ON DECOMPOSITION OF SEMINEARRINGS IN TERMS OF NEAR-RINGS

THESIS SUBMITTED TO JADAVPUR UNIVERSITY

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY (SCIENCE)

2024

BY

TUHIN MANNA

INDEX NO-35/18/Maths./25

DEPARTMENT OF MATHEMATICS

JADAVPUR UNIVERSITY

KOLKATA- 700032

WEST BENGAL, INDIA

JADAVPUR UNIVERSITY
DEPARTMENT OF MATHEMATICS
KOLKATA - 700032

CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled **"ON DECOMPOSITION OF SEMINEARRINGS IN TERMS OF NEAR-RINGS"** submitted by **Sri. Tuhin Manna**, who got his name registered on 09.02.2018 (Index No: 35/18/Maths./25) for the award of Ph. D. (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of Dr. Sujit Kumar Sardar, Professor, Department of Mathematics, Jadavpur University and co-supervision of Dr. Rajlaxmi Mukherjee, Assistant Professor, Department of Mathematics, Garhbeta College and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

19.01.24

PROFESSOR
Kolkata 700032
Dept. of Mathematics, Jadavpur University

Rajlaxmi Mukherjee 19.01.2024

Assistant Professor
Dept. of Mathematics
Garhbeta College, Garhbeta

(Signature of the Supervisor(s)
and date with official seal)

Dedicated to

my daughter

Ahana

Acknowledgement

This thesis is indeed a product of arduous work. Completion of this journey would not have been possible for me without the aid and support of countless people, who in one way or the other, contributed and extended their valuable assistance.

I am indebted to my supervisors whose guidance, support and expertise have enabled me to complete my Ph. D. within the time frame. First and foremost, I would like to express my sincere gratitude to my respected teacher and supervisor Dr. Sujit Kumar Sardar, Professor, Department of Mathematics, Jadavpur University, for his patience, motivation, enthusiasm and immense knowledge. Since my early college days, Dr. Sardar's captivating lectures had motivated me in pursuing further research in this specific field of mathematics. His insightful guidance helped me during the entire research work. I could not have imagined having a better advisor and mentor for my Ph. D. study.

I owe a deep sense of gratitude to my co-supervisor Dr. Rajlaxmi Mukherjee, Assistant Professor, Department of Mathematics, Garhbeta College. All through it has been a greatly enriching experience for me to work under her guidance. She has taken keen interest in every phase of the study, has gone through the draft patiently and has advised me for necessary modifications. Her unconditional help and encouragement also has become an aid for completing this work.

Besides my supervisors, I would like to express my gratitude to Prof. M. K. Sen, University of Calcutta. Without his guidance and constant feedback, this work would not have been accomplished.

I am profoundly grateful to Dr. Pavel Pal, Assistant Professor, Department of

Mathematics, Bankura University, who acted as an unofficial mentor throughout the entire journey. His thoughtful feedback enriched the quality of the thesis.

I am thankful to fellow research scholars for their constant co-operations in both academic and non-academic aspects. I would specially like to acknowledge Dr. Kaushik Chakraborty, Dr. Kamalika Chakraborty, Dr. Krishanu Dey, Dr. Sayan Sengupta, Dr. Monali Das, Dr. Jahangir Chowdhury and Dr. Sudip Mazumder.

I am extremely grateful to all the faculty members and staff members of Department of Mathematics, Jadavpur University for all kinds of intellectual and logistical support.

Without the ongoing support of a few special people in my life, completing this thesis would have seemed unattainable. My friend cum wife has endured my crankiness, unavailability and endless amount of panic with a smile and a kind word. Thank you for being there for me in every single moment. I also convey my love and affection to my beloved daughter. The most significant acknowledgement is reserved for my parents. It is their love and support which has encouraged me to become what I am today. Thanks to my family and friends for their love, belief and encouragement. Their constant support has provided the emotional sustenance needed to complete this challenging task.

I am thankful to my colleagues and well-wishers for their best wishes and co-operation in completing my work.

January, 2024 Tuhin Manna

Abstract

As the title suggests, the focus of the thesis is on decompositions of different classes of seminearrings comprising various characterization theorems. The study of decomposition of additively regular seminearrings has been initiated by Sardar and Mukherjee. One nice aspect of studying additively regular seminearrings is to obtain semigroup theoretic analogues. Sardar and Mukherjee obtained some decomposition theorems which are analogues of “A semigroup is completely regular if and only if it is a semilattice of completely simple semigroups” and “A semigroup is Clifford if and only if it is a semilattice of groups” in the setting of seminearrings.

In this thesis, as a continuation of the above mentioned study, (i) the seminearrings decomposable as strong bi-semilattice (distributive lattice) of near-rings have been characterized which is the analogue of “A semigroup is Clifford if and only if it is strong semilattice of groups”, (ii) the seminearrings decomposable as union of near-rings have been characterized which is the analogue of “A semigroup is completely regular if and only if it is a union of groups”, (iii) the seminearrings decomposable as union of various types of regular near-rings have been characterized, in the class of additively completely regular seminearrings.

The decomposition theorems established in (i) and (ii) have their counterparts in semirings. But the main difference between the study of decompositions in the setting of seminearrings and that in the semiring setting is that left completely regular, right completely regular concepts coincide in semirings. In this context one natural question arises - what class of seminearrings can be obtained if the main axioms leading to left, right completely regular seminearrings are made to coincide? The thesis is concluded with a study related with possible answer to this question.

Contents

Introduction

If one takes a group $(G, +)$ (not necessarily abelian), then the set $M(G)$ of all maps from G to G under point-wise addition and composition gives an algebraic structure where $M(G)$ under addition is a group (not necessarily abelian), $M(G)$ under composition is a semigroup (not necessarily commutative) and composition distributes over addition only from the right side. If one adds commutativity on $(G, +)$, composition in $M(G)$ still distributes over addition only from the right side. This algebraic structure is known as a near-ring. A *near-ring* $(N, +, \cdot)$ [101] is an algebraic structure where $(N, +)$ is a group, $(N, \cdot)$ is a semigroup and $(n_1+n_2)\cdot n_3 = n_1\cdot n_3+n_2\cdot n_3$, for all $n_1, n_2, n_3 \in N$ (*i.e.*,'$\cdot$' distributes over '$+$' from the right side which is known as the "right distributive law"). In this definition if we replace the "right distributive law" by the "left distributive law" (*i.e.*, if for all $n_1, n_2, n_3 \in N$, $n_1\cdot(n_2+n_3) = n_1\cdot n_2+n_1\cdot n_3$) then the algebraic structure we get is known as a 'left near-ring'. The theory runs completely parallel in both cases. Throughout the thesis 'near-ring' stands for 'right near-ring'. Influenced by the work of Dickson [26] related with the existence of fields with only one distributive law, many researchers such as J. R. Clay, W. M. L. Holcombe, C. J. Maxson and S. D. Scott developed the theory of near-rings in both theoretical and practical aspects (*cf.* [21, 22, 42, 83, 84, 111]).

In the natural near-ring $M(G)$, if we replace the group $(G, +)$ by a semigroup $(S, +)$ (not necessarily commutative), then the set $M(S)$ of all self maps from S to S under point-wise addition and composition gives a new algebraic structure called a 'seminearring'. A *seminearring* $(S, +, \cdot)$ is formally defined to be an algebraic structure where $(S, +)$, $(S, \cdot)$ are semigroups which are not necessarily commutative and $(a+b)\cdot c$

$= a \cdot c + b \cdot c$ for all $a, b, c \in S$ (right distributive law). One may call this structure as a 'right seminearring'. In this definition if we consider 'left distributive law' instead of 'right distributive law' then the resulting structure is known as a 'left seminearring'. Like the theory of near-rings, here also the seminearring theory runs completely parallel in both cases. Throughout this thesis, seminearring stands for 'right seminearring'. Some authors call seminearring by 'near-semiring'.

From the above discussion it is evident that a seminearring is a near-ring if and only if its additive reduct is a group. Again a seminearring is a semiring (*cf.* Section 1.3) if and only if it is left distributive. Thus seminearring generalizes the notion of near-rings as well as semirings with semigroup as one of its building blocks. As a consequence, we may develop the theory of seminearrings by taking impetus from the evolution of the theory of semirings, near-rings as well as semigroups.

The notion of seminearrings was introduced by van Hoorn et al. in 1967 (*cf.* [46]). In 70's and 80's van Hoorn [46, 47], Hoogewijs [45], Weinert [39, 120, 121, 122] studied radicals, representation theorems, seminearfields, interrelationships between seminearrings and different types of semigroups of right quotients, etc. In 90's, the theory of seminearrings was developed in many directions. In 1994, Ayaragarnchanakul and Mitchell [6] established that any finite division seminearring is uniquely determined by the Zappa-Szép product of two multiplicative subgroups. In 1995, J. Ahsan [2, 3] introduced the notion of $\mathcal{S}$-ideals by generalizing 'semiring ideals' in the setting of seminearrings and characterized seminearrings in terms of these ideals. In 1997, Ahsan and Zhongkui [4] introduced and studied 'strongly idempotent seminearring' which is an analogue of 'fully idempotent ring'. Meanwhile Blackett [11] took the study of seminearrings to another significant direction by showing that under the operations 'pseudosum' and 'pseudoproduct', the set of probability generating functions forms a seminearring with commutative addition and an additive identity and by explaining how the algebra of seminearrings of probability generating functions helps to understand the probability theory of non-negative integer-valued random variables. This work was then extended by Boykett in [13] where he studied seminearrings of all polynomials over a commutative semifield with zero under the operations of multiplication and composition. 'Distributively generated seminearring' is an important tool in the study of seminearrings. In 1997, Meldrum and Samman [85] defined the notion of distributively generated seminearrings keeping analogy with the notion of ‘distributively generated near-rings’ [101]. After that the study of distributively generated seminearrings and seminearrings of endomorphisms has been evolved through [33, 34, 85, 105, 106, 107, 108].

In early 2000, Changphas and Denecke [19, 20] gave a full characterization of Green’s relations on a sub-seminearring of the seminearring $Hyp\ (n)$ of all hypersubstitutions of type (n) and used seminearrings to study complexity of hypersubstitutions and lattices of varieties. K. V. Krishna and N. Chatterjee [62, 66, 67] studied the algebraic structure of seminearrings from different aspects. In their seminearring $(S, +, \cdot)$ they asumed $(S, +)$ to be a monoid with identity ‘0’ satisfying $0 \cdot s = 0$ for all $s \in S$. They studied categorical representations of seminearrings and extended the result of Holcombe of near-rings to seminearrings. In 2005, seminearrings of bivariate polynomials was studied by Neuerburg in [93]. In 2007, Shabir and Ahmed [117] characterized weakly regular seminearrings and studied the topology of the space of irreducible ideals of those seminearrings. In 2010, Zulfiqar [125] focused on the radicals of seminearrings and generalized several results of ring theory. Kornthorng et al. [61] introduced the notions of ‘k-ideal’, ‘full k-ideal’ and explored the lattice structure of right full k-ideals in an additively inverse seminearring. Kumar and Krishna [69, 70, 71, 72, 73] studied affine near-semirings over Brandt semigroups. They classified the elements, cardinality of an affine near-semiring over a Brandt semigroup and characterized the Green’s relations on both of its semigroup reducts. In 2012, Balakrishnan et al. [9] studied left duo seminearrings. Since 2012, Perumal et al. [95, 96, 97, 98, 115, 116] have studied left bipotent seminearrings, normal seminearrings, medial left bipotent seminearrings, prime ideals, minimal prime ideals in seminearrings, noetherian seminearrings and right duo seminearrings. In 2016, Hussain et al. [51] discussed isomorphism theorems of seminearrings. Since 2019, Manikandan et al. [80, 81, 82] have defined the notions of mate and mutual mate functions, mid units in duo seminearrings, strong (k, r)-seminearrings and characterized these seminearrings. In 2020, Khachorncharoenkul et al. [54] introduced the notion of left almost seminearrings which generalizes left almost semirings, near left almost rings and left almost rings and investigated some related properties of these seminearrings, Koppula et al. [59, 60] introduced the notions of prime strong ideals, perfect ideals, perfect homomorphisms in a seminearring and established some relations between them, Khan et al. [56, 57] introduced the notions of soft near-semirings, soft int-nearsemirings, soft subnear-semirings, soft ideals, soft int-ideals, idealistic soft near-semirings based on soft set theory and discussed related properties of nearsemirings and SI-nearsemirings. In 2022, Khan, Arif and Taouti [55] introduced the notion of group seminearrings and studied ideals and homomorphisms there. Taking impetus from development of semigroup theory, Sardar et al. have studied various kinds of regularity in seminearrings and established analogues of some structure theorems of semigroup/semiring theory in the setting of seminearrings (*cf.* [90, 109, 110]) since 2014. S. K. Sardar, R. Mukherjee and K. Chakraborty also studied various types of congruences, such as near-ring congruences, additively commutative near-ring congruences in different types of seminearrings along with their connection with ideals (*cf.* [15, 16, 17, 87, 92]). In continuation to this, R. Mukherjee, P. Pal and M. Ghosh [94] introduced and characterized rees matrix seminearrings to extend the notion of rees matrix semigroups in the setting of seminearrings in 2022.

Seminearrings have various applications. K. Chakraborty in her Ph. D. thesis [18] has rightly noted that, the theory of seminearrings is not only drawing the attention of many researchers from the theoretical point of view but also from the practical point of view. It is well known that the process algebra is an active area of research in computer science. From last century, many process algebras have been formulated, extended with data, time, mobility, probability and stochastic (see [7, 8]). A process algebra is based upon seminearrings where '+' is idempotent and commutative. Seminearring is also a useful tool in the study of reversible computation [14]. It also appears in generalized linear sequential machines. In [68], the authors obtained a necessary condition to test the minimality of the machines using α-radicals. Desharnais and Struth [25], Droste et al. [27], Armstrong et al. [5], Rivas et al. [102], Jenila et al. [52] have utilized the concept of seminearring in various applications.

The study of decomposition of additively regular seminearrings has been initiated in [109] and continued in [90]. One nice aspect of studying additively regular seminearrings is to obtain semigroup theoretic analogues. In this context it may be recalled that there are many significant structure theorems for completely regular semigroups as well as for Clifford semigroups. While studying additively completely regular seminearrings in [90] the authors obtained some characterization of a left (right) completely regular seminearrings which are the semigroup theoretic analogues of "A semigroup is completely regular if and only if it is a semilattice of completely simple semigroups" and "A semigroup is Clifford if and only if it is a semilattice of groups". But they could not obtain the analogues of "A semigroup is completely regular if and only if it is a union of groups" and "A semigroup is Clifford if and only if it is strong semilattice of groups". This thesis makes an attempt to accomplish this unfinished work. As a consequence, we get some significant decompositions of seminearrings in terms of near-rings in **Chapters 2** and **3** whereas last two chapters mainly aim in refining/modifying these decompositions.

To extend the structure theory of completely regular semigroups in the setting of seminearrings, Mukherjee et al. formulated the notions of *left completely regular (LCR) seminearrings* (*cf.* Definition 1.5.23), *right completely regular (RCR) seminearrings* (*cf.* Definition 1.5.24), *left completely simple seminearrings* (*cf.* Definition 1.5.27), *right completely simple seminearrings* (*cf.* Definition 1.5.27), *left Clifford seminearrings* (*cf.* Definition 1.5.30) and *right Clifford seminearrings* (*cf.* Definition 1.5.30) in [90] and obtained some structure theorems (*cf.* Theorems 1.5.28, 1.5.29, 1.5.36, 1.5.37 and Corollaries 1.5.33, 1.5.35) which are indeed the semigroup theoretic analogue of "A semigroup is completely regular if and only if it is a semilattice of completely simple semigroups" and "A semigroup is Clifford if and only if it is a semilattice of groups". As mentioned in the previous paragraph, this study has been extended in [89], which constitutes **Chapter 2**. There we obtain some more structure theorems for left (right) Clifford seminearrings (*cf.* Theorems 2.1.16, 2.1.21, 2.2.4 and Corollaries 2.3.2, 2.3.3) which are the semigroup theoretic analogues of "A semigroup is Clifford if and only if it is strong semilattice of groups". Another attempt to extend the study of [89] is to obtain some suitable characterization of union of near-rings in the class of additively completely regular seminearrings (*cf.* Theorems 3.1.17 and 3.1.20) as a semigroup theoretic analogue of "A semigroup is completely regular if and only if it is a union of groups". This work is accomplished in [88], which constitutes **Chapter 3**. So far, we mainly deal with regularity of the additive reduct of the seminearring under consideration. In **Chapter 4**, we intend to study what happens if we impose various kinds of regularity on the multiplicative reducts of the component near-rings obtained through the decompositions studied in **Chapter 3** (*cf.* Theorems 4.1.3, 4.1.4, 4.1.7, 4.1.8, 4.1.9, 4.1.10, 4.1.11 and 4.1.12). The decomposition theorems established in

Chapters 2 and **3** have their counterparts in semirings (*cf.* [113, 114]). But the main difference between the study of decompositions in the setting of seminearrings and that in semiring setting is that left, right concepts coincide in semirings. In this context one natural question arises - what class of seminearrings can be obtained if the main axioms leading to left, right completely regular seminearrings are made to coincide? The **Chapter 5** is the outcome of an attempt to find an answer to this question. The types of seminearrings we introduce and study in this chapter are called *completely regular seminearring* and *completely simple seminearring*. There the analogy between the decomposition of completely regular seminearrings and the decomposition of completely regular semigroups has been studied (*cf.* Theorems 5.1.13 and 5.1.14).

CHAPTER 1

PRELIMINARIES

1 Preliminaries

In this chapter certain basic definitions and results are presented for their use in the sequel or for some historical connections.

1.1 Semigroups

We recall the following preliminary notions of semigroup theory from [23], [48], [49], [75], [86], [99] and [100].

Definition 1.1.1. A non-empty set S together with a binary operation is called a *groupoid*. A groupoid S satisfying the associative law is a *semigroup*. A semigroup having only one element is *trivial*.

Definition 1.1.2. A non-empty subset A of a semigroup S is called a *subsemigroup* of S if $A^2 \subseteq A$.

Definition 1.1.3. Let S be a semigroup. A non-empty subset A of S is called a *left (right) ideal* of S if $SA \subseteq A$ (resp. $AS \subseteq A$). A *two-sided ideal* (or simply an *ideal*) of S is a subset of S which is both a left and a right ideal of S.

Definition 1.1.4. A semigroup S is said to be *left (right) simple* if S has no proper left (resp. right) ideals.

Definition 1.1.5. A semigroup S is said to be *simple* if S has no proper ideals.

Definition 1.1.6. Let S be a semigroup. An equivalence relation σ on S is said to be a *congruence* on S if $(a, b) \in \sigma$ implies $(ac, bc),\ (ca, cb) \in \sigma$ for all $a, b, c \in S$.

Definition 1.1.7. Let ρ be a congruence on a semigroup $(S,\cdot)$. Then

$$S/\rho := \{[s] : [s] \text{ is the congruence class of } s \text{ under } \rho \text{ where } s \in S\}$$

forms a semigroup w.r.t. '$\cdot$' defined by

$$[x]\cdot[y] = [xy] \text{ for all } x, y \in S.$$

Definition 1.1.8. For any two binary relations ρ, σ on a non-empty set S

$$\rho \circ \sigma := \{(x,y) \in S \times S \mid (x,z) \in \rho \text{ and } (z,y) \in \sigma \text{ for some } z \in S\}$$

is again a binary relation on S.

Definition 1.1.9. Let ρ be a relation on a non-empty set S. Then the *transitive closure of* ρ, denoted by ρ^∞, is defined as follows

$$\rho^\infty := \bigcup_{n=1}^{\infty} \rho^n.$$

Result 1.1.10. If ρ is a relation on a non-empty set S then ρ^∞ is the smallest transitive relation on S containing ρ.

Theorem 1.1.11. *Let ρ and σ be two equivalence relations on a non-empty set S (congruences on a semigroup $(S,\cdot)$) and $\rho \vee \sigma$ denote the smallest equivalence relation on S (the smallest congruence on $(S,\cdot)$) containing both ρ and σ. If $a, b \in S$, then $(a,b) \in \rho \vee \sigma$ if and only if for some $n \in \mathbb{N}$ there exist elements $x_1, x_2, ..., x_{2n-1}$ in S such that*

$$(a, x_1) \in \rho,\ (x_1, x_2) \in \sigma,\ (x_2, x_3) \in \rho, \ldots, (x_{2n-1}, b) \in \sigma.$$

Remark 1.1.12. Theorem 1.1.11 says effectively that

$$\rho \vee \sigma = (\rho \circ \sigma)^\infty.$$

Corollary 1.1.13. *Let ρ and σ be two equivalence relations on a non-empty set S (congruences on a semigroup $(S,\cdot)$) such that $\rho \circ \sigma = \sigma \circ \rho$. Then*

$$\rho \vee \sigma = \rho \circ \sigma.$$

Definition 1.1.14. Let $(S,\cdot)$ be a semigroup. Then the following equivalence relations are called *Green's relations* on S.

(i) $\mathcal{L}$ on S is defined by $a\mathcal{L}b$ if and only if $S^1a = S^1b$

(ii) $\mathcal{R}$ on S is defined by $a\mathcal{R}b$ if and only if $aS^1 = bS^1$

(iii) $\mathcal{J}$ on S is defined by $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$

(iv) $\mathcal{H}$ on S is defined by $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$

(v) $\mathcal{D}$ on S is defined by $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$,

where S^1 denotes the semigroup obtained by adjoining 1 with S.

Remark 1.1.15. A semigroup $(S, \cdot)$ is simple if and only if it has only one $\mathcal{J}$-class.

Definition 1.1.16. A semigroup $(S, \cdot)$ is called *regular* if for each $a \in S$, there exists $x \in S$ such that $a = axa$.

Result 1.1.17. In a regular semigroup $(S, \cdot)$

(i) $a\mathcal{L}b$ if and only if $Sa = Sb$

(ii) $a\mathcal{R}b$ if and only if $aS = bS$

(iii) $a\mathcal{J}b$ if and only if $SaS = SbS$.

