GENERALIZED SKEW DERIVATIONS AND RELATED ADDITIVE MAPS IN PRIME AND SEMIPRIME RINGS

Thesis submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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CERTIFICATE FROM THE SUPERVISOR(S)

This is to certify that the thesis entitled "**Generalized Skew Derivations and Related Additive Maps in Prime and Semiprime Rings**" submitted by Sri **Swarup Kuila**, who got his name registered on 2^{nd} September, 2019 for the award of Ph.D.(Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of Prof. Sukhendu Kar, Dept. of Mathematics, Jadavpur University, and Dr. Basudeb Dhara, Dept. of Mathematics, Belda College, Paschim Medinipur, and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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To My Parents

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Preface

For as long as rings and algebras have been studied for their own sakes, it has been a problem of interest to determine the consequences of various special identities and conversely, to find sufficient conditions on a given ring which ensure that a specified identity holds. Ring derivation is a branch of algebra in which we study about the structure of additive maps as well as structure of rings by analyzing some functional identities involving additive maps. These additive maps are derivation, skew derivation, generalized derivation, generalized skew derivation, b-generalized derivation, multiplicative generalized derivation, multiplicative (generalized) derivation, etc. It is well known that there is a strong relationship among the functional identities involving derivations and generalized derivations and the structure of the rings. This thesis is mainly intended to find out the structure of above mentioned additive maps satisfying some functional identities on different subsets of prime and semiprime rings. A simple and well known functional identity $[d(x), x] = 0$ for all $x \in R$, where R is a prime ring and d is a nonzero derivation on R, was studied by Posner [80]. On that article, he got a wonderful structure of R. Henceforth, several researchers studied several functional identities and got marvelous results. We have also studied some problems in this line of investigation. This thesis contains seven chapters. Chapter-wise brief information is given bellow:

Chapter 1 is basically devoted for introductory purpose. Some basic definitions, preliminaries and prerequisites have been collected from other references which are needed for the development of the subsequent chapters in this thesis.

Dhara and Ali [31] gave a precise definition of multiplicative (generalized) derivation and studied some standard situations. In Chapter 2, we have studied some identities of multiplicative (generalized) derivation of a semiprime ring. Some examples have been given at the end of this chapter concluding that semiprimeness hypothesis in the theorems are not superfluous.

The study of commuting and centralizing maps was initiated by Posner [80]. There, he proved that if a prime ring has a nonzero centralizing derivation, then the prime ring is commutative. Brešar [9] generalized Posner's result by considering co-centralizing derivations and also proved that, if two nonzero derivations of a prime ring R are co-centralizing on R, then R must be commutative. De Filippis et al.[49] and Carini et al.[13] generalized result of Brešar [9] acting on a noncentral Lee ideal. In Chapter 3, we have inspected an identity involving three generalized derivations on a noncentral Lee ideal of R, which extends the above results.

A number of authors have studied some functional identities involving noncentral valued multilinear polynomial. In chapter 4, we have studied the identity $d(F^2(x)x) = xG^2(x)$ for all $x \in f(I)$; where f is a noncentral valued multilinear polynomial, I is a two sided ideal of a prime ring R, d is a derivation and F, G are generalized derivations of R.

Chapter 5 has dealt with another identity involving derivations and generalized derivations acting on multilinear polynomials in prime ring.

Eroğlu and Argaç [43] studied the identity $F^2(x)x \in C$ for all $x \in f(R)$, where C is the extended centroid of R, F is a generalized derivation of a prime ring R and f is a multilinear polynomial. Recently, Yadav [90] has described all possible forms of the maps, when $F^2(x)d(x) = 0$ for all $x \in f(R)$, where d is a derivation and F is a generalized derivation of R. Chapter 6 has been dedicated to study the above identity with central value.

Lastly, in Chapter 7, we have studied an identity involving X-generalized skew derivation in prime rings.

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List of Publications

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[2] B. Dhara, S. Kar, S. Kuila : A note on generalized derivations of order 2 and multilinear polynomials in prime rings, *Acta Math. Vietnam. (Springer)*, 47 (2022), 755-773.

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[4] B. Dhara, S. Kar, S. Kuila : Generalized derivations commuting on Lie ideals in prime rings, *Ann. Univ. Ferrara (Springer)*, 69 (2023), 159-181.

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Chapter 1

Introduction, Preliminaries and Prerequisites

This chapter is devoted for preliminary materials and notations which are used in the subsequent chapters. Since all basic notations are not possible to mention here, we refer the books by Herstein [59] and Jacobson [63, 64] for more details.

Throughout this thesis, by R we mean an associative ring and ideal of a ring means two sided ideal. A functional identity on a ring R is an identity that involves some functions as well as elements of R. The usual goal, when considering these identities is to describe the form of the mappings appearing in the identity or, when this is not possible, to determine the structure of the ring admitting this identity.

1.1 Some Special Classes of Rings

Now we recall the definions of prime and semiprime ideals in rings.

Definition 1.1.1. *Let R be a ring. An ideal P of R is said to be a prime ideal if for any two ideals A, B in R, $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.*

Definition 1.1.2. *Let R be a ring. An ideal P of R is said to be a semiprime ideal of R if for any ideal A in R, $A^2 \subseteq P$ implies $A \subseteq P$.*

The following well known theorems provide characterizations of prime ideal and semiprime ideal in a ring :

Theorem 1.1.1. *Let P be an ideal of a ring R. Then the following are equivalent :*

(i) P is a prime ideal of R;

(ii) If $a, b \in R$ such that $aRb \subseteq P$, then either $a \in P$ or $b \in P$;

(iii) If $< a >$, $< b >$ are principal ideals in R such that $< a >< b > \subseteq P$, then either $a \in P$ or $b \in P$;

(iv) If U and V are right ideals in R, then $UV \subseteq P$ implies either $U \subseteq P$ or $V \subseteq P$;

(v) If U and V are left ideals in R, then $UV \subseteq P$ implies either $U \subseteq P$ or $V \subseteq P$.

Theorem 1.1.2. *Let Q be an ideal of a ring R. Then the following are equivalent :*

(i) Q is a semiprime ideal of R;

(ii) If $a \in R$ be such that $aRa \subseteq Q$, then $a \in Q$;

(iii) If $< a >$ is a principal ideal in R such that $< a >^2 \subseteq Q$, then $a \in Q$;

(iv) If U is a right ideal in R, then $U^2 \subseteq Q$ implies $U \subseteq Q$;

(v) If U is a left ideal in R, then $U^2 \subseteq Q$ implies $U \subseteq Q$.

Definition 1.1.3. *A ring R is said to be a prime ring if the zero ideal is a prime ideal in R.*

Example 1.1.1. *1. Every domain is a prime ring, since zero ideal is a prime ideal.*

2. Every simple ring is a prime ring.

3. Matrix ring over prime ring is a prime ring.

Theorem 1.1.3. *The following conditions are equivalent in a ring:*

1. R is a prime ring;

2. *If A, B are two ideals of R such that $AB = 0$, then either $A = 0$ or $B = 0$;*

3. *If $a, b \in R$ such that $aRb = (0)$, then either $a = 0$ or $b = 0$.*

Similarly, semiprime ring is defined as follows :

Definition 1.1.4. *A ring R is said to be a semiprime ring if the zero ideal is a semiprime ideal in R.*

Example 1.1.2. 1. *Every prime ring is a semiprime ring.*

2. *If R_1 and R_2 are nonzero prime rings, then $R_1 \bigoplus R_2$ is a semiprime ring.*

It is clear from the definition, every prime ring is semiprime ring, but the converse is not true always. For example, $\mathbb{Z} \bigoplus \mathbb{Z}$ is a semiprime ring, but not a prime ring.

Theorem 1.1.4. *The following conditions are equivalent in a ring:*

1. *R is a semiprime ring;*

2. *If A is an ideal of R such that $A^2 = 0$, then $A = 0$;*

3. *If $a \in R$ such that $aRa = (0)$, then $a = 0$.*

Lemma 1.1.5. *[60, Lemma 1.1.1] If R is a prime ring with no nonzero nilpotent elements, then R has no zero divisor.*

Recall that the center of R, is denoted by $Z(R)$ and defined by $Z(R) = \{x \in R : xr = rx \text{ for all } r \in R\}$.

Lemma 1.1.6. *[60, Lemma 1.1.5] Let R be a semiprime ring and I be a one-sided ideal of R. Then $Z(I) \subseteq Z(R)$. Further, if I is commutative, then $I \subseteq Z(R)$.*

Lemma 1.1.7. *[60] Let I be a nonzero one-sided ideal of a prime ring R. If $I \subseteq Z(R)$, then R is a commutative ring.*

Lemma 1.1.8. *Center of a prime ring does not contain zero divisor.*

Lemma 1.1.9. *Let R be a prime ring with center $Z(R)$. If $zr \in Z(R)$ for some $0 \neq z \in Z(R)$ and $r \in R$, then $r \in Z(R)$.*

Let us now define annihilator of a ring R. Let X be any nonempty subset of R. The subsets $r(X) := \{y \in R : xy = 0 \text{ for all } x \in X\}$ and $l(X) := \{y \in R : yx = 0 \text{ for all } x \in X\}$ of R are called right annihilator of X and left annihilator of X respectively. An element p of R is called annihilator of X, if $p \in r(X)$ as well as $p \in l(X)$.

Now we have the following results:

Lemma 1.1.10. *[60] A ring R is prime if and only if the right annihilator of a nonzero right ideal of R is (0).*

Lemma 1.1.11. *[60, p. 6] Let R be a semiprime ring and I be an ideal of R. Then $r(I) = l(I)$ and $I \cap r(I) = (0)$.*

Definition 1.1.5. *A ring R is said to be $n-$torsion free, where n is a positive integer, if for any $x \in R$, $nx = 0$ implies $x = 0$.*

It is easy to say that whenever a ring R is $n-$torsion free, then $char(R) \neq n$. But converse of the above result is not true for all rings. The primeness is required for the converse statement to be true. Thus for a prime ring R, $char(R) \neq n$ if and only if R is $n-$torsion free.

1.2 Commutator Identities in Rings

For any $x, y \in R$, the symbol $[x, y]$ stands for the *commutator or Lie product* $xy - yx$ and the symbol $x \circ y$ stands for the *anti-commutator or Jordan product* $xy + yx$.

We recall some basic commutator identities in a ring R as follows : For any $x, y, z \in R$,

$$[xy, z] = x[y, z] + [x, z]y; \quad [x, yz] = y[x, z] + [x, y]z;$$

$$(xy \circ z) = x(y \circ z) - [x, z]y = x[y, z] + (x \circ z)y;$$

$$(x \circ yz) = (x \circ y)z - y[x, z] = [x, y]z + y(x \circ z).$$

Moreover,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

The last identity is called as Jacobi Identity.

For $x, y \in R$, set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$, and then an Engel type polynomial $[x, y]_k = [[x, y]_{k-1}, y]$, $k = 1, 2, \ldots$.

Definition 1.2.1. *An additive subgroup L of R is said to be a Lie ideal of R, if $[u, r] \in L$ for all $u \in L$ and $r \in R$.*

Clearly, every ideal of a ring R is a Lie ideal of R. It is noted that a Lie ideal of a ring R may not be an ideal of R.

• A Lie ideal L is said to be square closed if $u^2 \in L$ for all $u \in L$.

Definition 1.2.2. *Let S be a nonempty subset of a ring R. A map $f : R \to R$ is called commuting (resp. centralizing) on S if $[f(x), x] = 0$ for all $x \in S$ (resp. $[f(x), x] \in Z(R)$ for all $x \in S$).*

Definition 1.2.3. *Let S be a nonempty subset of a ring R. Two maps $f, g : R \to R$ are called co-commuting (resp. co-centralizing) on S if $f(x)x - xg(x) = 0$ for all $x \in S$ (resp. $f(x)x - xg(x) \in Z(R)$ for all $x \in S$).*

1.3 Ring of Quotients

In the study of generalized identities in prime and semiprime rings it will be seen that ring of quotients plays a crucial role. For us the most important ring of quotients is the *maximal right ring of quotients* or *Utumi ring of quotients*. It was first constructed by Y. Utumi [87]. Another important ring of quotients is used here, *two-sided ring of quotients* or *Martindale ring of quotients*. This ring of quotients was introduced by Martindale [78] as a tool to study prime rings satisfying a generalized polynomial identity.

1.3.1 Utumi Ring of Quotients

Let R be a prime ring and $\mathcal{D} = \{J\}$ be the collection of all dense right ideals of R, and consider T to be the set of all R-homomorphisms $f : J_R \to R_R$, where J ranges over $\mathcal{D}$ and J and R are regarded as right R-modules. So $T = \{(f;J) |\ \ J \in \mathcal{D},\ \ f : J_R \to R_R\}$, where $(f;J)$ denotes f acting on J.

We define $(f;J) \sim (g;K)$ if there exists $L \subseteq J \cap K$ such that $L \in \mathcal{D}$ and $f = g$ on L. One readily check that '$\sim$' is indeed an equivalence relation, let $[f;J]$ denote the equivalence class determined $(f;J) \in \mathcal{D}$ and we let U denote the collection of all equivalence classes of T with respect to '$\sim$'. We then define addition and multiplication of equivalence classes as follows:

$$[f;J] + [g;K] = [f+g; J \cap K]$$

and

$$[f;J][g;K] = [fg; g^{-1}(J)].$$

One easily checks that the addition and multiplication is well-defined [6, pp. 55]. Under these operations it is readily seen that U forms a ring with respect to above addition and multiplication. This ring U is called *Utumi ring of quotients*. Some important properties are given in the following proposition:

Proposition 1.3.1. *[6, Proposition 2.1.7] U satisfies:*

1. *R is a subring of U;*

2. *For any $q \in U$ there exists $J \in \mathcal{D}$ such that $qJ \subseteq R$;*

3. *For any $q \in U$ and $J \in \mathcal{D}$, $qJ = 0$ if and only if $q = 0$;*

4. *For any $J \in \mathcal{D}$ and $f : J_R \to R_R$ there exists $q \in U$ such that $f(x) = qx$ for all $x \in J$.*

Furthermore, properties $(1) - (4)$ characterize ring U up to isomorphism.

1.3.2 Martindale Ring of Quotients

For a prime ring R, a nonzero two-sided ideal is obviously a dense right ideal of R. In the above construction if we consider only nonzero two-sided ideals instead of dense right ideals, then we obtain the Martindale ring of quotients(see [78]). Here we shall denote this ring by Q.

Proposition 1.3.2. *[6, Proposition 2.2.1] Let R be a semiprime ring. Then Q satisfies:*

1. *R is a subring of Q;*

2. *For any $q \in Q$ there exists $J \in \mathcal{D}$ such that $qJ \subseteq R$;*

3. *For any $q \in Q$ and $J \in \mathcal{D}$, $qJ = 0$ if and only if $q = 0$;*

4. *For any $J \in \mathcal{D}$ and $f : J_R \to R_R$ there exists $q \in Q$ such that $f(x) = qx$ for all $x \in J$.*

Furthermore, properties $(1) - (4)$ characterize ring Q up to isomorphism.

Some important facts are as follows:

• Q can be naturally regarded as a subring of U [6, Proposition 2.2.2] and can be characterized as follows: for $a \in U$, we have $a \in Q$ if and only if $aI \subseteq R$ for some nonzero two-sided ideal I of R.

• Also for a prime ring R, the corresponding rings of quotients Q and U both are prime ring [44, p. 74].

• $Z(Q) = Z(U)$, where $Z(Q)$ and $Z(U)$ are centers of Q and U respectively [6, Remark 2.3.1].

Definition 1.3.1. *The center of the Martindale ring of quotients as well as the Utumi ring of quotients is called the extended centroid of R. And $S = RC$ is called the central closure of R.*

The extended centroid of R is denoted by C. It is very well known that C forms a field, when R is prime ring [6, p. 70]. In fact, S is a prime ring containing R. Further S is contained in $Q \subseteq U$. If R has a unit element then $C = Z(S)$. If R is a simple ring with unit element then $Q = S = R$ i.e., R is its own central closure. We refer to [60, 78] for more details.

1.4 Some Special Type of Additive Maps in Rings

Let R be a ring. A map $f : R \to R$ is called additive map if it preserves the additive structure of R, i.e. $f(x + y) = f(x) + f(y)$ for all $x, y \in R$.

Definition 1.4.1. *Let R be a ring. An additive mapping $d : R \to R$ is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.*

An example of derivation is the usual derivation d on the polynomial ring $R = F[x]$

given by

$$d(\sum_{i=0}^{t} a_i x^i) = \sum_{i=1}^{t} i a_i x^{i-1}$$

Let us consider the ring R ,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ : \ a, b, c \in Z \right\},$$

where Z is the set of all integers. Let us define a mapping $d : R \to R$ by

$$d \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & nb \\ 0 & 0 \end{pmatrix} \ : \ a, b, c \in Z,$$

where n is any fixed integer. Then it is obvious that d is a derivation on R.

However, there is another fundamental class of derivations. The mapping $d_a : R \to R$ defined by $d_a(x) = [a, x]$ for all $x \in R$, for fixed $a \in R$ is a derivation of R. This kind of derivations are called as ***inner derivations*** of R. When a derivation of a ring is not inner, called ***outer derivation***.

Remark 1.4.1. *Let d be a derivation of a ring R. Then $d(Z(R)) \subseteq Z(R)$.*

The notion of derivation was extended by Brešar [8] in 1991. He first introduced the concept of generalized derivation which was further studied algebraically by Hvala [62] in 1998.

Definition 1.4.2. *Let R be a ring. An additive mapping $F : R \to R$ is called a generalized derivation, if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.*

Here this mentioned derivation d is called an associated derivation of the generalized derivation F. Thus it is evident from the above definition that every derivation is a generalized derivation of R with respect to a associated derivation itself. When $d = 0$, then generalized derivation becomes $F(xy) = F(x)y$ for all $x, y \in R$, which is called a left multiplier mapping of R. Some authors have used the notion of left centralizer instead of left multiplier in a ring. Thus generalized derivation of a ring covers the concept of derivation as well as the concept of left multiplier mapping in a ring. We now recall the following definitions:

Definition 1.4.3. *Let R be an associative ring and α be an automorphism of R. An additive mapping $d : R \to R$ is called a skew derivation of R if*

$$d(xy) = d(x)y + \alpha(x)d(y)$$

holds for all $x, y \in R$. Here α is called the associated automorphism of d.

Definition 1.4.4. *An additive mapping $G : R \to R$ is said to be a generalized skew derivation of R, if there exists a skew derivation d of R with associated automorphism α such that*

$$G(xy) = G(x)y + \alpha(x)d(y)$$

holds for all $x, y \in R$. Here d is said to be an associated skew derivation of G and α is called an associated automorphism of G.

The concept of the map X-generalized skew derivations was introduced by De Filippis and Wei in [54].

Definition 1.4.5. *Let R be an associative ring, $b \in Q$, $d : R \to R$ a linear mapping and α be an automorphism of R. A linear mapping $F : R \to R$ is called an $X-$generalized skew derivation of R, with associated term (b, α, d) if $F(xy) = F(x)y + b\alpha(x)d(y)$ for all $x, y \in R$.*

It is very easy to check that the concept of $X-$generalized skew derivation generalizes the concept of generalized skew derivation as well as b-generalized derivation. The map

$x \mapsto ax + b\alpha(x)c$ is an example of X-generalized skew derivation of R with associated map (b, α, d), where $a, b, c \in R$ are fixed elements and $d(x) = \alpha(x)c - cx$ for all $x \in R$. Such X-generalized skew derivations of R are called as **inner X-generalized skew derivations** of R.

1.5 Generalized Polynomial Identity (GPI)

Let R be an associative ring and let $X = \{x_1, x_2, \ldots\}$ be an infinite set of non-commutative indeterminates. The classical approach to the theory of polynomial identities of a ring R was to consider identical relations in R of the form $p[x] = 0$, where $p[x] = \sum \alpha_{(i)} x_{i_1} x_{i_2} \cdots x_{i_n}$ is a polynomial in the x_j with coefficients $\alpha_{(i)}$ which are integers or belong to a commutative field F over which R is an algebra. The main result in the theory of these identities is due to Kaplansky [63, p. 226] which states that a primitive ring satisfying a polynomial identity of degree d is a finite-dimensional algebra over its center, and its dimension is $\leq [d/2]^2$.

The generalized polynomial identities to be dealt with are of the form:

$$P[x] = \sum \alpha_{i_1} \pi_{j_1} \alpha_{i_2} \pi_{j_2} \cdots \alpha_{i_k} \pi_{j_k} \alpha_{i_{k+1}} = 0,$$

where the π_j are monomials in the indeterminates x_j and the elements a_{i_λ} appear both as coefficients and between the monomials π_j. More precisely, one considers a prime ring R and $S = RC$, its central closure. Consider $S < x >= S *_C \{X\}$, the free product of S and $\{X\}$ over C. The elements of $S < x >$ are called the **generalized polynomials**. By a nontrivial generalized polynomial, we mean a nonzero element of $S < x >$. An element $m \in S < x >$ of the form $m = q_0 y_1 q_1 y_2 q_2 \ldots y_n q_n$, where $\{q_0, q_1, \ldots, q_n\} \subseteq S$ and $\{y_1, y_2, \ldots, y_n\} \subseteq X$, is called a monomial (some of the q_i can be 1 also). $q_0, q_1, \ldots, q_n$ are called the coefficients of m. Each $f \in S < x >$ can be represented as a finite sum of monomials. Such a representation is certainly not unique.

Let B be a set of C-independent vectors of S. By a B-monomial, we mean a monomial of the form $u_0y_1u_1y_2u_2\ldots y_nu_n$, where $\{u_0,\ldots,u_n\}\subseteq B$ and $\{y_1,y_2,\ldots,y_n\}\subseteq X$. Let $V=BC$, the C-subspace spanned by B. Then any V-generalized polynomial f can be written in the form $\sum \alpha_i m_i$, where $\alpha_i\in C$ and m_i are B-monomials, in the following manner: First fix a representation of f with all of its coefficients in V and express each coefficient of the given representation as a linear combination of elements of B. Then substitute these linear combinations into the representation of f and expand the resulting expression using the distributive law. Finally, we collect similar terms to get our desired form.

It is also obvious that such representation of a given f in terms of B-monomials is unique. If B is chosen to be a basis of S over C, the B-monomials span the whole $S<x>$.

The uniqueness of representation in terms of B-monomials gives a practical criterion to decide whether a given generalized polynomial f is trivial or not: Pick a basis B for the C-subspace spanned by the coefficients of a given representation of f. Express f as a linear combination of B-monomials in the way explained above. Let us say $f=\sum \alpha_i m_i$, where $\alpha_i\in C$ and m_i are B-monomials. Then f is trivial if and only if $\alpha_i=0$ for each i. This simple criterion will be used frequently in several chapters.

Remark 1.5.1. *If we consider $T=U*_C C\{X\}$, the free product of U and the free algebra $C\{X\}$ over C. And if $a_1,a_2\in U$ are linearly independent over C and $a_1g_1(x_1,\ldots,x_n)+a_2g_2(x_1,\ldots,x_n)=0\in T$, where*

$$g_1(x_1,\ldots,x_n)=\sum_{i=1}^{n} x_ih_i(x_1,\ldots,x_n)$$

and

$$g_2(x_1,\ldots,x_n)=\sum_{i=1}^{n} x_ik_i(x_1,\ldots,x_n)$$

for $h_i(x_1,\ldots,x_n),k_i(x_1,\ldots,x_n)\in T$, then both $g_1(x_1,\ldots,x_n)$ and $g_2(x_1,\ldots,x_n)$ are zero element of T.

Definition 1.5.1. *S is said to satisfy generalized polynomial identity if there exists an $0 \neq f \in S < x >$ such that $f(s_1, s_2, \ldots, s_n) = 0$ for all $s_i \in S$.*

Definition 1.5.2. *The polynomial with n variables*

$$s_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)},$$

where $(-1)^{\sigma}$ is $+1$ or -1 according as σ being an even or odd permutation in symmetric group S_n, is called the standard polynomial of degree n.

Theorem 1.5.1. ***Amitsur-Levitzki Theorem:*** *Let R be a commutative ring. Then $M_n(R)$ satisfies s_{2n}.*

Theorem 1.5.2. *[17, Theorem 2] Let R be a prime ring with its Utumi ring of quotients U. For any dense submodule M of U, the GPIs satisfied by M are the same as the GPIs satisfied by U.*

Theorem 1.5.3. *[17, Theorem 3] Let R be a prime ring with its Utumi ring of quotients U. Let M and N be two dense submodules of U. If M satisfies a GPI, then M satisfies a GPI of N.*

Let R be a prime ring with its Utumi ring of quotients U. Let $Der(U)$ be the set of all derivations of U. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 \ldots d_n$ with each $d_i \in Der(U)$.

A differential polynomial is a generalized polynomial of the form $\Phi(\Delta_j(x_i))$ involving non-commutative indeterminates x_i which are acted by derivation words Δ_j as uniary operation and with coefficients from U. $\Phi(\Delta_j(x_i))$ is said to be *differential identity* on $S \subseteq U$, if $\Phi(\Delta_j(x_i))$ assumes the constant value 0 for any assignment of values from S to its indeterminates x_i.

Theorem 1.5.4. *[69, Theorem 3] Let R be a semiprime ring, U its Utumi ring of quotients and I_R a dense R-submodule of U_R. Then I and U satisfy the same differential identities.*

1.6 Some Important Results

Theorem 1.6.1. *[6, Proposition 2.5.1] Every derivation of a prime ring R can be uniquely extended to a derivation of the Utumi ring of quotients U.*

Theorem 1.6.2. *[71, Theorem 3] Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + \delta(x)$ for some $a \in U$ and a derivation δ on U.*

Theorem 1.6.3. *[52, Lemma 1.5] Suppose that $A_1, \ldots, A_k$ are non scalar matrices in $M_t(C)$, where $t \geq 2$ and C is infinite field. Then there exists an invertible matrix $P \in M_m(C)$ such that any matrices $PA_1P^{-1}, \ldots, PA_kP^{-1}$ have all nonzero entries.*

Theorem 1.6.4. *[65, Theorem 2]**Kharchenko's Theorem:***
Let R be a prime ring, U be its Utumi ring of quotient and I an ideal of R. Let $\Phi(\Delta_j(x_i)) = 0$ be a reduced differential identity for I. Then $\Phi(z_{ij}) = 0$ is GPI for U, where $z_{ij} = 0$ are distinct indeterminates.

In particular, we have:
If d is a nonzero outer derivation of R and $\Phi(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)) = 0$ is a differential identity on R, then U satisfies GPI $\Phi(x_1, \ldots, x_n, z_1, \ldots, z_n) = 0$, where $x_1, \ldots, x_n$, $z_1, \ldots, z_n$ are distinct indeterminates.

Theorem 1.6.5. *[63]**Jacobson Density Theorem:***
Let R be a (left) primitive ring with R^V a faithful irreducible R-module and $D = End(R^V)$. Then for any natural number n, if $v_1, \ldots, v_n$ are D-independent in V and $w_1, \ldots, w_n$ are arbitrary in V, then there exists $r \in R$ such that $rv_i = w_i, i = 1, \ldots, n$.

Theorem 1.6.6. *[78, Theorem 3]**Martindale Theorem:***
Let R be a prime ring with its extended centroid C. Then $S = RC$ satisfies a GPI over

C if and only if S contains a minimal right ideal eS(hence S is primitive) and eSe is a finite dimensional division algebra over C, where e is idempotent.

Theorem 1.6.7. *[6, 45, 63]**Litoff's Theorem:***

Let R be a primitive ring with nonzero socle $H = Soc(R)$ and $b_1, \ldots, b_m \in H$. Then there exists an idempotent $e \in H$ such that $b_1, \ldots, b_m \in eRe$ and the ring eRe is isomorphic to $M_n(C)$.

Fact 1.6.8. *Let R be a prime ring and $\Phi(x_1, \ldots, x_n) = 0$ be a nontrivial GPI for R. By Theorem 1.5.2, U also satisfies the same GPI $\Phi(x_1, \ldots, x_n) = 0$. When C is infinite, we know that $U \otimes_C \overline{C}$ also satisfies the GPI $\Phi(x_1, \ldots, x_n) = 0$, where $\overline{C}$ is the algebraic closure of C. By [42, Theorems 2.5 and 3.5], both U and $U \otimes_C \overline{C}$ are prime and centrally closed, and hence we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite. Then R is centrally closed over C. By [78] R is a primitive ring having nonzero socle soc(R) with C as its associated division ring. Hence R is isomorphic to a dense ring of linear transformations of a vector space V over C (By Jacobson's theorem [63, p.75]).*

Chapter 2

Multiplicative (Generalized)-Derivations and Left Ideals in Semiprime Rings

2.1 Introduction

Over the last several years, a number of authors studied the commutativity in prime and semiprime rings admitting derivations and generalized derivations. Daif and Bell [22] proved that if R is a semiprime ring, I a nonzero ideal of R and d a nonzero derivation of R such that $d([x, y]) = \pm[x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. In particular, if $I = R$, then R is commutative. Herstein [61] proved that if R is a 2-torsion free prime ring with a nonzero derivation d of R such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Further, these results ware extended by replacing derivations with generalized derivations. Ashraf et al. [5] studied for a prime ring R with a generalized derivation F associated to a nonzero derivation d that if R satisfies any one of the following conditions: (1) $d(x) \circ F(y) = 0$, (2) $[d(x), F((y)] = 0$, (3) $d(x) \circ F(y) = x \circ y$, (4) $d(x) \circ F(y) + x \circ y = 0$, (5) $[d(x), F(y)] = [x, y]$, (6) $[d(x), F(y)] + [x, y] = 0$, (7) $d(x)F(y) \pm xy \in Z(R)$, for all $x, y \in I$, where I is a nonzero ideal of R, then R must be commutative. Dhara et al.[32]

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studied these identities in semiprime rings.

In the present chapter, we want to investigate these identities satisfied by left sided ideals in a semiprime ring. Moreover, the maps involved in the identities not necessarily to be additive. A multiplicative derivation of R is a mapping $D : R \to R$ which satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. So a multiplicative derivation will be a derivation when it is an additive map. Daif [21] introduced the concept of multiplicative derivation, which was motivated by the work of Martindale [79]. Further, Goldmann and Šemrl gave the complete description of these maps in [58]. Further, the notion of multiplicative derivation was extended in [23] to multiplicative generalized derivation. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative generalized derivation of R, if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. In the above definition, if we take d is any map (not necessarily additive or derivation), then F is said to be a multiplicative (generalized)-derivation which was introduced by Dhara and Ali [31]. In [31], Dhara and Ali gave the precise definition of multiplicative (generalized)-derivation as follows: A map $F : R \to R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation, if there exists a mapping $d : R \to R$ (not necessarily additive nor a derivation) such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

Example 2.1.1. *Let $\mathbb{Z}$ be the set of all integers. Next, let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$.

Define the maps F and $d : R \to R$ as follows: $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & ac^3 \\ 0 & 0 & 0 \end{pmatrix}$ *and*

$d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2 & b^2 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$. *Then it is straightforward to verify that F is a multiplicative (generalized)-derivation associated with a map d. Since F and d are not additive, thus F can not be generalized derivation as well as multiplicative generalized derivation.*

Thus multiplicative (generalized)-derivation covers the concept of generalized derivations and multiplicative generalized derivations. Therefore, multiplicative (generalized)-derivations are the large number of maps containing derivations, generalized derivations, multiplicative generalized derivation etc. Recently, few papers investigated identities involving multiplicative (generalized)-derivations (see [1], [3], [31], [38], [39], [57], [86]).

In the present chapter, our motivation is to study the identities replacing generalized derivation by multiplicative (generalized)-derivation acting on left sided ideals in semiprime rings. More precisely, we investigate the following identities: (1) $[d(x), F(y)] = \pm[x, y]$, (2) $[d(x), F(y)] = \pm x \circ y$, (3) $[d(x), F(y)] = 0$, (4) $F([x, y]) \pm [\delta(x), \delta(y)] \pm [x, y] = 0$, (5) $d'([x, y]) \pm [\delta(x), \delta(y)] \pm [x, y] = 0$, (6) $d'([x, y]) \pm [\delta(x), \delta(y)] = 0$, (7) $F(x \circ y) \pm \delta(x) \circ \delta(y) \pm x \circ y = 0$, (8) $d'(x \circ y) \pm \delta(x) \circ \delta(y) \pm x \circ y = 0$, (9) $d'(x \circ y) \pm \delta(x) \circ \delta(y) = 0$ for all $x, y \in \lambda$, where F is a multiplicative (generalized)-derivation of R associated to the map d, and δ, d' are multiplicative derivations of R.

