Some Algebraic and Graph Theoretical Problems Related to Semigroup Theory

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CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled "**Some Algebraic and Graph Theoretical Problems Related to Semigroup Theory**" submitted by Sri **Biswaranjan Khanra** who got his name registered on **12 th october, 2020 (Index No: 32/20/Maths./27)** for the award of **Ph. D.** (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Dr. Manasi Mandal** and **Dr. Sarbani Mukherjee** and that neither this thesis nor any part of it has been submitted for either any degree/ diploma or any other academic award anywhere before.

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Dedicated to

my beloved parents

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Abstract

In the thesis, some characterization problems related to ideals of a commutative semigroup have been studied. Also some new graphs over semigroups have been defined and various aspects of these graphs, especially the interdependence of the graph-theoretic properties and the algebraic properties have been studied. This thesis contains 9 chapters.

In **Chapter 1**, an introductory ideas about the thesis have been discussed and some important preliminaries required in the entire thesis have been mentioned.

In **Chapter 2**, the notion of 2-absorbing ideals in a commutative semigroup has been considered. It is observed that every maximal ideal of a commutative semigroup is 2-absorbing but the converse is not true. Then a commutative semigroup in which 2-absorbing ideals are maximal has been characterized. The concept of 2-AB semigroup, in which 2-absorbing ideals are prime, has been introduced. Also a 2-AB semigroup has been characterized in terms of minimal prime ideal over a 2-absorbing ideal and some properties of these semigroups have been studied.

In **Chapter 3**, the notion of 2-absorbing primary ideals of a commutative semigroup has been introduced. The relation of 2-absorbing primary ideals with prime, maximal, semiprimary and 2-absorbing ideals has been established. Various characterization theorems on a commutative semigroup, in which 2-absorbing primary ideals are prime, maximal, semiprimary and 2-absorbing ideals have been obtained. Also some other important properties of 2-absorbing primary ideals of a commutative semigroup has been studied.

In **Chapter 4**, the concepts of 2-prime and weakly 2-prime ideals in a commutative semigroup have been introduced and studied. Then a semigroups where 2-prime ideals are prime has been characterized and also a semigroups where every ideal is weakly 2-prime has

been characterized.

In **Chapter 5**, the power graph $\mathcal{P}(\mathcal{S}_M)$ over a monogenic semigroup $\mathcal{S}_M$ with zero element has been considered and studied. Various graph parameters of $\mathcal{P}(\mathcal{S}_M)$ and topological indices based on distance of vertices have been determined. Finally, some graph parameters of the cartesian product $\mathcal{P}(\mathcal{S}_M^1)\Box\mathcal{P}(\mathcal{S}_M^2)$ of graphs $\mathcal{P}(\mathcal{S}_M^1)$ and $\mathcal{P}(\mathcal{S}_M^2)$ has been computed.

In **Chapter 6**, the inclusion ideal graph $In(S)$ of nontrivial right ideals of a semigroup S with zero element has been considered. A semigroup S for which the graph $In(S)$ is complete, connected has been characterized. Also various graph parameters of $In(S)$ have been obtained. The values of n for which the graph $In(Z_n)$ is complete, triangulated, split, unicyclic, thresold have been determined and also minimal embedding of $In(Z_n)$ into compact orientable (resp. non-orientable) surface have been studied. Both upper and lower bounds for metric and partition dimension of inclusion ideal graph of a completely 0-simple semigroup has been provided. Finally, some graph parameters of the cartesian product of inclusion ideal graph of two monoids have been computed.

In **Chapter 7**, the inclusion ideal graph of multiplicative semigroup R_S of a commutative ring R with unity, denoted by $In(R_S)$, has been considered. A commutative ring R has been characterised for which the graph $In(R_S)$ is null, complete, connected, bipartiate, split, co-graph, unicyclic and outerplanar, planar, toroidal and bitoroidal. Also the diameter and girth of the graph $In(R_S)$ have been characterized.

In **Chapter 8**, the notion of prime inclusion ideal graph $In_p(S)$ of nontrivial prime ideals of a commutative semigroup S has been introduced. Then a semigroup S has been characterized for which the graph $In_p(S)$ is null, complete or connected. Also various graph parameters, minimal embedding, metric and partition dimension of the prime inclusion ideal graph of the multiplicative semigroup Z_n of integers of modulo n have been studied.

In **Chapter 9**, three inclusion graphs related to ideals of commutative semigroup have been conidered and its relation with inclusion (resp. prime inclusion) ideal graph of a commutative semigroup have been observed. Also it is observed when the graphs related to ideals of a commutative semigroup are isomorphic.

Contents

List of symbols

$\mathbb{Z}$	Set of all integers
$\mathbb{N}$	Set of all natural numbers
M	Set of all composite natural numbers
$\mathbb{R}$	Set of all real numbers
$[k]$	$\{1, 2, \ldots, k\}$
$\mathbb{Z}_n$	Congruent classes of integers modulo n
$\sigma(n)$	Number of positive divisors of n
$\lfloor x \rfloor$	Greatest integer less or equal to x
$\lceil x \rceil$	Least integer greater or equal to x
n_p	Greatest prime less or equal to n
$\|A\|$	Cardinality of a set A
$A - B$	All elements which are in set A but not in set B
$I^*(R)$	Set of all non trivial ideals of R
$Z(R)$	Set of zero-divisors of a ring R
$J(R)$	Jacobson radical of a ring R
$\text{Nil}(R)$	Set of all nilpotent elements of R
$\text{Max}(R)$	Set of all maximal ideals of R
$\text{Min}(R)$	Set of all minimal ideals of R
$U(R)$	Set of units of a ring R
$(x_1, x_2, \ldots, x_k)$	Ideal grnerated by k elements x_1, x_2, $\ldots$, x_k
R_S	Multiplicative semigroup of a ring R
$\mathcal{D}$	Class of semigroups with unity having no proper essential right congrurence

$gcd(a, b)$	Greatest common divisor of integers a, b
$a \sim b$	Vertices a, b are adjacent
$deg(v)$	Degree of a vertex v
$diam(G)$	Diameter of a graph G
$girth(G)$	Girth of a graph G
$d(a, b)$	Length of the shortest path between vertices a, b
$\overline{G}$	Complement of a graph G
K_n	Complete graph of n vertices
$K_{m,n}$	Complete bipartite graph where partite sets have sizes m and n
P_n	Path with exactly n vertices
C_n	Cycle with exactly n vertices
$dom(G)$	Domination number of a graph G
$\alpha(G)$	The independence number of a graph G
$G \square H$	Cartesian product of two graphs G, H
$g(G)$	The genus of a graph G
$\theta(G)$	Thickness of a graph G
$N(v)$	The set of neighbours of a vertex v in a graph G
$\omega(G)$	Clique number of a graph G
$\chi(G)$	The chromatic number of a graph G
$\mathcal{S}_M$	Monogenic semigroup
$\mathcal{P}(\mathcal{S}_M)$	Power graph over monogenic semigroup $\mathcal{S}_M$

Chapter 1

Introduction, preliminaries and prerequisites

1.1 Introduction

From a study of various generalizations of groups, the theory of semigroup has become a separate scientific subject with a great number of published results, its own journal, a growing number of monographs and many open research fronts. The main reason for this development is that semigroups appear naturally in almost all mathematical contexts, and information about semigroup related to a mathematical object yields some information about the object itself. The monograph Howie (1976) is an excellent introduction to semigroup theory.

The concept of ideals, which arises in ring theory as a generalization of a special subset, the multiple of an integer in the ring of integers, play an highly important role in studying the structure of a ring. Also the concept of prime ideal, which arises in the theory of rings as a generalization of the concept of prime numbers in the ring of integers, occupied the central position in that theory similar as primes in arithmetic. There are many generalizations of prime ideals in ring theory, such as valution ideal, primary ideal, semiprimary ideal, semiprime ideal, f-prime ideal and etc. The notion of prime ideals and its generalizations have a vital role in commutative algebra since they have some applications to other areas such as Graph theory, Cryptology, Algebraic Geometry, General topology and etc. On the other hand, the prime ideals and its generalizations are used to classify certain class of rings such as fields, integral domain, von Neumann regular rings, Dedekind domains, Valuation domains and etc. The multiplicative theory of ideals in a commutative ring is a highly developed area of research as may be attested to by the numerous research articles and books. There are many similarities between the ideal theory of semigroups and that of rings. Many definitions and theorems from commutative ring theory have natural analogues in semigroup theory. All these concepts of ideals are defined in semigroup theory also and used to study the structure of a semigroup.

In recent times, several authors studied various generalizations of prime ideals in a commutative ring such as 2-absorbing ideals, 2-absorbing primary ideals, 2-prime ideals etc. In [11], Badawi introduce the notion of 2-absorbing ideal in a commutative ring as a generalization of prime ideal and later in [28], Cay et al. define the concept in a commutative semigroup as follows:

Definition 1.1.1. ([28], Definition 2.1) A proper ideal I of commutative semigroup is said to be a 2-absorbing ideal if for any elements a, b, $c \in S$, $abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$.

Then in [18], Bennis et al. characterize commutative rings in which 2-absorbing ideals are prime. The notion of 2-absorbing primary ideal of a commutative ring was introduced as a generalization of 2-absorbing ideal by Badawi [12]. The concept 2-absorbing primary ideals have been considered in various algebraic structures (ring, semiring) by many authors [12, 59, 66, 84] and 2-absorbing primary ideals in lattices were defined and studied in [87].

The concept of 2-prime (resp. weakly 2-prime) ideals as a generalizations of prime ideals (resp. weakly 2-prime ideals) in a commutative ring was introduced in [15](resp.[53]) and rings in which 2-prime ideals are prime was characterized in [67]. These observations prompted us to consider these concept in a commutative semigroup.

Leonhard Euler's paper on the historical Konigsberg's Bridges problem marked the beginning of graph theory. Euler showed the old and famous Konigsberg's Bridges puzzle graphically, and was able to show that it has no solution. In 1878, the term 'graph' was coined by J. J. Sylvester. Nowdays, it is one of the most extensively studied mathematical disciplines. Considered as one of the most important branches of combinatorics, graph theory has itself given rise to newer subfields, such as geometric graph theory, extremal graph theory, probabilistic graph theory and algebraic graph theory. Graph theory also has a wide range of applications in practical situations as well as in other fields like computer science, linguistics, genetics, social sciences etc.

Nowdays in algebric combinatorics, a major part of research is attached to the application of graph theory and combinatorics in abstract algebra. The study of algebraic structure via associating graphs is a very large and rising area of research, leading to many interesting evolutions. In these cases, usually some subset of an algebraic structure is taken as the vertex set, and an algebraic operation is used to define the adjacency conditions for the graph. The purpose of associating graphs to algebraic structures is to characterize the algebraic structure via graphs and vice versa. The opportunity to study this interplay has motivated the mathematicians to define various graphs over different algebraic structures and it has now become a very intense field of study.

Various graphs over semigroup have been defined and studied. In 1878, Cayley presented a graphical representation of groups, which gave rise to the idea of Cayley graphs defined over groups. Later on, Cayley graphs have been also generalized for semigroups. The non-commutating graph of a semigroup, zero divisor graph of a semigroup and the power graph of a semigroup are other examples of graphs defined over semigroups. Again, inclusion ideal graph of a semigroup and the intersection graph of ideals of a semigroup are instances of graphs defined over ideals of semigroups. Some results pertaining these graphs will be discussed in last section of this chapter.

In this thesis, we consider some generalizations of prime ideals in a commutative semigroup and study various characterization problem. Also we define some new graphs over semigroups and study the various aspects of these graphs, especially the interplay of the graph-theoretic properties and the algebraic properties.

Also we briefly give an outline of the earlier studies on graphs over various algebraic structures which motivate us to define the graphs studied in this thesis.

Among the graphs defined over rings, we first mention the graphs other than zero-divisor graphs or its variants. In 2009, Chakrabarty et al studied the intersection graphs of (non-trivial) ideals of rings [24]. The study of intersection graph of ideals is very much an active field in recent times (*cf*. [5, 74]). Another graph which uses the ideals of a ring is the annihilating-ideal graph of a ring, which was introduced by Behboodi and Rakeei [16]. Some constructions of graphs defined over rings can also be found in [34, 43, 83]. Comaximal graphs of commutative rings were studied in [63, 82].

Regarding the graphs defined over groups, Cayley graphs [21], the non-commuting graphs [1], intersection graphs of non-trivial subgroups of finite abelian groups [91] and power graphs [22] deserve special mention. Also Cayley graphs have been defined over semigroups [52]. In fact, various graphs have been defined over semigroups. In 1964, Bosak initiated certain graphs defined over semigroups [20]. Then in 2002, Kelarev and Quinn defined the directed power graphs on semigroups [54]. The divisibility graph over a semigroup was also introduced by Kelarev and Quinn [54]. Then Chakrabarty et al. define the concept of an undirected power graphs of semigroups [31]. Since then, the study of power graphs was continued by several authors (*cf*. [29, 64, 58]).

In [24], the power graph $\mathcal{J}(\mathcal{S})$ of a semigroup S is defined as the simple undirected graph with vertex set S and two distinct vertices $x, y \in S$ are adjacent if and only if $x = y^i$ or $y = x^i$ for some $i \in \mathbb{N}$. In [35], Das et al. define a graph $\Gamma(\mathcal{S}_M)$ over a finite monogenic semigroup $\mathcal{S}_M$ with zero element, by changing the adjacency rule of vertices and not destroying the main idea, the zero divisor graphs [3]. These observations prompted us to study the power graph over a monogenic semigroup.

As the ideal structure reflects ring (resp. semigroup) properties, several graphs that are based on the ideals were defined (*cf*. [2, 3, 16, 24]). The notion of inclusion ideal graph $In(R)$ of a ring R, defined by Akbari et al. [3], is a graph with vertices are non-trivial left ideals of R and two distinct vertices I and J are adjacent if and only if $I \subset J$ or $J \subset I$. In [4], Akbari et al. have studied some graph parameters like diameters, girth, connectedness, perfectness of $In(R)$ and in [51] Kavitha et al. characterized non-local Artinian ring for which the graph $In(R)$ is unicyclic, outer-planar and planar. Then in [49], Jahanbaksh et al.

computed diameter, girth of inclusion ideal graph $In(P)$ of a poset P and also characterize a poset P for which $In(P)$ is tree, cycle and star graph. In [36], Das introduced the subspace inclusion graph $In(V)$ of a vector space V and determined diameter, girth, clique number and chromatic number of the graph. Further in [37], Das studied planarity, perfectness, independence number, matchings and also proved necessary and sufficient condition for which $In(V)$ is Eulerian. Recently, Baloda et al.[14] introduced the concept of inclusion ideal graph of a semigroup and studied connectedness, diameter, girth, perfectness, planarity of the graph. Also they determined independence number, matching number and automorphism group of the inclusion ideal graph of a completely simple semigroup. The multiplicative semigroup Z_n of residue classes of integers modulo n have been studied by several authors in ([46], [69]) and in [73] puninagool et al. prove that non-zero ideal of Z_n is of the form $\cup\{m_iZ_n : i = 1, 2, ..., k\}$, where $m_1, m_2, ..., m_k$ are divisors of n such that m_i does not divide m_j for $i \neq j$. The prime objective of topological graph theory is to embed a graph into a surface and also the concept of metric dimension of a graph has some applications in chemistry for representing chemical compounds as well as in problems of pattern recognition, image processing, navigation of robots in networks. These observations prompted us to solve the following problems related to the inclusion ideal graph of a semigroup. For example:

(1) When the graph $In(\mathbb{Z}_n)$ is split, planar, toroidal or bitoroidal?

(2) How about the metric dimension, partition dimension of the inclusion ideal graph of a completely 0-simple semigroup and also when the graph is planar, toroidal or projective?

For other related works on inclusion ideal graph, we refer [56] . The multipliactive semigroup of a ring have been considered by several authors [8, 9, 23, 47, 69] and also it play an highly important role in studying the structure of a ring. These obsevations prompted us to consider the inclusion ideal graph of the multiplicative semigroup of a ring. Also we know that prime ideals play an highly important role in studying the structure of a ring (resp. semigroup). These observations prompted us to consider the prime inclusion ideal graph of a commutative semigroup S, denoted by $In_p(S)$, is a graph with vertices are nontrivial prime ideals of S and two distinct vertices P_1 and P_2 are adjacent if and only if $P_1 \subset P_2$ or $P_2 \subset P_1$. Now we recall some results which is useful in this thesis.

1.2 Some preliminaries on algebraic structure

In this section, we discuss some preliminary definitions and notions of semigroup and ring theory which are relevant for this thesis.

Definition 1.2.1. [27] A groupoid is a system $(S, .)$ consisting of a non-empty set S together with a binary operation '.' on S.

Definition 1.2.2. [27] A semigroup is a groupoid $(S, .)$, such that the operation '.' is associative.

Definition 1.2.3. [27] A semigroup is said to have a zero element z if $az = za = z$, for all $a \in S$ and identity e if $ae = ea = a$, for all $a \in S$. A semigroup with identity is said to be a monoid .

Definition 1.2.4. [27] A semigroup S is said to be commutative if $ab = ba$ for all $a, b \in S$.

Definition 1.2.5. [27] A semigroup S is called regular if for all $a \in S$, there exists an element $x \in S$ such that $axa = a$.

Definition 1.2.6. [27] A non-empty subset I of a semigroup S is said to be a right ideal (resp. left ideal) of S if $IS \subseteq I$ (resp. $SI \subseteq I$). A non-empty subset of S which is both left and right ideal is said to be an ideal of S.

Definition 1.2.7. [27] For an element $a \in S$, $(a)_r = \{a\} \cup\ aS$ (resp. $(a) = \{a\} \cup Sa \cup aS \cup SaS$) is called the principal right ideal(resp. ideal) of S generated by a.

Definition 1.2.8. [79] A semigroup S is said to be a principal right ideal semigroup if every right ideal of S is principal.

Lemma 1.2.9. *[27] If I and J are right ideals of a semigroup S, then $I \cup J$ and $I \cap J$ are right ideals of S.*

Definition 1.2.10. [14] A right ideal M of a semigroup S is said to be maximal if it is not contained in any other nontrivial right ideal of S.

Lemma 1.2.11. *[78] If S is a commutative semigroup with identity which is not a group, then it has unique maximal ideal, which is the union of all nontrivial ideal of S.*

Definition 1.2.12. [27] A right ideal I of a semigroup S is called 0-minimal if $I \neq \{0\}$ and $\{0\}$ is the only right ideal properly contained in I.

Lemma 1.2.13. *([14], Remark 2.1) If I_1 and I_2 be any two distinct 0-minimal right ideal, then $I_1 \cap I_2 = \{0\}$.*

Proposition 1.2.14. *[14] If S has a 0-minimal right ideal then every nontrivial right ideal contains at least one 0-minimal right ideal and if J is a nontrivial right ideals of S other than 0-minimal right ideal I, then either $I \subseteq J$ or $I \cap J = \{0\}$.*

Definition 1.2.15. [81] An ideal P of a semigroup S is said to be prime if $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, A, B being ideals of S. An ideal P is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$, a, b being elements of S.

Remark 1.2.16. The concepts of prime and completely prime ideal are coincide if S is commutative.

Definition 1.2.17. [77] For an ideal A of a semigroup S, radical of A, denoted as $\sqrt{A}$, is the set of all $x \in S$ such that some power of x is in A. An ideal A of S is called primary if $ab \in A$ implies either $a \in A$ or $b \in \sqrt{A}$.

Definition 1.2.18. [60] An ideal I of a semigroup S is said to be semiprimary ideal if $\sqrt{I}$ is a prime ideal of S.

Definition 1.2.19. [60] A commutative semigroup S is called fully prime semigroup if every ideal of S is prime and primary if every ideal of S is primary. Also a semigroup S is said to be semiprimary if every ideal of S is a semiprimary ideal of S.

Theorem 1.2.20. *[77] If I and J are any two ideals of a commutative semigroup S, then the following statements about S are true*

(1) $IJ \subseteq I \cap J \subseteq I$

(2) $I \subseteq \sqrt{I}$

(3) $I \subseteq J \Rightarrow \sqrt{I} \subseteq \sqrt{J}$

(4) $\sqrt{IJ} = \sqrt{(I \cap J)} = \sqrt{I} \cap \sqrt{J}$.

(5) *If A is a prime ideal of S, then $\sqrt{A} = A$ and if A is a primary ideal of S, then $\sqrt{A}$ is a prime ideal of S.*

Theorem 1.2.21. *[77] Let I be an ideal of a commutative semigroup S with unity. If $\sqrt{I} = M$, where M is a maximal ideal of S, then I is a primary ideal of S.*

Theorem 1.2.22. *[77] In a commutative semigroup S with unity, the unique maximal ideal M is prime, which is the union of all proper ideals of S; $\sqrt{M^n} = M$ for every positive integer n and M^n is a primary ideal for every positive integer n.*

Theorem 1.2.23. *[77] The radical of an ideal I in a commutative semigroup is the intersection of all prime ideals containing I.*

Theorem 1.2.24. *[77] Any prime ideal containing an ideal I in a semigroup contains a minimal prime ideal belonging to I.*

Lemma 1.2.25. *[57] Let P be a prime ideal of S and I be an ideal of S with $I \subseteq P$. Then the following statements are equivalent:*
(1) *P is a minimal prime ideal over I,*
(2) *For every element $p \in P$, there is an element $s \in S - P$ and a non negative integer n such that $sp^n \in I$.*

Theorem 1.2.26. *[81] If M is a maximal ideal of a semigroup S such that $S - M$ contains either more than one element, or an idempotent, then M is a prime ideal of S.*

Definition 1.2.27. [27] If a is any element of a semigroup S, then the subsemigroup $< a >$ of S generated by a consists of all positive integral powers of a:

$$< a >= \{a, a^2, a^3, \dots\}$$

If $< a >= S$, then S is called a cyclic semigroup or a monogenic semigroup.

Definition 1.2.28. [27] An element e of a semigroup S is said to be an idempotent element if $e^2 = e$.

Definition 1.2.29. [27] An idempotent element f of S is said to be primitive if $f \neq 0$ and if $e \leqq f$ implies $e = 0$ or $e = f$, where " $\leqq$ " is the natural ordering on idempotents of S.

Definition 1.2.30. [27] A semigroup S with 0 is said to be right 0-simple (resp. 0-simple) if it has no right ideals (resp. ideals) other than 0 and S.

Definition 1.2.31. [27] A Semigroup S with 0 element is said to be completely 0-simple if it is 0-simple and contains a primitive idempotent element.

Definition 1.2.32. [27] Two elements of a semigroup S are said to be $\mathcal{L}$ (resp. $\mathcal{R}$) equivallent if they generate the same principal left (resp. right) ideal of S.

Definition 1.2.33. [27] The join of the equivalence relations $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{D}$ and their intersections by $\mathcal{H}$.

Lemma 1.2.34. *([14], Lemma* 2.2*) A left ideal K of a semigroup S is maximal if and only if $S - K$ is an an $\mathcal{L}$-class.*

Lemma 1.2.35. *[44] If S is a commutative semigroup then we have $\mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{H}$.*

Theorem 1.2.36. *([27], Theorem* 2.16*) If a, b and ab all belong to same $\mathcal{H}$-class H of a semigroup S, then H is a subgroup of S.*

Definition 1.2.37. [55] A semigroup in which every ideal is idempotent is called fully idempotent semigroup.

Theorem 1.2.38. *[60] Let S be a commutative semigroup. Then S is a semiprimary semigroup if and only if prime ideals of S are linearly ordered.*

Definition 1.2.39. [81] A commutative semigroup S is said to be Archimedian if, for any two elements of S, each divides some power of the other. Moreover, a commutative semigroup is archimedian if and only if S has no proper prime ideals.

Theorem 1.2.40. *([27], Theorem* 4.13*) Every commutative semigroup S is uniquely expressible as a semilattice Y of archimedian semigroups S_α $(\alpha \in Y)$.*

Definition 1.2.41. [27] A right (resp. left) congrurence σ on a semigroup S is an equivalence relation on S such that if $(a, b) \in \sigma$ then for any $c \in S$ we have $(ac, bc) \in \sigma$ (resp. $(ca, cb) \in \sigma$). A congrurence on a semigroup is an equivalence relation which is both left and right congrurence.

Definition 1.2.42. [55] A congrurence δ on a semigroup S is said to essential if for every congrurence $\alpha \neq i$ we have $\alpha \cap \delta \neq i$, where i is the identity relation on S.

We denote the class of all semigroups with unity having no proper essential right congrurence by D.

Theorem 1.2.43. *([55], Theorem 2.5) If $S \in D$, then S is regular.*

Theorem 1.2.44. *([68], Theorem 2) If $S \in D$, then the set of ideals of S are linearly ordered.*

Definition 1.2.45. [81] An ideal I of a semigroup S is said to be a semiprime ideal if $a^2 \in I$ for some $a \in S$ implies $a \in I$.

Definition 1.2.46. [39] Let S be a semigroup and x be an indeterminate. Then $S[x] = \{sx^i : s \in S, i \geq 0\}$ forms a semigroup wiith respect to the multiplication defined as : $(sx^i)(tx^j) = (st)x^{i+j}$, where $s, t \in S$ and $i, j \geq 0$, called the polynomial semigroup over S. Similarly we can define the polynomial semigroup $S[x_1, x_2, \ldots, x_n]$ in n variables.

Definition 1.2.47. [42] Let C be a non-empty subsemigroup of a commutative semigroup S. Let ρ be the relation defined on $S \times C$ by $(x, a)\rho(y, b)$ if and only if $cay = cxy$ for some $c \in C$. Clearly ρ is a congrurence on $S \times C$. Then $C^{-1}S = \{\frac{s}{c} : s \in S, c \in C\}$ is the quoitent semigroup of $(S \times C)$ modulo ρ, called the semigroup of fractions.
The composition on $C^{-1}S$ is defined as $(\frac{x}{a})(\frac{y}{b}) = \frac{xy}{ab}$.

Lemma 1.2.48. *[42] If I is an ideal of S, then $C^{-1}I = \{\frac{i}{c} : i \in I, c \in C\}$ is an ideal of $C^{-1}S$. Moreover $\sqrt{C^{-1}I} = C^{-1}\sqrt{I}$.*

Theorem 1.2.49. *([76], Theorem 2.2) Let S be a commutative semigroup. Then every ideal of S is prime if and only if S is regular and idempotents of S are linearly ordered.*

Theorem 1.2.50. *([60], Theorem 2) Let S be a regular commutative semigroup.Then the following statements about S are equivallent:*

(1) *Every ideal in S is prime,*

(2) *S is a primary semigroup,*

(3) *S is a semi-primary semigroup,*

(4) *Idempotents in S form a chain under natural ordering,*

(5) *Principal ideals of S are totally ordered,*

(6) *All ideals of S are linearly ordered.*

Theorem 1.2.51. *([57], Theorem* 5.1*) The commutative semigroup S is regular if and only if every ideal in S coincides with its radical.*

Theorem 1.2.52. *([27], Theorem* 2.41*) Every ideal of an ideal of a semisimple semigroup S is an ideal of S.*

Lemma 1.2.53. *([27], 4(a), p.75) If A is an ideal of a semigroup S, and if B is an ideal of A such that $B = B^2$, then B is an ideal of S.*

Theorem 1.2.54. *([79], Theorem* 1.1*) If S is a principal right ideal semigroup, then right ideals form a chain under set inclusion.*

Definition 1.2.55. [80] A commutative semigroup is said to be chained semigroup if its ideals are linearly ordered by set inclusion.

Lemma 1.2.1. *([27],* 2(a)*, p. 76) A semigroup S is semisimple if and only if $A = A^2$ for every ideal A of S*

Theorem 1.2.56. *([38], Theorem* 3*) A commutative semigroup S satisfies ACC and DCC on its ideals if and only if S contains only only finitely many principal ideals.*

Theorem 1.2.57. *([48], Theorem* 2*). A commutative semigroup S is regular if and only if every ideal of S is idempotent.*

Theorem 1.2.58. *([70], Theorem* 1*) A commutative semigroup is archimedian if and only if it has no proper prime ideals.*

Theorem 1.2.59. *([78], Theorem* 2.1*) Let (S, M) be a commutative semigroup containing 0 and identity. Then every non-zero primary ideal is prime as well as maximal iff S is one of the following types.*
(1) *$S = G \cup M$, where G is the group of units and $M = \{0, ag : a \in M, a^2 = 0\}$*
(2) *S is the union of two groups with 0 adjoined.*

Theorem 1.2.60. *([78], Theorem* 3.1*) Let (S, M) be a commutative semigroup containing 0 and identity. Suppose that every non-zero primary ideal is prime or every non-zero ideal is prime. Then S satisfies either one of the following conditions.*

(1) $S = G \cup M$, *where* G *is the group of units and* $M = \{0, ag : a \in M, a^2 = 0\}$,

(2) $M^n = M$ *for every positive integer* n.

Theorem 1.2.61. *([73], Theorem 2) If* I *is a non-zero ideal of the multiplicative semigroup* $\mathbb{Z}_n$*, then* $I = \cup\{m_i\mathbb{Z}_n : i = 1, \ldots, k\}$*, where* m_1*,* m_2*,* $\ldots$*,* m_k *are divisors of* n *such that* m_i *does not divide* m_j *if* $i \neq j$*.*

Definition 1.2.62. [10] A non empty set R together with two binary operations "+" and ". " defined on it is said to be a ring if

(1) $(R, +)$ is an abelian group,

(2) $(R, .)$ is a semigroup,

(3) $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$, for all x, y, $z \in R$.

Definition 1.2.63. [10] A ring R is said to be commutative if $xy = yx$ for all x, $y \in R$.

Definition 1.2.64. [10] A ring R is said to be a ring with unity if there exists an element $1 \in R$ such that $1x = x1 = x$ for all $x \in R$.

Definition 1.2.65. [10] An element u is called a unit in R if there exists an element $v \in R$ such that $uv = vu = 1$.

Definition 1.2.66. [10] A commutative ring with unity in which every element is a unit is called a field .

Definition 1.2.67. [10] An element x in R is said to be nilpotent of index k if $x^k = 0$. We write the nilpotency of x by $n(x)$.

Definition 1.2.68. [10] A non-empty subset I of a ring R is said to be an ideal of R if for all $a, b \in I$ and for all $r \in R$, we have that $a - b \in I$, ra, $ar \in I$. An ideal I of a ring R is called a proper ideal if $I \neq R$.

Definition 1.2.69. [10] For a non-empty subset S of a ring R, (S) denotes the ideal generated by S, which is defined to be the intersection of all ideals of R containing S.

Definition 1.2.70. [10] A proper ideal I is said to be maximal ideal of R if for any ideal J with $I \subseteq J$, either $J = I$ or $J = R$.

Definition 1.2.71. [10] A commutative ring R is called a local ring if R has unique maximal ideal.

Definition 1.2.72. ([10], page 89) A commutative ring R is said to be Artinian if it satisfies minimal conditions on its ideals.

Definition 1.2.73. [10] A ring R is said to be special if $\mathrm{Nil}(R) = J(R)$, where $\mathrm{Nil}(R)$ is the set of all nilpotent elements of R and $J(R)$ is the Jacobson radical of R.

Definition 1.2.74. [50] A commutative ring R with unity is called arithmetical if lattice of ideals of R is distributive.

Definition 1.2.75. [50] A commutative ring R with unity is called Bezout if every finitely generated ideal of R is principal. Clearly every Bezout ring is arithmetical but converse is not true.

Theorem 1.2.76. *([10], Theorem 8.7) An Artinian ring R is uniquely (up to isomorphism) a finite direct product of Artinian local rings.*

Proposition 1.2.77. *([10], Proposition 8.8) Let R be a commutative Artinian local ring with unity such that the maximal ideal is principal. Then the following are equivalent:*

(1) *every ideal in R is principal,*

(2) *the maximal ideal is principal,*

(3) $dim(\frac{m}{m^2}) \leq 1$.

Theorem 1.2.78. *([50], Theorem 1) A commutative local ring with unity is arithmetical if and only if its ideals are totally ordered.*

Theorem 1.2.79. *([50], Theorem 5) A commutative semi-local arithmetical ring with unity is a Bezout ring.*

Theorem 1.2.80. *([72], Proposition 3) Let R be a commutative local ring with unity such that maximal ideal m minimally generated by two elements x_1 and x_2 and $m^2 = 0$. Then nontrivial ideals of R other than m is of the form (x_1), (x_2) and $(x_1 + tx_2)$, where t is a unit of R.*

Theorem 1.2.81. *([12], Theorem 2.2) If I is a 2-absorbing primary ideal of a commutative ring R, then $\sqrt{I}$ is 2-absorbing ideal of R.*

Theorem 1.2.82. *([12], Theorem 2.4) Let R be a commutative ring with unity. Suppose that I_1 is a P_1-primary ideal of R for some prime ideal P_1 of R_1, and I_2 is a P_2-primary ideal of R for some prime ideal P_2 of R_2. Then the following statements holds.*
(1) $I_1 \cap I_2$ is a 2-absorbing primary ideal of R,
(2) $I_1 I_2$ is a 2-absorbing primary ideal of R.

Theorem 1.2.83. *([12], Theorem 2.20(1)) Let $f : R \longrightarrow R'$ be a ring homomorphism of commutative rings. Now if I' is a 2-absorbing primary ideal of R', then $f^{-1}(I')$ is a 2-absorbing primary ideal of R.*

Theorem 1.2.84. *([28], Theorem 2.18) Let S and T be commutative multiplicative semigroup with 1 and 0. Let $f : S \longrightarrow T$ be a homomorphism of semigroups and I be a proper ideal of S such that $\{(x, y) \in kerf : x \neq y\} \subseteq I \times I$. Then*
(i) If $f(I)$ is a strongly 2-absorbing ideal of T, then I is a strongly 2-absorbing ideal of S.
(ii) If f is surjective and I is a strongly 2-absorbing ideal o S, then $f(I)$ is a strongly 2-absorbing ideal of T.

Theorem 1.2.85. *([55], Theorem 2.17) Let $R = R_1 \times R_2$, where R_1, R_2 be commutative semirings and $I = I_1 \times I_2$ is an ideal of R such that I_1 and I_2 are ideals of R_1 and R_2 respectively. Then the following statements are equivallent.*
(i) I is a 2-absorbing primary ideal of R.
(ii) $I = I_1 \times R_2$ for some 2-absorbing primary ideal I_1 of R_1 or $I = R_1 \times I_2$ for some 2-absorbing primary ideal I_2 of R_2 or $I = I_1 \times I_2$ for some primary ideal I_1 of R_1 and for primary ideal I_2 of R_2.

1.3 Some notions related to graph-theory

Here we discuss some definitions and terminologies related to graph theory.

A directed graph or digraph D consists of a finite nonempty set V of points together with a prescribed collection X of ordered pairs of distinct points. The elements of X are directed lines or arcs. A oriented graph is a digraph having no symmetric pair of directed lines. A graph G is an ordered pair (V, E), where V is a non-empty set called the set of vertices or vertex set $V(G)$, E is a set called the set of edges $E(G)$. If $e = \{v, w\} \in E(G)$, we call v and w the endpoints of e. If u and v are two endpoints of an edge e, the vertices u and v are said to be adjacent and we write $e = uv$ (or $e = vu$). The notation $u \sim v$ denotes that u, v are adjacent to each other, otherwise by $u \nsim v$.

A graph is finite if its vertex set and edge set are finite. A finite graph can be pictorially represented. In the usual pictorial representation, each vertex is represented by a dot in the plane, and the dots corresponding to any two adjacent vertices are joined by drawing a line. Sometimes, the diagram of a graph is identified as the graph itself, but one should keep in mind that it is only a representation of a graph.

The set of vertices adjacent to a particular vertex v is called the set of neighbors of v in G, and it is denoted by $N(v)$. The *closed neighbouhood* of a vertex v of G is $N[v] = N(v) \cup \{v\}$. If u is an endpoint of an edge e, then e is said to be incident on u. A loop is an edge whose endpoints are the same. Multiple edges are those edges which have the same pair of endpoints. A simple graph is a graph having no loops or multiple edges. If a graph has no edges, it is called a *null graph*.

The *degree* of a vertex v, denoted by $deg(v)$, is the number of edges incident on v (counting a loop twice). A vertex v is isolated if the degree of v is zero and pendent if $\deg(v)$ is one. Note that $deg(v)$ can be infinite if infinite graphs are considered. A graph G is called a *regular graph* if all the vertices of G have the same degree. If $deg(v) = k$ for all vertices v of a graph G, then G is called *k-regular*.

A walk of a graph G is an alternating sequence $(v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ of vertices and edges, beginning and ending with vertices, in which each edge is incident on the two points immediately preceding and following it. The length of a walk is the number of occurrences of edges in it. A walk is closed if $v_0 = v_n$ and open, otherwise. A trail is a walk if all its edges are distinct. A path is a walk if there is no repetition of vertices (and hence, no repetition of edges) are distinct. The existence of a walk between two vertices ensures the existence of a path between them.

A graph is *connected* if there is a path between any two distinct vertices. A maximal connected subgraph of G is a connected component or simply a component of G. A graph is disconnected if it is not connected.

Let u, v be two vertices of a graph G. The length of the shortest path between u and v is called the distance between u and v, which is denoted by $d(u, v)$. If there is no path between u and v then $d(u, v)$ is denoted by ∞, and also $d(w, w) = 0$ for any vertex w. The *diameter* of a graph G, denoted by $diam(G)$, is defined as the supremum of the set $\{d(u, v) \mid u, v \in V(G)\}$. Clearly, if G is disconnected, then $diam(G) = \infty$.

A *cycle* is a closed path and a circuit is a closed trail. A cycle with an odd number of edges (or vertices) is an odd cycle, whereas a cycle with an even number of edges (or vertices) is an even cycle. A graph having no cycle is called *acyclic*. The length of the shortest cycle in a graph G is called the *girth* of the graph, denoted by $gr(G)$. If G is acyclic, $gr(G)$ is denoted by ∞. A graph G is called a *tree* if it is a connected acyclic graph. A graph G is said to be *triangulated* if for any vertex u in $V(G)$, there exists v, w in $V(G)$, such that (u, v, w) is a triangle in G.

A vertex v in a graph G is called a *cut vertex* if deleting V from G increases the number of components of G. A *bridge* is an edge of a graph G whose deletion increases the number of connected components of G. Equivalently, an edge of a graph is a bridge if and only if it is not contained in any cycle. The number of edges of a graph G which is not contained in any cycle is called the *bridge number* of G and denoted by $\mathrm{br}(G)$.

A graph H is called a subgraph of a graph G if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and the assignment of endpoints to edges in H is same as in G. A subgraph H of G is called an *induced subgraph* of G if and only if $E(H)$ consists of all edges of G whose endpoints belong to $V(H)$. A subgraph H of a graph G is called a *spanning subgraph* of G if the vertex set of H is the same as the vertex set of G. A *complete graph* K_n with n vertices is a simple graph in which any two vertices are adjacent. The *complement* of a simple graph G is another simple graph $\overline{G}$ such that $V(G) = V(\overline{G})$ and for any two distinct vertices u and v, $u \sim v$ in $\overline{G}$ if and only if $u \nsim v$ in G.

A clique is a complete subgraph of a graph G. The number of vertices in a largest clique of a graph G is its *clique number*, denoted by $\omega(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colours required to colour every vertex of G in a way such that no two adjacent vertices are assigned the same colour. From [32], it follows that $\chi(G) \geq \omega(G)$ for a general graph G. A graph G is said to be *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. An orientation D of a graph G is called transitive if (x, y), $(y, z) \in E(D)$ implies $(x, z) \in E(D)$. A graph is a *comparability graph* if its edges can be oriented in such a way, that the resulting digraph is transitive and acyclic. The *chromatic index* $\chi'(G)$ of a graph G is the least number of colours needed to colour the edges of G so that any two edges that share a vertex have different colours.

Theorem 1.3.1. *(Strong Perfect graph theorem)([26], Theorem 1.2) A graph G is perfect if and only if it has no induced subgraph isomorphic either to a cycle of odd length at least five, or to the complement of such a cycle.*

Theorem 1.3.2. *([19], Theorem 17) Every comparability graph is a perfect graph.*

Theorem 1.3.3. *([19], Theorem 20) Complement of every perfect graph is perfect.*

Lemma 1.3.4. *([3], Remark 1) Let G be a graph and $D = \{x \in V(G) : d(x) = \Delta(G)\}$. If for every vertex $x \in D$, there exists an edge $\{x, w\}$ of G such that $\Delta(G) - deg(w) + 2 > |D|$, then $\chi'(G) = \Delta(G)$.*

Lemma 1.3.5. *([19], Chapter V, p.152) The chromatic index of K_n is*

$$\chi'(K_n) = \begin{cases} n-1 \ , & \textit{if } n \textit{ is even,} \\ n \ , & \textit{if } n \ \geq 3 \textit{ and odd} \end{cases}$$

Let G and H be two graphs. A mapping $f : V(G) \to V(H)$ is called *homomorphism* from G to H if for any $a, b \in V(G)$, a and b are adjacent in G implies $f(a)$ and $f(b)$ are adjacent in H. If f is bijective and its inverse mapping is homomorphism, then f is *isomorphism*. A homomorphism (resp. an isomorphism) from G to itself is called endomorphism (resp. automorphism).

A subgraph H of a graph G is said to be a *core* if every endomorphism of H is an automorphism and there exists a homomorphism from G to H. We know that every graph has a core, which is an induced subgraph and is unique upto isomorphism ([41], Lemma 2.2.2). Also a graph is called a core if its core is itself and the core of a graph G is complete if and only if $\chi(G) = \omega(G)$.

Let G be a graph. A subset U of $V(G)$ is called complete if every two distinct vertices of U are adjacent and is called independent or independent set of vertices if no two vertices in U are adjacent. A graph G is called split if there is a partition $V(G) = K \cup S$ of its vertex set into a complete set K and an independent set S. The cardinality of a maximal independent set of a graph G is called *independence number*. It is denoted by $\alpha(G)$.

A graph G is *bipartite* if $V(G)$ is the union of two disjoint sets V_1 and V_2, such that each edge consists of one vertex from each set. $V_1 \cup V_2$ is said to give the bipartition of $V(G)$. A *complete bipartite graph* $K_{m,n}$ is a bipartite graph with bipartition $V_1 \cup V_2$ (where $|V_1| = m$ and $|V_2| = n$) such that for any $v \in V_1, w \in V_2$, v and w are adjacent. A *star graph* is a complete bipartite graph $K_{1,n}$.