Definition 1.1.18. Let $(S, \cdot)$ be a semigroup. Then $a \in S$ is said to be an *idempotent element* of S if $a \cdot a = a$.

Theorem 1.1.19. *If e is an idempotent in a semigroup S, then $\mathcal{H}_e$, the $\mathcal{H}$-class containing e, is a subgroup of S. No $\mathcal{H}$-class in S can contain more than one idempotent.*

Definition 1.1.20. A semigroup S is said to a *band* if every element of S is idempotent.

Definition 1.1.21. A semigroup $(S, \cdot)$ is said to be *left (right) normal band* if S is a band in which $abc = acb$ (resp. $abc = bac$) for all $a, b, c \in S$. S is said to be a *normal band* if S is a band in which $abca = acba$ for all $a, b, c \in S$.

Definition 1.1.22. A commutative band is called a *semilattice*.

Definition 1.1.23. If a is an element of a semigroup $(S, \cdot)$ we say that a' is an *inverse* of a if

$$aa'a = a \text{ and } a'aa' = a'.$$

Definition 1.1.24. A semigroup $(S, \cdot)$ is called an *inverse semigroup* if every a in S possesses a unique inverse *i.e.*, if for every $a \in S$ there exists a unique element a^* in S such that

$$aa^*a = a, a^*aa^* = a^*.$$

Notation 1.1.25. Throughout this thesis, in an inverse semigroup $(S, \cdot)$, for each $a \in S$, a^* denotes the unique element of S satisfying

$$aa^*a = a, a^*aa^* = a^*.$$

Theorem 1.1.26. *The following statements about a semigroup S are equivalent:*

(a) S is an inverse semigroup;

(b) S is regular and idempotent elements commute;

(c) each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains a unique idempotent.

Proposition 1.1.27. *Let $(S, \cdot)$ be an inverse semigroup with $E(S)$ as the set of all idempotents of S. Then*

(a) $(a^*)^* = a$ for every $a \in S$;

(b) $e^* = e$ for every $e \in E(S)$;

(c) $(ab)^* = b^*a^*$ for every $a, b \in S$;

(d) $aea^* \in E(S)$, $a^*ea \in E(S)$ for every $a \in S$ and for every $e \in E(S)$.

(e) $a\mathcal{R}b$ if and only if $aa^* = bb^*$; $a\mathcal{L}b$ if and only if $a^*a = b^*b$.

Result 1.1.28. In an inverse semigroup $(S, \cdot)$, $(E(S), \cdot)$ forms a subsemigroup.

Theorem 1.1.29. *If $(S, \cdot)$ is an inverse semigroup with semilattice of idempotents $E(S)$, then the relation*

$$\sigma := \{(x, y) \in S \times S : xe = ye \text{ for some } e \in E(S)\}$$

is the minimum group congruence on S.

Definition 1.1.30. An element a of a semigroup $(S,\cdot)$ is *completely regular* if there exists an element $x \in S$ such that

$$a = axa \text{ and } ax = xa.$$

Definition 1.1.31. A semigroup S is *completely regular* if all of its elements are completely regular.

Definition 1.1.32. A semigroup S is *completely simple* if it is a completely regular, simple semigroup.

Definition 1.1.33. Let ρ be a congruence on a semigroup S. If S/ρ is a semilattice then ρ is called a *semilattice congruence* on S. In such a case S is a *semilattice* $Y = S/\rho$ *of semigroups* S_α, $\alpha \in Y$, where S_α are the ρ-classes, or briefly a *semilattice of semigroups* S_α.

Theorem 1.1.34. *In a semigroup $(S,\cdot)$ the following conditions are equivalent.*

(i) S is completely regular.

(ii) For each $a \in S$, there exists a unique $a' \in V(a)$ such that $aa' = a'a$ where $V(a)$ denotes the set of all inverses of a (cf. Definition 1.1.23).

(iii) Every $\mathcal{H}$-class of S is a group.

(iv) S is a union of (disjoint) groups.

(v) For every $a \in S$, $a \in aSa^2$.

(vi) S is a semilattice of completely simple semigroups.

Theorem 1.1.35. *If a semigroup S is union of groups then the intersection of all semilattice congruences on S coincides with $\mathcal{J}$ i.e., $\mathcal{J}$ is the unique minimum semilattice congruence on S.*

Definition 1.1.36. A semigroup $(S,\cdot)$ is said to be a *Clifford semigroup* if S is a regular semigroup in which the idempotents are central *i.e.*, $ex = xe$ for every idempotent e and every $x \in S$.

Result 1.1.37. In a Clifford semigroup $(S,\cdot)$, $aa^* = a^*a$ for all $a \in S$.

Result 1.1.38. In a Clifford semigroup $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{J}$.

Definition 1.1.39. Let Y be a semilattice. Suppose for each $\alpha \in Y$ there is a semigroup S_α such that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y$, $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism satisfying the following conditions:

(1) $\phi_{\alpha,\alpha} = i_{S_\alpha}$ where $\alpha \in Y$ and i_{S_α} denotes the identity morphism of S_α,

(2) $\phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha > \beta > \gamma$ (here functions are written from right).

On the set $\bigcup\limits_{\alpha \in Y} S_\alpha$, a multiplication is defined by

$$a * b = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})$$

for $a \in S_\alpha, b \in S_\beta$. This multiplication '$*$' is associative and the new multiplication coincides with the given one on each S_α. The semigroup so defined is denoted by $[Y; S_\alpha, \phi_{\alpha,\beta}] = S$ and is a *strong semilattice Y of semigroups S_α* determined by the homomorphisms $\phi_{\alpha,\beta}$ or briefly a *strong semilattice of semigroups S_α*.

Theorem 1.1.40. *If S is a semigroup with set $E(S)$ of idempotents, then the following statements are equivalent:*

(A) S is a Clifford semigroup;

(B) S is a semilattice of groups;

(C) S is a strong semilattice of groups.

Theorem 1.1.41. *In a completely regular semigroup which is inverse, every idempotent commutes with every element of the semigroup.*

Remark 1.1.42. In view of Definition 1.1.36 and Theorem 1.1.41 a semigroup is Clifford if and only if it is a completely regular, inverse semigroup *i.e.*, a completely regular semigroup in which idempotents commute with each other (*cf.* Theorem 1.1.26).

Definition 1.1.43. [49] A subset A of a semigroup $(S, \cdot)$ is called *right unitary* if

for any $a \in A$, for any $s \in S$, $sa \in A$ implies $s \in A$

and *left unitary* if

$$\text{for any } a \in A, \text{ for any } s \in S, \ as \in A \text{ implies } s \in A.$$

$(S, \cdot)$ is called *unitary* if it is both left and right unitary.

Definition 1.1.44. [49] Let $(S, \cdot)$ be a regular semigroup. Then S is called *E-unitary* if the set E of idempotents is a unitary subsemigroup of S.

1.2 Lattices and related structures

The following preliminaries are mainly collected from Davey and Priestley [24], Hebisch and Weinert [41] and Romanowska [103].

Definition 1.2.1. [24] Let L be an ordered set[1] and $S \subseteq L$. An element $x \in L$ is an *upper bound* of S if $s \leq x$ for all $s \in S$. A *lower bound* is defined dually. x is called the *least upper bound of S (supremum of S)* and denoted by sup S if x is an upper bound of S and $x \leq y$ for all upper bounds y of S. Dually, x is called the *greatest lower bound of S (infimum of S)* and denoted by inf S if x is a lower bound of S and $x \geq y$ for all lower bounds y of S.

L is called a *lattice* if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist for any two elements x, y in L. We often write $a \wedge b$ or *meet* of a, b instead of $\inf\{a, b\}$ and $a \vee b$ or *join* of a, b instead of $\sup\{a, b\}$.

A non empty subset M of a lattice L is called a *sublattice* of L if $a, b \in M$ implies $a \vee b$, $a \wedge b \in M$.

Theorem 1.2.2. *[41] (a) Let $(V, \leq)$ be a lattice. Considering $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$ as binary operations on V, we obtain the following:*

(1) $(V, \vee)$ and $(V, \wedge)$ are semilattices (cf. Definition 1.1.22).

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$ for all $a, b \in V$.

(b) Conversely, let $(V, \vee, \wedge)$ be a non-empty set with binary operations $\vee$ and $\wedge$ which satisfy (1) and (2). Then

$$a \leq b \Leftrightarrow a \vee b = b \text{ for all } a, b \in V$$

[1] Ordered set means a partially ordered set

or equivalently

$$a \leq b \Leftrightarrow a \wedge b = a \text{ for all } a, b \in V$$

defines a relation '$\leq$' on V for which $(V, \leq)$ is a lattice satisfying $\sup\{a, b\} = a \vee b$ *and* $\inf\{a, b\} = a \wedge b$ *for all* $a, b \in V$.

Definition 1.2.3. [24] A lattice L is said to be a *distributive lattice* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

or equivalently

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

for all $a, b, c \in L$.

Definition 1.2.4. [103] A non-empty set S with two semilattice operations '$\cdot$', '$+$' is called a *bi-semilattice*. One regards $(S, \cdot)$ as a *meet-semilattice* and $(S, +)$ as a *join-semilattice*. A bi-semilattice is called a *meet-distributive bi-semilattice* if the meet operation '$\cdot$' distributes over the join operation '$+$'. Throughout this thesis '*bi-semilattice*' stands for '*meet-distributive bi-semilattice*'.

Example 1.2.5. [103] For a given semilattice $(V, \cdot)$ let us consider the set of all finite non-empty subsemilattices of $(V, \cdot)$, denoted by $S(V, \cdot)$. If we define '$+$' and '$\cdot$' on $S(V, \cdot)$ in the following manner

$$S \cdot T = \{s \cdot t | s \in S, t \in T\}$$

and

$$S + T = S \cup T \cup S \cdot T$$

then $(S(V, \cdot), +, \cdot)$ forms a bi-semilattice where '$+$' is the join operation and '$\cdot$' is the meet operation.

Example 1.2.6. [103] Let us consider a three-element semilattice $(\{0, 1, 2\}, \cdot)$ with $0 \leq 1 \leq 2$. Now $X = \{0, 2\}$, $Y = \{0\}$, $Z = \{0, 1, 2\} \in S(\{0, 1, 2\}, \cdot)$. Then

$$\begin{aligned} X + Y \cdot Z &= \{0, 2\} + \{0\} \\ &= \{0, 2\} \cup \{0\} \cup \{0, 2\} \cdot \{0\} \\ &= \{0, 2\}, \end{aligned}$$

while

$$\begin{aligned}(X+Y)\cdot(X+Z) &= (\{0,2\}+\{0\})\cdot(\{0,2\}+\{0,1,2\}) \\ &= (\{0,2\}\cup\{0\}\cup\{0,2\}\cdot\{0\})\cdot(\{0,2\}\cup\{0,1,2\}\cup\{0,2\}\cdot\{0,1,2\}) \\ &= \{0,2\}\cdot\{0,1,2\} \\ &= \{0,1,2\}\end{aligned}$$

whence the join is not distributive over meet. Hence $S(\{0,1,2\},\cdot)$ is a bi-semilattice which is not a distributive lattice.

1.3 Semirings

Semiring is a generalization of both rings and distributive lattices. The first mathematical structure we encounter 'the set of all natural numbers' under usual addition and multiplication is a semiring. Other semirings arise naturally in diverse areas of mathematics such as combinatorics, functional analysis, topology, graph theory, optimization theory. In the literature of semiring theory, various definitions of semirings followed by several authors are found. Various versions of the definition of semiring are prevalent in the literature. We present below three versions, one from Hebisch and Weinert [40], one from Golan [36] and one from Hebisch and Weinert [41].

Definition 1.3.1. [40] An algebra $(R,+,\cdot)$ is said to be a *semiring* if it satisfies the following axioms:

(1) $(R,+)$ is a semigroup (not necessarily commutative),

(2) $(R,\cdot)$ is a semigroup (not necessarily commutative),

(3) multiplication distributes over addition from either side.

$(S,+,\cdot)$ is said to be *additively commutative semiring* if $(S,+)$ is commutative and *multiplicatively commutative semiring* if $(S,\cdot)$ is commutative. $(S,+,\cdot)$ is said to be a *commutative semiring* if both $(S,+)$ and $(S,\cdot)$ are commutative.

Each subalgebra of a semiring is a *subsemiring.*

Definition 1.3.2. [36] A *semiring* is a non-empty set R on which two binary operations, say '+' and '·' are defined such that $(R,+)$ is a commutative semigroup with identity element 0, $(R, \cdot)$ is a semigroup with identity element 1, '$\cdot$' distributes over '$+$' from either side and $0.r = 0 = r.0$ for all $r \in R$.

Definition 1.3.3. [41] A non-empty set S along with two binary operations, say '$+$' and '$\cdot$' is said to be a *semiring* if $(S, +)$ is a commutative semigroup, $(S, \cdot)$ is a semigroup and '$\cdot$' distributes over '$+$' from either side.

Remark 1.3.4. A semiring in the sense of [36] and [41] are *additively commutative* in the sense of [40]. In the present thesis, unless otherwise mentioned, a semiring is assumed to be in the sense of [40].

Definition 1.3.5. [1] Let $(S, +, \cdot)$ be an additively commutative semiring and $a \in S$. Then a is called *regular* or *multiplicatively regular* if $a = axa$ for some $x \in S$ and *k-regular* if there exist $x, y \in S$ satisfying $a + aya = axa$. If all the elements of S have the corresponding property then S is called a *(multiplicatively) regular* or *k-regular semiring.*

Definition 1.3.6. [114] A semiring $(S, +, \cdot)$ is called a *b-lattice* if $(S, \cdot)$ is a band and $(S, +)$ is a semilattice.

Definition 1.3.7. [38] A semiring $(S, +, \cdot)$ is called a *skew-ring* if its additive reduct $(S, +)$ is a group, not necessarily an abelian group.

Definition 1.3.8. [114] A semiring $(S, +, \cdot)$ is said to be a *completely regular semiring* if for each $a \in S$ there exists $x \in S$ satisfying $a = a + x + a$, $a + x = x + a$ and $(a + x)a = a + x = a(a + x)$.

Theorem 1.3.9. *[114] The following conditions on a semiring $(S, +, \cdot)$ are equivalent:*

(1) S is completely regular semiring;

(2) Every $\mathcal{H}^+$ - class is a skew-ring;

(3) S is a union of skew-rings;

(4) S is a b-lattice of completely simple semirings.

1.4 Near-rings

For the following preliminaries on near-rings, mainly [101] is consulted.

Definition 1.4.1. [101] A *near-ring* is a set N together with two binary operations '+' and '·' such that

(1) $(N, +)$ is a group (not necessarily abelian),

(2) $(N, \cdot)$ is a semigroup (not necessarily commutative), and

(3) for all $a, b, c \in N$, $(a + b) \cdot c = a \cdot c + b \cdot c$ ('right distributive law').

Remark 1.4.2. [101] In view of (3) of Definition 1.4.1, one speaks more precisely of a '*right near-ring*'. Postulating

($3'$) for all $a, b, c \in N$, $a \cdot (b + c) = a \cdot b + a \cdot c$ ('left distributive law')

instead of (3) one gets '*left near-rings*'. The theory runs completely parallel in both cases, of course; so one can decide to use just one version.

Notation 1.4.3. Throughout the present thesis '*near-ring*' stands for '*right near-ring*'.

Example 1.4.4. [101] Let G be an additively written (but not necessarily abelian) group. Then the following sets of mappings from $G \to G$ are near-rings under pointwise addition and composition of functions:

(i) $M(G) := \{f : G \to G\}$,

(ii) $M_0(G) := \{f : G \to G | f(0) = 0\}$,

(iii) $M_c(G) := \{f : G \to G | f \text{ is constant}\}$,

(iv) $M_{cont}(G) := \{f : G \to G | f \text{ is continuous}\}$ where G is a topological group.

Definition 1.4.5. [101] Let $(N, +, \cdot)$ be a near-ring. Then $N_0 := \{a \in N | a \cdot 0 = 0\}$ is called the *zero-symmetric part* of N (it may be noted that from the definition of near-ring $0 \cdot a = 0$ for all $a \in N$).

A near-ring N is called *zero-symmetric* if $N = N_0$.

Example 1.4.6. [101] $M_0(G)$ of Example 1.4.4 is a zero-symmetric near-ring but $M(G)$ is not a zero-symmetric near-ring.

Definition 1.4.7. [101] Let $(N, +, \cdot)$ be a near-ring. Then a subgroup M of $(N, +)$ is called a *subnear-ring* of N if $m_1 m_2 \in M$ for all $m_1, m_2 \in M$.

Definition 1.4.8. [101] In a near-ring $(N, +, \cdot)$ an element d is said to be a *distributive element* if $d \cdot (a + b) = d \cdot a + d \cdot b$ for all $a, b \in N$.

Definition 1.4.9. [101] Let $(N, +, \cdot)$ be a near-ring and $N_d := \{d \in N | d \text{ is distributive}\}$. Then N is called a *distributively generated near-ring* if N_d generates the group $(N, +)$, *i.e.*, for $x \in N$ there exists $d_1, d_2, ..., d_k \in N_d (k \in \mathbb{N})$ such that $x = \sum_{i=1}^{k} d_i$.

Definition 1.4.10. [101] A near-ring $(N, +, \cdot)$ is called *regular near-ring* if for each $n \in N$ there exists $x \in N$ such that $nxn = n$.

Example 1.4.11. [101] Let $(G, +)$ be a group (not necessarily abelian). Consider the set of all endomorphisms on G, denoted by $End(G)$. Then

$$< End(G) >:= \{\sum_{i=1}^{n} \sigma_i e_i | n \in \mathbb{N}, \sigma_i \in \{-1, 1\}, e_i \in End(G)\}$$

is a subnear-ring of $M(G)$, distributively generated by $(End(G), \cdot)$ and called the *endomorphism near-ring on G*. It can be verified that $< End(G) >$ is not a ring.

Remark 1.4.12. [101] Let $(N, +, \cdot)$ be a distributively generated near-ring. Then N is a ring if and only if $(N, +)$ is abelian.

1.5 Seminearrings

Definition 1.5.1. According to Weinert [119], an algebraic structure $(S, +, \cdot)$ is said to be a *(left distributive) seminearring* if it satisfies the following axioms:

(1) $(S, +)$ is a semigroup (not necessarily commutative),

(2) $(S, \cdot)$ is a semigroup (not necessarily commutative),

(3) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in S$ ("left distributive law"[2]).

[2] Pilz [101] considered right distributive law

This definition was adopted from [104] where $(S, +)$ was assumed to be with the identity 0 satisfying absorbing property with multiplication *i.e.*, $s \cdot 0 = 0 = 0 \cdot s$ for all $s \in S$ along with the above mentioned conditions (1), (2) and (3). Even if $(S, +)$ has the identity 0, according to Weinert [119], it need not be left absorbing or right absorbing. We call $(S, +, \cdot)$ to be a *(right distributive) seminearring with zero* if 0 is the additive identity of $(S, +)$ and 0 satisfies the property $0 \cdot a = 0$ for all $a \in S$ *i.e.*, 0 is right absorbing. K. V. Krishna called it *near-semiring*[3] in [62, 67]. Throughout the thesis seminearring is assumed to be right distributive. Throughout our work, unless mentioned otherwise, the term 'seminearring' will stand for (right distributive) seminearring without zero. Since the theory of left distributive seminearrings runs parallel to that of right distributive seminearrings, we do not mention further in the following preliminaries on seminearrings which distributive property of seminearring has been considered.

A seminearring S is said to be *zero-symmetric* if S is a seminearring with 0 (*i.e.*, $0 \cdot s = 0$ for all $s \in S$) in which $s \cdot 0 = 0$ for each $s \in S$ (*cf.* [67]).

A subset M of a seminearring $(S, +, \cdot)$ is called a *subseminearring* of S if $(M, +)$ is a subsemigroup of $(S, +)$ and $m_1 m_2 \in M$ for all $m_1, m_2 \in M$.

Example 1.5.2. [101] $M(S)$, the set of all self maps of an additively written semigroup S forms a seminearring w.r.t. pointwise addition and composition of maps.

Example 1.5.3. [62] $M_c(S)$, the set of all constant self maps of an additively written semigroup S forms a seminearring w.r.t. pointwise addition and composition of maps.

Definition 1.5.4. [85, 119] A seminearring $(S, +, \cdot)$ is said to be a *distributively generated seminearring* if S contains a multiplicative semigroup $(D, \cdot)$ of distributive elements (*cf.* Definition 1.4.8) which generates $(S, +)$.

Remark 1.5.5. In a distributively generated seminearring $(S, +, \cdot)$ any element can be expressed as a finite sum of distributive elements of S.

Example 1.5.6. [106] For a semigroup $(S, +)$, let $End(S)$ denote the set of all endomorphisms of S. Since $(S, +)$ is not commutative, $End(S)$ need not be closed w.r.t. the

[3] The algebraic structure under consideration may be obtained from a near-ring (*cf.* Definition 1.4.1) or from a semiring (*cf.* Definition 1.3.1) by removing one distributive property and accordingly may be called a *seminearring* or *near-semiring*

pointwise addition of functions. But $End(S)$ forms a semigroup w.r.t. the composition of functions. Then

$$< End(S) >:= \{\sum_{i=1}^{n} f_i | i \in \mathbb{N}, f_i \in End(S)\}$$

is a subseminearring of $M(S)$, the seminearring of self maps of S. Clearly each element of $End(S)$ is a distributive element in $M(S)$ and so from the construction it is clear that $< End(S) >$ is distributively generated by $End(S)$. So $< End(S) >$ forms a distributively generated seminearring w.r.t. pointwise addition and composition of functions.

Remark 1.5.7. In view of Remark 1.4.12 it is evident that a distributively generated and additively commutative seminearring is an additively commutative semiring.

Definition 1.5.8. In a seminearring $(S, +, \cdot)$, an element x is said to be an *additive idempotent* if $x+x = x$ and a *multiplicative idempotent* if $x \cdot x = x$. The set of all additive idempotents in S is denoted by $E^+(S)$ and the set of all multiplicative idempotents in S is denoted by $E^\times(S)$.