2.2 Main Results

Theorem 2.2.1. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $d : R \to R$. If $[d(x), F(y)] = \pm[x, y]$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ for all $x \in \lambda$.*

In particular, when R is prime ring and d is a derivation of R, then $\lambda[\lambda, \lambda] = (0)$.

Proof. By our assumption, we have

$$[d(x), F(y)] = \pm[x, y] \tag{2.2.1}$$

for all $x, y \in \lambda$. Putting $x = xz$, where $z \in \lambda$, in (2.2.1) we get

$$[d(x)z + xd(z), F(y)] = \pm([x, y]z + x[z, y]) \tag{2.2.2}$$

for all $x, y, z \in \lambda$. This implies

$$\begin{aligned} d(x)[z, F(y)] + [d(x), F(y)]z + x[d(z), F(y)] \\ +[x, F(y)]d(z) = \pm([x, y]z + x[z, y]) \end{aligned} \tag{2.2.3}$$

for all $x, y, z \in \lambda$. Applying (2.2.1), (2.2.3) yields that

$$d(x)[z, F(y)] + [x, F(y)]d(z) = 0 \tag{2.2.4}$$

for all $x, y, z \in \lambda$. Substituting zx for x in (2.2.4), we get

$$(d(z)x + zd(x))[z, F(y)] + z[x, F(y)]d(z) + [z, F(y)]xd(z) = 0 \tag{2.2.5}$$

for all $x, y, z \in \lambda$. Left multiplying (2.2.4) by z, we obtain

$$zd(x)[z, F(y)] + z[x, F(y)]d(z) = 0 \tag{2.2.6}$$

for all $x, y, z \in \lambda$. Subtracting (2.2.6) from (2.2.5) we get

$$d(z)x[z, F(y)] + [z, F(y)]xd(z) = 0 \tag{2.2.7}$$

for all $x, y, z \in \lambda$, that is,

$$d(z)x[z, F(y)] = -[z, F(y)]xd(z) \tag{2.2.8}$$

for all $x, y, z \in \lambda$. Replacing x with $xd(z)t$ we have

$$d(z)xd(z)t[z, F(y)] = -[z, F(y)]xd(z)td(z) \tag{2.2.9}$$

for all $x, y, z \in \lambda$. Right multiplying (2.2.8) by $td(z)x[z, F(y)]$, we get

$$d(z)x[z, F(y)]td(z)x[z, F(y)] = -[z, F(y)]xd(z)td(z)x[z, F(y)] \tag{2.2.10}$$

for all $x, y, z \in \lambda$. By using (2.2.9), it yields

$$d(z)x[z, F(y)]td(z)x[z, F(y)] = d(z)xd(z)t[z, F(y)]x[z, F(y)] \tag{2.2.11}$$

for all $x, y, z \in \lambda$. By (2.2.8), we can write $d(z)t[z, F(y)] = -[z, F(y)]td(z)$ and hence (2.2.11) reduces to

$$d(z)x[z, F(y)]td(z)x[z, F(y)] = -d(z)x[z, F(y)]td(z)x[z, F(y)] \tag{2.2.12}$$

that is $2d(z)x[z, F(y)]td(z)x[z, F(y)] = 0$ for all $x, y, z, t \in \lambda$. Since R is 2-torsion free ring, $d(z)x[z, F(y)]td(z)x[z, F(y)] = 0$ for all $x, y, z, t \in \lambda$. This gives $td(z)x[z, F(y)]$ $Rtd(z)x[z, F(y)] = (0)$ for all $x, y, z, t \in \lambda$. Since R is semiprime ring, $td(z)x[z, F(y)] = 0$ for all $x, y, z, t \in \lambda$. So we can say that

$$\lambda d(z)\lambda[z, F(y)] = (0)$$

for all $y, z \in \lambda$. Let $\{P_\alpha | \alpha \in I\}$ be a family of prime ideals of R such that $\bigcap P_\alpha = (0)$. We can say either $\lambda d(z) \subseteq P_\alpha$ or $\lambda[z, F(y)] \subseteq P_\alpha$, so that $[z, F(y)]\lambda d(z) \subseteq P_\alpha$ or $d(z)\lambda[z, F(y)] \subseteq P_\alpha$. By (2.2.8), $[z, F(y)]\lambda d(z) \subseteq P_\alpha$ implies that $d(z)\lambda[z, F(y)] \subseteq P_\alpha$ and hence $d(z)\lambda[z, F(y)] \subseteq \bigcap P_\alpha$, that is, $d(z)\lambda[z, F(y)] = (0)$, for all $y, z \in \lambda$.

Hence we have $d(z)x[z, F(y)] = 0$ for all $x, y, z \in \lambda$. Putting $y = yz$, we get

$$0 = d(z)x[z, F(yz)] = d(z)x[z, F(y)z + yd(z)] = d(z)x[z, yd(z)],$$

for all $x, y, z \in \lambda$. Hence $[z, yd(z)]x[z, yd(z)] = 0$ for all $x, y, z \in \lambda$. Since R is a semiprime ring, it follows that $x[z, yd(z)] = 0$ for all $x, y, z \in \lambda$. Replacing y with $d(z)y$, we have

$$x[z, d(z)yd(z)] = 0 \tag{2.2.13}$$

for all $x, y, z \in \lambda$, that is,

$$x(zd(z)yd(z) - d(z)yd(z)z) = 0 \tag{2.2.14}$$

for all $x, y, z \in \lambda$. Replacing y by $yd(z)u$

$$x(zd(z)yd(z)ud(z) - d(z)yd(z)ud(z)z) = 0 \tag{2.2.15}$$

for all $x, y, z, u \in \lambda$. Using (2.2.14), we obtain

$$x(d(z)yd(z)zud(z) - d(z)yzd(z)ud(z)) = 0 \tag{2.2.16}$$

for all $x, y, z, u \in \lambda$, which gives

$$xd(z)y[d(z), z]ud(z) = 0 \tag{2.2.17}$$

for all $x, y, z, u \in \lambda$. This implies that $x[d(z), z]y[d(z), z]u[d(z), z] = 0$ for all $x, y, z, u \in \lambda$, that is, $(\lambda[d(z), z])^3 = (0)$ for all $z \in \lambda$. Since a semiprime ring contains no nonzero nilpotent left ideals, it follows that $\lambda[d(z), z] = (0)$ for all $z \in \lambda$.

In case R is prime ring and d is a derivation of R, we have from above that $x[z, yd(z)] = 0$ for all $x, y, z \in \lambda$. Replacing y with ty, where $t \in \lambda$, we have $0 = x[z, yd(z)] = x[z, t]yd(z)$. Since R is prime ring, for each $z \in \lambda$, either $\lambda[z, \lambda] = (0)$ or $\lambda d(z) = (0)$. Since $z \in \lambda$, for which $\lambda[z, \lambda] = (0)$ and $\lambda d(z) = (0)$ holds, form two additive subgroups of λ, union of which is λ, therefore, either $\lambda[\lambda, \lambda] = (0)$ or $\lambda d(\lambda) = (0)$.

In case $\lambda d(\lambda) = (0)$, we have by (2.2.1) replacing x with xt that $[d(x)t, F(y)] = \pm[xt, y]$ which yields $d(x)[t, F(y)] = \pm x[t, y]$. Left multiplying by λ, we obtain $\lambda x[t, y] = (0)$ for all $x, t, y \in \lambda$. Since R is prime, $\lambda[\lambda, \lambda] = (0)$. □

Corollary 2.2.2. *Let R be a 2-torsion free semiprime ring, $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $d : R \to R$. If $[d(x), F(y)] = \pm[x, y]$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$.*

Theorem 2.2.3. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $d : R \to R$. If $[d(x), F(y)] = \pm x \circ y$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ for all $x \in \lambda$.*

In particular, when R is prime ring and d is a derivation of R, then $\lambda[\lambda, \lambda] = (0)$.

Proof. By the hypothesis, we have

$$[d(x), F(y)] = \pm x \circ y \tag{2.2.18}$$

for all $x, y \in \lambda$. Replacing x by xz in (2.2.18) we obtain

$$[d(x)z + xd(z), F(y)] = \pm xz \circ y \tag{2.2.19}$$

which implies that

$$\begin{aligned} d(x)[z, F(y)] + [d(x), F(y)]z + x[d(z), F(y)] + [x, F(y)]d(z) \\ = \pm\{(x \circ y)z + x[z, y]\} \end{aligned} \tag{2.2.20}$$

for all $x, y, z \in \lambda$. In view of (2.2.18), the expression reduces to

$$d(x)[z, F(y)] + x[d(z), F(y)] + [x, F(y)]d(z) = \pm x[z, y] \tag{2.2.21}$$

for all $x, y, z \in \lambda$. Putting $x = zx$, we have

$$\begin{aligned} d(z)x[z, F(y)] + zd(x)[z, F(y)] + zx[d(z), F(y)] + z[x, F(y)]d(z) \\ +[z, F(y)]xd(z) = \pm zx[z, y]. \end{aligned} \tag{2.2.22}$$

Left multiplication of (2.2.21) by z yields

$$zd(x)[z, F(y)] + zx[d(z), F(y)] + z[x, F(y)]d(z) = \pm zx[z, y] \quad (2.2.23)$$

for all $x, y, z \in \lambda$. Subtracting (2.2.23) from (2.2.22), we have

$$d(z)x[z, F(y)] + [z, F(y)]xd(z) = 0 \quad (2.2.24)$$

for all $x, y, z \in \lambda$. The last expression is the same as the relation (2.2.7) and hence using the similar argument as used in the Theorem2.2.1, we get the required result. □

Corollary 2.2.4. *Let R be a 2-torsion free semiprime ring, $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $d : R \to R$. If $[d(x), F(y)] = \pm x \circ y$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$.*

Similarly, following theorem is straightforward.

Theorem 2.2.5. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $d : R \to R$. If $[d(x), F(y)] = 0$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ for all $x \in \lambda$.*

Theorem 2.2.6. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $\delta : R \to R$. If $d : R \to R$ is a multiplicative derivation of R such that $F([x, y]) \pm [d(x), d(y)] \pm [x, y] = 0$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ and $\lambda[\delta(x), x] = (0)$ for all $x \in \lambda$.*

Proof. First, we begin with the situation

$$F([x, y]) + [d(x), d(y)] + [x, y] = 0 \quad (2.2.25)$$

for all $x, y \in \lambda$. Replacing yx in place of y in (2.2.25), we obtain

$$F([x,y])x + [x,y]\delta(x) + [d(x), d(y)x] + [d(x), yd(x)] + [x,y]x = 0 \tag{2.2.26}$$

for all $x, y \in \lambda$. Right multiplying (2.2.25) by x, we get

$$F([x,y])x + [d(x), d(y)]x + [x,y]x = 0 \tag{2.2.27}$$

for all $x, y \in \lambda$. Now subtracting (2.2.27) from (2.2.26), we get

$$[x,y]\delta(x) + d(y)[d(x), x] + [d(x), yd(x)] = 0 \tag{2.2.28}$$

for all $x, y \in \lambda$. Substituting xy instead of y in (2.2.28), we obtain

$$x[x,y]\delta(x) + (xd(y) + d(x)y)[d(x), x] + x[d(x), yd(x)] + [d(x), x]yd(x) = 0 \tag{2.2.29}$$

for all $x, y \in \lambda$. Left multiplying (2.2.28) by x and then subtracting from (2.2.29), we get

$$d(x)y[d(x), x] + [d(x), x]yd(x) = 0 \tag{2.2.30}$$

for all $x, y \in \lambda$, that is

$$d(x)y[d(x), x] = -[d(x), x]yd(x) \tag{2.2.31}$$

for all $x, y \in \lambda$. Replacing y with $yd(x)t$, where $t \in \lambda$, we have

$$d(x)yd(x)t[d(x), x] = -[d(x), x]yd(x)td(x) \tag{2.2.32}$$

for all $x, y, t \in \lambda$. Right multiplying (2.2.31) by $td(x)y[d(x), x]$, we get

$$d(x)y[d(x), x]td(x)y[d(x), x] = -[d(x), x]yd(x)td(x)y[d(x), x] \tag{2.2.33}$$

for all $x, y, t \in \lambda$. By using (2.2.32), it yields

$$d(x)y[d(x), x]td(x)y[d(x), x] = d(x)yd(x)t[d(x), x]y[d(x), x] \tag{2.2.34}$$

for all $x, y, t \in \lambda$. By (2.2.31), we can write $d(x)t[d(x), x] = -[d(x), x]td(x)$ and hence (2.2.34) reduces to

$$d(x)y[d(x), x]td(x)y[d(x), x] = -d(x)y[d(x), x]td(x)y[d(x), x] \tag{2.2.35}$$

that is $2d(x)y[d(x), x]td(x)y[d(x), x] = 0$ for all $x, y, t \in \lambda$. Since R is 2-torsion free ring, $d(x)y[d(x), x]td(x)y[d(x), x] = 0$ for all $x, y, t \in \lambda$.

This implies that $td(x)y[d(x), x]Rtd(x)y[d(x), x] = (0)$ for all $x, y, t \in \lambda$. Since R is semiprime ring, $td(x)y[d(x), x] = 0$ for all $x, y, t \in \lambda$. This implies $y[d(x), x]Ry[d(x), x] = (0)$ for all $x, y \in \lambda$. Again, by semiprimeness of R, $\lambda[d(x), x] = (0)$ for all $x \in \lambda$.

Then replacing y with ty, $t \in R$ in (2.2.28), we obtain

$$\begin{aligned} t[x, y]\delta(x) + [x, t]y\delta(x) + (td(y) + d(t)y)[d(x), x] \\ +t[d(x), yd(x)] + [d(x), t]yd(x) = 0 \end{aligned} \tag{2.2.36}$$

for all $x, y \in \lambda$, $t \in R$. Left multiplying (2.2.28) by t and then subtracting from (2.2.36), we get

$$[x, t]y\delta(x) + d(t)y[d(x), x] + [d(x), t]yd(x) = 0 \tag{2.2.37}$$

for all $x, y \in \lambda$, $t \in R$. Using the fact $\lambda[d(x), x] = (0)$ for all $x \in \lambda$, we get

$$[x, t]y\delta(x) + [d(x), t]yd(x) = 0 \tag{2.2.38}$$

for all $x, y \in \lambda$, $t \in R$. Put $y = yx$ in (2.2.38) and get

$$[x, t]yx\delta(x) + [d(x), t]yxd(x) = 0 \tag{2.2.39}$$

for all $x, y \in \lambda$, $t \in R$. Right multiplying (2.2.38) by x and then subtracting from (2.2.39), we obtain by using $\lambda[d(x), x] = (0)$ for all $x \in \lambda$ that $[x, t]y[\delta(x), x] = 0$ for all $x, y \in \lambda$,

$t \in R$. In particular, $[\delta(x), x]y[\delta(x), x] = 0$, that is $y[\delta(x), x]Ry[\delta(x), x] = (0)$ for all $x, y \in \lambda$. By semiprimeness of R, $\lambda[\delta(x), x] = (0)$ for all $x \in \lambda$.

Using similar approach we conclude that the same result holds for all other cases. □

Corollary 2.2.7. *Let R be a 2-torsion free semiprime ring and $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $\delta : R \to R$. If $d : R \to R$ is a multiplicative derivation of R such that $F([x, y]) \pm [d(x), d(y)] \pm [x, y] = 0$ for all $x, y \in R$, then $[d(x), x] = 0$ and $[\delta(x), x] = 0$ for all $x \in R$.*

Replacing $F(x) = \delta(x)$ or $F(x) = \mp x + \delta(x)$, following Corollaries are straightforward.

Corollary 2.2.8. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $d, \delta : R \to R$ two multiplicative derivations of R. If $\delta([x, y]) \pm [d(x), d(y)] \pm [x, y] = 0$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ and $\lambda[\delta(x), x] = (0)$ for all $x \in \lambda$.*

Corollary 2.2.9. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $d, \delta : R \to R$ two multiplicative derivations of R. If $\delta([x, y]) \pm [d(x), d(y)] = 0$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ and $\lambda[\delta(x), x] = (0)$ for all $x \in \lambda$.*

Theorem 2.2.10. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $\delta : R \to R$. If $d : R \to R$ is a multiplicative derivation of R such that $F(x \circ y) \pm d(x) \circ d(y) \pm x \circ y = 0$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ and $\lambda[\delta(x), x] = (0)$ for all $x \in \lambda$.*

Proof. We begin with the situation

$$F(x \circ y) + d(x) \circ d(y) + x \circ y = 0 \tag{2.2.40}$$

for all $x, y \in \lambda$. Replacing yx by y in (2.2.40), we have

$$F(x \circ y)x + (x \circ y)\delta(y) + d(x) \circ (d(y)x + yd(x)) + (x \circ y)x = 0 \tag{2.2.41}$$

that is

$$F(x \circ y)x + (x \circ y)\delta(x) + (d(x) \circ d(y))x - d(y)[d(x), x]$$
$$+(d(x) \circ y)d(x) + (x \circ y)x = 0 \tag{2.2.42}$$

for all $x, y \in \lambda$. Right multiplying (2.2.40) by x and then subtracting from (2.2.42), we get

$$(x \circ y)\delta(x) - d(y)[d(x), x] + (d(x) \circ y)d(x) = 0 \tag{2.2.43}$$

for all $x, y \in \lambda$. Substituting xy instead of y in (2.2.43), we obtain

$$x(x \circ y)\delta(x) - xd(y)[d(x), x] - d(x)y[d(x), x]$$
$$+x(d(x) \circ y)d(x) - [d(x), x]yd(x) = 0 \tag{2.2.44}$$

for all $x, y \in \lambda$. Left multiplying (2.2.43) by x and then subtracting from (2.2.44), we get

$$d(x)y[d(x), x] + [d(x), x]yd(x) = 0 \tag{2.2.45}$$

for all $x, y \in \lambda$. This is same as the relation (2.2.30) and hence using the similar argument, we get the required result.

Using similar techniques with some necessary variations we can conclude that the same result holds for all other cases. □

Corollary 2.2.11. *Let R be a 2-torsion free semiprime ring and $F : R \to R$ a multiplicative (generalized)-derivation of R associated with the map $\delta : R \to R$. If $d : R \to R$ is a multiplicative derivation of R such that $F(x \circ y) \pm d(x) \circ d(y) \pm (x \circ y) = 0$ for all $x, y \in R$, then $[d(x), x] = 0$ and $[\delta(x), x] = 0$ for all $x \in R$.*

Replacing $F(x) = \delta(x)$ or $F(x) = \mp x + \delta(x)$, following Corollaries are straightforward.

Corollary 2.2.12. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $d, \delta : R \to R$ two multiplicative derivations of R. If $\delta(x \circ y) \pm d(x) \circ d(y) \pm x \circ y = 0$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ and $\lambda[\delta(x), x] = (0)$ for all $x \in \lambda$.*

Corollary 2.2.13. *Let R be a 2-torsion free semiprime ring, λ a nonzero left ideal of R and $d, \delta : R \to R$ two multiplicative derivations of R. If $\delta(x \circ y) \pm d(x) \circ d(y) = 0$ for all $x, y \in \lambda$, then $\lambda[d(x), x] = (0)$ and $\lambda[\delta(x), x] = (0)$ for all $x \in \lambda$.*

Example 2.2.1. *Let $\mathbb{Z}$ be the set of all integers. Next, let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$.

Note that R is not semiprime for $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = (0)$.

Define the maps $F, d : R \to R$ as follows: $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$ *and* $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b^2 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$. *Then F is a multiplicative (generalized)-derivation associated with the map d. Then we find that $[d(x), F(y)] = \pm[x, y]$ for all $x, y \in R$. It is easy to check that $[d(x), x] \neq 0$ for all $x \in R$. So semiprimeness hypothesis in Corollary 2.2.2 is not superfluous.*

Example 2.2.2. *Let $\mathbb{Z}$ be the set of all integers. Next, let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$.

It is clear from above that R is not semiprime ring.

Define the maps $F, d : R \to R$ *as follows:* $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ *and* $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b^2 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$. *Then* F *is a multiplicative (generalized)-derivation associated with the map* d. *Then we find that* $[d(x), F(y)] = \pm x \circ y$ *for all* $x, y \in R$. *Since* $[d(x), x] \neq 0$ *for all* $x \in R$ *the semiprimeness assumption in Corollary 2.2.4 is crucial.*

Example 2.2.3. *Let* $\mathbb{Z}$ *be the set of all integers. Next, let* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$. *It is clear from above that* R *is not semiprime ring.*

Define the maps $F, d, \delta : R \to R$ *as follows:* $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & ac^3 \\ 0 & 0 & 0 \end{pmatrix}$, $\delta\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2 & b^2 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$ *and* $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$. *Then* F *is a multiplicative (generalized)-derivation associated with the map* δ *and* d *is a multiplicative derivation of* R. *We find that (1)* $F([x, y]) + [d(x), d(y)] + [x, y] = 0$ *and (2)* $F(x \circ y) + d(x) \circ d(y) + x \circ y = 0$ *for all* $x, y \in R$. *It is easy to check that* $[d(x), x] \neq 0$ *and* $[\delta(x), x] \neq 0$ *for all* $x \in R$. *So the semiprimeness hypothesis in Corollary 2.2.7 and Corollary 2.2.11 is not superfluous.*

Chapter 3

Generalized Derivations Commuting on Lie Ideals in Prime Rings

3.1 Introduction

The study of commuting and centralizing maps was initiated by Posner [80]. The Posner's result [80] states that existence of a nonzero centralizing derivation in a prime ring implies the ring to be commutative. A number of authors have extended Posner's result in several ways. Brešar in [9], generalized Posner's result by considering co-centralizing derivations. Brešar proved in [9], that if d and δ are two nonzero co-centralizing derivations in a prime ring R (i.e., $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$), then R must be commutative. Later, Lee and Wong [68] studied the co-centralizing derivations in a noncentral Lie ideal of a prime ring R. More precisely Lee and Wong proved that if $d(x)x - x\delta(x) \in Z(R)$ for all $x \in L$, then either $d = \delta = 0$ or R satisfies s_4, where L is a noncentral Lie ideal of R.

In [11], Carini et al. studied a result with left annihilator condition and replacing derivations by generalized derivations. More precisely, authors proved in [11] that if R is a prime ring with char $(R) \neq 2$, L is a noncentral Lie ideal of R and H, G are two nonzero generalized derivations of R such that $a(H(u)u - uG(u)) = 0$ for all $u \in L$ and

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for some $0 \neq a \in R$, then one of the following holds:

1. there exist $b', c' \in U$ such that $G(x) = c'x$ and $H(x) = b'x + xc'$ with $ab' = 0$;
2. R satisfies s_4 and there exist $b', c', q' \in U$ such that $G(x) = c'x + xq'$ and $H(x) = b'x + xc'$ with $a(b' - q') = 0$.

Let $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C and F, G are two nonzero generalized derivations of R. For some $S \subseteq R$, denote the set $f(S) = \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in S\}$. In [4] Argaç and De Filippis studied the case $F(x)x - xG(x) = 0$ for all $x \in f(I)$ in prime ring R, where I is an ideal of R and then obtained the structures of the maps F and G. In [47], De Filippis and Dhara investigated the above result with central values i.e., $G(x)x - xH(x) \in C$ for all $x \in f(I)$ in prime ring and then determined the structures of the maps, where I is a nonzero right ideal of R. In a recent paper [82], Tiwari studied the situation $F(u)G(u) - uH(u) = 0$ for all $u \in f(R)$ and obtain all possible forms of the maps F and G. Like previous papers, it is natural to investigate the situation of [82] with central values i.e., $F(u)G(u) - uH(u) \in C$ for all $u \in f(R)$. In the present chapter, we study more general case on Lie ideal in prime ring. More precisely, we prove the following:

Theorem 3.1.1. *Let R be a prime ring of characteristic different from 2, U its Utumi ring of quotients, C its extended centroid, L a non-central Lie ideal of R and F, G and H three generalized derivations of R. If*

$$[F(u)G(u) - uH(u), u] = 0$$

for all $u \in L$, then one of the following holds:

(1) there exist $a, c, q, p, p' \in U$ such that $F(x) = ax$, $G(x) = cx + xq$ and $H(x) = px + xp'$, for all $x \in R$, with $a, ac - p, p' - aq \in C$;

(2) there exist $a, c, p \in U$ *such that* $F(x) = xa$, $G(x) = cx$ *and* $H(x) = px$, *for all* $x \in R$, *with* $ac - p \in C$;

(3) there exist $c, p \in U$, $\lambda, \mu \in C$ *and a derivation* h *of* R *such that* $F(x) = \mu x$, $G(x) = cx + \lambda h(x)$ *and* $H(x) = px + h(x)$, *for all* $x \in R$, *with* $\mu c - p \in C$ *and* $\lambda\mu = 1$;

(4) R *satisfies* s_4, *the standard identity of degree* 4.

As an immediate application of Theorem 3.1.1, when $G(x) = x$ for all $x \in R$, we have the following corollary. This gives particular case of Theorem 1 in [49].

Corollary 3.1.2. *Let* R *be a prime ring with char*$(R) \neq 2$, U *be the Utumi ring of quotients of* R *and* $C = Z(U)$ *be the extended centroid of* R. *Let* L *be a non-central Lie ideal of* R, F *and* H *be two non-zero generalized derivations of* R. *If*

$$[F(u)u - uH(u), u] = 0$$

for all $u \in L$, *then one of the following holds:*

(1) there exist $a, p \in U$ *such that* $F(x) = xa$ *and* $H(x) = px$ *for all* $x \in R$, *with* $a - p \in C$;

(2) R *satisfies* s_4.

As an immediate application of Theorem 3.1.1, when $H = 0$, we have the following corollary. This gives Theorem 1 in [13].

Corollary 3.1.3. *Let* R *be a prime ring with char*$(R) \neq 2$, U *be the Utumi ring of quotients of* R *and* $C = Z(U)$ *be the extended centroid of* R. *Let* L *be a non-central Lie ideal of* R, F *and* G *be two non-zero generalized derivations of* R. *If*

$$[F(u)G(u), u] = 0$$

for all $u \in L$, then one of the following holds:

(1) there exist $a, c \in U$ such that $F(x) = xa$ and $G(x) = cx$ with $ac \in C$;

(2) R satisfies s_4.

3.2 The Case of Inner Generalized Derivations

In this section, we consider the case when F, G and H are three inner generalized derivations of R, that is, $F(x) = ax + xb$, $G(x) = cx + xq$ and $H(x) = px + xp'$ for all $x \in R$, where $a, b, c, q, p, p' \in U$. Moreover, we consider the case when $L = [R, R]$. Then $[F(u)G(u) - uH(u), u] = 0$ for all $u \in [R, R]$ becomes

$$\begin{aligned}\big[a[x_1,x_2]c[x_1,x_2] + a[x_1,x_2]^2q + [x_1,x_2]m[x_1,x_2]\\ +[x_1,x_2]b[x_1,x_2]q - [x_1,x_2]^2p\prime, [x_1,x_2]\big] = 0 \qquad (3.2.1)\end{aligned}$$

for all $x_1, x_2 \in R$ and $m = bc - p$. This is a generalized polynomial identity (GPI) for R. Now we investigate this generalized polynomial identity.

Lemma 3.2.1. *Let K be a field with char$(K) \neq 2$, $R = M_m(K)$ be the ring of all $m \times m$ matrices over K, $Z(R)$ the center of R, a, b, c, m, p', q elements of R. If $m \geq 3$ and R satisfies (3.2.1), then one of the following holds:*

(1) $a \in Z(R)$;

(2) $c, q \in Z(R)$.

Proof. Say $a = \sum_{kl} a_{kl}e_{kl}$, $c = \sum_{kl} c_{kl}e_{kl}$ and $q = \sum_{kl} q_{kl}e_{kl}$ for $0 \neq a_{kl}, c_{kl}, q_{kl} \in K$. Let i, j, k be three different indices. Choose $[x_1, x_2] = [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$ in (3.2.1) and then we have

$$\big[a(e_{ii}-e_{jj})c(e_{ii}-e_{jj})+a(e_{ii}+e_{jj})q+(e_{ii}-e_{jj})m(e_{ii}-e_{jj})$$
$$+(e_{ii}-e_{jj})b(e_{ii}-e_{jj})q-(e_{ii}+e_{jj})p\prime, e_{ii}-e_{jj}\big]=0. \tag{3.2.2}$$

Right multiplying by e_{jj} and left multiplying by e_{kk} in the above relation, we get

$$a_{ki}(c_{ij}-q_{ij})-a_{kj}(c_{jj}+q_{jj})=0. \tag{3.2.3}$$

For any inner automorphism φ of $M_m(K)$, we have

$$\big[\varphi(a)[x_1,x_2]\varphi(c)[x_1,x_2]+\varphi(a)[x_1,x_2]^2\varphi(q)+[x_1,x_2]\varphi(m)[x_1,x_2]$$
$$+[x_1,x_2]\varphi(b)[x_1,x_2]\varphi(q)-[x_1,x_2]^2\varphi(p'),[x_1,x_2]\big] \tag{3.2.4}$$

is a GPI for R. In particular, let $\varphi(x)=(1+e_{ik})x(1-e_{ik})$ for all $x\in R$. If we denote $\varphi(a)=\sum_{kl}a'_{kl}e_{kl}$, $\varphi(c)=\sum_{kl}c'_{kl}e_{kl}$ and $\varphi(q)=\sum_{kl}q'_{kl}e_{kl}$ for a'_{kl}, c'_{kl}, $q'_{kl}\in K$, and using relation (3.2.3), it follows that

$$a'_{ki}(c'_{ij}-q'_{ij})-a'_{kj}(c'_{jj}+q'_{jj})=0$$

that is

$$a_{ki}(c_{kj}-q_{kj})=0. \tag{3.2.5}$$

We consider the automorphism $\psi(x)=(1+e_{ij})x(1-e_{ij})$ for all $x\in R$ and if we denote $\psi(a)=\sum_{kl}a''_{kl}e_{kl}$, $\psi(c)=\sum_{kl}c''_{kl}e_{kl}$ and $\psi(q)=\sum_{kl}q''_{kl}e_{kl}$ for a''_{kl}, c''_{kl}, $q''_{kl}\in K$, and using above we have

$$a''_{ki}(c''_{kj}-q''_{kj})=0. \tag{3.2.6}$$

This implies

$$a_{ki}(c_{ki}-q_{ki})=0. \tag{3.2.7}$$

Then by [51, Proposition 1], either $a \in Z(R)$ or $c - q \in Z(R)$.

Now we consider the automorphism $\chi(x) = (1 + e_{jk})x(1 - e_{jk})$ for all $x \in R$ and if we denote $\chi(a) = \sum_{kl} a'''_{kl} e_{kl}$, $\chi(c) = \sum_{kl} c'''_{kl} e_{kl}$ and $\chi(q) = \sum_{kl} q'''_{kl} e_{kl}$ for a'''_{kl}, c'''_{kl}, $q'''_{kl} \in K$, then using relation (3.2.3), it follows that

$$a'''_{ki}(c'''_{ij} - q'''_{ij}) - a'''_{kj}(c'''_{jj} + q'''_{jj}) = 0$$

that is

$$a_{kj}(c_{kj} + q_{kj}) = 0. \tag{3.2.8}$$

Then by [51, Proposition 1], either $a \in Z(R)$ or $c + q \in Z(R)$.

Thus if $a \notin Z(R)$, then $c + q, c - q \in Z(R)$, implying $c, q \in Z(R)$. Thus we conclude that either $a \in Z(R)$ or $c, q \in Z(R)$. □

Lemma 3.2.2. *Let R be a prime ring with $char(R) \neq 2$. If (3.2.1) is a trivial generalized polynomial identity for R, then one of the following holds:*

(1) $c = -q \in C$, $p', m + bq \in C$;

(2) $a, q, p', ac + m + bq \in C$.

Proof. By Theorem 1.5.2, we know that R and U satisfy the same generalized polynomial identities (GPIs). Therefore, U satisfies

$$\Big[a[x_1, x_2]c[x_1, x_2] + a[x_1, x_2]^2 q + [x_1, x_2]m[x_1, x_2] + [x_1, x_2]b[x_1, x_2]q - [x_1, x_2]^2 p', [x_1, x_2]\Big] = 0. \tag{3.2.9}$$

Since this is a trivial GPI for U, either $\{a, 1\}$ is linearly C-dependent or U satisfies

$$\Big(a[x_1, x_2]c[x_1, x_2] + a[x_1, x_2]^2 q\Big)[x_1, x_2] = 0 \tag{3.2.10}$$

that is

$$a[x_1,x_2]\big(c[x_1,x_2]+[x_1,x_2]q\big)[x_1,x_2]=0. \tag{3.2.11}$$

In the last case, $q \in C$ and $c+q=0$. Then by relation (3.2.9), U satisfies

$$\Big[[x_1,x_2](m+bq)[x_1,x_2]-[x_1,x_2]^2p\prime,[x_1,x_2]\Big]=0 \tag{3.2.12}$$

which is

$$[x_1,x_2]\Big\{(m+bq)[x_1,x_2]^2-[x_1,x_2](p\prime+m+bq)[x_1,x_2]+[x_1,x_2]^2p\prime\Big\}=0.$$

Since this is a trivial GPI for U, $p\prime, m+bq \in C$. This is our conclusion (1).