Here by a surface, we mean a compact connected topological space such that each point has a neighbourhood homeomorphic to an open disc in $\mathbb{R}^2$. We recall that a map $\phi : G \longrightarrow S$ is an embedding of a graph G into a surface S if ϕ represents a drawing of G on S without any crossings. It is well known that every surface is homeomorphic to S_k or N_l for some k, $l \in \mathbb{N}$, where S_k(resp. N_l) denote the sphere with k handles(resp. l mobius bands) attached to it. That is S_k(resp. N_l) is an orientable(resp. non-orientable) surface of genus k(resp. crosscap l). The *genus* (resp. crosscap) of a graph G, denoted as $g(G)$(resp. $\overline{g}(G)$), is the minimal integer k such that the graph can be embedded in S_k(resp. N_l). The graphs of genus 0, 1, 2 are called *planar*, *toroidal*, *bitoroidal* respectively and graphs of crosscap 1 and 2 are called *projective*, *biprojective*, respectively. An *outerplanar graph* is a planar graph that can be embedded in the plane without crossings in such a way that all the vertices lie in the boundary of the unbounded face of the embedding. The *thickness* (resp. *outerthickness*) of a graph G, denoted as $\theta(G)$(resp. $\theta_0(G)$), is the minimum number of decomposition of G into it's planar (resp. outerplanar) subgraphs. For more on embedding and decomposition of a graph, see ([65], [88], [90]).

Lemma 1.3.6. *([90], Lemma* 2.1*) If G_1 is a subgraph of a graph G_2, then $g(G_1)$(resp. $\overline{g}(G_1)$) $\leq g(G_2)$(resp. $\overline{g}(G_2)$) ([65]) and $\theta(G_1)$(resp. $\theta_0(G_1)$) $\leq \theta(G_2)$(resp. $\theta_0(G_2)$).*

Lemma 1.3.7. *([65], Proposition* 4.4.4*) Let G be a finite connected graph with $p(\geq 3)$ vertices and q edges. Then $g(G)$(resp. $\overline{g}(G)$) $\geq \lceil \frac{q}{6} - \frac{p}{2} + 1 \rceil$(resp. $\lceil \frac{q}{3} - p + 2 \rceil$). Moreover, if G has genus g, then $p - q + f = 2 - 2g$(resp. $p - q + f = 2 - k$), where f is the number of faces created when G is minimally embedded on a surface of genus g(resp. crosscap k).*

Lemma 1.3.8. (1)*([88], Theorem* 6.38*) $g(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$ if $n \geq 3$.*

(2)

$$\overline{g}(K_n) = \begin{cases} \lceil \frac{(n-3)(n-4)}{6} \rceil & \text{if } n \geq 3 \\ 3 & \text{if } n = 7 \end{cases}$$

(3)*([85], Lemma 2.2)*

$$\theta(K_n) = \begin{cases} \lfloor \frac{n+7}{6} \rfloor & \text{if } n \neq 9, 10 \\ 3 & \text{if } n = 9, 10 \end{cases}$$

.

(4)

$$\theta_0(K_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil & \text{if } n \neq 7 \\ 3 & \text{if } n = 7 \end{cases}$$

Lemma 1.3.9. *[17] If G is a graph with n vertices and m edges, then $\theta(G) \geq \lceil \frac{m}{3n-6} \rceil$ and $\theta_0(G) \geq \lfloor \frac{m}{2n-3} \rfloor$.*

Lemma 1.3.10. *([62], Theorem* 1*) A graph G is outer planar if and only if it has no subgraph homeomorphic to $K_{2,3}$ or K_4.*

Lemma 1.3.11. *([65], Theorem* 2.4.1*) A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G. Clearly every outer-planar graph is planar.*

Definition 1.3.12. [40] Let G be a graph with n vertices and q edges. A cycle C in G is said to be primitive if it has no chord. A graph G has the primitive cycle property (PCP) if any two primitive cycles intersect at most one edge. The number frank(G) is called the free rank of G, is the number of primitive cycles of G and the number rank$(G) = q - n + r$ is called the cycle rank of G, where r is the number of connected components of G. These two numbers satisfy rank$(G) \leq$ frank(G) The family of graph where equality occurs is called the *ring graph* .

Lemma 1.3.13. *([40], Proposition* 2.2*) Let G be a graph. Then the following statements are equivalent*

(1) *G is a ring graph,*

(2) *rank(G) = frank(G),*

(3) *G satisfies PCP and G does not contain a subdivision of K_4 as a subgraph.*

Definition 1.3.14. [25] For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the metric representations of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)\}$. The set W is a resolving set for G if $r(u|W) = r(v|W)$ implies $u = v$ for all pairs u, v of vertices of G. The *metric dimension* of a graph G,denoted by $\dim(G)$, is the minimum cardinality of a resolving set for G.

Definition 1.3.15. [61] Given a graph G, for $x, y, z \in V(G)$, we say that z strongly resolves x and y if there exists a shortest path from z to x containing y, or a shortest path from z to y containing y. A subset S of $V(G)$ is a strong resolving set of G if every pair of vertices of G is strongly resolved by some vertex of G. The smallest cardinality of a strong resolving set of G is called the *strong metric dimension* of G and denoted by sdim(G).

Definition 1.3.16. [25] Given $P = \{P_1, P_2, \ldots, P_k\}$ an ordered partition of $V(G)$ we say P is a resolving if for each pair of distinct u, $v \in V(G)$ there is a part P_i where $d(u, P_i) \neq d(v, P_i)$. The *partition dimension* , denoted by $pd(G)$, is the minimum order of all resolving partitions of $V(G)$.

Theorem 1.3.17. *([25], Theorem* 1*) If G is a connected graph of order n and diameter*

d, then $f(n,d) \leq dim(G) \leq n-d$, where $f(n,d)$ is the least positive integer k such that $k + d^k \geq n$.

Theorem 1.3.18. *([25], Theorem 2) A connected graph G of order n has metric dimension 1 if and only if $G = P_n$.*

Theorem 1.3.19. *([61], Theorem 2.2) For any graph G with diameter 2, $sdim(G) = |V(G)| - \omega(\mathbb{R}_G)$, where $\mathbb{R}_G$ is the reduced graph of G.*

Theorem 1.3.20. *([25], Theorem 3.1) If G is a graph of order $n(\geq 3)$ and diameter d, then*

$$g(n,d) \leq pd(G) \leq n - d + 1$$

where $g(n,d)$ is the least positive integer k for which $(d+1)^k \geq n$.

Theorem 1.3.21. *([25], Theorem 1.1) If G is a nontrivial connected graph then*

$$pd(G) \leq dimG + 1$$

A circuit containing all edges of a graph is an Eulerian circuit. A graph is called *Eulerian* if it has an Eulerian circuit or if it is edgeless. A Hamiltonian path in a graph is a spanning path that contains all vertices of the graph. A Hamiltonian cycle is a spanning cycle (i.e., a cycle passing through every vertex) in a graph. A simple graph is called *Hamiltonian* if it has a Hamiltonian cycle.

Let $G = (V, E)$ be a graph and $|V| \geq 3$. If for any vertex x of G, $deg(x) \geq \frac{|V|}{2}$, then G is Hamiltonian.

Theorem 1.3.1. *(Theorem 7.1, [45])The following statements are equivalent for a connected graph G:*
(1) G is Eulerian,
(2) Every vertex of G has even degree,
(3) The set of lines of G can be partitioned into cycles.

The union of two graphs G and H, denoted by $G + H$, is a graph with vertex set $V(G) \cup V(H)$ and edge set $E(H) \cup E(G)$. For a graph G, a non-empty subset S of $V(G)$ is called a dominating set of G if every vertex of $V(G)-S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

A subset S of $V(G)$ is called an *accessible set* of the graph G if each vertex $x \in V(G)-S$ is adjacent to $N[S]$, where $N[S]$ is the closed neighborhood of S . The *accessibile number* of G is defined as the minimal number of vertices over all accessible sets of G and is denoted by $\eta(G)$.

For given arbitrary graphs G_1 and G_2, the *cartesian product* $G_1 \Box G_2$ is defined as the graph on vertex set $V(G_1) \times V(G_2)$ with vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are adjacent if and only if either $x_1 = y_1$ and $x_2y_2 \in E(G_2)$ or $x_1y_1 \in E(G_1)$ and $x_2 = y_2$.

Remark 1.3.22. ([35], Remark 2) For any two graphs G_1 and G_2, we have $\omega(G_1 \Box G_2) \geq max\{\omega(G_1), \omega(G_2)\}$.

Remark 1.3.23. ([75]) For any two graphs G_1 and G_2, we have $\chi(G_1 \Box G_2) \geq max$ $\{\chi(G_1), \chi(G_2)\}$.

The *Weiner* index is of certain importance in chemistry and most thoroughly studied graph-based molecular structure descriptors. The Wiener index of a graph G [7] is defined as

$$W = W(G) = \sum_{\{x,y\} \subseteq V(G)} d(x, y)$$

Let $\gamma(G, k)$ be the number of vertex pairs of a finite graph G that are at distance k. Then *Harary index* of G is defined as [7]

$$H(G) = \sum_{k \geq 1} \frac{1}{k} \gamma(G, k)$$

The concept of *Zagreb ecentricity* indices were introduced in chemical graph theory recently. The first Zagreb ecentricity (E_1) and the second Zagreb ecentricity (E_2) indices of a finite graph G are defined as

$$E_1 = E_1(G) = \sum_{x_i \in V(G)} e_i^2 \text{ and } E_2 = E_2(G) = \sum_{x_i x_j \in E(G)} e_i e_j$$

where $E(G)$ is the edge set and e_i is the ecentricity of the vertex x_i in G.

The *eccentric* connectivity index of a finite graph G is defined as [7]

$$\xi^C(G) = \sum_{i=1}^{k} d_i e_i$$

where d_i is the degree of the vertex x_i.

Also the *degree distance* of a finite graph G, denoted by $D'(G)$, is defined as [6]

$$D'(G) = \frac{1}{2} \sum_{1 \leq i < j \leq n} d(x_i, x_j)(d_i + d_j)$$

where d_i is the degee of the vertex x_i. For other results, terminologies and definitions of graph theory, one may look at [33], [45] and [86]. In [14], Baloda et al. studied the inclusion ideal graph of a completely simple semigroup. Let S_n be a completely 0-simple semigroup with n 0-minimal right ideals and $\Lambda = \{1, 2, \ldots, k\} \subseteq [n]$. For a completely 0-simple semigroup S_n with n 0-minimal right ideals, we write a non-trivial right ideals $I_\Lambda = I_{12\ldots k} = I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_k}$ such that $i_1, i_2, \ldots i_k \in [n]$ and $1 \leq k \leq n-1$, where I_{i_1}, I_{i_2}, $\ldots I_{i_k}$ are 0-minimal right ideals of S_n. The graph is connected if and only if $n \neq 2$.

Lemma 1.3.24. *([14], Lemma 4.1) For $n \geq 3$, $In(S_n)$ is a connected graph of order $2^n - 2$ and has diameter 3.*

Lemma 1.3.25. *([14], Lemma 4.3) if $I_{12\ldots k}$ is a non-trivial right ideals of S_n then the degree of the vertex is $deg(I_{12\ldots k}) = 2^k - 2 + 2^{n-k} - 2$.*

Theorem 1.3.26. *([14], Theorem 3.12(ii)) The graph $In(S_n)$ is planar if and only if $n = 3, 4$.*

Chapter 2

Semigroups in which 2-absorbing ideals are prime

In this chapter the 2-absorbing ideal in a commutative semigroup S, defined by Cay et al. in [28], has been considered. The family of commutative semigroups in which 2-absorbing ideals are maximal and also the class of commutative semigroups where 2-absorbing ideals are prime have been characterized.

In **Section** 2.1, we observe that every maximal ideal of a commutative semigroup is 2-absorbing (cf. Theorem 2.1.4). Then we characterize the class of semigroups with unity (cf. Theorem 2.1.8) and without unity (cf. Theorem 2.1.12), in which 2-absorbing ideals are maximal.

In **Section** 2.2, we define the notion of 2-AB semigroup, in which 2-absorbing ideals are prime and an example of this semigroup is given (cf. Definition 2.2.1 and Example 2.2.2). We study many properties of a 2-AB semigroup S such as 2-absorbing ideals are linearly ordered, S has atmost one maximal ideal, S is semiprimary and prime ideals of S are idempotent (cf. Theorem 2.2.3). Then we characterize 2-AB semigroup in terms of minimal prime ideal over a 2-absorbing ideal (cf. Theorem 2.2.5), some other characterizations have also been established (cf. Theorem 2.2.6, Theorem 2.2.7 and Theorem 2.2.9). We study some equivalent conditions for a regular semigroup S to be 2-AB semigroup (cf. Theorem 2.2.11). Finally, we prove that a semigroup S will be 2-AB if S is with unity and having no essential congrurence (cf. Corollary 2.2.12) or every 2-absorbing ideal of S generated by idempotent (cf. Theorem 2.2.13).

Throughout this chapter, by a semigroup S we mean a commutative semigroup, prime ideals are proper and whenever speaking about maximal ideal, we suppose, of course it exists.

2.1 Semigroups where 2-absorbing ideals are maximal

Lemma 2.1.1. *Let S be a semigroup. Then every prime ideal of S is a 2-absorbing ideal of S.*

Proof. Let I be a prime ideal of S and $abc \in I$ with $ab \notin I$ for some $a, b, c \in S$. Since I is prime ideal, so $c \in I$, which implies $ac \in I$ and $bc \in I$. Therefore I is a 2-absorbing ideal of S. □

Remark 2.1.2. The following example shows that the converse of the above lemma is not true.

Example 2.1.3. Consider the principal ideal $I = (6)$ in the semigroup $S = (\mathbb{N}, .)$, which is 2-absorbing but not prime as $2.3 \in (6)$ but neither $2 \in (6)$ nor $3 \in (6)$.

A commutative semigroup with unity has a unique maximal ideal, which is prime also and so 2-absorbing. But in a commutative semigroup without unity maximal ideal need not be prime for example we consider the ideal $I = \{m \in \mathbb{N} : m \geq 2\}$ in the semigroup $S = (\mathbb{N}, +)$, which is maximal but not prime. But the following theorem shows that in a commutative semigroup without unity every maximal ideal is a 2-absorbing ideal.

Theorem 2.1.4. *Let S be a semigroup without unity. Then every maximal ideal of S is a 2-absorbing ideal of S.*

Proof. Let M be a maximal ideal of a semigroup S without unity and $abc \in M$ with $ab \notin M$ for some $a, b, c \in S$.

Case(1): If $c \in M$ then $ac \in M$ and $bc \in M$, since M is an ideal of S. Hence M is a 2-absorbing ideal of S.

Case(2): Let $c \notin M$. Since $ab \notin M$, then both a, b belongs to $S - M$. Now if $c \neq ab$, then $S - M$ contains two distinct elements c and ab. Again if $c = ab$ and $a \neq b$ then $S - M$

contains two distinct elements a and b and if $a = b$ then $\{a, a^2\}$ belongs to $S - M$, moreover if $a = a^2$, then a is an idempotent element of S. Thus in either case $S - M$ contains more than one elemenet or an idempotent, hence M is a prime ideal of S (*cf.* Theorem 1.2.26). Consequently, M is a 2-absorbing ideal of S (*cf.* Lemma 2.1.1). $\square$

Remark 2.1.5. The following example shows that the converse is not true even if S has unity. Consider the ideal $I = \{m \in S : m \geq 2\}$ in the commutative semigroup $S = (\mathbb{N} \cup \{0\}, +)$ with unity 0. Here I is 2-absorbing but not a maximal ideal of S.

Lemma 2.1.6. *Let P_1 and P_2 be two prime ideals of a semigroup S. Then $P_1 \cap P_2$ is a 2-absorbing ideal of S.*

Proof. Let $abc \in P_1 \cap P_2$ for some $a, b, c \in S$. Then $abc \in P_1$ and $abc \in P_2$. Since P_1 and P_2 are prime ideals so either $a \in P_1$ or $b \in P_1$ or $c \in P_1$ and also either $a \in P_2$ or $b \in P_2$ or $c \in P_2$. Thus in either case ab or bc or ac belongs to $P_1 \cap P_2$. Hence $P_1 \cap P_2$ is a 2-absorbing ideal of S. $\square$

Theorem 2.1.7. *Let S be a semigroup in which 2-absorbing ideals are maximal. Then S has at most one prime ideals, if exists it is a maximal ideal of S.*

Proof. If possible let P_1 and P_2 be two prime ideals of S. By Lemma 2.1.6, $P_1 \cap P_2$ is a 2-absorbing ideal of S so maximal and is contained in both maximal ideal P_1 and P_2. Hence $P_1 = P_2$, so S has at most one prime ideal and consequently maximal ideal in S. $\square$

The following is a characterization of a semigroup with unity in which every 2-absorbing ideal is maximal.

Theorem 2.1.8. *Let S be a semigroup with unity. Then 2-absorbing ideals of S are maximal if and only if S is either a group or S has a unique 2-absorbing ideal A such that $S = A \cup H$, where H is the group of units of S and A is an archimedian subsemigroup of S.*

Proof. Let S be a semigroup with unity in which every 2-absorbing ideal is maximal. If S is not a group, then S has a unique maximal ideal A (*cf.* Lemma 1.2.11) which is the only prime as well as 2-absorbing ideal of S. Therefore $S = A \cup H$, where A is unique 2-absorbing ideal of S and H is the group of units. Since A is the unique prime ideal in S, for any

p,$q \in A$, $\sqrt{(p)} = \sqrt{(q)} = A$. Then there exist positive integers m and n such that $p^m = qx$ and $q^n = py$ for some x, $y \in S$. So $p^{m+1} = q(px)$ and $q^{n+1} = p(qy)$, where px and $qy \in M$. Hence A is an archimedian subsemigroup of S.

Conversely, let A be the unique 2-absorbing ideal of S. Since a semigroup with unity has unique maximal ideal and maximal ideals are 2-absorbing, therefore by uniquenes of 2-absorbing ideal A is maximal, as desired. $\square$

Theorem 2.1.9. *Let S be a regular semigroup with unity such that every 2-absorbing ideal is of the form M^n, where n is any positive integer and M is the unique maximal ideal of S. Then an ideal I of S is primary if and only if I is a 2-absorbing ideal of S.*

Proof. Let I be a 2-absorbing ideal of a semigroup S with unity, which is of the form M^n, where n is any positive integer and M is the unique maximal ideal of S. Then $\sqrt{I} = \sqrt{M^n} = M$ (*cf.* Theorem 1.2.22). Hence I is a primary ideals of S.

Conversely, let I be a primary ideal of S. Since S is regular so $I = \sqrt{I}$. Cosequently I is prime and hence I is 2-absorbing ideal of S. $\square$

The following is an obvious consequence of Theorem 2.1.9 and Theorem 1.2.59

Corollary 2.1.10. *Let S be a regular semigroup containing zero, identity and every 2-absorbing ideal of S is of the form M^n, where $n \in \mathbb{N}$ and M is the maximal ideal of S. Then every non-zero 2-absorbing ideal of S is maximal if and only if S is one of the following types:*

(i) $S = H \cup M$, where H is the group of units and $M = \{0, xh : h \in H, x^2 = 0, x \in M\}$.

(ii) S is the union of two groups with zero adjoined.

Theorem 2.1.11. *Let S be a semigroup with unity in which 2-absorbing ideal is maximal. Then the following statements about S are true:*

(1) *S is a primary semigroup.*

(2) *$M^2 = M$, where M is the maximal ideal of S.*

(3) *S has atmost one idempotent different from identity.*

Proof. (1) Let S be a semigroup with unity in which 2-absorbing ideal is maximal. Then S has a unique maximal ideal say M which is the union of all proper ideals of S and is also the unique prime ideal of S. Then for any proper ideal I of S, $\sqrt{I} = M$, hence I is a primary ideal of S. Therefore S is a primary semigroup.

(2) Let $abc \in M^2 \subseteq M$ for some $a, b, c \in S$. Since M is a prime ideal of S either a or b or c belongs to M. Let $a \in M$. Then $bc \in M$, implies b or c belongs to M. Hence ac or ab belongs to M^2 and so M^2 is a 2-absorbing ideal of S. Since 2-absorbing ideal of S is maximal so M^2 is a maximal ideal of S. Therefore $M^2 = M$.

(3) If possible let e and f be idempotents different from identity. Therefore $\sqrt{(eS)} = \sqrt{(fS)} = M$, where M is the unique prime as well as unique maximal ideal of S. Therefore $e = ef = f$. $\square$

The following is a characterization of a semigroup without unity in which 2-absorbing ideals are maximal:

Theorem 2.1.12. *Let S be a semigroup without unity. Then 2-absorbing ideals of S are maximal if and only if complement of 2-absorbing ideals contain exactly one non-idempotent element or forms a subgroup of S.*

Proof. Let S be a semigroup without unity in which 2-absorbing ideals are maximal. Then S has at most one prime ideal (cf. Theorem 2.1.7). Let I be a 2-absorbing ideal of S but not prime. Now if complement of I in S contains more than one element or an idempotent, then I is prime (cf. Theorem 1.2.26), a contradiction. Hence in this case complement of a 2-absorbing ideal contains exactly one non idempotent element of S. Again, let a 2-absorbing ideal J is prime. Then $a, b \in S - I$ implies $ab \in S - I$, since I is a prime ideal of S. We know that complement of a maximal ideal in a commutative semigroup is a $\mathcal{H}$-class (Green's) (cf. Lemma 1.2.34 and 1.2.35), and a, b and ab all belong to same $\mathcal{H}$-class $S - I$ of the semigroup S. Hence $S - I$ is a subgroup of S (cf. Theorem 1.2.36), as desired.

Conversely, if complement of a 2-absorbing ideal contains exactly one element then clearly it is maximal. Now let complement of a 2-absorbing ideal J forms a subgroup of S. If possible, let J is not maximal, then J is contained in a proper ideal K of S. Let i be the identity element of $S - J$. Since $J \neq K$, there exists $p \in K - J$ such that $pq = i$ for some $q \in S$. Hence $i \in K$. Since $K \neq S$, there exists $m \in S - K$ such that $m = mi \in K$, a contradiction. Consequently, J is a maximal ideal of S. $\square$

Since an archimedian semigroup has no proper prime ideal, the following is an immediate consequence of the above theorem.

Corollary 2.1.13. *Let S be an archimedian semigroup without unity. Then 2-absorbing ideals of S are maximal if and only if complement of every 2-absorbing ideal contains exactly one non-idempotent element.*

Corollary 2.1.14. *Let S be a semigroup without unity. Then 2-absorbing ideals of S are prime as well as maximal if and only if complement of 2-absorbing ideal forms a subgroup of S.*

The following is an example of a semigroup without unity in which 2-absorbing ideal is maximal.

Example 2.1.15. Consider the ideal $I = \{m \in \mathbb{N} : m \geq 2\}$ in the archimedian semigroup $S = (\mathbb{N}, +)$ without unity, which is the only 2-absorbing as well as maximal ideal of S.

2.2 Semigroups in which 2-absorbing ideals are prime

In this section we characterize the class of semigroup in which 2-absorbing ideals are prime and study some properties of this semigroup.

Definition 2.2.1. A commutative semigroup S is said to be a 2-AB semigroup if every 2-absorbing ideal of S is prime.

Example 2.2.2. Consider a semigroup $S = \{a, b\}$ with the following multiplication table $a^2 = a$, $b^2 = b$, $ab = ba = a$. Here $\{a\}$ is the only 2-absorbing ideal of S and which is prime also. Hence S is a 2-AB semigroup.

Theorem 2.2.3. *Let S be a 2-AB semigroup. Then the following statements hold:*

(1) *2-absorbing ideals of S are linearly ordered.*

(2) *S has at most one maximal ideal, if exists then it is prime.*

(3) *S is a semiprimary semigroup.*

(4) *Idempotents in S form a chain under natural ordering.*

(5) $P = P^2$ *for every prime ideal* P *of* S.

(6) *Semiprime ideals of* S *are prime.*

Proof. (1) Let P and Q be any two distinct 2-absorbing ideals of a 2-AB semigroup S. So $P \cap Q$ is 2-absorbing ($cf.$ Lemma 2.1.6) and hence prime, which implies either $P \subseteq Q$ or $Q \subseteq P$.

(2) Let M_1 and M_2 be two maximal ideal of S. Since every maximal ideal of S is 2-absobing ($cf.$ Theorem 2.1.4), so $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$ which implies $M_1 = M_2$. Hence S has atmost one maximal ideal and if exists clearly it is prime.

(3) By Theorem 1.2.38, a commutative semigroup is semiprimary if and only if prime ideals are linearly ordered. Hence S is a semiprimary semigroup.

(4) Since S is a semiprimary semigroup, then for any ideal A of S, $\sqrt{A}$ is prime. Let e and f are any two idempotents of S. Then $\sqrt{eS}$ and $\sqrt{fS}$ are prime ideals, so either $\sqrt{eS} \subseteq \sqrt{fS}$ or $\sqrt{fS} \subseteq \sqrt{eS}$. Thus $e \in fS$ or $f \in eS$ i.e., $e = fe$ or $f = ef$.which implies that the idempotents form a chain under natural ordering ($cf.$ Definition 1.2.29)

(5) Let P be a prime ideal of S and $abc \in P^2 \subseteq P$ for some $a, b, c \in S$. Since P is a prime ideal of S, either $a \in P$ or $b \in P$ or $c \in P$. Let $a \in P$. Then $bc \in P$, implies b or c belogs to P and so ac or ab belongs to P^2. Hence P^2 is a 2-absorbing ideal of S and so P^2 is a prime ideal of S. Let $x \in P$. Then $x^2 \in P^2$ implies $x \in P^2$ so $P \subseteq P^2$. Therefore $P = P^2$.

(6) Let I be a semiprime ideal of S. Then $I = \sqrt{I} = \bigcap P_i = P_\beta$, for some $\beta \in \Lambda$ where $\{P_i : i \in \Lambda\}$ are prime ideals containing I ($cf.$ Theorem 1.2.23) and hence prime, as desired. □

Lemma 2.2.4. *Let* S *be a semigroup with unity and unique maximal ideal* M. *Then for every prime ideal* P, PM *is a 2-absorbing ideal of* S.
Moreover, PM *is prime if and only if* $PM = P$.

Proof. Let $xyz \in PM \subseteq P$. Since P is prime, either $x \in P$ or $y \in P$ or $z \in P$. Let $x \in P$. Then either $y \in M$ or $z \in M$, since M is also prime. Hence $xy \in PM$ or $xz \in PM$. Consequently, PM is a 2-absorbing ideal of S.

Let PM is prime and $x \in P$. Then $x^2 \in PM$ and hence $x \in PM$. Therefore $P \subset PM$. Clearly $PM \subset P$ and hence $PM = P$. Converse is clear. □

The following is a characterization of a 2-AB semigroup in terms of minimal prime ideal over a 2-absorbing ideal.

Theorem 2.2.5. *A semigroup S with unity is 2-AB if and only if prime ideals of S are linearly ordered and if P is a minimal prime ideal over a 2-absorbing ideal I, then $IM = P$, where M is the unique maximal ideal ideal of S.*

Proof. Let I be a 2-absorbing ideal of a 2-AB semigroup S with unity. Then prime ideals of S are linearly ordered ($cf.$ Theorem 2.2.3). Then $IM = I = P$ ($cf.$ Lemma 2.2.4).
Conversely, Let I be a 2-absorbing ideal of S. Since prime ideals are linearly ordered and $P = IM$, where P is a minimal prime ideal over I, hence $P = IM \subseteq I \cap M = I \subseteq P$ implies $I = P$, as desired. □

The following is an another characterization of a 2-AB semigroup

Theorem 2.2.6. *Let S be a semigroup. Then S is 2-AB if and only if $P = P^2$ for every prime ideal P of S and every 2-absorbing ideal of S is of the form A^2, where A is a prime ideal of S.*

Proof. Let P be a 2-absorbing ideal of a 2-AB semigroup. Then P is prime and so $P = P^2$ ($cf.$ Theorem 2.2.3(6)).
Conversely, let I be a 2-absorbing ideal of S. Then $I = A^2 = A$, where A is a prime ideal of S. □

Theorem 2.2.7. *Let S be a semigroup. Then S is a 2-AB if and only if prime ideals are linearly ordered and $A = A^2$ for every 2-absorbing ideal A of S.*

Proof. let A be any 2-absorbing ideal of S and $x \in \sqrt{A}$. Then $x^2 \in A = A^2$, since A is 2-absorbing ideal of S. This implies $x \in A$, so $A = \sqrt{A}$. Since prime ideals are linearly ordered so A is prime and hence S is a 2-AB semigroup.
Conversely, let S be a 2-AB semigroup. Then prime ideals of S are linearly ordered ($cf.$ Theorem 2.2.3(2)). Since any 2-absorbing ideal A of S is prime, we have $A = A^2$ ($cf.$ Theorem 2.2.3(6)). □

Since in a fully idempotent semigroup S, $A = A^2$ for every ideal A of S, the following is a simple consequence of above theorems:

Corollary 2.2.8. *Let S be a fully idempotent semigroup. Then S is 2-AB if and only if one of the following conditions holds:*
(1) Prime ideals are linearly ordered.
(2) Every 2-absorbing ideal is of the form P^2, where P is a prime ideal of S.

Theorem 2.2.9. *A semigroup S is 2-AB if and only if prime ideals are linearly ordered and $A = \sqrt{A}$ for every 2-absorbing ideal A of S.*

Proof. Let S be a 2-AB semigroup. Then prime ideals of S are linearly ordered ($cf.$ Theorem 2.2.3). Again any 2-absorbing ideal A of S is prime so $A = \sqrt{A}$.
Conversely, let A be a 2-absorbing ideal of S. Then $A = \sqrt{A} = \bigcap P_i = P_\beta$, for some $\beta \in \Lambda$ where $\{P_i : i \in \Lambda\}$ are prime ideals containing A and therefore A is prime. Hence S is a 2-AB semigroup. □

Since in a semiprimary semigroup prime ideals are linearly ordered ($cf.$ Theorm 1.2.38), the following corollary is an obvious consequence of the above theorem:

Corollary 2.2.10. *Let S be a semiprimary semigroup. Then S is 2-AB if and only if $A = \sqrt{A}$ for every 2-absorbing ideal A of S.*

Theorem 2.2.11. *Let S be a regular semigroup. Then the following statements about S are equivalent:*
(1) *S is a 2-AB semigroup,*
(2) *S is a semiprimary semigroup.*

Proof. (1) $\Leftrightarrow$ (2) Let A be a 2-absorbing ideal of a commutative regular semigroup S. Then $A = \sqrt{A} = \bigcap P_\alpha$, where $\{P_\alpha : \alpha \in \Lambda\}$ are the prime ideals of S contaning A. Since S is semiprimary, so prime ideals are linearly ordered, which implies $A = \sqrt{A} = P_\beta$ for some $\beta \in \Lambda$. Therefore S is a 2-AB semigroup.
Conversely, let A be a 2-absorbing ideal of S. Since S is regular semiprimary semigroup, we have $A = \sqrt{A}$ is a prime ideal of S and hence a 2-AB semigroup. □

Let D be the class of commutative semigroups with an identity element and having no proper essential congrurences. We know that if $S \in D$, then the set of ideals of S are linearly ordered by inclusion (cf. Theorem 1.2.44) and hence the set of prime ideals of S are linearly ordered. Also we know that if $S \in D$, then S is regular (cf. Theorem 1.2.43) i.e., $A = \sqrt{A}$ for every ideal A of S. So as a simple consequence of Theorem 2.2.9, we have the following result:

Corollary 2.2.12. *If $S \in D$, then S is a 2-AB semigroup.*

Theorem 2.2.13. *If every 2-absorbing ideal of a semigroup has idempotent generator, then S is a 2-AB semigroup.*

Proof. Let I be a 2-absorbing ideal of S generated by idempotent say e i.e., $I = (e) = eS$. Since S is commutative so $I = I^2$. It is clear that $I \subseteq \sqrt{I}$. Let $x \in \sqrt{I}$. Then $x^2 \in I = I^2$, since I is 2-absorbing. This implies $x \in I$, so $\sqrt{I} \subseteq I$. Hence $I = \sqrt{I}$. Again, let P, Q be two prime ideals of S. Then the prime ideals $P \cup Q$ is 2-absorbing, has idempotent generator say e, i.e., $P \cup Q = eS$. But then $e \in P$ or $e \in Q$. This implies either $P = eS$ or $Q = eS$ and either $P \subseteq Q$ or $Q \subseteq P$. Hence by Theorem 2.2.9, S is 2-AB semigroup. □

Since every principal ideal of a commutative regular semigroup has an idempotent generator, the following is an obvious consequence of the above theorem:

Corollary 2.2.14. *If every 2-absorboing ideal of a commutative regular semigroup S is principal, then S is a 2-AB semigroup.*

Chapter 3

2-absorbing primary ideals of a semigroup

In this chapter, the notion of 2-absorbing primary ideals in a commutative semigroup S has been introduced and it's relation with maximal ideal, prime ideal, semiprimary ideal and 2-absorbing ideals have been observed. Then the family of commutative semigroups where 2-absorbing primary ideals are prime, maximal and semiprimary have been characterized and also various properties of 2-absorbing primary ideals in a commutative semigroup have been studied.

First we define 2-absorbing primary ideal in a commutative semigroup ($cf.$ Definition 3.1.1). Clearly 2-absorbing ideals, prime ideals and maximal ideals are 2-absorbing primay ideals ($cf.$ Theorem 3.1.2) and we prove that every semiprimary ideals of S are 2-absorbing primay ideal ($cf.$ Theorem 3.1.11). But the converses are not true ($cf.$ Remark 3.1.4, 3.1.5, 3.1.13). Then we characterize a commutative semigroup where 2-absorbing primary ideals are prime, maximal, 2-absorbing and semiprimary ($cf.$ Theorem 3.1.7, 3.1.9, 3.1.6, 3.1.14), as a result we obtain that semigroup in which 2-absorbing primary ideals are semiprimary is equivalent to a semiprimary semigroup ($cf.$ Theorem 3.1.14) and semigroup in which 2-absorbing primary ideals are maximal is either a group or union of two groups ($cf.$ Corollary 3.1.10). Then we prove that a proper ideal I of a semigroup S is 2-absorbing primary ideal of S if and only if $I[x]$ ($I[x_1, x_2, \ldots, x_n]$) is a 2-absorbing primary ideal of the polynomial semigroup $S[x]$ ($S[x_1, x_2, \ldots, x_n]$) ($cf.$ Theorem 3.1.28) and for any ideal I of S, $\sqrt{I}$ is a 2-absorbing primary ideal of S if and only if $\sqrt{I}$ is a 2-absorbing ideal of S ($cf.$ Theorem 3.1.22). As a consequence we prove that radical of a 2-absorbing primary ideal I is a 2-absorbing ideal, moreover if $\sqrt{I} = P$, a prime ideal of S then the residual $(I : x) = \{s \in S : sx \in I\}$ of I by $x \in S - \sqrt{I}$ is a 2-absorbing primary ideal with $\sqrt{(I : x)} = P$ ($cf.$ Theorem 3.1.20). We prove that arbitary union of 2-absorbing primary ideals is 2-absorbing primary but intersections of 2-absorbing primary ideals need not be 2-absorbing primary ideal ($cf.$ Example 3.1.23). Also we find equivalence classes in the semigroup of all 2-absorbing primary ideals of a semigroup S and each class is closed under finite intersections ($cf.$ Theorem 3.1.24). We observe that under certain conditions 2-absorbing primary ideals remains invariant under homomorphism of semigroups and it's inverse mapping ($cf.$ Theorem 3.1.32). Lastly we also study 2-absorbing primary ideals in direct product of semigrops ($cf.$ Theorem 3.1.35, 3.1.36) . Throughout this chapter, unless otherwise mentioned, S stands for a commutative semigroup.

3.1 Some properties of 2-absorbing primary ideals

Definition 3.1.1. A proper ideal I of a commutative semigroup S is said to be 2-absorbing primary if $abc \in I$ implies either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$ for any $a, b, c \in S$.

Since $I \subseteq \sqrt{I}$ for any ideal I of a semigroup S so we have the following result

Theorem 3.1.2. *Let S be a commutative semigroup. Then every 2-absorbing ideal of S is a 2-absorbing primary ideal of S.*

In view of Lemma 2.1.1, Lemma 2.1.4 and Lemma 2.1.6, the following proposition easily hold.

Proposition 3.1.3. *Let S be a commutative semigroup. Then*

(1) *if P_1 and P_2 are two prime ideals of S then $P_1 \cap P_2$ is a 2-absorbing primary ideal of S,*

(2) *every maximal ideal of S is a 2-absorbing primary ideal of S,*

(3) *every prime ideal of S is a 2-absorbing primary ideal of S.*

Remark 3.1.4. The following example shows that converse of Proposition 3.1.3 are not true. Consider the ideal $I_2 = \{n \in \mathbb{N} : n \geq 2\}$ in the semigroup $S = (\mathbb{N} \cup \{0\}, +)$, which is 2-absorbing primary (as well 2-absorbing) but neither prime nor a maximal ideal of S.

Remark 3.1.5. The following example shows that converse of Theorem 3.1.2 is not true. Consider the ideal $I = (m \in \mathbb{N} : m \geq 6)$ in the semigroup $S = (\mathbb{N}, +)$. Then $1 + 2 + 3 \in I$ but neither $1 + 2 \in I$ nor $2 + 3 \in I$ nor $1 + 3 \in I$. Clearly, I is a 2-absorbing primary ideal of S but not a 2-absorbing ideal of S.

Since in a commutative regular semigroup every ideal coincide with its radical ($cf.$ Theorem 1.2.51), we have the following result

Corollary 3.1.6. *Let S be a commutative regular semigroup. Then an ideal I of S is a 2-absorbing primary ideal of S if and only if I is a 2-absorbing ideal of S.*

The following is a characterization of a semigroup in which 2-absorbing primary ideals are prime:

Theorem 3.1.7. *Let S be a commutative semigroup. Then every 2-absorbing primary ideals of S are prime if and only if prime ideals of S are linearly ordered and $A = \sqrt{A}$ for every 2-absorbing primary ideal A of S.*

Proof. Let P_1 and P_2 be two prime ideals of S. Then $P_1 \cap P_2$ is a 2-absorbing primary ideal of S ($cf.$ Proposition 3.1.3(1)) and so prime by hypothesis. Hence prime ideals are linearly ordered. Again let A be a 2-absorbing primary ideal of S and so prime ideal of S. Therefore $A = \sqrt{A}$.

Conversely, let A be a 2-absorbing primary ideal of S. Since prime ideals are linearly ordered so $A = \sqrt{A} = \bigcap_{\alpha \in \Lambda} P_\alpha = P_\beta$ for some $\beta \in \Lambda$, where $\{P_\alpha\}_{\alpha \in \Lambda}$ are prime ideals of S containing A. Hence the result follows. □

Since every primary ideals of a commutative semigroup S is 2-absorbing primary, we have the following immediate result by applying Theorem 1.2.60.

Corollary 3.1.8. *Let S be a commutative semigroup with zero and identity in which nonzero 2-absorbing primary ideals are prime. Then S satisfies one of the following conditions.*

(i) $S = H \cup M$, where H is the group of units in S and $M = \{0, ah : a \in M, a^2 = 0, h \in H\}$.

(ii) $M^n = M$ for every positive integer n.

The following is a characterization of a semigroup in which 2-absorbing primary ideals are maximal:

Theorem 3.1.9. *Let S be a commutative semigroup with unity. Then 2-absorbing primary ideals of S are maximal if and only if S is either a group or has a unique 2-absorbing primary ideals A such that $S = A \cup H$, where H is the group of units of S.*

Proof. Let S be commutative semigroup with unity in which 2-absorbing primary ideals are maximal. If S is not a group, it has unique maximal ideal say A and since maximal ideals are 2-absorbing primary ($cf.$ Proposition 3.1.3(2)) so has unique 2-absorbing primary ideal A. Therefore $S = A \cup H$, where H is the group of units of S.
Conversely, if S is a group then it has no 2-absorbing primary ideal so the condition satisfied vacously. Again if S has unique 2-absorbing primary ideal then clearly it is maximal. □

Moreover, we prove that A is also a group. Clearly A is the unique prime ideal of S. Then for any $a \in A$, $\sqrt{aS} = A$. Hence aS is a 2-absorbing primary ideal of S. Hence $aS = A$ for every $a \in A$, by hypothesis. Then $aS = a^2S = A$ implies $a = a^2x \Rightarrow ax = a^2x^2$. Thus ax is an idempotent element of A. If possible let e, f be two idempotent element of S. Then $eS = fS \Rightarrow e = fe = ef = f$. Consequently $eS = aS = A$. Therefore A is a group. So we can conclude the following:

Corollary 3.1.10. *Let S be a commutative semigroup with unity. Then 2-absorbing primary ideals are maximal if and only if either S is a group or S is a union of two groups.*

Theorem 3.1.11. *Let S be a commutative semigroup. Then every semiprimary ideal of S is a 2-absorbing primary ideal of S.*

Proof. Let I be a semiprimary ideal of a semigroup S and $abc \in I \subseteq \sqrt{I}$ with $ab \notin I$ for some $a, b, c \in S$. Hence $\sqrt{I}$ is a prime ideal of S.
Case(1) Suppose $ab \notin \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal of S so $c \in \sqrt{I}$. Hence $ac \in \sqrt{I}$ and $bc \in \sqrt{I}$.
Case(2) Suppose $ab \in \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal, we have either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Hence either $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.
Therefore I is a 2-absorbing primary ideals of S. □

The following are obvious consequence of above theorem:

Corollary 3.1.12. *Let I be an ideal of a commutative semigroup S. Then*
(1) *if $\sqrt{I}$ is a prime ideal of S, then I is a 2-absorbing primary ideal of S.*
(2) *if I is a prime ideal of S, then I^n is a 2-absorbing primary ideal of S for each natural number n.*
(3) *every primary ideal of S is a 2-absorbing primary ideal of S.*

Remark 3.1.13. The converse of Theorem 3.1.11 is not true. Consider the principal ideal $I = (6)$ generated by 6 in the semigroup $S = \{\mathbb{Z}, .\}$, which is clearly 2-absorbing primary but $\sqrt{I} = (6)$ is not a prime ideal of S and hence not a semiprimary ideal of S.

