\xi = 0 \ \ \forall X, Y \in \chi(M)$.

Definition 7.2.7. [98] A $(2n+1)$-dimensional manifold M is called *pseudo projectively flat* with respect to Zamkovoy connection if $\bar{P}^*(X,Y)Z = 0 \ \ \forall X, Y, Z \in \chi(M)$.

Definition 7.2.8. [98] A $(2n+1)$-dimensional manifold M is called *ξ-pseudo projectively flat* with respect to Zamkovoy connection if $\bar{P}^*(X,Y)\xi = 0 \ \forall X, Y \in \chi(M)$.

Here $S^*, C^*, P^*, \bar{M}^*, \bar{P}^*$ are the Ricci tensor, the concircular curvature tensor given by (1.40), the projective curvature tensor given by (1.41), the M-projective curvature tensor given by (1.42), the pseudo projective curvature tensor given by (1.43) of M respectively with respect to the Zamkovoy connection ∇^* represented by (1.39), where R^*, Q^*, r^* are the curvature tensor, Ricci operator, scalar curvature with respect to ∇^*.

Next, let us consider a trans-Sasakian manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, g)$ of type (α, β). Then, $\tilde{M}$ satisfies the following relations [44]−

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta[g(X,Y) - \eta(X)\eta(Y)], \tag{7.2.1}$$

$$R(X,Y)\xi = (\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y]$$
$$+[(Y\alpha)\phi X-(X\alpha)\phi Y+(Y\beta)\phi^2X-(X\beta)\phi^2Y], \quad (7.2.2)$$
$$R(\xi,Y)X = (\alpha^2-\beta^2)[g(X,Y)\xi-\eta(X)Y]+2\alpha\beta[g(\phi X,Y)\xi+\eta(X)\phi Y]+(X\alpha)\phi Y$$
$$+g(\phi X,Y)(grad\ \alpha)-g(\phi X,\phi Y)(grad\ \beta)+(X\beta)[Y-\eta(Y)\xi], \quad (7.2.3)$$
$$S(X,\xi) = [2n(\alpha^2-\beta^2)-\xi\beta]\eta(X)-(\phi X)\alpha-(2n-1)(X\beta), \quad (7.2.4)$$
$$Q\xi = [2n(\alpha^2-\beta^2)-\xi\beta]\xi+\phi(grad\ \alpha)-(2n-1)(grad\ \beta), \quad (7.2.5)$$

$\forall X,Y \in \chi(\tilde{M})$, where R,S,Q are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator of $\tilde{M}$ respectively.

Moreover, we state a lemma which later proves to be quite useful in getting the results of the sections following this chapter.

Lemma 7.2.1. [133] *In a $(2n+1)$-dimensional trans-Sasakian manifold of type (α,β), if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$, then $\xi\beta = 0$.*

Using (1.7), (7.2.1) on (1.39) we get the expression of Zamkovoy connection on $\tilde{M}$ as−

$$\nabla^*_X Y = \nabla_X Y+\eta(X)\phi Y+\alpha\eta(Y)\phi X-\beta\eta(Y)X+\beta g(X,Y)\xi-\alpha g(\phi X,Y)\xi \quad (7.2.6)$$

with torsion tensor

$$T^*(X,Y) = (1-\alpha)[\eta(X)\phi Y-\eta(Y)\phi X]+\beta[\eta(X)Y-\eta(Y)X]+2\alpha g(X,\phi Y)\xi. \quad (7.2.7)$$

Again, we have

$$(\nabla^*_X g)(Y,Z) = \nabla^*_X g(Y,Z)-g(\nabla^*_X Y,Z)-g(Y,\nabla^*_X Z).$$

Then, using (7.2.6) in the above equation we obtain $\nabla^* g = 0$, i.e. Zamkovoy connection is a metric compatible connection on $\tilde{M}$.

Now applying (1.30) in (7.2.6) and (7.2.7) respectively we get the expression of Zamkovoy connection on an anti-invariant submanifold M of $\tilde{M}$ as−

$$\nabla^*_X Y = \nabla_X Y+\eta(X)\phi Y+\alpha\eta(Y)\phi X-\beta\eta(Y)X+\beta g(X,Y)\xi \quad (7.2.8)$$

with torsion tensor

$$T^*(X,Y) = (1-\alpha)[\eta(X)\phi Y-\eta(Y)\phi X]+\beta[\eta(X)Y-\eta(Y)X].$$

Setting $Y=\xi$ in (7.2.8) and then using (1.7) we obtain

$$\nabla^*_X \xi = 0. \quad (7.2.9)$$

Applying (1.6), (1.7) and (7.2.8) on the following equation

$$R^*(X,Y)Z = \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z,$$

we get $\forall X, Y, Z \in \chi(M)$,

$$\begin{aligned} R^*(X,Y)Z =& R(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) + \beta[\eta(X)\phi Y - \eta(Y)\phi X]\eta(Z) \\ &+ \beta^2[g(Y,Z)X - g(X,Z)Y] + \alpha\beta[g(X,Z)\phi Y - g(Y,Z)\phi X] \\ &+\beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi. \end{aligned} \tag{7.2.10}$$

Consequently, if $\xi\beta = 0$ and $\dim(M) = 2n+1$, then we have

$$S^*(Y,Z) = S(Y,Z) - 2n\alpha^2\eta(Y)\eta(Z) + 2n\beta^2 g(Y,Z), \tag{7.2.11}$$

which implies that

$$Q^*Y = QY - 2n\alpha^2\eta(Y)\xi + 2n\beta^2 Y, \tag{7.2.12}$$

$$r^* = r - 2n\alpha^2 + 2n(2n+1)\beta^2. \tag{7.2.13}$$

Here R^* & R, S^* & S, Q^* & Q, r^* & r are the curvature tensors, Ricci tensors, Ricci operators, scalar curvatures of M with respect to the Levi-Civita connection ∇ and the Zamkovoy connection ∇^* given by (7.2.8) respectively.

7.3 Ricci flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section consists of the study of the nature of a $(2n+1)$-dimensional Ricci flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the Zamkovoy connection ∇^* given by (7.2.8) and further a Ricci soliton on it.

Since M is Ricci flat with respect to ∇^*, then $S^*(Y,Z) = 0$, hence, (7.2.11) implies−

$$S(Y,Z) = 2n\alpha^2\eta(Y)\eta(Z) - 2n\beta^2 g(Y,Z). \tag{7.3.1}$$

Thus, using Lemma 7.2.1 and (7.3.1) we state that−

Theorem 7.3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is Ricci flat with respect to the Zamkovoy connection ∇^*, then M is η-Einstein if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Next, let us consider a Ricci soliton (g, ξ, λ) on M, then from (1.36) we get

$$(L_\xi g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) = 0$$
$$\Rightarrow g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) + 2S(Y, Z) + 2\lambda g(Y, Z) = 0.$$

Using (1.7) and (1.30) on the above equation we obtain

$$2S(Y, Z) + 2(\lambda + \beta)g(Y, Z) - 2\beta\eta(Y)\eta(Z) = 0.$$

Setting $Z = \xi$ we get

$$S(Y, \xi) = -\lambda\eta(Y). \tag{7.3.2}$$

Putting $Z = \xi$ in (7.3.1) we obtain

$$S(Y, \xi) = 2n(\alpha^2 - \beta^2)\eta(Y). \tag{7.3.3}$$

Now, equating (7.3.2) and (7.3.3) we get

$$\lambda = 2n(\beta^2 - \alpha^2),$$

which is < 0, $= 0$ or > 0 according as $|\beta| < |\alpha|$, $|\beta| = |\alpha|$ or $|\beta| > |\alpha|$ respectively.

Thus, using Lemma 7.2.1 we state that−

Theorem 7.3.2. *If a $(2n + 1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is Ricci flat with respect to the Zamkovoy connection ∇^*, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according as $|\beta| < |\alpha|$, $|\beta| = |\alpha|$ or $|\beta| > |\alpha|$ respectively, provided $\phi(grad\ \alpha) = (2n - 1)(grad\ \beta)$.*

7.4 Concircularly flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the study of the nature of a $(2n + 1)$-dimensional concircularly flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the Zamkovoy connection ∇^* given by (7.2.8) and also a Ricci soliton on it.

From (7.2.11) we get

$$r^* = r - 2n\alpha^2 + 2n(2n + 1)\beta^2. \tag{7.4.1}$$

As M is concircularly flat with respect to ∇^*, from (1.40) we have

$$R^*(X, Y)Z = \frac{r^*}{2n(2n + 1)}[g(Y, Z)X - g(X, Z)Y]$$
$$\Rightarrow S^*(Y, Z) = \frac{r^*}{2n + 1} g(Y, Z).$$

Using (7.2.11) and (7.4.1) on the above equation we obtain

$$S(Y,Z) = \frac{r - 2n\alpha^2}{2n+1} g(Y,Z) + 2n\alpha^2 \eta(Y)\eta(Z), \tag{7.4.2}$$

which, on using Lemma 7.2.1, shows that−

Theorem 7.4.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is concircularly flat with respect to the Zamkovoy connection ∇^*, then M is η-Einstein if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Setting $Z = \xi$ in (7.4.2) we get

$$S(Y,\xi) = \frac{r + 4n^2\alpha^2}{2n+1} \eta(Y). \tag{7.4.3}$$

Next, let us consider a Ricci soliton (g, ξ, λ) on M, then equating (7.3.2) and (7.4.3) we get

$$\lambda = -\frac{r + 4n^2\alpha^2}{2n+1},$$

which is < 0, $= 0$ or > 0 according as $r > -4n^2\alpha^2, r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$ respectively.

Hence, using Lemma 7.2.1 we have the following theorem−

Theorem 7.4.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is concircularly flat with respect to the Zamkovoy connection ∇^*, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according as $r > -4n^2\alpha^2, r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

7.5 ξ-projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

In this section, it will be proved that a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ is ξ-projectively flat with respect to the Zamkovoy connection ∇^* given by (7.2.8) if and only if it is so with respect to Levi-Civita connection under certain conditions.

If $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$, then using Lemma 7.2.1 from (1.41), (7.2.10) and (7.2.11) we have

$$\begin{aligned}
P^*(X,Y)Z =& R^*(X,Y)Z - \frac{1}{2n}[S^*(Y,Z)X - S^*(X,Z)Y] \\
=& R(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) \\
&+ \beta[\eta(X)\phi Y - \eta(Y)\phi X]\eta(Z) + \beta^2[g(Y,Z)X - g(X,Z)Y] \\
&+ \alpha\beta[g(X,Z)\phi Y - g(Y,Z)\phi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi \\
&- \frac{1}{2n}[S(Y,Z)X - 2n\alpha^2\eta(Y)\eta(Z)X + 2n\beta^2 g(Y,Z)X \\
&- S(X,Z)Y + 2n\alpha^2\eta(X)\eta(Z)Y - 2n\beta^2 g(X,Z)Y] \\
=& P(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) \\
&+ \beta[\eta(X)\phi Y - \eta(Y)\phi X]\eta(Z) + \beta^2[g(Y,Z)X - g(X,Z)Y] \\
&+ \alpha\beta[g(X,Z)\phi Y - g(Y,Z)\phi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi \\
&- [-\alpha^2\eta(Y)\eta(Z)X + \beta^2 g(Y,Z)X + \alpha^2\eta(X)\eta(Z)Y - \beta^2 g(X,Z)Y],
\end{aligned}$$

which implies

$$P^*(X,Y)\xi = P(X,Y)\xi + (\alpha\beta + \beta)[\eta(X)\phi Y - \eta(Y)\phi Y] + \beta[\nabla_Y\eta(X) - \nabla_X\eta(Y)]\xi.$$

Again, using (1.7), (1.30) on (7.2.1) we get

$$\nabla_X\eta(Y) = 0, \tag{7.5.1}$$

and applying it on the preceding equation we obtain

$$P^*(X,Y)\xi = P(X,Y)\xi + \beta(\alpha+1)[\eta(X)\phi Y - \eta(Y)\phi X]$$

$\Rightarrow P^*(X,Y)\xi = P(X,Y)\xi$ if $\alpha = -1$ or $\beta = 0$ or X, Y are horizontal vector fields.

Therefore we state that−

Theorem 7.5.1. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is ξ-projectively flat with respect to the Zamkovoy connection ∇^* if and only if it is so with respect to the Riemannian connection ∇ if $\alpha = -1$ or $\beta = 0$ or the vector fields are horizontal, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

7.6 M-projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

In this section, a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) has been taken and its nature is studied when it is M-projectively flat and ξ-M-projectively flat with respect to the Zamkovoy connection ∇^* given by (7.2.8). Also a Ricci soliton on M is discussed.

If M is M-projectively with respect to ∇^* then from (1.42) we have

$$R^*(X,Y)Z = \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] + \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y]$$

$$\Rightarrow S^*(Y,Z) = \frac{r^*}{2n+1}g(Y,Z). \tag{7.6.1}$$

If $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$, then using (7.2.11) and (7.4.1) on (7.6.1) and by Lemma 7.2.1 we obtain

$$S(Y,Z) = \frac{r - 2n\alpha^2}{2n+1}g(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z), \tag{7.6.2}$$

from which we conclude that−

Theorem 7.6.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is M-projectively flat with respect to the Zamkovoy connection ∇^*, then M is η-Einstein if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Putting $Z = \xi$ in (7.6.2) we have

$$S(Y,\xi) = \frac{r + 4n^2\alpha^2}{2n+1}\eta(Y). \tag{7.6.3}$$

Let us consider a Ricci soliton (g, ξ, λ) on M, then equating (7.3.2) and (7.6.3) we get

$$\lambda = -\frac{r + 4n^2\alpha^2}{2n+1},$$

which is < 0, $= 0$ or > 0 according as $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$ respectively.

Hence, by using Lemma 7.2.1 we state the following theorem−

Theorem 7.6.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is M-projectively flat with respect to the Zamkovoy connection ∇^*, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according as $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$, then applying (7.2.10), (7.2.11) and (7.2.12) on (1.42) we have

$$\begin{aligned}\bar{M}^*(X,Y)Z =& \bar{M}(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) + \beta[\eta(X)\phi Y - \eta(Y)\phi X]\eta(Z) \\ &+ \alpha\beta[g(X,Z)\phi Y - g(Y,Z)\phi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi \\ &+ \frac{\alpha^2}{2}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] + \frac{\alpha^2}{2}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi.\end{aligned}$$

Putting $Z = \xi$ in the above equation and using (7.5.1) we obtain

$$\bar{M}^*(X,Y)\xi = \bar{M}(X,Y)\xi + \beta(\alpha+1)[\eta(X)\phi Y - \eta(Y)\phi X] + \frac{\alpha^2}{2}[\eta(X)Y - \eta(Y)X],$$

which implies that $\bar{M}^*(X,Y)\xi = \bar{M}(X,Y)\xi$ if X, Y are horizontal vector fields.

Thus, we have the following theorem−

Theorem 7.6.3. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is ξ-M-projectively flat with respect to the Zamkovoy connection ∇^* if and only if it is so with respect to the Riemannian connection ∇ if the vector fields are horizontal, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

7.7 Pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the study of a pseudo projectively flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) admitting the Zamkovoy connection ∇^* given by (7.2.8) along with a Ricci soliton on it. Also the condition is established under which M is ξ-pseudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection.

Since M is pseudo projectively flat with respect to ∇^*, from (1.43) we have

$$aR^*(X,Y)Z = b[S^*(X,Z)Y - S^*(Y,Z)X] + cr^*[g(X,Z)Y - g(Y,Z)X]$$

$$\Rightarrow (a+2nb)S^*(Y,Z) = -2cnr^*g(Y,Z). \tag{7.7.1}$$

Applying (1.44), (7.2.11), (7.4.1) and the condition $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$ of Lemma 7.2.1 on (7.7.1) we obtain

$$S(Y,Z) = 2n\alpha^2\eta(Y)\eta(Z) + \frac{r - 2n\alpha^2}{2n+1}g(Y,Z). \tag{7.7.2}$$

Thus, we state that−

Theorem 7.7.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to the Zamkovoy connection ∇^*, then M is η-Einstein if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Setting $Z = \xi$ in (7.7.2) we have

$$S(Y, \xi) = \frac{r + 4n^2\alpha^2}{2n+1}\eta(Y). \tag{7.7.3}$$

Now, considering a Ricci soliton (g, ξ, λ) on M we have (7.3.2) and then equating it with (7.7.3) we get

$$\lambda = -\frac{r + 4n^2\alpha^2}{2n+1},$$

which is < 0, $= 0$ or > 0 according as $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$ respectively. Thus, we get the following theorem−

Theorem 7.7.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to the Zamkovoy connection ∇^*, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according as $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now, if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$, then putting $Z = \xi$ in (1.43) and using (1.44), (7.2.10), (7.2.11), (7.4.1), (7.5.1) we obtain

$$\begin{aligned} \bar{P}^*(X,Y)\xi =& \bar{P}(X,Y)\xi + a\beta(\alpha+1)[\eta(X)\phi Y - \eta(Y)\phi X] \\ &+ \frac{2n}{2n+1}(a+2nb)\alpha^2[\eta(X)Y - \eta(Y)X] \end{aligned}$$

$$\Rightarrow \bar{P}^*(X,Y)\xi = \bar{P}(X,Y)\xi \text{ if } X, Y \text{ are horizontal vector fields.}$$

Hence, we state that−

Theorem 7.7.3. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is ξ-pseudo projectively flat with respect to the Zamkovoy connection ∇^* if and only if it is so with respect to the Riemannian connection ∇ if the vector fields are horizontal, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

From Theorems 7.3.1, 7.4.1, 7.6.1 and 7.7.1 we make the following conclusion−

Conclusion 7.7.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) admitting Zamkovoy connection is*
(i) Ricci flat,
(ii) concircularly flat,
(iii)M-projectively flat or
(iv) pseudo projectively flat,
then M is η-Einstein if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

Next, observing Theorems 7.4.2, 7.6.2 and 7.7.2 we reach to the following interesting conclusion—

Conclusion 7.7.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is*
(i) concircularly flat,
(ii) M-projectively flat or
(iii) pseudo projectively flat
with respect to Zamkovoy connection, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according as $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

Again, observing Theorems 7.5.1, 7.6.3 and 7.7.3 we conclude that—

Conclusion 7.7.3. *For horizontal vector fields, a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is*
(i) ξ-projectively flat,
(ii) ξ-M-projectively flat and
(iii) ξ-pseudo projectively flat
with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

7.8 η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the topic of η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection. This section is further subdivided into four subsections but before proceeding to these subsections, here two theorems are proved concerning the nature of a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) when an η-Ricci-Yamabe soliton of type (p, q) is considered on it with respect to the Zamkovoy connection ∇^* given by (7.2.8).

Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on M with respect to ∇^*, then from (1.37) we have $\forall Y, Z \in \chi(M)$

$$\begin{aligned}
&(L^*_\xi g)(Y, Z) + 2pS^*(Y, Z) + (2\lambda - qr^*)g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0 \\
\Rightarrow &g(\nabla^*_Y \xi, Z) + g(\nabla^*_Z \xi, Y) + 2pS^*(Y, Z) + (2\lambda - qr^*)g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.
\end{aligned}$$

Using (7.2.9) in the above equation we get

$$S^*(Y,Z) = \left(\frac{qr^* - 2\lambda}{2p}\right)g(Y,Z) - \left(\frac{\mu}{p}\right)\eta(Y)\eta(Z). \tag{7.8.1}$$

Hence we state the following theorem–

Theorem 7.8.1. *Let (g,ξ,λ,μ,p,q) be an η-Ricci-Yamabe soliton on an anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to the Zamkovoy connection ∇^*, then M is η-Einstein with respect to ∇^*.*

Again if $\xi\beta = 0$, then using (7.2.11) and (7.2.13) in (7.8.1) we obtain

$$\begin{aligned} S(Y,Z) =& \left[\frac{q\{r - 2n\alpha^2 + 2n(2n+1)\beta^2 - 2\lambda\}}{2p} - 2n\beta^2\right]g(Y,Z) \\ &+ \left[2n\alpha^2 - \left(\frac{\mu}{p}\right)\right]\eta(Y)\eta(Z). \end{aligned}$$

Thus applying Lemma 7.2.1 we have the following theorem–

Theorem 7.8.2. *Let (g,ξ,λ,μ,p,q) be an η-Ricci-Yamabe soliton on a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to the Zamkovoy connection ∇^*, then M is η-Einstein with respect to Riemannian connection, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

7.8.1 η-Ricci-Yamabe solitons on Ricci flat anti-invariant submanifolds

Here η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) are discussed, where M is Ricci flat with respect to the Zamkovoy connection ∇^* given by (7.2.8).

Let (g,ξ,λ,μ,p,q) be an η-Ricci-Yamabe soliton on M, then from (1.37) we have $\forall Y,Z \in \chi(M)$

$$\begin{aligned} &(L_\xi g)(Y,Z) + 2pS(Y,Z) + (2\lambda - qr)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0 \\ \Rightarrow & g(\nabla_Y\xi,Z) + g(\nabla_Z\xi,Y) + 2pS(Y,Z) + (2\lambda - qr)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0. \end{aligned}$$

Using (1.7) and then applying (1.30) in the above equation we get

$$pS(Y,Z) + \left(\lambda + \beta - \frac{qr}{2}\right)g(Y,Z) + (\mu - \beta)\eta(Y)\eta(Z) = 0.$$

Setting $Z = \xi$ in the above equation we obtain

$$pS(Y,\xi) = \left(\frac{qr}{2} - \lambda - \mu\right)\eta(Y). \tag{7.8.2}$$

Now if $\xi\beta = 0$ and M is Ricci flat with respect to ∇^*, then from (7.2.11) we have

$$S(Y, Z) = 2n\alpha^2\eta(Y)\eta(Z) - 2n\beta^2 g(Y, Z).$$

Setting $Z = \xi$ in the above equation and multiplying both sides by p we obtain

$$pS(Y, \xi) = 2np(\alpha^2 - \beta^2)\eta(Y). \tag{7.8.3}$$

Equating (7.8.2) and (7.8.3) we get

$$\lambda = \frac{qr}{2} - \mu - 2np(\alpha^2 - \beta^2). \tag{7.8.4}$$

Hence from (7.8.4) and applying Lemma 7.2.1 we conclude the following theorem−

Theorem 7.8.1.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is Ricci flat with respect to the Zamkovoy connection ∇^*, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < 2np(\alpha^2 - \beta^2)$, $\frac{qr}{2} - \mu = 2np(\alpha^2 - \beta^2)$ or $\frac{qr}{2} - \mu > 2np(\alpha^2 - \beta^2)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now from (7.8.4) we get, when $p = 0$ then $\lambda = \frac{qr}{2} - \mu$, and when $q = 0$ then $\lambda = -\mu + 2np(\beta^2 - \alpha^2)$. Thus from Theorem 7.8.1.1 we respectively conclude the following results−

Corollary 7.8.1.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is Ricci flat with respect to the Zamkovoy connection ∇^*, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 7.8.1.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is Ricci flat with respect to the Zamkovoy connection ∇^*, then a p-η-Ricci soliton $(g, \xi, \lambda, \mu, p)$ on M is shrinking, steady or expanding according as $2np(\beta^2 - \alpha^2) < \mu$, $2np(\beta^2 - \alpha^2) = \mu$ or $2np(\beta^2 - \alpha^2) > \mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

7.8.2 η-Ricci-Yamabe solitons on concircularly flat anti-invariant submanifolds

This subsection deals with the study of η-Ricci-Yamabe soliton of type (p, q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional concircularly flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the Zamkovoy connection ∇^* given by (7.2.8).

Since M is concircularly flat with respect to ∇^*, from (1.40) we have

$$R^*(X,Y)Z = \frac{r^*}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y],$$

which implies that

$$S^*(Y,Z) = \left(\frac{r^*}{2n+1}\right)g(Y,Z). \tag{7.8.5}$$

Let $\xi\beta = 0$, hence using (7.2.11) and (7.2.13) in (7.8.5) we obtain

$$S(Y,Z) = \left(\frac{r-2n\alpha^2}{2n+1}\right)g(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z). \tag{7.8.6}$$

Putting $Z = \xi$ in (7.8.6) and then multiplying both sides by p we get

$$pS(Y,\xi) = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)\eta(Y). \tag{7.8.7}$$

Next, let (g,ξ,λ,μ,p,q) be an η-Ricci-Yamabe soliton on M, then equating (7.8.2) and (7.8.7) we obtain

$$\lambda = \frac{qr}{2} - \mu - p\left(\frac{r+4n^2\alpha^2}{2n+1}\right). \tag{7.8.8}$$

Thus, applying Lemma 7.2.1, from (7.8.8) we state the following theorem−

Theorem 7.8.2.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to the Zamkovoy connection ∇^*, then an η-Ricci-Yamabe soliton (g,ξ,λ,μ,p,q) on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p(\frac{r+4n^2\alpha^2}{2n+1})$, $\frac{qr}{2} - \mu = p(\frac{r+4n^2\alpha^2}{2n+1})$ or $\frac{qr}{2} - \mu > p(\frac{r+4n^2\alpha^2}{2n+1})$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now from (7.8.8) we have, when $p = 0$ then $\lambda = \frac{qr}{2} - \mu$, and when $q = 0$ then $\lambda = -\mu - p(\frac{r+4n^2\alpha^2}{2n+1})$. Thus from Theorem 7.8.2.1 we respectively conclude the following results−

Corollary 7.8.2.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to the Zamkovoy connection ∇^*, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 7.8.2.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to the Zamkovoy connection ∇^*, then a p-η-Ricci soliton (g,ξ,λ,μ,p) on M is shrinking, steady or expanding according as $-\mu < p(\frac{r+4n^2\alpha^2}{2n+1})$, $-\mu = p(\frac{r+4n^2\alpha^2}{2n+1})$ or $-\mu > p(\frac{r+4n^2\alpha^2}{2n+1})$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

7.8.3 η-Ricci-Yamabe solitons on M-projectively flat anti-invariant submanifolds

Here η-Ricci-Yamabe soliton of type (p, q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) are discussed, where M is M-projectively flat with respect to the Zamkovoy connection ∇^* given by (7.2.8).

Since M is M-projectively flat with respect to ∇^*, from (1.42) we have

$$R^*(X,Y)Z = \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] + \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y],$$

from which we have

$$S^*(Y,Z) = \left(\frac{r^*}{2n+1}\right)g(Y,Z),$$

which is same as the equation (7.8.5). Hence, proceeding similarly as the previous subsection we get the following results−

Theorem 7.8.3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is M-projectively flat with respect to the Zamkovoy connection ∇^*, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $\frac{qr}{2} - \mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $\frac{qr}{2} - \mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 7.8.3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is M-projectively flat with respect to the Zamkovoy connection ∇^*, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 7.8.3.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is M-projectively flat with respect to the Zamkovoy connection ∇^*, then a p-η-Ricci soliton $(g, \xi, \lambda, \mu, p)$ on M is shrinking, steady or expanding according as $-\mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $-\mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $-\mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

7.8.4 η-Ricci-Yamabe solitons on pseudo projectively flat anti-invariant submanifolds

This subsection deals with the study of η-Ricci-Yamabe soliton of type (p, q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional pseudo projectively flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to the Zamkovoy connection ∇^* given by (7.2.8).

Since M is pseudo projectively flat with respect to ∇^*, from (1.43) we have

$$aR^*(X,Y)Z = b[S^*(X,Z)Y - S^*(Y,Z)X] + cr^*[g(X,Z)Y - g(Y,Z)X],$$

which implies that

$$(a + 2nb)S^*(Y,Z) = -2cnr^*g(Y,Z).$$

Let $\xi\beta = 0$, then applying (1.44), (7.2.11) and (7.2.13) in the above equation we get

$$S(Y,Z) = \left(\frac{r - 2n\alpha^2}{2n+1}\right)g(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z),$$

which is same as the equation (7.8.6). Hence proceeding similarly as the Subsection 7.8.2 we reach to the following results−

Theorem 7.8.4.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to the Zamkovoy connection ∇^*, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $\frac{qr}{2} - \mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $\frac{qr}{2} - \mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 7.8.4.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to the Zamkovoy connection ∇^*, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 7.8.4.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to the Zamkovoy connection ∇^*, then a p-η-Ricci soliton $(g, \xi, \lambda, \mu, p)$ on M is shrinking, steady or expanding according as $-\mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $-\mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $-\mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now, after carefully looking into the results of the four subsections of third section we reach to the following three conclusions.

First, observing Theorems 7.8.2.1, 7.8.3.1 and 7.8.4.1 we get the following−

Conclusion 7.8.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is*
(i) concircularly flat,

(ii) M-projectively flat or
(iii) pseudo projectively flat
with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $\frac{qr}{2} - \mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $\frac{qr}{2} - \mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

Next, observing Corollaries 7.8.1.1, 7.8.2.1, 7.8.3.1 and 7.8.4.1 we have

Conclusion 7.8.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is*
(i) Ricci flat,
(ii) concircularly flat,
(iii) M-projectively flat or
(iv) pseudo projectively flat
with respect to Zamkovoy connection, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

At last, observing Corollaries 7.8.2.2, 7.8.3.2 and 7.8.4.2 we reach to the third conclusion−

Conclusion 7.8.3. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is*
(i) concircularly flat,
(ii) M-projectively flat or
(iii) pseudo projectively flat
with respect to Zamkovoy connection, then a p-η-Ricci soliton $(g, \xi, \lambda, \mu, p)$ on M is shrinking, steady or expanding according as $-\mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $-\mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $-\mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

7.9 Example of an anti-invariant submanifold of a trans-Sasakian manifold

Finally, we give the following example in which all the results proved in this chapter can be verified very easily.

Unit sphere S^5 is a trans-Sasakian manifold of type $(-1, 0)$ [142]. We here state an example of an anti-invariant submanifold of S^5 from [151] as:−

Let $J = (a_{ts})$ $(t, s = 1,2,3,4,5,6)$ be the almost complex structure of $\mathbb{C}^3$ such that $a_{2i,2i-1} = 1$, $a_{2i-1,2i} = -1$ $(i = 1,2,3)$ and all the other components are 0.

Let $S^1(\frac{1}{\sqrt{3}}) = \{z \in \mathbb{C} : |z|^2 = \frac{1}{3}\}$. We consider $S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}})$ in S^5 in $\mathbb{C}^3$. The position vector X of $S^1 \times S^1 \times S^1$ in S^5 in $\mathbb{C}^3$ has components given by−

$$X = \frac{1}{\sqrt{3}}(\cos\ u^1, \sin\ u^1, \cos\ u^2, \sin\ u^2, \cos\ u^3, \sin\ u^3),$$

where u^1, u^2, u^3 are parameters on each $S^1(\frac{1}{\sqrt{3}})$.

Let $X_i = \frac{\partial X}{\partial u^i}$, then we have

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{3}}(-\sin\ u^1, \cos\ u^1, 0, 0, 0, 0), \\ X_2 &= \frac{1}{\sqrt{3}}(0, 0, -\sin\ u^2, \cos\ u^2, 0, 0), \\ X_3 &= \frac{1}{\sqrt{3}}(0, 0, 0, 0, -\sin\ u^3, \cos\ u^3). \end{aligned}$$

The vector field ξ on S^5 is given by−

$$\xi = JX = \frac{1}{\sqrt{3}}(-\sin\ u^1, \cos\ u^1, -\sin\ u^2, \cos\ u^2, -\sin\ u^3, \cos\ u^3).$$

Since $\xi = X_1 + X_2 + X_3$, ξ is tangent to $S^1 \times S^1 \times S^1$. Also the structure tensors (ϕ, ξ, η) of S^5 satisfy

$$\phi X_i = JX_i + \eta(X_i)X,\ i = 1, 2, 3,$$

which shows that ϕX_i is normal to $S^1 \times S^1 \times S^1$ for all i. Thus, $S^1 \times S^1 \times S^1$ is an anti-invariant submanifold of S^5.

BIBLIOGRAPHY

[1] Abutuqayqah, H.I.: *Geometry of Hemi-slant Submanifolds of Almost Contact Manifolds*, King Abdulaziz University, 2012.

[2] Ahmad, M.: *CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with quarter symmetric metric connection*, Bull. Korean Math. Soc. **49**(2012), 25-32.

[3] Al-Solamy, F.R., Khan, M.A., Uddin, S.: *Totally umbilical hemi-slant submanifolds of Kaehler manifolds*, Abstr. Appl. Anal. **2011**(2011), Article ID 987157, 9 pages.

[4] Atceken, M., Dirik, S.: *On the geometry of pseudo-slant submanifolds of a Kenmotsu manifold*, Gulf J. Math. **2**(2)(2014), 51-66.

[5] Bagewadi, C.S.: *On totally real submanifolds of a Kahlerian manifold admitting semi-symmetric metric F-connection*, Indian J. Pure Appl. Math. **13**(5)(1982), 528-536.

[6] Bagewadi, C.S., Venkatesha, S.: *Anti-invariant submanifolds of* (ϵ)*-Sasakian manifold*, Int. J. Math. Archive **9**(3)(2018), 148-152.

[7] Bagewadi, C.S., Venkatesha, S.: *Anti-invariant submanifolds of S-manifold*, Int. J. Eng. Sci. Comput. **8**(5)(2018), 17368-17371.

[8] Bejan, C.L., Duggal, K.L.: *Global lightlike manifolds and harmonicity*, Kodai Math. J. **28**(1)(2005), 131-145.

[9] Bejancu, A.: *CR submanifolds of a Kaehler manifold. I*, Proc. Am. Math. Soc. **69**(1)(1978), 135-142.

[10] Bejancu, A.: *CR submanifolds of a Kaehler manifold. II*, Trans. Am. Math. Soc. **250**(1979), 333-345.

[11] Bejancu, A.: *Geometry of CR-submanifolds*, D. Reidel Publishing Company, Dordrecht, Holland, 1986.

[12] Bejancu, A., Duggal, K.L.: *Lightlike submanifolds of semi-Riemannian manifolds*, Acta Appl. Math. **38**(2)(1995), 197-215.

[13] Bejancu, A., Papaghuic, N.: *Semi-invariant submanifolds of a Sasakian manifold*, An. Stiint. Univ. Al. I. Cuza Iasi, Tomul **27**(1981), 163-170.

[14] Biswas, A., Baishya, K.K.: *A general connection on Sasakian manifolds and the case of almost pseudo symmetric Sasakian manifolds*, Sci. Stud. Res., Ser. Math. Inform. **29**(1)(2019), 59-72.

[15] Biswas, A., Baishya, K.K: *Study on generalized pseudo (Ricci) symmetric Sasakian manifold admitting general connection*, Bull. Transilv. Univ. Brasov **12**(2)(2020), 233-246.

[16] Blaga, A.M.: *The geometry of golden conjugate connections*, Sarajevo J. Math. **10**(23)(2014), 237-245.

[17] Blaga, A.M.: *Canonical connections on para-Kenmotsu manifold*, Novi Sad J. Math. **45**(2)(2015), 131-142.

[18] Blaga, A.M.: *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20**(1)(2015), 1-13.

[19] Blaga, A.M.: *On gradient η-Einstein solitons*, Kragujevac J. Math. **42**(2)(2018), 229-237.

[20] Blaga, A.M., Hretcanu, C.E.: *Metallic conjugate connections*, Rev. Union Mat. Argent. **59**(1)(2018), 179-192.

[21] Blaga, A.M., Hretcanu, C.E.: *Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold*, arXiv:1803.01415v2, 2018.

[22] Blaga, A.M., Perktas, S.Y.: *Remarks on almost η-Ricci solitons in (ϵ)-para Sasakian manifolds*, Commun. Fac. Sci. Univ. Ank., Ser. A1, Math. Stat. **68**(2)(2019), 1621-1628.

[23] Blair, D.E.: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Vol. **509**, Springer-Verlag, Berlin, 1976.

[24] Bonome, A., Castro, R., Garcia-Rio, E., Hervella, L.: *Curvature of indefinite almost contact manifolds*, J. Geom. **58**(1)(1997), 66-86.

[25] Brasil, A., Lobos, G.A., Mariano, M.: *C-Totally Real submanifolds with parallel mean curvature in λ-Sasakian space forms*, Mat. Contemp. **34**(2008), 83-102.

[26] Carriazo, A.: *New Developments in Slant Submanifolds Theory*, Narosa Publishing House, New Delhi, India, 2002.

[27] Catino, G., Cremaschi, L., Djadli, Z., Mantegazza, C., Mazzieri, L.: *The Ricci-Bourguignon flow*, Pac. J. Math. **287**(2)(2017), 337-370.

[28] Catino, G., Mazzieri, L.: *Gradient Einstein solitons*, Nonlinear Anal. **132**(2016), 66-94.

[29] Chaubey, S.K., Ojha, R.H.: *On the M-projective curvature tensor of a Kenmotsu manifold*, Differ. Geom. Dyn. Syst. **12**(2010), 52-60.

[30] Chaubey, S.K., Prakash, S., Nivas, S.: *Some properties of M-projective curvature tensor in Kenmotsu manifolds*, Bull. Math. Anal. Appl. **4**(3)(2012), 48-56.

[31] Chen, B.Y.: *Differential geometry of real submanifolds in a Kähler manifold*, Monatsh. Math. **91**(1981), 257-274.

[32] Chen, B.Y.: *CR-submanifolds of a Kaehler manifold. I*, J. Differ. Geom. **16**(2)(1981), 305-322.

[33] Chen, B.Y.: *Slant immersions*, Bull. Aust. Math. Soc. **41**(1)(1990), 135-147.

[34] Chen, B.Y.: *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, 1990.

[35] Chen, B.Y., Deshmukh, S.: *Yamabe and quasi-Yamabe solitons on Euclidean submanifolds*, Mediterr. J. Math. **15**(5)(2018), 1-9.

[36] Chen, B.Y., Ogiue, K.: *On totally real submanifolds*, Trans. Am. Math. Soc. **193**(1974), 157-266.

[37] Cho, J.T., Kimura, M.: *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. **61**(2)(2009), 205-212.

[38] Crasmareanu, M.: *A new approach to gradient Ricci solitons and generalizations*, Filomat **32**(9)(2018), 3337-3346.

[39] Crasmareanu, M., Hretcanu, C.E.: *Golden differential geometry*, Chaos Solitons Fractals **38**(5)(2008), 1229-1238.

[40] Das, A., Mandal, A.: *Study of Ricci solitons on concircularly flat Sasakian manifolds admitting Zamkovoy connection*, Aligarh Bull. Math. **39**(2)(2020), 47-61.

[41] De, U.C., Sarkar, A.: *On* (ϵ)*-Kenmotsu manifolds*, Hadronic J., **32**(2009), 231-242.

[42] De, U.C., Sarkar, A.: *On pseudo-slant submanifolds of trans-Sasakian manifolds*, Proc. Est. Acad. Sci. **60**(1)(2011), 1-11.

[43] De, U.C., Sengupta, J.: *Quarter-symmetric metric connection on a Sasakian manifold*, Commun. Fac. Sci. Univ. Ank., Ser. A1, Math. Stat. **49**(2000), 7-13.

[44] De, U.C., Shaikh, A.A.: *Complex Manifolds and Contact Manifolds*, Narosa Publishing House, New Delhi, India, 2009.

[45] De Spinadel, V.W.: *The metallic means family and forbidden symmetries*. Int. Math. J. **2**(3)(2002), 279-288.

[46] Dey, S., Roy, S.: $*$-η*-Ricci soliton within the framework of Sasakian manifold*, J. Dyn. Syst. Geom. Theor. **18**(2)(2020), 163-181.

[47] Dogan, B., Sahin, B., Yasar, E.: *Screen generic lightlike submanifolds*, Mediterr. J. Math. **16**(4)(2019), 104(1-21).

[48] Duggal, K.L., Bejancu, A.: *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Mathematics and its Applications, Vol. **364**, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[49] Duggal, K.L., Jin, D.H.: *Totally umbilical lightlike submanifolds*, Kodai Math. J. **26**(1)(2003), 49-68.

[50] Duggal, K.L., Jin, D.H.: *Generic lightlike submanifolds of an indefinite Sasakian manifold*, Int. Electron. J. Geom. **5**(1)(2012), 108-119.

[51] Duggal, K.L., Sahin, B.: *Screen Cauchy Riemann lightlike submanifolds*, Acta Math. Hungar. **106**(1)(2005), 137-165.

[52] Duggal, K.L., Sahin, B.: *Lightlike submanifolds of indefinite Sasakian manifolds*, Int. J. Math. Math. Sci. **2007**(2007), Article ID 57585, 21 pages.

[53] Duggal, K.L., Sahin, B.: *Generalized Cauchy-Riemann lightlike submanifolds of indefinite Sasakian manifolds*, Acta Math. Hungar. **122**(1-2)(2009), 45-58.

[54] Duggal, K.L., Sahin, B.: *Differential Geometry of Lightlike Submanifolds*, Frontiers in Mathematics, Birkhauser, Basel, 2010.

[55] Etayo, F., Santamaria, R., Upadhyay, A.: *On the geometry of almost golden Riemannian manifolds*, Mediterr. J. Math. **14**(5)(2017), 1-14.

[56] Friedmann, A.: *Uber die geometric derhalbsymmetrischen Ubertragungen*, Math. Z. **21**(1)(1924), 211-223.

[57] Gezer, A., Cengiz, N., Salimov, A.: *On integrability of golden Riemannian structures*, Turk. J. Math. **37**(4)(2013), 693-703.

[58] Golab, S.: *On semi-symmetric and quarter-symmetric linear connections*, Tensor, N.S. **29**(1975), 249-254.

[59] Goldberg, S.I., Petridis, N.C.: *Differentiable solutions of algebraic equations on manifolds*, Kodai Math. Semin. Rep. **25**(1973), 111-128.

[60] Goldberg, S.I., Yano, K.: *Polynomial structures on manifolds*, Kodai Math. Semin. Rep. **22**(1970), 199-218.

[61] Guler, S.: *On a class of gradient almost Ricci solitons*, Bull. Malays. Math. Sci. Soc. **43**(5)(2020), 3635-3650.

[62] Guler, S., Crasmareanu, M.: *Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy*, Turk. J. Math. **43**(5)(2019), 2631-2641.

[63] Gupta, R.S.: *Screen generic lightlike submanifolds of indefinite Sasakian manifolds*, Mediterr. J. Math. **17**(5)(2020), 141(1-23).

[64] Gupta, R.S., Sharfuddin, A.: *Screen transversal lightlike submanifolds of indefinite cosymplectic manifolds*, Rend. Sem. Mat. Univ. Padova **124**(2010), 145-156.

[65] Gupta, R.S., Sharfuddin, A.: *Screen transversal lightlike submanifolds of indefinite Kenmotsu manifolds*, Sarajevo J. Math. **7**(19)(2011), 103-113.

[66] Gupta, R.S., Sharfuddin, A.: *Slant lightlike submanifolds of indefinite Kenmotsu manifolds*, Turk. J. Math. **35**(2011), 115-127.

[67] Gupta, R.S., Upadhyay, A.: *Screen slant lightlike submanifolds of indefinite Kenmotsu manifolds*, Kyungpook Math. J. **50**(2)(2010), 267-279.

[68] Haider, S.M.K., Advin, Thakur, M.: *Screen slant lightlike submanifolds of indefinite Sasakian manifolds*, Kyungpook Math. J. **52**(4)(2012), 443-457.

[69] Haider, S.M.K, Thakur, M., Advin: *Totally contact umbilical screen transversal lightlike submanifolds of an indefinite Sasakian manifold*, Note Mat. **32**(2)(2012), 123-134.

[70] Hamilton, R.S.: *Three-manifolds with positive Ricci curvature*, J. Differ. Geom. **17**(2)(1982), 255-306.

[71] Hamilton, R.S.: *The Ricci flow on surfaces, Mathematics and General Relativity*, Contemp. Math. **71**(1988), 237-261.

