DOCTORAL THESIS

Classical and quantum analysis of gravitational singularity: A study of Raychaudhuri Equation

by

M A D H U K R I S H N A   C H A K R A B O R T Y

This thesis is submitted in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy (Science) of Jadavpur University

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**Classical and quantum analysis of gravitational singularity: A study of Raychaudhuri Equation**" submitted by **Smt. Madhukrishna Chakraborty** who got her name registered on 17.02.2022 [Index No. 44/22/Maths./27 and Registration No. SMATH1104422] for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon her own work under my supervision and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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DECLARATION BY THE AUTHOR

I hereby declare that the thesis is based on my own work carried out at the Department of Mathematics, Jadavpur University, Kolkata - 700032, India. Also, I declare that no part of it has not been submitted for any degree/diploma/fellowship/some other qualification at any other university or institution.

The author has produced all the figures presented in this thesis using Maple, Mathematica, and CLASS software. The thesis has been checked several times with extreme care to free it from all discrepancies and typos. Even then vigilant readers may find some mistakes, and several portions of this thesis may seem unwarranted, mistaken, or incorrect. The author takes the sole responsibility for these unwanted errors which have resulted from his inadequate knowledge of the subject or escaped his notice.

Finally, I state that, to the best of my knowledge, all the assistance taken to prepare this thesis has been properly cited and acknowledged.

Madhukrishna Chakraborty

Madhukrishna Chakraborty

19.04.2024

To

My parents

Banani Chakraborty

and

Bimalendranath Chakraborty

ABSTRACT

The thesis consists of seven chapters and is entirely devoted to the classical and quantum aspects of the Raychaudhuri equation in different background geometries. The work presented in the thesis not only shows the application of the Raychaudhuri equation in identifying the gravitational singularity but also suggests some classical and quantum mechanical tools to mitigate the singularity problem. The novelty of the work lies in the fact that it considers a plethora of background geometries and modified gravity theories namely, $f(R)$ gravity in inhomogeneous and isotropic background; $f(T)$ gravity in homogeneous and isotropic background; anisotropic universe characterized by Kantowski-Sachs model; Bouncing model of Universe etc. where the modified Raychaudhuri equation and Focusing Theorem (that follows as a consequence of the Raychaudhuri equation) has been investigated. The work also points out the quantum aspects of Raychaudhuri equation via its Lagrangian and Hamiltonian formulation and gives an account of Wheeler-DeWitt quantization in quantum cosmology alongwith the formulation of quantum Bohmian trajectories as two important quantum mechanical tools to find the possible escape routes from these problemmatic singularities at the classical level. Chapter 1 deals with the overview of relativistic cosmology while the subsequent chapters (2-8) comprise of the research work done. Finally, Chapter 9 summarizes the whole content and discusses some of its future prospects.

Einstein's General Theory of Relativity is the most well accepted theory of gravity to describe physical reality. Nevertheless, the problem of space-time singularity sets limitation to this theory which gained immense popularity after the discovery of gravitational waves. Even the simplest solution of Einstein's field equations namely, the Schawrzschild Black hole solution is singular. Therefore, it has remained puzzle over the decades about how to resolve this singularity! Even Einstein was worried about the fate of his gravity theory. Meanwhile, in the early 1950s Prof. Amal Kumar Raychaudhuri addressed this issue of space-time singularity being fascinated by cosmology. Later, in 2020 Roger Penrose was one of the recipient of the prestigious Nobel Prize for the seminal singularity theorems jointly formulated by him and Stephen Hawking. It is very interesting to note that the Raychaudhuri equation was the key ingredient behind those singularity theorems. Raychaudhuri equation using the notion of geodesic focusing (Focusing Theorem) proved the inevitable existence of singularity in Einstein gravity. Penrose's theorems use a minimal set of assumptions like the existence of trapped surfaces and weak energy condition on matter which results in its wider applicability.

On the other hand, recent observations are in favor of accelerated expansion of the Universe. Understanding the reason of this late time acceleration is one of the intriguing issues

of modern cosmology. To find its possible reason, two opinions have surfaced. Some suggest a modification of Einstein gravity giving rise to extended theories of gravity like $f(R)$, $f(T)$, Scalar-Tensor theory, $f(R,T)$, $f(Q)$, $f(G)$ etc. while others prescribe some Dark Energy models like Cosmological constant, K-essence, Quintessence, Tachyon, Holographic Dark energy model, Modified Chaplygin gas etc. However, in the present work we have explored the Raychaudhuri equation and resolution of singularities in modified gravity theories. Raychaudhuri equation hints that singularity is inevitable in Einstein gravity via the Focusing theorem which requires positiveness of the Raychaudhuri scalar for possible convergence (also known as Convergence Condition CC) of a bundle of geodesic. If there is a singularity in a space-time manifold, then a congruence of geodesic tend to focus at the singularity. Thus a singularity always implies focusing. So if we can avoid this focusing by making the Raychaudhuri scalar negative we can avoid singularity also. Motivated by this, Raychaudhuri equation and corresponding CC have been formulated in modified gravity theories where the Raychaudhuri scalar may be made negative under certain physical assumptions. This is because the field equations for modified gravity differ from those of Einstein gravity. Hence there may arise certain conditions under which focusing and hence singularity might be avoided. The role of anisotropy in Focusing theorem in case of anisotropic model described by Kantowski-Sachs metric has been explored. The thesis further brings out some inherent mathematical and cosmological properties of bouncing model using the Raychaudhuri equation and discusses its consequences in emergent scenario and Wormholes. Moreover it shows the entire cosmic evolution from the point of view of Raychaudhuri equation.

Quantum formulation of RE has also got ample motivation. This is because, it is generally speculated that quantum effects which become prominent in strong gravity regime may alleviate the singularity problem at the classical level. To be more precise, a quantum replica of Raychaudhuri equation and classical geodesic might be helpful in identifying the existence of singularity and also to resolve them. The thesis gives emphasis on Wheeler DeWitt quantization and Bohmian trajectory formulation corresponding to the quantum Raychaudhuri equation of which the former quantize the geodesic flow and the later replaces the classical geodesics by quantum Bohmian trajectories.

LIST OF PUBLICATIONS

The work of this thesis has been carried out at the Department of Mathematics, Jadavpur University, Kolkata- 700032, India. The thesis is based on the following published papers:

Publications included in the thesis

- Chapter 2 has been published as *"The Raychaudhuri Equation in inhomogeneous FLRW space-time : A $f(R)$-gravity model"*, **M. Chakraborty**, A. Bose and S. Chakraborty, **Phys.Scripta 98**, no.2, 025007 (2023), DOI : 10.1088/1402-4896/acb020.

- Chapter 3 has been published as *"Raychaudhuri equation from Lagrangian and Hamiltonian formulation: A quantum aspect"*, **M. Chakraborty** and S. Chakraborty, **Annals Phys. 457**, 169403 (2023), DOI : 10.1016/j.aop.2023.169403.

- Chapter 4 has been published as *"The classical and quantum implications of the Raychaudhuri equation in $f(T)$-gravity"*, **M. Chakraborty** and S. Chakraborty, **Class. Quantum Grav. 40**, no.15, 155010 (2023), DOI : 10.1088/1361-6382/ace231.

- Chapter 5 has been published as *"Raychaudhuri equation and bouncing cosmology"*, **M. Chakraborty** and S. Chakraborty, **Mod.Phys.Lett.A 38**, no. 28n29 2350129 (2023), DOI : 10.1142/S0217732323501298.

- Chapter 6 has been published as *"On the consequences of Raychaudhuri equation in Kantowski-Sachs space-time"*, **M. Chakraborty** and S. Chakraborty, **Annals Phys. 460**, 169577 (2024), DOI : 10.1016/j.aop.2023.169577.

- Chapter 7 has been published as *" Raychaudhuri equation and dynamics of cosmic evolution"*, **M. Chakraborty** and S. Chakraborty, **Phys.Scripta 99**, no.4, 045203 (2024), DOI : 10.1088/1402-4896/ad2c4c.

- Chapter 8 is under review as *"Raychaudhuri equation and possible non-singular universe: Some unconventional geometric aspects"*, **M. Chakraborty** and S. Chakraborty.

Other Publications

- *"A Revisit to Classical and Quantum aspects of Raychaudhuri equation and possible resolution of Singularity",* S. Chakraborty and **M. Chakraborty, Int.J.Geom.Meth.Mod.Phys.** (in press, DOI:10.1142/S0219887824400188)(2024)

- *"Theoretical and observational prescription of warm-inflation in FLRW universe with torsion",* **M. Chakraborty**, G. Sardar, A. Bose and S. Chakraborty, **Eur.Phys.J.C 83** , no. 9, 860 (2023), DOI : 10.1140/epjc/s10052-023-12030-8.

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ABOUT PROF. AMAL KUMAR RAYCHAUDHURI

Amal Kumar Raychaudhuri (AKR), born on 14th september, 1923 in Barishal (now in Bangladesh) is one of the talented Indian scientists who is known for his novelty and innovation in the field of Physics and Mathematics. He was brought up in Kolkata, did his graduation from Presidency college and M.Sc. in physics at Calcutta University. After Masters, he joined Indian Association for the Cultivation of Science (IACS), Kolkata and spent four years being engrossed in research over there. In 1950, he joined Ashutosh College, Kolkata as a faculty. His role as a teacher is worth mentioning. In 1961, he joined as a Professor of Physics at Presidency college, Kolkata. Students were attracted by his teaching and innovative research ideas. They were highly inspired by AKR. His teaching left a great impression on a large body of students. After getting negative response from S.N.Bose, AKR chose to study in the field of relativity and cosmology. AKR's earlier work discussed about the physical nature of the classical Schwarzschild solution. He was fascinated by cosmology and addressed the issue of singularity in the early 1950s by forming a general equation describing kinematics of a deformable medium which is known as the *Raychaudhuri equation*. Later the Raychaudhuri equation paved the way for research into the *Singularity Theorems* and for this Roger Penrose got the Nobel Prize in physics. Apart from this work, Raychaudhuri carried out several works in gravity theories with variable gravitational constant. His involvement in research at IUCAA (Inter -University Centre for Astronomy and Astrophysics) developed research of that centre. He also wrote a monograph on *Theoretical Cosmology*, college texts on classical mechanics and electromagnetism and a book on *Dynamics* in Bengali. His marvelous contribution in the field is very commendable and the thesis is developed based on his famous equation, the *Raychaudhuri equation*.

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CHAPTER 1

INTRODUCTION

1.1 Prelude

"The universe is not only stranger than we imagine, it is stranger than we can imagine" *Sir Arthur Eddington, The Nature of the Physics World (1927)*

The above quote by Sir Arthur Eddington throws light to the mystery and complexity of the Universe. The quote gives emphasis to the fact that Universe unveils more as Man tries to comprehend it, encouraging us to explore more secrets of the cosmos. Since ancient times, Men have been fascinated by looking at the night sky where thousands of luminous objects move throughout years and years which led to the idea of the geocentric universe. But Man is the most intelligent living being in the world. They believe in logic as well as inventions, and as a result, science progresses. The remarkable step in cosmology and astronomy with the discovery of the telescope by Galileo (1609), gave us the idea of the heliocentric universe. Subsequently, both theoretical and experimental physics moved forward. The next pioneering work was Newton's law of gravity (1687) which nicely described planetary motion and many other things. Nevertheless the problem arose when a series of experiments were being carried out (Michelson–Morley experiment (1887)) to find out the existence of the luminiferous aether, a supposed medium permeating space that was believed to be the carrier of light waves. The astounding discovery revealed almost no significant difference between the speed of light in the direction of movement through the presumed aether, and the speed at right angles. Majority was of the opinion that they did not have a sophisticated setup and so is the result. Unlike others, Einstein came up with his Special Theory of Relativity (1905) which provided an escape route to this problem. Although, that time he could not incorporate the gravitational force for his relativity theory. 10 years later he eventually overcame this and introduced his General Theory of Relativity (1915) which is one of the cornerstones of Modern cosmology.

1.2 Some important tools from Differential Geometry

We begin with a small recap of important tools of differential geometry [1].

- Let V^n be a vector space over a field F. Then the vectors of the vector space V^n are called contravariant vectors or contravariant tensors of rank 1 or tensor of type (1,0). Let V^{n*} be the dual space of V^n over F. Then the elements of this dual space are called Covariant vectors.

- Contraction of tensors:
$$\tilde{B}^{ij}_{kim} = \frac{\partial \tilde{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \tilde{x}^k} \frac{\partial x^t}{\partial \tilde{x}^m} \, B^q_{rt}$$
B^j_{km} is a tensor of type $(1, 2)$ and is the contracted one. This operation is known as Contraction.

- Inner product of two tensors A^p_q and B^{ij}_l is a tensor of type $(2, 1)$.

- Magnitude or length of contravariant (A^i) and covariant (B_i) vectors: $A^2 = g_{ij}A^iA^j$ and $B^2 = g^{ij}B_iB_j$. If A^2 or $B^2 = 1$, then the vector is a unit vector. On the other hand, vectors with magnitude zero are called null vectors. A null vector and zero vector whose each component is zero are different.

- Manifolds: Given a topological space $\mathcal{M}$, if there exists a one-one mapping f from an open subset U of $\mathcal{M}$ onto an open subset U of R^n (the mapping is a homeomorphism), then $\mathcal{M}$ is said to be a manifold of dimension n. This definition of a manifold suggests that the set $\mathcal{M}$ looks locally like R^n but globally they are quite distinct. The pair (U, f) is called a chart. A collection of charts is called an Atlas. In fact, if any two overlapping charts in an Atlas are C^k-related then manifold is said to be a C^k-manifold. A manifold of class C^1 is called a differentiable manifold.

- For a general quadratic differential form $g_{ab}dx^adx^b$, the Christoffel symbol of first kind is given by $[ab, c] = \frac{1}{2}\left(\frac{\partial g_{ac}}{\partial x_b} + \frac{\partial g_{bc}}{\partial x_a} - \frac{\partial g_{ab}}{\partial x_c}\right)$ and Christoffel symbol of second kind is given by $g^{cm}[ab, c]$.

- Notion of differentiation of tensors: Covariant derivative of covariant vector is $\nabla_d B_a = \frac{\partial B_a}{\partial x^d} - \Gamma^c_{ad}B_c$, a tensor of rank $(0, 2)$. Covariant derivative of a scalar is same as the partial derivative. The covariant derivative of a contravariant vector is given by $\nabla_d A^b = \frac{\partial A^b}{\partial x^d} + \Gamma^b_{cd}A^c$. Differentiation for arbitrary tensors can be done in the similar manner, for example $\nabla_d T^a_{bc} = \frac{\partial T^a_{bc}}{\partial x^d} + \Gamma^a_{kd}T^k_{bc} - \Gamma^k_{bd}T^a_{kc} - \Gamma^k_{cd}T^a_{bk}$.

- Intrinsic Differentiation: The Intrinsic derivative of a vector A is given by $\frac{\delta A^a}{d\lambda} = \frac{\partial A^a}{\partial \lambda} + \Gamma^a_{bc} \, A^a \, \frac{dx^c}{d\lambda}$.

- Notion of parallel transport: A vector A is said to be parallely transported along a curve γ if $\frac{\delta A}{d\lambda} = 0$.

- Riemann Curvature tensor: The expression for Riemann curvature tensor is given by $R^a_{bcd} = \frac{\partial}{\partial x^c}\Gamma^a_{db} - \frac{\partial}{\partial x^d}\Gamma^a_{cb} + \Gamma^e_{db}\Gamma^a_{ec} - \Gamma^e_{cb}\Gamma^a_{de}$. The properties of Riemann curvature tensor are:

  1. $R^a_{bcd} = -R^a_{bdc}$
  2. $R_{(ab)cd} = 0$
  3. $R^a_{[bcd]} = 0$ i.e, $R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0$
  4. $R^a_{b[cd;e]} = 0$ i.e, $R^a_{bcd;e} + R^a_{bde;c} + R^a_{bec;d} = 0$

  3 and 4 are called Bianchi's Identities. From the Bianchi's Identity given by 3 we have $R_{abcd} = R_{cdab}$.

- Ricci tensor: It is denoted by R_{ab} and given by $R_{ab} = g^{cd}R_{cabd}$. A Riemannian space with $R_{cabd} = 0$ is a flat space.

- Scalar curvature or Ricci scalar $R = g^{ab}R_{ab}$.

1.3 Ingredients of Relativistic Cosmology

The word "cosmology" was first used in 1656 and is coined from the Greek words "kosmos" which means "world" and "logia" which means "study of". Theoretical astrophysicist David N. Spergel portrayed cosmology as "historical science". Modern cosmology is based on three pillars, namely The cosmological principle; Weyl's postulate and Einstein's General Relativity.

1.3.1 Cosmological Principle

According to the cosmological principle, the universe is homogeneous and isotropic on large scale (i.e. in cosmic scale: ≈ 100 Mpc[1]). Homogeneity means there is no preferential point or direction in space-time i.e. if we consider a portion of the universe, then the number of galaxies in it is in the vicinity of the number in another portion with the same volume at a given time. While, Isotropy means that all the directions are identical i.e, the space looks similar irrespective of in what direction we look. This implies that space-time must be spherically symmetric. The greatest support in favour of isotropy is Cosmic Microwave Background Radiation (CMBR) according to which the universe is currently in a bath of thermal radiation having 2.73 K with anisotropy $\mathcal{O}\left(10^{-5}\right)$. On the other hand, the homogeneity of the universe is partially supported by the counts of the galaxies and the linearity of the Hubble law.

1.3.2 Weyl's Postulate

Weyl's postulate states that particles of the substratum lie in space-time on a congruence of time-like geodesics diverging from a point in the finite or infinite past. This shows that cosmic fluid is such a fluid that it has a unique direction of flow i.e. it has a unique velocity.

[1] 1 Mpc $\approx 3.26 \times 10^6$ light years $\approx 3.08 \times 10^{19}$ Km

Particles of the substratum thus move in a time-like path (since the time-like path does not intersect each other). This property of the particle is valid only for the perfect fluid. Hence, the cosmic particle should be perfect fluid in nature. The matter content of the cosmos is identified by the Weyl's postulate.

1.3.3 Einstein's General theory of Relativity (GR)

Einstein's GR emerges as one of the most wonderful and pioneering concepts that shows a relation between the geometry of space-time and the matter content of the Universe. In other words, non-Euclidean Riemannian geometry is the basis of this theory and gravity is described in terms of geometry which makes

$$\Gamma^c_{ad} = \Gamma^c_{da}$$
$$\nabla_d \, g_{ab} = 0$$
$$\Gamma^a_{bc} = \frac{1}{2} \, g^{ad} \, \left(\frac{\partial g_{bd}}{\partial x^c} + \frac{\partial g_{cd}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d} \right) \tag{1.1}$$

GR is based on the principle of general covariance which says that laws of physics are generally covariant i.e, independent of co-ordinate choices. The physical laws are written in terms of differential equations involving vectors and tensors. By construction, these laws will be generally covariant if expressed in the language of tensors. According to the theory, matter dictates the space-time to curve and on the other way round curvature of space-time determines the motion of test particles. This nice interrelation between matter and geometry appear in the Einstein's field equation. There lies the beauty of the equation. Einstein's field equation can be written as

$$G_{ab} = \kappa T_{ab}, \tag{1.2}$$

where $\kappa = 8\pi G$ is the gravitational constant (assuming velocity of light c to be unity), G is the Newton's gravitational constant having value $G = 6.673 \times 10^{-11}$ $\mathrm{Nm^2kg^{-1}}$, T_{ab} is the Energy-momentum tensor of the matter component and the Einstein tensor G_{ab} describes the geometry of the space-time which is defined by

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}, \tag{1.3}$$

where R_{ab} and R are the Ricci tensor and Ricci scalar respectively and g_{ab} is the metric tensor that determines the geometry of the space-time. R_{ab} is obtained by contracting the Riemann curvature tensor $R^{\sigma}{}_{\lambda ab}$ and its further contraction gives the Ricci scalar as

$$R_{ab} = R^{\sigma}{}_{\sigma ab} \quad \text{and} \quad R = g^{ab} R_{ab}. \tag{1.4}$$

The above field equation (1.2) can be obtained by varying the Einstein-Hilbert action given by

$$\mathcal{S} = \frac{1}{2\kappa} \int R\sqrt{-g}\mathrm{d}^4x + \int \sqrt{-g}\mathrm{d}^4x\mathcal{L}_M \tag{1.5}$$

with respect to the metric tensor g_{ab}. Conventionally, metric with signature $(-, +, +, +)$ is used.

1.4 Modern Cosmology

In accordance with the cosmological principle, the space-time is described by the metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{1.6}$$

where $a(t)$ is the scale factor with cosmic time t. Here (r, θ, ϕ) are co-moving spherical polar co-ordinates, K is the constant of curvature of the spatial part of the space-time which may have values 0, +1, −1 for flat, closed, and open universe respectively. This is known as Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time [2, 3, 4, 5].

Weyl's Postulate implies that the energy-momentum tensor of cosmic fluid (perfect fluid) is given by

$$T_{ab} = (\rho + p)v_a v_b + p g_{ab}, \text{ with } v_a v^a = -1, \tag{1.7}$$

where ρ and p are the energy density and thermodynamic pressure of the cosmic fluid respectively and v^a is the four-velocity of the fluid.

Using equations (1.2), (1.6), and (1.7), Einstein's field equations for FLRW line element can be written in the form of, so-called Friedmann equations as

$$3\left(H^2 + \frac{K}{a^2}\right) = 8\pi G\rho, \tag{1.8}$$

$$2\left(\dot{H} - \frac{K}{a^2}\right) = -8\pi G(p + \rho). \tag{1.9}$$

where $H = \frac{\dot{a}}{a}$ is termed as Hubble parameter and 'overdot' represents the differentiation with respect to cosmic time t.

From the energy conservation equation, the energy-momentum tensor is conserved (indices a and b have been changed to μ and ν respectively as $a = a(t)$ has been used as the cosmic scale factor) i.e.

$$T^{\mu}_{\nu;\mu} = 0, \tag{1.10}$$

which gives

$$\dot{\rho} + 3H(p + \rho) = 0. \tag{1.11}$$

It can be shown that equations (1.8), (1.9), and (1.11) are not linearly independent. Rather, any one of these three equations can be obtained from the remaining two.

Equation (1.8) can also be written as

$$\Omega(t) = 1 + \frac{K}{a^2 H^2}, \tag{1.12}$$

where $\Omega(t) = \frac{\rho}{\rho_c}$ is the dimensionless density parameter, with $\rho_c = \frac{3H^2}{8\pi G}$, the critical density. The spatial geometry of the Universe is determined by the matter distribution as follows:

$$\Omega(t) > 1 \implies \rho > \rho_c \implies K = +1. \tag{1.13}$$
$$\Omega(t) = 1 \implies \rho = \rho_c \implies K = 0. \tag{1.14}$$
$$\Omega(t) < 1 \implies \rho < \rho_c \implies K = -1. \tag{1.15}$$

Observational results from CMB indicate that at present $\Omega(t) \simeq 1$ [6], i.e. the geometry of the present universe is nearly spatially flat. Without any loss of generality, one may assume the spatially flat ($K = 0$) space-time for simplicity.

From equations (1.8) and (1.9), one may write the amount of acceleration as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p + \rho), \tag{1.16}$$

This equation is known as the Raychaudhuri equation in FLRW space-time. In this context, one can define the deceleration parameter

$$q \equiv -\frac{a\ddot{a}}{\dot{a}^2} = -\left(1 + \frac{\dot{H}}{H^2}\right), \tag{1.17}$$

which characterizes the dynamics of the universe. The sign of the above two parameters indicates the nature of the expansion of the universe.

(i) $\ddot{a} > 0 \implies q < 0 \implies (3p + \rho) < 0 \implies$ universe accelerates.

(ii) $\ddot{a} < 0 \implies q > 0 \implies (3p + \rho) > 0 \implies$ universe decelerates.

Hence, if the universe is assumed to be filled with perfect fluid; then Strong Energy Condition (SEC) ($\equiv (3p + \rho) > 0$) holds, implying the universe decelerates. Now, if we assume a perfect fluid with constant barotropic equation of state

$$p = \omega\rho, \tag{1.18}$$

the cosmological solution for the flat ($K = 0$) FLRW model can be written as

$$\rho = \rho_0 a^{-3(1+\omega)}, \tag{1.19}$$
$$a = a_0 (t - t_0)^{\frac{2}{3(1+\omega)}}, \tag{1.20}$$
$$H = \frac{2}{3(1+\omega)(t - t_0)}, \tag{1.21}$$

where ρ_0, t_0, a_0 are integration constants. One must note that $\omega \neq -1$ for this solution.

Different values of ω representing various cosmic eras are as follows:

(i) $\omega = 0$ (Dust era): $\rho \propto a^{-3}$, $a \propto (t - t_0)^{\frac{2}{3}}$, $H = \frac{2}{3(t - t_0)}$.

(ii) $\omega = \frac{1}{3}$ (Radiation era): $\rho \propto a^{-4}$, $a \propto (t - t_0)^{\frac{1}{2}}$, $H = \frac{1}{2(t - t_0)}$.

(ii) $\omega = 1$ (Stiff fluid era): $\rho \propto a^{-6}$, $a \propto (t - t_0)^{\frac{1}{3}}$, $H = \frac{1}{3(t - t_0)}$.

1.5 Late-time acceleration

In the late 90's, two independent research teams, namely, "High-redshift Supernovae Search Team" (HSST) and "Supernovae Cosmology Project Team" (SCPT) were investigating type Ia Supernovae (SNIa). Both the teams, Riess *et al.* [7] from HSST, and Perlmutter *et al.* [8] from SCPT independently reported that our Universe has been going through an accelerated expanding phase as the main conclusion of their investigations. Subsequently, Cosmic Microwave Background Radiation (CMBR) [6, 9, 10, 11, 12, 13, 14, 15, 16], Large Scale Structure (LSS) [17, 18, 19, 20], Baryon Acoustic Oscillation (BAO) [21, 22, 23], weak lensing [24, 25, 26], galaxy cluster number counts [27], gravitational waves detection [28, 29] confirmed this present acceleration of the Universe. However, the standard model of the cosmos in the framework of GR fail to match the overwhelming abundance of observational evidences of cosmic speedup. The cosmologists are therefore trying to explain this observed phenomenon in mainly two ways:

(i) Modification of matter components,

(ii) Modification of Einstein gravity.

1.5.1 Acceleration due to matter modification: Dark Energy Model

To accommodate the observational results into the theoretical set up, one section of cosmologists have modified Weyl's postulate, i.e. they have considered the cosmic fluid to be no longer the perfect fluid; rather a fluid which violates the SEC (i.e. $3p + \rho < 0$). This causes universe to accelerate. This fluid is termed as exotic matter (dark energy). For such a fluid with barotropic equation of state ω, one has

$$\omega < -\frac{1}{3}. \tag{1.22}$$

1.5.1.1 Cosmological constant

The simplest candidate for dark energy is the Cosmological constant Λ [30]. Einstein himself introduced this to counterbalance the effect of gravity and achieve a static universe, a notion that was prevalent at that time. Subsequently, when Edwin Hubble (1929) discovered that the universe is expanding, Einstein rejected this (1931) by remarking "the biggest blunder in my entire life". Much later it was reinterpreted as the vacuum energy in quantum mechanics. Finally, after 1998, it was again brought back to explain the accelerating phase [31, 32]. In general, a positive constant value has been chosen for Λ.

Introducing Λ, the action can be modified as

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}\mathrm{d}^4x(R - 2\Lambda) + \int \sqrt{-g}\mathrm{d}^4x\mathcal{L}_M, \tag{1.23}$$

and the modified Einstein field equation can be written as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\left(T_{\mu\nu} + \frac{\Lambda}{8\pi G}g_{\mu\nu}\right). \tag{1.24}$$

Now in the background of FLRW space-time, the Friedmann equations are modified as

$$3\left(H^2+\frac{K}{a^2}\right) = 8\pi G\rho+\Lambda, \tag{1.25}$$
$$2\left(\dot{H}-\frac{K}{a^2}\right) = -8\pi G(p+\rho). \tag{1.26}$$

and the acceleration equation can be written as

$$\frac{\ddot{a}}{a}=-\frac{4\pi G}{3}(3p+\rho)+\frac{\Lambda}{3}. \tag{1.27}$$

Equation (1.27) shows that the cosmological constant acts as a repulsive force against gravity. Hence, one may write the energy density (ρ_Λ) and pressure (p_Λ) of the cosmological constant as

$$\rho_\Lambda=\frac{\Lambda}{8\pi G},\ p_\Lambda=-\frac{\Lambda}{8\pi G},\ \text{i.e., } p_\Lambda=-\rho_\Lambda, \tag{1.28}$$

which shows that in this case the equation of state parameter

$$\omega_\Lambda=-1, \tag{1.29}$$

i.e. it violates SEC. Thus, the cosmological constant may be a viable candidate for dark energy. Also, it has been found that Λ together with cold dark matter, i.e., ΛCDM model is consistent with the most number of observations. Although recently, it was shown that the ΛCDM model may also suffer from the age problem [33]. Also, there is another problem known as the cosmological constant problem [34]. There are other dark energy models like holographic dark energy, K-essence, Tachyon, Quintessence, modified chaplygin gas etc. For a detailed overview of different forms of dark energy one may refer [35]-[73]. Since the thesis contain chapters on modified theories of gravity we discuss the extended gravity models namely $f(R)$ gravity and $f(T)$ gravity in the following subsection.

1.5.2 Acceleration due to gravity modification: Modified Gravity models

Besides dark energy which contributes to modification of the energy-momentum tensor in Einstein's field equation in this section, we shall discuss the present accelerated phase in a different way. Here we shall modify the gravitational sector as compared to the General Relativity. The gravitational action of Einstein's general relativity is given by

$$\mathcal{S}=\frac{1}{2\kappa}\int\sqrt{-g}\mathrm{d}^4x\ R+\int\sqrt{-g}\mathrm{d}^4x L_M\left(g_{\mu\nu},\Psi_M\right) \tag{1.30}$$

where $\kappa=8\pi G$, L_M is the matter Lagrangian depending on the metric tensor $g_{\mu\nu}$, and the matter field Ψ_M, R is the Ricci scalar, $g=\det(g_{\mu\nu})$. In modified gravity theories, changing the action suitably, one may try to describe the present accelerating phase of the Universe. We shall now describe some relevant modified gravity theories.

1.5.2.1 $f(R)$ gravity

The simplest modification in General Relativity is the $f(R)$ gravity theory where $f(R)$ is any arbitrary function of Ricci scalar R. The action for $f(R)$ gravity is given by [74]-[76]

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}\mathrm{d}^4x\ f(R) + \int \sqrt{-g}\mathrm{d}^4x L_M\left(g_{\mu\nu}, \Psi_M\right) \tag{1.31}$$

Now we shall study $f(R)$ gravity theory in two different approaches, namely, in metric and Palatini formalisms.

- ***$f(R)$ gravity in Metric formalism***

In metric formalism, it is assumed that the affine connections $\Gamma^{\alpha}_{\beta\gamma}$ are the usual metric connections in terms of $g_{\mu\nu}$. Now, varying the action (1.31) with respect to $g_{\mu\nu}$, one obtains the field equations as

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = \kappa T^M_{\mu\nu} \tag{1.32}$$

where $F(R) \equiv \dfrac{\mathrm{d}f(R)}{\mathrm{d}r}$, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the D'ALembert's operator, and the energy-momentum tensor of the matter field $T^M_{\mu\nu}$ is defined by

$$T^M_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta L_M}{\delta g^{\mu\nu}} \tag{1.33}$$

which satisfies the continuity equation

$$\nabla^\mu T^M_{\mu\nu} = 0 \tag{1.34}$$

Further, considering the trace of the equation (1.32), one has,

$$\begin{aligned} 3\Box F(R) + RF(R) - 2f(R) &= \kappa g^{\mu\nu}T^M_{\mu\nu} \\ &= \kappa T \end{aligned} \tag{1.35}$$

In particular, if one considers $f(R) = R$ then one can retrieve the Einstein gravity. In general, $\Box F(R) \neq 0$ and this corresponds to some propagating degrees of freedom $\phi = F(R)$. The dynamics of the scalar field, namely, 'scalaron' ϕ is characterized by the equation (1.35).

So, the field equation (1.32) is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\left(T^M_{\mu\nu} + T^{eff}_{\mu\nu}\right) \tag{1.36}$$

where $T^{eff}_{\mu\nu}$ is termed as the effective energy-momentum tensor and defined as

$$\kappa T^{eff}_{\mu\nu} \equiv \frac{1}{2}(f(R) - R)g_{\mu\nu} + \nabla_\mu\nabla_\nu F(R) - g_{\mu\nu}\Box F(R) + (1 - F(R))R_{\mu\nu} \tag{1.37}$$

Since $\nabla^\mu G_{\mu\nu} = 0$, and, $\nabla^\mu T^M_{\mu\nu} = 0$, we have

$$\nabla^\mu T^{eff}_{\mu\nu} = 0 \tag{1.38}$$

which shows that the continuity equation for the effective energy-momentum tensor also holds.

Now, considering the flat FLRW space-time, the line element can be written in Cartesian coordinate as

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{1.39}$$

and the Ricci scalar R is given by

$$R = 6\left(\dot{H} + 2H^2\right) \tag{1.40}$$

The energy-momentum tensor for the perfect fluid takes the form

$$T^M_{\mu\nu} = (p_M + \rho_M)u_\mu u_\nu + p_M g_{\mu\nu} \tag{1.41}$$

where p_M and ρ_M are the pressure and energy density of the perfect fluid respectively, u_μ is the four-velocity of the perfect fluid with a barotropic equation of state $\omega_M = \dfrac{p_M}{\rho_M}$.

Hence, the field equations for flat FLRW space-time is given by

$$3H^2 = \frac{\kappa}{f'}\left[\rho_M + \frac{1}{2}(Rf' - f) - 3H\dot{R}f''\right] \tag{1.42}$$

$$2\dot{H} + 3H^2 = -\frac{\kappa}{f'}\left[p_M + \dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right] \tag{1.43}$$

In vacuum, the field equations can be rewritten as

$$3H^2 = \kappa\rho_{eff} \tag{1.44}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}(3p_{eff} + \rho_{eff}) \tag{1.45}$$

where the effective pressure p_{eff} and effective energy density ρ_{eff} are respectively given by,

$$p_{eff} = \frac{1}{f'}\left[\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right] \tag{1.46}$$

$$\rho_{eff} = \frac{1}{f'}\left[\frac{1}{2}(Rf' - f) - 3H\dot{R}f''\right] \tag{1.47}$$

The equation of state parameter of the effective fluid is given by

$$\omega_{eff} = \frac{2\dot{R}^2 f''' + 4H\dot{R}f'' + 2\ddot{R}f'' + f - Rf'}{Rf' - f - 6H\dot{R}f''} \tag{1.48}$$

So to obtain accelerating universe, one must have $\omega_{eff} < -\dfrac{1}{3}$. Since, $\rho_{eff} > 0$, hence p_{eff} determines the sign of ω_{eff}.

- ***$f(R)$ gravity in Palatini formalism***

Attilio Palatini, an Italian Mathematician proposed another approach to obtain the field equations for $f(R)$ gravity. In this formalism, the affine connection $\Gamma^{\alpha}_{\beta\gamma}$ and the metric tensor $g_{\mu\nu}$ are treated as independent variables. Now, varying the action (1.31) with respect to the metric tensor, the field equations are obtained as

$$F(R)R_{\mu\nu}(\Gamma) - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T^{M}_{\mu\nu} \tag{1.49}$$

Here, it is to be noted that $R_{\mu\nu}(\Gamma)$, the Ricci tensor corresponding to the affine connection, is different from the usual $R_{\mu\nu}(g)$, Ricci tensor corresponding to the metric connection. Now, taking the trace of equation (1.49), one has

$$F(R)R - 2f(R) = \kappa T \tag{1.50}$$

where $R \equiv R(g, \Gamma) = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ is the Ricci scalar in Palatini formalism and it is different from the Ricci scalar in metric formalism. The explicit form $R_{\mu\nu}(\Gamma)$ is given by

$$R(\Gamma) = R(g) + \frac{3}{2[f'(R(\Gamma))]^2}\nabla_\mu f'(R(\Gamma))\nabla^\mu f'(R(\Gamma)) + \frac{3}{f'(R(\Gamma))}\Box f'(R(\Gamma)) \tag{1.51}$$

Now, varying the action (1.31) with respect to the connection, and using the relation (1.51), one has

$$\begin{aligned} R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \frac{\kappa}{F}T_{\mu\nu} - \frac{1}{2F}(FR(\Gamma) - f(R))g_{\mu\nu} + \frac{1}{F}\left(\nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F\right) \\ -\frac{3}{2F^2}\left(\partial_\mu F\partial_\nu F - \frac{1}{2}g_{\mu\nu}(\nabla F)^2\right) \end{aligned} \tag{1.52}$$

If one considers, $f(R) = R$, then the field equations (1.52) in Palatini formalism leads to that of GR. The difference between these two formalisms arises for those $f(R)$ gravity model which contain nonlinear terms in R.

1.5.2.2 $f(T)$ gravity

It was around 1920's when Einstein himself introduced torsion as a gravitating interaction term. He did this to unify gravity and electromagnetism over Weitzenbock non-Riemannian manifold. Consequently, the Levi-Civita connection is replaced by Weitzenbock connection in the underlying Riemann-Cartan space-time. In $f(T)$ gravity, dynamical objects are the four linearly independent vierbein (tetrad) fields that form the orthogonal bases for the tangent space at each point of space-time. These vierbeins are parallel vector fields, that give the theory of descriptor "teleparallel". Now, analogous to $f(R)$ gravity, teleparallel gravity has been generalized by replacing torsion scalar T with a generic function $f(T)$ and Linder coined it as $f(T)$ gravity. The action for $f(T)$ gravity is given by

$$\mathcal{S} = \frac{1}{2\kappa}\int \sqrt{-g}\mathrm{d}^4x\ f(T) + \int \sqrt{-g}L_M\mathrm{d}^4x \tag{1.53}$$

where T is the torsion scalar, $f(T)$ is a differentiable function of torsion and L_M is the matter Lagrangian. The torsion scalar is defined as

$$T = S_\sigma^{\ \mu\nu}T^\sigma_{\ \mu\nu} \tag{1.54}$$

where super-potential, $S_{\sigma}{}^{\mu\nu}$, is given by

$$S_{\sigma}{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_{\sigma} + \delta^{\mu}_{\sigma}\, T^{\alpha\nu}{}_{\alpha} - \delta^{\nu}_{\sigma}\, T^{\alpha\mu}{}_{\alpha}\right), \tag{1.55}$$

the contortion tensor, $K^{\mu\nu}{}_{\sigma}$, is given by

$$K^{\mu\nu}{}_{\sigma} = -\frac{1}{2}\left(T^{\mu\nu}{}_{\sigma} - T^{\nu\mu}{}_{\sigma} - T_{\sigma}{}^{\mu\nu}\right) \tag{1.56}$$

and the torsion tensor, $T^{\sigma}{}_{\mu\nu}$ is defined as

$$T^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\mu\nu} = e^{\sigma}_{A}\left(\partial_{\mu} e^{A}_{\nu} - \partial_{\nu} e^{A}_{\mu}\right) \tag{1.57}$$

where the Weitzenböck connection $\Gamma^{\sigma}_{\mu\nu}$ is defined as $\Gamma^{\sigma}_{\mu\nu} = e^{\sigma}_{A}\partial_{\nu} e^{A}_{\mu}$. In teleparallel gravity, orthogonal tetrad components $e_A(x^{\mu})$ are considered as dynamical variables. Geometrically, they form an orthonormal basis for the tangent space at each point x^{μ} of the manifold i.e.

$$e_A e_B = \eta_{AB} = diag(+1, -1, -1, -1) \tag{1.58}$$

Further, in a co-ordinate basis one may write $e_A = e^{\mu}_{A}\partial_{\mu}$ where e^{μ}_{A} are the components of e_A, with $\mu = 0, 1, 2, 3$ and $A = 0, 1, 2, 3$. It is to be noted that capital letters refer to the tangent space while Greek indices label coordinates on the manifold. Hence the metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu}(x) = \eta_{AB}\, e^{A}_{\mu}(x) e^{B}_{\nu}(x) \tag{1.59}$$

Now, varying the action (1.53), the field equation reads

$$\left[e^{-1}\partial_{\mu}(e S_{A}{}^{\mu\nu}) - e^{\lambda}_{A} T^{\rho}{}_{\mu\lambda} S_{\rho}{}^{\nu\mu}\right] f_T + S_{A}{}^{\mu\nu}\partial_{\mu}(T) f_{TT} + \frac{1}{4} e^{\nu}_{A} f(T) = \frac{\kappa}{2} e^{\rho}_{A} T_{\rho}{}^{\nu} \tag{1.60}$$

where $|e| = \det\left(e^{A}_{\mu}\right) = \sqrt{-g}$ and suffix T denotes the differentiation with respect to torsion scalar T. Now, assuming flat, homogeneous and isotropic FLRW space-time having line element

$$ds^2 = dt^2 - a^2(t)\delta_{ij} dx^i dx^j, \tag{1.61}$$

the Modified Friedmann equations and the continuity equation become

$$F - 2T f_T = 2\kappa\rho \tag{1.62}$$

$$-8\dot{H} T f_{TT} + 2(T - 2\dot{H}) f_T - f = 2\kappa p \tag{1.63}$$

$$\dot{\rho} + 3H(p + \rho) = 0 \tag{1.64}$$

with $T = -6H^2$. Here one must note that for $f(T) = T$, the equations reduce to the usual Friedmann equations. Now, the modified Friedmann equations (1.62) and (1.63) can be rewritten as

$$3H^2 = \kappa(\rho + \rho_{eff}) \tag{1.65}$$

$$2\dot{H} + 3H^2 = -\kappa(p + p_{eff}) \tag{1.66}$$

where

$$2\kappa\rho_{eff} = 2T f_T - T - f \tag{1.67}$$

$$2\kappa p_{eff} = 8\dot{H} T f_{TT} - 2(T - 2\dot{H}) f_T + f + T - 4\dot{H} \tag{1.68}$$

are the energy density and pressure of the effective fluid due to the contribution of torsion. The equation of state of the effective fluid is given by

$$\omega_{eff} = -1 + \frac{8\dot{H}Tf_{TT} + 4\dot{H}f_T - 4\dot{H}}{2Tf_T - T - f} \tag{1.69}$$

So the generic function $f(T)$ determines whether the universe is accelerating or not. There are a plethora of extended theories of gravity, for details of which one may refer to [77]- [130]. However, there are two common questions that may come to mind, even though there is no alternative way to accommodate the observational data into the theoretical framework without introducing either dark energy or modified gravity theories. These are listed below as follows.

a) How does the exotic matter suddenly come into scenario as the dominant fluid and produce the late-time accelerated expansion?

b) Why does General Relativity need to be modified suddenly while it nicely predicts several observational results with high accuracy and describes the evolution of the Universe till the matter dominated era?

After the detection of gravitational waves, Einstein's GR emerges as the most successful theory of gravity to describe physical reality. Nevertheless, the existence of singularity in Einstein gravity is its biggest drawback.

1.6 Limitation of GR: Problem of Singularity and Raychaudhuri equation

The basis of Einstein's General Theory of Relativity (GR) [131]-[137] is an n- dimensional Lorentzian manifold (M, g) where g is symmetric, non-degenerate $(0, 2)$-tensor field assigning to any point p in the smooth manifold M, a scalar product g_p on the tangent space $T_p(M)$ at the point p. However, it is to be noted that the scalar product g_p is not positive definite, but has signature $(-1, +1, +1, ..., +1)$. Thus, a principal premise of general relativity is that space-time can be modeled as a four dimensional Lorentzian manifold of signature $(3, 1)$ or equivalently $(1, 3)$ together with a time orientation. $v \in T_p(M)$ is time-like (space-like) if $< v, v >= 0$ but $v \neq 0$ and causal if it is time-like or null. These notions extend to vector fields. The most attractive part of GR are the field equations that show an equivalence between matter and geometry. The Riemannian curvature tensor of (M, g) is given by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{1.70}$$

where ∇ denotes the Levi-Civita connection of g. Ricci tensor is given by

$$Ric(X, Y) = \sum_{i=1}^{n} < E_i, E_i >< R(E_i, X)Y, E_i > \tag{1.71}$$

where $\{E_1, E_2, ..., E_n\}$ represent a local orthonormal frame field. Einstein's field equations are given by

$$Ric - \frac{1}{2}Sg + \Lambda g = 8\Pi T \tag{1.72}$$

where T is the energy momentum tensor which encodes information on matter and energy content of the space-time and S is the contraction of Ricci tensor. Einstein's GR [131]-[137] is the most well accepted theory of gravity till date that successfully describes perihelion shift of Mercury's orbit which Newtonian gravity failed to account on it fully. Gravitational lensing, time-dilation, redshift, bending of light etc are the most remarkable predictions of GR. Its popularity widely increased after the phenomenal discovery of Gravitational waves [138], [139] and the image of shadow of the supermassive black hole $M87*$ at the centre of the galaxy Messier 87 [140]-[145]. GR also succeeds in explaining different observational aspects of the Universe. Nevertheless, among a few drawbacks of GR the problem of singularity is the most troublesome one. The appearance of singularity in GR puts limit to its validation. Singularity can be identified as a point in the space-time manifold where space-time curvature, energy density and all other physical laws break down/ diverge resulting in the breakage of the theory itself. Even the simplest and first ever solution of Einstein's field equations i.e, the Schawrzschild Solution [131] has a singularity at $r = 0$. Further, the cosmological Friedmann–Lemaître–Robertson-Walker (FLRW) [2, 3, 4, 5] space-time has an initial singularity. Earlier, singularity was assumed to be due to artefacts of symmetry assumptions made in these particular solutions (Schwarzschild and FLRW). However, Roger Penrose and Stephen Hawking showed in their famous singularity theorems [146]-[153] that if one assumes positiveness of energy, causality and existence of trapped surface then singularity is quite inevitable in a broader sense. In this connection, the notion of geodesic incompleteness and singularity came into picture in Penrose's work, the former being geometric definition of the later. This means, singularity of a space-time is related to the presence of incomplete time-like/null geodesics. This award winning work uses the Raychaudhuri equation as the key ingredient. In other words, it can be said that Raychaudhuri equation together with the Focusing theorem is the turning point in Einstein gravity as it hints the inherent existence of singularity in Einstein gravity via the notion of geodesic focusing.

1.6.1 Geodesics

To introduce the notion of geodesics we should first know about the definition of tangent vector to a curve and about parallel transport tensors. Let $\Gamma : x^a = f^a(\lambda)$, $a = 1, 2, 3, ..., n$ be a differentiable curve passing through a point P in a manifold $\mathcal{M}$. Suppose, $\{x_0^a = f_0^a,\ a = 1, 2, 3, ...n\}$ be the co-ordinate of P and a neighboring point Q is identified by the parameter $\triangle\lambda$. Then tangent vector to the curve Γ at P is defined as

$$t^a = \frac{dx^a}{d\lambda}|_{\lambda=0} \tag{1.73}$$

The set of all such tangent vectors corresponding to all curves through P forms a vector space of dimension n. This vector space is called the tangent space at P in $\mathcal{M}$ or simply T_P.
The notion of parallel transport tensor can be introduced using intrinsic differentiation. For details regarding differentiation of tensors see appendix. A tensor A is said to be parallelly transported along a curve Γ (parametrized by λ) if $\dfrac{\delta A}{\delta\lambda} = 0$.
A curve Γ is said to be a geodesic if the parallel transport of the tangent vector is along the curve itself. So by definition one has,

$$\frac{\delta \mathbf{v}}{\delta\lambda} = 0 \tag{1.74}$$

i.e, $\mathbf{v}^a_{;b}\mathbf{v}^b = 0$ where λ is the parameter along the curve Γ and $\mathbf{v}$ is the tangent vector to Γ. In local coordinates, $\{x^a\}$ the explicit form of the geodesic equation is represented by the quasi-linear differential equation in $x^a(\lambda)$ as follows

$$\frac{d\mathbf{v}^a}{d\lambda} + \Gamma^a_{bc}\mathbf{v}^b\mathbf{v}^c = 0 \tag{1.75}$$

or,

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0. \tag{1.76}$$

If we apply the standard existence theorem for ordinary differential equations in the above differential equation then it is possible to have a geodesic through any point say P of the manifold $\mathcal{M}$ such that tangent to the geodesic at P is a given vector from $T_P(\mathcal{M})$. This geodesic $\Gamma(\lambda)$ is unique and depends continuously both on the point P and the direction at P. Such a geodesic is called maximal geodesic. A space-time (M', g') is called an extension of a given space-time (M, g) if (M, g) may be isometrically embedded as an open proper subset of (M', g'). A space-time which has no such extensions are said to be *inextendible* or *maximal*. We call a geodesic $\Gamma : [a, b] \to M$ extendible if there exists a geodesic $\tilde{\Gamma} : [\alpha, \beta] \to M$ with $[a, b] \subset [\alpha, \beta]$ and $\tilde{\Gamma}|_{[a,b]} = \Gamma$, otherwise Γ is inextendible. Γ is complete if it can be extended for all values of the affine parameter. A space-time manifold is geodesically complete if all its inextendible geodesics are complete, otherwise the the space-time manifold is said to be incomplete.