On the other hand, when $\{a,1\}$ is linearly C-dependent, i.e., $a \in C$, we have by (3.2.9)

$$\Big[[x_1,x_2](ac+m)[x_1,x_2]+[x_1,x_2]^2(aq-p\prime)+[x_1,x_2]b[x_1,x_2]q,[x_1,x_2]\Big]=0 \tag{3.2.13}$$

which is

$$\begin{aligned}&[x_1,x_2]\Big((ac+m)[x_1,x_2]^2+[x_1,x_2](aq-p\prime)[x_1,x_2]+b[x_1,x_2]q[x_1,x_2]\\&-[x_1,x_2](ac+m)[x_1,x_2]-[x_1,x_2]^2(aq-p\prime)-[x_1,x_2]b[x_1,x_2]q\Big)=0.\end{aligned} \tag{3.2.14}$$

This implies that $\{1, aq-p\prime, q\}$ is linearly C-dependent. Let $\alpha_1 1+\alpha_2(aq-p\prime)+\alpha_3 q=0$. Let $\alpha_2=0$. Then $q \in C$. By (3.2.14),

$$\begin{aligned}&[x_1,x_2]\Big((ac+m)[x_1,x_2]^2+[x_1,x_2](aq-p\prime)[x_1,x_2]+bq[x_1,x_2]^2\\&-[x_1,x_2](ac+m)[x_1,x_2]-[x_1,x_2]^2(aq-p\prime)-[x_1,x_2]bq[x_1,x_2]\Big)=0.\end{aligned} \tag{3.2.15}$$

This implies $aq-p\prime \in C$ and hence U satisfies

$$[x_1,x_2]\Big((ac+m+bq)[x_1,x_2]-[x_1,x_2](bq+ac+m)\Big)[x_1,x_2]=0.$$

This implies $ac+m+bq \in C$. Thus conclusion (2) is obtained.

Let $\alpha_2 \neq 0$. Then $\alpha_1 1+\alpha_2(aq-p\prime)+\alpha_3 q=0$ gives $aq-p\prime=\beta_1 q+\beta_2$, where $\beta_1,\beta_2 \in C$. Then by (3.2.14)

$$[x_1,x_2]\Big((ac+m)[x_1,x_2]^2+[x_1,x_2]\beta_1 q[x_1,x_2]+b[x_1,x_2]q[x_1,x_2] \\ -[x_1,x_2](ac+m)[x_1,x_2]-[x_1,x_2]^2\beta_1 q-[x_1,x_2]b[x_1,x_2]q\Big)=0. \qquad (3.2.16)$$

Since this is trivial GPI for U, q must be in C and hence U satisfies

$$[x_1,x_2]\Big((ac+m+bq)[x_1,x_2]-[x_1,x_2](ac+m+bq)\Big)[x_1,x_2]=0. \qquad (3.2.17)$$

This implies $ac+m+bq \in C$. This is our conclusion (2).

□

Lemma 3.2.3. *[49, Lemma 3] Let R be a prime with char$(R)\neq 2$. If $a,b,c,q \in U$ such that*

$$[a[r_1,r_2]+[r_1,r_2]b,[r_1,r_2]]_k[r_1,r_2]-[r_1,r_2][c[r_1,r_2]+[r_1,r_2]q,[r_1,r_2]]_k=0$$

for all $r_1,r_2 \in R$, then one of the following holds:

(1) $a,q,b-c \in C$;

(2) R satisfies s_4 and $a-b+c-q \in C$.

Lemma 3.2.4. *Let R be a prime ring with char$(R)\neq 2$ and C be its extended centroid. Let $a,b,c,q \in R$. If R satisfies a nontrivial GPI*

$$\big[[x_1,x_2]a[x_1,x_2]+[x_1,x_2]^2c+[x_1,x_2]b[x_1,x_2]q,[x_1,x_2]\big]=0$$

then one of the following holds:

(1) $a, b, bq + c \in C$;

(2) $q, c, a + bq \in C$;

(3) R *satisfies* s_4.

Proof. First we aim to prove that either $b \in C$ or $q \in C$. By Theorem 1.5.2, U satisfies

$$\big[[x_1, x_2]a[x_1, x_2] + [x_1, x_2]^2 c + [x_1, x_2]b[x_1, x_2]q, [x_1, x_2]\big] = 0. \qquad (3.2.18)$$

Let $\overline{C}$ be the algebraic closure of C. Since U satisfies (3.2.18), $U \otimes_C \overline{C}$ also satisfies (3.2.18) (see Fact 4.2.1). In view of [42, Theorems 2.5 and 3.5], both U and $U \otimes_C \overline{C}$ are prime and centrally closed. We may replace R by U or $U \otimes_C \overline{C}$ according as C finite or infinite. Then R is centrally closed over C. By Martindale's theorem [78], R is then a primitive ring with a nonzero socle $soc(R)$ and C as its associated division ring. In light of Jacobson's theorem [63, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. If $\dim_C V = k$, by density of R, $R \cong M_k(C)$, the ring of all $k \times k$ matrices over C. In this case, we substitute $[x_1, x_2] = [e_{ij}, e_{jj}] = e_{ij}$ for any $i \neq j$ in above relation, we have $e_{ij}be_{ij}qe_{ij} = 0$, which implies $b_{ji}q_{ji} = 0$. Then by [51, Proposition 1], either $b \in Z(R)$ or $q \in Z(R)$.

Next, we assume that $\dim_C V = \infty$. If $b \in C$ or $q \in C$, then we are done. So we assume that $b \notin C$ and $q \notin C$. Hence there exists $r_1, r_2 \in soc(R)$ such that $[b, r_1] \neq 0$ and $[q, r_2] \neq 0$.

By Litoff's theorem (Theorem 1.6.7), there exists an idempotent $e^2 = e \in soc(R)$ such that $r_1, r_2, br_1, r_1b, qr_2, r_2q \in eRe$; where $eRe \cong M_k(C)$, the ring of all $k \times k$ matrices over C. Note that eRe satisfies $\big[[x_1, x_2]eae[x_1, x_2] + [x_1, x_2]^2 ece + [x_1, x_2]ebe[x_1, x_2]eqe, [x_1, x_2]\big] = 0$. Thus by above finite dimensional case, we conclude that $ebe, eqe \in Ce$. Then

$br_1 = (ebe)r_1 = r_1(ebe) = r_1b$ and $qr_2 = (eqe)r_2 = r_2eqe = r_2q$ contradicting with the choices of r_1 and r_2.

Thus we conclude that in any case, either $b \in C$ or $q \in C$. We consider the following two cases:

Case-I. Suppose $b \in C$.

Then U satisfies

$$[[x_1, x_2]a[x_1, x_2] + [x_1, x_2]^2(bq + c), [x_1, x_2]] = 0,$$

that is

$$\Big[[x_1, x_2]\Big(a[x_1, x_2] + [x_1, x_2](bq + c)\Big), [x_1, x_2]\Big] = 0.$$

Hence by Lemma 3.2.3 both a and $bq + c \in C$ or R satisfies s_4.

Case-II. Suppose $q \in C$.

Then U satisfies

$$[[x_1, x_2](a + bq)[x_1, x_2] + [x_1, x_2]^2c, [x_1, x_2]] = 0,$$

that is

$$\Big[[x_1, x_2]\Big((a + bq)[x_1, x_2] + [x_1, x_2]c\Big), [x_1, x_2]\Big] = 0.$$

Hence by Lemma 3.2.3 both $a + bq$ and $c \in C$ or R satisfies s_4. □

Lemma 3.2.5. *Let R be a prime ring with char$(R) \neq 2$ and a, b, c, m, p', q are elements of R. Assume that $a \in C$. If R satisfies a nontrivial GPI (3.2.1), then one of the following holds:*

(1) $b, ac + m, bq + aq - p' \in C$;

(2) $q, aq - p', ac + m + bq \in C$;

(3) R *satisfies* s_4.

Proof. Since $a \in Z(R)$, by (3.2.1) R and so U satisfies

$$\big[[x_1, x_2](ac + m)[x_1, x_2] + [x_1, x_2]^2(aq - p') + [x_1, x_2]b[x_1, x_2]q, [x_1, x_2]\big] = 0.$$

By Lemma 3.2.4, one of the following holds:

(i) $ac + m, b, bq + aq - p' \in C$;

(ii) $q, aq - p', ac + m + bq \in C$;

(iii) R satisfies s_4.

□

Lemma 3.2.6. *Let R be a prime ring with $char(R) \neq 2$ and a, b, c, m, p', q are elements of R. Assume that $c, q \in C$. If R satisfies (3.2.1), then one of the following holds:*

(1) $a, p', m + bq \in C$;

(2) $c + q = 0, p', m + bq \in C$;

(3) R *satisfies* s_4.

Proof. Since $c, q \in C$, by (3.2.1) R and so U satisfies

$$\Big[a(c + q)[x_1, x_2]^2 + [x_1, x_2](m + bq)[x_1, x_2] - [x_1, x_2]^2 p\prime, [x_1, x_2]\Big] = 0. \quad (3.2.19)$$

This is

$$\Big[a(c + q)[x_1, x_2] + [x_1, x_2](m + bq), [x_1, x_2]\Big][x_1, x_2]$$
$$-[x_1, x_2]\Big[[x_1, x_2]p', [x_1, x_2]\Big] = 0 \quad (3.2.20)$$

Hence by Lemma 3.2.3, one of the following holds:

(i) $a(c+q), p', m+bq \in C$; In this case, if $c+q \neq 0$, then $a \in C$. Thus we have conclusions (1) and (2).

(ii) R satisfies s_4. □

Proposition 3.2.7. *Let R be a prime ring with $char(R) \neq 2$. If (3.2.1) is a generalized polynomial identity for R, then one of the following holds:*

(1) $a, b, ac+m, bq+aq-p' \in C$;

(2) $a, q, p', ac+m+bq \in C$;

(3) $c, q, a, p', m+bq \in C$;

(4) $c, q, p', m+bq \in C$ with $c+q=0$;

(5) R satisfies s_4.

Proof. If (3.2.1) is a trivial generalized polynomial identity for R, then the conclusions follows by Lemma 3.2.2. So we assume that (3.2.1) is nontrivial GPI for R. By Fact 1.6.8, R is isomorphic to a dense ring of linear transformations of a vector space V over C. If $\dim_C V = k$, by density of R, $R \cong M_k(C)$, the ring of all $k \times k$ matrices over C. If $k = 2$, we have our conclusion (5), otherwise by Lemma 3.2.1, either $a \in C$ or $c, q \in C$. Then the conclusions follows by Lemma 3.2.5 and Lemma 3.2.6.

Next we assume that $\dim_C V = \infty$. If $a \in C$ or $c, q \in C$, then the conclusions follows by Lemma 3.2.5 and Lemma 3.2.6. Thus, we assume that

(i) $a \notin C$;

and (ii) either $c \notin C$ or $q \notin C$.

Because of infinite dimensionality of V, R and so $soc(R)$ can not satisfy any polynomial identities.

We know that if $[r,x]=0$ for all $x\in soc(R)$, then $r\in C$. Thus there exist $t_1,t_2,t_3,t_4,t_5,t_6,t_7\in soc(R)$ such that

(i) $[a,t_1]\neq 0$;

(ii) either $[c,t_2]\neq 0$ or $[q,t_3]\neq 0$;

(iii) $s_4(t_4,t_5,t_6,t_7)\neq 0$.

By Litoff's theorem (Theorem 1.6.7), there exists $e^2=e\in soc(R)$ such that

- $t_1,t_2,t_3,t_4,t_5,t_6,t_7\in eRe$;
- $at_1,t_1a,at_2,t_2a,at_3,t_3a,at_4,t_4a,at_5,t_5a,at_6,t_6a,at_7,t_7a\in eRe$;
- $ct_1,t_1c,ct_2,t_2c,ct_3,t_3c,ct_4,t_4c,ct_5,t_5c,ct_6,t_6c,ct_7,t_7c\in eRe$;
- $qt_1,t_1q,qt_2,t_2q,qt_3,t_3q,qt_4,t_4q,qt_5,t_5q,qt_6,t_6q,qt_7,t_7q\in eRe$,

where $eRe\cong M_k(C)$, the ring of all $k\times k$ matrices over C. Note that eRe satisfies

$$\big[eae[x_1,x_2]ece[x_1,x_2]+eae[x_1,x_2]^2eqe+[x_1,x_2]eme[x_1,x_2]$$
$$+[x_1,x_2]ebe[x_1,x_2]eqe-[x_1,x_2]^2ep'e,[x_1,x_2]\big]=0.$$

Then by Lemma 3.2.1, one of the following holds:

(i) $eae\in eC$, which contradicts with the choice of $t_1\in soc(R)$;

(ii) $ece,eqe\in eC$, which contradicts with the choices of $t_2,t_3\in soc(R)$;

(iii) eRe satisfies s_4, which contradicts with the choices of $t_4,t_5,t_6,t_7\in soc(R)$.

Thus the proof is completed. □

3.3 Proof of Theorem 3.1.1

To prove the theorem, we need the following Propositions.

Proposition 3.3.1. *Let R be a prime ring with $char(R)\neq 2$. Let U be the Utumi ring of quotients of R and $C=Z(U)$ be the extended centroid of R, L a non-central Lie ideal of*

R. If F and G are two non-zero generalized derivations of R and $H(x) = px + xp'$ for all $x \in R$, for some $p, p' \in U$ such that

$$[F(u)G(u) - uH(u), u] = 0$$

for all $u \in L$, then one of the following holds:

(1) there exist $a, c, q \in U$ such that $F(x) = ax$ and $G(x) = cx + xq$ for all $x \in R$, with $a, ac - p, -aq + p' \in C$;

(2) there exist $a, c \in U$ such that $F(x) = xa$ and $G(x) = cx$ for all $x \in R$, with $p', ac - p \in C$;

(3) R satisfies s_4.

Proof. By [7, Lemma 1], there exists a nonzero ideal I of R such that $0 \neq [I, I] \subseteq L$. By assumption, I satisfies

$$[F([x, y])G([x, y]) - [x, y]H([x, y]), [x, y]] = 0.$$

By Theorem 1.6.2, there exist $a, c \in U$ and derivations d, g of U such that $F(x) = ax + d(x)$ and $G(x) = cx + g(x)$. By Theorem 1.5.2, we have

$$\Big[(a[x, y] + d([x, y]))(c[x, y] + g([x, y])) - [x, y](p[x, y] + [x, y]p'), [x, y]\Big] = 0 \quad (3.3.1)$$

for all $x, y \in U$. If d and g are inner derivations, i.e., $d(x) = [b, x]$ and $g(x) = [q, x]$ for all $x \in U$, then $F(x) = (a + b)x - xb$ and $G(x) = (c + q)x - xq$ for all $x \in U$. Then by Proposition 3.2.7 one of the following holds:

- $a + b, b, (a + b)(c + q) - b(c + q) - p, bq - (a + b)q - p' \in C$; Thus $F(x) = ax$ and $G(x) = (c + q)x - xq$ with $a, a(c + q) - p, aq + p' \in C$. This is conclusion (1).

- $a+b, q, p', (a+b)(c+q) - b(c+q) - p + bq \in C$; Thus $F(x) = (a+b)x - xb = xa$ and $G(x) = cx$ with $p', ac - p \in C$. This is conclusion (2).
- $c+q, q, a+b, p', -b(c+q) - p + bq \in C$; Thus $F(x) = xa$ and $G(x) = cx$ with $c, p', ac - p \in C$. This gives conclusion (2).
- $c+q, q, p', -b(c+q) - p + bq \in C$ with $c = 0$; Thus $F(x) = (a+b)x - xb$ and $G(x) = cx = 0$, a contradiction.
- R satisfies s_4. This is conclusion (3).

Next assume that d and g are not both inner derivations. We consider the following two cases:

Case-I: Let d and g be C-dependent modulo inner derivations of U, that is $\alpha d + \beta g = ad_m$, where $\alpha, \beta \in C$ and $ad_m(x) = [m, x]$ for all $x \in U$.

Subcase-i: Let $\alpha = 0$. Then $g(x) = [m', x]$ for all $x \in U$, where $m' = \beta^{-1}m$. Since g is inner, d must be outer. By (3.3.1), we have

$$\Big[(a[x,y] + d([x,y]))(c[x,y] + [m',[x,y]]) - [x,y](p[x,y] + [x,y]p'), [x,y]\Big] = 0 \quad (3.3.2)$$

for all $x, y \in U$. Since d is outer, by Theorem 1.6.4,

$$\Big[(a[x,y] + [s,y] + [x,t])(c[x,y] + [m',[x,y]]) - [x,y](p[x,y] + [x,y]p'), [x,y]\Big] = 0$$

for all $x, y, s, t \in U$. In particular, U satisfies blended component

$$\Big[([s,y] + [x,t])(c[x,y] + [m',[x,y]]), [x,y]\Big] = 0$$

for all $x, y, s, t \in U$. Replacing s with $[a', x]$ and t with $[a', y]$ for some $a' \in U - C$, we have

$$\Big[[a',[x,y]](c[x,y] + [m',[x,y]]), [x,y]\Big] = 0$$

for all $x, y \in U$. By Proposition 3.2.7, one of the following holds: (i) $a' \in C$, a contradiction; (ii) $m' \in C$ and $c = 0$. In this case $G = 0$, a contradiction. (iii) R satisfies s_4, as desired.

Subcase-ii: Let $\alpha \neq 0$. Then $d(x) = \lambda g(x) + [b', x]$ for all $x \in U$, where $\lambda = -\alpha^{-1}\beta$ and $b' = \alpha^{-1}m$. By (3.3.1), we have

$$\Big[(a[x,y] + \lambda g([x,y]) + [b',[x,y]])(c[x,y] + g([x,y])) - [x,y](p[x,y] + [x,y]p'), [x,y]\Big] = 0 \tag{3.3.3}$$

for all $x, y \in U$. By Theorem 1.6.4, U satisfies

$$\Big[(a[x,y] + \lambda([s,y] + [x,t]) + [b',[x,y]])(c[x,y] + [s,y] + [x,t]) - [x,y](p[x,y] + [x,y]p'), [x,y]\Big] = 0. \tag{3.3.4}$$

In particular, U satisfies blended component (for $s = 0$)

$$\Big[\lambda[s,y](c[x,y] + [s,y] + [x,t]) + (a[x,y] + \lambda([s,y] + [x,t]) + [b',[x,y]])[s,y], [x,y]\Big] = 0.$$

Again from above U satisfies blended component (for $t = 0$)

$$\Big[\lambda[s,y][x,t] + \lambda[x,t][s,y], [x,y]\Big] = 0. \tag{3.3.5}$$

This is a PI. Hence for $s = e_{31}$, $y = e_{12}$, $x = e_{21}$, $t = e_{11}$, we have $\Big[\lambda[s,y][x,t] + \lambda[x,t][s,y], [x,y]\Big] = -\lambda e_{31} = 0$ implying $\lambda = 0$. Hence from (3.3.4),

$$\Big[(a[x,y] + [b',[x,y]])(c[x,y] + [s,y] + [x,t]) - [x,y](p[x,y] + [x,y]p'), [x,y]\Big] = 0. \tag{3.3.6}$$

Replacing s with $[q,x]$ and t with $[q,y]$, for some $q \notin C$ we have

$$\Big[(a[x,y]+[b',[x,y]])(c[x,y]+[q,[x,y]]) - [x,y](p[x,y]+[x,y]p'),[x,y]\Big]=0. \tag{3.3.7}$$

Then by Proposition 3.2.7, since $q \notin C$, we must have $b' \in C$. Hence $d=0$. Thus from (3.3.6), U satisfies

$$\Big[a[x,y](c[x,y]+[s,y]+[x,t]) - [x,y](p[x,y]+[x,y]p'),[x,y]\Big]=0. \tag{3.3.8}$$

In particular, U satisfies

$$\Big[a[x,y]([s,y]+[x,t]),[x,y]\Big]=0. \tag{3.3.9}$$

For $t=0$ and $s=x$, we have $[a,[x,y]][x,y]^2=0$ which implies $a \in C$ (see [34, Lemma 3]). Since $F(x)=ax$ and F is nonzero, we must have $a \neq 0$ and hence U satisfies $[[x,y]([s,y]+[x,t]),[x,y]]=0$. This is a polynomial identity (PI) and hence by [67, Lemma 2] there exists a field K such that $U \subseteq M_n(K)$, $n>1$ and the matrix ring $M_n(K)$ satisfies $[[x,y]([s,y]+[x,t]),[x,y]]=0$. But for $x=e_{12}, y=e_{22}, s=e_{21}, t=0$, we have $0=[[x,y]([s,y]+[x,t]),[x,y]]=-e_{12}$, a contradiction.

Case-II: Let d and g be C-independent modulo inner derivation of U.

Applying Theorem 1.6.4 to (3.3.1), U satisfies

$$\Big[(a[x,y]+[u,y]+[x,v])(c[x,y]+[s,y]+[x,t]) - [x,y](p[x,y]+[x,y]p'),[x,y]\Big]=0.$$

In particular, U satisfies blended components

$$\Big[([u,y]+[x,v])([s,y]+[x,t]),[x,y]\Big]=0. \tag{3.3.10}$$

Assuming $u=x, v=0$, U satisfies $[[x,y]([s,y]+[x,t]),[x,y]]=0$, which leads to a contradiction, as above.

□

Proposition 3.3.2. *Let R be a prime ring with $char(R) \neq 2$. Let U be the Utumi ring of quotients of R and $C = Z(U)$ be the extended centroid of R, L a non-central Lie ideal of R. If G and H are two non-zero generalized derivations of R and $F(x) = ax + xb$ for all $x \in R$, for some $a, b \in U$ such that*

$$[F(u)G(u) - uH(u), u] = 0$$

for all $u \in L$, then one of the following holds:

(1) there exist $a', c, q, p, p' \in U$ such that $F(x) = a'x$, $G(x) = cx + xq$ and $H(x) = px + xp'$ for all $x \in R$, with $a', a'c - p, -a'q + p' \in C$;

(2) there exist $a', c, p \in U$ such that $F(x) = xa'$, $G(x) = cx$ and $H(x) = px$ for all $x \in R$, with $a'c - p \in C$;

(3) there exist $p, c \in U$, $\lambda, \mu \in C$ and a derivation h of R such that $F(x) = \mu x$, $G(x) = cx + \lambda h(x)$ and $H(x) = px + h(x)$ for all $x \in R$, with $\mu c - p \in C$ and $\lambda\mu = 1$;

(4) R satisfies s_4.

Proof. If H is inner, then conclusion follows by Proposition 3.3.1. Thus we assume that H is outer. Again by [7, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, I] \subseteq L$. Thus I satisfies

$$[(a[x, y] + [x, y]b)G([x, y]) - [x, y]H([x, y]), [x, y]] = 0.$$

By Theorem 1.6.2, there exist $a, c \in U$ and derivations d, g of U such that $G(x) = cx+g(x)$ and $H(x) = px + h(x)$. By Theorem 1.5.2, we have

$$\Big[(a[x, y] + [x, y]b)(c[x, y] + g([x, y])) - [x, y](p[x, y] + h([x, y])), [x, y]\Big] = 0 \quad (3.3.11)$$

for all $x, y \in U$. We consider the following two cases:

Case-I: Let g and h be C-dependent modulo inner derivation of U, that is $\alpha g + \beta h = ad_m$, where $\alpha, \beta \in C$ and $ad_m(x) = [m, x]$ for all $x \in U$.

If $\alpha = 0$, then h is inner, a contradiction. Thus assume that $\alpha \neq 0$. Then $g(x) = \lambda h(x) + [b', x]$ for all $x \in U$, where $\lambda = -\alpha^{-1}\beta$ and $b' = \alpha^{-1}m$. By (3.3.11), we have

$$\Big[(a[x,y]+[x,y]b)(c[x,y]+\lambda h([x,y])+[b',[x,y]]) \\ -[x,y](p[x,y]+h([x,y])),[x,y]\Big]=0 \qquad (3.3.12)$$

for all $x, y \in U$. By Theorem 1.6.4, U satisfies

$$\Big[(a[x,y]+[x,y]b)(c[x,y]+\lambda([s,y]+[x,t])+[b',[x,y]]) \\ -[x,y](p[x,y]+[s,y]+[x,t]),[x,y]\Big]=0.$$

In particular, U satisfies blended component

$$\Big[(a[x,y]+[x,y]b)(\lambda([s,y]+[x,t]))-[x,y]([s,y]+[x,t]),[x,y]\Big]=0 \qquad (3.3.13)$$

and thus for $t = 0$

$$\Big[\lambda(a[x,y]+[x,y]b)[s,y]-[x,y][s,y],[x,y]\Big]=0. \qquad (3.3.14)$$

In particular for $s = x$, we have

$$\Big[\lambda(a[x,y]+[x,y]b),[x,y]\Big][x,y]=0.$$

Then by [27, Lemma 3] one of the following holds: (i) $a, b \in C$; (ii) R satisfies s_4 and $a - b \in C$. The last case gives our conclusion (4). In the first case, when $a, b \in C$, then by (3.3.14), $(\lambda(a + b) - 1)[x, y][[s, y], [x, y]] = 0$. This implies either $\lambda(a + b) = 1$ or $[x, y][[s, y], [x, y]] = 0$.

Let $[x,y][[s,y],[x,y]]=0$. This is a polynomial identity (PI) and hence by [67, Lemma 2] there exists a field K such that $U \subseteq M_n(K)$, $n>1$ and the matrix ring $M_n(K)$ satisfies $[x,y][[s,y],[x,y]]=0$. But for $x=e_{12}$, $y=e_{21}$ and $s=e_{11}-e_{22}$, we have $0=[x,y][[s,y],[x,y]]=4e_{21}$ which leads to a contradiction, as $char(R)\neq 2$.

Let $\lambda(a+b)=1$. Then by (3.3.12)

$$\Big[(a+b)[x,y](c[x,y]+[b',[x,y]])-[x,y]p[x,y],[x,y]\Big]=0. \tag{3.3.15}$$

Which gives

$$\Big[[x,y]((a+b)c+b'-p)[x,y]-[x,y]^2(a+b)b',[x,y]\Big]=0 \tag{3.3.16}$$

for all $x,y\in U$. We re-write it as

$$[x,y]\Big[((a+b)c+b'-p)[x,y]-[x,y](a+b)b',[x,y]\Big]=0 \tag{3.3.17}$$

for all $x,y\in U$. Then by Lemma 3.2.3, one of the following holds:

(i) $(a+b)c+b'-p\in C$ and $(a+b)b'\in C$; In this case if $a+b=0$ then $F=0$, a contradiction. Thus assume that $0\neq a+b=\mu\in C$. Then $b'\in C$ and $\mu c-p\in C$. Hence $F(x)=\mu x$, $G(x)=cx+\lambda h(x)$ and $H(x)=px+h(x)$ for all $x\in R$ with $\lambda,\mu\in C$, $\mu c-p\in C$ and $\lambda\mu=1$. This is our conclusion (3).

(ii) R satisfies s_4, as desired in conclusion (4).

Case-II: Let g and h be C-independent modulo inner derivations of U.

Applying Theorem 1.6.4 to (3.3.11), U satisfies

$$\Big[(a[x,y]+[x,y]b)(c[x,y]+([s,y]+[x,t]))-[x,y](p[x,y]+([u,y]+[x,v])),[x,y]\Big]=0.$$

In particular, U satisfies blended components

$$\Big[[x,y]([u,y]+[x,v]),[x,y]\Big]=0, \tag{3.3.18}$$

which leads to a contradiction, as above.

□

Proposition 3.3.3. *Let R be a prime ring with $char(R) \neq 2$. Let U be Utumi ring of quotients and $C = Z(U)$ be the extended centroid of R, L a non-central Lie ideal of R. If F and H are two non-zero generalized derivations of R and $G(x) = cx + xq$ for all $x \in R$ such that*

$$[F(u)G(u) - uH(u), u] = 0$$

for all $u \in L$, then one of the following holds:

(1) *there exist $a, c, q, p, p' \in U$ such that $F(x) = ax$, $G(x) = cx+xq$ and $H(x) = px+xp'$ for all $x \in R$, with $a, ac - p, -aq + p' \in C$;*

(2) *there exist $a, c', p \in U$ such that $F(x) = xa$, $G(x) = c'x$ and $H(x) = px$ for all $x \in R$, with $ac' - p \in C$;*

(3) *there exist $p, c' \in U$, $\lambda, \mu \in C$ and a derivation h such that $F(x) = \mu x$, $G(x) = c'x + \lambda h(x)$ and $H(x) = px + h(x)$ for all $x \in R$, with $\mu c' - p \in C$ and $\lambda\mu = 1$;*

(4) *R satisfies s_4.*

Proof. If H is inner or F is inner, then conclusion follows by Proposition 3.3.1 and Proposition 3.3.2. Thus we assume that H as well as F are outer. By [7, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, I] \subseteq L$. Thus I satisfies

$$[F([x, y])(c[x, y] + [x, y]q) - [x, y]H([x, y]), [x, y]] = 0.$$

By Theorem 1.6.2, there exist $a, c \in U$ and derivations d, g of U such that $F(x) = ax+d(x)$ and $H(x) = px + h(x)$. By Theorem 1.5.2, we have

$$\Big[(a[x, y] + d([x, y]))(c[x, y] + [x, y]q) - [x, y](p[x, y] + h([x, y])), [x, y]\Big] = 0 \quad (3.3.19)$$

for all $x, y \in U$. We consider the following two cases:

Case-I: Let d and h be C-dependent modulo inner derivations of U, that is $\alpha d + \beta h = ad_m$, where $\alpha, \beta \in C$ and $ad_m(x) = [m, x]$ for all $x \in U$.

If $\beta = 0$, then d is inner, a contradiction. Thus assume that $\beta \neq 0$. Then $h(x) = \lambda d(x) + [b', x]$ for all $x \in U$, $\lambda \in C$. By (3.3.19), we have

$$\Big[(a[x,y] + d([x,y]))(c[x,y] + [x,y]q) - [x,y](p[x,y] + \lambda d([x,y]) + [b',[x,y]]), [x,y]\Big] = 0$$

for all $x, y \in U$. By Theorem 1.6.4, U satisfies

$$\Big[(a[x,y] + [s,y] + [x,t])(c[x,y] + [x,y]q)$$
$$-[x,y](p[x,y] + \lambda([s,y] + [x,t]) + [b',[x,y]]), [x,y]\Big] = 0.$$

In particular, U satisfies blended component

$$\Big[([s,y] + [x,t])(c[x,y] + [x,y]q) - [x,y](\lambda([s,y] + [x,t])), [x,y]\Big] = 0.$$

Replacing s with $[a', x]$ and t with $[a', y]$, for some $a' \notin C$, from above U satisfies

$$\Big[[a',[x,y]](c[x,y] + [x,y]q) - [x,y][\lambda a', [x,y]], [x,y]\Big] = 0.$$

Then by Proposition 3.2.7, either $c, q, \lambda a' \in C$ with $c + q = 0$ or R satisfies s_4. We are to consider the first case, for which we have $\lambda = 0$. Thus h becomes inner, a contradiction.

Case-II: Let d and h be C-independent modulo inner derivations of U.

Applying Theorem 1.6.4 to (3.3.19), U satisfies

$$\Big[(a[x,y] + ([s,y] + [x,t]))(c[x,y] + [x,y]q) - [x,y](p[x,y] + ([u,y] + [x,v])), [x,y]\Big] = 0.$$

In particular, U satisfies blended components

$$\Big[[x,y]([u,y] + [x,v]), [x,y]\Big] = 0, \tag{3.3.20}$$

which leads to a contradiction, as above.