Definitions 1.5.9. [45] A seminearring $(S, +, \cdot)$ is called

(i) *additively regular* if $(S, +)$ is a regular semigroup,

(ii) *additively inverse* if $(S, +)$ is an inverse semigroup,

Definition 1.5.10. [45] A seminearring $(S, +, \cdot)$ is said to be a *multiplicatively regular seminearring* (*multiplicatively inverse seminearring*) if $(S, \cdot)$ is a regular semigroup (resp. an inverse semigroup).

Definition 1.5.11. [46, 62, 107] Let S and S' be two seminearrings (both with 0). Then a mapping f from S to S' is said to be a *homomorphism of seminearrings* or *seminearring homomorphism* if it (fixes 0 and) satisfies the following properties

(i) $f(x + y) = f(x) + f(y)$ and

(ii) $f(xy) = f(x)f(y)$

for all $x, y \in S$.

If S' has 0, *Kernel* of f is the set $\{s \in S : f(s) = 0\}$ and is denoted by $ker f$.

A seminearring homomorphism g of S is said to be an *isomorphism of seminearrings* or *seminearring isomorphism* if g is both surjective and injective.

Definition 1.5.12. [2] A subset I of a seminearring $(S, +, \cdot)$ is said to be a *right (left) $\mathcal{S}$-ideal* if

(i) for all $x, y \in I$, $x + y \in I$,

(ii) for all $x \in I$, and for all $s \in S$, $xs \in I$ (resp. $sx \in I$).

I is said to be an *$\mathcal{S}$-ideal* if it is both left and a right $\mathcal{S}$-ideal.

Definition 1.5.13. [101] An algebraic structure $(F, +, \cdot)$ is said to be a *near-field* if it satisfies the following axioms:

(i) $(F, +)$ is a group (not necessarily abelian),

(ii) $(F^*, \cdot)$ is a group (not necessarily abelian), where F^* denotes the set of all non-zero elements of F,

(iii) for all $f_1, f_2, f_3 \in F$, $(f_1 + f_2) \cdot f_3 = f_1 \cdot f_3 + f_2 \cdot f_3$ ("right distributive law").

Since our work in this thesis begins with the continuation of the work done in [90], we recall following notation and results from [90] as they would be use frequently in the chapters of the thesis.

Notations 1.5.14. Throughout this thesis, unless mentioned otherwise,

(i) for a seminearring S, $E^+(S)$ denotes the set of all additive idempotents;

(ii) in an additively completely regular seminearring S, for $a \in S$, an element $x \in S$ satisfying $a + x + a = a$ and $a + x = x + a$ (p-58 [100], Definition 2.2) is denoted by x_a;

(iii) for each $a \in S$, the additive inverse is denoted by a^* and $a + a^*$, $a^* + a \in E^+(S)$ are, respectively, denoted by a^0, a_0.

(iv) $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{H}^+$ and $\mathcal{J}^+$ denote the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{J}$ on the semigroup $(S,+)$, the additive reduct of the seminearring S;

(v) in a seminearring S, $\mathcal{L}^+_a$, $\mathcal{R}^+_a$, $\mathcal{H}^+_a$ and $\mathcal{J}^+_a$ denote the $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{H}^+$ and $\mathcal{J}^+$ classes of $a \in S$;

(vi) in an additively completely regular seminearring S, for each $a \in S$, $(\mathcal{H}^+_a, +)$ is a group. The identity element of this group is denoted by $0_{\mathcal{H}^+_a}$.

Definition 1.5.15. [90] A seminearring $(S,+,\cdot)$ is said to be an *additively completely regular* seminearring if $(S,+)$ is a completely regular semigroup.

Definition 1.5.16. [90] A seminearring $(S,+,\cdot)$ is said to be an *additively completely simple* seminearring if $(S,+)$ is a completely simple semigroup.

From the following propositions we see that there are plenty of additively completely regular and additively completely simple seminearrings.

Proposition 1.5.17. *[90] Let $(S,+)$ be a semigroup. Then the seminearring $M(S)$ is an additively completely regular seminearring if and only if S is a completely regular semigroup.*

Proposition 1.5.18. *[90] Let $(S,+)$ be a semigroup. Then the seminearring $M(S)$ is an additively completely simple seminearring if and only if S is a completely simple semigroup.*

Definition 1.5.19. [90] A congruence ρ on a seminearring $(S,+,\cdot)$ is said to be a *bi-semilattice congruence* on S if the factor seminearring S/ρ becomes a bi-semilattice.

Definition 1.5.20. [90] A seminearring $(S,+,\cdot)$ is called a *bi-semilattice B of seminearrings (near-rings) S_i $(i \in B)$* if S admits of a bi-semilattice congruence β such that $B = S/\beta$ with each S_i a β-class.

Each of the following two theorems respectively characterizes separate classes of additively completely regular seminearrings in which $\mathcal{J}^+$ is a bi-semilattice congruence.

Theorem 1.5.21. *[90] Let S be an additively completely regular seminearring in which for every a there exists an x_a satisfying $(a + x_a)a = a + x_a$. Then the following are equivalent.*

(i) For each $a \in S$ and $e \in E^+(S)$, $ae \mathcal{J}^+ ea$.

(ii) $\mathcal{J}^+$ is a bi-semilattice congruence on S.

Theorem 1.5.22. *[90] Let S be an additively completely regular seminearring in which for every a there exists an x_a satisfying $a(a + x_a) = a + x_a$. Then the following are equivalent.*

(i) For each $a \in S$ and $e \in E^+(S)$, $ae \mathcal{J}^+ ea$.

(ii) $\mathcal{J}^+$ is a bi-semilattice congruence on S.

Definition 1.5.23. [90] An additively completely regular seminearring $(S, +, \cdot)$ is said to be a *left completely regular seminearring* if it satisfies the hypothesis of Theorem 1.5.21 and any one of the conditions (i) and (ii) of this theorem.

Definition 1.5.24. [90] An additively completely regular seminearring $(S, +, \cdot)$ is said to be a *right completely regular seminearring* if it satisfies the hypothesis of Theorem 1.5.22 and any one of the conditions (i) and (ii) of this theorem.

Example 1.5.25. [90] Any non zero-symmetric near-ring N is a left completely regular seminearring but not a right completely regular seminearring. In particular, the canonical near-ring $(M(G), +, \cdot)$ of all self-maps on an additive group G is a left completely regular seminearring but not a right completely regular seminearring.

Example 1.5.26. [90] If N is a zero-symmetric near-ring then it is a left as well as a right completely regular seminearring.

Definition 1.5.27. [90] A seminearring $(S, +, \cdot)$ is said to be a *left (right) completely simple seminearring* if S is a left (resp. right) completely regular seminearring in which any two elements are $\mathcal{J}^+$ related.

Theorem 1.5.28. *[90] A seminearring is left completely regular if and only if it is a bi-semilattice of left completely simple seminearrings.*

Theorem 1.5.29. *[90] A seminearring is right completely regular if and only if it is a bi-semilattice of right completely simple seminearrings.*

Definition 1.5.30. [90] A seminearring $(S, +, \cdot)$ is said to be a *left Clifford seminearring* (*right Clifford seminearring*) if it satisfies the following conditions:

(A) S is a left completely regular seminearring (resp. (A') S is a right completely regular seminearring) and

(B) the additive idempotents of S commute additively with each other.

Proposition 1.5.31. *[90] Let S be a left (right) Clifford seminearring. Then $\mathcal{J}^+ = \mathcal{H}^+$ and hence $\mathcal{H}^+$ is a bi-semilattice congruence on S.*

Theorem 1.5.32. *[90] Let S be a seminearring. Then the following are equivalent.*

(i) S is a left Clifford seminearring.

(ii) $\mathcal{H}^+$ is a congruence on $(S, .)$, for each $a \in S$, $\mathcal{H}^+_a$ is a near-ring and the additive identity $0_{\mathcal{H}^+_a}$ of $\mathcal{H}^+_a$ is both additively and multiplicatively central in $E^+(S)$, where $\mathcal{H}^+_a$ denotes the $\mathcal{H}^+$-class in S containing a.

Corollary 1.5.33. *[90] Let S be a seminearring. Then S is a left Clifford seminearring if and only if it is a bi-semilattice of near-rings.*

Theorem 1.5.34. *[90] Let S be a seminearring. Then the following are equivalent.*

(i) S is a right Clifford seminearring.

(ii) $\mathcal{H}^+$ is a congruence on $(S, .)$, and for each $a \in S$, $\mathcal{H}^+_a$ is a zero-symmetric near-ring and the additive identity $0_{\mathcal{H}^+_a}$ of $\mathcal{H}^+_a$ is both additively and multiplicatively central in $E^+(S)$.

Corollary 1.5.35. *[90] A seminearring S is right Clifford if and only if it is a bi-semilattice of zero-symmetric near-rings.*

Theorem 1.5.36. *[90] Let S be a seminearring. Then the following are equivalent.*

(i) S is a left Clifford seminearring in which $a + a^0 b = a$ for all $a, b \in S$.

(ii) S is a distributive lattice of near-rings.

Theorem 1.5.37. *[90] Let S be a seminearring. Then the following are equivalent.*

(i) S is a right Clifford seminearring in which $a + a^0 b = a$ for all $a, b \in S$.

(ii) S is a distributive lattice of zero-symmetric near-rings.

CHAPTER 2

STRONG BI-SEMILATTICE OF SEMINEARRINGS

2

Strong Bi-semilattice of Seminearrings

The study of additively regular seminearrings has been initiated in [109] and continued in [90]. One nice aspect of studying additively regular seminearrings is to obtain semigroup theoretic analogues. In this connection it may be recalled that there are some important structure theorems, in general for regular semigroups, in particular for completely regular semigroups and for Clifford semigroups. In [90], while studying additively completely regular seminearrings, the authors obtained a characterization of a left (right) completely regular seminearring as a bi-semilattice (bi-semilattice is a suitable substitute of semilattice in the setting of seminearring) of left (right) completely simple seminearrings (*cf.* Theorems 1.5.28 and 1.5.29) which is the semigroup theoretic analogue of "A semigroup is completely regular if and only if it is a semilattice of completely simple semigroups". They also obtained a characterization of left (right) Clifford seminearring as a distributive lattice of near-rings (zero-symmetric near-rings) (*cf.* Theorems 1.5.36 and 1.5.37) which is the semigroup theoretic analogue of "A semigroup is Clifford if and only if it is a semilattice of groups". But they did not obtain the semigroup theoretic analogue of "A semigroup is Clifford if and only if it is strong semilattice of groups".

The main purpose of the present chapter is to make an attempt to accomplish this unfinished work of [90].

In **Section 1**, the notion of strong bi-semilattice of seminearrings has been introduced by adopting the concept of strong b-lattice of semirings. Then characterizations

---

This chapter is mainly based on the work published in the following paper:

Tuhin Manna et al., ***On additively completely regular seminearrings: II, Communications in Algebra*****, 47 (5), 1954-1963 (2019)**

of left (right) Clifford seminearrings which are strong bi-semilattice of near-rings (zero-symmetric near-rings) (*cf.* Theorems 2.1.16 and 2.1.21) have been obtained.

In **Section 2**, the characterizations of left (right) Clifford seminearrings which are strong distributive lattice of near-rings (zero-symmetric near-rings) (*cf.* Theorem 2.2.4) have been obtained.

Sometimes the notion of being distributively generated (*cf.* Definition 1.5.4) for seminearrings makes many situations (as it happens for near-rings as well) nice. This is again evident from the results obtained in **Section 3** (*cf.* Corollaries 2.3.2 and 2.3.3) which is what have been obtained from Theorems 2.1.16, 2.1.21 and 2.2.4 by taking the seminearring to be distributively generated.

2.1 Strong Bi-semilattice of Near-rings

To generalize the notion of strong semilattice of semigroups (*cf.* Definition 1.1.39) Maity et al. formulated strong b-lattice of semirings as follows.

Definition 2.1.1. [114] Let T be b-lattice (*cf.* Definition 1.3.6) and $\{S_\alpha : \alpha \in T\}$ be a family of pairwise disjoint semirings which are indexed by the elements of T. For each $\alpha \leq \beta$ in T, we now define a semiring monomorphism $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ satisfying the following conditions:

(1) $\phi_{\alpha,\alpha} = I_{S_\alpha}$, where I_{S_α} denotes the identity mapping on S_α,

(2) $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, if $\alpha \leq \beta \leq \gamma$,

(3) $\phi_{\alpha,\gamma}S_\alpha\phi_{\beta,\gamma}S_\beta \subseteq \phi_{\alpha\beta,\gamma}S_{\alpha\beta}$, if $\alpha + \beta \leq \gamma$.

On $S = \bigcup S_\alpha$ we define addition $\oplus$ and multiplication $\odot$ as follows:

(4) $a \oplus b = \phi_{\alpha,\alpha+\beta}a + \phi_{\beta,\alpha+\beta}b$

and

(5) $a \odot b = c \in S_{\alpha\beta}$ such that $\phi_{\alpha\beta,\alpha+\beta}c = \phi_{\alpha,\alpha+\beta}a \cdot \phi_{\beta,\alpha+\beta}b$

where $a \in S_\alpha$, $b \in S_\beta$. We denote the above system by $S =< T, S_\alpha, \phi_{\alpha,\beta} >$ and call it the *strong b-lattice T of the semirings S_α, $\alpha \in T$* .

Taking impetus from the above formulation we define strong bi-semilattice of seminearrings in Definition 2.1.4. Before giving the formal definition of strong bi-semilattice of seminearrings we discuss some basic properties of a bi-semilattice (*cf.* Definition 1.2.4) in the following result.

Notations 2.1.2. Throughout this chapter, unless mentioned otherwise,

(i) for a bi-semilattice $(B, +, \cdot)$, $\alpha \leq \beta$ in B stands for $\alpha \leq \beta$ in the semilattice $(B, +)$, *i.e.*, $\alpha \leq \beta$ if and only if $\alpha + \beta = \beta$.

(ii) for a seminearring S which is a bi-semilattie B of near-rings N_α $(\alpha \in B)$ (*cf.* Definition 1.5.20), 0_α denotes the additive identity of N_α.

Result 2.1.3. Let $(B, +, \cdot)$ be a bi-semilattice and α, β and $\gamma \in B$. Then

(1) $\alpha \leq \beta$ implies $\alpha\beta \leq \beta$;

(2) $\alpha + \beta + \alpha\beta = \alpha + \beta$;

(3) $\alpha(\beta + \gamma) \leq \alpha + \beta + \gamma$.

Proof. (1)

$$\begin{gathered} \alpha \leq \beta \\ \Rightarrow \alpha + \beta = \beta \\ \Rightarrow (\alpha + \beta)\beta = \beta \cdot \beta \\ \Rightarrow \alpha\beta + \beta^2 = \beta^2 \\ \Rightarrow \alpha\beta + \beta = \beta \\ \Rightarrow \alpha\beta \leq \beta \end{gathered}$$

(2)

$$\begin{aligned} (\alpha + \beta) &= (\alpha + \beta)^2 \\ &= (\alpha + \beta)(\alpha + \beta) \\ &= \alpha(\alpha + \beta) + \beta(\alpha + \beta) \\ &= \alpha^2 + \alpha\beta + \beta\alpha + \beta^2 \\ &= \alpha + \alpha\beta + \beta \end{aligned}$$

(3)

$$\begin{aligned}
&\alpha(\alpha+\beta+\gamma) \leq \alpha+\beta+\gamma \text{ (Using (1))}\\
&\Rightarrow \alpha(\alpha+\beta+\gamma)+\alpha+\beta+\gamma = \alpha+\beta+\gamma\\
&\Rightarrow \alpha^2+\alpha\beta+\alpha\gamma+\alpha+\beta+\gamma = \alpha+\beta+\gamma\\
&\Rightarrow \alpha\beta+\alpha\gamma+\alpha+\beta+\gamma = \alpha+\beta+\gamma\\
&\Rightarrow \alpha(\beta+\gamma)+\alpha+\beta+\gamma = \alpha+\beta+\gamma\\
&\Rightarrow \alpha(\beta+\gamma) \leq \alpha+\beta+\gamma
\end{aligned}$$

□

Definition 2.1.4. Let B be bi-semilattice and $\{S_\alpha : \alpha \in B\}$ be a family of seminearrings which are indexed by the elements of B. For each $\alpha \leq \beta$ (*cf.* Notations 2.1.2) in B, we now define a seminearring monomorphism $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ satisfying the following conditions:

(1) $\phi_{\alpha,\alpha} = I_{S_\alpha}$, where I_{S_α} denotes the identity mapping on S_α,

(2) $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, if $\alpha \leq \beta \leq \gamma$,

(3) $\phi_{\alpha,\gamma}S_\alpha\phi_{\beta,\gamma}S_\beta \subseteq \phi_{\alpha\beta,\gamma}S_{\alpha\beta}$, if $\alpha+\beta \leq \gamma$.

On $S = \dot{\bigcup} S_\alpha$ (the disjoint union of S_α's) we define addition $\oplus$ and multiplication $\odot$ as follows:

(4) $a \oplus b = \phi_{\alpha,\alpha+\beta}(a) + \phi_{\beta,\alpha+\beta}(b)$

and

(5) $a \odot b = c \in S_{\alpha\beta}$ such that $\phi_{\alpha\beta,\alpha+\beta}(c) = \phi_{\alpha,\alpha+\beta}(a) \cdot \phi_{\beta,\alpha+\beta}(b)$

where $a \in S_\alpha$, $b \in S_\beta$. We denote the above system by $S =< B, S_\alpha, \phi_{\alpha,\beta} >$ and call it the *strong bi-semilattice B of the seminearrings S_α, $\alpha \in B$.*

Theorem 2.1.5. *With the same notation as in Definition 2.1.4, the system $S =< B, S_\alpha, \phi_{\alpha,\beta} >$ is a seminearring.*

Proof. We first show that the operation of multiplication defined above is well defined. Let $a \in S_\alpha$ and $b \in S_\beta$ with $\alpha, \beta \in B$. Then by (3) of Definition 2.1.4, there exists an element $c \in S_{\alpha\beta}$ satisfying (5) and the uniqueness of the element follows directly from the injectivity of the function $\phi_{\alpha\beta,\alpha+\beta}$. We now check the associativity of the addition. For this purpose, we let $a \in S_\alpha$, $b \in S_\beta$ and $c \in S_\gamma$ with $\alpha, \beta, \gamma \in B$.

$$\begin{aligned} a \oplus (b \oplus c) &= a \oplus (\phi_{\beta,\beta+\gamma} b + \phi_{\gamma,\beta+\gamma} c) \\ &= \phi_{\alpha,\alpha+\beta+\gamma} a + \phi_{\beta+\gamma,\alpha+\beta+\gamma}(\phi_{\beta,\beta+\gamma} b + \phi_{\gamma,\beta+\gamma} c) \\ &= \phi_{\alpha,\alpha+\beta+\gamma} a + \phi_{\beta,\alpha+\beta+\gamma} b + \phi_{\gamma,\alpha+\beta+\gamma} c \end{aligned}$$

and

$$\begin{aligned} (a \oplus b) \oplus c &= (\phi_{\alpha,\alpha+\beta} a + \phi_{\beta,\alpha+\beta} b) \oplus c \\ &= \phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\alpha,\alpha+\beta} a + \phi_{\beta,\alpha+\beta} b) + \phi_{\gamma,\alpha+\beta+\gamma} c \\ &= \phi_{\alpha,\alpha+\beta+\gamma} a + \phi_{\beta,\alpha+\beta+\gamma} b + \phi_{\gamma,\alpha+\beta+\gamma} c. \end{aligned}$$

Therefore

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c.$$

Now to check the multiplicative associativity, let $a \in S_\alpha$, $b \in S_\beta$ and $c \in S_\gamma$ with $\alpha, \beta, \gamma \in B$. Let $x = a \odot b$ and $d = x \odot c = (a \odot b) \odot c$. Then by definition we have

$$\phi_{\alpha\beta,\alpha+\beta} x = \phi_{\alpha,\alpha+\beta} a \cdot \phi_{\beta,\alpha+\beta} b ....(1.1)$$

and

$$\phi_{\alpha\beta\gamma,\alpha\beta+\gamma} d = \phi_{\alpha\beta,\alpha\beta+\gamma} x \cdot \phi_{\gamma,\alpha\beta+\gamma} c ....(1.2).$$

In (1.1) applying $\phi_{\alpha+\beta,\alpha+\beta+\gamma}$ to both side of it

$$\phi_{\alpha\beta,\alpha+\beta+\gamma} x = \phi_{\alpha,\alpha+\beta+\gamma} a \cdot \phi_{\beta,\alpha+\beta+\gamma} b ....(1.3)$$

and applying $\phi_{\alpha\beta+\gamma,\alpha+\beta+\gamma}$ to both side of (1.2)

$$\begin{aligned} \phi_{\alpha\beta\gamma,\alpha+\beta+\gamma} d &= \phi_{\alpha\beta,\alpha+\beta+\gamma} x \cdot \phi_{\gamma,\alpha+\beta+\gamma} c \\ &= \phi_{\alpha,\alpha+\beta+\gamma} a \cdot \phi_{\beta,\alpha+\beta+\gamma} b \cdot \phi_{\gamma,\alpha+\beta+\gamma} c ....(1.4) \text{ [using (1.3)]}. \end{aligned}$$

Again let $y = b \odot c$ and $e = a \odot y = a \odot (b \odot c)$. Then by definition,

$$\phi_{\beta\gamma,\beta+\gamma} y = \phi_{\beta,\beta+\gamma} b \cdot \phi_{\gamma,\beta+\gamma} c ....(1.5)$$

and

$$\phi_{\alpha\beta\gamma,\alpha+\beta\gamma} e = \phi_{\alpha,\alpha+\beta\gamma} a \cdot \phi_{\beta\gamma,\alpha+\beta\gamma} y ....(1.6).$$