1.6.2 Hypersurface in a Riemannian space

An n dimensional hyper-surface $\mathcal{H}_n$ in an $(n+1)$ dimensional Riemannian space $\mathcal{M}$ is given by

$$y^a = f^a(x^1, x^2, x^3, ..., x^n) \tag{1.77}$$

$a = 1, 2, ..., n+1$, y^a represents the co-ordinate system in $\mathcal{M}$, x^j's are the n real variables such that the Jacobian matrix given by

$$\mathcal{J} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \cdots & \frac{\partial y^2}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y^{n+1}}{\partial x^1} & \frac{\partial y^{n+1}}{\partial x^2} & \cdots & \frac{\partial y^{n+1}}{\partial x^n} \end{bmatrix}$$

is of rank n. $\mathcal{H}_n$ is called the hyper-surface of the enveloping space $\mathcal{M}$.

1.6.2.1 Induced Metric on $\mathcal{H}_n$

Let h_{ab} be the components of metric tensor in $\mathcal{M}$ under some y-coordinate system. The elementary distance $d\mathcal{S}$ between two neighboring points in $\mathcal{H}_n \in \mathcal{M}$ is given by

$$\begin{aligned} d\mathcal{S}^2 &= h_{ab} dy^a dy^b \\ &= h_{ab} \left(\frac{\partial y^a}{\partial x^j} dx^j\right) \left(\frac{\partial y^b}{\partial x^k} dx^k\right) = g_{jk} dx^j dx^k \end{aligned} \tag{1.78}$$

where $g_{jk} = h_{ab} \dfrac{\partial y^a}{\partial x^j} \dfrac{\partial y^b}{\partial x^k}$ is the metric tensor in $\mathcal{H}_n$.

1.6.2.2 Hyper-surface Orthogonality

Let N^a be the contravariant components (in the y-coordinate system) of the unit normal $\mathbf{N}$ to $\mathcal{H}_n \in \mathcal{M}$. For fixed $j = 1, 2, 3, ..., n$ the vector $\dfrac{\partial y^a}{\partial x^j}$ is tangential to $\mathcal{H}_n$ and hence orthogonal to the normal vector $\mathbf{N}$ and the above tangent vector yields

$$h_{ab} \nabla_j y^a N^b = 0 \tag{1.79}$$

and the normalization of $\mathbf{N}$ gives

$$h_{ab} N^a N^b = 1 \tag{1.80}$$

with $\nabla_j y^a = \dfrac{\partial y^a}{\partial x^j}$.

1.6.2.3 Frobenius' theorem in Differential Geometry

A vector field v^a (which may be time-like, space-like or null and not necessarily geodesic) is hyper-surface orthogonal if there exists a scalar field say, Φ such that $v^a \propto \Phi_{,a}$ which implies $v_{[a;\ b}\ v_{c]} = 0$. If the velocity vector field is time-like and geodesic, then it is hyper-surface orthogonal if there exists a scalar field Ψ such that $v_a = -\Psi_{,a}$ (i.e, the proportionality constant is -1) which implies a vanishing rotation i.e, $\omega_{ab} = v_{[a;b]} = 0$.

1.6.3 A short account of the Raychaudhuri equation

The existence of singularity is troublesome and possible resolution is needed to get the complete description of a physical theory. Therefore removal of these singularities is a topic of central interest in the field of relativity, cosmology and astrophysics. Before addressing the resolution of these singularities, one should know what precisely does a singularity mean? A singularity is a point in the space-time manifold where all physical laws break down, i.e. the space-time becomes pathological. However, a formal geometric definition of what a singularity is, first appeared in the works of Penrose where he defined singularity from the point of view of geodesic incompleteness [146]. In the early 1950s, Raychaudhuri first addressed this issue of singularity by forming an evolution equation for the kinematic quantity namely, the expansion scalar in terms of shear scalar, vorticity scalar, a velocity vector field and Ricci tensor corresponding to a flow of congruence. This evolution equation is popularly known as the Raychaudhuri equation (RE) [154], [155]. Later, this equation gained immense popularity following the works of Hawking and Penrose to prove the singularity theorems where they used the RE as the major ingredient. RE together with Focusing theorem (FT) proves that singularity is inevitable in GTR putting limits to this theory as long as SEC holds on matter. Focusing of geodesic congruence is the most vital consequence of RE. Therefore, RE is a very important equation to deal with space-time geometry and singularities.

1.6.4 Flow as congruence of geodesic and Kinematics of Deformable medium: Geometry behind the RE

The celebrated RE deals with the kinematics of flows [156]-[169]. If a vector field is given, then flows are the integral curves generated by that vector field. The curves are basically of two types namely, geodesic and non-geodesic. In the following sections and subsections we shall mainly review the results on geodesic congruence. This is because, the geodesics play a pivotal role when it comes to gravity. Although the RE is generally true for any curves (time-like or null), its geodesic version can be deduced from the general form. Thus a flow is identified by a congruence of such curves which may be time-like, null or curves having tangent vectors with a positive definite norm in the Euclidean case.

Let us now introduce a formal definition of a *congruence*. Let $\mathcal{M}$ be a manifold and U be open in $\mathcal{M}$. A *congruence* is defined as a family of curves in U if precisely one curve in this family passes through each point $p \in U$. Now, we focus our attention to study the kinematic quantities associated with such flows and how Raychaudhuri equation is related to them [170]. In this context, it is to be noted that evolution equation of the kinematic quantities (which characterize the flow) in a given space-time background along the flow is of central importance.

Let τ be the parameter labeling points on the curves and v^a be the velocity vector field along the congruence. Let, R_{ab} be the $(0,2)$ Ricci tensor projected along the geodesic flow. Essentially one need to have various functions of τ in order to characterize the flow. To define the kinematics of a deformable medium, we consider the gradient of velocity vector field which can be geometrically represented by a $(0,2)$ deformation tensor, say $\mathcal{B}_{ab}$. Further, this tensor can be split into four fundamental and irreducible tensors as follows:

$$\mathcal{B}_{ab} = \nabla_b v_a = \frac{\Theta}{n-1}\,\eta_{ab} + \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - A_a v_b, \tag{1.81}$$

where, ∇ denotes the *covariant derivative* and n is the dimension of the space-time. One

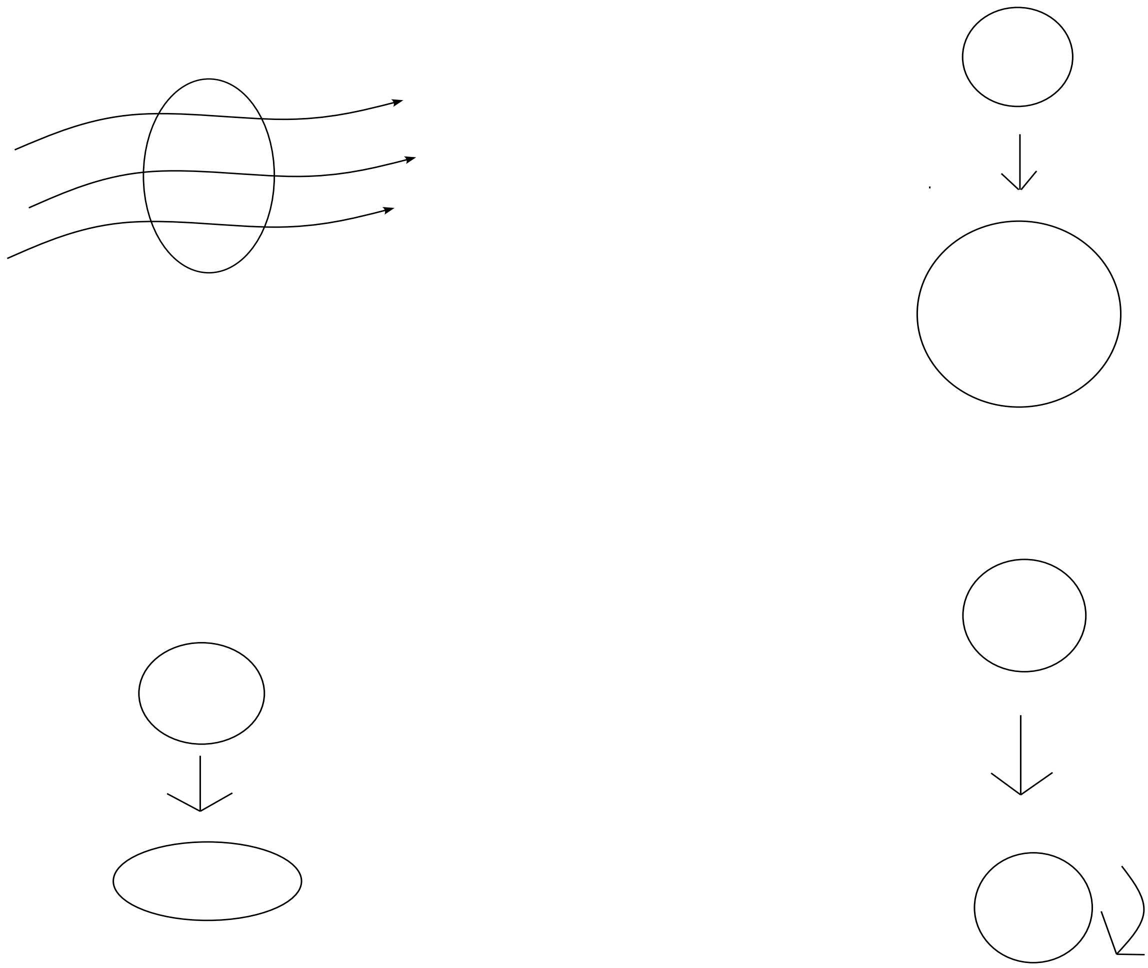


Figure 1.1: Figures representing (1) Area enclosing the flow lines (top left) (2) Isotropic Expansion (top right) (3) Shear (bottom left) and (4) Rotation (bottom right)

may check that $\mathcal{B}_{ab}v^b = \mathcal{B}_{ab}v^a = 0$ as v^a satisfies the geodesic equation and is normalized as $v^a v_a = -1$. Now, we discuss the geometric and physical meaning of the introduced tensors in eq.(1.81).

- To analyze the physical interpretation of $\mathcal{B}_{ab}$, a one parameter subfamily of geodesics in the congruence denoted by $\gamma_l(\tau)$ needs to be considered. Here l is the labeling parameter. The geodesics in the congruence can be collectively represented by $x^a(\tau, l)$. Let ξ^a be the orthogonal deviation vector from a reference geodesic say γ_0 in the subfamily. Using $v^a = \dfrac{\partial x^a}{\partial \tau}$ and $\xi^a = \dfrac{\partial x^a}{\partial l}$ we have

$$v^b \nabla_b \xi^a = \xi^b \nabla_b v^a = \mathcal{B}^a_b \xi^b. \tag{1.82}$$

$\mathcal{B}^a_b$ measures the failure of ξ^a to be transported parallelly. The symmetry of $\mathcal{B}_{ab}$ i.e, $\mathcal{B}_{ab} = \mathcal{B}_{ba}$ is ensured by the Frobenius' theorem.

- Θ is known as the expansion scalar. It is the trace part of $\mathcal{B}_{ab}$ i.e, $\Theta = \mathcal{B}^a_a = \nabla_a v^a$. It describes the average separation between the geodesic worldlines of the v_a–congruence, precisely the average expansion/contraction of the associated observers. This means, if one considers the congruence as a collection of flow lines (geodesics) then cross sectional area enclosing the geodesics may evolve along the congruence. If the geodesics go apart or come closer then area will increase or decrease accordingly. This phenomenon is demonstrated by Θ, the expansion scalar.

- σ_{ab} is known as the shear tensor defined by $\sigma_{ab} = \frac{1}{2}\left(\nabla_b v_a + \nabla_a v_b\right) - \frac{\Theta}{n-1}\eta_{ab}$. It is symmetric traceless part of $\mathcal{B}_{ab}$ i.e, $\sigma_{ab} = \sigma_{ba}$ and $\sigma^a_a = 0$. It measures the kinematic anisotropies. The shape of the area enclosing congruence of geodesics may be sheared and this is demonstrated by σ.

- ω_{ab} is the anti-symmetric part of $\mathcal{B}_{ab}$ (i.e, $\omega_{ab} = -\omega_{ba}$) and is defined by $\omega_{ab} = \frac{1}{2}\left(\nabla_b v_a - \nabla_a v_b\right)$. It is called the rotation/ vorticity tensor as it measures the kinematic rotation or monitors the rotational behavior of the v_a–vector field. The shape of the area enclosing the congruence of geodesics may be twisted. This is demonstrated by ω_{ab}. A congruence is hyper-surface orthogonal if and only if $\omega_{ab} = 0$. This is a consequence of the Frobenius' theorem discussed in the previous subsections in the context of differential geometry.

- $\eta_{ab} = g_{ab} \pm v_a v_b$ is called the induced metric/projection tensor that operates on the $(n-1)$ dimensional hyper-surface. η_{ab} satisfies the orthogonality condition i.e, $v^b \eta_{ab} = 0$. In expression for η_{ab}, '+' sign is for time-like curves ($v_a v^a = -1$) and '-' sign is for null curves ($v_a v^a = 0$).

- A_a is the 4-acceleration vector field defined by $A_a = v^b \nabla_b v_a$. This field guarantees the presence of non gravitational forces. Therefore, $A_a = 0$ in case of geodesic worldlines.

Thus, the above discussion shows that the expansion, rotation and shear are purely geometric characteristic of the cross sectional area enclosing a bundle of curves orthogonal to the flow lines. The shape of this area changes as one moves from one point to another along the flow but it still includes the same set of curves in the bundle. One thing which may change during the flow is that the bundle may be isotropically smaller or larger, sheared or twisted. This situation seems to be analogous to the elastic deformations or fluid flow.
The general Raychaudhuri equation [171], [172] for non-geodesic motion is nothing but the proper time (τ) evolution of the expansion scalar (Θ) as

$$\frac{d\Theta}{d\tau} = -\frac{\Theta^2}{n-1} - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} + \nabla_b A^b - R_{ab} v^a v^b \tag{1.83}$$

Raychaudhuri equation for null geodesic congruence is given by

$$\frac{d\Theta}{d\lambda} = -\frac{\Theta^2}{n-2} - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} + \nabla_b A^b - R_{ab} k^a k^b \tag{1.84}$$

where k^a is a null vector and λ is an affine parameter. Ricci tensor R_{ab} is a $(0,2)$ that carries the effect of local gravitational field as otherwise RE is a purely geometric identity and has nothing to do with gravity. The term $-R_{ab}v^a v^b$ encapsulates the contribution of space-time geometry and is independent of the derivatives of the vector field. Thus in comparison to the other terms present in eq. (1.83) and (1.84), this particular term possesses more general implications. It has a geometrical interpretation as a mean curvature in the direction of the v_a–congruence [173], [174]. The detailed derivation of the equations (1.83) and (1.84) are discussed in the following section.

1.6.5 An outline of the geometric derivation of the Raychaudhuri equation

Raychaudhuri addressed the issue of singularity in the early 1950's. At that time, he was working on the features of electronic energy bands in metals. The generic feature of GR and nature of gravitational singularities attracted his interest. Subsequently, motivated by cosmology he pointed out for the first time in his seminal paper that singularity is nothing more than an artifact of the symmetries of the matter distribution. Actually, he wanted to see the effect of spin (non-zero vorticity), anisotropy (shear) and (or) cosmological constant in avoiding the initial singularity. We first review the outline of the derivation of RE presented in the seminal 1955 paper. Motivation behind the original derivation was entirely devoted to cosmology. Raychaudhuri did not assume homogeneity or isotropy but a time dependent geometry which characterizes a universe i.e, he proposed a time dependent model of the Universe without assuming cosmological principle. The entire derivation was carried out in the synchronous/co-moving frame–the frame in which the observer is at rest in the fluid. In the 1955 paper, the space-time coordinates were labeled as x^1, x^2, x^3, x^4, where x^4 is t, the time coordinate. The quantity R^4_4 was evaluated both by using Einstein's field equations with Λ (cosmological constant) and again using the geometric definition of R^4_4 in terms of the metric and its derivatives. In order to do so, he used the geometric definition of Θ, σ and ω. Consequently, he equated these two ways of writing R^4_4 to get the evolution of expansion–which is the now famous RE. Motivated by this 1955 paper, many relativists and cosmologists published innumerable papers following Raychaudhuri's work. For example, Heckmann and Schucking [175] in the same year derived a set of equations while dealing with Newtonian cosmology. One of the equations resembled the RE in Newtonian case. They further showed a relativistic generalization of their work without any scientific issue. Later in 1961, Jordan et.al extensively wrote an article on the relativistic mechanics of continuous media where the derivation of these kinematic quantities appeared for the first time. Now we derive the RE for a congruence of time-like and null geodesics considering four dimensional space-time background.

1.6.5.1 Derivation of RE for time-like geodesic congruence

The evolution equation for the above kinematic quantities Θ, σ and ω can be derived from geometrically of which the evolution of expansion scalar Θ is popularly known as the Raychaudhuri equation and it is of central importance. We have,

$$v^c \nabla_c \mathcal{B}_{ab} = v^c \nabla_c \nabla_b v_a$$
$$= v^c \nabla_b \nabla_c v_a + R_{cba}{}^{d} v^c v_d = -B^c_b\, B_{ac} + R_{cba}{}^{d} v^c v_d, \tag{1.85}$$

where $R^a_{bcd} = \dfrac{\partial \Gamma^k_{ac}}{\partial x^d} - \dfrac{\partial \Gamma^k_{ad}}{\partial x^c} + \Gamma^m_{ac}\Gamma^k_{md} - \Gamma^m_{ad}\Gamma^k_{mc}$. Next we have,

$$[\nabla_b, \nabla_a]v^c = (\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = -R^c{}_{dab}\, v^d. \tag{1.86}$$

The trace of equation (1.85) gives,

$$v^c\nabla_c\Theta = \frac{d\Theta}{d\tau} = -\frac{\Theta^2}{3} - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}v^c v^d. \tag{1.87}$$

The Ricci tensor R_{cd} can be defined by contracting the Riemann curvature tensor as follows

$$R_{cd} = R^a_{cad}. \tag{1.88}$$

From the symmetric, trace-free part of equation (1.85) one has

$$v^c\nabla_c\sigma_{ab} = -\frac{2}{3}\Theta\sigma_{ab} - \sigma_{ac}\sigma^c{}_b - \omega_{ac}\omega^c{}_b + \frac{1}{3}h_{ab}(\sigma_{cd}\sigma^{cd} - \omega_{cd}\omega^{cd}) \tag{1.89}$$

$$+C_{cbad}u^c u^d + \frac{1}{2}\tilde{R}_{ab} \tag{1.90}$$

where $\tilde{R}_{ab}$ is the spatial, trace-free part of R_{ab} and C_{cbad} is the Weyl tensor. The expression for $\tilde{R}_{ab}$ is given by

$$\tilde{R}_{ab} = h_{ac}\, h_{bd}R^{cd} - \frac{1}{3}h_{ab}\, h_{cd}R^{cd} \tag{1.91}$$

and that of Weyl tensor is given by

$$C_{abcd} = R_{abcd} + \frac{1}{2}(R_{ad}\, g_{bc} - R_{ac}\, g_{bd} + R_{bc}\, g_{ad} - R_{bd}\, g_{ac}) + \frac{1}{6}R\,(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}) \tag{1.92}$$

From the anti-symmetric part of equation (1.85) one has

$$u^c\nabla_c\, \omega_{ab} = -\frac{2}{3}\Theta\omega_{ab} - 2\sigma^2_{[b\ \omega_a]c} \tag{1.93}$$

The evolution equation for Θ given by equation (1.87) is the RE for time-like geodesic congruence.

1.6.5.2 Derivation of RE for null geodesic congruence

Unlike the case of time-like geodesic, in case of null geodesic congruence the tangent vector field is denoted by k^a– a null vector. Let λ be the affine parameter. So one has $dx^a = k^a d\lambda$. Let η^a be the deviation vector. Similar to time-like case, one has the following

$$k^a k_a = 0 \tag{1.94}$$

$$k^b\nabla_b k^a = 0 \tag{1.95}$$

$$\eta^b\nabla_b k^a = k^b\nabla_b\eta^a \tag{1.96}$$

$$k^a\eta_a = 0 \tag{1.97}$$

Since k^a is a null vector, therefore $h_{ab} = g_{ab} + k_a k_b$ can no more act as transverse metric as $h_{ab}k^b = k^a \neq 0$. We thus introduce another null vector say N_a such that $k^a N_a \neq 0$. Using the arbitrary normalization of a null vector, the condition $k^a N_a = -1$ can always be imposed. We now introduce a purely transverse and effectively 2-dimensional metric

$$h_{ab} = g_{ab} + k_a N_b + N_a k_b \tag{1.98}$$

and we have

$$\begin{aligned} h_{ab}k^b &= h_{ab}N^b = 0 \\ h^a_a = 2, \quad & h^a{}_c h^c_b = h^a_b. \end{aligned} \tag{1.99}$$

The spatial tensor field is

$$\mathcal{B}_{ab} = \nabla_b k_a \tag{1.100}$$

This is the measurement of the failure of η^a to be transported parallelly along the congruence

$$k^b \nabla_b \eta^a = \mathcal{B}^a_b \eta^b \tag{1.101}$$

$\mathcal{B}_{ab}$ is orthogonal to the tangent vector field but not to N^a. Thus, η^a contains a non-transverse component that must be removed. The purely transverse part of the deviation vector is

$$\tilde{\eta}^a = h^a_c \eta^c = \eta^a + (N_c \eta^c) k^a \tag{1.102}$$

Now,

$$k^b \nabla_b \tilde{\eta}^c = h^c{}_d \mathcal{B}^d_b \eta^b + k^b \nabla_b h^c{}_d \eta^d = h^c{}_d \mathcal{B}^d{}_b \eta^b + (k^b \nabla_b N_d \eta^d) k^c \tag{1.103}$$

So, $k^b \nabla_b \tilde{\eta}^c$ has a component along k^c. It must be removed again by projecting with $h^a{}_c$. Using equation (1.99) one can write

$$\begin{aligned} \left(k^b \nabla_b \tilde{\eta}^a\right) &= h^a{}_c \left(k^b \nabla_b \tilde{\eta}^c\right) \\ = h^a{}_c \mathcal{B}^c{}_d \eta^d &= h^a{}_c \mathcal{B}^c{}_d \tilde{\eta}^d = h^a{}_c \ h^d{}_b \mathcal{B}^c{}_d \tilde{\eta}^b \end{aligned} \tag{1.104}$$

Hence the purely transverse behavior of the congruence is governed by the equation

$$\left(k^b \nabla_b \tilde{\eta}^a\right) = \tilde{\mathcal{B}}^a_a \ \tilde{\eta}_b \tag{1.105}$$

where $\tilde{\mathcal{B}}_{ab}$ denotes the purely transverse part of $\mathcal{B}_{ab}$,

$$\tilde{\mathcal{B}}_{ab} = h^c{}_a h^d{}_b \mathcal{B}_{cd} \tag{1.106}$$

The explicit expression for $\tilde{B}_{ab}$ can be written as

$$\tilde{\mathcal{B}}_{ab} = \mathcal{B}_{ab} + k_a N^c \mathcal{B}_{cb} + k_b \mathcal{B}_{ac} N^c + k_a k_b \mathcal{B}_{cd} N^c N^d \tag{1.107}$$

Following the same manner just like the time-like case the decomposition of $\mathcal{B}_{ab}$ is

$$\tilde{\mathcal{B}}_{ab} = \frac{1}{2}\Theta \ h_{ab} + \sigma_{(ab)} + \omega_{ab} \tag{1.108}$$

The expression for expansion scalar is given by

$$\Theta = \tilde{\mathcal{B}}^a_a \tag{1.109}$$

Shear and rotation tensor are given by

$$\sigma_{ab} = \tilde{\mathcal{B}}_{(ab)} - \frac{1}{2}\Theta h_{ab} \tag{1.110}$$

$$\omega_{ab} = \tilde{\mathcal{B}}_{[ab]} \tag{1.111}$$

The expansion scalar Θ is given by $\Theta = g^{ab}\tilde{\mathcal{B}}_{ab} = g^{ab}\mathcal{B}_{ab} = \nabla_a k^a$. Similar derivation follows in this case too which gives

$$k^c\nabla_c\mathcal{B}_{ab} + \mathcal{B}^c{}_b\mathcal{B}_{ac} = R_{cba}{}^d k_d k^c \tag{1.112}$$

The evolution equation for Θ or the null RE is given by

$$\frac{d\Theta}{d\lambda} = -\frac{\Theta^2}{2} - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}k^c k^d, \tag{1.113}$$

λ being an affine parameter. Evolution equations for shear and rotation are respectively given by

$$k^c\nabla_c\sigma_{ab} = -\Theta\ \sigma_{ab} + (C_{cbad}k^c k^d) \tag{1.114}$$

$$k^c\nabla_c\ \omega_{ab} = -\Theta\ \omega_{ab} \tag{1.115}$$

1.6.6 Significance of Raychaudhuri equation: Focusing Theorem

The general form of the RE (1.83) can be reduced to much simpler form if one assumes:
1. The congruence of curves to be time-like geodesic which in turn makes $A^b = 0$.
2. Congruence of time-like geodesics to be hyper surface orthogonal, which by virtue of Frobenius theorem [176] of differential geometry implies zero rotation i.e, $\omega_{ab} = 0$. Since σ_{ab} is spatial hence, $\sigma_{ab}\sigma^{ab} \geq 0$ (vanishing vorticity). If one is interested in geodesic focusing then zero vorticity congruence must be taken into consideration to avoid the effect of centrifugal forces.

Thus the simplified version of the original equation (1.83) upon assuming the above conditions is given by

$$\frac{d\Theta}{d\tau} = -\left(\frac{\Theta^2}{n-1} + 2\sigma^2 + \tilde{R}\right) \tag{1.116}$$

where $2\sigma^2 = \sigma_{ab}\sigma^{ab}$ and $\tilde{R} = R_{ab}v^a v^b$. In addition if $\tilde{R} \geq 0$ then $\frac{d\Theta}{d\tau} \leq 0$ which shows that the expansion gradually decreases with the evolution of the congruence. On an explicit manner we now analyse the Focusing Theorem as follows:
If matter content of the universe satisfies SEC i.e ,

$$T_{ab}v^a v^b + \frac{1}{2}T \geq 0, \tag{1.117}$$

where T_{ab} is the energy momentum tensor then Einstein's equation

$$R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab} \tag{1.118}$$

yields,

$$R_{ab}v^a v^b \geq 0. \tag{1.119}$$

Employing the condition (1.119) on (1.116) we get ,

$$\frac{d\Theta}{d\tau} + \frac{\Theta^2}{n-1} \leq 0. \tag{1.120}$$

Integrating the above inequality w.r.t proper time τ we get,

$$\frac{1}{\Theta(\tau)} \geq \frac{1}{\Theta_0} + \frac{\tau}{n-1}. \tag{1.121}$$

Thus, one can infer that any initially converging hyper-surface orthogonal congruence of time-like geodesics must continue to converge within a finite value of the proper time $\tau \leq -(n-1)\Theta_0^{-1}$ which leads to crossing of geodesics and formation of a congruence singularity (may or may not be a curvature singularity). This is called the Focusing Theorem (FT) and the condition (1.119) is the corresponding Convergence Condition (CC). Further, it is to be noted that the Strong Energy Condition (SEC) causes gravitation to be attractive and hence can't cause geodesic deviation, rather it increases the rate of convergence. Thus the Focusing Theorem (FT) inevitably proves the generic existence of singularity as a major drawback of Einstein gravity. As clear from the above discussion, FT follows as a consequence of the RE, this is the reason why RE is regarded as the fundamental equation of gravitational attraction as it proves the seemingly trivial statement that gravity is attractive which draws the geodesics closer and make them converge in some finite value of the affine parameter/ proper time. Focusing Theorem in terms of null geodesic can be derived in a similar manner by considering the RE in case of null geodesic congruence except there is a change in the inequality as $\frac{d\Theta}{d\lambda} + \frac{\Theta^2}{n-2} \leq 0$.

1.6.7 Focusing theorem for non zero curvature

Focusing theorem for zero curvature is well known in literature. However, if one includes curvature then what happens? Motivated by this question, in this section we try to rewrite the FT for open (curvature being negative) and closed (curvature being positive) model of the universe. The first Friedmann equation with non-zero curvature term K is given by

$$3\left(H^2 + \frac{K}{a^2}\right) = \rho. \tag{1.122}$$

In four dimensional FLRW background the expansion scalar Θ is given by $\Theta = 3H$ where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and $a(t)$ is the cosmic scale factor. Writing (1.122) in terms of Θ we have

$$\frac{\Theta^2}{3} = \rho - \frac{3K}{a^2}. \tag{1.123}$$

Using the RE in FLRW model ($\sigma = 0$) considering hyper-surface orthogonal ($\omega = 0$) congruence of time-like geodesic ($\nabla_b A^b = 0$) and $\tilde{R} = \frac{1}{2}(\rho + 3p)$ one has

$$\frac{d\Theta}{dt} = \frac{3K}{a^2} - \frac{3}{2}(\rho + p) \tag{1.124}$$

Now we discuss the following cases:
Case-I : $K < 0$

If $(\rho + p) \geq 0$ then equation (1.124) gives $\frac{d\Theta}{dt} < 0$. Thus the expansion of the congruence decreases with time. In other words if we consider an open model of universe with matter satisfying the Null Energy Condition (NEC) then congruence may focus either in finite or in infinite time. This may lead to formation of congruence singularity. However if $(\rho + p) < 0$ and $|\rho + p| > \frac{3K}{a^2}$ then $\frac{d\Theta}{dt} > 0$. Therefore open universe with exotic matter can avoid focusing and hence singularity formation.

Case II: $K > 0$
In this case $\frac{K}{a^2} > 0$ and using equation (1.124) one may find that if $(\rho + p) > \frac{2K}{a^2} > 0$ then $\frac{d\Theta}{dt} < 0$. Therefore in a closed model of universe if matter satisfies the NEC with a non zero lower bound then expansion of the congruence decreases with time. However singularity may be avoided even with the assumption of NEC on matter provided $\frac{2K}{a^2} > (\rho + p)$. Thus a high positive curvature and matter content satisfying NEC is required in a closed universe for the possible avoidance of singularity. However, in any case exotic matter always avoids the singularity.

1.6.8 An insight of the Singularity theorems

Although AKR pointed out the inherent existence of singularities via his celebrated equation in the original 1955 paper, yet more general results based on global techniques in Lorentzian space-time appeared as singularity theorems in Penrose's work and Hawking's contribution [146],[177],[178]. The key ingredient of their singularity theorems is that existence of singularity was established by considering Lorentz signature metrics and causality, existence of trapped surfaces and energy conditions on matter. Most importantly, the precise and formal (geometric) definition of a singularity first appeared in their works. The two concepts—notion of geodesic incompleteness and singularities (not necessarily curvature singularities) were aligned. To be precise, a space-time can be called singular if it contains an incomplete causal geodesic. It is to be noted that FT and RE could be completely benign as these are true irrespective of any singularity actually occurring in the space-time manifold. This gives us the knowledge that singularity would always imply focusing but focusing alone cannot imply a singularity. This was pointed out by Landau [179]. For more details regarding the singularity theorems, there are some phenomenal works [132],[178],[180] in this direction for general readership.

The 1965 Penrose singularity theorem states that: *"If the space-time contains a non-compact Cauchy hyper-surface Σ and a closed future-trapped surface, and if the convergence condition holds for null k^μ, then there are future incomplete null geodesics".* Now we briefly discuss the seminal Hawking-Penrose singularity theorem which may be stated as follows:

If the convergence condition (time-like or null) is satisfied, there are no closed future pointing time-like curves, a generic condition on the curvature holds and if there is one of the following

- *a closed achronal imbedded hyper-surface*
- *a closed trapped surface*
- *a point with re-converging light cone*

then the space-time is causal geodesically incomplete.

Geodesic incompleteness implies space-time incompleteness. This novel characterization of singularities was first pointed out by Penrose and later established by Hawking and Geroch. All assumptions in the Singularity theorem are physically reasonable, the most important among them is the convergence condition for causal geodesics. This condition leads to focusing of geodesics. Geodesic focusing is important for the proof of any Singularity theorem. Therefore, the thesis studies the time-like convergence condition in various background geometries to find the condition for possible resolution of singularities in GR and modified gravity models.

1.6.9 Singularity and Geodesic incompleteness: A general study by Raychaudhuri equation

A singularity has the property of geodesic incompleteness in which there is a failure to extend either some light path or some particle path beyond a certain affine parameter or proper time. Geodesics are paths of observers through space-time that can only be extended for a finite time as measured by an observer traveling along one. It is generally assumed that at the end of geodesic an observer has fallen into a singularity or has encountered some kind of pathology at which the laws of general relativity break. This geodesic incompleteness leads to the presence of infinite curvatures, making the right hand side of RE infinite. Thus there is a divergence of the expansion parameter Θ. However, there is a connection between "caustics" and "singularity". As a result of FT, the congruence will develop a "caustic" which is nothing but a point at which all geodesics focus. Thus, a caustic may also be called as a focal point which is nothing but a singularity of the congruence. However, this may not be a singularity of the space-time. This can be understood from two examples–(i) If one considers the null analog of RE in the context of gravitational lensing, then a caustic or focal point in this case is the intersection of trajectories representing light rays and is known as the caustic of the bundle of trajectories. (ii) The focusing condition given by $R_{\mu\nu}u^{\mu}u^{\nu} \geq 0$ is trivially satisfied by flat space-times but it has no singularities. This shows that "caustic" is not always a singularity of the space-time but essentially a congruence singularity. However some extra assumptions (discussed later in this section) and the notion of geodesic incompleteness are the additional criteria for the existence of space-time singularity. In this connection, the key turning point was the 1965 singularity theorem by Penrose and a subsequent work of Hawking. They proved the existence of black hole and cosmological singularity using the notion of incompleteness of null and time-like geodesics respectively. In general, no singularity but a caustic is formed along the flow lines of the u^{μ}–congruence. This property is usually called the focusing effect on causal geodesics. Although the singularity here is the congruence singularity and may not be a space-time singularity but these conditions along with some global arguments may lead to space-time singularity in certain cases. Although Raychaudhuri pointed out the connection of his equation to the existence of singularities in his 1955 article, however more general results based on global techniques in Lorentzian space-times appeared in the form of singularity theorems following the work of Penrose [149] and Hawking [177]. According to Penrose (for null geodesics) and Hawking (for time-like geodesics) the singularity to be a black hole or cosmological singularity, the incompleteness of geodesic is essential. They have shown that this incompleteness of geodesic is related to the causal structure of the space-time. They proved the existence of singularities by assuming Lorentz signature metrics and causality, the generic conditions on Riemann tensor components, the existence of trapped surfaces and energy conditions on matter.

A space-time manifold is causal geodesically incomplete if any one of the following holds:

- *a proper condition on the curvature i.e, a proper energy condition.*
- *an appropriate condition on the causality.*
- *a suitable initial or boundary condition.*

The initial condition serves the purpose that some causal geodesics start to focus towards each other then the energy condition implies that this focusing goes on till a causal geodesic develops a focal/conjugate point. As a result, the geodesic stops maximizing the Lorentzian distance. The causality condition on the other hand implies the existence of maximizing geodesic atleast in some region of space-time. Thus, to resolve the above contradiction it is expected that the geodesic should terminate before they reach a conjugate/focal point i.e, they are incomplete in nature. From the point of view of the singularity theorems the above curvature condition is necessary for focusing effect on causal geodesics. Mathematically, the central idea is to introduce the Jacobi field, a vector field along the geodesic γ which satisfies the Jacobi equation

$$\ddot{J} + R(J, \dot{\gamma})\dot{\gamma} = 0.$$

One can interpret this Jacobi field as a one-one correspondence with geodesic variation of γ to have a clear picture.

For focusing of geodesics, the notion of conjugate points is essential. Points $\gamma(a)$ and $\gamma(b)$ on the geodesic are called conjugate if there exists a non-trivial Jacobi field which vanishes at a and b. It has been established that a causal geodesic fails to maximize the Lorentzian distance after its first conjugate point. To have an analytic tool for determination of conjugate points, one may note that the relevant information on the conjugate points is contained in the $(n-1)$-dimensional subspace of the Jacobi field vanishing at a given point and taking values in the set $\dot{\gamma(t)}^{\perp} := \{v \in T_{\gamma(t)}(M) :< v, \dot{\gamma(t)} >= 0\}$.

Let us now define a class of $(1,1)$ tensor field matrix $[A] : [\dot{\gamma}]^{\perp} \to [\dot{\gamma}]^{\perp}$ for which the tensor Jacobi equation

$$[\ddot{A}] + [R][A] = 0.$$

Here R can be treated as a tidal force operator i.e, $[R] : [v] \to [R(v, \dot{\gamma})\dot{\gamma}]$. Now the analytic way to detect conjugate point can be obtained through Raychaudhuri equation

$$\dot{\Theta} = -Ric(\dot{\gamma}, \dot{\gamma}) - tr(\Sigma^2) - \frac{\Theta^2}{D}$$

where the expansion scalar Θ is given by $\Theta = tr([\dot{A}][A]^{-1}) = (det[A])^{-1}(det[A])^{\cdot}$ and shear Σ is defined as $\Sigma = \frac{1}{2}([B] + [B^{\dagger}]) - \frac{\Theta}{D} id$ with $B = [\dot{A}][A^{-1}]$. In this RE, the second and third term on the right hand side are negative definite. If the first term is also non positive through the SEC or NEC then one can generate conjugate points. In particular $\Theta(a) < 0$ at some parameter value a then it will diverge to $-\infty$ in finite parameter time. Further from the above RE we have $\ddot{\Theta} \leq -\frac{\Theta^2}{D}$ which on integration from a to some $t > a$ gives

$$\Theta \leq \frac{D}{t - a + \frac{D}{\Theta(a)}}$$

Thus, Θ diverges for some interval $a \leq t < a - \frac{D}{\Theta(a)}$. Thus if $[A]$ is a Jacobi tensor class with $[A](a) = 0$ and $[\dot{A}](a)$=id, the identity mapping and we have $|\Theta(t)| \rightarrow \infty$ for t to some b then $\det[A(b)] = 0$ and hence γ_b is conjugate to γ_a. Hence, the causal geodesic γ becomes inextendible after the first conjugate point. In this way, the notion of incomplete geodesic, existence of conjugate point and divergence of Θ can be associated using the geometrical RE.

1.6.10 Raychaudhuri equation and accelerated Universe

According to the recent observations, Universe is experiencing an accelerated expansion also known as late time acceleration and "Dark Energy" is assumed to be responsible for this observed acceleration of the Universe. However, there are no satisfactory and universally accepted explanation of this phenomena. The three ingredients of modern relativistic cosmology namely The cosmological principle, Weyl's postulate and GR require the description of the Universe by FLRW metric characterized by a scale factor a which is a function of t i.e, $a = a(t)$. This factor determines the spatial expansion or contraction of the Universe. The evolution of the Universe is thus characterized by the time variation of the scale factor $a(t)$. Naturally $\ddot{a}(t)$ will give acceleration or deceleration of the Universe depending on its sign. RE in FLRW model for hyper-surface orthogonal congruence of time-like geodesic is given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{1.125}$$

This is because, in FLRW case $\Theta = 3H = 3\frac{\dot{a}}{a}$, $\sigma_{ab} = 0$ and $R_{ab}v^a v^b = \frac{1}{2}(\rho + 3p)$. Hyper-surface orthogonality of time-like geodesic implies $\omega_{ab} = 0$. Thus matter satisfying $(\rho+3p) > 0$ can not ensure acceleration as $\ddot{a} < 0$. Negative pressure $p < -\frac{\rho}{3}$ can ensure acceleration. This puts forward the existence of an exotic fluid having a negative equation of state parameter (EoS). This fluid is known as dark energy. Thus, the Raychaudhuri equation gives an insight of Dark energy. A brief analysis of Dark energy has already been discussed in the previous section.

CHAPTER 2

RAYCHAUDHURI EQUATION AND GEODESIC CONGRUENCE IN INHOMOGENEOUS AND ISOTROPIC $F(R)$ GRAVITY MODEL

2.1 Prelude

In the previous chapter it has been discussed that the cosmologists have been facing a challenge for more than last two decades to accommodate the observed accelerated expansion in the theoretical framework of gravity theories. One of the two possible ways towards finding a resolution of the accelerated expansion is to choose an alternative theory of gravity (other than Einstein gravity) without introducing any exotic matter i.e dark energy (DE) [181]. This is because, there is no universally accepted explanation of the later and it is just a hypothetical fluid having large negative pressure. A common and natural generalization of Einstein gravity is obtained by replacing the Ricci scalar 'R' in the Einstein-Hilbert action by an arbitrary function $f(R)$ [75, 182, 183, 184, 185]. Also there are other well-known generalizations of Einstein gravity namely a non-minimally coupled scalar field theory [186, 187], $f(T)$ gravity theory [188, 189] etc.

From the derivation and consequences it is quite clear that the Raychaudhuri equation (RE) has an immense contribution in modern cosmology and is a fundamental tool to study exact solutions of Einstein's equations in general relativity [190]. On the other hand, after the detection of gravitational waves, general relativity (GR) [132, 191] is a universally accepted theory of gravity, despite the inherent existence of singularity in it as predicted by the famous singularity theorems of Hawking and Penrose [146, 177, 178]. The RE [154] is the main ingredient behind these singularity theorems. The RE as it stands is purely a geometric identity in Riemannian geometry. However, it becomes a physical equation showing an equivalence between geometry and matter when gravity theory is imposed to determine the Ricci Ten-

sor. In Einstein gravity, if the matter field satisfies the strong energy condition (SEC) then according to the RE, an initially converging congruence of time-like geodesics focuses within finite affine parameter value [176], leading to the formation of a congruence singularity (may or may not be a curvature singularity). The idea of Focusing theorem from RE together with some conditions on space-time geometry leads to the existence of singularity: The Singularity theorems [146, 177] (a brief discussion on Singularity theorems has been done in the previous chapter). In this context, one may say that the SEC or equivalent condition on Ricci tensor is termed as Convergence condition (CC) in Einstein gravity. This CC essentially indicates the attractive nature of gravity theory and leads to geodesic focusing. The thesis studies the time-like convergence condition in different geometric backgrounds. In the present chapter space-time is chosen as inhomogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) model and $f(R)$ gravity theory [75] has been constructed in this space-time. RE has been formulated for this model with the above $f(R)$ gravity theory. Finally, a congruence of time-like geodesics has been studied from the point of view of RE.
This chapter is organized as follows: Section 2.2 shows an analogy of RE with harmonic oscillator with some specific solutions. Section 2.3 shows the existence of Black-hole singularity using the harmonic oscillator approach. In section 2.4 RE and modified CC has been constructed in $f(R)$ gravity with inhomogeneous FLRW space-time. Section 2.5 deals with the study of geodesic congruence in inhomogeneous background from the point of view of RE. Finally the chapter ends with a brief discussion in section 2.6.

2.2 Raychaudhuri equation and Harmonic Oscillator

The RE [192] essentially characterizes the kinematics of flows in a geometrical space. Usually, flows are generated by a vector field and in turn integral curves of the vector field identify the flow. This congruence of integral curves may be geodesic or non-geodesic in nature. In the pseudo-Riemannian space these congruences are either time-like or null in nature. The REs are the evolution equations of the kinematic quantities which characterize the flow. Historically, only the evolution of the expansion scalar is the RE while the evolution of the other kinematic quantities are termed as Codazzi-Raychaudhuri equations.

The RE, the evolution equation for expansion scalar along the flow representing a time-like congruence is given by (1.83)

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} = -\frac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} + \nabla_b A^b - R_{ab}v^a v^b, \tag{2.1}$$

where we have taken $n = 4$ for four dimensional space-time, physical and geometric interpretation of rest of the terms appearing in the RE have also been discussed previously. The RE can be simplified if we assume i)the congruence of time-like geodesics (then the acceleration vector i.e $A^b = 0$) and ii)the congruence is chosen to be hypersurface orthogonal (by Frobenius theorem $w_{ab} = 0$). As a result, the simplified RE takes the form

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} = -\frac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}v^a v^b. \tag{2.2}$$

The mathematical expression for FT is given by (1.121). This implies that an initially converging time-like geodesic congruence develops a caustic (i.e $\Theta \to -\infty$) within finite proper time (choosing the parameter to be proportional to the proper time). The condition $R_{ab}v^a v^b \geqslant 0$ is termed as CC for time-like geodesics. One can consider Θ, the expansion scalar as the rate of change of volume of the transverse subspace of the congruence. So $\Theta \to -\infty$ implies a convergence of the congruence while $\Theta \to +\infty$ indicates a total divergence. The RE, a first order non-linear equation, is of central importance in the context of the *Singularity Theorems* [146, 177]. Further, mathematically the RE is known as a Riccati equation, and it becomes a second order linear equation in the form of a harmonic oscillator equation with varying frequency as [156, 170]

$$\frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} + \frac{1}{3}(R_{ab}v^a v^b + \sigma^2 - \omega^2)X = 0,$$

with

$$\Theta = \frac{3}{X}\frac{\mathrm{d}X}{\mathrm{d}l} = 3\frac{\mathrm{d}(\ln X)}{\mathrm{d}l}. \tag{2.3}$$

Now the above convergence condition can be stated as follows :
i)X is negative initially,
ii)$X = 0$ at a finite value of the parameter to have a negatively infinite expansion.

As Θ may be identified as the derivative of the geometric entropy (S) so one may identify S as $\ln X$. Here X can be chosen as an average or effective geodesic deviation. Using the well-known Sturm Comparison theorem in the theory of differential equations, the criterion for the existence of zeros in X at finite value of the affine parameter is given by

$$R_{ab}v^a v^b + \sigma^2 - \omega^2 \geqslant 0. \tag{2.4}$$

The above inequality for convergence of geodesic congruence shows that shear is in favour of convergence while rotation opposes the convergence. Hence for hyper-surface orthogonal congruence of geodesics the convergence condition is $R_{ab}v^a v^b > 0$ (as given earlier). Now we opt for some solutions of the Harmonic oscillator equation (2.2). Since $\tilde{R} = R_{ab}v^a v^b$ and σ^2 is in favor of convergence we call $R_c = \tilde{R} + \sigma^2$ as Convergence scalar. In four dimensional FLRW background, $R_c = \tilde{R}$ is the Raychaudhuri scalar. As already mentioned earlier that $\tilde{R}$ bears a geometric significance. It can be treated as the mean curvature geometrically. Motivated by this, we assume some physically reasonable forms of $\tilde{R}$ to solve the Harmonic Oscillator equation which turns out to be

$$\frac{d^2 a}{dt^2} + G(a)a = 0 \tag{2.5}$$

as $X = a(t)$, τ is replaced by cosmic time t in FLRW background. We now consider the following cases:

Case-I: $G(a) = l;\ l > 0$ i.e, we consider a positive constant mean curvature. The solution is given by

$$a(t) = A\ \sin(\sqrt{l}t) +\ B\ \cos(\sqrt{l}t). \tag{2.6}$$

The variation of the scale factor $a(t)$ with cosmic time t is shown graphically in FIG-(2.1) considering $l = 1$ and different choices of A and B.