The following theorem is a characterization of a semigroup in which 2-absorbing primary ideals are semiprimary:

Theorem 3.1.14. *Let S be a commutative semigroup. Then the following statements are equivalent:*
(1) 2-*absorbing primary ideals of S are semiprimary,*
(2) *Prime ideals of S are linearly ordered,*
(3) *S is a semiprimary semigroup,*
(4) *Semiprime ideals are linearly ordered,*
(5) *Semiprime ideals of S are prime.*

Proof. (1) $\Rightarrow$ (2) Let P_1 and P_2 be two prime ideals of S. Then $P_1 \cap P_2$ is a 2-absorbing primary ideal of S ($cf.$ Corollary 3.1.3(1)) and so semiprimary ideal of S, by hypothesis. Therefore $\sqrt{P_1 \cap P_2} = \sqrt{P_1} \cap \sqrt{P_2} = P_1 \cap P_2$, is a prime ideal of S. Therefore either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$.
(2) $\Rightarrow$ (1) Since prime ideals of S are linearly ordered, then for any ideal I of S, $\sqrt{I}$ is a prime ideal of S. Consequently, 2-absorbing primary ideals of S are semiprimary.
(2) $\Leftrightarrow$ (3) It follows from Theorem 1.2.38.
(2) $\Rightarrow$ (4) Let S_1 and S_2 are two distinct semiprime ideals of S. Then $S_1 \cap S_2$ is a semiprime ideal of S. Hence $\sqrt{S_1 \cap S_2} = S_1 \cap S_2$, is a prime ideal of S, since prime ideals are linearly ordered. Hence semiprime ideals of S are linearly ordered.
(4) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (5) Let I be a semiprime ideal of S. Then $I = \sqrt{I}$, is a prime ideal of S.

(5) $\Rightarrow$ (2) Let P_1 and P_2 be two distinct prime ideals of S. Then $P_1 \cap P_2$ is a semiprime ideals of S, hence prime ideals of S. Consequently, prime ideals of S are linearly ordered. $\square$

Definition 3.1.15. A commutative semigroup S is said to be 2-absorbing primary if every proper ideal of S is a 2-absorbing primary ideal of S.

Example 3.1.16. Let us consider the commuttive semigroup $S = \{\mathbb{N}, +\}$, which has no proper prime ideal. Clearly S is a primary semigroup and hence 2-absorbing primary semigroup.

Theorem 3.1.17. *Let S be a commutative semigroup with unity. Then S is a 2-absorbing primary semigroup if one the following holds*
(1) *every proper prime ideals are maximal.*
(2) *2-absorbing ideals are semiprimary.*

Proof. (1) If S is commutative semigroup with unity in which proper prime ideals are maximal. Then S has unique proper prime as well as maximal ideal say M. Clearly for any ideal I of S, $\sqrt{I} = M$, a prime ideal of S. Hence S is a 2-absorbing primary semigroup ($cf.$ Theorem 3.1.12(1)).
(2) Let P_1 and P_2 be two prime ideals of S. Then $P_1 \cap P_2$ is 2-absorbing ideal of S and hence semiprimary ideal of S. Then $\sqrt{P_1 \cap P_2} = \sqrt{P_1} \cap \sqrt{P_2} = P_1 \cap P_2$, is a prime ideal of S and hence P_1 and P_2 are linearly ordered and consequently prime ideals of S are linearly ordered. Then for any ideal I of S, $\sqrt{I}$ is a prime ideal of S and hence S is a 2-absorbing primary ideal of S ($cf.$ Theorem 3.1.12(1)). $\square$

Theorem 3.1.18. *Let S be a commutative semigroup. Then S can be written as disjoint union of 2-absorbing primary subsemigroup of S.*

Proof. Let S be a commutative semigroup. Then S can be written as disjount union of archimedian subsemigroups of S ($cf.$ Lemma 1.2.40). Clearly every archimedian semigroup is primary and every primary semigroup is 2-absorbing primary. Therefore $S = \bigcup S_\alpha$, where S_α is a 2-absorbing primary subsemigroup of S. $\square$

Lemma 3.1.19. *Let I be a 2-absorbing ideal of a semigroup S with unity. Then $(I : x)$ is a 2-absorbing ideal of S for all $x \in S - I$.*

Proof. Let $abc \in (I : x)$ for some $a, b, c \in S$. Since $x \notin I$, we have $(I : x) \neq S$. Then $abcx \in I$. Since I is a 2-absorbing ideal of S so either $ab \in I$ or $bcx \in I$ or $acx \in I$ that is $ab \in (I : x)$ or $bc \in (I : x)$ or $ac \in (I : x)$. Consequently $(I : x)$ is a 2-absorbing ideal of S. □

Theorem 3.1.20. *Let I be a 2-absorbing primary ideal of a semigroup S with unity. Then the following statements are true*

(1) *$\sqrt{I}$ is a 2-absorbing ideal of S.*

(2) *if $\sqrt{I}$ is a prime ideal of S then $(I : x)$ is a 2-absorbing primary ideal of S with $\sqrt{(I : x)} = \sqrt{I}$ for all $x \in S - \sqrt{I}$,*

(3) *if $\sqrt{I}$ is a maximal ideal of S then $(I : x)$ is a 2-absorbing primary ideal of S with $\sqrt{(I : x)} = \sqrt{I}$ for all $x \in S - \sqrt{I}$,*

(4) *$(\sqrt{I} : x)$ is a 2-absorbing ideal of S for all $x \in S - \sqrt{I}$.*

(5) *$(\sqrt{I} : x) = (\sqrt{I} : x^2)$ for all $x \in S - \sqrt{I}$.*

Proof. (1) The proof is similar to Theorem 1.2.81.

(2) Let $x \in S - \sqrt{I}$ and $p \in (I : x)$. Then $px \in I \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a prime ideal of S and $x \notin \sqrt{I}$ so $p \in \sqrt{I}$. Hence $I \subseteq (I : x) \subseteq \sqrt{I}$, which implies $\sqrt{I} \subseteq \sqrt{(I : x)} \subseteq \sqrt{I}$. Consequently, $\sqrt{(I : x)} = \sqrt{I}$, a prime ideal of S and hence $(I : x)$ is a 2-absorbing primary ideal of S (*cf*. Corollary 3.1.11(1)).

(3) If $\sqrt{I}$ is a maximal ideal of a semigroup S with unity then it is prime and hence the proof follows from (2).

(4) Since I is a 2-absorbing primary ideal of S so $\sqrt{I}$ is a 2-absorbing ideal of S. Hence $(\sqrt{I} : x)$ is a 2-absorbing ideal of S (*cf*. Lemma 3.1.19).

(5) Let $p \in (\sqrt{I} : x^2)$. Then $x^2 p \in \sqrt{I}$ and hence either $x^2 \in \sqrt{I}$ or $xp \in \sqrt{I}$, since $\sqrt{I}$ is a 2-absorbing ideal of S. If $x^2 \in \sqrt{I}$ then $x \in \sqrt{I}$, a contradiction. Hence $xp \in \sqrt{I}$ implies $p \in (\sqrt{I} : x)$. Therefore $(\sqrt{I} : x^2) \subseteq (\sqrt{I} : x)$. Clearly, $(\sqrt{I} : x) \subseteq (\sqrt{I} : x^2)$. Hence, $(\sqrt{I} : x) = (\sqrt{I} : x^2)$ for all $x \in S - \sqrt{I}$. □

Theorem 3.1.21. *Let S be a commutative regular semigroup. Then the following statements about S are equivalent:*

(1) 2-*absorbing primary ideals of S are semiprimary,*

(2) *S is a semiprimary semigroup,*

(3) *2-absorbing ideals of S are prime.*

Proof. (1) $\Rightarrow$ (2) It follows from Theorem 1.2.38).

(2) $\Rightarrow$ (3) Let I be a 2-absorbing ideal of S. Since S is semiprimary and regular so $I = \sqrt{I} = P$, a prime ideal of S, as desired.

(3) $\Rightarrow$ (1) Let I be a 2-absorbing primary ideal of a regular semigroup S. Then $I = \sqrt{I}$ is a 2-absorbing ideal of S (cf. Theorem 3.1.20(1)) and hence prime by hypothesis. Therefore I is a semiprimary ideal of S, as desired. $\square$

Theorem 3.1.22. *Let I be an ideal of a semigroup S. Then $\sqrt{I}$ is a 2-absorbing primary ideal of S if and only if $\sqrt{I}$ is a 2-absorbing ideal of S.*

Proof. Suppose $\sqrt{I}$ is a 2-absorbing primary ideal of S and $abc \in \sqrt{I}$ for some $a, b, c \in S$. So either $ab \in \sqrt{I}$ or $bc \in \sqrt{\sqrt{I}} = \sqrt{I}$ or $ca \in \sqrt{\sqrt{I}} = \sqrt{I}$. Hence $\sqrt{I}$ is a 2-absorbing ideal of S.

Conversely, let $\sqrt{I}$ be a 2-absorbing ideal of S. Since 2-absorbing ideals are 2-absorbing primary (cf. Theorem 3.1.2) so clearly $\sqrt{I}$ is a 2-absorbing primary ideal of S. $\square$

Clearly arbitrary union of 2-absorbing primary ideals of a semigroup S is 2-absorbing primary ideal but intersection of two 2-absorbing primary ideals need not be 2-absorbing primary ideal of S. We have the following example.

Example 3.1.23. Let $I_1 = 5\mathbb{Z}$ and $I_2 = 6\mathbb{Z}$ be two ideals of the semigroup $(\mathbb{Z}, .)$. Clearly I_1 and I_2 are 2-absorbing primary ideal of S. Then $\sqrt{I_1 \cap I_2} = 30\mathbb{Z}$, which is not a 2-absorbing ideal of S. Hence $I_1 \cap I_2$ is not a 2-absorbing primary ideal of S (cf. Theorem 3.1.20(1)).

Let M be the set of all 2-absorbing primary ideals of a semigroup S and we define a relation ρ on M by $I_1 \rho I_2$ if and only if $\sqrt{I_1} = A = \sqrt{I_2}$ for some 2-absorbing ideal A of S. Clearly ρ is a congrurence on M. So every element of a ρ-equivalence class A is a 2-absorbing primary ideal for some 2-absorbing ideal A of S. Clearly M forms a semigroup with respect to usual set union and each ρ-equivalence class of M is an element of the factor semigroup M/ρ.

Theorem 3.1.24. *Each ρ-class of M is closed under finte intersections.*

Proof. Let I_1,I_2,....I_n be elements of a ρ-class A for some 2-absorbing ideal A of S. Then $\sqrt{(I_1 \cap I_2 \cap ..... \cap I_n)} = A$. Let $abc \in I = \bigcap I_i$ and $ab \notin I$ for some $a, b, c \in S$. Then $ab \notin I_i$ for some $i \in \{1, 2, ..., n\}$. Hence $bc \in \sqrt{I_i} = A$ or $ac \in \sqrt{I_i} = A$. Hence the result follows. □

Therefore the semigroup M can be written as disjoint union of semigroups that is $M = \bigcup\{A : A \in M/\rho\}$.

Theorem 3.1.25. *Let I_1 and I_2 be two P_1-primary and P_2- primary ideals for some prime ideals P_1 and P_2 of a commutative a semigroup S. Then $I_1 \cap I_2$ and $I_1 I_2$ are 2-absorbing primary ideals of S.*

Proof. The proof is similar to Theorem 1.2.82. □

Proposition 3.1.26. *Let I be a proper ideal of a semigroup S. Then the following statements are equivalent*

(1) *For every ideals J, K, L of S such that $I \subseteq J$, $JKL \subseteq I$ implies $JK \subseteq I$ or $KL \subseteq \sqrt{I}$ or $JL \subseteq \sqrt{I}$,*

(2) *for every ideals J,K,L of S, $JKL \subseteq I$ implies either $JK \subseteq I$ or $KL \subseteq \sqrt{I}$ or $JL \subseteq \sqrt{I}$.*

Proof. (1) $\Rightarrow$ (2) Let $JKL \subseteq I$ for some ideals J, K, L of S. Then $(J \cup I)KL = JKL \cup IKL \subseteq I$. Setting $P = I \cup J$ we have $I \subseteq P$ and $PKL \subseteq I$ implies either $PK \subseteq I$ or $PL \subseteq \sqrt{I}$ or $KL \subseteq \sqrt{I}$. Therefore either $(J \cup I)K \subseteq I$ or $(J \cup I)L \subseteq \sqrt{I}$ or $KL \subseteq \sqrt{I}$ implies $JK \subseteq I$ or $KL \subseteq \sqrt{I}$ or $JL \subseteq \sqrt{I}$, as desired.

(2) $\Rightarrow$ (1) Straightforward. □

Let $S[x]$ be the polynomial semigroup of a semigroup S. Then if I is an ideal of S, then $I[x] = \{ax^i : a \in I, i \geq 0\}$ is an ideal of $S[x]$ and also we have the following result:

Lemma 3.1.27. *Let I be an ideal of a semigroup S. Then $\sqrt{I[x]} = \sqrt{I}[x]$.*

Proof. Let $f(x) = ax^i \in \sqrt{I[x]}$ for some $a \in S$ and $i \geq 0$. Then for some $n \in \mathbb{N}$, $a^n x^{in} \in I[x] \Rightarrow a^n \in I \Rightarrow a \in \sqrt{I} \Rightarrow f(x) = ax^i \in \sqrt{I}[x]$. Therefore $\sqrt{I[x]} \subseteq \sqrt{I}[x]$.

Again let $g(x) = bx^i \in \sqrt{I}[x]$. Then $b \in \sqrt{I} \Rightarrow b^n \in I$ for some $n \in \mathbb{N}$. Hence $(g(x))^n = b^n x^{in} \in I[x] \Rightarrow g(x) \in \sqrt{I[x]}$. Therefore $\sqrt{I}[x] \subseteq \sqrt{I[x]}$. Consequently, $\sqrt{I[x]} = \sqrt{I}[x]$. □

Theorem 3.1.28. *Let S be a commutative semigroup. Then a proper ideal I is a 2-absorbing primary ideal of S if and only if $I[x]$ is a 2-absorbing primary ideal of $S[x]$.*

Proof. Let I be a 2-absorbing primary ideal of S and $(ax^i)(bx^j)(cx^k) \in I[x]$, where $a, b, c \in S$ and $i, j, k \geq 0$. Then $abcx^{i+j+k} \in I[x] \Rightarrow abc \in I \Rightarrow ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$, since I is a 2-absorbing primary ideal of S. Hence $(ax^i)(bx^j) = abx^{i+j} \in I[x]$ or $(ax^i)(cx^j) = acx^{i+j} \in \sqrt{I}[x] = \sqrt{I[x]}$ or $(bx^j)(cx^k) = bcx^{j+k} \in \sqrt{I}[x] = \sqrt{I[x]}$, since $\sqrt{I[x]} = \sqrt{I}[x]$ (*cf.* Lemma 3.1.27). Therefore $I[x]$ is a 2-absorbing primary ideal of $S[x]$.
Conversely, let $I[x]$ be a 2-absorbing primary ideal of $S[x]$ and $abc \in I$ for some $a, b, c \in S$. Then $(ax^i)(bx^j)(cx^k) = abcx^{i+j+k} \in I[x]$, for some $i, j, k \geq 0$. Therefore $(ax^i)(bx^j) \in I[x]$ or $(ax^i)(cx^k) \in \sqrt{I[x]} = \sqrt{I}[x]$ (*cf.* Lemma 3.1.27) or $(bx^j)(cx^k) \in \sqrt{I[x]} = \sqrt{I}[x]$ (*cf.* Lemma 3.1.27) and hence $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. Therefore I is a 2-absorbing primary ideal of S. □

The Theorem 3.1.28 can be generalized as

Theorem 3.1.29. *Let I be a proper ideal of S. Then I is a 2-absorbing primary ideal of S if and only if $I[x_1, x_2, \ldots, x_n]$ is a 2-absorbing primary ideal of $S[x_1, x_2, \ldots, x_n]$.*

Theorem 3.1.30. *Let C be a subsemigroup of a semigroup S and I be an ideal of S such that $C \cap I = \phi$. If I is a 2-absorbing primary ideal of S then $C^{-1}I$ is a 2-absorbing primary ideal of $C^{-1}S$.*

Proof. Let I be a 2-absorbing primary ideal of S and $(\frac{a}{s})(\frac{b}{r})(\frac{c}{t}) \in C^{-1}I$ for some $a, b, c \in S$ and $s, r, t \in C$. Hence there exists some $p \in C$ such that $abcp \in I$. Therefore either $ab \in I$ or $bcp \in \sqrt{I}$ or $acp \in \sqrt{I}$. Now $ab \in I$ implies $(\frac{a}{s})(\frac{b}{r}) = \frac{ab}{sr} \in C^{-1}I$, $bcp \in \sqrt{I}$ implies $(\frac{b}{r})(\frac{c}{t}) = \frac{bcp}{rtp} \in C^{-1}\sqrt{I} = \sqrt{C^{-1}I}$ and $acp \in \sqrt{I}$ implies $(\frac{a}{s})(\frac{c}{t}) = \frac{acp}{stp} \in C^{-1}\sqrt{I} = \sqrt{C^{-1}I}$, consequently, $C^{-1}I$ is a 2-absorbing primary ideal of $C^{-1}S$. □

Lemma 3.1.31. *Let $f\colon S \to S'$ be a homomorphism of semigroups. Then the following statement holds*

(1) $f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$, *where I' is an ideal of S'.*

(2) *If f is an isomorphism, then* $f(\sqrt{I}) = \sqrt{f(I)}$, *where I is an ideal of S.*

Theorem 3.1.32. *Let $f\colon S \to S'$ be a homomorphism of semigroups. Then the following statements holds:*

(1) *If I' is a 2-absorbing primary ideal of S', then $f^{-1}(I')$ is a 2-absorbing primary ideal of S.*

(2) *Let I be a proper ideal of S such that $\{(x, y) \in ker f : x \neq y\} \subseteq I \times I$. Then*

(i) *if $f(I)$ is a 2-absorbing primary ideal of S', then I is a 2-absorbing primary ideal of S*

(ii) *if f is onto and I is a 2-absorbing primary ideal of S, then $f(I)$ is a 2-absorbing primary ideal of S'.*

(3) *If f is an isomorphism and I is a 2-absorbing primary ideal of S, then $f(I)$ is a 2-absorbing primary ideal of S'.*

Proof. (1) The proof is similar as that of Theorem 1.2.83.

(2) The proof of (i) and (ii) are similar as that of Theorem 1.2.84 by replacing 2-absorbing ideal I as 2-absorbing primary ideal of S and then use the result of (1).

(3) It is trivial. □

As a simple consequence of above theorem, we have the following result

Corollary 3.1.33. (1) *Let $S \subseteq S'$ be an extension of semigroup S and I be a 2-absorbing primary ideal of S'. Then $I \cap S$ is a 2-absorbing primary ideal of S.*

(2) *Let $I \subseteq J$ be two ideals of S. Then J is a 2-absorbing primary ideal of S if and only if J/I is a 2-absorbing primary ideal of S/I.*

Lemma 3.1.34. *Let $S = S_1 \times S_2$, where each S_i is a semigroup. Then the following statement holds*

(1) *If I_1 is an ideal of S_1, then* $\sqrt{I_1 \times S_2} = \sqrt{I_1} \times S_2$.

(2) *If I_2 is an ideal of S_2, then* $\sqrt{S_1 \times I_2} = S_1 \times \sqrt{I_2}$.

(3) $\sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$.

Theorem 3.1.35. *Let $S = S_1 \times S_2$, where each S_i is a semigroup. Then the following statement holds*

(1) *I_1 is a 2-absorbing primary ideal of S_1 if and only if $I_1 \times S_2$ is a 2-absorbing primary ideal of S.*

(2) *I_2 is a 2-absorbing primary ideal of S_2 if and only if $S_1 \times I_2$ is a 2-absorbing primary ideal of S.*

Proof. (1) Suppose $I_1 \times S_2$ is a 2-absorbing primary ideals of S. Let $abc \in I_1$ for some $a, b, c \in S_1$. Then $(abc, x^3) \in I_1 \times S_2$ for some $x \in S_2$. So $(a, x)(b, x)(c, x) \in I_1 \times S_2$ and hence either $(a, x)(b, x) \in I_1 \times S_2$ or $(b, x)(c, x) \in \sqrt{I_1} \times S_2$ or $(a, x)(c, x) \in \sqrt{I_1} \times S_2$ and so either $ab \in I_1$ or $bc \in \sqrt{I_1}$ or $ac \in \sqrt{I_1}$, which implies I_1 is a 2-absorbing primary ideal of S_1.

Conversely, Suppose I_1 is a 2-absorbing primary ideal of S_1. Let $(a, x)(b, x)(c, x) \in I_1 \times S_2$ for some $a, b, c \in I_1$ and $x \in S$. Then $abc \in I_1$ implies $ab \in I_1$ or $bc \in \sqrt{I_1}$ or $ac \in \sqrt{I_1}$ and hence either $(a, x)(b, x) \in I_1 \times S_2$ or $(b, x)(c, x) \in \sqrt{I_1} \times S_2$ or $(a, x)(c, x) \in \sqrt{I_1} \times S_2$, which implies $I_1 \times S_2$ is a 2-absorbing primary ideal of $S_1 \times S_2$.

(2) The proof is similar to (1). □

Theorem 3.1.36. *Let $S = S_1 \times S_2$, where each S_i is a semigroup. Let I_1 and I_2 are ideals of S_1 and S_2 respectively. If $I = I_1 \times I_2$ is a 2-absorbing primary ideal of S then I_1 and I_2 are 2-absorbing primary ideals of S_1 and S_2 respectively.*

Proof. Let $abc \in I_1$ for some $a, b, c \in S$. Then $(a, x)(b, x)(c, x) = (abc, x^3) \in I_1 \times I_2$ for some $x \in I_2$. Since $I_1 \times I_2$ is a 2-absorbing primary ideal of S, either $(a, x)(b, x) = (ab, x^2) \in I_1 \times I_2$ or $(b, x)(c, x) = (bc, x^2) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ or $(a, x)(c, x) = (ac, x^2) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ (*cf.* Lemma 3.1.34(3)). Hence $ab \in I_1$ or $bc \in \sqrt{I_1}$ or $ac \in \sqrt{I_1}$. Consequently I_1 is a 2-absorbing primary ideal of S_1.

Similarly, we can prove that I_2 is a 2-absorbing primary ideal of S_2. □

Remark 3.1.37. The following example shows that converse of Theorem 3.1.36 is not true. Consider the 2-absorbing primary ideals $I_1 = 6\mathbb{Z}$ and $I_2 = 5\mathbb{Z}$ of the semigroup $(\mathbb{Z}, .)$. Then $(2, 3)(3, 2)(5, 5) \in I_1 \times I_2$ but nether $(2, 3)(3, 2) \in I_1 \times I_2$ nor $(2, 3)(5, 5) \in \sqrt{I_1 \times I_2}$ nor $(3, 2)(5, 5) \in \sqrt{I_1 \times I_2}$. Hence $I_1 \times I_2$ is not a 2-absorbing primary ideal of $\mathbb{Z} \times \mathbb{Z}$.

Theorem 3.1.38. *Let $S = S_1 \times S_2$, where S_1, S_2 are commutative semigroup with zero and identity. Let I be a proper ideal of S. Then the following statements are equivalent*

(1) *I is a 2-absorbing primary ideal of S,*

(2) *either $I = I_1 \times S_2$ for some 2-absorbing primary ideal I_1 of S_1 or $I = S_1 \times I_2$ for some 2-absorbing primary ideal I_2 of S_2 or $I = I_1 \times I_2$ for some primary ideals I_1 of S_1 and I_2 of S_2 respectively.*

Proof. The proof is similar to Theorem 1.2.85. $\square$

Chapter 4

2-prime ideals of a commutative semigroup

In this chapter, the concept of 2-prime and weakly 2-prime ideal in a commutative semigroup has been introduced and studied. Then the family of commutative semigroups have been characterized where 2-prime ideals are prime and every proper ideal is weakly 2-prime.

In **Section** 4.1, we define 2-prime ideals in a commutative semigroup ($cf.$ Definition 4.1.1) and establish its relation with prime and semiprimary ideals ($cf.$ Lemma 4.1.2). We prove that every maximal ideal of a commutative semigroup is 2-prime ($cf.$ Theorem 4.1.7) but the converse is not true. Then we characterize semigroups in which 2-prime ideals are maximal ($cf.$ Theorem 4.1.9). We characterize 2-prime ideals in a commutative semigroup ($cf.$ Theorem 4.1.13) and also studied some properties of 2-prime ideals ($cf.$ Proposition 4.1.11, Theorem 4.1.19).

In **Section** 4.2, we characterize semigroup where 2-prime ideals are prime and in **Section** 4.3, we characterize semigroups in which every proper ideal is weakly 2-prime (resp. strong weakly 2-prime).

Throughout this chapter, unless otherwise mentioned, all semigroups are commutative, in particular S will denote such a semigroup.

4.1 2-prime ideals

Definition 4.1.1. A proper ideal I of a semigroup S is said to be a 2-prime ideal if $ab \in I$ for $a, b \in S$ implies $a^2 \in I$ or $b^2 \in I$.

The following lemma is obvious, hence we omit the proof.

Lemma 4.1.2. (1) *Every prime ideal of a semigroup S is a 2-prime ideal of S.*
(2) *Every 2-prime ideal of a semigroup S is a semiprimary ideal of S.*

The following examples show that converse of above lemma need not be true.

Example 4.1.3. Consider the ideal $I = \{m \in \mathbb{N} : m \geq 2\}$ in the semigroup $S = \{\mathbb{N}, +\}$, which is clearly 2-prime ideal but not a prime ideal of S.

Example 4.1.4. Consider the semigroup $S = \{a, b, c\}$ with the following multiplication $a^2 = a$, $b^2 = a$, $ab = bc = ac = c^2 = c$. Here $I = \{c\}$ is a semiprimary ideal but not a 2-prime ideal of S, since $ab \in I$ but neither $a^2 \in I$ nor $b^2 \in I$.

Theorem 4.1.5. *Let S be a semigroup such that $(\sqrt{I})^2 \subseteq I$ for every semiprimary ideal I of S. Then every semiprimary ideal of S is 2-prime.*

Proof. Let $ab \in I \subseteq \sqrt{I}$ for a,$b \in S$. Then either $a \in \sqrt{I}$ or $b \in \sqrt{I}$, since I is semiprimary. Let $a \in \sqrt{I}$. Then $a^2 \in (\sqrt{I})^2 \subseteq I$, by hypothesis. Consequently, I is a 2-prime ideal of S. $\square$

Theorem 4.1.6. *Let S be a semigroup in which prime ideals are maximal and $(\sqrt{I})^2 \subseteq I$ for every semiprimary ideal I of S. Then the followings are equivalent*
(1) *I is a 2-prime ideal of S,*
(2) *I is a semiprimary ideal of S,*
(3) *I is a primary ideal of S.*

If S is a semigroup with unity, then it has unique maximal ideal which is prime also and hence 2-prime. But if S is a semigroup without unity then maximal ideal need not be prime, for example consider the ideal $I = \{m \in \mathbb{N} : m \geq 2\}$ in the semigroup $S = \{\mathbb{N}, +\}$, which is maximal but not prime. But every maximal ideal of semigroup S without unity is a 2-prime ideal of S.

Theorem 4.1.7. *Let S be a semigroup without unity and assume maximal ideal exists. Then every maximal ideal of S is a 2-prime ideal of S.*

Proof. Let M be a maximal ideal of a semigroup S without unity and $ab \in M$ with $a^2 \notin M$ for some a, $b \in S$. Now if $b^2 \notin M$, then a, b, a^2, b^2 all belong to $S - M$. If any of a, a^2, b, b^2 are not equal, then $S - M$ contains more than one element and if all are equal then $S - M$ contains an idempotent element. Hence M is a prime ideal of S ($cf.$ Theorem 1.2.26). Consequently M is a 2-prime ideal of S. □

Remark 4.1.8. The converse of above theorem need not be true. Consider the ideal $I = \{m \in \mathbb{N} : m \geq 3\}$ in the semigroup $S = (\mathbb{N}, +)$, which is clearly 2-prime but not a maximal ideal of S.

The following is a characterization of a semigroup with unity in which 2-prime ideals are maximal, the result is obvious, hence we omit the proof.

Theorem 4.1.9. *Let S be a semigroup with unity. Then 2-prime ideals of S are maximal if and only if S has a unique 2-prime ideal A such that $S = A \cup H$, where H is the group of units and A is an archimedian subsemigroup of S.*

Theorem 4.1.10. *Let S be a semigroup without unity. Then 2-prime ideals of S are maximal if and only if complements of every 2-prime ideals contains exactly one non-idempotent element or forms a subgroup of S.*

Proof. Let 2-prime ideals of a semigroup S without unity are maximal. Let P be a 2-prime but not prime ideal of S. Hence complements of P contains exactly one non-idempotent element otherwise P is prime ($cf.$ Theorem 1.2.26). Again let a 2-prime J of S is prime. Then x, $y \in S - J$ implies $xy \in S - J$. Hence $S - J$ is a subgroup of S, since complements of maximal ideals of a commutative semigroup is Green's $\mathcal{H}$-class and all x, y, $xy \in S - J$ ($cf.$ Theorem 1.2.36).

Conversely, if complements of a 2-prime ideal contains exactly one element clearly it is maximal. Again let I be a 2-prime ideal whose complements in S forms a subgroup of S. Now let I is not maximal and is contained in some proper ideal P of S. Let e be the identity element of $S - I$. Since $I \neq P$, there exists $p \in P - I$ such that $pq = e$ for some $q \in S$.

Hence $e \in P$. Since $P \neq S$, there exists $l \in S - P$ such that $l = le \in P$, a contradiction. Consequently, I is a maximal ideal of S. $\square$

Arbitrary union of 2-prime ideals of a semigroup is a 2-prime ideal but intersection of two 2-prime ideals of a semigroup need not be 2-prime. For example consider the principal ideals $I_1 = (2)$ and $I_2 = (5)$ in the semigroup $S = (\mathbb{Z}, .)$, both are 2-prime ideals of S but $I_1 \cap I_2 = (10)$ is not a 2-prime ideal of S.

Proposition 4.1.11. *Let I be an ideal of a semigroup S with unity. Then the following statement are holds*

(1) *If I is a 2-prime ideal of S, then there is exactly one prime ideal of S that is minimal over I.*

(2) *If I is a 2-prime ideal of S, then $\sqrt{I} = P$ is a prime ideal of S, we say that I is P-2-prime.*

(3) *If I is a prime ideal of S, then I^2 is a 2-prime ideal of S.*

(4) *If I is a P-2-prime ideal of S, then $(I : a^2)$ is a 2-prime ideal, for all $a \in S$ such that $a^2 \notin I$. In particular, $(I : a^2)$ is a P-2-prime ideal of S for all $a \in S - \sqrt{I}$.*

(5) *If I is a 2-prime ideal and $(I : x) = (I : x^2)$ for all $x \in S - I$, then $(I : x)$ is a 2-prime ideal of S.*

(6) *Let I be a 2-prime ideal of S and C be a subsemigroup of S such that $C \cap I \neq \phi$. Then $C^{-1}I$ is a 2-prime ideal of $C^{-1}S$.*

(7) *An ideal I of a semigroup S is 2-prime if and only if $I[x]$ is a 2-prime ideal of $S[x]$.*

(8) *Let I_1, I_2,...,I_n be ideals of a semigroup S and I be a 2-prime ideals of S such that $\bigcap I_i \subseteq \sqrt{I}$. Then $I_i \subseteq \sqrt{I}$ for some $i \in \{1, 2, ..., n\}$.*

(9) *Let S_1 and S_2 be two semigroups with unity and $S = S_1 \times S_2$. Then I_1 (resp. I_2) is a 2-prime ideal of S_1 (resp. S_2) if and only if $I_1 \times S_2$ (resp. $S_1 \times I_2$) is a 2-prime ideals of S.*

Proof. (1) If possible, let P_1 and P_2 be two distinct prime ideal that are minimal over I. Hence there exists $x_1 \in P_1 - P_2$ and $x_2 \in P_2 - P_1$. By Lemma 1.2.25 there is $p_1 \notin P_1$ and $p_2 \notin P_2$ such that $p_1 x_1^n \in I$ and $p_2 x_2^m \in I$ for some integer m, $n \geq 1$. Since x_1, $x_2 \notin I \subseteq P_1 \cap P_2$ and I is 2-prime , hence $p_1^2 \in I \subseteq P_1 \cap P_2$ and $p_2^2 \in I \subseteq P_1 \cap P_2$. Therefore

$p_1^2 \in P_1$. Since P_1 is prime so $p_1 \in P_1$, a contradiction. Similarly if $p_2^2 \in P_2$ then $p_2 \in P_2$, a contradiction. Hence there is exactly one prime ideal which is minimal over I.

(2) Since every 2-prime ideal is semiprimary (cf. Lemma 4.1.2(2)), it is clear.

(3) Let $ab \in I^2 \subseteq I$ for some a,$b \in S$. Then either $a \in I$ or $b \in I$, since I is a prime ideal of S, which implies either $a^2 \in I^2$ or $b^2 \in I$ and hence I^2 is a 2-prime ideal of S.

(4) Let $xy \in (I : a^2)$ with $x^2 \notin (I : a^2)$ for x, $y \in S$. Then $xya^2 = (xa)(ya) \in I$. Hence $(ya)^2 = y^2a^2 \in I$, since I is a 2-prime ideal of S and $x^2a^2 \notin I$ and therefore $y^2 \in (I : a^2)$. Consequently $(I : a^2)$ is a 2-prime ideal of S.

Again let $a \in S - P$ and $x \in (I : a^2)$. Then $a^2x \in I \subseteq P$. Hence $x^2 \in I$, since $a \notin P$ and I is a 2-prime ideal of S. Thus $I \subseteq (I : a^2) \subseteq P$, which implies $P = \sqrt{I} \subseteq \sqrt{(I : a^2)} \subseteq \sqrt{P} = P$. Consequently $(I : a^2)$ is a P-2-prime ideal of S.

(5) It is clear from (4).

(6) Let $(a/s)(b/t) \in C^{-1}I$ for some a, $b \in S$ and s, $t \in C$. Then there exists $u \in C$ such that $abu \in I$. Then $a^2 \in I$ or $b^2u^2 \in I$, since I is a 2-prime ideal of S. If $a^2 \in I$, then $(a/s)^2 = (ua^2/us^2) \in C^{-1}I$ and if $b^2u^2 \in I$ then $(b/s)^2 = (b^2u^2/s^2u^2) \in C^{-1}I$. Therefore $C^{-1}I$ is a 2-prime ideal of $C^{-1}S$.

(7) Let I be a 2-prime ideal of S and $(ax^i)(bx^j) \in I[x]$ for some a, $b \in S$ and i $j \in \mathbb{N}$.Then $abx^{i+j} \in I[x]$. Hence $ab \in I$ implies $a^2 \in I$ or $b^2 \in I$, since I is a 2-prime ideal of S. Now $(ax^i)^2 = a^2x^{2i} \in I[x]$ or $(bx^j)^2 = b^2x^{2j} \in I[x]$. Hence $I[x]$ is a 2-prime ideal of $S[x]$. Similarly we can prove the converse.

(8) Let $I_i \nsubseteq \sqrt{I}$ for all $i \in \{1, 2, ..., n\}$. Then there exists $x_i \in I_i$ but $x_i \notin \sqrt{I}$ for all $i \in \{1, 2, ..., n\}$. Let $p = x_1x_2...x_n$. Then $p \in \bigcap I_i$ but $p \notin \sqrt{I}$, since $\sqrt{I}$ is a prime ideal of S, a contradiction . Hence $I_i \subseteq \sqrt{I}$ for some $i \in \{1, 2, ..., n\}$.

(9) Let I_1 be a 2-prime ideal of S_1 and $(a, b)(c, d) \in I_1 \times S_2$ for some (a, b),$(c, d) \in S$. Then $ac \in I_1$. Hence either $a^2 \in I_1$ or $c^2 \in I_1$, since I_1 is a 2-prime ideal of S_1. Hence $(a, b)^2 \in I_1 \times S_2$ or $(c, d)^2 \in I_1 \times S_2$. Consequently, $I_1 \times S_2$ is a 2-prime ideal of S.

Conversely, let $I_1 \times S_2$ be a 2-prime ideal of S and $ab \in I_1$ for some a,$b \in S_1$. Then $(a, 1)(b, 1) \in I_1 \times S_2$. Hence $a^2 \in I_1$ or $b^2 \in I_1$. Consequently, I_1 is a 2-prime ideal of S_1. □

Lemma 4.1.12. *Let I be a 2-prime ideal of a semigroup S. Then if $aJ \subseteq I$ for some $a \in S$ and ideal J of S, then $a^2 \in I$ or $\{x^2 : x \in J\} \subseteq I$.*

Proof. Let $aJ \subseteq I$ but $a^2 \notin I$ and $\{x^2 : x \in J\} \nsubseteq I$. Then there exists $j \in J$ such that $j^2 \notin I$. Also we have $aj \in I$. Since I is a 2-prime ideal of S either $a^2 \in I$ or $j^2 \in I$, a contradiction. Hence $a^2 \in I$ or $\{x^2 : x \in J\} \subseteq I$. □

The following is a characterization of a 2-prime ideal in a semigroup.

Theorem 4.1.13. *Let I be a proper ideal of a semigroup S with identity. Then I is a 2-prime ideal of S if and only if whenever $AB \subseteq I$ for some ideals A, B of S, then either $\{x^2 : x \in A\} \subseteq I$ or $\{y^2 : y \in B\} \subseteq I$.*

Proof. Let I be a 2-prime ideal of a semigroup S. Also assume $AB \subseteq I$ for some ideals A, B of S and $\{x^2 : x \in A\} \nsubseteq I$. Then there exists an element $a \in A$ such that $a^2 \notin I$. Since $aB \subseteq I$ and $a^2 \notin I$, we conclude $\{y^2 : y \in B\} \subseteq I$ (*cf*. Lemma 4.1.12).
Conversely, suppose the condition holds and let $ab \in I$ for some a,$b \in S$ and $a^2 \notin I$. Let $A = (a)$ and $B = (b)$. Then $AB \subseteq I$ and $\{x^2 : x \in A\} \nsubseteq I$, otherwise $a^2 \in I$. Hence $\{y^2 : y \in B\} \subseteq I$ implies $b^2 \in I$. Consequently, I is a 2-prime ideal of S. □

Theorem 4.1.14. *Let $S(R)$ be the multiplicative semigroup with zero and identity of a ring R. For a proper ideal I of R the following statements are equivalent:*
(1) *I is a 2-prime ideal of R,*
(2) *I is a 2-prime ideal of $S(R)$,*
(3) *if $AB \subseteq I$ for some ideals A, B of $S(R)$, then $\{x^2 : x \in A\} \subseteq I$ or $\{x^2 : x \in B\} \subseteq I$,*
(4) *if $JK \subseteq I$ for some ideals J, K of R, then $\{x^2 : x \in J\} \subseteq I$ or $\{x^2 : x \in K\} \subseteq I$.*

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (3) It follows from Theorem 4.1.13.
(3) $\Rightarrow$ (4) Since J, K are ideals of R so ideals of $S(R)$, hence $\{x^2 : x \in J\} \subseteq I$ or $\{x^2 : x \in K\} \subseteq I$.
(4) $\Rightarrow$ (1) Let $ab \in I$ for a,$b \in R$ with $a^2 \notin I$. Let $J = (a)$ and $K = (b)$. Then $JK \subseteq I$ and $\{x^2 : x \in J\} \nsubseteq I$, otherwise $a^2 \in I$. Hence $\{y^2 : y \in K\} \subseteq I$ implies $b^2 \in I$. Consequently, I is a 2-prime ideal of R. □

Definition 4.1.15. A semigroup S is said to be a 2-prime semigroup if every proper ideal of S is a 2-prime ideal of S.

Example 4.1.16. Every proper ideal of the semigroup $S = (\mathbb{N}, +)$ is 2-prime, hence S is a 2-prime semigroup.

Since every 2-prime ideal is semiprimary (*cf.* Lemma 4.1.2), we have the following result by Theorem 1.2.50.

Theorem 4.1.17. *Let S be a 2-prime semigroup. Then the following statements about S are true:*
(1) *Prime ideals of S are linearly ordered under set inclusion.*
(2) *Idempotents of S are linearly ordered under natural ordering.*

Theorem 4.1.18. *Let S be a regular semigroup. Then the following statements are equivalent.*
(1) *S is a 2-prime semigroup,*
(2) *every ideal of S is prime,*
(3) *every ideal of S is primary,*
(4) *every ideal of S is semiprimary,*
(5) *prime ideals of S are totally ordered,*
(6) *idempotents of S are linearly ordered,*
(7) *all ideals of S are linearly ordered.*

Proof. (1) $\Rightarrow$ (2) Let I be a proper ideal of a 2-prime semigroup S and $ab \in I$ for some a, $b \in S$. Then $a^2 \in I$ or $b^2 \in I$. Hence $a \in I$ or $b \in I$, since S is regular. Hence every ideal of S is prime.
(2) $\Rightarrow$ (1) It is clear.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (2) It follows from Theorem 1.2.50. $\square$

Theorem 4.1.19. *Let S and T be multiplicative semigroup with 1 and 0 and $f : S \to T$ be a semigroup homomorphism. Then the following statement holds*
(1) *If K is a 2-prime ideal of T, then $f^{-1}(K)$ is a 2-prime ideal of S.*
(2) *Let I be a proper ideal of S such that $\{(x, y) \in ker f : x \neq y\} \subseteq I \times I$.*
(*a*) *If $f(I)$ is a 2-prime ideal of T, then I is a 2-prime ideal of S.*
(*b*) *If f is onto and I is a 2-prime ideal of S, then $f(I)$ is a 2-prime ideal of S.*

Proof. (1) Let $ab \in f^{-1}(K)$ for some a, $b \in S$. Then $f(ab) = f(a)f(b) \in K$. Hence $f(a^2) \in K$ or $f(b^2) \in K$, since K is a 2-prime ideal of T. Therefore $a^2 \in f^{-1}(K)$ or $b^2 \in f^{-1}(K)$. Consequently, $f^{-1}(K)$ is a 2-prime ideal of S.

(2)(a) Let $f(I)$ is a 2-prime ideal of T. Clearly $f^{-1}(f(I)) = I$, since $\{(x, y) \in kerf : x \neq y\} \subseteq I \times I$. Hence I is a 2-prime ideal of S by (1).