□

Proof of Theorem 3.1.1 Since L is a noncentral Lie ideal of R, by [7, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, I] \subseteq L$. Thus I satisfies

$$[F([x, y])G([x, y]) - [x, y]H([x, y]), [x, y]] = 0$$

for all $x, y \in I$. By Theorem 1.6.2, there exist $a, b, c \in U$ and derivations d, g, h of U such that $F(x) = ax + d(x)$, $G(x) = bx + g(x)$ and $H(x) = cx + h(x)$. By Theorem 1.5.2, we have

$$\Big[(a[x, y] + d([x, y]))(b[x, y] + g([x, y])) - [x, y](c[x, y] + h([x, y])), [x, y]\Big] = 0 \quad (3.3.21)$$

for all $x, y \in U$. If any one of h, d and g is inner then conclusion follows by Proposition 3.3.1, Proposition 3.3.2 and Proposition 3.3.3 respectively. Our aim is now to prove that, if all derivations h, d, g are not inner, then a number of contradictions follows. We have the following two cases:

Case-I. *h, d and g are linearly C-independent modulo inner derivations of U.*
Applying Theorem 1.6.4 to (3.3.21), U satisfies

$$\Big[(a[x, y] + [s, y] + [x, s'])(b[x, y] + [t, y] + [x, t'])$$
$$-[x, y](c[x, y] + [m, y] + [x, m']), [x, y]\Big] = 0.$$

In particular, U satisfies blended component(for $s = s' = 0$),

$$\Big[([s, y] + [x, s'])(b[x, y] + [t, y] + [x, t']), [x, y]\Big] = 0.$$

Again from above U satisfies blended component(for $t = t' = 0$)

$$\Big[([s, y] + [x, s'])([t, y] + [x, t']), [x, y]\Big] = 0.$$

This is a polynomial identity (PI) and hence by [67, Lemma 2] there exists a field K such that $U \subseteq M_n(K)$, $n > 1$ and the matrix ring $M_n(K)$ satisfies $\Big[([s,y]+[x,s'])([t,y]+[x,t']),[x,y]\Big] = 0$. But for $x = s = t = e_{11}, y = e_{12}, s' = e_{21}, t' = 0$, we have $0 = \Big[([s,y]+[x,s'])([t,y]+[x,t']),[x,y]\Big] = e_{12}$, a contradiction.

Case-II. *h, d and g are linearly C-dependent modulo inner derivations of U.* Since in this case h, d and g are linearly C-dependent modulo inner derivations of U, that means there exist α, β, $\gamma \in C$ and $q \in U$ such that

$$\alpha d(x) + \beta g(x) + \gamma h(x) = [q,x], \tag{3.3.22}$$

for all $x \in R$. If both α and β are zero, then γ must be nonzero, which arises a contradiction, since h is not an inner derivation. So, at least one of α and β must be nonzero. Without loss of generality, we assume that $\alpha \neq 0$. Then by (3.3.22), we have that

$$d(x) = \beta' g(x) + \gamma' h(x) + [q',x] \tag{3.3.23}$$

for all $x \in R$ and $\beta' = -\alpha^{-1}\beta$, $\gamma' = -\alpha^{-1}\gamma$, $q' = \alpha^{-1}q$. Now from (3.3.23), using the value of d, in (3.3.21) we get,

$$\Big[(a[x,y]+\beta' g([x,y])+\gamma' h([x,y])+[q',[x,y]])(b[x,y]+g([x,y])) \\ -[x,y](c[x,y]+h([x,y])),[x,y]\Big] = 0. \tag{3.3.24}$$

We are now to consider the following two cases:

Sub-case-i. *When g and h are linearly C-independent modulo inner derivations of U.* Applying Theorem 1.6.4 to (3.3.24), U satisfies

$$\Big[(a[x,y]+\beta'([s,y]+[x,s'])+\gamma'([t,y]+[x,t'])+[q',[x,y]]). \\ (b[x,y]+[s,y]+[x,s'])-[x,y](c[x,y]+[t,y]+[x,t']),[x,y]\Big] = 0 \tag{3.3.25}$$

In particular, U satisfies blended component

$$\Big[(\gamma'([t,y]+[x,t']))(b[x,y]+[s,y]+[x,s'])-[x,y]([t,y]+[x,t']),[x,y]\Big]=0 \tag{3.3.26}$$

This again implies that U satisfies

$$\Big[(\gamma'([t,y]+[x,t']))([s,y]+[x,s']),[x,y]\Big]=0. \tag{3.3.27}$$

This is a polynomial identity (PI) and hence by [67, Lemma 2] there exists a field K such that $U\subseteq M_n(K)$, $n>1$ and the matrix ring $M_n(K)$ satisfies $\Big[(\gamma'([t,y]+[x,t']))([s,y]+[x,s']),[x,y]\Big]=0$. In particular, for $x=s=t=e_{11}$, $y=e_{12}$, $t'=e_{21}$ and $s'=0$, we get $0=\Big[(\gamma'([t,y]+[x,t']))([s,y]+[x,s']),[x,y]\Big]=\gamma' e_{13}$, which implies that $\gamma'=0$. Then by (3.3.26), U satisfies

$$\Big[[x,y]([t,y]+[x,t']),[x,y]\Big]=0. \tag{3.3.28}$$

This again a polynomial identity for U and hence there exists a field K such that $U\subseteq M_n(K)$, $n>1$ and the matrix ring $M_n(K)$ satisfies $\Big[[x,y]([t,y]+[x,t']),[x,y]\Big]=0$. But for $x=e_{12}, y=e_{22}, t=e_{21}, t'=0$, we have $0=\Big[[x,y]([t,y]+[x,t']),[x,y]\Big]=-e_{12}$, a contradiction.

Sub-case-ii. *When g and h are linearly C-dependent modulo inner derivations of U.*

Since in this case h and g are linearly C-dependent modulo inner derivations of U, that means there exist θ, $\eta\in C$ and $p\in U$ such that

$$\theta g(x)+\eta h(x)=[p,x],$$

for all $x\in R$. Since g and h are not inner derivations, here we may assume both $\theta\neq 0$ and $\eta\neq 0$. Since $\theta\neq 0$, we have that

$$g(x)=\eta' h(x)+[p',x],$$

for all $x \in R$ and $\eta' = -\theta^{-1}\eta$, $p' = \theta^{-1}p$. Putting the value of this $g(x)$ on (3.3.23) we get

$$d(x) = \beta'' h(x) + [q'', x]$$

for all $x \in R$ and $\beta'' = (\beta'\eta' + \gamma')$, $q'' = (\beta' p' + q')$. Thus we may write relation (3.3.24) as follows:

$$\Bigg[\bigg(a[x,y] + \beta'' h([x,y]) + [q'', [x,y]]\bigg)\bigg(b[x,y] + \eta' h([x,y]) + [p', [x,y]]\bigg) \\ -[x,y]\bigg(c[x,y] + h([x,y])\bigg), [x,y]\Bigg] = 0.$$

By Theorem 1.6.4, U satisfies

$$\Bigg[\bigg(a[x,y] + \beta''([t,y] + [x,t']) + [q'', [x,y]]\bigg)\bigg(b[x,y] + \eta'([t,y] + [x,t']) + [p', [x,y]]\bigg) \\ -[x,y]\bigg(c[x,y] + ([t,y] + [x,t'])\bigg), [x,y]\Bigg] = 0$$

In particular, U satisfies the blended component

$$\Bigg[\beta''\bigg([t,y] + [x,t']\bigg)\bigg(b[x,y] + \eta'([t,y] + [x,t']) + [p', [x,y]]\bigg) \\ +\bigg(a[x,y] + \beta''([t,y] + [x,t']) + [q'', [x,y]]\bigg)\bigg(\eta'([t,y] + [x,t']\bigg) \\ -[x,y]\bigg([t,y] + [x,t']\bigg), [x,y]\Bigg] = 0.$$

In particular for $t' = 0$, we have

$$\Bigg[\beta''[t,y]\bigg(b[x,y] + \eta'[t,y] + [p', [x,y]]\bigg) + \bigg(a[x,y] + \beta''[t,y] + [q'', [x,y]]\bigg)\eta'[t,y] \\ -[x,y][t,y], [x,y]\Bigg] = 0. \tag{3.3.29}$$

Replacing t with $-t$, we have

$$\Bigg[-\beta''[t,y]\bigg(b[x,y] - \eta'[t,y] + [p', [x,y]]\bigg) - \bigg(a[x,y] - \beta''[t,y] + [q'', [x,y]]\bigg)\eta'[t,y] \\ +[x,y][t,y], [x,y]\Bigg] = 0. \tag{3.3.30}$$

By adding above two relation and then using char $(R) \neq 2$, we have

$$\Big[\beta''[t,y]\eta'[t,y] + \beta''[t,y]\eta'[t,y], [x,y]\Big] = 0$$

that is

$$\Big[\beta''\eta'[t,y]^2, [x,y]\Big] = 0$$

for all $x, y, t \in U$. This is again a polynomial identity for U and hence there exists a field K such that $U \subseteq M_n(K)$, $n > 1$ and the matrix ring $M_n(K)$ satisfies $\Big[\beta''\eta'[t,y]^2, [x,y]\Big] = 0$. If $k = 2$ then R satisfies s_4 which is our conclusion. If $k \geq 3$, then for $t = e_{12}, y = e_{21}, x = e_{32}$, we have $0 = \Big[\beta''\eta'[t,y]^2, [x,y]\Big] = -\beta''\eta' e_{31}$. This implies that $\beta''\eta' = 0$. If $\beta'' = 0$, then d is inner, a contradiction. If $\eta' = 0$, g is inner, a contradiction. Thus the proof of the theorem is completed.

Chapter 4

Generalized Derivations of Order 2 and Multilinear Polynomials in Prime Rings

4.1 Introduction

The commuting and centralizing maps are tightly connected with the behavior of the ring. Wong in [89] prove that if $f(x_1, \ldots, x_n)$ is not central-valued multilinear polynomial on R and d, δ are derivations of R such that $d(u)u - u\delta(u) \in Z(R)$ for all $u \in f(R)$, then either $d = \delta = 0$ or $d = -\delta$ and $f(x_1, \ldots, x_n)^2$ is central-valued on R, except when char $(R) = 2$ and R satisfies s_4. In [40], Dhara and Sharma studied the case when char $(R) \neq 2$, d a derivation of R and I a right ideal of R such that $d^2(u)u - ud^2(u) = 0$ for all $u \in f(I)$, and then obtained that either $[f(x_1, \ldots, x_n), x_{n+1}]x_{n+2}$ is satisfied by I, or there exists $b \in U$ such that $d(x) = [b, x]$ for all $x \in R$, with $b^2 = 0$ and $bI = (0)$.

In [10], Carini and De Filippis proved that if char $(R) \neq 2$, d a nonzero derivation of R, F a nonzero generalized derivation of R such that $d(F(u)u) = 0$ for all $u \in f(R)$, then $f(x_1, \ldots, x_n)^2$ is central-valued on R and there exists $a \in U$ such that $F(x) = ax$ for all $x \in R$. Moreover, d is an inner derivation of R such that $\delta(a) = 0$. In [35], Dhara and De

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Filippis studied the situation $F^2(u)u - uG^2(u) = 0$ for all $u \in f(I)$ where F and G are two generalized derivations of R and I is an ideal of R and then obtain all possible forms of the maps.

In [85], Tiwari et al. considered the situation $d(F^2(u)u - uF^2(u)) = 0$ for all $u \in f(R)$, where d is a nonzero derivation of R and F is a nonzero generalized derivation of R. They proved that if char $(R) \neq 2$, then there exists $a \in U$ with $a^2 \in C$ such that either $F(x) = ax$ or $F(x) = xa$ for all $x \in R$. Recently in [43], Eroğlu and Argaç studied the case when $F^2(u)u \in C$ for all $u \in f(R)$, where F is a nonzero generalized derivation of R.

Motivated by above results, it is natural to ask what happen in case $d(F^2(u)u) = 0$ for all $u \in f(R)$. In the present chapter, we study more general case $d(F^2(u)u) = uG^2(u)$ for all $u \in f(I)$ and then determine forms of the maps, where d is a derivation, F, G two generalized derivations of R and I is a both sided ideal of R. More precisely, we prove the following theorem:

Theorem 4.1.1. *Let R be a prime ring of char $(R) \neq 2, 3$, U its Utumi ring of quotients and C its extended centroid, I be a two sided ideal of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, that is noncentral-valued on R, F, G be two generalized derivations of R and d be a derivation of R. If*

$$d(F^2(u)u) = uG^2(u)$$

for all $u \in f(I)$, then one of the following holds:

1. *there exist $b, q \in U$ such that $F(x) = xb$ and $G(x) = xq$ for all $x \in R$, with $b^2 = 0 = q^2$;*

2. *there exist $b, p \in U$ such that $F(x) = xb$ and $G(x) = px$ for all $x \in R$, with $b^2 = 0 = p^2$;*

3. *there exist* $b, q \in U$ *such that* $F(x) = bx$ *and* $G(x) = xq$ *for all* $x \in R$, *with* $b^2 = 0 = q^2$;

4. *there exist* $b, p \in U$ *such that* $F(x) = bx$ *and* $G(x) = px$ *for all* $x \in R$, *with* $b^2 = 0 = p^2$;

5. *there exist* $a, b, p \in U$ *such that* $d(x) = [a, x]$, $F(x) = xb$ *and* $G(x) = xp$ *for all* $x \in R$, *with* $p^2 = 0$, $b^2 \in C$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued;*

6. *there exist* $a, b, p \in U$ *such that* $d(x) = [a, x]$, $F(x) = xb$ *and* $G(x) = px$ *for all* $x \in R$, *with* $p^2 = 0$, $b^2 \in C$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued;*

7. *there exist* $a, b, q \in U$ *such that* $d(x) = [a, x]$, $F(x) = bx$ *and* $G(x) = xq$ *for all* $x \in R$, *with* $d(b^2) = q^2$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued;*

8. *there exist* $a, b, p \in U$ *such that* $d(x) = [a, x]$, $F(x) = bx$ *and* $G(x) = px$ *for all* $x \in R$, *with* $d(b^2) = p^2$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued.*

Corollary 4.1.2. *Let* R *be a prime ring of char* $(R) \neq 2, 3$, U *its Utumi ring of quotients and* C *its extended centroid,* I *be a two sided ideal of* R, $f(x_1, \ldots, x_n)$ *a noncentral-valued multilinear polynomial over* C *on* R, F *be a generalized derivation of* R *and* d *be a nonzero derivation of* R. *If*

$$d(F^2(u)u) = 0$$

for all $u \in f(I)$, *then one of the following holds:*

1. *there exists* $b \in U$ *such that* $F(x) = xb$ *for all* $x \in R$, *with* $b^2 = 0$;

2. *there exists* $b \in U$ *such that* $F(x) = bx$ *for all* $x \in R$, *with* $b^2 = 0$;

3. *there exist* $a, b \in U$ *such that* $d(x) = [a, x]$, $F(x) = bx$ *for all* $x \in R$, *with* $d(b^2) = 0$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued on* R.

Let F and G be two generalized derivations of R. In [2], Ali et al. studied the commutator $[G(u)u, G(v)v] = 0$ for all $u, v \in f(R)$. This result was further generalized by Dhara et al. [32] setting the commutator identity $[F(u)u, G(v)v] = 0$ for all $u, v \in f(R)$. In the same flavor, we have the following Corollary.

Corollary 4.1.3. *Let R be a prime ring of char $(R) \neq 2$, U its Utumi ring of quotients and C its extended centroid, I be a two sided ideal of R, $f(x_1, \ldots, x_n)$ a noncentral-valued multilinear polynomial over C on R, F and G be two generalized derivations of R. If*

$$[F^2(u)u, G^2(v)v] = 0$$

for all $u \in f(I)$, then one of the following holds:

1. *there exists $b \in U$ such that $F(x) = bx$ or $F(x) = xb$ for all $x \in R$, with $b^2 = 0$;*
2. *there exists $a \in U$ such that $G(x) = ax$ or $G(x) = xa$ for all $x \in R$, with $a^2 = 0$;*
3. *there exist $a, b \in U$ such that $F(x) = bx$ and $G(x) = ax$ for all $x \in R$, with $[a^2, b^2] = 0$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.*

Proof. Let $c = G^2(v)v$, $v \in f(I)$. Then applying above Corollary 4.1.2 (if derivation d in Corollary 4.1.2 is inner, then char $(R) \neq 3$ is not required as in proof in Theorem 4.1.1), we conclude one of the following:

1. there exists $b \in U$ such that $F(x) = xb$ for all $x \in R$, with $b^2 = 0$;
2. there exists $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $b^2 = 0$;
3. there exist $b \in U$ such that $F(x) = bx$ for all $x \in R$, with $[b^2, G^2(v)v] = 0$, where $v \in f(I)$ and $f(x_1, \ldots, x_n)^2$ is central valued on R. In this case again applying Corollary 4.1.2, we have one of the following:

(i) there exists $a \in U$ such that $G(x) = xa$ for all $x \in R$, with $a^2 = 0$;

(ii) there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$, with $a^2 = 0$;

(iii) there exist $a, b \in U$ such that $G(x) = ax$ for all $x \in R$, with $[b^2, a^2] = 0$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.

Thus the conclusion follows.

□

Corollary 4.1.4. *Let R be a prime ring of char $(R) \neq 2, 3$, I be a two sided ideal of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C on R, d, δ and g be three nonzero derivations of R. If*

$$d(\delta^2(u)u) = ug^2(u)$$

for all $u \in f(I)$, then $f(x_1, \ldots, x_n)$ must be central valued on R.

Corollary 4.1.5. *Let R be a prime ring of char $(R) \neq 2, 3$, I be a two sided ideal of R, d, δ and g be three nonzero derivations of R. If*

$$d(\delta^2(x)x) = xg^2(x)$$

for all $x \in I$, then R must be commutative.

4.2 Some Results

In all that follows, we assume that R is a prime ring, U be its Utumi ring of quotients, $C = Z(U)$ the center of U. Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C.

The following facts are very essential to prove our main result.

Fact 4.2.1. *[68, Proposition] Let A be an algebra over an infinite field k and K be a field extension over k. If A satisfies a GPI $p(r_1, \ldots, r_n) = 0$, so does $A \otimes_k K$.*

Fact 4.2.2. *[74, Lemma 2] Let T be a K-algebra with 1 and $R = M_m(T)$, $m \geq 2$. Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over K. Let $u = (A_1, \ldots, A_k)$ be a simple sequence of matrices from R.*

(a) If u is even, then $f(u)$ is a diagonal matrix.

(b) If u is odd, then $f(u) = ae_{ij}$ for some $a \in T$ and $i \neq j$.

Fact 4.2.3. *Let R be a prime ring, $d_1, \ldots, d_n \in Der(U)$, $\Phi(x_i^{\Delta_j})$ is a differential identity on R, involving n derivation words $\Delta_i, \ldots \Delta_n$. Assume that each Δ_i is a derivations words of the following form*

$$\Delta_j = d_1^{s_{1,j}} d_2^{s_{2,j}} \ldots d_m^{s_{m,j}} \qquad j = 1, \ldots, n$$

and let

$$s = max s_{i,j}, i = 1, \ldots, m \quad j = 1 \ldots, n.$$

If $d_1, \ldots, d_m$ are C-linearly independent modulo D_{int} and $s < p$, if $char(R) = p \neq 0$, then $\Phi(y_{ji})$ is a generalized polynomial identity on R, where y_{ji} are distinct indeterminates.

Fact 4.2.4. *Let $f(r_1, \ldots, r_n)$ be the multilinear polynomial over the field C and d, δ are derivations on R.*

We shall use the notation

$$f(r_1, \ldots, r_n) = r_1 r_2 \cdots r_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)}$$

for some $\alpha_\sigma \in C$, and S_n denotes the symmetric group of degree n.

Then we have

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n),$$

where $f^d(r_1,\ldots,r_n)$ *be the polynomials obtained from* $f(r_1,\ldots,r_n)$ *replacing each coefficients* α_σ *with* $\delta(\alpha_\sigma)$*. Similarly, by calculation, we have*

$$\begin{aligned} d^2(f(r_1,\ldots,r_n)) &= f^{d^2}(r_1,\ldots,r_n) + 2\sum_i f^d(r_1,\ldots,d(r_i),\ldots,r_n) \\ &\quad + \sum_i f(r_1,\ldots,d^2(r_i),\ldots,r_n) \\ &\quad + \sum_{i\neq j} f(r_1,\ldots,d(r_i),\ldots,d(r_j),\ldots,r_n), \end{aligned}$$

and

$$\begin{aligned} d^3(f(r_1,\ldots,r_n)) &= f^{d^3}(r_1,\ldots,r_n) + 3\sum_i f^{d^2}(r_1,\ldots,d(r_i),\ldots,r_n) \\ &\quad +3\sum_i f^d(r_1,\ldots,d^2(r_i),\ldots,r_n) + 6\sum_{i\neq j} f^d(r_1,\ldots,d(r_i),\ldots,d(r_j)\ldots,r_n) \\ &\quad + \sum_i f(r_1,\ldots,d^3(r_i),\ldots,r_n) + 3\sum_{i\neq j} f(r_1,\ldots,d(r_i),\ldots,d^2(r_j)\ldots,r_n) \\ &\quad +6\sum_{i\neq j\neq k} f(r_1,\ldots,d(r_i),\ldots,d(r_j)\ldots,d(r_k),\ldots,r_n) \end{aligned}$$

and

$$\begin{aligned} d\delta(f(r_1,\ldots,r_n)) &= f^{d\delta}(r_1,\ldots,r_n) + \sum_i f^\delta(r_1,\ldots,d(r_i),\ldots,r_n) \\ &\quad + \sum_i f^d(r_1,\ldots,\delta(r_i),\ldots,r_n) + \sum_i f(r_1,\ldots,d\delta(r_i),\ldots,r_n) \\ &\quad + \sum_{i\neq j} f(r_1,\ldots,d(r_i),\ldots,\delta(r_j),\ldots,r_n). \end{aligned}$$

and

$$\begin{aligned}
d\delta^2(f(r_1,\ldots,r_n)) &= f^{d\delta^2}(r_1,\ldots,r_n)+\sum_i f^{\delta^2}(r_1,\ldots,d(r_i),\ldots,r_n)\\
&+2\sum_i f^{d\delta}(r_1,\ldots,\delta(r_i),\ldots,r_n)+2\sum_i f^{\delta}(r_1,\ldots,d\delta(r_i),\ldots,r_n)\\
&+2\sum_{i\neq j} f^{\delta}(r_1,\ldots,d(r_i),\ldots,\delta(r_j),\ldots,r_n)\\
&+\sum_i f^{d}(r_1,\ldots,\delta^2(r_i),\ldots,r_n)+\sum_i f(r_1,\ldots,d\delta^2(r_i),\ldots,r_n)\\
&+\sum_{i\neq j} f(r_1,\ldots,d(r_i),\ldots,\delta^2(r_j),\ldots,r_n)\\
&+2\sum_{i\neq j} f^{d}(r_1,\ldots,\delta(r_i),\ldots,\delta(r_j),\ldots,r_n)\\
&+2\sum_{i\neq j} f(r_1,\ldots,d\delta(r_i),\ldots,\delta(r_j),\ldots,r_n)\\
&+2\sum_{i\neq j\neq k} f(r_1,\ldots,\delta(r_i),\ldots,d(r_j),\ldots,\delta(r_k),\ldots,r_n).
\end{aligned}$$

Fact 4.2.5. *[4, Lemma 3] Let R be a noncommutative prime ring and $f(x_1,\ldots,x_n)$ be a non-central valued multilinear polynomial on R. Suppose that there exist $a,b,c,q\in U$ such that $(af(r)+f(r)b)f(r)-f(r)(cf(r)+f(r)q)=0$ for all $r=(r_1,\ldots,r_n)\in R^n$. Then one of the following holds:*

(1) $a,q\in C$ and $q-a=b-c=\alpha\in C$;

(2) $f(x_1,\ldots,x_n)^2$ is central valued on R and there exists $\alpha\in C$ such that $q-a=b-c=\alpha$;

(3) char $(R)=2$ and R satisfies s_4.

In particular, we have the following facts.

Fact 4.2.6. *Let R be a noncommutative prime ring and $f(x_1,\ldots,x_n)$ be a non-central valued multilinear polynomial on R. Suppose that there exist $a,b,q\in U$ such that $af(r)+f(r)bf(r)-f(r)^2q=0$ for all $r=(r_1,\ldots,r_n)\in R^n$. Then one of the following holds:*

(1) $a, q \in C$ *and* $q - a = b = \alpha \in C$;

(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exists* $\alpha \in C$ *such that* $q - a = b = \alpha$;

(3) char $(R) = 2$ *and* R *satisfies* s_4.

Fact 4.2.7. *Let* R *be a noncommutative prime ring and* $f(x_1, \ldots, x_n)$ *be a non-central valued multilinear polynomial on* R. *Suppose that there exist* $a, q \in U$ *such that* $af(r)^2 - f(r)^2 q = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$. *Then one of the following holds:*

(1) $q = a \in C$;

(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exists* $\alpha \in C$ *such that* $q = a$;

(3) char $(R) = 2$ *and* R *satisfies* s_4.

Fact 4.2.8. *[33, Corollary 2.9] Let* R *be a prime ring of char* $(R) \neq 2$, d *and* δ *two nonzero derivations of* R. *If*

$$d(\delta(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n)) = 0$$

for all $x_1, \ldots, x_n \in I$, *then* $f(x_1, \ldots, x_n)$ *is central valued on* R.

Fact 4.2.9. *[29, Lemma 2.9] Let* R *be a noncommutative prime ring of char* $(R) \neq 2$, $a, b, c, c' \in U$, $p(x_1, \ldots, x_n)$ *be any polynomial over* C, *which is not an identity for* R. *If* $ap(r) + p(r)b + cp(r)c' = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$, *then one of the following holds:*

(1) $b, c' \in C$ *and* $a + b + cc' = 0$;

(2) $a, c \in C$ *and* $a + b + cc' = 0$;

(3) $a + b + cc' = 0$ *and* $p(x_1, \ldots, x_n)$ *is central valued on* R.

4.3 When Derivation and Generalized Derivations are Inner

Let $d(x) = [a, x]$, $F(x) = bx+xc$ and $G(x) = px+xq$ for all $x \in R$ for some $a, b, c, p, q \in U$. Then the hypothesis $d(F^2(x)x) = xG^2(x)$ for all $x \in f(R)$ becomes

$$[a, b^2x^2 + 2bxcx + xc^2x] = xp^2x + 2xpxq + x^2q^2 \tag{4.3.1}$$

for all $x \in f(R)$. This implies

$$\begin{aligned} &ab^2x^2 + 2abxcx + axc^2x - b^2x^2a - 2bxcxa - xc^2xa \\ &= xp^2x + 2xpxq + x^2q^2 \end{aligned} \tag{4.3.2}$$

which can be written as

$$\begin{aligned} &a_1x^2 + 2a_2xcx + axa_3x - a_4x^2a - 2bxcxa - xa_3xa \\ &= xa_5x + 2xpxq + x^2a_6 \end{aligned} \tag{4.3.3}$$

for all $x \in f(R)$, where $a_1 = ab^2$, $a_2 = ab$, $a_3 = c^2$, $a_4 = b^2$, $a_5 = p^2$, $a_6 = q^2$.

Lemma 4.3.1. *Let C be an infinite field and $R = M_m(C)$ be the ring of all $m \times m$ matrices over C, $m \geq 2$. Suppose that char $(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. If a_1, a_2, a_3, a_4, a_5, a_6, a, b, c, p, $q \in R$ such that (4.3.3) holds for all $x \in f(R)$, then either a or b or c are scalar matrices.*

Proof. Let a, b, c be not scalar matrices. Then by Theorem 1.6.3, there exists an invertible matrix P such that PaP^{-1}, PbP^{-1} and PcP^{-1} have all nonzero entries. Write $PaP^{-1} = \sum_{i,j} a'_{ij}e_{ij}$, $PbP^{-1} = \sum_{i,j} b'_{ij}e_{ij}$ and $PcP^{-1} = \sum_{i,j} c'_{ij}e_{ij}$. Then a'_{ij}, b'_{ij} and c'_{ij} are all nonzero for any i and j. Since $\Psi(x) = PxP^{-1}$ for all $x \in R$ is an automorphism, by (4.3.3), we have

$$Pa_1P^{-1}x^2 + 2Pa_2P^{-1}xPcP^{-1}x + PaP^{-1}xPa_3P^{-1}x - Pa_4P^{-1}x^2PaP^{-1}$$
$$-2PbP^{-1}xPcP^{-1}xPaP^{-1} - xPa_3P^{-1}xPaP^{-1}$$
$$= xPa_5P^{-1}x + 2xPpP^{-1}xPqP^{-1} + x^2Pa_6P^{-1} \tag{4.3.4}$$

for all $x \in f(R)$. Since $f(x_1, \ldots, x_n)$ is not central valued on R, by Fact 4.2.2, there exist some matrices $x_1, \ldots, x_n \in M_m(C)$ and $0 \neq \alpha \in C$ such that $f(x_1, \ldots, x_n) = \alpha e_{kl}, k \neq l$. Thus for $x = \alpha e_{kl} \in f(R)$, equation(4.3.4) gives

$$2Pa_2P^{-1}e_{kl}PcP^{-1}e_{kl} + PaP^{-1}e_{kl}Pa_3P^{-1}e_{kl}$$
$$-2PbP^{-1}e_{kl}PcP^{-1}e_{kl}PaP^{-1} - e_{kl}Pa_3P^{-1}e_{kl}PaP^{-1}$$
$$= e_{kl}Pa_5P^{-1}e_{kl} + 2e_{kl}PpP^{-1}e_{kl}PqP^{-1} \tag{4.3.5}$$

Left and right multiplying by e_{kl} in both side we get $2e_{kl}PbP^{-1}e_{kl}PcP^{-1}e_{kl}PaP^{-1}e_{kl} = 0$, that is $b'_{lk}c'_{lk}a'_{lk} = 0$. This contradicts with the fact that a'_{ij}, b'_{ij} and c'_{ij} are all nonzero for any i and j.

Thus either a or b or c are scalar matrices. □

Lemma 4.3.2. *Let C be a field and $R = M_m(C)$ be the ring of all $m \times m$ matrices over C, $m \geq 2$. Suppose that char $(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. If a_1, a_2, a_3, a_4, a_5, a_6, a, b, c, p, $q \in R$ such that (4.3.3) holds for all $x \in f(R)$, then either a or b or c are scalar matrices.*

Proof. If we assume C is infinite, then the conclusions follow by Lemma 4.3.1.

Now let C be finite and let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on R if and only if it is central-valued on $\overline{R}$. Suppose that the generalized polynomial $Q(x_1, \ldots, x_n)$ such that

$$Q(x_1, \ldots, x_n) = a_1 f(x_1, \ldots, x_n)^2 + 2a_2 f(x_1, \ldots, x_n) c f(x_1, \ldots, x_n)$$
$$+ a f(x_1, \ldots, x_n) a_3 f(x_1, \ldots, x_n) - a_4 f(x_1, \ldots, x_n)^2 a$$
$$-2b f(x_1, \ldots, x_n) c f(x_1, \ldots, x_n) a - f(x_1, \ldots, x_n) a_3 f(x_1, \ldots, x_n) a$$
$$-f(x_1, \ldots, x_n) a_5 f(x_1, \ldots, x_n) - 2f(x_1, \ldots, x_n) p f(x_1, \ldots, x_n) q - f(x_1, \ldots, x_n)^2 a_6.$$

Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $x_1, \ldots, x_n$. Hence the complete linearization of $Q(x_1, \ldots, x_n)$ is a multilinear generalized polynomial $\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $2n$ indeterminates and

$$\Theta(x_1, \ldots, x_n, x_1, \ldots, x_n) = 2^n Q(x_1, \ldots, x_n).$$

It is clear that the multilinear polynomial $\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a generalized polynomial identity for both R and $\overline{R}$ (see Fact 4.2.1). Since char $(C) \neq 2$, then $Q(x_1, \ldots, x_n) = 0$, for all $x_1, \ldots, x_n \in \overline{R}$ and then the conclusion follows from Lemma 4.3.1. □

Corollary 4.3.3. *Let C be a field and $R = M_m(C)$ be the ring of all $m \times m$ matrices over C, $m \geq 2$. Suppose that char $(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. If a_1, a_2, a_3, a_4, a_5, a_6, a, b, c, p, $q \in R$ such that (4.3.3) holds for all $x \in R$, then either a or b or c are scalar matrices.*

Lemma 4.3.4. *Let R be a noncommutative prime ring of char $(R) \neq 2$, U be its Utumi quotient ring and C its extended centroid. Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial* *over C, noncentral-valued on R. Suppose that $a_1, a_2, a_3, a_4, a_5, a_6, a, b, c, p, q \in R$ such that (4.3.3) holds for all $x \in f(R)$. Then either $a \in C$ or $b \in C$ or $c \in C$.*

Proof. By hypothesis we have

$$\Psi(r) = a_1 f(r)^2 + 2a_2 f(r)cf(r) + af(r)a_3 f(r) - a_4 f(r)^2 a - 2bf(r)cf(r)a$$
$$-f(r)a_3 f(r)a - f(r)a_5 f(r) - 2f(r)pf(r)q - f(r)^2 a_6 \tag{4.3.6}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. By Theorem 1.5.2, U satisfies $\Psi(r_1, \ldots, r_n)$.