Case-II: $G(a) = -p\ ; p > 0$ i.e, we consider constant negative mean curvature. In this case the solution is given by

$$a(t) = c\ \sinh(\sqrt{p}t) + d\ \cosh(\sqrt{p}t). \tag{2.7}$$

The graph of $a(t)$ vs t has been shown in FIG-(2.2) for $p = 1$ and different choices of the parameters involved.

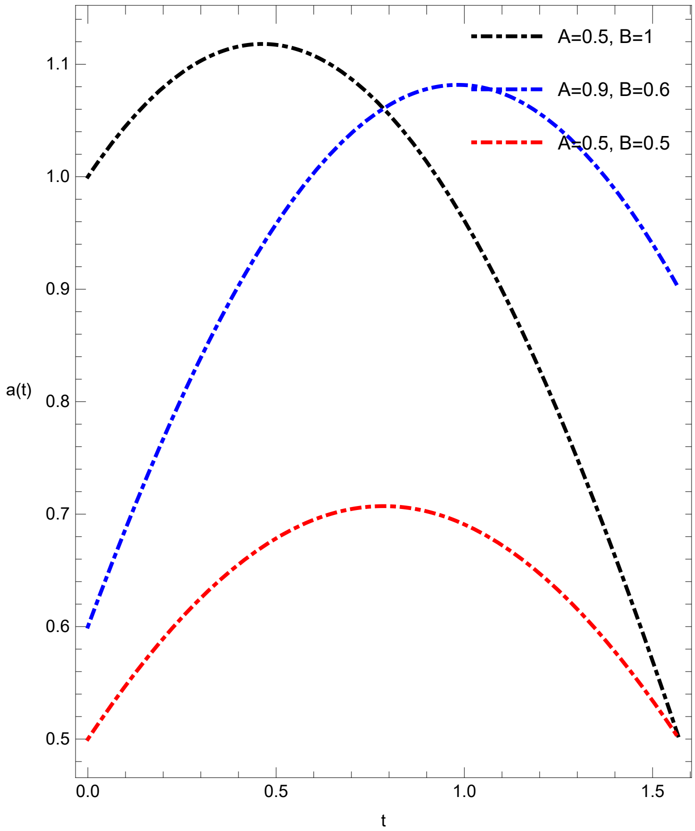


Figure 2.1: $a(t)$ vs t for constant positive mean curvature considering various parameters specified in the panel.

Case-III: $G(a) = G_0 a^n$. In this case the solution assumes the form

$$a(t) = -\frac{G_0 n^2}{2(n+2)} t^{-\frac{2}{n}} \tag{2.8}$$

and graphically it is depicted in FIG- (2.3) for different choices of the parameters involved.

The above analysis shows that

1. In case of positive constant curvature we get bouncing scale factor (there is an epoch with respect to which there are two phases- expanding and contracting phase). In this case although the CC holds yet there is no curvature singularity. This is in agreement with the result of Landau and Lifschitz which reveals that CC or focusing alone can not imply singularity.

2. In case of negative constant mean curvature, the CC is violated. The graph of $a(t)$ shows that there is no singularity. Thus in this case avoidance of geodesic focusing or violation of the CC leads to possible avoidance of singularity.

3. In case of variable mean curvature (positive definite) CC holds. This shows that at a strong curvature singularity, the gravitational tidal forces linked with the singularity

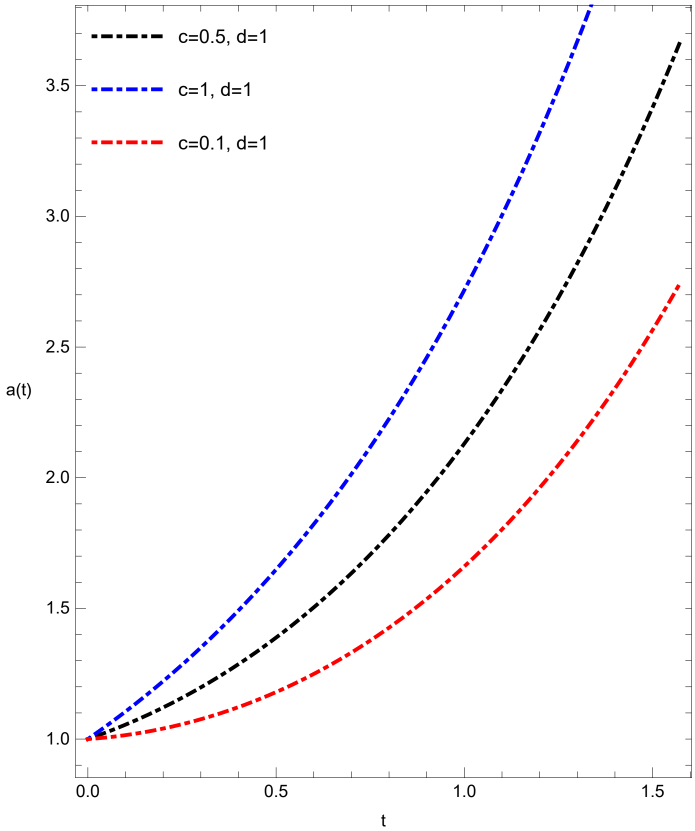


Figure 2.2: $a(t)$ vs t for constant negative mean curvature considering various parameters specified in the panel.

are so strong that any particle trying to cross it must be crushed to zero size which is evident from FIG 2.3.

Thus the above study shows that CC may or may not imply existence of singularity.

2.3 Black-Hole singularity from the Harmonic oscillator approach of RE

In this section, we address the question **Can Raychaudhuri equation identify Black-Hole singularity?** If the answer is "**Yes**" then "**How**?" In order to prove the existence of Black-Hole singularity using the Raychaudhuri equation we consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.9}$$

Schwarzschild metric is a vacuum solution of the Einstein's field equations and in general relativity a vacuum solution is a Lorentzian manifold whose Einstein tensor (and hence the Ricci tensor R_{ab}) vanishes identically and thus $\tilde{R} = 0$. Therefore the Convergence scalar $R_c = 2\sigma^2$. By the dynamics of classical Schwarzschild metric [193] and using the definition

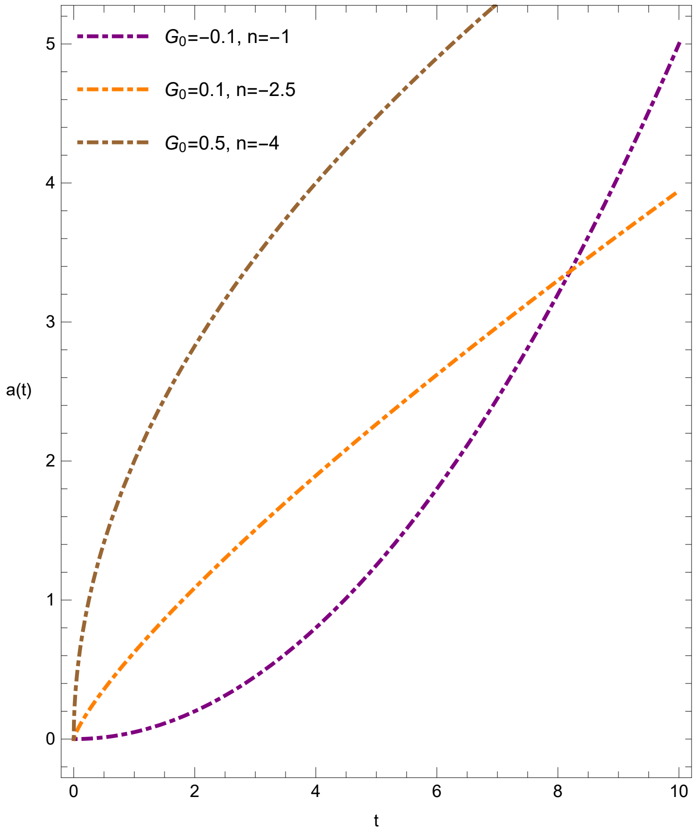


Figure 2.3: $a(t)$ vs t for variable mean curvature considering various parameters specified in the panel.

of anisotropy scalar (σ^2) one has

$$R_c = 2\sigma^2 = \frac{4}{3r^3}\frac{(3GM-r)^2}{(2GM-r)}.$$

Based on the above expression of R_c one has the following findings:

(i) $R_c > 0$ only when $r < 2GM$. So singularity may be possible only when $r < 2GM$. Further as $r \to 0$ a stage will come when R_c will predominate over ω^2 and $\nabla_c A^c$ so that $R_c - 2\omega^2 - \nabla_c A^c \geq 0$ (Convergence Condition) and this hints that convergence and hence formation of singularity may occur.

(ii) For $r > 2GM$, $R_c < 0$. So singularity is not possible in the region $r > 2GM$. Therefore $r = 2GM$ acts a boundary to distinguish the regions with and without singularity. Here $r = 2GM$ is nothing but the Event-Horizon.

(iii) $r = 0$ and $r = 2GM$ are the points where R_c diverges/ blows off. These are therefore identified as the points of singularity. However singularity at $r = 2GM$ is a co-ordinate singularity that arises due to bad choice of coordinates and can be removed in Eddington-Finkelstein co-ordinates. On the other hand $r = 0$ is the physical singularity or the Black-Hole singularity whose existence is inevitable from the point of view of the RE.

Using the Harmonic oscillator approach, we have $\omega_0^2 = R_c$, the frequency of the oscillator. In the expression for R_c if $r = o(\epsilon)$ then as $r \to 0$, ω_0^2 varies largely with $o(\epsilon)^3$ as compared to $r \to 2GM$ where the frequency varies with $o(\epsilon)$. This is in agreement with the prediction of General Relativity that the space-time near a cosmological singularity undergoes an infinite number of oscillations between different Kasner epochs with rapid transitions between them [194],[195].

Therefore, RE tells us that even if there is no matter (i.e vacuum) yet the space-time may develop singularity due to presence of anisotropy and Black-Hole singularity is an example of such a case.

2.4 RE in $f(R)$ gravity with inhomogeneous FLRW background

In $f(R)$ gravity theory the usual Einstein-Hilbert action is generalized as

$$\mathcal{A} = \frac{1}{2\kappa}\int \sqrt{-g}f(R)\mathrm{d}^4x + \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu},\sigma). \tag{2.10}$$

Here $\mathcal{L}_m$ stands for the Lagrangian of the matter field with σ denoting the coupling between geometry ($g_{\mu\nu}$) and matter source, $f(R)$ is an arbitrary continuous function of the Ricci scalar R and $\kappa = 8\pi G = c = 1$ denotes the usual gravitational coupling. Thus the field equations for $f(R)$-gravity, obtained by variation of the above action with respect to $g_{\mu\nu}$, are in the compact form as

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F(R) = \kappa T_{\mu\nu}, \tag{2.11}$$

with $F(R) = \dfrac{\mathrm{d}f(R)}{\mathrm{d}R}$ and

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g^{\mu\nu}}, \tag{2.12}$$

being the stress energy tensor of the matter field. The D'Alembertian Operator $\Box$ can be written as $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. The trace of the field equation (2.11) takes the form

$$3\Box F(R) + RF(R) - 2f(R) = \kappa T. \tag{2.13}$$

Now, combination of the field equation (2.11) with (2.13) gives the modified Einstein field equations (after some algebraic manipulation) as [196]

$$G_{\mu\nu} = \tilde{T}_{\mu\nu} + \frac{1}{F}(\nabla_\mu\nabla_\nu F - g_{\mu\nu}N), \tag{2.14}$$

where

$$\tilde{T}_{\mu\nu} = \frac{1}{F}T_{\mu\nu}, \quad N(t,r) = \frac{1}{4}(RF + \Box F + T). \tag{2.15}$$

Now from the field equations (2.11) and their trace equation (2.13) one gets for a unit time-like vector u^μ,

$$\tilde{R} = R_{\mu\nu}u^\mu u^\nu = \frac{1}{F(R)}\left[-\frac{1}{2}\Box F(R) + \frac{1}{2}(f(R) - RF(R)) + \kappa\left(T_{\mu\nu}u^\mu u^\nu + \frac{1}{2}T\right) + u^\mu u^\nu\nabla_\mu\nabla_\nu F(R)\right], \tag{2.16}$$

So for CC of a congruence of time-like curves having u^μ as the unit tangent vector field, the r.h.s of the above equation must be positive semi-definite. In the background of inhomogeneous FLRW space-time geometry having line-element [288]

$$ds^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\frac{b(r)}{r}} + r^2\mathrm{d}\Omega_2^2\right], \tag{2.17}$$

where $\mathrm{d}\Omega_2^2 = d\theta^2 + \sin^2\theta d\Phi^2$ is the metric on 2-sphere (θ being the polar angle), $a(t)$ is the scale factor, $b(r)$ is an arbitrary function of r, the scalar curvature has the form

$$R = 6(\dot{H} + 2H^2) + 2\frac{b'}{a^2r^2}. \tag{2.18}$$

The field equations for $f(R)$-gravity has the explicit form

$$3H^2 + \frac{b'(r)}{a^2r^2} = \frac{\rho(r,t)}{F(R)} + \frac{\rho_e(r,t)}{F(R)} \tag{2.19}$$

$$-\left(2\dot{H} + 3H^2\right) - \frac{b(r)}{a^2r^3} = \frac{p_r(r,t)}{F(R)} + \frac{p_{re}(r,t)}{F(R)} \tag{2.20}$$

$$-\left(2\dot{H} + 3H^2\right) - \frac{(b - rb')}{2a^2r^3} = \frac{p_t(r,t)}{F(R)} + \frac{p_{te}(r,t)}{F(R)} \tag{2.21}$$

where $H = \dfrac{\dot{a}}{a}$ is the usual Hubble parameter. Here the matter is in the form of cosmic anisotropic fluid with $\rho = \rho(r,t)$, $p_r = p_r(r,t)$, $p_t = p_t(r,t)$ as the energy density, radial and transverse pressures respectively. Also the expression for the hypothetical matter components are

$$\rho_e = N + \ddot{F},\ p_{re} = -N - H\dot{F} + \frac{(r-b)}{a^2r}F' - \frac{(b - rb')}{2a^2r^2}F',\ p_{te} = -N - H\dot{F} + \frac{(r-b)}{a^2r^2}F', \tag{2.22}$$

Now the conservation relations for the anisotropic fluid can be written as :

$$\frac{\partial\rho}{\partial t} + (3\rho + p_r + 2p_t)H = 0, \quad \text{and} \quad \frac{\partial p_r}{\partial r} = \frac{2}{r}(p_t - p_r) \tag{2.23}$$

Although from cosmological principle the universe is on the large scale homogeneous and isotropic and almost all models in cosmology has this property due to the elegance and simplicity of these models, still the universe is not fundamentally homogeneous on the scale of galaxies clusters and superclusters – there is clumping of matter. Further, at the very early phase of the universe, it is very likely to have a state of much disorder (for example in emergent era/inflationary epoch). Moreover, it is possible to have apparent acceleration of the universe due to the back reaction on the metric of the local inhomogeneities [198, 199]. Thus it is reasonable to consider the inhomogeneous model as an alternative to dark energy. For the present $f(R)$ gravity model it is reasonable to choose

$$b(r) = b_0\left(\frac{r}{r_0}\right)^3 + d_0 = \mu_0 r^3 + d_0 \tag{2.24}$$

so that

$$R = 6(\dot{H} + 2H^2) + \frac{6\mu_0}{a^2} \tag{2.25}$$

is a function of 't' alone. This choice of $b(r)$ includes two parameters, namely μ_0 and $d_0(\neq 0)$, where d_0 is identified as the inhomogeneity parameter. The motivation behind choosing a suppressed 3rd order polynomial (not any general degree polynomial or other analytic function) lies in the fact that this particular choice of $b(r)$ with $d_0 = 0$ reduces the present model to FLRW and hence successfully helps us to study an inhomogeneous model as an alternative to dark energy. Moreover with this choice of $b(r)$, the scalar curvature R given by (2.25) turns out to be homogeneous which leads to a lot of simplification in mathematical calculations. The above modified Friedmann equations (2.19)-(2.21) with $b(r)$ from equation (2.24) has a possible solution for the two matter components as

$$\rho = 3H^2 g(t)\ ,\ \rho_e = \frac{3\mu_0 g(t)}{a^2} \quad , \quad p_{re} = p_{te} = -\frac{\mu_0}{a^2} + H\dot{g(t)},$$
$$p_r = \psi(t)\left[-\left(2\dot{H} + 3H^2\right) - \frac{d_0}{a^2r^3}\right] - H\dot{g}(t) \quad , \quad p_t = g(t)\left[-\left(2\dot{H} + 3H^2\right) + \frac{d_0}{2a^2r^3}\right] - H\dot{g}(t) \tag{2.26}$$

The above choice shows that the usual matter component is inhomogeneous and anisotropic in nature while the hypothetical curvature fluid is both homogeneous and isotropic in nature with $\omega_e = \frac{1}{3}\left(1 + \frac{Ha^2\dot{g(t)}}{\mu_0}\right)$ as the expression for state parameter. Further, due to the inhomogeneity (i.e $d_0 \neq 0$) the equation of state parameters for the normal fluids i.e $\omega_r = \frac{p_r}{\rho}$ and $\omega_t = \frac{p_t}{\rho}$ are related linearly as

$$\omega_t - \omega_r = \frac{d_0}{2a^2H^2r^3} \tag{2.27}$$

It may be noted that both the conservation equations given by (2.23) will be satisfied identically for the choice

$$\omega_t = \frac{d_0}{6a^2H^2r^3}\ ,\ \omega_r = -\frac{d_0}{3H^2a^2r^3} \tag{2.28}$$

Moreover, this choice of the state parameters results a differential equation in $\psi(t) = F(R)$ as

$$\frac{\dot{\psi}}{\psi} + 2\frac{\dot{H}}{H} + 3H = 0, \tag{2.29}$$

which has the solution

$$\psi = \frac{\psi_0}{(a^3H^2)} \tag{2.30}$$

This solution shows that $f(R)$ will be in the power-law form of R if the power-law form of expansion of the universe is assumed. Subsequently, in the following sections we have used the power-law form of $f(R)$ particularly in graph plotting to study the CC. Also the power-law form of $f(R)$ over Einstein gravity is suitable for inflationary scenario. However, from the point of view of hypothetical curvature fluid the different components of the fluid (given by equation (2.23)) results a differential equation for ψ as

$$\ddot{\psi} - 2H\dot{\psi} - \frac{2\mu_0}{a^2r^3}\psi = 0, \tag{2.31}$$

having solution of the form (with $\mu_0 = 0$)

$$\psi(t) = \frac{\psi_0}{2}\int \frac{d(a^2)}{H} \tag{2.32}$$

Here also $f(R)$ is of the form $R^{-(n+\frac{1}{2})}$ for power law expansion of the universe. Finally for the choice (2.28) of the given inhomogeneous fluid after a little bit of algebra with the field equations one obtains the RE as

$$\frac{\ddot{a}}{a} = -\frac{1}{2\psi(t)}\left[\frac{\rho}{3} + H\dot{\psi}\right] \tag{2.33}$$

Thus the RE is a homogeneous equation although the spacetime geometry is inhomogeneous in nature. Further, the RE does not depend on the equation of state parameters, it depends only on the energy density of the physical fluid. Lastly, in this section, convergence condition ($R_{\mu\nu}u^\mu u^\nu \geq 0$) for a congruence of time-like curves in the present inhomogeneous space-time has been discussed. If $u^\mu = (1, 0, 0, 0)$ denotes the unit time-like vector field along the congruence then from (2.16) , for CC we must have

$$R_{\mu\nu}u^\mu u^\nu = \frac{1}{F(R)}\left[T_1 + T_2 + T_3\right] \geq 0 \tag{2.34}$$

where

$$T_1 = \kappa\left(T_{\mu\nu}u^\mu u^\nu + \frac{1}{2}T\right) \tag{2.35}$$

$$T_2 = -\frac{3}{2}\Box F(R) = \frac{3}{2}u^\mu u^\nu \nabla_\mu \nabla_\nu F(R) \tag{2.36}$$

$$T_3 = \frac{1}{2}(f(R) - RF(R)) \tag{2.37}$$

Note that if the matter satisfies SEC then $T_1 \geq 0$ for all time.

The time variation of T_1, T_2, T_3 and $R_{\mu\nu}u^\mu u^\nu$ has been shown graphically in FIG. 2.4 considering the cosmic fluid to be perfect fluid with barotropic equation of state $p = \omega\rho$. The figure shows that the CC is not universally satisfied, it depends on the choice of the parameters involved. Thus it is possible to avoid the singularity in the present model.

2.5 Raychaudhuri Equation and Geodesic congruences

The RE for a congruence of time-like geodesics [200] with velocity vector field v^a ($v^a v_a = -1$) is given by equation (2.2).

For the sake of simplicity, the congruence of time-like geodesics are chosen to be hyper-surface orthogonal (which by Frobenius Theorem implies zero rotation). So we have considered a metric, conformal to the original metric (2.17) by carrying out a conformal transformation

$$dT = \frac{dt}{a(t)}, \tag{2.38}$$

so that the transformed metric is given by

$$ds^2 = a^2(T)\left[-dT^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega_2^2\right] \tag{2.39}$$

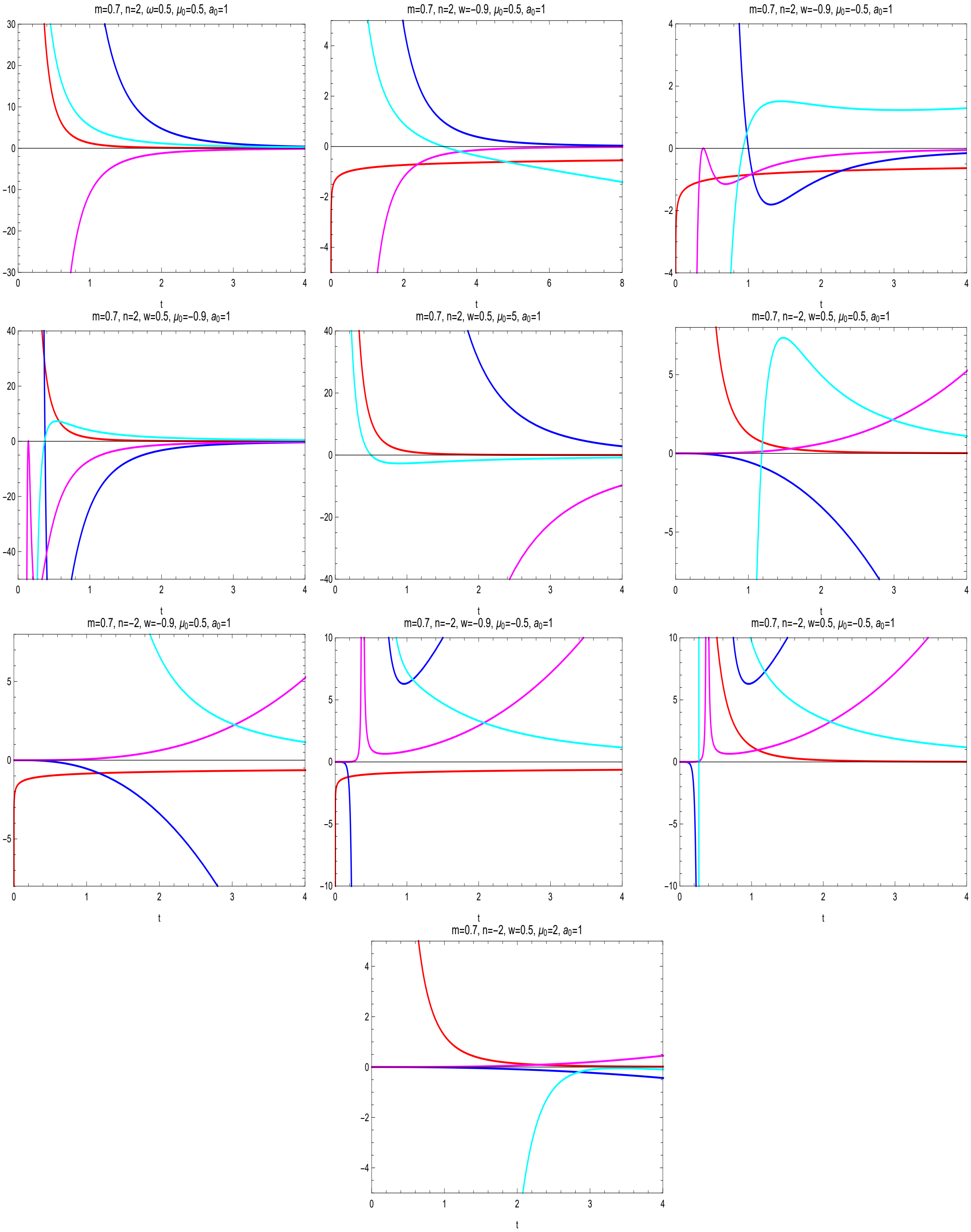


Figure 2.4: Time (t) variation of different terms: T_1 (red), T_2 (blue), T_3 (magenta) and $\tilde{R}= R_{\mu\nu}u^{\mu}u^{\nu}$ (cyan) for various choices of arbitrary parameters. Here we choose $a(t) = a_0 t^m$ and $f(R) = R^n$

For this conformal line element, corresponding to a timelike geodesics (choosing $\theta = \frac{\pi}{2}$ without any loss of generality) the components of the four velocity vector field are given by

$$\dot{t} = -E,\ \dot{\phi} = \frac{h}{r^2},\ \dot{r} = \sqrt{\left(1-\frac{b}{r}\right)\left(E^2-1-\frac{h^2}{r^2}\right)},\ \dot{\theta} = 0 \tag{2.40}$$

where E and h are identified as the conserved energy and angular momentum of the time-like particle (per unit mass). One may recall the definitions of the kinematic variables from the Introduction section (Chapter 1) for congruence of time-like geodesics. Thus the explicit form for the kinematic variables appearing on the r.h.s of the RE are given by

$$\Theta^2 = \frac{\left(1-\frac{b(r)}{r}\right)}{r^2\left(E^2-1-\frac{h^2}{r^2}\right)}\left[2(E^2-1)-\frac{h^2}{r^2}\right]^2, \tag{2.41}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab} = \frac{\left(1-\frac{b(r)}{r}\right)}{3r^2\left(E^2-1-\frac{h^2}{r^2}\right)}\left[(E^2-1)^2+\frac{h^4}{r^4}-\frac{h^2(E^2-1)}{r^2}\right], \tag{2.42}$$

and by our construction

$$\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab} = 0 \tag{2.43}$$

Also, the explicit expression for $R_{ab}v^a v^b$ is as follows :

$$R_{ab}v^a v^b = -\frac{h^2}{r^4} + \frac{b'(r)}{r^2}\left(E^2-1-\frac{h^2}{2r^2}\right) - \frac{b(r)}{r^3}\left(E^2-1-\frac{3h^2}{2r^2}\right) \tag{2.44}$$

where R_{ab} is the Ricci tensor projected along the congruence of geodesics and it has been evaluated from the metric (2.39). Now the radial variation of different kinematic parameters (Θ,σ), Raychaudhuri scalar ($\tilde{R}$) and $\frac{d\Theta}{d\tau}$ has been studied graphically in FIG.2.5 choosing parameters μ_0, h and E which gives realistic cases.

For the choice of $b(r)$ (given by (2.24)), the behavior of the above kinematic variables and $R_{ab}v^a v^b$ as $r \to \infty$ are given by

$$\Theta^2 \xrightarrow{r\to\infty} 4\mu_0\left(1-E^2\right) \tag{2.45}$$

$$\sigma^2 \xrightarrow{r\to\infty} \frac{1}{3}\mu_0\left(1-E^2\right) \tag{2.46}$$

$$R_{ab}v^a v^b \xrightarrow{r\to\infty} 2\mu_0\left(E^2-1\right) \tag{2.47}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} \xrightarrow{r\to\infty} 0 \tag{2.48}$$

The condition in equation (2.48) states that the expansion leads to a constant value at infinity, and hence, the congruence will be either convergent or divergent. From the expression of $\dot{r}$ given in equation (2.40), it can be inferred that for realistic $\dot{r}$ and r, one

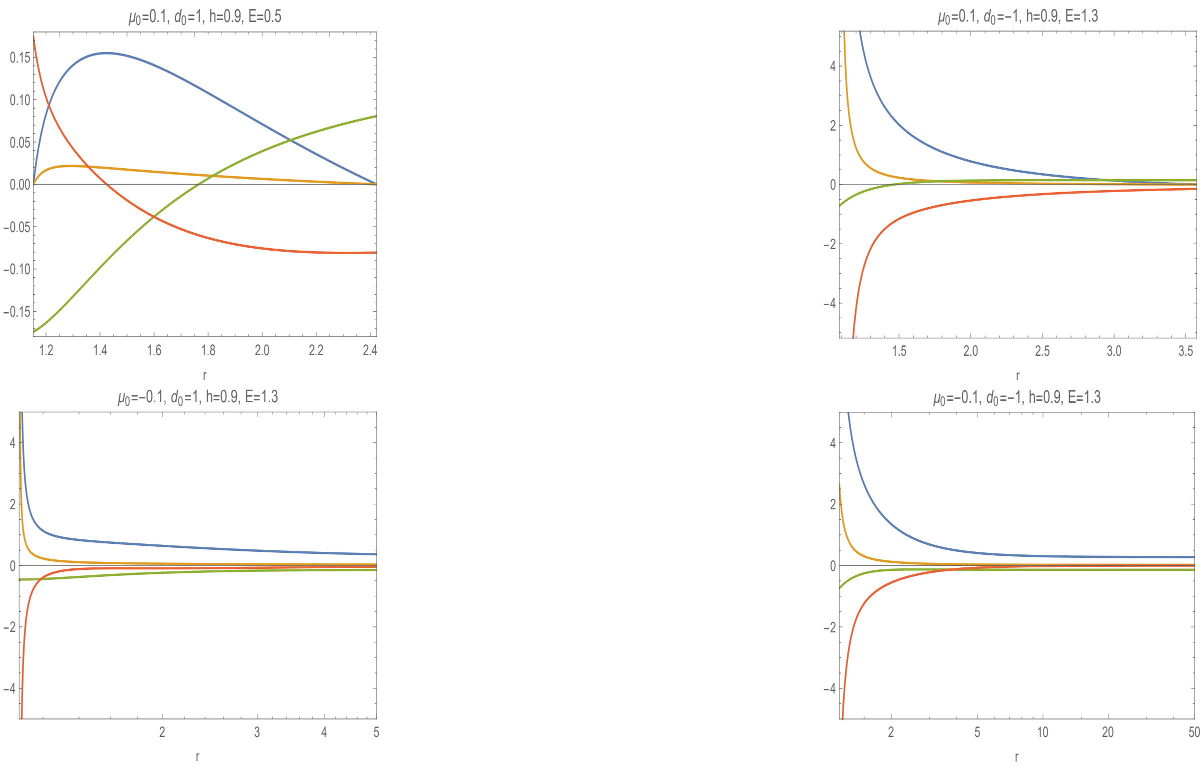


Figure 2.5: Radial (r) variation of different kinematic parameter: Θ^2 (blue), σ^2 (yellow), $R_{ab}u^a u^b$ (green) and $\frac{\mathrm{d}\Theta}{\mathrm{d}\tau}$ (red). From the figure it is clear that $\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} < 0$ in all cases except the first one on the top left which shows that $\frac{d\Theta}{d\tau}$ is positive upto some radial coordinate and then it goes negative. So Θ^2 should be increasing upto that radial coordinate and then it is decreasing as clear from their graphs.

must have $(E^2 - 1) > 0$, since $\left(1 - \frac{b(r)}{r}\right) > 0$. Further it is to be noted that for real expansion scalar Θ and shear σ at infinity μ_0 should be negative i.e it dictates open geometry. On the other hand, if $0 < \mu_0 < \frac{4}{27d_0^2}$ then geodesics are confined within a bounded region.

Now, to study the space-time topology and geodesic motion, let us consider the 2D hyper-surface $H : t =$constant, $\theta = \frac{\pi}{2}$. The geometry is characterized by [201]

$$\mathrm{d}S_H^2 = \frac{\mathrm{d}r^2}{1 - \frac{b(r)}{r}} + r^2 \mathrm{d}\Phi^2. \tag{2.49}$$

One may consider this 2D hyper surface H embedded as rotational surface $z = z(r, \Phi)$ into the Euclidean space with metric

$$\mathrm{d}S_H^2 = \left[1 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right] \mathrm{d}r^2 + r^2 \mathrm{d}\Phi^2, \tag{2.50}$$

in cylindrical co-ordinates (r, ϕ, z). Thus, comparing (2.49) and (2.50), one may obtain the expression for embedding function as

$$z(r) = \int_{r_0}^{r} \sqrt{\frac{\frac{b(r)}{r}}{1 - \frac{b(r)}{r}}} \mathrm{d}r \tag{2.51}$$

where r_0 is a non-zero constant related to μ_0 by the relation $\frac{b_0}{r_0^3} = \mu_0$ (2.24). In comparison to FLRW metric, $\mu_0 = \frac{b_0}{r_0^3}$ is related to the curvature scalar κ which takes values 0, +1 and -1 for flat, closed and open model respectively. The regions in which congruence of time-like geodesics exist for the choice of $b(r)$ (given by (2.24)) with different (feasible) signs of μ_0 and d_0 has been presented in Table 2.1 and the corresponding scenario is depicted graphically in Figures 2.6-2.9.

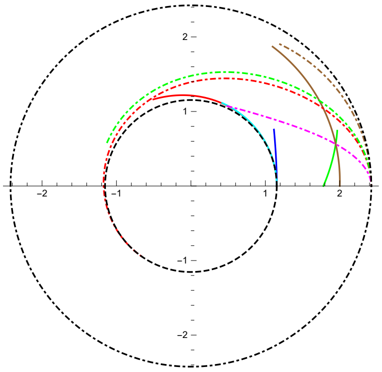
2.6 Brief Discussion

An extensive analysis of the Raychaudhuri equation has been done in the present chapter for $f(R)$ modified gravity theory in the background of inhomogeneous FLRW space-time. By suitable transformation, this first order nonlinear ordinary differential equation can be converted to a second order linear differential equation analogous to the evolution equation for simple harmonic oscillator (with varying frequency) and possible solutions of the Harmonic Oscillator differential equation have been found with physical interpretation. Though the present model is inhomogeneous but still the RE so constructed turns out to be a homogeneous differential equation. For congruence of time-like geodesic , convergence conditions have been analyzed graphically and one

Table 2.1: Regions for geodesics for different signs of μ_0 and d_0 with $b(r_1) = 0$, $r_\star = \dfrac{|h|}{\sqrt{E^2-1}}$ and suffix '+' indicates the positive root of the equation $b(r) = r$.

Case	Choice for μ_0 & d_0		Region for $\left(1-\frac{b(r)}{r}\right) > 0$	Region in which geodesics exists	Region in which geodesics are embedded
IA	$\mu_0 > 0,\ d_0 > 0$	$\mu_0 < \frac{4}{27d_0^2}$	$r_{+1} < r < r_{+2}$	$\max\{r_{+1}, r_\star\} < r < r_{+2}$	$r_{+1} < r < r_{+2}$
IB		$\mu_0 > \frac{4}{27d_0^2}$	not possible	not possible	not possible
II	$\mu_0 > 0,\ d_0 < 0$		$r < r_+$	$r_\star < r < r_+$	$r_1 < r < r_+$
III	$\mu_0 < 0,\ d_0 > 0$		$r > r_+$	$r > \max\{r_+, r_\star\}$	$r_+ < r < r_1$
IV	$\mu_0 < 0,\ d_0 < 0$		$r > 0$	$r > r_\star$	no embedded region

* In case IV i.e $\mu_0 < 0, d_0 < 0$, the integrand becomes imaginary so there is no real expression for the embedding function $z(r)$ given by (2.51). Hence embedding is not feasible.



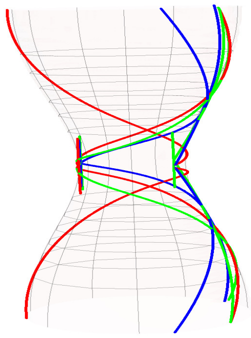


Figure 2.6: Polar plot for the congruence of geodesics for the choice of $b(r) = \mu_0 r^3 + d_0$ with $\mu_0 = 0.1$ and $d_0 = 1$ (left) for different initial values, namely $r_0 = 1.15348$ (Red ,Blue, Cyan) ; $r_0 = 1.5$ (Green , Magenta); $r_0 = 2$ (Brown, Green(dot-dashed)); $r_0 = 2.42361$ (Red,magenta, Brown dot-dashed). Also the solid/dot-dashed graphs are drawn considering $\frac{\mathrm{d}r}{\mathrm{d}\phi} > 0/ < 0$. In all cases ϕ has been chosen within ranges which are a proper subset of $(0, 2\pi)$. For the 3D embedding diagram (right) vertical axis is z and the horizontal plane is $r - \phi$.

may conclude that singularity may be avoided for specific choices for the parameters involved. Finally, congruence of time-like geodesics are studied for the present model , choosing the corresponding conformal metric for simplicity. Different kinematic parameters (involved in the RE) are evaluated and their asymptotic behaviours are examined at infinity. It is found that both bounded and unbounded geodesics are possible for different signs of the parameters involved in the metric function $b(r)$. Also, all possible types of geodesics are shown graphically.

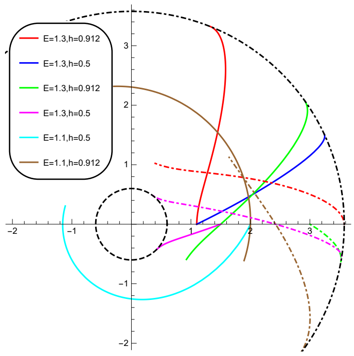


Figure 2.7: Polar plot for the congruence of geodesics for the choice of $b(r) = \mu_0 r^3 + d_0$ with $\mu_0 = 0.1$ and $d_0 = -1$ for different initial values,namely $r_0 = 1.098$ (Red,Blue); $r_0 = 1.5$ (Green,Magenta); $r_0 = 2$ (Cyan,Brown); $r_0 = 3.577$ (Red dot-dashed); $r_0 = 3$ (Green dot-dashed); $r_0 = 2.42361$ (Magenta and Brown dot-dashed) $\phi \in (0, 2\pi)$. Solid/ dot-dashed graphs are drawn considering $\frac{\mathrm{d}r}{\mathrm{d}\phi} > 0/ < 0$. In all cases ϕ has been chosen within ranges which are a proper subset of $(0, 2\pi)$.

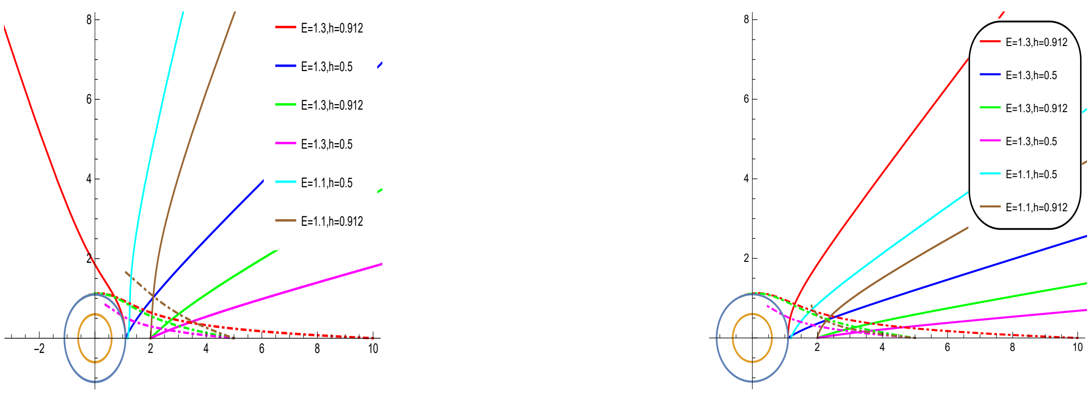


Figure 2.8: Polar plot for the congruence of geodesics for the choice of $b(r) = \mu_0 r^3 + d_0$ with (i) $\mu_0 = -0.1$ and $d_0 = 1$ (left) with initial values namely, $r_0 = 1.098$ (Red, Blue); $r_0 = 2$ (Green, Magenta, Brown); $r_0 = 1.18$ (Cyan); $r_0 = 10$ (Red dot-dashed); $r_0 = 5$ (Green,Magenta,Brown dot-dashed). (ii) $\mu_0 = -0.1$ and $d_0 = 1.5$ (right) for the initial values same as (i)(left). In all cases ϕ has been chosen within ranges which are a proper subset of $(0, 2\pi)$. Solid/ dot-dashed graphs are drawn considering $\frac{\mathrm{d}r}{\mathrm{d}\phi} > 0/ < 0$. From the figure it can be seen that the rate of divergence (convergence) increases with decrease of h and increase of E, $|\mu_0|$ and $|d_0|$.

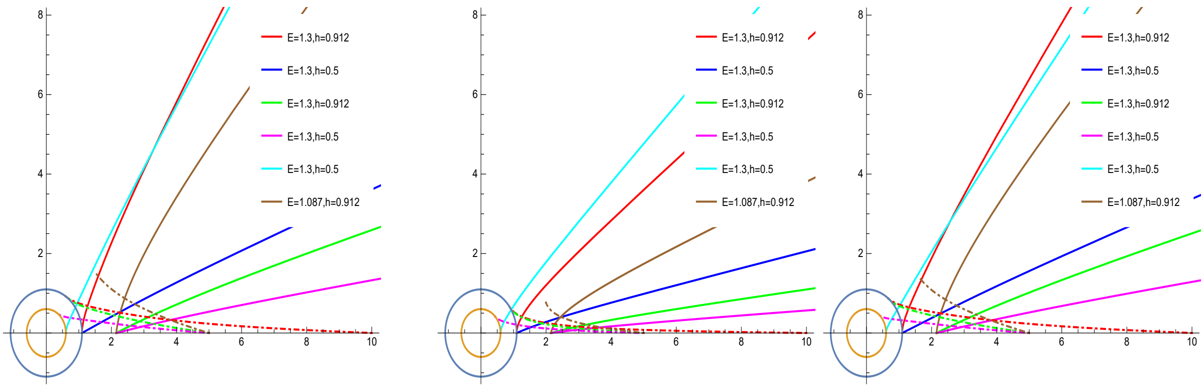


Figure 2.9: Polar plot for the congruence of geodesics for the choice of $b(r) = \mu_0 r^3 + d_0$ with (i) $\mu_0 = -0.1$ and $d_0 = -1$ (left) with initial values namely, $r_0 = 1.098$ (Red, Blue); $r_0 = 2.15$ (Green ,Magenta,Brown); $r_0 = 0.602$ (Cyan); $r_0 = 10$ (Red dot-dashed); $r_0 = 5$ (Green,Magenta,Brown dot-dashed) (ii) $\mu_0 = -1.1$ and $d_0 = -1$ (center) with the same initial values as (i)(left) and (iii) $\mu_0 = -0.1$ and $d_0 = -1.5$ (right) for the same set of initial values as in (i) and (ii). In all cases ϕ has been chosen within ranges which are a proper subset of $(0, 2\pi)$. Solid /dot-dashed graphs are drawn considering $\frac{\mathrm{d}r}{\mathrm{d}\phi} > 0/ < 0$. From the figure it can be seen that the rate of divergence (convergence) increases with decrease of h and increase of E, $|\mu_0|$ and $|d_0|$.

CHAPTER 3

QUANTUM ASPECTS OF RAYCHAUDHURI EQUATION FROM LAGRANGIAN AND HAMILTONIAN FORMULATION

3.1 Prelude

Now we are well aware of the fact that the appearance of singularity in Einstein's general theory of relativity is inevitable classically but may be avoided in modified gravity models under suitable physical assumptions. So the natural question that arises "How can this singularity in general relativity be avoided or it is inevitable?". Raychaudhuri in the early 1950's tried to address this issue by formulating an evolution equation for expansion scalar—the Raychaudhuri equation (RE) [154], [192]. Subsequently, Hawking and Penrose formulated the seminal singularity theorems [146],[177],[178] in General Relativity (GR) using this RE as the main ingredient through the notion of geodesic focusing. At the singularity, there is no structure of space-time and physical laws break down. It is generally speculated that quantum effects which become dominant in strong gravity regime may alleviate the singularity problem at the classical level. In particular a quantum version of the Raychaudhuri equation may probably be useful in the context of identifying the existence of a singularity in the quantum regime. Though there is no universally accepted theory of quantum gravity [202], there are at present two major approaches for formulating a quantum theory of gravity– canonical quantization [203] and path integral formulation [204]. In canonical quantization , the operator version of the Hamiltonian constraint (known as Wheeler-Dewitt (WD) equation) is a second order hyperbolic functional differential equation and its solution is known as the wave function of the universe [205]. However, even in simple minisuperspace models it is hard to find a solution of the WD equation. Also there is an

ambiguity in operator ordering and how to know the initial conditions of the universe to have a well defined wave function. However an important feature of the Hamiltonian in the operator version is that it admits a self adjoint extension in a general sense. As a result, the conservation of probability is ensured. On the other hand, the path integral formulation is more favourable due to some definite proposals for the sum over histories (namely by Hawking [204], Hartle [206] and by Vilenkin [207]).
RE may be considered as a key ingredient for the classical singularity theorems. So a general speculation about the behavior of the singularity at the quantum domain may be revealed by examining the RE in quantum settings. In this context it is worthy to mention that quantum version of the RE by Das [208] does not allow focusing of geodesics as long as there is non zero quantum potential in Bohmian trajectories. A general view about the avoidance of singularity is the existence of repulsive terms due to quantum effects. For better clarity one can go through the works [208]-[209]

The quantization scheme of RE has some issues. As RE is essentially an identity in the Riemannian geometry so naturally RE cannot be obtained from a variational principle as the equation of motion for a geodesic congruence. However when we express the curvature scalar in terms of Einstein's field equations (or modified field equations) then RE is no longer an identity, rather there is some meaning as Lagrangian formulation.
In the present chapter, the RE has been transformed to a second order differential equation in a general space-time and it is possible to have a first integral of this second order differential equation. This first integral gives a possible solution of the RE. A general form of the Lagrangian and Hamiltonian has been formulated so that the RE can be derived from them in any dimension. Finally, a canonical quantization scheme and formulation of Bohmian trajectories have been presented.
The layout of this chapter is: An analytic solution of RE has been presented in section 3.2. Section 3.3 deals with the Lagrangian and Hamiltonian formulation of RE. A canonical quantization scheme has been presented in section 3.4. Section 3.5 deals with the formulation of Bohmian trajectories. Finally the chapter ends with summary of the obtained results.

3.2 Solution of Raychaudhuri equation

Let us consider a congruence of time-like geodesics in an $(n+1)$-dimensional spacetime $\mathcal{M}$. Suppose Σ be an n-dimensional hyper-surface (space-like) such that the congruence of time-like geodesics are orthogonal to this hyper-surface Σ i.e the unit velocity vector u^μ to the congruence is orthogonal to the hyper-surface. Let q_{ab} be the induced metric on the n dimensional hyper-surface Σ. If the congruence of geodesics is treated as dynamical system [210] then it is convenient to consider

$$x(\tau) = \sqrt{q} \tag{3.1}$$

as the dynamical degree of freedom (τ is the proper time and q=det(q_{ab})). Essentially x is related to the volume of the hyper-surface and $x = 0$ identifies the singularity.