(b) Let $xy \in f(I)$ for some x, $y \in T$. Since f is surjective, there exists a, $b \in S$ such that $f(a) = x$ and $f(b) = y$. Hence $f(a)f(b) \in f(I)$ implies $f(ab) \in f(I)$, since f is a homomorphism. Again $\{(x, y) \in kerf : x \neq y\} \subseteq I \times I$ implies $ab \in I$. Hence $a^2 \in I$ or $b^2 \in I$, since I is a 2-prime ideal of S. Therefore $x^2 = (f(a))^2 = f(a^2) \in f(I)$ or $y^2 = (f(b))^2 = f(b^2) \in f(I)$. Consequently, $f(I)$ is a 2-prime ideal of T. □

Corollary 4.1.20. *Let S and T be commutative multiplicative semigroup with 0 and 1.*

(1) *If $S \subseteq T$ be an extension of semigroups and K is a 2-prime ideal of T, then $K \cap S$ is a 2-prime ideal of S.*

(2) *Let $A \subseteq B$ be two ideals of S. Then B is a 2-prime ideal of S if and only if B/A is a 2-prime ideal of S/A.*

4.2 2-P semigroup

Definition 4.2.1. A semigroup S is said to be 2-P semigroup if every 2-prime ideal of S is a prime ideal of S.

Example 4.2.2. Every commutative band is a 2-P semigroup.

The following is a characterization of a 2-P semigroup,

Theorem 4.2.3. *A semigroup S is 2-P semigroup if and only if prime ideals are idempotent and every 2-prime ideals of S is of the form A^2, where A is a prime ideal of S.*

Proof. Let S be a 2-P semigroup and $ab \in A^2 \subseteq A$ for some a, $b \in S$, where A is a prime ideal of S. Since A is prime either $a \in A$ or $b \in A$. Let $a \in A$. Then $a^2 \in A^2$ implies A^2 is a 2-prime ideal of S and hence prime ideal of S. Clearly $A^2 \subseteq A$. Let $x \in A$. Then $x^2 \in A^2$ implies $x \in A^2$, since A^2 is a prime ideal of S. Therefore $A = A^2$, as desired.

Conversely, Let I be a 2-prime ideal of a semigroup S. Then $I = A^2 = A$, for some prime ideal A of S. Consequently, S is a 2-P semigroup. $\square$

The following is a characterization of a prime ideals of a semigroup in terms of 2-prime ideals.

Theorem 4.2.4. *A proper ideal I of a semigroup S is prime if and only if it is both 2-prime and semiprime.*

Proof. Let I be a prime ideal of a semigroup S. Then clearly it is both semiprime and 2-prime ideals of S.
Conversely, Let I be a proper ideal of a semigroup which is both semiprime and 2-prime ideal and $ab \in I$ for some a,$b \in S$. Then either $a^2 \in I$ or $b^2 \in I$, since I is a 2-prime ideal of S, which implies either $a \in I$ or $b \in I$, since I is a semiprime ideal of S. Consequently, I is a prime ideal of S. $\square$

The following is an obvious consequence of above theorem.

Corollary 4.2.5. *A semigroup S is a 2-P semigroup if and only if 2-prime ideals of S are semiprime.*

Lemma 4.2.6. *Let (S, M) be a semigroup with unity, where M is the unique maximal ideal of S. Then for every prime ideal P of S, PM is a 2-prime ideal of S.*
In particular, PM is prime if and only if $PM = P$.

Proof. Let $ab \in PM \subseteq P$. Then either $a \in P$ or $b \in P$, since P is a prime ideal of S. Let $a \in P$ implies $a^2 \in PM$, since $P \subseteq M$. Hence PM is a 2-prime ideal of S.
Again, let PM is prime and $x \in P$. Then $x^2 \in PM$, since $P \subseteq M$. Hence $x \in PM$, since PM is a prime ideal of S. Therefore $P \subseteq PM$ and clearly $PM \subseteq P$. Hence $PM = P$.

$\square$

Theorem 4.2.7. *A semigroup (S, M) with unity is 2-P if and only if $IM = P$, where P is the minimal prime ideal over a 2-prime ideal I.*

Proof. Let I be a 2-prime ideal of a 2-P semigroup (S, M) with unity. Then clearly $IM = I$ (Lemma 4.2.6).
Conversely, let I be a 2-prime ideal of a semigroup (S, M) and P is the minimal prime ideal over I such that $IM = P$. Then $I \subseteq P = IM \subseteq I \cap M = I$ implies $I = P$, as desired. $\square$

4.3 Weakly 2-prime

Throughout this section all semigroups are commutative with zero and identity, in particular S will denote such a semigroup.

Definition 4.3.1. A proper ideal I of a semigroup S is said to be a weakly 2-prime ideal of S if $0 \neq ab \in I$ for any a,$b \in S$ implies $a^2 \in I$ or $b^2 \in I$.

The following lemma is obvious, hence we omit the proof.

Lemma 4.3.2. (1) *Every 2-prime ideal of S is a weakly 2-prime ideal of S.*
(2) *Every weakly prime ideal of S is a weakly 2-prime ideal of S.*

The following is a characterization of a semigroup in which every proper ideal is weakly 2-prime.

Theorem 4.3.3. *Every proper ideal of a semigroup S is weakly 2-prime if and only if $(a^2) \subseteq (ab)$ or $(b^2) \subseteq (ab)$ or $ab = 0$, for any a,$b \in S$ such that $(ab) \neq S$.*

Proof. Let every proper ideal of a semigroup S is weakly 2-prime and a, $b \in S$ such that $(ab) \neq S$. Then (ab) is weakly 2-prime ideal of S. If $ab \neq 0$, then $0 \neq ab \in (ab)$ implies $a^2 \in (ab)$ or $b^2 \in (ab)$. Consequently $(a^2) \subseteq (ab)$ or $(b^2) \subseteq (ab)$.
Conversely, let I be a proper ideal of S such that $0 \neq ab \in I$ for some a, $b \in S$. Then $0 \neq ab \in (ab) \subseteq I$ implies $a^2 \in (a^2) \subseteq (ab) \subseteq I$ or $b^2 \in (b^2) \subseteq (ab) \subseteq I$, as desired. $\square$

Definition 4.3.4. A proper ideal I of a semigroup S is said to be a strong weakly 2-prime ideal if $0 \neq AB \subseteq I$ implies $\{x^2 : x \in A\} \subseteq I$ or $\{y^2 : y \in B\} \subseteq I$ for any ideals A, B of S.

Theorem 4.3.5. *Every proper ideal of a semigroup S is strong weakly 2-prime ideal if and only if $\{x^2 : x \in A\} \subseteq AB$ or $\{y^2 : y \in B\} \subseteq AB$ or $AB = 0$ for any proper ideals A, B of S.*

Proof. Let every proper ideal of S is strong weakly 2-prime ideal and A, B be any two proper ideals of S. Then $AB \neq S$ and AB is a strong weakly 2-prime ideal of S. If $0 \neq AB \subseteq AB$, then $\{x^2 : x \in A\} \subseteq AB$ or $\{y^2 : y \in B\} \subseteq AB$.

Conversely, let I be a proper ideals of S and $0 \neq AB \subseteq I$ for any two proper ideals A, B of S. Then by hypothesis $\{x^2 : x \in A\} \subseteq AB \subseteq I$ or $\{y^2 : y \in B\} \subseteq AB \subseteq I$. Consequently every proper ideal of S is strong weakly 2-prime. $\square$

Chapter 5

Power graph of monogenic semigroup

In this chapter, we define a type of power graph, an undirected graph $\mathcal{P}(\mathcal{S}_M)$ over $\mathcal{S}_M = \{0, x, x^2, \ldots, x^n\}$, with vertex set $V = \mathcal{S}_M^* = \mathcal{S}_M - \{0\}$ and two distinct vertices x^i and x^j are adjacent with the rule $x^i = (x^j)^k$ or $x^j = (x^i)^k$ for some $k \in \mathbb{N}$ if and only if $i|j$ or $j|i$, where $1 \leq i \neq j \leq n$. Here we compute various graph parameters, topological indices and study perfectness property of the graph $\mathcal{P}(\mathcal{S}_M)$. Also we determine some graph parameters of the cartesian product of power graphs of monogenic semigroups.

In **Section** 5.1, we determine the diameter (*cf.* Corollary 5.1.2), girth (*cf.* Theorem 5.1.4), degree of vertices (*cf.* Theorem 5.1.6), maximum and minimum degrees (*cf.* Theorem 5.1.7), domination number (*cf.* Theorem 5.1.9), independence number (*cf.* Theorem 5.1.12), vertex cover number (*cf.* Theorem 5.1.11), bridge number (*cf.* Theorem 5.1.14) of $\mathcal{P}(\mathcal{S}_M)$.

Then in **Section** 5.2, we compute some topological indices such as Wiener index (*cf.* Theorem 5.2.1), Harary index (*cf.* Theorem 5.2.2), first and second Zagreb eccentricity (*cf.* Theorem 5.2.3), ecentric connectivity index (*cf.* Theorem 5.2.4) and degree distance (*cf.* Theorem 5.2.5) of the graph $\mathcal{P}(\mathcal{S}_M)$ based on distance of vertices.

In **Section** 5.3, we prove that the graph $\mathcal{P}(\mathcal{S}_M)$ and it's complement are perfect graph (*cf.* Theorem 5.3.2). Then we determine clique number (*cf.* Theorem 5.3.3), chromatic number (*cf.* Theorem 5.3.6) and chromatic index (*cf.* Theorem 5.3.7) of $\mathcal{P}(\mathcal{S}_M)$.

Finally, in **Section** 5.4, we present diameter (*cf.* Theorem 5.4.1), radius (*cf.* Theorem 5.4.2), girth (*cf.* Theorem 5.4.3), degree of vertices (*cf.* Theorem 5.4.4), domination number (*cf.* Theorem 5.4.5), clique number (*cf.* Theorem 5.4.6) and chromatic number (*cf.*

Theorem 5.4.8) of the cartesian product of power graphs of $\mathcal{P}(\mathcal{S}_M^1)$ and $\mathcal{P}(\mathcal{S}_M^2)$.

Throughout this chapter, all graphs are simple, connected and by $\mathcal{P}(\mathcal{S}_M)$, we denote the power graph over the finite monogenic semigroup $\mathcal{S}_M$ as defined above.

5.1 Some graph parameters

Theorem 5.1.1. *The graph $\mathcal{P}(\mathcal{S}_M)$ is a connected graph. Moreover, diam $(\mathcal{P}(\mathcal{S}_M)) \leq 2$.*

Proof. The vertex x of $\mathcal{P}(\mathcal{S}_M)$ is adjacent to every other vertex x^k, where $1 < k \leq n$. Let x^i and x^j be any two vertex of $\mathcal{P}(\mathcal{S}_M)$ such that $1 < i \neq j \leq n$. Then $x^i \sim x \sim x^j$ is a path and hence $\mathcal{P}(\mathcal{S}_M)$ is a connected graph. Clearly, diam $(\mathcal{P}(\mathcal{S}_M)) \leq 2$. $\square$

Since for $n \geq 3$, $d(x^2, x^3) = 2$, the following result is immediate.

Corollary 5.1.2.

$$diam \mathcal{P}(\mathcal{S}_M) = \begin{cases} 1\ , & if\ n = 2 \\ 2\ , & if\ n \geq 3 \end{cases}$$

Clearly ecentricity of the vertex x in $\mathcal{P}(\mathcal{S}_M)$ is 1 and ecentricity of other vertices is 2. Thus we have the following result about radius of $\mathcal{P}(\mathcal{S}_M)$.

Corollary 5.1.3. *rad* $(\mathcal{P}(\mathcal{S}_M)) = 1$.

We have the following result about girth of $\mathcal{P}(\mathcal{S}_M)$

Theorem 5.1.4.

$$gr(\mathcal{P}(\mathcal{S}_M)) = \begin{cases} \infty\ , & if\ n \leq 3 \\ 3, & if\ n \geq 4 \end{cases}$$

Proof. Clearly if $n \leq 3$, then $\mathcal{P}(\mathcal{S}_M)$ has no cycle and hence girth $\mathcal{P}(\mathcal{S}_M) = \infty$. Again if $n \geq 4$, then $x \sim x^2 \sim x^4 \sim x$ is a cycle of length 3 in $\mathcal{P}(\mathcal{S}_M)$ and hence girth $(\mathcal{P}(\mathcal{S}_M)) = 3$. $\square$

The following corollary is the immediate consequence of the Theorem 5.1.4.

Corollary 5.1.5. *The graph $\mathcal{P}(\mathcal{S}_M)$ is bipartiate if and only if $2 \leq n \leq 3$.*

Theorem 5.1.6. *The degree of the vertex x^k in $\mathcal{P}(\mathcal{S}_M)$ is $d_k(x^k) = \sigma(k) + \lfloor \frac{n}{k} \rfloor - 2$,*

Proof. Let x^k be any vertex of $\mathcal{P}(\mathcal{S}_M)$, where $1 \leq k \leq n$. Then x^k is adjacent to x^i, where $i|k$, $1 \leq i < k$ and also adjacent to x^j where $k|j$, $k < j \leq n$. Clearly the number of elements i satisfying $i|k$ and $1 \leq i < k$ is $\sigma(k) - 1$. Also the number of elements j satisfying $k|j$ and $k < j \leq n$ is $\lfloor \frac{n}{k} \rfloor - 1$. Therefore the degree of the vertex x^k is $\sigma(k) - 1 + \lfloor \frac{n}{k} \rfloor - 1$, as desired. □

Theorem 5.1.7. *The maximum and minimum degrees of $\mathcal{P}(\mathcal{S}_M)$ are $\Delta(\mathcal{P}(\mathcal{S}_M)) = n - 1$ and $\delta(\mathcal{P}(\mathcal{S}_M)) = 1$ respectively.*

Proof. Let us consider the vertex x of $\mathcal{P}(\mathcal{S}_M)$. Clearly the vertex x is adjacent to every other vertex x^j of $\mathcal{P}(\mathcal{S}_M)$, where $1 < j \leq n$. Hence $\Delta(\mathcal{P}(\mathcal{S}_M)) = n - 1$.
Again, consider the vertex x^{n_p} of $\mathcal{P}(\mathcal{S}_M)$, where n_p is the greatest prime less or equals to n. Then x^{n_p} is adjacent to only one vertex x of $\mathcal{P}(\mathcal{S}_M)$. Consequently, $\delta(\mathcal{P}(\mathcal{S}_M)) = 1$. □

Since the vertex x^{n_p} of $\mathcal{P}(\mathcal{S}_M)$ has degree one, we have the following result.

Corollary 5.1.8. *The graph $\mathcal{P}(\mathcal{S}_M)$ is neither Eulerian nor Hamiltonian. Also the graph is not triangulated.*

Theorem 5.1.9. *The domination number of $\mathcal{P}(\mathcal{S}_M)$ is $\gamma(\mathcal{P}(\mathcal{S}_M)) = 1$.*

Proof. Let us consider $D = \{x\}$. Note that the vertex x of $\mathcal{P}(\mathcal{S}_M)$ is adjacent to every other vertex x^j, where $1 < j \leq n$. Then clearly D is a smallest dominating set for $\mathcal{P}(\mathcal{S}_M)$ and hence $\gamma(\mathcal{P}(\mathcal{S}_M)) = 1$. □

Theorem 5.1.10. *The accessible number of $\mathcal{P}(\mathcal{S}_M)$ is $\eta(\mathcal{P}(\mathcal{S}_M)) = 1$.*

Proof. Let us choose $U = \{x\}$ in $\mathcal{P}(\mathcal{S}_M)$. Then $N[U] = V(\mathcal{P}(\mathcal{S}_M))$. Then each $v \in V(\mathcal{P}(\mathcal{S}_M)) - U$ is adjacent to $N[U]$. Hence $\eta(\mathcal{P}(\mathcal{S}_M)) = 1$, as desired. □

Theorem 5.1.11. *The covering number of $\mathcal{P}(\mathcal{S}_M)$ is $\tau(\mathcal{P}(\mathcal{S}_M)) = \lfloor \frac{n}{2} \rfloor$.*

Proof. To define a cover set C for $\mathcal{P}(\mathcal{S}_M)$, we start by adding the vertices in C that have heighest degree, which provides to cover heighest number of edge with less number of vertices. Since the vertex x is of heighest degree in $\mathcal{P}(\mathcal{S}_M)$, we add x in C. Now we consider two cases

Case (1) n is even : Then to cover all edges adjacent to x^2, we must have add it in C. Continuing this way we must have to add $x^{\frac{n}{2}}$ in C to cover the edge $x^{\frac{n}{2}}x^n \in E(\mathcal{P}(\mathcal{S}_M))$. Thus we have the cover set $C = \{x, x^2, \ldots, x^{\frac{n}{2}}\}$ for $\mathcal{P}(\mathcal{S}_M)$, clearly it is of minimum cardinality.

Case (2) n is odd : Continuing as in case (1), the last vertex we have to add in C is $x^{\frac{n-1}{2}}$ as $x^{\frac{n-1}{2}}x^{n-1} \in E(\mathcal{P}(\mathcal{S}_M)$. Therfore we have the cover set $C = \{x, x^2, \ldots, x^{\frac{n-1}{2}}\}$ for $\mathcal{P}(\mathcal{S}_M)$, which is of minimum cardinality. Considering both above cases, we have

$$\begin{aligned}\tau(\mathcal{P}(\mathcal{S}_M)) &= \begin{cases} \frac{n}{2}\ , & \text{if n is even} \\ \frac{n-1}{2}\ , & \text{if n is odd} \end{cases} \\ &= \lfloor \frac{n}{2} \rfloor\end{aligned}$$

□

Theorem 5.1.12. *The independence number of $\mathcal{P}(\mathcal{S}_M)$ is $\alpha(\mathcal{P}(\mathcal{S}_M)) = \lceil \frac{n}{2} \rceil$.*

Proof. By Theorem 5.1.11, we have $I = \{x^{\frac{n}{2}+1}, \ldots, x^n\}$ (when n is even) and $I = \{x^{\frac{n+1}{2}}, \ldots, x^n\}$ (when n is odd) as an independent set, which is of largest cardinality. Therefore

$$\begin{aligned}\alpha(\mathcal{P}(\mathcal{S}_M)) &= \begin{cases} \frac{n}{2}\ , & \text{if n is even} \\ \frac{n+1}{2}\ , & \text{if n is odd} \end{cases} \\ &= \lceil \frac{n}{2} \rceil\end{aligned}$$

. □

Theorem 5.1.13. *The only cut vertex of $\mathcal{P}(\mathcal{S}_M)$ with $n \geq 3$ is $\{x\}$.*

Proof. Since the vertex x of $\mathcal{P}(\mathcal{S}_M)$ is adjacent to all other vertices, it must be a cut vertex of $\mathcal{P}(\mathcal{S}_M)$. Moreover, we have the separation set of $\mathcal{P}[V(\mathcal{S}_M) - \{x\}]$ as $X = V(\mathcal{P}(\mathcal{S}_M)) - \{x, x^{n_p}\}$ and $Y = \{x^{n_p}\}$.

Now we prove that no other vertex of $\mathcal{P}(\mathcal{S}_M)$ is a cut vertex. If possible, let $x^j (j \neq 1)$ is a cut vertex of $\mathcal{P}(\mathcal{S}_M)$. So there exists subsets X and Y of $V(\mathcal{P}(\mathcal{S}_M)) - \{x^j\}$ such that there is no edge with one endpoints in X and other endpoints in Y. Without loss of generality, assume $x \in X$. Since $x \sim x^j$, where $1 < j \leq n$, then $x \sim x^k$ for some $x^k \in Y$, which is a contradiction. Hence $\{x\}$ is the only cut vertex of $\mathcal{P}(\mathcal{S}_M)$. $\square$

We have the following result on bridge number of $\mathcal{P}(\mathcal{S}_M)$, denoted as $br(\mathcal{P}(\mathcal{S}_M))$.

Theorem 5.1.14. *The bridge number of $\mathcal{P}(\mathcal{S}_M)$ is $br(\mathcal{P}(\mathcal{S}_M)) = k$, where k is the number of elements in the set $\{p : p$ is a prime and $\lfloor \frac{n}{2} \rfloor < p \leq n\}$.*

Proof. Let m be a composite number such that $1 < m \leq n$. Then any edge joining the vertex x^m must contain in some cycle $x \sim x^m \sim x^d \sim x$, where d is proper positive divisor of m. Again, let t be a prime such that $1 < t \leq \lfloor \frac{n}{2} \rfloor$. Then any edge joining x^t must lie in some cycle $x \sim x^t \sim x^r \sim x$, where r is a multiple of t and $r \leq n$. Now, let p be a prime such that $\lfloor \frac{n}{2} \rfloor < p \leq n$. Then x^p is adjacent to x only in $\mathcal{P}(\mathcal{S}_M)$. Hence the edges which is not contained in any cycle, is of the form $xx^p \in E(\mathcal{P}(\mathcal{S}_M))$, where p is prime and $\lfloor \frac{n}{2} \rfloor < p \leq n$, as desired. $\square$

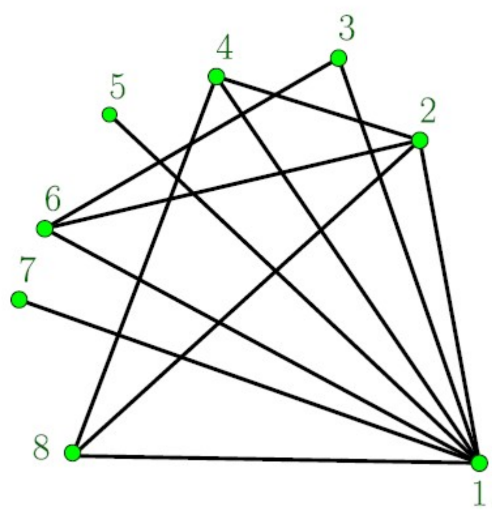


Figure 5.1: $\mathcal{P}(\mathcal{S}_M^8)$: Here, for $1 \leq i \leq 8$, each label i corresponds to the vertices x^i.

Example 5.1.15. The results presented in above theorems and also from the Figure 5.1 of the graph $\mathcal{P}(\mathcal{S}_M^8)$, we have $\mathrm{diam}(\mathcal{P}(\mathcal{S}_M^8)) = 2$, $\mathrm{girth}(\mathcal{P}(\mathcal{S}_M^8)) = 3$, $\Delta(\mathcal{P}(\mathcal{S}_M^8)) = 7$, $\delta(\mathcal{P}(\mathcal{S}_M^8)) = 1$, $\alpha(\mathcal{P}(\mathcal{S}_M^8)) = 4$ with $J = \{x^5, x^6, x^7, x^8\}$ as maximum independent set,

$\gamma(\mathcal{P}(\mathcal{S}_M^8)) = 1$ with $D = \{x\}$ as minimum dominating set. Also $\mathrm{br}(\mathcal{P}(\mathcal{S}_M^8)) = 2$, $\eta(\mathcal{P}(\mathcal{S}_M^8)) = 1$.

5.2 Topological indices

In this section we will investigate some topological indices based on the distance of vertices over $\mathcal{P}(\mathcal{S}_M)$.

Theorem 5.2.1. *The Wiener index of the graph $\mathcal{P}(\mathcal{S}_M)$ is $W(\mathcal{P}(\mathcal{S}_M)) = n^2 - \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor$.*

Proof. We have $W(\mathcal{P}(\mathcal{S}_M))$

$$
\begin{aligned}
&= \sum_{j=2}^{n} d(x, x^j) + \sum_{j=3}^{n} d(x^2, x^j) + \cdots + \sum_{j=n-1}^{n} d(x^{n-2}, x^j) + d(x^{n-1}, x^n) \\
&= (\lfloor \frac{n}{1} \rfloor - 1) + \{(\lfloor \frac{n}{2} \rfloor - 1) + 2(n - 2 - \lfloor \frac{n}{2} \rfloor + 1)\} + \cdots + \{(\lfloor \frac{n}{n-2} \rfloor - 1) + 2(3 - \lfloor \frac{n}{n-2} \rfloor)\} + 2 \\
&= n^2 - \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor.
\end{aligned}
$$

$\square$

Theorem 5.2.2. *The Harary index of the graph $\mathcal{P}(\mathcal{S}_M)$ is $H(\mathcal{P}(\mathcal{S}_M)) = \frac{n^2-3n}{4} + \frac{1}{2} \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor$.*

Proof. We have $H(\mathcal{P}(\mathcal{S}_M))$

$$
\begin{aligned}
&= \{(\lfloor \frac{n}{1} \rfloor - 1) + \{(\lfloor \frac{n}{2} \rfloor - 1) + \frac{1}{2}(n - 2 - \lfloor \frac{n}{2} \rfloor + 1)\} + \cdots + \{(\lfloor \frac{n}{n} \rfloor - 1) + \frac{1}{2}(n - n - \lfloor \frac{n}{n} \rfloor + 1)\} \\
&= \{\sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor - n\} + \frac{1}{2}\{n^2 - \frac{n(n+1)}{2} - \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor + n\} \\
&= \frac{n^2 - 3n}{4} + \frac{1}{2} \sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor.
\end{aligned}
$$

$\square$

Theorem 5.2.3. *The first Zagreb ecentricity* (E_1) *and second Zagreb ecentricity* (E_2) *indices of the graph* $\mathcal{P}(\mathcal{S}_M)$ *are*

$$E_1(\mathcal{P}(\mathcal{S}_M)) = 4n - 3 \qquad \textit{and} \qquad E_2(\mathcal{P}(\mathcal{S}_M)) = 2[\sum_{i=1}^{n}\{\sigma(i) + \lfloor \frac{n}{i} \rfloor\} + 1 - 3n].$$

Proof. Since the vertex x is adjacent to every other vertices of $\mathcal{P}(\mathcal{S}_M)$, the ecentricity of the vertex x is $1(e_1 = 1)$ and the ecentricity of the other vertices are $2(e_i = 2$ for $i \neq 1)$.
Hence we have

$$E_1(\mathcal{P}(\mathcal{S}_M)) = 1 + 2^2 + \cdots + 2^2 = 4n - 3.$$

The number of edges in $\mathcal{P}(\mathcal{S}_M)$ is

$$m = \frac{1}{2}\sum_{i=1}^{n} d_i = \frac{1}{2}[\sum_{i=1}^{n}\{\sigma(i) + \lfloor \frac{n}{i} \rfloor\} - 2n].$$

Therefore $E_2(\mathcal{P}(\mathcal{S}_M))$

$$= 2(n-1) + 4[\frac{1}{2}\{\sum_{i=1}^{n}\{\sigma(i) + \lfloor \frac{n}{i} \rfloor\} - 2n\} - (n-1)]$$

$$= 2[\sum_{i=1}^{n}\{\sigma(i) + \lfloor \frac{n}{i} \rfloor\} + 1 - 3n].$$

□

Theorem 5.2.4. *The eccentric connectivity index of the graph* $\mathcal{P}(\mathcal{S}_M)$ *is*

$$\xi^C(\mathcal{P}(\mathcal{S}_M)) = 2\{\sum_{i=2}^{n}(\sigma(i) + \lfloor \frac{n}{i} \rfloor)\} - 3(n-1).$$

Proof. We have

$$\xi^C(\mathcal{P}(\mathcal{S}_M)) = \sum_{i=1}^{n} d_i e_i = \{\sigma(1) - \lfloor \frac{n}{1} \rfloor - 2\} + 2\{\sigma(2) - \lfloor \frac{n}{2} \rfloor - 2\} + \cdots + 2\{\sigma(n) - \lfloor \frac{n}{n} \rfloor - 2\}$$

$$= 2\{\sum_{i=2}^{n}(\sigma(i) + \lfloor \frac{n}{i} \rfloor)\} - 3(n-1).$$

□

Theorem 5.2.5. *The degree distance of $\mathcal{P}(\mathcal{S}_M)$ is*

$$D'(\mathcal{P}(\mathcal{S}_M)) = \sum_{i=1}^{n}\{\sigma(i) + \lfloor\frac{n}{i}\rfloor - 2\}\{2n - \sigma(i) - \lfloor\frac{n}{i}\rfloor\}$$

.

Proof. Here $d_k = \sigma(k) + \lfloor\frac{n}{k}\rfloor - 2$ and $D_k = 2n - \{\sigma(k) + \lfloor\frac{n}{k}\rfloor\}$, where d_i is the degree of the vertex x^i, $D_i = \sum_{j=1, j\neq i}^{n} d(x^i, x^j)$ and $1 \leq k \leq n$ (see [7]).
Hence we have

$$D'(\mathcal{P}(\mathcal{S}_M)) = \sum_{i=1}^{n}\{\sigma(i) + \lfloor\frac{n}{i}\rfloor - 2\}\{2n - \sigma(i) - \lfloor\frac{n}{i}\rfloor\}.$$

□

5.3 Perfectness property of $\mathcal{P}(\mathcal{S}_M)$

In this section, we prove the perfectness of $\mathcal{P}(\mathcal{S}_M)$ and compute the clique number $\omega(\mathcal{P}(\mathcal{S}_M))$, chromatic number $\chi(\mathcal{P}(\mathcal{S}_M))$ and chromatic index $\chi'(\mathcal{P}(\mathcal{S}_M))$ of $\mathcal{P}(\mathcal{S}_M)$.

Lemma 5.3.1. *Let $\mathcal{S}_M$ be a monogenic semigroup. Then neither $\mathcal{P}(\mathcal{S}_M)$ nor its complement contains an induced odd cycle of length at least* 5.

Proof. We first show that the graph $\mathcal{P}(\mathcal{S}_M)$ does not contain any induced odd cycle of length at least 5. If possible, let $L : x^{i_1} \sim x^{i_2} \sim x^{i_3} \sim \cdots \sim x^{2t+1} \sim x^{i_1}$ be an induced cycle in $\mathcal{P}(\mathcal{S}_M)$, where $t \geq 2$. We denote $a \to b$ if $a|b$ and $a \leftarrow b$ if $b|a$. Since $x^{i_1} \sim x^{i_2}$, either $i_1 \to i_2$ or $i_1 \leftarrow i_2$. Without loss of generality, let $i_1 \to i_2$. Again as $x^{i_2} \sim x^{i_3}$ either $i_2 \to i_3$ or $i_2 \leftarrow i_3$. If $i_2 \to i_3$, then by transitivity of divides relation, we have $i_1 \to i_3$, which contradicts that L is an induced cycle in $\mathcal{P}(\mathcal{S}_M)$. So we assume $i_2 \leftarrow i_3$. Keep following the same procedure, we have $i_1 \to i_2 \leftarrow i_3 \to i_4 \leftarrow i_5 \to \cdots \leftarrow i_{2t+1} \to i_1$ which implies $i_{2t+1} \to i_1 \to i_2$ i.e. $x^{2t+1} \sim x^{i_2}$ is a chord, a contradiction. Hence $\mathcal{P}(\mathcal{S}_M)$ does not contain any induced cycle of odd length at least five.
Again if possible, let $L' : x^{i_1} \sim x^{i_2} \sim x^{i_3} \sim \cdots \sim x^{2t+1} \sim x^{i_1}$ be an induced cycle in complements of $\mathcal{P}(\mathcal{S}_M)$, where $t \geq 2$. Fixing t, let x^{2t+1} be the final vertex in the cycle

L'. Since $x^{i_1} \sim x^{i_2}$, neither $i_1|i_2$ nor $i_2|i_1$. Since $x^{i_1} \nsim x^{i_3}$, then either $i_1 \to i_3$ or $i_1 \leftarrow i_3$. Without loss of generality, we assume $i_1 \to i_3$. Again as $x^{i_1} \nsim x^{i_4}$, either $i_1 \to i_4$ or $i_1 \leftarrow i_4$. If $i_1 \leftarrow i_4$, then by transitivity of divides relation, we have $i_3 \leftarrow i_4$. This implies $x^{i_3} \nsim x^{i_4}$, a contradiction. Therefore $i_1 \to i_4$. Again as $x^{i_2} \nsim x^{i_4}$, we have either $i_2 \to i_4$ or $i_2 \leftarrow i_4$. But if $i_2 \leftarrow i_4$, then by transitivity of divides relation, $i_1 \to i_2$ which implies $x^{i_1} \nsim x^{i_2}$, a contradiction. Hence $i_2 \to i_4$. Also as $x^{i_2} \nsim x^{i_5}$, we have either $i_2 \to i_5$ or $i_2 \leftarrow i_5$. If $i_2 \leftarrow i_5$, then by arguing as above we have $i_4 \leftarrow i_5$. Thus $x^{i_4} \nsim x^{i_5}$, a contradiction. Therefore $i_2 \to i_5$. Again as $x^{i_3} \nsim x^{i_5}$, either $i_3 \to i_5$ or $i_3 \leftarrow i_5$. Now if $i_3 \leftarrow i_5$, then by arguing as above we have $x^{i_2} \nsim x^{i_3}$, which is a contradiction. Again if $i_3 \to i_5$, then by transitivity of divides relation, we have $x^{i_1} \nsim x^{i_5}$ in L'. Therefore x^{i_5} is not the final vertex in L', there must exists at least two vertex x^{i_6} and x^{i_7} in L'. Now keep following the same process, we have $x^{i_1} \nsim x^{i_7}$. Hence x^{i_7} is not the final vertex in L'. It is clear to observe that we cannot find a final vertex in L' as this process continues indefinitely, which contradicts that L' is of finite length. Therefore complements of $\mathcal{P}(\mathcal{S}_M)$ has no induced cycle of odd length at least five. □

By applying Lemma 5.3.1, Theorem 1.3.1 and Theorem 1.3.3 we have the following result.

Theorem 5.3.2. *The graph $\mathcal{P}(\mathcal{S}_M)$ and its complement $\overline{\mathcal{P}(\mathcal{S}_M)}$ are perfect.*

Theorem 5.3.3. *Let $\mathcal{S}_M$ be a monogenic semigroup. Then $\omega(\mathcal{P}(\mathcal{S}_M)) = k+1$, where $2^k \leq n < 2^{k+1}$, $k \in \mathbb{N} \cup \{0\}$.*

Proof. Let $\mathcal{S}_M = \{0, x, x^2, \ldots, x^n\}$ be a monogenic semigroup where $2^k \leq n < 2^{k+1}$, $k \in \mathbb{N} \cup \{0\}$. Then clearly $C = \{x, x^2, x^{2^2}, \ldots, x^{2^k}\}$ is a clique of size $k+1$ in $\mathcal{P}(\mathcal{S}_M)$ and there does not exists any clique containing C. Now we prove that there does not exists any clique of size greater than $k+1$. If possible, let $C' = \{x^{i_1}, x^{i_2}, \ldots, x^{i_m}\}$ be a clique of size $m(> k+1)$ such that $i_1 \to i_2 \to i_3 \to \cdots \to i_m$. Since x is adjacent to every other vertex of $\mathcal{P}(\mathcal{S}_M)$, we have $i_1 = 1$.

Therefore $i_m = k_1.i_{m-1}$ for some $k_1 \geq 2$.

$= k_1.k_2 \ldots k_{m-1}.i_1$, where each $k_i \geq 2$.

$\geq 2^{m-1}.i_1 = 2^{m-1}$.

Since $m > k+1$, $i_m \geq 2^{k+1}$, which contradicts that $n < 2^{k+1}$.

Therefore $\omega(\mathcal{P}(\mathcal{S}_M)) = k+1$, where $2^k \leq n < 2^{k+1}$, $k \in \mathbb{N} \cup \{0\}$. $\square$

Corollary 5.3.4. *The graph $\mathcal{P}(\mathcal{S}_M)$ is outer planar if and only if $n < 8$.*

Proof. Case (1) Let $n \geq 8$. Then $\mathcal{P}(\mathcal{S}_M)$ has a subgraph isomorphic to K_4 ($cf.$ Theorem 5.3.3). Therefore $\mathcal{P}(\mathcal{S}_M)$ is not outer planar.

Case (2) Let $n < 8$. Clearly $\mathcal{P}(\mathcal{S}_M)$ has no subgraph isomorphic to K_4 ($cf.$ Theorem 5.3.3). Also it is easy to observe that $\mathcal{P}(\mathcal{S}_M)$ has no subgraph isomorphic to $K_{2,3}$, which is possible if $n \geq 18$, a contradiction. Hence the graph $\mathcal{P}(\mathcal{S}_M)$ is outer planar. $\square$

Corollary 5.3.5. *The graph $\mathcal{P}(\mathcal{S}_M)$ is non planar if $n \geq 16$.*

Proof. Let $n \geq 16$. Then clearly $\mathcal{P}(\mathcal{S}_M)$ has a subgraph isomorphic to K_5 ($cf.$ Theorem 5.3.3). Hence the graph $P(S_M)$ is not planar. $\square$

Theorem 5.3.6. *Let $\mathcal{S}_M$ be a monogenic semigroup. Then $\chi(\mathcal{P}(\mathcal{S}_M)) = k+1$, where $2^k \leq n < 2^{k+1}$, $k \in \mathbb{N} \cup \{0\}$.*

Proof. We know that a graph G is perfect if and only if $\omega(H) = \chi(H)$ for every induced subgraph H of G. Hence by Theorem 5.3.2 and Theorem 5.3.3, we have $\omega(\mathcal{P}(\mathcal{S}_M)) = \chi(\mathcal{P}(\mathcal{S}_M)) = k+1$, where $2^k \leq n < 2^{k+1}$, $k \in \mathbb{N} \cup \{0\}$. $\square$

However, in the following, we develop an algorithm to label the vertices with minimum number of colors so that adjacent vertices of $\mathcal{P}(\mathcal{S}_M)$ will receive different colors, where $2^k \leq n < 2^{k+1}$, $k \in \mathbb{N} \cup \{0\}$.

An Algorithm : Coloring the vertices of $\mathcal{P}(\mathcal{S}_M)$

Since we know $\chi(\mathcal{P}(\mathcal{S}_M)) \geq \omega(\mathcal{P}(\mathcal{S}_M)) = k+1$, where $2^k \leq n < 2^{k+1}$, $k \in \mathbb{N} \cup \{0\}$ ($cf.$ Theorem 6.2.3), here we demonstrate a $k+1$ coloring of $\mathcal{P}(\mathcal{S}_M)$:

A_1: Since the vertex x is adjacent to every other vertices of $\mathcal{P}(\mathcal{S}_M)$, the color used to label the vertex x cannot be used for other vertex. So first we label the vertex x with color L_1.

A_2: Let us take the vertices of the form x^i, where $2 \leq i < 2^2$. Since no two of them are adjacent and both are adjacent to x, label the vetices with colour L_2.

A_3: Now we take the vertices of the form x^i, where $2^2 \leq i < 2^3$ and label the vertices with color L_3.

...

Continiuning in this way, the last step in this process we have

A_{k+1}: We consider the vertices of the form x^i, where $2^k \leq i < 2^{k+1}$ and label the vertices with color L_{k+1}.

Clearly this a proper coloring of $\mathcal{P}(\mathcal{S}_M)$ and hence $\chi(\mathcal{P}(\mathcal{S}_M)) = k+1$, where $2^k \leq n < 2^{k+1}$, $k \in \mathbb{N} \cup \{0\}$.

By using Lemma 1.3.4, we have the following result on chromatic index $\chi'(\mathcal{P}(\mathcal{S}_M))$ of $\mathcal{P}(\mathcal{S}_M)$.

Theorem 5.3.7. $\chi'(\mathcal{P}(\mathcal{S}_M)) = n - 1$.

Proof. If $n = 2$, then $\mathcal{P}(\mathcal{S}_M) \cong K_2$ and hence $\chi'(\mathcal{P}(\mathcal{S}_M)) = 1 = \Delta(\mathcal{P}(\mathcal{S}_M))$. Let $n \geq 3$. Then $\Delta(\mathcal{P}(\mathcal{S}_M)) = n - 1$ ($cf.$ Theorem 5.1.7) with the set of vertices of maximum degrees in $\mathcal{P}(\mathcal{S}_M)$ is $D = \{x\}$. Then for any vertex w in $\mathcal{P}(\mathcal{S}_M)$ such that $\{x, w\}$ is and edge in $\mathcal{P}(\mathcal{S}_M))$, we have $\Delta(\mathcal{P}(\mathcal{S}_M)) - \deg(w) + 2 = n - 1 - \deg(w) + 2 > 1 = |D|$, as $\deg(w) < n - 1$. Therefore by Lemma 1.3.4, we have $\chi'(\mathcal{P}(\mathcal{S}_M)) = \Delta(\mathcal{P}(\mathcal{S}_M)) = n - 1$. □

Example 5.3.8. Depending on the results obtained in this chapter and also from the graph drawn in Figure 5.1, we have $\omega(\mathcal{P}(\mathcal{S}_M^8)) = \chi(\mathcal{P}(\mathcal{S}_M^8)) = 4$ and $\chi'(\mathcal{P}(\mathcal{S}_M^8)) = 7$.

5.4 Cartesian product of $\mathcal{P}(\mathcal{S}_M^1)$ and $\mathcal{P}(\mathcal{S}_M^2)$

Let $\mathcal{S}_M^1 = \{0, x_1, x_1^2, x_1^3, \ldots, x_1^n\}$ and $\mathcal{S}_M^2 = \{0, x_2, x_2^2, x_2^3, \ldots, x_2^m\}$ be two monogenic semigroups with zero and $n \geq m \geq 3$. Here we compute some graph parameters of the cartesian product graph $\mathcal{P}(\mathcal{S}_M^1)\square\mathcal{P}(\mathcal{S}_M^2)$ and denote it by Σ. The results of this section is as follows.