[72] Haseeb, A., Prakasha, D.G., Harish, H.: $*$*-Conformal* η*-Ricci solitons on* α*-cosymplectic manifolds*, Int. J. Anal. Appl. **19**(2)(2021), 165-179.

[73] Hretcanu, C.E., Blaga, A.M.: *Submanifolds in metallic Riemannian manifolds*, Differ. Geom. Dyn. Syst. **20**(2018), 83-97.

[74] Hretcanu, C.E., Blaga, A.M.: *Slant and semi-slant submanifolds in metallic Riemannian manifolds*, J. Funct. Spaces **2018**(2018), Article ID 2864263, 13 pages.

[75] Hretcanu, C.E., Crasmareanu, M.: *On some invariant submanifolds in a Riemannian manifold with golden structure*, An. Stiint. Univ. Al. I. Cuza Iasi (S. N.) Mat., Tomul **53**(1)(2007), 199-211.

[76] Hretcanu, C.E., Crasmareanu, M.: *Applications of the golden ratio on Riemannian manifolds*, Turk. J. Math. **33**(2)(2009), 179-191.

[77] Hretcanu, C.E., Crasmareanu, M.: *Metallic structures on Riemannian manifolds*, Rev. Union. Mat. Argent. **54**(2)(2013), 15-27.

[78] Hu, C., Wang, Y.: *A note on invariant submanifolds of trans-Sasakian manifolds*, Int. Electron. J. Geom. **9**(2)(2016), 27-35.

[79] Hui, S.K., Pal, T.: *Totally real submanifolds of* $(LCS)_n$*-manifolds*, Ser. Math. Inform. **33**(2)(2018), 141-152.

[80] Hui, S.K., Atceken, M., Pal, T., Mishra, L.N.: *On contact CR-submanifolds of* $(LCS)_n$*-manifolds*, Thai J. Math. **19**(1)(2021), 175-186.

[81] Ishihara, I., Kon,M.: *Contact totally umbilical submanifolds of a Sasakian space form*, Ann. Mat. Pura Appl. **114**(1)(1977), 351-364.

[82] Janssens, D., Vanhecke, L.: *Almost contact structures and curvature tensors*, Kodai Math. J. **4**(1)(1981), 1-27.

[83] Jin, D.H., Lee, J.W.: *Generic lightlike submanifolds of an indefinite Kaehler manifold*, Int. J. Pure Appl. Math. **101**(4)(2015), 543-560.

[84] Kenmotsu, K.: *A class of almost contact Riemannian manifolds*, Tohoku Math. J. **24**(1)(1972), 93-103.

[85] Khan, K.A., Khan, V.A. and Sirajuddin: *Warped product contact CR-submanifolds of trans-Sasakian manifolds*, Filomat **21(2)**(2007), 55-62.

[86] Khan, M.A.: *Totally umbilical hemi-slant submanifolds of cosymplectic manifolds*, Math. Aeterna **3**(8)(2013), 645-653.

[87] Khan, M.A., Singh, K., Khan, V.A.: *Slant submanifolds of LP-contact manifolds*, Differ. Geom. Dyn. Syst. **12**(2010), 102-108.

[88] Khan, M.A., Uddin, S., Singh, K.: *A classification on totally umbilical proper slant and hemi-slant submanifolds of a nearly trans-Sasakian manifold*, Differ. Geom. Dyn. Syst. **13**(2011), 117-127.

[89] Khan, V.A., Khan, M.A.: *Semi-slant submanifolds of trans-Sasakian manifolds*, Sarajevo J. Math. **2**(14)(2006), 83-93.

[90] Laha, B., Bhattacharyya, A.: *On CR-submanifolds of (ϵ)-paracontact Sasakian manifold*, Int. J. Pure Appl. Math. **94(3)**(2014), 403-417.

[91] Laha, B., Bhattacharyya, A.: *Totally umbilical hemi-slant submanifolds of LP-Sasakian manifold*, Lobachevskii J. Math. **36**(2)(2015), 127-131.

[92] Laha, B., Das, B., Bhattacharyya, A.: *Contact CR-submanifolds of an Indefinite Lorentzian para-Sasakian manifold*, Acta Univ. Sapientiae Math. **5**(2)(2013), 157-168.

[93] Laha, B., Das, B., Bhattacharyya, A.: *Submanifolds of some indefinite contact and paracontact manifolds*, Gulf J. Math. **1**(2)(2013), 180-188.

[94] Lone, M.A., Lone, M.S., Shahid, M.H.: *Hemi-slant submanifolds of cosymplectic manifolds*, Cogent Math. Stat. **3**(1)(2016), 1204143.

[95] Lotta, A.: *Slant submanifolds in contact geometry*, Bull. Math. Soc. Sci. Math. Roum. (N.S) **39**(1996), 183-198.

[96] Mandal, A., Das, A.: *On M-projective curvature tensor of Sasakian manifolds admitting Zamkovoy Connection*, Adv. Math. Sci. J. **9**(10)(2020), 8929-8940.

[97] Mandal, A., Das, A.: *Projective curvature tensor with respect to Zamkovoy connection in Lorentzian para-Sasakian manifolds*, J. Indones. Math. Soc. **26**(3)(2020), 369-379.

[98] Mandal, A., Das, A.: *Pseudo projective curvature tensor on Sasakian manifolds admitting Zamkovoy connection*, Bull. Calcutta Math. Soc. **112**(5)(2020), 431-450.

[99] Mandal, A., Das, A.: *LP-Sasakian manifolds equipped with Zamkovoy connection and conharmonic curvature tensor*, J. Indones. Math. Soc. **27**(2)(2021), 137-149.

[100] Marrero, J.C.: *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. **162**(1)(1992), 77-86.

[101] Matsumoto, K.: *On contact CR-submanifolds of Sasakian manifolds*, Int. J. Math. Math. Sci. **6(2)**(1983), 313-326.

[102] Munteanu, M.I.: *A note on doubly warped product contact CR-submanifolds in trans-Sasakian manifolds*, arXiv:math/0604008v2, 2007.

[103] Nagaraja, H.G., Somashekhara, G.: *On psudo projective curvature tensor in Sasakian manifolds*, Int. J. Contemp. Math. Sci. **6**(27)(2011), 1319-1328.

[104] Narain, D., Prakash, A., Prasad, B.: *A pseudo projective curvature tensor on a Lorentzian para-Sasakian manifold*, An. Stiint. Univ. Al. I. Cuza Iasi (S. N.) Mat., Tomul **55**(2009), 275-284.

[105] Ojha, R.H.: *On Sasakian manifold*, Kyungpook Math. J. **13**(1973), 211-215.

[106] Ojha, R.H.: *A note on the M-projective curvature tensor*, Indian J. Pure Appl. Math. **8**(12)(1975), 1531-1534.

[107] Ojha, R.H.: *M-projectively flat Sasakian manifolds*, Indian J. Pure Appl. Math. **17**(4)(1986), 481-484.

[108] Oubina, J.A.: *New classes of almost contact metric structures*, Publ. Math. Debrecen **32**(1985), 187-193.

[109] Pal, T., Shahid, M.H., Hui, S.K.: *CR-submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric non-metric connection*, Filomat **33**(11)(2019), 3337-3349.

[110] Pandey, H.B., Kumar, A.: *Anti-invariant submanifolds of almost paracontact manifolds*, Indian J. Pure Appl. Math. **16**(1985), 586-590.

[111] Park, K.S., Prasad, R.: *Semi-slant submersions*, Bull. Korean Math. Soc. **50**(3)(2013), 951-962.

[112] Patra, C., Laha, B., Bhattacharyya, A.: *On hemi-slant submanifold of Kenmotsu manifold*, Int. J. Math. Comb. **1**(2019), 62-72.

[113] Pitis, G.: *On some submanifolds of a locally product manifold*, Kodai Math. J. **9**(3)(1986), 327-333.

[114] Pokhariyal, G.P., Mishra, R.S.: *Curvature tensors and their relativistic significance II*, Yokohama Math. J. **19**(2)(1971), 97-103.

[115] Pokhariyal, G.P., Yadav, S., Chaubey, S.K.: *Ricci solitons on trans-Sasakian manifolds*, Differ. Geom. Dyn. Syst. **20**(2018), 138-158.

[116] Prakasha, D.G., Mirji, K.: *On the M-projective curvature tensor of a* (k, μ)*-contact metric manifold*, Ser. Math. Inform. **32**(1)(2017), 117-128.

[117] Prasad, B.: *A pseudo projective curvature tensor on a Riemannian manifold*, Bull. Calcutta Math. Soc. **94**(3)(2002), 163-166.

[118] Prasad, R., Kumar, S.: *Semi-slant Riemannian maps from cosymplectic manifolds into Riemannian manifolds*, Gulf J. Math. **9**(1)(2020), 62-80.

[119] Prasad, R., Shukla, S.S., Haseeb, A., Kumar, S.: *Quasi hemi-slant submanifolds of Kaehler manifolds*, Honam Math. J. **42**(4)(2020), 795-809.

[120] Prasad, R., Singh, P.K., Rai, A.K.: *On quasi hemi-slant submanifolds of nearly Kaehler manifolds*, Differ. Geom. Dyn. Syst. **23**(2021), 188-202.

[121] Prasad, R., Verma, S.K., Kumar, S.: *Quasi hemi-slant submanifolds of Sasakian manifolds*, J. Math. Comput. Sci. **10**(2)(2020), 418-435.

[122] Prasad, R., Verma, S.K., Kumar, S., Chaubey, S.K.: *Quasi hemi-slant submanifolds of cosymplectic manifolds*, Korean J. Math. **28**(2)(2020), 257-273.

[123] Roy, S., Dey, S., Bhattacharyya, A., Siddiqi, M.D.: $*$*-*η*-Ricci-Yamabe solitons on* α*-Cosymplectic manifolds with a quarter-symmetric metric connection*, arXiv:2109.04700v1, 2021.

[124] Sahin, B.: *Transversal lightlike submanifolds of indefinite Kaehler manifolds*, An. Univ. Vest Timis., Ser. Mat.-Inform. **44**(1)(2006), 119-145.

[125] Sahin, B.: *Screen transversal lightlike submanifolds of indefinite Kaehler manifolds*, Chaos Solitons Fractals **38**(5)(2008), 1439-1448.

[126] Sahin, F.: *Cohomology of hemi-slant submanifolds of a Kaehler manifold*, J. Adv. Stud. Topol. **5**(2)(2014), 27-31.

[127] Sarkar, A., Bhakta, P.: *Some characterizations of slant submanifolds of trans-Sasakian manifolds*, BSG Proc. **26**(2018), 102-111.

[128] Shahid, M.H.: *CR-submanifolds of a trans-Sasakian manifold*, Indian J. Pure Appl. Math. **22**(12)(1991), 1007-1012.

[129] Shahid, M.H.: *On some semi-invariant submanifolds of a trans-Sasakian manifold*, Riv. Math. Univ. Parma. **5**(2)(1993), 257-265.

[130] Shahid, M.H.: *CR-submanifolds of a trans-Sasakian manifold II*, Indian J. Pure Appl. Math. **25**(3)(1994), 299-307.

[131] Shahid, M.H., Sharfuddin, A.: *Anti-invariant submanifolds of trans-Sasakian manifolds*, **735/93**, International Centre for Theoritical Physics, 1993.

[132] Shaikh, A.A., Kundu, H.: *On equivalency of various geometric structures*, J. Geom. **105**(1)(2014), 139-165.

[133] Shukla, S.S., Singh, D.D.: *On* (ϵ)*-trans-Sasakian manifolds*, Int. J. Math. Anal. **4**(49)(2010), 2401-2414.

[134] Siddesha, M.S., Bagewadi, C.S., Venkatesha, S.: *On the geometry of hemi-slant submanifolds of LP-cosymplectic manifold*, Asian J. Math. Appl. **2018**(2018), Article ID ama0473, 11 pages.

[135] Siddiqui, A.N.: *Warped product and doubly warped product bi-slant submanifolds in trans-Sasakian manifolds*, JMI Int. J. Math. Sci. **9**(2018), 15-27.

[136] Siddiqi, M.D.: η*-Ricci solitons in 3-dimensional normal almost contact metric manifolds*, Bull. Transilv. Univ. Brasov, Ser. III, Math. Inform. Phys. **11**(2)(2018), 215-234.

[137] Siddiqi, M.D.: η*-Einstein solitons in* δ*-Lorentzian trans-Sasakian manifolds*, Math. Adv. Pure Appl. Sci. **1**(1)(2018), 27-38.

[138] Siddiqi, M.D., Akyol, M.A.: η*-Ricci-Yamabe soliton on Riemannian submersions from Riemannian manifolds*, arXiv:2004.14124v1, 2020.

[139] Singh, J.P.: *On M-relatively flat almost pseudo Ricci symmetric manifolds*, Acta Math. Univ. Comenianae **86**(2)(2017), 335-343.

[140] Somashekhara, G., Reddy, P.S.K., Pavani, N., Manjula, G.J.: η*-Ricci-Yamabe solitons on submanifolds of some indefinite almost contact manifolds*, J. Math. Comput. Sci. **11**(3)(2021), 3775-3791.

[141] Tachibana, S.: *On almost-analytic vectors in almost-Kählerian manifolds*, Tohoku Math. J. **11**(2)(1959), 247-265.

[142] Takahashi, T.: *Sasakian manifold with pseudo-Riemannian metric*, Tohoku Math. J. **21**(2)(1969), 271-290.

[143] Tastan, H.M., Gerdan, S.: *Hemi-slant submanifolds of a locally conformal Kähler manifold*, Int. Electron. J. Geom. **8**(2)(2015), 46-56.

[144] Tastan, H.M., Ozdemir, F.: *The geometry of hemi-slant submanifolds of a locally product Riemannian manifold*, Turk. J. Math. **39**(2)(2015), 268-284.

[145] Tripathi, M.M., Gupta, P.: *On* τ*-curvature tensor in K-contact and Sasakian manifolds*, Int. Electron. J. Geom. **4**(1)(2011), 32-47.

[146] Yadav, A., Shankar, G., Kaur, R.: *An investigation on the existence of totally contact umbilical screen-slant lightlike submanifolds of indefinite Sasakian manifolds*, Differ. Geom. Dyn. Syst. **23**(2021), 244-254.

[147] Yadav, S., Suthar, D.: *Semi-invariant ξ-submanifolds of trans-Sasakian manifolds*, Casp. J. Appl. Sci. Res. **2**(2)(2013), 23-28.

[148] Yano, K.: *Concircular geometry I. concircular transformations*, Proc. Imp. Acad. Tokyo **16**(6)(1940), 195-200.

[149] Yano, K.: *On structure defined by a tensor field of type* $(1,1)$ *satisfying* $f^3 + f = 0$, Tensor **14**(1963), 99-109.

[150] Yano, K., Bochner, S.: *Curvature and Betti Numbers*, Annals of Mathematics Studies, Vol. **32**, Princeton University Press, Princeton, New Jersey, 1953.

[151] Yano, K., Kon, M.: *Anti-invariant submanifolds of Sasakian space forms I*, Tohoku Math. J. **29**(1)(1977), 9-23.

[152] Yano, K., Kon, M.: *Generic submanifolds of Sasakian manifolds*, Kodai Math. J. **3**(2)(1980), 163-196.

[153] Yano, K., Kon, M.: *Stuctures on Manifolds*, Series in Pure Mathematics, Vol. **3**, World Scientific Publishing Company, Singapore, 1984.

[154] Zamkovoy, S.: *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom. **36**(1)(2008), 37-60.

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ANTI-INVARIANT SUBMANIFOLDS OF SOME INDEFINITE ALMOST CONTACT AND PARACONTACT MANIFOLDS

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Abstract. In this paper we discuss about anti-invariant submanfolds of indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold, indefinite LP-Sasakian manifold and obtain some results regarding the relation between the structure vector field of a manifold and the anti-invariance of the submanifold. Also we obtain some results on totally umbilical, totally geodesic submanifolds.

Mathematics Subject Classification[2010] : 53C15, 53C20, 53C25, 53C40, 53C50.

Keywords : Anti-invariant submanifold, indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold, indefinite LP-Sasakian manifold.

1. Introduction. A (2n+1)-dimensional semi-Riemannian manifold $(\tilde{M}, \tilde{g})$ is called an indefinite almost contact manifold if it admits an indefinite almost contact structure (ϕ, ξ, η), where ϕ is a tensor field of type (1,1), ξ is a vector field and η is a 1-form, satisfying for all vector fields X, Y on $\tilde{M}$ (Blair, 1976),

$$\phi^2 X = -X + \eta(X)\xi, \eta \circ \phi = 0, \phi\xi = 0, \eta(\xi) = 1, \tag{1.1}$$

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{1.2}$$

$$\tilde{g}(X, \xi) = \epsilon\eta(X), \tilde{g}(\phi X, Y) = -\tilde{g}(X, \phi Y). \tag{1.3}$$

Here $\epsilon = \tilde{g}(\xi, \xi) = \pm 1$ and $\tilde{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\tilde{g}$.

Based on the structure equations manifolds are classified as follows (Blair, 1976) :

● An indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite Sasakian structure if for all vector fields Z, W on $\tilde{M}$,

$$(\bar{\nabla}_Z \phi)W = \epsilon\eta(W)Z - \tilde{g}(Z, W)\xi, \tag{1.4}$$

$$\bar{\nabla}_Z \xi = -\epsilon\phi Z \tag{1.5}$$

• An indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite trans-Sasakian structure of type (α, β) if

$$(\tilde{\nabla}_Z \phi)W = \alpha[\tilde{g}(Z, W)\xi - \epsilon\eta(W)Z] + \beta[\tilde{g}(\phi Z, W)\xi - \epsilon\eta(W)\phi Z], \tag{1.6}$$

$$\tilde{\nabla}_Z \xi = -\epsilon\alpha\phi Z + \epsilon\beta[Z - \eta(Z)\xi]. \tag{1.7}$$

for functions α, β on $\tilde{M}$ and for all vector fields Z, W on $\tilde{M}$.

• In 2009, U.C. De and A.Sarkar introduced and studied the notion of ϵ-Kenmotsu manifolds with indefinite metric by giving an example of $\alpha = 0, \beta = 1$, then indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite Kenmotsu structure. The structure equation thus becomes

$$(\bar{\nabla}_Z \phi)W = \tilde{g}(\phi Z, W)\xi - \epsilon\eta(W)\phi Z, \tag{1.8}$$

$$\tilde{\nabla}_Z \xi = \epsilon Z - \epsilon\eta(Z)\xi \tag{1.9}$$

The study of geometry of anti-invariant submanifolds is carried out by (Bagewadi, 1982, Brasil, Lobos and Mariano, 2008, Chen, 1974, Shahid, 1993 Ishihara and Kon, 1977 and Yano, 1963) in various contact manifolds. In 1976–77, anti-invariant submanifolds of Sasakian space forms (Yano, 1977) were discussed by K.Yano and M.Kon. In 1985, H.B.Pandey and A.Kumar investigated about anti-invariant submanifolds of almost para-contact manifolds (Pandey and Kumar, 1989). In 2018, C.S.Bagewadi and S.Venkatesha studied extensively anti-invariant submanifolds of S-manifold and (ϵ)-Sasakian manifold (Bagewadi and Venkatesha, 2018). Motivated from their work we have established some new results on anti-invariant submanifolds of indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold and indefinite LP-Sasakian manifold.

A differentiable manifold $\tilde{M}$ of dimension n is called indefinite Lorentzian para-Sasakian manifold (briefly, indefinite LP-Sasakian manifold) if it admits the structure (ϕ, ξ, η) along with a Lorentzian metric (which is almost para-contact) $\tilde{g}$ which satisfy for all vector fields X, Y on $\tilde{M}$,

$$\begin{aligned} \phi^2 X &= X + \eta(X)\xi, \\ \eta(\xi) &= -1, \\ \tilde{g}(\xi, \xi) &= -\epsilon, \\ \eta(X) &= \epsilon\tilde{g}(X, \xi), \\ \phi\xi &= 0, \end{aligned}$$

$$\eta(\phi X) = 0,$$
$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y).$$

In this manifold we also have,

$$(\tilde{\nabla}_X \phi)Y = \tilde{g}(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{1.10}$$

$$\tilde{\nabla}_X \xi = \epsilon\phi X. \tag{1.11}$$

Let ϕ be a differentiable map from a manifold M into a manifold M' and let the dimensions of M, M' be n, m respectively. If at each point p of M, $(\varphi_*)_p$ is a 1-1 map, i.e., if rankφ=n, then φ is called an immersion of M into M'.

If an immersion ϕ is one-one, i.e., if $\phi(p) \neq \phi(q)$ for $p \neq q$, then ϕ is called an imbedding of M into M'.

If the manifolds $M, \tilde{M}$ satisfy the following two conditions, then M is called a submanifold of $\tilde{M}$

i) $M \subset \tilde{M}$,

ii) the identity map i from M into $\tilde{M}$ is an imbedding of M into $\tilde{M}$.

Let $\phi : (M, g) \to (\tilde{M}, \tilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M, g) into m-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. We denote by ∇ and $\tilde{\nabla}$ as Levi-Civita connection of M^n and $\tilde{M}^m$ respectively. Then the formulas of Gauss and Weingarten are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.12}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V. \tag{1.13}$$

where $h, A, \nabla^\perp$ are the second fundamental form, the shape operator and the normal connection respectively.

Moreover the shape operator A_V and the second fundamental form h are related by

$$g(h(X, Y), V) = g(A_V X, Y) \tag{1.14}$$

for $X, Y \in T_x(M)$ and $V \in T^\perp_x(M)$.

A submanifold M is called

i) totally geodesic in $\tilde{M}$ if $h = 0$ or equivalently $A_V = 0$ for all $V \in T^\perp M$,

ii) minimal in $\tilde{M}$ if the mean curvature vector H satisfies $H = \dfrac{Trh}{\dim M} = 0$,

iii) totally umbilical if

$$h(X,Y) = g(X,Y)H, \tag{1.15}$$

and

iv) anti-invariant if

$$X \in T_x(M) \Rightarrow \phi X \in T_x^{\perp}(M). \tag{1.16}$$

On a smooth manifold M, a distribution D is the assignment, to each point $x \in M$, of a subspace D_x of the tangent space $T_x(M)$. A distribution D is smooth at a point x if any tangent vector $X_{(x)} \in D_x$ can be locally extended to a smooth vector field X on some open set $U \in M$ such that $X_{(y)} \in D_y$ for every $y \in U$.

A connected submanifold N of M is called an integral manifold of the distribution D if $(\phi_*)_x N = D_x \ \forall\ x \in N$, ϕ being the imbedding of N into M. If there is no other integral manifold of D which contains N, then N is called a maximal integral manifold of D.

Thus $D = \bigcup_{x\in M} D_x$ is a distribution on M. We say that a distribution D is involute for every two sections X, Y of D, the commutator $[X, Y]$ is a section of D.

Let D denote an integrable distribution on M, then the collection of integrable manifolds of D is called a foliation.

A maximal connected integral manifold of D is called a leaf of the foliation.

2. Anti-invariant submanifolds of indefinite Sasakian manifold, indefinite Kenmotsu manifold, indefinite trans-Sasakian manifold, indefinite LP-Sasakian manifold. This chapter consists of four sections which are devoted to the study of anti-invariant submanifolds of indefinite Sasakian manifold ,indefinite Kenmotsu manifold, indefinite trans-Sasakian manifod and indefinite LP-Sasakian manifold.

2.1 Anti-invariant submanifolds of an indefinite Sasakian manifold. In this section we obtain some results on anti-invariant submanifolds of indefinite Sasakian manifold. They are as follows

THEOREM 2.1.1 *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If M is totally umbilical, then M is totally geodesic.*

Proof: Since ξ is tangent to M, from (1.12) we have,

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi),$$

where the vector fields X and Y are tangent to M.

Using (1.5) we get, $-\epsilon\phi X = \nabla_X \xi + h(X, \xi)$.

Equating tangential and normal components we obtain,

$$\begin{aligned} \epsilon(\phi X)^T &= -\nabla_X \xi, \\ \epsilon(\phi X)^\perp &= -h(X, \xi). \end{aligned}$$

Putting $X = \xi$ in second equation(as $\phi\xi = 0$) we get $h(\xi, \xi) = 0$.

Let us assume that M is totally umbilical, then $h(X, Y) = g(X, Y)H$. Putting $X = Y = \xi$ we get,

$$\begin{aligned} & g(\xi, \xi)H = \pm H = h(\xi, \xi) = 0 \\ \Rightarrow \ & H = 0 \text{ and so, } h = 0. \end{aligned}$$

Thus M is totally geodesic.

THEOREM 2.1.2 *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. Then ξ is parallel with respect to the induced connection on M if and only if M is an anti-invariant submanifold in $\tilde{M}$.*

Proof: Let the structure vector field ξ be tangent to M, then from (1.5) and (1.12),

$$-\epsilon\phi X = \bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{2.1}$$

Now let ξ be parallel with respect to the induced connection on M, then we have $\nabla_X \xi = 0$ and $h(X, \xi) \in T_x^\perp(M)$.

From (2.1),

$$\begin{aligned} -\epsilon\phi X &= h(X, \xi) \\ \Rightarrow \ \phi X &= -\epsilon h(X, \xi). \end{aligned}$$

Hence ϕX is normal to M, $\phi X \in T_x^\perp(M)$. Thus M is anti-invariant.

Conversely, let M be anti-invariant, then by (1.16) if $X \in T_x(M)$, then $\phi X \in T_x^\perp(M)$, so from (2.1) and as $h(X, \xi) \in T_x^\perp(M)$,we have,

$$\begin{aligned} & -\epsilon\phi X = h(X, \xi) \\ \Rightarrow \ & \nabla_X \xi = 0 \ \text{(by (2.1))} \end{aligned}$$

$\Rightarrow$ ξ is parallel with respect to the induced connection on M.

THEOREM 2.1.3 *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$.*

If ξ is normal to M, then M is totally geodesic and consequently M is anti-invariant.

Proof: Let ξ be normal to M, then (1.13) implies

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi. \tag{2.2}$$

Using (1.5) and (2.2), and taking inner product with Y we obtain,

$$\epsilon g(\phi X, Y) = g(A_\xi X, Y) \text{ for any } X, Y \text{ tangent to } M. \tag{2.3}$$

Interchanging X, Y in (2.3) and then adding we get,

$$\begin{aligned} g(A_\xi X, Y) + g(A_\xi Y, X) &= \epsilon[g(\phi X, Y) + g(\phi Y, X)] \\ &= 0 \quad \text{(by (1.3))} \end{aligned}$$

$$\begin{aligned} &\Rightarrow g(A_\xi X, Y) = 0 \quad \text{(since } A_\xi \text{ is symmetric)} \\ &\Rightarrow g(h(X, Y), \xi) = 0 \quad \text{(by (1.14))} \\ &\Rightarrow h = 0 \\ &\Rightarrow A_\xi X = 0. \end{aligned}$$

Thus M is totally geodesic.

Thus by (2.2),

$$\begin{aligned} \nabla^\perp_X \xi &= \bar{\nabla}_X \xi \\ &= -\epsilon\phi X \quad \text{(by (1.5))} \\ \Rightarrow \phi X &\in T^\perp_x(M). \end{aligned}$$

Hence M is anti-invariant.

THEOREM 2.1.4 *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If $A_V X = 0$ for any $V \in T^\perp_x(M)$, then $\phi(T_x(M))$ is parallel with respect to the normal connection.*

Proof: To show that $\phi(T_x(M))$ is parallel with respect to the normal connection $\nabla^\perp$, we have to show that for every local section $\phi Y \in \phi(T_x(M)), \nabla^\perp_X(\phi Y)$ is also a local section in $\phi(T_x(M))$.

Now,

$$
\begin{aligned}
& (\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y) \\
\Rightarrow \quad & \epsilon\eta(Y)X - \tilde{g}(X,Y)\xi = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \phi(\nabla_X Y + h(X,Y)) \\
& \hfill \text{(by (1.4), (1.12), (1.13))} \\
\Rightarrow \quad & \nabla^{\perp}_X \phi Y = -\tilde{g}(X,Y)\xi + \epsilon\eta(Y)X + A_{\phi Y}X + \phi(\nabla_X Y + h(X,Y)).
\end{aligned}
$$

Taking inner product with $V \in T_x^{\perp}(M)$ we have,

$$
\begin{aligned}
g(\nabla^{\perp}_X \phi Y, V) &= -\tilde{g}(X,Y)g(\xi,V) + \epsilon\eta(Y)g(X,V) + g(A_{\phi Y}X, V) \\
&\quad + g(\phi\nabla_X Y, V) + g(\phi h(X,Y), V) \\
&= 0.
\end{aligned}
$$

Hence the proof.

THEOREM 2.1.5 *Let M be a submanifold of an indefinite Sasakian manifold $\tilde{M}$, then M is anti-invariant if and only if D is integrable.*

Proof: Let $X, Y \in D$, then $X, Y \in T\tilde{M}$,

$$
\begin{aligned}
g([X,Y],\xi) &= g(\bar{\nabla}_X Y, \xi) - g(\bar{\nabla}_Y X, \xi) \\
&= Xg(Y,\xi) - g(Y, \bar{\nabla}_X \xi) - Yg(X,\xi) + g(X, \bar{\nabla}_Y \xi) \\
&= -g(Y, -\epsilon\phi X) + g(X, -\epsilon\phi Y) \quad \text{(by (1.5))} \\
&= \epsilon g(\phi X, Y) - \epsilon g(X, \phi Y) \\
&= \epsilon g(\phi X, Y) + \epsilon g(\phi X, Y) \quad \text{(by (1.3))} \\
&= 2\epsilon g(\phi X, Y).
\end{aligned}
$$

Hence $[X,Y] \in D$ if and only if ϕX is normal to Y, i.e., $\phi X \in T_x^{\perp}(M)$ i.e., M is anti-invariant. Hence D is integrable if and only if M is anti-invariant.

2.2 Anti-invariant submanifolds of an indefinite Kenmotsu manifold. Here we consider an anti-invariant submanifold of an indefinite Kenmotsu manifold and using it's structure equation we state and prove the following theorem:

THEOREM 2.2.1 *Let M be a submanifold of an indefinite Kenmotsu manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If $A_V X = 0$ for any $V \in T_x^{\perp}(M)$, then $\phi(T_x(M))$ is parallel with respect to the normal connection.*

Proof: To show that $\phi(T_x^\perp(M))$ is parallel with respect to the normal connection $\nabla^\perp$, we have to show that for every local section $\phi Y \in \phi(T_x(M))$, $\nabla_x^\perp(\phi Y)$ is also a local section in $\phi(T_x(M))$.

Now,

$$
\begin{aligned}
& (\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y) \\
\Rightarrow\ & \tilde{g}(\phi X, Y)\xi - \epsilon\eta(Y)\phi X = -A_{\phi Y}X + \nabla_X^\perp \phi Y - \phi(\nabla_X Y + h(X,Y)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by } (1.8),\ (1.12),(1.13)) \\
\Rightarrow\ & \nabla_X^\perp \phi Y = A_{\phi Y}X + \tilde{g}(\phi X, Y)\xi - \epsilon\eta(Y)\phi X + \phi(\nabla_X Y) + \phi(h(X,Y)).
\end{aligned}
$$

Taking inner product with $V \in T_x^\perp(M)$ we have,

$$
\begin{aligned}
g(\nabla_X^\perp \phi Y, V) &= g(A_{\phi Y}X, V) + \tilde{g}(\phi X, Y)g(\xi, V) - \epsilon\eta(Y)g(\phi X, V) \\
&\quad + g(\phi\nabla_X Y, V) + g(\phi(h(X,Y)), V) \\
&= 0.
\end{aligned}
$$

Hence the result.

• **Example of an indefinite Kenmotsu manifold.** We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$, where (x,y,z) are standard coordinates of $\mathbb{R}^3$.

The vector fields $e_1 = \epsilon z\frac{\partial}{\partial x}$, $e_2 = \epsilon z\frac{\partial}{\partial y}$, $e_3 = -\epsilon z\frac{\partial}{\partial z}$ are linearly independent at each pt. of M.

Indefinite metric $g = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$.

$$
\begin{aligned}
& \phi e_1 = -e_2, \phi e_2 = e_1, \phi e_3 = 0. \\
& \phi^2 Z = -Z + \eta(Z)\xi, \eta(Z) = \epsilon g(Z, e_3), \eta(\xi) = 1. \\
& g(\phi Z, \phi W) = g(Z,W) - \epsilon\eta(Z)\eta(W); Z, W \in \Gamma(TM). \\
& [e_1, e_3] = \epsilon e_1, [e_2, e_3] = \epsilon e_2, [e_1, e_2] = 0.
\end{aligned}
$$

By Koszul's formula,

$$
\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) \\
&\quad - g(Y,[X,Z]) + g(Z,[X,Y]),
\end{aligned}
$$

We calculate

$$\nabla_{e_1}e_3 = \epsilon e_1, \ \ \nabla_{e_1}e_2 = 0, \ \ \nabla_{e_1}e_1 = -\epsilon e_3.$$
$$\nabla_{e_2}e_3 = \epsilon e_2, \ \ \nabla_{e_2}e_2 = \epsilon e_3, \ \ \nabla_{e_2}e_1 = 0.$$
$$\nabla_{e_3}e_3 = 0, \ \ \nabla_{e_3}e_2 = 0, \ \ \nabla_{e_3}e_1 = 0.$$
$$\nabla_X\xi = \epsilon(X - \eta(X)\xi) \ \text{ for } \ \xi = e_3.$$

(ϕ, ξ, η, g) forms an indefinite Kenmotsu structure in $\mathbb{R}^3$.

2.3 Anti-invariant submanifolds of an indefinite trans-Sasakian manifold. In this section state and prove two theorems related to the anti-invariant submanifolds of an indefinite trans-Sasakian manifold.

THEOREM 2.3.1 *Let M be a submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If M is totally umbilical, then M is totally geodesic.*

Proof: Since ξ is tangent to M,

$$\bar{\nabla}_X\xi = \nabla_X\xi + h(X,\xi).$$

Using (1.7) we get,

$$-\epsilon\alpha\phi X + \epsilon\beta[X - \eta(X)\xi] = \nabla_X\xi + h(X,\xi).$$

Equating tangential and normal components,

$$\begin{aligned} &\nabla_X\xi = -\epsilon\alpha(\phi X)^T + \epsilon\beta X - \epsilon\beta\eta(X)\xi, \\ &h(X,\xi) = -\epsilon\alpha(\phi X)^{\perp} \\ \Rightarrow\ &h(\xi,\xi) = 0 \quad (\text{since } \ \phi\xi = 0) \\ \Rightarrow\ &g(\xi,\xi)H = 0 \\ \Rightarrow\ &H = 0. \end{aligned}$$

Hence $h(X,Y) = g(X,Y)H = 0$ and so, $h = 0 \Rightarrow M$ is totally geodesic.

THEOREM 2.3.2 *Let M be a submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If $A_V X = 0$ for any $V \in T_x^{\perp}(M)$, then $\phi(T_x(M))$ is parallel with respect to the normal connection.*

Proof: To show that $\phi(T_x(M))$ is parallel with respect to the normal connection $\nabla^{\perp}$, we have to show that for every local section $\phi Y \in \phi(T_x(M))$, $\nabla^{\perp}_X(\phi Y)$ is also a local section in $\phi(T_x(M))$.

Using (1.6), (1.12), (1.13) we get,

$$
\begin{aligned}
&(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi Y - \phi(\bar{\nabla}_X Y)\\
\Rightarrow\ & -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \phi(\nabla_X Y) - \phi(h(X,Y)) = \alpha[\tilde{g}(X,Y)\xi - \epsilon\eta(Y)X]\\
&\quad +\beta[\tilde{g}(\phi X,Y)\xi - \epsilon\eta(Y)\phi X]\\
\Rightarrow\ & \nabla^{\perp}_X\phi Y = A_{\phi Y}X + \phi(\nabla_X Y) + \phi(h(X,Y)) + \alpha[\tilde{g}(X,Y)\xi\\
&\quad -\epsilon\eta(Y)X] + \beta[\tilde{g}(\phi X,Y)\xi - \epsilon\eta(Y)\phi X].
\end{aligned}
$$

Since $A_V = 0$ for any $V \in T^{\perp}_x(M)$, we have,

$$
\begin{aligned}
g(\nabla^{\perp}_X\phi Y, V) =\ & g(A_{\phi Y}X, V) + g(\phi(\nabla_X Y), V) + g(\phi h(X,Y), V) + \alpha[\tilde{g}(X,Y)g(\xi, V)\\
& -\epsilon\eta(Y)g(X,V)] + \beta[\tilde{g}(\phi X,Y)g(\xi,V) - \epsilon\eta(Y)g(\phi X, V)].
\end{aligned}
$$

Since $\phi V \in T^{\perp}_x(M)$ also, we have,

$$g(\nabla^{\perp}_X\phi Y, V) = 0.$$

Hence the proof.

• **Example of an indefinite trans-Sasakian manifold.** Let $\mathbb{R}^3$ be a 3-dimensional Euclidean space with regular coordinates (x,y,z). In $\mathbb{R}^3$ we define,

$$\eta = dz - ydx,\ \ \xi = \frac{\partial}{\partial z},$$
$$\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y},\ \ \phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ \ \phi\left(\frac{\partial}{\partial z}\right) = 0.$$

The semi-Riemannian metric g is defined by the matrix: $g = \begin{pmatrix} \epsilon y^2 & 0 & -\epsilon y \\ 0 & 0 & 0 \\ -\epsilon y & 0 & \epsilon \end{pmatrix}$.

(ϕ, ξ, η, g) forms an indefinite trans-Sasakian structure in $\mathbb{R}^3$.

2.4 Anti-invariant submanifolds of an indefinite LP-Sasakian manifold. In this section we first state and prove two theorems regarding submanifolds of an indefinite LP-Sasakian manifold and then in the last theorem we get an important result concerning anti-invariant submanifolds of an indefinite LP-Sasakian manifold.

THEOREM 2.4.1 *Let M be a submanifold of an indefinite LP-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If M is totally umbilical, then M is totally geodesic.*

Proof: Since ξ is tangent to M,

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

Using (1.11) we get,

$$\epsilon\phi X = \nabla_X \xi + h(X, \xi)$$

Equating tangential and normal components we obtain,

$$\epsilon(\phi X)^T = \nabla_X \xi,$$
$$\epsilon(\phi X)^\perp = h(X, \xi).$$

Putting $X = \xi$ in second equation we get,

$$h(\xi, \xi) = 0 \ \text{(as } \phi\xi = 0).$$

Let us assume that M is totally umbilical, then

$$h(X, Y) = g(X, Y)H.$$

Putting $X = Y = \xi$ we get,

$$h(\xi, \xi) = g(\xi, \xi)H \Rightarrow 0 = H$$

and hence $h(X, Y) = 0$.

Hence M is totally geodesic.

THEOREM 2.4.2 *Let M be a submanifold of an indefinite LP-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. Then ξ is parallel with respect to the induced connection on M if and only if M is an anti-invariant submanifold in $\tilde{M}$.*

Proof: Using (1.11), (1.12),

$$\epsilon\phi X = \bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{2.4}$$

Now let ξ be parallel with respect to induced connection on M, then we have,

$$\nabla_X \xi = 0.$$

From (2.4) we get,

$$\begin{aligned} \epsilon\phi X &= h(X,\xi) \\ \Rightarrow \quad \phi X &= \epsilon h(X,\xi). \end{aligned}$$

Hence ϕX is normal to M, $\phi X \in T_x^{\perp}(M)$. Thus M is anti-invariant.

Conversely, let M be anti-invariant, then by definition of anti-invariant if $X \in T_x(M)$, then $\phi X \in T_x^{\perp}(M)$ and so from (2.4) we get as $h(X,\xi) \in T_x^{\perp}(M)$,

$$\epsilon\phi X = h(X,\xi)$$

and so, from (2.4) we have, $\nabla_X \xi = 0 \Rightarrow \xi$ is parallel with respect to the induced connection on M.

THEOREM 2.4.3 *Let M be a submanifold of an indefinite LP-Sasakian manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M. If $A_V X = 0$ for any $V \in T_x^{\perp}(M)$, then $\phi(T_x(M))$ is parallel with respect to the normal connection.*

Proof: To show that $\phi(T_x(M))$ is parallel with respect to the normal connection $\nabla^{\perp}$, we have to show that for every local section $\phi Y \in \phi(T_x(M))$, $\nabla_X^{\perp}(\phi Y)$ is also a local section in $\phi(T_x(M))$.

Using (1.10), (1.12), (1.13) we obtain,

$$\begin{aligned} &\bar{g}(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi \\ &\qquad = (\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y) \\ &\qquad = -A_{\phi Y} X + \nabla_X^{\perp}\phi Y - \phi(\nabla_X Y + h(X,Y)) \end{aligned}$$

$$\Rightarrow \quad \nabla_X^{\perp}\phi Y = \bar{g}(X,Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi + A\phi Y X + \phi(\nabla_X Y) + \phi(h(X,Y)).$$

Since $A_V = 0$ for any $V \in T_x^{\perp}(M)$,

$$\begin{aligned} g(\nabla_X^{\perp}\phi Y, V) = {} & g(X,Y)g(\xi,V) + \epsilon\eta(Y)g(X,V) + 2\epsilon\eta(X)\eta(Y)g(\xi,V) \\ & + g(A_{\phi Y}X, V) + g(\phi(\nabla_X Y), V) + g(\phi(h(X,Y)), V). \end{aligned}$$

Since $\phi V \in T_x^{\perp}(M)$ for any $V \in T_x^{\perp}(M)$, by hypothesis, $g(\nabla_X^{\perp}\phi Y, V) = 0$.

Hence the proof.

• **Example of an indefinite LP-Sasakian manifold.** Let $\mathbb{R}^3$ be a 3-dimensional Euclidean space with regular coordinates (x, y, z). In $\mathbb{R}^3$ we define,

$$\eta = -dz - ydx, \quad \xi = \frac{\partial}{\partial z}$$

$$\text{and} \quad \phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad \phi\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad \phi\left(\frac{\partial}{\partial z}\right) = 0.$$

The Lorentzian metric g is defined by the matrix:

$$g = \begin{pmatrix} -\epsilon y^2 & 0 & \epsilon y \\ 0 & 0 & 0 \\ \epsilon y & 0 & -\epsilon \end{pmatrix}.$$

(ϕ, ξ, η, g) forms an indefinite Lorentzian Para-Sasakian structure in $\mathbb{R}^3$.

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References

Bagewadi, C.S. and **Venkatesha, S.** (2018) : Anti-invariant submanifolds of (ϵ)-Sasakian manifold, *Int. J. Math. Archive*, **9**(3), 148.

Bagewadi, C.S. and **Venkatesha, S.** (2018) : Anti-invariant submanifolds of S-manifold, *Int. J. Eng. Sci. Comput.*, Vol. **8**, Issue No. 5, 17368.

Bagewadi, C.S. (1982) : On totally real submanifolds of a Kahlerian manifold admitting semi-symmetric metric, *Indian J. Pure Appl. Math.*, **13**(5), 528.

Blair, D.E. (1976) : Contact manifolds in Riemannian geometry, *Lecture Notes in Math.*, Vol. **509**, Springer Verlag, Berlin, 1.

Brasil, A.; Lobos, G.A. and **Mariano, M.** (2008) : C-Totally Real submanifolds with parallel mean curvature in λ-Sasakian space forms, *Math. Contemp.*, Vol. **34**, Sociedade Brasileira de Matematica, 83.

Chen, B.Y. and **Ogiue, K.** (1974) : On totally real submanifolds, *Trans. Amer. Math. Soc.*, **193**, 157.

De, U.C. and **Sarkar, A.** (2009) : On (ϵ)-Kenmotsu manifolds, *Hadronic J.*, **32**, 231.

Shahid, M.H. (1993) : On some semi-invariant submanifolds of a trans-Sasakian manifold, *Riv. Math. Univ. Parma.*, **(5)**2, 257.