Using the definition of the expansion scalar Θ of the congruence as

$$\frac{d}{d\tau}\ln\sqrt{q} = \Theta = \nabla_\mu u^\mu \tag{3.2}$$

the evolution of x takes the form,

$$\frac{dx}{d\tau} = x\Theta \tag{3.3}$$

Due to orthogonality of each member of the congruence to the hyper-surface, the rotation tensor $\omega_{ab} = \frac{1}{2}(\nabla_b u_a - \nabla_a u_b) = 0$ as a consequence of Frobenius theorem of differential geometry. So, the Raychaudhuri equation simplifies to

$$\frac{d\Theta}{d\tau} = -\frac{\Theta^2}{n} - 2\sigma^2 - \tilde{R} \tag{3.4}$$

where $2\sigma^2 = \sigma_{ab}\sigma^{ab}$, $\sigma_{ab} = \frac{1}{2}(\nabla_a u_b + \nabla_b u_a) - \frac{1}{n}\Theta q_{ab}$ is the shear tensor and $\tilde{R} = R_{ab}u^a u^b$ is an effective scalar curvature, also known as the Raychaudhuri scalar. Using the evolution equation (3.3) for the dynamical degree one may write the above Raychaudhuri equation as a second order nonlinear differential equation as

$$x\frac{d^2x}{d\tau^2} + \left(\frac{dx}{d\tau}\right)^2\left(\frac{1}{n} - 1\right) + (2\sigma^2 + \tilde{R})x^2 = 0 \tag{3.5}$$

which has a first integral

$$\left(\frac{dx}{d\tau}\right)^2 = z_0 x^{2(1-\frac{1}{n})} - 2x^{2(1-\frac{1}{n})}\int x^{(\frac{2}{n}-1)}(2\sigma^2 + \tilde{R})dx \tag{3.6}$$

with z_0, a constant of integration. So using (3.3), we have the solution of the Raychaudhuri equation as

$$\Theta^2 = z_0 x^{-\frac{2}{n}} - 2x^{-\frac{2}{n}}\int x^{(\frac{2}{n}-1)}(2\sigma^2 + \tilde{R})dx \tag{3.7}$$

One may note that unlike the field equations, the Raychaudhuri equation has nothing to do with any gravity theory. It is purely a geometric identity, but the role of gravity comes into picture through the Ricci tensor (R_{ab}) projected along the geodesics. Therefore, one may find the explicit expression for Θ for a particular gravity theory in a four dimensional spacetime.

3.3 Lagrangian and Hamiltonian Formulation of Raychaudhuri equation

In the previous section, it is found that the Raychaudhuri equation can be expressed as a second order differential equation. So, it is a natural search for a Lagrangian corresponding to which the Euler-Lagrange equation gives (3.5). According to Helmholtz

[211],[212],[213],[214],[215],[216], for a system of 'r' second order differential equations of the form

$$\mu_\alpha(\tau, y_\delta, \dot{y}_\delta, \ddot{y}_\delta) = 0, \quad \alpha, \delta = 1, 2, ..., r \tag{3.8}$$

('.' indicates differentiation w.r.t proper time τ), the necessary and sufficient conditions for being the Euler-Lagrange equations corresponding to a Lagrangian $L(\tau, y_\delta, \dot{y}_\delta)$, termed as Helmholtz conditions [211]-[216] are given by

$$\frac{\partial \mu_\alpha}{\partial \ddot{y}_\delta} = \frac{\partial \mu_\delta}{\partial \ddot{y}_\alpha} \tag{3.9}$$

$$\frac{\partial \mu_\alpha}{\partial y_\delta} - \frac{\partial \mu_\delta}{\partial y_\alpha} = \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial \mu_\alpha}{\partial \dot{y}_\delta} - \frac{\partial \mu_\delta}{\partial \dot{y}_\alpha}\right) \tag{3.10}$$

and

$$\frac{\partial \mu_\alpha}{\partial \dot{y}_\delta} + \frac{\partial \mu_\delta}{\partial \dot{y}_\alpha} = 2\frac{d}{d\tau}\left(\frac{\partial \mu_\delta}{\partial \ddot{y}_\alpha}\right) \tag{3.11}$$

with $(\alpha, \delta) = 1, 2, ..., r$. In the present context, we have a single second order differential equation (3.5) so the above conditions reduce to

$$\frac{d\mu}{d\dot{x}} = \frac{d}{d\tau}\left(\frac{d\mu}{d\ddot{x}}\right) \tag{3.12}$$

with,

$$\mu(\tau, x, \dot{x}, \ddot{x}) = x\ddot{x} + \left(\frac{1}{n} - 1\right)\dot{x}^2 + (2\sigma^2 + \tilde{R})x^2 \tag{3.13}$$

A simple algebraic calculation shows that equation (3.12) is satisfied for μ given in equation (3.13) only for $n = \frac{2}{3}$, which is not possible as n is the dimension of the hyper-surface. (If one chooses, $\mu = \frac{\ddot{x}}{x} + \left(\frac{1}{n} - 1\right)\frac{\dot{x}^2}{x^2} + (2\sigma^2 + \tilde{R})$ then (3.12) implies $n = 2$). Thus for general 'n', (3.12) will be satisfied for $\tilde{\mu}$, provided $\tilde{\mu} = x^\alpha \mu$ with $\alpha = \left(\frac{2}{n} - 3\right)$. Therefore, one has

$$\begin{aligned}\tilde{\mu} &= x^{2\left(\frac{1}{n}-1\right)}\ddot{x} + \left(\frac{1}{n} - 1\right)x^{\left(\frac{2}{n}-3\right)}\dot{x}^2 + (2\sigma^2 + \tilde{R})x^{\left(\frac{2}{n}-1\right)} \\ &= \frac{d}{d\tau}\left[x^{2\left(\frac{1}{n}-1\right)}\dot{x}\right] - \left(\frac{1}{n} - 1\right)x^{\left(\frac{2}{n}-3\right)}\dot{x}^2 + h(x)x^{\left(\frac{2}{n}-1\right)}\end{aligned} \tag{3.14}$$

provided $(2\sigma^2 + \tilde{R})$ is a function of x alone, say $h(x)$. Thus one may construct the Lagrangian as

$$\mathcal{L} = \frac{1}{2}x^{2\left(\frac{1}{n}-1\right)}\dot{x}^2 - V[x] \tag{3.15}$$

with,

$$\frac{\delta V[x]}{\delta x} = x^{\left(\frac{2}{n}-1\right)}h(x) \tag{3.16}$$

Now the momentum conjugate to the variable 'x' is

$$\Pi_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = x^{2(\frac{1}{n}-1)}\dot{x} \tag{3.17}$$

At this point one may check that the Euler Lagrange equation corresponding to the Lagrangian in eq. (3.15) gives back the RE in eq. (3.5). So, the Hamiltonian of the system is given by

$$\mathcal{H} = \frac{1}{2}x^{-2(\frac{1}{n}-1)}\Pi_x^2 + V[x] \tag{3.18}$$

One may note that one of the Hamilton's equation of motion gives the Raychaudhuri equation (3.5) while the other one yields the definition of momentum (3.17). From the above formulation of $\tilde{\mu}$, it is clear that $\tilde{\mu}$ satisfies all the Helmholtz conditions provided $2\sigma^2 + \tilde{R}$ is a sole function of x. Now we recall the RE in eq. (3.4) and denote the scalar $2\sigma^2 + \tilde{R}$ by R_c. Therefore eq.(3.4) can be written as

$$\frac{d\Theta}{d\tau} + \frac{\Theta^2}{n} = -R_c \tag{3.19}$$

If $R_c > 0$ then $\frac{d\Theta}{d\tau} + \frac{\Theta^2}{n} < 0$. Integrating this inequality w.r.t τ we get,

$$\frac{1}{\Theta(\tau)} > \frac{1}{\Theta_0} + \frac{\tau}{n}. \tag{3.20}$$

This shows that an initially converging hyper-surface orthogonal congruence of time-like geodesics must continue to converge within a finite value of the proper time $\tau < n\Theta_0^{-1}$ thereby forming caustics. This leads to crossing/focusing of geodesics. Although the singularity here is the congruence singularity and may not be a space-time singularity. In most of the situations these caustics do not lead to any spacetime singularities, but under certain circumstances they do, leading to formation of black hole or cosmological singularities. Removal of these curvature singularities has remained a puzzle for decades. This vital consequence of the RE ultimately played a key role in the proof of the seminal singularity theorems furnished by Hawking and Penrose. The condition $R_c > 0$ is called the Convergence Condition (CC) and hence we name the scalar R_c as the Convergence scalar. In this context $R_c = h(x)$. L.H.S of eq. (3.16) is the gradient of the potential corresponding to the dynamical system representing the congruence and has to be constructed using gravitational field equations. Further one finds that $R_c > (< 0)$ implies force is attractive (repulsive) in nature. This hints that convergence will occur (i.e. $R_c > 0$) if the matter is attractive. This is the reason why RE is regarded as the fundamental equation of gravitational attraction.
In quantum correction of the RE, some extra terms are added with $(-R_c)$ in the R.H.S of the classical RE so that they act as repulsive force to prevent focusing. Appearance of singularity implies convergence/focusing, hence if one can prevent focusing using this quantum correction of RE avoidance of singularity might be guaranteed. (for ref see [208]-[209]).

3.4 Wheeler-Dewitt quantization: The canonical approach

To formulate the operator version of the Hamiltonian from the classical one as obtained in the previous section we carry out the canonical quantization of the system under consideration where x and Π_x are promoted to operators so that $[\hat{x}, \hat{\Pi}_x] = i\hbar$. These operators act on the geometric flow state $\Psi[x,\tau]$. In x- representation we have $\hat{x} = x$, $\hat{\Pi}_x = -i\hbar\frac{\partial}{\partial x}$. Thus the Hamiltonian in terms of operators is given by $\tilde{H} = -\frac{\hbar^2}{2}x^{2(1-\frac{1}{n})}\frac{\partial^2}{\partial x^2} + V[x]$. The operator version of the Hamiltonian is given by $\tilde{H} = -\frac{\hbar^2}{2}x^{2(1-\frac{1}{n})}\frac{\partial^2}{\partial x^2} + V[x]$. Consequently the evolution equation of the physical state Ψ can be described by the Schrodinger equation $\tilde{H}\Psi = i\hbar\frac{\partial}{\partial\lambda}\Psi$. One may consider this as the quantized version of the evolution of a time-like geodesic congruence having classical analogue as the Raychaudhuri equation but this is applicable to only a limited class of geometries. However in the context of cosmology, there is notion of Hamiltonian constraint and operator version of it acting on the wave function of the universe Ψ i.e, $\tilde{H}\Psi = 0$, known as the Wheeler Dewitt (WD) equation. This is because $\tilde{H}$ generates infinitesimal gauge transformations and since physical states should be invariant under gauge transformations, they should be invariant under the action of the group member associated to $\tilde{H}$ (its exponential). Alternatively it is because classical $H = 0$ on the constraint surface and hence quantum mechanically $\tilde{H}\Psi = 0$. In the quantization process due to Dirac, the physical states or physical quantum states of the associated Hilbert Space must be annihilated by the operator version of $\mathcal{H}$. In this case the WD equation takes the form

$$\frac{d^2\Psi}{dx^2} - \frac{2}{\hbar^2}x^{2(\frac{1}{n}-1)}V(x)\Psi(x) = 0 \tag{3.21}$$

($\because$ for spatially homogeneous and isotropic cosmological models $V[x] = V(x)$ and due to operator ordering Π_x^2 is replaced by $[-\frac{d^2}{dx^2} - \frac{p}{x}\frac{d}{dx}]$, with $p = 0$ as the ordering factor). However for spatially anisotropic models there is a problem of non unitary evolution which can be resolved by the proper choice of operator ordering in the first term of the Hamiltonian. The operator form $\tilde{H} = -\frac{\hbar^2}{2}x^{2(1-\frac{1}{n})}\frac{\partial^2}{\partial x^2} + V[x]$ is symmetric with norm $|\Psi|^2 = \int_0^\infty dx\; x^{2(\frac{1}{n}-1)}\Psi^*\Psi$ but fails to be self adjoint . However a self adjoint extension is guaranteed by Friedrichs [217]. For example if we consider the operator ordering as $\tilde{H} = -\frac{\hbar^2}{2}x^{(1-\frac{1}{n})}\frac{\partial}{\partial x}x^{(1-\frac{1}{n})}\frac{\partial}{\partial x} + V[x]$ and a change in minisuperspace variable as $u = nx^{\frac{1}{n}}$ then the WD becomes

$$\left[\frac{-\hbar^2}{2}\frac{d^2}{du^2} + V[u]\right]\Psi(u) = 0, \tag{3.22}$$

with symmetric norm as

$$|\Psi|^2 = \int_0^\infty du\ \Psi^* \Psi, \tag{3.23}$$

provided the integral exists and finite. Hence the Hamiltonian admits a self adjoint extension resolving the non unitary evolution of the geodesic congruence. Therefore for the sake of singularity analysis in the quantum regime in case of homogeneous and isotropic/ anisotropic cosmological models (a larger class of models) we have invoked the WDW equation and inclined to find its solution in those models. The takeaway from the above formulation is listed below:

1. The WD equation (3.22) can be interpreted as time-independent Schrödinger equation of a point particle of unit mass moving along u direction in a potential field $V[u]$ and it has zero eigen value of the Hamiltonian and the wave function of the universe is identified as the energy eigen function.

2. The key ingredient to solve the WD equation (3.21) is $V(x)$, the classical potential. In case of any modified gravity theory constructed in the homogeneous background, one can obtain the expression for the classical potential $V(x)$. Further, using this classical potential if one can solve the WD equation to find the wave function of the universe and hence its norm $|\Psi|^2$ (probability measure on the minisuperspace), then this is an important tool for the singularity analysis in the quantum regime.

3. If $|\Psi|^2 = 0$ at zero volume ($x^3 = 0$) then singularity is avoided in the sense that probability of having zero volume (singularity) is zero, otherwise the singularity still persists in the quantum description. Therefore an immediate application of this canonical quantization for the present quantum system lies in the singularity analysis of spatially homogeneous and isotropic cosmological models at quantum level.

4. Thus the existence (or non existence) of singularity is not a generic one, it depends on the gravity theory under consideration.

Remark: Here $|\Psi|^2$ is proportional to the probability measure on the minisuperspace (hence we can normalize it by proper scaling and can apply in the models where the norm is finite). Further, the solution of the Wheeler-Dewitt (WD) equation may be interpreted as the propagation amplitude of the congruence of geodesics. Norm of this solution (wave function) can be interpreted as the probability distribution of the system. If the wave packet so constructed by this solution is peaked along the classical solution at the early era then the singularity may be avoided so that the geodesics will never converge. This means, if $|\Psi|^2 \to 0$ as volume of the minisuperspace $\to 0$ then it implies that the quantum description may avoid the initial big-bang singularity (in the sense that probability of the universe (or the minisuperspace model in quantum cosmology) to have zero volume (singularity) is zero). There lies the motivation behind using $|\Psi|^2$ as a quantity proportional to the probability density in the present context. Moreover Wheeler-DeWitt formalism is expected to find application in the investigation of the singularities in the quantum regime for a collapse of homogeneous systems, such as the Datt-Oppenheimer-Snyder collapse.

WKB Approximation
The WKB approximation is a process of transition from quantum solutions to the classical regime. Here the wave function is written as

$$\Psi = \exp\left(\frac{i}{\hbar}S\right) \tag{3.24}$$

with power series expansion in $\hbar$ for S i.e,

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + ... \quad , \tag{3.25}$$

One recovers the classical solution by constructing a wave packet from S_0 as,

$$\Psi = \int A(\mathbf{k}) \ \exp\left(\frac{i}{\hbar}\ S_0\right) \ d\mathbf{k} \tag{3.26}$$

with $\mathbf{k}$, a parameter. Now substituting (3.24) (with S from (3.25)) into the WD equation (3.21) and equating power of $\hbar$, one gets

$$\left(\frac{dS_0}{dx}\right)^2 = 2x^{2(\frac{1}{n}-1)}V(x) \tag{3.27}$$

Thus from (3.27),

$$S_0 = \int \sqrt{2}\ x^{(\frac{1}{n}-1)}\sqrt{V(x)}dx + k_0 \tag{3.28}$$

where k_0 is a constant of integration. So a wave can be constructed as

$$\Psi(x) = \int A(\mathbf{k}) \ \exp\left[\frac{i}{\hbar}S_0(\mathbf{k},x)\right]d\mathbf{k} \tag{3.29}$$

with $A(\mathbf{k})$, a sharply peaked Gaussian function. Now a constructive interference occurs if $\dfrac{\partial S_0(x)}{\partial \mathbf{k}} = 0$ which implies a relation between $\mathbf{k}$ and x i.e. $\mathbf{k} = \mathbf{k}(x)$. So the wave function can now be written as

$$\Psi(x) = \int A(\mathbf{k}(x)) \ \exp\left[\frac{i\sqrt{2}}{\hbar}\int x^{(1-\frac{1}{n})}\sqrt{V(x)}dx + k_0\right]\frac{d\mathbf{k}}{dx}dx \tag{3.30}$$

3.5 Bohmian trajectories: Causal interpretation

Here, we adopt an alternative interpretation of quantum mechanics to cosmology. In this ontological interpretation of quantum mechanics [218], the quantum effects are carried out by a quantum potential and it is applicable to the minisuperspaces upon

replacing the Schrödinger equation by the Wheeler- DeWitt (WD) equation. The quantum trajectories (known as Bohmian trajectories) are the time evolution of the metric and field variables, obeying the quantum Hamilton-Jacobi equation. These Bohmian trajectories are purely classical for large values of the scale factor and quantum effects become dominant for small value of the scale factor. Some typical superposition of the wave functions may resolve the initial singularity but in any case these trajectories will not grow to the size of our universe.
In the metric formulation of Einstein gravity there are four constraints : Super momentum constraints or vector constraints and Hamiltonian constraint or scalar constraint. Due to cosmological principle as the space-time is homogeneous and isotropic so vector constraints are identically satisfied and the quantum version of the scalar constraint/ Hamiltonian constraint equation is nothing but the WD equation i.e,

$$\mathcal{H}\ [\tilde{q_\alpha}(t)\ ,\ \tilde{p^\alpha}(t)]\ \Psi(q_\alpha) = 0 \tag{3.31}$$

where, $p^\alpha(t)$ and $q_\alpha(t)$ represent the homogeneous degree of freedom obtained from the three metric q_{ij} and the conjugate momenta Π^{ij}. Now, similar to WKB ansatz one may write,

$$\Psi(q_\alpha) = R\ (q_\alpha)\ \exp\left[\frac{i}{\hbar}\ S(q_\alpha)\right] \tag{3.32}$$

Now using (3.32) in the WD equation (3.31) one gets the Hamilton-Jacobi (H-J) equation

$$\frac{1}{2}h_{\alpha\beta}(q_\rho)\frac{\partial S}{\partial q_\alpha}\frac{\partial S}{\partial q_\beta} + U(q_\rho) + W(q_\rho) = 0 \tag{3.33}$$

where,

$$W(q_\rho) = -\frac{1}{R}\ h_{\alpha\beta}\ \frac{\partial^2 R}{\partial q_\alpha \partial q_\beta} \tag{3.34}$$

is termed as the quantum potential, $h_{\alpha\beta}$ denotes the reduction of the supermetric to the given minisuperspace [219] and $U(q_\rho)$ is the particularization of the scalar curvature density $(-q^{\frac{1}{2}}R)$ of the space-like hyper-surfaces. It may be noted that, due to causal interpretation, the trajectories $q_\alpha(t)$ in quantum cosmology must be real and observer independent and the H-J equation will classify them as follows.
The momentum corresponding to q_α can be obtained from the above H-J equation (3.33) as

$$p^\alpha = \frac{\partial S}{\partial q_\alpha}. \tag{3.35}$$

Now comparing the above momentum with the usual momentum-velocity relation (i.e $p^\alpha = f^{\alpha\beta}\ \frac{\partial q_\beta}{\partial t}$) one obtains the quantum trajectories as

$$p^\alpha = \frac{\partial S}{\partial q_\alpha} = f^{\alpha\beta}\frac{\partial q_\beta}{\partial t} \tag{3.36}$$

These first order differential equations are also known as Bohmian trajectories and are invariant under time reparametrization [219]. Hence there will be no problem of time

for causal interpretation of minisuperspace quantum cosmology.
Now to obtain the Bohmian trajectories for the present quantum system the ansatz for the wave function in eq. (3.32) reduces to

$$\Psi(x) = R(x)\ \exp\left(\frac{i}{\hbar}\ S(x)\right). \tag{3.37}$$

Using this ansatz into the WD- equation (3.21) one gets the Hamilton-Jacobi equation as

$$\frac{-1}{2\ x^{2(\frac{1}{n}-1)}}\ \left(\frac{dS}{dx}\right)^2 + W(x) + V(x) = 0, \tag{3.38}$$

where $W(x)$, the quantum potential has the expression as

$$W(x) = \frac{1}{2R(x)x^{2(\frac{1}{n}-1)}}\frac{d^2R(x)}{dx^2}. \tag{3.39}$$

Thus the Hamilton-Jacobi function S is given by

$$S = s_0 \pm \int \left(\frac{1}{R(x)}\frac{d^2R(x)}{dx^2} + 2x^{2(\frac{1}{n}-1)}\right)^{\frac{1}{2}}\ dx. \tag{3.40}$$

s_0 is the constant of integration.
It may be noted that the trajectories $x(t)$ due to causal interpretation should be real, independent of any observation and are classified by the above H-J equation. In fact, the quantum trajectories i.e the Bohmian trajectories are first order differential equations characterized by the equivalence of the usual definition of momentum with that from the Hamilton-Jacobi function S as

$$\frac{dS(x)}{dx} = -2x^{2(\frac{1}{n}-1)}x' \tag{3.41}$$

i.e

$$2x^{2(\frac{1}{n}-1)}x' = \mp\left(\frac{1}{R(x)}\frac{d^2R(x)}{dx^2} + 2x^{2(\frac{1}{n}-1)}\right)^{\frac{1}{2}} \tag{3.42}$$

or,

$$2\int \frac{x^{2(\frac{1}{n}-1)}\ dx}{\left(\frac{1}{R(x)}\frac{d^2R(x)}{dx^2} + 2x^{2(\frac{1}{n}-1)}\right)^{\frac{1}{2}}} = \mp(t-t_0). \tag{3.43}$$

Now we construct the trajectories first without quantum potential and then with quantum potential as follows:
Case-I
$R(x) = R_0$, a constant then one has,

$$S = s_0 \pm \sqrt{2}\int x^{(\frac{1}{n}-1)}\ dx, \tag{3.44}$$

or

$$S = s_0 \pm \sqrt{2}\ n\ x^{\frac{1}{n}} \tag{3.45}$$

and the quantum trajectory is described as,

$$\sqrt{2}\; n\; x^{\frac{1}{n}} = \pm(t-t_0). \tag{3.46}$$

Here the quantum potential is zero and the H-J equation (3.38) coincides with the classical one. Thus Bohmian trajectory corresponds to classical power law form of expansion and it can't avoid the initial big-bang singularity.
Case-II
Now we study the nature of the trajectories for non-zero quantum potential. Therefore, one may choose $R(x) = x^N$, $N \neq 0, 1$ (for this choice $W(x) \neq 0$) and substituting this in equation (3.43) one gets the quantum trajectories as,

$$x^{\frac{2}{n}} = \frac{(t-t_0)^2}{2n^2} - \frac{N(N-1)}{2}. \tag{3.47}$$

Hence for $N \in (0,1)$ volume is non-zero as $t \to t_0$. Therefore initial big-bang singularity is avoided with non-zero quantum potential and proper fractional power law choice of $R(x)$.
Case-III $R(x) = R_0 \exp(-\alpha x),\ \alpha \neq 0$. The quantum trajectory is described by

$$\frac{6}{\alpha x^{\frac{1}{3}}} 2^{F_1}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{2}{\alpha^2 x^{\frac{4}{3}}}\right) = (t-t_0). \tag{3.48}$$

where 2^{F1} is the Gauss- Hypergeometric function. Clearly the quantum trajectory is a one parameter family of curves which never pass through classical singularity. The trajectories for the last case is shown graphically in FIG-(3.1).

One may note that formulation of Bohmian trajectories is independent of any gravity theory and the key role in avoiding singularity is played by quantum potential. This is unlike the Wheeler-DeWitt quantization scheme which involves the classical potential corresponding to the congruence. This classical potential has to be constructed using the field equations via the Raychaudhuri scalar ($\tilde{R}$) and employing it in the WD equation one can proceed to singularity analysis.

3.6 Conclusion

This chapter deals with RE both from classical and quantum aspects. The issue of quantization of the RE is a bit tricky if it is treated as a geometric identity in space-time. However in relation to some gravity theory the curvature scalar can be expressed in terms of the energy momentum tensor (and/ or effective energy momentum tensor, in modified gravity theory). In that case Lagrangian formulation due to the Helmholtz conditions has some definite meaning. Further it is generally speculated that x (defined in eq.(3.1)) is a function of proper time τ (for time-like geodesic congruence) or cosmic time t (in cosmology) and so its derivatives. Therefore it is reasonable to consider x and its derivative as functionally related. As a consequence the functional $V[x]$ becomes simply a function $V(x)$.

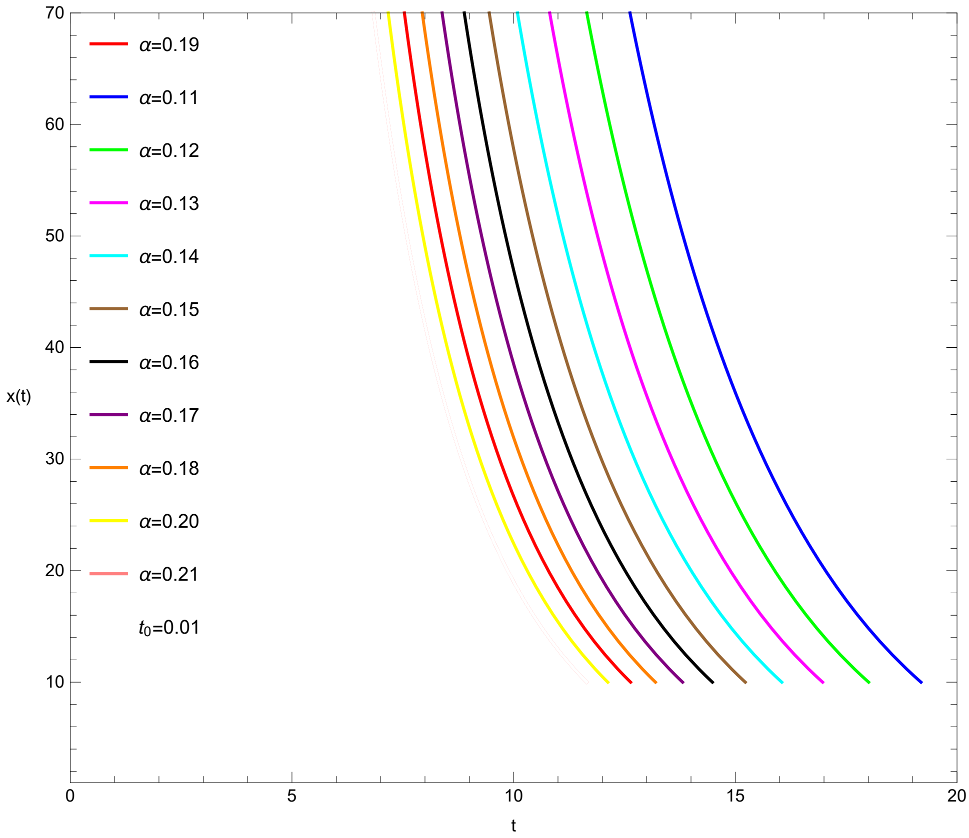


Figure 3.1: Bohmian trajectories for non zero quantum potential.

Moreover there is another issue related to quantization of RE, applicable to cosmological models, particularly for spatially an-isotropic space-time. In canonical quantization scheme the evolution will be non-unitary in nature. However this may be resolved (i.e. evolution may be made unitary) by suitable choice of the operator ordering. However for homogeneous models it is always possible to have a self adjoint extension using the result of Friedrichs [217]. Hence it is always expected to have an unitary evolution of geodesic congruence due to self-adjoint extension of the Hamiltonian.
In order to have a Lagrangian and Hamiltonian formulation, the RE has been transformed into a second order nonlinear differential equation by a suitable transformation related to the metric of the hyper-surface. A first integral of this transformed second order differential equation may be considered as a solution to the RE. As the RE can be written as a second order differential equation so it has been shown to have a general formulation of Lagrangian and Hamiltonian for it.

A canonical quantization scheme with construction of WD equation and WKB approximation has been carried out. Finally, Bohmian trajectories have been derived by constructing the Hamilton-Jacobi equation with quantum potential and applied to the present quantum system. Here the quantum Bohmian trajectories unlike classical geodesics are able to obviate the initial big-bang singularity in the presence of non-zero quantum potential and with proper power-law choice of the pre factor present in the ansatz for the wave function.

So this chapter aims at showing two possible pathways of avoiding singularity quantum mechanically: (i) By Canonical quantization method where the wave function (most specifically norm of the wave function) is determined by solving the WD equation for spatially homogeneous and isotropic cosmological models for the sake of singularity analysis and, (ii) Bohiman formulation, where suitable choice of the wave function of the universe helps to avoid the classical singularity in the presence of quantum potential.

CHAPTER 4

RAYCHAUDHURI EQUATION AND SINGULARITY ANALYSIS IN $F(T)$ GRAVITY

4.1 Prelude

In Chapter 2, we studied the RE in $f(R)$ gravity. However, studies of $f(R)$ theories are obstructed by the complication in the fourth order field equations in the metric framework [220]. Although Palatini formalism [221], [222] for $f(R)$ theory leads to second order field equations, yet there is difficulty to get both exact and numerical solutions which can be compared with observations in many cases. Recently, another gravity theory known as teleparallel gravity [189] has attracted the interest of the literature. Einstein proposed such a gravity model with an aim to unify gravity and electromagnetism over Weitzenböck non-Riemannian manifold. Hence the Levi-civita connection is replaced by Weitzenböck connection in Riemann-cartan space-time. Though teleparallel gravity and GR differ conceptually, yet both of them have equivalent dynamics at classical level [223]. Analogous to $f(R)$-gravity theory, a generalization to teleparallel gravity [189] has been done by replacing T, the torsion scalar by a generic function $f(T)$ leading to $f(T)$-gravity theory [188], [189], [224], [225], [226], [227] and Linder coined the name [228]. In these theories, torsion [229] (instead of curvature) is the driving force for the late time accelerated expansion, and the field equations are of second order.

This motivates us to formulate the modified Raychaudhuri equation and the corresponding convergence condition in some $f(T)$ gravity models. In the present chapter, $f(T)$ gravity theory in the background of homogeneous and isotropic FLRW space-time has been studied. RE has been formulated in this model and the corresponding CC has been analyzed graphically for mainly two popular choices of $f(T)$ in literature

[230], [231] as a tool to avoid singularity in the respective models. Further, a quantum description of the RE as discussed in Chapter 3 has been dealt in the present model as another probable approach of avoiding singularity where the solution of the Wheeler-Dewitt (WD) equation may be interpreted as the propagation amplitude of the congruence of geodesics. Norm of this solution (wave function) can be interpreted as the probability distribution of the system. If the wave packet so constructed by this solution is peaked along the classical solution at the early era then the singularity may be avoided so that the geodesics will never converge. It is generally speculated that quantum effects which become dominant in the strong gravity regime may alleviate the singularity problem at classical level. Hence in quantum description, canonical quantization and formulation of Bohmian trajectories are used to analyse the classical singularity at quantum level. This chapter aims to show two possible pathways (classical and quantum) of avoiding singularity in some $f(T)$ gravity models.
The layout of the chapter is: Section 4.2 deals with the formulation of RE in $f(T)$ gravity. Modified CC in $f(T)$ gravity has been discussed in Section 4.3. A quantum description of RE is presented in Section 4.4. and possible solutions of the Wheeler-DeWitt equation is found.

4.2 Raychaudhuri equation in $f(T)$ gravity

Let us begin with the action for $f(T)$ gravity [232]

$$\mathcal{A}_m = \frac{1}{2}\int \mathrm{d}^4x\ e\left[T + f(T) + \mathcal{L}_m\right] \tag{4.1}$$

where T is the torsion scalar , $f(T)$ is an arbitrary differentiable function of the torsion T, $e = \sqrt{-g} = det(e_\mu^{\mathcal{A}_m})$, and $\mathcal{L}_m$ corresponds to the matter Lagrangian, $\kappa = 8\pi G = 1$. The above torsion scalar T is defined as

$$T = S_\sigma^{\mu\nu}\ T_{\mu\nu}^\sigma \tag{4.2}$$

where $T_{\mu\nu}^\sigma$, the torsion tensor is defined as

$$T_{\mu\nu}^\sigma = \Gamma_{\nu\mu}^\sigma - \Gamma_{\mu\nu}^\sigma = e_A^\sigma\ (\partial_\mu e_\nu^A - \partial_\nu e_\mu^A) \tag{4.3}$$

The Weitzenbock connection $\Gamma_{\mu\nu}^\sigma$ is defined as

$$\Gamma_{\mu\nu}^\sigma = e_A^\sigma\ \partial_\nu e_\mu^A, \tag{4.4}$$

and the super-potential, $S_\sigma^{\mu\nu}$ is defined as

$$S_\sigma^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_\sigma + \delta_\sigma^\mu\ T^{\alpha\nu}{}_\alpha - \delta_\sigma^\nu\ T^{\alpha\mu}{}_\alpha\right), \tag{4.5}$$

where the contortion tensor takes the form

$$K^{\mu\nu}{}_\sigma = -\frac{1}{2}\left(T^{\mu\nu}{}_\sigma - T^{\nu\mu}{}_\sigma - T_\sigma^{\mu\nu}\right). \tag{4.6}$$

Geometrically, the orthogonal tetrad components $e_{\mathcal{A}_m}(x^\mu)$ (considered as dynamical variables), form an orthonormal basis for the tangent space at each point x^μ of the manifold i.e

$$e_i{\cdot}e_j = \eta_{ij} \quad , \quad \eta_{ij} = diag\ (+1, -1, -1, -1) \tag{4.7}$$

In a coordinate basis, we may write

$$e_i = e_i^\mu\ \partial_\mu \tag{4.8}$$

where e_i^μ are the components of e_i , with $\mu,\ i = 0, 1, 2, 3$. The metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu}(x) = \eta_{ij}\ e_\mu^i(x)\ e_\nu^j(x) \tag{4.9}$$

The present work deals with $f(T)$ gravity in the framework of homogeneous and isotropic FLRW space-time having line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right] \tag{4.10}$$

where $a(t)$ is the scale factor, $H = \frac{\dot{a}}{a}$ is the Hubble parameter, '.' denotes differentiation w.r.t cosmic time t.
'k', the curvature index dictates the model of the universe. To be precise,

$$k = \begin{cases} -1, & \text{open model} \\ 0, & \text{flat model} \\ +1, & \text{closed model} \end{cases} .$$

Further it has been assumed that the universe is filled with perfect fluid having barotropic equation of state

$$p = \omega\rho \tag{4.11}$$

where $\omega = \gamma - 1$ ($0 \leq \gamma \leq 2$) being the equation of state parameter.
Now using equations (4.2), (4.3), (4.5), (4.6) and (4.7) we have ,

$$T = -6H^2 \tag{4.12}$$

It is to be noted that during cosmic evolution T is negative. Varying the action (4.1) we get the modified Einstein field equations as

$$\left[e^{-1}\partial_\mu\left(e\ e_A^\rho\ S_\rho^{\mu\nu}\right) - e_A^\lambda\ T_{\mu\lambda}^\rho\ S_\rho^{\nu\mu}\right][1+f_T] + e_A^\rho\ S_\rho^{\mu\nu}\ \partial_\mu(T) f_{TT} + \frac{1}{4}\ e_A^\nu\left[T + f(T)\right] = 4\pi G\ e_A^\rho\ T_\rho^\nu \tag{4.13}$$

where $f_T = \frac{df}{dT}$, $f_{TT} = \frac{d^2f}{dT^2}$, and T_ρ^ν is the energy momentum tensor of the total matter – baryonic matter and dark energy. Thus for FLRW model, the modified Friedmann equations can be written as :

$$H^2 = \frac{1}{2f_T + 1}\left[\frac{\rho}{3} - \frac{f}{6}\right] \tag{4.14}$$

$$2\dot{H} = \frac{-(p+\rho)}{1 + f_T + 2Tf_{TT}} \tag{4.15}$$

where p and ρ are the thermodynamic pressure and density of the matter fluid having conservation equation

$$\dot{\rho} + 3H(p+\rho) = 0. \tag{4.16}$$

Based on our assumption (4.11) , the solution of the differential equation (4.16) can be written in the form

$$\rho = \rho_0 \; a^{-3\gamma}. \tag{4.17}$$

Finally, using the expression for H and equations (4.14), (4.15) after some algebraic manipulation one gets the Raychaudhuri equation in $f(T)$ gravity as

$$\frac{\ddot{a}}{a} = \rho_0 a^{-3\gamma}\left[\frac{1}{3(2f_T+1)} - \frac{\gamma}{2(1+f_T+2Tf_{TT})}\right] - \frac{f(T)}{6(2f_T+1)}. \tag{4.18}$$

Therefore, the Raychaudhuri equation is homogeneous and depends on γ which is related to the equation of state parameter ω by the relation $\gamma = \omega + 1$ and choice of the torsion function $f(T)$. This hints that the CC essentially depends on the function $f(T)$.

Table showing the RE for various choices of f(T)		
Choice of f(T)	Gravity Theory	Raychaudhuri Equation
0	Einstein Gravity	$\frac{\ddot{a}}{a} = \rho_0 a^{-3\gamma}\left(\frac{1}{3} - \frac{\gamma}{2}\right)$
c, a non-zero constant	Einstein gravity with cosmological constant	$\frac{\ddot{a}}{a} = \rho_0 a^{-3\gamma}\left(\frac{1}{3} - \frac{\gamma}{2}\right) - \frac{c}{6}$
$f(T) = f_0 T$, (f_0, a non zero constant)	Einstein gravity with reconstruction of gravitational constant	$\frac{\ddot{a}}{a} = \rho_0 a^{-3\gamma}\left[\frac{1}{3(2f_0+1)} - \frac{\gamma}{2(1+f_0)}\right] - \frac{f_0 T}{6(2f_0+1)}$

Table 4.1: Raychaudhuri equation for various forms of the generic function $f(T)$

4.3 Convergence Condition in $f(T)$ gravity

The field equations for $f(T)$ gravity given by equations (4.14) and (4.15) may be expressed as :

$$3H^2 = (\rho + \rho^{(e)}) \tag{4.19}$$

$$2\dot{H} = -(p+\rho) - (p^{(e)} + \rho^{(e)}) \tag{4.20}$$

where

$$\rho^{(e)} = -\left(\frac{f(T)}{2} + 6H^2 f_T\right) \tag{4.21}$$

$$p^{(e)} = 2\left(\dot{H} + 3H^2\right) f_T + 4Tf_{TT}\dot{H} + \frac{f(T)}{2} \tag{4.22}$$

are the energy density and thermodynamic pressure of the effective fluid. The Raychaudhuri scalar $\tilde{R}$=$R_{\mu\nu}u^\mu u^\nu$ in this modified gravity turns out to be ,

$$R_{\mu\nu}u^\mu u^\nu = \left(J_{\mu\nu}u^\mu u^\nu + \frac{1}{2}J\right) + \left(J^{(e)}_{\mu\nu}u^\mu u^\nu + \frac{1}{2}J^{(e)}\right), \tag{4.23}$$

J being the trace of $J_{\mu\nu}$ i.e $J = g^{\mu\nu}J_{\mu\nu}$. Energy momentum tensor for perfect fluid having unit time-like vector u^μ (so that $u^\mu u_\mu = -1$) is given by

$$J_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu} \tag{4.24}$$

Thus, the expression for the Raychaudhuri scalar ($\tilde{R}$) is

$$\tilde{R} = R_{\mu\nu}u^\mu u^\nu = \frac{1}{2}(\rho + 3p) + \frac{1}{2}\left(\rho^{(e)} + 3p^{(e)}\right) \tag{4.25}$$

Using the equations (4.14), (4.15), (4.17), (4.21) and (4.22), the explicit expression for $\tilde{R}$ in terms of $f(T)$, f_T and f_{TT} can be written as

$$\tilde{R} = R_{\mu\nu}u^\mu u^\nu = \frac{3\gamma\rho_0 a^{-3\gamma}}{2(1 + f_T + 2Tf_{TT})} + \frac{\left(-\rho_0 a^{-3\gamma} + \frac{f(T)}{2}\right)}{(1 + 2f_T)}. \tag{4.26}$$

Therefore, the expression for the Raychaudhuri scalar ($\tilde{R}$) shows that the CC essentially depends on the choice of $f(T)$. Therefore we shall analyze the CC for the following choices of $f(T)$, assuming power-law expansion of the universe, i.e $a = t^m$. The argument behind the power law choice of the scale factor is that if we choose $f(T)$ as a power law form as given in model 1 or as a linear combination of T and T^{-1} in model 2, then solving the field equations (4.14) and (4.15) one may get $a(t)$ as a power law form only for the choices $n = 1$ and $\frac{1}{2}$. Further it should be mentioned that analytic solution for $a(t)$ is possible only for the above choices of n. Even if there are a plethora of models in $f(T)$–gravity cosmology, the following models are taken into consideration for simplicity in mathematical calculations and they favor accelerating universe as per recent observation (see for ref. [230], [231]).

Model 1 : $f(T) = \alpha(-T)^n$, α and n are two model parameters [230]. This model has the same background evolution equation as some phenomenological models [233], [234]. Further, the model reduces to ΛCDM model at $n = 0$ and to the DGP model [95] at $n = \frac{1}{2}$. Thus, for this model (setting $\alpha = 1$ and varying n) one has

$$\tilde{R} = 1.5\gamma\rho_0 t^{-3m\gamma}\left[1 - n(6m^2)^{(n-1)}t^{-2(n-1)} + 2n(n-1)(-1)^n(6m^2)^{(n-2)}t^{-2(n-2)}\right]^{-1} + \left[-\rho_0 t^{-3m\gamma} + \frac{(6m^2)^n}{2t^{2n}}\right]\left[1 - \frac{2n(6m^2)^{(n-1)}}{t^{2(n-1)}}\right]^{-1} \tag{4.27}$$

Model 2 : $f(T) = \alpha T + \frac{\beta}{T}$, α,β are the model parameters [231]. The expression for Raychaudhuri scalar in this model is given by

$$\tilde{R} = 1.5\gamma\rho_0 t^{-3m\gamma}\left[1+\alpha+\frac{\beta t^4}{12m^4}\right]^{-1} + \left[-\rho_0 t^{-3m\gamma} - \frac{3\alpha m^2}{t^2} - \frac{\beta t^2}{12m^2}\right]\left[1+2\left(\alpha-\frac{\beta t^4}{36m^4}\right)\right]^{-1} \tag{4.28}$$

Now, $\tilde{R}$ has been split into two terms R_1 and R_2 for both the cases and time variation of R_1,R_2 and $\tilde{R}$ has been shown graphically in FIG. 4.1 and FIG. 4.2 to study the Convergence Condition (CC) in the two models.

Based on the graphs one has the following findings:

- For the model $f(T) = \alpha(-T)^n$, the CC i.e $\tilde{R} \geq 0$ holds for negative m and n while singularity may be avoided for positive exponents. It may be noted that in all cases SEC ($\rho + 3p = (3\gamma - 2)\rho_0 t^{-3m\gamma} \geq 0$) holds good. Therefore unlike Einstein gravity, it is possible to obviate singularity in Model 1.

- For the model $f(T) = \alpha T + \frac{\beta}{T}$, $\tilde{R}$ is either positive or indefinite in sign whenever SEC holds good, while negative ρ_0 or violation of SEC ($(3\gamma - 2)\rho_0 t^{-3m\gamma} < 0$) yields $\tilde{R} < 0$. Hence singularity may be avoided for any negative choice of ρ_0 irrespective of the exponent m.

  Remark:
  From the above two choices of $f(T)$ we see that avoidance of singularity is very much related to the choice of the generic function $f(T)$ as well as the nature of the physical fluid under consideration. From the above study we find that in the first case with the positive power law choice of $f(T)$ it is possible to avoid singularity even with the normal/usual fluid as the matter content of the universe, however for the second choice of $f(T)$ (a linear combination of linear power law and its inverse) it is found that there must be some ghost field that may possibly lead to avoidance of singularity. Therefore the choice of $f(T)$ has a crucial role in identifying the singularity free nature of the space-time models considered.