On contrary we assume that $a \notin C$, $b \notin C$ and $c \notin C$. Let $\Psi(r_1, \ldots, r_n) = 0$ is a trivial GPI for U. Then as $a \notin C$ the term $2bf(r)cf(r)a$ can not be cancelled with $a_1 f(r)^2$ or $2a_2 f(r)cf(r)$ or $af(r)a_3 f(r)$ or $-f(r)a_5 f(r)$ and as $b \notin C$, the term $2bf(r)cf(r)a$ can not be cancelled with $-f(r)a_3 f(r)a$ or $-2f(r)pf(r)q$ or $-f(r)^2 a_6$. Thus $-a_4 f(r)^2 a - 2bf(r)cf(r)a = 0$, that is $\Big(- a_4 f(r) - 2bf(r)c\Big) f(r)a = 0$ is a trivial GPI for $r = (r_1, \ldots, r_n) \in U^n$. Since $c \notin C$, the term $2bf(r)cf(r)a$ can not be cancelled with $-a_4 f(r)^2 a$ and hence the term $2bf(r)cf(r)a$ appears nontrivially, a contradiction.

Next assume that, $\Psi(r_1, \ldots, r_n) = 0$ is a non-trivial GPI for U. In this case by Fact 1.6.8, R is isomorphic to a dense ring of linear transformations of a vector space V over C.

Assume first that V is finite dimensional over C, that is, $dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case by Lemma 4.3.2, we get either a or b or c are in C. If V is infinite dimensional over C, then by [88, Lemma 2], the set $f(R)$ is dense on R and hence R satisfies

$$a_1 r^2 + 2a_2 rcr + ara_3 r - a_4 r^2 a - 2brcra$$
$$-ra_3 ra - ra_5 r - 2rprq - r^2 a_6 = 0. \tag{4.3.7}$$

For any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = dim_C Ve$. We want to show that in this case also either a or b or c are in C. To prove this, let none of a, b, c be in C. Then a, b, c do not centralize the nonzero ideal $soc(R)$. Hence there exist h_1, h_2, $h_3 \in soc(R)$ such that $[a, h_1] \neq 0$, $[b, h_2] \neq 0$, $[c, h_3] \neq 0$. By Litoff's Theorem (Theorem 1.6.7), there exists an idempotent $e \in soc(R)$ such that ah_1, h_1a, bh_2, h_2b, ch_3, h_3c, h_1, h_2, $h_3 \in eRe$. We have $eRe \cong M_t(C)$ with $t = dim_C Ve$. Since

$$ea_1er^2 + 2ea_2erecer + eaerea_3er - ea_4er^2eae - 2eberecereae$$
$$-rea_3ereae - rea_5er - 2repereqe - r^2ea_6e = 0 \qquad (4.3.8)$$

for all $r \in eRe$, by Corollary 4.3.3, either eae or ebe or ece are central elements of eRe. Thus

$$ah_1 = eah_1e = (eae)h_1e = eh_1eae = eh_1ae = h_1a$$

or

$$bh_2 = ebh_2e = (ebe)h_2e = eh_2ebe = eh_2be = h_2b$$

or

$$ch_3 = ech_3e = (ece)h_3e = eh_3ece = eh_3ce = h_3c,$$

a contradiction.

Thus we have proved that either a or b or c are in C.

□

In the same manner as above, we can prove the following Lemmas.

Lemma 4.3.5. *Let R be a noncommutative prime ring of char $(R) \neq 2$, U be its Utumi quotient ring and C its extended centroid. Let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C, noncentral-valued on R. Suppose that a_1, a_2, a_3, a_4, a_5, a_6, a_7, $a_8 \in U$ such that*

$$a_1xa_2x - x\{a_3xa_4 + a_5x + 2a_6xa_7 + xa_8\} = 0$$

holds for all $x \in f(R)$. *Then either* $a_1 \in C$ *or* $a_2 \in C$.

Lemma 4.3.6. *Let* R *be a noncommutative prime ring of char* $(R) \neq 2$, U *be its Utumi quotient ring and* C *its extended centroid. Let* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C, *noncentral-valued on* R. *Suppose that* a_1, a_2, a_3, a_4, $a_5 \in U$ *such that*

$$a_1x^2 - x^2a_1 = xa_2x + 2xa_3xa_4 + x^2a_5$$

holds for all $x \in f(R)$. *Then either* $a_3 \in C$ *or* $a_4 \in C$.

Lemma 4.3.7. *Let* R *be a noncommutative prime ring of char* $(R) \neq 2$, U *be its Utumi quotient ring and* C *its extended centroid. Let* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C, *noncentral-valued on* R. *Suppose that* a_1, a_2, a_3, a_4, $a_5, a_6, a_7 \in U$ *such that*

$$a_1x^2 - a_2x^2a_3 - xa_4x - 2xa_5xa_6 - x^2a_7 = 0$$

holds for all $x \in f(R)$. *Then either* $a_5 \in C$ *or* $a_6 \in C$.

Lemma 4.3.8. *Let* R *be a noncommutative prime ring of char* $(R) \neq 2$, U *be its Utumi quotient ring and* C *its extended centroid. Let* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C, *noncentral-valued on* R. *Suppose that* $a_1, a_2, a_3, a_4 \in U$ *such that*

$$a_1x^2 + a_2x^2a_3 + xa_4x = 0$$

holds for all $x \in f(R)$. *Then* $a_4 \in C$.

Lemma 4.3.9. *Let* R *be a noncommutative prime ring of char* $(R) \neq 2$ *with Utumi quotient ring* U *and extended centroid* C, *and* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C, *which is not central valued on* R. *Suppose that* F *and* G *are two inner generalized derivations of* R *and* d *is a nonzero inner derivation of* R *such that* $d(F^2(f(r))f(r)) = f(r)G^2(f(r))$ *for all* $r = (r_1, \ldots, r_n) \in R^n$, *then one of the following holds:*

1. *there exist* $a, p \in U$ *such that* $F(x) = xa$ *and* $G(x) = xp$ *for all* $x \in R$*, with* $a^2 = 0 = p^2$*;*

2. *there exist* $a, p \in U$ *such that* $F(x) = xa$ *and* $G(x) = px$ *for all* $x \in R$*, with* $a^2 = 0 = p^2$*;*

3. *there exist* $a, p \in U$ *such that* $F(x) = ax$ *and* $G(x) = xp$ *for all* $x \in R$*, with* $a^2 = 0 = p^2$*;*

4. *there exist* $a, p \in U$ *such that* $F(x) = ax$ *and* $G(x) = px$ *for all* $x \in R$*, with* $a^2 = 0 = p^2$*;*

5. *there exist* $a, b, p \in U$ *such that* $d(x) = [a, x]$*,* $F(x) = xb$ *and* $G(x) = xp$ *for all* $x \in R$*, with* $p^2 = 0$*,* $b^2 \in C$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued;*

6. *there exist* $a, b, p \in U$ *such that* $d(x) = [a, x]$*,* $F(x) = xb$ *and* $G(x) = px$ *for all* $x \in R$*, with* $p^2 = 0$*,* $b^2 \in C$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued;*

7. *there exist* $a, b, p \in U$ *such that* $d(x) = [a, x]$*,* $F(x) = bx$ *and* $G(x) = xp$ *for all* $x \in R$ *with,* $[a, b^2] = p^2$ *and* $f(x_1, \ldots, x_n)^2$ *is central valued;*

8. *there exist* $a, b, p \in U$ *such that* $d(x) = [a, x]$*,* $F(x) = bx$ *and* $G(x) = px$ *for all* $x \in R$*, with* $[a, b^2] = p^2 \in C$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued.*

Proof. Suppose that for some $a, b, c, p, q \in U$, $d(x) = [a, x]$, $F(x) = bx + xc$, $G(x) = px + xq$ for all $x \in R$. By hypothesis, we have

$$\begin{aligned} &[a, b^2 f(r)^2 + 2bf(r)cf(r) + f(r)c^2 f(r)] \\ &= f(r)p^2 f(r) + 2f(r)pf(r)q + f(r)^2 q^2 \end{aligned} \tag{4.3.9}$$

that is,

$$\begin{aligned} &ab^2f(r)^2 + 2abf(r)cf(r) + af(r)c^2f(r) \\ &-b^2f(r)^2a - 2bf(r)cf(r)a - f(r)c^2f(r)a \\ &= f(r)p^2f(r) + 2f(r)pf(r)q + f(r)^2q^2 \end{aligned} \tag{4.3.10}$$

for all $r = (r_1, \dots, r_n) \in R^n$. By Theorem 1.5.2, above identity is also satisfied by U.

By Lemma 4.3.4, either $a \in C$ or $b \in C$ or $c \in C$. Since d is nonzero, $a \notin C$.

Now we consider the following cases:

Case 1: *When $b \in C$.* In this case, we have

$$[a, f(r)(b+c)^2f(r)] = f(r)p^2f(r) + 2f(r)pf(r)q + f(r)^2q^2 \tag{4.3.11}$$

that is,

$$af(r)(b+c)^2f(r) - f(r)\{(b+c)^2f(r)a - p^2f(r) - 2pf(r)q - f(r)q^2\} = 0$$

for all $r = (r_1, \dots, r_n) \in U^n$. By Lemma 4.3.5, it yields $(b+c)^2 \in C$, since $a \notin C$. Then (4.3.11) reduces to

$$a(b+c)^2f(r)^2 - f(r)^2a(b+c)^2 = f(r)p^2f(r) + 2f(r)pf(r)q + f(r)^2q^2$$

for all $r = (r_1, \dots, r_n) \in U^n$, which again implies by Lemma 4.3.6, that either $p \in C$ or $q \in C$.

- If $p \in C$, then

$$a(b+c)^2f(r)^2 - f(r)^2\{a(b+c)^2 + (p+q)^2\} = 0$$

for all $r = (r_1, \dots, r_n) \in U^n$.

By Fact 4.2.7, one of the following holds:

(1) $a(b+c)^2 = a(b+c)^2 + (p+q)^2 \in C$. This yields $(p+q)^2 = 0$. Moreover, since $a \notin C$, $a(b+c)^2 \in C$ implies $(b+c)^2 = 0$. Thus $F(x) = x(b+c)$ and $G(x) = x(p+q)$ for all $x \in R$, with $(b+c)^2 = 0 = (p+q)^2$.

(2) $f(x_1, \ldots, x_n)^2$ is central valued on R and $(p+q)^2 = 0$. Thus $F(x) = x(b+c)$ and $G(x) = x(p+q)$ for all $x \in R$, with $(p+q)^2 = 0$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.

• If $q \in C$, then

$$[a(b+c)^2, f(r)^2] = f(r)(p+q)^2 f(r) \tag{4.3.12}$$

that is

$$a(b+c)^2 f(r)^2 - f(r)^2 a(b+c)^2 - f(r)(p+q)^2 f(r) = 0 \tag{4.3.13}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By Fact 4.2.6, one of the following holds:

(1) $a(b+c)^2 \in C$ and $(p+q)^2 = 0$. Moreover, since $a \notin C$, $a(b+c)^2 \in C$ implies $(b+c)^2 = 0$. Thus $F(x) = x(b+c)$ and $G(x) = (p+q)x$ for all $x \in R$, with $(b+c)^2 = 0 = (p+q)^2$.

(2) $f(x_1, \ldots, x_n)^2$ is central valued on R and $(p+q)^2 = 0$. Thus $F(x) = x(b+c)$ and $G(x) = (p+q)x$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ is central valued on R and $(p+q)^2 = 0$.

Case 2: *When $c \in C$.*

In this case, we have

$$[a, (b+c)^2 f(r)^2] = f(r)p^2 f(r) + 2f(r)pf(r)q + f(r)^2 q^2 \tag{4.3.14}$$

which is

$$a(b+c)^2 f(r)^2 - (b+c)^2 f(r)^2 a - f(r)p^2 f(r) - 2f(r)pf(r)q - f(r)^2 q^2 = 0$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By Lemma 4.3.7, either $p \in C$ or $q \in C$.

• If $p \in C$, we have

$$[a, (b+c)^2 f(r)^2] = f(r)^2 (p+q)^2 \tag{4.3.15}$$

that is

$$a(b+c)^2 f(r)^2 - (b+c)^2 f(r)^2 a - f(r)^2 (p+q)^2 = 0 \tag{4.3.16}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By Fact 4.2.9, one of the following holds:

(i) $a(b+c)^2, (b+c)^2 \in C$ with $a(b+c)^2 - (b+c)^2 a - (p+q)^2 = 0$. Now $0 \neq (b+c)^2 \in C$ and $a(b+c)^2 \in C$ yields $a \in C$, a contradiction. Hence $(b+c)^2 = 0$. Thus $F(x) = (b+c)x$ and $G(x) = x(p+q)$ for all $x \in R$ with $(b+c)^2 = 0 = (p+q)^2$.

(ii) $a, (p+q)^2 \in C$ with $a(b+c)^2 - (b+c)^2 a - (p+q)^2 = 0$, a contradiction as $a \notin C$.

(iii) $a(b+c)^2 - (b+c)^2 a - (p+q)^2 = 0$ with $f(x_1, \ldots, x_n)^2$ is central valued. Thus $F(x) = (b+c)x$ and $G(x) = x(p+q)$ for all $x \in R$ with $[a, (b+c)^2] = (p+q)^2$ and $f(x_1, \ldots, x_n)^2$ is central valued.

• If $q \in C$, we have

$$[a, (b+c)^2 f(r)^2] = f(r)(p+q)^2 f(r) \tag{4.3.17}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By Lemma 4.3.8, $(p+q)^2 \in C$ and so

$$\{a(b+c)^2 - (p+q)^2\} f(r)^2 - (b+c)^2 f(r)^2 a = 0.$$

By Fact 4.2.9, one of the following holds:

(i) $a \in C$ with $a(b+c)^2 - (p+q)^2 - (b+c)^2 a = 0$, a contradiction as $a \notin C$.

(ii) $a(b+c)^2 - (p+q)^2, (b+c)^2 \in C$ with $a(b+c)^2 - (p+q)^2 - (b+c)^2 a = 0$. In this case $(p+q)^2 = 0$, $a(b+c)^2, (b+c)^2 \in C$. Again if $(b+c)^2 \neq 0$, then $a \in C$, a contradiction. Therefore, $(b+c)^2 = 0$. Thus $F(x) = (b+c)x$ and $G(x) = (p+q)x$ for all $x \in R$, with $(b+c)^2 = 0 = (p+q)^2$.

(iii) $a(b+c)^2 - (p+q)^2 - (b+c)^2 a = 0$ with $f(x_1,\ldots,x_n)^2$ is central valued. Thus $F(x) = (b+c)x$ and $G(x) = (p+q)x$ for all $x \in R$, with $[a,(b+c)^2] = (p+q)^2 \in C$ with $f(x_1,\ldots,x_n)^2$ is central valued.

□

4.4 Proof of Theorem 4.1.1

By Theorem 1.6.2, there exist derivations d' and δ on U, $b, p \in U$ such that $F(x) = bx + d'(x)$ and $G(x) = px + \delta(x)$. Thus $F^2(x) = b(bx+d'(x)) + d'(bx+d'(x)) = b^2x + bd'(x) + d'(b)x + bd'(x) + d'^2(x) = F(b)x + 2bd'(x) + d'^2(x)$ and $G^2(x) = G(p)x + 2p\delta(x) + \delta^2(x)$.

By hypothesis

$$d\Big(F(b)f(r)^2 + 2bd'(f(r))f(r) + d'^2(f(r))f(r)\Big)$$

$$= f(r)\Big(G(p)f(r) + 2p\delta(f(r)) + \delta^2(f(r))\Big)$$

for all $r = (r_1,\ldots,r_n) \in R^n$. Since I, R and U satisfy the same GPIs (see Theorem 1.5.2) as well as the same differential identities (see Theorem 1.5.4)

$$d\Big(F(b)f(r)^2 + 2bd'(f(r))f(r) + d'^2(f(r))f(r)\Big) = f(r)\Big(G(p)f(r) + 2p\delta(f(r)) + \delta^2(f(r))\Big) \tag{4.4.1}$$

for all $r = (r_1,\ldots,r_n) \in U^n$.

If either $d = 0$ or $F = 0$, then by [35], we get our conclusions.

Thus we assume that d and F are nonzero maps. If d, d' and δ are all inner maps, then by Lemma 4.3.9, we have our all conclusions of Theorem 4.1.1. Thus, to prove our Theorem 4.1.1, we need to consider when all of d, F and G are not inner. Therefore, we have the following cases:

- d is inner, d' and δ are not both inner.
- d' is inner, d and δ are not both inner.
- δ is inner, d and d' are not both inner.
- d, d' and δ all are outer.

Case-I: *When d is inner, d' and δ are not both inner.*

In this case, let $d(x) = [a, x]$ for all $x \in U$. Then

$$\Big[a, \Big\{F(b)f(r) + 2bd'(f(r)) + d'^2(f(r))\Big\}f(r)\Big] = f(r)\Big\{G(p)f(r) + 2p\delta(f(r)) + \delta^2(f(r))\Big\} \tag{4.4.2}$$

for all $r = (r_1, \ldots, r_n) \in U^n$, where d', δ are two derivations on U.

Now we consider the following two sub-cases:

Sub-case-i: *Assume that δ is inner derivation of U*

Let $\delta(x) = [q, x]$ for all $x \in U$, for some $q \in U$.

By (4.4.2), U satisfies

$$\Big[a, \Big\{F(b)f(r) + 2bd'(f(r)) + d'^2(f(r))\Big\}f(r)\Big] = f(r)\Big\{G(p)f(r) + 2p[q, f(r)] + [q, [q, f(r)]]\Big\}. \tag{4.4.3}$$

We know that $d'(f(r_1, \ldots, r_n)) = f^{d'}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d'(r_i), \ldots, r_n)$, and

$$d'^2(f(r_1, \ldots, r_n)) = f^{d'^2}(r_1, \ldots, r_n) + 2\sum_i f^{d'}(r_1, \ldots, d'(r_i), \ldots, r_n)$$
$$+\sum_i f(r_1, \ldots, d'^2(r_i), \ldots, r_n) + \sum_{i \neq j} f(r_1, \ldots, d'(r_i), \ldots, d'(r_j), \ldots, r_n).$$

By Fact 4.2.3, since char $(R) \neq 2$, we can replace $d'(r_i)$ with x_i and $d'^2(r_i)$ with y_i in (4.4.3), and then U satisfies blended component

$$[a, \sum_i f(r_1, \ldots, y_i, \ldots, r_n)f(r_1, \ldots, r_n)] = 0. \tag{4.4.4}$$

In particular, replacing y_i with $[p', r_i]$ for some $p' \notin C$, we have

$$[a, [p', f(r_1, \ldots, r_n)]f(r_1, \ldots, r_n)] = 0 \tag{4.4.5}$$

which implies by Fact 4.2.8, $a \in C$, a contradiction.

Sub-case-ii: *Assume that δ is outer derivation of U*

Let d' and δ are C-dependent modulo inner derivations of U, say $\alpha d' + \beta\delta = ad_{q'}$, where $\alpha, \beta \in C$, $q \in U$ and $ad_{q'}(x) = [q', x]$ for all $x \in U$. Since δ is outer, $\alpha \neq 0$ and hence $d'(x) = \lambda\delta(x) + [q'', x]$ for all $x \in U$, for some $\lambda \in C$ and $q'' \in U$. Moreover,

$$\begin{aligned} d'^2(x) = d'(\lambda\delta(x) + [q'', x]) &= d'(\lambda)\delta(x) + \lambda d'\delta(x) + [d'(q''), x] + [q'', d'(x)] \\ &= d'(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda[q'', \delta(x)] + [d(q''), x] + [q'', [q'', x]]. \end{aligned} \tag{4.4.6}$$

From (4.4.2), we obtain

$$\begin{aligned} &\Bigg[a, \Bigg\{F(b)f(r) + 2b\lambda\delta(f(r)) + 2b[q'', f(r)] + d'(\lambda)\delta(f(r)) \\ &+\lambda^2\delta^2(f(r)) + 2\lambda[q'', \delta(f(r))] + [d(q''), f(r)] + [q'', [q'', f(r)]]\Bigg\}f(r)\Bigg] \\ &= f(r)\Bigg\{G(p)f(r) + 2p\delta(f(r)) + \delta^2(f(r))\Bigg\} \end{aligned} \tag{4.4.7}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By Fact 4.2.3, since char $(R) \neq 2$, we can replace $\delta(r_i)$ with x_i and $d'^2(r_i)$ with y_i in (4.4.7), and then U satisfies blended component

$$\Bigg[a, \lambda^2 \sum_i f(r_1, \ldots, y_i, \ldots, r_n)f(r_1, \ldots, r_n)\Bigg] = f(r_1, \ldots, r_n)\sum_i f(r_1, \ldots, y_i, \ldots, r_n).$$

In particular,

$$[a, \lambda^2 f(r_1, \ldots, r_n)^2] = f(r_1, \ldots, r_n)^2, \tag{4.4.8}$$

that is $(a\lambda^2 - 1)f(r_1, \ldots, r_n)^2 - f(r_1, \ldots, r_n)^2(a\lambda^2) = 0$. By Fact 4.2.7, $f(r_1, \ldots, r_n)$ is central valued, a contradiction.

Case-II: *When d' is inner, d and δ are not both inner.*

In this case, let $d'(x) = [a', x]$ for all $x \in U$. Then F is also inner. Let $F(x) = bx + xc$ for all $x \in R$, and for some $b, c \in U$. Then U satisfies

$$d\Big\{b^2 f(r)^2 + 2bf(r)cf(r) + f(r)c^2 f(r)\Big\}$$
$$= f(r)\Big\{G(p)f(r) + 2p\delta(f(r)) + \delta^2(f(r))\Big\} \tag{4.4.9}$$

for all $r = (r_1, \ldots, r_n) \in U^n$, where d, δ are two derivations on U. If d is inner, then we have contradiction by Case-I(ii) (when $\lambda = 0$).

So we assume that d is outer.

<u>Sub-case-i</u>: *Let d and δ be C-dependent modulo inner derivations of U*

Let $\alpha d + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in R$. Since d is outer, $\beta \neq 0$ and hence $\delta(x) = \lambda d(x) + [q', x]$ for all $x \in U$, for some $\lambda \in C$ and $q' \in U$.

$$\delta^2(x) = \delta(\lambda d(x) + [q', x]) = \delta(\lambda)d(x) + \lambda\delta d(x) + [\delta(q'), x] + [q', \delta(x)]$$
$$= \delta(\lambda)d(x) + \lambda^2 d^2((x)) + 2\lambda[q', d(x)] + [\delta(q'), x] + [q', [q', x]]. \tag{4.4.10}$$

From (4.4.9), we obtain

$$d\Big\{b^2 f(r)^2 + 2bf(r)cf(r) + f(r)c^2 f(r)\Big\}$$
$$= f(r)\Big\{G(p)f(r) + 2p\lambda d(f(r)) + 2p[q', f(r)] + \delta(\lambda)d(f(r))$$
$$+\lambda^2 d^2((f(r))) + 2\lambda[q', d(f(r))] + [\delta(q'), f(r)] + [q', [q', f(r)]]\Big\} \tag{4.4.11}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By Fact 4.2.3, we can replace $d(r_i)$ with x_i and $d^2(r_i)$ with y_i in (4.4.11), and then U satisfies blended component

$$0 = f(r_1, \ldots, r_n)\lambda^2 \sum_i f(r_1, \ldots, y_i, \ldots, r_n). \tag{4.4.12}$$

In particular,

$$\lambda^2 f(r_1, \ldots, r_n)^2 = 0. \tag{4.4.13}$$

As $f(r_1,\dots,r_n)=0$ is not an identity for U, $\lambda=0$ and so δ is inner. By applying Theorem 1.6.4 in (4.4.11), we can replace $d(r_i)$ with x_i and $d^2(r_i)$ with y_i, and then U satisfies blended component

$$\begin{aligned} b^2\sum_i f(r_1,\dots,x_i,\dots,r_n)f(r_1,\dots,r_n)+b^2f(r_1,\dots,r_n)\sum_i f(r_1,\dots,x_i,\dots,r_n)\\ +2b\sum_i f(r_1,\dots,x_i,\dots,r_n)cf(r_1,\dots,r_n)+2bf(r_1,\dots,r_n)c\sum_i f(r_1,\dots,x_i,\dots,r_n)\\ +\sum_i f(r_1,\dots,x_i,\dots,r_n)c^2f(r_1,\dots,r_n)\\ +f(r_1,\dots,r_n)c^2\sum_i f(r_1,\dots,x_i,\dots,r_n)=0. \end{aligned} \tag{4.4.14}$$

In particular,

$$b^2f(r_1,\dots,r_n)^2+2bf(r_1,\dots,r_n)cf(r_1,\dots,r_n)+f(r_1,\dots,r_n)c^2f(r_1,\dots,r_n)=0,$$

that is, $F^2(f(r))f(r)=0$ for all $r=(r_1,\dots,r_n)\in U^n$. Hence by (4.4.9), $f(r)G^2(f(r))=0$ for all $r=(r_1,\dots,r_n)\in U^n$. Thus by [35], we have our conclusions.

Sub-case-ii: *Let d and δ be C-independent modulo inner derivations of U*

Then by applying Theorem 1.6.4 in (4.4.9), we can replace $d(r_i)$ with x_i, $d^2(r_i)$ with y_i, $\delta(r_i)$ with s_i and $\delta^2(r_i)$ with t_i, and then U satisfies blended component

$$f(r_1,\dots,r_n)\sum_i f(r_1,\dots,t_i,\dots,r_n)=0 \tag{4.4.15}$$

which implies in particular $f(r_1,\dots,r_n)^2=0$ and so U satisfies identity $f(r_1,\dots,r_n)=0$, a contradiction.

Case-III: *When δ is inner, d and d' are not both inner.*

In this case G is inner and so we may assume $G(x)=px+xq$ for all $x\in R$, for some $p,q\in U$. Then U satisfies

$$\begin{aligned} d\Big\{F(b)f(r)^2+2bd'(f(r))f(r)+d'^2(f(r))f(r)\Big\}\\ =\Big\{f(r)p^2f(r)+2f(r)pf(r)q+f(r)^2q^2\Big\} \end{aligned} \tag{4.4.16}$$

for all $r = (r_1, \ldots, r_n) \in U^n$, where d, d' are two derivations on U. If d is inner, then d' must be outer and hence conclusion follows by Case-I(i). If d' is inner, then d must be outer and hence conclusions follows by Case-II(i) (when $\lambda = 0$).

So we assume that both d and d' are outer.

Sub-case-i: *Let d and d' be C-dependent modulo inner derivations of U*

Assume $\alpha d + \beta d' = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in R$. Since d' is outer, $\alpha \neq 0$ and so $d(x) = \lambda d'(x) + [q', x]$ for all $x \in U$, for some $\lambda \in C$ and $q' \in U$.

From (4.4.16), we obtain

$$\begin{aligned} &\lambda d'\Big\{F(b)f(r)^2 + 2bd'(f(r))f(r) + d'^2(f(r))f(r)\Big\} \\ &+\Big[q', F(b)f(r)^2 + 2bd'(f(r))f(r) + d'^2(f(r))f(r)\Big] \\ &= f(r)\Big\{f(r)p^2f(r) + 2f(r)pf(r)q + f(r)^2q^2\Big\} \end{aligned} \tag{4.4.17}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. By applying Fact 4.2.3, since char $(R) \neq 2, 3$, we can replace $d'(r_i)$ with x_i, $d'^2(r_i)$ with t_i and $d'^3(r_i)$ with y_i in (4.4.17), and then U satisfies blended component

$$\lambda \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0. \tag{4.4.18}$$

In particular,

$$\lambda f(r_1, \ldots, r_n)^2 = 0. \tag{4.4.19}$$

As $f(r_1, \ldots, r_n) = 0$ is not an identity for U, $\lambda = 0$ and so d is inner, a contradiction.

Sub-case-ii: *Let d and d' be C-independent modulo inner derivations of U*

Then by applying Fact 4.2.3 in (4.4.16), we can replace $d(r_i)$ with x_i, $d'(r_i)$ with y_i, $d'^2(r_i)$ with z_i and $dd'^2(r_i)$ with s_i, and then U satisfies blended component

$$\sum_i f(r_1, \ldots, s_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0 \tag{4.4.20}$$

which implies in particular $f(r_1, \ldots, r_n)^2 = 0$ and so U satisfies identity $f(r_1, \ldots, r_n) = 0$, a contradiction.

Case-IV: *When δ, d and d' are all outer.*

Sub-case-i: *When d, δ and d' be C-independent modulo inner derivations of U.*

By 4.2.3, we replace $d(r_i)$,$d'(r_i)$, ${d'}^2(r_i)$, $d{d'}^2(r_i)$, $\delta(r_i)$ and $\delta^2(r_i)$ with m_i, n_i, n'_i, t_i, x_i and y_i respectively in (4.4.1) and then U satisfies the blended component:

$$\sum_i f(r_1, \ldots, t_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0.$$

In particular, replacing t_1 with r_1 and $t_2 = \ldots = t_n = 0$, we get the identity $f(r_1, \ldots, r_n)^2 = 0$ for all $r_1, \ldots, r_n \in U$. This implies the contradiction $f(r_1, \ldots, r_n) = 0$ for any $r_1, \ldots, r_n \in R$.

Sub-case-ii: *When d, δ and d' be C-dependent modulo inner derivations of U.*

This implies there exist λ, μ, $\gamma \in C$ and $q \in U$ such that

$$\lambda d(x) + \mu d'(x) + \gamma \delta(x) = [q, x] \tag{4.4.21}$$

for all $x \in R$. If we consider $\mu = \gamma = 0$, then definitely $\lambda \neq 0$, which implies the contradiction that d is inner derivation.

So to move forward we have to consider $(\mu, \gamma) \neq (0, 0)$.

Without loss of generality, we assume that $\mu \neq 0$.

By (4.4.21), we have that

$$d'(x) = \lambda' d(x) + \gamma' \delta(x) + [q', x] \tag{4.4.22}$$

where $\lambda' = -\mu^{-1}\lambda$, $\gamma' = -\mu^{-1}\gamma$, $q' = \mu^{-1}q$ and for all $x \in R$. If the expression of $d'(x)$, has been calculated on (4.4.22) we put on (4.4.1), leads a differential identity in which the derivation words those appear of the type: d, δ, d^2, δ^2, $d^2\delta$, $d\delta^2$, $d\delta d$, d^3.

Now let d, δ be linearly $C-$independent modulo inner derivations, using the Fact 4.2.3 again, we may replace each $d^3(r_i)$ and $d\delta^2(r_i)$ with y_i and t_i respectively and U satisfies the blended components

$$\lambda'^2 \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0$$

and

$$\gamma'^2 \sum_i f(r_1, \ldots, t_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0.$$

In above two expressions, putting $y_1 = r_1, r_2 = \ldots = r_n = 0$ on the former and $t_1 = r_1, r_2 = \ldots = r_n = 0$ on the latter, we get

$$\lambda'^2 f(r_1, \ldots, r_n)^2 = 0$$

and

$$\gamma'^2 f(r_1, \ldots, r_n)^2 = 0$$

for all $r_1, \ldots, r_n \in U$. Being a nonzero multilinear polynomial f, we get $\lambda' = 0$ and $\gamma' = 0$. From the values of $\lambda' = 0$ and $\gamma' = 0$ we get $\lambda = 0$ and $\gamma = 0$, which force d' to be inner derivation, this leads to a contradiction.

Now consider the case d, δ are linearly $C-$dependent modulo inner derivations. Then there exist suitable η, $\vartheta \in C$ and $p \in U$ such that

$$\eta d(x) + \vartheta \delta(x) = [p, x] \tag{4.4.23}$$

for all $x \in R$. Here both η and ϑ are nonzero, otherwise we get the contradiction of one of d and δ is inner.

Consider $\vartheta \neq 0$, then we may write

$$\delta(x) = \eta' d(x) + [p', x] \tag{4.4.24}$$

where $\eta' = -\vartheta^{-1}\eta$, $p' = \vartheta^{-1}p$ and for all $x \in R$. Hence (4.4.22) and (4.4.23) lead to

$$d'(x) = (\lambda' + \gamma'\eta')d(x) + [q' + \gamma'p', x] \tag{4.4.25}$$

for all $x \in R$. Once again, we substitute d' in (4.4.1). In this case we arrive at a differential identity in which the derivation words that appear are of the type: d, d^2, d^3. Since d is not inner, we may replace each $d(r_i)$, $d^2(r_i)$, $d^3(r_i)$ with y_i, t_i, z_i respectively. In particular, U satisfies the blended component

$$(\lambda' + \gamma'\eta')^2 \sum_i f(r_1, \ldots, z_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0.$$

In particular,

$$(\lambda' + \gamma'\eta')^2 f(r_1, \ldots, r_n)^2 = 0$$

for all $r_1, \ldots, r_n \in U$. This says that $\lambda' + \gamma'\eta' = 0$ and d' is an inner derivation, which is again a contradiction.