Ishihara, I. and **Kon, M.** (1977) : Contact totally umbilical submanifolds of a Sasakian space form, *Ann. Mat. Pura Appl. Series* IV Tohoku, **114**(1), 351.

Pandey, H.B. and **Kumar, A.** (1989) : Anti-invariant submanifolds of almost para contact manifolds, *Indian J. Pure Appl. Math.*, **20**(11), 1119. **Yano, K.** (1963) : On structure defined by a tensor field of type $(1, 1)$ satisfying $f^3 + f = 0$, **Tensor 14**, 99.

Yano, K. and **Kon, M.** (1977) : Anti-invariant submanifolds of Sasakian space forms, *Tohoku Math. J.*, **9**, 9.

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$*$-η-RICCI-YAMABE SOLITONS ON ANTI-INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLD WITH RESPECT TO A QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. In this paper, we discuss $*$-η-Ricci-Yamabe solitons on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection. We obtain some results regarding a Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton. Further, we prove some curvature properties of anti-invariant submanifolds of Kenmotsu manifold admitting a quarter symmetric metric connection. Next, we obtain a result regarding anti-invariant submanifolds of Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton with respect to a quarter symmetric metric connection. Then, we study the nature of a $*$-η-Ricci-Yamabe soliton and solitons appeared as its particular cases on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection when the vector field becomes a conformal Killing vector field. Finally, we give an example of a 3-dimensional Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton and verify a relation on it.

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1. Introduction. The concepts of Ricci flow and Yamabe flow were introduced simultaneously by R. S. Hamilton (1982, 1988). Ricci soliton emerged as a self-similar solution to the Ricci flow (Hamilton, 1988) and as did Yamabe soliton. These solitons are equivalent in dimension 2 but in greater dimensions, these two do not agree since Yamabe soliton preserves the conformal class of the metric but Ricci soliton does not in general. In 2019, S. Guler and M. Crasmareanu (2019) introduced a new geometric flow called Ricci-Yamabe flow as a scalar combination of Ricci flow and Yamabe flow. Ricci-Yamabe flow of type (p, q) is an evolution for the metrics on Riemannian or semi-Riemannian manifolds defined as (Guler and Crasmareanu, 2019)

$$\frac{\partial}{\partial t} g(t) = -2p \text{ Ric } (t) + qr(t)g(t),\ g(0) = g_0,$$

where p, q are scalars. Due to the signs of p, q, this flow can also be a Riemannian flow or semi-Riemannian flow or singular Riemannian flow. A Ricci-Yamabe flow of type (p, q) is called (Guler and Crasmareanu, 2019):

(i) Ricci flow if $p = 1,\ q = 0$;

(ii) Yamabe flow if $p = 0,\ q = 1$;

(iii) Einstein flow (Catino and Mazzieri, 2016) if $p = 1,\ q = -1$.

Naturally, a soliton to the Ricci-Yamabe flow is called Ricci-Yamabe soliton. Ricci-Yamabe soliton of type (p, q) on a Riemannian complex (M, g) is represented by the quintuplet (g, V, λ, p, q) satisfying the following equation

$$L_V g + 2pS + (2\lambda - qr)g = 0,$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, r is the scalar curvature, S is the Ricci curvature tensor and λ, p, q are scalars. This soliton is called shrinking, steady or expanding according as $\lambda < 0,\ \lambda = 0$ or $\lambda > 0$ respectively.

J.T. Cho and M. Kimura introduced the notion of η-Ricci soliton as an advance extension of Ricci soliton in 2009 (Cho and Kimura, 2009). Analogously in 2020 (Siddiqi and Akyol, 2020), M. D. Siddiqi and M. A. Akyol introduced the concept of η-Ricci-Yamabe soliton as a generalization of Ricci-Yamabe soliton. η-Ricci-Yamabe soliton of type (p, q) is represented by the sextuplet $(g, V, \lambda, \mu, p, q)$ on a Riemannian manifold M satisfying the following equation

$$L_V g + 2pS + (2\lambda - qr)g + 2\mu\eta \otimes \eta = 0,$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, r is the scalar curvature, S is the Ricci curvature tensor, $\eta \otimes \eta$ is a $(0, 2)$-tensor field and λ, μ, p, q are scalars. The soliton is called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively.

In 2020, S. Dey and S. Roy (2020) defined $*$-η-Ricci soliton as a generalizalion of η-Ricci soliton as follows

$$L_\xi g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where S^* is the $*$-Ricci tensor which was first introduced by S. Tachibana (1959).

Recently, S. Roy et al. (2021) introduced the notion of $*$-η-Ricci-Yamabe soliton as a generalizalion of η-Ricci-Yamabe soliton and defined it as follows

$$L_\xi g + 2pS^* + (2\lambda - qr^*)g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where $r^* = Tr(S^*)$ is the $*$-scalar curvature. A $*$-η-Ricci-Yamabe soliton is called

(i) $*$-η-Ricci soliton if $p = 1,\ q = 0$;

(ii) $*$-η-Yamabe soliton if $p = 0,\ q = 1$;

(iii) $*$-η-Einstein soliton if $p = 1,\ q = -1$.

A linear connection $\tilde{\nabla}$ on a Riemannian manifold (M, g) is called a quarter symmetric connection (Golab, 1975) if its torsion tensor T defined by

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \ \ \forall X, Y \in \chi(M) \tag{1.2}$$

satisfies the following equation

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{1.3}$$

Moreover, if the connection $\tilde{\nabla}$ satisfies

$$(\tilde{\nabla}_X g)(Y, Z) = 0 \ \ \forall X, Y, Z \in \chi(M), \tag{1.4}$$

then $\tilde{\nabla}$ is called a quarter symmetric metric connection (De and Sengupta, 2000).

Motivated by the works mentioned above, in this paper, we discuss $*$-η-Ricci-Yamabe solitons on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection. This paper is divided into three sections. After introduction, in the second section, we mention some definitions and relations which are used in the following section. The last, i.e. the third section consists of the main results. In this section, we obtain some results regarding a Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton. Further, we prove some curvature properties of anti-invariant submanifolds of Kenmotsu manifold admitting a quarter symmetric metric connection. Next, we obtain a result regarding anti-invariant submanifolds of Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton with respect to a quarter symmetric metric connection. Then, we study the nature of a $*$-η-Ricci-Yamabe soliton and solitons appeared as its particular cases on anti-invariant submanifolds of Kenmotsu manifold with respect to a quarter symmetric metric connection when the vector field becomes a conformal Killing vector field. Finally, we give an example of a 3-dimensional Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton and verify a relation on it.

2. Preliminaries. Let M be an odd dimensional differentiable manifold equipped with a metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the following relations$-$

$$\phi^2 X = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \eta \circ \phi = 0, \ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \ \eta(X) = g(X, \xi) \ \ \forall X, Y \in \chi(M), \tag{2.3}$$

then M is called almost contact metric manifold (Blair, 1976).

Kenmotsu manifold is named after K. Kenmotsu who introduced its notion in 1972 (Kenmotsu, 1972). An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is called Kenmotsu manifold if $\forall X, Y \in \chi(M)$,

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Also on an n-dimensional Kenmotsu manifold M, the following relations hold (Kenmotsu, 1972) $\forall X, Y \in \chi(M)$,

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.7}$$

$$R(X,\xi)\xi = \eta(X)\xi - X, \tag{2.8}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.9}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.10}$$

$$Q\xi = -(n-1)\xi. \tag{2.11}$$

Now from (Haseeb, Prakasha and Harish, 2021) we have that the $*$-Ricci tensor S^* on an n-dimensional α-cosymplectic manifold M is given by

$$S^*(X,Y) = S(X,Y) + \alpha^2(n-2)g(X,Y) + \alpha^2\eta(X)\eta(Y) \ \ \forall X, Y \in \chi(M).$$

Hence putting $\alpha = 1$ we obtain that on an n-dimensional Kenmotsu manifold M, S^* is given by $\forall X, Y \in \chi(M)$,

$$S^*(X,Y) = S(X,Y) + (n-2)g(X,Y) + \eta(X)\eta(Y), \tag{2.12}$$

and hence we get the $*$-scalar curvature r^* is given by

$$r^* = r + (n-1)^2. \tag{2.13}$$

Let ψ be a differentiable map from a manifold M into a manifold $\tilde{M}$ and let the dimensions of $M, \tilde{M}$ be m, n respectively $(n > m)$. If at each point p of M, $(\psi_*)_p$ is a 1-1 map, i.e., if rank $\psi = m$, then ψ is called an immersion of M into $\tilde{M}$.

If an immersion ψ is one-one, i.e., if $\psi(p) \neq \psi(q)$ for $p \neq q$, then ψ is called an imbedding of M into $\tilde{M}$.

If the manifolds $M, \tilde{M}$ satisfy the following two conditions, then M is called a submanifold of $\tilde{M}$

(i) $M \subset \tilde{M}$,

(ii) the inclusion map i from M into $\tilde{M}$ is an imbedding of M into $\tilde{M}$.

A submanifold M is called anti-invariant (Yano and Kon, 1977) if $X \in T_x(M) \Rightarrow \phi X \in T_x^{\perp}(M)\ \ \forall x \in M$, where $T_x(M)$, $T_x^{\perp}(M)$ are respectively the tangent space and the normal space at $x \in M$. Thus in an anti-invariant submanifold M, we have $\forall X, Y \in \chi(M)$,

$$g(X, \phi Y) = 0. \tag{2.14}$$

Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connection and a linear connection on an almost contact metric manifold M respectively such that $\forall X, Y \in \chi(M)$,

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{2.15}$$

where H is a (1,1) tensor field. For $\tilde{\nabla}$ to be a quarter symmetric metric connection on M we have (De and Sengupta, 2000),

$$H(X, Y) = \tfrac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X)], \tag{2.16}$$

where $\forall X, Y, Z \in \chi(M)$,

$$g(T'(X, Y), Z) = g(T(X, Y), Z). \tag{2.17}$$

Now, using (1.3) in (2.17) we get

$$T'(X, Y) = g(X, \phi Y)\xi - \eta(X)\phi Y. \tag{2.18}$$

Then, applying (1.3) and (2.18) on (2.16) we obtain

$$H(X, Y) = -\eta(X)\phi Y. \tag{2.19}$$

Hence from (2.15) we have, a quarter symmetric metric connection $\tilde{\nabla}$ on a Kenmotsu manifold M is given by $\forall X, Y \in \chi(M)$,

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{2.20}$$

A vector field V is called conformal Killing vector field if and only if the following relation holds

$$L_V g = 2\kappa g, \tag{2.21}$$

where κ is a function of the co-ordinates. If κ is not a constant, then V is called proper. Also, if κ is a constant, then V is called homothetic vector field and if κ is non-zero, then V is called proper homothetic vector field. If $\kappa = 0$, then V becomes Killing vector field.

3. Main Results. Let $\tilde{M}(\phi, \xi, \eta, g)$ be an n-dimensional Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton given by (1.1). Applying (2.5) on the following equation

$$(L_\xi g)(Y,Z) = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) \quad \forall Y, Z \in \chi(\tilde{M})$$

we get

$$(L_\xi g)(Y,Z) = 2[g(Y,Z) - \eta(Y)\eta(Z)]. \tag{3.1}$$

Now, from (1.1) we have

$$(L_\xi g)(Y,Z) + 2pS^*(Y,Z) + (2\lambda - qr^*)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0. \tag{3.2}$$

Applying (2.12), (2.13) on (3.2) we get

$$\begin{aligned}(L_\xi g)(Y,Z) &+ 2pS(Y,Z) + [2\lambda - q\{r + (n-1)^2\} \\ &+ 2p(n-2)]g(Y,Z) + 2(p+\mu)\eta(Y)\eta(Z) = 0.\end{aligned} \tag{3.3}$$

Using (3.1) in (3.3) we get

$$\begin{aligned}2pS(Y,Z) &+ [2(\lambda+1) - q\{r + (n-1)^2\} + 2p(n-2)]g(Y,Z) \\ &+ 2(p+\mu-1)\eta(Y)\eta(Z) = 0.\end{aligned} \tag{3.4}$$

Putting $Y = Z = \xi$ in (3.4) and then using (2.10) we obtain

$$\lambda + \mu = \frac{q}{2}\{r + (n-1)^2\}. \tag{3.5}$$

Hence we can state that

THEOREM 3.1 *If an n-dimensional Kenmotsu manifold admits a $*$-η-Ricci-Yamabe soliton, then the soliton scalars λ, μ, q are related by the following equation–*

$$\lambda + \mu = \frac{q}{2}\{r + (n-1)^2\}.$$

Let $\{e_i\}_{i=1}^n$ be an orthonormal frame of $T\tilde{M}$, then putting $Y = Z = e_i$ and replacing the value of μ from (3.5) in (3.3) we get

$$\begin{aligned}2\,\mathrm{div}\,(\xi) &+ 2pr + [2\lambda - q\{r + (n-1)^2\} + 2p(n-2)]n \\ &+ 2p + q\{r + (n-1)^2\} - 2\lambda = 0\end{aligned}$$

$$\Rightarrow \quad \lambda = -\frac{\operatorname{div}(\xi)}{n-1} + \left(\frac{q}{2} - \frac{p}{n-1}\right)\{r + (n-1)^2\}, \tag{3.6}$$

where $div(\xi)$ is the divergence of ξ.

Replacing the value of λ from (3.6) in (3.5) we obtain

$$\mu = \frac{1}{n-1}[\operatorname{div}(\xi) + p\{r + (n-1)^2\}]. \tag{3.7}$$

Thus we have the following corollary

COROLLARY 3.1 *If an n-dimensional Kenmotsu manifold admits a $*$-η-Ricci-Yamabe soliton, then the soliton scalars λ, μ, p, q are related by the equations* (3.6) *and* (3.7).

Now, let us consider an m-dimensional anti-invariant submanifold M of an n-dimensional ($n > m$) Kenmotsu manifold $\tilde{M}$ admitting a quarter symmetric metric connection $\tilde{\nabla}$ given by (2.20). Let R, S, r be the Riemannian curvature tensor, Ricci tensor and scalar curvature of M with respect to the Levi-Civita connection ∇ respectively, and $\tilde{R}$, $\tilde{S}$, $\tilde{r}$ be the Riemannian curvature tensor, Ricci tensor and scalar curvature of M with respect to $\tilde{\nabla}$ respectively.

Using (2.4), (2.5), (2.14), (2.20) in

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z \quad \forall X,Y,Z \in \chi(M)$$

we obtain

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X. \tag{3.8}$$

Taking inner product of (3.8) with $W \in \chi(M)$ and applying (2.14) in the resultant equation we get

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W). \tag{3.9}$$

Contracting (3.9) over X and W we obtain

$$\tilde{S}(Y,Z) = S(Y,Z), \tag{3.10}$$

and again contracting (3.10) over Y and Z we obtain

$$\tilde{r} = r. \tag{3.11}$$

Hence we can state that

THEOREM 3.2 *For an anti-invariant submanifold M of a Kenmotsu manifold admitting a quarter symmetric metric connection $\tilde{\nabla}$, we have*

(i) *the Riemannian curvature tensor $\tilde{R}$ of M with respect to $\tilde{\nabla}$ is given by*

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X \quad \forall X,Y,Z \in \chi(M).$$

(ii) $\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) \quad \forall X,Y,Z,W \in \chi(M)$,

(iii) *the Ricci tensors S and $\tilde{S}$ of M with respect to ∇ and $\tilde{\nabla}$ respectively are equal,*

(iv) *the scalar curvatures r and $\tilde{r}$ of M with respect to ∇ and $\tilde{\nabla}$ respectively are equal.*

Now, let M admit a $*$-η-Ricci-Yamabe soliton with respect to $\tilde{\nabla}$. Replacing n by m in (3.3) we get $\forall Y, Z \in \chi(M)$,

$$(L_\xi g)(Y,Z) + 2pS(Y,Z) + [2\lambda - q\{r + (m-1)^2\} \\ +2p(m-2)]g(Y,Z) + 2(p+\mu)\eta(Y)\eta(Z) = 0. \tag{3.12}$$

Similarly, with respect to $\tilde{\nabla}$ we have

$$(\tilde{L}_\xi g)(Y,Z) + 2p\tilde{S}(Y,Z) + [2\lambda - q\{\tilde{r} + (m-1)^2\} \\ +2p(m-2)]g(Y,Z) + 2(p+\mu)\eta(Y)\eta(Z) = 0, \tag{3.13}$$

where $\tilde{L}_\xi$ is the Lie derivative along ξ with respect to $\tilde{\nabla}$.

Using (3.10), (3.11) in (3.13) and comparing the resultant equation with (3.12) we obtain

$$(\tilde{L}_\xi g)(Y,Z) = (L_\xi g)(Y,Z). \tag{3.14}$$

Thus we have the following theorem

THEOREM 3.3 *For an anti-invariant submanifold of a Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton with respect to a quarter symmetric metric connection $\tilde{\nabla}$, the Lie derivatives along ξ with respect to ∇ and $\tilde{\nabla}$ are equal.*

Next, let M be an m-dimensional anti-invariant submanifold of an n-dimensional $(n > m)$ Kenmotsu manifold $\tilde{M}$ admitting a $*$-η-Ricci-Yamabe soliton $(g, V, \lambda, \mu, p, q)$ with respect to a quarter symmetric metric connection $\tilde{\nabla}$ given by (2.20) such that V is a conformal Killing vector field. Then, replacing ξ by V in (3.13) we get $\forall\, Y, Z \in \chi(M)$,

$$(\tilde{L}_V g)(Y,Z) + 2p\tilde{S}(Y,Z) + [2\lambda - q\{\tilde{r} + (m-1)^2\} \\ +2p(m-2)]g(Y,Z) + 2(p+\mu)\eta(Y)\eta(Z) = 0. \tag{3.15}$$

Applying (3.14) (by replacing ξ by V), (2.21), (3.10), (3.11) on (3.15) we obtain

$$[2\kappa + 2\lambda - q\{r + (m-1)^2\} + 2p(m-2)]g(Y,Z) \\ +2pS(Y,Z) + 2(p+\mu)\eta(Y)\eta(Z) = 0. \tag{3.16}$$

Taking $Y = Z = \xi$ in (3.16) and using (2.10) (by replacing n by m) we get

$$[2\kappa + 2\lambda - q\{r + (m-1)^2\} + 2p(m-2)] - 2p(m-1) + 2(p+\mu) = 0$$

$$\Rightarrow \quad \kappa = -(\lambda+\mu) + \frac{q}{2}\{r + (m-1)^2\}. \tag{3.17}$$

Therefore, we have the following theorem

THEOREM 3.4 *Let an m-dimensional anti-invariant submanifold of a Kenmotsu manifold admit a $*$-η-Ricci-Yamabe soliton $(g, V, \lambda, \mu, p, q)$ with respect to a quarter symmetric metric connection $\tilde{\nabla}$. If V is a conformal Killing vector field, then*

$$\kappa = -(\lambda+\mu) + \frac{q}{2}\{r + (m-1)^2\}.$$

Now, putting $q = 0$ in (3.17) we get

$$\kappa = -(\lambda+\mu) \tag{3.18}$$

and hence we can state that

COROLLARY 3.2 *Let an m-dimensional anti-invariant submanifold of a Kenmotsu manifold admit a $*$-η-Ricci soliton with respect to a quarter symmetric metric connection $\tilde{\nabla}$. If V is a conformal Killing vector field, then* (3.18) *holds.*

Again, putting $q = 1$ in (3.17) we have

$$\kappa = -(\lambda + \mu) + \frac{1}{2}\{r + (m-1)^2\} \tag{3.19}$$

and thus we have the following corollary

COROLLARY 3.3 *Let an m-dimensional anti-invariant submanifold of a Kenmotsu manifold admit a $*$-η-Yamabe soliton with respect to a quarter symmetric metric connection $\tilde{\nabla}$. If V is a conformal Killing vector field, then* (3.19) *holds.*

Finally, putting $q = -1$ in (3.17) we obtain

$$\kappa = -(\lambda + \mu) - \frac{1}{2}\{r + (m-1)^2\} \tag{3.20}$$

and hence we have another corollary

COROLLARY 3.4 *Let an m-dimensional anti-invariant submanifold of a Kenmotsu manifold admit a $*$-η-Einstein soliton with respect to a quarter symmetric metric connection $\tilde{\nabla}$. If V is a conformal Killing vector field, then* (3.20) *holds.*

At last, we give the following example of a 3-dimensional Kenmotsu manifold admitting a $*$-η-Ricci-Yamabe soliton and verify the relation (3.5) on it.

EXAMPLE. *Let us consider the 3-dimensional manifold $\tilde{M}$ $=\{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where (x, y, z) are standard co-ordinates of $\mathbb{R}^3$.*

The vector fields $e_1 = z\frac{\partial}{\partial x}$, $e_2 = z\frac{\partial}{\partial y}$, $e_3 = -z\frac{\partial}{\partial z}$ form a linearly independent frame on $\tilde{M}$.

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = 1 \text{ if } i = j \text{ and } g(e_i, e_j) = 0 \text{ if } i \neq j, \quad i, j = 1, 2, 3.$$

Let ϕ be the (1,1) tensor field defined by

$$\phi e_1 = -e_2, \phi e_2 = e_1, \phi e_3 = 0.$$

Hence we have $\phi^2 Z = -Z + \eta(Z)\xi$ and $g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W) \ \ \forall Z, W \in \chi(\tilde{M})$, where η is the 1-form defined by $\eta(Z) = g(Z, e_3) \ \ \forall Z \in \chi(\tilde{M})$ so that $\xi = e_3$ satisfying $\eta(\xi) = 1$.

Let ∇ be the Levi-Civita connection with respect to g, then we have

$$[e_1, e_3] = e_1, [e_2, e_3] = e_2 \text{ and } [e_i, e_j] = 0 \text{ for all other values of } i, j.$$

By Koszul's formula given by $\forall X, Y, Z \in \chi(\tilde{M})$,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &\quad -g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned}$$

we calculate

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = e_1, \\ &\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = -e_3, && \nabla_{e_2} e_3 = e_2, \\ &\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0. \end{aligned}$$

Hence it can be easily verified that $\nabla_X \xi = X - \eta(X)\xi$ and $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \ \ \forall X, Y \in \chi(\tilde{M})$ for $\xi = e_3$.

Therefore, $\tilde{M}(\phi, \xi, \eta, g)$ is a Kenmotsu manifold.

Now, let $\tilde{M}$ admit a $*$-η-Ricci-Yamabe soliton given by (1.1). We now verify the relation (3.5).

Let us take $Y = Z = e_3$. Then, from (3.1) we have

$$(L_\xi g)(e_3, e_3) = 0. \tag{3.21}$$

The Riemannian curvature tensor R is given by $\forall X, Y, Z \in \chi(\tilde{M})$,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Therefore, we calculate

$$\begin{aligned} &R(e_1, e_1)e_1 = 0, && R(e_2, e_1)e_1 = -e_2, && R(e_3, e_1)e_1 = -e_3, \\ &R(e_1, e_2)e_2 = -e_1, && R(e_2, e_2)e_2 = 0, && R(e_3, e_2)e_2 = -e_3, \\ &R(e_1, e_3)e_3 = -e_1, && R(e_2, e_3)e_3 = -e_2, && R(e_3, e_3)e_3 = 0. \end{aligned}$$

From the above relations we obtain

$$S(e_1, e_1) = -2, \ S(e_2, e_2) = -2, \ S(e_3, e_3) = -2. \tag{3.22}$$

Hence the scalar curvature is given by

$$r = \sum_{i=1}^{3} S(e_i, e_i) = -6. \tag{3.23}$$

Now, putting $n = 3$ and $Y = Z = e_3$ in (3.3) we get

$$(L_\xi g)(e_3, e_3) + 2pS(e_3, e_3) + [2\lambda - q\{r + 2^2\} + 2p]g(e_3, e_3) + 2(p + \mu) = 0.$$

Using (3.21), (3.22), (3.23) in the above equation we obtain

$$\lambda + \mu = -q = \frac{q}{2}\{-6 + 2^2\} = \frac{q}{2}\{r + (n-1)^2\}$$

which shows that λ, μ, q satisfy the relation (3.5).

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References

Blair, D.E. (1976) : Contact manifolds in Riemannian geometry, *Lecture Notes in Mathematics*, **509**, Springer-Verlag, Berlin, 1–16.

Catino, G. and **Mazzieri, L.** (2016) : Gradient Einstein solitons, *Nonlinear Anal.*, **132**, 66–94.

Cho, J.T. and **Kimura, M.** (2009) : Ricci solitons and real hypersurfaces in a complex space form, *Tohoku Math. J.*, **61**(2), 205–212.

De, U.C. and **Sengupta, J.** (2000) : Quarter-symmetric metric connection on a Sasakian manifold, *Commun. Fac. Sci. Univ. Ank. Ser. A1, Math. Stat.*, **49**, 7–13.

Dey, S. and **Roy, S.** (2020) : $*$-η-Ricci soliton within the framework of Sasakian manifold, *J. Dyn. Syst. Geom. Theor.*, **18**(2), 163–181.

Golab, S. (1975) : On semi-symmetric and quarter-symmetric linear connections, *Tensor, N.S.*, **29**, 249–254.

Guler, S. and **Crasmareanu, M.** (2019) : Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy, *Turk. J. Math.*, **43**(5), 2631–2641.

Hamilton, R.S. (1982) : Three-manifolds with positive Ricci curvature, *J. Differ. Geom.*, **17**(2), 255–306.

Hamilton, R.S. (1988) : The Ricci flow on surfaces, Mathematics and General Relativity (Santa Cruz, CA, 1986), *Contemporary Mathematics*, **71**, *Am. Math. Soc.*, Providence, RI, 237–262.

Haseeb, A., **Prakasha, D.G.** and **Harish, H.** (2021) : $*$-Conformal η-Ricci solitons on α-cosymplectic manifolds, *Int. J. Anal. Appl.*, **19**(2), 165–179.

Kenmotsu, K. (1972) : A class of contact Riemannian manifolds, *Tohoku Math. J.*, **24**(1), 93–103.

Roy, S., **Dey, S.**, **Bhattacharyya, A.** and **Siddiqi, M.D.** (2021) : $*$-η-Ricci-Yamabe solitons on α-Cosymplectic manifolds with a quarter-symmetric metric connection, *arXiv preprint*, arXiv: 2109.04700.

Siddiqi, M.D. and **Akyol, M.A.** (2020) : η-Ricci-Yamabe soliton on Riemannian submersions from Riemannian manifolds, *arXiv preprint*, arXiv: 2004.14124.

Tachibana, S. (1959) : On almost-analytic vectors in almost-Kahlerian manifolds, *Tohoku Math. J.*, **11**(2), 247–265.

Yano, K. and **Kon, M.** (1977) : Anti-invariant submanifolds of Sasakian space forms I., *Tohoku Math. J.*, **29**(1), 9–23.

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CONTACT CR-SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLDS WITH RESPECT TO QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. The present paper deals with the study of contact CR-submanifolds of trans-Sasakian manifolds with respect to quarter symmetric non-metric connection. We investigate totally geodesic leaves and integrability of the distributions and also study the totally umbilical contact CR-submanifolds of trans-Sasakian manifolds. At last we give an example to verify a relation.

1. Introduction and preliminaries

Trans-Sasakian manifold was introduced by J.A. Oubina in 1985 [15] and the local structure of trans-Sasakian manifolds of dimension $n \geq 5$ was completely characterized by J.C. Marrero [12]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu. Trans-Sasakian structures of type (0,0), (α,0) and (0,β) are cosymplectic, α-Sasakian and β-Kenmotsu respectively.

An n-dimensional (n is odd and $n > 1$) manifold $\tilde{M}$ with the almost contact metric structure (ϕ, ξ, η, g) (where ϕ, ξ, η are tensor fields on $\tilde{M}$ of type (1,1), (1,0), (0,1) respectively and g is a compatible metric with the almost structure) is called a *trans-Sasakian manifold of type* (α, β) if the following equation holds

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{1.1}$$

for smooth functions α and β on $\tilde{M}$ and $\forall X, Y \in \Gamma(T\tilde{M})$. Hence a trans-Sasakian manifold of type (0,0), (1,0) and (0,1) is called *cosymplectic manifold* [18], *Sasakian manifold* [13] and *Kenmotsu manifold* [2] respectively.

On a trans-Sasakian manifold $\tilde{M}$, authors in [12] and [15] have obtained

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{1.2}$$

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also due to the almost contact metric structure we have $\forall X, Y \in \Gamma(T\tilde{M})$,

$$\eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \phi^2(X) = -X + \eta(X)\xi, \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ g(\phi X, Y) = -g(X, \phi Y). \tag{1.4}$$

Let M be an m-dimensional submanifold of an n-dimensional trans-Sasakian manifold $\tilde{M}$ $(m < n)$ with induced metric g and induced connections ∇ and $\nabla^\perp$ on TM and $T^\perp M$ respectively. Then for $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.5}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{1.6}$$

respectively, where h and A_V are second fundamental form and shape operator respectively for the immersion of M satisfying the relation [21]

$$g(h(X, Y), V) = g(A_V X, Y). \tag{1.7}$$

M is *totally umbilical* if $\forall X, Y \in \Gamma(TM)$,

$$h(X, Y) = g(X, Y)H, \tag{1.8}$$

where H is the mean curvature vector on M and M becomes *minimal* if $H = 0$ and *totally geodesic* if $h = 0$.

CR-submanifold was introduced by A. Bejancu [3, 4, 5]. There are several research papers on geometry of CR-submanifolds [9, 10, 11, 13]. Some research papers on CR-submanifolds of trans-Sasakian manifolds are given by [8, 14, 19, 20].

A submanifold M of a trans-Sasakian manifold $\tilde{M}$ is called a *contact CR-submanifold* if ξ is tangent to M and there is a differential distribution D and its orthogonal complementary distribution $D^\perp$ such that [5]
(i) $\phi(D) \subseteq D$ and
(ii) $\phi(D^\perp) \subseteq T^\perp M$,
where D (respectively $D^\perp$) is called horizontal (respectively vertical) distribution.

M is called *ξ-horizontal* (respectively *ξ-vertical*) if $\xi \in D$ (respectively $\xi \in D^\perp$).

Now,

$$TM = D \oplus D^\perp \ \textit{and}\ T^\perp M = \phi(D^\perp) \oplus \mu, \tag{1.9}$$

where μ is a normal sub-bundle invariant to ϕ. For $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$ we write

$$X = PX + QX \tag{1.10}$$

and

$$\phi V = BV + CV, \tag{1.11}$$

where $PX \in D,\ QX \in D^\perp,\ BV = tan(\phi V)$ and $CV = nor(\phi V)$.

A contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ is called a *contact CR-product* [13] if M is locally a Riemannian product of M^T and $M^\perp$, where M^T and $M^\perp$ denote the leaves of D and $D^\perp$ respectively.

Quarter symmetric linear connection on a smooth manifold $\tilde{M}$ introduced by S. Golab [7], is a linear connection $\bar{\nabla}$ such that its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y.$$

If $\phi X = X$ in particular, then it reduces to *semi-symmetric connection* introduced by A. Friedmann and J.A. Shouten [6]. Further, if $(\bar{\nabla}_X g)(Y,Z) \neq 0$ $\forall X,Y,Z \in \chi(\tilde{M})$, then $\bar{\nabla}$ is called a *quarter symmetric non-metric connection.*

M. Ahmad studied CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with quarter symmetric metric connection [1] and T. Pal et al. discussed CR-submanifolds of $(LCS)_n$-manifolds with respect to quarter symmetric non-metric connection [16] in detail.

Motivated from all the research papers mentioned above and specially from the work done by T. Pal et al. [16], in this paper we have studied contact CR-submanifolds of trans-Sasakian manifolds with respect to quarter symmetric non-metric connection. We have investigated totally geodesic leaves and integrability of the distributions and also have studied the totally umbilical contact CR-submanifolds of trans-Sasakian manifolds. At last we have given an example to verify a relation.

Let us consider a linear connection $\bar{\tilde{\nabla}}$ on a trans-Sasakian manifold $\tilde{M}$ by [16]

$$\bar{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X + a(X)\phi Y, \tag{1.12}$$

where a is a 1-form associated to a vector field A on $\tilde{M}$ by

$$g(X,A) = a(X) \tag{1.13}$$

$\forall X,Y \in \chi(\tilde{M})$. If $\bar{\tilde{T}}$ be the torsion tensor of $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$, then from (1.12) we find

$$\bar{\tilde{T}}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y + a(X)\phi Y - a(Y)\phi X. \tag{1.14}$$

Furthermore $\forall X,Y,Z \in \chi(\tilde{M})$,

$$(\bar{\tilde{\nabla}}_X g)(Y,Z) = -\eta(Y)g(\phi X,Z) - \eta(Z)g(\phi X,Y) - 2a(X)g(\phi Y,Z). \tag{1.15}$$

Thus $\bar{\tilde{\nabla}}$ given in (1.12) satisfying (1.14) and (1.15) is a quarter symmetric non-metric connection.

Now, for $\tilde{M}$ with respect to $\bar{\tilde{\nabla}}$ we get,

$$(\bar{\tilde{\nabla}}_X \phi)Y = \alpha g(X,Y)\xi + \beta g(\phi X,Y)\xi + (1-\alpha)\eta(Y)X - \beta\eta(Y)\phi X - \eta(X)\eta(Y)\xi \tag{1.16}$$

and

$$\bar{\tilde{\nabla}}_X \xi = (1-\alpha)\phi X + \beta[X - \eta(X)\xi]. \tag{1.17}$$

We have used all the above equations in the next chapter of the paper.

2. Trans-Sasakian manifolds with respect to quarter symmetric non-metric connection

This chapter consists of three sections but before proving the results of those sections we now first compute the curvature tensor, Ricci tensor and scalar curvature of a trans-Sasakian manifold with respect to the quarter symmetric non-metric connection given in (1.12).

Let with respect to $\bar{\tilde{\nabla}}$ and $\tilde{\nabla}$, the curvature tensors of an n-dimensional trans-Sasakian manifold $\tilde{M}$ of type (α, β) be $\bar{\tilde{R}}$ and $\tilde{R}$ respectively, the Ricci tensors of $\tilde{M}$ be $\bar{\tilde{S}}$ and $\tilde{S}$ respectively and the scalar curvatures of $\tilde{M}$ be $\bar{\tilde{r}}$ and $\tilde{r}$ respectively. Then we obtain $\forall X, Y, Z \in \chi(\tilde{M})$,

$$\bar{\tilde{R}}(X,Y)Z = \tilde{R}(X,Y)Z + da(X,Y)\phi Z + (1-\alpha)[a(Y)\eta(Z)X - a(X)\eta(Z)Y] + \alpha[-\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y - a(X)g(Y,Z)\xi + a(Y)g(X,Z)\xi + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y] + \beta[2\eta(Z)g(\phi X,Y)\xi - a(X)g(\phi Y,Z)\xi + a(Y)g(\phi X,Z)\xi + a(X)\eta(Z)\phi Y - a(Y)\eta(Z)\phi X - g(Y,Z)\phi X + g(X,Z)\phi Y] + a(X)\eta(Y)\eta(Z)\xi - a(Y)\eta(X)\eta(Z)\xi, \tag{2.1}$$

after contraction we obtain,

$$\bar{\tilde{S}}(Y,Z) = \tilde{S}(Y,Z) + da(Y,\phi Z) + [(n-1)(1-\alpha) + \alpha - \lambda\beta - 1]a(Y)\eta(Z) + [-\alpha n + a(\xi)]\eta(Y)\eta(Z) + [\alpha\{1-a(\xi)\} - \lambda\beta]g(Y,Z) + [-\alpha\lambda + \beta\{1-a(\xi)\}]g(\phi Y,Z) + \beta a(\phi Y)\eta(Z) \tag{2.2}$$

and

$$\bar{\tilde{r}} = \tilde{r} + \mu + [(n-1)(1-2\alpha) - 2\lambda\beta]a(\xi) - \lambda\beta(n-1) - \alpha\lambda^2, \tag{2.3}$$

where $\lambda = trace(\phi)$ and $\mu = trace(da)$. Thus we have the following:

Theorem 2.1. *$\bar{\tilde{R}}$, $\bar{\tilde{S}}$ and $\bar{\tilde{r}}$ of an n-dimensional trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)) are given in (2.1), (2.2) and (2.3) respectively.*

2.1. Contact CR-submanifolds of a trans-Sasakian manifold with respect to quarter symmetric non-metric connection.

In this section we state and prove some results regarding a contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ with respect to the quarter symmetric non-metric connection given in (1.12).

Let ∇ be the induced connection on M from the connection $\tilde{\nabla}$ and $\bar{\nabla}$ be the induced connection on M from the connection $\bar{\tilde{\nabla}}$. Let h and $\bar{h}$ be second fundamental forms with respect to ∇ and $\bar{\nabla}$ respectively. Then we have for $X, Y \in \Gamma(TM)$,

$$\bar{\tilde{\nabla}}_X Y = \bar{\nabla}_X Y + \bar{h}(X,Y). \tag{2.4}$$

From (1.5), (1.12) and (2.4) we get

$$\bar{\nabla}_X Y + \bar{h}(X,Y) = \nabla_X Y + h(X,Y) + \eta(Y)\phi X + a(X)\phi Y. \quad (2.5)$$

Using (1.10) in (2.5) we get

$$P\bar{\nabla}_X Y + Q\bar{\nabla}_X Y + \bar{h}(X,Y) = P\nabla_X Y + Q\nabla_X Y + h(X,Y) + \eta(Y)\phi PX + \eta(Y)\phi QX + a(X)\phi PY + a(X)\phi QY. \quad (2.6)$$

Comparing horizontal, vertical and normal parts from both sides of (2.6) we get

$$P\bar{\nabla}_X Y = P\nabla_X Y + \eta(Y)\phi PX + a(X)\phi PY, \quad (2.7)$$

$$Q\bar{\nabla}_X Y = Q\nabla_X Y, \quad (2.8)$$

$$\bar{h}(X,Y) = h(X,Y) + \eta(Y)\phi QX + a(X)\phi QY. \quad (2.9)$$

Now, for $X, Y \in D$ from (2.5) we get

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X + a(X)\phi Y \quad (2.10)$$

and

$$\bar{h}(X,Y) = h(X,Y). \quad (2.11)$$

Hence we have the following:

Theorem 2.2. *If M is a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ admitting $\bar{\tilde{\nabla}}$ (given in (1.12)), then*
(i) the induced connection $\bar{\nabla}$ on M is also a quarter symmetric non-metric connection,
(ii) the second fundamental forms h and $\bar{h}$ are equal.

Again for $X \in TM$ and $V \in T^{\perp}M$ from Weingarten formula for quarter symmetric non-metric connection $\bar{\tilde{\nabla}}$ we have

$$\bar{\tilde{\nabla}}_X V = -\bar{A}_V X + \bar{\nabla}^{\perp}_X V. \quad (2.12)$$

Also from (1.6) and (1.12) we get

$$\bar{\tilde{\nabla}}_X V = -A_V X + \nabla^{\perp}_X V + a(X)\phi V. \quad (2.13)$$

From (2.12) and (2.13) we obtain

$$-\bar{A}_V X + \bar{\nabla}^{\perp}_X V = -A_V X + \nabla^{\perp}_X V + a(X)\phi V. \quad (2.14)$$

Now, for $Z \in D^{\perp}$ we have $\phi Z \in T^{\perp}M$ and hence from (2.14) we obtain

$$-\bar{A}_{\phi Z} X + \bar{\nabla}^{\perp}_X \phi Z = -A_{\phi Z} X + \nabla^{\perp}_X \phi Z + a(X)[-Z + \eta(Z)\xi]$$

from which we get

$$\bar{A}_{\phi Z} X = A_{\phi Z} X - a(X)[-Z + \eta(Z)\xi] \quad (2.15)$$

and

$$\bar{\nabla}^{\perp}_X \phi Z = \nabla^{\perp}_X \phi Z. \quad (2.16)$$

Theorem 2.3. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to the quarter symmetric non-metric connection $\bar{\tilde{\nabla}}$ (given in (1.12)), then $\forall X, Y \in TM$,*

$$P\bar{\nabla}_X\phi PY - P\bar{A}_{\phi QY}X = \phi P(\bar{\nabla}_X Y) + \alpha g(X,Y)P\xi + \beta g(\phi X,Y)P\xi + (1-\alpha)\eta(Y)PX - \beta\eta(Y)\phi PX - \eta(X)\eta(Y)P\xi, \tag{2.17}$$

$$Q\bar{\nabla}_X\phi PY - Q\bar{A}_{\phi QY}X = B\bar{h}(X,Y) + \alpha g(X,Y)Q\xi + (1-\alpha)\eta(Y)QX + \beta g(\phi X,Y)Q\xi - \eta(X)\eta(Y)Q\xi, \tag{2.18}$$

$$\bar{h}(X,\phi PY) + \bar{\nabla}^{\perp}_X\phi QY = \phi Q(\bar{\nabla}_X Y) + C\bar{h}(X,Y) - \beta\eta(Y)\phi QX. \tag{2.19}$$

Proof. From (1.16) we have $\forall X, Y \in TM$,

$$\bar{\tilde{\nabla}}_X\phi Y - \phi(\bar{\tilde{\nabla}}_X Y) = \alpha g(X,Y)\xi + \beta g(\phi X,Y)\xi + (1-\alpha)\eta(Y)X - \beta\eta(Y)\phi X - \eta(X)\eta(Y)\xi. \tag{2.20}$$

Applying (1.10), (1.11), (2.4) and (2.12) on (2.20) we get

$$P\bar{\nabla}_X\phi PY + Q\bar{\nabla}_X\phi PY + \bar{h}(X,\phi PY) - P\bar{A}_{\phi QY}X - Q\bar{A}_{\phi QY}X + \bar{\nabla}^{\perp}_X\phi QY - \phi P(\bar{\nabla}_X Y) - \phi Q(\bar{\nabla}_X Y) - B\bar{h}(X,Y) - C\bar{h}(X,Y) = \alpha g(X,Y)P\xi + \alpha g(X,Y)Q\xi + \beta g(\phi X,Y)P\xi + \beta g(\phi X,Y)Q\xi + (1-\alpha)\eta(Y)PX + (1-\alpha)\eta(Y)QX - \beta\eta(Y)\phi PX - \beta\eta(Y)\phi QX - \eta(X)\eta(Y)P\xi - \eta(X)\eta(Y)Q\xi. \tag{2.21}$$

Equating horizontal, vertical and normal parts from both sides of (2.21) we get (2.17), (2.18) and (2.19) respectively.

□

2.2. **Totally geodesic leaves and integrability of the distributions.** In this section we obtain the necessary and sufficient conditions of integrability of the distributions D and $D^{\perp}$ of a contact CR-submanifold M of a trans-Sasakian manifold $\tilde{M}$ in different cases. We also discuss the cases where the leaves of D and $D^{\perp}$ are totally geodesic.

Lemma 2.4. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)). Then $\forall Z, W \in D^{\perp}$,*

$$\phi P[W,Z] = A_{\phi W}Z - A_{\phi Z}W - a(Z)[-W + \eta(W)\xi] + a(W)[-Z + \eta(Z)\xi] + (\alpha-1)[\eta(Z)W - \eta(W)Z].$$

Proof. $\forall Z, W \in D^{\perp}$, $\bar{\tilde{\nabla}}_Z\phi W = (\bar{\tilde{\nabla}}_Z\phi)W + \phi(\bar{\tilde{\nabla}}_Z W)$.