Theorem 5.4.1. *diam* $(\Sigma) = 4$

Proof. It is clear that diameter of Σ can be obtained by considering the distance between $(x_1^{n_p}, x_2^{m_p})$ and one of the other vertices of Σ. The vertex $(x_1^{n_p}, x_2^{m_p})$ of Σ is adjacent to $(x_1^{n_p}, x_2)$ and $(x_1, x_2^{m_p})$ only. Also both these vertices are adjacent to the vertex (x_1, x_2). Again $(x_1, x_2) \sim (x_1, x_2^j)$, where $1 < j \leq m$ and $(x_1, x_2) \sim (x_1^i, x_2)$, where $1 < i \leq n$. Let us consider the distance between the vertices $(x_1^{n_p}, x_2^{m_p})$ and $(x_1^{t_1}, x_2^{t_2})$, where $1 < t_1 \neq n_p \leq n$ and $1 < t_2 \neq m_p \leq m$. These facts can be shown systematically as

$$(x_1^{n_p}, x_2^{m_p}) \sim (x_1^{n_p}, x_2) \sim (x_1, x_2) \sim (x_1^{t_1}, x_2) \sim (x_1^{t_1}, x_2^{t_2}) \qquad \text{and}$$
$$(x_1^{n_p}, x_2^{m_p}) \sim (x_1, x_2^{m_p}) \sim (x_1, x_2) \sim (x_1, x_2^{t_2}) \sim (x_1^{t_1}, x_2^{t_2}).$$

Therefore diam $(\Sigma) = 4$, as desired. □

Theorem 5.4.2. *rad* $(\Sigma) = 2.$

Proof. Let us consider the vertex (x_1, x_2) of Σ. The maximum distance from (x_1, x_2) to any other vertex is at most 2, which can be shown as $(x_1, x_2) \sim (x_1, x_2^{t_2}) \sim (x_1^{t_1}, x_2^{t_2})$ or $(x_1, x_2) \sim (x_1^{t_1}, x_2) \sim (x_1^{t_1}, x_2^{t_2})$, where $1 < t_1, t_2 \leq n$. Clearly it is of minimum ecentricity in Σ, as no other vertices in Σ is adjacent to every other vertex. Therefore, rad $(\Sigma) = 2$, as desired. □

Theorem 5.4.3.

$$gr(\Sigma) = \begin{cases} 4\ , & \textit{if } m = n = 3 \\ 3\ , & \textit{otherwise} \end{cases} \tag{5.4.1}$$

Proof. To prove the result we consider the following two cases:

Case (1) Let $m = n = 3$. Then it is clear that the graph Σ has no 3-cycle and $(x_1, x_2) \sim (x_1, x_2^2) \sim (x_1^2, x_2^2) \sim (x_1^2, x_2) \sim (x_1, x_2)$ is a 4-cycle. Hence $\mathrm{gr}(\Sigma) = 4$

Case (2) Let $n \geq 4$. Then $(x_1^4, x_2) \sim (x_1, x_2) \sim (x_1^2, x_2) \sim (x_1^4, x_2)$ is a 3-cycle in Σ. Consequently, gr $(\Sigma) = 3$, as desired. □

Theorem 5.4.4. *Let (x_1^i, x_2^j) be a vertex of Σ, where $1 \leq i \leq n$ and $1 \leq j \leq m$. Then the degree of the vertex (x_1^i, x_2^j) is*

$$d_{i,j}(x_1^i, x_2^j) = \sigma(i) + \sigma(j) + \lfloor \frac{n}{i} \rfloor + \lfloor \frac{m}{j} \rfloor - 4.$$

Moreover, maximum and minimum degrees of Σ are $\Delta(\Sigma) = m + n - 2$ and $\delta(\Sigma) = 2$ respectively.

Proof. Clearly the vertex (x_1^i, x_2^j) is adjacent to (x_1^t, x_2^j), where $x_1^i \sim x_2^t$, $1 \leq t \leq n$ and (x_1^i, x_2^k), where $x_2^j \sim x_2^k$, $1 \leq k \leq m$. Therefore

$$\begin{aligned} d_{i,j}(x_1^i, x_2^j) &= \sigma(i) + \lfloor \frac{n}{i} \rfloor - 2 + \sigma(j) + \lfloor \frac{m}{j} \rfloor - 2 \\ &= \sigma(i) + \sigma(j) + \lfloor \frac{n}{i} \rfloor + \lfloor \frac{m}{j} \rfloor - 4. \end{aligned}$$

Let us consider the vertex (x_1, x_2) in Σ. Then from the result presented above, we have $d_{1,1}(x_1, x_2) = m + n - 2$, which is of maximum degree. Hence $\Delta(\Sigma) = m + n - 2$.

Again consider the vertex $(x_1^{n_p}, x_2^{m_p})$ of Σ, is adjacent to $(x_1^{n_p}, x_2)$ and $(x_1, x_2^{m_p})$ only. Hence $\delta(\Sigma) = 2$. □

Theorem 5.4.5. $\gamma(\Sigma) = m$.

Proof. Clearly $D = \{(x_1, x_2^{t_i}) : 1 \leq t_i \leq m\}$ is a dominating set Σ. Then no proper subset of D is a dominating set for Σ, if possible, let $D' = D - \{(x_1, x_2^k)\}$ is a dominating set for Σ, for some $k \in \{1, 2, \ldots, m\}$. Then the vertex (x_1^j, x_2^k) in $V(\Sigma) - D'$ where $j \neq 1$, is not adjacent to any element of D', a contradiction.

Now we prove that there does not exists any dominating set for Σ of cardinality less than m. If possible, let L be one such. Hence there exists $x_1^{t_1} \in V(\mathcal{P}(\mathcal{S}_M^1))$ and $x_2^{t_2} \in V(\mathcal{P}(\mathcal{S}_M^2))$ such that $(x_1^{t_1}, x_2^j) \notin L$ for all $j \in \{1, 2, \ldots, m\}$ and $(x_1^i, x_2^{t_2}) \notin L$ for all $i \in \{1, 2, \ldots, n\}$. Then the vertex $(x_1^{t_1}, x_2^{t_2})$ of $V(\Sigma) - L$ is not adjacent to any vertex of L, contradicting that L is a dominating set for Σ. Therefore, $\gamma(\Sigma) = m$. $\square$

Theorem 5.4.6. *$\omega(\Sigma) = k + 1$, where $2^k \leq n < 2^k$, $k \in \mathbb{N}$.*

Proof. Clearly $\{x_1, x_1^2, x_1^{2^2}, \ldots, x_1^{2^k}\}$ is a clique of size $k+1$ in $P(S_M^1)$. Then $C = \{(x_1, x_2^j), (x_1^2, x_2^j), \ldots, (x_1^{2^k}, x_2^j)\}$ is a clique of size $k+1$ in Σ, for some $j \in \{1, 2, \ldots, m\}$. If possible, let $C \cup \{J\}$ be a clique in Σ. Then every element of J is of the form $(x_1^{2^q}, x_2^j)$ and hence $\{x_1, x_1^2, \ldots, x_1^{2^k}, x_1^{2^q}\}$ is a clique of size $k+2$ in $\mathcal{P}(\mathcal{S}_M^1)$, a contradiction. Therefore there is no clique in Σ containing C. Now we prove that there does not exists a clique of size greater than $k+1$ in Σ. If possible, let $\{(x_1^{t_1}, x_2^l), (x_1^{t_2}, x_2^l), \ldots, (x_1^{t_p}, x_2^l)\}$ be a clique of size $p(> k+1)$ in Σ. Then $\{x_1^{t_1}, x_1^{t_2}, \ldots, x_1^{t_p}\}$ is a clique of size $p(> k+1)$ in $\mathcal{P}(\mathcal{S}_M^1)$, a contradiction. Consequently, $\omega(\Sigma) = k + 1$. $\square$

Remark 5.4.7. We know $\omega(G_1 \Box G_2) \geq max\{\omega(G_1), \omega(G_2)\}$, as presented in Remark 1.3.22. But here we found the strict equality $\omega(\Sigma) = max\{\omega(\mathcal{P}(\mathcal{S}_M^1)), \omega(\mathcal{P}(\mathcal{S}_M^2))\}$ (*cf.* Theorem 5.4.6).

Theorem 5.4.8. *$\chi(\Sigma) = \omega(\Sigma) = k + 1$, where $2^k \leq n < 2^k$, $k \in \mathbb{N}$.*

Proof. We know that for simple graphs G_1 and G_2, $\chi(G_1 \Box G_2) = \max\{\chi(G_1), \chi(G_2)\}$ (*cf.* Theorem 1.3.23). Since $n \geq m$ and $2^k \leq n < 2^{k+1}$, we have $\chi(\Sigma) = \max\{\chi(P(S_M^1)), \chi(P(S_M^2))\}$ $= k + 1 = \omega(\Sigma)$, as desired. $\square$

Example 5.4.9. For the semigroups $\mathcal{S}_M^4 = \{0, x, x^2, x^3, x^4\}$ and $\mathcal{S}_M^5 = \{0, y, y^2, y^3\}$, let us consider the graph $\Sigma = \mathcal{P}(\mathcal{S}_M^4) \Box \mathcal{P}(\mathcal{S}_M^3)$ as drawn in Figure 5.2. Depending on the result presented in this paper, we have the following results:

(a) diam $(\Sigma) = 4$ $(cf$, Theorem 5.4.1),

(i) rad $(\Sigma) = 2$ $(cf.$ Theorem 5.4.2),

(ii) girth $(\Sigma) = 3$ $(cf.$ Theorem 5.4.3),

(iii) $\Delta(\Sigma) = 5$ and $\delta(\Sigma) = 2$ $(cf.$ Theorem 5.4.4),

(iv) $\gamma(\Sigma) = 3$ $(cf.$ Theorem 5.4.5).

(v) $\omega(\Sigma) = \chi(\Sigma) = 3$ $(cf.$ Theorem 5.4.8).

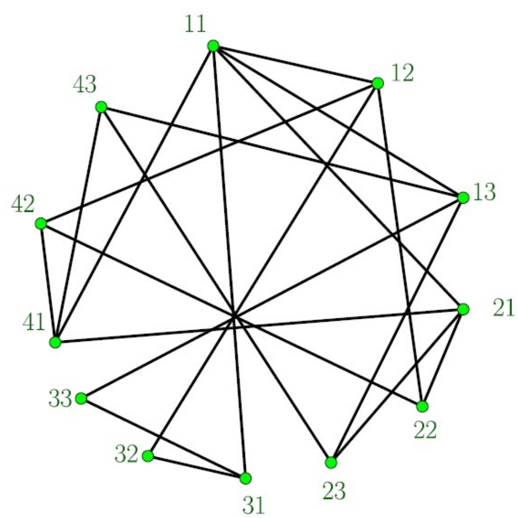


Figure 5.2: $\mathcal{P}(\mathcal{S}_M^4)\Box\mathcal{P}(\mathcal{S}_M^3)$: Here, for $1 \leq i \leq 4, 1 \leq j \leq 3$, each label ij corresponds to the vertices (x^i, x^j).

Chapter 6

Inclusion ideal graph of a semigroup

In this chapter, we consider the inclusion ideal graph $In(S)$ of nontrivial right ideals of a semigroup S with zero element. We characterize a semigroup S for which the graph $In(S)$ is complete, connected and also find various graph parameters of $In(S)$. We determine the values of n for which the graph $In(\mathbb{Z}_n)$ of multiplicative semigroup $\mathbb{Z}_n$ of integers modulo n is complete, triangulated, split, unicyclic, thresold and also study minimal embedding of $In(\mathbb{Z}_n)$ into compact orientable (resp. non-orientable) surface. We obtain both upper and lower bouds for metric and partition dimension of inclusion ideal graph of a completely 0-simple semigroup. Finally, we compute some graph parameters of the cartesian product of inclusion ideal graph of two monoids. This chapter is arranged as follows.

In **Section** 6.1, we first define inclusion ideal graph $In(S)$ of a semigroup S ($cf.$ Definition 6.1.1) and prove that $In(S)$ and it's complement are perfect graphs ($cf.$ Corollary 6.1.3). We characterize a semigroup S for which $In(S)$ is a connected ($cf.$ Theorem 6.1.5) and complete ($cf.$ Theorem 6.1.8). Also we determine the values of n for which $In(Z_n)$ is complete ($cf.$ Corollary 6.1.9), bipartiate ($cf.$ Corollary 6.1.17), triangulated ($cf.$ Theorem 6.1.20), split ($cf.$ Theorem 6.1.21), cograph, thresold ($cf.$ Theorem 6.1.22).

In **Section** 6.2, we determine clique number and chromatic number ($cf.$ Theorem 6.2.3) of $In(S)$, where S is a semigroup with unity having finitely many right ideals. Also we determine the clique number ($cf.$ Corollary 6.2.4) and chromatic index ($cf.$ Proposition 6.2.9) of $In(\mathbb{Z}_n)$.

In **Section** 6.3, we determine the values of n for which $In(\mathbb{Z}_n)$ is outer-planar ($cf.$ Theorem

6.3.1), planar ($cf.$ Theorem 6.3.2), toroidal and bitoroidal ($cf.$ Theorem 6.3.3). Also we compute the values of n, for which $In(Z_n)$ has thickness (resp. outerthickness) 1 and 2 ($cf.$ Theorem 6.3.5).

In **Section** 6.4, we compute the number of edges ($cf.$ Proposition 6.4.1) of the inclusion ideal graph of completely 0-simple semigroup and also study minimal embedding into orientable ($cf.$ Theorem 6.4.3) and non-orientable ($cf.$ Theorem 6.4.4) surfaces. We obtain both upper and lower bounds for partition dimension ($cf.$ Theorem 6.4.7) and metric dimension ($cf.$ Theorem 6.4.8) of the graph.

Finally in **Section** 6.5, we determine domination number ($cf.$ Theorem 6.5.1), diameter ($cf.$ Theorem 6.5.2), girth ($cf.$ Theorem 6.5.4), clique number ($cf.$ Theorem 6.5.6) and chromatic number ($cf.$ Theorem 6.5.7) of the cartesian product of inclusion ideal graphs of two monoids.

Throughout this chapter by a semigroup S, we mean a semigroup with zero which is not right 0-simple.

6.1 The inclusion ideal graph

Definition 6.1.1. Let S be a semigroup. We define the directed inclusion ideal graph of S, denoted by $\overrightarrow{In}(S)$, is a digraph with nontrivial right ideals as vertex set and for two distinct vertices I_1 and I_2, there is an arc from I_1 to I_2 if and only if $I_2 \subset I_1$. The underlying graph of $\overrightarrow{In}(S)$, denoted as $In(S)$, is said to be the inclusion ideal graph of S.

Theorem 6.1.2. *The graph $In(S)$ is a comparability graph.*

Proof. From the defination of $In(S)$ and $\overrightarrow{In}(S)$ it is easy to obseve that $\overrightarrow{In}(S)$ is an orientation of $In(S)$. Let (I_1, I_2), $(I_2, I_3) \in E(\overrightarrow{In}(S))$. Then $I_3 \subset I_2 \subset I_1$, which implies $(I_1, I_3) \in E(\overrightarrow{In}(S))$. Hence $\overrightarrow{In}(S)$ is a transitive orientation of $In(S)$. To prove that $\overrightarrow{In}(S)$ is acyclic, let $\{I_1, I_2, \ldots, I_k\}$ be a cycle in $\overrightarrow{In}(S)$. Then $I_1 \subset I_k \subset \cdots \subset I_2 \subset I_1$, which is a contradiction. Hence $In(S)$ is a comparability graph. □

As we know that every comparability graph is perfect ($cf.$ Theorem 1.3.2) and complement of a perfect graph is perfect ($cf.$ Theorem 1.3.3), we have

Corollary 6.1.3. *The graph $In(S)$ and it's complement $\overline{In(S)}$ are perfect graph.*

Proposition 6.1.4. *The following results about $In(S)$ are true.*

(1) *Let I be a nontrivial ideal of a commutative semigroup S. Then $In(I)$ is a subgraph of $In(S)$ if one of the following condition holds*

(a) S is semisimple. (b) $J = J^2$ for every ideal J of I.

(2) *The graph $In(S)$ and it's complement has no induced cycle of odd length at least 5.*

(3) *If $In(S)$ has a cycle of length 4 or 5, then $In(S)$ has a cycle of length 3.*

(4) *If S is a union of finite number of right ideals, then $\gamma(In(S)) = 2$. Moreover if S is a monoid, then $\gamma(In(S)) = 1$.*

Proof. (1) (a) Since S is semisimple, every ideal of I is an ideal of S ($cf.$ Theorem 1.2.52). Hence the result follows ($cf.$ Definition 2.2.1).

(b) Clearly every ideal of I is an ideal of S, since $J = J^2$ for every ideal J of S($cf.$ Lemma 1.2.53). Hence the result follows.

(2) It is clear from Corollary 6.1.3 and Theorem 1.3.1.

(3) Let the graph $In(S)$ has a cycle $C: I_1 \sim I_2 \sim I_3 \sim I_4 \sim I_5 \sim I_1$ of length 5. It is easy to check that there exists a chain $I_a \subset I_b \subset I_c$ in S, where a, b, c are integers between 1 and 5. Again, let $In(S)$ has a cycle C: $I_1 \sim I_2 \sim I_3 \sim I_4 \sim I_1$ of length 4. Assume I_1, I_3 and I_2, I_4 are not adjacent, otherwise it is clear. Hence either $I_1 \cup I_3 \subset I_2$, I_4 or I_2, $I_4 \subset I_1 \cap I_3$. Now if $I_1 \cap I_3$ or $I_1 \cup I_3 \in \{I_1, I_2, I_3, I_4\}$, then it is done. Otherwise, either $I_2 \sim I_1 \sim I_1 \cup I_3 \sim I_2$ or $I_2 \sim I_1 \sim I_1 \cap I_3 \sim I_2$ is a cycle of length 3 of $In(S)$, as desired.

(4) Suppose S is union of k nontrivial right ideals say $I_1, I_2, \ldots, I_k$. Now we show that the set of the form $D = \{I_k, I_1 \cup I_2 \cup \cdots \cup I_{k-1}\}$ is a dominating set for $In(S)$. Let $P = I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_t} \in V(In(S)) \setminus D$, where $x_i \in [k]$ and $t \in [k-1]$. If for some $x_r = k$, we have $I_k \sim P$, otherwise $I_1 \cup I_2 \cup \cdots \cup I_{k_1} \sim P$ for some integer lies between 1 and $k-2$. Hence D is a dominating set of $In(S)$. Since there is no dominating vertex, clearly $\gamma(In(S)) = 2$.

Now if S is a monoid, then it has unique maximal right ideal say, M. Consider $D = \{M\}$. Then every vertex of $In(S)$ which is not in D, is adjacent to M. Consequently, $\gamma(In(S)) = 1$. $\square$

Theorem 6.1.5. *Let S be a semigroup. Then the inclusion ideal graph $In(S)$ is disconnected if and only if S has at least two nontrivial right ideals and each nontrivial right ideal is* 0-*minimal (as well as maximal).*

Proof. Let the graph $In(S)$ is disconnected and $C_1, C_2, ..., C_k$ are components of G, where $k \geq 2$. Let $I \in C_1$ and $J \in C_2$. Clearly $I \cup J = S$ and $I \cap J = \{0\}$, otherwise there would be a path from I to J which is a contradiction. We now prove that both I and J are 0-minimal. On the contrary, assume $0 \neq A \subset I$. Then A and I are adjacent. Now $I \sim A \sim A \cup J \sim J$ is a path, a contradiction. Hence I is a 0-minimal. Again, let I is not maximal that is $I \subset M \neq S$, for some nontrivial right ideal M of S. Then $I \sim M \sim J \cap M \sim J$ is a path, a contradiction. Hence I is maximal also, as desired. Similarly we can show that J is 0-minimal as well as maximal. The proof of the converse part is immediate. □

Theorem 6.1.6. *The inclusion ideal graph of a semigroup S with identity is connected. Moreover, diam$(In(S)) \leq 2$.*

Proof. Since S has identity, it has unique maximal right ideal, say M, which is union of all proper right ideal of S. Let I and J be two nontrivial right ideals other than M. Then $I \sim M \sim J$ is a path. Consequently, $In(S)$ is a connected graph and clearly diam$(In(S)) \leq 2$. □

Since the semigroup $S = (Z_n, .)$ (where $n \in \mathbb{N}$) has the identity, the following is an immediate consequence of above theorem.

Example 6.1.7. The inclusion ideal graph $In(Z_n)$ of the semigroup $(Z_n, .)$ is a connected graph, where $n \in \mathbb{N}$ is not a prime number.

Theorem 6.1.8. *Let S be a semigroup with identity. Then the following statements are equivalent:*

(1) *$In(S)$ is a complete graph.*

(2) *S is a principal right ideal semigroup.*

Proof. (1) $\Rightarrow$ (2) Let I be a right ideal of S. Then $\bigcup_{i \in I} (i)_r = I$. Since $In(S)$ is a complete graph, $I = (j)_r$ for some $j \in I$. Consequently, S is a principal right ideal semigroup.

(2) $\Rightarrow$ (1) Since S is principal right ideal semigroup, right ideals of S forms a chain under set inclusion ($cf.$ Theorem 1.2.54). Hence the graph $In(S)$ is complete. $\square$

The following is an immediate consequence of Theorem 6.1.6 and Theorem 6.1.8.

Corollary 6.1.9. *Let $n \in M$. Then*

$$diam(In(Z_n)) = \begin{cases} 1\ , & \text{if } n = p^k, p \text{ is prime and } k \text{ is an integer} \geq 2 \\ 2\ , & \text{otherwise.} \end{cases} \tag{6.1.1}$$

If S is a semigroup with unity and having no proper essential right congrurence, then the set of right ideals of S is linearly ordered by inclusion ($cf.$ Theorem 1.2.44). So we have the following result.

Corollary 6.1.10. *Let S be a semigroup with unity and having no proper essential right congrurence. Then $In(S)$ is a complete graph.*

We know a commutative semigroup is said to be a chained semigroup if it's ideals are linearly ordered by set inclusion ($cf.$ Definition 1.2.55). So we have following immediate result.

Theorem 6.1.11. *Let S be a commutative semigroup. Then $In(S)$ is a complete graph if and only if S is a chained semigroup.*

Combining Corollary 6.1.3, Theorem 6.1.8 and Theorem 6.1.11, we have

Corollary 6.1.12. *The core of any induced subgraph of $In(S)$ is complete. Therefore $In(S)$ is a core if and only if one of the following condition holds.*
(1) *S is a principal right ideal monoid.*
(2) *S is a commutative chained semigroup.*

Lemma 6.1.13. *Let S be a commutative semigroup. Then S is a semisimple semigroup if and only if S is a regular semigroup.*

Proof. We know a semigroup S is semisimple if and only if $A = A^2$ for every ideal A of S ($cf.$ Lemma 1.2.1). Also Iseki characterize that a commutative semigroup S is regular if and only if every ideal of S is idempotent ($cf.$ Theorem 1.2.57). Hence the result follows. $\square$

Theorem 6.1.14. *Let S be a commutative semisimple semigroup. Then the followings are equivalent:*

(1) *$In(S)$ is a complete graph,*

(2) *Every ideal of S is prime.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal of S and $xy \in I$ for some x, $y \in S$. Since $In(S)$ is a complete graph, either $(x) \subseteq (y)$ or $(y) \subseteq (x)$. Let $(x) \subseteq (y)$. Since S is semisimple, $(x) = (x)^2$ ($cf.$ Lemma 1.2.1). Hence $x \in (x) = (x)^2 \subseteq (x)(y) \subseteq I$. Similarly, if $(y) \subseteq (x)$, then $y \in I$. Consequently, I is a prime ideal of S, as desired.

(2) $\Rightarrow$ (1) If every ideal of S is prime, then ideals of S are linearly ordered by inclusion ($cf.$ Theorem 1.2.50) and hence $In(S)$ is a complete graph. □

Theorem 6.1.15. *Let $n \in M$. Then*

$$gr(In(Z_n)) = \begin{cases} \infty\ , & \text{if } n = pq, p^2, p^3, \text{where } p,\ q \text{ are primes} \\ 3\ , & \text{otherwise} \end{cases} \tag{6.1.2}$$

Proof. Case (1) Let $n = pq$. Then only nontrivial ideals of Z_n are (p), (q) and $(p) \cup (q)$. Since (p) and (q) are not adjacent, so has no cycle. Also, if $n = p^2$ or p^3, then $In(Z_n)$ has less than three vertex and so has no cycle. Hence $gr(In(Z_n)) = \infty$.

Case (2) Let $n = pqr$ or p^2q, where p, q, r are distinct primes. Then $In(Z_n)$ has the 3-cycle $(p) \sim (p) \cup (q) \sim (pq) \sim (p)$. Again if $n = p^4$, then $In(Z_n)$ has the 3-cycle $(p) \sim (p^2) \sim (p^3) \sim (p)$. Hence $gr(In(Z_n)) = 3$.

Let $m = pqr, p^2q, p^4$. Then for any $n \in M$ other than $n = pq, p^2, p^3$ is of the form $n = mt$, where $t \in \mathbb{N}$. Thus the graph $In(Z_n)$ contains a subgraph isomorphic to $In(Z_m)$, as desired. □

Corollary 6.1.16. *Let $n \in M$. Then $In(Z_n)$ has a cycle if and only if $n = mt$ for $m = p^4, p^2q, pqr$, where p, q, r are distinct prime numbers, $t \in \mathbb{N}$.*

By applying Theorem 6.1.15 we have the following immediate result.

Corollary 6.1.17. *Let $n \in M$. Then $In(Z_n)$ is bipartiate if and only if $n = pq, p^3$, where p, q are distinct prime numbers.*

Theorem 6.1.18. *Let S be a semigroup with unity. Then $gr(In(S)) = \{3, \infty\}$ and $gr(\overrightarrow{In}(S))$ $= \{\infty\}$*

Proof. If there exists three nontrivial right ideals I_1, I_2 and I_3 such that $I_1 \subset I_2 \subset I_3$, then $gr(In(S)) = 3$. Thus otherwise every nontrivial right ideal is 0-minimal or maximal. Since S has identity, it has unique maximal right ideal. Hence every other nontrivial right ideals are 0-minimal. Consequently, $In(S)$ is a star graph and so has no cycle. Therefore $gr(In(S)) = \infty$.
If possible let there exists a directed cycle $J_1 \to J_2 \to \cdots \to J_k \to J_1$ of minimal length $k \geq 3$ in $\overrightarrow{In}(S)$. Then we have $J_1 \subset J_k \subset \cdots \subset J_2 \subset J_1$ which is impossible. Therefore the digraph $\overrightarrow{In}(S)$ does not contain any directed cycle and hence the result follows. $\square$

Corollary 6.1.19. *Let S be a semigroup with unity. Then $In(S)$ is either a star graph or has a cycle of length* 3.

Theorem 6.1.20. *Let $n \in M$. Then $In(Z_n)$ is triangulated if and only if $n \neq p^k(k = 2, 3), pq$, where p, q are prime numbers.*

Proof. For $n = p^2$, p^3, $In(Z_n)$ contains less than three vertex, hence $In(Z_n)$ is not triangulated. Again, if $n = pq$, then $In(Z_n)$ has three vertex (p), (q) and $(p) \cup (q)$ but (p) and (q) are not adjacent. Consequently, $In(Z_n)$ is not triangulated.
For $n = p^k(k \geq 4)$, it is clear that $In(Z_n)$ is triangulated ($cf.$ Theorem 6.1.9). Let $n = p_1^{t_1} p_2^{t_2} \dots p_k^{t_k} (\neq p_a p_b, 1 \leq a \neq b \leq k)$, where $k \geq 2$ and p_i's are distinct primes. Then nontrivial ideals of Z_n are of the form $\cup\{m_i Z_n : i = 1, 2, \dots, k\}$, where $m_1, m_2, \dots, m_k$ are divisors of n such that m_i does not divide m_j for $i \neq j$. Since the semigroup $(Z_n, .)$ has identity, it has unique maximal ideal say $M = \cup(p_i)$, where p_i's are distinct prime divisor of n. Then any vertex x of $In(Z_n)$ is either a principal ideal say (u) or union of some nontrivial principal ideal. Now if $x = (u)$, then the vertex $x = (u)$, M, $(x) \cup (y) (\neq M)$ forms a triangle, for some principal ideal (y) of Z_n and if x is union of principal ideals say $(u) \cup (v)$, then x, M, (u) forms a triangle. Hence the result follows. $\square$

Theorem 6.1.21. *Let $n \in M$. Then the following statements are equivalent:*
(1) *The graph $In(Z_n)$ is a split graph,*

(2) $n = p^k (k \geq 2)$, pq, *where* p, q *are distinct prime numbers,*

(3) $In(Z_n)$ *is a cograph,*

(4) $In(Z_n)$ *is a chordal graph.*

Proof. (1) $\Leftrightarrow$ (2) We complete the proof by considering the following cases.

Case (1) Clearly the graph $In(Z_n)$ does not contain any induced subgraph isomorphic to C_5 ($cf.$ Proposition 6.1.4).

Case (2) Here we compute the values of n for which $In(Z_n)$ has no induced subgraph isomorphic to C_4.

Since for $n = p^r (r \geq 2)$, $In(Z_n)$ is a complete graph($cf.$ Theorem 6.1.9), clearly $In(Z_{p^r})$ has no induced subgraph isomorphic to C_4. If $n = pq$, then $In(Z_{pq}) \cong P_3$, such is induced C_4 free. Let $n = p^2q$, then the vertices $\{(p), (p^2), (p^2) \cup (q), (pq)\}$ form an induced C_4 in $In(Z_{p^2q})$. If $n = pqr$, then the set of verices $\{(p), (pq), (q) \cup (r), (pr)\}$ form an induced C_4 in $In(Z_{pqr})$. Now for any $n \in M \setminus \{p^r, pq\}$ is of the form $n = mt$, where $m = p^2q$ or pqr and $t \in \mathbb{N}$. Therefore $In(Z_n)$ must have a subgraph isomorphic to $In(Z_m)$ and so there exists an induced subgraph isomorphic to C_4.

Case (3) Here we find the values of n for which $In(Z_n)$ has no induced subgraph isomorphic to $2K_2$. It is easy to observe that $n = p^r (r \geq 2)$, pq, the graph $In(Z_n)$ has no induced subgraph isomorphic to $2K_2$. Now if $n = p^2q$, then it is easy to observe that $In(Z_{p^2q})$ has no induced $2K_2$. For $n = pqr$, the set of vertices $\{(p), (pr), (q), (qr)\}$ forms an induced $2K_2$ in $In(Z_{pqr})$. Again for $n = p^3q$ and $n = p^2q^2$, the set of vertices $\{(p^2), (p^3), (pq), (q)\}$ and $\{(p), (p^2), (q), (q^2)\}$ forms an induced $2K_2$ in $In(Z_{p^3q})$ and $In(Z_{p^2q^2})$, respectively. Hence by applying the same procedure as in case (2), we have that $In(Z_n)$ is induced $2K_2$ free if and only if $n = p^r$, pq, p^2q, where p, q are prime numbers.

Combining all the cases, we have the desired result.

(3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) By applying the similar approch as in above cases, we have the required result. $\square$

By applying same procedure as in the proof of Theorem 6.1.21, we have the following result.

Theorem 6.1.22. (1) *The graph* $In(Z_n)$ *is unicyclic if and only if* $n = p^4$, *where* p *is a* *prime number.*

(2) *The graph $In(Z_n)$ is thresold if and only if $n = p^k (2 \leq k \leq 4)$, pq, where p, q are prime numbers.*

6.2 Clique number and chromatic number

By applying corollary 6.1.3 we have the following immediate result.

Theorem 6.2.1. *Let S be a semigroup with finite number of right ideals. Then $\chi(In(S)) = \omega(In(S))$.*

Theorem 6.2.2. *Let S be a principal right ideal semigroup with finite number of right ideals. Then clique number of S is the number of distinct nontrivial right ideals of S.*

Proof. Since S is a principal right ideal semigroup, $In(S)$ is a complete graph ($cf.$ Theorem 6.1.8). Since S has finite number of right ideals, we have $\chi(In(S)) = \omega(In(S)) = k$ ($cf.$ Theorem 6.2.1), where k is the number of distinct nontrivial right ideals of S. □

Theorem 6.2.3. *Let S be a semigroup with unity and has finite number of principal right ideals. Then clique number of $In(S)$ is the number of distinct nontrivial principal right ideals of S.*

Proof. Let $(a_1)_r$, $(a_2)_r, ..., (a_k)_r$ be the only k distinct nontrivial principal right ideals of a semigroup S. Then these ideals can rearranged as $(a_{t_1})_r$, $(a_{t_2})_r, \ldots, (a_{t_k})_r$ such that $(a_{t_i})_r \nsubseteq (a_{t_j})_r$, where $i > j$. We define $I_i = (a_{t_1})_r \cup (a_{t_2})_r \cup \cdots \cup (a_{t_i})_r$, where $i = 1, 2, \ldots, k$. Then $K = \{I_1, I_2, \ldots, I_k\}$ is a clique. If possible, let $K \cup \{I\}$ be a clique, where I is a nontrivial right ideal of S, different from I_i. Then there exists integer j such that $I_j \subset I \subset I_k$. Then $\bigcup_{x \in I}(x)_r = I = I_i$, for some i satisfying $1 \leq i \leq k$, a contradiction. Therefore K is a maximal clique. Now we prove that there does not exists any clique of size $k+1$ or more. If possibe, let there exists a clique $I_1 \subset I_2 \subset, \ldots, \subset I_{k+1}$ of size $k+1$. Let $j_t \in I_{t+1} - I_t$, where $0 \leq t \leq k$ and $I_0 = \{0\}$. Then $0 \neq (j_0)_r \neq (j_1)_r \neq ... \neq (j_k)_r$ be the list of $k+1$ distinct nontrivial principal right ideals of S, a contradiction. Hence $\omega(In(S)) = k$. □

Corollary 6.2.4. *Let $S = (Z_n, .)$, where $n \in M$. Then $\chi(In(S)) = \omega(In(S)) = \sigma(n) - 2$, where $\sigma(n)$ is the number of positive divisor of n.*

Proof. Since $S = (Z_n, .)$ has unity and finite number of ideals, by Theorem 6.2.1 and 6.2.3, $\chi(In(Z_n)) = \omega(In(Z_n) = k$, where k is the number of nontrivial principal ideals of $(Z_n, .)$. Clearly, principal ideals of the semigroup $(Z_n, .)$ is of the form $(\overline{m})$, where m is a divisor of n (*cf*. Theorem 1.2.61). Therefore $k = \sigma(n) - 2$, as desired. □

Since for any $n \in \mathbb{N}$, $\omega(In(Z_{p^{n+1}})) = \chi(In(Z_{p^{n+1}})) = n$, we have the following result

Corollary 6.2.5. *For any natural number n, there is a semigroup S such that $\omega(In(S)) = \chi(In(S)) = n$.*

Theorem 6.2.6. *Let S_1 and S_2 be two semigroups. Then if $S_1 \cong S_2$, then $In(S_1) \cong In(S_2)$.*

Proof. We define the map $f : V(In(S_1)) \longrightarrow V(In(S_2))$ by $f(I) = g(I)$, where $I \in V(In(S_1))$ and g is an isomorphism from S_1 to S_2. It is clear to check that f is an isomorphism. □

The following example shows that converse of above theorem is not true.

Example 6.2.7. Let $S_1 = (Z_{pq}, .)$ and $S_2 = (Z_{rs}, .)$, where p, q, r, s are distinct prime numbers. Then it is easy to see that $In(S_1) \cong P_3 \cong In(S_2)$ but $S_1 \ncong S_2$.

Example 6.2.8. Let $S_1 = (Z_{p^4}, .)$ and $S_2 = (Z_{q^4}, .)$, where p, q are distinct primes. Then $In(S_1) \cong K_3 \cong In(S_2)$ but $S_1 \ncong S_2$.

We know that chromatic index of a graph G satisfy $\chi'(G) \geq \Delta(G)$. The result stated in Lemma 1.3.4 is helpful to find chromatic index of $In(Z_n)$.

Proposition 6.2.9. *Let $n \in M$. Then*

$$\chi'(In(Z_n)) = \begin{cases} \Delta + 1 \ , & \text{if } n = p^{2r}, r \in \mathbb{N} \\ \Delta \ , & \text{otherwise.} \end{cases}$$

Proof. Case (1) Let $n = p^r$, where p is prime and r is a positive integer ≥ 2. Then $In(Z_n) \cong K_{r-1}$ (cf. Theorem 6.1.9 and Theorem 1.2.61). Now if r is even, then $\chi'(In(Z_n)) = r - 1 = \Delta + 1$ and if r is odd then $\chi'(In(Z_n)) = r - 2 = \Delta$ (cf. Lemma 1.3.5).

Case (2) Let p_1 and p_2 be two distinct prime divisors of n. Here $D = \{u \in V(In(Z_n)) : d(u) = \Delta(G)\} = \{M\}$, where M is the unique maximal ideal of Z_n. Let $In(Z_n)$ has r distinct nontrivial right ideals. Then $deg(M) = r - 1$ and $p_1 \sim M$ but $p_1 \nsim p_2$. Hence $deg((p_1)) \leq r - 2$. Therefore $\Delta(In(Z_n)) - deg((p_1)) + 2 > |D| = 1$. Hence by Lemma 1.3.4, we have $\chi'(In(Z_n)) = \Delta(In(Z_n))$.

Combining (1) and (2), we have the desired result. □

6.3 Embedding and decomposition of $In(Z_n)$

Theorem 6.3.1. *Let $n \in M$. Then the following statements are equivalent:*

(1) $\theta_0(In(Z_n)) = 1$,

(2) $In(Z_n)$ *is outer planar,*

(3) $n = p^k (2 \leq k \leq 4)$, pq, *where p, q are prime numbers.*

Proof. (1) $\Leftrightarrow$ (2) It is clear.

(2) $\Leftrightarrow$ (3) If $n = p^k (2 \leq k \leq 4)$, pq, then $In(Z_n)$ contains less than four vertex. Hence clearly $In(Z_n)$ is outer planar (cf. Lemma 1.3.10).

Conversely, for any $n \in M$ other than $n = p^k (2 \leq k \leq 4)$, pq is of the form $n = mt$, where $m = p^5$ or p^2q or pqr and $t \in \mathbb{N}$. Then the graph $In(Z_n)$ must contain a subgraph isomporphic to $In(Z_m)$. Now it is enough to prove that graphs of the form $In(Z_m)$ is not outerplanar.

Case(a) Let $m = p^5$, p^2q, where p, q are distinct prime numbers. Then the subgraph formed by the set of vertices $\{(p), (p^2), (p^3), (p^4)\}$ in $In(Z_{p^5})$(resp.$\{(q), (q) \cup (p^2), (p) \cup (q), (pq)\}$ in $In(Z_{p^2q})$) is isomorphic to K_4.

Case (b) Let $m = pqr$, where p, q, r are distinct prime numbers. Then the subgraph formed by the set of vertices $\{(pq), (pr), (p) \cup (r), (q) \cup (r), (p) \cup (q)\}$ in $In(Z_{pqr})$ is isomorphic to $K_{2,3}$.

Hence the graphs $In(Z_m)$ for $m = p^5$, p^2q, pqr is not outerplanar (cf. Lemma 1.3.10), as desired. $\square$

Theorem 6.3.2. *Let $n \in M$. Then the following statements are equivalent:*

(1) $\gamma(In(Z_n)) = 0$,

(2) $\theta(In(Z_n)) = 1$,

(3) *The graph $In(Z_n)$ is planar,*

(4) $n = p^k (2 \leq k \leq 5)$, pq, *where p, q are prime numbers.*

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2) It is clear.

(3) $\Leftrightarrow$ (4) If $n = p^k(2 \leq k \leq 5)$, pq, then $In(Z_n)$ contains at most four vertex, hence it is planar ($cf.$ Lemma 1.3.11).

Now for any $n \in M$ other than $p^k(2 \leq k \leq 5)$, pq, is of the form $n = mt$ where $m = p^6$,

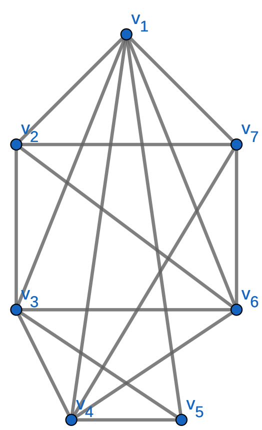


Figure 6.1: $In(Z_{p^2q})$

p^2q, pqr (where p, q, r are distinct primes), $t \in \mathbb{N}$. Then $In(Z_n)$ must contains a subgraph isomorphic to $In(Z_m)$. Now it is enough to prove that the graphs of the form $In(Z_m)$ is not planar.

Case (i) Let $m = p^6$. Then the subdivison formed by the set of vertices $\{v_i = (p^i)\}$,

$i \in \{1, 2, \ldots, 5\}$ in $In(Z_{p^6})$ is K_5.

Case (ii) Let $m = p^2q$, where p, q are distinct prime numbers. Now from Figure 6.1, it is clear that $In(Z_{p^2q})$ with vertices $v_1 = (p) \cup (q)$, $v_2 = (p^2)$, $v_3 = (p^2) \cup (q)$, $v_4 = (pq)$, $v_5 = (q)$, $v_6 = (p^2) \cup (pq)$, $v_7 = (p)$, has a subgraph homeomorphic to K_5, is obtained by first (v_4, v_5) edge contraction and then (v_4, v_7) edge contraction.

Case (iii) Let $m = pqr$, where p, q, r are distinct prime numbers. Then the subdivison formed by the set of vertices $\{(pq), (pr), (qr), (p) \cup (q), (q) \cup (r), (p) \cup (r)\}$ is $K_{3,3}$.

Hence the graphs $In(Z_m)$ for $m = p^6$, p^2q, pqr are not planar (*cf.* Lemma 1.3.11), as desired. □

Theorem 6.3.3. *Let $n \in M$. Then the following statements are hold.*

(1) $In(Z_n)$ is toroidal if and only if $n = p^k (6 \leq k \leq 8)$, p^2q, where p, q are prime numbers.

(2) $In(Z_n)$ is bitoroidal if and only if $n = p^9$, where p is a prime number.

Proof. Let $n = \prod_{i=1}^{k} P_i^{\alpha_i}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq 1$. We complete the proof by considering the following cases.

Case (1) Let $k = 1$.

Then $In(Z_n) \cong K_{\alpha_1 - 1} (\alpha_1 \geq 2)$(*cf.* Theorem 6.1.9). Now by applying Theorem 6.3.2 and Lemma 1.3.8, we have $g(In(Z_n)) = 1$ if and only if $6 \leq \alpha_1 \leq 8$ and $g(In(Z_n)) = 2$ if and only if $\alpha_1 = 9$.

Case (2) Let $k = 2$.

(i) If $\alpha_1 = \alpha_2 = 1$, then $g(In(Z_n)) = 0$ (*cf.* Theorem 6.3.2).

(ii) Let $\alpha_1 = 2$ and $\alpha_2 = 1$. Then the graph $In(Z_n)$ has 7 vertices and 16 edges (see Figure 6.1). Hence by Lemma 1.3.6 and Lemma 1.3.8, we have $1 \leq \lceil \frac{16}{6} - \frac{7}{2} + 1 \rceil \leq g(In(Z_{p_1^2 p_2})) \leq g(K_7) = 1$. Therefore $g(In(Z_{p_1^2 p_2})) = 1$.