This completes the proof of the theorem.

Chapter 5

Generalized Derivations Acting on Multilinear Polynomials in Prime Rings

5.1 Introduction

In [73], Lee and Shiue showed that if R is a prime ring, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C and d a nonzero derivation of R such that $d(u)u \in C$ for all $u \in f(R)$, then $char(R) = 2$ and R satisfies s_4. Further, Filippis and Dhara [48] extend the above result to generalized derivation by considering $F(u)u = 0$ for all $u \in f(\rho)$, where ρ a nonzero right ideal and F is a generalized derivation of R. They found out the all possible structures of F. In [25], Demir and Argaç give a generalization of the above result [48] by taking $F(u)u \in C$ for all $u \in f(\rho)$, where ρ a nonzero right ideal of R. Then $F(x) = ax$, where $a \in C$ and $f(r_1, \ldots, r_n)^2$ is central valued on R, except when $char(R) = 2$ and R satisfies s_4.

Furthermore, Eroğlu and Argaç [43] extended the above result [25] and determined all possible structures of F by considering $F^2(u)u \in C$ for all $u \in f(R)$ and F is a

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generalized derivation of R. Few years back Dhara and Filippis [35] studied the identity $F^2(u)u = uH^2(u)$ for all $u \in f(I)$, where F, H are two nonzero generalized derivations of R, I is an ideal of R and determined all possible forms of F and G.

In a paper, Tiwari [83] described two generalized derivations F and H of R satisfying the condition $F^2(u)u = H(u^2)$ for all $u \in f(R)$. In a recent paper, Yadav [90] described all possible forms of the maps when $F^2(u)d(u) = 0$ for all $u \in f(R)$, where F is generalized derivation of R and d is a nonzero derivation of R. Motivated by the above situations and combining [83, 90], in the present chapter our motivation is to consider the situation

$$F^2(u)d(u) = H(u^2)$$

for all $u \in f(I)$, where F, H are two generalized derivations and d is a derivation of R and I is a non-zero ideal of R. More precisely, we prove the following:

Theorem 5.1.1. *Let R be a prime ring of characteristic different from 2, I a nonzero ideal of R, U its Utumi ring of quotients and C its extended centroid. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, F and H be two generalized derivations of R and d be a nonzero derivation of R such that*

$$F^2(f(r))d(f(r)) = H(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in I^n$. Then one of the following holds:

1. *there exists $a \in U$ such that $F(x) = xa$, $H(x) = 0$ for all $x \in R$ with $a^2 = 0$;*
2. *there exist $a, p \in U$ such that $F(x) = xa$, $H(x) = [p, x]$ for all $x \in R$ with $a^2 = 0$ and $f(R)^2 \in C$;*
3. *there exists $a \in U$ such that $F(x) = ax$, $H(x) = 0$ for all $x \in R$ with $a^2 = 0$;*
4. *there exist $a, p \in U$ such that $F(x) = ax$, $H(x) = [p, x]$ for all $x \in R$ with $a^2 = 0$ and $f(R)^2 \in C$.*

5.2 Preliminary Results

In the sequel, unless specifically stated, R always denotes a prime ring with center $Z(R)$, Utumi ring of quotients U and extended centroid C.

Fact 5.2.1. *[4, Lemma 1] Let R be a noncommutative prime ring, $a, b \in U$, $p(x_1, \ldots, x_n)$ be any polynomial over C, which is not an identity for R. If $ap(r) - p(r)b = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

(1) $a = b \in C$,

(2) $a = b$ and $p(x_1, \ldots, x_n)$ is central valued on R,

(3) char $(R) = 2$ and R satisfies s_4.

Fact 5.2.2. *[42, Theorem 3.5] If A is a closed prime algebra over a field Φ and F is an extension field of Φ, then $A \bigotimes_\Phi F$ is a closed prime algebra over F.*

Fact 5.2.3. *[83, Corollary 1.3] Let R be a non commutative prime ring of characteristic different from 2, I an ideal of R and U be its Utumi ring of quotients with extended centroid C and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C. Suppose that F is a non zero generalized derivation of R such that $F^2(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in I$. Then one of the following holds:*

(i) there exists $a \in U$ such that $F(x) = xa$ for all $x \in R$ with $a^2 = 0$;

(ii) there exists $a \in U$ such that $F(x) = ax$ for all $x \in R$ with $a^2 = 0$.

Fact 5.2.4. *[80, Theorem 1] Let R be a prime ring of characteristic not 2 and d_1, d_2 derivations of R such that the iterate $d_1 d_2$ is also a derivation; then at least one of d_1, d_2 is zero.*

Fact 5.2.5. *[37, Lemma 2.2] Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and $f(x_1, \ldots, x_n)$*

be a multilinear polynomial over C, which is not central valued on R. Suppose that there exist $a, b, q \in U$ such that $af(r)^2 + f(r)^2q + f(r)bf(r) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following holds:

(1) $a, q \in C$ and $q + a = -b \in C$;

(2) $f(r_1, \ldots, r_n)^2$ is central valued on R and $q + a = -b \in C$.

5.3 The Case of Inner Generalized Derivations and Inner Derivation

In this section, we consider the case when F and G both are inner generalized derivations of R and d is a nonzero derivation of R. Let $F(x) = ax + xb$, $G(x) = px + xq$ and $d(x) = [c, x]$ for all $x \in R$, for some $a, b, p, q, c \in U$.

To study this situation, first we study the following generalized polynomial identity

$$a_1ua_2u + a_3ua_4u + ua_4u - a_1u^2a_2 - a_3ua_5ua_2 - ua_6ua_2 = a_7u^2 + u^2a_8 \quad (5.3.1)$$

for all $u \in f(R)$, where $a_1, \ldots, a_8 \in U$.

We analyze this generalized polynomial identity in prime ring. In all that follows, let R be a prime ring with extended centroid C, char $(R) \neq 2$ and $f(r_1, \ldots, r_n)$ a multilinear polynomial over C which is not central valued on R.

Proposition 5.3.1. *Let $R = M_m(C)$, $m \geq 2$, be the ring of all $m \times m$ matrices over the field C, $f(r_1, \ldots, r_n)$ a non-central multilinear polynomial over C. If for some $a_1, \ldots, a_8 \in R$, R satisfies*

$$a_1ua_2u + a_3ua_4u + ua_4u - a_1u^2a_2 - a_3ua_5ua_2 - ua_6ua_2 = a_7u^2 + u^2a_8$$

for all $u \in f(R)$, then either $a_2 \in C.I_m$ or $a_3 \in C.I_m$ or $a_5 \in C.I_m$.

Proof. We assume first that C is an infinite field. By hypothesis, R satisfies the generalized polynomial identity

$$\begin{aligned} a_1f(r_1,\dots,r_n)a_2f(r_1,\dots,r_n)+a_3f(r_1,\dots,r_n)a_4f(r_1,\dots,r_n)\\ +f(r_1,\dots,r_n)a_4f(r_1,\dots,r_n)-a_1f(r_1,\dots,r_n)^2a_2\\ -a_3f(r_1,\dots,r_n)a_5f(r_1,\dots,r_n)a_2-f(r_1,\dots,r_n)a_6f(r_1,\dots,r_n)a_2\\ =a_7f(r_1,\dots,r_n)^2+f(r_1,\dots,r_n)^2a_8. \end{aligned} \tag{5.3.2}$$

We assume first that $a_2\notin C.I_m$ and $a_3\notin C.I_m$ and $a_5\notin C.I_m$. By Theorem 1.6.3 there exists an invertible matrix P in $M_m(C)$ such that Pa_2P^{-1}, Pa_3P^{-1} and Pa_5P^{-1} have all non-zero entries. Note that $R=M_m(C)$ must satisfies

$$\begin{aligned} Pa_1P^{-1}f(r_1,\dots,r_n)Pa_2P^{-1}f(r_1,\dots,r_n)\\ +Pa_3P^{-1}f(r_1,\dots,r_n)Pa_4P^{-1}f(r_1,\dots,r_n)\\ +f(r_1,\dots,r_n)Pa_4P^{-1}f(r_1,\dots,r_n)-Pa_1P^{-1}f(r_1,\dots,r_n)^2Pa_2P^{-1}\\ -Pa_3P^{-1}f(r_1,\dots,r_n)Pa_5P^{-1}f(r_1,\dots,r_n)Pa_2P^{-1}\\ -f(r_1,\dots,r_n)Pa_6P^{-1}f(r_1,\dots,r_n)Pa_2P^{-1}\\ =Pa_7P^{-1}f(r_1,\dots,r_n)^2+f(r_1,\dots,r_n)^2Pa_8P^{-1}. \end{aligned} \tag{5.3.3}$$

Given that $f(x_1,\dots,x_n)$ is non central valued. Hence by Fact 4.2.2, there exist matrices $r_1,\dots,r_n\in M_m(C)$ and $0\neq\gamma\in C$ such that $f(r_1,\dots,r_n)=\gamma e_{ij}$, with $i\neq j$. Replacing this value of $f(r_1,\dots,r_n)$ in (5.3.3), we get

$$\begin{aligned} Pa_1P^{-1}e_{ij}Pa_2P^{-1}e_{ij}+Pa_3P^{-1}e_{ij}Pa_4P^{-1}e_{ij}\\ +e_{ij}Pa_4P^{-1}e_{ij}-Pa_3P^{-1}e_{ij}Pa_5P^{-1}e_{ij}Pa_2P^{-1}-e_{ij}Pa_6P^{-1}e_{ij}Pa_2P^{-1}\\ =0. \end{aligned} \tag{5.3.4}$$

Then multiplying by e_{ij} in above relation from both sides, it follows

$$e_{ij}Pa_3P^{-1}e_{ij}Pa_5P^{-1}e_{ij}Pa_2P^{-1}e_{ij} = 0.$$

This leads to a contradiction as Pa_2P^{-1}, Pa_3P^{-1} and Pa_5P^{-1} have all non-zero entries.

Next we assumes that C is finite field. Let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Since multilinear polynomial $f(x_1, \ldots, x_n)$ is non-central-valued on R, so it is also non-central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{gathered} P(r_1, \ldots, r_n) = \\ a_1f(r_1, \ldots, r_n)a_2f(r_1, \ldots, r_n) + a_3f(r_1, \ldots, r_n)a_4f(r_1, \ldots, r_n) \\ +f(r_1, \ldots, r_n)a_4f(r_1, \ldots, r_n) - a_1f(r_1, \ldots, r_n)^2a_2 \\ -a_3f(r_1, \ldots, r_n)a_5f(r_1, \ldots, r_n)a_2 - f(r_1, \ldots, r_n)a_6f(r_1, \ldots, r_n)a_2 \\ -a_7f(r_1, \ldots, r_n)^2 - f(r_1, \ldots, r_n)^2a_8 \end{gathered} \tag{5.3.5}$$

which is a generalized polynomial identity for R.

Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $r_1, \ldots, r_n$.

Linearizing the variable r_1, that is, replacing r_1 with $r_1 + s_1$ in $P(r_1, \ldots, r_n) = 0$, we obtain

$$P(r_1 + s_1, \ldots, r_n) = 0 \tag{5.3.6}$$

which implies

$$P(r_1, \ldots, r_n) + P(s_1, \ldots, r_n) + \Theta_1(r_1, \ldots, r_n, s_1) = 0 \tag{5.3.7}$$

where

$$\Theta_1(r_1,\ldots,r_n,s_1) =$$

$$a_1f(r_1,\ldots,r_n)a_2f(s_1,\ldots,r_n) + a_1f(s_1,\ldots,r_n)a_2f(r_1,\ldots,r_n)$$

$$+a_3f(r_1,\ldots,r_n)a_4f(s_1,\ldots,r_n) + +a_3f(s_1,\ldots,r_n)a_4f(r_1,\ldots,r_n)$$

$$+f(r_1,\ldots,r_n)a_4f(s_1,\ldots,r_n) + f(s_1,\ldots,r_n)a_4f(r_1,\ldots,r_n)$$

$$-a_1f(r_1,\ldots,r_n)f(s_1,\ldots,r_n)a_2 - a_1f(s_1,\ldots,r_n)f(r_1,\ldots,r_n)a_2$$

$$-a_3f(r_1,\ldots,r_n)a_5f(s_1,\ldots,r_n)a_2 - a_3f(s_1,\ldots,r_n)a_5f(r_1,\ldots,r_n)a_2$$

$$-f(r_1,\ldots,r_n)a_6f(s_1,\ldots,r_n)a_2 - f(s_1,\ldots,r_n)a_6f(r_1,\ldots,r_n)a_2$$

$$-a_7f(r_1,\ldots,r_n)f(s_1,\ldots,r_n) - a_7f(s_1,\ldots,r_n)f(r_1,\ldots,r_n)$$

$$-f(r_1,\ldots,r_n)f(s_1,\ldots,r_n)a_8 - f(s_1,\ldots,r_n)f(r_1,\ldots,r_n)a_8.$$

Since $P(r_1,\ldots,r_n) = 0$ and $P(s_1,\ldots,r_n) = 0$, we have $\Theta_1(r_1,\ldots,r_n,s_1) = 0$ for all $r_1,\ldots,r_n,s_1 \in R$. Note that $\Theta_1(r_1,\ldots,r_n,r_1) = 2P(r_1,\ldots,r_n)$.

Again linearizing the situation (i.e., replacing r_2 with r_2+s_2) in $\Theta_1(r_1,\ldots,r_n,s_1) = 0$ and using it, we obtain $\Theta_2(r_1,\ldots,r_n,s_1,s_2) = 0$ such that

$$\Theta_2(r_1,\ldots,r_n,r_1,r_2) = 2^2P(r_1,\ldots,r_n).$$

Continuing this process of linearization, after complete linearization of $P(r_1,\ldots,r_n) = 0$, we get

$$\Theta_n(r_1,\ldots,r_n,s_1,\ldots,s_n) = 0$$

for all $r_1,\ldots,r_n,s_1,\ldots,s_n \in R$, where $\Theta_n(r_1,\ldots,r_n,s_1,\ldots,s_n)$ is a multilinear generalized polynomial in $2n$ indeterminates, such that

$$\Theta_n(r_1,\ldots,r_n,s_1,\ldots,s_n) = 2^nP(r_1,\ldots,r_n).$$

Clearly the multilinear polynomial $\Theta_n(r_1, \dots, r_n, s_1, \dots, s_n)$ is a generalized polynomial identity for R and $\overline{R}$ too (see Fact 4.2.1). Since $char(C) \neq 2$ we obtain $P(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in \overline{R}$ and then conclusion follows from above when C was infinite. $\square$

Proposition 5.3.2. *Let R be a prime ring of char $(R) \neq 2$, U the Utumi ring of quotients, C the extended centroid of R and $f(r_1, \dots, r_n)$ a non-central multilinear polynomial over C. If for some $a_1, \dots, a_8 \in U$, R satisfies*

$$a_1ua_2u + a_3ua_4u + ua_4u - a_1u^2a_2 - a_3ua_5ua_2 - ua_6ua_2 = a_7u^2 + u^2a_8,$$

then either $a_2 \in C$ or $a_3 \in C$ or $a_5 \in C$.

Proof. Since R and U satisfy the same generalized polynomial identities (see Theorem 1.5.2), U satisfies

$$\begin{aligned} a_1f(r_1, \dots, r_n)a_2f(r_1, \dots, r_n) + a_3f(r_1, \dots, r_n)a_4f(r_1, \dots, r_n) \\ +f(r_1, \dots, r_n)a_4f(r_1, \dots, r_n) - a_1f(r_1, \dots, r_n)^2a_2 \\ -a_3f(r_1, \dots, r_n)a_5f(r_1, \dots, r_n)a_2 - f(r_1, \dots, r_n)a_6f(r_1, \dots, r_n)a_2 \\ -a_7f(r_1, \dots, r_n)^2 - f(r_1, \dots, r_n)^2a_8 = 0. \end{aligned} \quad (5.3.8)$$

Suppose that this is a trivial GPI for U. For sake of clearness, we denote $X = f(r_1, \dots, r_n)$. Then

$$\begin{aligned} a_1Xa_2X + a_3Xa_4X + Xa_4X - a_1X^2a_2 \\ -a_3Xa_5Xa_2 - Xa_6Xa_2 - a_7X^2 - X^2a_8 \end{aligned} \quad (5.3.9)$$

is zero element in $T = U *_C C\{r_1, \dots, r_n\}$, the free product of U and $C\{r_1, \dots, r_n\}$, the free C-algebra in noncommuting indeterminates $r_1, \dots, r_n$ (see Remark 1.5.1).

From Remark 1.5.1 $\{1, a_2, a_8\}$ is linearly C-dependent. There exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 + \alpha_2a_2 + \alpha_3a_8 = 0$. If $\alpha_3 = 0$, then α_2 can not be zero and hence $a_2 \in C$, as desired. Thus we assume that $\alpha_3 \neq 0$. Then $a_8 = \alpha a_2 + \beta$ for some $\alpha, \beta \in C$. Then by (5.3.9),

$$\begin{aligned} a_1 X a_2 X + a_3 X a_4 X + X a_4 X - a_1 X^2 a_2 \\ -a_3 X a_5 X a_2 - X a_6 X a_2 - a_7 X^2 - X^2(\alpha a_2 + \beta) = 0 \in T. \end{aligned} \tag{5.3.10}$$

Again, if $a_2 \in C$, we have our conclusion. So we assume that $a_2 \notin C$. Then by Remark 1.5.1 from equation (5.3.10) we get,

$$-a_1 X^2 a_2 - a_3 X a_5 X a_2 - X a_6 X a_2 - X^2 \alpha a_2 = 0 \in T \tag{5.3.11}$$

that is

$$\{a_1 X + a_3 X a_5 + X a_6 + \alpha X\} X a_2 = 0 \in T. \tag{5.3.12}$$

Again from Remark 1.5.1 $\{1, a_1, a_3\}$ is linearly C-dependent. Let $\beta_1 + \beta_2 a_1 + \beta_3 a_3 = 0$. If $\beta_2 = 0$, then $a_3 \in C$, as desired. So we assume that $\beta_2 \neq 0$. Then $a_1 = \lambda a_3 + \mu$ for some $\lambda, \mu \in C$. Then by (5.3.12),

$$\{\lambda a_3 X + \mu X + a_3 X a_5 + X a_6 + \alpha X\} X a_2 = 0 \in T. \tag{5.3.13}$$

Assume $a_3 \notin C$. Then by Remark 1.5.1

$$a_3 X(\lambda + a_5) X a_2 = 0 \in T. \tag{5.3.14}$$

This implies either $a_3 = 0$ or $a_5 = -\lambda \in C$ or $a_2 = 0$, as desired.

Next suppose that (5.3.8) is a non-trivial GPI for U. By Fact 1.6.8, R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$.

In this case, by Proposition 5.3.1, we get that either a_2 or a_3 or a_5 are in C. If V is infinite dimensional over C, then for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since a_2, a_3, a_5 are not in C, there exist $h_1, h_2, h_3 \in soc(R)$ such that $[a_2, h_1] \neq 0$ $[a_3, h_2] \neq 0$, $[a_5, h_3] \neq 0$. By Litoff's Theorem (see Theorem 1.6.7), there exists idempotent $e \in soc(R)$ such that $a_2h_1, h_1a_2, a_3h_2, h_2a_3, a_5h_3, h_3a_5, h_1, h_2, h_3 \in eRe$. Since R satisfies generalized identity

$$e\{a_1f(er_1e, \dots, er_ne)a_2f(er_1e, \dots, er_ne) + a_3f(er_1e, \dots, er_ne)a_4f(er_1e, \dots, er_ne)$$
$$+f(er_1e, \dots, er_ne)a_4f(er_1e, \dots, er_ne) - a_1f(er_1e, \dots, er_ne)^2a_2$$
$$-a_3f(er_1e, \dots, er_ne)a_5f(er_1e, \dots, er_ne)a_2 - f(er_1e, \dots, er_ne)a_6f(er_1e, \dots, er_ne)a_2$$
$$-a_7f(er_1e, \dots, er_ne)^2 - f(er_1e, \dots, er_ne)^2a_8\}e = 0,$$

the subring eRe satisfies

$$ea_1ef(r_1, \dots, r_n)ea_2ef(r_1, \dots, r_n) + ea_3ef(r_1, \dots, r_n)ea_4ef(r_1, \dots, r_n)$$
$$+f(r_1, \dots, r_n)ea_4ef(r_1, \dots, r_n) - ea_1ef(r_1, \dots, r_n)^2ea_2e$$
$$-ea_3ef(r_1, \dots, r_n)ea_5ef(r_1, \dots, r_n)ea_2e - f(r_1, \dots, r_n)ea_6ef(r_1, \dots, r_n)ea_2e$$
$$-ea_7ef(r_1, \dots, r_n)^2 - f(r_1, \dots, r_n)^2ea_8e = 0.$$

Then by the above finite dimensional case, either ea_2e or ea_3e or ea_5e are central elements of eRe. Thus either $a_2h_1 = (ea_2e)h_1 = h_1ea_2e = h_1a_2$ or $a_3h_2 = (ea_3e)h_2 = h_2(ea_3e) = h_2a_3$ or $a_5h_3 = (ea_5e)h_3 = h_3(ea_5e) = h_3a_5$, in any case we get a contradiction.

Thus, we have proved that either a_2 or a_3 or a_5 are in C. □

Lemma 5.3.3. *Let R be a noncommutative prime ring of characteristic different from 2, U the Utumi ring of quotients, C be its extended centroid and $f(r_1, \dots, r_n)$ be a noncentral multilinear polynomial over C. Suppose for some $a, b, p, q, c \in U$, $F(x) = ax+xb$, $H(x) =$*

$px + xq$ and $d(x) = [c, x]$ for all $x \in R$ with $c \notin C$. If

$$F^2(f(r))d(f(r)) = H(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

1. *$a \in C$, $(a + b)^2 = 0$, $p = -q \in C$;*
2. *$a \in C$, $(a + b)^2 = 0$, $f(R)^2 \in C$ and $p + q = 0$;*
3. *$b \in C$, $(a + b)^2 = 0$, $p = -q \in C$;*
4. *$b \in C$, $(a + b)^2 = 0$, $f(R)^2 \in C$ and $p + q = 0$.*

Proof. By our hypothesis, R satisfies

$$(a^2 f(r) + 2af(r)b + f(r)b)(cf(r) - f(r)c) = pf(r)^2 + f(r)^2 q$$

that is

$$a^2 f(r)cf(r) + 2af(r)bcf(r) + f(r)b^2 cf(r) - a^2 f(r)^2 c - 2af(r)bf(r)c - f(r)b^2 f(r)c = pf(r)^2 + f(r)^2 q \tag{5.3.15}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Proposition 5.3.2, either $a \in C$ or $b \in C$ or $c \in C$. Since $c \notin C$, either $a \in C$ or $b \in C$.

If $a \in C$, then $F(x) = x(a + b)$ and hence $F^2(x) = x(a + b)^2$ for all $x \in R$.

If $b \in C$, then $F(x) = (a + b)x$ and hence $F^2(x) = (a + b)^2 x$ for all $x \in R$.

Therefore, in any case F^2 becomes a generalized derivation of R. Assume $F^2 = G$, a generalized derivation of R. By hypothesis

$$G(f(r))d(f(r)) = H(f(r)^2)$$

for all $r = (r_1, \dots, r_n) \in R^n$. By [29, Main Theorem], $G = 0$ and $H(f(r)^2) = 0$ i.e., $pf(r)^2 + f(r)^2 q = 0$ for all $r = (r_1, \dots, r_n) \in R^n$.

Now $G = 0$ implies $F(x) = x(a + b)$ with $(a + b)^2 = 0$ or $F(x) = (a + b)x$ with $(a + b)^2 = 0$.

Moreover, $pf(r)^2 + f(r)^2 q = 0$ for all $r = (r_1, \dots, r_n) \in R^n$ implies by Fact 5.2.1, $p, q \in C$ with $p + q = 0$ or $f(R)^2 \in C$ with $p + q = 0$.

Thus in any cases we have our conclusions. □

5.4 Proof of Theorem 5.1.1

This is the final section which is dedicated to the proof of Theorem 5.1.1.
By Theorem 1.6.2, generalized derivations F and H have their forms $F(x) = ax + \delta(x)$, $H(x) = px + h(x)$ for some $a, p \in U$ and δ, h derivations on U.

Then $F^2(x) = F(ax + \delta(x)) = F(a)x + 2a\delta(x) + \delta^2(x)$.

By hypothesis, we have

$$\Big(F(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\Big)d(f(r)) = pf(r)^2 + h(f(r)^2)$$

for all $r = (r_1, \dots, r_n) \in I^n$. By Theorem 1.5.4, we have

$$\Big(F(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\Big)d(f(r)) = pf(r)^2 + h(f(r)^2) \tag{5.4.1}$$

for all $r = (r_1, \dots, r_n) \in U^n$.

If d, δ and h all are inner derivations, then by Lemma 5.3.3, we have our conclusions of Theorem 5.1.1. Thus, to prove Theorem 5.1.1, we need to consider the following cases.

- d, δ are inner, h is outer.

- d, h are inner, δ is outer.
- δ, h are inner, d is outer.
- d is inner, δ, h are outer.
- δ is inner, d, h are outer.
- h is inner, d, δ are outer.
- d, δ and h all are outer.

Case-1: *d, δ are inner, h is outer.*

Let $d(x) = [c, x]$ and $\delta(x) = [q, x]$ for all $x \in R$ and for some $c, q \in U$. By (5.4.1), U satisfies

$$\Big(F(a)f(r) + 2a[q, f(r)] + [q, [q, f(r)]]\Big)[c, f(r)] = pf(r)^2 + h(f(r)^2) \qquad (5.4.2)$$

By Theorem 1.6.4, we can replace $h(f(r_1, \ldots, r_n))$ by $f^h(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ in (5.4.2) and then U satisfies blended component

$$0 = \sum_i f(r_1, \ldots, y_i, \ldots, r_n)f(r_1, \ldots, r_n) + f(r_1, \ldots, r_n)\sum_i f(r_1, \ldots, y_i, \ldots, r_n).$$

In particular, for $y_1 = r_1$ and $y_2 = \cdots = y_n = 0$ we have $f(r_1, \ldots, r_n)^2 = 0$ implying $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in U$, a contradiction.

Case-2: *d, h are inner, δ is outer.*

Let $d(x) = [c, x]$ and $h(x) = [q, x]$ for all $x \in R$ and for some $p, q \in U$. By (5.4.1), U satisfies

$$\Big(F(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\Big)[c, f(r)] = pf(r)^2 + [q, f(r)^2]. \qquad (5.4.3)$$

Applying Theorem 1.6.4 to (5.4.3), we can replace $\delta(f(r_1,\ldots,r_n))$ with $f^{\delta}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)$ and $\delta^2(f(r_1,\ldots,r_n))$ with

$$f^{\delta^2}(r_1,\ldots,r_n)$$
$$+2\sum_i f^{\delta}(r_1,\ldots,y_i,\ldots,r_n)+\sum_i f(r_1,\ldots,t_i,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,y_j,\ldots,r_n)$$

in (5.4.3) and then U satisfies blended component

$$\sum_i f(r_1,\ldots,t_i,\ldots,r_n)[c,f(r_1,\ldots,r_n)]=0. \tag{5.4.4}$$

In particular, above relation yields $f(r_1,\ldots,r_n)[c,f(r_1,\ldots,r_n)]=0$ for all $r_1,\ldots,r_n\in U$, which implies $c\in C$, a contradiction.

Case-3: *δ, h are inner, d is outer.*

Let $\delta(x)=[p',x]$ and $h(x)=[q,x]$ for all $x\in R$ and for some $p',q\in U$. By (5.4.1), U satisfies

$$\Big(F(a)f(r)+2a[p',f(r)]+[p',[p',f(r)]]\Big)d(f(r))=pf(r)^2+[q,f(r)^2] \tag{5.4.5}$$

Since d is outer, by Theorem 1.6.4, we can replace $d(f(r_1,\ldots,r_n))$ by $f^{d}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)$ in (5.4.5) and then U satisfies blended component

$$\Big(F(a)f(r_1,\ldots,r_n)+2a[p',f(r_1,\ldots,r_n)]$$
$$+[p',[p',f(r_1,\ldots,r_n)]]\Big)\sum_i f(r_1,\ldots,y_i,\ldots,r_n)]=0. \tag{5.4.6}$$

In particular, we obtain from above relation

$$\Big(F(a)f(r)+2a[p',f(r)]+[p',[p',f(r)]]\Big)f(r)=0,$$

that is

$$F^2(f(r))f(r)=0$$

for all $r = (r_1, \ldots, r_n) \in U^n$. From Fact 5.2.3 we arrive the conclusions (1), (2) of our Theorem 5.1.1.

Case-4: *d is inner, δ, h are outer.*

Let $d(x) = [c, x]$ for all $x \in R$ and for some $c \in U$. By (5.4.1),

$$\Big(F(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\Big)[c, f(r)] = pf(r)^2 + h(f(r)^2) \tag{5.4.7}$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

Sub-case-i: Assume next that δ and h are C-independent modulo inner derivations of U. Then by Theorem 1.6.4, we can replace $h(f(r_1, \ldots, r_n))$ by $f^h(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, x_i, \ldots, r_n)$ and $\delta(f(r_1, \ldots, r_n))$ by $f^\delta(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ in (5.4.7) and then U satisfies blended components

$$\begin{aligned} 0 = &\sum_i f(r_1, \ldots, x_i, \ldots, r_n) f(r_1, \ldots, r_n) \\ &+ f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, x_i, \ldots, r_n). \end{aligned} \tag{5.4.8}$$

This is same as (5.4.4) and then by same argument it leads to a contradiction.

Sub-case-ii: Assume that δ and h are C-dependent modulo inner derivations of U, say $\alpha\delta + \beta h = ad'_q$, where $\alpha, \beta \in C$, $q' \in U$ and $ad'_q(x) = [q', x]$ for all $x \in R$.

Since δ is outer, $\alpha \neq 0$ and hence $\delta(x) = \lambda h(x) + [q, x]$ for all $x \in U$, where $\lambda = -\beta\alpha^{-1}$ and $q = \alpha^{-1}q'$. Then $\delta^2(x) = \lambda h(\delta(x)) + [q, \delta(x)]$ which yields $\delta^2(x) = \lambda h(\lambda)h(x) + \lambda^2 h^2(x) + \lambda h([q, x]) + \lambda[q, h(x)] + [q, [q, x]]$. Thus replacing the values of $\delta(x)$ and $\delta^2(x)$ in (5.4.7) and then applying Theorem 1.6.4, we can replace $h(f(r_1, \ldots, r_n))$ with $f^h(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ and $h^2(f(r_1, \ldots, r_n))$ with

$$\begin{gathered} f^{h^2}(r_1, \ldots, r_n) \\ +2\sum_i f^h(r_1, \ldots, y_i, \ldots, r_n) + \sum_i f(r_1, \ldots, t_i, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, y_j, \ldots, r_n) \end{gathered}$$

and then U satisfies blended component

$$\lambda^2 \sum_i f(r_1, \ldots, t_i, \ldots, r_n)[c, f(r_1, \ldots, r_n)] = 0. \tag{5.4.9}$$

Since $\lambda \neq 0$, we have from above

$$\sum_i f(r_1, \ldots, t_i, \ldots, r_n)[c, f(r_1, \ldots, r_n)] = 0. \tag{5.4.10}$$

In particular, $f(r_1, \ldots, r_n)[c, f(r_1, \ldots, r_n)] = 0$ for all $r_1, \ldots, r_n \in U$, which implies $c \in C$, a contradiction.

Case-5: *δ is inner, d, h are outer.*

Let $\delta(x) = [p', x]$ for all $x \in R$ and for some $p' \in U$. By (5.4.1), U satisfies

$$\Big(F(a)f(r) + 2a[p', f(r)] + [p', [p', f(r)]]\Big)d(f(r)) = pf(r)^2 + h(f(r)^2) \tag{5.4.11}$$

Sub-case-i: Let d and h be C-independent modulo inner derivations of U.

By Theorem 1.6.4, we can replace in (5.4.11) $d(f(r_1, \ldots, r_n))$ with $f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ and $h(f(r_1, \ldots, r_n))$ with $f^h(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, t_i, \ldots, r_n)$ in (5.4.11) and then U satisfies blended component

$$\begin{aligned} 0 &= f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, t_i, \ldots, r_n) \\ &+ \sum_i f(r_1, \ldots, t_i, \ldots, r_n) f(r_1, \ldots, r_n). \end{aligned} \tag{5.4.12}$$

This is same as (5.4.3) and thus by same argument we have our conclusions.