Using (1.10), (1.11), (1.16), (2.4) and (2.12) in the above equation we get

$$\bar{\tilde{\nabla}}^{\perp}_Z\phi W = \bar{A}_{\phi W}Z + \phi P(\bar{\nabla}_Z W) + \phi Q(\bar{\nabla}_Z W) + B\bar{h}(W,Z) + C\bar{h}(W,Z) + \alpha g(W,Z)\xi + (1-\alpha)\eta(W)Z - \beta\eta(W)\phi Z - \eta(Z)\eta(W)\xi. \tag{2.22}$$

Also from (2.19) we get

$$\bar{\bar{\nabla}}^{\perp}_{Z}\phi W = \phi Q(\bar{\nabla}_Z W) + C\bar{h}(W,Z) - \beta\eta(W)\phi Z. \tag{2.23}$$

Using (2.23) in (2.22) we obtain

$$\phi P(\bar{\nabla}_Z W) = -\bar{A}_{\phi W}Z - B\bar{h}(W,Z) - \alpha g(W,Z)\xi + (\alpha-1)\eta(W)Z + \eta(Z)\eta(W)\xi. \tag{2.24}$$

Interchanging Z and W in (2.24) and subtracting (2.24) from the resultant equation we get

$$\phi P[W,Z] = \bar{A}_{\phi W}Z - \bar{A}_{\phi Z}W + (\alpha-1)[\eta(Z)W - \eta(W)Z].$$

Using (2.15) in the above equation we obtain

$$\phi P[W,Z] = A_{\phi W}Z - A_{\phi Z}W - a(Z)[-W + \eta(W)\xi] + a(W)[-Z + \eta(Z)\xi]$$
$$+ (\alpha-1)[\eta(Z)W - \eta(W)Z].$$

□

Theorem 2.5. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*
$A_{\phi W}Z - A_{\phi Z}W = a(Z)[-W + \eta(W)\xi] - a(W)[-Z + \eta(Z)\xi] + (\alpha-1)[\eta(W)Z - \eta(Z)W]$.

Proof. It is obvious from Lemma 2.2.1.

□

Corollary 2.6. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). Then the distribution $D^{\perp}$ is integrable if and only if*

$$\forall Z, W \in D^{\perp},\ A_{\phi W}Z - A_{\phi Z}W = a(W)Z - a(Z)W.$$

Remark 2.7. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\tilde{\nabla}$. Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*

$$A_{\phi W}Z - A_{\phi Z}W = \alpha[\eta(W)Z - \eta(Z)W].$$

Remark 2.8. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\tilde{\nabla}$. Then the distribution $D^{\perp}$ is integrable if and only if $\forall Z, W \in D^{\perp}$,*

$$A_{\phi W}Z = A_{\phi Z}W.$$

Theorem 2.9. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). Then the distribution D is integrable if and only if $\forall X, Y \in D$,*

$$h(X,\phi Y) = h(\phi X, Y).$$

Proof. From (2.11) and (2.19) we have $\forall X, Y \in D$,

$$\phi Q(\bar{\nabla}_X Y) = h(X,\phi Y) - Ch(X,Y). \tag{2.25}$$

Interchanging X and Y in (2.25) and subtracting the resultant equation from (2.25) we get

$$\phi Q[X,Y] = h(X,\phi Y) - h(Y,\phi X).$$

Hence the distribution D is integrable if and only if $\forall X,Y \in D$, $h(X,\phi Y) = h(\phi X, Y)$.

□

Remark 2.10. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\tilde{\nabla}$. Then the distribution D is integrable if and only if $\forall X,Y \in D$, $h(X,\phi Y) = h(\phi X, Y)$.*

Theorem 2.11. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)). If the leaf of D is totally geodesic in M, then $\forall X,Y \in D, Z \in D^\perp$,*
$-g(h(X,Y),\phi Z) + (\alpha-1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X,Y) + \eta(X)\eta(Y)\eta(Z) = 0.$

Proof. As the leaf of D is totally geodesic in M, $\bar{\nabla}_X \phi Y \in D$ $\forall X,Y \in D$(since $\phi Y \in D$). Now, $\forall Z \in D^\perp$ from (2.21) we have
$$\phi P(\bar{\nabla}_X Z) = -\bar{A}_{\phi Z}X + \nabla^\perp_X \phi Z - \phi Q(\bar{\nabla}_X Z) - \phi\bar{h}(X,Z) + (\alpha-1)\eta(Z)X + \beta\eta(Z)\phi X + \eta(X)\eta(Z)\xi. \tag{2.26}$$

Using (1.4), (1.7), (1.10), (2.11) and (2.26) we get
$0 = g(\bar{\nabla}_X \phi Y, Z) = -g(\phi Y, \bar{\nabla}_X Z) = -g(\phi Y, P(\bar{\nabla}_X Z)) = g(Y, \phi P(\bar{\nabla}_X Z))$
$= -g(\bar{A}_{\phi Z}X, Y) + (\alpha-1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X, Y) + \eta(X)\eta(Z)\eta(Y)$
$= -g(h(X,Y),\phi Z) + (\alpha-1)\eta(Z)g(X,Y) + \beta\eta(Z)g(\phi X,Y) + \eta(X)\eta(Y)\eta(Z).$

□

Corollary 2.12. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)), then the leaf of D is totally geodesic in M if and only if $\forall X,Y \in D, Z \in D^\perp$,*
$$g(h(X,Y),\phi Z) = 0.$$

Proof. The direct part follows from Theorem 2.2.3.

Conversely, $\forall X,Y \in D, Z \in D^\perp$ (since $\phi Y \in D$),
$0 = g(h(X,\phi Y),\phi Z) = g(\bar{\tilde{\nabla}}_X \phi Y, \phi Z) = g(\phi\bar{\tilde{\nabla}}_X Y, \phi Z) = g(\bar{\tilde{\nabla}}_X Y, Z) = g(\bar{\nabla}_X Y, Z)$
which implies that $\forall X,Y \in D$, $\bar{\nabla}_X Y \in D$. Hence the leaf of D is totally geodesic in M.

□

Theorem 2.13. *Let M be a contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\tilde{\nabla}}$ (given in (1.12)). If the leaf of $D^\perp$ is totally geodesic in M, then $\forall Z,W \in D^\perp, X \in D$,*
$g(h(X,Z),\phi W) + a(X)g(Z,W) + \alpha\eta(X)g(Z,W) - a(X)\eta(Z)\eta(W) - \eta(X)\eta(Z)\eta(W) = 0.$

Proof. As the leaf of $D^\perp$ is totally geodesic in M, $\forall Z,W \in D^\perp, \bar{\nabla}_Z W \in D^\perp$. Now from (2.21) we have
$\phi P(\bar{\nabla}_Z W) = -\bar{A}_{\phi W}Z + \bar{\nabla}^\perp_Z \phi W - \phi Q(\bar{\nabla}_Z W) - \phi\bar{h}(Z,W) + (\alpha-1)\eta(W)Z$

$+\beta\eta(W)\phi Z+\eta(Z)\eta(W)\xi-\alpha g(Z,W)\xi.$ (2.27)

Taking inner product of (2.27) with $X \in D$ we get (since $\bar{\nabla}_Z W \in D^\perp$)
$0 = g(\phi P(\bar{\nabla}_Z W), X) = -g(\bar{A}_{\phi W} Z, X) - \alpha g(Z,W)\eta(X) + \eta(Z)\eta(W)\eta(X).$(2.28)

Using (1.4), (1.7) and $\bar{h}(X,Z) = h(X,Z) + a(X)\phi Z$ (obtained from (2.9)) in (2.28) we get
$0 = -[g(h(X,Z),\phi W) + a(X)g(Z,W) - a(X)\eta(Z)\eta(W)] - \alpha\eta(X)g(Z,W)$
$+ \eta(X)\eta(Z)\eta(W).$

□

Corollary 2.14. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). If the leaf of $D^\perp$ is totally geodesic in M, then $\forall Z, W \in D^\perp, X \in D$,*

$$g(h(X,Z),\phi W) + a(X)g(Z,W) + \alpha\eta(X)g(Z,W) = 0.$$

Corollary 2.15. *Let M be a ξ-vertical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). If the leaf of $D^\perp$ is totally geodesic in M, then $\forall Z, W \in D^\perp, X \in D$,*

$$g(h(X,Z),\phi W) + a(X)g(Z,W) - a(X)\eta(Z)\eta(W) = 0.$$

Theorem 2.16. *Let M be a ξ-horizontal contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\bar{\bar{\nabla}}$ (given in (1.12)). If M is a contact CR-product, then $\forall X \in D, W \in D^\perp$,*

$$A_{\phi W} X + a(X)W + \alpha\eta(X)W = 0.$$

Proof. As the leaf of $D^\perp$ is totally geodesic in M, from Corollary 2.2.3 we have $\forall Z, W \in D^\perp, X \in D$,

$$g(A_{\phi W} X + a(X)W + \alpha\eta(X)W, Z) = 0$$

which implies that

$$A_{\phi W} X + a(X)W + \alpha\eta(X)W \in D. \tag{2.29}$$

Again as the leaf of D is totally geodesic in M and $\forall Y \in D$, $\phi Y \in D$, we have $\bar{\nabla}_X \phi Y \in D$. Hence we get
$g(A_{\phi W} X + a(X)W + \alpha\eta(X)W, Y) = g(A_{\phi W} X, Y) = g(h(X,Y),\phi W)$
$= -g(\phi h(X,Y), W) = -g(\phi(\bar{\bar{\nabla}}_X Y - \bar{\nabla}_X Y), W) = -g(\phi(\bar{\bar{\nabla}}_X Y), W)$
$= -g(\bar{\bar{\nabla}}_X \phi Y, W) = -g(\bar{\nabla}_X \phi Y, W) = 0$ (using (1.4), (1.7), (1.16), (2.4), (2.11))
which implies that $A_{\phi W} X+a(X)W+\alpha\eta(X)W \in D^\perp$. (2.30)

From (2.29) and (2.30) we obtain $A_{\phi W} X + a(X)W + \alpha\eta(X)W = 0$.

□

2.3. **Totally umbilical contact CR-submanifolds.** In this section we study totally umbilical contact CR-submanifolds of a trans-Sasakian manifold with respect to $\tilde{\nabla}$.

Let M be a totally umbilical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\tilde{\nabla}$, then from (1.1) we have $\forall Z,W \in D^{\perp}$,

$$\tilde{\nabla}_Z \phi W - \phi(\tilde{\nabla}_Z W) = \alpha[g(Z,W)\xi - \eta(W)Z] - \beta\eta(W)\phi Z. \quad (2.31)$$

Using (1.5), (1.6) and (1.10) in (2.31) we get

$$-A_{\phi W}Z + \nabla^{\perp}_Z \phi W = \phi P(\nabla_Z W) + \phi Q(\nabla_Z W) + \phi h(Z,W) + \alpha[g(Z,W)\xi - \eta(W)Z] - \beta\eta(W)\phi Z. \quad (2.32)$$

Taking inner product of (2.32) with Z and using (1.7) we get

$$-g(h(Z,Z),\phi W) = g(\phi h(Z,W),Z) + \alpha[g(Z,W)\eta(Z) - \eta(W)\|Z\|^2]. \quad (2.33)$$

Using (1.8) in (2.33) we get

$$g(H,\phi W) = -\frac{1}{\|Z\|^2}[g(Z,W)g(\phi H,Z) + \alpha\{g(Z,W)\eta(Z) - \eta(W)\|Z\|^2\}]. \quad (2.34)$$

Interchanging Z and W in (2.34) and using (1.4) we get

$$g(\phi H,Z) = \frac{1}{\|W\|^2}[-g(Z,W)g(H,\phi W) + \alpha\{g(Z,W)\eta(W) - \eta(Z)\|W\|^2\}]. \quad (2.35)$$

Substituting (2.35) in (2.34) we obtain

$$g(H,\phi W) = -\tfrac{g(Z,W)}{\|Z\|^2}\Big[-\tfrac{1}{\|W\|^2}g(Z,W)g(H,\phi W) + \tfrac{\alpha}{\|W\|^2}\{g(Z,W)\eta(W) - \eta(Z)\|W\|^2\}\Big] - \tfrac{\alpha}{\|Z\|^2}\{g(Z,W)\eta(Z) - \eta(W)\|Z\|^2\}$$

$$\Rightarrow \Big[1 - \tfrac{g(Z,W)^2}{\|Z\|^2\|W\|^2}\Big]g(H,\phi W) + \alpha\tfrac{g(Z,W)}{\|Z\|^2}\Big[\tfrac{\eta(W)g(Z,W)}{\|W\|^2} - \eta(Z)\Big] + \alpha\Big[\tfrac{\eta(Z)g(Z,W)}{\|Z\|^2} - \eta(W)\Big] = 0. \quad (2.36)$$

Hence from (2.36) we get the following theorems:

Theorem 2.17. *Let M be a ξ-vertical totally umbilical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect $\tilde{\nabla}$, then dim$D^{\perp}$=1.*

Theorem 2.18. *Let M be a ξ-horizontal totally umbilical contact CR-submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to $\tilde{\nabla}$, then*
(i) M is minimal in $\tilde{M}$
or (ii) dim$D^{\perp}$=1
or (iii) $H \in \Gamma(\mu)$.

Remark 2.19. *Theorem 2.3.2 also holds good in case of considering $\tilde{M}$ with respect to the quarter symmetric non-metric connection $\bar{\tilde{\nabla}}$ (given in (1.12)).*

3. Example

Here we give an example of a 3-dimensional trans-Sasakian manifold from [17] and then verify the relation (2.1).

Let us consider a 3-dimensional manifold $\tilde{M} = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$, where (x,y,z) are the standard co-ordinates of $\mathbb{R}^3$. Let the vector fields

$$E_1 = e^{-2z}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right),\ E_2 = -e^{-2z}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right),\ E_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 0, & for\ i \neq j \\ 1, & for\ i = j \end{cases}$$

and η be the 1-form defined by $\eta(X) = g(X, E_3)\ \forall X \in \chi(\tilde{M})$. Let ϕ be the (1,1) tensor field defined by $\phi E_1 = E_2,\ \phi E_2 = -E_1,\ \phi E_3 = 0$. Then we have $\eta(E_3) = 1,\ \phi^2(X) = -X + \eta(X)E_3,\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\ \forall X, Y \in \chi(\tilde{M})$. Let $\tilde{\nabla}$ be the Riemannian connection on $\tilde{M}$ with respect to the metric g. Then we obtain $[E_1, E_2] = 0,\ [E_1, E_3] = 2E_1,\ [E_2, E_3] = 2E_2$. Now, using Koszul's formula for g, it can be calculated that

$$\tilde{\nabla}_{E_1}E_1 = -2E_3,\ \tilde{\nabla}_{E_1}E_2 = 0,\ \tilde{\nabla}_{E_1}E_3 = 2E_1,$$
$$\tilde{\nabla}_{E_2}E_1 = 0,\ \tilde{\nabla}_{E_2}E_2 = -2E_3,\ \tilde{\nabla}_{E_2}E_3 = 2E_2,$$
$$\tilde{\nabla}_{E_3}E_1 = 0,\ \tilde{\nabla}_{E_3}E_2 = 0,\ \tilde{\nabla}_{E_3}E_3 = 0.$$

Since $\{E_1, E_2, E_3\}$ forms a basis for $\tilde{M}$, then any vector field $X, Y \in \chi(\tilde{M})$ can be written as

$$X = x_1E_1 + x_2E_2 + x_3E_3,\ Y = y_1E_1 + y_2E_2 + y_3E_3,$$

where $x_i, y_i \in \mathbb{R},\ i = 1, 2, 3$. Hence $g(X, Y) = x_1y_1 + x_2y_2 + x_3y_3$.

Thus $\tilde{\nabla}_X Y = 2x_1y_3E_1 + 2x_2y_3E_2 - 2(x_1y_1 + x_2y_2)E_3. \quad (3.1)$

Therefore $\tilde{\nabla}_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi]\ \forall X \in \chi(\tilde{M})$ holds for $\alpha = 0,\ \beta = 2$ and $\xi = E_3$. Thus $(\tilde{M}, g)$ is a 3-dimensional trans-Sasakian manifold of type (0,2).

We set $A = E_1$. Then $a(X) = g(X, E_1) = x_1\ \forall X = x_1E_1 + x_2E_2 + x_3E_3 \in \chi(\tilde{M})$. Hence using (3.1) in (1.12) we get

$$\bar{\tilde{\nabla}}_X Y = (2x_1y_3 - x_2y_3 - x_1y_2)E_1 + (2x_2y_3 + x_1y_3 + x_1y_1)E_2 - 2(x_1y_1 + x_2y_2)E_3. \quad (3.2)$$

Also for $Z = z_1E_1 + z_2E_2 + z_3E_3 \in \chi(\tilde{M})$ we have

$$(\bar{\tilde{\nabla}}_X g)(Y, Z) = x_2y_3z_1 - x_1y_3z_2 + (x_2y_1 - x_1y_2)z_3 \neq 0.$$

Hence $\bar{\tilde{\nabla}}$(given by (3.2)) is a quarter symmetric non-metric connection on $\tilde{M}$.

Now, we will verify the relation (2.1) for $X = E_1, Y = E_2, Z = E_2$.

Using the values of $\tilde{\nabla}_{E_i}E_j\ (i, j = 1, 2, 3)$ given above, we obtain

$$\tilde{R}(E_1, E_2)E_2 = -4E_1, \quad (3.3)$$

and using (3.2) we obtain $\bar{\tilde{R}}(E_1, E_2)E_2 = -4E_1 - 2E_2. \quad (3.4)$

Now, from (2.1) we get $\bar{\tilde{R}}(E_1, E_2)E_2 = \tilde{R}(E_1, E_2)E_2 - 2E_2$ which is satisfied by (3.3) and (3.4). Hence the relation (2.1) holds for $X = E_1, Y = E_2, Z = E_2$. Similarly we can prove that the relation (2.1) holds for other values of $X, Y, Z \in \chi(\tilde{M})$.

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References

1. M. Ahmad, *CR-submanifolds of a Lorentzian para-Sasakian manifold endowed with quarter symmetric metric connection*, Bull. Korean Math. Soc. **49**(2012), 25-32.
2. M. Atceken and S. Dirik, *On the geometry of pseudo-slant submanifolds of a Kenmotsu manifold*, Gulf J. Math. **2(2)**(2014), 51-66.
3. A. Bejancu, *CR-submanifolds of a Kaehler manifold I*, Proc. Amer. Math. Soc. **69**(1978), 135-142.
4. A. Bejancu, *CR-submanifolds of a Kaehler manifold II*, Trans. Amer. Math. Soc. **250**(1979), 333-345.
5. A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Publ. Co., Dordrecht, Holland, 1986.
6. A. Friedmann, *Uber die geometric derhalbsymmetrischen Ubertragungen*, Math. Z. **21(1)**(1924), 211-223.
7. S. Golab, *On semi-symmetric and quarter symmetric linear connections*, Tensor (N.S.) **29**(1975), 249-254.
8. K.A. Khan, V.A. Khan and Sirajuddin, *Warped product contact CR-submanifolds of trans-Sasakian manifolds*, Filomat **21(2)**(2007), 55-62.
9. B. Laha and A. Bhattacharyya, *On CR-submanifolds of* (ϵ)*-paracontact Sasakian manifold*, Int. J. Pure Appl. Math. **94(3)**(2014), 403-417.
10. B. Laha, B. Das and A. Bhattacharyya, *Contact CR-submanifolds of an Indefinite Lorentzian para-Sasakian manifold*, Acta Univ. Sapientiae Math. **5(2)**(2013), 157-168.
11. B. Laha, B. Das and A. Bhattacharyya, *Submanifolds of some indefinite contact and paracontact manifolds*, Gulf J. Math. **1(2)**(2013), 180-188.
12. J.C. Marrero, *The local structure of trans-Sasakian manifolds*, Ann. Mat. Pura Appl. **162(1)**(1992), 77-86.
13. K. Matsumoto, *On contact CR-submanifolds of Sasakian manifolds*, Int. J. Math. Math. Sci. **6(2)**(1983), 313-326.

14. M.I. Munteanu, *A note on doubly warped product contact CR-submanifolds in trans-Sasakian manifolds*, arXiv preprint math/0604008, 2006.

15. J.A. Oubina, *New classes of almost contact metric structures*, Publ. Math. Debrecen **32**(1985), 187-193.

16. T. Pal, M.H. Shahid and S.K. Hui, *CR-submanifolds of* $(LCS)_n$*-manifolds with respect to quarter symmetric non-metric connection*, Filomat **33(11)**(2019), 3337-3349.

17. G.P. Pokhariyal, S. Yadav and S.K. Chaubey, *Ricci solitons on trans-Sasakian manifolds*, Differ. Geom. Dyn. Syst. **20**(2018), 138-158.

18. R. Prasad and S. Kumar, *Semi-slant Riemannian maps from cosymplectic manifolds into Riemannian manifolds*, Gulf J. Math. **9(1)**(2020), 62-80.

19. M.H. Shahid, *CR-submanifolds of a trans-Sasakian manifold*, Indian J. Pure Appl. Math. **22(12)**(1991), 1007-1012.

20. M.H. Shahid, *CR-submanifolds of a trans-Sasakian manifold II*, Indian J. Pure Appl. Math. **25(3)**(1994), 299-307.

21. K. Yano and M. Kon, *Structure on manifolds*, World Sci. Publ., 1984.

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HEMI-SLANT SUBMANIFOLD OF $(LCS)_n$-MANIFOLD

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Abstract. In this paper, we analyse briefly some properties of hemi-slant submanifold of $(LCS)_n$-manifold. Here we discuss about some necessary and sufficient conditions for distributions to be integrable and obtain some results in this direction. We also study the geometry of leaves of hemi-slant submanifold of $(LCS)_n$-manifold. At last, we give an example of a hemi-slant submanifold of an $(LCS)_n$-manifold.

Key words and Phrases: $(LCS)_n$-manifold, hemi-slant submanifold, integrablity, leaves of distribution.

1. INTRODUCTION

An n-dimensional Lorentzian manifold $\tilde{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $\tilde{g}$, that is $\tilde{M}$ admits a smooth symmetric tensor field $\tilde{g}$ of type (0,2) such that for each point. the tensor $\tilde{g}_p$: $T_p\tilde{M} \times T_p\tilde{M} \to \mathbb{R}$ is a non-degenerate inner-product of signature (-,+,...,+), $T_p\tilde{M}$ denotes the tangent vector space of $\tilde{M}$ at p and $\mathbb{R}$ is the real no. space. A non-zero vector $X_p \in T_p\tilde{M}$ is known to be spacelike, null or lightlike, or timelike according as $\tilde{g}_p(X_p, X_p) > 0, = 0$ or < 0 respectively.

If $\tilde{M}$ is a differentiable manifold of dimension n, and there exists a (ϕ, ξ, η) structure satisfying

$$\phi^2 = I + \eta \otimes \xi, \eta(\xi) = -1, \phi(\xi) = 0, \eta \circ \phi = 0,$$

then M is called an almost paracontact manifold.

In an almost paracontact structure $(\phi, \xi, \eta, \tilde{g})$,

$$\tilde{g}(X, \phi Y) = \tilde{g}(\phi X, Y),$$

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$$2g(\phi X,Y)=(\bar{\nabla}_X\eta)Y+(\bar{\nabla}_Y\eta)X,$$
$$\phi^2X=X+\eta(X)\xi,\eta\circ\phi=0,\phi(\xi)=0,\eta(\xi)=-1, \tag{1.1}$$

where ϕ is a tensor of type (1,1), ξ is a vector field, η is a 1-form and $\tilde{g}$ is Lorentzian metric satisfying

$$\tilde{g}(\phi X,\phi Y)=\tilde{g}(X,Y)+\eta(X)\eta(Y),\tilde{g}(X,\xi)=\eta(X) \tag{1.2}$$

for all vector fields X,Y on $\tilde{M}$.

In a Lorentzian manifold $(\tilde{M},\tilde{g})$, a vector field P defined by $\tilde{g}(X,P)=A(X)$ for any $X\in\Gamma(T\tilde{M})$, is called con-circular if

$$(\bar{\nabla}_XA)(Y)=\alpha\{\tilde{g}(X,Y)+\omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form and $\bar{\nabla}$ denotes the operator of covariant differentiation of $\tilde{M}$ with respect to $\tilde{g}$.

Let $\tilde{M}$ admits a unit timelike concircular vector field ξ, called the structure vector field of the manifold, then $\tilde{g}(\xi,\xi)=-1$, since ξ is a unit concircular vector field, it follows that $\exists$ a non-zero 1-form η such that $\tilde{g}(X,\xi)=\eta(X)$. The following equations hold−

$$(\bar{\nabla}_X\eta)Y=\alpha[\tilde{g}(X,Y)+\eta(X)\eta(Y)],\ \alpha\neq 0,$$
$$\bar{\nabla}_X\alpha=X\alpha=d\alpha(X)=\rho\eta(X),$$

for all vector fields X,Y on $\tilde{M}$ and α is a non-zero scalar function related to ρ, by $\rho=-(\xi\alpha)$.

Let $\phi X=\frac{1}{\alpha}\bar{\nabla}_X\xi$, from which it follows that ϕ is a symmetric (1,1) tensor and call it the structure tensor on the manifold. Thus the Lorentzian manifold $\tilde{M}$ together with unit timelike concircular vector field ξ, its associated 1-form η and a (1,1) tensor field ϕ is called a Lorentzian Concircular Structure manifold i.e. $(LCS)_n$-manifold. Specially, if $\alpha=1$, then we obtain LP-Sasakian structure of Matsumoto [15]. In an $(LCS)_n$-manifold $(n>2)$, the following relations hold−

$$\phi^2=I+\eta\otimes\xi,\eta(\xi)=-1,$$

where I denotes the identity transformation of the tangent space $T\tilde{M}$,

$$\phi\xi=0,\eta\circ\phi=0,\tilde{g}(X,\phi Y)=\tilde{g}(\phi X,Y),rank\phi=2n, \tag{1.3}$$

$$\tilde{g}(\phi X,\phi Y)=\tilde{g}(X,Y)+\eta(X)\eta(Y),\tilde{g}(X,\xi)=\eta(X), \tag{1.4}$$

$$\bar{R}(X,Y)\xi=(\alpha^2-\rho)[\eta(Y)X-\eta(X)Y] \tag{1.5}$$

$\forall\ X,Y\in T\tilde{M}$.

Also $(LCS)_n$-manifold satisfies−

$$(\bar{\nabla}_X\phi)Y=\alpha[\tilde{g}(X,Y)\xi+2\eta(X)\eta(Y)\xi+\eta(Y)X], \tag{1.6}$$

$$\bar{\nabla}_X\xi=\alpha\phi X. \tag{1.7}$$

Let M be a submanifold of $\tilde{M}$ with $(LCS)_n$-structure $(\phi, \xi, \eta, \tilde{g})$ with induced metric g and let ∇ is the induced connection on the tangent bundle TM and $\nabla^\perp$ is the induced connection on the normal bundle $T^\perp M$ of M.

The Gauss and Weingarten formulae are characterized by−

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.8}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{1.9}$$

$\forall\ X, Y \in TM, N \in T^\perp M, h$ is the 2nd fundamental form and A_N is the Weingarten mapping associated with N via

$$g(A_N X, Y) = g(h(X, Y), N). \tag{1.10}$$

The mean curvature H is given by

$$H = \frac{1}{k}\sum_{i=1}^{k} h(e_i, e_i), \tag{1.11}$$

where k is the dimension of M and $\{e_i\}_{i=1}^k$ is the local orthonormal frame on M.

For any $X \in \Gamma(TM)$,

$$\phi X = TX + FX, \tag{1.12}$$

where TX is the tangential component and FX is the normal component of ϕX.

Similarly, for any $V \in \Gamma(T^\perp M)$,

$$\phi V = tV + fV, \tag{1.13}$$

where tV, fV are the tangential component and the normal component of ϕV respectively.

The covariant derivatives of the tensor fields T, F, t, f are defined as−

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{1.14}$$

$$(\nabla_X F)Y = \nabla^\perp_X FY - F\nabla_X Y, \tag{1.15}$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla^\perp_X V, \tag{1.16}$$

$$(\nabla_X f)V = \nabla^\perp_X fV - f\nabla^\perp_X V \tag{1.17}$$

$\forall\ X, Y \in TM, V \in T^\perp M$.

A submanifold is called−
i) invariant if $\forall\ X \in \Gamma(TM), \phi X \in \Gamma(TM)$,
ii) anti-invariant if $\forall\ X \in \Gamma(TM), \phi X \in \Gamma(T^\perp M)$,
iii) totally umbilical if $h(X, Y) = g(X, Y)H$ (1.18)
$\forall\ X, Y \in \Gamma(TM), H$ is the mean curvature,
iv) totaly geodesic if $h(X, Y) = 0\ \forall\ X, Y \in \Gamma(TM)$,
v) minimal if $H = 0$ on M.

Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ and ξ be tangent to M. Then the tangent bundle TM decomposes as $TM = D\oplus < \xi >$, where D is the orthogonal distribution to ξ. Now for each non-zero vector X tangent to M at x, such that X is not proportional to ξ_x, we denote the angle between ϕX and D_x by $\theta(X)$. M is called slant submanifold if the angle $\theta(X)$ is constant, which is independent of the choice of $x \in M$ and $X \in T_xM- < \xi_x >$. The constant angle $\theta \in [0, \frac{\pi}{2}]$ is then called the slant angle of M in $\tilde{M}$. If $\theta = 0$, then the submanifold is invariant, if $\theta = \frac{\pi}{2}$, then the submanifold is anti-invariant and if $\theta \neq 0, \frac{\pi}{2}$, then the submanifold is proper slant.

According to A. Lotta [9], when M is a proper slant submanifold of $\tilde{M}$ with slant angle θ, then $\forall\ X \in \Gamma(TM)$,

$$T^2(X) = -cos^2\theta(X - \eta(X)\xi). \tag{1.19}$$

A. Carriazo [3] introduced hemi-slant submanifolds as a special case of bislant submanifolds and he called them pseudo-slant submanifolds.

A submanifold M of an $(LCS)_n$-manifold is called hemi-slant if there exist two orthogonal distributions D^θ and $D^\perp$ satisfying [5]−
i) $TM = D^\theta \oplus D^\perp \oplus < \xi >$,
ii) D^θ is a slant distribution with slant angle $\theta \neq \frac{\pi}{2}$,
iii) $D^\perp$ is totally real i.e., $\phi D^\perp \subseteq T^\perp M$.

A hemi-slant submanifold is called proper if $\theta \neq 0, \frac{\pi}{2}$.

CR-submanifolds and slant submanifolds are hemi-slant submanifolds with slant angle $\theta = \frac{\pi}{2}$ and $D^\theta = 0$ respectively.

In the rest of this paper, we use M as a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. If we denote the dimensions of the distributions $D^\perp$ and D^θ by m_1, m_2 respectively, then we have−
i) if $m_2 = 0$, then M is anti-invariant,
ii) if $m_1 = 0, \theta = 0$, then M is invariant,
iii) if $m_1 = 0, \theta \neq 0$, then M is proper-slant with slant angle θ,
iv) if $m_1 m_2 \neq 0, \theta \in (0, \frac{\pi}{2})$, then M is proper hemi-slant.

Let M be hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$, then for any $X \in TM$,

$$X = P_1X + P_2X + \eta(X)\xi, \tag{1.20}$$

where P_1, P_2 are projection maps on the distributions $D^\perp, D^\theta$ respectively. Now operating ϕ on (1.20), we get

$$\phi X = \phi P_1X + \phi P_2X + \eta(X)\phi\xi.$$

Using (1.1) and (1.12), we obtain

$$TX + FX = FP_1X + TP_2X + FP_2X.$$

On comparing, we get

$$TX = TP_2X,$$
$$FX = FP_1X + FP_2X.$$

If we denote the orthogonal complement of $\phi(TM)$ in $T^\perp M$ by μ, then the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = F(D^\perp) \oplus F(D^\theta) \oplus < \mu > . \tag{1.21}$$

Since $F(D^\perp)$ and $F(D^\theta)$ are orthogonal distributions, $g(X, Y) = 0$ for each $X \in D^\perp$ and $Y \in D^\theta$. Hence by (1.5) and (1.12), we have

$$\forall\ Z \in D^\perp, W \in D^\theta,\ g(FZ, FW) = g(\phi Z, \phi W) = g(Z, W) = 0,$$

which shows that $F(D^\perp), F(D^\theta)$ are mutually perpendicular. So, (1.21) is an orthogonal direct decomposition.

There are various types of works done on hemi-slant submanifolds. H. I. Abutuqayqah worked on geometry of hemi-slant submanifolds of almost contact manifolds [1]. M. A. Khan et al. discussed about totally umbilical hemi-slant submanifolds of Kahler manifolds [2] and of cosymplectic manifolds [4], and they also discussed about a classification on totally umbilical proper slant and hemi-slant submanifolds of a nearly trans-Sasakian manifold [6]. B. Laha et al. studied totally umbilical hemi-slant submanifolds of LP-Sasakian manifold [7] and hemi-slant submanifold of Kenmotsu manifold [10]. H. M. Tastan et al. discussed about hemi-slant submanifolds of a locally product Riemannian manifold [12] and of a locally conformal Kahler manifold [13]. Another important works on hemi-slant submanifolds were done by A. Lotta in 1996 [9], by M. A. Lone et al. in 2016 [8] and by M. S. Siddesha et al. in 2018 [11]. Motivated from these works, in this paper, we analyse some properties regarding distributions and leaves of hemi-slant submanifold of $(LCS)_n$-manifold.

2. MAIN RESULTS

In this section, we discuss about some necessary and sufficient conditions for distributions to be integrable and obtain some results in this direction. We also study the geometry of leaves of hemi-slant submanifold of $(LCS)_n$-manifold.

Theorem 2.1. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$, then $\forall\ Z, W \in D^\perp,\ A_{\phi W}Z = A_{\phi Z}W - \alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi$.*

Proof. On using (1.10), we have

$$g(A_{\phi W}Z, X) = g(h(Z, X), \phi W) = g(\phi h(Z, X), W) = g(\phi\tilde{\nabla}_X Z, W) - g(\phi\nabla_X Z, W)$$

$= g(\phi\tilde{\nabla}_X Z, W) = g(\tilde{\nabla}_X \phi Z, W) - g((\tilde{\nabla}_X \phi)Z, W)$.

Again using (1.6) and (1.9), we get
$g(A_{\phi W}Z, X) = g(A_{\phi Z}X + \nabla^{\perp}_X \phi Z, W) - \alpha g(g(X,Z)\xi + 2\eta(X)\eta(Z)\xi + \eta(Z)X, W)$
$= g(A_{\phi Z}X, W) - \alpha g(X,Z)\eta(W) - 2\alpha\eta(X)\eta(Z)\eta(W) - \alpha\eta(Z)g(X,W)$
$= g(h(W,X), \phi Z) - \alpha g(X,Z)\eta(W) - \alpha\eta(Z)g(X,W) - 2\alpha\eta(X)\eta(Z)\eta(W)$
$= g(A_{\phi Z}W - \alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi, X)$
$\Rightarrow A_{\phi W}Z = A_{\phi Z}W - \alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi$.

Theorem 2.2. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. Then the distribution $D^\theta \oplus D^\perp$ is integrable if and only if $g([X,Y],\xi) = 0 \ \forall \ X, Y \in D^\theta \oplus D^\perp$.*

Proof. For $X, Y \in D^\theta \oplus D^\perp$,
$g([X,Y],\xi) = g(\tilde{\nabla}_X Y, \xi) - g(\tilde{\nabla}_Y X, \xi)$
$= -g(\tilde{\nabla}_X \xi, Y) + g(\tilde{\nabla}_Y \xi, X)$
$= -g(\alpha\phi X, Y) + g(\alpha\phi Y, X)$
$= 0$. (by (1.4))

Since $TM = D^\theta \oplus D^\perp \oplus < \xi >$, therefore $[X,Y] \in D^\theta \oplus D^\perp$. So, $D^\theta \oplus D^\perp$ is integrable.

Conversely, let $D^\theta \oplus D^\perp$ is integrable. Then $\forall \ X, Y \in D^\theta \oplus D^\perp, [X,Y] \in D^\theta \oplus D^\perp$. As $TM = D^\theta \oplus D^\perp \oplus < \xi >$, therefore $g([X,Y],\xi) = 0$.

Theorem 2.3. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if $\forall \ W \in D^\perp, W$ is a scalar multiple of ξ.*

Proof. For $Z, W \in D^\perp$, from (1.6), we have

$$(\tilde{\nabla}_Z \phi)W = \alpha[g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z]. \quad (2.1)$$

After some calculations and using (1.12), (1.13), we get

$$-A_{FW}Z + \nabla^{\perp}_Z FW - T\nabla_Z W - F\nabla_Z W - th(Z,W) - fh(Z,W) = \alpha[g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z]. \quad (2.2)$$

Comparing tangential components, we have

$$-A_{FW}Z - T\nabla_Z W - th(Z,W) = \alpha[g(Z,W)\xi + 2\eta(Z)\eta(W)\xi + \eta(W)Z]. \quad (2.3)$$

Interchanging Z, W, we obtain

$$-A_{FZ}W - T\nabla_W Z - th(W,Z) = \alpha[g(W,Z)\xi + 2\eta(W)\eta(Z)\xi + \eta(W)Z]. \quad (2.4)$$

Subtracting (2.3) from (2.4) and using the fact that h is symmetric, we have

$$A_{FW}Z - A_{FZ}W + T(\nabla_Z W - \nabla_W Z) = \alpha[\eta(Z)W - \eta(W)Z]. \quad (2.5)$$

From (2.5), we have

$$A_{FW}Z - A_{FZ}W + T([Z,W]) = \alpha[\eta(Z)W - \eta(W)Z]. \tag{2.6}$$

Now $D^{\perp}$ is integrable if and only if $[Z,W] \in D^{\perp}$ and as $D^{\perp}$ is anti-invariant, $\phi D^{\perp} \subseteq T^{\perp}M$ and so, $T[Z,W] = 0$.

Hence from (2.6), $D^{\perp}$ is integrable if and only if $A_{FW}Z - A_{FZ}W = \alpha[\eta(Z)W - \eta(W)Z]$.

From Theorem 2.1, we have as $TW = 0 = TZ$,
$A_{\phi W}Z - A_{\phi Z}W = -\alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi$
$\Rightarrow \alpha[\eta(Z)W - \eta(W)Z] = -\alpha\eta(W)Z - \alpha\eta(Z)W - 2\alpha\eta(Z)\eta(W)\xi$
$\Rightarrow 2\alpha\eta(Z)W + 2\alpha\eta(Z)\eta(W)\xi = 0$
$\Rightarrow \eta(Z)W + \eta(Z)\eta(W)\xi = 0$
$\Rightarrow W + \eta(W)\xi = 0$. Hence the result is proved.

Theorem 2.4. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. Then the slant distribution D^{θ} is integrable if and only if $\forall\ X, Y \in D^{\theta}$,*

$$P_1(\nabla_X TY - \nabla_Y TX) = \alpha[\eta(Y)P_1X - \eta(X)P_1Y]. \tag{2.7}$$

Proof. We denote by P_1, P_2 the projections on $D^{\perp}, D^{\theta}$ respectively. $\forall\ X, Y \in D^{\theta}$, we have from (1.6),

$$(\tilde{\nabla}_X\phi)Y = \alpha[\tilde{g}(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]. \tag{2.8}$$

On applying (1.8), (1.9), (1.12), (1.13), we have
$(\tilde{\nabla}_X\phi)Y = \nabla_X TY + h(X,TY) - A_{FY}X + \nabla_X FY - (T\nabla_X Y + F\nabla_X Y) - (th(X,Y)$
$+ fh(X,Y)) = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]$. (2.9)

Comparing tangential components, we get
$\nabla_X TY - A_{FY}X - T\nabla_X Y - th(X,Y) = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X]$. (2.10)

Interchanging X, Y in (2.10) and subtracting the resultant from (2.10), we obtain
$\nabla_X TY - \nabla_Y TX - A_{FY}X + A_{FX}Y - T\nabla_X Y + T\nabla_Y X = \alpha[\eta(Y)X - \eta(X)Y]$. (2.11)

Since $X, Y \in D^{\theta}, FX = 0 = FY$, applying P_1 to both sides of (2.11), we have

$$P_1(\nabla_X TY - \nabla_Y TX) = \alpha[\eta(Y)P_1X - \eta(X)P_1Y].$$

Theorem 2.5. *Let M be a hemi-slant submanifold of an $(LCS)_n$-manifold $\tilde{M}$. If the leaves of $D^{\perp}$ are totally geodesic in M, then $\forall\ X \in D^{\theta}$ and $Z, W \in D^{\perp}$,*

$$g(h(Z,X), FW) + g(th(Z,W), X) = 0. \tag{2.12}$$

Proof. From (1.6), (1.8), (1.9), we have
$\nabla_Z \phi W + h(Z,\phi W) - A_{FW}Z + \nabla^{\perp}_Z FW - \phi\nabla_Z W - \phi h(Z,W)$
$= \alpha[g(Z,W)\xi + 2\eta(W)\eta(Z)\xi + \eta(W)Z]$.

Comparing tangential components and on taking inner product with $X \in D^{\theta}$, we obtain

$$-g(A_{FW}Z,X) - g(th(Z,W),X) - g(T\nabla_Z W,X) = 0.$$

The leaves of $D^{\perp}$ are totally geodesic in M if for $Z,W \in D^{\perp}, \nabla_Z W \in D^{\perp}$. So, $T\nabla_Z W = 0$.

Thus $g(A_{FW}Z,X) + g(th(Z,W),X) = 0$.

Example. Now we give an example of a hemi-slant submanifold of an $(LCS)_n$-manifold.

Let $\tilde{M}(\mathbb{R}^9,\phi,\xi,\eta,g)$ denote the manifold $\mathbb{R}^9$ with the (LCS)-structure given by−
$\xi = 3\frac{\partial}{\partial z}, \eta = \frac{1}{3}(-dz + \sum_{i=1}^{4} b^i da^i)$,
$g = \frac{1}{9}\sum_{i=1}^{4}(da^i \otimes da^i \oplus db^i \otimes db^i) - \eta \otimes \eta$,
$\phi(\frac{\partial}{\partial z}) = 0, \phi(\frac{\partial}{\partial a^i}) = \frac{\partial}{\partial b^i}, i = 1,2,3,4$, and
$\phi(\frac{\partial}{\partial b^i}) = \frac{\partial}{\partial a^i}$ for $i = 1,2$ and $\phi(\frac{\partial}{\partial b^i}) = -\frac{\partial}{\partial a^i}$ for $i = 3,4$,
where $(a^1,a^2,a^3,a^4,b^1,b^2,b^3,b^4,z) \in \mathbb{R}^9$.

Let us consider a 5-dimensional submanifold M of $\tilde{M}$ defined by
$(a^1,a^2,a^3,a^4,b^1,b^2,b^3,b^4,z) \mapsto (cos\alpha a^1 + sin\alpha a^2, cos\beta b^1 + sin\beta b^2, \frac{a^3-b^3}{\sqrt{3}}, \frac{a^4-b^4}{\sqrt{3}}, 3z)$.

Then it can be easily proved that M is a hemi-slant submanifold of $\tilde{M}$ by choosing the slant distribution $D_\theta =< e_1, e_2 >$ with slant angle $|\alpha - \beta|$ and the totally real distribution $D^{\perp} =< e_3, e_4 >$, where $e_1 = sin\alpha\frac{\partial}{\partial a^1} - cos\alpha\frac{\partial}{\partial a^2}$, $e_2 = sin\beta\frac{\partial}{\partial b^1} - cos\beta\frac{\partial}{\partial b^2}$, $e_3 = \frac{\partial}{\partial a^3} + \frac{\partial}{\partial b^3}$, $e_4 = \frac{\partial}{\partial a^4} + \frac{\partial}{\partial b^4}$ such that $\{e_1,e_2,e_3,e_4,\xi\}$ forms an orthogonal frame on TM so that $TM = D_\theta \oplus D^{\perp} \oplus < \xi >$.

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REFERENCES

[1] Abutuqayqah, H.I., *Geometry of hemi-slant submanifolds of almost contact manifolds*, King Abdulaziz University, 2012.