In (1.5) applying $\phi_{\beta+\gamma,\alpha+\beta+\gamma}$ to both side,

$$\phi_{\beta\gamma,\alpha+\beta+\gamma}y = \phi_{\beta,\alpha+\beta+\gamma}b \cdot \phi_{\gamma,\alpha+\beta+\gamma}c.................(1.7)$$

and applying $\phi_{\alpha+\beta\gamma,\alpha+\beta+\gamma}$ to both side of (1.6)

$$\begin{aligned}
\phi_{\alpha\beta\gamma,\alpha+\beta+\gamma}e &= \phi_{\alpha,\alpha+\beta+\gamma}a \cdot \phi_{\beta\gamma,\alpha+\beta+\gamma}y \\
&= \phi_{\alpha,\alpha+\beta+\gamma}a \cdot \phi_{\beta,\alpha+\beta+\gamma}b \cdot \phi_{\gamma,\alpha+\beta+\gamma}c \text{ [using (1.7)]} \\
&= \phi_{\alpha\beta\gamma,\alpha+\beta+\gamma}d \text{ [from (1.4)].}
\end{aligned}$$

Since the mapping $\phi_{\alpha\beta\gamma,\alpha+\beta+\gamma}$ is injective, we have $e = d$, i.e.,

$$a \odot (b \odot c) = (a \odot b) \odot c.$$

Finally to prove the right distributivity of the seminearring S, let $a \in S_\alpha$, $b \in S_\beta$ and $c \in S_\gamma$ with $\alpha, \beta, \gamma \in B$. Let

$$d = (a \oplus b) \odot c = (\phi_{\alpha,\alpha+\beta}a + \phi_{\beta,\alpha+\beta}b) \odot c.$$

Then by definition

$$\begin{aligned}
\phi_{(\alpha+\beta)\gamma,\alpha+\beta+\gamma}d &= \phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\alpha,\alpha+\beta}a + \phi_{\beta,\alpha+\beta}b) \cdot \phi_{\gamma,\alpha+\beta+\gamma}c \\
&= (\phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\alpha,\alpha+\beta}a) + \phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\beta,\alpha+\beta}b)) \cdot \phi_{\gamma,\alpha+\beta+\gamma}c \\
&= (\phi_{\alpha,\alpha+\beta+\gamma}a + \phi_{\beta,\alpha+\beta+\gamma}b) \cdot \phi_{\gamma,\alpha+\beta+\gamma}c \\
&= \phi_{\alpha,\alpha+\beta+\gamma}a \cdot \phi_{\gamma,\alpha+\beta+\gamma}c + \phi_{\beta,\alpha+\beta+\gamma}b \cdot \phi_{\gamma,\alpha+\beta+\gamma}c..................(1.8) \\
&[\quad \text{using right distributivity of seminearring } S_{\alpha+\beta+\gamma}].
\end{aligned}$$

Let $e = a \odot c$ and $f = b \odot c$. Then by definition

$$\phi_{\alpha\gamma,\alpha+\gamma}e = \phi_{\alpha,\alpha+\gamma}a \cdot \phi_{\gamma,\alpha+\gamma}c.............(1.9)$$

and

$$\phi_{\beta\gamma,\beta+\gamma}f = \phi_{\beta,\beta+\gamma}b \cdot \phi_{\gamma,\beta+\gamma}c.................(1.10).$$

Applying $\phi_{\alpha+\gamma,\alpha+\beta+\gamma}$ in both side of (1.9), we get

$$\phi_{\alpha\gamma,\alpha+\beta+\gamma}e = \phi_{\alpha,\alpha+\beta+\gamma}a \cdot \phi_{\gamma,\alpha+\beta+\gamma}c................(1.11)$$

and applying $\phi_{\beta+\gamma,\alpha+\beta+\gamma}$ in both side of (1.10) we get

$$\phi_{\beta\gamma,\alpha+\beta+\gamma}f = \phi_{\beta,\alpha+\beta+\gamma}b \cdot \phi_{\gamma,\alpha+\beta+\gamma}c................(1.12).$$

Using above two results in (1.8) we get

$$\phi_{(\alpha+\beta)\gamma,\alpha+\beta+\gamma}d = \phi_{\alpha\gamma,\alpha+\beta+\gamma}e + \phi_{\beta\gamma,\alpha+\beta+\gamma}f...........(1.13).$$

Now

$$e \oplus f = \phi_{\alpha\gamma,\alpha\gamma+\beta\gamma}e + \phi_{\beta\gamma,\alpha\gamma+\beta\gamma}f$$

Applying $\phi_{(\alpha+\beta)\gamma,\alpha+\beta+\gamma}$ both side we get,

$$\begin{aligned}\phi_{(\alpha+\beta)\gamma,\alpha+\beta+\gamma}(e \oplus f) &= \phi_{\alpha\gamma,\alpha+\beta+\gamma}e + \phi_{\beta\gamma,\alpha+\beta+\gamma}f \\ &= \phi_{(\alpha+\beta)\gamma,\alpha+\beta+\gamma}d \text{ [using (1.13)]}.\end{aligned}$$

Since $\phi_{(\alpha+\beta)\gamma,\alpha+\beta+\gamma}$ is injective, $d = e \oplus f$ whence

$$(a \oplus b) \odot c = a \odot c \oplus b \odot c.$$

□

Example 2.1.6. Let N be a zero symmetric near-ring and D be a distributive lattice. Let $S = N \times D$ be the seminearring direct product of N and D. Clearly S is a ditributive lattice D of near-rings $\{S_\alpha : \alpha \in D\}$ where each S_α =$\{(x, \alpha) : x \in N\}$. Let $\alpha, \beta \in D$ such that $\alpha \leq \beta$, we define $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ by

$$\begin{aligned}\phi_{\alpha,\beta}(a, \alpha) &= (a, \alpha) + (0, \beta), \ (0, \beta) \text{ is the zero of the near-ring } S_\beta \\ &= (a + 0, \alpha + \beta) \\ &= (a, \beta)[\text{As } \alpha \leq \beta, \alpha + \beta = \beta]\end{aligned}$$

We first show that $\phi_{\alpha,\beta}$ is injective. For this purpose, let $(a, \alpha), (b, \alpha) \in S_\alpha$ such that

$$\begin{aligned} & \phi_{\alpha,\beta}(a, \alpha) = \phi_{\alpha,\beta}(b, \alpha) \\ \Rightarrow \quad & (a, \alpha) + (0, \beta) = (b, \alpha) + (0, \beta) \\ \Rightarrow \quad & (a, \beta) = (b, \beta) \\ \Rightarrow \quad & a = b\end{aligned}$$

which shows that $\phi_{\alpha,\beta}$ is injective.

To show that $\phi_{\alpha,\beta}$ is a monomorphism, let $(a,\alpha),(b,\alpha) \in S_\alpha$. Then,

$$\begin{aligned}
\phi_{\alpha,\beta}(a,\alpha)+\phi_{\alpha,\beta}(b,\alpha) &= (a,\beta)+(b,\beta)\\
&= (a+b,\beta+\beta)\\
&= (a+b,\beta)\\
&= (a+b+0,\alpha+\beta) \text{ [As } \alpha\leq\beta, \text{ so } \alpha+\beta=\beta]\\
&= (a+b,\alpha)+(0,\beta)\\
&= \phi_{\alpha,\beta}(a+b,\alpha)
\end{aligned}$$

Also,

$$\begin{aligned}
\phi_{\alpha,\beta}(a,\alpha)\phi_{\alpha,\beta}(b,\alpha) &= (a,\beta)(b,\beta)\\
&= (ab,\beta)\\
&= \phi_{\alpha,\beta}(ab,\alpha)
\end{aligned}$$

Thus we have prove that $\phi_{\alpha,\beta}$ is a monomorphism.

Clearly $\phi_{\alpha,\alpha}=I_{S_\alpha}$. Now let $\alpha\leq\beta\leq\gamma$. To show $\phi_{\beta,\gamma}\phi_{\alpha,\beta}=\phi_{\alpha,\gamma}$, let $(a,\alpha)\in S_\alpha$. Then,

$$\begin{aligned}
\phi_{\beta,\gamma}\phi_{\alpha,\beta}(a,\alpha) &= \phi_{\beta,\gamma}(a,\beta)\\
&= (a,\gamma)\\
&= \phi_{\alpha,\gamma}(a,\alpha)
\end{aligned}$$

Let $\alpha,\beta,\gamma\in D$ with $\alpha+\beta\leq\gamma$. Then for $(a,\alpha)\in S_\alpha, (b,\beta)\in S_\beta$, we have

$$\begin{aligned}
\phi_{\alpha,\gamma}(a,\alpha)\phi_{\beta,\gamma}(b,\beta) &= ((a,\alpha)+(0,\gamma))((b,\beta)+(0,\gamma))\\
&= (a,\alpha+\gamma)(b,\beta+\gamma)\\
&= (a,\gamma)(b,\gamma) \text{ [As } \alpha+\beta\leq\gamma, \text{ so } \alpha\leq\gamma \text{ and } \beta\leq\gamma]\\
&= (ab,\gamma)\\
&= (ab+0,\alpha\beta+\gamma) \text{ [} \because \alpha+\beta\leq\gamma, \text{ so } \alpha\beta\leq\gamma]\\
&= (ab,\alpha\beta)+(0,\gamma)\\
&= \phi_{\alpha\beta,\gamma}(ab,\alpha\beta)
\end{aligned}$$

which shows $\phi_{\alpha,\gamma}S_\alpha\phi_{\beta,\gamma}S_\beta \subseteq \phi_{\alpha\beta,\gamma}S_{\alpha\beta}$ if $\alpha+\beta \leq \gamma$. Thus we prove that S is a strong distributive lattice of near rings.

Remark 2.1.7. Let S be a left Clifford seminearring (*cf.* Definition 1.5.30) (left Cllifford seminearring in which $a+a^0b=a$ for all $a,b\in S$). Then in view of Corollary 1.5.33 (Theorem 1.5.36), S is a bi-semilattice (distributive lattice) of near-rings. Suppose S is a bi-semilattice (distributive lattice) (*cf.* Definition 1.2.4 and 1.2.3) B of near-rings $N_\alpha(\alpha\in B)$. Now in view of Theorem 1.5.32, $\mathcal{H}^+$ is a congruence on S with each $\mathcal{H}^+_a$ $(a\in S)$ is a near-ring and the additive identity $0_{\mathcal{H}^+_a}$ of $\mathcal{H}^+_a$ is both additively and multiplicatively central in $E^+(S)$. Let $a\in S$. Then $a\in N_\alpha$ for some $\alpha\in B$. Let $b\in N_\beta$ for some $\beta\in B$ with $\alpha\neq\beta$. Then a is not $\mathcal{H}^+$ related to b otherwise $0_{\mathcal{H}^+_a}=0_{\mathcal{H}^+_b}$ which is absurd. On the other hand $a\ \mathcal{H}^+\ x$ for all $x\in N_\alpha$. Thus $N_\alpha=\mathcal{H}^+_a$. Hence for each $a\in S$, $\mathcal{H}^+_a$ can be seen as N_α where $a\in N_\alpha$ for some $\alpha\in B$. Consequently, $S/\mathcal{H}^+$ and B are isomorphic as bi-semilattices via the map $\mathcal{H}^+_s\mapsto N_\beta$ (where $s\in N_\beta$ for some $\beta\in B$) $\mapsto\beta$ for any $\mathcal{H}^+_s\in S/\mathcal{H}^+$. In such case we often write $S/\mathcal{H}^+=B$. Again let S be a right Clifford Seminearring (*cf.* Definition 1.5.30) (resp. right Clifford Seminearring in which $a+a^0b=a$ for all a, $b\in S$). Then in view of Corollary 1.5.35 (Theorem 1.5.37) S is a bi-semilattice (distributive lattice) of zero-symmetric near-rings. Suppose S is a bi-semilattice (distributive lattice) B of zero-symmetric near-rings $N_\alpha(\alpha\in B)$. Now in view of Theorem 1.5.34, $\mathcal{H}^+$ is a congruence on S with each $\mathcal{H}^+_a(a\in S)$ is a zero-symmetric near-ring and the additive identity $0_{\mathcal{H}^+_a}$ of $\mathcal{H}^+_a$ is both additively and multiplicatively central in $E^+(S)$. $N_\alpha=\mathcal{H}^+_a$ holds in this case, too. Hence for each $a\in S$, $\mathcal{H}^+_a$ can be seen as N_α where $a\in N_\alpha$ for some $\alpha\in B$. Consequently, $S/\mathcal{H}^+$ and B are isomorphic as bi-semilattices via the map $\mathcal{H}^+_s\mapsto N_\beta$ (where $s\in N_\beta$ for some $\beta\in B$) $\mapsto\beta$ for any $\mathcal{H}^+_s\in S/\mathcal{H}^+$. In such case we often write $S/\mathcal{H}^+=B$.

Lemma 2.1.8. *Let S be a bi-semilattice B of near rings N_α (cf. Definition 1.5.20) $(\alpha\in B)$. Then*

(1) $a+b\in N_\beta$ whenever $\alpha\leq\beta,\ a\in N_\alpha,\ b\in N_\beta$;

(2) $0_\alpha+0_\beta=0_\beta$ whenever $\alpha\leq\beta$;

(3) $0_\alpha\cdot 0_\beta=0_{\alpha\beta}$.

Proof. (1) Let S be a bi-semilattice B of near-rings $N_\alpha(\alpha \in B)$. Then S is a left clifford seminearring and equivalently $\mathcal{H}^+$ is a congruence on S with each $\mathcal{H}^+$ class a near-ring (*cf.* Remark 2.1.7). Let $a \in N_\alpha$ and $b \in N_\beta$ where α, $\beta \in B$ with $\alpha \leq \beta$. Then $[a]_{\mathcal{H}^+} = N_\alpha$ and $[b]_{\mathcal{H}^+} = N_\beta$. Now $\{[s]_{\mathcal{H}^+} | s \in S\}$ forms a bi-semilattice.

So,

$$\alpha \leq \beta$$

$$\Rightarrow \quad \alpha + \beta = \beta \ (\textit{cf.}\ \text{Notation 2.1.2})$$

$$\Rightarrow \quad [a]_{\mathcal{H}^+} + [b]_{\mathcal{H}^+} = [b]_{\mathcal{H}^+}, \ \text{ where } a \in N_\alpha, b \in N_\beta$$

(in view of the isomorphism between $S/\mathcal{H}^+$ and B as stated in Remark 2.1.7)

$$\Rightarrow \quad [a+b]_{\mathcal{H}^+} = [b]_{\mathcal{H}^+}$$

$$\Rightarrow \quad a + b \in N_\beta$$

(2) It follows from the previous part that $0_\alpha + 0_\beta \in N_\beta$. Now,

$$\begin{aligned} 0_\alpha + 0_\beta + 0_\alpha + 0_\beta &= 0_\alpha + 0_\alpha + 0_\beta + 0_\beta \quad (\because (S,+) \text{ is clifford}) \\ &= 0_\alpha + 0_\beta \\ \Rightarrow 0_\alpha + 0_\beta &\in N_\beta \cap E^+(S) \\ \Rightarrow 0_\alpha + 0_\beta &= 0_\beta \end{aligned}$$

(3) Clearly $0_\alpha \cdot 0_\beta \in N_{\alpha\beta}$. Now,

$$\begin{aligned} 0_\alpha \cdot 0_\beta &= (0_\alpha + 0_\alpha) \cdot 0_\beta \\ &= 0_\alpha \cdot 0_\beta + 0_\alpha \cdot 0_\beta \\ \Rightarrow 0_\alpha \cdot 0_\beta &\in N_{\alpha\beta} \cap E^+(S) \\ \Rightarrow 0_\alpha \cdot 0_\beta &= 0_{\alpha\beta} \end{aligned}$$

□

Definition 2.1.9. A seminearring $(S, +, .)$ is said to be E^+-unitary if whenever $a + b \in E^+(S)$, a, $b \in S$, then $a \in E^+(S)$ if and only if $b \in E^+(S)$.

Example 2.1.10. Let $(S, +)$ be an E-unitary (*cf.* Definition 1.1.44) semigroup. Then $M(S)$, the seminearring of self maps is E^+-unitary.

Proof. Let $f, g \in M(S)$ such that $f + g \in E^+(M(S))$ and $f \in E^+(M(S))$. Then

$$(f + g) + (f + g) = f + g \text{ and } f + f = f.$$

Let $a \in S$. Then

$$f(a) + f(a) = f(a) \text{ and } (f(a) + g(a)) + (f(a) + g(a)) = f(a) + g(a).$$

This implies $f(a), f(a) + g(a) \in E(S)$. Since $(S, +)$ is E-unitary, $g(a) \in E(S)$. Since $a \in S$ is arbitrary, $g(a) \in E(S)$ for all $a \in S$. Thus $g \in E^+(M(S))$. $\square$

Proposition 2.1.11. *Let S be a strong bi-semilattice of near-rings. Then S is a left Clifford E^+-unitary seminearring such that $a(b + e) = ab + a^0e$ (cf. Notations 1.5.14(iii)) for all a, $b \in S$ and for all $e \in E^+(S)$.*

Proof. Let $S =< B, N_\alpha, \phi_{\alpha,\beta} >$ be a strong bi-semilattice B of near rings $N_\alpha(\alpha \in B)$. Let us define a relation ρ on S by

$$a\rho b \text{ if and only if } a, b \in N_\alpha \text{ for some } \alpha \in B.$$

Then clearly ρ is reflexive and symmetric. To prove ρ is transitive, let $a\rho b$ and $b\rho c$. Then $a, b \in N_\alpha$ for some $\alpha \in B$ and $b, c \in N_\beta$ for some $\beta \in B$. This implies $b \in N_\alpha \cap N_\beta$. Therefore $N_\alpha = N_\beta$ ($\because \{N_\alpha | \alpha \in B\}$ is a disjoint family of near-rings). Hence $a, c \in N_\alpha$, whence $a\rho c$. Thus ρ is an equivalence relation on S. Now let $a\rho b$ and $c \in S$. Then $a, b \in N_\alpha$ for some $\alpha \in B$ and $c \in N_\gamma$ for some $\gamma \in B$.

Therefore

$$a \oplus c = \phi_{\alpha,\alpha+\gamma}(a) + \phi_{\gamma,\alpha+\gamma}(c) \in N_{\alpha+\gamma} \text{ and } b \oplus c = \phi_{\alpha,\alpha+\gamma}(b) + \phi_{\gamma,\alpha+\gamma}(c) \in N_{\alpha+\gamma}$$

whence $(a \oplus c)\ \rho\ (b \oplus c)$. Similarly, $(c \oplus a)\ \rho\ (c \oplus b)$. Again $a \odot c \in N_{\alpha\gamma}$ and $b \odot c \in N_{\alpha\gamma}$ implies that $(a \odot c)\ \rho\ (b \odot c)$. Similarly, $(c \odot a)\ \rho\ (c \odot b)$. Hence ρ is a congruence on S such that each ρ class is a near-ring. Let $x, y \in S$. Then $x \in N_\alpha$ and $y \in N_\beta$ for some $\alpha, \beta \in B$.