(iii) Let $\alpha_1 = 2$ and $\alpha_2 = 2$. Then it is easy to calculate that $In(Z_{p_1^2 p_2^2})$ has 16 vertices and 87 edges. Hence by Lemma 1.3.7, we have $g(In(p_1^2 p_2^2)) \geq \lceil \frac{87}{6} - \frac{16}{2} + 1 \rceil = 8$.

(iv) Let $\alpha_1 = 3$, $\alpha_2 = 1$. Then $In(Z_{p_1^3 p_2})$ has 12 vertices and 49 edges. So $g(In(Z_{p_1^3 p_2})) \geq \lceil \frac{49}{6} - \frac{12}{2} + 1 \rceil = 4$ (*cf.* Lemma 1.3.7).

Hence for $k = 2$, $g(In(Z_n)) = 1$ if and only if $n = p_1^2 p_2$ and no graph is of genus 2(*cf.* Lemma 1.3.6).

Case (3) Let $k \geq 3$.

Let $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Then $In(Z_{p_1p_2p_3})$ has 17 vertices and 95 edges. So $g(In(Z_{p_1p_2p_3})) \geq \lceil \frac{95}{6} - \frac{17}{2} + 1 \rceil = 9$ (cf. Lemma 1.3.7). So for $k \geq 3$, the graph $In(Z_n)$ must have a subgraph isomorphic to $In(Z_{p_1p_2p_3})$.

Hence by Lemma 1.3.6, we conclude that the graph $In(Z_n)$ has genus 1 if and only if $n = p^k(6 \leq k \leq 8)$, p^2q, where p, q are distinct prime numbers and has genus 2 if and only if $n = p^9$, as desired. □

By applying same arguments as in the proof of Theorem 6.3.3 in association with Lemma 1.3.8, we have the following result.

Theorem 6.3.4. *Let $n \in M$. Then $In(Z_n)$ is*

(1) *projective if and only if $n = p^6$, p^7, p^2q, where p, q are distinct prime numbers.*

(2) *never bi-projective.*

Theorem 6.3.5. (1) *Let $n \in M \setminus \{p^3q\}$. Then $\theta_0(In(Z_n)) = 2$ if and only if n is one of the form: $n = p^k(5 \leq k \leq 7)$, p^2q, where p, q are prime numbers.*

(2) *Let $n \in M \setminus \{p^3q, p^4q\}$. Then $\theta(In(Z_n)) = 2$ if and only if n is one of the form: $p^k(6 \leq k \leq 9)$, p^2q, p^3q, where p, q are prime numbers.*

Proof. (1) Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq 1$. We complete the proof by considering the following cases.

Case (a) Let $k = 1$. Then by Lemma 1.3.8, we have $\theta_0(In(Z_n)) = 1$ if and only if $2 \leq \alpha_1 \leq 4$ (cf. Theorem 6.3.1), $\theta_0(In(Z_n)) = 2$ if and only if $5 \leq \alpha_1 \leq 7$ and for $\alpha_1 \geq 8$, $\theta_0(In(Z_n)) \geq 3$.

Case (b) Let $k = 2$.

(i) Let $\alpha_1 = \alpha_2 = 1$. Then $\theta_0(In(Z_{p_1p_2})) = 1$ (cf. Theorem 6.3.1).

(ii) $\alpha_1 = 2$, $\alpha_2 = 1$. So by applying Lemma 1.3.9, we have $2 = \lceil \frac{16}{14-3} \rceil \leq \theta_0(In(Z_{p_1^2p_2})) \leq \theta_0(K_7) = 3$. An outerplanar decomposition of $In(Z_{p_1^2p_2})$ with vertices $v_1 = (p_1^2) \cup (p_2)$, $v_2 = (p_1^2) \cup (p_1p_2)$, $v_3 = (p_1) \cup (p_2)$, $v_4 = (p_2)$, $v_5 = (p_1p_2)$, $v_6 = (p_1)$, $v_7 = (p_1^2)$ is shown in the Figure 6.2. Hence we deduce that $\theta_0(In(Z_{p_1^2p_2})) = 2$.

(iii) Let $\alpha_1 = \alpha_2 = 2$. Then by applying Lemma 1.3.9, we have $\theta_0(In(Z_{p_1^2p_2^2})) \geq \lceil \frac{87}{32-3} \rceil = 3$.

(iv) Let $\alpha_1 = 3$, $\alpha_2 = 1$. Then by Lemma 1.3.8 and Lemma 1.3.9, we have $3 = \lceil \frac{49}{21} \rceil \leq$

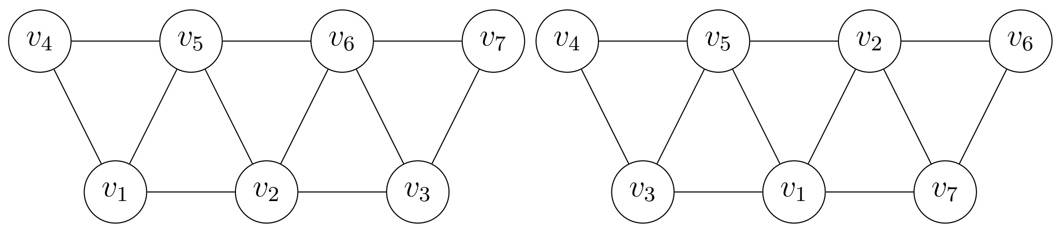


Figure 6.2: An outer-planar drawing of $In(\mathbb{Z}_{p_1^2 p_2})$

.

$\theta_0(In(Z_{p_1^3 p_2})) \leq \theta_0(K_{12}) = 4$.

(v) Let $\alpha_1 \geq 4$, $\alpha_2 = 1$. Then we have $\omega(In(Z_{p_1^3 p_2})) = 8$ (cf. Theorem 6.2.4). Hence by Lemma 1.3.8, we have $\theta_0(In(Zp_1^3 p_2)) \geq \theta_0(K_8) = 3$.

Case (c) Let $k \geq 3$. Now if $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Then $In(Z_{p_1 p_2 p_3})$ has 17 vertices and 95 edges. So by Lemma 1.3.7, we have $\theta_0(In(Z_{p_1 p_2 p_3})) \geq 4$. Hence for $k \geq 3$, there does not exist any graph $In(Z_n)$ of outerthickness 2 (cf. Lemma 1.3.6).

Hence for $n \in M \setminus \{p^3 q\}$, the graph $In(Z_n)$ has outerthickness 2 if and only if $n = p^k (6 \leq k \leq 9)$, $p^2 q$, where p, q are distinct prime numbers, as desired.

(2) Case (a) Let $k = 1$. Then $In(Z_n) \cong K_{\alpha_1 - 1}$ (cf. Theorem 6.1.9). Hence by applying Lemma 1.3.8, we have $\theta(In(Z_n)) = 1$ if $2 \leq \alpha_1 \leq 5$ (cf. Theorem 6.3.2), $\theta(In(Z_n)) = 2$ if $6 \leq \alpha_1 \leq 9$ and for $\alpha_1 \geq 10$, $\theta(In(Z_n)) \geq 3$.

Case (b) Let $k = 2$.

(i) $\alpha_1 = \alpha_2 = 1$. Then $\theta(In(Z_{p_1 p_2})) = 1$ (cf. Theorem 6.3.2).

(ii) $\alpha_1 = 2$, $\alpha_2 = 1$. Then $In(Z_{p_1^2 p_2}))$ has 7 vertices and 16 edges (see Figure. 6.1). So by applying Lemma 1.3.6, Lemma 1.3.8 and Lemma 1.3.9, we have $2 = \lceil \frac{16}{21-6} \rceil \leq \theta(In(Z_{p_1^2 p_2})) \leq \theta(K_7) = 2$. Hence $\theta(In(Z_{p_1^2 p_2})) = 2$.

(iii) Let $\alpha_1 = \alpha_2 = 2$. Then by applying Lemma 1.3.9, we have $\theta(In(Z_{p_1^2 p_2^2})) \geq \lceil \frac{87}{48-6} \rceil = 3$.

(iv) Let $\alpha_1 = 3$, $\alpha_2 = 1$. Then by Lemma 1.3.8 and Lemma 1.3.9, we have $2 = \lceil \frac{49}{36-6} \rceil \leq \theta(In(Z_{p_1^3 p_2})) \leq \theta(K_{12}) = 3$. In a similar way as the outerplanar decomposition of $In(Z_{p_1^2 p_2})$, one can obatin a planar decomposition of $In(Z_{p_1^3 p_2})$.

(v) Let $\alpha_1 = 4$, $\alpha_2 = 1$. Then we have $\omega(In(Z_{p_1^3 p_2})) = 8$ (cf. Theorem 6.2.4). Hence by Lemma 1.3.8, we have $\theta(In(Zp_1^3 p_2)) \geq \theta(K_8) = 2$. In a similar way as the outerplanar decomposition of $In(Z_{p_1^2 p_2})$, one can obatin a planar decomposition of $In(Z_{p_1^3 p_2})$.

(vi) Let $\alpha_1 \geq 5$, $\alpha_2 = 1$. Then we have $\omega(In(Z_{p_1^{\alpha_1}p_2}) \geq 10$ $(cf.$ Theorem 6.2.4). Hence by Lemma 1.3.8, we have $\theta(In(Zp_1^3p_2)) \geq \theta(K_{10}) = 3$.

Case (c) Let $k \geq 3$. Now if $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Then by similar arguments as above, we have $\theta(In(Z_{p_1p_2p_3})) \geq 3$. Therefore for $k \geq 3$, there does not exist any graph $In(Z_n)$ of thickness 2 $(cf.$ Lemma 1.3.6).

Hence for $n \in M \setminus \{p^3q, p^4q\}$, the graph $In(Z_n)$ has thickness 2 if and only if $n = p^k(6 \leq k \leq 9)$, p^2q, where p, q are distinct prime numbers, as desired. $\square$

6.4 Inclusion ideal graph of completely 0-simple semigroup

Here we consider the inclusion ideal graph $In(S_n)$ of a completely 0-simple semigroup S_n with n 0-minimal right ideals. The graph is k-partiate if and only if $n = k+1$ and $\omega(In(S_n)) = \chi(In(S_n)) = n-1$.

Proposition 6.4.1. *The number of edges of $In(S_n)$ is given by $E(In(S_n)) = \sum\limits_{i=1}^{n-1} {}^nC_i(2^{i-1} + 2^{n-i-1}) - 2^{n+1} + 4$.*

Proof. It is well known that the sum of degrees of vertices is twice the number of edges of the graph. Hence $2|E| = {}^nC_1(2-2+2^{n-1}-2)+{}^nC_2(2^2-2+2^{n-2}-2)+\cdots+{}^nC_{n-1}(2^{n-1}-2+2-2)$
$= \sum\limits_{i=1}^{n-1} {}^nC_i(2^i + 2^{n-i}) - 4\sum\limits_{i=1}^{n-1} {}^nC_i = \sum\limits_{i=1}^{n-1} {}^nC_i(2^i + 2^{n-i}) - 4(2^n - 2)$

Therefore $|E(In(S_n)| = \sum\limits_{i=1}^{n-1} {}^nC_i(2^{i-1} + 2^{n-i-1}) - 2(2^n - 2)$ $\square$

Lemma 6.4.2. *Let I_{Λ_1} and I_{Λ_2} be any two vertex of $In(S_n)$. Then deg(I_{Λ_1}) =deg(I_{Λ_2}) if and only if $|\Lambda_1| = |\Lambda_2|$ or $|\Lambda_1| + |\Lambda_2| = n$.*

Proof. Let deg$(I_{\Lambda_1}) = s$ and deg$(I_{\Lambda_2}) = t$, where s, $t \in [n]$. Then deg(I_{Λ_1}) =deg(I_{Λ_2}) implies $2^s - 2 + 2^{n-s} - 2 = 2^t - 2 + 2^{n-t} - 2 \Rightarrow 2^s - 2^t = 2^n.\frac{2^s-2^t}{2^{s+t}} = 2^{n-s-t}(2^s - 2^t)$.

Now if $2^s \neq 2^t$, then $2^{n-s-t} = 1 \Rightarrow s+t = n \Rightarrow |\Lambda_1| + |\Lambda_2| = n$.

Again if $2^s = 2^t$ then $s = t \Rightarrow |\Lambda_1| = |\Lambda_2|$.

The proof of the converse part is clear. $\square$

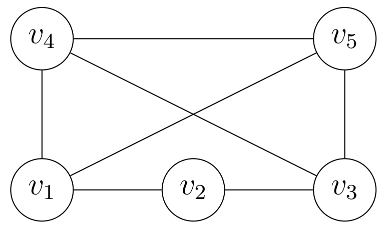


Figure 6.3: A subdivision of K_4

.

Theorem 6.4.3. (1) *The graph $In(S_n)$ is planar if and only if $n = 3, 4$.*

(2) *The graph $In(S_n)$ is outer-planar if and only if $n = 3$.*

(3) *The graph $In(S_n)$ is neither toroidal nor bi-toroidal.*

Proof. (1) It is clear from Theorem 1.3.26.

(2) Since every outer-planar graph is planar, we only concentrate on the cases for which $In(S_n)$ is planar.

Now if $n = 3$ then $In(S_3) \cong C_6$, a cycle on six veritces, which is clearly outer-planar.

Now if $n = 4$, then $In(S_n)$ contains a subdivision of the complete graph K_4 with vertices $v_1 = I_1$, $v_2 = I_{124}$, $v_3 = I_2$, $v_4 = I_{12}$, $v_5 = I_{123}$ (see Figure 6.3) and therefore not outer-planar (*cf*. Lemma 1.3.10). Hence the result follows.

(3) Since for $n \leq 4$ the graph $In(S_n)$ is planar and hence $g(In(S_n)) = 0$.

Let $n = 5$. Then $In(S_n)$ has 30 vertices and 150 edges (*cf*. Proposition 6.4.1) and hence $g(In(S_5)) \geq \lceil \frac{150}{6} - \frac{30}{2} + 1 \rceil = 11$ (*cf*. Lemma 1.3.7). Now if $n \geq 6$, then $In(S_n)$ must have a subgraph isomorphic to $In(S_5)$ and hence $g(In(S_n)) \geq 11$ (*cf*. Lemma 1.3.6). Therefore $In(S_n)$ is neither toroidal nor bitoroidal. $\square$

Theorem 6.4.4. *The graph $In(S_n)$ is projective if and only if $n = 3, 4$ and never bi-projective.*

Proof. We know that for every connected graph G which is not tree, we have $\overline{g}(G) \leq 2g(G)+1$ and the graphs $In(S_3)$ and $In(S_4)$ are connected planar graph. Since the graphs are not tree, we have $\overline{g}(In(S_n)) = 1$ for $n = 3, 4$. Let $n = 5$. Then $\overline{g}(In(S_5)) \geq \lceil \frac{150}{3} - 30 + 2 \rceil = 22$. Since for $n \geq 5$, $In(S_n)$ must have a subgraph isomorphic to $In(S_5)$, we have $\overline{g}(In(S_n)) \geq 22$. Hence the result follows. $\square$

Theorem 6.4.5. *Let S_n be a completely 0-simple semigroup with n 0-minimal right ideals. Then*

(1) $In(S_n)$ is unicyclic if and only if $n = 3$.

(2) The graph $In(S_n)$ is never chordal, split, co-graph and thresold.

(3) The graph $In(S_n)$ is Hamiltonian if $n = 3, 4$.

Proof. (1) It is easy to observe that for $n = 3$, $In(S_n) \cong C_6$, a cycle of length six and hence unicyclic. Let $n \geq 4$. Then $I_1 \sim I_{12} \sim I_{123} \sim I_1$ and $I_3 \sim I_{13} \sim I_{123} \sim I_3$ are two different cycles in $In(S_n)$ and therefore not unicyclic. Hence the result follows.

(2) For $n \geq 3$, there always exists a cycle of length six with vertices $v_1 = I_1$, $v_2 = I_{12}$, $v_3 = I_2$, $v_4 = I_{23}$, $v_5 = I_3$, $v_6 = I_{13}$ in $In(S_n)$ but there does not exist a chord and hence not chordal.
Also for $n \geq 3$, the set of vertices $v_1 = I_1$, $v_2 = I_{12}$, $v_3 = I_3$, $v_4 = I_{23}$ forms an induced $2K_2$ in $In(S_n)$ and hence not a split graph.

Again for $n \geq 3$, there exists an induced P_4 with vertices $v_1 = I_1$, $v_2 = I_{12}$, $v_3 = I_2$, $v_4 = I_{23}$ in $In(S_n)$ and hence not a co-graph.

Since for $n \geq 3$, the graph $In(S_n)$ induced $2K_2$ so not a thresold graph.

(3) Since for $n = 3$, $In(S_n)$ is a cycle and hence Hamiltonian. Also for $n = 4$, $I_1 \sim I_{12} \sim I_{123} \sim I_{13} \sim I_3 \sim I_{23} \sim I_{234} \sim I_{34} \sim I_{134} \sim I_{14} \sim I_4 \sim I_{24} \sim I_2 \sim I_{124} \sim I_1$ is a Hamiltonian cycle in $In(S_n)$. □

Since for $n = 3$ we have $In(S_n) \cong C_6$, a cycle of length six, so we have the following immediate result.

Corollary 6.4.6. *For $n = 3$ the partition dimension of $In(S_n)$ is $pd(In(S_n)) = 3$.*

Theorem 6.4.7. *The partition dimension of $In(S_n)$ satisfy the inequality $\lceil \frac{n}{2} \rceil \leq pd(In(S_n)) \leq n$ for $n \geq 4$.*

Proof. Since $In(S_n)$ is a graph of diameter 3, we have $g(n, 3) \leq$pd$In(S_n)$ ($cf.$ Theorem 1.3.20), where $g(n, 3)$ is the least positive integer l for which $4^l \geq 2^n - 2$. Clearly $l = \frac{n}{2}$ if n is even and $l = \frac{n+1}{2}$ if n is odd. Therefore we have

$$pd(In(S_n)) \geq \lceil \frac{n}{2} \rceil \tag{6.4.1}$$

Now we consider the n-partition $\Pi = \{S_1, S_2, \ldots, S_n\}$ of $V(In(S_n))$ where for $1 \leq t \leq n-1$ we consider $S_t = \{I_{\Lambda_t} : 1, 2, \ldots, t-1 \notin \Lambda_t$ but $t \in \Lambda_t\}$ and we show that Π is a resolving partition of $In(S_n)$.

Let $I_{\Lambda_1} \in S_1$ such that $1 \leq |\Lambda_1| \leq n-2$. Then $r(I_{\Lambda_1}|\Pi) = (0, a_2, \ldots, a_n)$ where $a_i = 1$ if $i \in \Lambda_1$ and $a_i = 2$ if $i \notin \Lambda_1$. Also if $|\Lambda_1| = n-1$ then $r(I_{\Lambda_1}|\Pi) = (0, a_2, \ldots, a_n)$ where $a_i = 1$ if $i \in \Lambda_1$ and $a_i = 3$ if $i \notin \Lambda_1$. Again let $I_{\Lambda_2} \in S_2$ and $1 \leq |\Lambda_2| \leq n-2$. Then $r(I_{\Lambda_2}|\Pi) = (1, 0, a_3 \ldots, a_n)$ where $a_i = 1$ if $i \in \Lambda_2$ and $a_i = 2$ if $i \notin \Lambda_2$. Similarly $r(I_{23\ldots n}|\Pi) = (2, 0, 1, \ldots, 1)$. Now for $3 \leq l \leq n$ let $I_{\Lambda_l} \in S_l$. Then $r(I_{\Lambda_l}|\Pi) = (1, 1, a_3, \ldots, a_n)$ where $a_i = 1$ if $i < l$ or $i \in \Lambda_l$, $a_i = 0$ if $i = l$, $a_i = 2$ if $i \notin \Lambda_l$. Clearly representations of all vertices of $In(S_n)$ with respect to the partition Π are distinct and hence $\Pi = \{S_1, S_2, \ldots, S_n\}$ is a resolving partition of $In(S_n)$. Therefore we have

$$pd(In(S_n)) \leq n \tag{6.4.2}$$

Hence combining equation (3) and (4) we have the required result. □

Theorem 6.4.8. *The metric dimension of $In(S_n)$ satisfy the inequality* $\lceil \frac{n}{2} \rceil - 1 \leq dim(In(S_n)) \leq n-1$.

Proof. By applying Theorem 6.4.7 and Theorem 1.3.21, we have

$$dim(In(S_n)) \geq \lceil \frac{n}{2} \rceil - 1 \tag{6.4.3}$$

Now we show that $W = \{I_1, I_2, \ldots, I_{n-1}\}$ is a resolving set of $In(S_n)$.

Let $2 \leq |\Lambda| \leq n-2$. Then $r(I_\Lambda|W) = (a_1, a_2, \ldots, a_{n-1})$ where $a_i = 1$ if $i \in \Lambda$ and $a_i = 2$ if $i \notin \Lambda$. Also we have $r(I_n|W) = (2, 2, \ldots, 2)$. Finally let $|\Lambda| = n-1$. Then $r(I_\Lambda|W) = (a_1, a_2, \ldots, a_n)$ where $a_i = 1$ if $i \in \Lambda$ and $a_i = 3$ if $i \notin \Lambda$. Clearly the representations of all vertices of $In(S_n)$ with respect to W are all distinct and hence W is a resolving set for $In(S_n)$. So we have

$$dim(In(S_n)) \leq n-1 \tag{6.4.4}$$

Therefore combining equation (5) and (6) we have the required result. □

6.5 Cartesian Product of inclusion ideal graphs

Throughout this section, S_1 and S_2 are monoids. Here we compute some graph parameters of the cartesian product graph $In(S_1)\Box In(S_2)$, for simplicity we write it as Γ.

Theorem 6.5.1. *Let S_1 and S_2 be two monoids with finitely many ideals. Then $\gamma(\Gamma) = min\{|V(In(S_1))|, |V(In(S_2))|\}$.*

Proof. Without loss of generality, we assume that $k = |V(In(S_1))| \leq |V(In(S_2))|$. Clearly the set $D = \{((I_j)_1, M_2) : 1 \leq j \leq k\}$ is a domination set for Γ, where $(I_j)_1$'s are nontrivial ideals of S_1 and M_2 is the maximal ideal of S_2. Then no proper subset of D be a domination set for Γ, if so, then let $D' = D - \{((I_t)_1, M_2)\}$ is a dominating set for Γ, for some $1 \leq t \leq k$. Then vertex $((I_t)_1, I_2)$ is not adjacent to any element of D', where $I_2(\neq M_2)$ is a vertex of $In(S_2)$, contradicting that D' is a dominating set for Γ. We now show that there does not exist any dominating set of cardinality less than k. If possible, let K be dominating set for Γ of cardinality less than k. Then there must exists vertex $(I_t)_1$ of $In(S_1)$ and J_2 of $In(S_2)$ such that $((I_t)_1, J) \notin K$ and $(I, J_2) \notin K$ for any nontrivial ideals I of S_1 and J of S_2, where $1 \leq t \leq k$. Then the vertex $((I_t)_1, J_2)$ is not adjacent to any element of K, a contradiction. Consequently, D is the smallest dominating set for Γ and hence $\gamma(\Gamma) = k$, as desired. $\square$

Theorem 6.5.2. *Let S_1 and S_2 be two monoids such that $|V(In(S_i))| \geq 2$, $i = 1, 2$. Then*

$$diam(\Gamma) = \begin{cases} 2\ , & \textit{ideals of both monoids are linearly ordered} \\ 3\ , & \textit{ideals of one monoids is linearly ordered but not others} \\ 4\ , & \textit{there exists } I_i,\ J_i \in V(In(S_i)) \textit{ such that } I_i \nsim J_i,\ i = 1, 2. \end{cases} \tag{6.5.1}$$

Proof. Since S_1 and S_2 are monoids, have unique maximal ideals, say M_1 and M_2. To prove the result we consider three cases

Case (1) Let ideals of both monoids are linearly ordered. Then any two vertex (I_1, I_2) and (J_1, J_2) of Γ are adjacent by the shortest path as $(I_1, I_2) \sim (I_1, J_2) \sim (J_1, J_2)$. Hence in this case diam$(\Gamma) = 2$.

Case(2) Let ideals of S_1 are linearly ordered and there exists ideals I_2 and J_2 of S_2 such that $I_2 \nsim J_2$. Then for any ideals I_1 and J_1 of S_1, the shortest path between (I_1, I_2) and (J_1, J_2)

is $(I_1, I_2) \sim (J_1, I_2) \sim (J_1, M_2) \sim (J_1, J_2)$. Consequently, here diam $(\Gamma) = 3$.

Case (3) Let there exists ideals I_1, J_1 of S_1 and I_2, J_2 of S_2 such that $I_1 \nsim J_1$ and $I_2 \nsim J_2$. Then the shortest path between (I_1, J_1) and (I_2, J_2) is $(I_1, I_2) \sim (I_1, M_2) \sim (I_1, J_2) \sim (M_1, J_2) \sim (J_1, J_2)$. Hence in this case diam$(\Gamma) = 4$, as desired. $\square$

Theorem 6.5.3. *Let S_1 and S_2 be two monoids such that $|V(In(S_1))| = 1$ and $|V(In(S_2))| \geq 2$. Then*

$$diam(\Gamma) = \begin{cases} 1\ , & \text{ideals of } S_2 \text{ are linearly ordered} \\ 2\ , & \text{ideals of } S_2 \text{ are not linearly ordered.} \end{cases} \tag{6.5.2}$$

Proof. To prove the result we consider two cases

Case (1) Let ideals of S_2 are linearly ordered. It is clear that all nontrivial ideals of Γ are adjacent. Hence diam $(\Gamma) = 1$.

Case (2) Let there exists ideals I_2 and J_2 of S_2 such that $I_2 \nsim J_2$. Then shortest path between (M_1, I_2) and (M_1, J_2) is $(M_1, I_2) \sim (M_1, M_2) \sim (M_1, J_2)$, where M_1 is the only nontrivial ideals of S_1 and M_2 is the maximal ideal of S_2. Consequently, diam $(\Gamma) = 2$, as desired. $\square$

Theorem 6.5.4. *Let S_1 and S_2 be two monoids such that $|V(S_i)| \geq 2$, $i \in \{1, 2\}$. Then*

$$gr(\Gamma) = \begin{cases} 4\ , & \text{nontrivial ideals of both monoids are either } 0\text{-minimal or maximal} \\ 3\ , & \text{otherwise.} \end{cases} \tag{6.5.3}$$

Proof. To prove the result we consider two cases

Case (1) Let nontrivial ideals of both monoids are either 0-minimal or maximal. Then $(I_1, M_2) \sim (M_1, M_2) \sim (M_1, I_2) \sim (I_1, I_2) \sim (I_1, M_2)$ is a cycle of length four in Γ, where I_i are nontrivial ideals of S_i and M_i are maximal ideals of S_i, $i = 1, 2$. Now if possible let there is a cycle say $(I_1, I_2) \sim (J_1, J_2) \sim (K_1, K_2) \sim (I_1, I_2)$ of length three in Γ. Since each nontrivial ideals are either maximal or 0-minimal, we have the only three following subcases. subcase(i) Let $I_i = M_i$, $i \in \{1, 2\}$. Then $J_1 = M_1$ or $J_2 = M_2$. Let $J_1 = M_1$ and $J_2 \subset M_2$. Now if $K_1 = M_1$, then $J_2 \sim K_2$ which implies $J_2 \subset K_2$ or $K_2 \subset J_2$. Now if $K_2 \neq M_2$, either K_2 or J_2 is neither maximal nor 0-minimal, which is a contradiction. Also if $K_2 = M_2$, then $(M_1, J_2) \sim (K_1, M_2)$ implies $K_1 = M_1$, which is again a contradiction. Therefore there is no three cycle in Γ.

subcase (ii) Let $I_1 = M_1$ and I_2 is a 0-minimal ideal of S_2. Since $(J_1, J_2) \sim (M_1, I_2)$, if $J_1 = M_1$ then $J_2 = M_2$. So in a similar way as in subcase (i) we will arrive at a contradiction. So let $J_1 \neq M_1$. Then $I_2 = J_2$. Also as $(I_1, I_2) \sim (K_1, K_2)$, we have $K_1 \neq M_1$ and $K_2 = I_2$. This implies $J_1 \sim K_1$ which is impossible as they are both 0-minimal. Hence there does not exist a 3-cycle in γ.

subcase (iii) Let I_1 is a 0-minimal ideal of S_1 and I_2 is maximal. Then in a similar way as in subcase (ii), we will arrive at a contradiction.

Hence $\mathrm{gr}(\Gamma) = 4$.

Case (2) If there exist nontrivial nonmaximal ideals I_i, J_i in one of the monoids S_i such that $I_i \subset J_i$ or $J_i \subset I_i$, $i = \{1, 2\}$. Then $(I_1, M_2) \sim (M_1, M_2) \sim (J_1, M_2) \sim (I_1, M_2)$ is a cycle of length three or symmetrically $(M_1, I_2) \sim (M_1, M_2) \sim (M_1, J_2) \sim (M_1, I_2)$. Consequently, $\mathrm{gr}(\Gamma) = 3$, as desired. $\square$

The following result is immediate, hence we omit the proof.

Theorem 6.5.5. *Let S_1 and S_2 be two monoids such that $|V(S_1)| = 1$ and $|V(S_2)| \geq 3$, then*

$$gr(\Gamma) = \begin{cases} \infty \ , & \text{nontrivial ideals of } S_2 \text{ are either } 0\text{-minimal or maximal} \\ 3 \ , & \text{otherwise.} \end{cases} \tag{6.5.4}$$

Theorem 6.5.6. *Let S_1 and S_2 be two monoids with finitely many ideals. Then* $\omega(\Gamma) = max\{\omega(In(S_1)), \omega(In(S_2))\}$.

Proof. Without loss of generality, we assume $In(S_1)$ has the maximal clique $\{V_1, V_2, ..., V_k\}$ of size k. Then $M = \{(V_1, J_2), (V_2, J_2), ..., (V_k, J_2)\}$ is a clique in Γ, for some nontrivial ideal J_2 of S_2. If possible, let $M \cup I$ be a clique in Γ. Then I is of the form (V_p, J_2) and $V_p \sim V_i$, where $1 \leq i \leq k$ and hence $\{V_1, V_2, ..., V_k, V_p\}$ is a clique of size $k+1$ in $In(S_1)$, a contradiction. Therefore there is no clique in Γ containing M. Now we prove that there does not exists a clique of size greater than k in Γ. If possible let $\{((I_{t_1})_1, I_2), ((I_{t_2})_1, I_2), ..., ((I_{t_m})_1, I_2)\}$ is a clique of size $m \geq k+1$. Then $\{(I_{t_1})_1, (I_{t_2})_1, ..., (I_{t_m})_1\}$ is a clique of size $m \geq k+1$ in $In(S_1)$, a contradiction. Hence $\omega(\Gamma) = k$, as desired. $\square$

Theorem 6.5.7. *Let S_1 and S_2 be two monoids with finitely many ideals. Then $\chi(\Gamma) = \omega(\Gamma)$.*

Proof. we first note that the chromatic number of the cartesian product of simple graphs G_1 and G_2 satisfy the equality $\chi(G_1 \square G_2) = max\{\chi(G_1), \chi(G_2)\}$ ($cf.$ Theorem 1.3.22). Therefore $\chi(\Gamma) = max\{\chi(In(S_1)), \chi(In(S_2))\} = max\{\omega(In(S_1)), \omega(In(S_2))\}$ ($cf.$ Theorem 6.2.1) $= \omega(\Gamma)$ ($cf.$ Theorem 6.5.6), as desired. □

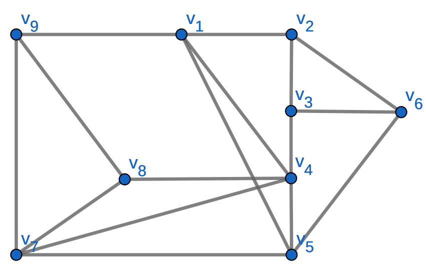
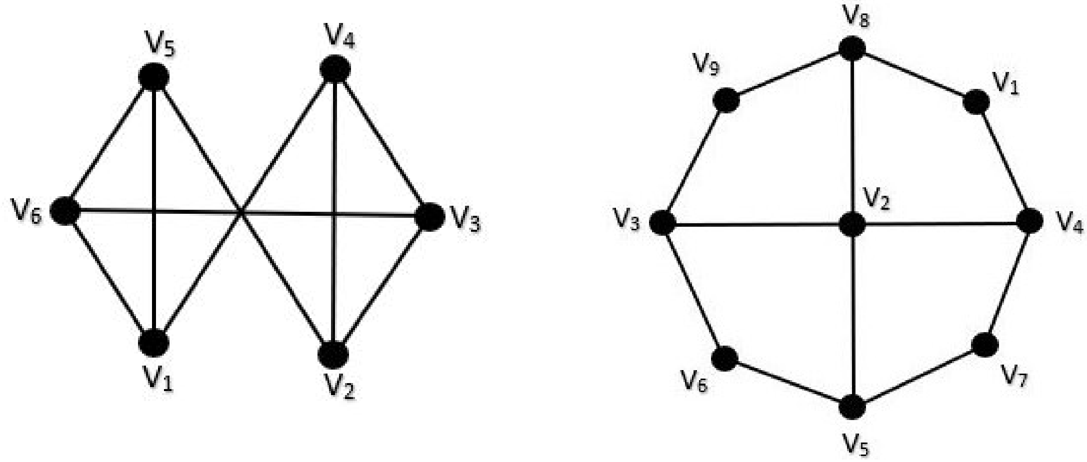


Figure 6.4: $\Gamma = In(S_1) \square In(S_2)$

Example 6.5.8. Let us consider the semigroups $S_1 = (Z_{t^4}, .)$, $S_2 = (Z_{pq}, .)$, $S_3 = (Z_{rs}, .)$ and $S_4 = (Z_{x^3}, .)$, where p, q, r, s, t, x are distinct primes and give our attention to the graphs $\Gamma = In(S_1) \square In(S_2)$, $\Delta = In(S_1) \square In(S_4)$ and $\Sigma = In(S_2) \square In(S_3)$ as drawn in Figure 6.4, 6.5(a), 6.5(b) respectively. The vertices of the graph Γ are $v_1 = ((t), (p) \cup (q))$, $v_2 = ((t), (p))$, $v_3 = ((t^2), (p))$, $v_4 = ((t^2), (p) \cup (q))$, $v_5 = ((t^3), (p) \cup (q))$, $v_6 = (t^3), (p))$, $v_7 = ((t^3), (q))$, $v_8 = ((t^2), (q))$, $v_9 = ((t), (q))$. The vertices of the graph Δ are $v_1 = ((t), (x^2))$, $v_2 = ((t^3), (x))$, $v_3 = ((t^2), (x))$, $v_4 = ((t), (x))$, $v_5 = ((t^3), (x^2))$, $v_6 = ((t^2), (x^2)$, whereas the vertices of the graph Σ are $v_1 = ((p), (r))$, $v_2 = ((p) \cup (q), (r) \cup (s))$, $v_3 = ((p) \cup (q), (s))$,



(a) (b)

Figure 6.5: (a) $\triangle = In(S_1)\Box In(S_4)$, $(b)\Sigma = In(S_2)\Box In(S_3)$

$v_4 = ((p) \cup (q), (r))$, $v_5 = ((q), (r) \cup (s))$, $v_6 = ((q), (s))$, $v_7 = ((q), (r)$, $v_8 = ((p), (r) \cup (s))$, $v_9 = ((p), (s))$. Depending on the results presented in this chapter and from the figure of the graphs, we can check the following equalities:

(i) $\gamma(\Gamma) = 3$, $\gamma(\Sigma) = 3$, $\gamma(\Delta) = 2$.

(ii) diam $(\Gamma) = 3$, diam $(\Sigma) = 4$, diam $(\Delta) = 2$.

(iii) girth $(\Gamma) = 3$, girth $(\Sigma) = 4$, girth $(\Delta) = 3$.

(iv) $\omega(\Gamma) = \chi(\Gamma) = 3$, $\omega(\Sigma) = \chi(\Sigma) = 2$, $\omega(\Delta) = \chi(\Delta) = 3$.

Chapter 7

Inclusion ideal graph of multiplicative semigroup of a ring

In this chapter, we consider the inclusion ideal graph of multiplicative semigroup R_S of a commutative ring R with unity. We characterize a commutative ring R for which the graph $In(R_S)$ is complete and study perfectess property of the graph. Also we study minimal embedding of the graph in orientable surfaces.

In **Section** 7.1, first we define inclusion ideal graph of the multiplicative semigroup of a commutative ring R with unity ($cf.$ Definition 7.1.1). Then we characterize a ring R for which the graph $In(R_S)$ is complete ($cf.$ Theorem 7.1.3), bipartiate ($cf.$ Corollary 7.1.6), unicyclic ($cf.$ Theorem 7.1.7), split ($cf.$ Theorem 7.1.13) and characterize the girth ($cf.$ Theorem 7.1.5) of $In(R_S)$. Also we prove that $In(R_S)$ is a perfect graph ($cf.$ Theorem 7.1.9) and determine the clique number ($cf.$ Theorem 7.1.10) of $In(R_S)$.

In **Section** 7.2, we characterize the family of rings R for which $In(R_s)$ is planar ($cf.$ Theorem 7.2.2), outerplanar and ring graph ($cf.$ Proposition 7.2.3).

Finally in **Section** 7.3, we characterize a ring R for which $In(R_S)$ is of genus one and two ($cf.$ Theorem 7.3.2).

Throughout this chapter by a ring R we mean a commutative ring with unity which is not a field. Clearly every ideal of R is also an ideal of R_S but converse is not true and every principal ideal of R_S is an ideal of R.

7.1 Some basic properties

Definition 7.1.1. The inclusion ideal graph of multiplicative semigroup R_S of a commutative ring R with unity, denoted by $In(R_S)$, is a graph with vertices are non-trivial ideals of R_S and two distinct vertices I_1 and I_2 are adjacent if and only if $I_1 \subset I_2$ or $I_2 \subset I_1$.

Since the semigroup R_S has unique maximal ideal ($cf.$ Lemma 1.2.11), it is a universal vertex in $In(R_S)$. Therefore we have the following immediate results :

Theorem 7.1.2. *For a ring R, the following statements about $In(R_S)$ are true.*
(1) *$In(R_S)$ is an empty graph if and only if R is a field.*
(2) *$In(R_S)$ is a null graph if and only if $In(R_S) \cong K_1$ if and only if R is an Artinian local ring with maximal ideal $m = (x)$ for some non-zero $x \in R$ such that $I^*(R) = \{m\}$.*
(3) *$In(R_S)$ is a connected graph. Moreover, diam($In(R_S)$) ≤ 2.*
(4) *The domination number of $In(R_S)$ is $\gamma(In(R_S)) = 1$.*

The following is a characterization of a ring R for which $In(R_S)$ is complete.

Theorem 7.1.3. *Let R be a ring. Then the following statements are equivallent:*
(1) *$In(R_S)$ is a complete graph,*
(2) *$In(R)$ is a complete graph,*
(3) *R is local and one of the following holds,*
(*i*) *R is a special principal ideal ring,*
(*ii*) *R is a principal ideal domain,*
(*iii*) *R is an arithmetical ring,*
(*iv*) *R is a bezout ring.*

Proof. (1) $\Leftrightarrow$ (2) Let $In(R)$ be a complete graph. We prove that $I^*(R) = I^*(R_S)$. Clearly $I^*(R) \subseteq I^*(R_S)$. Let $I \in I^*(R_S)$ and $x, y \in I$. Since every principal ideal of R_S is an ideal of R and $In(R)$ is complete, we have either $(x) \subset (y)$ or $(y) \subset (x)$. Without loss of generality, let $(x) \subset (y)$. Then $x, y \in (y) \subseteq I$ and hence $x - y \in (y) \subseteq I$. Therefore $I \in I^*(R)$ and hence $I^*(R_S) \subset I^*(R)$. Therefore $I^*(R) = I^*(R_S)$ and hence $In(R_S)$ is a complete graph. Since $In(R)$ is a subgraph of $In(R_S)$, the proof of the converse part is immediate.
(2) $\Leftrightarrow$ (3) Let $In(R)$ be a complete graph. If R is not local then there exist distinct maximal

ideals M_1 and M_2 in R but $M_1 \nsim M_2$, which is a contradiction. Therefore R is a local ring. If x and y are two distinct elements in a minimal generating set of maximal ideal M then $(x) \nsim (y)$, again a contradiction. Therefore $M = (x)$ for some non-zero $x \in R$. By similar argument as above, it is easy to observe that every ideal of R is principal. Now if $\{0\}$ is the only annhiliator of x then R is a integral domain and hence principal ideal domain. We know that $\mathrm{Nil}(R) \subseteq J(R)$. Again, if x has non-zero annhiliator then there exists $sx^i \in M$ such that $sx^{i+1} = 0$, where s is a unit of R. This implies x is a nilpotent element of R and hence $J(R) = \mathrm{Nil}(R)$. Therefore in this case R is a special principal ideal ring. Conversely, let R be a local principal ideal domain or a local special principal ideal ring with maximal ideal $m = (x)$ for some non-zero $x \in R$. If $I = (sx^i)$ and $J = (tx^j)$ are two distinct non-trivial ideals of R , where $i > j$, then $I \subset J$ and hence $In(R)$ is a complete graph. Therefore if and only if conditions for statements (i) and (ii) holds.

The necessary and sufficient condition for statement (iii) is clear by just recalling that a local ring is arithmetical if and only if its ideals are totally ordered by set inclusion (see Theorem 1.2.78). Since every Bezout ring is arithmetical and every semi-local arithmetical ring is Bezout (see Theorem 1.2.79), the if and only if condition for statements (iv) is immediate. Hence the result follows. □

Corollary 7.1.4. *Let $n \in M$. Then $In((\mathbb{Z}_n)_S)$ is complete if and only if $n = p^k$, where p is prime and k is a natural number ≥ 2.*

In the following theorem we characterize the girth of $In(R_S)$.

Theorem 7.1.5. *Let R be a ring. Then*

$$gr(In(R_S)) = \begin{cases} \infty & R \cong R_1 \times R_2, \text{ where } R_1 \text{ and } R_2 \text{ are fields or} \\ & R \text{ is Artinian local with maximal ideal } m = (x) \text{ such that } 2 \leq n(m) \leq 3 \\ 3 & \text{otherwise.} \end{cases}$$

Proof. If there exists a chain of non-trivial ideals of length at least three in R, then clearly $\mathrm{gr}(In(R_S)) = 3$. If not, then R must be Artinian and hence $R \cong \prod_{i=1}^{n} R_i$, where each R_i is an Artinian local ring with maximal ideal m_i ($cf.$ Theorem 1.2.76).