[2] Al-Solamy, F.R., Khan, M.A., and Uddin, S., Totally umbilical hemi-slant submanifolds of Kahler manifold, *Abstr. Appl. Anal.*, (2011).

[3] Carriazo, A., *New developments in slant submanifold theory*, Narosa Publishing House, New Delhi, 2002.

[4] Khan, M.A., Totally umbilical hemi-slant submanifolds of cosymplectic manifolds, *Math. Aeterna*, **3:8** (2013), 645-653.

[5] Khan, M.A., Singh, K., and Khan, V.A., Slant-submanifold of LP-contact manifolds, *Differ. Geom. Dyn. Syst.*, **12** (2010), 102-108.

[6] Khan, M.A., Uddin, S., and Singh, K., A classification on totally umbilical proper slant and hemi-slant submanifolds of a nearly trans-Sasakian manifold, *Differ. Geom. Dyn. Syst.*, **13** (2011), 117-127.

[7] Laha, B. and Bhattacharyya, A., Totally umbilical hemi-slant submanifolds of LP-Sasakian manifold, *Lobachevskii J. Math.*, **36:2** (2015), 127-131.

[8] Lone, M.A., Lone, M.S., and Shahid, M.H., Hemi-slant submanifolds of cosymplectic manifolds, *Cogent Math. Stat.*, **3:1** (2016), 1204143.

[9] Lotta, A., Slant submanifolds in contact geometry, *Bull. Math. Soc. Sci. Math. Roum. (N.S)*, **39** (1996), 183-198.

[10] Patra, C., Laha, B., and Bhattacharyya, A., On hemi-slant submanifold of Kenmotsu manifold, *Int. J. Math. Comb.*, **1** (2019), 62-72.

[11] Siddesha, M.S., Bagewadi, C.S., and Venkatesha, S., On the geometry of hemi-slant submanifolds of LP-cosymplectic manifold, *Asian J. Math. Appl.*, (2018).

[12] Tastan, H.M., and Ozdemir, F., The geometry of hemi-slant submanifolds of a locally product Riemannian manifold, *Turk. J. Math.*, **39:2** (2015), 268-284.

[13] Tastan, H.M., and Gerdan, S., Hemi-slant submanifolds of a locally conformal Kahler manifold, *Int. Electron. J. Geom.*, **8:2** (2015), 46-56.

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QUASI HEMI-SLANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLD

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Abstract In this paper, we study quasi hemi-slant (QHS) submanifolds of trans-Sasakian manifold. We obtain various results satisfied by these submanifolds. Further, we obtain necessary and sufficient conditions for integrability of distributions related to these submanifolds, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. Moreover, we conclude the necessary and sufficient condition for a QHS submanifold of a trans-Sasakian manifold to be a local product Riemannian manifold. At last, we construct an example of a QHS submanifold of a trans-Sasakian manifold.

1 Introduction

The notion of slant submanifold was introduced by B. Y. Chen in 1990 [5] as a generalization of holomorphic and totally real immersions. Later he collected many consequent results in his book [2]. Further slant submanifold was generalized as semi-slant, pseudo-slant, bi-slant and hemi-slant submanifolds etc. in different types of differentiable manifolds.

Many geometers studied invariant [8], anti-invariant [17], semi-invariant [19], slant [16], semi-slant [9], pseudo-slant [7] and bi-slant [18] submanifolds of trans-Sasakian manifolds in different times.

The concept of quasi hemi-slant submanifold was introduced recently by R. Prasad et al. in 2020 [13] as a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant and semi-slant submanifolds. Later in 2020-2021, R. Prasad along with some other researchers discussed this submanifold in various types of manifolds ([11], [12], [14]).

Motivated from the works mentioned above, in this paper, we study quasi hemi-slant (QHS) submanifolds of trans-Sasakian manifold. This paper consists of four sections. After introduction, in the second section, we mention some definitions and properties related to the main topic. The third section deals with some results satisfied by a QHS submanifold of a trans-Sasakian manifold. In the fourth section, we obtain necessary and sufficient conditions for integrability of distributions related to this submanifold, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. At the end of the fourth section, we conclude the necessary and sufficient condition for a QHS submanifold of a trans-Sasakian manifold to be local product Riemannian manifold and also make two other conclusions after observing the results. Finally, at last, we construct an example of a QHS submanifold of a trans-Sasakian manifold.

2 Preliminaries

In this section, we mention some definitions and properties related to quasi hemi-slant submanifolds of trans-Sasakian manifold.

Let $\tilde{M}$ be an odd dimensional differentiable manifold equipped with a metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the following relations−

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0,\ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y),\ \eta(X) = g(X, \xi)\ \ \forall X, Y \in \chi(\tilde{M}), \tag{2.3}$$

then $\tilde{M}$ is called *almost contact metric manifold* [6].

An odd dimensional almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$ is called *trans-Sasakian manifold of type* (α, β) (α, β are smooth functions on $\tilde{M}$) if [6] $\forall X, Y \in \chi(\tilde{M})$

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{2.5}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$.

Let φ be a differentiable map from a manifold M into a manifold $\tilde{M}$ and let the dimensions of $M,\ \tilde{M}$ be $n,\ m$ respectively. If at each point p of M, $(\varphi_*)_p$ is a 1-1 map i.e., if rank(φ)=n, then φ is called an *immersion* of M into $\tilde{M}$.

If an immersion φ is one-one i.e., if $\varphi(p) \neq \varphi(q)$ for $p \neq q$, then φ is called an *imbedding* of M into $\tilde{M}$.

If the manifolds $M, \tilde{M}$ satisfy the following two conditions, then M is called a *submanifold* of $\tilde{M}$−
(i) $M \subset \tilde{M}$,
(ii) the inclusion map i from M into $\tilde{M}$ is an imbedding of M into $\tilde{M}$.

Let M be a submanifold of $\tilde{M}$ and $A,\ h$ denote the *shape operator*, *second fundamental form* respectively of the immersion of M into $\tilde{M}$, then the Gauss and Weingarten formulae of M into $\tilde{M}$ are given by [3]

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.6}$$

$$\tilde{\nabla}_X V = A_V X + \nabla^{\perp}_X V \tag{2.7}$$

$\forall X, Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$, where ∇ is the induced connection on M, $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$ of M and A_V is the shape operator of M with respect to the normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, A_V and h are related by the following equation−

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.8}$$

The *mean curvature vector* is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where $\dim(M) = n$ and $\{e_i\}_{i=1}^n$ is an orthonormal basis of the tangent space of M.

We have $\forall X \in \Gamma(TM)$,

$$\phi X = TX + NX, \tag{2.9}$$

where $TX,\ NX$ are the tangential and normal components of ϕX on M respectively.

Similarly, we have $\forall V \in \Gamma(T^{\perp}M)$,

$$\phi V = tV + nV, \tag{2.10}$$

where $tV,\ nV$ are the tangential and normal components of ϕV on M respectively.

A submanifold M is called−

(i) *totally umbilical* if

$$h(X, Y) = g(X, Y)H \ \ \forall X, Y \in \Gamma(TM); \tag{2.11}$$

(ii) *totally geodesic* if [4]

$$h(X, Y) = 0 \ \ \forall X, Y \in \Gamma(TM); \tag{2.12}$$

(iii) *minimal* if

$$H = 0. \tag{2.13}$$

The covariant derivatives of the tangential and normal components given in the equations (2.9), (2.10) are given by $\forall X, Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.14}$$

$$(\tilde{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{2.15}$$

$$(\tilde{\nabla}_X t)V = \nabla_X tV - t\nabla^{\perp}_X V, \tag{2.16}$$

$$(\tilde{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V. \tag{2.17}$$

Quasi hemi-slant submanifold M of a trans-Sasakian manifold $\tilde{M}$ is a submanifold that admits three orthogonal complementary distributions $D,\ D_\theta,\ D^{\perp}$ such that [13]

(i) TM admits the orthogonal direct decomposition

$$TM = D \oplus D_\theta \oplus D^{\perp} \oplus < \xi >, \tag{2.18}$$

(ii) the distribution D is invariant i.e., $\phi D = D$,
(iii) the distribution D_θ is slant with constant angle θ and hence θ is called *slant angle*,
(iv) the distribution $D^{\perp}$ is ϕ anti-invariant i.e., $\phi D^{\perp} \subseteq T^{\perp}M$.

In the above case, θ is called the *quasi hemi-slant angle* of M, and M is called *proper* [13] if $D \neq \{0\},\ D_\theta \neq \{0\},\ D^{\perp} \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Let the dimensions of the distributions $D,\ D_\theta,\ D^{\perp}$ be $n_1,\ n_2,\ n_3$ respectively, then we obtain the following particular cases [13]−
(i) if $n_1 = 0$, then M is a hemi-slant submanifold,
(ii) if $n_2 = 0$, then M is a semi-invariant submanifold,

(iii) if $n_3 = 0$, then M is a semi-slant submanifold.

Now, it can be concluded from the definitions of invariant [8], anti-invariant [17], semi-invariant [1], slant [16], hemi-slant [15] and semi-slant [9] submanifolds that, quasi hemi-slant submanifold is a generalization of all these kinds of submanifolds [13].

From the definition of quasi hemi-slant submanifold given above, it is clear that if $D \neq \{0\},\ D_\theta \neq \{0\},\ D^\perp \neq \{0\}$, then dim$(D) \geq 2$, dim$(D_\theta) \geq 2$ and dim$(D^\perp) \geq 1$. Thus, we have the following remark [13]–

Remark 2.1. *For a proper quasi hemi-slant submanifold M, dim$(M) \geq 6$.*

Note. From now on, in this paper, we will write the term *quasi hemi-slant* in its abbreviated form i.e., *QHS*.

Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ and the projections of $X \in \Gamma(TM)$ on the distributions $D,\ D_\theta,\ D^\perp$ be $P,\ Q,\ R$ respectively, then we have $\forall X \in \Gamma(TM)$,

$$X = PX + QX + RX + \eta(X)\xi. \tag{2.19}$$

Using (2.9) in (2.19) we get

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since $\phi D = D,\ \phi D^\perp \subseteq T^\perp M$, we have $NPX = 0,\ TRX = 0$ and hence we obtain

$$\phi X = TPX + TQX + NQX + NRX. \tag{2.20}$$

Comparing (2.20) with (2.9) we have

$$TX = TPX + TQX, \tag{2.21}$$

$$NX = NQX + NRX. \tag{2.22}$$

From (2.20) we have the following decomposition–

$$\phi(TM) = TD \oplus TD_\theta \oplus ND_\theta \oplus ND^\perp. \tag{2.23}$$

Again, since $ND_\theta \subseteq \Gamma(T^\perp M),\ ND^\perp \subseteq \Gamma(T^\perp M)$, we have another decomposition–

$$T^\perp M = ND_\theta \oplus ND^\perp \oplus \mu, \tag{2.24}$$

where μ is the orthogonal complement of $ND_\theta \oplus ND^\perp$ in $\Gamma(T^\perp M)$ and it is anti-invariant with respect to ϕ [13].

3 QHS submanifolds of trans-Sasakian manifold

This section deals with some results satisfied by a QHS submanifold of a trans-Sasakian manifold.

Theorem 3.1. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X, Y \in \Gamma(TM)$,*

$$\nabla_X TY - A_{NY}X - T(\nabla_X Y) - th(X,Y) = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)TX], \tag{3.1}$$

$$h(X,TY) + \nabla^\perp_X NY - N(\nabla_X Y) - nh(X,Y) = -\beta\eta(Y)NX. \tag{3.2}$$

Proof. Using (2.9) in (2.4) we get

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)(TX + NX)]. \tag{3.3}$$

Again, using (2.6), (2.7), (2.9) and (2.10) in $(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y)$ we obtain

$$(\tilde{\nabla}_X \phi)Y = \nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY - th(X,Y) - nh(X,Y) - T(\nabla_X Y) - N(\nabla_X Y). \tag{3.4}$$

Equating tangential and normal components of (3.3), (3.4) we obtain (3.1) and (3.2) respectively. □

Using (2.14) and (2.15) respectively in (3.1) and (3.2), we can conclude the following−

Corollary 3.1. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X, Y \in \Gamma(TM)$,*

$$(\tilde{\nabla}_X T)Y = A_{NY}X + th(X,Y) + \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(TX,Y)\xi - \eta(Y)TX], \tag{3.5}$$

$$(\tilde{\nabla}_X N)Y = -h(X,TY) + nh(X,Y) - \beta\eta(Y)NX. \tag{3.6}$$

Next, we state the following theorem [13]−

Theorem 3.2. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have*

$$TD = D,\ TD_\theta = D_\theta,\ TD^{\perp} = \{0\},\ tND_\theta = D_\theta,\ tND^{\perp} = D^{\perp}.$$

Now, using (2.9) and (2.10) on $\phi^2 = -I + \eta \otimes \xi$ we get the following theorem−

Theorem 3.3. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we get*
(i) $T^2 + nN = -I + \eta \otimes \xi$ *on* TM,
(ii) $NT + tN = 0$ *on* TM,
(iii) $Tt + n^2 = -I$ *on* $T^{\perp}M$,
(iv) $Nt + tn = 0$ *on* $T^{\perp}M$,
where I is the identity operator.

Next, we have the following theorem [10]−

Theorem 3.4. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X, Y \in \Gamma(D_\theta)$,*
(i) $T^2X = -(cos^2\theta)X$,
(ii) $g(TX,TY) = (cos^2\theta)g(X,Y)$,
(iii) $g(NX,NY) = (sin^2\theta)g(X,Y)$.

Theorem 3.5. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,*

$$\nabla_X tV - A_{nV}X + T(A_V X) - t\nabla^{\perp}_X V = \beta g(NX,V)\xi, \tag{3.7}$$

$$h(X,tV) + \nabla^{\perp}_X nV + N(A_V X) - n\nabla^{\perp}_X V = 0. \tag{3.8}$$

Proof. Using (2.6), (2.7), (2.9) and (2.10) in $(\tilde{\nabla}_X \phi)V = \tilde{\nabla}_X \phi V - \phi(\tilde{\nabla}_X V)$ we get

$$(\tilde{\nabla}_X \phi)V = \nabla_X tV + h(X,tV) - A_{nV}X + \nabla^{\perp}_X nV + T(A_V X) + N(A_V X) - t\nabla^{\perp}_X V - n\nabla^{\perp}_X V.$$

Again, applying (2.4) and then (2.9) in the left hand side of the above equation we obtain

$$\beta[g(NX,V)\xi] = \nabla_X tV + h(X,tV) - A_{nV}X + \nabla^{\perp}_X nV + T(A_V X) + N(A_V X) - t(\nabla^{\perp}_X V) - n(\nabla^{\perp}_X V). \tag{3.9}$$

Equating tangential and normal components from both sides of (3.9) we get (3.7) and (3.8) respectively. □

Now, using (2.16) and (2.17) in (3.7) and (3.8) respectively we conclude the following−

Corollary 3.2. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we get $\forall X \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,*

$$(\tilde{\nabla}_X t)V = A_{nV}X - T(A_V X) + \beta g(NX, V)\xi, \tag{3.10}$$

$$(\tilde{\nabla}_X n)V = -h(X, tV) - N(A_V X). \tag{3.11}$$

Theorem 3.6. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X \in \Gamma(TM)$,*

$$\nabla_X \xi = -\alpha TX - \beta T^2 X, \tag{3.12}$$

$$h(X, \xi) = -\alpha NX - \beta nNX. \tag{3.13}$$

Proof. Using (2.6), (2.9) and Theorem 3.3.(i) in (2.5) we obtain

$$\nabla_X \xi + h(X, \xi) = -\alpha(TX + NX) + \beta[-T^2 - nN]X.$$

Equating tangential and normal components from both sides of the above equation we get (3.12) and (3.13) respectively. □

Theorem 3.7. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X, Y \in \Gamma(D^{\perp})$,*

$$A_{\phi X}Y = A_{\phi Y}X \ if \ and \ only \ if \ \phi[X, Y] = 2\beta g(X, \phi Y)\xi. \tag{3.14}$$

Proof. Replacing V by ϕY in (2.7) and then applying (2.4), (2.6) and the fact that $Y \in \Gamma(D^{\perp})$ we get

$$\alpha g(X, Y)\xi + \beta g(\phi X, Y)\xi + \phi(\nabla_X Y) + \phi h(X, Y) = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y.$$

Equating tangential components from both sides of the above equation we obtain

$$A_{\phi Y}X = -\alpha g(X, Y)\xi - \beta g(\phi X, Y)\xi - \phi(\nabla_X Y). \tag{3.15}$$

Interchanging X, Y in (3.15) and then subtracting (3.15) from the resultant equation we have

$$A_{\phi X}Y - A_{\phi Y}X = \phi[X, Y] - 2\beta g(X, \phi Y)\xi \tag{3.16}$$

from which we get (3.14). □

Theorem 3.8. *Let M be a QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then we have $\forall X, Y \in \Gamma(D \oplus D_\theta \oplus D^{\perp})$,*

$$g([X, Y], \xi) = 2\alpha g(TX, Y), \tag{3.17}$$

$$g(\tilde{\nabla}_X Y, \xi) = \alpha g(TX, Y) - \beta cos^2\theta g(X, Y). \tag{3.18}$$

Proof. Applying (3.12) and Theorem 3.4.(i) on the following equation

$$g([X, Y], \xi) = g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) = -g(Y, \nabla_X \xi) + g(X, \nabla_Y \xi)$$

and after simplifying we obtain (3.17).

Again, using (2.6) we have

$$g(\tilde{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) + h(X, Y)\eta(\xi) = -g(Y, \nabla_X \xi) + h(X, Y).$$

Now, applying (3.12) and Theorem 3.4.(i) on the above equation we get (3.18).

Thus the proof is completed. □

4 Integrability of distributions and decomposition theorems

In this section, we obtain necessary and sufficient conditions for integrability of distributions related to the QHS submanifolds of a trans-Sasakian manifold, for these distributions to define totally geodesic foliations and also for a submanifold of a trans-Sasakian manifold to be totally geodesic. At the end, we make three conclusions after observing the results.

Theorem 4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the invariant distribution D is not integrable.*

Proof. Let $X, Y \in \Gamma(D)$, then using (2.6), $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$ and then (2.5), $g(\phi X, Y) = -g(X, \phi Y)$ in the following equation

$$g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi)$$

we get on simplifying,

$$g([X, Y], \xi) = 2\alpha g(\phi X, Y). \tag{4.1}$$

Applying (2.19), (2.20) on (4.1) we obtain $g([X, Y], \xi) = 2\alpha g(TPX, PY) \neq 0$. Thus, D is not integrable. □

Theorem 4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D\oplus <\xi>$ is integrable if and only if $\forall X, Y \in \Gamma(D\oplus <\xi>), Z \in \Gamma(D_\theta \oplus D^\perp)$,*

$$g(T\nabla_X Y - T\nabla_Y X, TQZ) + g(nh(X, Y) - nh(Y, X), NQZ + NRZ) = 0. \tag{4.2}$$

Proof. Using (2.2) in $g([X, Y], Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X, Y], Z) = g(\phi\tilde{\nabla}_X Y, \phi Z) - g(\phi\tilde{\nabla}_Y X, \phi Z)$$

on which applying (2.6), (2.9), (2.10), (2.20) and after simplifying we get

$$g([X, Y], Z) = g(T\nabla_X Y - T\nabla_Y X, TQZ) + g(nh(X, Y) - nh(Y, X), NQZ + NRZ).$$

Hence $g([X, Y], Z) = 0$ if and only if (4.2) holds and thus the proof is completed. □

Theorem 4.3. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the slant distribution D_θ is not integrable.*

Proof. Let $X, Y \in \Gamma(D_\theta)$. Applying (2.19) and (2.20) in (4.1) we have $g([X, Y], \xi) = 2\alpha g(TQX + NQX, QY) \neq 0$ and hence the proof is completed. □

Theorem 4.4. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D_\theta\oplus <\xi>$ is integrable if and only if $\forall X, Y \in \Gamma(D_\theta\oplus <\xi>), Z \in \Gamma(D \oplus D^\perp)$,*

$$g(n(\nabla^\perp_X Y) - n(\nabla^\perp_Y X), NRZ) + cos^2\theta g(A_X Y - A_Y X, PZ) = 0. \tag{4.3}$$

Proof. Using (2.2) in $g([X, Y], Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X,Y],Z) = g(\phi\tilde{\nabla}_X Y, \phi Z) - g(\phi\tilde{\nabla}_Y X, \phi Z)$$

on which applying (2.7), (2.9), (2.10), (2.20), Theorem 3.4.(ii) and after simplifying we get

$$g([X,Y], Z = cos^2\theta g(A_X Y - A_Y X, PZ) + g(n(\nabla^{\perp}_X Y) - n(\nabla^{\perp}_Y X), NRZ).$$

Therefore, $g([X,Y],Z) = 0$ if and only if (4.3) holds and hence the proof is completed. □

From the above theorem, using (2.18) and (2.24) respectively we conclude the following–

Corollary 4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the distribution $D_\theta \oplus <\xi>$ is integrable if $\forall X,Y \in \Gamma(D_\theta \oplus <\xi>)$,*

$$A_X Y - A_Y X \in \Gamma(D_\theta \oplus D^{\perp}), \tag{4.4}$$

$$n(\nabla^{\perp}_X Y) - n(\nabla^{\perp}_Y X) \in \Gamma(ND_\theta \oplus \mu). \tag{4.5}$$

Theorem 4.5. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the anti-invariant distribution $D^{\perp}$ is integrable if and only if $\forall X,Y \in \Gamma(D^{\perp}), Z \in \Gamma(D \oplus D_\theta)$*

$$g(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X, NQZ) = 0. \tag{4.6}$$

Proof. Using (2.2) in $g([X,Y],Z) = g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z)$ we get

$$g([X,Y],Z) = g(\tilde{\nabla}_X \phi Y, \phi Z) - g(\tilde{\nabla}_Y \phi X, \phi Z)$$

on which applying (2.7), (2.8), (2.20) and Theorem 3.2 we get after simplification,

$$g([X,Y],Z) = g(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X, NQZ).$$

Thus $g([X,Y],Z) = 0$ if and only if (4.6) holds and hence the proof is completed. □

Using (2.24) in the above theorem we conclude the following–

Corollary 4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the anti-invariant distribution $D^{\perp}$ is integrable if $\forall X,Y \in \Gamma(D^{\perp})$, $\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X \in \Gamma(ND^{\perp} \oplus \mu)$.*

Theorem 4.6. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then M is totally geodesic if and only if $\forall X,Y \in \Gamma(TM), V \in \Gamma(T^{\perp}M)$,*

$$g(\nabla_X TY - A_{NY}X, tV) + g(h(X,TY) + \nabla^{\perp}_X NY, nV) = 0. \tag{4.7}$$

Proof. Applying (2.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$.

Further, using (2.6), (2.7), (2.9), (2.10) in the above equation we obtain on simplifying,

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TY - A_{NY}X, tV) + g(h(X,TY) + \nabla^{\perp}_X NY, nV). \tag{4.8}$$

Now, M is totally geodesic $<=> h = 0 <=> \forall X,Y \in \Gamma(TM), \tilde{\nabla}_X Y = \nabla_X Y$ (from (2.6))$<=> g(\tilde{\nabla}_X Y, V) = 0\ \forall V \in \Gamma(T^{\perp}M)$. Hence from (4.8) we have, M is totally geodesic if and only if (4.7) holds. Thus the proof is completed. □

Theorem 4.7. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α,β), then the invariant distribution D does not define a totally geodesic foliation on M.*

Proof. Let $X, Y \in \Gamma(D)$. Using (2.5) and the fact that $X \in \Gamma(D)$ in $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$ we get $g(\tilde{\nabla}_X Y, \xi) = -\beta g(X, Y) + \alpha g(Y, \phi X) \neq 0$, and hence the proof is completed. □

Theorem 4.8. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D \oplus < \xi >$ defines a totally geodesic foliation on M if and only if $\forall X, Y \in \Gamma(D), Z \in \Gamma(D_\theta \oplus D^\perp), V \in \Gamma(T^\perp M)$,*

$$g(\nabla_X TY, TQZ) = -g(h(X, TY), NZ), \tag{4.9}$$

$$g(\nabla_X TY, tV) = -g(h(X, TY), nV). \tag{4.10}$$

Proof. Applying (2.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (2.6) and (2.20) we get

$$g(\tilde{\nabla}_X Y, Z) = g(\nabla_X TY, TQZ) + g(h(X, TY), NZ)$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (4.9) holds.

Again, applying (2.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (2.6), (2.10) and (2.20) we obtain

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TY, tV) + g(h(X, TY), nV).$$

Hence we have $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (4.10) holds.

Thus the proof is completed. □

Theorem 4.9. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the slant distribution D_θ does not define a totally geodesic foliation on M.*

Proof. Let $X, Y \in \Gamma(D_\theta)$. Applying (2.5) and the fact that $X \in \Gamma(D_\theta)$ on $g(\tilde{\nabla}_X Y, \xi) = -g(Y, \tilde{\nabla}_X \xi)$ we get $g(\tilde{\nabla}_X Y, \xi) = -\beta g(X, Y) + \alpha g(\phi X, Y) \neq 0$.

Hence the proof is completed. □

Theorem 4.10. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the distribution $D_\theta \oplus < \xi >$ defines a totally geodesic foliation on M if and only if $\forall X, Y \in \Gamma(D_\theta \oplus < \xi >), Z \in \Gamma(D \oplus D^\perp), V \in \Gamma(T^\perp M)$,*

$$g(\nabla_X TQY - A_{NQY} X, TPZ) + g(h(X, TQY) + \nabla^\perp_X NQY, NRZ) = 0, \tag{4.11}$$

$$g(\nabla_X TQY - A_{NQY} X, tV) + g(h(X, TQY) + \nabla^\perp_X NQY, nV) = 0. \tag{4.12}$$

Proof. Applying (2.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (2.6), (2.7) and (2.20) we get

$$g(\tilde{\nabla}_X Y, Z) = g(\nabla_X TQY - A_{NQY} X, TPZ) + g(h(X, TQY) + \nabla^\perp_X NQY, NRZ)$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (4.11) holds.

Again, applying (2.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (2.6), (2.7), (2.10) and (2.20) we obtain

$$g(\tilde{\nabla}_X Y, V) = g(\nabla_X TQY - A_{NQY} X, tV) + g(h(X, TQY) + \nabla^\perp_X NQY, nV),$$

which implies that $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (4.12) holds.

Thus the proof is completed. □

Theorem 4.11. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then the anti-invariant distribution $D^{\perp}$ defines a totally geodesic foliation on M if and only if $\forall X, Y \in \Gamma(D^{\perp}), Z \in \Gamma(D \oplus D_\theta), V \in \Gamma(T^{\perp}M)$,*

$$g(A_{NY}X, TZ) = g(\nabla^{\perp}_X NY, NQZ), \tag{4.13}$$

$$g(A_{NY}X, tV) = g(\nabla^{\perp}_X NY, nV). \tag{4.14}$$

Proof. Applying (2.2) we have $g(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X \phi Y, \phi Z)$ on which using (2.7) and (2.20) we obtain

$$g(\tilde{\nabla}_X Y, Z) = -g(A_{NY}X, TZ) + g(\nabla^{\perp}_X NY, NQZ)$$

which implies that $g(\tilde{\nabla}_X Y, Z) = 0$ if and only if (4.13) holds.

Now, applying (2.2) we have $g(\tilde{\nabla}_X Y, V) = g(\tilde{\nabla}_X \phi Y, \phi V)$ on which using (2.7), (2.10) and (2.20) we get

$$g(\tilde{\nabla}_X Y, V) = -g(A_{NY}X, tV) + g(\nabla^{\perp}_X NY, nV)$$

which implies that $g(\tilde{\nabla}_X Y, V) = 0$ if and only if (4.14) holds.

Thus the proof is completed. □

From theorems 4.8, 4.10 and 4.11, we reach to the following conclusion−

Conclusion 4.1. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then M is a local product Riemannian manifold of the form $M_D \times M_{D_\theta} \times M_{D^{\perp}}$ if and only if equations (4.9)-(4.14) hold, where $M_D,\ M_{D_\theta},\ M_{D^{\perp}}$ are leaves of the distributions $D,\ D_\theta,\ D^{\perp}$ respectively.*

Next, theorems 4.1 and 4.3 give us the following conclusion−

Conclusion 4.2. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then both of the invariant distribution D and the slant distribution D_θ are not integrable.*

Again, observing theorems 4.7 and 4.9 we can conclude the following−

Conclusion 4.3. *Let M be a proper QHS submanifold of a trans-Sasakian manifold $\tilde{M}$ of type (α, β), then both of the invariant distribution D and the slant distribution D_θ do not define a totally geodesic foliation on M.*

Example. Now, we construct an example of a QHS submanifold of a trans-Sasakian manifold.

Let $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$-dimensional Euclidean space endowed with the almost contact metric structure (ϕ, ξ, η, g) defined by

$$\phi(x^1, x^2, ..., x^{2n}, t) = (-x^{n+1}, -x^{n+2}, ..., -x^{2n}, x^1, x^2, ..., x^n, 0),$$

$$\xi = e^t \tfrac{\partial}{\partial t},\ \eta = e^{-t}dt,\ g = e^{-2t}k,$$

where $(x^1, x^2, ..., x^{2n}, t)$ are cartesian coordinates and k is the Euclidean Riemannian metric on $\mathbb{R}^{2n+1}$. Then (ϕ, ξ, η, g) is a trans-Sasakian structure on $\mathbb{R}^{2n+1}$ which is neither cosymplectic nor Sasakian.

For $\theta \in \left(0, \frac{\pi}{2}\right)$, we have, the map given by

$$x(u_1,u_2,u_3,u_4,u_5,u_6,u_7)=(u_1,u_2\cos\theta,0,u_2\sin\theta,u_3,u_4,u_5,u_6,0,0,u_7)$$

defines a 7-dimensional submanifold M of $\mathbb{R}^{11}$ with the trans-Sasakian structure described above. Further, let

$$E_1=e^t\frac{\partial}{\partial x^1},\ E_2=e^t\frac{\partial}{\partial x^6},$$

$$E_3=e^t\big(\cos\theta\frac{\partial}{\partial x^2}+\sin\theta\frac{\partial}{\partial x^4}\big),\ E_4=e^t\frac{\partial}{\partial x^7},$$

$$E_5=e^t\frac{\partial}{\partial x^5},\ E_6=e^t\frac{\partial}{\partial x^8},\ E_7=e^t\frac{\partial}{\partial t}=\xi,$$

then $\{E_i\}_{i=1}^7$ is an orthonormal frame of TM.

If we define the distributions as

$$D=<E_1,E_2>,\ D_\theta=<E_3,E_4>,\ D^\perp=<E_5,E_6>,$$

then it is clear that

$$TM=D\oplus D_\theta\oplus D^\perp\oplus<\xi>$$

and D is an invariant distribution since $\phi E_1=E_2$ and $\phi E_2=-E_1$, D_θ is a slant distribution with slant angle $\theta\in\big(0,\frac{\pi}{2}\big)$ since $g(\phi E_3,E_4)=\cos\theta=-g(E_3,\phi E_4)$, $D^\perp$ is an anti-invariant distribution since $\phi E_5=e^t\frac{\partial}{\partial x^{10}}$ and $\phi E_6=-e^t\frac{\partial}{\partial x^3}$.

Therefore, M is a QHS submanifold of the trans-Sasakian manifold $(\mathbb{R}^{11},\phi,\xi,\eta,g)$.

References

[1] A. Bejancu and N. Papaghuic, Semi-invariant submanifolds of a Sasakian manifold, *An. Stiint. Univ. Al. I. Cuza Iasi, Tomul* **27**, 163-170 (1981).

[2] B. Y. Chen, Geometry of slant submanifolds, *Katholieke Univ. Leuven* (1990).

[3] B. Y. Chen, Differential geometry of real submanifold in a Kaehler manifold, *Monatsh. Math.* **91** (4), 257-274 (1981).

[4] B. Y. Chen, CR-submanifolds of a Kaehler manifold. I, *J. Differ. Geom.* **16** (2), 305-323 (1981).

[5] B. Y. Chen, Slant immersions, *Bull. Aust. Math. Soc.* **41** (1), 135-147 (1990).

[6] U. C. De and A. A. Shaikh, Complex Manifolds and Contact Manifolds, *Narosa Publ. House* (2009).

[7] U. C. De and A. Sarkar, On pseudo-slant submanifolds of trans-Sasakian manifolds, *Proc. Est. Acad. Sci.* **60** (1), 1-11 (2011).

[8] C. Hu and Y. Wang, A note on invariant submanifolds of trans-Sasakian manifolds, *Int. Electron. J. Geom.* **9** (2), 27-35 (2016).

[9] V. A. Khan and M. A. Khan, Semi-slant submanifolds of trans-Sasakian manifolds, *Sarajevo J. Math.* **2** (14), 83-93 (2006).

[10] K. S. Park and R. Prasad, Semi-slant submersions, *Bull. Korean Math. Soc.* **50** (3), 951-962 (2013).

[11] R. Prasad, S. S. Shukla, A. Haseeb and S. Kumar, Quasi hemi-slant submanifolds of Kaehler manifolds, *Honam Math. J.* **42** (4), 795-809 (2020).

[12] R. Prasad, P. K. Singh and A. K. Rai, On quasi hemi-slant submanifolds of nearly Kaehler manifolds, *Differ. Geom. Dyn. Syst.* **23**, 188-202 (2021).

[13] R. Prasad, S. K. Verma and S. Kumar, Quasi hemi-slant submanifolds of Sasakian manifolds, *J. Math. Comput. Sci.* **10** (2), 418-435 (2020).

[14] R. Prasad, S. K. Verma, S. Kumar and S. K. Chaubey, Quasi hemi-slant submanifolds of cosymplectic manifolds, *Korean J. Math.* **28** (2), 257-273 (2020).

[15] F. Sahin, Cohomology of hemi-slant submanifolds of a Kaehler manifold, *J. Adv. Stud. Topol.* **5** (2), 27-31 (2014).

[16] A. Sarkar and P. Bhakta, Some characterizations of slant submanifolds of trans-Sasakian manifolds, *BSG Proc.* **26**, 102-111 (2018).

[17] M. H. Shahid and A. Sharfuddin, Anti-invariant submanifolds of trans-Sasakian manifolds, *Int. Cent. Theor. Phys.* **735/93** (1993).

[18] A. N. Siddiqui, Warped product and doubly warped product bi-slant submanifolds in trans-Sasakian manifolds, *JMI Int. J. Math. Sci.* **9**, 15-27 (2018).

[19] S. Yadav and D. Suthar, Semi-invariant ξ-submanifolds of trans-Sasakian manifolds, *Casp. J. Appl. Sci. Res.* **2** (2), 23-28 (2013).

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SCIENTIFIC RESEARCH PAPER

Quasi Hemi-Slant Submanifolds of Metallic Riemannian Manifolds

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Abstract In this paper, we present the general notion of quasi hemi-slant submanifolds of metallic Riemannian manifolds. We study some properties of submanifolds, specially quasi hemi-slant submanifolds of metallic and golden Riemannian manifolds. We obtain some necessary and sufficient conditions for submanifolds to be quasi hemi-slant in metallic and golden Riemannian manifolds and also obtain integrability conditions for the distributions. At last, we construct an example of a quasi hemi-slant submanifold of a metallic Riemannian manifold.

Keywords Quasi hemi-slant (QHS) submanifold · Metallic Riemannian manifold · Golden Riemannian manifold

Significance of the work The theory of submanifolds has the origin in the study of geometry of plane curves initiated by Fermat. Nowadays, it has gained prominence in computer design, image processing, economic modeling, mathematical physics and mechanics except modern differential geometry. Also, according to De Spinadel, besides carrying the name of metals, the metallic means family have common mathematical properties that attach a fundamental importance to them in modern investigations about the search of universal roads to chaos, and the metallic numbers found many applications in researches that analyse the behaviour of non linear dynamical systems when they proceed from a periodic regime to a chaotic one. Golden mean is known from ancient times as an expression of harmony of many constructions, paintings and music. It also appears as an expression of the objects from the natural world (flowers, trees, fruits) possessing pentagonal symmetry.

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1 Introduction

The notion of golden structure on a Riemannian manifold was introduced by Crasmareanu and Hretcanu [1]. They also investigated the properties of golden structure related to the almost product structure [2] and on some invariant submanifolds in a Riemannian manifold [3]. Later they generalized metallic structures as golden structures on Riemannian manifolds [4]. Blaga studied the properties of the conjugate connections by a golden structure and expressed their virtual and structural tensor fields and their behaviour on invariant distributions. Also, she studied the impact of the duality between the golden and almost product structures on golden and product conjugate connections [5]. Further, she along with Hretcanu discussed the properties of the metallic conjugate connections [6] where they expressed the virtual and structural tensor fields and analysed their behaviour on invariant distributions. Recently, they worked on invariant, anti-invariant and slant submanifolds [7], and also semi-slant submanifolds [8] in metallic Riemannian manifolds. Some properties regarding the integrability of the golden Riemannian structures were investigated by Gezer et al. [9]. The connection adapted on the almost golden Riemannian structure was studied by Etayo et al. [10].

The metallic structure J, a polynomial structure defined by Goldberg and Yano [11], is inspired by the metallic number $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$, which is the positive solution of the equation $x^2 - px - q = 0$ for $p, q \in \mathbb{N}$ [12]. These $\sigma_{p,q}$ numbers are members of the metallic means family or metallic proportions (as generalizations of the golden number $\phi = \frac{1+\sqrt{5}}{2} = 1.618...$)

which was introduced by De Spinadel [13]. Golden mean, silver mean, bronze mean, copper mean, nickel mean etc. are examples of the members of the metallic means family.

On the other hand, the concept of quasi hemi-slant submanifold of Sasakian manifold was introduced by Prasad et al. [14] as a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant and semi-slant submanifolds. Later, he along with some other researchers discussed this submanifold in various types of manifolds. Recently, Karmakar and Bhattacharyya [15] discussed about quasi hemi-slant submanifolds of trans-Sasakian manifold. However, the general notion of quasi hemi-slant submanifolds of metallic Riemannian manifolds has not been introduced yet. Therefore, motivated by the works mentioned above, in this paper, we introduce and study quasi hemi-slant submanifolds of metallic Riemannian manifolds.

2 Preliminaries

Let $\tilde{M}$ be an n-dimensional manifold endowed with a (1, 1) tensor field J. This structure J is called *metallic structure* if it satisfies the following relation

$$J^2 = pJ + qI \tag{1}$$

for $p, q \in \mathbb{N}$, where I is the identity operator on $\Gamma(T\tilde{M})$. Then the pair $(\tilde{M}, J)$ is called *metallic manifold*. In particular, if $p = q = 1$, then this manifold is called *golden manifold*.

Moreover, if $\tilde{g}$ is J-compatible for the metallic (or golden) structure J, i.e. if

$$\tilde{g}(JX, Y) = \tilde{g}(X, JY) \tag{2}$$

$\forall X, Y \in \Gamma(T\tilde{M})$, then the triplet $(\tilde{M}, \tilde{g}, J)$ is called *metallic (or golden) Riemannian manifold*. Then we have

$$\tilde{g}(JX, JY) = \tilde{g}(J^2X, Y) = p\tilde{g}(JX, Y) + q\tilde{g}(X, Y). \tag{3}$$

Next, let M be an m-dimensional submanifold of the n-dimensional metallic (or golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with $n, m \in \mathbb{N}$ and $n > m$. Let T_xM, $T_x^{\perp}M$ be the tangent space and normal space of M at $x \in M$ respectively. Then the tangent space $T_x\tilde{M}$ of $\tilde{M}$ can be decomposed as $T_x\tilde{M} = T_xM \oplus T_x^{\perp}M$. Let for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, the tangential parts are $TX, tV \in \Gamma(TM)$ and the normal parts are $NX, nV \in \Gamma(T^{\perp}M)$ of JX, JV respectively so that

$$JX = TX + NX, \tag{4}$$

$$JV = tV + nV. \tag{5}$$

T, N, t, n satisfy the following relations $\forall X, Y \in \Gamma(TM)$, $U, V \in \Gamma(T^{\perp}M)$,

$$\tilde{g}(TX, Y) = \tilde{g}(X, TY), \tag{6}$$

$$\tilde{g}(nU, V) = \tilde{g}(U, nV), \tag{7}$$

$$\tilde{g}(NX, U) = \tilde{g}(X, tU). \tag{8}$$

Let $r = n - m$ be the co-dimension of M in $\tilde{M}$ and $\{N_i\}_{i=1}^r$ be a local orthonormal basis of $T_x^{\perp}M$ for $x \in M$. We assume that the indices α, β run over the range $\{1, \dots, r\}$. Then JX and JN_α can be decomposed into tangential and normal components as [4]

$$JX = TX + \sum_{\alpha=1}^{r} u_\alpha(X)N_\alpha, \tag{9}$$

$$JN_\alpha = \xi_\alpha + \sum_{\beta=1}^{r} a_{\alpha\beta}N_\beta, \tag{10}$$

where ξ_α are vector fields, u_α are 1-forms and $(a_{\alpha\beta})_r$ is an $r \times r$ matrix of smooth real functions on M.

Using Eqs. (4), (5) in Eqs. (9), (10) we get

$$NX = \sum_{\alpha=1}^{r} u_\alpha(X)N_\alpha, \tag{11}$$

$$tN_\alpha = \xi_\alpha, \tag{12}$$

$$nN_\alpha = \sum_{\beta=1}^{r} a_{\alpha\beta}N_\beta. \tag{13}$$

Let $\tilde{\nabla}$ and ∇ be the Levi-Civita connection on $\tilde{M}$ and M respectively. Then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_XY = \nabla_XY + h(X, Y), \tag{14}$$

$$\tilde{\nabla}_XV = -A_VX + \nabla_X^{\perp}V \tag{15}$$

$\forall X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ is the induced connection on $T^{\perp}M$ called the normal connection, h is the second fundamental form and A is the shape operator of M related by

$$\tilde{g}(h(X, Y), V) = \tilde{g}(A_VX, Y). \tag{16}$$

Remark 1 [8] If $\{N_i\}_{i=1}^r$ be a local orthonormal basis of $T_x^{\perp}M$, where r is the co-dimension of M in $\tilde{M}$, and $A_\alpha = A_{N_\alpha}$ for any $\alpha \in \{1, \dots, r\}$, then we obtain $\forall X, Y \in \Gamma(TM)$,

$$\tilde{\nabla}_XN_\alpha = -A_\alpha X + \nabla_X^{\perp}N_\alpha, \tag{17}$$

$$h_\alpha(X, Y) = \tilde{g}(A_\alpha X, Y). \tag{18}$$

Remark 2 [8] The normal connection $\nabla^{\perp}_X N_\alpha$ has the decomposition $\nabla^{\perp}_X N_\alpha = \sum_{\beta=1}^{r} l_{\alpha\beta}(X) N_\beta$ for $\alpha \in \{1, \dots, r\}$ and $\forall X \in \Gamma(TM)$, where $(l_{\alpha\beta})_r$ is an $r \times r$ matrix of 1-forms on M. Moreover [16] $\tilde{g}(N_\alpha, N_\beta) = \delta_{\alpha\beta}$ implies that $\tilde{g}(\nabla^{\perp}_X N_\alpha, N_\beta) + \tilde{g}(N_\alpha, \nabla^{\perp}_X N_\beta) = 0$ which is equivalent to $l_{\alpha\beta} = -l_{\beta\alpha}$ for any $\alpha, \beta \in \{1, \dots, r\}$ and $\forall X \in \Gamma(TM)$.