$x \oplus x = \phi_{\alpha,\alpha}(x) + \phi_{\alpha,\alpha}(x) \in N_\alpha$ and $x \odot x \in N_{\alpha^2} = N_\alpha$ ($\because (B, +, \cdot)$ is a bi-semilattice)

Therefore $x\rho x \oplus x$ and $x\rho x \odot x$. Also,

$$x \oplus y = \phi_{\alpha,\alpha+\beta}(x) + \phi_{\beta,\alpha+\beta}(y) \in N_{\alpha+\beta} \text{ and}$$
$$y \oplus x = \phi_{\beta,\beta+\alpha}(y) + \phi_{\alpha,\beta+\alpha}(x) \in N_{\beta+\alpha} = N_{\alpha+\beta}\ (\because (B, +, \cdot) \text{ is a bi-semilattice})$$

which shows that $(x \oplus y)\rho(y \oplus x)$. Again $x \odot y \in N_{\alpha\beta}$ and $y \odot x \in N_{\beta\alpha} = N_{\alpha\beta}$. Therefore $(x \odot y)\rho(y \odot x)$. Whence S is a bi-semilattice of near-rings. Thus S is a left Clifford seminearring (*cf.* Corollary 1.5.33). Let $a, b \in S$ and $e \in E^+(S)$. Then $a \in N_\alpha$, $b \in N_\beta$ and $e \in N_\gamma$ for some $\alpha, \beta, \gamma \in B$. Now the following relations hold:

$$\begin{aligned}
\alpha &\leq \alpha + \beta + \gamma \\
\beta &\leq \alpha + \beta + \gamma \\
\gamma &\leq \alpha + \beta + \gamma \\
\alpha\beta &\leq \alpha + \beta + \gamma \quad \text{(Using Result 2.1.3)} \\
\beta + \gamma &\leq \alpha + \beta + \gamma \\
\alpha\gamma &\leq \alpha + \beta + \gamma \quad \text{(Using Result 2.1.3)} \\
\alpha(\beta + \gamma) &\leq \alpha + \beta + \gamma \quad \text{(Using Result 2.1.3)}
\end{aligned}$$

for all $\alpha, \beta, \gamma \in B$. So the corresponding mappings such as $\phi_{\alpha,\alpha+\beta+\gamma}$, $\phi_{\beta,\alpha+\beta+\gamma}$, $\phi_{\gamma,\alpha+\beta+\gamma}$, $\phi_{\alpha\beta,\alpha+\beta}$, $\phi_{\alpha+\beta,\alpha+\beta+\gamma}$, $\phi_{\beta+\gamma,\alpha+\beta+\gamma}$, $\phi_{\alpha\gamma,\alpha+\beta+\gamma}$ and $\phi_{\alpha(\beta+\gamma),\alpha+\beta+\gamma}$ are meaningful. It is known that $E^+(S) = \{0_\alpha | \alpha \in B\}$. Therefore $e = 0_\gamma$. Let $a \odot (b \oplus 0_\gamma) = c$. Then $c \in N_{\alpha(\beta+\gamma)}$. Now

$$\begin{aligned}
\phi_{\alpha(\beta+\gamma),\alpha+\beta+\gamma}(c) &= \phi_{\alpha,\alpha+\beta+\gamma}(a) \cdot \phi_{\beta+\gamma,\alpha+\beta+\gamma}(b \oplus 0_\gamma) \\
&= \phi_{\alpha,\alpha+\beta+\gamma}(a) \cdot \phi_{\beta+\gamma,\alpha+\beta+\gamma}(\phi_{\beta,\beta+\gamma}(b) + \phi_{\gamma,\beta+\gamma}(0_\gamma)) \\
&(\because b \oplus 0_\gamma = \phi_{\beta,\beta+\gamma}(b) + \phi_{\gamma,\beta+\gamma}(0_\gamma)) \\
&= \phi_{\alpha,\alpha+\beta+\gamma}(a) \cdot (\phi_{\beta,\alpha+\beta+\gamma}(b) + \phi_{\gamma,\alpha+\beta+\gamma}(0_\gamma)) \\
&= \phi_{\alpha,\alpha+\beta+\gamma}(a) \cdot \phi_{\beta,\alpha+\beta+\gamma}(b) \ (\because \phi_{\gamma,\alpha+\beta+\gamma}(0_\gamma) = 0_{\alpha+\beta+\gamma}) \ ....(2).
\end{aligned}$$

Again let $a \odot b = d \in N_{\alpha\beta}$. Then

$$\phi_{\alpha\beta,\alpha+\beta}(d) = \phi_{\alpha,\alpha+\beta}(a) \cdot \phi_{\beta,\alpha+\beta}(b) \ ....(3).$$

Now

$$\begin{aligned}
x &= d \oplus 0_{\alpha\gamma} \\
&= \phi_{\alpha\beta,\alpha\beta+\alpha\gamma}(d) + \phi_{\alpha\gamma,\alpha\beta+\alpha\gamma}(0_{\alpha\gamma}) \ ....(4).
\end{aligned}$$

Now applying $\phi_{\alpha\beta+\alpha\gamma,\alpha+\beta+\gamma}$ to both side of (4) we get,

$$\begin{aligned}
\phi_{\alpha\beta+\alpha\gamma,\alpha+\beta+\gamma}(x) &= \phi_{\alpha\beta+\alpha\gamma,\alpha+\beta+\gamma}(\phi_{\alpha\beta,\alpha\beta+\alpha\gamma}(d)+\phi_{\alpha\gamma,\alpha\beta+\alpha\gamma}(0_{\alpha\gamma}))\\
&= \phi_{\alpha\beta,\alpha+\beta+\gamma}(d)+\phi_{\alpha\gamma,\alpha+\beta+\gamma}(0_{\alpha\gamma})\\
&= \phi_{\alpha\beta,\alpha+\beta+\gamma}(d)\ (\because \phi_{\alpha\gamma,\alpha+\beta+\gamma}(0_{\alpha\gamma}) = 0_{\alpha+\beta+\gamma})\ \ldots\ldots\ldots\ldots\ldots..(5).
\end{aligned}$$

Now applying $\phi_{\alpha+\beta,\alpha+\beta+\gamma}$ to both side of (3) we get,

$$\begin{aligned}
\phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\alpha\beta,\alpha+\beta}(d)) &= \phi_{\alpha+\beta,\alpha+\beta+\gamma}(\phi_{\alpha,\alpha+\beta}(a)\cdot\phi_{\beta,\alpha+\beta}(b))\\
\Rightarrow \qquad \phi_{\alpha\beta,\alpha+\beta+\gamma}(d) &= \phi_{\alpha,\alpha+\beta+\gamma}(a)\cdot\phi_{\beta,\alpha+\beta+\gamma}(b)\ \ \ldots\ldots\ldots\ldots\ldots\ldots..(6).
\end{aligned}$$

From (5) and (6) we obtain the following

$$\phi_{\alpha(\beta+\gamma),\alpha+\beta+\gamma}(x) = \phi_{\alpha,\alpha+\beta+\gamma}(a)\cdot\phi_{\beta,\alpha+\beta+\gamma}(b) = \phi_{\alpha(\beta+\gamma),\alpha+\beta+\gamma}(c)\ \text{(Using (2))}$$

whence $x = c$ using injectivity of $\phi_{\alpha(\beta+\gamma),\alpha+\beta+\gamma}$. Hence in S,

$$\begin{aligned}
a\odot(b\oplus e) &= a\odot b\oplus 0_{\alpha\gamma}\\
&= a\odot b\oplus 0_{\alpha}\odot 0_{\gamma}\\
&= a\odot b\oplus a^0\odot e
\end{aligned}$$

Now to prove $(E^+(S),\oplus)$ is E-unitary. Let $a\oplus b\in E^+(S)$ and $b\in E^+(S)$. It is known that $E^+(S)=\{0_\alpha|\alpha\in B\}$. So, $b=0_\gamma$ for some $\gamma\in B$.
Case I : Let $a\in N_\gamma$. Then $a+0_\gamma+a+0_\gamma = a+0_\gamma \Rightarrow a+a=a$ [$\because a\in N_\gamma$ and 0_γ is the zero element of N_γ] $\Rightarrow a\in E^+(S)$
Case II : Let $a\in N_\beta$ for some $\beta\neq\gamma$ in B. Therefore,

$$\begin{aligned}
a\oplus 0_\gamma &= \phi_{\beta,\beta+\gamma}(a)+\phi_{\gamma,\beta+\gamma}(0_\gamma)\\
&= \phi_{\beta,\beta+\gamma}(a)+0_{\beta+\gamma}\in N_{\beta+\gamma}\cap E^+(S)
\end{aligned}$$

whence

$$\begin{aligned}
\phi_{\beta,\beta+\gamma}(a)+0_{\beta+\gamma} &= 0_{\beta+\gamma}\\
\Rightarrow \qquad \phi_{\beta,\beta+\gamma}(a) &= 0_{\beta+\gamma}\\
\Rightarrow \qquad a &= 0_\beta\ \text{(since } \phi_{\beta,\beta+\gamma} \text{ is one-one)}\\
\Rightarrow \qquad a\in E^+(S)
\end{aligned}$$

So, S is E^+-unitary, which completes the proof.

$\square$

Remark 2.1.12. The above result shows that the following three conditions are necessary for a seminearring S to be a strong bi-semilattice of near-rings:

(i) it is a bi-semilattice of near-rings (i.e., left Clifford),

(ii) it is E^+-unitary seminearing, and

(iii) $a(b+e) = ab + a^0e$ for all a, $b \in S$ and for all $e \in E^+(S)$.

It is natural to see if these are also sufficient for a seminearring S to be a strong bi-semilattice of near-rings which is shown to be true in Proposition 2.1.14. But one question still remains − which is, if conditions (ii) and (iii) are inherent in a left Clifford seminearrings (i.e., in a bi-semilattice of near-rings). The following example shows that this is not true.

Example 2.1.13. Let $S = \{0, a, b\}$ be a seminearring with the following Cayley tables :

+	0	a	b
0	0	a	b
a	a	a	a
b	b	a	0

.	0	a	b
0	0	0	0
a	0	a	0
b	0	b	0

Let $a, b, c \in S$. If at least one of a, b, c is the zero element then clearly $a + (b + c) = (a + b) + c$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$. Suppose none of a, b, c is zero. Then we have

- $a + (a + b) = a + a = a = a + b = (a + a) + b$
- $a + (b + a) = a + a = a = a + a = (a + b) + a$
- $a + (b + b) = a + 0 = a = a + b = (a + b) + b$
- $b + (a + b) = b + a = a = a + b = (b + a) + b$
- $b + (b + a) = b + a = a = 0 + a = (b + b) + a$
- $b + (a + a) = b + a = a = a + a = (b + a) + a$
- $a \cdot (a \cdot b) = a \cdot 0 = 0 = a \cdot b = (a \cdot a) \cdot b$

- $a \cdot (b \cdot a) = a \cdot b = 0 = 0 \cdot a = (a \cdot b) \cdot a$
- $a \cdot (b \cdot b) = a \cdot 0 = 0 = 0 \cdot b = (a \cdot b) \cdot b$
- $b \cdot (a \cdot b) = b \cdot 0 = 0 = b \cdot b = (b \cdot a) \cdot b$
- $b \cdot (b \cdot a) = b \cdot b = 0 = 0 \cdot a = (b \cdot b) \cdot a$
- $b \cdot (a \cdot a) = b \cdot a = b = b \cdot a = (b \cdot a) \cdot a$
- $(a + a) \cdot b = a \cdot b = 0 = 0 + 0 = a \cdot b + a \cdot b$
- $(a + b) \cdot b = a \cdot b = 0 = 0 + 0 = a \cdot b + b \cdot b$
- $(b + a) \cdot b = a \cdot b = 0 = 0 + 0 = b \cdot b + a \cdot b$
- $(b + b) \cdot b = 0 \cdot b = 0 = 0 + 0 = b \cdot b + b \cdot b$
- $(a + b) \cdot a = a \cdot a = a = a + b = a \cdot a + b \cdot a$
- $(b + a) \cdot a = a \cdot a = a = b + a = b \cdot a + a \cdot a$
- $(b + b) \cdot a = 0 \cdot a = 0 = b + b = b \cdot a + b \cdot a$
- $(a + a) \cdot a = a \cdot a = a = a + a = a \cdot a + a \cdot a.$

Thus $(S, +, \cdot)$ is a seminearing. But $(S, +, \cdot)$ is not left distributive since

$$b \cdot (a + a) = b \cdot a = b \neq 0 = b + b = b \cdot a + b \cdot a.$$

$(S, +, \cdot)$ is additively completely regular since $a + a + a = a$, $b + b + b = b$. Also $(a + a) \cdot a = a + a$ and $(b + b) \cdot b = b + b$. Now, $E^+(S) = \{0, a\}$. Since

- $a \cdot 0 = 0 \cdot a = 0$
- $b \cdot 0 = 0 \cdot b = 0$
- $(b \cdot a =) b \mathcal{J}^+ 0 (= a \cdot b)$ ($\because$ $0 = 0 + b + b$ and $b = b + 0 + 0$)

$(S, +, \cdot)$ is left completely regular and additive idempotents additively commute with each other. So $(S, +, \cdot)$ is left Clifford. But $(S, +, \cdot)$ is not E^+-unitary because

$$a + b \in E^+(S),\ a \in E^+(S) \text{ but } b \notin E^+(S).$$

Routine verification shows that $b^* = b$. Then

$$\begin{aligned} b^0 &= b + b^* \\ &= b + b \\ &= 0. \end{aligned}$$

But

$$\begin{aligned} b \cdot b + b^0 \cdot a &= 0 + 0 \\ &= 0 \\ &\neq b \\ &= b \cdot a \\ &= b \cdot (b + a). \end{aligned}$$

Hence $(S, +, \cdot)$ does not satisfy the condition (iii) of Remark 2.1.12.

Proposition 2.1.14. *Let S be a left Clifford E^+-unitary seminearring such that $a(b+e) = ab + a^0e$ for all a, $b \in S$ and for all $e \in E^+(S)$. Then S ia a strong bi-semilattice of near rings.*

Proof. In view of Remark 2.1.7, S is a bi-semilattice $B = S/\mathcal{H}^+$ of near rings $N_\alpha = [a]_{\mathcal{H}^+}$ where $a \in S$ and $\alpha \in B$. For $\alpha, \beta \in B$ with $\alpha \leq \beta$ let us define maps

$$\phi_{\alpha,\beta} : N_\alpha \to N_\beta \text{ by } \phi_{\alpha,\beta}(a) = a + 0_\beta.$$

In view of Lemma 2.1.8 the definition of the map is meaningful. Now let $a, b \in N_\alpha$ and $\alpha \leq \beta$. Then

$$\begin{aligned} \phi_{\alpha,\beta}(a) + \phi_{\alpha,\beta}(b) &= (a + 0_\beta) + (b + 0_\beta) \\ &= a + \{0_\beta + (b + 0_\beta)\} \\ &= a + (b + 0_\beta) + 0_\beta \ (\because (S, +) \text{ is clifford}) \\ &= a + b + 0_\beta \ (\because 0_\beta, (b + 0_\beta) \in N_\beta \ (cf. \text{ Lemma 2.1.8})) \\ &= \phi_{\alpha,\beta}(a + b) \end{aligned}$$

and

$$\begin{aligned} \phi_{\alpha,\beta}(a) \cdot \phi_{\alpha,\beta}(b) &= (a + 0_\beta) \cdot (b + 0_\beta) \\ &= a(b + 0_\beta) + 0_\beta(b + 0_\beta) \\ &= a(b + 0_\beta) + 0_\beta \ (\because 0_\beta, (b + 0_\beta) \in N_\beta \ ). \end{aligned}$$

By the hypothesis

$$\begin{aligned} a(b+0_\beta) &= ab + a^0 0_\beta \\ &= ab + 0_\alpha 0_\beta \quad (a^0 \in N_\alpha \cap E^+(S) = \{0_\alpha\}) \\ &= ab + 0_{\alpha\beta} \quad (cf. \text{ Lemma 2.1.8 }) \end{aligned}$$

which gives

$$\begin{aligned} \phi_{\alpha,\beta}(a)\cdot\phi_{\alpha,\beta}(b) &= ab + 0_{\alpha\beta} + 0_\beta \\ &= ab + 0_\beta \ (cf. \text{ Lemma 2.1.8}) \\ &= \phi_{\alpha,\beta}(ab). \end{aligned}$$

Thus $\phi_{\alpha,\beta}$ becomes seminearring morphism. Again let $a, b \in N_\alpha$ such that

$$\begin{aligned} \phi_{\alpha,\beta}(a) &= \phi_{\alpha,\beta}(b), \text{ for some } \beta \in B \text{ with } \alpha \leq \beta. \\ i.e., \ a + 0_\beta &= b + 0_\beta \\ i.e., \ b^* + a + 0_\beta &= b^* + b + 0_\beta \\ i.e., \ b^* + a + 0_\beta &= 0_\alpha + 0_\beta \ (\because b^* + b \in E^+(S) \cap N_\alpha = \{0_\alpha\}) \\ &= 0_\beta \ (\text{Using Lemma 2.1.8}) \end{aligned}$$

Therefore $b^* + a + 0_\beta \in E^+(S)$. Since S is E^+-unitary, $b^* + a \in E^+(S)$. As $b \in N_\alpha$ and N_α is a near-ring, $b^* \in N_\alpha$. So $b^* + a \in E^+(S) \cap N_\alpha$. Then $b^* + a = 0_\alpha = a^* + a$. Thus $b^* = a^*$ as N_α is a near ring. We then obtain

$$\begin{aligned} a &= a + a^* + a \\ &= b + b^* + a \ (a + a^* = 0_\alpha = b + b^*) \\ &= b + 0_\alpha \\ &= b \ (\because b \in N_\alpha) \end{aligned}$$

whence $\phi_{\alpha,\beta}$ is a monomorphism. Clearly, $\phi_{\alpha,\alpha}$ is I_{N_α}, the identity morphism on N_α. Now let $\alpha, \beta, \gamma \in B$ with $\alpha \leq \beta \leq \gamma$. Let $a \in N_\alpha, b \in N_\beta$. Then

$$\begin{aligned} \phi_{\beta,\gamma}\phi_{\alpha,\beta}(a) &= \phi_{\beta,\gamma}(a + 0_\beta) \\ &= a + 0_\beta + 0_\gamma \\ &= a + 0_\gamma \ (cf. \text{ Lemma 2.1.8}) \\ &= \phi_{\alpha,\gamma}(a) \end{aligned}$$

i.e., $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$. Again let $\alpha, \beta, \gamma \in B$ with $\alpha + \beta \leq \gamma$ and $a \in N_\alpha$, $b \in N_\beta$. Again by hypothesis

$$\begin{aligned} a(b+0_\gamma) &= ab + a^0 0_\gamma \\ &= ab + 0_\alpha 0_\gamma \quad (a^0 \in N_\alpha \cap E^+(S) = \{0_\alpha\}) \\ &= ab + 0_{\alpha\gamma} \ \ldots\ldots\ldots\ldots\ldots\ldots(1) \end{aligned}$$

Now

$$\begin{aligned} \phi_{\alpha,\gamma}(a) \cdot \phi_{\beta,\gamma}(b) &= (a+0_\gamma)(b+0_\gamma) \\ &= a(b+0_\gamma) + 0_\gamma(b+0_\gamma) \\ &= a(b+0_\gamma) + 0_\gamma \ (\because 0_\gamma, (b+0_\gamma) \in N_\gamma\ ) \\ &= ab + 0_{\alpha\gamma} + 0_\gamma \ (\text{Using } (1)) \\ &= ab + 0_\gamma \ (\textit{cf.}\ \text{Lemma 2.1.8}) \\ &= \phi_{\alpha\beta,\gamma}(ab) \end{aligned}$$

whence $\phi_{\alpha,\gamma}N_\alpha\phi_{\beta,\gamma}N_\beta \subseteq \phi_{\alpha\beta,\gamma}N_{\alpha\beta}$. Thus S is a strong bi-semilattice of near-rings. $\square$

In the above result we can take the last condition in the hypothesis for a smaller subset of S which is evident from the following result.

Proposition 2.1.15. *Let S be a left Clifford E^+-unitary seminearring such that $a(b+e) = ab + a^0e$ whenever $a^0 + e = e$ for a, $b \in S$ and $e \in E^+(S)$. Then S is a strong bi-semilattice of near-rings.*

Proof. In view of Remark 2.1.7, S is a bi-semilattice $B = S/\mathcal{H}^+$ of near-rings $N_\alpha = [a]_{\mathcal{H}^+}$ where $a \in S$ and $\alpha \in B$. For $\alpha, \beta \in B$ with $\alpha \leq \beta$ let us define maps

$$\phi_{\alpha,\beta} : N_\alpha \to N_\beta \text{ by } \phi_{\alpha,\beta}(a) = a + 0_\beta.$$

In view of Lemma 2.1.8 the definition of the map is meaningful. In a similar manner as shown in the proof of Proposition 2.1.14 here also $\phi_{\alpha,\beta}(a)+\phi_{\alpha,\beta}(b) = \phi_{\alpha,\beta}(a+b)$ for all $a, b \in N_\alpha$, $\alpha, \beta \in B$ with $\alpha \leq \beta$. Now

$$\phi_{\alpha,\beta}(a) \cdot \phi_{\alpha,\beta}(b) = a(b + 0_\beta) + 0_\beta \ (\textit{cf.}\ \text{proof of the Proposition 2.1.14})$$

In B,

$$\begin{array}{rc} & \alpha \leq \beta \\ \Rightarrow & [a]_{\mathcal{H}^+} + [0_\beta]_{\mathcal{H}^+} = [0_\beta]_{\mathcal{H}^+} \text{ where } a \in N_\alpha. \\ \Rightarrow & (a+0_\beta), 0_\beta \text{ are in same } \mathcal{H}^+ \text{ class} \\ \Rightarrow & (a+0_\beta)^*, 0_\beta^* \text{ are in same } \mathcal{H}^+ \text{ class} \\ \Rightarrow & (a+0_\beta)+(a+0_\beta)^* = (a+0_\beta)^0, 0_\beta \text{ are in same } \mathcal{H}^+ \text{ class.} \end{array}$$

Therefore $(a+0_\beta)^0$ and 0_β are equal, being elements of $E^+(S)$ ($\because$ each $\mathcal{H}^+$ class is a near-ring). Now

$$\begin{aligned} (a+0_\beta)^0 &= (a+0_\beta)+(a+0_\beta)^* \\ &= a+0_\beta+0_\beta^*+a^* \ (\because (S,+) \text{ is Clifford}) \\ &= a+0_\beta+a^* \\ &= a+a^*+0_\beta \ (\because (S,+) \text{ is Clifford}) \\ &= a^0+0_\beta \\ &= 0_\beta. \end{aligned}$$

Hence by the hypothesis

$$\begin{aligned} a(b+0_\beta) &= ab+a^0 0_\beta \\ &= ab+0_\alpha 0_\beta \quad (a^0 \in N_\alpha \cap E^+(S) = \{0_\alpha\}) \\ &= ab+0_{\alpha\beta} \quad (cf. \text{ Lemma 2.1.8 }) \end{aligned}$$

which gives

$$\phi_{\alpha,\beta}(a) \cdot \phi_{\alpha,\beta}(b) = \phi_{\alpha,\beta}(ab) \ (cf. \text{ proof of the Proposition 2.1.14}).$$

Thus $\phi_{\alpha,\beta}$ becomes a seminearring morphism. In view of proof of the Proposition 2.1.14 $\phi_{\alpha,\beta}$ is a monomorphism for any $\alpha, \beta \in B$ with $\alpha \leq \beta$ as well as $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ for any $\alpha, \beta, \gamma \in B$ with $\alpha \leq \beta \leq \gamma$. Again let $\alpha, \beta, \gamma \in B$ with $\alpha + \beta \leq \gamma$ and $a \in N_\alpha$, $b \in N_\beta$. Then $\alpha \leq \gamma$ and $\beta \leq \gamma$, whence $[a]_{\mathcal{H}^+} \leq [0_\gamma]_{\mathcal{H}^+}$. So $[a]_{\mathcal{H}^+} + [0_\gamma]_{\mathcal{H}^+} = [0_\gamma]_{\mathcal{H}^+}$. Thus $a^0 + 0_\gamma = 0_\gamma$. So by hypothesis

$$\begin{aligned} a(b+0_\gamma) &= ab+a^0 0_\gamma \\ &= ab+0_\alpha 0_\gamma \quad (a^0 \in N_\alpha \cap E^+(S) = \{0_\alpha\}) \\ &= ab+0_{\alpha\gamma} \ \text{.................(1)} \end{aligned}$$

Now

$$\begin{aligned}\phi_{\alpha,\gamma}(a)\cdot\phi_{\beta,\gamma}(b) &= (a+0_\gamma)(b+0_\gamma)\\ &= a(b+0_\gamma)+0_\gamma(b+0_\gamma)\\ &= a(b+0_\gamma)+0_\gamma \ (\because 0_\gamma,(b+0_\gamma)\in N_\gamma\ )\\ &= ab+0_{\alpha\gamma}+0_\gamma \text{ (Using (1))}\\ &= ab+0_\gamma \ (\textit{cf.}\text{ Lemma 2.1.8})\\ &= \phi_{\alpha\beta,\gamma}(ab)\end{aligned}$$

whence $\phi_{\alpha,\gamma}N_\alpha\phi_{\beta,\gamma}N_\beta \subseteq \phi_{\alpha\beta,\gamma}N_{\alpha\beta}$. Thus S is a strong bi-semilattice of near-rings. □

For convenience of future reference we combine Propositions 2.1.11, 2.1.14 and 2.1.15 in the following theorem.