If $n \geq 3$, then $|Max(R)| \geq 3$. Let m_1, m_2, $m_3 \in Max(R)$. Then $m_1 \sim m_1 \cup m_2 \sim m_1 \cup m_2 \cup m_3 \sim m_1$ is a cycle of length three in $In(R_S)$ and hence $\mathrm{gr}(In(R_S)) = 3$.

Now let $n = 2$. If $m_1 = m_2 = 0$, then $In(R_S) \cong P_3$ and hence $\mathrm{gr}(In(R_S)) = \infty$. Let $m_1 \neq 0$. Then $m_1 \times 0 \sim R_1 \times 0 \sim (R_1 \times 0) \cup (m_1 \times R_2) \sim m_1 \times 0$ is a cycle of length three in $In(R_S)$ and hence $\mathrm{gr}(In(R_S)) = 3$.

Let $n = 1$. Since R is Artinian, clearly $m = (x_1, x_2, \ldots, x_k)$ for some $k \in \mathbb{N}$. If $k \geq 2$, then (x_1), (x_2), $(x_1 + x_2)$, (x_1, x_2) are non-trivial ideals of R. If one of x_1^2, x_2^2 or x_1x_2 is nonzero, then $\mathrm{gr}(In(R_S)) = 3$. So let, $m^2 = 0$. Then non-trivial ideals of R are of the form (x_1), (x_2), (x_1, x_2), $(x_1 + ex_2)$, where e is a unit of R and each non-maximal ideal are minimal (*cf.* Theorem 1.2.80). Now we prove that every minimal ideal of R is a 0-minimal ideal of R_S. If not, then there exists a non-trivial ideal I_1 of R_S such that $I_1 \subset I$. Let $x \in I_1$ be a non-zero element of R. Then $(x) \subseteq I_1 \subset I$, contradicts that I is a minimal ideal of R. Therefore every minimal ideal of R is a 0-minimal ideal of R_S. Therefore $(x_1) \cap (x_2) = (x_1) \cap (x_1 + x_2) = (x_2) \cap (x_1 + x_2) = 0$. Now if $(x_1) \cup (x_2) = (x_1, x_2)$, then $(x_1) \cap (x_1 + x_2) \neq 0$, a contradiction. Hence $(x_1) \cup (x_2) \neq (x_1, x_2)$. Therefore $(x_1) \sim (x_1) \cup (x_2) \sim (x_1, x_2) \sim (x_1)$ is a cycle in $In(R_S)$ and hence $\mathrm{gr}(In(R_S)) = 3$.

Finally, let $k = 1$. Then non-trivial ideals of R are of the form $\{m^t : 1 \leq t \leq r - 1\}$, where r is the smallest positive integer such that $m^r = 0$ (*cf.* Proposition 1.2.77). Therefore if $r \geq 4$, then $\mathrm{gr}(In(R_S)) = 3$. Hence the result follows. □

Corollary 7.1.6. *The graph $In(R_S)$ is bipartiate if and only if one of the following holds.*

(1) *$R \cong R_1 \times R_2$, where R_1 and R_2 are fields.*

(2) *R is an Artinian local ring with maximal ideal $m = (x)$ such that $I^*(R) = \{m, m^2\}$.*

Theorem 7.1.7. *Let R be a ring. Then $In(R_S)$ is a unicyclic graph if and only if R is an Artinian local ring with maximal ideal $m = (x)$ such that $I^*(R) = \{m, m^2, m^3\}$.*

Proof. Let $In(R_S)$ be an unicyclic graph. Then clearly R is Artinian and hence $R \cong \prod_{i=1}^{n} R_i$, where each R_i is an Artinian local ring with maximal ideal m_i. Now if $n \geq 3$, then $|Max(R)| \geq 3$. So let, m_1, m_2, $m_3 \in Max(R)$. Then $m_1 \sim m_1 \cup m_2 \sim m_1 \cup m_2 \cup m_3 \sim m_1$ and $m_2 \sim m_1 \cup m_2 \sim m_1 \cup m_2 \cup m_3 \sim m_2$ are two distinct cycles in $In(R_S)$, a contradiction. Therefore $n \leq 2$. Let $n = 2$. If $m_1 = m_2 = 0$, then $In(R_S) \cong P_3$, which contradicts that $In(R_S)$ is unicyclic. So let at least one m_i is nonzero. Without loss of generality, let $m_1 \neq 0$. Then $m_1 \times 0 \sim R_1 \times 0 \sim (R_1 \times 0) \cup (m_1 \times R_2) \sim m_1 \times 0$ and $0 \times R_2 \sim m_1 \times R_2 \sim (R_1 \times 0) \cup (m_1 \times R_2) \sim 0 \times R_2$ are two distinct cycles in $In(R_S)$, a contradiction. Let $n = 1$. Since R is Artinian, let $m_1 = (x_1, x_2, \ldots, x_k)$ for some $k \in \mathbb{N}$. If $k \geq 2$, then $(x_1) \sim (x_1) \cup (x_2) \sim (x_1, x_2) \sim (x_1)$ and $(x_2) \sim (x_1) \cup (x_2) \sim (x_1, x_2) \sim (x_2)$ are two distinct cycles in $In(R_S)$, a contradiction. Let $k = 1$. Then non-trivial ideals of R are of the form $\{m^t : 1 \leq t \leq r - 1\}$, where r is the smallest positive integer such that $m^r = 0$ (*cf.* Proposition 1.2.77). Therefore $In(R_S)$ is unicyclic if and only if $r = 4$. Hence the result follows. □

Corollary 7.1.8. *Let $n \in M$. Then $In((\mathbb{Z}_n)_S)$ is unicyclic if and only if $n = p^4$, where p is a prime number.*

Theorem 7.1.9. *Let R be a ring. Then $In(R_S)$ and it's complement are perfect graph.*

Proof. We first define a digraph $\overrightarrow{In}(R_S)$ with vertex set as non-trivial ideals of R_S and for two distinct vertices I_1 and I_2, there is an arc from I_1 to I_2 if $I_1 \subset I_2$. Clearly $\overrightarrow{In}(R_S)$ is an orientation of $In(R_S)$. To prove that $\overrightarrow{In}(R_S)$ is transitive, let (I_1, I_2), $(I_2, I_3) \in E(\overrightarrow{In}(R_S))$. Then $I_1 \subset I_2 \subset I_3$ and hence $(I_1, I_3) \in E(\overrightarrow{In}(R_S))$. Therefore $\overrightarrow{In}(R_S)$ is transitive. Now if possible, let $I_1 \to I_2 \to \cdots \to I_n \to I_1$ be a cycle in $\overrightarrow{In}(R_S)$. Then (I_1, I_n), $(I_n, I_1) \in E(\overrightarrow{In}(R_S))$ which implies $I_1 = I_n$, a contradiction. Therefore $\overrightarrow{In}(R_S)$ is acyclic and hence $In(R_S)$ is a comparability graph. Since every comparability graph is perfect (*cf.* Theorem 1.3.2) and complements of a perfect graph is perfect (*cf.* Theorem 1.3.3), we have the desired result. □

Proposition 7.1.10. *Let R be a ring such that $|I^*(R)| < \infty$. Then $\omega(In(R_S)) = \chi(In(R_s)) = t$, where t is the number of non-trivial principal ideal of R.*

Proof. Let $P = \{(a_i) : 1 \leq i \leq t\}$ be the set of all distinct non-trivial principal ideals of R_S. Clearly this principal ideals can be rearranged as (a_{r_1}), (a_{r_2}), $\ldots$, (a_{r_t}) such that $(a_{r_i}) \nsubseteq (a_{r_j})$ for $i > j$. Then we define $I_i = (a_{r_1}) \cup \cdots \cup (a_{r_i})$, where $1 \leq i \leq t$. Then $C = \{I_1, I_2, \ldots, I_t\}$ is a clique in $In(R_S)$. Now if possible, let $C \cup I$ is a clique in $In(R_S)$, where $I \neq I_i$ for every $i \in [t]$. But $I = \bigcup_{x \in I}(x) = I_i$ for some $i \in [t]$, which is a contradiction. Therefore there is no clique properly containing C. Now we prove that there is no clique of cardinality strictly greater than t. If possible, let the vertices $J_1,\ J_2,\ \ldots,\ J_t,\ J_{t+1}$ forms a clique in $In(R_S)$. Without loss of generality, let $J_1 \subset J_2 \subset \cdots \subset J_t \subset J_{t+1}$ and $j_i \in J_{i+1} - J_i$, where $0 \leq i \leq t$ and $J_0 = \{0\}$. Then $P_1 = \{(j_i) : 0 \leq i \leq t\}$ is a set of distinct non-trivial principal ideal of R_S of cardinality $t+1$, a contradiction. Therefore C is a clique of largest cardinality in $In(R_S)$ and hence $\omega(In(R_S)) = t$. Since the graph $In(R_S)$ is perfect, we have $\omega(In(R_S)) = \chi(In(R_S)) = t$. □

Since $\mathbb{Z}_n$ is a principal ideal ring and every ideal of $\mathbb{Z}_n$ is of the form $(\overline{m})$, where m is a divisor of n, by Proposition 7.1.10 we have the following immediate result.

Corollary 7.1.11. *Let $n \in M$. Then $\chi(In((\mathbb{Z}_n)_S)) = \omega(In((\mathbb{Z})_S)) = \sigma(n) - 2$, where $\sigma(n)$ is the number of positive divisor of n.*

Theorem 7.1.12. *Let R be a ring. Then $\omega(In(R_S)) < \infty$ if and only if $In(R_S)$ is finite. Moreover, $\omega(In(R_S)) = \chi(In(R_S)) = t$, where t is the number of distinct non-trivial principal ideal of R.*

Proof. Let $\omega(In(R_S)) < \infty$. Then clearly R_S satisfies ascending and descending chain condition on its ideals and hence R_S has finitely many ideals ($cf.$ Theorem 1.2.56). Therefore $In(R_S)$ is finite. The proof of the converse part is clear.
Since R_S has finitely many ideals, clearly $\omega(In(R_S)) = \chi(In(R_S)) = t$, where t is the number of distinct non-trivial principal ideal of R ($cf.$ Theorem 7.1.10). □

Theorem 7.1.13. *Let R be an Artinian ring. Then $In(R_S)$ is a split graph if and only if one the following holds.*
(1) *$R \cong R_1 \times R_2$, where R_1 and R_2 are fields.*
(2) *R is an Artinian local ring such that the maximal ideal is principal.*

Proof. Since R is Artinian, clearly $R \cong \prod_{i=1}^{n} R_i$, where each R_i is an Artinian local ring with maximal ideal m_i. Let $n \geq 3$. Then the set of vertices $v_1 = m_1 \times R_2 \times R_3 \times \cdots \times R_n$, $v_2 = m_1 \times R_2 \times m_3 \times \cdots \times R_n$, $v_3 = R_1 \times m_2 \times R_3 \times \cdots \times R_n$, $v_4 = R_1 \times m_2 \times m_3 \times \cdots \times R_n$ induce a $2K_2$ in $In(R_S)$ and hence not a split graph. Let $n = 2$. If both $m_1 = m_2 = 0$, then $In(R_S) \cong P_3$ and hence a split graph. Now let $m_1 \neq 0$. Then the vertices $v_1 = 0 \times R_2$,

$v_2 = m_1 \times R_2$, $v_3 = m_1 \times 0$, $v_4 = (R_1 \times 0) \cup (0 \times R_2)$ induce a C_4 in $In(R_S)$ and hence not a split graph. Let $n = 1$. Since R is Artinian, let $m = (x_1, x_2, \ldots, x_k)$ for some $k \in \mathbb{N}$. Now if $k \geq 2$, then the vertices (x_1), $(x_1) \cup (x_2)$, $(x_1 + x_2)$ and $(x_1 + x_2) \cup (x_2)$ induce a $2K_2$ in $In(R_S)$ and hence not a split graph. If $k = 1$, then $In(R_S)$ is a complete graph and hence a split graph. Therefore combining all the above cases we have the desired result. □

Corollary 7.1.14. *Let $n \in M$. Then $In((\mathbb{Z}_n)_S)$ is a split graph if and only if $n = p^t$, pq, where p and q are prime numbers.*

By applying the similar process as in Theorem 7.1.13, we have the following results.

Theorem 7.1.15. *For an Artinian ring R, the following statements are equivallent:*
(1) *$In(R_S)$ is a split graph,*
(2) *$In(R_S)$ is a co-graph,*
(3) *$In(R_S)$ is chordal.*

7.2 Planar, outerplanar and ring graphs of $In(R_S)$

Proposition 7.2.1. *If $In(R_S)$ is planar, then R is an Artinian ring. Moreover, $|Min(R)| \leq 3$ and $|Max(R)| = |Spec(R)| \leq 2$..*

Proof. Let $In(R_S)$ be a planar graph. Then any chain of non-trivial ideals of R has length at most four and hence R is an Artinian ring. Since R is Artinian, clearly it has a minimal ideal, say I and every minimal ideal of R is a 0-minimal ideal of R_S. Now if $|Min(R)| \geq 4$, let I_1, I_2, I_3, $I_4 \in Min(R)$. Then the ideals I_1, I_2, I_3, $I_1 \cup I_4$, $I_1 \cup I_3$, $I_1 \cup I_2$, $I_1 \cup I_2 \cup I_3$, $I_1 \cup I_2 \cup I_3 \cup I_4$, $I_1 \cup I_2 \cup I_4$, $I_1 \cup I_3 \cup I_4$ of R_S forms a subdivision of the complete bipartiate graph $K_{3,3}$, contradicts that $In(R_S)$ is planar.
Now if $|Max(R)| \geq 3$, let m_1, m_2, $m_3 \in Max(R)$. Then the vertices $m_1 \cup m_2 \cup m_3$, $m_1 \cup m_2$, m_1, $m_1 \cap m_2$, m_2, $m_2 \cap m_3$, m_3, $m_1 \cap m_3$ forms a subdivision of K_5 in $In(R_S)$, a contradiction. Hence the result follows. □

The following is a characterization of the family of rings for which $In(R_S)$ is planar.

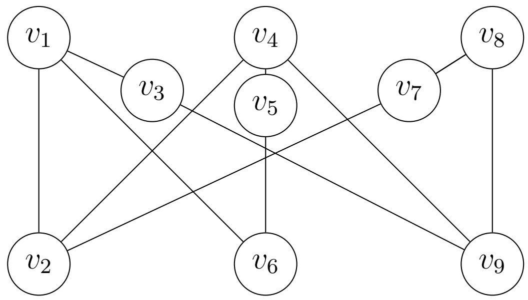


Figure 7.1: A subdivision of $K_{3,3}$

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Theorem 7.2.2. *Let R be a ring. Then $In(R_S)$ is planar if and only if one of the following holds*

(1) *$R \cong R_1 \times R_2$, where R_1 and R_2 are fields.*

(2) *R is an Artinian local ring with maximal ideal $m = (x_1, x_2)$ such that $m^2 = 0$ and $|\frac{R}{m}| = 2$.*

(3) *R is an Artinian local ring with maximal ideal $m = (x)$ such that $2 \leq n(x) \leq 5$.*

Proof. Let $In(R_S)$ be a planar graph. Then R is Artinian and hence $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is an Artinian local ring with maximal ideal m_i. Suppose that $n \geq 3$. Then $In(R_S)$ contain a subdivision of the complete bipartiate graph $K_{3,3}$ with vertices $v_1 = m_1 \times R_2 \times R_3 \times \cdots \times R_n$, $v_2 = m_1 \times m_2 \times R_3 \times \cdots \times R_n$, $v_3 = (m_1 \times R_2 \times R_3 \times \cdots \times R_n) \cup (R_1 \times m_2 \times R_3 \times \cdots \times R_n)$, $v_4 = R_1 \times m_2 \times R_3 \times \cdots \times R_n$, $v_5 = (R_1 \times m_2 \times R_3 \times \cdots \times R_n) \cup (R_1 \times R_2 \times m_3 \times \cdots \times R_n)$, $v_6 = m_1 \times R_2 \times m_3 \times \cdots \times R_n$, $v_7 = (m_1 \times R_2 \times R_3 \times \cdots \times R_n)) \cup (R_1 \times R_2 \times m_3 \times \cdots \times R_n)$, $v_8 = R_1 \times R_2 \times m_3 \times \cdots \times R_n$, $v_9 = R_1 \times m_2 \times m_3 \times \cdots \times R_n$ (see Figure 7.1), contradicts that $In(R_S)$ is planar. Therefore we conclude that $n \leq 2$.

Let $n = 2$. Now if possible, let one of the m_i is non-zero. Without loss of generality, let $m_1 \neq 0$. Then $In(R_S)$ contain a subdivision of the complete graph K_5 with vertices $v_1 = (m_1 \times R_2) \cup (R_1 \times 0)$, $v_2 = 0 \times R_2$, $v_3 = (m_1 \times 0) \cup (0 \times R_2)$, $v_4 = (m_1 \times 0)$, $v_5 = (R_1 \times 0) \cup (0 \times R_2)$, $v_6 = m_1 \times R_2$, a contradiction. Therefore, let $m_1 = m_2 = 0$. Then $In(R_S) \cong P_3$ and hence planar.

Finally, let $n = 1$. Since R is Artinian, let $m = (x_1, x_2, \ldots, x_k)$ for some $k \in \mathbb{N}$. If $k \geq 3$, then $In(R_S)$ contain a subdivision of the complete graph K_5 with vertices $v_1 = (x_1, x_2, x_3)$,

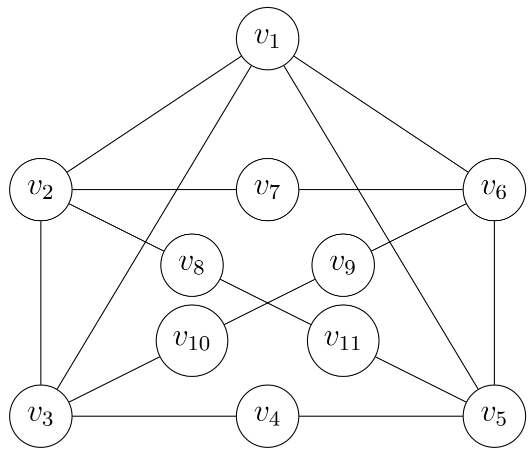

Figure 7.2: A subdivision of K_5

$v_2 = (x_1, x_2)$, $v_3 = (x_2)$, $v_4 = (x_2, x_3)$, $v_5 = (x_3)$, $v_6 = (x_1, x_3)$, $v_7 = (x_1)$, $v_8 = (x_1 + x_2)$, $v_9 = (x_1 + x_3)$, $v_{10} = (x_2, x_1 + x_2)$, $v_{11} = (x_3, x_1 + x_2)$ (*cf.* Figure 7.2), a contradiction. So we assume that $k \leq 2$.

Let $k = 2$. Then $m = (x_1, x_2)$. If $x_2^2 \neq 0$, then the ideals $v_1 = (x_1, x_2)$, $v_2 = (x_1)$, $v_3 = (x_1) \cup (x_1 + x_2)$, $v_4 = (x_1 + x_2)$, $v_5 = (x_1 + x_2, x_2^2)$, $v_6 = (x_2^2)$, $v_7 = (x_2)$, $v_8 = (x_1) \cup (x_2)$, $v_9 = (x_1, x_2^2)$, $v_{10} = (x_2) \cup (x_1 + x_2)$ of R_S forms a subdivision of K_5, a contradiction. Similarly, if $x_1^2 \neq 0$, then we arrive at a contradiction. Therefore $x_1^2 = x_2^2 = 0$. Now if $x_1 x_2 \neq 0$, then the ideals $v_1 = (x_1, x_2)$, $v_2 = (x_1)$, $v_3 = (x_1 x_2)$, $v_4 = (x_1 + x_2)$, $v_5 = (x_2) \cup (x_1 + x_2)$, $v_6 = (x_2)$, $v_7 = (x_1) \cup (x_2)$, $v_8 = (x_1) \cup (x_1 + x_2)$ of R_S forms a subdivision of K_5, a contradiction. Therefore $x_1 x_2 = 0$ and hence $m^2 = 0$. Then non-trivial ideals of R is of the form (x_1), (x_2), (x_1, x_2) and $(x_1 + e x_2)$, where e is a unit of R (*cf.* Proposition 1.2.80). Now if $|\frac{R}{m}| \geq 3$, then the ideals $v_1 = (x_1)$, $x_2 = (x_2)$, $v_3 = (x_1 + x_2)$, $v_4 = (x_1) \cup (x_1 + e x_2)(e \neq 1$ is a unit of $R)$, $v_5 = (x_1) \cup (x_1 + x_2)$, $v_6 = ((x_1) \cup (x_2)$, $v_7 = (x_1) \cup (x_2) \cup (x_1 + x_2)$, $v_8 = (x_1) \cup (x_2) \cup (x_1 + x_2) \cup (x_1 + e x_2)$, $v_9 = (x_1) \cup (x_2) \cup (x_1 + e x_2)$, $v_{10} = (x_1) \cup (x_1 + x_2) \cup (x_1 + e x_2)$ of R_S forms a subdivision of K_5, a contradiction. If $|\frac{R}{m}| = 2$, then non-trivial ideals of R_S are (x_1), (x_2), $(x_1 + x_2)$, (x_1, x_2), $(x_1) \cup (x_2)$, $(x_1) \cup (x_1 + x_2)$, $(x_2) \cup (x_1 + x_2)$. Then it is easy to observe that $In(R_S)$ can be drawn in the plane without crossings of any edges and hence planar.

Lastly, let $k = 1$. Then m is principal and hence ideals are of the form $\{m^t : 1 \leq t \leq r - 1\}$, where r is the smallest positive integer such that $m^r = 0$. Clearly $I^*(R) = I^*(R_S)$ and hence $In(R_S)$ is planar if and only if $2 \leq n(m) \leq 5$. Hence the result follows. $\square$

We know that every outerplanar graph is a ring graph and every ring graph is outerplanar. The following is a characterization of family of ring R for which the graph $In(R_S)$ is ring graph and outerplanar.

Proposition 7.2.3. *Let R be a ring. Then $In(R_S)$ is a ring graph if and only if $In(R_S)$ is an outerplanar graph if and only if one of the following condition holds:*

(1) *$R \cong R_1 \times R_2$, where R_1 and R_2 are fields.*

(2) *R is an Artinian local ring with maximal ideal $m = (x)$ such that $2 \leq n(x) \leq 4$.*

Proof. Let $In(R_S)$ be a ring graph. Since every ring graph is planar, we only consider the case for which $In(R_S)$ is planar ($cf.$ Theorem 7.2.2).

Case (i) Let $R \cong R_1 \times R_2$, where each R_1 and R_2 are fields. Then $In(R_S) \cong P_3$ and hence both outerplanar and ring graph.

Case (ii) R is an Artinian local ring with maximal ideal $m = (x_1, x_2)$ such that $m^2 = 0$ and $|\frac{R}{m}| = 2$. Then $In(R_S)$ contains a subdivision of the complete graph K_4 with vertices $v_1 = (x_1, x_2)$, $v_2 = (x_1)$, $v_3 = (x_1) \cup (x_2)$, $v_4 = (x_2)$, $v_5 = (x_1 + x_2) \cup (x_2)$, $v_6 = (x_1 + x_2)$, $v_7 = (x_1) \cup (x_1 + x_2)$ and hence neither a ring graph nor outerplanar.

Case (iii) If R is an Artinian local ring with maximal ideal $m = (x)$ then $In(R_S)$ is outerplanar if and only if $2 \leq n(m) \leq 4$. $\square$

7.3 Genus of $In(R_S)$

Theorem 7.3.1. *Let R be a ring. Then $In(R_S)$ has finite genus if and only if $I^*(R_S) < \infty$.*

Proof. Let the graph $In(R_S)$ has finite genus. Then clearly R_S satisfies ascending and decending chain condition on its ideals and hence R_S has finitely many ideals ($cf.$ Theorem 1.2.56). Therefore $I^*(R_S) < \infty$. The converse is clear. $\square$

Theorem 7.3.2. *Let R be a non-local ring. Then $In(R_S)$ is*

(1) *toroidal if and only if R is isomorphic to direct product of a field and an Artinian local*

ring with maximal ideal m is principal such that $I^*(R) = \{m\}$.

(2) *never bitoroidal. Moreover, $g(In(R_S) \geq 4$.*

Proof. For finite genus of $In(R_S)$, R must be Artinian (*cf.* Theorem 7.3.1). Therefore $R \cong R_1 \times R_2 \times \cdots \times R_n$ ($n \geq 2$), where each R_i is an Artinian local ring with maximal ideal m_i.

Let us assume that $n \geq 3$. Then $In(R_S)$ contains a subgraph H induced by $v_1 = 0 \times R_2 \times R_3 \times \cdots \times R_n$, $v_2 = R_1 \times 0 \times R_3 \times \cdots \times R_n$, $v_3 = R_1 \times R_2 \times 0 \times \cdots \times R_n$, $v_4 = 0 \times 0 \times R_3 \times \cdots \times R_n$, $v_5 = 0 \times R_2 \times 0 \times \cdots \times R_n$, $v_6 = R_1 \times 0 \times 0 \times \cdots \times R_n$, $v_7 = (R_1 \times 0 \times R_3 \times \cdots \times R_n) \cup (0 \times R_2 \times 0 \times \cdots \times R_n)$, $v_8 = (0 \times 0 \times R_3 \times \cdots \times R_n) \cup (0 \times R_2 \times 0 \times \cdots \times R_n)$, $v_9 = (0 \times 0 \times R_3 \times \cdots \times R_n) \cup (0 \times R_2 \times 0 \times \cdots \times R_n) \cup (R_1 \times 0 \times 0 \times \cdots \times R_n)$, $v_{10} = (0 \times 0 \times R_3 \times \cdots \times R_n) \cup (R_1 \times 0 \times 0 \times \cdots \times R_n)$, $v_{11} = (R_1 \times R_2 \times 0 \times \ldots R_n) \cup (0 \times 0 \times R_3 \times \cdots \times R_n)$, $v_{12} = (0 \times R_2 \times R_3 \times \ldots R_n) \cup (R_1 \times R_2 \times 0 \times \cdots \times R_n)$, $v_{13} = (0 \times R_2 \times R_3 \times \cdots \times R_n) \cup (R_1 \times 0 \times R_3 \times \cdots \times R_n) \cup (R_1 \times R_2 \times 0 \times \cdots \times R_n)$, $v_{14} = (R_1 \times 0 \times R_3 \times \ldots R_n) \cup (R_1 \times R_2 \times 0 \times \cdots \times R_n)$, $v_{15} = (0 \times R_2 \times 0 \times \ldots R_n) \cup (R_1 \times 0 \times 0 \times \cdots \times R_n)$, $v_{16} = (0 \times R_2 \times R_3 \times \ldots R_n) \cup (R_1 \times 0 \times 0 \times \cdots \times R_n)$, $v_{17} = (0 \times R_2 \times R_3 \times \ldots R_n) \cup (R_1 \times 0 \times R_3 \times \cdots \times R_n)$. This subgraph H has 17 vertices and greater or equals to 94 edges. Hence $g(In(R_S)) \geq g(H) \geq 9$ (*cf.* Lemma 1.3.7). Therefore if $n \geq 3$, then $In(R_S)$ is neither toroidal nor bitoroidal.

Let $n = 2$. Now if $m_1 = m_2 = 0$, then $In(R_S)$ is planar (*cf.* Theorem 7.2.2). So, let at least one m_i is non-zero. Without loss of generality, let $m_1 \neq 0$. Now if $m_2 \neq 0$, then $In(R_S)$ contains a subgraph H' induced by $v_1 = R_1 \times m_2$, $v_2 = m_1 \times R_2$, $v_3 = R_1 \times 0$, $v_4 = m_1 \times m_2$, $v_5 = m_1 \times 0$, $v_6 = 0 \times R_2$, $v_7 = 0 \times m_2$, $v_8 = (R_1 \times m_2) \cup (m_1 \times R_2)$, $v_9 = (R_1 \times m_2) \cup (0 \times R_2)$, $v_{10} = (m_1 \times R_2) \cup (R_1 \times 0)$, $v_{11} = (R_1 \times 0) \cup (m_1 \times m_2)$, $v_{12} = (R_1 \times 0) \cup (0 \times R_2)$, $v_{13} = (R_1 \times 0) \cup (0 \times m_2)$, $v_{14} = (m_1 \times m_2) \cup (0 \times R_2)$, $v_{15} = (m_1 \times 0) \cup (0 \times R_2)$, $v_{16} = (m_1 \times 0) \cup (0 \times m_2)$. This subgraph H' has 16 vertices and greater or equals to 87 edges. Therefore $g(In(R_S)) \geq g(H') \geq 8$ (*cf.* Lemma 1.3.7)

Also, if I_1 is a non-trivial ideal of R_1 other than m_1, then $In(R_S)$ contains a subgraph H'' induced by $v_1 = R_1 \times 0$, $v_2 = m_1 \times R_2$, $v_3 = m_1 \times 0$, $v_4 = I_1 \times R_2$, $v_5 = I_1 \times 0$, $v_6 = 0 \times R_2$, $v_7 = (R_1 \times 0) \cup (m_1 \times R_2)$, $v_8 = (R_1 \times 0) \cup (I_1 \times R_2)$, $v_9 = (R_1 \times 0) \cup (0 \times R_2)$, $v_{10} = (m_1 \times 0) \cup (I_1 \times R_2)$, $v_{11} = (m_1 \times 0) \cup (0 \times R_2)$, $v_{12} = (I_1 \times 0) \cup (0 \times R_2)$. This subgraph H'' has 12 vertices and greater or equals to 49 edges. Therefore $g(In(R_S)) \geq 4$ (*cf.* Lemma

1.3.7).

Finally, consider the case when m_1 is the only nontrivial ideal of R_1 and $m_2 = 0$. Then nontrivial ideals of R_S are $v_1 = R_1 \times 0$, $v_2 = m_1 \times R_2$, $v_3 = m_1 \times 0$, $v_4 = 0 \times R_2$, $v_5 = (R_1 \times 0) \cup (m_1 \times R_2)$, $v_6 = (R_1 \times 0) \cup (0 \times R_2)$, $v_7 = (m_1 \times 0) \cup (0 \times R_2)$. Clearly $In(R_S)$ is non-planar (cf. Theorem 7.2.2) and hence $g(In(R_S)) \geq 1$. Since $In(R_S)$ has seven vertices, we have $g(In(R_S)) \leq g(K_7) = 1$. Hence $g(In(R_S)) = 1$. Therefore combining all the above cases we have the desired result. □

By applying Proposition 1.2.77 we have the following immediate result.

Theorem 7.3.3. *Let R be an local ring such that the maximal ideal m is principal. Then $In(R_S)$ is toroidal if and only if $6 \leq n(m) \leq 8$ and bitoroidal if and only if $n(x) = 9$.*

Corollary 7.3.4. *Let $n \in M$. Then $In((\mathbb{Z}_n)_S)$ is toroidal if and only if $n = p^r$ $(6 \leq r \leq 8)$, p^2q and bitoroidal if and only if $n = p^9$, where p and q are distinct prime numbers.*

Chapter 8

Prime inclusion ideal graph of a semigroup

In this chapter, we consider the prime inclusion ideal graph $In_p(S)$ of a commutative semigroup S, is a graph with vertices are nontrivial prime ideals of S and two distinct vertices P_1, P_2 are adjacent if and only if $P_1 \subset P_2$ or $P_2 \subset P_1$. Here we characterize a commutative semigroup S for which the graph $In_p(S)$ is empty, null and complete. Then we study various graph parameters of the prime inclusion ideal graph $In_p(\mathbb{Z}_n)$ of the multiplicative semigroup $\mathbb{Z}_n$ of integers modulo n.

In **Section** 8.1, we characterize a commutative semigroup S for which $In_p(S)$ is null ($cf.$ Theorem 8.1.3), connected ($cf.$ Theorem 8.1.6) and complete ($cf.$ Theorem 8.1.7).

In **Section** 8.2, we determine the prime ideals of the multiplicative semigroup $\mathbb{Z}_n$ of integers modulo n ($cf.$ Theorem 8.2.1). Then we compute the degree of a vertex ($cf.$ Theorem 8.2.6), degree sequence ($cf.$ Theorem 8.2.9), vertex connectivity ($cf.$ Theorem 8.2.13), vertex cover number ($cf.$ Theorem 8.2.18) of the graph $In_p(\mathbb{Z}_n)$ and also determine the values of n for which $In_p(\mathbb{Z}_n)$ has thickness 1 and 2 ($cf.$ Theorem 8.2.22). We determine the metric dimension ($cf.$ Theorem 8.2.24), strong metric dimension ($cf.$ Theorem 8.2.25) and partition dimension ($cf.$ Theorem 8.2.26) of the graph $In_p(\mathbb{Z}_n)$.

Throughout this chapter, by a semigroup S we mean a commutative semigroup.

8.1 Prime inclusion ideal graph of semigroup

It is well known that a commutative semigroup S is archimedian if and only if it has no proper prime ideals (cf. Theorem 1.2.58). So we have the following immediate result.

Theorem 8.1.1. *Let S be a semigroup. Then the following statements are equivalent*

(1) *$In_p(S)$ is an null graph.*

(2) *S is an archimedian semigroup.*

Example 8.1.2. It is easy to observe that non-trivial ideals of the monogenic semigroup $S_M = \{0, x, x^2, \ldots, x^{m-1}\}$ with zero element is of the form $I_t = \{0, x^t, \ldots, x^{m-1}\}$, where t is an integer such that $2 \leq t \leq m-1$. Therefore S_M has no nontrivial prime ideals and consequently $In_p(S_M)$ is an empty graph.

Theorem 8.1.3. *Let S be a semigroup with zero element. Then $In_p(S)$ is a null graph if and only if each nontrivial prime ideal is minimal.*

Proof. Let S be a semigroup with zero such that $In_p(S)$ is a null graph. Now if S has exactly one nontrivial prime ideal then it is clear. So let P_1 and P_2 be two distinct nontrivial prime ideals of S. If P_1 is not minimal, then there exists a prime ideal P of S such that $0 \neq P \subset P_1$. Hence $P \sim P_1$, which contradicts that $In_p(S)$ is a null graph. Consequently, each nontrivial prime ideal of S is minimal. The proof of the converse part is obvious. $\square$

Example 8.1.4. Let us consider the semigroup $S = \{0, a, b, ab : a^2 = a, b^2 = b, ab = ba\}$ with zero element. The only prime ideals of S are $\{0, a, ab\}$ and $\{0, b, ab\}$, both are minimal prime. Consequently $In_p(S)$ is a null graph.

We know that if a semigroup has unity but not a group, then it has unique maximal ideal, which is prime also. So we have the following immediate result.

Corollary 8.1.5. *Let S be a semigroup with unity. Then the following statements are equivalent:*

(1) *$In_p(S)$ is a null graph,*

(2) *Each nontrivial prime ideals of S are maximal,*

(3) *S has unique nontrivial prime ideal.*

Since every ideal in a semigroup S with unity is contained in a unique maximal ideal of S, we have the following result:

Theorem 8.1.6. *The prime inclusion ideal graph $In_p(S)$ of a semigroup S with unity is connected. Moreover, diam $(In_p(S)) \leq 2$.*

Theorem 8.1.7. *Let S be a semigroup. Then the following statements are equivalent:*
(1) *$In_p(S)$ is a complete graph,*
(2) *S is a semiprimary semigroup,*
(3) *Prime ideals of S are linearly ordered,*
(4) *Semiprime ideals of S are linearly ordered,*
(5) *Semiprime ideals of S are prime,*
(6) *For each a, $b \in S$, there exists $k \in \mathbb{N}$ such that $a|b^k$ or $b|a^k$.*

Proof. (1) $\Leftrightarrow$ (2) Let I be a semiprimary ideal of S. Then $\sqrt{I} = \bigcap P_\alpha$ ($cf.$ Theorem 1.2.23), where P_α's are minimal prime ideal of S containing I. Since the graph $In_p(S)$ is complete we have $\sqrt{I} = P$ for some prime ideals P of S and hence S is a semiprimary semigroup.
Conversely, let P_1, $P_2 \in V(In_p(S))$ but $P_1 \nsim P_2$. Then there exist x, $y \in S$ such that $x \in P_1 - P_2$ and $y \in P_2 - P_1$. Then $xy \in P_1 \cap P_2 = \sqrt{P_1P_2}$, a prime ideal of S as S is semiprimary. Hence $x \in P_1 \cap P_2$ or $y \in P_1 \cap P_2$, which is a contradiction. Consequently $In_p(S)$ is a complete graph.
(2) $\Leftrightarrow$ (3) It follows from Theorem 1.2.38.
(3) $\Leftrightarrow$ (4) Let I_1 and I_2 be two semiprime ideals of S. Clearly $I_1 \cap I_2$ is a semiprime ideal of S. Then $I_1 \cap I_2 = \sqrt{I_1 \cap I_2}$ is a prime ideal of S. Consequently semiprime ideals are linearly ordered. Converse is clear.
(4) $\Leftrightarrow$ (5) Let I be a semiprime ideal of S. Then $I = \sqrt{I}$ is a prime ideal of S. For converse part, let I_1 and I_2 are two distinct semiprime ideals of S. Clearly $I_1 \cap I_2$ is a prime ideal of S as semiprime ideals are prime. Now in a similar way as in the proof of above cases it is clear that semiprime ideals are linearly ordered.
(3) $\Leftrightarrow$ (6) Let a, $b \in S$ and prime ideals of S are linearly ordered. Then $\sqrt{(a)} \subseteq \sqrt{(b)}$ or $\sqrt{(b)} \subseteq \sqrt{(a)}$ which implies either $a^s \in (b)$ or $b^t \in (a)$ for some s, $t \in \mathbb{N}$. Let $k = max\{s, t\}$. Therefore either $a|b^k$ or $b|a^k$.

Conversely, let P_1 and P_2 be two distinct prime ideals of S and $ab \in P_1 \cap P_2$ for some a, $b \in S$. Then $ab \in P_1$ and $ab \in P_2$ which implies $a \in P_1$ or $b \in P_1$ and $a \in P_2$ or $b \in P_2$. Now if $a \in P_1 \cap P_2$ or $b \in P_1 \cap P_2$ then it is clear that $P_1 \subset P_2$ or $P_2 \subset P_1$. Now we consider the remaining cases $a \in P_1$ and $b \in P_2$ or $a \in P_2$ and $b \in P_1$. Without loss of generality let $a \in P_1$ and $b \in P_2$. Then by assumption there exists $k \in \mathbb{N}$ such that $a|b^k$ or $b|a^k$. Now if $a|b^k$ then $b^k = as \in P_1$ for some $s \in S$, implies $b \in P_1$ and hence $b \in P_1 \cap P_2$. Similarly $b|a^k$ implies $a \in P_1 \cap P_2$. Consequently $P_1 \cap P_2$ is a prime ideal of S. Hence the result follows. □

8.2 The prime inclusion ideal graph of the semigroup $\mathbb{Z}_n$

Theorem 8.2.1. *Every non-trivial prime ideal P of $\mathbb{Z}_n$ is of the form $P = \cup\{p_i\mathbb{Z}_n : i \in [k]\}$, where $p_1, p_2, \dots p_k$ are k distinct prime divisor of n.*

Proof. It is well known that a non-zero ideal of $\mathbb{Z}_n$ is of the form $\cup\{m_i\mathbb{Z}_n : i \in [k]\}$, where $m_1, m_2, \dots m_k$ are divisors of n such that $m_i \nmid m_j$ if $i \neq j$ (cf. Theorem 1.2.61). Let P be a non-trivial prime ideal of $\mathbb{Z}_n$. Then $P \in \cup\{m_i\mathbb{Z}_n : i \in [k]\}$, where $m_1, m_2, \dots m_k$ are divisors of n such that $m_i \nmid m_j$ if $i \neq j$. Now in the expression of P if one of m_i's is composite, say m_t, then $m_t = m_a m_b$ for some proper divisor m_a and m_b of m_t. Then $m_a m_b \in P$ but neither $m_a \in P$ nor $m_b \in P$, contradicts that P is a prime ideal of $\mathbb{Z}_n$. Since ideal generated by prime divisor of n is a prime ideal and union of any collection of prime ideals of a semigroup is a prime ideal, we have $P = \cup\{p_i\mathbb{Z}_n : i \in [k]\}$, where p_i's ($i \in [k]$) are distinct prime divisor of n. Hence the result follows. □

Proposition 8.2.2. *The number of non-trivial prime ideal of $\mathbb{Z}_n$ is equal to $2^k - 1$, where $n = \prod\limits_{i=1}^{k} p_i^{\alpha_i}$.*

Proof. The non-trivial prime ideal of $\mathbb{Z}_n$ is of the form $\cup\{p_i\mathbb{Z}_n : i \in [k]\}$ (cf. Theorem 8.2.1), where p_i's ($i \in [k]$) are distinct prime divisor of n. Therefore the number of non-trivial prime ideals of $\mathbb{Z}_n$ is

$= {}^kC_1 + {}^kC_2 + \cdots + {}^kC_k = 2^k - 1.$ □

Corollary 8.2.3. *The order of the graph $In_p(Z_n)$ is equal to $2^k - 1$, where $n = \prod_{i=1}^{k} p_i^{\alpha_i}$.*

Let $\Lambda = \{1, 2, \dots, t\} \subseteq [k]$. We use the sign P_Λ or $P_{12\dots t}$ to denote the prime ideal $(p_1) \cup (p_2) \cup \dots \cup (p_t)$ of $\mathbb{Z}_n$. Clearly for $k = 1$, $In_p(\mathbb{Z}_n) \cong K_1$.
So throughout this chapter we consider the graph $In_p(\mathbb{Z}_n)$, where $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ with $k \geq 2$.
Since the multiplicative semigroup $\mathbb{Z}_n$ has identity, by Theorem 8.1.6 we have the following immediate result.

Corollary 8.2.4. *The graph $In_p(\mathbb{Z}_n)$ is a connected with diam$(In_p(\mathbb{Z}_n)) = 2$.*

Since the proof of the following results are immediate so we omit the proof.