4.4 Quantum description of Raychaudhuri equation

We start by considering a family of hyper-surface orthogonal congruence of time-like geodesics in a $(r+1)$ dimensional spacetime-$\mathcal{M}$. Let $\eta_{\mu\nu}$ (orthogonal to time-like unit velocity vector field u^μ of the congruence) be the induced metric on the r- dimensional hyper-surface. One can define the dynamical degree of freedom as [210]

$$\Lambda(\tau) = \sqrt{\eta} \tag{4.29}$$

where η =det($\eta_{\mu\nu}$) and τ is the proper time. Λ is essentially related to the volume of the hyper-surface and $\Lambda = 0$ identifies the singularity. The dynamical evolution of η is given by [176]

$$\frac{1}{\sqrt{\eta}}\frac{d\sqrt{\eta}}{d\tau} = \Theta \tag{4.30}$$

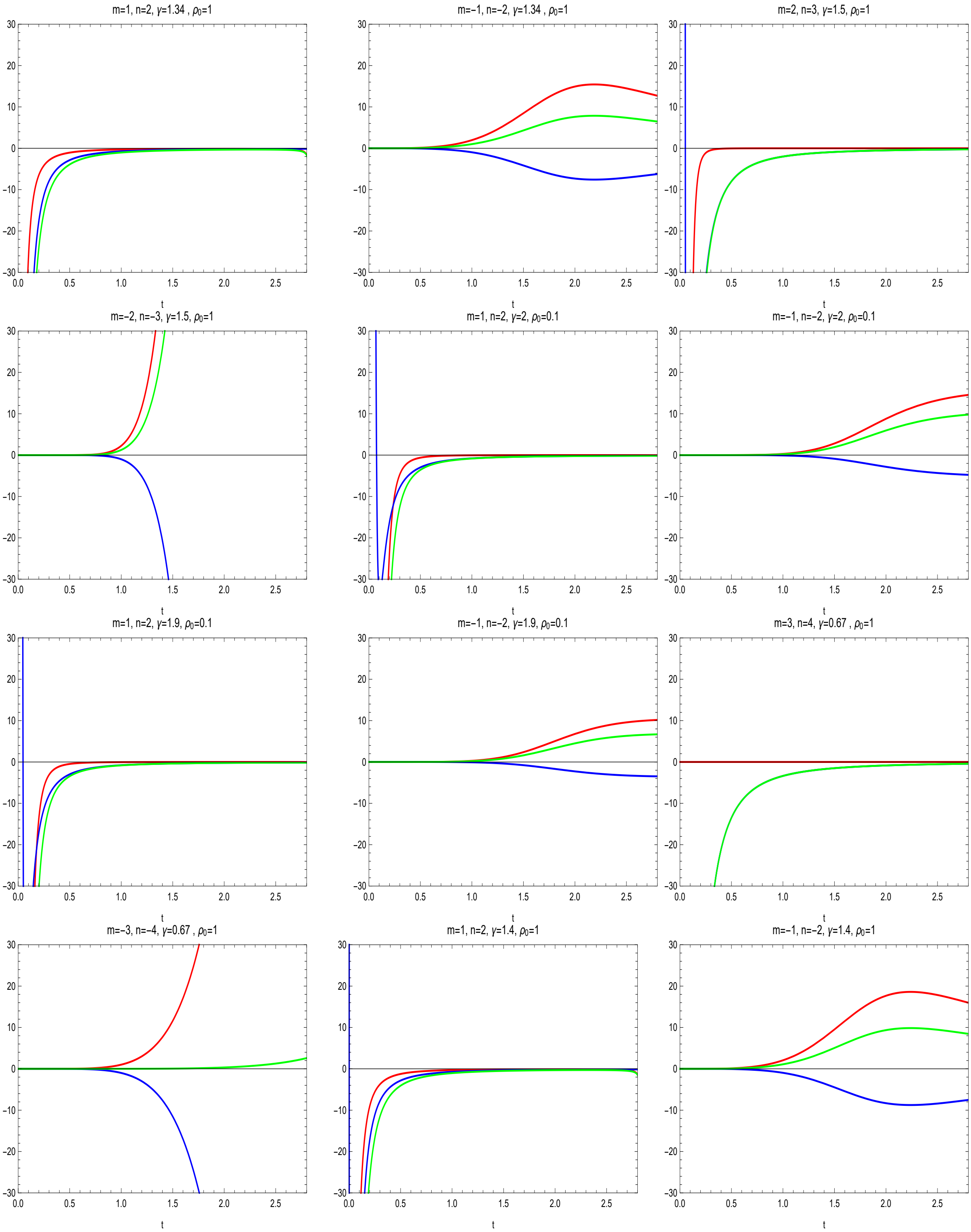


Figure 4.1: Time variation of different terms: R_1 (red), R_2 (blue) and $R_{\mu\nu}u^\mu u^\nu = \tilde{R}$ (green) choosing $0 \leq \gamma \leq 2$, for the model $a(t) = t^m$, $f(T) = \alpha(-T)^n, \alpha = 1$.

where Θ, is the volume scalar of the congruence. Thus,

$$\Lambda^{'} = \Lambda\Theta \tag{4.31}$$

where $'$ denotes differentiation with respect to τ. The Raychaudhuri equation which essentially gives the evolution of the congruence can be written as

$$\frac{d\Theta}{d\tau} + \frac{\Theta^2}{r} + 2\sigma^2 + \tilde{R} = 0 \tag{4.32}$$

where the expression for Θ and σ are given in Chapter 1. The Raychaudhuri scalar is given by

$$\tilde{R} = R_{\mu\nu}u^{\mu}u^{\nu}. \tag{4.33}$$

Since hyper-surface orthogonal congruence of time-like geodesics are taken into consideration, by virtue of Frobenius Theorem $\omega_{\mu\nu} = 0$. Using (4.31) and (4.32) one may obtain the RE as :

$$\frac{\Lambda''}{\Lambda} + \left(\frac{1}{r} - 1\right)\frac{\Lambda'^2}{\Lambda^2} + 2\sigma^2 + \tilde{R} = 0 \tag{4.34}$$

We can write the above second order differential equation in functional form as follows

$$\mathcal{F} = \frac{\Lambda''}{\Lambda} + \left(\frac{1}{r} - 1\right)\frac{\Lambda'^2}{\Lambda^2} + 2\sigma^2 + \tilde{R} = 0 \tag{4.35}$$

The necessary and sufficient conditions which (4.35) must satisfy for being the Euler-Lagrange equation corresponding to a Lagrangian $\mathcal{L}$ are known as the Helmholtz conditions [211]-[216]. Therefore, with an aim to formulate a Lagrangian corresponding to which the Euler-Lagrange equation gives back the RE, it has been found that $\mathcal{F}$ in the above form fails to satisfy all the Helmholtz conditions [211]-[216]. However if one multiplies $\mathcal{F}$ by $\Lambda^{(\frac{2}{r}-1)}$ then

$$\tilde{\mathcal{F}} = \Lambda^{(\frac{2}{r}-1)}\left[\frac{\Lambda''}{\Lambda} + \left(\frac{1}{r} - 1\right)\frac{\Lambda'^2}{\Lambda^2} + 2\sigma^2 + \tilde{R}\right], \tag{4.36}$$

satisfies all the Helmholtz conditions, since in the present model $2\sigma^2 + \tilde{R}$ is a function of Λ only. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\Lambda^{(\frac{2}{r}-2)}\Lambda'^2 - V[\Lambda]. \tag{4.37}$$

A variation of the lagrangian $\mathcal{L}$ with respect to dynamical variable Λ gives,

$$\delta\mathcal{L} = -\Lambda^{(\frac{2}{r}-1)}\left[\frac{\Lambda''}{\Lambda} + \left(\frac{1}{r} - 1\right)\frac{\Lambda'^2}{\Lambda^2}\right]\delta\Lambda - \delta V + \frac{d}{d\tau}\left(\Lambda^{(\frac{2}{r}-2)}\Lambda'\delta\Lambda\right). \tag{4.38}$$

Hence to get $\tilde{\mathcal{F}} = 0$ from the principle of least action one needs,

$$\frac{\delta V[\Lambda]}{\delta\Lambda} = \Lambda^{(\frac{2}{r}-1)}\left(2\sigma^2 + \tilde{R}\right). \tag{4.39}$$

In the present model ($f(T)$ gravity constructed in the background of homogeneous and isotropic FLRW spacetime) one has $\Lambda = a^3$, $2\sigma^2 = 0$ and for the two choices of $f(T)$, the expression for $\tilde{R}$ are given by (4.27) and (4.28). Finally solving the differential equation (4.39) for V one has the potential in the form

$$V = V_0 + \frac{t^{\frac{2}{r}}}{12} g(t) \tag{4.40}$$

where, $$g(t) = \left(\frac{2c_0 t^{-\gamma}}{2l - m\gamma}\right) - \left(\frac{3t^{\left(\frac{-2}{3m}-\gamma\right)}\ 2^{F_1}\ \left(1, -1+3l-\frac{3m\gamma}{2}; 3l-\frac{3m\gamma}{2}; -2t^{\left(\frac{2}{3m}\right)}\right) c_1}{2-6l+3m\gamma}\right) + \frac{3c_2 t^{\left(-\frac{2}{3m}-\gamma\right)}}{(2-6l+3m\gamma)} - \left(\frac{12t^{\left(\frac{-2}{3m}-\gamma\right)}\ 2^{F_1}\ \left(1, -1+3l-\frac{3m\gamma}{2}; 3l-\frac{3m\gamma}{2}; -t^{\left(\frac{2}{3m}\right)}\right) c_3}{2-6l+3m\gamma}\right) + \frac{6\ c_4\ 2^{F_1}\ \left(1, -1+3l;\ 3l; -t^{\frac{2}{3m}}\right)}{(-1+3l)\ t^{\frac{2}{3m}}},$$

$l = \dfrac{m}{r}$ and V_0, c_0, c_1, c_2, c_3, and c_4 are arbitrary constants. 2^{F_1} is the Gauss Hyper-geometric function.

For general n in $f(T) = \alpha(-T)^n$, it is difficult to solve for V. Therefore V has been found for $\alpha = 1$ and $n = 2$ for simplicity, while $f(T) = \alpha T + \dfrac{\beta}{T}$ with $\alpha = -1, \beta = 1$ yields the expression for V as

$$V = V_0 + \frac{t^{\frac{2}{r}}}{6} h(t) \tag{4.41}$$

where,
$$h(t) = \left(\frac{2t^{-\gamma}}{2l - m\gamma}\right) - \left(\frac{6t^{\left(\frac{4}{3m}-\gamma\right)}\ 2^{F_1}\ \left(1, 1+\frac{3l}{2}-\frac{3m\gamma}{4}; 2+\frac{3l}{2}-\frac{3m\gamma}{4}; -t^{\left(\frac{4}{3m}\right)}k_0\right) k_0}{4+6l-3m\gamma}\right) + \frac{3k_2}{(3l-1)t^{\left(\frac{2}{3m}\right)}} + \frac{3t^{\left(\frac{2}{3m}\right)}\ 2^{F_1}\ \left(1, \frac{1+3l}{2}; \frac{3(l+1)}{2}; -t^{\left(\frac{4}{3m}\right)}k_4\right) k_3}{1+3l} + 2k_1 t^{-\gamma}\left(\frac{l}{2l-m\gamma} - \frac{3t^{\left(\frac{4}{3m}\right)}\ 2^{F_1}\ \left(1, 1+\frac{3l}{2}-\frac{3m\gamma}{4}; 2+\frac{3l}{2}-\frac{3m\gamma}{4}; -t^{\left(\frac{4}{3m}\right)}k_4\right) k_4}{4+6l-3m\gamma}\right), l =$$
$\dfrac{m}{r}$ and V_0, k_0, k_1, k_2, k_3 and k_4 are constants. 2^{F_1} is the Gauss Hyper-geometric function.

Now, we aim to find the Hamiltonian in operator version. This is because in this form, it admits a self-adjoint extension quite generally and therefore the conservation of probability is ensured. The momentum conjugate to the configuration variable Λ is given by

$$\Pi_\Lambda = \frac{\partial \mathcal{L}}{\partial \Lambda} = \Lambda^{2(\frac{1}{r}-1)}\Lambda'. \tag{4.42}$$

Hence, the Hamiltonian is given by:

$$\mathcal{H} = \frac{1}{2}\Lambda^{2(1-\frac{1}{r})}\Pi_\Lambda^2 + V[\Lambda]. \tag{4.43}$$

It may be verified that, the Hamilton's equation of motion gives the RE and definition of momentum. For canonical quantization of the system under consideration, Λ and Π_Λ are considered as operators acting on the state vector $\Psi(\Lambda, \lambda)$ of the geometric flow. In Λ-representation, the operators assume the form:

$$\tilde{\Lambda} \longrightarrow \Lambda \tag{4.44}$$

and

$$\tilde{\Pi}_\Lambda \longrightarrow -i\hbar\frac{\partial}{\partial\Lambda}. \tag{4.45}$$

One may verify,

$$[\tilde{\Lambda}, \tilde{\Pi}_\Lambda] = i\hbar \tag{4.46}$$

So the evolution of the state vector is given by

$$i\hbar\frac{\partial\Psi}{\partial\lambda} = \tilde{H}\Psi \tag{4.47}$$

with ,

$$\tilde{H} = -\frac{\hbar^2}{2}\Lambda^{2(1-\frac{1}{r})}\frac{\partial^2}{\partial\Lambda^2} + V[\Lambda] \tag{4.48}$$

being , the operator version of the Hamiltonian. Equation (4.48) is known as the quantized RE. In the context of cosmology, there is notion of Hamiltonian constraint and operator version of it acting on the wave function of the universe (Ψ) i.e,

$$\tilde{\mathcal{H}}\Psi = 0, \tag{4.49}$$

known as the WD equation [235], [236]. However, there is a problem of non-unitary evolution [237], [238] which can be resolved by the proper choice of operator ordering in the first term of the Hamiltonian. One may note that the operator form (4.48) is symmetric with norm given by

$$|\Psi|^2 = \int_0^\infty d\Lambda\Lambda^{2\left(\frac{1}{r}-1\right)}\Psi^*\Psi, \tag{4.50}$$

but it fails to be self-adjoint. However, one may extend it as a self-adjoint operator with the following operator ordering [217]

$$\tilde{H} = -\frac{\hbar^2}{2}\Lambda^{(1-\frac{1}{r})}\frac{\partial}{\partial\Lambda}\Lambda^{(1-\frac{1}{r})}\frac{\partial}{\partial\Lambda} + V[\Lambda] \tag{4.51}$$

Further, if a change of minisuperspace variable is carried out as

$$v = r\Lambda^{\frac{1}{r}} \tag{4.52}$$

then the transformed WD equation is written as:

$$\left[\frac{-\hbar^2}{2}\frac{d^2}{dv^2}+V(v)\right]\Psi(v)=0, \tag{4.53}$$

with symmetric norm as

$$|\Psi|^2=\int_0^\infty dv\Psi^*\Psi. \tag{4.54}$$

The above WD equation (4.53) can be interpreted as time-independent Schrodinger equation of a point particle of unit mass moving along v direction in a potential field $V(v)$ and it has zero eigen value of the Hamiltonian and the wave function is the corresponding energy eigen function.

Possible Solutions of the Wheeler-DeWitt (WD) equation in the present model:

For the first model, the expression for the potential V is given by (4.40). Now we opt for particular choice of the arbitrary integration constants involved in the expression of potential. This is because, it has been found post calculation that other choices lead to either unrealistic cases or complicated calculations. Therefore for the choice $c_0=\frac{2l-m\gamma}{2}, c_1=c_2=c_3=c_4=0$, one has

$$V=V_0+v_0v^\delta, \tag{4.55}$$

where $\delta=\frac{\left(\frac{2}{3}-\gamma\right)}{3l}=\frac{\left(\frac{2}{3}-\gamma\right)}{m}$, $v_0=\frac{1}{12\times 3^{\frac{\left(\frac{2}{3}-\gamma\right)}{3l}}}=\frac{1}{12\times 3^{\frac{\left(\frac{2}{3}-\gamma\right)}{m}}}$.

Since in the general quantum description we have considered $(r+1)$ dimensional space-time, so in the present model $r=3$ and $3l=m$.

The WD equation in this case is written as

$$\frac{d^2\Psi(v)}{dv^2}-\frac{2v_0}{\hbar^2}\left(\frac{V_0}{v_0}+v^\delta\right)\Psi(v)=0 \tag{4.56}$$

i.e,

$$\left(-\frac{\hbar^2}{2v_0}\frac{d^2}{dv^2}+v^\delta\right)\Psi(v)=\left(-\frac{V_0}{v_0}\right)\Psi(v). \tag{4.57}$$

This form of the WD equation can be interpreted as the energy eigen value equation of a particle of mass v_0 moving in a potential field $V(v)=v^\delta$ with energy eigen value $\left(-\frac{V_0}{v_0}\right)$ and the wave function of the universe is nothing but the corresponding energy eigen function.

The solution to the above equation for $\delta=-1$ is,

$$\Psi(v)=K_1\;M_{\frac{-1}{\sqrt{2V_0}}\frac{v_0}{\hbar},\frac{1}{2}}\left(\frac{2}{\hbar}\sqrt{2V_0}v\right)+K_2\;W_{\frac{-1}{\sqrt{2V_0}}\frac{v_0}{\hbar},\frac{1}{2}}\left(\frac{2}{\hbar}\sqrt{2V_0}v\right), \tag{4.58}$$

where K_1 and K_2 are arbitrary integration constants. M and W are the usual Whittaker functions of 1st and 2nd kind.

The solution for $\delta = 1$ is given by

$$\Psi(v) = C_1\ Ai\left[\left(\frac{2v_0}{\hbar^2}\right)^{\frac{1}{3}}\left(\frac{V_0}{v_0}+v\right)\right] + C_2\ Bi\left[\left(\frac{2v_0}{\hbar^2}\right)^{\frac{1}{3}}\left(\frac{V_0}{v_0}+v\right)\right], \tag{4.59}$$

where C_1, C_2 are arbitrary integration constants. Ai and Bi are the Airy and Bairy functions.

For $\delta = -2$, the solution is given by

$$\Psi(v) = B_1\ \sqrt{v}\ J_{\frac{1}{2}\sqrt{1+\frac{8v_0}{\hbar^2}}}\left(\sqrt{-2\frac{V_0}{\hbar^2}v}\right) + B_2\ \sqrt{v}\ Y_{\frac{1}{2}\sqrt{1+\frac{8v_0}{\hbar^2}}}\left(\sqrt{-2\frac{V_0}{\hbar^2}v}\right), \tag{4.60}$$

where B_1 and B_2 are arbitrary integration constants. J and Y are the Bessel functions of 1st and 2nd kind.

For $\delta = 2$, the solution is given by

$$\Psi(v) = A_1\frac{M_{\frac{-1}{4}\sqrt{\frac{2}{v_0}}\frac{V_0}{\hbar},\ \frac{1}{4}}\left(\sqrt{2v_0}\frac{v^2}{\hbar}\right)}{\sqrt{v}} + A_2\frac{W_{\frac{-1}{4}\sqrt{\frac{2}{v_0}}\frac{V_0}{\hbar},\ \frac{1}{4}}\left(\sqrt{2v_0}\frac{v^2}{\hbar}\right)}{\sqrt{v}}, \tag{4.61}$$

where A_1, A_2 are arbitrary integration constants. M and W are the usual Whittaker functions of 1st and 2nd kind.
For the second model, the expression for V is given by (4.41). The choice $k_0 = k_1 = k_2 = k_3 = k_4 = 0$ yields

$$V = V_0 + \ \epsilon_0 v^{\epsilon}, \tag{4.62}$$

where $\epsilon = \dfrac{\left(\frac{2}{3}-\gamma\right)}{m}$, $\epsilon_0 = \dfrac{1}{\epsilon m^2 \times 3^{\left(1+\frac{\epsilon}{3}\right)}}$.
The WD equation in this case turns out to be

$$\frac{d^2\Psi(v)}{dv^2} - \frac{2\epsilon_0}{\hbar^2}\left(\frac{V_0}{\epsilon_0}+v^{\epsilon}\right)\Psi(v) = 0 \tag{4.63}$$

or,

$$\left(-\frac{\hbar^2}{2\epsilon_0}\frac{d^2}{dv^2}+v^{\epsilon}\right)\Psi(v) = \left(-\frac{V_0}{\epsilon_0}\right)\Psi(v). \tag{4.64}$$

The above equation can be compared to energy eigen value equation of a particle having mass ϵ_0 and moving in a potential field $V(v) = v^{\epsilon}$ with energy eigen value $\left(-\frac{V_0}{\epsilon_0}\right)$ and the wave function can be interpreted as the corresponding energy eigen function. The solution corresponding to $\epsilon = -1$ is given by

$$\Psi(v) = F_1\ M_{\frac{-1}{\sqrt{2V_0}}\frac{\epsilon_0}{\hbar},\frac{1}{2}}\left(\frac{2}{\hbar}\sqrt{2V_0}v\right) + F_2\ W_{\frac{-1}{\sqrt{2V_0}}\frac{\epsilon_0}{\hbar},\frac{1}{2}}\left(\frac{2}{\hbar}\sqrt{2V_0}v\right), \tag{4.65}$$

where F_1 and F_2 are arbitrary integration constants. M and W are the usual Whittaker functions of 1st and 2nd kind.
The solution for $\epsilon = 1$ is given by

$$\Psi(v) = D_1\ Ai\left[\left(\frac{2\epsilon_0}{\hbar^2}\right)^{\frac{1}{3}}\left(\frac{V_0}{\epsilon_0} + v\right)\right] + D_2\ Bi\left[\left(\frac{2\epsilon_0}{\hbar^2}\right)^{\frac{1}{3}}\left(\frac{V_0}{\epsilon_0} + v\right)\right], \tag{4.66}$$

where D_1 and D_2 are arbitrary integration constants. Ai and Bi are the Airy and Bairy functions.
For $\epsilon = -2$, one has the solution,

$$\Psi(v) = G_1\ \sqrt{v}\ J_{\frac{1}{2}\sqrt{1+\frac{8\epsilon_0}{\hbar^2}}}\left(\sqrt{-2\frac{V_0}{\hbar^2}}v\right) + G_2\ \sqrt{v}\ Y_{\frac{1}{2}\sqrt{1+\frac{8\epsilon_0}{\hbar^2}}}\left(\sqrt{-2\frac{V_0}{\hbar^2}}v\right), \tag{4.67}$$

where G_1 and G_2 are arbitrary integration constants. J and Y are the Bessel functions of 1st and 2nd kind.
For $\epsilon = 2$, the wave function is given by

$$\Psi(v) = E_1\frac{M_{\frac{-1}{4}\sqrt{\frac{2}{\epsilon_0}}\frac{V_0}{\hbar},\ \frac{1}{4}}\left(\sqrt{2\epsilon_0}\frac{v^2}{\hbar}\right)}{\sqrt{v}} + E_2\frac{W_{\frac{-1}{4}\sqrt{\frac{2}{\epsilon_0}}\frac{V_0}{\hbar},\ \frac{1}{4}}\left(\sqrt{2\epsilon_0}\frac{v^2}{\hbar}\right)}{\sqrt{v}} \tag{4.68}$$

where E_1 and E_2 are arbitrary integration constants. M and W are the usual Whittaker functions of 1st and 2nd kind.
It follows from (4.52) that in the present model one has, $v = 3a(t)$ (since $\Lambda = a^3, r = 3$). Now, to have an idea about probability measure on the minisuperspace $|\Psi|^2$ has been plotted against v as a tool to perform singularity analysis in the quantum regime for both the models in FIG.4.3 (Model 1) and FIG.4.4 (Model 2). Based on the plots one has the following findings:

- From FIG.4.3, it is clear that the probability of approaching singularity (i.e zero volume) is zero for the choice of the parameter $\delta = -1, -2$, while for $\delta = +1, +2$, there is a finite non-zero probability for the existence of initial big-bang singularity.

- From FIG.4.4, the probability of having zero volume is zero for all the cases except for a typical choice of $\epsilon = 1$. Therefore, similar to the previous model it is possible to obviate big-bang singularity by the present quantum description.

Remark: In this section, canonical quantization technique has been used to study the singularity. The basic question that we have attempted to investigate is whether quantum formulation may overrule the singularity particularly the initial big-bang singularity. Essentially, we have examined it by studying the probability function near the classical singularity. Similar to classical analysis we have found that probability is zero at big-bang singularity (i.e. avoidance of singularity) if the potential corresponding to the to the dynamical system representing the congruence of time-like geodesics is in (i) inverse power law (linear and quadratic) form in case of model 1, and (ii) in linear, quadratic and inverse linear power law form for model 2. Therefore, we may conclude that the present quantum description may eliminate the classical singularity.

4.5 Concluding Remarks

An investigation of modified gravity theory namely $f(T)$ gravity has been done in this chapter by analyzing the RE both classically and quantum mechanically. At first, a general formulation of RE has been performed for arbitrary $f(T)$. Subsequently, the existence of singularity has been examined by studying the CC in the modified gravity model with two physically motivated specified choices for $f(T)$. The Raychaudhuri scalar has been plotted graphically in FIG. 4.1 and FIG. 4.2 for various choices of the parameters involved.

For Model 1, it is possible to avoid the classical singularity although SEC is satisfied while for Model 2, avoidance of singularity is possible by violation of SEC or the matter field should be ghost in nature. By choosing $\Lambda = \sqrt{\eta}$ (η is the metric scalar of the space-like hyper-surface) it is possible to write the RE as a second order differential equation and a Lagrangian (hence a Hamiltonian) formulation can be done.

Finally, quantum cosmology has been furnished and the wave function of the universe has been found to be the solution of the one dimensional time independent Schrödinger equation associating the energy eigen function to the wave function of the universe. From probabilistic description, it is found that the initial big-bang singularity may be avoided for both the models with proper choice of the classical potential corresponding to the dynamical system representing the congruence of geodesics.

Therefore in the present chapter, corresponding to the RE in $f(T)$ gravity, both classical and quantum description have been furnished and the initial big-bang singularity has been examined both at the classical as well as at the quantum level. Classically, the avoidance of singularity is more realistic in model 1 as it involves normal fluid as the matter content of the universe but this is not possible for model 2. For quantum description the avoidance of the big-bang singularity is obtained by probabilistic description from WD formalism.

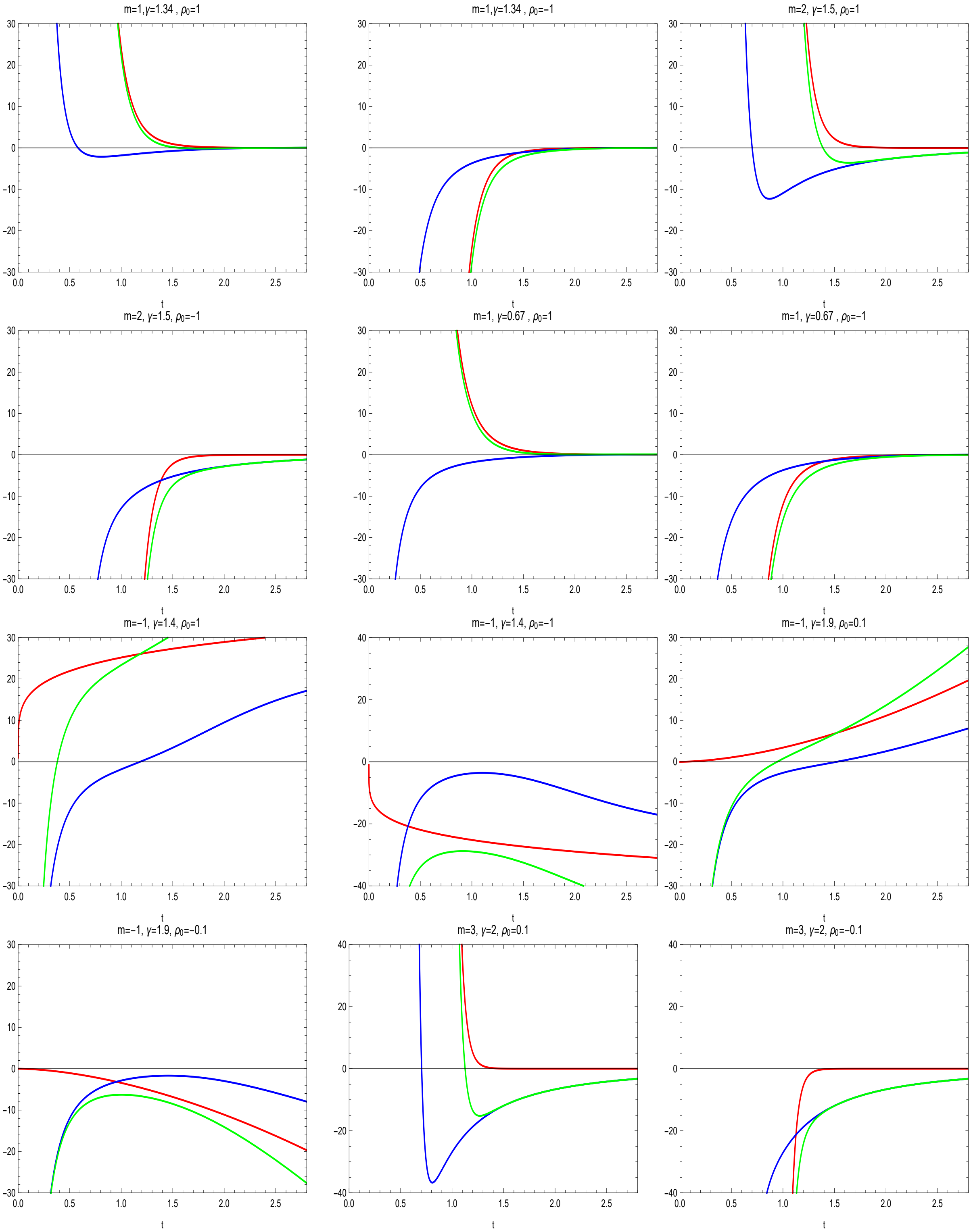
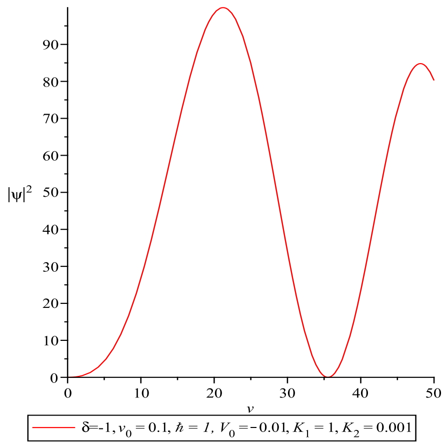
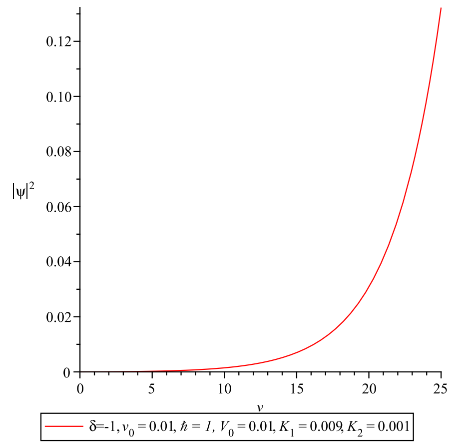
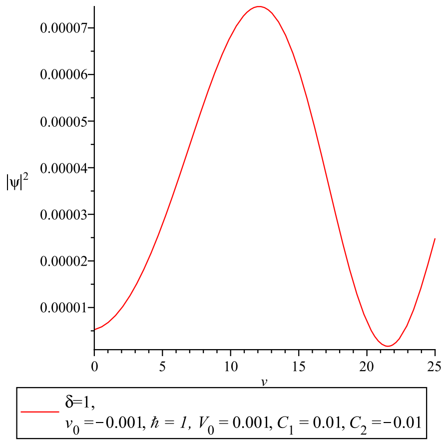
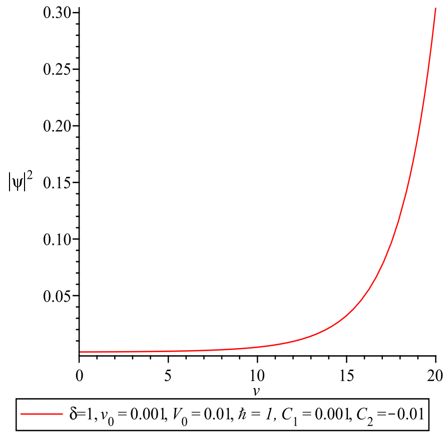


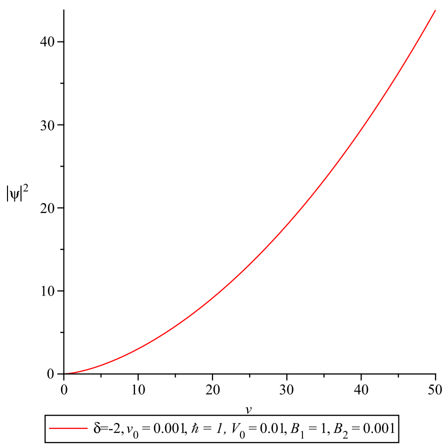
Figure 4.2: Time variation of different terms: R_1 (red), R_2 (blue) and $R_{\mu\nu}u^{\mu}u^{\nu} = \tilde{R}$ (green) for $0 \leq \gamma \leq 2$. Here we choose $a(t) = t^m$ and $f(T) = \alpha T + \dfrac{\beta}{T}$, $\alpha = -1$, $\beta = 1$.

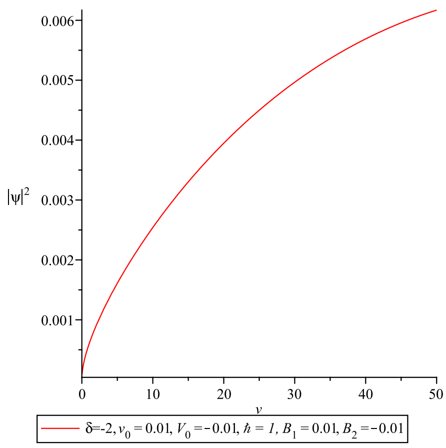

|ψ|²
v
δ=-1, v0 = 0.1, ℏ = 1, V0 = -0.01, K1 = 1, K2 = 0.001


|ψ|²
v
δ=-1, v0 = 0.01, ℏ = 1, V0 = 0.01, K1 = 0.009, K2 = 0.001


|ψ|²
v
δ=1,
v0 = -0.001, ℏ = 1, V0 = 0.001, C1 = 0.01, C2 = -0.01


|ψ|²
v
δ=1, v0 = 0.001, V0 = 0.01, ℏ = 1, C1 = 0.001, C2 = -0.01


|ψ|²
v
δ=-2, v0 = 0.001, ℏ = 1, V0 = 0.01, B1 = 1, B2 = 0.001


|ψ|²
v
δ=-2, v0 = 0.01, V0 = -0.01, ℏ = 1, B1 = 0.01, B2 = -0.01

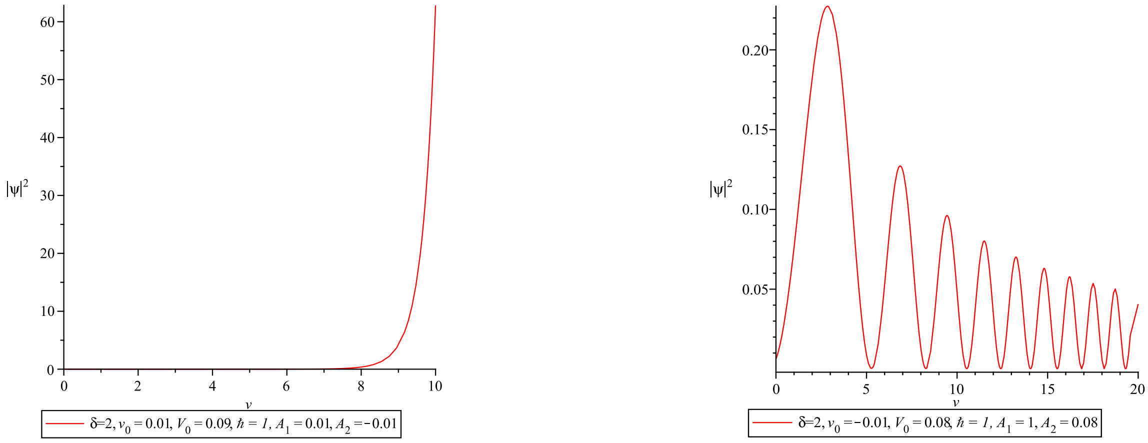
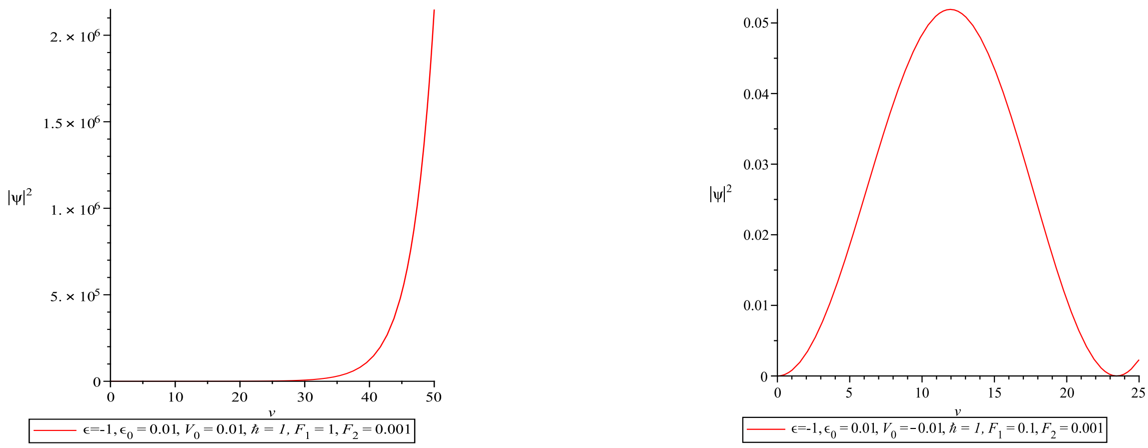
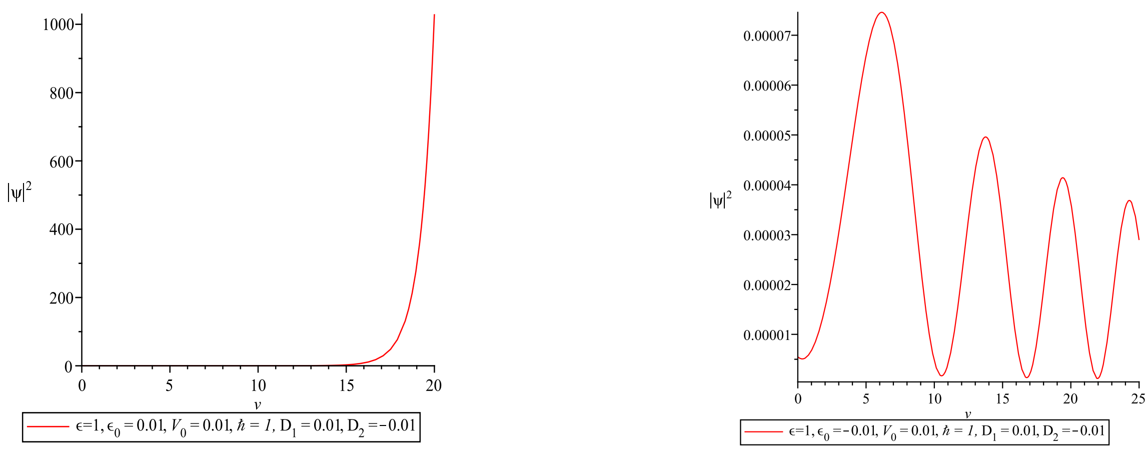


Figure 4.3: $|\Psi|^2$ vs v for Model 1 for various choices of parameters specified in each panel.





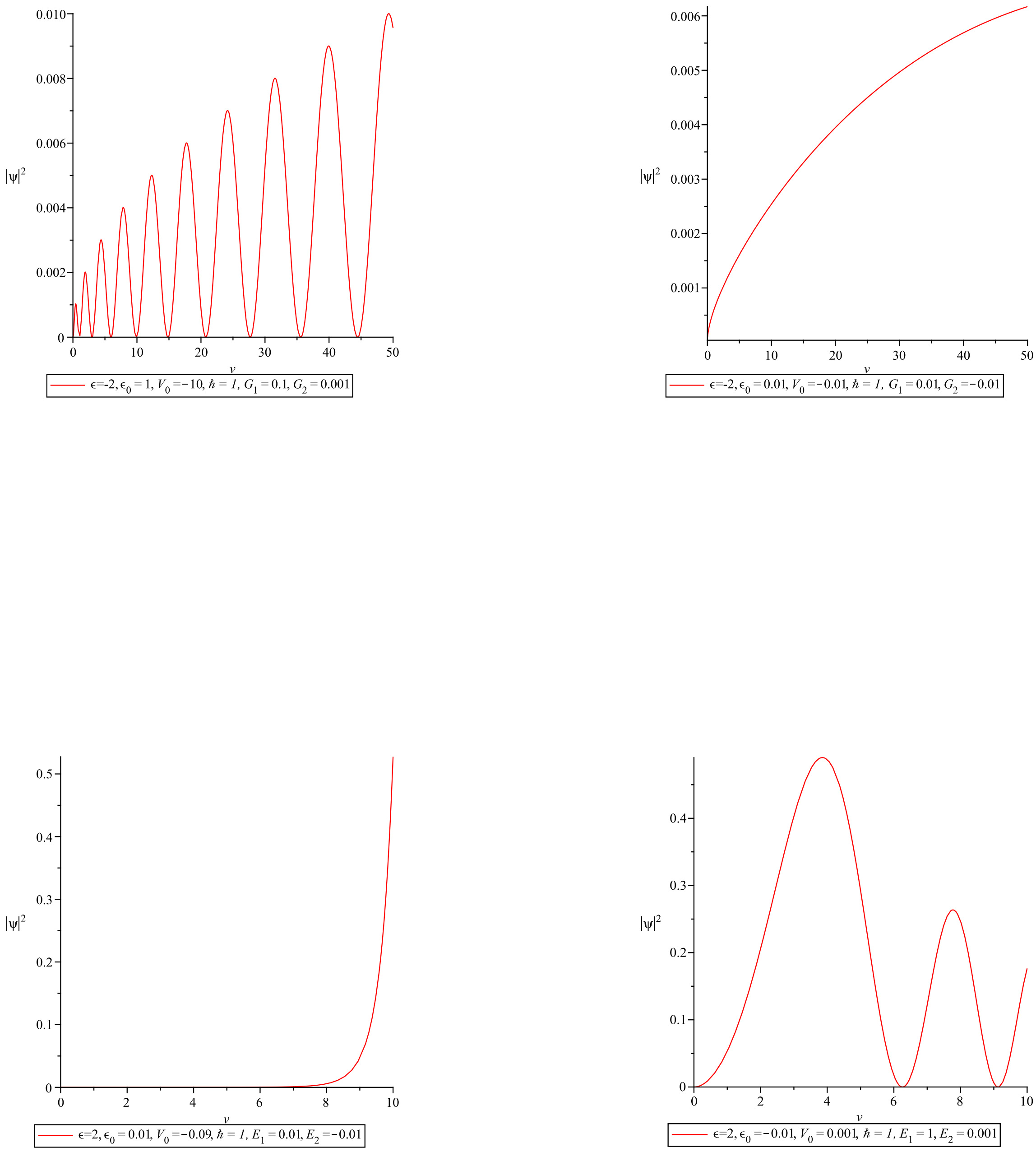


Figure 4.4: $|\Psi|^2$ vs v for Model 2 for various choices of parameters specified in each panel.

CHAPTER 5

IMPLICATIONS OF RAYCHAUDHURI EQUATION IN BOUNCING MODEL OF UNIVERSE

5.1 Prelude

In the recent times, there has been an increasing inclination of the cosmologists towards the cosmological models that replace the cosmological singularity (or big bang) with a "big bounce"—a smooth transition from contraction to expansion in order to resolve fundamental issues in cosmology. The cosmological evolution in the early universe is usually described by the standard Big-Bang cosmology. However, the standard cosmology suffers from a couple of issues such as the Horizon problem, Flatness problem, baryon asymmetry and initial singularity. Although inflationary scenario [239], successfully resolved some of the issues related to early universe standard model, yet it suffers from the singularity problem and hence fails to reconstruct the complete past history of the universe.

An alternative model to the standard big bang scenario, describing inflation without initial singularity had been proposed. This model is known as the Emergent Universe model [240]. Another non-singular approach without inflation known as the matter bounce scenario had been proposed [241], [242]. For further information regarding non-singular models, one may refer to [243], [244]. The initial singularity occurring in the standard Big Bang cosmology and the inflationary cosmology can be suitably avoided in the matter bounce scenario. Bouncing cosmologies have been investigated in extended theories of gravity such as $f(R)$ theory [245], [246], modified Gauss-Bonnet gravity [247], [248], $f(R,T)$ gravity [249], [250], $f(Q,T)$ gravity [251] and $f(T)$ gravity [252]. Resolution of the initial singularity by applying Loop Quantum gravity approach gives rise to a hybrid cosmological scenario of the emergent and bouncing universe [253]. The present chapter deals with bouncing cosmology from RE point of view. It gives light on a plethora of bouncing models and the nature of the bouncing points.

This chapter is organized as follows: Section 5.2 gives an overview of various bouncing models in literature. Section 5.3 discusses the implications of RE in bouncing cosmology. Oscillatory bounce has been demonstrated using the harmonic oscillator approach of RE in Section 5.4. In section 5.5, bouncing point has been classified mathematically as a point of inflection, cusp or corner along with their physical implications. This section also discusses some observational aspects related to bouncing models. Finally, the chapter ends with a conclusion in section 5.6.

5.2 An overview of Bouncing scenario

Now, we examine the consequences of the RE for two types of bouncing models, namely **B1** and **B2**. The present section deals with the details of the two models as follows:

B1: Bouncing Point is a local minima for $a(t)$

Behavior of cosmic scale factor $a(t)$ and Hubble parameter H:

- $a(t)$ decreases before bounce, attains a minimum at the bouncing point and then increases after the bounce. This is illustrated in figures 5.1-5.4 (The variation of scale factor is shown)
- $H < 0$ before bounce, $H = 0$ at the bouncing point and $H > 0$ after the bounce.
- Continuity of H can be shown graphically in figures 5.1-5.4 which clearly depict that H is an increasing function i.e. $\dot{H} > 0$ in the deleted neighborhood of bounce.

Some examples of **B1** are:

1. Symmetric Bounce [254]: $a(t) = B\exp\left(\beta\,\frac{t^2}{t_0^2}\right)$, $H = \frac{2\beta t}{t_0^2}$, t_0 is some arbitrary time $B > 0$, $\beta > 0$.
2. Matter Bounce [255], [256]: $a(t) = A\left(1.5\rho_c t^2 + 1\right)^{\frac{1}{3}}$, $H = \frac{2\rho_c t}{3\rho_c t^2 + 2}$ where $A > 0$ is a constant, $0 < \rho_c << 1$ is the critical density whose value stems from LQC.
3. Type I-IV (past/future) singularities and little rip cosmologies [257]: $a(t) = A\exp\left[\frac{f_0}{\alpha+1}(t-t_s)^{\alpha+1}\right]$, $H = f_0(t-t_s)^{\alpha}$, $\alpha \neq -1, 0, 1$ and α must be odd. t_s is the time at which bounce occurs.
4. A bounce characterized by $a(t) = b\exp(k\exp(t-\alpha) - t)$, $H = k(\exp(t-\alpha) - 1)$ $(k, \alpha, b > 0)$

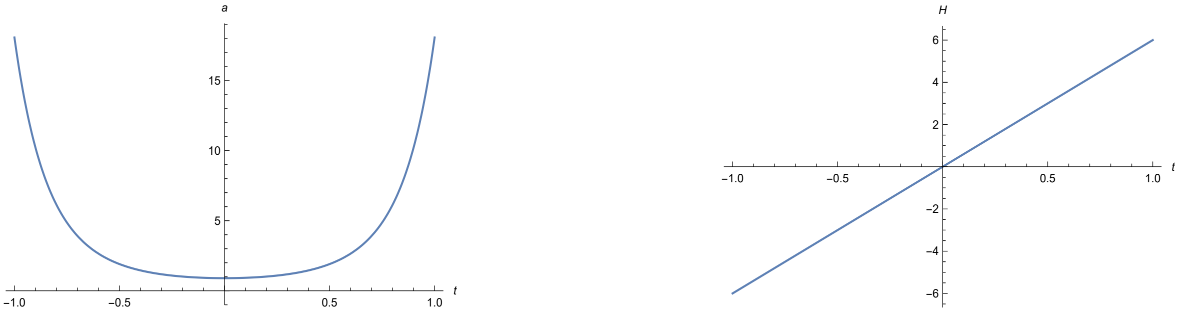


Figure 5.1: Scale factor $a(t)$ vs t (left) and Hubble parameter H vs t (right) representing example 1 of **B1**

Figure 5.2: Scale factor $a(t)$ vs t (left) and Hubble parameter H vs t (right) representing example 2 of **B1**

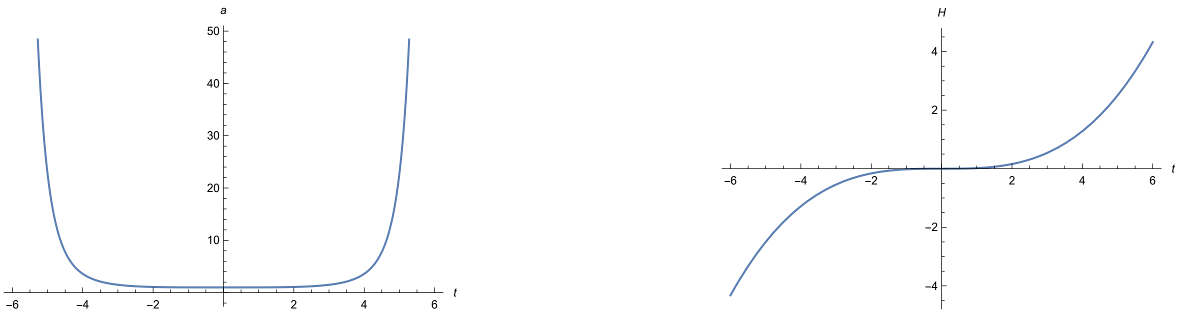


Figure 5.3: Scale factor $a(t)$ vs t (left) and Hubble parameter H vs t (right) representing example 3 of **B1**

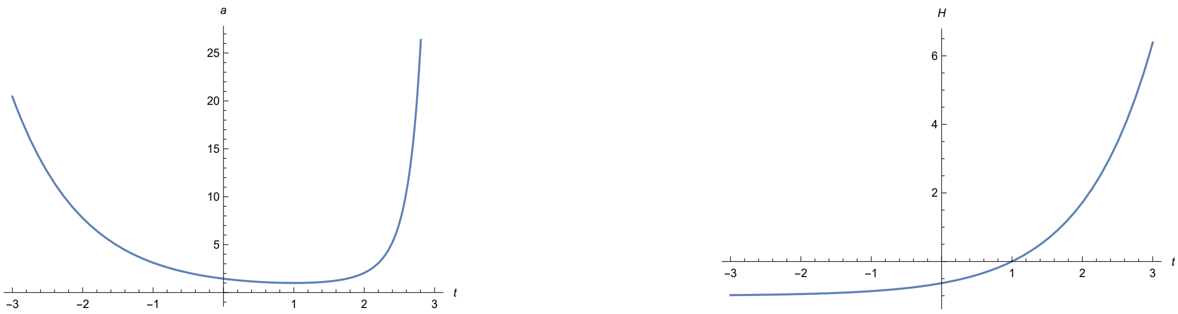


Figure 5.4: Scale factor $a(t)$ vs t (left) and Hubble parameter H vs t (right) representing example 4 of **B1**

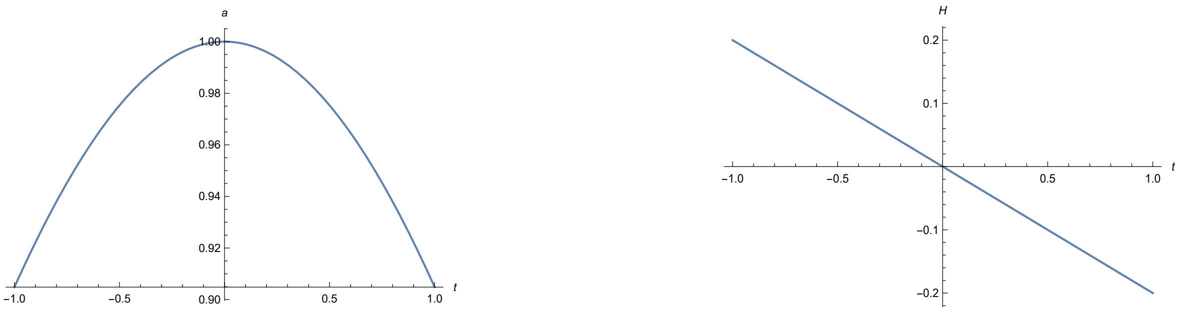


Figure 5.5: Scale factor $a(t)$ vs t (left) and Hubble parameter H vs t (right) representing a case of **B2**

B2 : Bouncing Point is a local maxima for $a(t)$

Behavior of cosmic scale factor $a(t)$ and Hubble parameter H:

- $a(t)$ increases before bounce, attains a maximum at the bouncing point and then decreases after the bounce. This is illustrated in Fig 5.5 (an example).
- $H > 0$ before bounce, $H = 0$ at the bouncing point and $H < 0$ after the bounce.
- Continuity of H can be shown graphically in Fig 5.5 which clearly depicts that H is a decreasing function i.e. $\dot{H} < 0$ in the deleted neighborhood of bounce.