Theorem 2.1.16. *Let S be a seminearring. Then the following are equivalent.*

(1) S ia strong bi-semilattice of near-rings.

(2) S is a left Clifford E^+-unitary seminearring such that $a(b+e) = ab + a^0e$ whenever $a^0 + e = e$ for a, $b \in S$ and $e \in E^+(S)$.

(3) S is a left Clifford E^+-unitary seminearring such that $a(b+e) = ab + a^0e$ for all a, $b \in S$ and for all $e \in E^+(S)$.

The following is the right Clifford analogue of Proposition 2.1.14

Proposition 2.1.17. *Let S be a strong bi-semilattice of zero-symmetric near-rings. Then S is a right Clifford E^+-unitary seminearring such that $a(b+e) = ab + a^0e$ for all a, $b \in S$ and for all $e \in E^+(S)$.*

Proof. Let $S =< B, N_\alpha, \phi_{\alpha,\beta} >$ be a strong bi-semilattice B of zero-symmetric near-rings $N_\alpha(\alpha \in B)$. Then in view of the proof of Proposition 2.1.14, the relation ρ on S defined by

$$a\rho b \text{ if and only if } a, b \in N_\alpha \text{ for some } \alpha \in B,$$

becomes a bi-semilattice congruence with each class a near-ring and S is an E^+-unitary seminearring in which $a\odot(b\oplus e) = a\odot b \oplus a^0\odot e$ for all a, $b \in S$ and for all $e \in E^+(S)$. Since each N_α, $\alpha \in B$ is zero-symmetric, S becomes a bi-semilattice of zero-symmetric near-rings, whence S is a right Clifford seminearring (*cf.* Corollary 1.5.35) . □

Remark 2.1.18. The above result shows that the following three conditions are necessary for a seminearring S to be a strong bi-semilattice of zero-symmetric near-rings:

(i) it is a bi-semilattice of zero-symmetric near-rings (i.e., right Clifford),

(ii) it is E^+-unitary seminearing, and

(iii) $a(b+e) = ab + a^0e$ for all a, $b \in S$ and for all $e \in E^+(S)$.

It is natural to see if these are also sufficient for a seminearring S to be a strong bi-semilattice of zero-symmetric near-rings which is shown to be true in Proposition 2.1.19. But one question still remains − which is, if conditions (ii) and (iii) are inherent in a right Clifford seminearrings (i.e., in a bi-semilattice of zero-symmetric near-rings). The following example shows that this is not true. In the seminearring $(S, +, \cdot)$ of Example 2.1.13,

- $a + a + a = a$,
- $b + b + b = b$,
- $a(a + a) = (a + a)a = a + a$,
- $b(b + b) = (b + b)b = b + b$,

S is also right Clifford. But S does not satisfy the condition (ii) and (iii).

The following is the right Clifford analogue of Proposition 2.1.11

Proposition 2.1.19. *Let S be a right Clifford E^+-unitary seminearring such that $a(b + e) = ab + a^0e$ for all a, $b \in S$ and for all $e \in E^+(S)$. Then S ia a strong bi-semilattice of zero-symmetric near-rings.*

Proof. In view of Remark 2.1.7, S is a bi-semilattice $B = S/\mathcal{H}^+$ of zero-symmetric near-rings $\{N_\alpha : \alpha \in B\}$. The rest follows from the proof of Proposition 2.1.11. □

The following is the right Clifford analogue of Proposition 2.1.15

Proposition 2.1.20. *Let S be a right Clifford E^+-unitary seminearring such that $a(b + e) = ab + a^0e$ whenever $a^0 + e = e$ for a, $b \in S$ and $e \in E^+(S)$. Then S ia a strong bi-semilattice of zero-symmetric near-rings.*

Proof. In view of Remark 2.1.7, S is a bi-semilattice $B = S/\mathcal{H}^+$ of zero-symmetric near-rings $N_\alpha = [a]_{\mathcal{H}^+}$ where $a \in S$ and $\alpha \in B$. The rest follows from the proof of Proposition 2.1.15. $\square$

Now Propositions 2.1.17, 2.1.19 and 2.1.20 together gives the following theorem.

Theorem 2.1.21. *Let S be a seminearring. Then the following are equivalent.*

(1) S is a strong bi-semilattice of zero-symmetric near-rings.

(2) S is a right Clifford E^+-unitary seminearring such that $a(b+e) = ab + a^0e$ whenever $a^0 + e = e$ for a, $b \in S$ and $e \in E^+(S)$.

(3) S is a right Clifford E^+-unitary seminearring such that $a(b+e) = ab + a^0e$ for all a, $b \in S$ and for all $e \in E^+(S)$.

2.2 Strong Distributive Lattice of Near-rings

Now we investigate below what happens if we replace bi-semilattice by distributive lattice in Theorems 2.1.16 and 2.1.21. In this direction we first obtain the following results which is the combined (distributive lattice) analogue of Propositions 2.1.14 and 2.1.17.

Proposition 2.2.1. *Let S be a strong disributive lattice of near-rings (zero-symmetric near-rings). Then S is a left (right) Clifford E^+-unitary seminearring in which $a + a^0b$ $= a$ for all $a, b \in S$ and $c(d+e) = cd + c^0e$ for all c, $d \in S$ and for all $e \in E^+(S)$.*

Proof. Let $S =< D, N_\alpha, \phi_{\alpha,\beta} >$ be a strong distributive lattice D of near-rings (zero-symmetric near-rings) $N_\alpha(\alpha \in D)$. So S is a strong bi-semilattice of near-rings (zero-symmetric near-rings), too. So S is a left (right, respectively) Clifford E^+-unitary seminearring in which $c \odot (d \oplus e) = c \odot d \oplus c^0 \odot e$ for all c, $d \in S$ and for all $e \in E^+(S)$. In view of the proof of Proposition 2.1.14 (respectively 2.1.17), the relation ρ on S defined by

$$a \rho b \text{ if and only if } a, b \in N_\alpha \text{ for some } \alpha \in D,$$

becomes a bi-semilattice congruence with each class a near-ring (zero-symmetric near-ring). Now let x, $y \in S$ such that $x \in N_\alpha$ and $y \in N_\beta$, α, $\beta \in D$. Then $x \oplus x \odot y \in N_{\alpha+\alpha\beta}$

$= N_\alpha$ since D is a distributive lattice. Hence $x \oplus x \odot y \ \rho \ x$. So ρ becomes a distributive lattice congruence which implies that S is a distributive lattice of near-rings (zero-symmetric near-rings). Thus S is a left (right) Clifford seminearring in which $a + a^0 b$ $= a$ for all $a, b \in S$ in view of Theorem 1.5.36 (Theorem 1.5.37). □

The following is the combined (distributive lattice) analogue of Propositions 2.1.11 and 2.1.19.

Proposition 2.2.2. *Let S be a left (right) Clifford E^+-unitary seminearring such that $a+a^0b = a$ for all $a, b \in S$. If $c(d+e) = cd + c^0e$ for all c, $d \in S$ and for all $e \in E^+(S)$, then S is a strong distributive lattice of near-rings (zero-symmetric near-rings).*

Proof. In view of Remark 2.1.7, S is a distributive lattice $D = S/\mathcal{H}^+$ of near-rings (zero-symmetric near-rings) $\{N_\alpha : \alpha \in D\}$ and for any $\alpha \in D$, $N_\alpha = \mathcal{H}^+_a$ for some $a \in S$. The rest of the proof follows from the proof of Proposition 2.1.11 (respectively 2.1.19). □

The following is the combined (distributive lattice) analogue of Propositions 2.1.15 and 2.1.20.

Proposition 2.2.3. *Let S be a left (right) Clifford E^+-unitary seminearring such that $a + a^0b = a$ for all $a, b \in S$. If $c(d + e) = cd + c^0e$ whenever $c^0 + e = e$ for c, $d \in S$ and $e \in E^+(S)$, then S is a strong distributive lattice of near-rings (zero-symmetric near-rings).*

Proof. In view of Remark 2.1.7, S is a distributive lattice $D = S/\mathcal{H}^+$ of near-rings (zero-symmetric near-rings) $\{N_\alpha : \alpha \in D\}$ and for any $\alpha \in D$, $N_\alpha = \mathcal{H}^+_a$ for some $a \in S$. The rest of the proof follows from the proof of Proposition 2.1.11 (respectively 2.1.19). □

Combining Propositions 2.2.1, 2.2.2 and 2.2.3 we obtain the following theorem.

Theorem 2.2.4. *Let S be a seminearring. Then the following are equivalent.*

(1) S ia strong distributive lattice of near-rings (zero-symmetric near-rings).

(2) S is a left (right) Clifford E^+-unitary seminearring in which $a + a^0b = a$ for all $a, b \in S$ and $c(d+e) = cd + c^0e$ whenever $c^0 + e = e$ for c, $d \in S$ and $e \in E^+(S)$.

(3) *S is a left (right) Clifford E^+-unitary seminearring in which $a + a^0b = a$ for all $a, b \in S$ and $c(d+e) = cd + c^0e$ for all c, $d \in S$ and for all $e \in E^+(S)$.*

2.3 Strong Bi-semilattice of Near-rings in the class of Distributively Generated Seminearrings

Sometimes the notion of being distributively generated (*cf.* Definition 1.5.4) for seminearrings makes many situations (as it happens for near-rings as well) nice. This is again evident from the result obtained in Corollary 2.3.2 below which is what has been obtained from Theorems 2.1.16 and 2.1.21 by taking the seminearring to be distributively generated. In order to obtain Corollary 2.3.2 we need the following lemma as well.

Lemma 2.3.1. *Let S be an additively inverse seminearring. Then for any a, $b \in S$,*

(i) $(ab)^* = a^*b$, *and*

(ii) $(ab)^0 = a^0b$.

Proof. (i) For any $a, b \in S$, $ab + a^*b + ab = (a + a^* + a)b = ab$ and $a^*b + ab + a^*b = (a^* + a + a^*)b = a^*b$. Hence $(ab)^* = a^*b$.

(ii) For any $a, b \in S$,

$$\begin{aligned} (ab)^0 &= ab + (ab)^* \\ &= ab + a^*b \text{ (using } (i)) \\ &= (a + a^*)b \\ &= a^0b. \end{aligned}$$

□

Corollary 2.3.2. *Let S be a distributively generated seminearring. Then S is a strong bi-semilattice of near-rings (zero-symmeric near-rings) if and only if S is a left (right) Clifford E^+-unitary seminearring.*

Proof. The direct implication follows immediately from Theorem 2.1.16 (respectively Theorem 2.1.21).

Conversely, let S be a left (right) Clifford E^+-unitary seminearring. Let $a \in S$ and $e \in E^+(S)$. Since S is distributively generated, $a = c_1 + c_2 + ... + c_n$ for some distributive elements $c_1, c_2, ..., c_n$. Now

$$\begin{aligned}
a(b+e) &= (\sum_{i=1}^{n} c_i)(b+e) \\
&= (c_1 + c_2 + ... + c_n)(b+e) \\
&= c_1(b+e) + c_2(b+e) + ... + c_n(b+e) \\
&= \sum_{i=1}^{n}(c_i b + c_i e) \ (\because \text{each } c_i \text{ is a distributive})
\end{aligned}$$

Since each c_i is a distributive element,

$$\begin{aligned}
c_i e + c_i e &= c_i(e+e) \\
&= c_i e \in E^+(S) \text{ for } i = 1, 2, ..., n.
\end{aligned}$$

So in view of the fact that the semigroup $(S, +)$ is Clifford, we deduce that $a(b+e) = \sum\limits_{i=1}^{n} c_i b + \sum\limits_{i=1}^{n} c_i e$. In view of Lemma 2.3.1 and the fact that $c_i e \in E^+(S)$, ${c_i}^0 e = c_i e$ for $i = 1, 2, ..., n$. Thus $a(b+e) = ab + \sum\limits_{i=1}^{n} {c_i}^0 e$. Now,

$$\begin{aligned}
(\sum_{i=1}^{n} c_i) + (\sum_{j=n}^{1} c_j^*) + (\sum_{i=1}^{n} c_i) &= c_1 + c_2 + ... + (c_n + c_n^*) + ... + (c_1^* + c_1) + ... + c_n \\
&= c_1 + (c_1^* + c_1) + c_2 + (c_2^* + c_2) + ... + c_n + (c_n^* + c_n) \\
&\qquad (\because (S, +) \text{ is Clifford, idempotents are central}) \\
&= c_1 + c_2 + ... + c_n
\end{aligned}$$

and

$$\begin{aligned}
(\sum_{j=n}^{1} c_j^*) + (\sum_{i=1}^{n} c_i) + (\sum_{j=n}^{1} c_j^*) &= c_n^* + ... + (c_1^* + c_1) + c_2 + ... + (c_n + c_n^*) + ... + c_1^* \\
&= c_n^* + (c_n + c_n^*) + c_{n-1}^* + (c_{n-1} + c_{n-1}^*) + ... + c_1^* + (c_1 + c_1^*) \\
&\qquad (\because (S, +) \text{ is Clifford, idempotents are central}) \\
&= c_n^* + c_{n-1}^* + ... + c_1^*.
\end{aligned}$$

Thus, $(\sum\limits_{i=1}^{n} c_i)^* = \sum\limits_{j=n}^{1} c_j^*$. So,

$$\begin{aligned}
a^0 &= (\sum_{i=1}^{n} c_i)^0 \\
&= (\sum_{i=1}^{n} c_i) + (\sum_{i=1}^{n} c_i)^* \\
&= (\sum_{i=1}^{n} c_i) + (\sum_{j=n}^{1} c_j^*) \\
&= c_1 + c_2 + ... + (c_n + c_n^*) + c_{n-1}^* + ... + c_1^* \\
&= (c_1 + c_1^*) + (c_2 + c_2^*) + ... + (c_n + c_n^*) \ (\because (S, +) \text{ is Clifford}) \\
&= c_1^0 + c_2^0 + ... + c_n^0 \\
&= \sum_{i=1}^{n} c_i^0
\end{aligned}$$

whence $a(b+e) = ab + a^0e$. Thus in view of Theorem 2.1.16 (Theorem 2.1.21), S is a strong bi-semilattice of near-rings (respectively zero-symmetric near-rings). □

If we consider the seminearring S to be distributively generated then the results obtained in Theorem 2.2.4 reduces to what we obtain below.

Corollary 2.3.3. *Let S be a distributively generated seminearring. Then S is a strong distributive lattice of near-rings (zero-symmetric near-rings) if and only if S is a left (right) Clifford E^+-unitary seminearring with $a + a^0b = a$ for all $a, b \in S$.*

Proof. The direct implication follows immediately from Theorem 2.2.4. Conversely, let S be a left (right) Clifford E^+-unitary seminearring with $a + a^0b = a$ for all $a, b \in S$. Applying the argument similar to that of Corollary 2.3.2, we deduce that $c(d + e) = cd + c^0e$ for all c, $d \in S$ and for all $e \in E^+(S)$. Hence in view of Theorem 2.2.4, S is a strong distributive lattice of near-rings (zero-symmetric near-rings). □

CHAPTER 3

UNION OF NEAR-RINGS

3

Union of Near-rings

This chapter is a continuation of the study on additively completely regular seminearrings accomplished in the previous chapter. The main purpose of this chapter is to characterize those additively completely regular seminearrings which can be decomposed as a union of near-rings (zero-symmetric near-rings). This study is also motivated by a question raised in the Concluding Remark of [90] as well as by the intention of obtaining the semigroup theoretic analogue of "A semigroup is completely regular if and only if it is a union of groups" in the seminearring setting.

Since in semigroup theory, the following three classes — the class of completely regular semigroups, the class of semigroups which are union of groups and the class of semigroups which are semilattice of completely simple semigroups coincide, to accomplish the above mentioned study the possible first move is to investigate whether the class of seminearrings which are bi-semilattice of left (right) completely simple seminearrings *i.e.*, the class of left (right) completely regular seminearrings (*cf.* Theorems 1.5.28 and 1.5.29) serves the purpose or not. In this direction in Example 3.1.1 a seminearring $(S, +, \cdot)$ has been obtained which is union of near-rings but neither a left nor a right completely regular seminearring. In particular, S satisfies the hypothesis of Theorem 1.5.21 but neither of the properties (i) and (ii) of the respective theorem holds

---

This chapter is mainly based on the work published in the following paper:

Tuhin Manna et al., ***A note on additively completely regular seminearrings, Semigroup Forum*****, 100 (1), 339-347 (2020)**

here. This motivates us to remove the following restriction: "$ae \mathcal{J}^+ ea$ for all $a \in S$, for all $e \in E^+(S)$" (*cf.* Theorem 1.5.21) from the notion of left (right) completely regular seminearrings and to introduce a new notion called *generalized left (right) completely regular seminearrings* (*cf.* Definitions 3.1.2 and 3.1.3) which is finally proved to be the desired class of seminearrings.

3.1 Generalized Completely Regular Seminearrings

In order to complete the task of obtaining the analogue of "A semigroup is completely regular if and only if it is a union of groups" in the setting of seminearring it is natural to determine the type of seminearings, in the class of additively regular seminearrings, which are union of near-rings. The following example motivates us to define (*cf.* Definition 3.1.2) a class of seminearrings which happens to be our desired class (*cf.* Theorem 3.1.17).

Example 3.1.1. Let us define '+' on $T = \{u, a, b, c\}$ as follows.

$+$	u	a	b	c
u	u	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	b	a	u

$(T, +)$ is nothing but the full transformation semigroup on a set of two elements written in the additive notation. $(T, +)$ is a completely regular semigroup since

- $u + u + u = u$
- $a + a + a = a$
- $b + b + b = b$
- $c + c + c = c.$

Let S be the semigroup direct product of T with itself. Then clearly $(S, +)$ is again a completely regular semigroup. Therefore in view of Theorem 1.1.34, $(S, +)$ is a union of group. Let us define '$*$' on S by $x * y = x$ for all $x,\ y \in S$. Clearly $(S, +, *)$ is a seminearring. S is not a semiring because

$$\begin{aligned} (c,c) * ((c,c) + (c,c)) &= (c,c) \\ &\neq (u,u) \\ &= (c,c) + (c,c) \\ &= (c,c) * (c,c) + (c,c) * (c,c). \end{aligned}$$

In $(S, +, *)$ we have

$$x + x + x = x \text{ and } (x + x) * x = x + x \text{ for any } x \in S.$$

Now from the definition of '$*$' and the fact that $(S, +)$ is a union of groups it follows that $(S, +, *)$ is a union of near-rings. We now observe that,

- $(a,u) = (a,u) + (b,u) + (a,u)$, $(b,u) = (b,u) + (a,u) + (b,u)$
- $(a,u) = (a,u) + (b,c) + (b,c)$, $(b,c) = (b,c) + (a,u) + (a,u)$
- $(a,u) = (a,c) + (a,c) + (a,u)$, $(a,c) = (a,c) + (a,u) + (a,u)$

and

- $(a,b) = (a,b) + (a,a) + (a,b)$, $(a,a) = (a,a) + (a,b) + (a,a)$
- $(a,b) = (a,b) + (b,b) + (b,b)$, $(b,b) = (b,b) + (a,b) + (a,b)$
- $(a,b) = (a,b) + (b,a) + (b,a)$, $(b,a) = (b,a) + (a,b) + (a,b)$.

Again if $(a,u)\mathcal{J}^+(a,a)$ then $u\mathcal{J}a$ in $(T, +)$ which is absurd since there exist no $x, y \in T$ satisfying $x + a + y = u$. So in $(S, +, *)$, $\{(a,u),\ (b,u),\ (b,c),\ (a,c)\}$ and $\{(a,b),\ (a,a),\ (b,b),\ (b,a)\}$ are two different $\mathcal{J}^+$ classes.

Now, $E^+(S) = \{(u,u), (u,a), (u,b), (a,u), (a,b), (a,a), (b,u), (b,a), (b,b)\}$. Since $(a,b) * (b,c) = (a,b)$ and $(b,c) * (a,b) = (b,c)$ belong to two different $\mathcal{J}^+$-classes, $(S, +, *)$ is nither a left completely regular seminearring nor a right left completely regular seminearring(*cf.* Definitions 1.5.23 and 1.5.24). The seminearring $(S, +, *)$ is additively regular, decomposable as union of near-rings and satisfies the hypothesis of Theorem 1.5.21 but S satisfies none of the conditions (i), (ii) of Theorem 1.5.21. Motivated by this we formulate the following definition.

Definition 3.1.2. A seminearring $(S, +, \cdot)$ is called a *generalized left completely regular* (GLCR) if for each $a \in S$ there exists an element $x_a \in S$ satisfying the following conditions :

(i) $a = a + x_a + a$

(ii) $a + x_a = x_a + a$

(iii) $(a + x_a)a = a + x_a$

The defining conditions of Definition 3.1.2 are nothing but the hypothesis of Theorem 1.5.21. Since Theorem 1.5.22 is the right analogue of the Theorem 1.5.21, so for the notion of generalized right completely regular (GRCR) seminearrings, we have the following definition.

Definition 3.1.3. A seminearring $(S, +, \cdot)$ is called a *generalized right completely regular* (GRCR) if for each $a \in S$ there exists an element $x_a \in S$ satisfying the following conditions :

(i) $a = a + x_a + a$

(ii) $a + x_a = x_a + a$

(iii) $a(a + x_a) = a + x_a$

Observation 3.1.4. While studying additively regular seminearrings in order to obtain analogues of various structure Theorems of regular semigroups, so far four types of additively completely regular seminearrings *viz.*, generalized left completely regular (GLCR), generalized right completely regular (GRCR), left completely regular (LCR) (*cf.* Definition 1.5.23) and right completely regular seminearrings (RCR) (*cf.* Definition 1.5.24) have been introduced. Regarding the relationships among these four notions of seminearrings it is relevent to ask some questions which we list below in a tabular form. Each row of the table contains one question. The way to read the question for row (*ii*) is as follows: "Does there exist a seminearring which is RCR as well as GLCR as well as LCR but not GRCR?". In the last column of the table we answer the questions and refer to supporting Examples/Remarks/Results.