The covariant derivatives of T, N, t, n (given in Eqs. 4, 5) are given by $\forall X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$,

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{19}$$

$$(\tilde{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{20}$$

$$(\tilde{\nabla}_X t)V = \nabla_X tV - t\nabla^{\perp}_X V, \tag{21}$$

$$(\tilde{\nabla}_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V. \tag{22}$$

From Eq. (2) it follows that $\forall X, Y, Z \in \Gamma(TM)$,

$$\tilde{g}((\tilde{\nabla}_X J)Y, Z) = \tilde{g}(Y, (\tilde{\nabla}_X J)Z). \tag{23}$$

Hence if M is an isometrically immersed submanifold of the metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then [6] $\forall X, Y, Z \in \Gamma(TM)$,

$$\tilde{g}((\tilde{\nabla}_X T)Y, Z) = \tilde{g}(Y, (\tilde{\nabla}_X T)Z). \tag{24}$$

Now we can define locally metallic (or locally golden) Riemannian manifold analogously as a locally product manifold [17] in the following manner [16].

A metallic (or golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ is called *locally metallic (or locally golden) Riemannian manifold* if J is parallel with respect to $\tilde{\nabla}$, i.e. $\tilde{\nabla} J = 0$.

We now state some propositions regarding submanifolds of locally metallic (or locally golden) Riemannian manifolds [8]:

Proposition 1 *Let M be a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then* $\forall X, Y \in \Gamma(TM)$,

$$T[X, Y] = \nabla_X TY - \nabla_Y TX - A_{NY}X + A_{NX}Y, \tag{25}$$

$$N[X, Y] = h(X, TY) - h(TX, Y) + \nabla^{\perp}_X NY - \nabla^{\perp}_Y NX. \tag{26}$$

Proposition 2 *Let M be a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then* $\forall X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$,

$$(\nabla_X T)Y = A_{NY}X + th(X, Y), \tag{27}$$

$$(\tilde{\nabla}_X N)Y = nh(X, Y) - h(X, TY), \tag{28}$$

$$(\nabla_X t)V = A_{nV}X - TA_V X, \tag{29}$$

$$(\tilde{\nabla}_X n)V = -h(X, tV) - NA_V X. \tag{30}$$

Proposition 3 *Let M be a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with co-dimension r, then the structure $(T, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on M by the metallic (or golden) Riemannian structure $(\tilde{g}, J)$ satisfies the following properties* [16] $\forall X, Y \in \Gamma(TM)$

$$(\nabla_X T)Y = \sum_{\alpha=1}^{r} h_\alpha(X, Y)\xi_\alpha + \sum_{\alpha=1}^{r} u_\alpha(Y) A_\alpha X, \tag{31}$$

$$(\nabla_X u_\alpha)Y = -h_\alpha(X, TY) + \sum_{\beta=1}^{r}[u_\beta(Y) l_{\alpha\beta}(X) + h_\beta(X, Y) a_{\beta\alpha}]. \tag{32}$$

Proposition 4 *Let M be a submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then* $\forall X, Y \in \Gamma(TM)$,

$$T[X, Y] = \nabla_X TY - \nabla_Y TX - \sum_{\alpha=1}^{r}[u_\alpha(Y) A_\alpha X - u_\alpha(X) A_\alpha Y], \tag{33}$$

$$N[X, Y] = \sum_{\alpha=1}^{r}[\{(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y\} + \sum_{\beta=1}^{r}\{u_\alpha(X) l_{\alpha\beta}(Y) - u_\alpha(Y) l_{\alpha\beta}(X)\}]N_\alpha. \tag{34}$$

Now, we introduce the following definition:

Quasi hemi-slant submanifold M of a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ is a submanifold that admits three orthogonal complementary distributions D, D_θ, $D^{\perp}$ such that

(i) TM admits the orthogonal direct decomposition

$$TM = D \oplus D_\theta \oplus D^{\perp}, \tag{35}$$

(ii) The distribution D is invariant i.e., $JD = D$,

(iii) The distribution D_θ is slant with constant angle θ and hence θ is called *slant angle*,

(iv) The distribution $D^{\perp}$ is anti-invariant i.e., $JD^{\perp} \subseteq T^{\perp}M$.

In the above case, θ is called the *quasi hemi-slant angle* of M, and M is called *proper* if $D \neq \{0\}$, $D_\theta \neq \{0\}$, $D^{\perp} \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Let the dimensions of the distributions D, D_θ, $D^\perp$ be n_1, n_2, n_3 respectively, then we obtain the following particular cases

(i) If $n_1 = 0$, then M is a hemi-slant submanifold,
(ii) If $n_2 = 0$, then M is a semi-invariant submanifold,
(iii) If $n_3 = 0$, then M is a semi-slant submanifold.

Note. From now on, in this paper, we will write the term *quasi hemi-slant* in its abbreviated form i.e., *QHS*.

Let M be a QHS submanifold of a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ and the projections of $X \in \Gamma(TM)$ on the distributions D, D_θ, $D^\perp$ be P, Q, R respectively, then, we have $\forall X \in \Gamma(TM)$,

$$X = PX + QX + RX. \tag{36}$$

Using Eq. (4) in Eq. (36) we get

$$JX = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since $JD = D$, $JD^\perp \subseteq T^\perp M$, we have

$$JPX = TPX,\ NPX = 0,\ TRX = 0 \tag{37}$$

and hence we obtain

$$JX = TPX + TQX + NQX + NRX. \tag{38}$$

Comparing Eq. (38) with Eq. (4) we have

$$TX = TPX + TQX, \tag{39}$$

$$NX = NQX + NRX. \tag{40}$$

From Eq. (38) we have the following decomposition

$$J(TM) = TD \oplus TD_\theta \oplus ND_\theta \oplus ND^\perp. \tag{41}$$

Again, since $ND_\theta \subseteq \Gamma(T^\perp M)$, $ND^\perp \subseteq \Gamma(T^\perp M)$, we have another decomposition

$$T^\perp M = ND_\theta \oplus ND^\perp \oplus \mu, \tag{42}$$

where μ is the orthogonal complement of $ND_\theta \oplus ND^\perp$ in $\Gamma(T^\perp M)$ and it is anti-invariant with respect to J.

Moreover, for any $X \in \Gamma(TM)$ we have

$$\cos\theta(X) = \frac{\tilde{g}(JQX, TQX)}{\|TQX\|.\|JQX\|} = \frac{\|TQX\|}{\|JQX\|}. \tag{43}$$

3 QHS Submanifolds of Metallic (or golden) Riemannian Manifolds

In this section, we obtain some necessary and sufficient conditions for submanifolds to be quasi hemi-slant in metallic and golden Riemannian manifolds and also obtain the integrability conditions for the associated distributions along with some properties satisfied by them. At last, we construct an example of a quasi hemi-slant submanifold of a metallic Riemannian manifold.

Theorem 1 *If M is a QHS submanifold of a metallic Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with the quasi hemi-slant angle θ, then $\forall X, Y \in \Gamma(TM)$,*

$$\tilde{g}(TQX, TQY) = \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)], \tag{44}$$

$$\begin{aligned}\tilde{g}(NX, NY) = &- \sin^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)] \\ &- [p\tilde{g}(JRX, RY) + q\tilde{g}(RX, RY)].\end{aligned} \tag{45}$$

Proof Replacing X by $X + Y$ in Eq. (43) and then using Eq. (3) we get

$$\begin{aligned}\tilde{g}(TQX, TQY) &= \cos^2\theta\tilde{g}(JQX, JQY) \\ &= \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)]\end{aligned}$$

which gives Eq. (44).

Again from Eqs. (38) and (40) we have

$$TQX = JX - TPX - NX$$

on which applying Eqs. (36) and (37) we obtain

$$TQX = JQX + JRX - NX$$

and using it in Eq. (44) we get

$$\begin{aligned}&\tilde{g}(JQX, JQY) + \tilde{g}(JRX, JRY) + \tilde{g}(NX, NY) \\ &\quad = \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)].\end{aligned}$$

Now applying Eq. (3) on the above equation we get

$$\begin{aligned}&[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)] \\ &\quad + [p\tilde{g}(JRX, RY) + q\tilde{g}(RX, RY)] + \tilde{g}(NX, NY) \\ &= \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)]\end{aligned}$$

which gives Eq. (45). □

Corollary 1 *If M is a QHS submanifold of a golden Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with the quasi hemi-slant angle θ, then $\forall X, Y \in \Gamma(TM)$,*

$$\tilde{g}(TQX, TQY) = \cos^2\theta[\tilde{g}(JQX, QY) + \tilde{g}(QX, QY)], \tag{46}$$

$$\begin{aligned}\tilde{g}(NX, NY) = &- \sin^2\theta[\tilde{g}(JQX, QY) + \tilde{g}(QX, QY)] \\ &- [\tilde{g}(JRX, RY) + \tilde{g}(RX, RY)].\end{aligned} \tag{47}$$

Proof Putting $p = q = 1$ in Theorem 1, we get the required results. □

Theorem 2 *If M is a QHS submanifold of a metallic Riemannian manifold* $(\tilde{M}, \tilde{g}, J)$ *with the quasi hemi-slant angle* θ, *then*

$$T^2 Q = \cos^2\theta[pJQ + qQ]. \tag{48}$$

Proof Using Eq. (6) in Eq. (44) we get

$$\tilde{g}(T^2 QX, QY) = \cos^2\theta[p\tilde{g}(JQX, QY) + q\tilde{g}(QX, QY)]$$

which gives Eq. (48). □

Corollary 2 *If M is a QHS submanifold of a golden Riemannian manifold* $(\tilde{M}, \tilde{g}, J)$ *with the quasi hemi-slant angle* θ, *then*

$$T^2 Q = \cos^2\theta(J + I)Q, \tag{49}$$

where I is the identity mapping on $\Gamma(D_\theta)$.

Proof Putting $p = q = 1$ in Theorem 2, we get the required result. □

Theorem 3 *An immersed submanifold M of a metallic Riemannian manifold* $(\tilde{M}, \tilde{g}, J)$ *is QHS if and only if there exists a constant* $\lambda \in [0, 1)$ *such that* $D^* = \{X \in \Gamma(TM) : T^2 X = \lambda(pJX + qX)\}$ *is a distribution and* $D^{*\perp} = D$.

Proof If M is QHS, then putting $\cos^2\theta = \lambda \in [0, 1)$ in Eq. (48) we get $\forall X \in \Gamma(TM)$,

$$T^2 QX = \lambda[pJQX + qQX]$$

which implies that $QX \in \Gamma(D^*)$ and hence $D_\theta \subseteq D^*$.

Again, since from Eq. (37) we have $TRX = 0$, $T^2 RX = \lambda(pJRX + qRX)$ holds for $\lambda = 0$ and thus, $RX \in \Gamma(D^*)$ which implies that $D^\perp \subseteq D^*$.

Therefore, $D_\theta \oplus D^\perp \subseteq D^*$.

Next, let $X \in \Gamma(D^*)$ be a non-zero vector field, then by Eq. (36) we have

$$X = PX + QX + RX.$$

Now using Eqs. (37) and (1) we have

$$\begin{aligned} &pTPX + qPX = pJPX + qPX = J^2 PX = J(JPX) \\ &\quad = J(TPX) = T(TPX) = T^2 PX. \end{aligned}$$

Again, as $X \in \Gamma(D^*)$, from the above equation, we get on using Eq. (37),

$$\begin{aligned} &pTPX + qPX = \lambda(pTPX + qPX) \\ &\quad \Rightarrow (\lambda - 1)(pTPX + qPX) = 0 \\ &\quad \Rightarrow pTPX + qPX = 0 \text{ (since } \lambda \neq 1) \\ &\quad \Rightarrow TPX = \frac{-q}{p}PX \Rightarrow PX = 0 \text{ (since } \frac{q^2}{p^2} \neq 0 \text{ for } p, q \in \mathbb{N}) \\ &\quad \Rightarrow X = QX + RX \in D_\theta \oplus D^\perp \\ &\quad \Rightarrow D^* \subseteq D_\theta \oplus D^\perp. \end{aligned}$$

Thus we conclude that $D^* = D_\theta \oplus D^\perp$ and consequently $D^{*\perp} = D$.

Conversely, let there exists a constant $\lambda \in [0, 1)$ such that $D^* = \{X \in \Gamma(TM) : T^2 X = \lambda(pJX + qX)\}$ is a distribution and $D^{*\perp} = D$. Then from Eq. (48) we get for $X \in \Gamma(D^*)$,

$$\cos^2\theta(X) = \lambda \Rightarrow \cos\theta(X) = \sqrt{\lambda} \Rightarrow \theta(X) = \cos^{-1}(\sqrt{\lambda})$$

which does not depend on X.

We can consider the orthogonal decomposition $TM = D \oplus D_\theta \oplus D^\perp$.

Now, for $X \in \Gamma(D^*)$, $Y \in \Gamma(D^{*\perp}) = \Gamma(D)$ we have on applying Eqs. (4) and (6),

$$\begin{aligned} &\tilde{g}(X, J^2 Y) = \tilde{g}(X, J(JY)) = \tilde{g}(X, TJY) = \tilde{g}(TX, JY) \\ &\quad = \tilde{g}(TX, TY) = \tilde{g}(T^2 X, Y). \end{aligned}$$

Again, since $X \in \Gamma(D^*)$, using Eq. (3) we get from the above equation

$$\begin{aligned} &p\tilde{g}(X, JY) + q\tilde{g}(X, Y) = \lambda[p\tilde{g}(JX, Y) + q\tilde{g}(X, Y)] \\ &\quad \Rightarrow \tilde{g}(X, JY) = \lambda\tilde{g}(JX, Y) \\ &\quad \Rightarrow \tilde{g}(X, JY) = \lambda\tilde{g}(X, JY) \ \ (\textit{by Eq. } (2.2)) \\ &\quad \Rightarrow \tilde{g}(X, JY) = 0 \ \ (\textit{since } \lambda \neq 1) \\ &\quad \Rightarrow JY \in \Gamma(D^{*\perp}) = \Gamma(D) \\ &\quad \Rightarrow JD \subseteq D \ \ \textit{i.e. } D \textit{ is invariant.} \end{aligned}$$

Next, let $X \in \Gamma(D^\perp) \subseteq \Gamma(D_\theta \oplus D^\perp) = \Gamma(D^*)$ and $Y \in \Gamma(TM) = \Gamma(D \oplus D_\theta \oplus D^\perp)$. Then as before we get

$$\begin{aligned} &\tilde{g}(X, JY) = 0 \\ &\quad \Rightarrow \tilde{g}(JX, Y) = 0 \ \ (\textit{by Eq. } (2.2)) \\ &\quad \Rightarrow JX \in \Gamma(T^\perp M) \\ &\quad \Rightarrow JD^\perp \subseteq T^\perp M \ \ \textit{i.e. } D^\perp \textit{ is anti-invariant.} \end{aligned}$$

Therefore M is a QHS submanifold of $(\tilde{M}, \tilde{g}, J)$. □

Corollary 3 *An immersed submanifold M of a golden Riemannian manifold* $(\tilde{M}, \tilde{g}, J)$ *is QHS if and only if there exists a constant* $\lambda \in [0, 1)$ *such that* $D^* = \{X \in \Gamma(TM) : T^2 X = \lambda(JX + X)\}$ *is a distribution and* $D^{*\perp} = D$.

Proof Putting $p = q = 1$ in Theorem 3, we get the required results.

We now state and prove some results on QHS submanifolds of locally metallic (or locally golden) Riemannian manifolds: □

Theorem 4 *If M is a QHS submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$ with co-dimension r, then*

(i) The distribution D is integrable if and only if $\forall X, Y \in \Gamma(D)$,

$$(\nabla_Y u_\alpha)X = (\nabla_X u_\alpha)Y \ \ \forall \alpha \in \{1, \ldots, r\}, \tag{50}$$

(ii) The distribution D_θ is integrable if and only if $\forall X, Y \in \Gamma(D_\theta)$,

$$P(\nabla_X TY - \nabla_Y TX) = \sum_{\alpha=1}^{r}[u_\alpha(Y)PA_\alpha X - u_\alpha(X)PA_\alpha Y], \tag{51}$$

(iii) The distribution $D^\perp$ is integrable if and only if $\forall X, Y \in \Gamma(D^\perp)$,

$$u_\alpha(X)A_\alpha Y = u_\alpha(Y)A_\alpha X \ \ \forall \alpha \in \{1, \ldots, r\}. \tag{52}$$

Proof (i) Let $X, Y \in \Gamma(D)$, then $X = PX,\ Y = PY$.

Now, D is integrable if and only if $[X, Y] \in \Gamma(D)$ and $[X, Y] \in \Gamma(D)$ if and only if $N[X, Y] = 0$. Thus, D is integrable if and only if $N[X, Y] = 0$.

As $JD \subseteq D$, $NX = 0 = NY$. Hence from Eq. (11) we have $u_\alpha(X)l_{\alpha\beta}(Y) = u_\alpha(Y)l_{\alpha\beta}(X) = 0$.

Thus, from Eq. (34) we get

$$N[X, Y] = 0 \iff \sum_{\alpha=1}^{r}[(\nabla_Y u_\alpha)X - (\nabla_X u_\alpha)Y] = 0$$

$$\Rightarrow D \text{ is integrable if and only if } (\nabla_Y u_\alpha)X = (\nabla_X u_\alpha)Y \ \ \forall \alpha \in \{1, \ldots, r\}.$$

(ii) Let $X, Y \in \Gamma(D_\theta)$, then $X = QX,\ Y = QY$.

Now, D_θ is integrable if and only if $[X, Y] \in \Gamma(D_\theta)$ and $[X, Y] \in \Gamma(D_\theta)$ if and only if $PT[X, Y] = 0$. Therefore, D_θ is integrable if and only if $PT[X, Y] = 0$.

Again, from Eq. (33) we obtain

$$PT[X, Y] = 0 \iff P(\nabla_X TY - \nabla_Y TX) = \sum_{\alpha=1}^{r}[u_\alpha(Y)PA_\alpha X - u_\alpha(X)PA_\alpha(Y)].$$

(iii) Let $X, Y \in \Gamma(D^\perp)$, then $X = RX,\ Y = RY$.

Now, $D^\perp$ is integrable if and only if $[X, Y] \in \Gamma(D^\perp)$ and $[X, Y] \in \Gamma(D^\perp)$ if and only if $T[X, Y] = 0$. Thus, $D^\perp$ is integrable if and only if $T[X, Y] = 0$.

As $JD^\perp \subseteq T^\perp M$, $TX = 0 = TY$. Hence from Eq. (33) we have

$$T[X, Y] = 0 \iff u_\alpha(X)A_\alpha Y = u_\alpha(Y)A_\alpha X \ \ \forall \alpha \in \{1, \ldots, r\}.$$

□

Theorem 5 *If M is a QHS submanifold of a locally metallic (or locally golden) Riemannian manifold $(\tilde{M}, \tilde{g}, J)$, then*

(i) *The distribution D is integrable if and only if* $\forall X, Y \in \Gamma(D)$,

$$h(X, TY) = h(TX, Y), \tag{53}$$

(ii) *The distribution D is integrable if and only if* $\forall X \in \Gamma(D),\ V \in \Gamma(T^\perp M)$,

$$JA_V X = A_V JX, \tag{54}$$

(iii) *The distribution D_θ is integrable if and only if* $\forall X, Y \in \Gamma(D_\theta)$,

$$P(\nabla_X TY - \nabla_Y TX) = P(A_{NY}X - A_{NX}Y), \tag{55}$$

(iv) *The distribution $D^\perp$ is integrable if and only if* $\forall X, Y \in \Gamma(D^\perp)$,

$$A_{NX}Y = A_{NY}X. \tag{56}$$

Proof (i) D is integrable if and only if $[X, Y] \in \Gamma(D)$ and $[X, Y] \in \Gamma(D)$ if and only if $N[X, Y] = 0$ which implies that D is integrable if and only if $N[X, Y] = 0$.

As $JD \subseteq D$, $NX = 0 = NY$. Hence from Eq. (26) we have

$$N[X, Y] = 0 \iff h(X, TY) = h(TX, Y).$$

(ii) Again, for $X, Y \in \Gamma(D)$ and $V \in \Gamma(T^\perp M)$,

$$\begin{aligned} 0 &= \tilde{g}(h(X, TY) - h(TX, Y), V) \\ &= \tilde{g}(h(X, JY), V) - \tilde{g}(h(JX, Y), V) \ \ (\textit{since } TX = JX,\ TY = JY) \\ &= \tilde{g}(A_V X, JY) - \tilde{g}(A_V JX, Y) \ \ (\textit{by Eq.}\ (2.16)) \\ &= \tilde{g}(JA_V X, Y) - \tilde{g}(A_V JX, Y) \ \ (\textit{by Eq.}\ (2.2)) \\ &\Rightarrow \tilde{g}(h(X, TY) - h(TX, Y), V) = 0 \iff JA_V X = A_V JX. \end{aligned}$$

Hence from Theorem 5.(i) we obtain Eq. (54).

(iii) D_θ is integrable if and only if $[X, Y] \in \Gamma(D_\theta)$ and $[X, Y] \in \Gamma(D_\theta)$ if and only if $PT[X, Y] = 0$ which implies that D_θ is integrable if and only if $PT[X, Y] = 0$.

Hence from Eq. (25) we obtain

$$PT[X,Y]=0 \iff P(\nabla_X TY-\nabla_Y TX)$$
$$=P(A_{NY}X-A_{NX}Y).$$

(iv) $D^\perp$ is integrable if and only if $[X,Y]\in\Gamma(D^\perp)$ and $[X,Y]\in\Gamma(D^\perp)$ if and only if $T[X,Y]=0$ which implies that $D^\perp$ is integrable if and only if $T[X,Y]=0$.

As $JD^\perp\subseteq T^\perp M$, $TX=0=TY$. Hence from Eq. (25) we get

$$T[X,Y]=0 \iff A_{NX}Y=A_{NY}X.$$

□

Theorem 6 *If M is a QHS submanifold of a locally metallic (or locally golden) Riemannian manifold* $(\tilde{M},\tilde{g},J)$, *then N is parallel if and only* $\forall X\in\Gamma(TM),\ V\in\Gamma(T^\perp M)$,

$$A_{nV}X=TA_VX=A_VTX. \tag{57}$$

Proof Now, for $X,Y\in\Gamma(TM),\ V\in\Gamma(T^\perp M)$,

$$\tilde{g}(nh(X,Y),V)=\tilde{g}(Jh(X,Y),V)$$
$$=\tilde{g}(h(X,Y),nV) \text{ (by Eq. (2.7))}.$$

From Eq. (28) we have

$$\begin{aligned}
\tilde{g}((\tilde{\nabla}_X N)Y,V)&=\tilde{g}(nh(X,Y),V)-\tilde{g}(h(X,TY),V)\\
&=\tilde{g}(h(X,Y),nV)-\tilde{g}(h(X,TY),V)\ \ (\textit{by Eq.}\ (2.7))\\
&=\tilde{g}(A_{nV}X,Y)-\tilde{g}(A_VX,TY)\ \ (\textit{by Eq.}\ (2.16))\\
&=\tilde{g}(A_{nV}X,Y)-\tilde{g}(TA_VX,Y)\ \ (\textit{by Eq.}\ (2.6))\\
&=\tilde{g}(A_{nV}X-TA_VX,Y)=0\\
&\Rightarrow A_{nV}X=TA_VX.
\end{aligned} \tag{58}$$

Again from Eq. (28) we have

$$\begin{aligned}
\tilde{g}((\tilde{\nabla}_X N)Y,V)&=\tilde{g}(nh(X,Y),V)-\tilde{g}(h(X,TY),V)\\
&=\tilde{g}(h(X,Y),nV)-\tilde{g}(h(X,TY),V)\ \ (\textit{by Eq.}\ (2.7))\\
&=\tilde{g}(A_{nV}Y,X)-\tilde{g}(A_VTY,X)\ \ (\textit{by Eq.}\ (2.16))\\
&=\tilde{g}(A_{nV}Y-A_VTY,X)=0\\
&\Rightarrow A_{nV}Y=A_VTY.
\end{aligned} \tag{59}$$

Combining Eqs. (58) and (59) we obtain Eq. (57). □

Example We now construct an example of a QHS submanifold of a metallic Riemannian manifold.

Let us consider the Euclidean space $\mathbb{R}^{10}$ with usual Euclidean metric. Let $J:\mathbb{R}^{10}\to\mathbb{R}^{10}$ be the metallic structure defined by

$$J(X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9,X_{10})$$
$$=(\sigma X_1,\sigma X_2,\bar\sigma X_3,\bar\sigma X_4,\sigma X_5,\sigma X_6,\bar\sigma X_7,\bar\sigma X_8,\sigma X_9,\sigma X_{10}),$$

where $\sigma=\sigma_{pq}=\frac{p+\sqrt{p^2+4q}}{2}>0$ is a metallic number, $\bar\sigma=\frac{p-\sqrt{p^2+4q}}{2}=p-\sigma<0$, and $p,q\in\mathbb{N}$.

Now, by simple calculations we get $\sigma^2=p\sigma+q$ and similarly $\bar\sigma^2=p\bar\sigma+q$.

Hence, we have

$$J^2(X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9,X_{10})$$
$$=(pJ+qI)(X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9,X_{10})$$

so that $\mathbb{R}^{10}$ forms a metallic Riemannian manifold together with the usual Euclidean metric and the metallic structure J defined above.

Next, let $M=\{(u,\alpha_1,\alpha_2,\alpha_3): u>0, \alpha_i\in(0,\frac{\pi}{2})\}$ and $f:M\to\mathbb{R}^{10}$ be the immersion given by

$$f(u,\alpha_1,\alpha_2,\alpha_3)$$
$$=(u\cos\alpha_1,u\sin\alpha_1,u\cos\alpha_2,u\sin\alpha_2,u\cos\alpha_3,u\sin\alpha_3,u,\alpha_1,\alpha_2,\alpha_3).$$

We consider an orthonormal frame $\{Z_1,Z_2,Z_3,Z_4\}$ on TM such that

$$\begin{aligned}
Z_1=&\frac{1}{2}\Big(\cos\alpha_1\frac{\partial}{\partial x_1}+\sin\alpha_1\frac{\partial}{\partial x_2}+\cos\alpha_2\frac{\partial}{\partial x_3}+\sin\alpha_2\frac{\partial}{\partial x_4}\\
&+\cos\alpha_3\frac{\partial}{\partial x_5}+\sin\alpha_3\frac{\partial}{\partial x_6}+\frac{\partial}{\partial x_{10}}\Big),\\
Z_2=&\frac{1}{\sqrt{2}}\Big(-\sin\alpha_1\frac{\partial}{\partial x_1}+\cos\alpha_1\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_9}\Big),\\
Z_3=&\frac{1}{\sqrt{2}}\Big(-\sin\alpha_2\frac{\partial}{\partial x_3}+\cos\alpha_2\frac{\partial}{\partial x_4}+\frac{\partial}{\partial x_8}\Big),\\
Z_4=&\frac{1}{\sqrt{q(\sigma-\bar\sigma)}}\Big(-\bar\sigma\sqrt{\sigma}\frac{\partial}{\partial x_5}\sin\alpha_3+\bar\sigma\sqrt{\sigma}\frac{\partial}{\partial x_6}\cos\alpha_3\\
&+\sigma\sqrt{-\bar\sigma}\frac{\partial}{\partial x_7}\Big).
\end{aligned}$$

Thus we have

$$\begin{aligned}
JZ_1=&\frac{1}{2}\Big(\sigma\cos\alpha_1\frac{\partial}{\partial x_1}+\sigma\sin\alpha_1\frac{\partial}{\partial x_2}+\bar\sigma\cos\alpha_2\frac{\partial}{\partial x_3}\\
&+\bar\sigma\sin\alpha_2\frac{\partial}{\partial x_4}+\sigma\cos\alpha_3\frac{\partial}{\partial x_5}+\sigma\sin\alpha_3\frac{\partial}{\partial x_6}+\sigma\frac{\partial}{\partial x_{10}}\Big),\\
JZ_2=&\sigma Z_2,\ JZ_3=\bar\sigma Z_3,\\
JZ_4=&\frac{1}{\sqrt{q(\sigma-\bar\sigma)}}\Big(q\sqrt{\sigma}\frac{\partial}{\partial x_5}\sin\alpha_3-q\sqrt{\sigma}\frac{\partial}{\partial x_6}\cos\alpha_3\\
&-q\sqrt{-\bar\sigma}\frac{\partial}{\partial x_7}\Big).
\end{aligned}$$

Hence taking the distributions as

$$D=<Z_2,Z_3>,\ D_\theta=<Z_1>,\ D^\perp=<Z_4>$$

we have that the distribution D is invariant, the distribution $D^{\perp}$ is anti-invariant and the distribution D_{θ} is slant with the slant angle θ given by

$$\theta = \cos^{-1} \frac{\tilde{g}(JZ_1, Z_1)}{\|Z_1\|.\|JZ_1\|} = \cos^{-1} \frac{2p + \sqrt{p^2 + 4q}}{\sqrt{3p\sigma + p\bar{\sigma} + 4q}}.$$

Therefore, $TM = D \oplus D_{\theta} \oplus D^{\perp}$ and hence M is a QHS submanifold of $\mathbb{R}^{10}$.

4 Conclusion

Although the concept of quasi hemi-slant submanifold has been already introduced for contact metric manifolds, in this paper, we have modified that previous concept and introduced the general notion of quasi hemi-slant (QHS) submanifolds of metallic Riemannian manifolds, and further studied about it in detail. Also, we have constructed an example of such submanifold. Hence, the extensive applications of the topic of this paper (discussed in "Significance of the work") makes it an active and interesting field for researchers of various fields.

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References

1. Crasmareanu M, Hretcanu CE (2008) Golden differential geometry. Chaos Solitons Fractals 38(5):1229–1238
2. Hretcanu CE, Crasmareanu M (2009) Applications of the golden ratio on Riemannian manifolds. Turk J Math 33(2):179–191
3. Hretcanu CE, Crasmareanu M (2007) On some invariant submanifolds in a Riemannian manifold with golden structure. An Stiint Univ Al I Cuza Iasi (SN) Mat Tomul 53(1):199–211
4. Hretcanu CE, Crasmareanu M (2013) Metallic structures on Riemannian manifolds. Rev Union Mat Argent 54(2):15–27
5. Blaga AM (2014) The geometry of golden conjugate connections. Sarajevo J Math 10(23):237–245
6. Blaga AM, Hretcanu CE (2018) Metallic conjugate connections. Rev Union Mat Argent 59(1):179–192
7. Blaga AM, Hretcanu CE (2018) Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold. arXiv:1803.01415v2
8. Hretcanu CE, Blaga AM (2018) Slant and semi-slant submanifolds in metallic Riemannian manifolds. J Funct Spaces
9. Gezer A, Cengiz N, Salimov A (2013) On integrability of golden Riemannian structures. Turk J Math 37(4):693–703
10. Etayo F, Santamaria R, Upadhyay A (2017) On the geometry of almost golden Riemannian manifolds. Mediterr J Math 14(5):1–14
11. Goldberg SI, Yano K (1970) Polynomial structures on manifolds. Kodai Math Semin Rep 22:199–218
12. Goldberg SI, Petridis NC (1973) Differentiable solutions of algebraic equations on manifolds. Kodai Math Semin Rep 25:111–128
13. De Spinadel VW (2002) The metallic means family and forbidden symmetries. Int Math J 2(3):279–288
14. Prasad R, Verma SK, Kumar S (2020) Quasi hemi-slant submanifolds of Sasakian manifolds. J Math Comput Sci 10(2):418–435
15. Karmakar P, Bhattacharyya A (2023) Quasi hemi-slant submanifolds of trans-Sasakian manifold. Palest J Math 12(1):745–756
16. Hretcanu CE, Blaga AM (2018) Submanifolds in metallic Riemannian manifolds. Differ Geom Dyn Syst 20:83–97
17. Pitis G (1986) On some submanifolds of a locally product manifold. Kodai Math J 9(3):327–333

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TOTALLY CONTACT UMBILICAL SCREEN-SLANT AND SCREEN-TRANSVERSAL LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KENMOTSU MANIFOLD

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Abstract. We study totally contact umbilical screen-slant lightlike submanifolds and totally contact umbilical screen-transversal lightlike submanifolds of an indefinite Kenmotsu manifold. We prove a characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. We further prove some results on a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold, such as the necessary and sufficient conditions for the screen distribution $S(TM)$ to be integrable and for the induced connection ∇ to be a metric connection.

Keywords: indefinite Kenmotsu manifold; lightlike submanifold; totally contact umbilical screen-slant lightlike submanifold; totally contact umbilical radical screen-transversal lightlike submanifold

MSC 2020: 53C15, 53C20, 53C25, 53C40, 53C50

1. Introduction

The general theory of lightlike submanifolds of a semi-Riemannian manifold was developed by Duggal and Bejancu in 1996 (see [3]). Later, Sahin characterized lightlike submanifolds in many ways. In 2006, he introduced the notion of transversal lightlike submanifolds and studied some differential geometric properties of those submanifolds (see [11]). In 2008, he initiated the study of screen transversal lightlike submanifolds (see [12]). Gupta introduced the notions of slant and screen slant submanifolds in indefinite Kenmotsu manifolds, respectively, in 2011 with Sharfuddin

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(see [7]) and in 2010 with Upadhyay (see [8]). Gupta and Sharfuddin also conceptualised screen transversal lightlike submanifolds in the context of indefinite cosymplectic manifolds in 2010 (see [5]) and later in the context of indefinite Kenmotsu manifolds in 2011 (see [6]). In 2012, Haider et al. in [10] studied totally contact umbilical screen transversal lightlike submanifolds of an indefinite Sasakian manifold and recently, in 2021, Yadav et al. investigated the existence of totally contact umbilical screen-slant lightlike submanifolds of indefinite Sasakian manifolds (see [13]).

Motivated by the works mentioned above, in this paper we study totally contact umbilical screen-slant lightlike submanifolds and totally contact umbilical screen-transversal lightlike submanifolds of indefinite Kenmotsu manifold. This paper is divided into five sections. After introduction (first section) and preliminaries (second section), in the third section, we prove some results regarding screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. In the fourth section, we prove a characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold. In the last, i.e., the fifth section, we further prove some results on a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold, such as the necessary and sufficient conditions for the screen distribution $S(TM)$ to be integrable and for the induced connection ∇ to be a metric connection.

2. Preliminaries

A submanifold (M^m, g) which is immersed in a proper semi-Riemannian manifold $(\widetilde{M}^{m+n}, \widetilde{g})$ is called a *lightlike submanifold* (see [3]) if the metric g induced from $\widetilde{g}$ is degenerate and the radical distribution $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$ is of rank r such that $1 \leqslant r \leqslant m$. Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, i.e.,

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM).$$

Let us consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$, i.e.,

$$TM^{\perp} = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM^{\perp}).$$

Since for any local basis $\{\xi_i\}$ of $\mathrm{Rad}(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $\widetilde{g}(\xi_i, N_j) = \delta_{ij}$ and $\widetilde{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $\operatorname{ltr}(TM)$ locally spanned by $\{N_i\}$. Let $\operatorname{tr}(TM)$ be the complementary (not orthogonal) vector bundle to TM in $T\widetilde{M}$. Now we have the following decompositions (see [3]):

$$T\widetilde{M}|_M = TM \oplus \operatorname{tr}(TM), \quad \operatorname{tr}(TM) = S(TM^\perp) \oplus_{\mathrm{orth}} \operatorname{ltr}(TM),$$
$$T\widetilde{M}|_M = S(TM) \oplus_{\mathrm{orth}} [\operatorname{Rad}(TM) \oplus \operatorname{ltr}(TM)] \oplus_{\mathrm{orth}} S(TM^\perp).$$

A submanifold $(M, g, S(TM), S(TM^\perp))$ of $\widetilde{M}$ is called

▷ *r-lightlike* if $r < \min\{m, n\}$,
▷ *co-isotropic* if $r = n < m$, $S(TM^\perp) = \{0\}$,
▷ *isotropic* if $r = m < n$, $S(TM) = \{0\}$,
▷ *totally lightlike* if $r = m = n$, $S(TM) = \{0\} = S(TM^\perp)$.

An odd dimensional semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is called an *indefinite almost contact metric manifold* (see [1]) if it admits an indefinite almost contact structure (φ, ξ, η), where φ is a tensor field of type $(1,1)$, ξ is a vector field and η is a 1-form satisfying for all $X, Y \in \chi(\widetilde{M})$

$$\widetilde{g}(\varphi X, \varphi Y) = \widetilde{g}(X, Y) - \varepsilon\eta(X)\eta(Y), \quad \widetilde{g}(\xi, \xi) = \varepsilon = \pm 1, \tag{2.1}$$
$$\varphi^2 X = -X + \eta(X)\xi, \quad \widetilde{g}(X, \xi) = \varepsilon\eta(X), \tag{2.2}$$
$$\widetilde{g}(X, \varphi Y) = -\widetilde{g}(\varphi X, Y), \tag{2.3}$$
$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1. \tag{2.4}$$

De and Sarkar in [2] introduced the notion of ε-Kenmotsu manifolds with indefinite metric. An *indefinite Kenmotsu manifold* $\widetilde{M}(\varphi, \xi, \eta, \widetilde{g})$ satisfies the following structure equations for all $X, Y \in \chi(\widetilde{M})$:

$$(\widetilde{\nabla}_X \varphi)Y = \widetilde{g}(\varphi X, Y)\xi - \varepsilon\eta(Y)\varphi X, \tag{2.5}$$
$$\widetilde{\nabla}_X \xi = \varepsilon[X - \eta(X)\xi], \tag{2.6}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\widetilde{g}$.

A lightlike submanifold M of an indefinite Kenmotsu manifold $\widetilde{M}$, with the structure vector field ξ tangent to M, is called a *totally contact umbilical lightlike submanifold* (see [14]) if for a vector field α transversal to M and for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\alpha + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \tag{2.7}$$

where h is a symmetric bilinear form on $\Gamma(TM)$ with values in $\Gamma(\operatorname{tr}(TM))$ known as the *second fundamental form*. If $\alpha = 0$, then M is called a *totally contact geodesic lightlike submanifold*.

Now, equating components of (2.7) belonging to $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively, we have (see [4])

$$h^l(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_l + \eta(X)h^l(Y,\xi) + \eta(Y)h^l(X,\xi), \tag{2.8}$$
$$h^s(X,Y) = [g(X,Y) - \eta(X)\eta(Y)]\alpha_s + \eta(X)h^s(Y,\xi) + \eta(Y)h^s(X,\xi), \tag{2.9}$$

where $h^l(X,Y) = L(h(X,Y))$, $h^s(X,Y) = S(h(X,Y))$ (L, S are the projection morphisms of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$, $S(TM^{\perp})$, respectively) and $\alpha_l \in \Gamma(\mathrm{ltr}(TM))$, $\alpha_s \in \Gamma(S(TM^{\perp}))$. h^l and h^s are called the *lightlike second fundamental form* and the *screen second fundamental form* of M, respectively.

Let M be a lightlike submanifold of an indefinite Kenmotsu manifold $\widetilde{M}$ and ∇, $\widetilde{\nabla}$ be the Levi-Civita connections on M, $\widetilde{M}$, respectively. The Gauss and Weingarten formulae are given by:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \forall\, X,Y \in \Gamma(TM), \tag{2.10}$$
$$\widetilde{\nabla}_X V = -A_V X + \nabla^t_X V \quad \forall\, X \in \Gamma(TM),\ V \in \Gamma(\mathrm{tr}(TM)), \tag{2.11}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X,Y)$, $\nabla^t_X V \in \Gamma(\mathrm{tr}(TM))$. Here A is a linear operator on TM known as the *shape operator* and ∇^t is a linear connection on $\mathrm{tr}(TM)$ known as the *transversal linear connection* on M.

Now, the equations (2.10) and (2.11) further reduce to

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h^l(X,Y) + h^s(X,Y) \quad \forall\, X,Y \in \Gamma(TM),\\ \widetilde{\nabla}_X V &= -A_V X + D^l(X,V) + D^s(X,V) \quad \forall\, X \in \Gamma(TM),\ V \in \Gamma(\mathrm{tr}(TM)), \end{aligned} \tag{2.12}$$

where $D^l(X,V) = L(\nabla^t_X V)$, $D^s(X,V) = S(\nabla^t_X V)$.

In particular, we have

$$\widetilde{\nabla}_X U = -A_U X + \nabla^l_X U + D^s(X,U) \quad \forall\, U \in \Gamma(\mathrm{ltr}(TM)), \tag{2.13}$$
$$\widetilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W) \quad \forall\, W \in \Gamma(S(TM^{\perp})), \tag{2.14}$$

where ∇^l and ∇^s are linear connections on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$ called the *lightlike transversal connection* and the *screen transversal connection* on M, respectively.

Again, from (2.12)–(2.14) we get

$$\widetilde{g}(h^s(X,Y),W) + \widetilde{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.15}$$
$$\widetilde{g}(D^s(X,U),W) = \widetilde{g}(U, A_W X). \tag{2.16}$$

Let $\overline{P}$ be the projection morphism of TM on $S(TM)$, then we have for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(\mathrm{Rad}(TM))$,

$$\nabla_X \overline{P}Y = \nabla^*_X \overline{P}Y + h^*(X, \overline{P}Y), \tag{2.17}$$

$$\nabla_X V = -A^*_V X + \nabla^{*t}_X V, \tag{2.18}$$

where h^* is the local second fundamental form on $S(TM)$ and A^* is the shape operator of $\mathrm{Rad}(TM)$, $\nabla^*_X \overline{P}Y, A^*_V X \in \Gamma(S(TM))$ and $h^*(X, \overline{P}Y), \nabla^{*t}_X V \in \Gamma(\mathrm{Rad}(TM))$. Here ∇^* and ∇^{*t} are induced connections on $S(TM)$ and $\mathrm{Rad}(TM)$, respectively.

3. Screen-slant lightlike submanifolds

In this section, we prove some results regarding screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold.

Let M be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\widetilde{M}$ of index $2q$ such that $2q < \dim(M)$ with structure vector field ξ tangent to M, then M is called a *screen-slant lightlike submanifold* of $\widetilde{M}$ if the following conditions are satisfied (see [8]):

(i) $\mathrm{Rad}(TM)$ is invariant with respect to φ, i.e., $\varphi(\mathrm{Rad}(TM)) \subseteq \mathrm{Rad}(TM)$,

(ii) for any nonzero vector field Y tangent to $S(TM) = D \oplus_{\mathrm{orth}} \langle\xi\rangle$ at $y \in M$, the angle $\theta(Y)$ (known as the *slant angle*) between φY and $S(TM)$ is constant, where D is the complementary distribution to $\langle\xi\rangle$ in $S(TM)$ and Y, ξ are linearly independent.

M is called *proper* if $D \neq \{0\}$, $\theta \neq 0, \frac{1}{2}\pi$, and is called a *screen real lightlike submanifold* if $\theta = \frac{1}{2}\pi$. Then we have the decomposition

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} D \oplus_{\mathrm{orth}} \langle\xi\rangle .$$

Let P, Q be the projection morphisms of TM on $\mathrm{Rad}(TM)$, D, respectively, then for any $X \in \Gamma(TM)$, we have

$$X = PX + QX + \eta(X)\xi, \tag{3.1}$$

where $PX \in \Gamma(\mathrm{Rad}(TM))$, $QX \in \Gamma(D)$.

Again, for any $X \in \Gamma(TM)$, we have

$$\varphi X = TX + \omega X, \tag{3.2}$$

where $TX \in \Gamma(TM)$ and $\omega X \in \Gamma(\mathrm{tr}(TM))$ are the tangential and transversal components of φX, respectively.

Now, applying φ on (3.1) we get

$$\varphi X = TPX + TQX + \omega QX. \tag{3.3}$$

$S(TM^{\perp})$ can be decomposed as

$$S(TM^{\perp}) = \omega Q(S(TM)) \oplus_{\text{orth}} \mu,$$

where μ is an invariant subspace of $T\widetilde{M}$. Then for any $W \in \Gamma(S(TM^{\perp}))$, we have

$$\varphi W = BW + CW, \tag{3.4}$$

where $BW \in \Gamma(S(TM))$, $CW \in \Gamma(S(TM^{\perp}))$.