Example: A bounce characterized by $a(t) = A \exp\left(b\left(\frac{t}{t_0}\right)^2\right)$, $H = \frac{2bt}{t_0^2}$ where $A > 0$, $b < 0$ $t_0 > 0$ is an arbitrary time. This bounce is illustrated in Fig 5.5.

Some special cases:

- 1.1 Super bounce [258],[259]: $a(t) = \left(\frac{t_s - t}{t_0}\right)^{\frac{2}{c^2}}$, $H = -\frac{2}{c^2}\left(\frac{1}{t_s - t}\right)$ where $c > \sqrt{6}$ is a constant, t_s is the time at which bounce occurs. This bouncing model is shown in Fig 5.6.
- 1.2 Bounce characterized by scale factor $a(t) = |t|$. This bounce is shown in Fig 5.7 (left).
- 1.3 Bounce characterized by

$$a(t) = \begin{cases} \sqrt{-t} & t \leq 0 \\ t^2 & t \geq 0 \end{cases}$$

This bounce is shown in Fig 5.7 (right).

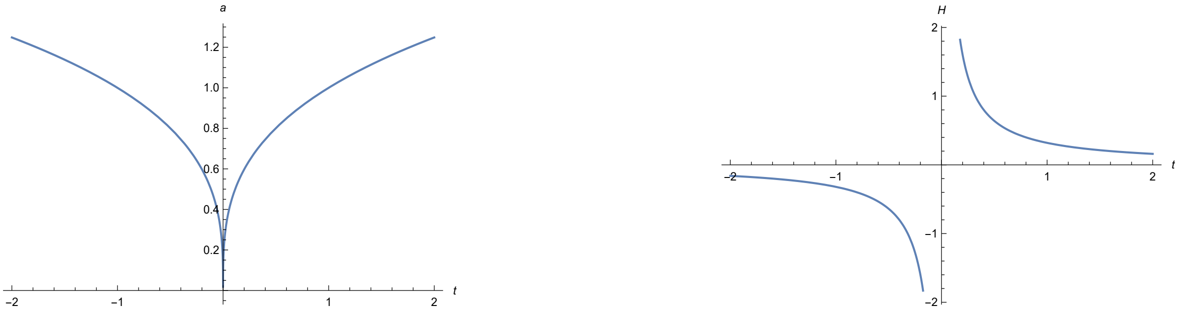


Figure 5.6: Scale factor $a(t)$ vs t (left) and Hubble parameter H vs t (right) for super bounce 1.1

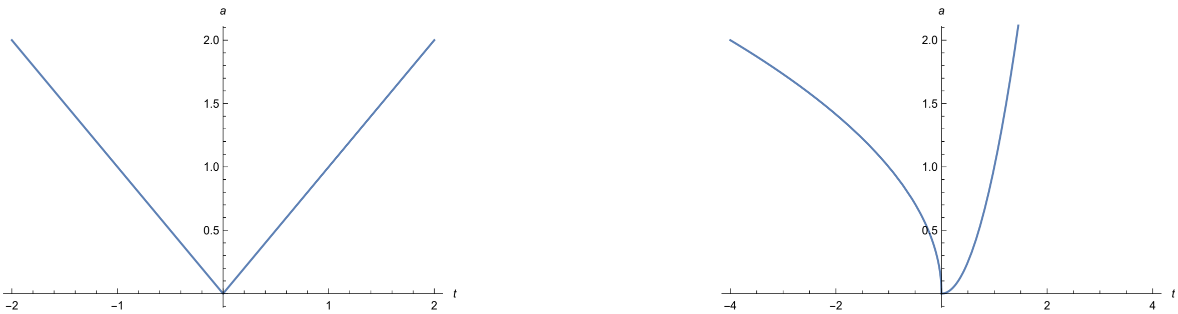


Figure 5.7: Scale factor $a(t)$ vs t for the bounce 1.2 (left) and 1.3 (right)

5.3 Raychaudhuri equation in FLRW model and bouncing scenario

For Friedmann–Lemaître–Robertson-Walker (FLRW) model, $\Theta = 3H = 3\frac{\dot{a}}{a}$ and $\sigma^2 = 0$. Thus the RE (2.2) takes the form

$$3\dot{H} = -3H^2 - R_{ab}v^a v^b. \tag{5.1}$$

In the context of cosmology τ can be treated as the cosmic time t and '.' represents differentiation w.r.t t.

RE and B1: In the previous section we have seen that $\dot{H} > 0$ in the deleted neighborhood of the bouncing point in **B1**. Therefore from the RE (5.1) one has
(i) $R_{ab}v^a v^b < 0$ and,
(ii) $|R_{ab}v^a v^b| > 3H^2$ in the deleted neighborhood of **B1**.
If the matter content of the universe is a perfect fluid having energy density ρ and pressure p then $R_{ab}v^a v^b = \dfrac{1}{2}(\rho + 3p)$. The RE in FLRW space-time is given by

$$\frac{\ddot{a}}{a} = -\frac{4}{3}(\rho + 3p) \tag{5.2}$$

Thus using (i) in equation (5.2) one may conclude that in this bouncing model there is always acceleration. Violation of SEC indicates that matter is exotic in nature and singularity can be avoided near the bounce (in the sense that CC needed for focusing of geodesic congruence is violated here). Further in this type of bounce, the behavior of energy density ρ can be studied using the RE. From the first Friedmann equation we have $3H^2 = \rho$ and from the RE (5.1) we have $2\dot{H} = -(\rho + p)$ which is nothing but the second Friedmann-equation. Hence for **B1** in the deleted neighborhood of bounce $(\rho + p) < 0$ i.e. Null Energy Condition (NEC) is violated. To study the continuity of ρ we shall use the matter conservation equation

$$\dot{\rho} + 3H(p + \rho) = 0 \tag{5.3}$$

Thus before bounce one as $H < 0$ and hence $\dot{\rho} < 0$. At bouncing point $H = 0$ and hence from the first Friedmann equation $\rho = 0$. After bounce $H > 0$ and hence $\dot{\rho} > 0$. Thus continuity of ρ can be studied as: ρ is decreasing before bounce, attains zero value at bouncing point and increases after bounce. If we assume further that the matter content of the universe is a perfect fluid with barotropic equation of state $p = \omega\rho$, ω being the equation of state (EoS) parameter then in the neighborhood of **B1**, $\omega < -1$ i.e. phantom energy favors this type of bounce. From the second Friedmann equation $\dot{H} = 0$ at the bouncing point. But q, the deceleration parameter is not defined at the bouncing point. Although this type of bounce can avoid the initial big-bang singularity as evident from the RE, yet the peculiarity observed at the bouncing point hints that the bouncing point may be regarded as a higher order singularity in the sense that a, H, $\dot{H}$ are defined at the bouncing point but no other higher order parameters like q and j are defined at the bouncing point. However there is always acceleration in this type of model except at the bouncing point where $\ddot{a} = 0$.

RE and B2: For **B2**, $\dot{H} < 0$ in the neighborhood of the bouncing point and $H = 0$ at the bouncing point. Therefore from the RE (5.1) there are two possibilities: (i) $R_{ab}v^a v^b > 0$ or (ii) $R_{ab}v^a v^b < 0$ but $|R_{ab}v^a v^b| < 3H^2$ in the neighborhood of the bouncing point in **B2**.

1. In the first case convergence condition (CC) or precisely the Strong Energy Condition (SEC) holds. Therefore, bounce occurs even with normal/usual matter. There is always deceleration except at the bouncing epoch as clear from the RE (5.2). Since $a_b \neq 0$, hence at the bouncing point there is non-zero volume (hence no singularity). However focusing/convergence condition holds in the neighborhood of bounce. Therefore, focusing alone does not always leads to singularity formation. However the converse is true i.e, if there is a singularity of the space-time then a congruence of time-like/ null geodesic will focus there. This seemingly trivial statement is proved by the Focusing theorem.

2. For the second case bounce occurs with exotic matter having EoS parameter $-1 < \omega < -\frac{1}{3}$ and violation of CC results in the avoidance of big-bang singularity. There is always acceleration except at the bouncing point where $\ddot{a} = 0$.

The energy density ρ shows a similar behavior just like in **B1**. $\dot{H} = 0$ at the bouncing point. In **B2** also a, H and $\dot{H}$ are defined at the bouncing point but next higher order derivatives are undefined at the bouncing point. This absurdness of the bouncing point again hints the existence of 2nd order singularity. Therefore the RE essentially depicts

the existence of higher order singularities at the bouncing point.
The full analysis done till now has been carried out for $\kappa = 0$. Now let us investigate which value of κ favors the bouncing scenario. For $\kappa \neq 0$, the Einstein's field equations are

$$3\left(H^2 + \frac{\kappa}{a^2}\right) = \rho \tag{5.4}$$

$$2\left(\dot{H} - \frac{\kappa}{a^2}\right) = -(\rho + p) \tag{5.5}$$

Since $H^2 > 0$, from (5.4) $\rho > \frac{3\kappa}{a^2}$. For $\kappa = +1$, ρ is positive definite but for $\kappa = -1$, $\rho = 3\left(H^2 - \frac{1}{a^2}\right)$, therefore one must have a lower bound on H i.e. $H^2 > \dfrac{1}{a^2}$ in order to have a normal matter in the neighborhood of bouncing point. More specifically, $H > \dfrac{1}{a}$ after the bounce and $H < -\dfrac{1}{a}$ before the bounce i.e. $\dot{a} > 1$ after bounce, $\dot{a} < -1$ before bounce and $\dot{=} 0$ at the bouncing point. At the bouncing point $H = 0$ which implies $\rho_b = \frac{3\kappa}{a_b^2} \neq 0$ (ρ_b and a_b are the energy density and scale factor at the bouncing epoch t_b). Further it is to be noted that for $\kappa = -1$ energy density at bouncing point (ρ_b) is negative which says that matter must be ghost type at the bouncing point. From (5.5), $\dot{H} = \dfrac{\kappa}{a^2} - \dfrac{1}{2}(\rho + p)$. For **B1**, $\dot{H} > 0 \implies \rho + p < \frac{2\kappa}{a^2}$. So for $\kappa = -1$, the matter is of phantom type, while for $\kappa = +1$, the matter may not be exotic in nature. Therefore the above analysis shows that for bouncing scenario $\kappa = +1$ is more suitable than $\kappa = -1$.
For $\kappa = 0$ we have seen that perfect fluid with barotropic equation of state can not give the true picture of a bouncing model. This is because we do not have any idea about the behavior of q, j and other parameters involving higher derivatives of H. Therefore let us choose two different types of equation of state as follows:
(i) van der Waals equation of state : The EoS is given by

$$p = \frac{A\rho}{1 - B\rho} - C\rho^2 \tag{5.6}$$

where A, B and C are constants. So $\omega = \dfrac{p}{\rho} = \dfrac{A}{1 - B\rho} - C\rho \to A$ as $\rho \to 0$ i.e. at the bouncing point $\omega = A$, a constant hence defined. Also $2\dot{H} = -(\rho + p) = -\left(\rho + \dfrac{A\rho}{1 - B\rho} - C\rho^2\right)$ and $3H^2 = \rho$. $\therefore q = -(1 + \frac{\dot{H}}{H^2}) = \dfrac{(1 + 3A)}{2}$. Therefore q is defined at the bouncing point for these bouncing models with perfect fluid having van der waals EoS. Further the sign of q depends on A. Thus acceleration (deceleration) near bounce is determined by A.
(ii) Polytropic equation of state: The EoS is given by

$$p = k\rho^{(1+\frac{1}{n})} \tag{5.7}$$

where k is a proportionality constant and n is the polytropic index (n is any real number). Here $\omega = \dfrac{p}{\rho} = k\rho^{\frac{1}{n}} \to 0$ as $\rho \to 0$ at the bouncing point for $n > 0$ and $q = \dfrac{1}{2}$

at the bouncing point indicating deceleration near bounce. However for $n \leq 0$ both q and ω are undefined. Hence it shows that q and ω are defined at bouncing point of these bouncing models with perfect fluid having polytropic EoS with a positive polytropic index.

5.4 Raychaudhuri equation: A Linear Harmonic Oscillator and Oscillatory bounce

In this section, we aim to explore whether/how the Raychaudhuri equation can explain the existence and avoidance of initial big-bang singularity in case of oscillatory bouncing cosmological model. To do so, we recall the general form of the RE in (2.2) and look at the RE from the point of view of an evolution equation for a real harmonic oscillator. Geometrically $R_{ab}v^a v^b = \tilde{R}$ can be interpreted as mean curvature in the direction of $\mathbf{v}$, the velocity vector field [174]. From mathematical point of view, the RE can be treated as Riccati equation and it becomes a linear second order equation or a Hill Type differential equation of the form:

$$\frac{d^2Y}{d\tau^2} + \omega_0^2 \, Y = 0, \tag{5.8}$$

where

$$\Theta = (n-1)\frac{d}{d\tau}\ln Y \tag{5.9}$$

and

$$\omega_0^2 = \frac{1}{n-1}(\tilde{R} + 2\sigma^2 - 2\omega^2 - \nabla_b A^b). \tag{5.10}$$

Thus the RE can be identified as a linear harmonic oscillator equation with time varying frequency ω_0. As Θ may be defined as the derivative of the geometric entropy (S) or an average (or effective) geodesic deviation so one may identify $S = \ln Y$. The expansion Θ is nothing but the rate of change of volume of the transverse subspace of the congruence/bundle of geodesics. Therefore, the expansion approaching negative infinity (i.e. $\Theta \to -\infty$) implies a convergence of the bundle, whereas a value of positive infinity (i.e. $\Theta \to +\infty$) would imply a complete divergence. Thus the Convergence Condition (CC) can be stated as follows:

(i) Initially Y is positive but decreases with proper time i.e $\frac{dY}{d\tau} < 0$.

(ii) Subsequently $Y = 0$ at a finite proper time to have negative infinite expansion.

From the above interrelation: $\Theta = \frac{(n-1)}{Y}\frac{dY}{d\tau}$, it is clear that there should be an initially negative expansion (i.e. $\Theta(\tau = 0) < 0$) and subsequently $\Theta \to -\infty$ as $Y \to 0$ at a finite proper time. Therefore the CC essentially coincides with the condition for the existence of zeroes of Y in finite proper time. However the Sturm comparison theorem (in differential equation) shows that the existence of zeros in Y at finite value of the proper time τ requires

$$(\tilde{R} + 2\sigma^2 - 2\omega^2 - \nabla_b A^b) \geq 0. \tag{5.11}$$

Therefore (5.11) is the CC for a congruence of time-like curves (may be geodesic or non geodesic). Further, the above inequality shows that, the Raychaudhuri scalar $\tilde{R}$ and the shear/anisotropy scalar $2\sigma^2$ are in favor of convergence of the congruence of time-like curves while rotation and acceleration terms oppose the convergence. The CC reduces to $\tilde{R} \geq 0$ if we consider the congruence of time-like curves to be geodesic and orthogonal to the space-like hyper-surface. This leads to **Geodesic Focusing** and hence the **Focusing Theorem**. In other words, rotation and acceleration terms act against the focusing but shear and Raychaudhuri scalar are in favor of it. Thus from physical point of view, if the RE corresponds to a realistic linear harmonic oscillator then it is inevitable to have a singularity. We name the scalar: $R_c = \tilde{R} + 2\sigma^2$, as the Convergence scalar. In FLRW background, we have $R_c = \tilde{R}$. Thus avoidance of singularity may be guaranteed by avoiding the focusing of geodesics or by disregarding the CC.

In the present context, $Y = a(t),\ n = 4,\ \sigma^2 = \omega^2 = \nabla_b A^b = 0,\ \tau = t$. The RE in real linear harmonic oscillator form can be written as:

$$\ddot{a} = -\frac{1}{3}\,\tilde{R}\,a. \tag{5.12}$$

We now consider an oscillatory bouncing cosmological model [244] characterized by $a(t) = A\ \sin^2\left(\frac{C\,t}{t_*}\right)$, $H = \frac{2C}{t_*}\cot\left(\frac{C\,t}{t_*}\right)$ where $A,\ C > 0$, t_* is some reference time. For the sake of convenience we choose $t_* > 0$. This model represents the behavior of a cyclic universe, which treats the universe as a continuous sequence of contraction and expansion. The model is shown graphically in Fig 5.8. For this bounce, the RE (5.12) gives the expression of the Raychaudhuri scalar $\tilde{R}$ as:

$$\tilde{R} = -6\left(\frac{C}{t_*}\right)^2 \frac{\cos\left(\frac{2Ct}{t_*}\right)}{\sin^2\left(\frac{Ct}{t_*}\right)}. \tag{5.13}$$

The bounce occurs at $t = 0$ and $\tilde{R}$ is singular at the bouncing epoch. However the plot of $\tilde{R}$ vs t shows that in very small neighborhood of the bouncing point singularity is avoided since $\tilde{R} \leq 0$ avoids the focusing of geodesics over there (see Fig 5.9). This type of bounce avoids the geodesic focusing in a small neighborhood of the bouncing point as evident from the RE. However at the bouncing point a vanishes and H becomes singular. This singularity is experienced throughout each cycle when the scale factor goes to zero. The first bounce occurs ar $t = \frac{n\pi t_*}{C}$ for n, an integer and this corresponds to a Big Crunch singularity. This can be resolved by constructing a non zero scale factor through other mechanisms. The second bounce occurs when the universe reaches its maximal size at $t = \frac{(2n+1)\pi t_*}{2C}$ for an integer n leading to a cosmological turnaround. This represents the instance when the universe stops expanding and starts to contract towards the Big Crunch singularity.

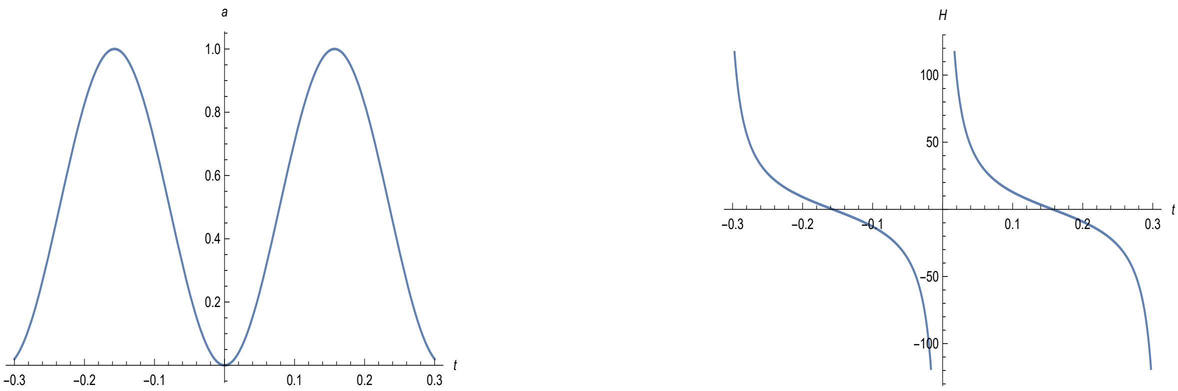


Figure 5.8: Scale factor $a(t)$ vs t (left) and Hubble parameter H vs t (right) representing an oscillatory bounce.

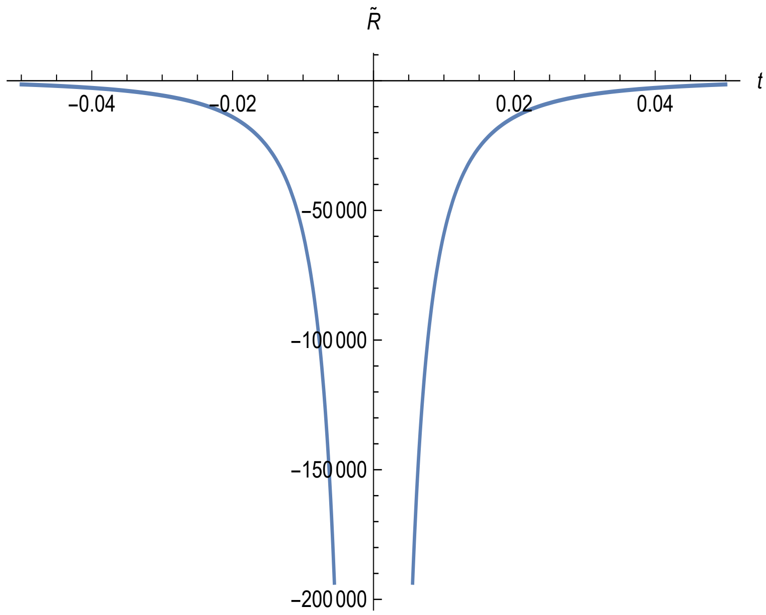


Figure 5.9: Variation of $\tilde{R}$ with cosmic time t for the oscillatory bouncing cosmological model.

5.5 Bouncing Point : A point of Inflection/ Cusp/ Corner

In this section we examine whether the bouncing point in these models namely **B1** and **B2** is a point of inflection or a point of different nature and discuss its consequences wherever applicable/ possible.

B1: Let $t = t_b$ be the bouncing epoch. In this model from the graph of H vs t (see Fig 5.2 and 5.3), it is clear that the curve $H(t)$ changes its concavity at $t = t_b$. Hence from the definition of point of inflection we can say that the bouncing epoch is a point of inflection of $H(t)$ in some cases of **B1**, namely matter bounce and Type I-IV (past/future) singularities and little rip cosmologies. Using the necessary and sufficient condition for a function to have a point of inflection we have the following:

1. $\ddot{H}(t) = 0$ at $t = t_b$.
2. $\ddot{H}(t) < 0$ for $t < t_b$ (if H is concave downwards in $t < t_b$) or in other words $\dot{H}$ is decreasing in $t < t_b$.
3. $\ddot{H}(t) > 0$ for $t > t_b$ (if H is concave upwards in $t > t_b$) or in other words $\dot{H}$ is increasing in $t > t_b$.

The above findings help us to draw the graph of $\dot{H}$ vs t in **B1** model. For example the above analysis holds good in case of Type I-IV (past/future) singularities and little rip cosmologies (see Fig 5.3). The continuity of $\dot{H}$ can be written as: $\dot{H}$ is decreasing before bounce, attains a zero value at the bouncing point (follows from the RE using $\rho = 0$ at bouncing point) and then increases after the bounce. The graph for $\dot{H}$ vs t in case of **B1** has been shown in Fig . It further shows that $\dot{H} > 0$ in the neighborhood of bounce which is consistent with our analysis of **B1** in Section II. The bouncing point is a rising point of inflection for the curve $H = H(t)$ in some cases of **B1**. Similar analysis can be performed in case of **B2**.

To sum up, the advantage of getting the bouncing point as a point of inflection for the models lies in the information regarding the continuity of $\dot{H}$ and value of $\ddot{H}$ at the bouncing point. Further from the graph of $\dot{H}$ vs t in **B1** and **B2** one can draw the graph of $\rho + p$ vs t. This further helps to distinguish the models from the point of view of cosmology.
In Bounce 1.3 we have two distinct directions of the tangent. So H is singular at the bouncing point. Since the curve $a(t)$ changes its concavity across the bouncing point in this case bouncing epoch is a point of inflection w.r.t the curve $a(t)$. But no further analysis can be done due to the singular nature of the Hubble parameter H and hence higher order cosmographic parameters like j, s etc.

Some cases where the bouncing point is not a point of inflection:

1. In example 1 i.e, in symmetric bounce the Hubble parameter turns out to be linear. Hence the bouncing point is not a point of inflection in this case. Further

in example 4 the variation of Hubble parameter is concave upward throughout. Therefore in this case also the bouncing point fails to be a point of inflection.

2. In case of Super bounce 1.1, the bouncing point is a cusp. It is an infinitely sharp corner. The vertical line at the bouncing point is the tangent at the bouncing point. On one side derivative is $+\infty$ and on the other side the derivative is $-\infty$. So H does not exist at the bouncing epoch as clear from Fig 6.

3. For the bounce 1.2, the bouncing point is a corner point. Both left and right hand derivatives exist but they are not equal. There are two distinct tangents to the two branches at the bouncing point.

Both types 1 and 2 have first order singularities at the bouncing point.

Finally, though the present work is a theoretical (more specifically mathematical) study of bouncing scenario, still it is worthy to mention some areas regarding whether a bounce can actually occur in practice or what are the challenges faced by these models to be practically relevant. In the present chapter, we have considered three general type of bouncing cosmological models namely, **B1**, **B2** and Oscillatory bouncing cosmological model. Subsequently, we have implemented the Raychaudhuri equation (RE) in these models to deduce the criteria for occurrence of such bounce and discuss the geometry of bouncing point as regular or singular point. Based on the theoretical formulation of the models we have:

1. The theoretical implications of **B1** demands to break a series of singularity theorems by Hawking and Penrose which uses RE as a key ingredient. Such an issue is accompanied by violation of NEC since we restricted our study within the framework of GR. Thus construction of **B1** in reality without theoretical pathologies is not easy as the scenario is associated with NEC violation which is again accompanied by quantum instabilities. The cosmological model involving a contraction phase (e.g **B1**) suffers from BKL (Belinsky-Khalatinov-Lifschitz) instability issue. Also the examples of **B1** are non-singular bouncing model. This means they eradicate the singularity by constructing a universe that begins with a contracting phase and then bounces back to an expanding phase. After years of continuous efforts, it is proposed that an effective field theoretic description combining the benefits of matter bounce and Ekpyrotic scenarios can give rise to a non singular cosmological model without pathologies through a Galileon-like Lagrangian.

2. For RE in **B2** there are two possibilities. In the first case, occurrence of bounce is realistic with usual matter. For the second case bounce occurs with exotic matter

3. The third bouncing model is Oscillatory bounce. If the universe did experience a bounce, this may necessitate another bounce in the future. It may be possible that it is of cyclical nature. Motivated by this, we consider bounce model embedded in cyclic theories of the universe in which bounce occurs at regular

intervals (oscillatory bounce). These theories do not just describe the early evolution phase of the universe, but its entire history. Consequently, recent stages such as the dark matter and dark energy domination are naturally closely tied to bounces both past and future imposing novel qualitative and quantitative constraints that can make bouncing cosmology more powerfully predictive [260]. For example, one immediate prediction of cyclic theories is that the current dark energy dominating phase must be meta stable or slowly decaying ultimately transitioning to a state of low energy density that will initiate a contraction period. Cycling may also explain the magnitude of the dark energy density and other fundamental parameters. However, our results in Oscillatory bounce deals with the behavior of time-like geodesic congruence in the neighborhood of bouncing point using RE and FT (Focusing Theorem).

As a consequence, the phenomenologies of a non singular bounce in the very early universe could be associated with the quintom scenario as inspired by the dark energy study of late time acceleration. Although the present study is a theoretical or rather a mathematical study of bounce, yet there are some observational aspects of the models under consideration. In this context it is worthy to refer [261] and [262].

1. For example in [261], the author proposed some possible mechanisms of generating a red tilt for primordial curvature perturbations and confront its general predictions with current CMB observations. Non singular bounce that attempts to address the issue of big-bang singularity went through a series of considerable developments which led to brand new predictions of cosmological signatures, visible in many observations.

2. From the perspective of phenomenological considerations there is “matter bounce” scenario that gives rise to almost scale invariant power spectra of primordial perturbations and thus can fit to observations very well. For ref. see [261]

3. In the paper [262], it was found that the Big Bounce predictions do not conflict with the observational data rather they agree with it.

5.6 Conclusion

This chapter has examined bouncing scenario or particularly the bouncing point both geometrically and physically. General prescription of bouncing cosmologies along with examples of some peculiar bounces have been dealt with in more details. It has been found that the bouncing point may be (i) a point of local maximum/minimum for the scale factor, (ii) a cusp in case of Super bounce, (iii) an oscillatory bouncing point, (iv) a point of inflection (w.r.t either the curve of scale factor or the curve of Hubble parameter) or (v) a corner point. Subsequently Raychaudhuri equation has been implemented to analyse the bouncing scenario and to dictate the favourable conditions for different types of bounce to occur. Besides, the role of curvature in bouncing has been investigated and it has been found that positive curvature favors the bouncing scenario.

Further it has been shown that by a suitable transformation of variable it is possible to identify the Raychaudhuri equation with the evolution equation for a classical linear Harmonic oscillator and it is found that the convergence condition can be restated in terms of the frequency of the oscillator. Further in this context, behavior of a congruence of time-like geodesics near the bouncing point (precisely in the deleted neighbourhood of the bouncing point) of an oscillatory bouncing model has been investigated using the Focusing Theorem.

Moreover the Raychaudhuri equation in case of **B1** hints the existence of second order singularity at the bouncing point hence posing problem in defining higher order cosmographic parameters like j, s etc. Therefore to define them two different equation of states have been considered namely the van der Waals EoS and Polytropic EoS as an alternative to barotropic EoS and some probable conditions have been determined to define the cosmographic parameters at least upto q. However Raychaudhuri equation in **B2** shows that focusing alone does not imply the formation of singularity or in other words for convergence condition there may be bouncing scenario without the existence of singularity. On the other hand, bouncing scenario is also possible for violation of convergence condition with exotic matter.

Thus extensive analysis of the Raychaudhuri equation in various bouncing models shows that the cosmological singularity (i.e, big-bang singularity) and bouncing scenario are two independent notions in cosmology and one does not imply the other generally. Behavior (continuity) of energy density and pressure for the bouncing models have also been studied using the Raychaudhuri equation. Moreover, the advantage of getting the bouncing point as a point of inflection over being cusp or corner lies in the information regarding existence and continuity of the higher order derivatives of H in the neighbourhood of bouncing point including the bouncing point itself. Finally some observational aspects and challenges faced by different bouncing models to become practically relevant have been discussed.

CHAPTER 6

ROLE OF ANISOTROPY IN SINGULARITY ANALYSIS USING RAYCHAUDHURI EQUATION IN KANTOWSKI-SACHS SPACE-TIME

6.1 Prelude

RE and modified CC have been studied in $f(R)$ gravity with inhomogeneous background [263] (Chapter 2), $f(T)$ gravity in homogeneous background [264] (Chapter 4), scalar tensor theory [265] etc. and certain physical conditions have been determined for the possible avoidance of singularity. This shows that existence of singularity is not a generic one, it depends on the gravity theory and the background space-time under consideration. There lies the importance of RE in modified gravity theories in homogeneous and isotropic background. The common link in these works is that all have isotropic background. However, it is to be noted that anisotropy plays a very important role in early stages of evolution of the universe and hence the study of spatially homogeneous but anisotropic cosmological models is physically as well as cosmologically significant. Observations state that the Universe is homogeneous and isotropic when the inflationary phase was successfully produced (see [266], [267] for details of inflation). However, CMBR anomalies [268] concluded that there was an anisotropic phase in the early Universe which does not make it exactly uniform. There lies the motivation of writing the present chapter, where we consider anisotropic Universe described by Kantowski-Sachs (KS) space-time model [269] and explore the effect/role of anisotropy in RE and CC.

The exact solutions for homogeneous space-times in GR belongs to either Kantowski-Sachs model or the Bianchi Types. KS is the only anisotropic but spatially homogeneous cosmology that does not fall under the Bianchi classification [270], [271]. These models gained popularity with the publication by Kantowski and Sachs [272]. KS space-time has some exciting features. Firstly, its classical and quantum solutions are well known in different contexts [273],[274]. Secondly, these models exhibit spherical and transnational symmetry and can be treated as non empty analogs of a part of the extended Schwarzschild metric [275]. Moreover these models help to study the behavior of the added degrees of freedom in quantum cosmological models. Finally, it may be possible that constructing a KS quantum cosmological model may suggest modifications and adaptions in the quantization methods applied to cosmology. In the present chapter, anisotropy described by KS model has been analyzed as geometric and physical property of matter. Further which property of anisotropy favors or avoids formation of singularity has also been discussed.

The layout of this chapter is as follows: Section 6.2 deals with the derivation of RE in Kantowski-Sachs model. CC has been analyzed in section 6.3 for Kantowski-Sachs space-time model. In section 6.4, existence and possible resolution of singularity has been attempted. Section 6.5 brings out a dual nature of anisotropy using harmonic oscillator approach. Finally, the chapter ends with some concluding remarks in section 6.6.

6.2 Raychaudhuri equation in Kantowski-Sachs model

With an aim to formulate the RE in anisotropic background and to find the effect of anisotropy in CC we consider a general metric for an homogeneous and anisotropic space-time with spatial section topology $\mathbf{R} \times \mathbf{S^2}$. This is the Kantowski-Sachs (KS) space-time described by the metric [276]

$$ds^2 = -dt^2 + a^2(t)dr^2 + b^2(t)(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.1}$$

where $a(t)$ and $b(t)$ are two arbitrary and independent functions of cosmic time t. The generic form of the energy-momentum tensor in support of this geometry is given by

$$T^{\mu}_{\nu} = diag\ (-\rho, p_r, p_t, p_t) \tag{6.2}$$

where ρ is the energy density of the physical fluid, p_r is the radial and p_t is the lateral pressure of the physical fluid. The Einstein's field equations defining the above metric (6.1) and energy-momentum source (6.2) can be written as [277], [278]

$$\frac{\dot{b}^2}{b^2} + 2\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{b}}{b}\right) + \frac{1}{b^2} = \kappa\rho \tag{6.3}$$

$$2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2 + \frac{1}{b^2} = -\kappa p_r \tag{6.4}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{b}}{b}\right) = -\kappa p_t \tag{6.5}$$

where $\kappa = 8\pi G$ is the four dimensional gravitational coupling constant and in units $8\pi G = 1$. Further $p_r = \omega_r\rho$, $p_t = \omega_t\rho$ and $\omega_r \neq \omega_t$ i.e, for the sake of generality we consider distinct EoS for the radial and lateral pressures p_r and p_t respectively. Thus we introduce metric (6.2) and corresponding field equations (6.3)-(6.5) in four dimensions. The average Hubble parameter H in this case is given by

$$H = \frac{1}{3}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right) = \frac{1}{3}\left(H_a + 2H_b\right) \tag{6.6}$$

The expansion scalar and anisotropy scalar are given by

$$\Theta = \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)$$
$$\sigma = \frac{1}{\sqrt{3}}\left(\frac{\dot{b}}{b} - \frac{\dot{a}}{a}\right) = \frac{1}{\sqrt{3}}(H_b - H_a) \tag{6.7}$$

where $H_a = \dfrac{\dot{a}}{a}$ and $H_b = \dfrac{\dot{b}}{b}$. As the present model is Kantowski-Sachs, the metric (6.1) is locally rotationally symmetric (LRS) and it belongs to the LRS Class II having rotation tensor $\omega_{ab} = 0$ [279], [280]. Thus the only non vanishing kinematic quantities are Θ and σ. Considering the following transformations:

$$V^3 = ab^2 \tag{6.8}$$
$$Z = \frac{b}{a} \tag{6.9}$$

Equations (6.3), (6.8) and (6.9) gives (after simplification)

$$3\frac{\dot{V}^2}{V^2} - \frac{1}{3}\frac{\dot{Z}^2}{Z^2} + \frac{1}{V^2 Z^{\frac{2}{3}}} = \kappa\rho \tag{6.10}$$

Using the definition of Θ in (6.7) and (6.8) one gets

$$\Theta = 3\frac{\dot{V}}{V} \tag{6.11}$$

and using definition of σ in (6.7) and (6.9) one obtains

$$\sigma = \frac{1}{\sqrt{3}}\frac{\dot{Z}}{Z} \tag{6.12}$$

From equations (6.10), (6.11) and (6.12) one has

$$\frac{\Theta^2}{3} - \sigma^2 + \frac{1}{V^2 Z^{\frac{2}{3}}} = \kappa\rho \tag{6.13}$$

The measure of acceleration corresponding to the two scale factors $a(t)$ and $b(t)$ in terms of the transformed variables V and Z are

$$\frac{\ddot{b}}{b} = \frac{\ddot{V}}{V} - \frac{2}{9}\frac{\dot{Z}^2}{Z^2} + \frac{1}{3}\frac{\ddot{Z}}{Z} + \frac{2}{3}\left(\frac{\dot{V}}{V}\right)\left(\frac{\dot{Z}}{Z}\right) \tag{6.14}$$

$$\frac{\ddot{a}}{a} = \frac{\ddot{V}}{V} + \frac{10}{9}\frac{\dot{Z}^2}{Z^2} - \frac{2}{3}\frac{\ddot{Z}}{Z} - \frac{4}{3}\left(\frac{\dot{V}}{V}\right)\left(\frac{\dot{Z}}{Z}\right) \tag{6.15}$$

Therefore the field equations (6.4) and (6.5) in terms of the transformed variables can be rewritten as

$$2\frac{\ddot{V}}{V}+\frac{2}{3}\frac{\ddot{Z}}{Z}+\frac{\dot{V}^2}{V^2}+2\frac{\dot{V}}{V}\frac{\dot{Z}}{Z}-\frac{1}{3}\frac{\dot{Z}^2}{Z^2}+\frac{1}{V^2Z^{\frac{2}{3}}}=-\kappa p_r \tag{6.16}$$

$$2\frac{\ddot{V}}{V}-\frac{1}{3}\frac{\ddot{Z}}{Z}+\frac{\dot{V}^2}{V^2}-\frac{\dot{V}}{V}\frac{\dot{Z}}{Z}+\frac{2}{3}\frac{\dot{Z}^2}{Z^2}=-\kappa p_t \tag{6.17}$$

Also we have,

$$\frac{d\Theta}{d\tau}+\frac{\Theta^2}{3}=3\frac{\ddot{V}}{V} \tag{6.18}$$

and,

$$\frac{d\sigma}{d\tau}=\frac{1}{\sqrt{3}}\left(\frac{\ddot{Z}}{Z}-\frac{\dot{Z}^2}{Z^2}\right) \tag{6.19}$$

By some algebraic manipulation with the field equations we get

$$\frac{\ddot{Z}}{Z}+3\frac{\dot{V}}{V}\frac{\dot{Z}}{Z}-\frac{\dot{Z}^2}{Z^2}+\frac{1}{V^2Z^{\frac{2}{3}}}=\kappa(p_t-p_r) \tag{6.20}$$

and,

$$\frac{\ddot{V}}{V}+\frac{2}{9}\frac{\dot{Z}^2}{Z^2}=-\frac{\kappa}{6}(\rho+p_r+2p_t) \tag{6.21}$$

Thus the evolution equations for the expansion scalar (Θ) and shear scalar σ are given by

$$\frac{d\Theta}{d\tau}+\frac{\Theta^2}{3}+2\sigma^2=-\frac{\kappa}{2}(\rho+p_r+2p_t) \tag{6.22}$$

and,

$$\frac{d\sigma}{d\tau}+\Theta\sigma+\frac{\sigma^2}{\sqrt{3}}-\frac{\Theta^2}{3\sqrt{3}}=-\frac{\kappa}{\sqrt{3}}(\rho-\frac{1}{3}(p_t-p_r)) \tag{6.23}$$

One may note that the above equations (6.22) and (6.23) are first order, coupled and non-linear in nature. Since the present work is in the background of anisotropic space-time, we have shown the evolution of anisotropy scalar in equation (6.23). Further, it is to be noted that the rotation tensor vanishes identically in the present model due to locally rotational symmetry of the KS metric and consideration of hyper-surface orthogonal congruence of time-like geodesic which by virtue of Frobenius theorem implies a zero rotation tensor. However, the evolution of expansion scalar Θ is of central interest in the context of singularity theorems. Mathematically, the evolution equation for expansion scalar is known as Riccati equation (for ref. see a review of Raychaudhuri equations by Kar and Sengupta [156]). Historically, the Riccati equation or equation (6.22) is known as the Raychaudhuri equation.

6.3 Convergence Condition in KS space-time models

In order to study the CC in KS space-time we consider the RE in KS space-time given by equation (6.22). Let us write

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} + \frac{\Theta^2}{3} = -\tilde{R}_c \tag{6.24}$$

where

$$\tilde{R}_c = \frac{\kappa}{2}(\rho + p_r + 2p_t) + 2\sigma^2 \tag{6.25}$$

Here we consider $p_r = \omega_r \rho$ and $p_t = \omega_t \rho$, ω_r and ω_t are the radial and transverse EoS parameters. From the modified form of the RE (6.24) the CC now becomes $\tilde{R}_c \geq 0$. Thus the possibilities of focusing are listed as follows:

- $\frac{\kappa}{2}(\rho + p_r + 2p_t) \geq 0$ i.e, matter satisfies SEC. In this case $\tilde{R}_c \geq 0$ (follows from equation 6.25). Since the positive semi definiteness of $\tilde{R}_c$ leads to convergence of the bundle of geodesics therefore we name it as Convergence Scalar and we shall use this term in the following sections.
- Matter violates SEC i.e, $\frac{\kappa}{2}(\rho + p_r + 2p_t) < 0$ but $2\sigma^2 > \frac{\kappa}{2}|(\rho + p_r + 2p_t)|$.

Based on the above discussion we conclude that the anisotropy term favors the convergence/ formation of congruence singularity. However this term alone does not lead to CC. If the matter is attractive in nature i.e, $1 + \omega_r + 2\omega_t \geq 0$ then CC is automatically satisfied. However if the matter is repulsive in nature i.e, $1 + \omega_r + 2\omega_t < 0$ then anisotropy can not alone lead to CC but needs an extra condition namely $2\sigma^2 > \frac{\kappa}{2}|(\rho + p_r + 2p_t)|$. Thus in comparison to FLRW model the anisotropy term may be interpreted as a matter part which is attractive in nature. This interpretation can be justified from Einstein's idea of gravity where geometry and matter are equivalent quantities. Thus to avoid formation of singularity in KS space-time we need $\frac{\kappa}{2}(\rho + p_r + 2p_t) < 0$ and $2\sigma^2 < \frac{\kappa}{2}|(\rho + p_r + 2p_t)|$. In other words, to avoid focusing matter cannot be usual in nature. Further violation of SEC must be dominant over the anisotropy term in order to avoid singularity formation. Hence with usual matter singularity is inevitable in a general KS model.

Thus the general KS space-time classically has a past and a future singularity, which can be an anisotropic structure such as a barrel, cigar, a pancake or an isotropic point like structure depending on the initial conditions on anisotropic shear and matter [281]. Evolution of geodesics terminates at these classical singularities which is identified by the divergence of expansion and shear scalars in the presence of matter (which contributes to the energy density). Existence of singularities pushes GR to the limits of its validity and hence a quantum gravitational treatment which becomes dominant in strong gravity regimes may alleviate the classical singularity.

6.4 Analysis of singularity in a constrained KS model

In order to examine the existence and possible avoidance of singularity we shall study the sign of the Convergence Scalar R_c in this section. We know that $R_c \geq 0$ ensures convergence i.e, it is the condition for focusing. Also it is known that if a space time has a singularity, it means that a bundle of geodesic will tend to focus at the singularity. Thus if we can avoid the focusing of geodesic by making $R_c < 0$, we can avoid the formation of a singularity in a classical space-time. This is because if there were a singularity, focusing would have inevitably happened there. In this section we find the physical conditions under which R_c can be made negative. For this we consider the anisotropic relation i.e, the physical condition that the expansion scalar $\Theta(t)$ is proportional to the shear scalar $\sigma(t)$ i.e, $\Theta \propto \sigma$ [272], [282], [283]. This results in the relation between the scale factors as

$$a = b^m \tag{6.26}$$

where m is an arbitrary real number and $m \neq 0, 1$ to ensure non triviality and anisotropy. Thorne [284] justified this physical law based on the observations of the velocity Redshift relation for extra-galactic sources which suggest that the Hubble expansion of the Universe is isotropic at present time with 30% [272]. More precisely, the Redshift studies put the limit $\frac{\sigma}{\Theta} \leq 0.3$, the ratio of the shear to expansion scalar in the vicinity of our galaxy at present time. Further it was pointed out by Collins et al. [275] that the normal congruence to the homogeneous expansion for spatially homogeneous metric satisfies the condition $\frac{\sigma}{\Theta}$ = constant. Bunn et al. [285] did statistical analysis on 4-yr CMB data and set a limit for primordial anisotropy to be less than 10^{-3} in Planck epoch. In Literature many researchers have thus used this condition motivated by the above physical implications while dealing with KS space-time models. This is the physical motivation behind choosing this constraint. Now using equations (6.7) and (6.26) we have

$$2\sigma^2 = \frac{2}{3}(1-m)^2\left(\frac{\dot{b}}{b}\right)^2 \tag{6.27}$$

From equation (6.27) and (6.25) the expression for $\tilde{R}_c$ is given by

$$\tilde{R}_c = \frac{\kappa}{2}(\rho+p_r+2p_t)+\frac{2}{3}(1-m)^2\left(\frac{\dot{b}}{b}\right)^2 = \frac{\kappa\rho}{2}(1+\omega_r+2\omega_t)+\frac{2}{3}(1-m)^2\left(\frac{\dot{b}}{b}\right)^2 \tag{6.28}$$

Let us assume,

$$\omega_r = \alpha\,\omega_t \tag{6.29}$$

where $\alpha \neq 1$ to maintain anisotropy. Finally by doing some algebraic manipulation with the field equations (6.3)-(6.5) (to eliminate $\dot{b}$ and b) and using equations (6.26), (6.28), expression for $\tilde{R}_c$ takes the form

$$\tilde{R}_c = \kappa\rho R_c \tag{6.30}$$

where

$$R_c = \left[\left(\alpha + \frac{1}{2}\right) + \frac{(1-m)^2}{3m}\left(1 - \frac{2\alpha}{m+1}\right)\right]\omega_r + \frac{2(1-m)^2}{3m(m+1)} + \frac{1}{2} + \frac{(1-m)^3}{3m(m+1)} \quad (6.31)$$

Thus the sign of $\tilde{R}_c$ depends upon the sign of R_c provided $\kappa\rho > 0$. We assume that $\kappa\rho > 0$ and plot R_c w.r.t ω_r and m (the power appears in equation (6.26)) in the following FIG.6.1. We consider $\omega_r \in (-1, 1)$ and $m \in (-10, -5)$ or $m \in (2, 10)$ and see the variation of R_c for different values of α. Based on the graphs the range/ values of α, m and ω_r are represented in a tabular form in tables (6.1) and (6.2) for which $R_c < 0$.