Question	LCR	RCR	GLCR	GRCR	Answer
(i)	Yes	Yes	Yes	Yes	exists (*cf.* Remark 3.1.5 (ii))
(ii)	Yes	Yes	Yes	No	does not exist (*cf.* Remark 3.1.5 (i))
(iii)	Yes	Yes	No	Yes	does not exist (*cf.* Remark 3.1.5 (i))
(iv)	Yes	No	Yes	Yes	does not exist (*cf.* Remark 3.1.5 (iii))
(v)	No	Yes	Yes	Yes	does not exist (*cf.* Remark 3.1.5 (iii))
(vi)	Yes	Yes	No	No	does not exist (*cf.* Remark 3.1.5 (i))
(vii)	Yes	No	Yes	No	exists (*cf.* Example 3.1.6)
(viii)	Yes	No	No	Yes	does not exist (*cf.* Remark 3.1.5 (i))
(ix)	No	Yes	Yes	No	does not exist (*cf.* Remark 3.1.5 (i))
(x)	No	Yes	No	Yes	exists (*cf.* Example 3.1.8)
(xi)	No	No	Yes	Yes	exists (*cf.* Example 3.1.10)
(xii)	No	No	No	Yes	exists (*cf.* Example 3.1.9)
(xiii)	No	No	Yes	No	exists (*cf.* Example 3.1.1)
(xiv)	No	Yes	No	No	does not exist (*cf.* Remark 3.1.5 (i))
(xv)	Yes	No	No	No	does not exist (*cf.* Remark 3.1.5 (i))
(xvi)	No	No	No	No	exists (*cf.* Proposition 3.1.11)

The following remark is useful for finding answers to the Questions (i) - (vi), $(viii)$, (ix), (xiv) and (xv).

Remark 3.1.5. (i) If a seminearring S is LCR, then it is GLCR also and if a seminearring S is RCR, then it is GRCR also.

(ii) Any zero-symmetric near-ring, being an LCR as well as an RCR seminearring, is a GLCR as well as a GRCR seminearring.

(iii) If S is an LCR (RCR) as well as GRCR (respectively, GLCR), then it is RCR (respectively, LCR), too.

The following example provides the answer to the Question (vii).

Example 3.1.6. Any non zero-symmetric near-ring N is a left completely regular seminearring but not a right completely regular seminearring. In particular, the canonical near-ring $(M(G), +, \cdot)$ of all self-maps on an additive group G is a left completely regular seminearring but not a right completely regular seminearring, but neither GRCR nor RCR.

Before presenting the next example we wish to make the following observation.

Observation 3.1.7. [90, 91] If '+' is defined on $\mathbb{R} \setminus \{0\}$, the set of all non-zero real numbers, by $a + b = |a| \cdot b$ where "$\cdot$" denotes the usual multiplication of real numbers, then $(\mathbb{R} \setminus \{0\}, +)$ becomes a semigroup in which for any two elements a, b,

$$1 + a + (\tfrac{1}{a} + b) = b \text{ and } 1 + b + (\tfrac{1}{b} + a) = a$$

whence $a \mathcal{J} b$. Therefore $(\mathbb{R} \setminus \{0\}, +)$ is a simple semigroup. It is clearly completely regular and hence completely simple. Also $1 \mathcal{H} (-1)$ implies that $1 \mathcal{L} (-1)$ whence $-1 = x + 1$ for some $x \in \mathbb{R} \setminus \{0\}$ *i.e.*, $-1 = |x|$ for some $x \in \mathbb{R} \setminus \{0\}$ which is absurd. Therefore 1 and -1 lie in two different $\mathcal{H}$ classes.

Now we see below that $\mathcal{H}$ class of 1 and that of -1 both contain infinitely many elements. Let a be an element of $\mathbb{R} \setminus \{0\}$ such that $a > 0$. Then $1 + a = a$ and $a + \frac{1}{a}$ $= |a| \cdot \frac{1}{a} = 1$ (since $a > 0$) whence $1 \mathcal{R} a$. Again $a + 1 = |a| \cdot 1 = a$ (since $a > 0$) and $\frac{1}{a} + a = \frac{1}{|a|} \cdot a = 1$ (since $a > 0$) whence $1 \mathcal{L} a$. Thus $1 \mathcal{H} a$. Therefore $\mathcal{H}$ class of 1 contains all $a \in \mathbb{R} \setminus \{0\}$ with $a > 0$. Similarly for any $b \in \mathbb{R} \setminus \{0\}$ with $b < 0$, $-1 \mathcal{H} b$. Then $(\mathbb{R} \setminus \{0\}, +)$ is a completely simple semigroup having at least two $\mathcal{H}$ classes such that at least one of them has more than two members.

In the following example besides giving the answer to the question (x), we recall some key features of seminearrings of idempotent fixing self maps of an additively written semigroup S from [90] and [91].

Example 3.1.8. Let $(S, +)$ be a completely simple semigroup having more than one $\mathcal{H}$ classes such that at least one of them has more than two members. Let

$$IF(S) := \{f \in M(S) | f(e) = e \text{ for all } e \in E(S)\}.$$

Let $f, g \in IF(S)$ and $e \in E(S)$. Then $(f + g)(e) = f(e) + g(e) = e$ and $(f \circ g)(e) = f(g(e)) = e$ whence $f + g$, $f \circ g \in IF(S)$. Then $(IF(S), +, \circ)$ is a subseminearring of $(M(S), +, \circ)$. Let $a, b \in S$ such that $a, b \notin E(S)$ and $b + b \notin E(S)$. Let $f, g, h \in IF(S)$ be such that

$$f(b) = b = f(b + b), g(a) = b, h(a) = b.$$

Then

$$\begin{aligned}(f\circ(g+h))(a) &= f(g(a)+h(a))\\ &= f(b+b)\\ &= b\\ &\neq b+b \text{ (since } b\notin E(S))\\ &= f(b)+f(b)\\ &= f(g(a))+f(h(a))\\ &= (f\circ g+f\circ h)(a)\end{aligned}$$

implying that $(IF(S),+,\circ)$ is not a left distributive seminearring.

Let $f\in IF(S)$. Since $(S,+)$ is completely regular, for each $f(a)\in S\setminus E(S)$ there exists $x_{f(a)}\in S$ satisfying

$$f(a)+x_{f(a)}+f(a)=f(a),\ f(a)+x_{f(a)}=x_{f(a)}+f(a).$$

Using axiom of choice we define $x_f : S\to S$ by

$$x_f(a)=\begin{cases} x_{f(a)}, & \text{if } f(a)\notin E(S)\\ f(a), & \text{if } f(a)\in E(S).\end{cases}$$

Let $e\in E(S)$. Then $f(e)=e\in E(S)$. Thus $x_f(e)=f(e)=e$ whence $x_f\in IF(S)$. Let $s\in S$. If $f(s)\notin E(S)$ then

$$\begin{aligned}(f+x_f)(s) &= f(s)+x_f(s)\\ &= f(s)+x_{f(s)}\\ &= x_{f(s)}+f(s)\\ &= x_f(s)+f(s)\\ &= (x_f+f)(s)\end{aligned}$$

and

$$\begin{aligned}(f+x_f+f)(s) &= f(s)+x_f(s)+f(s)\\ &= f(s)+x_{f(s)}+f(s)\\ &= f(s).\end{aligned}$$

Again if $f(s) \in E(S)$ then

$$\begin{aligned}
(f + x_f)(s) &= f(s) + x_f(s) \\
&= f(s) + f(s) \\
&= x_f(s) + f(s) \\
&= (x_f + f)(s)
\end{aligned}$$

and

$$\begin{aligned}
(f + x_f + f)(s) &= f(s) + x_f(s) + f(s) \\
&= f(s) + f(s) + f(s) \\
&= f(s)
\end{aligned}$$

whence $f + x_f + f = f$ and $f + x_f = x_f + f$. Now for any $s \in S$,

$$\begin{aligned}
(f \circ (f + x_f))(s) &= f((f + x_f)(s)) \\
&= f(f(s) + x_f(s)) \\
&= f(s) + x_f(s) \text{ (since } f(s) + x_f(s) \in E(S)) \\
&= (f + x_f)(s)
\end{aligned}$$

whence $f \circ (f + x_f) = f + x_f$.

Let $f, g \in IF(S)$. Then using the fact that S is completely simple and Result 1.1.17 we deduce that for each $a \in S \setminus E(S)$ there exist $L^a_{f,g}$, $L^a_{g,f}$, $R^a_{f,g}$ and $R^a_{g,f}$ in S satisfying

$$f(a) = L^a_{f,g} + g(a) + R^a_{f,g},\ g(a) = L^a_{g,f} + f(a) + R^a_{g,f}.$$

Then by axiom of choice we define four self maps $L_{f,g}$, $L_{g,f}$, $R_{f,g}$ and $R_{g,f}$ of S by

$$L_{f,g}(x) = \begin{cases} L^x_{f,g}, & \text{if } x \notin E(S) \\ x, & \text{if } x \in E(S), \end{cases}$$

$$L_{g,f}(x) = \begin{cases} L^x_{g,f}, & \text{if } x \notin E(S) \\ x, & \text{if } x \in E(S), \end{cases}$$

$$R_{f,g}(x) = \begin{cases} R^x_{f,g}, & \text{if } x \notin E(S) \\ x, & \text{if } x \in E(S), \end{cases}$$

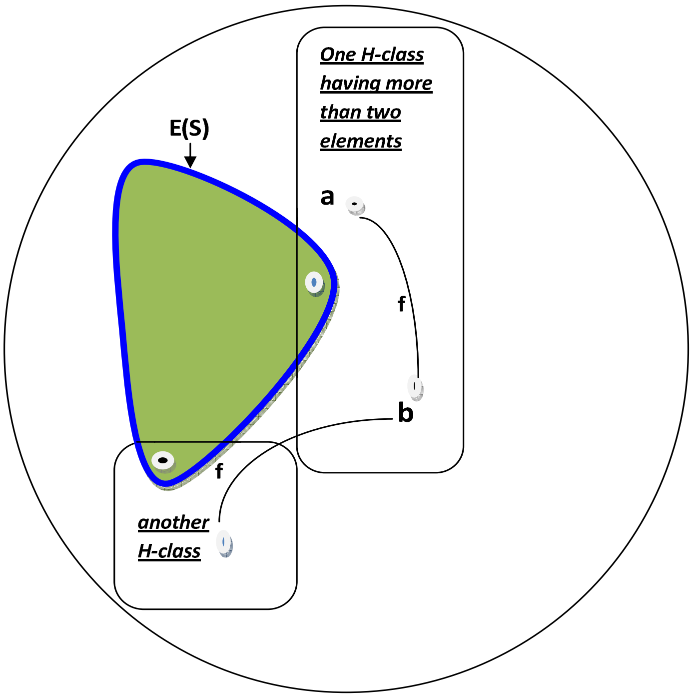
and

$$R_{g,f}(x) = \begin{cases} R^x_{g,f}, & \text{if } x \notin E(S) \\ x, & \text{if } x \in E(S). \end{cases}$$

It is clear from the construction that $L_{f,g}$, $L_{g,f}$, $R_{f,g}$ and $R_{g,f} \in IF(S)$ and

$$f = L_{f,g} + g + R_{f,g},\ g = L_{g,f} + f + R_{g,f}.$$

Hence any two elements of $IF(S)$ are $\mathcal{J}^+$ related. So $(IF(S), +, \circ)$ is a right completely regular seminearring. Let $a \in S \setminus E(S)$. Let us choose f from $IF(S)$ so that $f(a) \notin E(S)$ and $f(a)$, $(f \circ f)(a)$ lie in different $\mathcal{H}$ classes of S (as shown in the figure).



Suppose there exists $f' \in IF(S)$ satisfying $f + f' + f = f$, $f + f' = f' + f$, $(f + f') \circ f = f + f'$. Then

$$\begin{aligned} &((f + f') \circ f)(a) = (f + f')(a) \\ &\Rightarrow (f + f')(f(a)) = f(a) + f'(a) \\ &\Rightarrow f^2(a) + f'(f(a)) = f(a) + f'(a) \quad (\text{where } f \circ f = f^2) \\ &\Rightarrow f^2(a) + f'(f(a)) + f(a) = f(a) + f'(a) + f(a) \\ &\Rightarrow f^2(a) + [f'(f(a)) + f(a)] = f(a) \ldots . (I^*) \end{aligned}$$

and

$$\begin{aligned}
&((f+f')\circ f)(a) = (f+f')(a)\\
&\Rightarrow f^2(a) + f'(f(a)) + f^2(a) = f(a) + f'(a) + f^2(a)\\
&\Rightarrow (f+f'+f)(f(a)) = f(a) + f'(a) + f^2(a)\\
&\Rightarrow f(f(a)) = f(a) + f'(a) + f^2(a)\\
&\Rightarrow f^2(a) = f(a) + [f'(a) + f^2(a)]....(II^*)
\end{aligned}$$

whence $f(a)\ \mathcal{R}\ f^2(a)$ combining (I^*) and (II^*). Again

$$\begin{aligned}
&((f+f')\circ f)(a) = (f+f')(a)\\
&\Rightarrow (f'+f)(f(a)) = f'(a) + f(a) \quad (\text{since } f+f' = f'+f)\\
&\Rightarrow f'(f(a)) + f^2(a) = f'(a) + f(a)\\
&\Rightarrow f(a) + f'(f(a)) + f^2(a) = f(a) + f'(a) + f(a)\\
&\Rightarrow [f(a) + f'(f(a))] + f^2(a) = f(a)....(III^*)
\end{aligned}$$

and

$$\begin{aligned}
&((f+f')\circ f)(a) = (f+f')(a)\\
&\Rightarrow (f'+f)(f(a)) = f'(a) + f(a) \quad (\text{since } f+f' = f'+f)\\
&\Rightarrow f'(f(a)) + f^2(a) = f'(a) + f(a)\\
&\Rightarrow f^2(a) + f'(f(a)) + f^2(a) = f^2(a) + f'(a) + f(a)\\
&\Rightarrow (f+f'+f)(f(a)) = [f^2(a) + f'(a)] + f(a)\\
&\Rightarrow f^2(a) = [f^2(a) + f'(a)] + f(a)....(IV^*).
\end{aligned}$$

Hence $f(a)\ \mathcal{L}\ f^2(a)$ combining (III^*) and (IV^*). Therefore $f(a)\ \mathcal{H}\ f^2(a)$ which is a contradiction to our assumption that $f(a)$, $f^2(a)$ lie in different $\mathcal{H}$ classes of S.

Hence $(IF(S), +, \circ)$ is a right completely regular seminearring which is not left completely regular. In view of Remark 3.1.5 (ii), $IF(S)$ is not also GLCR.

The following example provides the answer to the Question (xii).

Example 3.1.9. Let us consider the subseminearring $IF(S)$ (as defined in Example 3.1.8) of $M(S)$ where $(S, +)$ is the completely regular semigroup as considered in Example 3.1.1. Let $f, g, h \in IF(S)$ be such that

$$f(b, c) = (b, u) = f(c, b), g(a, c) = (b, c), h(a, c) = (c, b).$$

Then

$$\begin{aligned}
(f \circ (g+h))(a,c) &= f(g(a,c)+h(a,c)) \\
&= f((b,c)+(c,b)) \\
&= f(b,a) \\
&= (b,a) \text{ (since } (b,a) \in E(S)) \\
&\neq (b,u) \\
&= (b,u)+(b,u) \\
&= f(b,c)+f(c,b) \\
&= f(g(a,c))+f(h(a,c)) \\
&= (f \circ g + f \circ h)(a,c)
\end{aligned}$$

implying that $(IF(S),+,\circ)$ is not a left distributive seminearring *i.e.* $(IF(S),+,\circ)$ is not a semiring.

Following calculation shown in Example 3.1.8, $(IF(S),+,\circ)$ is a GRCR seminearring. We now observe that,

- $(c,u)+(c,u)=(u,u)$
- $(u,u)+(u,u)=(u,u)$
- $(c,c)+(c,c)=(u,u)$
- $(u,c)+(u,c)=(u,u)$

and

- $(c,a)+(c,a)=(u,a)$
- $(u,a)+(u,a)=(u,a)$

So in $(S,+)$, the $\mathcal{H}$-class of (u,u) is $\{(u,u),(c,u),(u,c),(c,c)\}$ and the $\mathcal{H}$-class of (u,a) is $\{(u,a),(c,a)\}$. Let us choose f from $IF(S)$ so that

$$f(c,u)=(u,c) \text{ and } f(u,c)=(c,a).$$

If $IF(S)$ is a GLCR seminearing then $f(a)\ \mathcal{H}\ f^2(a)$ for all $a \in S$ (*cf.* Lemma 3.1.13) which contradicts the fact that $f(c,u)=(u,c)$, $(f \circ f)(c,u)=f(f(c,u))=(c,a)$ lie in different $\mathcal{H}$ classes of S. Hence $(IF(S),+,\circ)$ is not a GLCR.

In S, $\{(a,u),(b,u),(b,c),(a,c)\}$ and $\{(a,b),(a,a),(b,b),(b,a)\}$ are two different $\mathcal{J}$ classes (*cf.* Example 3.1.1). Let us construct two functions $g, h : S \to S$ in the following manner.

$$g(x,y) = \begin{cases} (a,u), & \text{if } (x,y) = (c,c) \\ (x,y), & \text{where } (x,y) \in E(S) \\ (u,y), & \text{where } x = c \text{ and } y \in \{u,a,b\} \\ (x,u), & \text{where } y = c \text{ and } x \in \{u,a,b\} \end{cases}$$

and

$$h(x,y) = \begin{cases} (a,b), & \text{if } (x,y) = (c,c) \\ (x,y), & \text{if } (x,y) \neq (c,c). \end{cases}$$

Now, $E(S) = \{(u,u),(u,a),(u,b),(a,u),(a,b),(a,a),(b,u),(b,a),(b,b)\}$. Let $e \in E(S)$ then

$$h(e) = e \text{ and } g(e) = e \text{ for all } e \in E(S)$$

So $h, g \in IF(S)$. Since

$$\begin{aligned} (g+g)(c,c) &= g(c,c) + g(c,c) \\ &= (a,u) + (a,u) \\ &= (a,u) \\ &= g(c,c) \end{aligned}$$

$$\begin{aligned} (g+g)(x,y) &= g(x,y) + g(x,y), \text{ where } (x,y) \in E(S) \\ &= (x,y) + (x,y) \\ &= (x,y) \\ &= g(x,y) \end{aligned}$$

$$\begin{aligned} (g+g)(c,y) &= g(c,y) + g(c,y), \text{ where } y \in \{u,a,b\} \\ &= (u,y) + (u,y) \\ &= (u,y) \\ &= g(c,y) \end{aligned}$$

and

$$\begin{aligned}(g+g)(x,c) &= g(x,c)+g(x,c), \text{ where } x\in\{u,a,b\}\\ &= (x,u)+(x,u)\\ &= (x,u)\\ &= g(x,c)\end{aligned}$$

$g \in E^+(IF(S))$. Now $(g\circ h)(c,c)=(a,b)$ and $(h\circ g)(c,c)=(a,u)$. Consequently, $g\circ h$ and $h\circ g$ are not $\mathcal{J}^+$ related in $IF(S)$. Hence $IF(S)$ is a GRCR seminearring which is neither a RCR nor GLCR seminearring.

The following example provides the answer to the Question (xi).

Example 3.1.10. Let us consider a semilattice $(L,+)$ having at least two elements. Let us define '$*$' on L by $x*y=x$ for all x, $y\in L$. It is a matter of routine verification that $(L,+,*)$ is a GLCR as well as a GRCR seminearring which is also a semiring. Now let $a,b\in L$ such that $a\neq b$. Then as $a*b=a$ and $b*a=b$, being two different additive idempotents, belong to two different $\mathcal{J}^+$-classes of $(L,+,*)$. Hence $(L,+,*)$ is neither an LCR nor an RCR seminearring.

The following proposition, together with Remark 3.1.5, provides the answer to the Question (xvi).

Proposition 3.1.11. *Let S_L be an LCR seminearring which is not RCR and S_R be an RCR seminearring which is not LCR. Then $S_L\times S_R$, the seminearring direct product of S_L and S_R is neither GLCR nor GRCR.*

Proof. Since S_L is an LCR seminearring which is not RCR, there exists an element $a\in S_L$ for which $x_a\in S_L$ satisfying $a(a+x_a)=(a+x_a)$ does not exist in S_L. Again since S_R is an RCR seminearring which is not LCR, there exists an element $b\in S_R$ for which $x_b\in S_R$ satisfying $(b+x_b)b=(b+x_b)$ does not exist in S_R. Now consider $(a,b)\in S_L\times S_R$. Then there does not exist any $(x_a,x_b)\in S_L\times S_R$ satisfying any of the following two conditions:

- (A) $(a,b)[(a,b)+(x_a,x_b)]=(a,b)+(x_a,x_b)$
- (B) $[(a,b)+(x_a,x_b)](a,b)=(a,b)+(x_a,x_b)$.