Also, for any $N \in \Gamma(\text{ltr}(TM))$,

$$\varphi N = CN, \tag{3.5}$$

where $CN \in \Gamma(\text{ltr}(TM))$.

Now, we state and prove some results.

Theorem 3.1. *Let M be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\widetilde{M}$ with constant index $2q < \dim(M)$, then M is a screen-slant lightlike submanifold if and only if there exists a constant $\lambda \in [-1, 0]$ such that for all $X \in \Gamma(S(TM))$,*

$$(P \circ T)^2 X = \lambda[-X + \eta(X)\xi], \tag{3.6}$$

where $\lambda = \cos^2 \theta|_{S(TM)}$.

P r o o f. The proof follows from Theorem 3.1 in [9]. □

Corollary 3.2. *Let (M, g) be a screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then for all $X, Y \in \Gamma(TM)$,*

$$g(TQX, TQY) = \cos^2 \theta|_{S(TM)}[g(X, Y) - \varepsilon\eta(X)\eta(Y)], \tag{3.7}$$

$$\widetilde{g}(\omega QX, \omega QY) = \sin^2 \theta|_{S(TM)}[g(X, Y) - \varepsilon\eta(X)\eta(Y)]. \tag{3.8}$$

P r o o f. The proof follows from Corollary 3.2 in [9]. □

Theorem 3.3. *Let (M, g) be a screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then for all $X, Y \in \Gamma(TM)$,*

$$(\nabla_X T)Y = A_{\omega Y}X + Bh^s(X, Y) + \widetilde{g}(\varphi X, Y)\xi - \varepsilon\eta(Y)TX, \tag{3.9}$$

$$(\nabla_X \omega)Y = Ch^s(X, Y) + Ch^l(X, Y) - h^s(X, TY) - h^l(X, TY) - D^l(X, \omega Y) - \varepsilon\eta(Y)\omega X, \tag{3.10}$$

where $(\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y)$ and $(\nabla_X \omega)Y = \nabla^s_X \omega Y - \omega(\nabla_X Y)$.

Proof. From (2.5) we get

$$\widetilde{\nabla}_X\varphi Y=\varphi\widetilde{\nabla}_XY+\widetilde{g}(\varphi X,Y)\xi-\varepsilon\eta(Y)\varphi X. \tag{3.11}$$

Applying (3.2) on (3.11) we obtain

$$\widetilde{\nabla}_X(TY+\omega Y)=\varphi\widetilde{\nabla}_XY+\widetilde{g}(\varphi X,Y)\xi-\varepsilon\eta(Y)(TX+\omega X),$$

on which applying (2.12), (2.14), (3.2), (3.4), (3.5), we get

$$\begin{aligned}\nabla_XTY&+h^l(X,TY)+h^s(X,TY)-A_{\omega Y}X+\nabla^s_X\omega Y+D^l(X,\omega Y)\\&=T\nabla_XY+\omega\nabla_XY+Ch^l(X,Y)+Bh^s(X,Y)+Ch^s(X,Y)\\&\quad+\widetilde{g}(\varphi X,Y)\xi-\varepsilon\eta(Y)(TX+\omega X).\end{aligned} \tag{3.12}$$

Equating tangential and transversal components of (3.12) we obtain (3.9) and (3.10), respectively. □

4. Totally contact umbilical screen-slant lightlike submanifolds

In this section, we prove the following characterization theorem of totally contact umbilical screen-slant lightlike submanifolds of an indefinite Kenmotsu manifold.

Theorem 4.1. *Let (M,g) be a totally contact umbilical screen-slant lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M},\widetilde{g})$, then at least one of the following statements is true:*

(i) *M is a screen real lightlike submanifold,*
(ii) *$D=\{0\}$,*
(iii) *if M is a proper screen-slant lightlike submanifold, then $\alpha_s\in\Gamma(\mu)$.*

Proof. For any $Y=QY\in\Gamma(D)$, from (2.7) we have

$$h(TQY,TQY)=g(TQY,TQY)\alpha,$$

on which applying (2.3), (2.5), (2.10), (2.12), (2.14), (3.1), (3.2), (3.7), we get

$$\begin{aligned}\varphi(\nabla_{TQY}QY&+h^l(TQY,QY)+h^s(TQY,QY))+A_{\omega QY}TQY-\nabla^s_{TQY}\omega QY\\&-D^l(TQY,\omega QY)-\nabla_{TQY}TQY-g(TQY,TQY)\xi=\cos^2\theta g(Y,Y)\alpha,\end{aligned}$$

which (by the help of (2.8), (2.9), (3.2)) reduces to

$$
\begin{aligned}
(4.1)\qquad & T\nabla_{TQY}QY + \omega\nabla_{TQY}QY + A_{\omega QY}TQY - \nabla^s_{TQY}\omega QY - D^l(TQY, \omega QY) \\
& - \nabla_{TQY}TQY - g(TQY, TQY)\xi = \cos^2\theta g(Y, Y)\alpha,
\end{aligned}
$$

since $g(TQY, QY) = \widetilde{g}(\varphi Y, Y) = -\widetilde{g}(Y, \varphi Y) = -g(TQY, QY) \Rightarrow g(TQY, QY) = 0$.

Equating transversal components of (4.1) we obtain

$$
(4.2)\qquad \omega\nabla_{TQY}QY - \nabla^s_{TQY}\omega QY - D^l(TQY, \omega QY) = \cos^2\theta g(Y, Y)\alpha.
$$

Now, taking covariant derivative of (3.8) with respect to TQY we get

$$
(4.3)\qquad \widetilde{g}(\nabla^s_{TQY}\omega QY, \omega QY) = \sin^2\theta g(\nabla^s_{TQY}Y, Y).
$$

Again, from (3.8) we have

$$
(4.4)\qquad \widetilde{g}(\omega\nabla_{TQY}QY, \omega QY) = \sin^2\theta g(\nabla^s_{TQY}Y, Y).
$$

Now, taking inner product of (4.2) with ωQY we obtain

$$
\begin{aligned}
\widetilde{g}(\omega\nabla_{TQY}QY, \omega QY) - \widetilde{g}(\nabla^s_{TQY}\omega QY, \omega QY) &= \cos^2\theta g(Y, Y)\widetilde{g}(\alpha_s, \omega QY) \\
&\Rightarrow \cos^2\theta g(Y, Y)\widetilde{g}(\alpha_s, \omega QY) = 0 \text{ (by (4.3), (4.4))} \\
&\Rightarrow \theta = \frac{\pi}{2} \text{ or } Y = 0 \text{ or } \alpha_s \in \Gamma(\mu),
\end{aligned}
$$

which gives that either M is a screen real lightlike submanifold or $D = \{0\}$ or $\alpha_s \in \Gamma(\mu)$ if M is proper. This completes the proof. □

5. Totally contact umbilical radical screen-transversal lightlike submanifolds

In this section, we prove some results on a totally contact umbilical radical screen-transversal lightlike submanifold M of an indefinite Kenmotsu manifold $\widetilde{M}$, such as the necessary and sufficient conditions for the screen distribution $S(TM)$ to be integrable and for the induced connection ∇ to be a metric connection.

First we state the following definitions from [12].

▷ An r-lightlike submanifold M of an indefinite Kenmotsu manifold $\widetilde{M}$ is called a *screen-transversal lightlike submanifold* if $\varphi(\mathrm{Rad}(TM)) \subseteq S(TM^\perp)$.

▷ A screen-transversal lightlike submanifold M of an indefinite Kenmotsu manifold $\widetilde{M}$ is called a *radical screen-transversal lightlike submanifold* if $S(TM)$ is invariant with respect to φ, i.e., $\varphi(S(TM)) \subseteq S(TM)$.

Next, we prove the following results.

Theorem 5.1. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then $S(TM)$ is integrable if and only if α_s has no component in $\varphi(\mathrm{Rad}(TM))$.*

P r o o f. For any $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(\mathrm{Rad}(TM))$, using (2.1), (2.3), (2.5), (2.9), (2.12) we get

$$\widetilde{g}([X, Y], N) = \widetilde{g}(h^s(X, \varphi Y) - h^s(Y, \varphi X), \varphi N) = 2g(X, \varphi Y)\widetilde{g}(\alpha_s, \varphi N),$$

which implies that $[X, Y] \in \Gamma(S(TM))$ for all $X, Y \in \Gamma(S(TM))$ if and only if $\widetilde{g}(\alpha_s, \varphi N) = 0$ for all $N \in \Gamma(\mathrm{Rad}(TM))$.

This completes the proof. □

Theorem 5.2. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then $h^* = 0$ if and only if α_s has no component in $\varphi(\mathrm{Rad}(TM))$.*

P r o o f. For any $X, Y \in \Gamma(S(TM))$, using (2.5), (2.12) we have

$$\begin{aligned} \nabla_X \varphi Y &+ h^l(X, \varphi Y) + h^s(X, \varphi Y) \\ &= \widetilde{g}(\varphi X, Y)\xi - \varepsilon\eta(Y)\varphi X + \varphi(\nabla_X Y + h^l(X, Y) + h^s(X, Y)). \end{aligned} \tag{5.1}$$

Taking inner product of (5.1) with φN for any $N \in \Gamma(\mathrm{Rad}(TM))$, we obtain

$$\widetilde{g}(h^s(X, \varphi Y), \varphi N) = \widetilde{g}(\varphi \nabla_X Y, \varphi N). \tag{5.2}$$

Now, using (2.1), (2.9), (2.17) in (5.2) we get

$$\widetilde{g}(\alpha_s, \varphi N)g(X, \varphi Y) = \widetilde{g}(h^*(X, Y), N),$$

which implies our assertion. □

Theorem 5.3. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then the induced connection ∇ on M is a metric connection if and only if α_s has no component in $\varphi(\mathrm{Rad}(TM))$.*

P r o o f. For any $X \in \Gamma(TM)$ and $N \in \Gamma(\mathrm{Rad}(TM))$, using (2.5) we get

$$\widetilde{\nabla}_X \varphi N - \varphi(\widetilde{\nabla}_X N) = \widetilde{g}(\varphi X, N)\xi,$$

on which applying φ and then using (2.2), (2.4), we obtain

$$\widetilde{\nabla}_X N = -\varphi(\widetilde{\nabla}_X \varphi N). \tag{5.3}$$

Using (2.12), (2.14) in (5.3) and then taking inner product with $Y \in \Gamma(S(TM))$ and then using (2.3) we get

$$g(\nabla_X N, Y) = -g(A_{\varphi N}X, \varphi Y) + \widetilde{g}(\nabla^s_X \varphi N, \varphi Y) + \widetilde{g}(D^l(X, \varphi N), \varphi Y),$$

in which using (2.9), (2.15), we obtain

$$g(\nabla_X N, Y) = -g(X, \varphi Y)\widetilde{g}(\alpha_s, \varphi N).$$

Therefore, ∇ is a metric connection on M if and only if $\mathrm{Rad}(TM)$ is parallel if and only if $\nabla_X N \in \Gamma(\mathrm{Rad}(TM))$ for all $X \in \Gamma(TM)$, $N \in \Gamma(\mathrm{Rad}(TM))$ if and only if $\widetilde{g}(\alpha_s, \varphi N) = 0$ for all $N \in \Gamma(\mathrm{Rad}(TM))$. This completes the proof. □

Theorem 5.4. *Let (M, g) be a totally contact umbilical radical screen-transversal lightlike submanifold of an indefinite Kenmotsu manifold $(\widetilde{M}, \widetilde{g})$, then*

(i) $A_{\varphi N}X = [X - \varepsilon\eta(X)\xi]\widetilde{g}(\alpha_s, \varphi N) + \varepsilon\eta(X)\varphi N + D^l(X, \varphi N)$ *for all* $X \in \Gamma(S(TM)), N \in \Gamma(\mathrm{ltr}(TM))$,

(ii) $A_{\varphi N}X = X\widetilde{g}(\alpha_s, \varphi N) + D^l(X, \varphi N)$ *for all* $X \in \Gamma(\mathrm{Rad}(TM))$, $N \in \Gamma(\mathrm{ltr}(TM))$.

P r o o f. Replacing W by φN in (2.15) we have

$$g(A_{\varphi N}X, Y) = \widetilde{g}(h^s(X, Y), \varphi N) + \widetilde{g}(Y, D^l(X, \varphi N)),$$

on which applying (2.6), (2.9), (2.12), we get

$$\begin{aligned} g(A_{\varphi N}X, Y) = [g(X, Y) - \eta(X)\eta(Y)]\widetilde{g}(\alpha_s, \varphi N) + \varepsilon\eta(X)\widetilde{g}(Y, \varphi N) \\ + \varepsilon\eta(Y)\widetilde{g}(X, \varphi N) + \widetilde{g}(Y, D^l(X, \varphi N)), \end{aligned}$$

which gives

$$A_{\varphi N}X = [X - \varepsilon\eta(X)\xi]\widetilde{g}(\alpha_s, \varphi N) + \varepsilon\eta(X)\varphi N + \widetilde{g}(X, \varphi N)\xi + D^l(X, \varphi N). \tag{5.4}$$

Then (i) and (ii) immediately follow from (5.4) restricting X to $S(TM)$ and $\mathrm{Rad}(TM)$, respectively. □

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References

[1] *A. Bonome, R. Castro, E. García-Río, L. Hervella*: Curvature of indefinite almost contact manifolds. J. Geom. *58* (1997), 66–86. zbl MR doi
[2] *U. C. De, A. Sarkar*: On (ε)-Kenmotsu manifolds. Hadronic J. *32* (2009), 231–242. zbl MR
[3] *K. L. Duggal, A. Bejancu*: Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. Mathematics and Its Applications 364. Kluwer Academic, Dordrecht, 1996. zbl MR doi
[4] *K. L. Duggal, B. Sahin*: Lightlike submanifolds of indefinite Sasakian manifolds. Int. J. Math. Math. Sci. *2007* (2007), Article ID 57585, 21 pages. zbl MR doi
[5] *R. S. Gupta, A. Sharfuddin*: Screen transversal lightlike submanifolds of indefinite cosymplectic manifolds. Rend. Semin. Mat. Univ. Padova *124* (2010), 145–156. zbl MR doi
[6] *R. S. Gupta, A. Sharfuddin*: Screen transversal lightlike submanifolds of indefinite Kenmotsu manifolds. Sarajevo J. Math. *7* (2011), 103–113. zbl MR
[7] *R. S. Gupta, A. Sharfuddin*: Slant lightlike submanifolds of indefinite Kenmotsu manifolds. Turk. J. Math. *35* (2011), 115–127. zbl MR
[8] *R. S. Gupta, A. Upadhyay*: Screen slant lightlike submanifolds of indefinite Kenmotsu manifolds. Kyungpook Math. J. *50* (2010), 267–279. zbl MR doi
[9] *S. M. K. Haider, Advin, M. Thakur*: Screen slant lightlike submanifolds of indefinite Sasakian manifolds. Kyungpook Math. J. *52* (2012), 443–457. zbl MR doi
[10] *S. M. K. Haider, M. Thakur, Advin*: Totally contact umbilical screen transversal lightlike submanifolds of an indefinite Sasakian manifold. Note Mat. *32* (2012), 123–134. zbl MR doi
[11] *B. Sahin*: Transversal lightlike submanifolds of indefinite Kähler manifolds. An. Univ. Vest Timiş., Ser. Mat.-Inform. *44* (2006), 119–145. zbl MR
[12] *B. Sahin*: Screen transversal lightlike submanifolds of indefinite Kähler manifolds. Chaos Solitons Fractals *38* (2008), 1439–1448. zbl MR doi
[13] *A. Yadav, G. Shanker, R. Kaur*: An investigation on the existence of totally contact umbilical screen-slant lightlike submanifolds of indefinite Sasakian manifolds. Differ. Geom. Dyn. Syst. *23* (2021), 244–254. zbl MR
[14] *K. Yano, M. Kon*: Structures on Manifolds. Series in Pure Mathematics 3. World Scientific, Singapore, 1984. zbl MR doi

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CURVATURE TENSORS AND RICCI SOLITONS WITH RESPECT TO ZAMKOVOY CONNECTION IN ANTI-INVARIANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLD

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Abstract. The present paper deals with the study of some properties of anti-invariant submanifolds of trans-Sasakian manifold with respect to a new non-metric affine connection called Zamkovoy connection. The nature of Ricci flat, concircularly flat, ξ-projectively flat, M-projectively flat, ξ-M-projectively flat, pseudo projectively flat and ξ-pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection are discussed. Moreover, Ricci solitons on Ricci flat, concircularly flat, M-projectively flat and pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold admitting the aforesaid connection are studied. At last, some conclusions are made after observing all the results and an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified easily.

Keywords: anti-invariant submanifold; trans-Sasakian manifold; Zamkovoy connection; η-Einstein manifold; Ricci curvature tensor; concircular curvature tensor; projective curvature tensor; M-projective curvature tensor; pseudo projective curvature tensor; Ricci soliton

MSC 2020: 53C05, 53C15, 53C20, 53C25, 53C40

1. Introduction

The notion of Zamkovoy connection was introduced by Zamkovoy in 2009, see [32]. Later Biswas and Baishya applied this connection on generalized pseudo Ricci symmetric Sasakian manifolds (see [1]) and on almost pseudo symmetric Sasakian manifolds (see [2]). This connection was further studied by Blaga in 2015, see [3]. In 2020, Mandal and Das worked in detail on various curvature tensors of Sasakian and Lorentzian para-Sasakian manifolds admitting this new connection (see [11], [12],

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[13], [6]), and recently in 2021, they discussed LP-Sasakian manifolds equipped with this new connection and conharmonic curvature tensor, see [14].

For an n-dimensional almost contact metric manifold $M(\varphi, \xi, \eta, g)$ consisting of a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g with the Riemannian connection ∇, *Zamkovoy connection* ∇^* is defined as (see [32])

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\varphi Y. \tag{1.1}$$

Ricci flow was introduced by Hamilton in 1982 (see [8]). He observed that it can be used well in simplifying the structure of a manifold. He developed the concept to answer Thurston's geometric conjecture stating that each closed 3-manifold admits a geometric decomposition. The *Ricci flow equation* (see [8]) is given by

$$\frac{\partial g}{\partial t} = -2S,$$

where g, S, t are, respectively, the Riemannian metric, Ricci curvature tensor and time. *Ricci soliton*, which is a self similar solution of the above equation, was also introduced by Hamilton in [9]. It is represented by the triplet (g, V, λ) (where V, λ are, respectively, a vector field and a constant) satisfying the equation

$$L_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $L_V g$ is the Lie derivative of g along V (see [9]). Ricci soliton is called *shrinking*, *steady* or *expanding* according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively.

Curvature is the central subject in Riemannian geometry. It measures the distance between an manifold and a Euclidean space.

Yano introduced the notion of *concircular curvature tensor* C of type (1,3) on Riemannian manifold for an n-dimensional manifold M as

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type (1,3) and r is the scalar curvature (see [29]).

Hence, if we consider C^* as the concircular curvature tensor with respect to Zamkovoy connection, then for a $(2n+1)$-dimensional manifold we have

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.3}$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where R^* is the curvature tensor and r^* is the scalar curvature with respect to Zamkovoy connection.

Definition 1.1. A $(2n+1)$-dimensional manifold M is called *Ricci flat* with respect to Zamkovoy connection if $S^*(X,Y) = 0$ for all $X, Y \in \chi(M)$.

Definition 1.2 ([6]). A $(2n+1)$-dimensional manifold M is called *concircularly flat* with respect to Zamkovoy connection if $C^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$.

Yano and Bochner introduced the notion of *projective curvature tensor* P of type (1,3) for an n-dimensional manifold M as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]$$

for all smooth vector fields $X, Y, Z \in \chi(M)$, where S is the Ricci tensor of type (0,2) (see [30]). Thus, for dimension $(2n+1)$ we have

$$P^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{2n}[S^*(Y,Z)X - S^*(X,Z)Y], \tag{1.4}$$

where we consider P^* and S^*, respectively, as the projective curvature tensor and the Ricci curvature tensor with respect to Zamkovoy connection. Both of the above curvature tensors represent the deviation of a manifold from being a manifold of constant curvature (see [30], [29]).

Definition 1.3 ([12]). A $(2n+1)$-dimensional manifold M is called *projectively flat* with respect to Zamkovoy connection if $P^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$.

Definition 1.4 ([12]). A $(2n+1)$-dimensional manifold M is called ξ*-projectively flat* with respect to Zamkovoy connection if $P^*(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$.

Pokhariyal and Mishra introduced the notion of M-projective curvature tensor on a Riemannian manifold in 1971 (see [21]). Later Ojha studied its properties in [17], [18], [19]. This curvature tensor was further discussed by many researchers, see [4], [5], [11], [22], [26]. The *M-projective curvature tensor* $\overline{M}$ of rank 3 on an n-dimensional manifold M is given by

$$\begin{aligned}\overline{M}(X,Y)Z = R(X,Y)Z &- \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y]\\ &- \frac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY]\end{aligned}$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where Q is the Ricci operator (see [21]).

Thus, for a $(2n+1)$-dimensional manifold, considering $\overline{M}^*$ as the M-projective curvature tensor with respect to Zamkovoy connection we get

$$\begin{aligned}\overline{M}^*(X,Y)Z = R^*(X,Y)Z &- \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y]\\ &- \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y],\end{aligned} \tag{1.5}$$

where Q^* is the Ricci operator with respect to Zamkovoy connection.

Definition 1.5 ([11]). A $(2n+1)$-dimensional manifold M is called *M-projectively flat* with respect to Zamkovoy connection if $\overline{M}^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$.

Definition 1.6 ([11]). A $(2n+1)$-dimensional manifold M is called *ξ-M-projectively flat* with respect to Zamkovoy connection if $\overline{M}^*(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$.

Prasad introduced the notion of pseudo projective curvature tensor in a Riemannian manifold of dimension $n > 2$ in 2002, see [23]. Its properties were further studied by many researchers on various manifolds (see [13], [15], [16], [24], [28]). The *pseudo projective curvature tensor* $\overline{P}$ of rank 3 on an n-dimensional manifold M is given by

$$\overline{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] + cr[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where a, b, c are nonzero constants related as $c = -n^{-1}(a(n-1)^{-1} + b)$, see [23].

Thus, for a $(2n+1)$-dimensional manifold, considering $\overline{P}^*$ as the pseudo projective curvature tensor with respect to Zamkovoy connection, we get

$$\begin{aligned}\overline{P}^*(X,Y)Z = {}& aR^*(X,Y)Z + b[S^*(Y,Z)X - S^*(X,Z)Y] \\ & + cr^*[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.6}$$

where a, b, c are nozero constants related as

$$c = -\frac{1}{2n+1}\Big(\frac{a}{2n} + b\Big). \tag{1.7}$$

Definition 1.7 ([13]). A $(2n+1)$-dimensional manifold M is called *pseudo projectively flat* with respect to Zamkovoy connection if $\overline{P}^*(X,Y)Z = 0$ for all $X, Y, Z \in \chi(M)$.

Definition 1.8 ([13]). A $(2n+1)$-dimensional manifold M is called *ξ-pseudo projectively flat* with respect to Zamkovoy connection if $\overline{P}^*(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$.

Motivated by the works mentioned above, in this paper the study was done on Ricci flat, concircularly flat, ξ-projectively flat, M-projectively flat, ξ-M-projectively flat, pseudo projectively flat and ξ-pseudo projectively flat anti-invariant submanifolds of a trans-Sasakian manifold with respect to Zamkovoy connection. This paper consists of seven sections. After introduction, the second section consists of a short description of trans-Sasakian manifold and anti-invariant submanifold. In the third, fourth, sixth and seventh section, Ricci flat, concircularly flat, M-projectively flat and pseudo projectively flat anti-invariant submanifolds of a trans-Sasakian manifold admitting Zamkovoy connection are discussed, respectively. Ricci solitons on those submanifolds are also studied. Also we have found out the conditions under which an anti-invariant submanifold of a trans-Sasakian manifold is ξ-projectively, ξ-M-projectively and ξ-pseudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection in the fifth, sixth and seventh section, respectively. At last, three conclusions are made after observing all the results and an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified easily.

2. Preliminaries

Let M be an odd dimensional differentiable manifold equipped with a metric structure (φ, ξ, η, g) consisting of a (1,1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the relations

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi) \quad \forall\, X, Y \in \chi(M). \tag{2.3}$$

Then M is called *almost contact metric manifold* (see [7]). An almost contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is called *trans-Sasakian manifold of type* (α, β) (α, β are smooth functions on M) if for all $X, Y \in \chi(M)$ (see [7])

$$(\nabla_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \tag{2.4}$$

$$\nabla_X \xi = -\alpha\varphi X + \beta[X - \eta(X)\xi]. \tag{2.5}$$

In a trans-Sasakian manifold of type (α, β), we have the following relations (see [7])

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta[g(X, Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$\begin{aligned} R(X, Y)\xi = {} & (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] \\ & + [(Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y], \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(\xi, Y)X = {} & (\alpha^2 - \beta^2)[g(X, Y)\xi - \eta(X)Y] + 2\alpha\beta[g(\varphi X, Y)\xi + \eta(X)\varphi Y] \\ & + g(\varphi X, Y)(\operatorname{grad}\alpha) - g(\varphi X, \varphi Y)(\operatorname{grad}\beta) \\ & + (X\alpha)\varphi Y + (X\beta)[Y - \eta(Y)\xi], \end{aligned} \tag{2.8}$$

$$S(X, \xi) = [2n(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (\varphi X)\alpha - (2n-1)(X\beta), \tag{2.9}$$

$$Q\xi = [2n(\alpha^2 - \beta^2) - \xi\beta]\xi + \varphi(\operatorname{grad}\alpha) - (2n-1)(\operatorname{grad}\beta). \tag{2.10}$$

Now we state the following lemma.

Lemma 2.1 ([25]). *In a* $(2n+1)$*-dimensional trans-Sasakian manifold of type* (α, β)*, if* $\varphi(\operatorname{grad} \alpha) = (2n-1)(\operatorname{grad} \beta)$*, then* $\xi\beta = 0$*.*

In 1977, anti-invariant submanifolds of Sasakian space forms were discussed by Yano and Kon (see [31]). In 1985, Pandey and Kumar investigated anti-invariant submanifolds of almost para-contact manifolds (see [20]). Recently, Karmakar and Bhattacharyya studied anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds (see [10]).

Let φ be a differentiable map from a manifold M into a manifold $\widetilde{M}$ and let the dimensions of M, $\widetilde{M}$ be n, m, respectively. If at each point p of M, $(\varphi_*)_p$ is a 1-1 map, i.e., if rank $\varphi{=}n$, then φ is called an *immersion* of M into $\widetilde{M}$.

If an immersion φ is one-one, i.e., if $\varphi(p) \neq \varphi(q)$ for $p \neq q$, then φ is called an *imbedding* of M into $\widetilde{M}$.

If the manifolds M, $\widetilde{M}$ satisfy the following two conditions, then M is called a *submanifold* of $\widetilde{M}$:

(i) $M \subset \widetilde{M}$,

(ii) the identity map i from M into $\widetilde{M}$ is an imbedding of M into $\widetilde{M}$.

A submanifold M is called *anti-invariant* if $X \in T_x(M) \Rightarrow \varphi X \in T_x^{\perp}(M)$ for all $x \in M$, where $T_x(M)$, $T_x^{\perp}(M)$ are, respectively, the tangent space and the normal space at $x \in M$. Thus, in an anti-invariant submanifold M, we have for all $X, Y \in \chi(M)$,

$$g(X, \varphi Y) = 0. \tag{2.11}$$

3. Ricci flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section consists of the study of the nature of a $(2n+1)$-dimensional Ricci flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) with respect to Zamkovoy connection and further a Ricci soliton on it.

Using (2.5), (2.6) on (1.1) we get the expression of Zamkovoy connection on $\widetilde{M}$ as

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\varphi Y + \alpha\eta(Y)\varphi X - \beta\eta(Y)X + \beta g(X, Y)\xi - \alpha g(\varphi X, Y)\xi \tag{3.1}$$

with torsion tensor

$$T^*(X, Y) = (1-\alpha)[\eta(X)\varphi Y - \eta(Y)\varphi X] + \beta[\eta(X)Y - \eta(Y)X] + 2\alpha g(X, \varphi Y)\xi. \tag{3.2}$$

Again, we have

$$(\nabla^*_X g)(Y,Z) = \nabla^*_X g(Y,Z) - g(\nabla^*_X Y, Z) - g(Y, \nabla^*_X Z).$$

Then, using (3.1) in the above equation we obtain $\nabla^* g = 0$, i.e., Zamkovoy connection is a metric compatible connection on $\widetilde{M}$.

Now applying (2.11) in (3.1) and (3.2), respectively, we get the expression of Zamkovoy connection on M as

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\varphi Y + \alpha\eta(Y)\varphi X - \beta\eta(Y)X + \beta g(X,Y)\xi \tag{3.3}$$

with torsion tensor

$$T^*(X,Y) = (1-\alpha)[\eta(X)\varphi Y - \eta(Y)\varphi X] + \beta[\eta(X)Y - \eta(Y)X].$$

Applying (2.4), (2.5) and (3.3) on the equation

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z$$

we get for all $X, Y, Z \in \chi(M)$,

$$\begin{aligned} R^*(X,Y)Z = R(X,Y)Z &+ \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) \\ &+ \beta[\eta(X)\varphi Y - \eta(Y)\varphi X]\eta(Z) + \beta^2[g(Y,Z)X - g(X,Z)Y] \\ &+ \alpha\beta[g(X,Z)\varphi Y - g(Y,Z)\varphi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi. \end{aligned} \tag{3.4}$$

Consequently, if $\xi\beta = 0$, then we have

$$S^*(Y,Z) = S(Y,Z) - 2n\alpha^2\eta(Y)\eta(Z) + 2n\beta^2 g(Y,Z), \tag{3.5}$$

which implies that

$$Q^*Y = QY - 2n\alpha^2\eta(Y)\xi + 2n\beta^2 Y. \tag{3.6}$$

Now if M is Ricci flat with respect to Zamkovoy connection, then $S^*(Y,Z) = 0$, hence (3.5) implies

$$S(Y,Z) = 2n\alpha^2\eta(Y)\eta(Z) - 2n\beta^2 g(Y,Z). \tag{3.7}$$

Thus, using Lemma 2.1 and (3.7) we can state the following theorem.

Theorem 3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is Ricci flat with respect to Zamkovoy connection, then M is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Next, let us consider a Ricci soliton (g, ξ, λ) on M, then from (1.2) we get

$$\begin{aligned} &(L_\xi g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0 \\ &\qquad \Rightarrow g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) + 2S(Y,Z) + 2\lambda g(Y,Z) = 0. \end{aligned}$$

Using (2.5) and (2.11) on the above equation we obtain

$$2S(Y,Z) + 2(\lambda+\beta)g(Y,Z) - 2\beta\eta(Y)\eta(Z) = 0.$$

Setting $Z = \xi$ we get

$$S(Y,\xi) = -\lambda\eta(Y). \tag{3.8}$$

Putting $Z = \xi$ in (3.7) we obtain

$$S(Y,\xi) = 2n(\alpha^2 - \beta^2)\eta(Y). \tag{3.9}$$

Now, equating (3.8) and (3.9) we get $\lambda = 2n(\beta^2 - \alpha^2)$, which is < 0, $= 0$ or > 0 according to $|\beta| < |\alpha|$, $|\beta| = |\alpha|$ or $|\beta| > |\alpha|$. Thus, using Lemma 2.1 we can state the following theorem.

Theorem 3.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is Ricci flat with respect to Zamkovoy connection, then a Ricci soliton (g,ξ,λ) on M is shrinking, steady or expanding according to $|\beta| < |\alpha|$, $|\beta| = |\alpha|$ or $|\beta| > |\alpha|$, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

4. Concircularly flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the study of the nature of a $(2n+1)$-dimensional concircularly flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) with respect to Zamkovoy connection given by (3.3) and also a Ricci soliton on it. From (3.5) we get

$$r^* = r - 2n\alpha^2 + 2n(2n+1)\beta^2. \tag{4.1}$$

As M is concircularly flat with respect to Zamkovoy connection, from (1.3) we have

$$R^*(X,Y)Z = \frac{r^*}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \Rightarrow S^*(Y,Z) = \frac{r^*}{2n+1}g(Y,Z).$$

Using (3.5) and (4.1) on the above equation we obtain

$$S(Y,Z) = \frac{r-2n\alpha^2}{2n+1}g(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z), \tag{4.2}$$

which, using Lemma 2.1, shows the following theorem.

Theorem 4.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then M is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Setting $Z=\xi$ in (4.2) we get

$$S(Y,\xi)=\frac{r+4n^2\alpha^2}{2n+1}\eta(Y). \tag{4.3}$$

Next, let us consider a Ricci soliton (g,ξ,λ) on M, then equating (3.8) and (4.3) we get

$$\lambda=-\frac{r+4n^2\alpha^2}{2n+1},$$

which is $<0,=0$ or >0 according to $r>-4n^2\alpha^2$, $r=-4n^2\alpha^2$ or $r<-4n^2\alpha^2$.

Hence, using Lemma 2.1 we have the following theorem.

Theorem 4.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then a Ricci soliton (g,ξ,λ) on M is shrinking, steady or expanding according to $r>-4n^2\alpha^2$, $r=-4n^2\alpha^2$ or $r<-4n^2\alpha^2$, provided $\varphi(\operatorname{grad}\alpha)=(2n-1)(\operatorname{grad}\beta)$.*

5. ξ-projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

In this section, it will be proved that a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ is ξ-projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Levi-Civita connection under certain conditions.

If $\varphi(\operatorname{grad}\alpha)=(2n-1)(\operatorname{grad}\beta)$, then using Lemma 2.1 from (1.4), (3.4) and (3.5) we have

$$\begin{aligned}
P^*(X,Y)Z &= R^*(X,Y)Z-\frac{1}{2n}[S^*(Y,Z)X-S^*(X,Z)Y]\\
&= R(X,Y)Z+\alpha^2[\eta(X)Y-\eta(Y)X]\eta(Z)\\
&\quad+\beta[\eta(X)\varphi Y-\eta(Y)\varphi X]\eta(Z)+\beta^2[g(Y,Z)X-g(X,Z)Y]\\
&\quad+\alpha\beta[g(X,Z)\varphi Y-g(Y,Z)\varphi X]+\beta[\nabla_Y g(X,Z)-\nabla_X g(Y,Z)]\xi\\
&\quad-\frac{1}{2n}[S(Y,Z)X-2n\alpha^2\eta(Y)\eta(Z)X+2n\beta^2 g(Y,Z)X\\
&\qquad-S(X,Z)Y+2n\alpha^2\eta(X)\eta(Z)Y-2n\beta^2 g(X,Z)Y]\\
&= P(X,Y)Z+\alpha^2[\eta(X)Y-\eta(Y)X]\eta(Z)\\
&\quad+\beta[\eta(X)\varphi Y-\eta(Y)\varphi X]\eta(Z)+\beta^2[g(Y,Z)X-g(X,Z)Y]\\
&\quad+\alpha\beta[g(X,Z)\varphi Y-g(Y,Z)\varphi X]+\beta[\nabla_Y g(X,Z)-\nabla_X g(Y,Z)]\xi\\
&\quad-[-\alpha^2\eta(Y)\eta(Z)X+\beta^2 g(Y,Z)X+\alpha^2\eta(X)\eta(Z)Y-\beta^2 g(X,Z)Y].
\end{aligned}$$

That implies

$$P^*(X,Y)\xi = P(X,Y)\xi + (\alpha\beta+\beta)[\eta(X)\varphi Y - \eta(Y)\varphi Y] + \beta[\nabla_Y\eta(X) - \nabla_X\eta(Y)]\xi.$$

Again, using (2.5), (2.11) on (2.6) we get

$$\nabla_X\eta(Y) = 0, \tag{5.1}$$

and applying it on the above equation we obtain

$$P^*(X,Y)\xi = P(X,Y)\xi + \beta(\alpha+1)[\eta(X)\varphi Y - \eta(Y)\varphi X] \Rightarrow P^*(X,Y)\xi = P(X,Y)\xi$$

if $\alpha = -1$ or $\beta = 0$ or X, Y are horizontal vector fields. Therefore we can state the following theorem.

Theorem 5.1. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is ξ-projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection if $\alpha=-1$ or $\beta=0$ or the vector fields are horizontal, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

6. M-projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

In this section, a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is taken and its nature is studied when it is M-projectively flat and ξ-M-projectively flat with respect to Zamkovoy connection. Also a Ricci soliton on M is discussed.

If M is M-projectively flat with respect to Zamkovoy connection, then from (1.5) we have

$$R^*(X,Y)Z = \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] + \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y]$$

and then

$$S^*(Y,Z) = \frac{r^*}{2n+1}g(Y,Z). \tag{6.1}$$

If $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$, then using (3.5) and (4.1) on (6.1) and by Lemma 2.1 we obtain

$$S(Y,Z) = \frac{r-2n\alpha^2}{2n+1}g(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z), \tag{6.2}$$

from which we can conclude the following theorem.

Theorem 6.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is M-projectively flat with respect to Zamkovoy connection, then M is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Putting $Z = \xi$ in (6.2) we have

$$S(Y,\xi) = \frac{r + 4n^2\alpha^2}{2n+1}\eta(Y). \tag{6.3}$$

Let us consider a Ricci soliton (g,ξ,λ) on M. Then equating (3.8) and (6.3) we get

$$\lambda = -\frac{r + 4n^2\alpha^2}{2n+1},$$

which is $< 0, = 0$ or > 0 according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$. Hence, by using Lemma 2.1 we can state the following theorem.

Theorem 6.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is M-projectively flat with respect to Zamkovoy connection, then a Ricci soliton (g,ξ,λ) on M is shrinking, steady or expanding according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Now if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$, then applying (3.4), (3.5) and (3.6) on (1.5) we have

$$\begin{aligned}\overline{M}^*(X,Y)Z = {} & \overline{M}(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) + \beta[\eta(X)\varphi Y - \eta(Y)\varphi X]\eta(Z) \\ & + \alpha\beta[g(X,Z)\varphi Y - g(Y,Z)\varphi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi \\ & + \frac{\alpha^2}{2}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] + \frac{\alpha^2}{2}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi.\end{aligned}$$

Putting $Z = \xi$ in the above equation and using (5.1) we obtain

$$\overline{M}^*(X,Y)\xi = \overline{M}(X,Y)\xi + \beta(\alpha+1)[\eta(X)\varphi Y - \eta(Y)\varphi X] + \frac{\alpha^2}{2}[\eta(X)Y - \eta(Y)X],$$

which implies that $\overline{M}^*(X,Y)\xi = \overline{M}(X,Y)\xi$ if X, Y are horizontal vector fields. Thus, we have the following theorem.

Theorem 6.3. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is ξ-M-projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection if the vector fields are horizontal, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

7. Pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the study of a pseudo projectively flat anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) admitting Zamkovoy connection along with a Ricci soliton on it. Also the condition is established under which M is ξ-pseudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection.

Since M is pseudo projectively flat with respect to Zamkovoy connection, from (1.6) we have

$$aR^*(X,Y)Z = b[S^*(X,Z)Y - S^*(Y,Z)X] + cr^*[g(X,Z)Y - g(Y,Z)X] \\ \Rightarrow (a+2nb)S^*(Y,Z) = -2cnr^*g(Y,Z). \tag{7.1}$$

Applying (1.7), (3.5), (4.1) and the condition $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$ of Lemma 2.1 on (7.1) we obtain

$$S(Y,Z) = 2n\alpha^2\eta(Y)\eta(Z) + \frac{r - 2n\alpha^2}{2n+1}g(Y,Z). \tag{7.2}$$

Thus, we can state the following theorem.

Theorem 7.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then M is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Setting $Z = \xi$ in (7.2) we have

$$S(Y,\xi) = \frac{r + 4n^2\alpha^2}{2n+1}\eta(Y). \tag{7.3}$$

Now, considering a Ricci soliton (g, ξ, λ) on M we have (3.8) and then equating it with (7.3) we get

$$\lambda = -\frac{r + 4n^2\alpha^2}{2n+1},$$

which is $< 0, = 0$ or > 0 according to $r > -4n^2\alpha^2, r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$. Thus, we get the following theorem.

Theorem 7.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then a Ricci soliton (g, ξ, λ) on M is shrinking, steady or expanding according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

Now, if $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$, then putting $Z = \xi$ in (1.6) and using (1.7), (3.4), (3.5), (4.1), (5.1) we obtain

$$\overline{P}^*(X,Y)\xi = \overline{P}(X,Y)\xi + a\beta(\alpha+1)[\eta(X)\varphi Y - \eta(Y)\varphi X] + \frac{2n}{2n+1}(a+2nb)\alpha^2[\eta(X)Y - \eta(Y)X]$$

implies $\overline{P}^*(X,Y)\xi = \overline{P}(X,Y)\xi$ if X, Y are horizontal vector fields. Hence, we can state the following theorem.

Theorem 7.3. *A $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is ξ-pseudo projectively flat with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection if the vector fields are horizontal, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.*

From Theorems 3.1, 4.1, 6.1 and 7.1 we make the following conclusion.

Conclusion 7.1.
(i) Ricci flat,
(ii) concircularly flat,
(iii) M-projectively flat or
(iv) pseudo projectively flat
$(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) admitting Zamkovoy connection is η-Einstein if $\varphi(\operatorname{grad}\alpha) = (2n-1)\times(\operatorname{grad}\beta)$.

Next, observing Theorems 4.2, 6.2 and 7.2 we reach the following interesting conclusion.

Conclusion 7.2. If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is
(i) concircularly flat,
(ii) M-projectively flat or
(iii) pseudo projectively flat
with respect to Zamkovoy connection, then a Ricci soliton (g,ξ,λ) on M is shrinking, steady or expanding according to $r > -4n^2\alpha^2$, $r = -4n^2\alpha^2$ or $r < -4n^2\alpha^2$, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.

Again, observing Theorems 5.1, 6.3 and 7.3 we can conclude the following.

Conclusion 7.3. For horizontal vector fields, a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\widetilde{M}$ of type (α,β) is
(i) ξ-projectively flat,
(ii) ξ-M-projectively flat and
(iii) ξ-pseudo projectively flat

with respect to Zamkovoy connection if and only if it is so with respect to Riemannian connection, provided $\varphi(\operatorname{grad}\alpha) = (2n-1)(\operatorname{grad}\beta)$.

Now we give the following example.

Example 7.1. Unit sphere S^5 is a trans-Sasakian manifold of type $(-1, 0)$ (see [27]). We here state an example of an anti-invariant submanifold of S^5 from [31] as:

Let $J = (a_{ts})$ $(t, s = 1, 2, 3, 4, 5, 6)$ be the almost complex structure of $\mathbb{C}^3$ such that $a_{2i,2i-1} = 1$, $a_{2i-1,2i} = -1$ $(i = 1, 2, 3)$ and all the other components are 0. Let $S^1\big(\frac{1}{\sqrt{3}}\big) = \{z \in \mathbb{C}\colon\ |z|^2 = \frac{1}{3}\}$. We consider $S^1\big(\frac{1}{\sqrt{3}}\big) \times S^1\big(\frac{1}{\sqrt{3}}\big) \times S^1\big(\frac{1}{\sqrt{3}}\big)$ in S^5 in $\mathbb{C}^3$. The position vector X of $S^1 \times S^1 \times S^1$ in S^5 in $\mathbb{C}^3$ has components given by

$$X = \frac{1}{\sqrt{3}}(\cos u^1, \sin u^1, \cos u^2, \sin u^2, \cos u^3, \sin u^3),$$

where u^1, u^2, u^3 are parameters on each $S^1\big(\frac{1}{\sqrt{3}}\big)$.