We observed that focusing theorem (1.121) follows from the evolution equation for expansion (6.22) by assuming SEC on matter. Since in the previous section we found that shear is in favor of focusing, this motivates us to deduce the mathematical statement of focusing theorem or whether it actually holds in terms of σ. However the general evolution equation for shear (6.23) is highly non-linear and coupled posing difficulties to fulfill the aim. Therefore we attempt to see whether the focusing theorem holds following the evolution of shear in the physically motivated constrained KS model (6.26). For this we need the explicit relation between Θ and σ. It is to be noted that the relation between Θ and σ for the choice in equation (6.26) can be explicitly written as

$$\Theta = \pm 3\sqrt{3}\left(\frac{m+2}{m-1}\right)\sigma \quad (6.32)$$

The above relation follows from equations (6.8), (6.9), (6.11), (6.12) and (6.26). Hence, the evolution equation for shear follows from eq.(6.22) and eq.(6.32) as

$$\frac{d\sigma}{d\tau} + \beta\sigma^2 = -\left(\frac{(m-1)}{3\sqrt{3}(m+2)}\right)\frac{\kappa}{2}(\rho + p_r + 2p_t) \quad (6.33)$$

where

$$\beta = \frac{\left(\left(\frac{m+2}{m-1}\right)^2 + 2\right)(m-1)}{3\sqrt{3}(m+2)} \quad (6.34)$$

if $\Theta = +3\sqrt{3}\left(\frac{m+2}{m-1}\right)\sigma$. Now considering expanding model of universe i.e, $m > 1$ and matter, usual in nature which satisfies the SEC ($\frac{\kappa}{2}(\rho + p_r + 2p_t) \geq 0$) then eq.(6.33) gives rise to the inequality

$$\frac{1}{\sigma} \geq \beta\tau + \frac{1}{\sigma_0} \quad (6.35)$$

where σ_0 is the constant of integration and physically $\sigma_0 = \sigma(\tau = 0)$. We do not interpret this result physically since the choice $\Theta = +3\sqrt{3}\left(\frac{m+2}{m-1}\right)\sigma$ is not physically feasible for $m > 1$. This is because $\Theta \to -\infty$ implies convergence which will make $\sigma \to -\infty$. However anisotropy is in favor of convergence which necessitates $\sigma \to +\infty$

for convergence. On the other hand if we consider the physically feasible choice $\Theta = -3\sqrt{3}\left(\frac{m+2}{m-1}\right)\sigma$, then we have

$$\frac{d\sigma}{d\tau} - \beta\sigma^2 = \left(\frac{(m-1)}{3\sqrt{3}(m+2)}\right)\frac{\kappa}{2}(\rho + p_r + 2p_t) \tag{6.36}$$

Assuming an expanding universe with matter satisfying SEC we arrive at the inequality

$$\frac{1}{\sigma} \leq -\beta\tau + \frac{1}{\sigma_0} \tag{6.37}$$

Recall the definition of Θ which is the rate of change of the cross-sectional area orthogonal to the bundle of geodesics. Thus $\Theta \to +\infty$ implies a divergence of the bundle while $\Theta \to -\infty$ implies a complete convergence. Also, for convergence there must be a negative expansion initially. Therefore in this particular case with $\Theta = -3\sqrt{3}\left(\frac{m+2}{m-1}\right)\sigma$, as $\Theta_0 < 0$ we have $\sigma_0 > 0$ for expanding model with $m > 1$. Inequality (6.37) shows that an initially converging bundle of geodesic will diverge after some finite time τ. Therefore in this constrained KS model with $\Theta = -3\sqrt{3}\left(\frac{m+2}{m-1}\right)\sigma$, although there is an initial singularity but in the course of evolution of the universe there may not be any future singularity in the sense that focusing does not happen and it is reflected in the inequality (6.37). However the possible resolution of the initial singularity is attempted via the signature of R_c in FIG. (6.1).

$\alpha = 1.1$	$\omega_r \in [-1, -0.5]$	$m \in [2, 10]$
$\alpha = 1.1$	ω_r close to $+1$	higher negative power of m ($m \to -10$)
$\alpha = 0.1$	ω_r close to -1	$m \in [2, 10]$
$\alpha = 0.1$	ω_r close to $+1$	$m \in [-10, -5]$
$\alpha = 1.5$	$\omega_r \in [-1, -0.5]$	$m \in [2, 10]$
$\alpha = 1.5$	ω_r close to $+1$	$m \in [-10, -5]$
$\alpha = 2.0$	ω_r close to -1	$m \in [2, 10]$
$\alpha = 2.0$	ω_r close to $+1$	$m \in [-10, -5]$
$\alpha = 0.9$	$\omega_r \in [-1, -0.5]$	$m \in [2, 10]$
$\alpha = 0.9$	ω_r close to $+1$	$m \in [-10, -5]$

Table 6.1: Table showing the positive values of α and corresponding range of ω_r and m which make $R_c < 0$

$\alpha = -1$	ω_r close to -1	higher positive power of m ($m \to 10$)
$\alpha = -1$	ω_r close to $+1$	higher negative power of m ($m \to -10$)
$\alpha = -1.5$	$\omega_r \in [-1, -0.5]$	higher positive power of m ($m \to 10$)
$\alpha = -1.5$	$\omega_r \in [0.5, 1]$	higher negative power of m ($m \to -10$)
$\alpha = -2$	$\omega_r \in [-1, -0.5]$	higher positive power of m ($m \to 10$)
$\alpha = -2$	$\omega_r \in [0.5, 1]$	higher negative power of m ($m \to -10$)

Table 6.2: Table showing the negative values of α and corresponding range of ω_r and m which make $R_c < 0$

The two tables Table.6.1 and Table.6.2 cover up all possibilities which make $R_c < 0$ as depicted in graphs. From Table.6.1, we find that when $\alpha = 0.9/1.1/1.5$ (closer to $+1$) i.e, a little effect of anisotropy and positive value of m (which accounts for expanding model since $a = b^m$, $m > 0$) is taken into consideration then both cosmological constant having EoS parameter $= -1$ and fluid having EoS parameter $-1 \leq \omega_r < -\frac{1}{3}$ may avoid singularity. Such type of fluids are non phantom in nature. Also according to the observations, the expansion of the universe is accelerating (both Θ and $\dot{\Theta} > 0$) for any EoS$< -\frac{1}{3} \in [-1, -0.5]$ and this does not allow geodesics to focus by ensuring $R_c < 0$. Thus our results are at par with the observations with little effect of anisotropy which may be considered as comparable to isotropic universe. On the other hand, if we substantially increase the effect of anisotropy by choosing $\alpha = 0.1/2.0$ (far from $+1$) then avoidance of singularity is guaranteed for an expanding model ($m > 0$) if and only if the EoS approaches to that of Cosmological Constant. However, if we consider $m < 0$ i.e, as $b(t)$ increases $a(t)$ will decrease then whatever be the effect of anisotropy a stiff fluid (having EoS parameter close to $+1$) is always able to avoid the singularity formation.

From Table.6.2, we find that both negative values of α and m (higher negative powers) may avoid singularity formation provided the fluid is a stiff fluid or a fluid that has a positive EoS $\in [0.5, 1]$. On the other hand whatever be the value of $\alpha < 0$, either cosmological constant or fluid having EoS $\in [-1, -\frac{1}{3}]$ may avoid the singularity formation in an expanding universe (provided m is highly positive).

6.5 Role of Anisotropy in convergence: A Harmonic Oscillator approach

In this section we aim to show how the RE which is a first order non linear differential equation can be converted to a second order Hill-Type equation or differential equation for a Harmonic Oscillator. For this, we eliminate $2\sigma^2$ between equations (6.13) and (6.22) so that one has

$$\frac{d\Theta}{d\tau} + \Theta^2 + \frac{2}{V^2 Z^{\frac{2}{3}}} = \frac{\kappa\rho}{2}(5 + \omega_r + 2\omega_t) \tag{6.38}$$

Now we consider a transformation

$$\Theta = \frac{d \ln Y}{d\tau} = \frac{1}{Y}\frac{dY}{d\tau} \tag{6.39}$$

Under this transformation equation (6.38) can be written as

$$\frac{d^2Y}{d\tau^2} + \left(\frac{2}{V^2 Z^{\frac{2}{3}}} - \frac{\kappa\rho}{2}(5 + \omega_r + 2\omega_t)\right) Y = 0 \tag{6.40}$$

Thus, equation (6.40) can be identified as a Hill-Type equation or the differential equation for a harmonic oscillator in the transformed variable Y with time varying frequency ω_0 where

$$\omega_0^2 = \left(\frac{2}{V^2 Z^{\frac{2}{3}}} - \frac{\kappa\rho}{2}(5 + \omega_r + 2\omega_t)\right) \geq 0 \tag{6.41}$$

Now we show how equation (6.40) can be used to deduce the criterion for convergence. To do so, we recall the physical meaning of Θ (by Kar and SenGupta [156]). Θ is nothing but the rate of change of the cross-sectional area orthogonal to the bundle of geodesics. Thus $\Theta \to -\infty$ implies a convergence of the bundle while $\Theta \to +\infty$ implies a complete divergence. For convergence, there must be a negative expansion initially. Finally with $\dot{Y} < 0$ we should end up at a root of Y at some finite time say, τ to have a negatively infinite expansion. Thus the requirement for convergence reduces to the criterion for the existence of roots of Y at some finite value of τ. This can be linked with the famous Strum-Comparison theorem of second order differential equations which necessitates $\left(\frac{2}{V^2 Z^{\frac{2}{3}}} - \frac{\kappa\rho}{2}(5 + \omega_r + 2\omega_t)\right) \geq 0$ to be the condition for convergence or simply the Convergence Condition (CC).

Using equations (6.11) and (6.12) in equation (6.32) we have

$$\frac{\dot{V}}{V} = \pm\left(\frac{m+2}{3(m-1)}\right)\frac{\dot{Z}}{Z} \tag{6.42}$$

Solving it we get

$$V = V_0 Z^{\pm\frac{(m+2)}{3(m-1)}} \tag{6.43}$$

with V_0, a constant of integration. Thus, the first term on the r.h.s of equation (6.41) can be written as

$$\frac{2}{V^2 Z^{\frac{2}{3}}} = \frac{2}{V_0 Z^{\frac{2}{3}\left(\frac{m+2}{m-1}\right)}} \tag{6.44}$$

if positive sign is considered on the r.h.s of equation (6.32) and,

$$\frac{2}{V^2 Z^{\frac{2}{3}}} = \frac{2}{V_0 Z^{\frac{2}{1-m}}} \tag{6.45}$$

if the corresponding sign is negative. It is to be noted from equations (6.12), (6.44) and (6.45) that the first term on the r.h.s of equation (6.41) is indicative of shear/

anisotropy. On the other hand if we look at the second term on the r.h.s of equation (6.41) anisotropy occurs for $\omega_r \neq \omega_t$ which is true according to our assumption in equation (6.29) . Further if we look at the CC, we find that anisotropy represented by the term $\frac{2}{V^2 Z^{\frac{2}{3}}}$ is in favor of convergence while anisotropy represented by the term $\frac{\kappa\rho}{2}(5+\omega_r+2\omega_t)$ is against convergence if the matter satisfies SEC $(\rho(1+\omega_r+2\omega_t) \geq 0)$ i.e, in case of usual matter. Based on the above discussion, we can interpret these two types of anisotropy as geometric and physical anisotropy so that the former assists convergence while the later opposes it provided, matter is attractive in nature. Thus the above analysis shows the role of anisotropy in convergence explicitly.

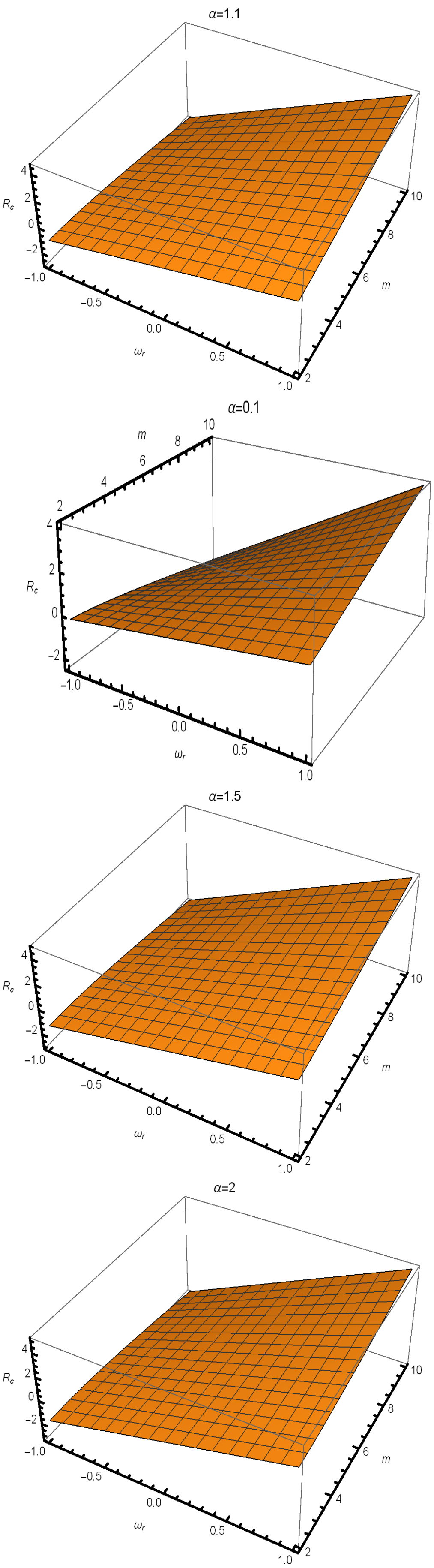
6.6 Conclusion

The present chapter vividly describes the formulation of Raychaudhuri equation (evolution equation for the expansion scalar) in the background of homogeneous and anisotropic Kantowski Sachs (KS) space-time model. An extensive analysis of the kinematic quantities that appear in the RE has been carried out purely from geometric point of view. The evolution of shear or anisotropy has been found to be a first order non linear coupled differential equation in a general KS metric. Although Einstein's GR is the most successful theory of gravity to describe physical reality, yet the Focusing theorem which follows as a consequence of RE is the turning point in GR. This is because it hints the inevitable existence of singularity in a classical space-time. Focusing or simply the convergence of a bundle of geodesic requires a Convergence condition (CC) to happen. Thus in the present work we attempted to study the role of anisotropy in focusing both in a general KS model as well as in a physically motivated constrained KS model.

It has been found that a general KS space-time has an initial singularity and anisotropy assists the convergence in the presence of usual matter. However, with repulsive matter anisotropy alone can not lead to convergence but some extra conditions related to the matter part are required. Thus, in comparison to FLRW class of models, anisotropy may be considered as a matter part which is attractive in nature and hence facilitates convergence. From the evolution equation for shear it can be shown that focusing does not occur in some finite time in the future. However, the constrained KS model may avoid the initial singularity with suitable choice of the parameters involved. This leads to a conclusion that an expanding universe with either cosmological constant or non-phantom energy may avoid the formation of singularity in the presence of anisotropy. Although there are possibilities to avoid singularity with negative values of α and m mathematically, yet we avoid discussing them for their physical irrelevance.

The chapter also gives a transformation under which the RE, a first order non linear differential equation in Θ can be converted to a second order differential equation in a transformed variable say, Y. The second order differential equation so formed is analogous to the evolution equation for a harmonic oscillator. CC has been restated in

this context using the physical definition of Θ along with some initial assumptions and it is found that the CC is associated with the time varying frequency of the oscillator. Explicit expression for the frequency of the harmonic oscillator shows the dependence of anisotropy. This analysis further points out a two fold feature of anisotropy namely geometric and physical anisotropy of which the former assists convergence while the latter defies it as long as there is usual matter.
Thus the present chapter studies the consequences of RE and corresponding convergence condition via the signature of the convergence scalar in the presence of anisotropy and brings out an interesting dual behavior of anisotropy towards convergence via a Harmonic oscillator approach.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ω_r
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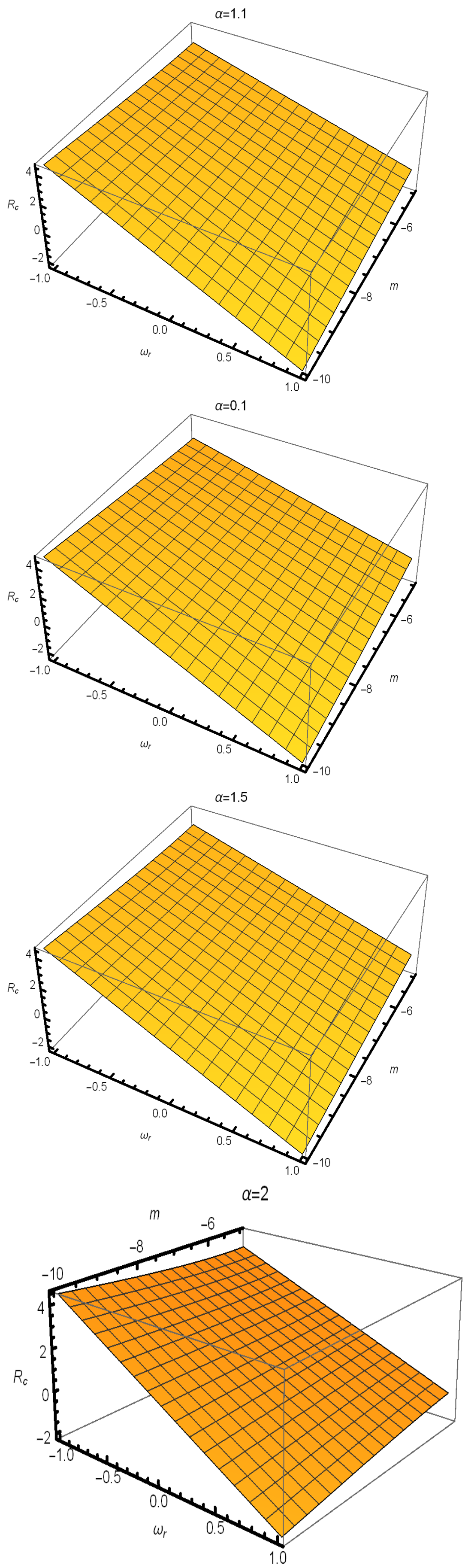
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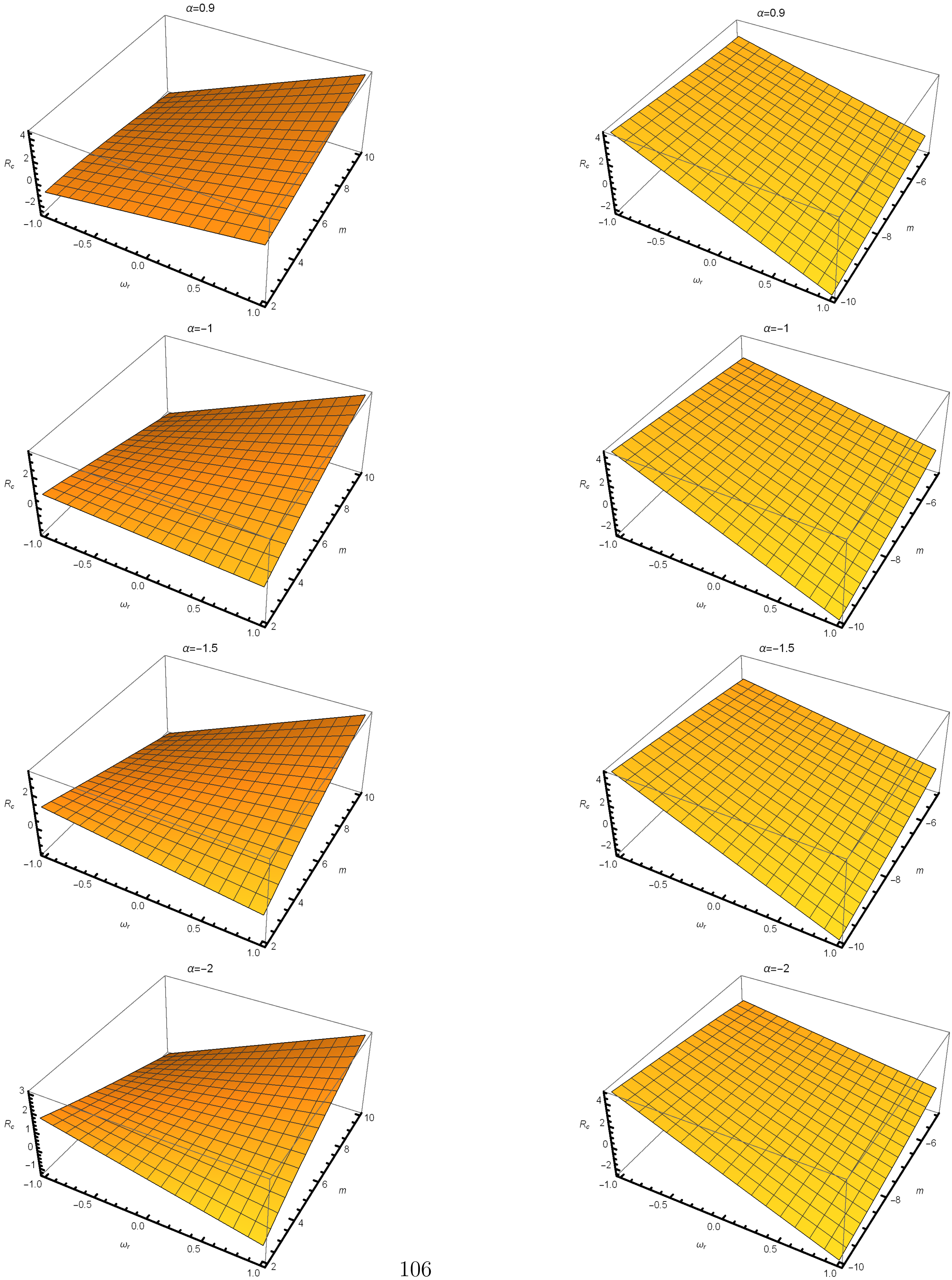

Figure 6.1: Graphs showing the variation of R_c with m and ω_r for arbitrary α (fixed after choice)

CHAPTER 7

RAYCHAUDHURI EQUATION AND DYNAMICS OF COSMIC EVOLUTION

7.1 Prelude

The phenomena of accelerated expansion of the Universe may be studied by considering congruence of geodesics in a coordinate independent way. A useful tool to analyze the evolution or behavior of geodesics is the Focusing Theorem (FT) [176] which follows as a consequence of the RE (1.83). It is a non-linear differential equation and hence it is not so easy to find out its analytic solution. In this chapter, we have adopted a technique to find the analytic solution and used this solution to describe the dynamics of cosmic evolution. The universal attractive nature of gravity in General Relativity is a feature embodied by the RE which requires the non-increasing expansion of a congruence of geodesics. How the entire cosmic evolution can be studied by studying the evolution of geodesics is shown in this chapter.

Further this chapter explores and analyzes the Focusing Theorem and how cosmic parameters affect it. Different era of cosmic evolution have been studied in the context of RE and FT. Further a three fold interpretation of the Convergence scalar also known as Raychaudhuri scalar has been given in cosmology which interestingly *converges* to the same conclusion. Subsequently, the non-linear RE has been converted to a second order differential equation in a transformed variable by a suitable transformation related to the metric scalar of the hyper-surface to obtain a first integral from it for a general $n+1$-dimensional space-time. An analogy of this first integral with the first Friedmann equation has been found in Einstein gravity. Finally, using the first integral cosmological solution corresponding to each era of cosmic evolution has been explicitly determined in the background of Friedmann–Lemaître–Robertson-Walker (FLRW) space-time model. Since, the solution of the non-linear RE using the first integral method has been applied to understand the dynamics of cosmic evolution therefore this particular treatment adopted in this chapter shows an application of non-linear dynamics to cosmology. Further, this chapter also illustrates a similar behavior of $\tilde{R}$ and q in the context of convergence via the geometric and cosmological forms of the RE. In this connection, a cosmic harmonic oscillator has been dealt with using the cosmological form of the RE.

The layout of this chapter is: Section 7.2 deals with RE and FT from the point of view of cosmology. Section 7.3 deals with the conversion of RE to a second order differential equation and formation of a first integral out of it. Further this section shows cosmological solutions at various epochs using this first integral. Finally the chapter ends with discussion of the obtained results.

7.2 Raychaudhuri equation and Focusing theorem in terms of cosmic parameters

In Einstein gravity or in usual modified gravity the field equations for gravity can be written as

$$G_{\mu\nu} = \kappa\, T'_{\mu\nu}\ , \tag{7.1}$$

where $T'_{\mu\nu} = T_{\mu\nu}$ is the usual energy-momentum tensor for the matter field in Einstein gravity while $T'_{\mu\nu} = T_{\mu\nu} + T^{(e)}_{\mu\nu}$ for most of the modified gravity theories with $T^{(e)}_{\mu\nu}$ containing the extra geometric/physical terms in the field equations. Thus the Raychaudhuri scalar $\tilde{R}$ takes the following form in terms of the energy-momentum tensor or/and the effective energy-momentum tensor in Einstein gravity (EG) and in Modified gravity (MG) as:

$$\tilde{R} = \kappa\ (T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu})u^{\mu}u^{\nu} = \frac{1}{2}\,(\rho + 3p)\,,\ EG \tag{7.2}$$

and,

$$\tilde{R} = \kappa\ \left[(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu})u^{\mu}u^{\nu} + (T^{(e)}_{\mu\nu} - \frac{1}{2}T^{(e)} g_{\mu\nu})u^{\mu}u^{\nu}\right] = \frac{1}{2}\,(\rho + 3p) + \frac{1}{2}\left(\rho^{(e)} + 3p^{(e)}\right),\ MG \tag{7.3}$$

<u>**A three fold interpretation of $\tilde{R}$ in cosmology is given below as**</u>:

1. In FLRW space-time the (effective) Einstein field equations are

$$3H^2 = \rho,\quad 2\dot{H} = -(\rho + p) \tag{7.4}$$

$$3H^2 = (\rho + \rho_e),\quad 2\dot{H} = -[(\rho + p) + (\rho_e + p_e)] \tag{7.5}$$

where equation (7.2) is for Einstein gravity and (7.3) is for modified gravity. So the deceleration parameter $q = -\left(1 + \dfrac{\dot{H}}{H^2}\right)$ takes the form (7.6) in Einstein gravity and (7.7) in Modified gravity as follows :

$$q = \frac{\rho + 3p}{2\rho} \tag{7.6}$$

$$q = \frac{(\rho + 3p) + (\rho_e + 3p_e)}{2(\rho + \rho_e)} \tag{7.7}$$

Hence

$$\tilde{R} = q\rho \text{ , } Einstein\ gravity$$
$$\tilde{R} = q(\rho + \rho_e) \text{ , } Modified\ gravity$$

Thus,

$$\tilde{R} = 3qH^2 \tag{7.8}$$

for both the cases. Now for convergence $\tilde{R} > 0$ so one may conclude that convergence will occur during the evolution of the universe if $q > 0$ i.e, CC occurs only in decelerating phase. In other words, q behaves as convergence scalar. This formulation brings out an inherent feature of the deceleration parameter as convergence scalar. This shows that formation of singularity is not possible both in the early inflationary era and in the present late time accelerated era of evolution, while the matter dominated era of evolution favors convergence.

2. In Einstein gravity expression of $\tilde{R}$ is given by equation (7.2). Let us write,

$$\rho = \rho_1 + \rho_2, \quad p = p_1 + p_2. \tag{7.9}$$

Then $\tilde{R} = \frac{1}{2}\{(\rho_1 + 3p_1) + (\rho_2 + 3p_2)\}$. If we assume (ρ_1, p_1), the energy density and pressure of normal matter component (that satisfies SEC i.e. $\rho_1 + 3p_1 \geq 0$) then in order to prevent convergence (focusing) we must have

$$\rho_2 + 3p_2 < 0, \quad |\rho_2 + 3p_2| > \rho_1 + 3p_1.. \tag{7.10}$$

This shows that the component having energy density and pressure (ρ_2, p_2) is dark energy. Hence dominance of dark energy over normal matter (having density and pressure $(\rho_1,\ p_1)$) may prevent focusing. Therefore the era dominated by dark energy namely the inflationary era and the present accelerated era of expansion are against the formation of singularity. Further if $\rho_1 + 3p_1 \geq 0$ and $|\rho_1 + 3p_1| \geq \rho_2 + 3p_2$ then $\tilde{R} \geq 0$. Thus the matter dominated era is in favor of convergence. This matches with the conclusion of the previous case.

3. Now, we consider the expression of $\tilde{R}$ in modified gravity in eq. (7.3). To make $\tilde{R} < 0$ we need

$$\rho^{(e)} + 3p^{(e)} < 0, \quad |\rho^{(e)} + 3p^{(e)}| > \rho + 3p. \tag{7.11}$$

Again it hints that $(\rho^{(e)}, p^{(e)})$ corresponds to the density and pressure of dark energy if we assume that (ρ, p) corresponds to the energy density and pressure of normal/usual matter that satisfies the SEC (i.e. $\rho + 3p \geq 0$). Thus we arrive at the same conclusion as in the former cases.

Therefore the above analysis is at par with the observation that Focusing theorem does not hold for the present accelerated era of expansion of the universe. The above three points are consistent from cosmological point of view. The first point reveals that existence of singularity is not possible whenever there is accelerated expansion. This conclusion is supported in points 2 and 3. In point 2 we have Einstein gravity with two fluid system with one matter component that behaves as dark energy and it has been found that convergence of geodesic is possible only in the matter dominated era. In point 3 we have similar conclusion but using the modified gravity theory.

We shall now examine how Raychaudhuri scalar constraints the cosmological parameters. For flat FLRW model, if the modified gravity theory is equivalent to Einstein gravity with normal fluid (satisfying the SEC) and the effective matter is assumed to be in Dark energy form (not satisfying Weak Energy Condition (WEC)) then it is possible to have some interrelation between the observationally measurable quantities and the Raychaudhuri scalar as follows:
For normal matter considered as Dark matter with constant equation of state (ω), the energy density from the conservation equation has the expression in terms of the redshift parameter z as

$$\rho = \rho_0 (1+z)^{3(1+\omega)}. \tag{7.12}$$

where ρ_0 is the energy density at present. Now the first Friedmann equation

$$H^2(z) = \frac{8\pi G}{3}(\rho + \rho_e) \tag{7.13}$$

can be written in terms of the density parameter as

$$\tilde{H}^2 = \Omega(1+z)^{3(1+\omega)} + (1-\Omega)\left(\frac{\rho_e}{\rho_{e0}}\right). \tag{7.14}$$

Here, $\Omega = \dfrac{\rho_0}{\rho_e}$ is the density parameter for the dark matter, $\rho_e = \dfrac{3H_0^2}{8\pi G}$ is the critical density and $\tilde{H} = \dfrac{H}{H_0}$. Due to WEC for the effective fluid, $\dfrac{\rho_e}{\rho_{e0}} \geq 1$ for $z > 0$, hence from equation (7.14) $\tilde{H}$ has a lower bound

$$H^2(z) \geq \Omega(1+z)^{3(1+\omega)} + (1-\Omega) \tag{7.15}$$

As $\tilde{R} = q(\rho + \rho_e) = q\rho_e \tilde{H}^2$, so the above inequality puts a restriction on $\tilde{R}$ as

$$\tilde{R} \geq q\rho_e \left[\Omega(1+z)^{3(1+\omega)} + (1-\Omega)\right] \tag{7.16}$$

Now, the luminosity distance d_L, a measurable quantity in the supernova red-shift survey is related to the coordinate distance $r(z)$ (defined as $r(z) = \int_0^z \dfrac{dz'}{H(z')}$) by the relation:

$$d_L = c(1+z)\frac{r(z)}{H_0}. \tag{7.17}$$

Thus the density parameter Ω and the above luminosity distance d_L are restricted by the inequality (7.16) as

$$\Omega \leq \left\{(1+z)^{3(1+\omega)} - 1\right\}^{-1} \min\left[\frac{1}{(H_0 r'(z))^2}, \frac{\tilde{R}}{q\rho_e}\right] - 1, \tag{7.18}$$

and

$$d_L \leq \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{[\Omega(1+z')^{3(1+\omega)} + (1-\Omega)]^{\frac{1}{2}}} \tag{7.19}$$

For a hyper-surface orthogonal congruence of time-like geodesic in isotropic background, the geometric form of the RE is given by

$$\frac{d\Theta}{d\tau} = -\frac{\Theta^2}{n} - \tilde{R} \tag{7.20}$$

Using $\Theta = 3H$ and equation (7.8) the RE in terms of cosmic parameters can be written as

$$\dot{H} = -(1+q)H^2 \tag{7.21}$$

Thus RE can be written in two ways: one in terms of geometric or kinematic variables namely equation (1.84) or in particular form as equation (7.20) and the other in terms of cosmic parameters given by equation (7.21). We call the former as geometric form of RE and the later as cosmological form of RE. In this context, one may observe a nice analogy in these two forms. In geometric form $\tilde{R}$ or the Raychaudhuri scalar plays the role in convergence or focusing while in cosmological form q, the deceleration parameter does it. Moreover, the effect of gravity theory comes through $\tilde{R}$ in the geometric form while in the cosmological form q carries the effect of gravity and this is clear from equations (7.6) and (7.7). Therefore q has a similar role as $\tilde{R}$. Further the geometric form of the RE can be converted to the evolution equation of a Harmonic Oscillator using a suitable transformation of variable. One may refer to [263] for details of Harmonic oscillator equation derived from the geometric form of RE. The work [263] gives a transformation under which the first order RE can be converted to the Harmonic oscillator equation and it can be shown that the convergence condition, avoidance of singularity everything are related to the time varying frequency of the oscillator. Following this approach, in this paper we attempt to show the Harmonic oscillator equation from the cosmological form of the RE. For this, we consider the cosmological RE given by equation (7.21). Using,

$$H = \frac{\dot{Y}}{Y} \tag{7.22}$$

the first order cosmological form of the RE can be converted to a second order differential equation analogous to the evolution equation of a classical real harmonic oscillator as

$$\ddot{Y} + qH^2 Y = 0. \tag{7.23}$$

The frequency of the real harmonic oscillator is given by $W^2 = qH^2$. This harmonic oscillator equation is very much analogous to the Harmonic oscillator presented in

[156] and [263] where it was derived purely from the geometric form of the RE with suitable transformation of variables. Further, the harmonic oscillator equation (7.23) can be termed as cosmic Harmonic oscillator as it is expressed in terms of cosmological parameter H and deceleration parameter q. Moreover, the above harmonic oscillator is a realistic one only in the matter dominated era and from [263], realistic harmonic oscillator is associated with the convergence of geodesics. Hence convergence of geodesic is only feasible in the decelerated phase—a conclusion that we have already obtained earlier in this section.

7.3 Integrability of the Raychaudhuri equation and cosmological solutions

The Raychaudhuri equation for hyper-surface orthogonal congruence of time-like geodesics in FLRW space-time is given by equation (7.20). In order to study the integrability of the RE we consider the following transformation

$$Z = \sqrt{h} = a^3, \tag{7.24}$$

where h=det($h_{\mu\nu}$) is the determinant of the metric of the n-dimensional space-like hyper-surface. The dynamical evolution of h is given by,

$$\frac{1}{\sqrt{h}}\frac{d\sqrt{h}}{d\tau} = \Theta, \tag{7.25}$$

so that

$$\frac{dZ}{d\tau} = Z\Theta. \tag{7.26}$$

Hence for (n+1)-dimensional space-time manifold the RE can be written as a second order nonlinear ordinary differential equation as,

$$\frac{Z''}{Z} + \left(\frac{1}{n} - 1\right)\left(\frac{Z'}{Z}\right)^2 + \tilde{R} = 0. \tag{7.27}$$

' $'$ ' denotes differentiation w.r.t τ. The above second order non-linear differential equation has a first integral of the form

$$H^2 = \frac{a^{-\frac{6}{n}}}{9}\left[u_0 - 6\int a^{(\frac{6}{n}-1)}\tilde{R}\, da\right], \tag{7.28}$$

where $H = \dfrac{\dot{a}}{a}$ is the Hubble parameter, $a(t)$ is the scale factor and u_0 is a constant of integration.

The above first integral for 4-D space-time ($n = 3$) can be identified as the first Friedmann equation

$$3H^2 + \frac{\kappa}{a^2} = \rho \tag{7.29}$$

with

$$\kappa = -\frac{u_0}{3} \tag{7.30}$$

and,

$$\rho = -\frac{2}{a^2}\int a\ \tilde{R}\ da. \tag{7.31}$$

Equation (7.30) hints that u_0 is not merely a constant of integration but is related to the geometry of space-time as $u_0 >=< 0$ for open/flat/closed model. Further one may show that (7.31) holds in Einstein gravity. To show the equivalence between the first integral of the RE and the first Friedmann equation in FLRW model, we consider the Einstein field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\ R\ g_{\mu\nu} = T_{\mu\nu}, \tag{7.32}$$

or

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu} \tag{7.33}$$

$$\therefore \tilde{R} = R_{\mu\nu}u^{\mu}u^{\nu} = T_{\mu\nu}u^{\mu}u^{\nu} + \frac{1}{2}T. \tag{7.34}$$

$$(T = g^{\mu\nu}T_{\mu\nu})$$

If the matter content is assumed to be perfect fluid then the energy-momentum tensor is given by

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u^{\mu}u^{\nu}. \tag{7.35}$$

From the solution of energy-conservation equation $\dot{\rho} + 3H(p+\rho) = 0$ one has

$$\rho = \rho_0 a^{-3(1+\omega)}, \tag{7.36}$$

where ω is the equation of state parameter ($p = \omega\rho$).

Hence $\tilde{R} = \dfrac{\rho+3p}{2} = \dfrac{(1+3\omega)}{2}\rho_0 a^{-3(1+\omega)}$. Substituting this $\tilde{R}$ in the R.H.S of equation (7.31) one gets ρ, the L.H.S of equation (7.31) i.e. $\frac{-2}{a^2}\int a\tilde{R}\,da = \frac{-2}{a^2}\frac{(1+3\omega)\rho_0}{2}\int a^{-(2+3\omega)}\ da = \frac{-2}{a^2} \times \frac{-\rho_0}{2} \times a^{-(1+3\omega)} = \rho_0 a^{-3(1+\omega)} = \rho$. Thus (7.31) holds good in Einstein gravity for a 4-D space-time as a particular case.

In the above derivation we have obtained a first integral (7.28) of the RE and it matches with the first Friedmann equation. Thus essentially, study of cosmology either by Einstein field equations or by RE seems to be identical. But RE seems to have an extra advantage. This is because it is a geometric theory, so it may hold not only in Einstein gravity but also in any other modified theories of gravity and it is reflected through equation (7.31) where ρ can be obtained using the geometric scalar $\tilde{R}$. It seems to be a general one as in the light of the current analysis the standard evolution is recovered (discussed later in this section). This may be treated as an advantage of this approach.

Now we find the cosmological solutions or the scenario of cosmic evolution using the above first integral (7.28).

CaseI : Matter in the form of perfect fluid with equation of state $p = \omega(a)\rho$.
In this case, $\tilde{R} = \frac{1}{2}\rho(1 + 3\omega(a))$. The energy-momentum conservation equation is

$$\dot{\rho} + 3\rho(1 + \omega(a))H = 0, \tag{7.37}$$

the solution of which is given by

$$\rho = \rho_0 a^{-3} \exp\left(-3\int \frac{w(a)}{a} da\right). \tag{7.38}$$

So,

$$\tilde{R} = \frac{\rho_0 a^{-3}}{2}(1 + 3\omega(a))\ \exp\left(-3\int \frac{w(a)}{a} da\right). \tag{7.39}$$

Hence from the first integral (7.28) we have,

$$H^2 = \frac{a^{-\frac{6}{n}}}{9}\left[u_0 - 3\rho_0 \int a^{(\frac{6}{n}-4)}\ (1 + 3\omega(a))\ \exp\left(-3\int \frac{w(a)}{a} da\right) da\right]. \tag{7.40}$$

Subcase (i) $\omega(a) = 0$ i.e dust era of evolution.

$$H^2 = \frac{u_0}{9} a^{\frac{-6}{n}} - \frac{\rho_0 a^{-3}}{3(\frac{6}{n} - 3)}. \tag{7.41}$$

For 4D- space-time $n = 3$, so

$$H^2 = \frac{u_0}{9} a^{-2} + \frac{\rho_0}{3} a^{-3}. \tag{7.42}$$

Using $H = \frac{\dot{a}}{a}$ one has the solution as,

$$(t - t_0) = 3\int \frac{\sqrt{a}}{\sqrt{(3\rho_0 + u_0 a)}} da \tag{7.43}$$

or,

$$(t - t_0) = \frac{6}{u_0^{\frac{3}{2}}}\left[\frac{a u_0}{2}\sqrt{(3\rho_0 + a u_0)} - \frac{3\rho_0}{2}\cosh^{-1}\left(\sqrt{1 + \frac{u_0 a}{3\rho_0}}\right) + k\right], \tag{7.44}$$

where k is the constant of integration. Putting $u_0 = 0$ (flat space-time), for matter dominated era we have from equation (7.42)

$$H^2 = \frac{\rho_0}{3} a^{-3}, \tag{7.45}$$

solving which we get the variation of scale factor $a(t)$ to cosmic time t as $a(t) \propto t^{\frac{2}{3}}$, the standard result for matter dominated era.

Subcase (ii) $\omega(a) = \omega_0$, a non-zero constant.

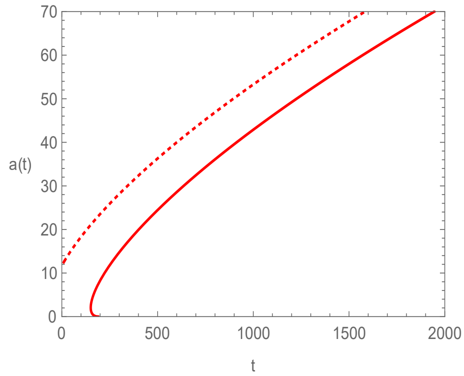

Figure 7.1: $a(t)$ vs t for $\omega_0 = 0$ and (i) $u_0 = 0.1, k = 1, \rho_0 = 1, t_0 = 0$ (Solid Red line); (ii) $u_0 = 0.01, k = -0.001, \rho_0 = 0.01, t_0 = 0$ (Dotted Red line)

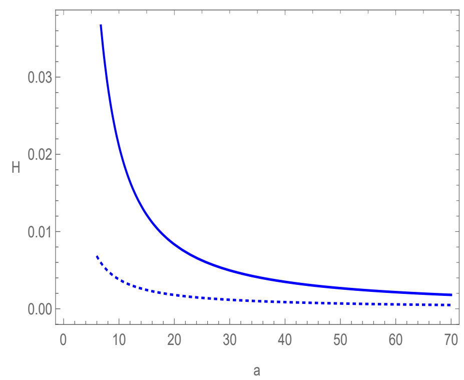

Figure 7.2: H vs a for $\omega_0 = 0$ and (i) $u_0 = 0.1, \rho_0 = 1$ (Solid Blue line); (ii) $u_0 = 0.01, \rho_0 = 0.01$ (Dotted Blue line)

$$H^2 = \frac{a^{-\frac{6}{n}}}{9}\left[u_0 - 3\rho_0(1+3\omega_0)\frac{a^{(\frac{6}{n}-3-3\omega_0)}}{(\frac{6}{n}-3-3\omega_0)}\right]. \tag{7.46}$$

For 4-D space-time again putting $n = 3$, one gets

$$H^2 = \frac{u_0}{9}a^{-2} + \frac{\rho_0}{3}a^{-3(1+\omega_0)}, \tag{7.47}$$

or

$$(t - t_0) = 3\int \frac{da}{\sqrt{u_0 + 3\rho_0 a^{-(1+3\omega_0)}}}. \tag{7.48}$$

For the radiation dominated era characterized by $\omega_0 = \frac{1}{3}$, using equation (7.40) we have,

$$H^2 = 3\rho_0 a^{-4} \tag{7.49}$$

which gives the variation of scale factor to cosmic time as $a(t) \propto t^{\frac{1}{2}}$. To find the vacuum dominated solution we put $\kappa = -\frac{u_0}{3} = 0$ and $\rho = \rho_0$, a constant in equation (7.29). Then we get

$$3H^2 = \rho_0 \tag{7.50}$$

Solution for various non-zero choices of ω_0	
ω_0	Solution
$\frac{1}{3}$	$a(t) = \left[\frac{(t-t_0)^2 u_0}{9} - \frac{3\rho_0}{u_0}\right]^{\frac{1}{2}}$
1	$(t-t_0) = \frac{a^3\ {}_2F_1\ \left(\frac{1}{2}, \frac{3}{4}; \frac{7}{4}; \frac{-u_0}{3\rho_0} a^4\right)}{\sqrt{3\rho_0}}$
$-\frac{1}{3}$	$a(t) = \frac{\sqrt{u_0 + 3\rho_0}}{3}(t-t_0)$
-1	$a(t) = \sqrt{\frac{u_0}{3\rho_0}}\left[\coth^2(\sqrt{\frac{\rho_0}{3}}(t-t_0)) - 1\right]^{\frac{-1}{2}}$

Table 7.1: Cosmic scale factor for various non zero choices of $\omega = \omega_0$

so that the variation of scale factor to cosmic time is $a(t) \propto \exp(\Lambda t)$ where $\Lambda = \sqrt{\frac{\rho_0}{3}}$. The solution for various non-zero choices (of course that make the integration solvable) of ω_0 are given below in tabular form.