Theorem 8.2.5. (1) *The girth of $In_p(\mathbb{Z}_n)$ is given by*

$$gr(In_p(\mathbb{Z}_n)) = \begin{cases} \infty \ , & \text{if } k = 2 \\ 3, & \text{if } k \geq 3. \end{cases}$$

(2) *The graph is triangulated if and only if $k \geq 3$.*
(3) *The graph is k-partiate if and only if $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ with $k \geq 2$.*

Theorem 8.2.6. *The degree of a vertex P_Λ such that $|\Lambda| = t(t \in [k])$ is deg$(P_\Lambda) = 2^t + 2^{k-t} - 3$.*

Proof. Let P_Λ be any vertex of $In_p(\mathbb{Z}_n)$ such that $|\Lambda| = t(t \in [k])$. Then the number of non-trivial prime ideals properly contained in P_Λ is
$= {}^tC_1 + {}^tC_2 + \dots + {}^tC_{t-1} = 2^t - 2$.
Also the number of nontrivial prime ideals properly containing P_Λ is
$= {}^{k-t}C_1 + {}^{k-t}C_2 + \dots + {}^{k-t}C_{k-t} = 2^{k-t} - 1$.
Therefore the total number of vertices adjacent to P_Λ is $= 2^t - 2 + 2^{k-t} - 1 = 2^t + 2^{k-t} - 3$.
Therefore $\deg(P_\Lambda) = 2^t + 2^{k-t} - 3$. □

It is well known that a simple connected graph G is Eulerian if and only if every vertex of G is of even degree. Since for $k \neq t$, $\deg(P_\Lambda) = 2^t + 2^{k-t} - 3$ is an odd number, where $|\Lambda| = t$, we have the following immediate result

Corollary 8.2.7. *The graph $In_p(\mathbb{Z}_n)$ is not Eulerian.*

Lemma 8.2.8. *Let P_{Λ_1} and P_{Λ_2} be any two vertex of $In_p(\mathbb{Z}_n)$. Then deg(P_{Λ_1}) =deg(P_{Λ_2}) if and only if $|\Lambda_1| = |\Lambda_2|$ or $|\Lambda_1| + |\Lambda_2| = k$.*

Proof. Let $|\Lambda_1| = s$ and $|\Lambda_2| = t$.
Then deg(P_{Λ_1}) =deg(P_{Λ_2}) $\Rightarrow 2^s + 2^{k-s} - 3 = 2^t + 2^{k-t} - 3 \Rightarrow 2^s - 2^t = \frac{2^k(2^s-2^t)}{2^{s+t}}$.
So either $2^s = 2^t$ or $2^s \neq 2^t$.
If $2^s = 2^t$, then $s = t \Rightarrow |\Lambda_1| = |\Lambda_2|$.
Again if $2^s \neq 2^t$, then $2^k = 2^{s+t} \Rightarrow k = s + t \Rightarrow |\Lambda_1| + |\Lambda_2| = k$.
Conversely, If $|\Lambda_1| = |\Lambda_2|$, then clearly deg($P_{\Lambda_1}$) = deg ($P_{\Lambda_2}$).
Now let $|\Lambda_1| + |\Lambda_2| = k \Rightarrow s + t = k$
Therefore deg(P_{Λ_2}) $= 2^t + 2^{k-t} - 3 = 2^{k-s} + 2^s - 3 =$ deg (P_{Λ_1}). □

Theorem 8.2.9. *The maximum and minimum degrees of $In_p(\mathbb{Z}_n)$ are $\Delta(In_p(\mathbb{Z}_n)) = 2^k - 2$ and $\delta(In_p(\mathbb{Z}_n)) = 2^{t+1} - 3$ (if $k = 2t$) and $\delta(In_p(\mathbb{Z}_n)) = 3(2^t - 1)$ (if $k = 2t + 1$). Moreover, the degree sequence $DS(In_p(\mathbb{Z}_n))$ is*
$2^t + 2^{k-t} - 3$ (${}^kC_t$ times), $\ldots, 2 + 2^{k-1} - 3$ ($2.{}^kC_1$ times), $2^k - 2$ when $k = 2t$ and
$2^t + 2^{k-t} - 3$ ($2.{}^kC_t$ times), $\ldots, 2 + 2^{k-1} - 3$ ($2.{}^kC_1$ times), $2^k - 2$ when $k = 2t + 1$.

Proof. Since the vertex $P_{12\ldots k}$ is adjacent to every other vertex in $In_p(\mathbb{Z}_n)$, we have $\Delta(In_p(\mathbb{Z}_n))$ =deg($P_{12\ldots k}$) $= 2^k - 2$. To find the minimum degree vertices in $In_p(\mathbb{Z}_n)$, we consider the function $f : [1, k] \longrightarrow \mathbb{R}$ defined by

$$f(x) = 2^x + 2^{k-x} - 3 \tag{8.2.1}$$

Now $f'(x) = 0 \Rightarrow \frac{2^x}{log_e 2} - \frac{2^{k-x}}{log_e 2} = 0 \Rightarrow x = \frac{k}{2}$.
Also $f''(x) = \frac{2^x}{(log_e 2)^2} + \frac{2^{k-x}}{(log_e 2)^2}$. Hence $f''(\frac{k}{2}) = \frac{2^{\frac{k}{2}+1}}{(log_e 2)^2} > 0$. Therefore f has a minimimun value at $x = \frac{k}{2}$. Since we are interseted in integer solutions and also by combining Lemma 8.2.8, it is easy to observe that vertices of the form P_Λ such that $|\Lambda| = \frac{k}{2}$ (if k is even) and $|\Lambda| = \frac{k+1}{2}$ or $\frac{k-1}{2}$ (if k is odd) have minimum degrees. So we have $\delta(In^p(\mathbb{Z}_n)) = 2^{t+1} - 3$ (if $k = 2t$) and $\delta(In^p(\mathbb{Z}_n)) = 3(2^t - 1)$ (if $k = 2t + 1$).
It is clear to observe that for $m > n$ in $[1, t]$ we have $f(m) < f(n)$ and for $a > b$ in $[t, k]$

we have $f(a) > f(b)$ when $k = 2t$. Also for $a > b$ in $[p+1, k]$ we have $f(a) > f(b)$ and $f(t) = f(t+1)$ when $k = 2t+1$. $\square$

Therefore by Theorem 8.2.6 and Lemma 8.2.8, we have the required degree sequence.

Corollary 8.2.10. *The irregularity index of $In_p(\mathbb{Z}_n)$ is $MWB(In_p(\mathbb{Z}_n)) = t+1$, where $k = 2t$ or $2t+1$.*

Now we are interested in finding the number of edges of $In_p(\mathbb{Z}_n)$.

Theorem 8.2.11. *The number of edges of $In_p(\mathbb{Z}_n)$ is given by the equation $2|E_n| = \sum\limits_{i=1}^{k} {}^kC_i(2^i + 2^{k-i}) - 3|V_n|$, where $|V_n|$ and $|E_n|$ denotes the number of vertices and edges of $In_p(\mathbb{Z}_n)$ respectively.*

Proof. We know that sum of degrees of vertices of a graph is twice the number of edges, hence the total no of edges of $In_p(\mathbb{Z}_n)$ is

$$2|E_n| = {}^kC_1(2 + 2^{k-1} - 3) + {}^kC_2(2^2 + 2^{k-2} - 3) + \cdots + {}^kC_k(2^k + 2^0 - 3)$$

$$= \sum_{i=1}^{k} {}^kC_i(2^i + 2^{k-i}) - 3(2^k - 1) = \sum_{i=1}^{k} {}^kC_i(2^i + 2^{k-i}) - 3|V_n|$$

Hence the result follows. $\square$

Example 8.2.12. Let $n = \prod\limits_{i=1}^{3} p_i^{\alpha_i}$, then $k = 3$ and $|V_n| = 7$. Hence by Theorem 8.2.11 we have $2|E_n| = \sum\limits_{i=1}^{3} {}^3C_i(2^i + 2^{3-i}) - 3.7 = 24 \Rightarrow |E_n| = 12$.

Theorem 8.2.13. *The vertex connectivity of $In_p(\mathbb{Z}_n)$ is $\kappa(In_p(\mathbb{Z}_n)) = 2^t + 2^{k-t} - 3$, where $k = 2t$ or $2t+1$, $t \in \mathbb{N}$.*

Proof. Let $k = 2t$ or $2t+1$. Then the minimum degree of $In_p(\mathbb{Z}_n)$ is $2^t + 2^{k-t} - 3$, in fact, every vertex of the form $\{P_\Lambda : |\Lambda| = t\}$ is of minimum degree. Now consider the set $N(P_\Lambda)$ of neighborhoods of any vertex of the form $\{P_\Lambda : |\Lambda| = t\}$. Clearly $|N(P_\Lambda)| = 2^t + 2^{k-t} - 3$. We claim that $N(P_\Lambda)$ is a minimal vertex cut of $In_p(\mathbb{Z}_n)$. It is easy to observe that $In_p(\mathbb{Z}_n)$ has two components C_1 and C_2 where $C_1 = \{P_\Lambda\}$ and C_2 is the induced subgraph of $In_p(\mathbb{Z}_n)$ with vertex set $V(In_p(\mathbb{Z}_n)) - \{N(P_\Lambda) \cup P_\Lambda\}$. Now if possible let $In_p(\mathbb{Z}_n) - S$ is a disconnected graph where $S \subset N(P_\Lambda)$ be a vertex cut of $In_p(\mathbb{Z}_n)$. Then there exists $P_{\Lambda_1} \in N(P_\Lambda) - S$ and hence $P_{\Lambda_1} \sim P_\Lambda$. Now it is easy to observe that $In_p(\mathbb{Z}_n) - S$ is a connected graph, a contradiction. Hence the result follows. $\square$

Theorem 8.2.14. *The graphs $In_p(\mathbb{Z}_n)$ and $In_p(\mathbb{Z}_m)$ are isomorphic if and only if n and m have same number of prime factors.*

Proof. Let the two graphs $In_p(\mathbb{Z}_n)$ and $In_p(\mathbb{Z}_m)$ are isomorphic but n and m have different number of prime factors. So without loss of generality we assume $s > t$, where s and t are number of prime factors of n and m respectively. Then $|V(In_p(\mathbb{Z}_n)| > |V(In_p(\mathbb{Z}_m)|$, a contradiction. Therefore n and m have same number of prime factors.
Conversely, Let n and m have same number of prime factors. Now we define a map f : $In_p(\mathbb{Z}_n) \longrightarrow In_p(\mathbb{Z}_m)$ by $f(P_\Lambda) = Q_\Lambda$, where Λ is an indexed subset of $[s] = [t]$ and Q_Λ is the prime ideal $\bigcup_{i=1}^{t}(q_i)$. One can easily see that f is a bijection and two vertices in $In_p(\mathbb{Z}_n)$ are adjacent if and only if their f-images are adjacent in $In_p(\mathbb{Z}_m)$. Hence the two graphs are isomorphic. $\square$

Lemma 8.2.15. *Let Λ_1 and Λ_2 be two indexed subset of $[k]$. If $|\Lambda_1| = |\Lambda_2|$, then $P_{\Lambda_1} \nsim P_{\Lambda_2}$.*

Proof. For different Λ_1 and Λ_2 there exists $s \in \Lambda_1 - \Lambda_2$ and $t \in \Lambda_2 - \Lambda_1$. Hence $P_{\Lambda_1} \nsim P_{\Lambda_2}$. $\square$

Theorem 8.2.16. *The clique number of the graph $In_p(\mathbb{Z}_n)$ is $\omega(In_p(\mathbb{Z}_n)) = k$.*

Proof. Clearly $L = \{P_1, P_{12}, \ldots, P_{12\ldots k}\}$ is a clique of size k in $In_p(\mathbb{Z}_n)$. If possible let $L \cup \{P\}$ be a clique containing L. Then $P = P_\Lambda$, where Λ is an indexed subset of $[k]$. Since P_Λ is different from elements of L, we have either $1 \notin \Lambda$ or there exists s, $t \in [k]$ with $s < t$ and $s \notin \Lambda$ but $t \in \Lambda$. Now if $1 \in \Lambda$, then $P_1 \nsim P_\Lambda$, contradicts that $L \cup \{P\}$ is a clique in $In_p(\mathbb{Z}_n)$. Again considering the second case we have $P_{12\ldots s} \nsim P_\Lambda$, a contradiction. Hence L is a maximal clique in $In_p(\mathbb{Z}_n)$.
Now if possible let L' be a clique of size greater than equals to $k+1$. Then there exists i, $J \in L'$ such that $I = P_{\Lambda_1}$ and $J = P_{\Lambda_2}$ where $|\Lambda_1| = |\Lambda_2| = t$ for some $1 \leq t \leq k$. Hence from Lemma 8.2.15, we have $I \nsim J$, a contradiction. Therefore $\omega(In_p(\mathbb{Z}_n)) = k$. $\square$

Theorem 8.2.17. *The chromatic number of $In_p(\mathbb{Z}_n))$ is $\chi(In_p(\mathbb{Z}_n)) = k$.*

Proof. Since the inclusion ideal graph is perfect (*cf.* Corollary 6.1.3), clearly $In_p(\mathbb{Z}_n)$ is perfect. Therefore we have $\chi(In_p(\mathbb{Z}_n)) = \omega(In_p(\mathbb{Z}_n)) = k$.
Also we know $\chi(In_p(\mathbb{Z}_n)) \geq \omega(In_p(\mathbb{Z}_n)) = k$. Here we demonstrate a proper k-coloring of $In_p(\mathbb{Z}_n)$. Let us define $A_i = \{P_\Lambda : \Lambda \text{ is an indexed subset of cardinality } i \in [k]\}$. Now put the colour i to the set of vertices in A_i, this is a proper k-coloring. Hence $\chi(In_p(\mathbb{Z}_n)) \leq k$. Consequently, $\chi(In_p(\mathbb{Z}_n)) = \omega(In_p(\mathbb{Z}_n)) = k$. □

A vertex cover of a graph G is a set of vertices that covers all the edges of G. The minimum cardinality of a vertex cover in G is called the vertex cover number of G, denoted by $\tau(G)$.

Theorem 8.2.18. *The vertex cover number of $In_p(\mathbb{Z}_n)$ is given by $\tau(In_p(\mathbb{Z}_n)) = 2^k - 1 -{}^kC_r$, where $k = 2r$ or $2r+1$ and $r \in \mathbb{N}$.*

Proof. First we need to define a cover set C for the graph $In_p(\mathbb{Z}_n)$. Now to cover highest number of edges with less number of vertices, we need to start adding vertices in C one by one that having highest number of neighbourhoods.
Since the vertex $P_{12\ldots k}$ is of highest degree it must be in C. Now we look into the vertices which is of second highest degree and these vertices is of the form $\{P_\Lambda : |\Lambda| = 1 \text{ or } k-1\}$ (*cf.* Theorem 8.2.9). Now to cover the rest of the edges incident with the vertices $\{P_\Lambda : |\Lambda| = 1\}$ and $\{P_\Lambda : |\Lambda| = k-1\}$, we have to add all these vertices. By continuing this way, the final step of adding the vertices in C is the set of vertices $\{P_\Lambda : |\Lambda| = r-1 \text{ and } r+1\}$ if k is even or $\{P_\Lambda : |\Lambda| = r \text{ or } r+1\}$ if k is odd. Hence the vertex cover of $In_p(\mathbb{Z}_n)$ is given by $C = V(In_p(\mathbb{Z}_n)) - \{P_\Lambda : |\Lambda| = r\}$. Consequently the vertex cover number of $In_p(\mathbb{Z}_n)$ is given by
$\tau(In_p(\mathbb{Z}_n) = {}^kC_1 + \cdots + {}^kC_{r-1} + {}^kC_{r+1} + \ldots {}^kC_k = \sum\limits_{i=1}^{k} {}^kC_i -^k C_r = 2^k - 1 -^k C_r.$ □

We know a set I is independent if and only if it's complement is a vertex cover. Hence the number of vertices of a graph G is the sum of independence number and vertex cover number of G. So we have the following immediate result.

Corollary 8.2.19. *The independence number of $In_p(\mathbb{Z}_n)$ is ${}^kC_r$, where $k = 2r$ or $2r+1$.*

If $k=2$ then $In_p(\mathbb{Z}_n) \cong P_3$, a path of length three and hence not a hamiltonian graph. If $k=3$ then $P_1 \sim P_{12} \sim P_2 \sim P_{23} \sim P_3 \sim P_{13} \sim P_{123} \sim P_1$ is a Hamiltonian cycle in $In_p(\mathbb{Z}_n)$ and if $k=4$ then $P_1 \sim P_{12} \sim P_{123} \sim P_{13} \sim P_3 \sim P_{23} \sim P_{234} \sim P_{34} \sim P_4 \sim P_{24} \sim P_2 \sim P_{124} \sim P_{14} \sim P_{134} \sim P_{1234} \sim P_1$ is a Hamiltonian cycle in $In_p(\mathbb{Z}_n)$. Again if $k=5$ then $P_1 \sim P_{12} \sim P_{123} \sim P_{1234} \sim P_{124} \sim P_{24} \sim P_2 \sim P_{23} \sim P_{234} \sim P_{2345} \sim P_{235} \sim P_{35} \sim P_3 \sim P_{34} \sim P_{345} \sim P_{1345} \sim P_{134} \sim P_{14} \sim P_4 \sim P_{45} \sim P_{145} \sim P_{1245} \sim P_{245} \sim P_{25} \sim P_5 \sim P_{15} \sim P_{125} \sim P_{1235} \sim P_{135} \sim P_{13} \sim P_{12345} \sim P_1$ is a Hamiltonian cycle in $In_p(\mathbb{Z}_n)$.

Theorem 8.2.20. *The following statements about $In_p(\mathbb{Z}_n)$ are equivalent:*
(1) *$In_p(\mathbb{Z}_n)$ is an outer-planar graph,*
(2) *$k=2$,*
(3) *$In_p(\mathbb{Z}_n)$ is a ring graph.*

Proof. (1) $\Leftrightarrow$ (2) If $k=2$, then $In_p(\mathbb{Z}_n) \cong P_3$, a path of order three and hence outer-planar. Now it is easy to observe from Figure 8.1 with $v_1 = P_1$, $v_2 = P_{13}$, $v_3 = P_3$, $v_4 = P_{23}$, $v_5 = P_{123}$, $v_6 = P_{12}$, $v_7 = P_2$ that if $k=3$, then $In_p(\mathbb{Z}_n)$ contains a subdivision of the complete graph K_4 and hence not an outer-planar graph. Again if $k \geq 4$, then the induced subgraph formed by the set of vertices $\{P_1, P_{12}, P_{123}, P_{1234}\}$ is the complete graph K_4 and hence not outer-planar.
(2) $\Leftrightarrow$ (3) Since every outer-planar graph is a ring graph, it is clear that $In_p(\mathbb{Z}_n)$ is a ring graph for $k=2$. Now if $k=3$, then it is clear that rank$(In_p(\mathbb{Z}_n)) = 6 \neq 7 =$frank$(In_p(\mathbb{Z}_n))$ ($cf.$ Figure 8.1) and hence not a ring graph ($cf.$ Lemma 1.3.13). Also it is clear that for $k \geq 3$, $In_p(\mathbb{Z}_n)$ contains a subdivision of the complete graph K_4 and hence not a ring graph. □

Theorem 8.2.21. *The graph $In_p(\mathbb{Z}_n)$ is planar if and only if $2 \leq k \leq 3$.*

Proof. Since every ring graph is planar, it is clear that $In_p(\mathbb{Z}_n)$ is planar for $k=2$. Since there is no crossing of edges in drawing the graph $In_p(\mathbb{Z}_n)$ for $k=3$ in the plane ($cf.$ Figure 8.1, clearly it is planar. Now if $k=4$, then $In_p(\mathbb{Z}_n)$ contains a complete bipartiate graph $K_{3,3}$ as a minor (see Figure 8.1 with $v_1 = P_1$, $v_2 = P_2$, $v_3 = P_3$, $v_4 = P_{14}$, $v_5 = P_{13}$, $v_6 = P_{12}$, $v_7 = P_{123}$, $v_8 = P_{1234}$, $v_9 = P_{124}$, $v_{10} = P_{134}$) and hence not planar. Again if $k \geq 5$, then the

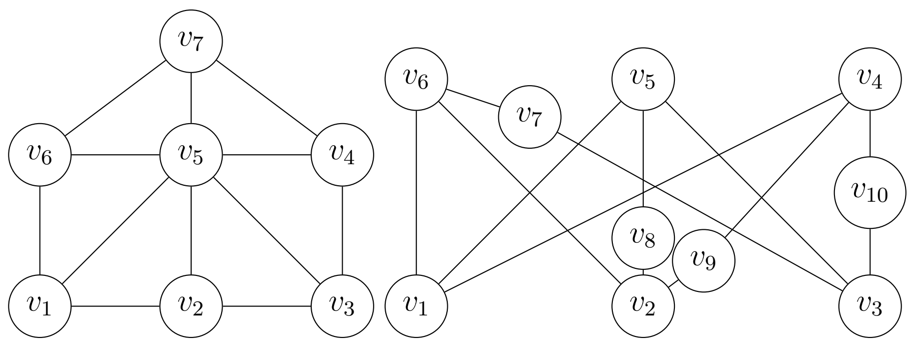

Figure 8.1: (a) $In_p(\mathbb{Z}_{\prod_{i=1}^{3} p_i^{\alpha_i}})$, (b) Subgraph of $In_p(\mathbb{Z}_{\prod_{i=1}^{4} p_i^{\alpha_i}})$ homeomorphic to $K_{3,3}$

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subgraph forms by the set of vertices $\{P_1, P_{12}, P_{123}, P_{1234}, P_{12345}\}$ is the complete graph K_5 and hence not planar. □

Theorem 8.2.22. *The graph $In_p(\mathbb{Z}_n)$ has thickness one if and only if $2 \leq k \leq 3$ and has thickness two if and only if $k = 4$.*

Proof. We know that a graph has thickness one if and only if it is planar. Therefore $In_p(\mathbb{Z}_n)$ has thickness one if and only if $2 \leq k \leq 3$ ($cf.$ Theorem 8.2.21).
Let us consider the case for $k = 4$. It is easy to calculate that for $k = 4$, $In_p(\mathbb{Z}_n)$ has 15 vertices and 50 edges ($cf.$ Theorem 8.2.11). Therefore we have $\theta(In_p(\mathbb{Z}_n))_{k=4} \geq 2$ ($cf.$ Lemma 1.3.9). The planar decomposition of $In_p(\mathbb{Z}_n))_{k=4}$ as shown in Figure 8.2, we have $\theta(In_p(\mathbb{Z}_n))_{k=4} = 2$.
Now if $k = 5$, then $In_p(\mathbb{Z}_n)$ has 31 vertices and 180 edges (see Theorem 8.2.11). Therefore $\theta(In_p(\mathbb{Z}_n))_{k=5} \geq 3$ ($cf.$ Lemma 1.3.9). Also for $k \geq 5$, the graph $In_p(\mathbb{Z}_n)$ has a subgraph isomorphic to $In_p(\mathbb{Z}_m)$ where $m = \prod_{i=1}^{5} p_i^{\alpha_i}$. Therefore for $k \geq 5$, $\theta(In_p(\mathbb{Z}_n)) \geq 3$ ($cf.$ Lemma 1.3.6). Hence the result follows. □

Proposition 8.2.23. *The graph $In_p(\mathbb{Z}_n)$ is never toroidal and is not bitoroidal for $k \geq 5$.*

Proof. We complete the proof by considering the following cases :
Case (1) Let $k \leq 3$. Then we have $\gamma(In_p(\mathbb{Z}_n)) = 0$ ($cf.$ Theorem 8.2.21).
Case (2) Let $k = 4$. Then the graph $In_p(\mathbb{Z}_n)$ has $n = 15$ vertices and $e = 50$ edges ($cf.$

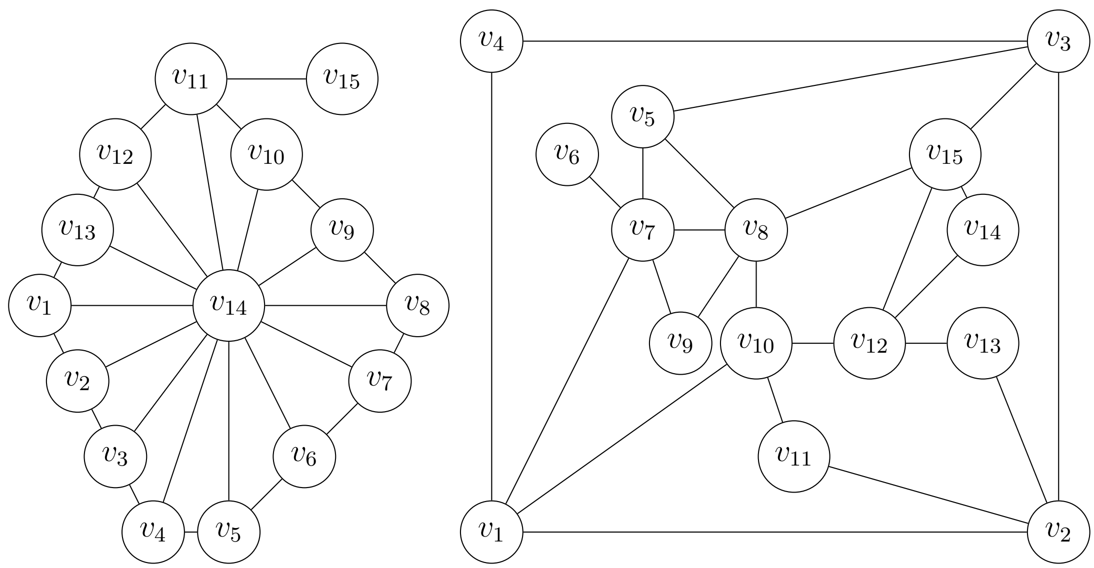

Figure 8.2: A planar decomposition of $In_p(\mathbb{Z}_{\prod_{i=1}^{4} p_i^{\alpha_i}})$

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Theorem 8.2.11). Therefore by applying Lemma 1.3.7, we have

$$g(In_p(\mathbb{Z}_n)) \geq \lceil \frac{50}{6} - \frac{15}{2} + 1 \rceil = 2 \tag{8.2.2}$$

Case (3) Let $k = 5$. Then $In_p(\mathbb{Z}_n)$ has $n = 31$ vertices and $e = 180$ edges ($cf.$ Theorem 8.2.11). Therefore $g(In_p(\mathbb{Z}_{\prod_{i=1}^{5} p_i^{\alpha_i}})) \geq 16$ ($cf.$ Lemma 1.3.7). Now if $k \geq 6$, then $In_p(\mathbb{Z}_n)$ contains a subgraph isomorphic to $In_p(\mathbb{Z}_m)$ where $m = \prod_{i=1}^{5} P_i^{\alpha_i}$ and hence $g(In_p(\mathbb{Z}_n)) \geq 16$. Combining all the above cases we have the desired result. $\square$

Theorem 8.2.24. *The metric dimension of* $In_p(\mathbb{Z}_n)$ *is*

$$dim(In_p(\mathbb{Z}_n)) = \begin{cases} 1 & \textit{if } k = 2 \\ k, & \textit{if } k \geq 3. \end{cases}$$

Proof. Case (1) Let $k = 2$. Then $In_p(\mathbb{Z}_n) \cong P_3$. Hence $\dim(In_p(\mathbb{Z}_n)) = 1$ ($cf.$ Theorem 1.3.18). Moreover any of the pendent vetrices P_1 or P_2 is a metric basis for $In_p(\mathbb{Z}_n)$.

Case (2) Let $k \geq 3$. Since $In_p(\mathbb{Z}_n)$ has $2^k - 1$ vertices and is of diameter 2, so by Theorem 1.3.17 we have $f(n, d) \leq \dim(In_p(\mathbb{Z}_n)) \leq 2^k - 3$, where $f(n, d)$ is the least positive integers l such that $l + 2^l \geq 2^k - 1$. Clearly k is the least positive integers such that $k + 2^k \geq 2^k - 1$. Therefore

$$k \leq \dim(In_p(\mathbb{Z}_n)) \leq 2^k - 3, \qquad \text{for } k \geq 3. \tag{8.2.3}$$

Now we prove that $W = \{P_1, P_2, \ldots, P_k\}$ is a resolving set for $In_p(\mathbb{Z}_n)$. On the contarary, if possible let there exists distinct vertices P_{Λ_1} and $P_{\Lambda_2} \in V(In_p(\mathbb{Z}_n)) - W$ such that $r(P_{\Lambda_1}, W) = (a_1, \ldots, a_k) = (b_1, \ldots, b_k) = r(P_{\Lambda_2}, W)$ where $a_i = b_i = 1$ or 2 for $i \in [k]$.
We claim that $|\Lambda_1| = |\Lambda_2|$. If not, without loss of generality, let $|\Lambda_1| < |\Lambda_2|$. Then there exists $t \in \Lambda_2 - \Lambda_1$, $t \in [k]$.Then $(a_1, \ldots, a_t = 2, \ldots a_k) \neq (b_1, \ldots, b_t = 1, \ldots b_k)$, a contradiction. Therefore $|\Lambda_1| = |\Lambda_2|$. Now since P_{Λ_1} and P_{Λ_2} are distinct there exists $i_1 \in \Lambda_1 - \Lambda_2$ and $i_2 \in \Lambda_2 - \Lambda_1$, where i_1, $i_2 \in [k]$.
Then $(a_1, \ldots, a_{i_1} = 1, \ldots a_{i_2} = 2, \ldots, a_k) = (b_1, \ldots, b_{i_1} = 2, \ldots, b_{i_2} = 1, \ldots b_k)$, a contradiction. Hence $P_{\Lambda_1} = P_{\Lambda_2}$. Therefore distinct vertices of $In_p(\mathbb{Z}_n)$ has distinct representations with respect to W. So W is a resolving set of $In_p(\mathbb{Z}_n)$, which implies

$$dim(In_p(\mathbb{Z}_n)) \leq k \tag{8.2.4}$$

Now combining equation (3) and (4) we have $\dim(In_p(\mathbb{Z}_n)) = k$ for $k \geq 3$ with W as metric basis. $\square$

In the following theorem we determine the strong metric dimension of $In_p(\mathbb{Z}_n)$.

Theorem 8.2.25. *The strong metric dimension is sdim*$(In_p(\mathbb{Z}_n)) = 2^k - k - 1$ *where* $k \geq 2$.

Proof. Let P_{Λ_1} and P_{Λ_2} be any two vertices of $In_p(\mathbb{Z}_n)$. Now if $N[P_{\Lambda_1}] = N[P_{\Lambda_2}]$ then we must have deg(P_{Λ_1}) =deg(P_{Λ_2}), which is possible only if $|\Lambda_1| = |\Lambda_2|$ or $|\Lambda_1| + |\Lambda_2| = k$ (*cf.* Lemma 8.2.8).
Case (1) Let $|\Lambda_1| = |\Lambda_2|$. Then $P_{\Lambda_1} \in N[P_{\Lambda_1}]$ but $P_{\Lambda_2} \notin N[P_{\Lambda_1}]$. Also $P_{\Lambda_2} \in N[P_{\Lambda_2}]$ but $P_{\Lambda_2} \notin N[P_{\Lambda_2}]$. Therefore $N[P_{\Lambda_1}] \neq N[P_{\Lambda_2}]$.
Case (2) Let $|\Lambda_1| + |\Lambda_2| = k$. Without loss of generality, let $|\Lambda_1| < |\Lambda_2|$. Then there exists $a \in [k]$ such that $a \in \Lambda_2 - \Lambda_1$. Then $P_a \in N[P_{\Lambda_2}]$ but $P_a \notin N[P_{\Lambda_1}]$ and hence $N[P_{\Lambda_1}] \neq N[P_{\Lambda_2}]$.
Thus in any cases $N[P_{\Lambda_1}] \neq N[P_{\Lambda_2}]$ and since P_{Λ_1} and P_{Λ_2} are arbitrary, we have $\mathbb{R}_{In_p(\mathbb{Z}_n)} =$

$In_p(\mathbb{Z}_n)$. We know that $In_p(\mathbb{Z}_n)$ is a graph with $2^k - 1$ vertices and $\omega(In_p(\mathbb{Z}_n)) = k$. Therefore by Theorem 1.3.19 we have $\text{sdim}(In_p(\mathbb{Z}_n)) = 2^k - k - 1$. $\square$

The above result can be proved in a different way, which is as follows :
If $N[P_{\Lambda_1}] = N[P_{\Lambda_2}]$ then we must have $P_{\Lambda_1} \sim P_{\Lambda_2}$ otherwise $P_{\Lambda_1} \in N[P_{\Lambda_1}]$ but $P_{\Lambda_2} \notin N[P_{\Lambda_1}]$ and $P_{\Lambda_1} \notin N[P_{\Lambda_2}]$ but $P_{\Lambda_2} \in N[P_{\Lambda_2}]$. Now since $P_{\Lambda_1} \sim P_{\Lambda_2}$, either $\Lambda_1 \subset \Lambda_2$ or $\Lambda_2 \subset \Lambda_1$. Without loss of generality , let $\Lambda_1 \subset \Lambda_2$. So there exists $t \in \Lambda_2$ but $t \notin \Lambda_1$ and hence $P_t \sim P_{\Lambda_2}$ but $P_t \nsim P_{\Lambda_1}$ which implies $N[P_{\Lambda_1}] \neq N[P_{\Lambda_2}]$, a contradiction. Therefore $\mathbb{R}_{In_p(\mathbb{Z}_n)} = In_p(\mathbb{Z}_n)$. Hence the result follows.

Theorem 8.2.26. *The partition dimension of the graph $In_p(\mathbb{Z}_n)$ satisfy the inequality $k - 1 \leq pd(In_p(\mathbb{Z}_n)) \leq k$ for $k \geq 3$.*

Proof. Let $k \geq 3$. Since $In_p(\mathbb{Z}_n)$ is a graph of diameter 2, by applying Theorem 8.2.26 we have $g(n, 2) \leq \text{pd}(In_p(\mathbb{Z}_n))$, where $g(n, 2)$ is the least positive integer l for which $(2+1)^l = 3^l \geq n = 2^k - 1$. Clearly $l = k - 1$. Therefore we have the inequality

$$pd(In_p(\mathbb{Z}_n)) \geq k - 1 \tag{8.2.5}$$

Now here we present a k-resolving partition $\Pi = (S_1, S_2, \dots, S_k)$ of $V(In_p(\mathbb{Z}_n))$ as
$S_1 = N[P_1]$
$S_2 = \{P_2, \{P_{\Lambda_2} : |\Lambda_2| = 2 \text{ and } 2 \in \Lambda_2 \text{ but } 1 \notin \Lambda_2\}, \dots, \{P_{\Lambda_{k-1}} : |\Lambda_{k-1}| = k - 1 \text{ and } 2 \in \Lambda_{k-1} \text{ but } 1 \notin \Lambda_{k-1}\}\}$
$\dots$
$S_{k-1} = \{P_{k-1}, P_{k-1k}\}$
$S_k = \{P_k\}$.
Let $v_1 \in S_1$. Then $v_1 = P_\Lambda$ such that $1 \in \Lambda$ and
$r(v_1, \Pi) = (0, a_2, \dots, a_k)$, where $a_i = 1$ if $i \in \Lambda$ otherwise $a_i = 2$.
Let $v_2 \in S_2$. Then $v_2 = P_{\Lambda_2}$ such that $2 \in \Lambda_2$ but $1 \notin \Lambda_2$. Then
$r(v_2, \Pi) = (1, 0, a_3, \dots, a_k)$ where $a_i = 1$ if $i \in \Lambda_2$ otherwise $a_i = 2$.
$\dots$
Let $v_t \in S_t$ where $t \in [k]$. Then $v_t = P_{\Lambda_t}$ such that $t \in \Lambda_t$ but $1, 2, \dots, t - 1 \notin \Lambda_t$. Then $r(v_t, \Pi) = (1, a_2, \dots, a_t = 0, \dots, a_k)$ where $a_i = 1$ if $i \in \Lambda_t$ otherwise $a_i = 2$.

...

Similarly as above for $v_k = P_k \in S_k$ we have

$r(v_k, \Pi) = (0, 1, 1 \ldots, 1)$.

Since representations of all vertices with respect to the partition Π of $V(In_p(\mathbb{Z}_n))$ are distinct, clearly Π is a resolving partition of $In_p(\mathbb{Z}_n)$. Therefore

$$pd(In_p(\mathbb{Z}_n)) \leq k \tag{8.2.6}$$

Combining equation 3 and 4 we have the required result.

□

Remark 8.2.27. Also note that for $k = 3$, no 2-partition can be a resolving partition of $In_p(\mathbb{Z}_n)_{k=3}$. Since for $k = 3$, $In_p(\mathbb{Z}_n)$ has seven vertices so one set of a 2-partitions of $V(In_p(\mathbb{Z}_n))$ must contain at least four vertex but we cannot have four distinct representations with respect to Π for this four vertices. So $\mathrm{pd}(In_p(\mathbb{Z}_n))_{k=3} = 3$.

In a similar way as above we can prove that $\mathrm{pd}(In_p(\mathbb{Z}_n))_{k=4} = 4$.

Chapter 9

Conclusion

9.1 Concluding Remark

In this chapter we first consider three graphs related to 2-absorbing, 2-absorbing primary and 2-prime ideals of a commutative semigroup S and then discuss the relations of these graphs and the graphs considered in Chapter 6 and 8, the inclusion (resp. prime inclusion) ideal graph of S. All the graphs related to ideals of commutative semigroup S considered in this thesis are isomorphic if and only if S is regular and idempotents are linearly ordered ($cf.$ Theorem 9.1.4).

First we define the graphs as follows:

Definition 9.1.1. The 2-absorbing inclusion ideal graph over S, denoted by $In_{2-AB}(S)$, is a graph with vertices are 2-absorbing ideals of S and two distinct vertices are adjacent if and only if one is contained in the other.

Definition 9.1.2. The 2-absorbing primary inclusion ideal graph over S, denoted by $In_{2-ABP}(S)$, is a graph with vertices are 2-absorbing primary ideals of S and two distinct vertices are adjacent if and only if one is contained in the other.

Definition 9.1.3. The 2-prime inclusion ideal graph over S, denoted by $In_{2-P}(S)$, is a graph with vertices are 2-prime ideals of S and two distinct vertices are adjacent if and only if one is contained in the other.

Clearly $In_p(S)$ is a subgraph of $In(S)$. By applying Theorem 1.2.49, we have the following immediate result.

Theorem 9.1.4. *Let S be a commutative semigroup. Then $In(S) \cong In_p(S)$ if and only if S is regular and the idempotents in S form a chain.*

Since every prime ideal of a commutative semigroup S is 2-prime ($cf.$ Lemma 4.1.2), clearly $In_p(S)$ is a subgraph of $In_{2-P}(S)$. Also by applying Theorem 4.2.6, we have the following immediate result.

Theorem 9.1.5. *Let S be a commutative semigroup. Then $In_P(S) \cong In_{2-P}(S)$ if and only if one of the following conditions hold.*
(1) *S is a $2-P$ semigroup.*
(2) *2-prime ideals of S are semiprime.*
(3) *Prime ideals of S are idempotent and every 2-prime ideals is of the form A^2, where A is a prime ideal of S.*

Since every prime ideals of S is a 2-absorbing ideal, clearly $In_p(S)$ is a subgraph of $In_{2-AB}(S)$. By applying Theorem 2.2.6 and Theorem 2.2.7, we have the following result.

Theorem 9.1.6. *Let S be a commutative semigroup. Then $In_p(S) \cong In_{2-AB}(S)$ if and only if one of the following statement holds*
(1) *S is a $2-AB$ semigroup.*
(2) *$P = P^2$ for every prime ideal P of S and every 2-absorbing ideal of S is of the form A^2, where A is a prime ideal of S.*
(3) *Prime ideals of S are linearly ordered and $A = A^2$ for every 2-absorbing ideal A of S.*

Similarly as above by applying Theorem 3.1.7, we have the following result.

Theorem 9.1.7. *Let S be a commutative semigroup. Then $In_p(S) \cong In_{2-ABP}(S)$ if and only if prime ideals of S are linearly ordered and $I = \sqrt{I}$ for every 2-absorbing ideal I of S.*

9.2 Emanating questions

Work from this thesis raises the following questions:

Problem 1. Is the concept of weakly 2-prime and strong weakly 2-prime are equivalent in a semigroup with zero and identity?

Problem 2. In which class of semigroups every weakly 2-prime ideals are 2-prime?

Problem 3. In which class of rings 2-prime ideals are maximal?

Problem 4. When the graph $\mathcal{P}(\mathcal{S}_M)$ is planar? Also determine the values of n for which the graph $\mathcal{P}(\mathcal{S}_M)$ has genus (resp. crosscap) one and two.

Problem 5. Determine the automorphism group and fixing number of the graph $\mathcal{P}(\mathcal{S}_M)$.

Problem 6. Determine the decycling number and metric dimension of the graph $\mathcal{P}(\mathcal{S}_M)$.

Problem 7. Find the number of vertices of $In(Z_n)$, which is the number of nontrivial ideals of the semigroup Z_n.

Problem 8. Classify the semigroups whose inclusion ideal graph is Eulerian, Hamiltonian.

Problem 9. Find classes of semigroups such that if S_1 and S_2 are two semigroups in that class and $In(S_1) \cong In(S_2)$, then $S_1 \cong S_2$.

Problem 10. When the graph $In_p(\mathbb{Z}_n)$ is Hamiltonian?

List of Published / Accepted/ Communicated papers

1) **B. Khanra**, M. Mandal, Semigroups in which 2-absorbing ideals are prime and maximal, Quasigroups and related systems, 29 (2021) 213-222.

2) M. Mandal, **B. Khanra**, On 2-absorbing primary ideals of a commutaive semigroup, Kyungpook Math. J. 62(2022) 425-436.

3) **B. Khanra**, M, Mandal, On 2-prime ideals in commutative semigroups, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) Tomul LXVIII, 2022 f. 1, 49-59.

4) **B. Khanra**, M. Mandal, B. Ghosh, On the power graph of a monogenic semigroup, Discrete Mathematics, Algorithms and Applications, doi:10.1142/S1793830923500052.

5) **B. Khanra**, M. Mandal, The inclusion ideal graph of a semigroup, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) Tomul LXIX, 2023 f. 2.

6) **B. Khanra**, The inclusion ideal graph of multiplicative semigroup of a ring (Communicated).

7) **B. Khanra**, M. Mandal, S. Mukherjee, On the prime inclusion ideal graph of semigroup and ring (Communicated).

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