Sub-case-ii: Let d and h be C-dependent modulo inner derivations of U. Then $h(x) = \lambda d(x) + [q', x]$ for all $x \in U$, for some $0 \neq \lambda \in C$. From (5.4.11), U satisfies

$$\begin{aligned} &\Big(F(a)f(r) + 2a[p', f(r)] + [p', [p', f(r)]]\Big)d(f(r)) \\ &= pf(r)^2 + \lambda d(f(r)^2) + [q', f(r)^2]. \end{aligned} \tag{5.4.13}$$

Applying Theorem 1.6.4 to (5.4.13), we can replace $d(f(r_1,\ldots,r_n))$ with $f^d(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)$ in (5.4.13) and then U satisfies blended component

$$\begin{aligned}&\Big(F(a)f(r_1,\ldots,r_n)+2a[p',f(r_1,\ldots,r_n)]\\ &+[p',[p',f(r_1,\ldots,r_n)]]\Big)\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\\ &=\lambda\sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n)\\ &+\lambda f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n).\end{aligned} \tag{5.4.14}$$

In particular, for $t_1=r_1$ and $t_2=\cdots=t_n=0$, we have

$$\Big(F(a)f(r)+2a[p',f(r)]+[p',[p',f(r)]]\Big)f(r)=2\lambda f(r)^2 \tag{5.4.15}$$

that is

$$F^2(f(r))f(r)=2\lambda f(r)^2. \tag{5.4.16}$$

Then by [83] either $F(x)=ax$ or $F(x)=xa$, for some $a\in U$ with $a^2\in C$.
When $F(x)=ax$, then by (5.4.16) we get $(a^2-2\lambda)x^2=0$ for all $x\in f(R)$. This implies $a^2=2\lambda$, as $f(R)\notin C$.
Similarly, when $F(x)=xa$ we get $a^2=2\lambda$.

Now from the main identity $F^2(u)d(u)=H(u^2)$ and using $a^2=2\lambda$ from above two cases we get

$$\lambda[d(u),u]+(p+q')u^2-u^2q'=0 \tag{5.4.17}$$

for all $u\in f(R)$. Using Theorem 1.6.4 to (5.4.17), then U satisfies a blended component

$$\lambda\Big[\sum_i f(r_1,\ldots,y_i,\ldots,r_n),f(r_1,\ldots,r_n)\Big]=0.$$

Putting $y_i = [q'', x_i]$, for some $q'' \notin C$ and for all $i \in \{1, \ldots, n\}$ in the above blended component we get $\big[\lambda q'', f(r)\big]_2 = 0$, for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Fact 5.2.4 we get $\lambda q'' \in C$, implies $\lambda = 0$ as $q'' \notin C$.

Hence, from (5.4.17) we get $(p + q')u^2 - u^2 q' = 0$. So from Fact 5.2.5 we have one of the following:

(a) $q' \in C$ and $p = 0$;

(b) $f(R)^2 \in C$ and $p = 0$.

Using the case-(a), we have that $F(x) = ax$ or $F(x) = xa$ with $a^2 = 0$ and

$$\begin{aligned} H(x) &= px + h(x) \\ &= px + \lambda d(x) + [q', x] \\ &= px + [q', x] \\ &= 0, \end{aligned}$$

for all $x \in R$, which is nothing but our conclusions (1) and (3) of the Theorem 5.1.1.

Using the case-(b), we conclude $F(x) = ax$ or $F(x) = xa$ with $a^2 = 0$ and

$$\begin{aligned} H(x) &= px + h(x) \\ &= px + \lambda d(x) + [q', x] \\ &= [q', x] \end{aligned}$$

for all $x \in R$ along with $f(R)^2 \in C$, which gives our conclusions (2) and (4) of the

Theorem 5.1.1.

Case-6: *h is inner, d, δ are outer.*

Let $h(x) = [p', x]$ for all $x \in R$, for some $p' \in U$. By (5.4.1), U satisfies

$$\Big(F(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\Big)d(f(r)) = pf(r)^2 + [p', f(r)^2]. \tag{5.4.18}$$

Sub-case-i: Let d and δ be C-independent modulo inner derivations of U. By applying Theorem 1.6.4 to the above relation, we can replace $d(f(r_1, \dots, r_n))$ with $f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$ and $\delta^2(f(r_1, \dots, r_n))$ with

$$f^{\delta^2}(r_1, \dots, r_n) + 2\sum_i f^{\delta}(r_1, \dots, s_i, \dots, r_n)$$
$$+\sum_i f(r_1, \dots, t_i, \dots, r_n) + \sum_i f(r_1, \dots, s_i, \dots, y_j, \dots, r_n)$$

in (5.4.18) and then U satisfies blended component

$$\sum_i f(r_1, \dots, t_i, \dots, r_n) \sum_i f(r_1, \dots, y_i, \dots, r_n) = 0. \tag{5.4.19}$$

In particular, for $t_1 = r_1$, $t_2 = \dots = t_n = 0$, $y_1 = r_1$ and $y_2 = \dots = y_n = 0$ we have $f(r_1, \dots, r_n)^2 = 0$ which implies $f(r_1, \dots, r_n) = 0$, a contradiction.

Sub-case-ii: Let d and δ be C-dependent modulo inner derivations of U. Then $d(x) = \alpha\delta(x) + [q, x]$ for all $x \in U$, for some $0 \neq \alpha \in C$. From (5.4.18), U satisfies

$$\Big(F(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\Big)(\alpha\delta(f(r)) + [q, f(r)])$$
$$= pf(r)^2 + [p', f(r)^2]. \tag{5.4.20}$$

By applying Theorem 1.6.4 to (5.4.20), we can replace $\delta(f(r_1, \dots, r_n))$ with $f^{\delta}(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)$ and $\delta^2(f(r_1, \dots, r_n))$ with

$$f^{\delta^2}(r_1, \dots, r_n)$$
$$+2\sum_i f^{\delta}(r_1, \dots, y_i, \dots, r_n) + \sum_i f(r_1, \dots, t_i, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, y_j, \dots, r_n)$$

in (5.4.20) and then U satisfies blended component

$$\alpha \sum_i f(r_1, \ldots, t_i, \ldots, r_n) \sum_i f(r_1, \ldots, y_i, \ldots, r_n) = 0. \tag{5.4.21}$$

In particular, for $t_1 = r_1$, $y_1 = r_1$, $y_2 = \cdots = y_n = 0$ and $t_2 = \cdots = t_n = 0$, we have $\alpha f(r_1, \ldots, r_n)^2 = 0$ which implies $f(r_1, \ldots, r_n) = 0$, a contradiction.

Case-7: *d, δ and h all are outer.*

Sub-case-i: Let d, d' and δ be C-independent modulo inner derivations of U.

We can replace $h(f(r_1, \ldots, r_n))$ with $f^h(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, t_i, \ldots, r_n)$ in (5.4.1) and then U satisfies blended component

$$\begin{aligned} 0 = {} & f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, t_i, \ldots, r_n) \\ & + \sum_i f(r_1, \ldots, t_i, \ldots, r_n) f(r_1, \ldots, r_n). \end{aligned} \tag{5.4.22}$$

This is same as (5.4.3) and thus by same argument we have our conclusions.

Sub-case-ii: Let d, δ and h be C-dependent modulo inner derivations of U i.e., $\alpha_1 d + \alpha_2 \delta + \alpha_3 h = ad_{a'}$ for some $\alpha_1, \alpha_2, \alpha_3 \in C$ and $a' \in U$. Then at least one of $\alpha_1, \alpha_2, \alpha_3$ must be nonzero. Let $\alpha_1 \neq 0$. Then we can write $d = \beta_1 \delta + \beta_2 h + ad_{a''}$ for some $\beta_1, \beta_2 \in C$ and $a'' \in U$.

Then by (5.4.1), we have

$$\begin{aligned} & \Big(F(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\Big)\Big(\beta_1 \delta(f(r)) + \beta_2 h(f(r)) + [a'', f(r)]\Big) \\ & \qquad = pf(r)^2 + h(f(r)^2) \end{aligned} \tag{5.4.23}$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

If δ and h are C-independent modulo inner derivations, then by Theorem 1.6.4, we substitute the following values, $\delta(f(r_1, \ldots, r_n))$ by

$$f^{\delta}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, t_i, \ldots, r_n),$$

$\delta^2(f(r_1,\ldots,r_n))$ by

$$f^{\delta^2}(r_1,\ldots,r_n)+2\sum_i f^{\delta}(r_1,\ldots,t_i,\ldots,r_n)$$

$$+\sum_i f(r_1,\ldots,z_i,\ldots,r_n)+\sum_{i\neq j} f(r_1,\ldots,t_i,\ldots,t_j,\ldots,r_n),$$

and $h(f(r_1,\ldots,r_n))$ by

$$f^{h}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n),$$

in (5.4.23) and then U satisfies blended components

$$\left(\sum_i f(r_1,\ldots,z_i,\ldots,r_n)\right)\left(\beta_1\sum_i f(r_1,\ldots,t_i,\ldots,r_n)\right)=0 \tag{5.4.24}$$

and

$$\left(\sum_i f(r_1,\ldots,z_i,\ldots,r_n)\right)\left(\beta_2\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\right)=0. \tag{5.4.25}$$

Both the situations imply $\beta_1=\beta_2=0$, which implies that d is inner derivation, a contradiction.

If δ and h are C-dependent modulo inner derivations, then assume that $h(x)=\lambda\delta(x)+[q'',x]$ for all $x\in U$, where $\lambda\in C$ and $q''\in U$. Then by (5.4.23)

$$\begin{aligned}\Big(F(a)f(r)+2a\delta(f(r))+\delta^2(f(r))\Big)\Big((\beta_1+\beta_2\lambda)\delta(f(r))+[\beta_2 q''+a'',f(r)]\Big)\\ =pf(r)^2+\lambda\delta(f(r)^2)+[q'',f(r)^2]\end{aligned} \tag{5.4.26}$$

for all $r=(r_1,\ldots,r_n)\in U^n$.

Using Theorem 1.6.4, we substitute the following values in (5.4.26)

$\delta(f(r_1,\ldots,r_n))$ by

$$f^{\delta}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,t_i,\ldots,r_n),$$

and $\delta^2(f(r_1,\ldots,r_n))$ by

$$f^{\delta^2}(r_1,\ldots,r_n)+2\sum_i f^{\delta}(r_1,\ldots,t_i,\ldots,r_n)$$

$$+\sum_i f(r_1,\ldots,z_i,\ldots,r_n)+\sum_{i\neq j} f(r_1,\ldots,t_i,\ldots,t_j,\ldots,r_n).$$

Therefore, U satisfies the blended component

$$\sum_i f(r_1,\ldots,z_i,\ldots,r_n)(\beta_1+\beta_2\lambda)\sum_i f(r_1,\ldots,t_i,\ldots,r_n)=0.$$

This gives that $\beta_1+\beta_2\lambda=0$. Thus $d(x)=\beta_1\delta(x)+\beta_2 h(x)+[a'',x]=(\beta_1+\beta_2\lambda)\delta(x)+[\beta_2 q''+a'',x]=[\beta_2 q''+a'',x]$, that is, d is inner, a contradiction.

This completes the proof of the theorem.

Chapter 6

Generalized Derivations of Order 2 with Derivations Acting on Multilinear Polynomials with Centralizing Conditions

6.1 Introduction

In [56], it is proved that if F_1 and F_2 are generalized derivations of a prime ring R having $char(R) \neq 2$, such that $F_1(x)F_2(x) = 0$ for all $x \in R$, then there exist elements $p, q \in U$ such that $F_1(x) = xp$ and $F_2(x) = qx$ for all $x \in R$ and $pq = 0$. Moreover the above identity is studied by Carini et al [12] replacing x with multilinear polynomial and then obtained the structures of F_1 and F_2.

Furthermore, Eroğlu and Argaç [43] determined all possible structures of F by considering $F^2(u)u \in C$ for all $u \in f(R)$, where F is a generalized derivation of R and $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C. More recently, Yadav [90] described all possible forms of the maps when $F^2(u)d(u) = 0$ for all $u \in f(R)$, where F is generalized derivation of R and d is a nonzero derivation of R. He proved the following:

Let R be a noncommutative prime ring of $char(R) \neq 2$, U be its Utumi quotient ring,

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C be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C. Suppose that d is a nonzero derivation of R and G is a generalized derivation on R. If

$$G^2(f(r))d(f(r)) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

1. there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$ with $a^2 = 0$;
2. there exists $a \in U$ such that $G(x) = xa$ for all $x \in R$ with $a^2 = 0$.

In this Chapter we extend Yadav's result [90] in central case. More precisely, we study the following:

Theorem 6.1.1. *Let R be a noncommutative prime ring of $char(R) \neq 2$, U be its Utumi quotient ring, C be its extended centroid and $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C. Suppose that d is a nonzero derivation of R and G is a generalized derivation on R. If*

$$G^2(f(r))d(f(r)) \in C$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

1. *there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$ with $a^2 = 0$;*
2. *there exists $a \in U$ such that $G(x) = xa$ for all $x \in R$ with $a^2 = 0$.*

6.2 When Derivation and Generalized Derivations are Inner

We dedicate this section to prove the Theorem 6.1.1 in case both the generalized derivation G and the derivation d are inner, that is, there exist $a, b, c \in U$ such that $G(x) = ax + xb$

and $d(x) = [c, x]$ for all $x \in R$. Then $G^2(f(r))d(f(r)) \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$ implies

$$a^2 f(r)cf(r) + 2af(r)bcf(r) + f(r)b^2cf(r) - a^2 f(r)^2 c - 2af(r)bf(r)c - f(r)b^2 f(r)c \in C.$$

This gives

$$\begin{aligned} & a'f(r)cf(r)^2 + 2af(r)pf(r)^2 + f(r)p'f(r)^2 \\ & +f(r)a'f(r)^2c + 2f(r)af(r)bf(r)c + f(r)^2b'f(r)c \\ & -a'f(r)^2cf(r) - 2af(r)bf(r)cf(r) - f(r)b'f(r)cf(r) \\ & -f(r)a'f(r)cf(r) - 2f(r)af(r)pf(r) - f(r)^2p'f(r) = 0 \end{aligned} \quad (6.2.1)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, where $a' = a^2$, $b' = b^2$, $p = bc$ and $p' = b^2c$.

Proposition 6.2.1. *Let C be a field and $R = M_m(C)$ be the ring of all $m \times m$ matrices over C, $m \geq 2$. Suppose that char $(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C. If a, b and $c \in R$ such that (6.2.1) holds for all $r = (r_1, \ldots, r_n) \in R^n$, then either a or b or c is scalar matrix.*

Proof. By our assumption, (6.2.1) is a generalized polynomial identity of R. Suppose that all of a, b and c are not scalar matrices.

Case-I: Suppose that C is infinite field.

As we assumed $a \notin C.I_m$ and $b \notin C.I_m$ and $c \notin C.I_m$, by Theorem 1.6.3 there exists an invertible matrix P in $M_m(C)$ such that PaP^{-1}, PbP^{-1} and PcP^{-1} have all non-zero entries. Clearly R satisfies

$$\begin{aligned}
&Pa'P^{-1}f(r)PcP^{-1}f(r)^2+2PaP^{-1}f(r)PpP^{-1}f(r)^2\\
&+f(r)Pp'P^{-1}f(r)^2+f(r)Pa'P^{-1}f(r)^2PcP^{-1}\\
&+2f(r)PaP^{-1}f(r)PbP^{-1}f(r)PcP^{-1}+f(r)^2Pb'P^{-1}f(r)PcP^{-1}\\
&-Pa'P^{-1}f(r)^2PcP^{-1}f(r)-2PaP^{-1}f(r)PbP^{-1}f(r)PcP^{-1}f(r)\\
&-f(r)Pb'P^{-1}f(r)PcP^{-1}f(r)-f(r)Pa'P^{-1}f(r)PcP^{-1}f(r)\\
&-2f(r)PaP^{-1}f(r)PpP^{-1}f(r)-f(r)^2Pp'P^{-1}f(r)=0 \qquad (6.2.2)
\end{aligned}$$

for all $r=(r_1,\ldots,r_n)\in R^n$. By hypothesis $f(x_1,\ldots,x_n)$ is non central valued. Hence by Fact 4.2.2, there exist matrices $r_1,\ldots,r_n\in M_m(C)$ and $0\neq\gamma\in C$ such that $f(r_1,\ldots,r_n)=\gamma e_{ij}$, with $i\neq j$. We replace this value of $f(r_1,\ldots,r_n)$ in (6.2.2), we get

$$\begin{aligned}
&2e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}-2PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij}\\
&-e_{ij}Pb'P^{-1}e_{ij}PcP^{-1}e_{ij}-e_{ij}Pa'P^{-1}e_{ij}PcP^{-1}e_{ij}\\
&-2e_{ij}PaP^{-1}e_{ij}PpP^{-1}e_{ij}=0. \qquad (6.2.3)
\end{aligned}$$

Now multiplying by e_{ij} in (6.2.3) from right side, we get
$2e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij}=0$, this implies $e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij}=0$, as $char(R)\neq 2$. This is a contradiction as PaP^{-1}, PbP^{-1} and PcP^{-1} have all non-zero entries.

Case-II: When C is finite field.

Let K be an infinite field which is an extension of the field C. Let $\overline{R}=M_m(K)\cong R\otimes_C K$.

Since multilinear polynomial $f(x_1, \dots, x_n)$ is non-central-valued on R, so it is also non-central-valued on $\overline{R}$. Consider the generalized polynomial

$$\phi(r_1, \dots, r_n) =$$
$$a' f(r) c f(r)^2 + 2a f(r) p f(r)^2 + f(r) p' f(r)^2$$
$$+ f(r) a' f(r)^2 c + 2 f(r) a f(r) b f(r) c + f(r)^2 b' f(r) c$$
$$- a' f(r)^2 c f(r) - 2a f(r) b f(r) c f(r) - f(r) b' f(r) c f(r)$$
$$- f(r) a' f(r) c f(r) - 2 f(r) a f(r) p f(r) - f(r)^2 p' f(r)$$

which is a generalized polynomial identity for R.

Moreover, it is a multi-homogeneous of multi-degree $(2, \dots, 2)$ in the indeterminates $r_1, \dots, r_n$.

Hence the complete linearization of $\phi(r_1, \dots, r_n)$ is a multilinear generalized polynomial $\Theta(r_1, \dots, r_n, s_1, \dots, s_n)$ in $2n$ indeterminates, moreover

$$\Theta(r_1, \dots, r_n, s_1, \dots, s_n) = 2^n \phi(r_1, \dots, r_n).$$

Clearly the multilinear polynomial $\Theta(r_1, \dots, r_n, s_1, \dots, s_n)$ is a generalized polynomial identity for R and $\overline{R}$ too (see Fact 4.2.1). Since $char(C) \neq 2$ we obtain $\phi(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in \overline{R}$ and then conclusion follows from above when C was infinite.

□

Proposition 6.2.2. *Let R be a prime ring of char $(R) \neq 2$, C the extended centroid of R and $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C. If R satisfies (6.2.1), then either a or b or c is scalar matrix.*

Proof. Since R and U satisfy the same generalized polynomial identities (see Theorem

1.5.2), U satisfies

$$\begin{aligned} &a'f(r)cf(r)^2 + 2af(r)pf(r)^2 + f(r)p'f(r)^2 \\ &+f(r)a'f(r)^2c + 2f(r)af(r)bf(r)c + f(r)^2b'f(r)c \\ &-a'f(r)^2cf(r) - 2af(r)bf(r)cf(r) - f(r)b'f(r)cf(r) \\ &-f(r)a'f(r)cf(r) - 2f(r)af(r)pf(r) - f(r)^2p'f(r) \end{aligned} \tag{6.2.4}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. Suppose that this is a trivial GPI for U. So,

$$\begin{aligned} &a'f(r)cf(r)^2 + 2af(r)pf(r)^2 + f(r)p'f(r)^2 \\ &+f(r)a'f(r)^2c + 2f(r)af(r)bf(r)c + f(r)^2b'f(r)c \\ &-a'f(r)^2cf(r) - 2af(r)bf(r)cf(r) - f(r)b'f(r)cf(r) \\ &-f(r)a'f(r)cf(r) - 2f(r)af(r)pf(r) - f(r)^2p'f(r) \end{aligned} \tag{6.2.5}$$

is zero element in $T = U *_C C\{r_1, \ldots, r_n\}$, the free product of U and $C\{r_1, \ldots, r_n\}$, the free C-algebra in noncommuting indeterminates $r_1, \ldots, r_n$. This implies $\{1, c\}$ is linearly $C-$dependent, that is $c \in C$, as desired. Let us assume $c \notin C$, then by (6.2.5)

$$\{f(r)a'f(r) + 2f(r)af(r)b + f(r)^2b'\}f(r)c = 0 \in T. \tag{6.2.6}$$

This again implies that $\{1, b, b'\}$ is linearly $C-$dependent. There exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 + \alpha_2 b + \alpha_3 b' = 0$. If $\alpha_3 = 0$, then $\alpha_2 \neq 0$ and hence $b \in C$, as desired. Thus we assume $\alpha_3 \neq 0$ and $b \notin C$. Then by (6.2.6)

$$\{f(r)a'f(r) + 2f(r)af(r)b + \alpha f(r)^2b + \beta f(r)^2\}f(r)c = 0 \in T. \tag{6.2.7}$$

Assume $a \notin C$, then $2f(r)af(r)bf(r)c$ appears nontrivially in (6.2.7), which is a contradiction. So, either a or b or c is central, as desired.

Next suppose that (6.2.4) is a non-trivial GPI for Q. By Fact 1.6.8, R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Proposition 6.2.1, we get that either a or b or c is in C. If V is infinite dimensional over C, then for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since a_2, a_3, a_5 are not in C, there exist $h_1, h_2, h_3 \in soc(R)$ such that $[a, h_1] \neq 0$, $[b, h_2] \neq 0$, $[c, h_3] \neq 0$. By Litoff's Theorem (see Theorem 1.6.7), there exists idempotent $e \in soc(R)$ such that $ah_1, h_1a, bh_2, h_2b, ch_3, h_3c, h_1, h_2, h_3 \in eRe$. Since R satisfies generalized identity

$$
\begin{aligned}
e\Big\{ & a'f(er_1e, \ldots, er_ne)cf(er_1e, \ldots, er_ne)^2 + 2af(er_1e, \ldots, er_ne)pf(er_1e, \ldots, er_ne)^2 \\
& + f(er_1e, \ldots, er_ne)p'f(er_1e, \ldots, er_ne)^2 + f(er_1e, \ldots, er_ne)a'f(er_1e, \ldots, er_ne)^2c \\
& + 2f(er_1e, \ldots, er_ne)af(er_1e, \ldots, er_ne)bf(er_1e, \ldots, er_ne)c \\
& + f(er_1e, \ldots, er_ne)^2b'f(er_1e, \ldots, er_ne)c - a'f(er_1e, \ldots, er_ne)^2cf(er_1e, \ldots, er_ne) \\
& - 2af(er_1e, \ldots, er_ne)bf(er_1e, \ldots, er_ne)cf(er_1e, \ldots, er_ne) \\
& - f(er_1e, \ldots, er_ne)b'f(er_1e, \ldots, er_ne)cf(er_1e, \ldots, er_ne) \\
& - f(er_1e, \ldots, er_ne)a'f(er_1e, \ldots, er_ne)cf(er_1e, \ldots, er_ne) \\
& - 2f(er_1e, \ldots, er_ne)af(er_1e, \ldots, er_ne)pf(er_1e, \ldots, er_ne) \\
& - f(er_1e, \ldots, er_ne)^2p'f(er_1e, \ldots, er_ne)\Big\}e = 0,
\end{aligned}
$$

then the subring eRe satisfies

$$ea'ef(r_1,\dots,r_n)ecef(r_1,\dots,r_n)^2 + 2eaef(r_1,\dots,r_n)epef(r_1,\dots,r_n)^2$$
$$+f(r_1,\dots,r_n)ep'ef(r_1,\dots,r_n)^2 + f(r_1,\dots,r_n)ea'ef(r_1,\dots,r_n)^2ece$$
$$+2f(r_1,\dots,r_n)eaef(r_1,\dots,r_n)ebef(r_1,\dots,r_n)ece$$
$$+f(r_1,\dots,r_n)^2eb'ef(r_1,\dots,r_n)ece - ea'ef(r_1,\dots,r_n)^2ecef(r_1,\dots,r_n)$$
$$-2eaef(r_1,\dots,r_n)ebef(r_1,\dots,r_n)ecef(r_1,\dots,r_n)$$
$$-f(r_1,\dots,r_n)eb'ef(r_1,\dots,r_n)ecef(r_1,\dots,r_n)$$
$$-f(r_1,\dots,r_n)ea'ef(r_1,\dots,r_n)ecef(r_1,\dots,r_n)$$
$$-2f(r_1,\dots,r_n)eaef(r_1,\dots,r_n)epef(r_1,\dots,r_n)$$
$$-f(r_1,\dots,r_n)^2ep'ef(r_1,\dots,r_n) = 0.$$

Then by the above finite dimensional case, either eae or ebe or ece is central element of eRe. Thus either $ah_1 = (eae)h_1 = h_1(eae) = h_1a$ or $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$ or $ch_3 = (ece)h_3 = h_3(ece) = h_3c$, in any case we get a contradiction.

Hence, we say that either a or b or c is in C. □

By the same way as above we can prove the following propositions.

Proposition 6.2.3. *Let R be a prime ring of char $(R) \neq 2$, C the extended centroid of R and $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C. If c and $k \in R$ such that*

$$f(r)kcf(r)^2 - f(r)kf(r)cf(r) - f(r)^2kcf(r) + f(r)^2kf(r)c = 0$$

for all $r = (r_1,\dots,r_n) \in R^n$, then either $k \in C$ or $c \in C$.

Proposition 6.2.4. *Let R be a prime ring of char $(R) \neq 2$, C the extended centroid of R and $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C. If c and $k \in R$ such that*

$$kf(r)cf(r)^2 - kf(r)^2cf(r) - f(r)kf(r)cf(r) + f(r)kf(r)^2c = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$*, then either* $k \in C$ *or* $c \in C$*.*

Lemma 6.2.5. *Let* R *be a noncommutative prime ring of* $char(R) \neq 2$*,* U *be its Utumi quotient ring,* C *be its extended centroid and* $f(x_1, \ldots, x_n)$ *be a noncentral multilinear polynomial over* C*. Suppose for some* $a, b, c \in U$*,* $G(x) = ax + xb$*, and* $d(x) = [c, x]$ *for all* $x \in R$ *with* $c \notin C$*. If*

$$G^2(f(r))d(f(r)) \in C$$

for all $r = (r_1, \ldots, r_n) \in R^n$*, then one of the following holds:*

1. $G(x) = (a + b)x$ *for all* $x \in R$ *with* $(a + b)^2 = 0$*;*

2. $G(x) = x(a + b)$ *for all* $x \in R$ *with* $(a + b)^2 = 0$*.*

Proof. By the hypothesis, we have

$$\big(a^2f(r) + 2af(r)b + f(r)b^2\big)\big(cf(r) - f(r)c\big) \in C \tag{6.2.8}$$

that is

$$\big[\big(a^2f(r) + 2af(r)b + f(r)b^2\big)\big(cf(r) - f(r)c\big), f(r)\big] = 0 \tag{6.2.9}$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Proposition 6.2.2, either $a \in C$ or $b \in C$ or $c \in C$. Since $c \notin C$, so either $a \in C$ or $b \in C$.

If $a \in C$, it follows by (6.2.8) that

$$f(r)(a + b)^2\big(cf(r) - f(r)c\big) \in C.$$

Commuting both sides with $f(r)$, we get

$$f(r)(a+b)^2cf(r)^2 - f(r)(a+b)^2f(r)cf(r) - f(r)^2(a+b)^2cf(r) + f(r)^2(a+b)^2f(r)c = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Proposition 6.2.3, $(a + b)^2 \in C$.

If $b \in C$, it follows by (6.2.8) that

$$(a+b)^2 f(r)\big(cf(r) - f(r)c\big) \in C.$$

Commuting both sides with $f(r)$, we get

$$(a+b)^2 f(r)cf(r)^2 - (a+b)^2 f(r)^2 cf(r) - f(r)(a+b)^2 f(r)cf(r) + f(r)(a+b)^2 f(r)^2 c = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. Then by Proposition 6.2.4, $(a+b)^2 \in C$. Thus in both the above cases we have $(a+b)^2 \in C$ and hence we can write $G(x) = (a+b)x$ or $G(x) = x(a+b)$ for all $x \in R$ with $(a+b)^2 \in C$.

Thus our hypothesis $G^2(f(r))d(f(r)) \in C$ gives $f(r)\big[(a+b)^2 c, f(r)\big] \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$ or $\big[(a+b)^2 c, f(r)\big] f(r) \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$. Then by [73] we have $(a+b)^2 c \in C$. This implies $(a+b)^2 = 0$ as $c \notin C$.

Thus we arrive either $G(x) = (a+b)x$ or $x(a+b)$, with $(a+b)^2 = 0$. These are our required conclusions. □

6.3 Proof of Theorem 6.1.1

In light of the notion in Theorem 1.6.2, generalized derivation G has its form $G(x) = ax + \delta(x)$ for some $a \in U$ and δ is a derivation on U.

By hypothesis, we have

$$\big(G(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\big)d(f(r)) \in C$$

for all $r = (r_1, \ldots, r_n) \in R^n$. By Theorem 1.5.4, we have

$$\big(G(a)f(r) + 2a\delta(f(r)) + \delta^2(f(r))\big)d(f(r)) \in C \tag{6.3.1}$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

If d and δ both are inner derivations, then by Proposition 6.2.2, we have our conclusions of Theorem 6.1.1. Thus, to prove our Theorem 6.1.1, we need to consider the case when not both of d and δ are inner. Indeed we have to consider the two following cases.

- d and δ are linearly C-independent modulo inner derivations of U.
- d and δ are linearly C-dependent modulo inner derivations of U.

Case-1: *When d and δ are linearly C-independent modulo inner derivations of U.*

By (6.3.1) U satisfies

$$\Big\{F(a)f(r_1,\ldots,r_n)+2a\big\{f^{\delta}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,\delta(r_i),\ldots,r_n)\big\}$$
$$+f^{\delta^2}(r_1,\ldots,r_n)+2\sum_i f^{\delta}(r_1,\ldots,\delta(r_i),\ldots,r_n)+\sum_i f(r_1,\ldots,\delta^2(r_i),\ldots,r_n)$$
$$+\sum_{i\neq j} f(r_1,\ldots,\delta(r_i),\ldots,\delta(r_j),\ldots,r_n)\Big\}\big\{f^{d}(r_1,\ldots,r_n)$$
$$+\sum_i f(r_1,\ldots,d(r_i),\ldots,r_n)\big\}\in C,$$

for all $r_1,\ldots,r_n\in U$. Since d and δ are not inner, by Kharchenko's theorem (Theorem 1.6.4), U satisfies

$$\Big\{F(a)f(r_1,\ldots,r_n)+2a\big\{f^{\delta}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,x_i,\ldots,r_n)\big\}$$
$$+f^{\delta^2}(r_1,\ldots,r_n)+2\sum_i f^{\delta}(r_1,\ldots,x_i,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)$$
$$+\sum_{i\neq j} f(r_1,\ldots,x_i,\ldots,x_j,\ldots,r_n)\Big\}\big\{f^{d}(r_1,\ldots,r_n)$$
$$+\sum_i f(r_1,\ldots,z_i,\ldots,r_n)\big\}\in C.$$

In particular U satisfies the blended component

$$\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\sum_i f(r_1,\ldots,z_i,\ldots,r_n)\in C. \tag{6.3.2}$$

Putting $y_i = [q', r_i]$ for each $i \in \{1, \dots, n\}$, for some $q' \notin C$ and $z_1 = r_1$, $z_2 = \dots, z_n = 0$, we get

$$[q', f(r_1, \dots, r_n)]f(r_1, \dots, r_n) \in C$$

for all $r_1, \dots, r_n \in U$. Then by [73] we get $q' \in C$, which is a contradiction.