Let $X_i = \frac{\partial X}{\partial u^i}$, then we have

$$\begin{aligned} X_1 &= \frac{1}{\sqrt{3}}(-\sin u^1, \cos u^1, 0, 0, 0, 0),\\ X_2 &= \frac{1}{\sqrt{3}}(0, 0, -\sin u^2, \cos u^2, 0, 0),\\ X_3 &= \frac{1}{\sqrt{3}}(0, 0, 0, 0, -\sin u^3, \cos u^3). \end{aligned}$$

The vector field ξ on S^5 is given by

$$\xi = JX = \frac{1}{\sqrt{3}}(-\sin u^1, \cos u^1, -\sin u^2, \cos u^2, -\sin u^3, \cos u^3).$$

Since $\xi = X_1 + X_2 + X_3$, ξ is tangent to $S^1 \times S^1 \times S^1$. Also the structure tensors (φ, ξ, η) of S^5 satisfy

$$\varphi X_i = JX_i + \eta(X_i)X, \quad i = 1, 2, 3,$$

which shows that φX_i is normal to $S^1 \times S^1 \times S^1$ for all i. Thus, $S^1 \times S^1 \times S^1$ is an anti-invariant submanifold of S^5.

Now the results proved in this paper can be verified in the example given above very easily.

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References

[1] *K. K. Baishya, A. Biswas*: Study on generalized pseudo (Ricci) symmetric Sasakian manifold admitting general connection. Bull. Transilv. Univ. Braşov, Ser. III, Math. Inform. Phys. *12* (2019), 233–246. MR doi

[2] *A. Biswas, K. K. Baishya*: A general connection on Sasakian manifolds and the case of almost pseudo symmetric Sasakian manifolds. Sci. Stud. Res., Ser. Math. Inform. *29* (2019), 59–72. MR

[3] *A. M. Blaga*: Canonical connections on para-Kenmotsu manifolds. Novi Sad J. Math. *45* (2015), 131–142. zbl MR doi

[4] *S. K. Chaubey, R. H. Ojha*: On the m-projective curvature tensor of a Kenmotsu manifold. Differ. Geom. Dyn. Syst. *12* (2010), 52–60. zbl MR

[5] *S. K. Chaubey, S. Prakash, R. Nivas*: Some properties of m-projective curvature tensor in Kenmotsu manifolds. Bull. Math. Anal. Appl. *4* (2012), 48–56. zbl MR

[6] *A. Das, A. Mandal*: Study of Ricci solitons on concircularly flat Sasakian manifolds admitting Zamkovoy connection. Aligarh Bull. Math. *39* (2020), 47–61.

[7] *U. C. De, A. A. Shaikh*: Complex Manifolds and Contact Manifolds. Narosa Publishing House, New Delhi, 2009. zbl MR

[8] *R. S. Hamilton*: Three-manifolds with positive Ricci curvature. J. Differ. Geom. *17* (1982), 255–306. zbl MR doi

[9] *R. S. Hamilton*: The Ricci flow on surfaces. Mathematics and General Relativity. Contemporary Mathematics 71. AMS, Providence, 1988, pp. 237–262. zbl MR doi

[10] *P. Karmakar, A. Bhattacharyya*: Anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds. Bull. Calcutta Math. Soc. *112* (2020), 95–108.

[11] *A. Mandal, A. Das*: On M-projective curvature tensor of Sasakian manifolds admitting Zamkovoy connection. Adv. Math., Sci. J. *9* (2020), 8929–8940. doi

[12] *A. Mandal, A. Das*: Projective curvature tensor with respect to Zamkovoy connection in Lorentzian para-Sasakian manifolds. J. Indones. Math. Soc. *26* (2020), 369–379. zbl MR doi

[13] *A. Mandal, A. Das*: Pseudo projective curvature tensor on Sasakian manifolds admitting Zamkovoy connection. Bull. Cal. Math. Soc. *112* (2020), 431–450.

[14] *A. Mandal, A. Das*: LP-Sasakian manifolds equipped with Zamkovoy connection and conharmonic curvature tensor. J. Indones. Math. Soc. *27* (2021), 137–149. MR doi

[15] *H. G. Nagaraja, G. Somashekhara*: On pseudo projective curvature tensor in Sasakian manifolds. Int. J. Contemp. Math. Sci. *6* (2011), 1319–1328. zbl MR

[16] *D. Narain, A. Prakash, B. Prasad*: A pseudo projective curvature tensor on a Lorentzian para-Sasakian manifold. An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat. *55* (2009), 275–284. zbl MR

[17] *R. H. Ojha*: On Sasakian manifold. Kyungpook Math. J. *13* (1973), 211–215. zbl MR

[18] *R. H. Ojha*: A note on the M-projective curvature tensor. Indian J. Pure Appl. Math. *8* (1977), 1531–1534. zbl MR

[19] *R. H. Ojha*: M-projectively flat Sasakian manifolds. Indian J. Pure Appl. Math. *17* (1986), 481–484. zbl MR

[20] *H. B. Pandey, A. Kumar*: Anti-invariant submanifolds of almost para-contact manifolds. Indian J. Pure Appl. Math. *16* (1985), 586–590. zbl MR

[21] *G. P. Pokhariyal, R. S. Mishra*: Curvature tensors and their relativistic significance. II. Yokohama Math. J. *19* (1971), 97–103. zbl MR

[22] *D. G. Prakasha, K. Mirji*: On the M-projective curvature tensor of a (k,μ)-contact metric manifold. Facta Univ., Ser. Math. Inf. *32* (2017), 117–128. zbl MR doi

[23] *B. Prasad*: A pseudo projective curvature tensor on a Riemannian manifold. Bull. Calcutta Math. Soc. *94* (2002), 163–166. zbl MR

[24] *A. A. Shaikh, H. Kundu*: On equivalency of various geometric structures. J. Geom. *105* (2014), 139–165. zbl MR doi
[25] *S. S. Shukla, D. D. Singh*: On (ε)-trans-Sasakian manifolds. Int. J. Math. Anal., Ruse *4* (2010), 2401–2414. zbl MR
[26] *J. P. Singh*: On m-projectively flat almost pseudo Ricci symmetric manifolds. Acta Math. Univ. Comen., New Ser. *86* (2017), 335–343. zbl MR
[27] *T. Takahashi*: Sasakian manifold with pseudo-Riemannian metric. Tohoku Math. J., II. Ser. *21* (1969), 271–290. zbl MR doi
[28] *M. M. Tripathi, P. Gupta*: On τ-curvature tensor in K-contact and Sasakian manifolds. Int. Electron. J. Geom. *4* (2011), 32–47. zbl MR
[29] *K. Yano*: Concircular geometry I. Concircular transformations. Proc. Acad., Tokyo *16* (1940), 195–200. zbl MR doi
[30] *K. Yano, S. Bochner*: Curvature and Betti Numbers. Annals of Mathematics Studies 32. Princeton University Press, Princeton, 1953. zbl MR doi
[31] *K. Yano, M. Kon*: Anti-invariant submanifolds of Sasakian space forms. I. Tohoku Math. J., II. Ser. *29* (1977), 9–23. zbl MR doi
[32] *S. Zamkovoy*: Canonical connections on paracontact manifolds. Ann. Global Anal. Geom. *36* (2009), 37–60. zbl MR doi

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η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold admitting Zamkovoy connection

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Abstract. The present paper deals with the study of η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to a new non-metric affine connection called Zamkovoy connection. An η-Ricci-Yamabe soliton and also two more solitons arose as its particular cases are studied on Ricci flat, concircularly flat, M-projectively flat and pseudo projectively flat anti-invariant submanifolds of trans-Sasakian manifold with respect to the aforesaid connection. At last, some conclusions are made after observing all the results and an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified very easily.

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Key words: η-Ricci-Yamabe soliton; q-η-Yamabe soliton; p-η-Ricci soliton; anti-invariant submanifold; trans-Sasakian manifold; Zamkovoy connection; η-Einstein manifold; Ricci flat; concircularly flat; M-projectively flat; pseudo projectively flat.

1 Introduction

The concepts of Ricci flow and Yamabe flow were introduced simultaneously by R. S. Hamilton in 1988[17]. Ricci soliton and Yamabe soliton emerged as limits of solutions of Ricci flow and Yamabe flow respectively. These solitons are equivalent in dimension 2 but in greater dimensions, these two do not agree since Yamabe soliton preserves the conformal class of the metric but Ricci soliton does not in general. In 2019, S. Guler and M. Crasmareanu[16] introduced a new geometric flow called Ricci-Yamabe flow as a scalar combination of Ricci flow and Yamabe flow. *Ricci-Yamabe flow of type* (p, q) is an evolution for the metrics on Riemannian or semi-Riemannian manifolds defined as[16]

$$\frac{\partial}{\partial t}g(t) = -2pRic(t) + qr(t)g(t), \; g(0) = g_0, \tag{1.1}$$

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where $p,\ q$ are scalars. Due to the signs of $p,\ q$, this flow can also be a Riemannian flow or semi-Riemannian flow or singular Riemannian flow. Naturally, Ricci-Yamabe soliton emerged as the limit of solutions of Ricci-Yamabe flow. An interpolation soliton, called Ricci-Bourguignon soliton corresponding to Ricci-Bourguignon flow, was considered and further studied by G. Catino and L. Mazzieri ([7], [8]), but this soliton depends on a single scalar.

Ricci-Yamabe soliton of type (p, q) on Riemannian complex (M, g) is represented by the quintuplet (g, V, λ, p, q) satisfying the following equation−

$$L_V g + 2pS + (2\lambda - qr)g = 0, \tag{1.2}$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, r is the scalar curvature, S is the Ricci curvature tensor and $\lambda,\ p,\ q$ are scalars. This soliton is called shrinking, steady or expanding according as $\lambda < 0,\ \lambda = 0$ or $\lambda > 0$ respectively. Ricci-Yamabe soliton of type $(0, q)$ and $(p, 0)$ are called *q-Yamabe soliton* and *p-Ricci soliton* respectively. These solitons are studied by many geometers([4], [5], [6], [11], [13], [15], [37], [38]).

J. T. Cho and M. Kimura introduced the notion of η-Ricci soliton as an advance extension of Ricci soliton in 2009[12]. Analogously in 2020[39], M. D. Siddiqi and M. A. Akyol introduced the concept of η-Ricci-Yamabe soliton as a generalization of Ricci-Yamabe soliton. *η-Ricci-Yamabe soliton of type* (p, q) is represented by the sextuplet $(g, V, \lambda, \mu, p, q)$ on a Riemannian manifold M satisfying the following equation−

$$L_V g + 2pS + (2\lambda - qr)g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where $L_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, r is the scalar curvature, S is the Ricci curvature tensor, $\eta \otimes \eta$ is a $(0, 2)$-tensor field and $\lambda,\ \mu,\ p,\ q$ are scalars. The soliton is called shrinking, steady or expanding according as $\lambda < 0,\ \lambda = 0$ or $\lambda > 0$ respectively. η-Ricci-Yamabe soliton of type $(0, q)$ and $(p, 0)$ are called *q-η-Yamabe soliton* and *p-η-Ricci soliton* respectively. Recently, in 2021, G. Somashekhara et al. studied η-Ricci-Yamabe solitons on submanifolds of some indefinite almost contact manifolds concerning Riemannian and quarter symmetric metric connection[40].

The notion of Zamkovoy connection was introduced by S. Zamkovoy in 2009[45]. Later A. Biswas and K. K. Baishya applied this connection on generalized pseudo Ricci symmetric Sasakian manifolds[1] and on almost pseudo symmetric Sasakian manifolds[2]. This connection was further studied by A. M. Blaga in 2015[3]. In 2020, A. Mandal and A. Das worked in detail on various curvature tensors of Sasakian and Lorentzian para-Sasakian manifolds admitting this new connection([20], [21], [22], [23]), and recently in 2021, they discussed LP-Sasakian manifolds equipped with this connection and conharmonic curvature tensor[24]. Also, most recently in 2021, the author studied curvature tensors and Ricci solitons with respect to this connection in anti-invariant submanifolds of trans-Sasakian manifold[18].

For an n-dimensional almost contact metric manifold $M(\phi, \xi, \eta, g)$ consisting of a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g with the Riemannian connection ∇, *Zamkovoy connection* ∇^* is defined as[45]

$$\nabla^*_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y. \tag{1.4}$$

Curvature is the central subject in Riemannian geometry. It measures distance between a manifold and a Euclidean space.

K. Yano introduced the notion of *concircular curvature tensor* C of type (1,3) on Riemannian manifold for an n-dimensional manifold M as[43]

$$C(X,Y)Z = R(X,Y)Z - \tfrac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor of type (1,3) and r is the scalar curvature.

Hence if we consider C^* as the concircular curvature tensor with respect to Zamkovoy connection, then for a $(2n+1)$-dimensional manifold M we have

$$C^*(X,Y)Z = R^*(X,Y)Z - \tfrac{r^*}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.5}$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where R^* is the curvature tensor and r^* is the scalar curvature with respect to Zamkovoy connection.

Definition 1.1. A $(2n+1)$-dimensional manifold M is called *Ricci flat* with respect to Zamkovoy connection if $S^*(X,Y) = 0 \;\; \forall X, Y \in \chi(M)$, where S^* is the Ricci curvature tensor with respect to Zamkovoy connection.

Definition 1.2. [23] A $(2n+1)$-dimensional manifold M is called *concircularly flat* with respect to Zamkovoy connection if $C^*(X,Y)Z = 0 \;\; \forall X, Y, Z \in \chi(M)$.

G. P. Pokhariyal and R. S. Mishra introduced the notion of M-projective curvature tensor on a Riemannian manifold in 1971[31]. Later R. H. Ojha studied its properties([27], [28], [29]). This curvature tensor was further discussed by many geometers([9], [10], [20], [32], [35]). The *M-projective curvature tensor* $\bar{M}$ of rank 3 on an n-dimensional manifold M is given by[31]

$$\bar{M}(X,Y)Z = R(X,Y)Z - \tfrac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y] - \tfrac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where Q is the Ricci operator.

Thus for a $(2n+1)$-dimensional manifold, considering $\bar{M}^*$ as the M-projective curvature tensor with respect to Zamkovoy connection we get

(1.6) $\bar{M}^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] - \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y],$

where Q^* is the Ricci operator with respect to Zamkovoy connection.

Definition 1.3. [20] A $(2n+1)$-dimensional manifold M is called *M-projectively flat* with respect to Zamkovoy connection if $\bar{M}^*(X,Y)Z = 0 \ \ \forall X,Y,Z \in \chi(M)$.

B. Prasad introduced the notion of pseudo projective curvature tensor in a Riemannian manifold of dimension $n > 2$ in 2002[33]. Its properties were further studied by many geometers on various manifolds([22], [25], [26], [34], [42]). The *pseudo projective curvature tensor* $\bar{P}$ of rank 3 on an n-dimensional manifold M is given by[33]

$$\bar{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] + cr[g(Y,Z)X - g(X,Z)Y]$$

for all smooth vectors fields $X, Y, Z \in \chi(M)$, where $a,\ b,\ c$ are non-zero constants related as $c = -\frac{1}{n}\big(\frac{a}{n-1} + b\big)$.

Thus for a $(2n+1)$-dimensional manifold, considering $\bar{P}^*$ as the pseudo projective curvature tensor with respect to Zamkovoy connection we get

(1.7) $$\bar{P}^*(X,Y)Z = aR^*(X,Y)Z + b[S^*(Y,Z)X - S^*(X,Z)Y] + cr^*[g(Y,Z)X - g(X,Z)Y],$$

where $a,\ b,\ c$ are non-zero constants related as

(1.8) $$c = -\frac{1}{2n+1}\big(\frac{a}{2n} + b\big).$$

Definition 1.4. [22] A $(2n+1)$-dimensional manifold M is called *pseudo projectively flat* with respect to Zamkovoy connection if $\bar{P}^*(X,Y)Z = 0 \ \ \forall X,Y,Z \in \chi(M)$.

In 1977, anti-invariant submanifolds of Sasakian space forms[44] were discussed by K. Yano and M. Kon. In 1985, H. B. Pandey and A. Kumar investigated about anti-invariant submanifolds of almost para-contact manifolds[30]. Recently in 2020, the author and A. Bhattacharyya studied anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds[19].

Let φ be a differentiable map from a manifold M into a manifold $\tilde{M}$ and let the dimensions of $M, \tilde{M}$ be n, m respectively. If at each point p of M, $(\varphi_*)_p$ is a 1-1 map, i.e., if rankφ=n, then φ is called an *immersion* of M into $\tilde{M}$.

If an immersion φ is one-one, i.e., if $\varphi(p) \neq \varphi(q)$ for $p \neq q$, then φ is called an *imbedding* of M into $\tilde{M}$.

If the manifolds $M, \tilde{M}$ satisfy the following two conditions, then M is called a *submanifold* of $\tilde{M}-$
(i) $M \subset \tilde{M}$,

(ii) the inclusion map i from M into $\tilde{M}$ is an imbedding of M into $\tilde{M}$.

A submanifold M is called *anti-invariant* if $X \in T_x(M) \Rightarrow \phi X \in T_x^{\perp}(M)\ \ \forall x \in M$, where $T_x(M),\ T_x^{\perp}(M)$ are respectively the tangent space and the normal space at $x \in M$. Thus in an anti-invariant submanifold M, we have $\forall X, Y \in \chi(M)$,

$$g(X, \phi Y) = 0. \tag{1.9}$$

Motivated by the works mentioned above, in this paper, the study has been done on η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection. This paper is divided into three sections. After the introduction, in the preliminaries section, definition and some properties of a trans-Sasakian manifold of type (α, β) are given. After preliminaries, there remains the third section which concerns the main topic and it is further subdivided into four subsections dealing with the study of η-Ricci-Yamabe soliton, q-η-Yamabe soliton and p-η-Ricci soliton on an anti-invariant submanifold of a trans-Sasakian manifold, where the submanifold is (i) Ricci flat, (ii) concircularly flat, (iii) M-projectively flat and (iv) pseudo projectively flat respectively with respect to Zamkovoy connection. At last, three conclusions are made after observing all the results of the four subsections and an example of an anti-invariant submanifold of a trans-Sasakian manifold is given in which all the results can be verified very easily.

2 Preliminaries

Let $\tilde{M}$ be an odd dimensional differentiable manifold equipped with a metric structure (ϕ, ξ, η, g) consisting of a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the following relations−

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0,\ \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y),\ \eta(X) = g(X, \xi)\ \ \forall X, Y \in \chi(\tilde{M}), \tag{2.3}$$

then $\tilde{M}$ is called *almost contact metric manifold*[14].

An odd dimensional almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$ is called *trans-Sasakian manifold of type* (α, β) (α, β are smooth functions on $\tilde{M}$) if $\forall X, Y \in \chi(\tilde{M})$ [14]

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.4}$$

$$\nabla_X \xi = -\alpha\phi X + \beta[X - \eta(X)\xi]. \tag{2.5}$$

In a trans-Sasakian manifold $\tilde{M}^{2n+1}$ of type (α, β), we have the following relations[14]−

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta[g(X,Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ +[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y], \tag{2.7}$$

$$R(\xi,Y)X = (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(X)Y] + 2\alpha\beta[g(\phi X,Y)\xi + \eta(X)\phi Y] + (X\alpha)\phi Y \\ +g(\phi X,Y)(grad\ \alpha) - g(\phi X,\phi Y)(grad\ \beta) + (X\beta)[Y - \eta(Y)\xi], \tag{2.8}$$

$$S(X,\xi) = [2n(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (\phi X)\alpha - (2n-1)(X\beta), \tag{2.9}$$

$$Q\xi = [2n(\alpha^2 - \beta^2) - \xi\beta]\xi + \phi(grad\ \alpha) - (2n-1)(grad\ \beta). \tag{2.10}$$

Now we state the following lemma−

Lemma 2.1. [36] *In a $(2n+1)$-dimensional trans-Sasakian manifold of type (α,β), if $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$, then $\xi\beta = 0$.*

Using (2.5), (2.6) on (1.4) we get the expression of Zamkovoy connection on $\tilde{M}$ as−

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\phi Y + \alpha\eta(Y)\phi X - \beta\eta(Y)X + \beta g(X,Y)\xi - \alpha g(\phi X,Y)\xi \tag{2.11}$$

with torsion tensor

$$T^*(X,Y) = (1-\alpha)[\eta(X)\phi Y - \eta(Y)\phi X] + \beta[\eta(X)Y - \eta(Y)X] + 2\alpha g(X,\phi Y)\xi. \tag{2.12}$$

Again, we have

$$(\nabla^*_X g)(Y,Z) = \nabla^*_X g(Y,Z) - g(\nabla^*_X Y, Z) - g(Y, \nabla^*_X Z).$$

Then, using (2.11) in the above equation we obtain $\nabla^* g = 0$, i.e. Zamkovoy connection is a metric compatible connection on $\tilde{M}$.

Now applying (1.9) in (2.11) and (2.12) respectively we get the expression of Zamkovoy connection on an anti-invariant submanifold M of $\tilde{M}$ as−

$$\nabla^*_X Y = \nabla_X Y + \eta(X)\phi Y + \alpha\eta(Y)\phi X - \beta\eta(Y)X + \beta g(X,Y)\xi \tag{2.13}$$

with torsion tensor

$$T^*(X,Y) = (1-\alpha)[\eta(X)\phi Y - \eta(Y)\phi X] + \beta[\eta(X)Y - \eta(Y)X].$$

Setting $Y = \xi$ in (2.13) and then using (2.5) we obtain

$$\nabla^*_X \xi = 0. \tag{2.14}$$

Applying (2.4), (2.5) and (2.13) on the following equation

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z$$

we get $\forall X, Y, Z \in \chi(M)$,

$$(2.15) \quad R^*(X,Y)Z = R(X,Y)Z + \alpha^2[\eta(X)Y - \eta(Y)X]\eta(Z) + \beta[\eta(X)\phi Y - \eta(Y)\phi X]\eta(Z) + \beta^2[g(Y,Z)X - g(X,Z)Y] + \alpha\beta[g(X,Z)\phi Y - g(Y,Z)\phi X] + \beta[\nabla_Y g(X,Z) - \nabla_X g(Y,Z)]\xi.$$

Consequently, if $\xi\beta = 0$ and $\dim(M) = 2n+1$, then we have

$$S^*(Y,Z) = S(Y,Z) - 2n\alpha^2\eta(Y)\eta(Z) + 2n\beta^2 g(Y,Z) \tag{2.16}$$

which implies that

$$Q^*Y = QY - 2n\alpha^2\eta(Y)\xi + 2n\beta^2 Y, \tag{2.17}$$

$$r^* = r - 2n\alpha^2 + 2n(2n+1)\beta^2. \tag{2.18}$$

3 η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection

This section deals with the main topic, i.e. η-Ricci-Yamabe solitons on anti-invariant submanifolds of trans-Sasakian manifold with respect to Zamkovoy connection. This section is further subdivided into four subsections but before proceeding to these subsections, here two theorems are proved concerning the nature of a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) when an η-Ricci-Yamabe soliton of type (p, q) is considered on it with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on M with respect to Zamkovoy connection ∇^*, then from (1.3) we have $\forall Y, Z \in \chi(M)$

$$(L^*_\xi g)(Y,Z) + 2pS^*(Y,Z) + (2\lambda - qr^*)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0$$

$$\Rightarrow g(\nabla^*_Y \xi, Z) + g(\nabla^*_Z \xi, Y) + 2pS^*(Y,Z) + (2\lambda - qr^*)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

Using (2.14) in the above equation we get

$$S^*(Y,Z) = \left(\tfrac{qr^* - 2\lambda}{2p}\right)g(Y,Z) - \left(\tfrac{\mu}{p}\right)\eta(Y)\eta(Z). \tag{3.1}$$

Hence we can state the following theorem−

Theorem 3.1. *Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on an anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to Zamkovoy connection, then M is η-Einstein with respect to Zamkovoy connection.*

Again if $\xi\beta = 0$, then using (2.16) and (2.18) in (3.1) we obtain

$$S(Y,Z) = \big[\tfrac{q\{r-2n\alpha^2+2n(2n+1)\beta^2-2\lambda\}}{2p} - 2n\beta^2\big]g(Y,Z) + \big[2n\alpha^2 - \big(\tfrac{\mu}{p}\big)\big]\eta(Y)\eta(Z).$$

Thus applying lemma 2.1 we have the following theorem−

Theorem 3.2. *Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on a $(2n + 1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) with respect to Zamkovoy connection, then M is η-Einstein with respect to Riemannian connection, provided $\phi(grad\ \alpha) = (2n - 1)(grad\ \beta)$.*

3.1 η-Ricci-Yamabe solitons on Ricci flat anti-invariant submanifolds

Here η-Ricci-Yamabe soliton of type (p, q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is discussed, where M is Ricci flat with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci-Yamabe soliton on M, then from (1.3) we have $\forall Y, Z \in \chi(M)$

$$(L_\xi g)(Y,Z) + 2pS(Y,Z) + (2\lambda - qr)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0$$

$$\Rightarrow g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) + 2pS(Y,Z) + (2\lambda - qr)g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

Using (2.5) and then applying (1.9) in the above equation we get

$$pS(Y,Z) + \big(\lambda + \beta - \tfrac{qr}{2}\big)g(Y,Z) + (\mu - \beta)\eta(Y)\eta(Z) = 0.$$

Setting $Z = \xi$ in the above equation we obtain

$$pS(Y,\xi) = \big(\tfrac{qr}{2} - \lambda - \mu\big)\eta(Y). \tag{3.2}$$

Now if $\xi\beta = 0$ and M is Ricci flat with respect to ∇^*, then from (2.16) we have

$$S(Y,Z) = 2n\alpha^2\eta(Y)\eta(Z) - 2n\beta^2 g(Y,Z).$$

Setting $Z = \xi$ in the above equation and multiplying both sides by p we obtain

$$pS(Y,\xi) = 2np(\alpha^2 - \beta^2)\eta(Y). \tag{3.3}$$

Equating (3.2) and (3.3) we get

$$\lambda = \tfrac{qr}{2} - \mu - 2np(\alpha^2 - \beta^2). \tag{3.4}$$

Hence from (3.4) and applying lemma 2.1 we conclude the following theorem−

Theorem 3.1.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is Ricci flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton (g,ξ,λ,μ,p,q) on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < 2np(\alpha^2 - \beta^2)$, $\frac{qr}{2} - \mu = 2np(\alpha^2 - \beta^2)$ or $\frac{qr}{2} - \mu > 2np(\alpha^2 - \beta^2)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now from (3.4) we get, when $p = 0$ then $\lambda = \frac{qr}{2} - \mu$, and when $q = 0$ then $\lambda = -\mu + 2np(\beta^2 - \alpha^2)$. Thus from theorem 3.1.1 we respectively conclude the following results−

Corollary 3.1.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is Ricci flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.1.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is Ricci flat with respect to Zamkovoy connection, then a p-η-Ricci soliton (g,ξ,λ,μ,p) on M is shrinking, steady or expanding according as $2np(\beta^2 - \alpha^2) < \mu$, $2np(\beta^2 - \alpha^2) = \mu$ or $2np(\beta^2 - \alpha^2) > \mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

3.2 η-Ricci-Yamabe solitons on concircularly flat anti-invariant submanifolds

This subsection deals with the study of η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional concircularly flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Since M is concircularly flat with respect to ∇^*, from (1.5) we have

$$R^*(X,Y)Z = \tfrac{r^*}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]$$

which implies that

$$S^*(Y,Z) = \left(\tfrac{r^*}{2n+1}\right)g(Y,Z). \tag{3.5}$$

Let $\xi\beta = 0$, hence using (2.16) and (2.18) in (3.5) we obtain

$$S(Y,Z) = \left(\tfrac{r-2n\alpha^2}{2n+1}\right)g(Y,Z) + 2n\alpha^2\eta(Y)\eta(Z). \tag{3.6}$$

Putting $Z = \xi$ in (3.6) and then multiplying both sides by p we get

$$pS(Y,\xi) = p\left(\tfrac{r+4n^2\alpha^2}{2n+1}\right)\eta(Y). \tag{3.7}$$

Next, let (g,ξ,λ,μ,p,q) be an η-Ricci-Yamabe soliton on M, then equating (3.2) and (3.7) we obtain

$$\lambda = \tfrac{qr}{2} - \mu - p\left(\tfrac{r+4n^2\alpha^2}{2n+1}\right). \tag{3.8}$$

Thus, applying lemma 2.1, from (3.8) we state the following theorem−

Theorem 3.2.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton (g,ξ,λ,μ,p,q) on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $\frac{qr}{2} - \mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $\frac{qr}{2} - \mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Now from (3.8) we have, when $p = 0$ then $\lambda = \frac{qr}{2} - \mu$, and when $q = 0$ then $\lambda = -\mu - p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$. Thus from theorem 3.2.1 we respectively conclude the following results−

Corollary 3.2.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.2.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is concircularly flat with respect to Zamkovoy connection, then a p-η-Ricci soliton (g,ξ,λ,μ,p) on M is shrinking, steady or expanding according as $-\mu < p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$, $-\mu = p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ or $-\mu > p\left(\frac{r+4n^2\alpha^2}{2n+1}\right)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

3.3 η-Ricci-Yamabe solitons on M-projectively flat anti-invariant submanifolds

Here η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is discussed, where M is M-projectively flat with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Since M is M-projectively flat with respect to ∇^*, from (1.6) we have

$$R^*(X,Y)Z = \frac{1}{4n}[S^*(Y,Z)X - S^*(X,Z)Y] + \frac{1}{4n}[g(Y,Z)Q^*X - g(X,Z)Q^*Y]$$

from which we have

$$S^*(Y,Z) = \big(\tfrac{r^*}{2n+1}\big)g(Y,Z)$$

which is same as the equation (3.5). Hence, proceeding similarly as the previous subsection we get the following results−

Theorem 3.3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is M-projectively flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton (g,ξ,λ,μ,p,q) on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $\frac{qr}{2} - \mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $\frac{qr}{2} - \mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.3.1. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is M-projectively flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

Corollary 3.3.2. *If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is M-projectively flat with respect to Zamkovoy connection, then a p-η-Ricci soliton (g,ξ,λ,μ,p) on M is shrinking, steady or expanding according as $-\mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $-\mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $-\mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.*

3.4 η-Ricci-Yamabe solitons on pseudo projectively flat anti-invariant submanifolds

This subsection deals with the study of η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional pseudo projectively flat anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) with respect to Zamkovoy connection ∇^* given by the equation (2.13).

Since M is pseudo projectively flat with respect to ∇^*, from (1.7) we have

$$aR^*(X,Y)Z = b[S^*(X,Z)Y - S^*(Y,Z)X] + cr^*[g(X,Z)Y - g(Y,Z)X]$$

which implies that

$$(a+2nb)S^*(Y,Z) = -2cnr^*g(Y,Z).$$

Let $\xi\beta = 0$, then applying (1.8), (2.16) and (2.18) in the above equation we get

$$S(Y, Z) = \big(\tfrac{r-2n\alpha^2}{2n+1}\big)g(Y, Z) + 2n\alpha^2\eta(Y)\eta(Z)$$

which is same as the equation (3.6). Hence proceeding similarly as the subsection 3.2 we reach to the following results−

Theorem 3.4.1. *If a $(2n + 1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $\frac{qr}{2} - \mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $\frac{qr}{2} - \mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided $\phi(grad\ \alpha) = (2n - 1)(grad\ \beta)$.*

Corollary 3.4.1. *If a $(2n + 1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n - 1)(grad\ \beta)$.*

Corollary 3.4.2. *If a $(2n + 1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is pseudo projectively flat with respect to Zamkovoy connection, then a p-η-Ricci soliton $(g, \xi, \lambda, \mu, p)$ on M is shrinking, steady or expanding according as $-\mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $-\mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $-\mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided $\phi(grad\ \alpha) = (2n - 1)(grad\ \beta)$.*

Now, after carefully looking into the results of the four subsections of third section we reach to the following three conclusions.

First, observing theorems 3.2.1, 3.3.1 and 3.4.1 we get the following−

Conclusion 1. If a $(2n + 1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is (i) concircularly flat, (ii) M-projectively flat or (iii) pseudo projectively flat with respect to Zamkovoy connection, then an η-Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, p, q)$ on M is shrinking, steady or expanding according as $\frac{qr}{2} - \mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $\frac{qr}{2} - \mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $\frac{qr}{2} - \mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided $\phi(grad\ \alpha) = (2n - 1)(grad\ \beta)$.

Next, observing corollaries 3.1.1, 3.2.1, 3.3.1 and 3.4.1, we have

Conclusion 2. If a $(2n + 1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α, β) is (i) Ricci flat, (ii) concircularly flat, (iii) M-projectively flat or (iv) pseudo projectively flat with respect to Zamkovoy connection, then a q-η-Yamabe soliton $(g, \xi, \lambda, \mu, q)$ on M is shrinking, steady or expanding according as $qr < 2\mu$, $qr = 2\mu$ or $qr > 2\mu$ respectively, provided $\phi(grad\ \alpha) = (2n - 1)(grad\ \beta)$.

At last, observing corollaries 3.2.2, 3.3.2 and 3.4.2 we obtain the third conclusion:

Conclusion 3. If a $(2n+1)$-dimensional anti-invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ of type (α,β) is (i) concircularly flat, (ii) M-projectively flat or (iii) pseudo projectively flat with respect to Zamkovoy connection, then a p-η-Ricci soliton (g,ξ,λ,μ,p) on M is shrinking, steady or expanding according as $-\mu < p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$, $-\mu = p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ or $-\mu > p\big(\frac{r+4n^2\alpha^2}{2n+1}\big)$ respectively, provided that $\phi(grad\ \alpha) = (2n-1)(grad\ \beta)$.

Finally, we give the following example in which all the results proved in this paper can be verified very easily.

Example. Unit sphere S^5 is a trans-Sasakian manifold of type (-1,0)[41]. We here state an example of an anti-invariant submanifold of S^5 from [44] as:−

Let $J = (a_{ts})$ $(t,s = 1,2,3,4,5,6)$ be the almost complex structure of $\mathbb{C}^3$ such that $a_{2i,2i-1} = 1$, $a_{2i-1,2i} = -1$ $(i = 1,2,3)$ and all the other components are 0. Let $S^1(\frac{1}{\sqrt{3}}) = \{z \in \mathbb{C} : |z|^2 = \frac{1}{3}\}$. We consider $S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}}) \times S^1(\frac{1}{\sqrt{3}})$ in S^5 in $\mathbb{C}^3$. The position vector X of $S^1 \times S^1 \times S^1$ in S^5 in $\mathbb{C}^3$ has components given by

$$X = \frac{1}{\sqrt{3}}(cos\ u^1, sin\ u^1, cos\ u^2, sin\ u^2, cos\ u^3, sin\ u^3),$$

where u^1, u^2, u^3 are parameters on each $S^1(\frac{1}{\sqrt{3}})$.

Let $X_i = \frac{\partial X}{\partial u^i}$, then we have

$$X_1 = \frac{1}{\sqrt{3}}(-sin\ u^1, cos\ u^1, 0, 0, 0, 0),$$

$$X_2 = \frac{1}{\sqrt{3}}(0, 0, -sin\ u^2, cos\ u^2, 0, 0),$$

$$X_3 = \frac{1}{\sqrt{3}}(0, 0, 0, 0, -sin\ u^3, cos\ u^3).$$

The vector field ξ on S^5 is given by

$$\xi = JX = \frac{1}{\sqrt{3}}(-sin\ u^1, cos\ u^1, -sin\ u^2, cos\ u^2, -sin\ u^3, cos\ u^3).$$

Since $\xi = X_1 + X_2 + X_3$, ξ is tangent to $S^1 \times S^1 \times S^1$. Also the structure tensors (ϕ,ξ,η) of S^5 satisfy

$$\phi X_i = JX_i + \eta(X_i)X,\ i = 1,2,3,$$

which shows that ϕX_i is normal to $S^1 \times S^1 \times S^1$ for all i. Thus $S^1 \times S^1 \times S^1$ is an anti-invariant submanifold of S^5.

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References

[1] A. Biswas, K. K. Baishya, *Study on generalized pseudo (Ricci) symmetric Sasakian manifold admitting general connection*, Bull. Transilv. Univ. Brasov 12, 2 (2020), 233-246.

[2] A. Biswas, K. K. Baishya, *A general connection on Sasakian manifolds and the case of almost pseudo symmetric Sasakian manifolds*, Sci. Stud. Res., Ser. Math. Inform. 29, 1 (2019), 59-72.

[3] A. M. Blaga, *Canonical connections on para-Kenmotsu manifold*, Novi Sad J. Math. 45, 2 (2015), 131-142.

[4] A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. 20, 1 (2015), 1-13.

[5] A. M. Blaga, *On gradient η-Einstein solitons*, Kragujevac J. Math. 42, 2 (2018), 229-237.

[6] A. M. Blaga, S. Y. Perktas, *Remarks on almost η-Ricci solitons in (ϵ)-para Sasakian manifolds*, Commun. Fac. Sci. Univ. Ank., Ser. A1, Math. Stat. 68, 2 (2019), 1621-1628.

[7] G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, L. Mazzieri, *The Ricci-Bourguignon flow*, Pac. J. Math. 287, 2 (2017), 337-370.

[8] G. Catino, L. Mazzieri, *Gradient Einstein solitons*, Nonlinear Anal. 132 (2016), 66-94.

[9] S. K. Chaubey, R. H. Ojha, *On the m-projective curvature tensor of a Kenmotsu manifold*, Differ. Geom. Dyn. Syst. 12 (2010), 52-60.

[10] S. K. Chaubey, S. Prakash, S. Nivas, *Some properties of M-projective curvature tensor in Kenmotsu manifolds*, Bull. Math. Anal. Appl. 4, 3 (2012), 48-56.

[11] B. Y. Chen, S. Deshmukh, *Yamabe and quasi-Yamabe solitons on Euclidean submanifolds*, Mediterr. J. Math. 15, 5 (2018), 1-9.

[12] J. T. Cho, M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. 61, 2 (2009), 205-212.

[13] M. Crasmareanu, *A new approach to gradient Ricci solitons and generalizations*, Filomat 32, 9 (2018), 3337-3346.

[14] U. C. De, A. A. Shaikh, *Complex Manifolds and Contact Manifolds*, Narosa Publishing House, 2009.

[15] S. Guler, *On a class of gradient almost Ricci solitons*, Bull. Malays. Math. Sci. Soc. 43, 5 (2020), 3635-3650.

[16] S. Guler, M. Crasmareanu, *Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy*, Turk. J. Math. 43, 5 (2019), 2631-2641.

[17] R. S. Hamilton, *The Ricci flow on surfaces, Mathematics and General Relativity (Santa Cruz, CA, 1986)*, Contemp. Math. 71 (1988), Am. Math. Soc., Providence, RI, 237-262.

[18] P. Karmakar, *Curvature tensors and Ricci solitons with respect to Zamkovoy connection in anti-invariant submanifolds of trans-Sasakian manifold*, Math. Bohem., published online on October 18, 2021.

[19] P. Karmakar, A. Bhattacharyya, *Anti-invariant submanifolds of some indefinite almost contact and paracontact manifolds*, Bull. Calcutta Math. Soc. 112, 2 (2020), 95-108.

[20] A. Mandal, A. Das, *On M-projective curvature tensor of Sasakian manifolds admitting Zamkovoy Connection*, Adv. Math. Sci. J. 9, 10 (2020), 8929-8940.

[21] A. Mandal, A. Das, *Projective curvature tensor with respect to Zamkovoy connection in Lorentzian para Sasakian manifolds*, J. Indones. Math. Soc. 26, 3 (2020), 369-379.

[22] A. Mandal, A. Das, *Pseudo projective curvature tensor on Sasakian manifolds admitting Zamkovoy connection*, Bull. Calcutta Math. Soc. 112, 5 (2020), 431-450.

[23] A. Mandal, A. Das, *Study of Ricci solitons on concircularly flat Sasakian manifolds admitting Zamkovoy connection*, Aligarh Bull. Math. 39, 2 (2020), 47-61.

[24] A. Mandal, A. Das, *LP-Sasakian manifolds equipped with Zamkovoy connection and conharmonic curvature tensor*, J. Indones. Math. Soc. 27, 2 (2021), 137-149.

[25] H. G. Nagaraja, G. Somashekhara, *On psudo projective curvature tensor in Sasakian manifolds*, Int. J. Contemp. Math. Sci. 6, 27 (2011), 1319-1328.

[26] D. Narain, A. Prakash, B. Prasad, *A pseudo projective curvature tensor on a Lorentzian para-Sasakian manifold*, An. Stiint. Univ. Al. I. Cuza Iasi (S. N.) Mat., Tomul 55 (2009), 275-284.

[27] R. H. Ojha, *On Sasakian manifold*, Kyungpook Math. J. 13 (1973), 211-215.

[28] R. H. Ojha, *A note on the M-projective curvature tensor*, Indian J. Pure Appl. Math. 8, 12 (1975), 1531-1534.

[29] R. H. Ojha, *M-projectively flat Sasakian manifolds*, Indian J. Pure Appl. Math. 17, 4 (1986), 481-484.

[30] H. B. Pandey, A. Kumar, *Anti-invariant submanifolds of almost para-contact manifolds*, Indian J. Pure Appl. Math. 20, 11 (1989), 1119-1125.

[31] G. P. Pokhariyal, R. S. Mishra, *Curvature tensors and their relativistic significance II*, Yokohama Math. J. 19, 2 (1971), 97-103.

[32] D. G. Prakasha, K. Mirji, *On the M-projective curvature tensor of a (k,μ)-contact metric manifold*, Ser. Math. Inform. 32, 1 (2017), 117-128.

[33] B. Prasad, *A pseudo projective curvature tensor on a Riemannian manifold*, Bull. Calcutta Math. Soc. 94, 3 (2002), 163-166.

[34] A. A. Shaikh, H. Kundu, *On equivalency of various geometric structures*, J. Geom. 105, 1 (2014), 139-165.

[35] J. P. Shing, *On M-relatively flat almost pseudo Ricci symmetric manifolds*, Acta Math. Univ. Comenianae 86, 2 (2017), 335-343.

[36] S. S. Shukla, D. D. Singh, *On (ϵ)-trans-Sasakian manifolds*, Int. J. Math. Anal. 4, 49 (2010), 2401-2414.

[37] M. D. Siddiqi, *η-Ricci solitons in 3-dimensional normal almost contact metric manifolds*, Bull. Transilv. Univ. Brasov, Ser. III, Math. Inform. Phys. 11, 2 (2018), 215-234.

[38] M. D. Siddiqi, *η-Einstein solitons in δ-Lorentzian trans-Sasakian manifolds*, Math. Adv. Pure Appl. Sci. 1, 1 (2018), 27-38.

[39] M. D. Siddiqi, M. A. Akyol, *η-Ricci-Yamabe soliton on Riemannian submersions from Riemannian manifolds*, arXiv preprint, arXiv: 2004.14124, 2020.

[40] G. Somashekhara, P. S. K. Reddy, N. Pavani, G. J. Manjula, *η-Ricci-Yamabe solitons on submanifolds of some indefinite almost contact manifolds*, J. Math. Comput. Sci. 11, 3 (2021), 3775-3791.

[41] T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric*, Tohoku Math. J. 21, 2 (1969), 271-290.

[42] M. M. Tripathi, P. Gupta, *On τ-curvature tensor in K-contact and Sasakian manifolds*, Int. Electron. J. Geom. 4, 1 (2011), 32-47.

[43] K. Yano, *Concircular geometry I. concircular transformations*, Proc. Imp. Acad. Tokyo 16, 6 (1940), 195-200.

[44] K. Yano, M. Kon, *Anti-invariant submanifolds of Sasakian space forms I*, Tohoku Math. J. 29, 1 (1977), 9-23.

[45] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Global Anal. Geom. 36, 1 (2008), 37-60.

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