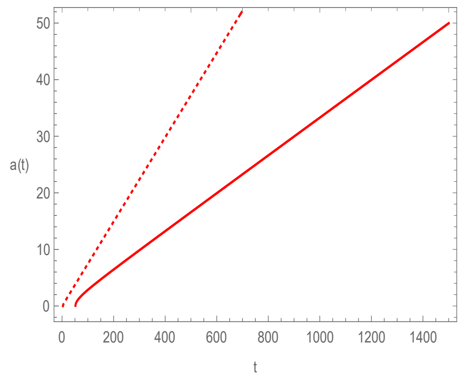

Figure 7.3: $[a(t)$ vs t for $\omega_0 = \frac{1}{3}$ and (i) $u_0 = 0.01, \rho_0 = 0.01, t_0 = 0$ (Solid Red line); (ii) $u_0 = 0.05, \rho_0 = 0.001, t_0 = 0$ (Dotted Red line)]

The solution for a general $w(a)$ of the form [for ref. see [286]]

$$\omega(a) = \omega_0 + \omega' \left(\frac{a}{a-1}\right), \tag{7.51}$$

is given by

$$\dot{a} = \frac{1}{3}\left[u_0 - 3\rho_0 \int ((1+3\omega_0)(a-1) + 3\omega' a)(a-1)^{-1-3\omega'} a^{-(2+3\omega_0)} da\right]^{\frac{1}{2}} \tag{7.52}$$

which upon further simplification yields

$$3\int \frac{da}{[u_0 - 3\rho_0\ A]^{\frac{1}{2}}} = (t-t_0), \tag{7.53}$$

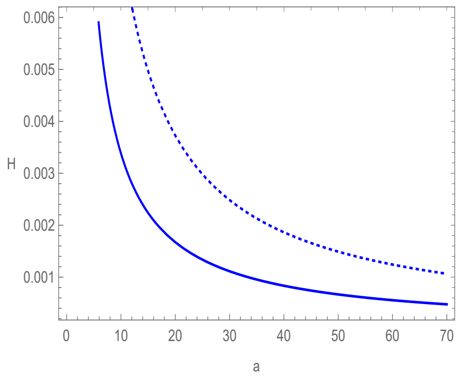

Figure 7.4: H vs a for $\omega_0 = \frac{1}{3}$ and (i) $u_0 = 0.01, \rho_0 = 0.01$ (Solid Blue line); (ii) $u_0 = 0.05, \rho_0 = 0.001$ (Dotted Blue line)

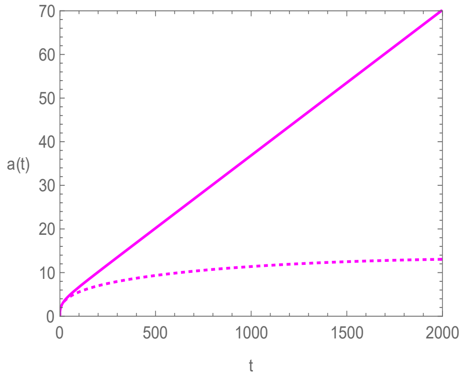

Figure 7.5: $a(t)$ vs t for $\omega_0 = 1$ and (i) $u_0 = 0.01, \rho_0 = 1, k = 0.1, t_0 = 0$ (Solid Magenta line); (ii) $u_0 = -0.0001, \rho_0 = 1, k = 0.1, t_0 = 0$ (Dotted Magenta line)

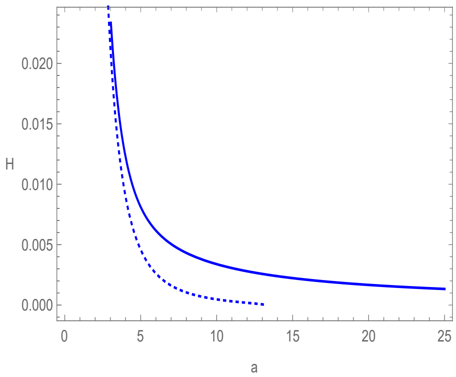

Figure 7.6: H vs a for $\omega_0 = 1$ and (i) $u_0 = 0.01, \rho_0 = 1$ (Solid Blue line); (ii) $u_0 = -0.0001, \rho_0 = 1$ (Dotted Blue line)

where A is given by

$$A = \frac{(a-1)^{-3\omega'}}{3\omega'} \left[-(1+3\omega_0+3\omega') \ 2^{F_1}(1+3\omega_0, -3\omega'; 1-3\omega'; 1-a) + (1+3\omega_0) \ 2^{F_1} \ (2+3\omega_0, -3\omega'; 1-3\omega'; 1-a) \right]. \tag{7.54}$$

Figure 7.7: $a(t)$ vs t for $\omega_0 = -\dfrac{1}{3}$ (left) and for $\omega_0 = -1$ (right)

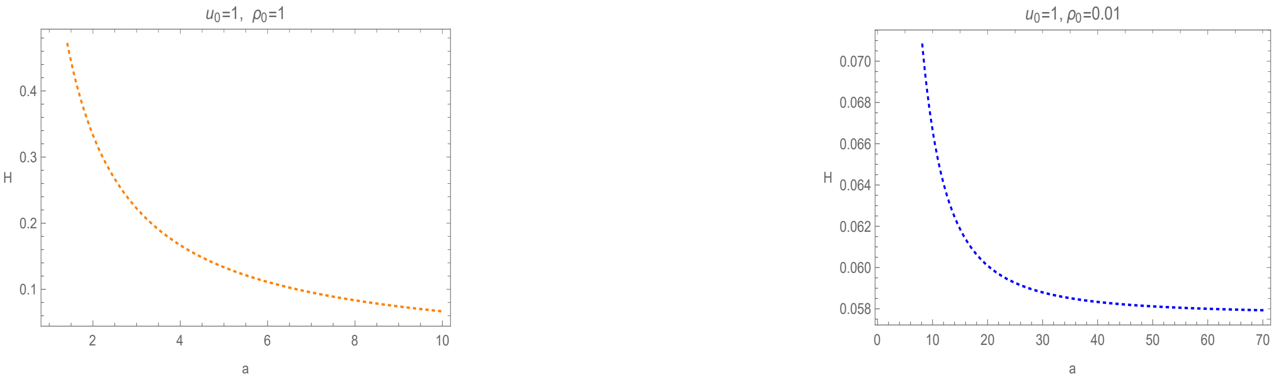

Figure 7.8: H vs a for $\omega_0 = -\dfrac{1}{3}$ (left) and for $\omega_0 = -1$ (right)

and 2^{F_1} is the Gauss-Hypergeometric function.

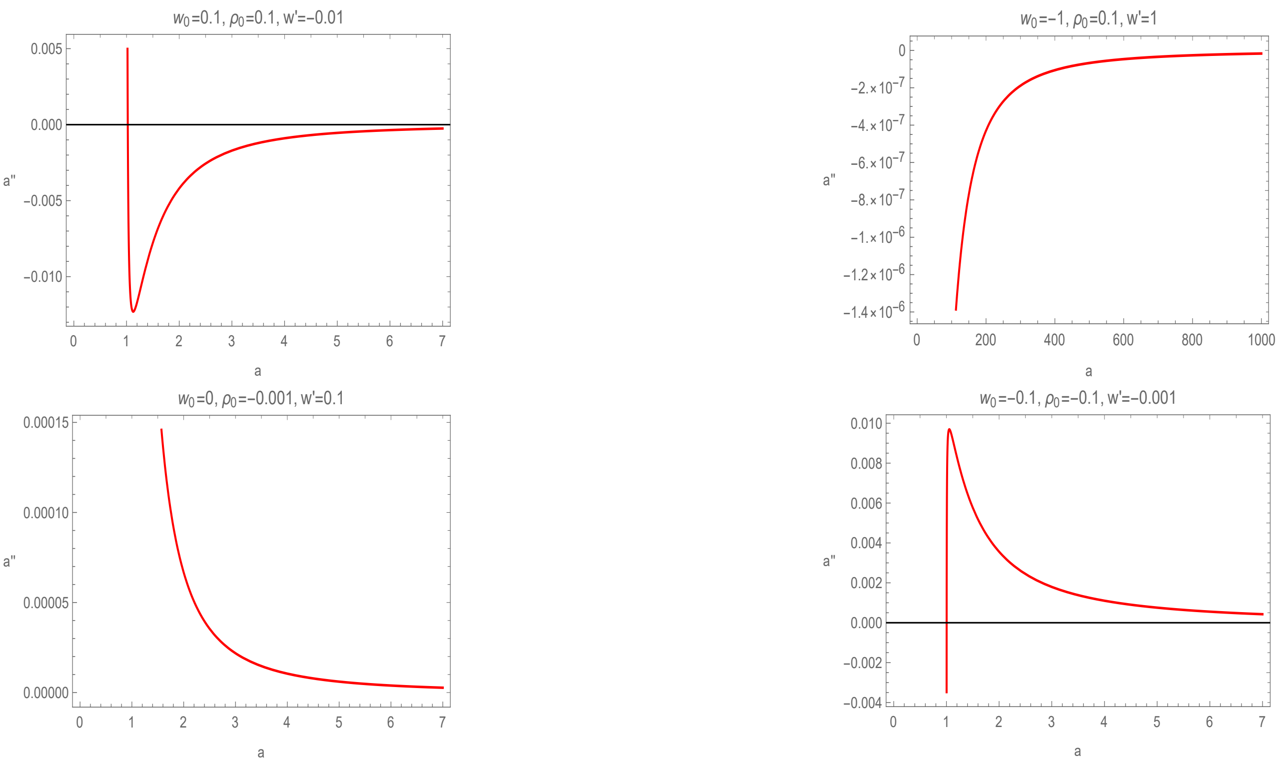

Figure 7.9: a'' vs a for $\omega(a) = \omega_0 + \omega' \left(\frac{a}{a-1}\right)$ and various choices of the parameters specified in each panel

Case II- Inflation:

In case of inflation $\tilde{R} = \frac{1}{2}(\rho_\phi + 3p_\phi) \simeq -V_0$. So one has

$$H^2 = \frac{nV_0}{9} + \frac{u_0}{9} a^{-\frac{6}{n}}. \tag{7.55}$$

For 4-D space-time the solution becomes,

$$(t - t_0) = \int \frac{da}{\sqrt{\frac{u_0}{9} + \frac{V_0 a^2}{3}}}, \tag{7.56}$$

or

$$(t - t_0) = \sqrt{\frac{3}{V_0}} \ \ln|a| + \left(a^2 + \frac{u_0}{3V_0}\right)^{\frac{1}{2}} + C, \tag{7.57}$$

C, being the constant of integration.

7.4 Brief discussion

The present chapter is an example where the RE has been used in cosmological context. A general formulation of the Raychaudhuri scalar (i.e.Curvature scalar) has been done

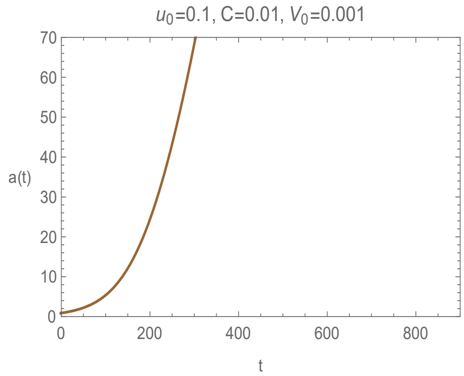

Figure 7.10: Graphical representation of scale factor $a(t)$ with cosmic time t during inflation

in terms of cosmological parameters both in Einstein gravity as well as in modified gravity theories. It is found that the positiveness of the curvature scalar or precisely the convergence condition demands that q, the deceleration parameter should be positive. Hence, the deceleration parameter for the present universe being negative does not allow focusing theorem to hold. Again since $q = -1(< 0)$ during the inflationary era of evolution, therefore one may conclude that matter dominated era of evolution is in favor of convergence while the formation singularity can be avoided both in early inflationary era and in the present accelerated era of expansion.

Further, a nice analogy of RE through geometry and cosmology has been elaborated and the role of $\tilde{R}$ and q has been shown to be similar in the context of focusing. Thus the deceleration parameter plays the role of convergence scalar in the context of cosmology. Subsequently, using a suitable transformation the cosmological form of RE has been converted a Harmonic oscillator equation and condition for the formation of real Harmonic oscillator has been shown to be possible only in the matter dominated era of cosmic evolution. The observable quantities namely luminosity distance (d_L) and density parameter (Ω) are shown to be related to the convergence scalar ($\tilde{R}$).

A suitable transformation related to geometric variable (metric scalar of the hypersurface), transforms the RE to a second order non-linear differential equation whose first integral is easily obtained. It is found that this first integral is nothing but the 1st Friedmann equation. By choosing perfect fluid with barotropic EoS as the matter content of the universe, cosmological solutions are obtained for various choices of the parameters appearing in the equation of state (EoS). In most of the choices (where scale factor and Hubble parameter have explicit form), the cosmological parameters namely the scale factor (a) and the Hubble parameter (H) behave in accordance with the observational data at least qualitatively i.e. the universe is in an expanding phase with the rate of expansion gradually decreases. However, for the specific choices of the EoS as a function of the scale factor (variable equation of state) only acceleration ($\ddot{a}$) can be evaluated as a function of the scale factor (a) in FIG.s (7.9). The graphs show that due to some choices of the parameters involved: (i) there is a sharp fall from acceleration to deceleration and asymptotically it goes to zero acceleration (FIG. 7.9 top left), (ii) the universe is totally in a decelerated phase with rate of decrease of deceleration being sharp in the initial stage and then gradually becomes a constant (FIG. 7.9 top right), (iii) the universe experiences only accelerating phase but it gradually decreases with the evolution and finally reaches a constant value (FIG. 7.9 bottom left), (iv) there is a sharp rise from deceleration to acceleration and then gradually the acceleration parameter goes to zero asymptotically (FIG. 7.9 bottom right).

Finally, it is to be noted that throughout this chapter we have used congruence of time-like geodesics not the congruence of null geodesic. In principle, there is no basic difference between the use of these two kinds of geodesics. However, in the context of cosmology as the evolution is characterized by time variation so time-like geodesic is important where proper time or the cosmic time is being used. In case of null geodesics since we have only an affine parameter, so time evolution will be rather ambiguous. One may interpret an affine parameter to be the null analog of proper time. It is worthy to mention that there is a nice study in the literature [174] where the convergence scalar has been geometrically interpreted as the mean curvature. Using this interpretation, in the present context we can say that the frequency of the harmonic oscillator formed out of the cosmological RE is associated to the mean curvature. It would be interesting to find the solution of this harmonic oscillator equation subject to some realistic choices of the mean curvature in future work. Also the authors in [174], have analyzed cosmic evolution from the expression of deceleration parameter q based on ΛCDM model. On the other hand, this chapter shows the complete evolution of the universe by analyzing the first integral of the Raychaudhuri equation.

CHAPTER 8

RAYCHAUDHURI EQUATION AND NON-SINGULAR UNIVERSE: SOME UNCONVENTIONAL ASPECTS

8.1 Prelude

Existence of singularities in a physical theory either indicates about some unknown information of the theory itself or demand its possible modification. Therefore, the study of singularities, non singular models of the universe has attracted interest of the relativists and cosmologists. In this chapter we explore the effects of RE in non-singular models of the universe. In literature, several models have been proposed of which in this paper we discuss wormholes and emergent universe model and explore the role of RE and FT in them.
Among the non-singular models, emergent scenario [287]-[299] is very much fascinating in standard cosmology. This model is free of any time-like singularity throughout $-\infty < t < +\infty$. Asymptotically, this model matches with Einstein static universe, characterized by positive spatial curvature, dust and a positive cosmological constant. In Einstein gravity, this type of singularity free solution is possible in the context of non-equilibrium thermodynamic prescription with particle creation mechanism. In this model, the initial Big-Bang singularity is replaced by an Einstein static era with $a = a_0 \neq 0$ so that the physical quantities namely energy density, pressure etc. all are finite at the beginning.

On the other hand, Wormholes (WH) are usually defined as smooth bridges between different universes or topological handles (i.e. throats having no horizons) between remote parts of a single universe. Earlier, wormholes were purely of theoretical curiosity [300] but recently, this theoretical aspect has gained significant importance due to the present accelerating phase of our universe. There is a nice similarity between this theoretical phenomenon and the recent observational aspects. A traversable wormhole [301] is supported by a so-called exotic matter which violates the null energy condition (i.e. $\rho + p < 0$) at least in a neighborhood of the wormhole throat [300]. Exotic matter or matter that violated the strong energy condition (i.e. $\rho + 3p < 0$) has started gaining importance in the last two decades because of the observed acceleration of the present-day universe. It is speculated that the recent acceleration in the expansion of the universe could only be explained using matter that is characterized by the negative equation of state, called dark energy (DE). A useful choice of DE is the cosmological constant having equation of state $\omega = -1$, thereby violating the null energy condition (NEC). Several papers on wormholes show that such phantom energy could be the nature of the matter of wormhole throat [300]. Recently, numerous wormhole solutions are investigated in different modified gravity theories such as Bumblebee gravity [302], Teleparallel theories of gravity [303] etc. using various matter content. Gravitational lensing, light deflection angle, various thin-shell wormhole models have also been extensively studied. Also, several works have been carried out on the existence of a wormhole solution with non-exotic matter. Readers may refer to [301], [304],[305],[306],[307] for more details.

In the recent times, there has been an increasing inclination of the cosmologists towards the cosmological models that replace the cosmological singularity (or big bang) with a “big bounce”—a smooth transition from contraction to expansion in order to resolve fundamental issues in cosmology. The implication of RE in bouncing cosmology has been studied in Chapter 5, [308]. Actually, there is violation of singularity theorem in framing the non-singular models.

The chapter is dedicated to the answers of the following questions:

Q1. Will the Focusing Theorem give any new idea if we consider some meaningful transformations?

Q2. What about Harmonic Oscillator approach of analyzing the convergence w.r.t the transformed variables?

Q3. What does the Raychaudhuri equation imply for an emergent scenario to exist?

Q4. For a WH what is the relation between flaring out condition and convergence condition for both time-like as well as null geodesic congruence?

Q5. Is it possible that null geodesics will converge but not the time-like geodesics in case of a WH or vice-versa?

Q6. What is the role of shape function and red-shift function of WH geometry in convergence of null or time-like congruence?

This chapter is organized as follows: Section 8.2 deals with an unconventional way of stating the Focusing theorem. Section 8.3 discusses the implication of RE in emergent universe scenario. Wormhole geometry has been studied from the point of view of RE in section 8.4. Finally the chapter ends with a brief discussion.

8.2 An unconventional analysis of Focusing Theorem

The Raychaudhuri equation (RE) for a congruence of hyper-surface orthogonal time-like geodesic can be written as [154], [309]

$$\frac{d\Theta}{d\tau} = -\frac{\Theta^2}{3} - 2\sigma^2 - R_{\mu\nu}u^\mu u^\nu \tag{8.1}$$

where Θ is the expansion scalar, σ is the shear scalar, $R_{\mu\nu}$ is the Ricci tensor, u^μ is the velocity vector and τ is the proper time. Further, we denote $R_{\mu\nu}u^\mu u^\nu = \tilde{R}$ and name it as Raychaudhuri scalar. Using the Einstein's field equations and assuming Strong Energy Condition (SEC) on matter the RE gives rise to a very important theorem known as the Focusing Theorem (FT). The FT is the turning point of Einstein gravity as it hints that singularity is inevitable in GR. Mathematically, the inequality

$$\frac{1}{\Theta} \geq \frac{1}{\Theta_0} + \frac{\tau}{3} \tag{8.2}$$

gives the FT which states that an initially converging bundle of time-like geodesic will develop a caustic within finite value of τ provided SEC ($\tilde{R} \geq 0$) holds. The condition $\tilde{R} \geq 0$ is known as the Convergence Condition (CC). The focusing of geodesic implies a congruence singularity or the singularity of the space-time. However sometimes with some more assumptions this may lead to cosmological or black-hole singularity. We consider v to be the volume of the hyper-surface. Then w.r.t v, the RE (8.1) takes the form

$$\frac{v}{2}\frac{d\Theta^2}{dv} = -\frac{\Theta^2}{3} - 2\sigma^2 - \tilde{R} \tag{8.3}$$

Suppose the matter satisfies SEC i.e, $\tilde{R} \geq 0$, then RE (8.3) gives

$$\frac{v}{2}\frac{d\Theta^2}{dv} + \frac{\Theta^2}{3} = -(2\sigma^2 + \tilde{R}) \leq 0 \tag{8.4}$$

Upon integrating we get,

$$\Theta \leq (v_0 v)^{-\frac{1}{3}} \tag{8.5}$$

v_0, being the constant of integration. From this inequality, the FT can be restated in terms of volume as follows: If $v \to 0$, $\Theta \leq \pm\infty$ depending on the sign of v_0. There is no notion of time here unlike the FT of which we are familiar in literature. The inequality shows that if volume of the hyper-surface approaches to zero, there is either convergence or divergence of the bundle depending on whether initially the bundle converged or diverged. This shows that a singularity of the space-time (determined by $v \to 0$) will always imply a convergence of the bundle. From the definition of Θ we have,

$$\Theta = \frac{1}{v}\frac{dv}{d\tau} \tag{8.6}$$

We can think of another transformation given by

$$T = \ln v \tag{8.7}$$

Then RE is given by

$$\frac{1}{2}\frac{d\Theta^2}{dT}+\frac{\Theta^2}{3}=-(2\sigma^2+\tilde{R})=-R_c \tag{8.8}$$

If $R_c \geq 0$, RE gives the inequality

$$\frac{d\Theta^2}{dT}+2\frac{\Theta^2}{3}\leq 0 \tag{8.9}$$

which upon integration yields,

$$\Theta \leq \Theta_0 \exp\left(-\frac{T}{3}\right) \tag{8.10}$$

Here, T behaves as time as clear from the transformation $T=\ln v$. We get similar kind of inference as the previous case. That answers Q1. Further the RE can be identified as the differential equation for a real linear harmonic oscillator as

$$\left(\frac{d^2}{dT^2}+2R_c\right)Y=0 \tag{8.11}$$

with the frequency of the oscillator as $2R_c$ if one considers a transformation

$$\begin{aligned}\Theta^2 &= \frac{1}{Y}\frac{dY}{dT}\\ Y &= \sqrt{3\ln v}=\sqrt{3T}\\ \Theta^2 &= \frac{1}{2T}\exp(-2T)\end{aligned} \tag{8.12}$$

Thus we see that if under some suitable transformation RE, a first order differential equation can be converted to a second order differential equation analogous to the evolution equation of a real linear harmonic oscillator then CC is always satisfied as it is associated to the frequency of the oscillator. That answers Q2.

8.3 Raychaudhuri equation and Emergent Universe

The RE for a hyper-surface orthogonal congruence of time-like geodesic is given by (8.1). In FLRW (homogeneous and isotropic) case, $\Theta=3H$, $H=\frac{\dot{a}}{a}$ (Hubble parameter) and due to isotropy $2\sigma^2=0$. In cosmology, τ is nothing but the cosmic time t. So RE in terms of the cosmic parameter H and cosmic time t takes the form

$$\dot{H}+H^2=-\frac{1}{3}\tilde{R} \tag{8.13}$$

which in terms of scale factor can be written as

$$\frac{\ddot{a}}{a}=-\frac{1}{3}\tilde{R} \tag{8.14}$$

Previously we have learned that $\tilde{R}\geq 0$ is the CC. In order to examine the sign of Raychaudhuri scalar for emergent scenario we consider the following asymptotic behavior for the Hubble parameter and scale factor as [307],[310]

1. $a \to a_0$, $H \to 0$ as $t \to -\infty$.
2. $a \approx a_0$, $H \approx 0$ for $t \leq t_0$.
3. $a = a_0 e^{H_0(t-t_0)}$, $H \approx H_0$ for $t \leq t_0$.

Here a_0, a constant is the tiny value of the scale factor during emergent era of evolution and $H_0 > 0$ is the value of the Hubble parameter at $t = t_0$. Thus from the RE, employing the above conditions we can find the nature of $\tilde{R}$ in emergent scenario. This is because it is the sign of $\tilde{R}$ that plays a significant role in Focusing theorem. We find that:

1. as $t \to -\infty$ (infinite past), $\tilde{R} \to 0$
2. For $t \leq t_0$, $\tilde{R} \approx -3H_0^2 < 0$

Thus throughout the time axis $(-\infty, +\infty)$ we get $\tilde{R} \leq 0$. Thus CC is violated. This leads to the avoidance of singularity. This is why, emergent universe is a non-singular model of universe where the initial big-bang singularity is replaced by an Einstein static era with $a = a_0 \neq 0$. In the emergent era RE leads to the violation of Focusing theorem as $\tilde{R} \leq 0 \implies \dfrac{d\Theta}{d\tau} + \dfrac{\Theta^2}{3} \geq 0$. This allows the defocusing of geodesics thereby guaranteeing the singularity free nature of emergent universe. This is a case which shows that in a singularity free model, geodesics are complete and there is no focusing. The formal definition of singularity first appeared in the works of Penrose where he defined singularity from the point of view of geodesic incompleteness. Thus, from the definition itself it is also clear that a singularity free model has complete geodesics. That answers Q3. As from the RE it is clear that emergent era is described by $\tilde{R} \leq 0$ so from Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}, \tag{8.15}$$

using the expression for energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \tag{8.16}$$

we have

$$\tilde{R} = \frac{1}{2} (\rho + 3p) \tag{8.17}$$

Thus for an emergent model of the universe, we need that type of matter which satisfies $(\rho + 3p) \leq 0$ or simply, $\omega \leq -\dfrac{1}{3}$ $(\because p = \omega\rho)$. Thus RE hints that exotic matter is needed for an emergent model of the universe.

8.4 Raychaudhuri equation and Wormholes

Under very general conditions, a traversable wormhole violates the average Null Energy Condition (NEC) in the nearby region of the throat. This will be examined using the Raychaudhuri equation (RE), together with the fact that a wormhole throat by definition defocuses light rays. We know that Focusing theorem is the most vital consequence of the RE which needs $\tilde{R} \geq 0$ (CC) for geodesic focusing. If a space-time has a singularity, then a bundle of geodesic will tend to focus at the singularity. Thus a violation of the CC may possibly avoid singularity. The condition that the wormhole be traversable, in particular means that there are no event horizons or curvature singularities. Now to construct a traversable wormhole we consider the Morris-Thorne line element given by [311], [312]

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{8.18}$$

where

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2, \tag{8.19}$$

is the metric on unit 2-sphere, $\Phi(r)$ is the red-shift function and $b(r)$ is the shape function. For the line element (8.18), the components of the Einstein tensor are

$$G_{00} = \frac{b'(r)}{r^2} \tag{8.20}$$

$$G_{11} = -\frac{b}{r^3} + 2\left(1 - \frac{b(r)}{r}\right)\frac{\Phi'}{r} \tag{8.21}$$

$$G_{22} = G_{33} = \left(1 - \frac{b(r)}{r}\right)\left(\Phi'' + \Phi'^2 + \left(\frac{-rb' + 2r - b}{2r(r-b)}\right)\Phi' - \frac{(rb' - b)}{2r^2(r-b)}\right) \tag{8.22}$$

The RE for hyper-surface orthogonal null geodesics is given by

$$\frac{d\Theta}{d\lambda} = -\frac{\Theta^2}{2} - 2\sigma^2 + R_{\mu\nu}k^\mu k^\nu \tag{8.23}$$

where k^μ is the null vector so that $k_\mu k^\mu = 0$. In this context the CC becomes $R_{\mu\nu}k^\mu k^\nu \geq 0$. Now from the Einstein's field equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \tag{8.24}$$

one has $\tilde{R} = R_{\mu\nu}k^\mu k^\nu = G_{\mu\nu}k^\mu k^\nu$ using the property of null vector k^μ. For infalling observer we can take the null vector as $(\sqrt{-g^{00}}, \pm\sqrt{g^{11}}, 0, 0)$ i.e, $k^0 = e^{-\Phi}$, $k^1 = \pm\frac{1}{\left(1 - \frac{b(r)}{r}\right)^{\frac{-1}{2}}}$ $k^2 = k^3 = 0$. The Raychaudhuri scalar turns out to be

$$\tilde{R} = G_{\mu\nu}k^\mu k^\nu = \frac{b'}{r^2}e^{-2\Phi} - \left(\frac{-b(r)}{r^3} + 2\left(1 - \frac{b(r)}{r}\right)\frac{\Phi'}{r}\right)\left(1 - \frac{b(r)}{r}\right) \tag{8.25}$$

Now to have traversable wormhole we need the following:

- Wormhole is a bridge between two asymptotically flat regions connected by a throat. The throat radius is defined by a global minimum $r = r_0$, so the radial coordinate runs in $r \geq r_0$.

- The redshift function $\Phi(r)$ must be finite everywhere in order to avoid the presence of horizons and singularity. So, $e^{\Phi(r)} > 0$ and finite for $r > r_0$. In this context, the ultrastatic wormhole is a particular point of interest which defines the zero-tidal force wormhole (i.e. $\Phi(r) = 0$ so that $e^{2\Phi(r)} = 1$). So in case of ultrastatic wormhole one has,

$$\tilde{R} = \frac{b'}{r^2} - \frac{b(r)}{r^3}\left(1 - \frac{b(r)}{r}\right). \tag{8.26}$$

- Flairing out condition: $\left(\frac{-rb'(r) + b(r)}{b^2(r)} > 0\right)$ must hold near the throat. Clearly, the flare-out condition is independent of the red-shift function Φ and only depends on the wormhole shape function $b(r)$. The flare-out condition is more understandable through the embedding geometry which is given by the minimality of the size of the wormhole at the throat.

- $b(r) < r$ and $b'(r) \leq 1$ for all $r \geq r_0$ and at $r = r_0$, $b(r) = r$, $b'(r) = 1$.

- The asymptotic flatness implies that $\Phi(r) \to 0$ and $\frac{b(r)}{r} \to 0$ as $r \to \infty$

Equation (8.25) can be rewritten as

$$\tilde{R} = \frac{1}{r^2}\left(\left(b' - \frac{b(r)}{r}\right) + \left(e^{-2\Phi} - 1\right) b(r)\right) - \frac{2}{r}\left(1 - \frac{b(r)}{r}\right)^2 \Phi' \tag{8.27}$$

Therefore, using the flair-out condition it is easy to check that, CC is violated for $\Phi > 0$ and $\Phi' > 0$ i.e. Φ, the red-shift function must be positive and increasing in order to avoid singularity. Now we consider the expression for the Raychaudhuri scalar given in equation (8.26) i.e, $\tilde{R} = \frac{b^2(r)}{r^4} + \frac{rb' - b}{r^3} = R_1 + R_2$. R_1 is always positive. To satisfy the flair out conditions we need $(rb' - b) < 0$ i.e. $R_2 < 0$. Hence, there is a possibility of violation of CC which is a necessary condition for FT to hold if R_2 dominates in the expression for $\tilde{R}$. However, if R_1 predominates there is focusing of null geodesics inside the wormhole. In that case the kind of wormhole is not traversable as the geodesics are incomplete there.

This is illustrated in the following example where we consider an incoming light travelling along geodesics. It then crosses a wormhole and again expands on the other side of the wormhole. Since the neck of the wormhole is of finite length, thus Focusing theorem does not hold there and hence formation of singularity at least within the vicinity of the neck is forbidden. According to the optical Raychaudhuri's theorem this phenomena requires a violation of the NEC. To speak lucidly, the existence of a traversable wormhole is plausible only if the geodesics entering the wormhole on one side (and thus converging as they approach the throat) will emerge on the other side diverging away from each other. By Raychaudhuri's equation this is possible if specific energy conditions are violated. Thus the RE hints why exotic matter or configurations which violate energy conditions are needed for a traversable wormhole to exist. The above statements can be illustrated by considering an example of a wormhole described by the metric [313]

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta d\phi^2), \ -\infty < t < +\infty, \ -\infty < r < +\infty \quad (8.28)$$

The space-time associated with this metric is asymptotically flat and hosts a stable wormhole of radius b at $r = 0$, the origin. The metric does not possess a singularity nor there is an existence of event horizon. Now we see the implications of RE or more precisely the Raychaudhuri scalar for this particular type of wormhole. The expression for $\tilde{R}$ turns out to be

$$\tilde{R} = R_{\mu\nu}k^{\mu}k^{\nu} = R_{rr}(k^r)^2 \quad (8.29)$$

as the only non zero component of Ricci tensor is $R_{rr} = -\frac{2b^2}{(b^2+r^2)^2}$. Thus $\tilde{R} < 0$ and CC is violated. This proves that there is no singularity. This is because, if there were a singularity focusing would have occured there. Also $\tilde{R} < 0$ stands for violation of energy conditions. Thus the theoretical description of a traversable wormhole from the point of view of RE matches with the implication of RE in the above wormhole. This shows that the above wormhole is a traversable one and in literature its named as Ellis Wormhole. Thus, RE is responsible for the existence of a traversable wormhole. One can find the Raychaudhuri scalar $\tilde{R}$ and expansion scalar Θ for a general wormhole metric and see the consequences of RE in terms of average energy conditions. Wormhole is such an astrophysical object for which a number of singularity theorems fail to hold true. Thus not only in black hole but also RE is equally important in the study of wormholes. Next we consider the ultrastatic Morris-Thorne wormhole described by the line element

$$ds^2 = -dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \quad (8.30)$$

Without loss of generality we consider $\theta = \frac{\pi}{2}$ and the components of the time-like velocity vector field u^{μ} is given by

$$\dot{t} = -E, \ \dot{\phi} = \frac{h}{r^2}, \ \dot{r} = \sqrt{\left(1 - \frac{b}{r}\right)\left(E^2 - 1 - \frac{h^2}{r^2}\right)}, \ \dot{\theta} = 0 \quad (8.31)$$

where E and h are identified as the conserved energy and angular momentum of the time-like particle (per unit mass). The explicit expression for $R_{\mu\nu}u^{\mu}u^{\nu}$ is as follows :

$$\tilde{R} = R_{\mu\nu}u^{\mu}u^{\nu} = -\frac{h^2}{r^4} + \frac{b'(r)}{r^2}\left(E^2 - 1 - \frac{h^2}{2r^2}\right) - \frac{b(r)}{r^3}\left(E^2 - 1 - \frac{3h^2}{2r^2}\right) \quad (8.32)$$

where $R_{\mu\nu}$ is the Ricci tensor projected along the congruence of geodesics and it has been evaluated from the metric. For realistic $\dot{r}$, we need

$$(E^2 - 1) > \frac{h^2}{r^2} \quad (8.33)$$

From the metric we have

$$\left(1-\frac{b(r)}{r}\right)>0. \tag{8.34}$$

The flairing out condition gives

$$\frac{-rb'+b}{b^2}>0$$

which upon simplification yields,

$$\frac{b(r)}{r^3}-\frac{b'(r)}{r^2}>0 \tag{8.35}$$

After some algebraic manipulation, one can write $\tilde{R}$ as

$$\tilde{R}=R_1+R_2+R_3 \tag{8.36}$$

where,

$$R_1=\left(E^2-1-\frac{h^2}{r^2}\right)\left(\frac{b'(r)}{r^2}-\frac{b(r)}{r^3}\right) \tag{8.37}$$

$$R_2=\frac{1}{2}\frac{h^2}{r^2}\left(\frac{b'(r)}{r^2}-\frac{b(r)}{r^3}\right) \tag{8.38}$$

$$R_3=\frac{h^2}{r^4}\left(\frac{b(r)}{r}-1\right) \tag{8.39}$$

Now equations (8.33), (8.34) and (8.35) show that R_1, R_2, R_3 all are negative and hence $\tilde{R}<0$. Thus CC is violated inside the wormhole with time-like test particle unlike null geodesics. Thus we have proved that flaring out condition implies a violation of CC and thus the FT in case of time-like geodesics while CC may hold for null geodesics inside a wormhole under certain conditions. That answers Q4 and Q5. Now we examine the role of red-shift function and shape function in convergence to answer Q6. For this we consider the general wormhole metric given by (8.18) and the expression for $\tilde{R}$ given by (8.25). If $\tilde{R}=\mathcal{R}_1+\mathcal{R}_2$. Then,

$$\mathcal{R}_1=\frac{b'}{r^2}e^{-2\Phi} \tag{8.40}$$

$$\mathcal{R}_2=\left(-\frac{b(r)}{r^3}+2\left(1-\frac{b(r)}{r}\right)\frac{\Phi'}{r}\right)\left(1-\frac{b(r)}{r}\right) \tag{8.41}$$

Here $'$ is differentiation w.r.t r. To avoid singularity we need $\tilde{R}<0$. Now we consider the following cases:

Case-I: $b(r)$ is constant, say $b(r)=b_0$. Then $\mathcal{R}_1=0$ and since from the metric $\left(1-\frac{b(r)}{r}\right)>0$, thus sign of $\tilde{R}$ depends on the sign of $\left(-\frac{b_0}{r^3}+2\left(1-\frac{b_0}{r}\right)\frac{\Phi'}{r}\right)$. After some mathematical calculation the restriction on the red-shift function for the possible avoidance of singularity reduces to

$$\Phi<\ln\sqrt{\Phi_0\left(1-\frac{b_0}{r}\right)} \tag{8.42}$$

Therefore for a constant shape function, singularity is avoided with the above restriction of red-shift function.

Case-II: $b(r)$ is an increasing function of r so that $b' > 0$ and hence $\mathcal{R}_1 > 0$. Avoidance of singularity might be guaranteed on a restriction of Φ as

$$\Phi' < \frac{\frac{b(r)}{r^2}}{2\left(1 - \frac{b(r)}{r}\right)} \tag{8.43}$$

which shows that Φ is increasing or decreasing according as $b(r)$ is +ve or -ve.

Case-III: $b(r)$ is a decreasing function of r so that $b' < 0$ and hence $\mathcal{R}_1 < 0$. Avoidance of singularity is guaranteed with the same restriction on Φ as in equation (8.43). Finally one can explicitly find the red shift function assuming some phenomenological choices of $b(r)$ and vice versa to find a singularity free model. The choice $b(r) = b_0 \left(\frac{r}{r_0}\right)^n$ is a popular choice of $b(r)$ in literature which can be applied here with b_0, r_0 and n, being the model parameters which varies from case to case.

8.5 Discussion

The present chapter is an example where RE has been used to examine the formation of singularity in various cosmological and astrophysical context. The analysis of RE demands that for the formation of emergent phase in the early cosmic epoch, exotic matter should be essential. In literature, Focusing theorem and analogy of RE with Harmonic Oscillator equation is well known in terms of proper time. In this chapter, those have been analyzed in a general context by transforming the proper time to the volume element of the space-time. Subsequently, RE has been analyzed in Wormhole (WH) configuration, particularly in the vicinity of the throat. Finally, the possibility of a time-like geodesic to move across the throat has been examined for a ultrastatic traversable WH. Thus the chapter studies two non-singular models of the universe namely Emergent scenario and wormholes using the celebrated Raychaudhuri equation where essentially there leads to violation of the singularity theorems.

CHAPTER 9

SUMMARY AND FUTURE WORK

This chapter summarizes the whole research work carried out to address the most critical problem in classical gravitational physics—the problem of singularity. All physical laws break down and physical quantities diverge at a singularity. On the other hand, from geometric point of view existence of singularity is associated with the notion of geodesic incompleteness. We have explored this problem and attempted its possible mitigation using the Raychaudhuri equation and Focusing theorem as the main tool both classically as well as quantum mechanically. In all those models, we have deduced the criteria or conditions to violate the Focusing of geodesics with an aim to mitigate the issue of singularity. Moreover, we have framed the Raychaudhuri equation in quantum settings motivated by the fact that quantum effects which become dominant in strong gravity regime may alleviate the singularity problem at the classical level. The thesis also discusses the cosmological implications of the Raychaudhuri scalar, Black Hole singularity and various non-singular models like Wormholes, Bouncing models and emergent scenario—all from the point of view of Raychaudhuri equation. Thus the thesis can be summarized chapter wise as follows:

Chapter 2 deals with the formulation of Raychaudhuri equation and the corresponding convergence condition in $f(R)$ gravity model with inhomogeneous FLRW space-time. This is because, it is possible to have apparent acceleration of the universe due to the metric of local inhomogeneities and thus the inhomogeneous model may serve as an alternative to Dark Energy. We have found that although the background is inhomogenenous yet the Raychaudhuri equation so constructed is homogeneous in nature and is a function of the energy density of the physical fluid. Convergence condition for a congruence of time-like geodesic in the inhomogeneous background is not universally satisfied. Under suitable choice of the parameters involved in the inhomogeneous metric function, it is possible to avoid singularity unlike General Relativity. This chapter also encounters asymptotic behavior of the various kinematic quantities like expansion, shear, rotation etc. and using their expressions in the Raychaudhuri equation it is found that expansion leads to a constant value at infinity and hence the congruence either converge or diverge. Further, we have studied the nature of time-like geodesic congruence graphically by considering a conformal line element to the given inhomogeneous FLRW metric and found that both bounded and unbounded geodesics are possible considering inhomogeneity and isotropy at the outset.

Chapter 3 is entirely devoted to the Raychaudhuri equation in quantum settings. In this chapter, we have dealt with the quantization of the Raychaudhuri equation and deriving a quantum replica of the classical geodesics for possible resolution of singularity that persist at the classical level. An analytic solution of the Raychaudhuri equation has been found by a suitable transformation of variable obtained by a first integral formulation. Lagrangian and Hamiltonian formulation have been carried out. Moreover, Wheeler DeWitt quantization has been presented which is expected to find application in the investigation of singularities in quantum regime for the collapse of homogeneous system like Datt-Oppenhiemer Synder collapse. Subsequently, WKB approximation followed by formulation of Quantum Bohmian trajectories have been done. The trajectories for the present quantum system obliterate the initial big-bang singularity as and when the quantum potential is included. Thus, the chapter shows whether singularity may be mitigated in the quantum description or not using WD and Bohmian formalism.

Chapter 4 deals with the classical and quantum implications of the Raychaudhuri equation in $f(T)$-gravity theory. Classically we have deduced the modified Raychaudhuri equation in $f(T)$ gravity and studied the focusing condition for two physically motivated $f(T)$ gravity models—Model 1: $f(T) = \alpha(-T)^n$ and Model 2: $f(T) = \alpha T + \dfrac{\beta}{T}$. We have found that avoidance of singularity is very much associated with the choice of $f(T)$ and nature of cosmic fluid. For Model 1 unlike General Relativity singularity may be avoided even with the assumption of Strong Energy Condition on matter i.e, with usual fluid. However, possible avoidance of singularity is guaranteed by some ghost field in case of Model 2. This chapter also shows an application of the WD quantization scheme presented in chapter 3 in homogeneous $f(T)$ gravity model in the context of singularity analysis. We have studied the probability function near the classical singularity for both the models and found that choice of the potential corresponding to the dynamical system representing the congruence of time-like geodesic has a very important role to play in resolution of singularities quantum mechanically. For inverse power law (linear and quadratic) choice of potential in case of Model 1 and linear, quadratic and inverse linear power law choice of potential in Model 2, the probability is zero at the big-bang singularity. Thus we have addressed the mitigation of singularity in some $f(T)$ gravity models in this chapter.

Chapter 5 deals with an exhaustive study of bouncing cosmology in the background of homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker space-time. we have studied the geometry of the bouncing point extensively and used as a tool to classify the models from the point of view of cosmology. Further we have furnished Raychaudhuri equation in these models to classify the bouncing point as regular point or singular point. Next we have discussed the behavior of time-like geodesic congruence in the neighbourhood of the bouncing point using the Focusing Theorem which follows as a consequence of the RE. Finally we have made an analogy of the RE with the evolution equation for a linear harmonic oscillator and discussed an oscillatory bouncing model in this context.

Chapter 6 aims to study the geometry and physics of the Raychaudhuri equation (RE) in the background of a homogeneous and anisotropic space-time described by Kantowski-Sachs (KS) metric. We have analyzed the role of anisotropy/shear in the context of convergence and possible avoidance of singularity subject to a physically motivated constraint. We have obtained that the constrained KS model may avoid the initial singularity with suitable choice of the parameters involved and an expanding universe with either cosmological constant or non-phantom energy may avoid the formation of singularity in the presence of anisotropy. Moreover, using a suitable transformation the first order RE has been converted to a second order differential equation analogous to a Harmonic Oscillator and criterion for convergence has been shown to be associated with the time varying frequency of the Oscillator. Finally, the chapter points out a geometric and physical notion of anisotropy along with their corresponding behavior towards convergence. It is found that geometric anisotropy favors convergence and physical anisotropy defies it.

Chapter 7 deals with the solution of Raychaudhuri equation and how it helps to describe the entire cosmic scenario. We have restated the Focusing theorem which follows as a consequence of the Raychaudhuri equation in terms of the cosmic parameter q (deceleration parameter) both for Einstein gravity and for modified gravity theory. Measurable quantities namely the luminosity distance and density parameter are shown to have an upper bound using the Raychaudhuri scalar. The chapter also gives a transformation (related to the metric scalar of the hyper-surface) which converts the first order Raychaudhuri equation to a second order differential equation which has a first integral. This first integral determines an analytic solution of the Raychaudhuri equation. Using this first integral we have found cosmological solutions for different choices of the equation of state parameters (both constant as well as variable). In most of the choices (where scale factor and Hubble parameter have explicit form), the cosmological parameters namely the scale factor (a) and the Hubble parameter (H) behave in accordance with the observational data at least qualitatively i.e. the universe is in an expanding phase with the rate of expansion gradually decreases. However, for the specific choices of the equation of state as a function of the scale factor (variable equation of state) only acceleration ($\ddot{a}$) can be evaluated as a function of the scale factor (a).

Chapter 8 deals with the implications of the Raychaudhuri equation in two non-singular model of universe namely Wormholes and Emergent scenario. Raychaudhuri equation states that exotic matter is required for emergent universe. We have shown the behavior of time-like as well as null geodesics near the throat of traversable wormholes. The chapter also gives two transformations which can be used to restate the Focusing theorem. We have also examined the role of red-shift function and wormhole shape function in convergence using the Raychaudhuri equation.

To conclude, this thesis aims at the classical and quantum consequences of the RE and singularity analysis in GR as well as in Modified theories of gravity. To mitigate the cosmological singularities namely the initial big-bang singularity, different techniques have been adopted along with further scope of application. The wide range of applicability of the RE may be attributed to the fact that the equation rather the identity encode geometric statements about flows. The scope of Raychaudhuri equation is beyond gravity and cosmology. This is because, it is an identity in Pseudo-Riemannian geometry and the effect of gravity comes into picture through the Ricci tensor projected along the congruence. In physics, flows appear in diverse contexts and so is the applicability of the RE, as the later is nothing but an evolution for the geodesic flow. Understanding the singularity theorems classically and restating them in a different approach can be a good piece of future work. That means, it is exciting to find how the classical singularity theorems will look like if restated and if anything more can be inferred by putting or removing certain assumptions?

Although the quantum corrections to RE has been studied extensively as a tool to avoid the Black-Hole singularity, more applications can be studied in greater details. We want to pursue future research to understand the nature of gravitational interaction in the quantum regime. Since, RE is associated with geodesic flow, it can also be made analogous to geometric flows particularly the Ricci flow so that if the analogy is established then properties and theorems of Ricci flow can be applied for more exciting results to come up. Since the kinematic quantities that appear on the r.h.s of the RE are derivatives of a vector field. This hints that wherever there is a vector field, we can deduce some analogous form of the RE. The universal attractive nature of gravity in General Relativity is a feature embodied by the RE which requires the non-increasing expansion of a congruence of geodesics. In most of the works, vorticity or rotation has been considered to be zero. Reformulating the RE and CC in the presence of rotation might be of central interest.

On the other hand, there are a plethora of gravity theories where one may formulate the RE and modified CC for examining singularity free nature of underlying theory and background space-time. Specifically, formulation of RE in $f(Q)$ gravity might be interesting to study the role of the non-metricity scalar Q in focusing/convergence. Further the application of RE and more importantly the singularity theorems in studying collapse of a star and its stability is of greater importance since these include energy conditions. In the thesis, we have discussed the implications of the Raychaudhuri equation in case of non singular models like bouncing model, emergent scenario and Wormholes. Further, the shadow of wormhole [314],[315] which has attracted the interest of astrophysicists and cosmologists may be studied from the point of view of Raychaudhuri equation [316]. Also, one may study the effect of null Raychaudhuri equation in gravitational lensing, one of the finest prediction of General Relativity [317]. A few reconstruction of the cosmological models along with their stability analysis have been done in literature using the Raychaudhuri equation [318]-[320]. More such reconstructions may be carried out to see if Raychaudhuri equation can provide some valuable insights. In quantum mechanics there is no idea as "*paths*". So, Bohmian trajectories can be made more elaborate in the sense that generalization of the Bohmian formalism may come up with interesting comments on the nature of the trajectories near the classical Big-Bang singularity. In addition, studying the nature of these trajectories at black-hole singularity and wormholes might be an interesting piece of future work. Recently, Breno Barreto da Silva et al. have studied the kinematic parameters and Raychaudhuri equation in Kantowski-Sachs model [321]. In this study, rotation term vanishes as Kantowski Sachs (KS) model is Locally Rotationally Symmetric (LRS). One may study the kinematics and associated Raychaudhuri equation in Bianchi models where one can take into account the effect of rotation or voticity unlike the KS metric. The motivation of this study comes from the inference that anisotropy favors convergence while rotation defies it. Avoiding focusing might avoid the singularity and in this connection the vorticity/rotation might play an important role. Most importantly, reformulation of the RE and famous singularity theorems in GR might be carried out from a different geometric concept namely, the matrix representation of tensor fields. These are some possible scope of future work. We end this thesis here, stating clearly the fact that there are diverse fields where the seminal Raychaudhuri equation can be employed, of which we have touched only a few.

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