Case-2: *When d and δ are linearly C-dependent modulo inner derivations of U.*

In this case we get $\alpha, \beta \in C$ and $q \in U$ such that $\alpha d + \beta\delta = ad_q$. It is clear from the context that $(\alpha, \beta) \neq (0, 0)$. So without loss of generality we arrive the following two sub-cases:

Sub-case-i: *When $\alpha = 0$.*

Then we get $\delta(x) = [p, x]$, where $p = \beta^{-1}q$. It is obvious that d is not inner, otherwise we get contradiction. Now from (6.3.1) we have

$$(a'^2 f(r_1, \dots, r_n) + 2a' f(r_1, \dots, r_n)b' + f(r_1, \dots, r_n)b'^2)d(f(r_1, \dots, r_n)) \in C$$

for all $r_1, \dots, r_n \in U$ and $a' = a + p, b' = -p \in U$. Now from above we can write

$$\begin{aligned}&\big(a'^2 f(r_1, \dots, r_n) + 2a' f(r_1, \dots, r_n)b' + f(r_1, \dots, r_n)b'^2\big).\\&\big(f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, d(r_i), \dots, r_n)\big) \in C.\end{aligned} \tag{6.3.3}$$

Since d is not inner, by Kharchenko's theorem (Theorem 1.6.4)

$$\begin{aligned}&\big(a'^2 f(r_1, \dots, r_n) + 2a' f(r_1, \dots, r_n)b' + f(r_1, \dots, r_n)b'^2\big).\\&\big(f^d(r_1, \dots, r_n) + \sum_i f(r_1, \dots, y_i, \dots, r_n)\big) \in C.\end{aligned}$$

In particular U satisfies the blended component

$$\begin{aligned}&\big(a'^2 f(r_1, \dots, r_n) + 2a' f(r_1, \dots, r_n)b' + f(r_1, \dots, r_n)b'^2\big).\\&\sum_i f(r_1, \dots, y_i, \dots, r_n) \in C,\end{aligned}$$

that is

$$\Big[\big(a'^2 f(r_1,\ldots,r_n) + 2a' f(r_1,\ldots,r_n)b' + f(r_1,\ldots,r_n)b'^2\big).$$
$$\sum_i f(r_1,\ldots,y_i,\ldots,r_n), f(r_1,\ldots,r_n)\Big] = 0. \tag{6.3.4}$$

Replacing y_i by $[q, r_i]$, for some $q \notin C$ in (6.3.4) we have

$$\Big[\big(a'^2 f(r_1,\ldots,r_n) + 2a' f(r_1,\ldots,r_n)b' + f(r_1,\ldots,r_n)b'^2\big)$$
$$\big[q, f(r_1,\ldots,r_n)\big], f(r_1,\ldots,r_n)\Big] = 0, \tag{6.3.5}$$

which is similar as (6.2.9) of Lemma 6.2.5, so from there we get our conclusions (1) and (2) of Theorem 6.1.1.

<u>Sub-case-ii:</u> *When* $\alpha \neq 0$.

Then we have $d = \mu\delta + ad_c$, for some $\mu \in C$ and $c \in U$. Here δ never be an inner derivation, otherwise both d and δ will be inner, a contradiction. Then from (6.3.1) we have

$$\big(G(a)f(r_1,\ldots,r_n) + 2a\delta(f(r_1,\ldots,r_n)) + \delta^2(f(r_1,\ldots,r_n))\big)$$
$$\big(\mu\delta\big(f(r_1,\ldots,r_n)\big) + \big[c, f(r_1,\ldots,r_n)\big]\big) \in C \tag{6.3.6}$$

for all $r = (r_1,\ldots,r_n) \in U^n$. This is a differential identity containing the terms of the type δ and δ^2. As, δ and δ^2 are outer, by Kharchenko's theorem (Theorem 1.6.4) $\delta(r_i)$ and $\delta^2(r_i)$ can be replaced by x_i and y_i respectively in (6.3.6). And hence U satisfies the blended component

$$\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\mu \sum_i f(r_1,\ldots,x_i,\ldots,r_n) \in C. \tag{6.3.7}$$

Replacing y_i by $[q, r_i]$, where $q \notin C$ and $x_1 = r_1$, $x_2 = \ldots = x_n = 0$ in (6.3.7) we get

$$\mu\big[q, f(r_1,\ldots,r_n)\big] f(r_1,\ldots,r_n) \in C,$$

that is,

$$\big[\mu q, f(r_1,\ldots,r_n)\big] f(r_1,\ldots,r_n) \in C.$$

So by [73] we get $\mu q \in C$, this says $\mu = 0$. Then from (6.3.6) we get

$$\Big(G(a)f(r_1,\ldots,r_n) + 2a\delta(f(r_1,\ldots,r_n)) + \delta^2(f(r_1,\ldots,r_n))\Big)\big[c, f(r_1,\ldots,r_n)\big] \in C \tag{6.3.8}$$

for all $r_1,\ldots,r_n \in U$. Again from above, putting the expressions of $\delta(f(r_1,\ldots,r_n))$ and $\delta^2(f(r_1,\ldots,r_n))$ and then using Kharchenko's theorem (Theorem 1.6.4) we will find a blended component satisfied by U as follows:

$$\sum_i f(r_1,\ldots,y_i,\ldots,r_n)[c, f(r_1,\ldots,r_n)] \in C. \tag{6.3.9}$$

In particular $y_1 = r_1$ and $y_2 = \cdots y_n = 0$ we get

$$f(r_1,\ldots,r_n)[c, f(r_1,\ldots,r_n)] \in C$$

for all $r_1,\ldots,r_n \in R$. Then from [73] we get $c \in C$. Finally we get $\mu = 0$ and $c \in C$, which implies $d = 0$, a contradiction.

This completes the proof.

Chapter 7

X-Generalized Skew Derivations and Commutators with Central Values in Prime Rings

7.1 Introduction

Let F, G be two generalized derivations of R. Recently, De Filippis and Rania [50] studied the commutator $[A, B] = 0$ in prime rings, where $A = \{[F(u), u] | u \in f(R)\}$, $B = \{[G(v), v] | v \in f(R)\}$ and $f(R) = \{f(r_1, \ldots, r_n) | r_1, \ldots, r_n \in R\}$. In the paper authors described the possible forms of the additive maps F and G.

In the present Chapter, our motivation is to study the above situation when the additive maps F and G are X-generalized skew derivations of R. Recently, the concept of the map X-generalized skew derivations was introduced by De Filippis and Wei in [54]. The concept of X-generalized skew derivations generalizes the concept of generalized skew derivations as well as b-generalized derivations in R.

We recall now the following definitions. Let R be an associative ring and α be an

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automorphism of R. An additive mapping $d : R \to R$ is called a skew derivation of R if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

holds for all $x, y \in R$. Here α is called the associated automorphism of d. An additive mapping $G : R \to R$ is said to be a generalized skew derivation of R, if there exists a skew derivation d of R with associated automorphism α such that

$$G(xy) = G(x)y + \alpha(x)d(y)$$

holds for all $x, y \in R$. Here d is said to be an associated skew derivation of G and α is called an associated automorphism of G. Obviously, every skew derivations are generalized skew derivations, but the converse need not be true. The mapping $x \mapsto ax - \alpha(x)b$ is an example of generalized skew derivation which is called inner generalized skew derivation of R. Chang first introduced this notion and studied the situation involving generalized skew derivation. In fact, in the paper Chang studied the situation $h(x) = af(x) + g(x)b$ for all $x \in R$, where $a, b \in R$ and f, g and h are generalized (α, β)-derivations of a prime ring R. Many researchers have investigated generalized skew derivations from various points of view. Eremita et al. [41] determined the structure of generalized skew derivations implemented by elementary operators. In [77] Liu et al. described the structure of generalized skew derivations with nilpotent values on rings and Banach Algebras and analogous results are also obtained in C^*-algebras and standard operator algebras on Banach spaces (see [75]). Qi and Hou [81] characterized generalized skew derivations on nest algebras.

On the other hand recently, Kosan and Lee [66] introduced the notion of b-generalized derivations. Let $d : R \to Q_r$ be an additive map and $b \in Q_r$. Then an additive map $F : R \to Q_r$ is said to be a b-generalized derivation with associated map d if

$F(xy) = F(x)y + bxd(y)$ for all $x, y \in R$. Obviously, a generalized derivation is a 1-generalized derivation of R. Thus we write that F is associated with the pair of maps (b, d). For some $a, b, c \in Q_r(R)$, the map $x \mapsto ax + bxc$ is a b-generalized derivation with the pair of maps (b, d), where $d(x) = [x, c]$ for all $x \in R$; which is called inner b-generalized derivation of R. It is proved in [66] that if R is a prime ring and $b \neq 0$, then the associated map d must be a derivation of R. The b-generalized derivations were introduced and studied recently in few papers (viz. [28], [72], [76]).

Let F be an inner generalized skew derivation of R with associated inner automorphism $\alpha(x) = bxb^{-1}$ for all $x \in R$ and associated inner skew derivation $d(x) = ax - \alpha(x)a$ for all $x \in R$. Then $F(xy) = F(x)y + \alpha(x)d(y) = F(x)y + bxb^{-1}(ay - byb^{-1}a) = F(x)y + bx[b^{-1}a, y]$ for all $x, y \in R$, which is a b-generalized derivation of R with associated derivation $d(x) = [b^{-1}a, x]$ for all $x \in R$. It is very easy to prove that any generalized skew-derivation of R with associated skew-derivation d, where $\alpha(x) = bxb^{-1}$ for all $x \in R$ an inner automorphism, is a b-generalized derivation of R with the associated map $b^{-1}d$. Thus it is natural to consider the common generalization of these maps. In view of this idea De Filippis [46] introduced the new map as follows:

Definition. Let R be an associative ring, $b \in Q_r$, $d : R \to R$ a linear mapping and α be an automorphism of R. A linear mapping $F : R \to R$ is called an $X-$generalized skew derivation of R, with associated term (b, α, d) if $F(xy) = F(x)y + b\alpha(x)d(y)$ for all $x, y \in R$.

It is very easy to check that $X-$generalized skew derivation generalizes the concept of generalized skew derivation as well as b-generalized derivation. The map $x \mapsto ax + b\alpha(x)c$ is an example of X-generalized skew derivation of R with associated map (b, α, d), where

$a, b, c \in R$ are fixed elements and $d(x) = \alpha(x)c - cx$ for all $x \in R$. Such X-generalized skew derivations of R are called as inner X-generalized skew derivations of R. There are few papers which recently introduced and studied the X-generalized skew derivations (viz. [53], [54], [55]). In the present paper our motivation is to study X-generalized skew derivation in prime rings.

De Filippis and Di Vincenzo [52] studied a situation for generalized derivation as follows:

Theorem A. *Let R be a prime ring of char $(R) \neq 2$ and let $f(r_1, \ldots, r_n)$ be a multilinear polynomial over C, not central valued on R. If d is a nonzero derivation of R and F is a nonzero generalized derivation of R such that*

$$d([F(f(r)), f(r)]) = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $a \in U$, Utumi ring of quotients, such that $F(x) = ax + xa + \lambda x$ for all $x \in R$, and $f(r_1, \ldots, r_n)^2$ is central valued on R.*

This result further studied by Dhara [28] assuming F as b-generalized derivation of R and obtained the same structure of the maps. Moreover, Dhara [30] studied the case when F is generalized skew derivation of R and d is an inner derivation of R.

Recently, Tiwari and Prajapati [84] studied the result of [28] with central values. Tiwari and Prajapati [84] proved the following:

Theorem B. *Let R be a prime ring of char $(R) \neq 2$ and F a b-generalized derivation on R. Let $f(r_1, \ldots, r_n)$ be a multilinear polynomial over C which is not central valued on R.*

If d is a nonzero derivation on R such that

$$d([F(f(r)), f(r)]) \in C$$

for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $a \in U$, Utumi ring of quotients, and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$ and $f(r_1, \dots, r_n)^2$ is central valued on R.*

The main goal of the present chapter is to investigate the situation of [30] with central values assuming F as an X-generalized skew derivation. Then we determine the form of the map. More precisely, we prove the following theorem.

Theorem 7.1.1. *Let R be a noncommutative prime ring of char $(R) \neq 2$, Q_r be its right Martindale quotient ring and C be its extended centroid. Let $f(r_1, \dots, r_n)$ be a noncentral multilinear polynomial over C. If F is an X-generalized skew derivation of R and $a \in R - C$ such that*

$$[a, [F(f(r)), f(r)]] \in C$$

for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $b \in Q_r$ and $\lambda \in C$ such that $F(x) = bx + xb + \lambda x$ for all $x \in R$ and $f(r_1, \dots, r_n)^2$ is central valued on R.*

As an application of Theorem 7.1.1, the following Corollaries are straightforward.

Corollary 7.1.2. *Let R be a noncommutative prime ring of char $(R) \neq 2$. Let $f(r_1, \ldots, r_n)$ be a noncentral multilinear polynomial over C. If F is an X-generalized skew derivation of R such that*

$$[F(f(r)), f(r)] \in C$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $b \in Q_r$ and $\lambda \in C$ such that $F(x) = bx + xb + \lambda x$ for all $x \in R$ and $f(r_1, \ldots, r_n)^2$ is central valued on R.*

The next Corollary generalizes the well known result of De Filippis and Rania [50].

Corollary 7.1.3. *Let R be a noncommutative prime ring of char $(R) \neq 2$ and Q_r be its right Martindale quotient ring. Let $f(r_1, \ldots, r_n)$ be a noncentral multilinear polynomial over C. If F and G are two X-generalized skew-derivations of R such that*

$$[[F(u), u], [G(v), v]] = 0$$

for all $u, v \in f(R)$, then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exists $\mu \in C$ such that $G(x) = \mu x$ for all $x \in R$;*
3. *$f(r_1, \ldots, r_n)^2$ is central valued on R and either there exist $b \in Q_r$ and $\lambda \in C$ such that $F(x) = bx + xb + \lambda x$ for all $x \in R$ or there exist $c \in Q_r$ and $\mu \in C$ such that $G(x) = cx + xc + \mu x$ for all $x \in R$.*

Proof. If $[F(u), u] \in C$ for all $u \in f(R)$, then by Corollary 7.1.2, we have our conclusions (1) and (3).

Similarly, if $[G(v), v] \in C$ for all $v \in f(R)$, we have our conclusions (2) and (3).

If $[F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]$ is not central valued on R, then we choose $r_1, \ldots, r_n \in R$ such that $[F(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = a \notin C$. By hypothesis

$$[a, [G(f(r)), f(r)]] = 0$$

for all $r = (r_1, \ldots, r_n) \in R^n$. By Theorem 7.1.1, we have our conclusions (2) and (3). $\square$

7.2 Preliminaries

We recall some known facts, which will be useful to prove our Theorem 7.1.1.

Fact 7.2.1. *For prime ring R, following statements hold:*

1. *Every generalized derivation of R can be uniquely extended to Q_r [70, Theorem 3].*
2. *Any automorphism of R can be uniquely extended to Q_r [18, Fact 2].*
3. *Every generalized skew derivation of R can be uniquely extended to Q_r [15, Lemma 2].*

Fact 7.2.2. *[55, Lemma 3.2] Let $F : R \to R$ be an $X-$generalized skew derivation of R, with associated term (b, α, d). If R is prime ring, then d is a skew derivation of R with associated automorphism α.*

Fact 7.2.3. *Let R be a prime ring and $F : R \to R$ be an $X-$generalized skew derivation of R, with associated term (b, α, d). Then F can be uniquely extended to Q_r. Moreover, the form of F will be $F(x) = ax + bd(x)$, where $a \in Q_r$.*

Proof. As above by Fact 7.2.2, d must be a skew derivation. Define $T : R \to R$ by $T(x) = F(x) - bd(x)$. We have

$$T(xy) = F(x)y + b\alpha(x)d(y) - bd(x)y - b\alpha(x)d(y) = (F(x) - bd(x))y = T(x)y$$

for all $x, y \in R$. Thus T is a left multiplier map of R. By [62, Lemma 2], T can be extended to Q_r with its form $T(x) = ax$ for all $x \in R$, and for some $a \in Q_r$. Since $F(x) = T(x) + bd(x)$ and both T and d can be uniquely extended to Q_r, we conclude that F can also be uniquely extended to Q_r and its form will be $F(x) = ax + bd(x)$ with $a \in Q_r$.

Fact 7.2.4. *Let R be a prime ring and $F : R \to R$ be an $X-$generalized skew derivation of R. Suppose that F is associated to inner skew derivation $d(x) = ux - \alpha(x)v$ and inner automorphism $\alpha(x) = pxp^{-1}$ for all $x \in R$. Then as above $F(x) = ax + bd(x) = ax + b(ux - \alpha(x)v) = ax + b(ux - pxp^{-1}v) = (a + bu)x - bpxp^{-1}v = a'x + b'xc'$, where $a' = a + bu, b' = -bp, c' = p^{-1}v \in Q_r$.*

Fact 7.2.5. *Chuang and Lee [20] studied polynomial identities with a single skew derivation. They proved that if $G(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D is an outer skew derivation of R, then R also satisfies the generalized polynomial identity $G(x_i, y_i)$, where x_i and y_i are distinct indeterminates. Furthermore, they observe [20, Theorem 1] that in the case $G(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, D is an outer skew derivation of R and α is an outer automorphism of R, then R also satisfies the generalized polynomial identity $G(x_i, y_i, z_i)$, where x_i, y_i, and z_i are distinct indeterminates.*

Fact 7.2.6. *By [20], if d is a non-zero skew-derivation of R and*

$$\Phi(x_1, \dots x_n, d(x_1), \dots d(x_n))$$

is a skew-differental polynomial identity of R, then one of the following statements holds:

1. *either d is inner skew derivation of R;*
2. *or R satisfies the generalized polynomial identity $\Phi(x_1, \dots x_n, y_1, \dots y_n)$.*

7.3 Proof of Theorem 7.1.1

In this section, first we deals with the situation when F is an inner X-generalized skew derivation of R with associated inner automorphism. By Fact 7.2.4, F has its form $F(x) = bx + pxc$ for all $x \in R$, for some $b, c, p \in Q_r$. So we need the following proposition:

Proposition 7.3.1. *[85, Proposition 4.1] Let R be a noncommutative prime ring of char $(R) \neq 2$ and C be its extended centroid. Suppose that $f(r_1, \dots, r_n)$ be a noncentral multilinear polynomial over C and $F(x) = bx + pxc$ for all $x \in R$, for some $b, c, p, b', p', c' \in Q_r$. If $a \in R - C$ such that*

$$[a, [F(f(r)), f(r)]] \in C$$

for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $b \in Q_r$ and $\lambda \in C$ such that $F(x) = bx + xb + \lambda x$ for all $x \in R$ and $f(r_1, \dots, r_n)^2$ is central valued on R.*

Lemma 7.3.2. *Let R be a noncommutative prime ring with char $(R) \neq 2$, $a, b, c \in Q_r$, $p(r_1, \dots, r_n)$ be any polynomial over C, which is not central valued in R. If $[c, [a, p(r)]] \in C$ for all $r = (r_1, \dots, r_n) \in R^n$, then either $c \in C$ or $a \in C$.*

Proof. Let $S = \{p(r_1, \ldots, r_n) | r_1, \ldots, r_n \in R\}$. Let G be the additive subgroup of R generated by the set S. Then $S \neq \{0\}$, because $p(r_1, \ldots, r_n)$ is nonzero valued on R. By hypothesis, $[c, [a, x]] \in C$ for any $x \in G$. By [16], either $G \subseteq Z(R)$ or G contains a noncentral Lie ideal L of R. Since $p(r_1, \ldots, r_n) \in G$ and $p(r_1, \ldots, r_n)$ is not central valued on R, $G \nsubseteq Z(R)$. Thus G contains a noncentral Lie ideal L of R. By [7, Lemma 1], there exists a noncentral two sided ideal I of R such that $[I, R] \subseteq L$. Then by assumption, $[c, [a, [x_1, x_2]]] \in C$ for all $x_1, x_2 \in I$. By [84, Lemma 3.6], either $c \in C$ or $a \in C$. □

Lemma 7.3.3. *Let R be a noncommutative prime ring with char $(R) \neq 2$, $a(\neq 0), b(\neq 0) \in R$, $p(r_1, \ldots, r_n)$ be any polynomial over C, which is not an identity for R. If $ap(r)b \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

1. *$ab \in C$ and $p(r_1, \ldots, r_n)$ is central valued on R;*
2. *$b \in C$ with $ab = 0$.*

Proof. If $p(r_1, \ldots, r_n)$ is central valued on R, then our assumption $ap(r)b \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$ yields $ab \in C$ and hence we obtain our conclusion (1).

Hence, assume next that $p(r_1, \ldots, r_n)$ is not central valued on R. Let G be the additive subgroup of R generated by the set $S = \{p(r_1, \ldots, r_n) | r_1, \ldots, r_n \in R\}$. Then $S \neq \{0\}$. By our assumption we get $axb \in C$ for any $x \in G$. By [16], either $G \subseteq Z(R)$ or G contains a noncentral Lie ideal L of R. Since $p(r_1, \ldots, r_n)$ is not central valued on R, $G \nsubseteq Z(R)$ and hence G contains a noncentral Lie ideal L of R. By [7, Lemma 1], there exists a noncentral two sided ideal I of R such that $[I, R] \subseteq L$. In particular, $a[x_1, x_2]b \in C$ for all $x_1, x_2 \in I$ and so by [17] $a[x_1, x_2]b \in C$ for all $x_1, x_2 \in Q_r$. If $a[x_1, x_2]b = 0$ for all $x_1, x_2 \in Q_r$, then by [36, Lemma 2.8], $b \in C$ with $ab = 0$, as desired in conclusion (2).

If $0 \neq a[x_1, x_2]b \in C$, then by Theorem 1 in [14], Q_r is a PI-ring and so it is a finite dimensional central simple C-algebra. It follows from Lemma 2 in [67] that there exists a suitable field K such that $Q_r \subseteq M_t(K)$, the ring of all $t \times t$ matrices over the field K, and moreover $M_t(K)$ satisfies $a[x_1, x_2]b \in K \cdot I_t$. Since there exist $x_1, x_2 \in M_t(K)$ such that $0 \neq a[x_1, x_2]b \in K \cdot I_t$, a must be invertible. Hence $[x_1, x_2]b \in K \cdot a^{-1}$ for all $x_1, x_2 \in M_t(K)$. For $x_1 = e_{ij}, x_2 = e_{jj}$, we have $[x_1, x_2] = e_{ij}$ and so $e_{ij}b \in K \cdot a^{-1}$. Since rank of $e_{ij}b$ is ≤ 1, it can not be invertible in $M_t(K)$ for $t \geq 2$ and hence it must be zero. Therefore $e_{ij}b = 0$ implying b is diagonal matrix. Let $b = \sum_{i=1}^{t} b_{ii}e_{ii}$. Since for any automorphism ϕ of R, $0 \neq \phi(a)[x_1, x_2]\phi(b) \in C$, we can write $\phi(b)$ is also diagonal matrix. Thus for each $j \neq 1$, we have $(1 + e_{1j})b(1 - e_{1j}) = \sum_{i=1}^{t} b_{ii}e_{ii} + (b_{jj} - b_{11})e_{1j}$ is diagonal. This implies $b_{jj} = b_{11}$ for any $j \neq 1$. Therefore, b is central.

Thus we have $b \in C$ and then we have $ab[x_1, x_2] \in C$, i.e., $[ab[x_1, x_2], [x_1, x_2]] = 0$ for all $x_1, x_2, x_3 \in R$. This implies $[ab, [x_1, x_2]][x_1, x_2] = 0$ for all $x_1, x_2 \in R$. By [26], $ab \in C$ and so $a \in C$. Since $ab \neq 0$, $ap(r)b \in C$ for all $r = (r_1, \ldots, r_n) \in R^n$ yields $p(r_1, \ldots, r_n)$ is central valued on R, a contradiction. □

Lemma 7.3.4. *Let R be a noncommutative prime ring with char $(R) \neq 2$. Let $f(r_1, \ldots, r_n)$ be a multilinear polynomial over C, not central valued on R. If $F(x) = ax + b\alpha(x)c$ is an inner X-generalized skew derivation of R and $q \in R - C$ such that*

$$[q, [F(f(r)), f(r)]] \in C$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

1. *there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$;*
2. *there exist $b \in Q_r$ and $\lambda \in C$ such that $F(x) = bx + xb + \lambda x$ for all $x \in R$ and $f(r_1, \ldots, r_n)^2$ is central valued on R.*

Proof. If α is an inner automorphism, then the result follows by Proposition 7.3.1. So we assume that α is outer automorphism of R. By hypothesis, we have

$$[q,[af(r_1,\dots,r_n)+b\alpha(f(r_1,\dots,r_n))c,f(r_1,\dots,r_n)]]\in C \qquad (7.3.1)$$

for all $r_1,\dots,r_n\in Q_r$. If $c=0$, then it yields $[q,[a,f(r_1,\dots,r_n)]f(r_1,\dots,r_n)]\in C$ for all $r_1,\dots,r_n\in Q_r$. By [24, Lemma 2.4], $a\in C$, which is our conclusion (1).

So we assume that $c\neq 0$. Since the degree of any $(x_i)^\alpha$-word in (7.3.1) is equal to one, by [19, Theorem 3] Q_r satisfies generalized polynomial identity

$$[q,[af(r_1,\dots,r_n)+bf^\alpha(y_1,\dots,y_n)c,f(r_1,\dots,r_n)]]\in C. \qquad (7.3.2)$$

In particular, Q_r satisfies the blended component

$$[q,[bf^\alpha(y_1,\dots,y_n)c,f(r_1,\dots,r_n)]]\in C. \qquad (7.3.3)$$

Then by Lemma 7.3.2, $bf^\alpha(y_1,\dots,y_n)c\in C$ for all $y_1,\dots,y_n\in Q_r$. Since $f(r_1,\dots,r_n)$ is noncentral valued, $f^\alpha(r_1,\dots,r_n)$ is also noncentral valued. Thus by Lemma 7.3.3, we have $b=0$. Then (7.3.1) yields $[q,[a,f(r_1,\dots,r_n)]f(r_1,\dots,r_n)]\in C$ for all $r_1,\dots,r_n\in Q_r$. By [24, Lemma 2.4], $a\in C$, which is our conclusion (1). □

As a particular case we have the following Corollary.

Corollary 7.3.5. *Let R be a noncommutative prime ring of char $(R)\neq 2$ and C be its extended centroid. Let $f(r_1,\dots,r_n)$ be a multilinear polynomial over C, not central valued on R. If $\delta(x)=a(x-\alpha(x))$ is an inner X-generalized skew derivation of R and $q\in R-C$ such that*

$$[q,[\delta(f(r)),f(r)]]\in C$$

for all $r=(r_1,\dots,r_n)\in R^n$, then $\delta=0$, that is, $a(\alpha(x)-x)=0$ for all $x\in R$.

Proof of Theorem 7.1.1:

As we remarked in Fact 7.2.2 and Fact 7.2.3, the X-generalized skew derivation F has its form $F(x) = bx + pd(x)$ for all $x \in R$, where $b, p \in Q_r$ and d is a skew-derivation of R. Since any skew derivation of R can be uniquely extended in Q_r and by [20, Theorem 2] R and Q_r satisfy the same generalized polynomial identities with a single skew derivation, Q_r satisfies

$$[a, [bf(r_1, \ldots, r_n) + pd(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]] \in C. \tag{7.3.4}$$

If $d(x) = cx - \alpha(x)c$ is an inner skew derivation of Q_r, then $F(x) = (b + pc)x - p\alpha(x)c$ for all $x \in Q_r$. By Lemma 7.3.4, we have our conclusions.

Thus we assume that d is an outer skew derivation of Q_r. We denote

$$f(r_1, \ldots, r_n) = \sum_{\sigma \in S_n} \gamma_\sigma r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)},$$

where $\gamma_\sigma \in C$. Let $f^d(r_1, \ldots, r_n)$ be the polynomial obtained from $f(r_1, \ldots, r_n)$ by replacing each coefficients γ_σ with $d(\gamma_\sigma)$. Hence

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n)$$
$$+ \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(r_{\sigma(1)} \cdots r_{\sigma(j)}) d(r_{\sigma(j+1)}) r_{\sigma(j+2)} \cdots r_{\sigma(n)}.$$

Thus Q_r satisfies

$$[a, [bf(r_1, \ldots, r_n) + pf^d(r_1, \ldots, r_n)$$
$$+p \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(r_{\sigma(1)} \cdots r_{\sigma(j)}) d(r_{\sigma(j+1)}) r_{\sigma(j+2)} \cdots r_{\sigma(n)}, f(r_1, \ldots, r_n)]] \in C.$$

In this case by [20, Theorem 1], Q_r satisfies

$$[a, [bf(r_1, \ldots, r_n) + pf^d(r_1, \ldots, r_n)$$
$$+p \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(r_{\sigma(1)} \cdots r_{\sigma(j)}) y_{\sigma(j+1)} r_{\sigma(j+2)} \cdots r_{\sigma(n)}, f(r_1, \ldots, r_n)]] \in C.$$

In particular, Q_r satisfies the blended component

$$[a,[p\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(r_{\sigma(1)}\cdots r_{\sigma(j)})y_{\sigma(j+1)}r_{\sigma(j+2)}\cdots r_{\sigma(n)}, f(r_1,\ldots,r_n)]]\in C. \quad (7.3.5)$$

Then two cases arises:

Case-I: α is an inner automorphism of Q_r.

There exists $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$ for all $x \in Q_r$. Thus, $\alpha(\gamma_\sigma) = \gamma_\sigma$ for all coefficients involved in $f(r_1,\ldots,r_n)$. Then replacing y_i by $x_i - \alpha(x_i)$ in above relation, we have that Q_r satisfies

$$[a,[p(f(r_1,\ldots,r_n)-\alpha(f(r_1,\ldots,r_n))), f(r_1,\ldots,r_n)]]\in C,$$

that is

$$[a,[p\delta(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)]]\in C,$$

where $\delta(x) = x - \alpha(x)$ for all $x \in Q_r$ is a skew derivation of Q_r. Then by Corollary 7.3.5, $p\delta = 0$, that is $p(\alpha(x) - x) = 0$ for all $x \in Q_r$. Since $\alpha(x) = qxq^{-1}$, we have $p(qxq^{-1} - x) = 0$ for all $x \in Q_r$. Multiplying by q from right side, we get $p[q,x] = 0$ for all $x \in Q_r$. This implies either $p = 0$ or $q \in C$. If $p = 0$, then F becomes generalized derivation. If $q \in C$, then α becomes identity map and hence F becomes b-generalized derivation. In any case, conclusion follows by Theorem A.

Case-II: α is an outer automorphism of Q_r.

Since the degree of any $(x_i)^\alpha$-word in (7.3.5) is equal to one, by [19, Theorem 3] Q_r satisfies generalized polynomial identity

$$[a,[p\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}z_{\sigma(1)}\cdots z_{\sigma(j)}y_{\sigma(j+1)}r_{\sigma(j+2)}\cdots r_{\sigma(n)}, f(r_1,\ldots,r_n)]]\in C. (7.3.6)$$

In particular, Q_r satisfies blended component

$$[a,[p\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)z_{\sigma(1)}\cdots z_{\sigma(n-1)}y_{\sigma(n)}, f(r_1,\ldots,r_n)]]\in C. \quad (7.3.7)$$

Replacing z_i by $\alpha(z_i)$ and y_i by $\alpha(z_i)$ for $i = 1, 2, \ldots, n$, we get that Q_r satisfies

$$[a, [p\alpha(f(z_1, \ldots, z_n)), f(r_1, \ldots, r_n)]] \in C, \tag{7.3.8}$$

that is

$$[a, [c', f(r_1, \ldots, r_n)]] \in C, \tag{7.3.9}$$

where $c' = p\alpha(f(z_1, \ldots, z_n))$. Now as $f(r_1, \ldots, r_n)$ is noncentral valued, by Lemma 7.3.2, either $a \in C$ or $c' = p\alpha(f(z_1, \ldots, z_n)) \in C$ for all $z_1, \ldots, z_n \in Q_r$. By hypothesis $a \notin C$ and hence $c' = p\alpha(f(z_1, \ldots, z_n)) \in C$ for all $z_1, \ldots, z_n \in Q_r$ which yields $pf(z_1, \ldots, z_n) \in C$ for all $z_1, \ldots, z_n \in Q_r$. This implies $[p, f(z_1, \ldots, z_n)]f(z_1, \ldots, z_n) = 0$ for all $z_1, \ldots, z_n \in Q_r$, implying $p \in C$ (see [73]). If $p = 0$, then $F(x) = bx$ for all $x \in R$. In this case, $[a, [b, f(r_1, \ldots, r_n)]f(r_1, \ldots, r_n)] = 0$ for all $r_1, \ldots, r_n \in Q_r$, which implies by [33, Corollary 2.9] $b \in C$. This is our conclusion (1). On the other hand if $0 \neq p \in C$, then $p\alpha(f(z_1, \ldots, z_n)) \in C$ for all $z_1, \ldots, z_n \in Q_r$ implies $f(r_1, \ldots, r_n) \in C$ for all $r_1, \ldots, r_n \in Q_r$, a contradiction. □

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