Hence $S_L \times S_R$ is neither GLCR nor GRCR. $\square$

Lemma 3.1.12. *Let S be a generalized left completely regular (GLCR) seminearring. Then $\mathcal{H}^+$ is a right congruence (i.e., an equivalence relation which is right compatible w.r.t. multiplication) on S.*

Proof. Let $a\mathcal{H}^+b$. Then

$$\begin{aligned} a &= x+b \\ &= b+y \end{aligned}$$

and

$$\begin{aligned} b &= s+a \\ &= a+t \end{aligned}$$

for some $x, y, s, t \in S$. Let $c \in S$. Then $ac = (x+b)c = xc+bc$. Also $bc = (s+a)c = sc+ac$. This implies that $ac\ \mathcal{L}^+\ bc$. Again $ac = (b+y)c = bc+yc$. Also $bc = (a+t)c = ac+tc$. So, we have $ac\ \mathcal{R}^+\ bc$. Hence $ac\ \mathcal{H}^+\ bc$. Therefore $\mathcal{H}^+$ is a right congruence. $\square$

Lemma 3.1.13. *In a generalized left completely regular seminearring S, $a^2\mathcal{H}^+a$ for all $a \in S$.*

Proof. Let $a \in S$. Then

$$\begin{aligned} a &= a+x_a+a \\ &= a+(a+x_a) \\ &= a+(a+x_a)a \\ &= a+(x_a+a)a \\ &= (a+x_aa)+a^2 \end{aligned}$$

and

$$\begin{aligned} a^2 &= (a+x_a+a)a \\ &= a^2+(x_a+a)a \\ &= a^2+(a+x_a)a \\ &= a^2+a+x_a \\ &= (a^2+x_a)+a \end{aligned}$$

This implies $a^2 \ \mathcal{L}^+ \ a$. Again,

$$\begin{aligned} a &= a + x_a + a \\ &= (a + x_a)a + a \\ &= a^2 + (x_a a + a) \end{aligned}$$

and

$$\begin{aligned} a^2 &= (a + x_a + a)a \\ &= (a + x_a)a + a^2 \\ &= a + (x_a + a^2). \end{aligned}$$

So, $a^2 \ \mathcal{R}^+ \ a$. Consequently $a^2 \ \mathcal{H}^+ \ a$. □

Lemma 3.1.14. *Let S be a generalized left completely regular seminearring and $a \in S$. Then for each $y \in \mathcal{H}_a^+$ there exists a unique element $y^* \in \mathcal{H}_a^+$ such that $y + y^* + y = y$, $y + y^* = y^* + y$ and $(y + y^*)y = y + y^*$.*

Proof. Let $y \in \mathcal{H}_a^+$. In view of 1.1.34(iii), $\mathcal{H}_a^+$ is a group. Hence there exists a unique $y^* \in \mathcal{H}_a^+$ such that

$$\begin{aligned} y + y^* &= y^* + y \\ &= 0_{\mathcal{H}_a^+} \\ &= e, \text{ say} \end{aligned}$$

and $y + y^* + y = y$, where e is the unique additive idempotent of the group $\mathcal{H}_a^+$. Now, $(y + y^*)y = ey$ where $ey \in E^+(S)$ (since $ey + ey = ey$). Now $y\mathcal{H}^+e$. As $\mathcal{H}^+$ is a right congruence (*cf.* Lemmma 3.1.12), $y^2\mathcal{H}^+ey$. Since $\mathcal{H}^+$ is an equivalence relation and $y^2\mathcal{H}^+y$ (*cf.* Lemma 3.1.13), $y\mathcal{H}^+ey$. ey is an additive idempotent which is an element of $\mathcal{H}_a^+$. This implies, $ey = e$, so $(y + y^*)y = e = y + y^*$. □

Theorem 3.1.15. *Let $(S, +, \cdot)$ be a generalized left completely regular seminearring. Then every $\mathcal{H}^+$ -class is a near-ring.*

Proof. Consider $\mathcal{H}^+$ on S. Clearly $\mathcal{H}^+$ is an equivalence relation on S such that each $\mathcal{H}_a^+$ is a group for all $a \in S$ (*cf.* Theorem 1.1.34(iii)). Let $b, c \in \mathcal{H}_a^+$. Then $b + b^* =$

$c + c^* = e$, where e is the unique additive idempotent of $\mathcal{H}_a^+$. Now

$$\begin{aligned} bc &= (b + b^* + b)c \\ &= (b + b^*)c + bc \\ &= (c + c^*)c + bc \\ &= c + (c^* + bc) \end{aligned}$$

and

$$\begin{aligned} c &= (c + c^* + c) \\ &= (c + c^*)c + c \\ &= (b + b^*)c + c \\ &= bc + (b^*c + c). \end{aligned}$$

Hence $bc\ \mathcal{R}^+ c$. Again,

$$\begin{aligned} bc &= (b + b^* + b)c \\ &= bc + (b + b^*)c \\ &= bc + (c + c^*)c \\ &= bc + c + c^* \\ &= (bc + c^*) + c \end{aligned}$$

and

$$\begin{aligned} c &= (c + c^* + c) \\ &= c + c + c^* \\ &= c + (c + c^*)c \\ &= c + (b + b^*)c \\ &= c + (b^* + b)c \\ &= (c + b^*c) + bc. \end{aligned}$$

Hence $bc\ \mathcal{L}^+ c$. So $bc\ \mathcal{H}^+ c$. Since $\mathcal{H}^+$ is an equivalence relation on S, $bc\ \mathcal{H}^+ c$ and $c\ \mathcal{H}^+ a$ implies $bc\ \mathcal{H}^+ a$. This shows that $(\mathcal{H}_a^+, \cdot)$ is a semigroup. Consequently, $(\mathcal{H}_a^+, +, \cdot)$ is a near-ring. $\square$

Theorem 3.1.16. *If S is a union of near-rings then S is generalized left completely regular.*

Proof. Let S be a union of near-rings $\{N_\alpha : \alpha \in \Lambda\}$. Then for each $a \in S$, there exists $\alpha \in \Lambda$ such that $a \in N_\alpha$. Let x_a be the unique additive inverse of a within the near-ring $(N_\alpha, +, \cdot)$. Then it is clear that

$$a + x_a + a = a,\ a + x_a = x_a + a \text{ and } (a + x_a)a = a + x_a.$$

So S is generalized left completely regular. $\square$

Combining above results we get the following theorem:

Theorem 3.1.17. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is generalized left completely regular;

(2) Every $\mathcal{H}^+$ -class is a near-ring;

(3) S is a union (disjoint) of near-rings.

Remark 3.1.18. The right analogue of the above theorem does not hold, *i.e.*, if S is a generalized right completely regular (GRCR) seminearring then it need not be decomposed as a union (disjoint) of near-rings which is evident from the following example.

Example 3.1.19. Let us consider the seminearring $(IF(S), +, \circ)$ considered in Example 3.1.9 which is generalized right completely regular. Let us choose f from $IF(S)$ so that $f(c, u) = (u, c)$ and $f(u, c) = (c, a)$. Then in view of Example 3.1.9, $f(c, u)$ and $(f \circ f)(c, u)$ lie in different $\mathcal{H}^+$ classes of S. So $\mathcal{H}^+_f$ is not a near-ring as it does not contain f^2. Hence $IF(S)$ can not be decomposed as a union of near-rings in view of Theorem 3.1.17.

In the following result we characterize those seminearrings which are unions of zero-symmetric near-rings in the class of additively completely regular seminearrings.

Theorem 3.1.20. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is generalized left (GLCR) as well as generalized right completely regular (GRCR);

(2) Every $\mathcal{H}^+$ -class is a zero-symmetric near-ring;

(3) S is a union (disjoint) of zero-symmetric near-rings.

Proof. (1) $\Rightarrow$ (2) Suppose (1) holds. Let $a \in S$. Then by Theorem 3.1.17, $\mathcal{H}_a^+$ is a near-ring. So there exists $a^* \in \mathcal{H}_a^+$ such that $a + a^* = a^* + a = 0_{\mathcal{H}_a^+}$. Again by Definition 3.1.3, there exists $x_a \in S$ such that $a + x_a + a = a$, $a + x_a = x_a + a$ and $a(a + x_a) = a + x_a$. Now

$$\begin{aligned} a + x_a &= x_a + a + 0_{\mathcal{H}_a^+} \\ &= a + x_a + a + a^* \\ &= a + a^* \\ &= 0_{\mathcal{H}_a^+}. \end{aligned}$$

Hence $a \cdot 0_{\mathcal{H}_a^+} = 0_{\mathcal{H}_a^+}$. Consequently, $\mathcal{H}_a^+$ is a zero-symmetric near-ring.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Let S be a (disjoint) union of zero-symmetric near rings $\{N_\alpha : \alpha \in \Lambda\}$. Let $a \in S$. Then $a \in N_\alpha$ for some $\alpha \in \Lambda$. Let x_a be the unique additive inverse of a in N_α. Then $a + x_a = x_a + a = 0_{N_\alpha}$. Since N_α is a zero-symmetric near-ring, $a(a + x_a)$ $= (a + x_a)a = a + x_a$. Consequently, S is GLCR as well as GRCR. $\square$

CHAPTER 4

UNION OF REGULAR NEAR-RINGS

4

Union of Regular Near-rings

In the previous chapter, the analogue of a structure theorem of completely regular semigroups "a semigroup is completely regular if and only if it is a union of groups" has been obtained in the setting of seminearrings. In this direction, the notion of "*generalized left completely regular (**GLCR**) and generalized right completely regular (**GRCR**) seminearrings*" have been introduced and it has been established that a seminearring S is a generalized left (as well as right) completely regular seminearring if and only if S is a union of (zero-symmetric) near-rings (*cf.* Theorems 3.1.17 and 3.1.20). The purpose of the present chapter is to study the class of seminearrings that can be obtained by replacing, in the above results (*cf.* Theorems 3.1.17 and 3.1.20), near-rings by regular near-rings (*cf.* Definition 1.4.10), in general and inverse near-rings, completely regular near-rings and Clifford near-rings (*cf.* Definition 4.1.1 and Note 4.1.2), in particular. To accomplish the said study it has been observed that,

(1) the multiplicative reduct of a union of near-rings (*i.e.,* GLCR seminearings)

  (i) is a completely regular semigroup if and only if each component near-ring is completely regular (*cf.* Theorem 4.1.3)

  (ii) is an inverse semigroup if and only if each component near-ring is an inverse near-ring (*cf.* Theorem 4.1.9)

This chapter is mainly based on the work of the following paper:

Tuhin Manna et al., *On Union of Regular Near-rings*, Communicated.

(iii) is a Clifford semigroup if and only if each component is a Clifford near-ring (*cf.* Theorem 4.1.11)

(2) the multiplicative reduct of a union of zero-symmetric near-rings (*i.e.,* GLCR as well as GRCR seminearings)

(i) is a completely regular semigroup if and only if each component of zero-symmetric near-ring is completely regular (*cf.* Theorem 4.1.4)

(ii) is an inverse semigroup if and only if each component is a zero-symmetric inverse near-ring (*cf.* Theorem 4.1.10)

(iii) is a Clifford semigroup if and only if each component is a zero-symmetric Clifford near-ring (*cf.* Theorem 4.1.12)

Thus the multiplicative reduct of seminearrings which are union of near-rings or zero-symmetric near-rings behave nicely with the component near-rings in respect of being completely regular, inverse and Clifford. But in respect of being regular the situation is not so nice. The component near-rings need not be regular for a multiplicative regular seminearring which is union of near-rings (*cf.* Example 4.1.5). This motivates us to find necessary and sufficient conditions for a union of near-rings (*i.e.,* GLCR seminearring) or a zero-symmetric near-rings (*i.e.,* both GLCR as well as GRCR seminearring) to be a union of regular near-rings and a union of zero-symmetric regular near-rings (*cf.* Theorems 4.1.7 and 4.1.8).

4.1 On Union of Near-rings

Definition 4.1.1. Suppose $(S, +, \cdot)$ is a seminearring. Then S is said to be a *multiplicatively completely regular seminearring* (*multiplicatively Clifford seminearring*) if the multiplicative reduct is completely regular (resp. Clifford).

Note 4.1.2. A near-ring is always not only an additively regular, but also an additively inverse as well as an additively completely regular and an additively Clifford seminearring. So in what follows, a near-ring, whose multiplicative reduct is an inverse or a completely regular or a Clifford semigroup, is called an *inverse near-ring* or a *completely regular near-ring* or a *Clifford near-ring*, respectively, instead of calling them a *multiplicatively inverse near-ring* or a *multiplicatively completely regular near-ring* or a *multiplicatively Clifford near-ring.*

In Theorems 3.1.17 and 3.1.20 characterization of generalized left completely regular (GLCR) seminearrings (generalized right completely regular (GRCR) seminearrings) have been obtained as union (disjoint) of near-rings (zero-symmetric near-rings respectively).

In the following two theorems the intention is to study the case of taking the multiplicative reduct of a union of near-rings (zero-symmetric near-rings) to be a completely regular semigroup.

Theorem 4.1.3. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is a generalized left completely regular (GLCR) as well as multiplicatively completely regular;

(2) Every $\mathcal{H}^+$ -class is a completely regular near-ring;

(3) S is a union (disjoint) of completely regular near-rings.

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds. Then by Theorem 3.1.15 every $\mathcal{H}^+$ -class is a near-ring. Let $a \in S$ and $y \in \mathcal{H}_a^+$. Then by 3.1.14 there exists a unique element $y^* \in \mathcal{H}_a^+$ such that $y + y^* + y = y$, $y + y^* = y^* + y$ and $(y + y^*)y = y + y^*$. Since S is multiplicatively completely regular, there exists a unique $c_y \in S$ such that $yc_yy = y$, $c_yyc_y = c_y$ and $c_yy = yc_y$. Now

$$\begin{aligned} & y + y^* + y = y \\ \Rightarrow \quad & (y + y^* + y)c_yy = yc_yy \\ \Rightarrow \quad & yc_yy + y^*c_yy + yc_yy = yc_yy \\ \Rightarrow \quad & y + y^*c_yy + y = y. \end{aligned}$$

Also,

$$\begin{aligned} & y^* + y + y^* = y^* \\ \Rightarrow \quad & (y^* + y + y^*)c_yy = y^*c_yy \\ \Rightarrow \quad & y^*c_yy + yc_yy + y^*c_yy = y^*c_yy \\ \Rightarrow \quad & y^*c_yy + y + y^*c_yy = y^*c_yy. \end{aligned}$$

Now $y\mathcal{H}^+y^*$. Since $\mathcal{H}^+$ is a right congruence (*cf.* Lemma 3.1.12), $yc_yy\mathcal{H}^+y^*c_yy$. So $y\mathcal{H}^+y^*c_yy$. As y^*, $y \in \mathcal{H}_a^+$ and $(\mathcal{H}_a^+, +, \cdot)$ is a near-ring, so $y^*y \in \mathcal{H}_a^+$. Now y^*y $\mathcal{H}^+$ y implies that y^*yc_y $\mathcal{H}^+$ yc_y ($\because$ $\mathcal{H}^+$ is right congruence). Hence y^*c_yy $\mathcal{H}^+$ yc_y ($\because$ $yc_y = c_yy$). Thus y $\mathcal{H}^+$ yc_y ($\because$ $y\mathcal{H}^+y^*c_yy$). Therefore, y $\mathcal{H}^+$ c_yy whence yc_y $\mathcal{H}^+$ c_yyc_y. So y $\mathcal{H}^+$ yc_y and yc_y $\mathcal{H}^+$ c_y Therefore $c_y \in \mathcal{H}_a^+$. Hence $(\mathcal{H}_a^+, +, \cdot)$ is a completely regular near-ring. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) holds in view of Theorem 3.1.17. □

The following result is the zero-symmetric counterpart of Theorem 4.1.3 whose proof follows from Theorem 3.1.20 and Theorem 4.1.3.

Theorem 4.1.4. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is both generalized left completely regular (GLCR) and generalized right completely regular (GRCR) as well as multiplicatively completely regular;

(2) Every $\mathcal{H}^+$ -class is a completely regular, zero-symmetric near-ring;

(3) S is a union (disjoint) of completely regular, zero-symmetric near-rings.

In the above results it has been observed that if a seminearring S is a union of near-rings (*i.e.*, GLCR seminearring) together with S is multiplicatively completely regular then each component near-ring is also multiplicatively completely regular. But the case is not so nice if completely regularity is replaced by regularity, *i.e.*, if S is a seminearring which is a union of near-rings (*i.e.*, GLCR seminearring) and which is multiplicatively regular then the component near-rings are not necessarily regular. This is illustrated in the following example.

Example 4.1.5. Consider '+' on $T = \{u, a, b, c\}$ defined as follows

$+$	u	a	b	c
u	u	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	b	a	u

$(T, +)$ is semigroup considered in Example 3.1.1. $(T, +)$ is a completely regular semigroup. Let us define '$*$' on T by $x * y = x$ for all x, $y \in T$. Then $(T, +, *)$ is a additively completely regular seminearring. Now for every $a \in T$, $(a+a)*a = a+a$ and $a*a*a = a$. Hence $(T, +, *)$ is a GLCR seminearring which is also multiplicatively regular. Consider the regular ring $(M_2(\mathbb{R}), +, \cdot)$ and one of its right ideal $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in \mathbb{R} \right\}$. Now let us define $S = S_1 \cup S_2 \cup S_3 \subset T \times M_2(\mathbb{R})$ (seminearring direct product of T and $M_2(\mathbb{R})$), where $S_1 = \{u, c\} \times I$, $S_2 = \{a\} \times M_2(\mathbb{R})$ and $S_3 = \{b\} \times M_2(\mathbb{R})$. Clearly S_1, S_2 and S_3 are near-rings. In view of Theorem 3.1.17, S is a GLCR seminearring. Let $s \in S$. Then $s \in S_1$ or $s \in S_2$ or $s \in S_3$. Let $s \in S_2$ or $s \in S_3$. Then $s = (x, A)$ for some $A \in M_2(\mathbb{R})$ and $x \in \{a, b\}$. Since $M_2(\mathbb{R})$ is regular, there exists $B \in M_2(\mathbb{R})$ such that $ABA = A$. Then $t = (x, B) \in S$ satisfying $sts = s$. Now if $s \in S_1$ then $s = (y, X)$ where $X \in I$ and $y \in \{u, c\}$. Let $X = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ where $a, b \in \mathbb{R}$. Now if $a \neq 0$, take $t = (y, Y)$ where $Y = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{pmatrix}$. Then $t \in S$ satisfying $sts = s$. Similarly if $b \neq 0$, take $t = (y_1, Z)$ where $Z = \begin{pmatrix} 0 & 0 \\ \frac{1}{b} & 0 \end{pmatrix}$ and $y_1 \in \{a, b\}$. Then $t \in S$ satisfying $sts = s$. Hence S is multiplicatively regular. Let $s_1 = (y, C) \in S_1$ where $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y \in \{u, c\}$. Now for any $D = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in I$, $CDC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus there is no $D \in I$ which satisfy $CDC = C$. Hence there exist no such $t \in S_1$ such that $s_1 t s_1 = s_1$. So, S_1 is not a multiplicatively regular near-ring.

The above situation (as discussed immediately preceding the above example) does not improve even after adding zero-symmetricity and take a semiring instead of a seminearring. This is illustrated in the following example.

Example 4.1.6. Let us define '+' on $L = \{\alpha, \beta\}$ as follows.

$+$	α	β
α	α	β
β	β	β

$(L, +)$ is nothing but a semilattice with two elements. Let us define '$*$' on L by $x*y = x$ for all x, $y \in L$. It is a matter of routine verification that $(L, +, *)$ is a semiring which is also multiplicatively regular. Consider the regular ring $(M_2(\mathbb{R}), +, \cdot)$ and one of its right ideal $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in \mathbb{R} \right\}$. Now let us define $S = T_1 \cup T_2 \subset L \times M_2(\mathbb{R})$ (semiring direct product of L and $M_2(\mathbb{R})$), where $T_1 = \{\alpha\} \times I$ and $T_2 = \{\beta\} \times M_2(\mathbb{R})$. Clearly T_1 and T_2 are skew-rings. In view of Theorem 1.3.9, S is a completely regular semiring.

Let $s \in S$. Then $s \in T_1$ or $s \in T_2$. Let $s \in T_2$. Then $s = (\beta, A)$ for some $A \in M_2(\mathbb{R})$. Since $M_2(\mathbb{R})$ is regular so there exists $B \in M_2(\mathbb{R})$ such that $ABA = A$. Let us consider $t = (\beta, B)$, then $t \in S$ satisfying $sts = s$. Now if $s \in T_1$ then $s = (\alpha, X)$ where $X \in I$. Let $X = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ where $a, b \in \mathbb{R}$. Now if $a \neq 0$, take $t = (\alpha, Y)$ where $Y = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{pmatrix}$. Then $t \in S$ satisfying $sts = s$. Similarly if $b \neq 0$, take $t = (\beta, Z)$ where $Z = \begin{pmatrix} 0 & 0 \\ \frac{1}{b} & 0 \end{pmatrix}$. Then $t \in S$ satisfying $sts = s$. Hence S is multiplicatively regular.

Let $s_1 = (\alpha, C) \in T_1$ where $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then for any $D \in I$, $CDC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, *i. e.*, there exist no such $t \in T_1$ such that $s_1 t s_1 = s_1$. So, T_1 is not a multiplicatively regular skew-ring.

So it is the time to find a necessary and sufficient condition for a seminearring S which is a union of near-rings, *i.e.*, a GLCR seminearring, to be such that the component near-rings become regular. This is accomplished in the following result.

Theorem 4.1.7. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is generalized left completely regular (GLCR) in which for each $a \in S$ there exists $b \in S$ such that

(A1) $aba = a$,

(A2) $bab = b$, and

(A3) $(a + a^*)b$, $b(a + a^*) \in \mathcal{H}^+_{ba}$,

where according to Lemma 3.1.14, for each $a \in S$, a^* denotes the unique element in $\mathcal{H}^+_a$ such that $a + a^* + a = a$, $a + a^* = a^* + a$ and $(a + a^*)a = a + a^*$

(2) Every $\mathcal{H}^+$ -class is a regular near-ring;

(3) S is a union (disjoint) of regular near-rings.

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds. Then by Theorem 3.1.15 every $\mathcal{H}^+$ -class is a near-ring. Let $a \in S$. Then there exists $b \in S$ satisfying the conditions $(A1)$, $(A2)$ and $(A3)$. Then as $\mathcal{H}^+$ is a right congruence (*cf.* Lemma 3.1.12), we have

$$\begin{aligned} & (a+a^*)b \;\; \mathcal{H}^+ \;\; b(a+a^*) \\ \Rightarrow & (a+a^*)ba \;\; \mathcal{H}^+ \;\; b(a+a^*)a \\ \Rightarrow & (a+a^*)ba \;\; \mathcal{H}^+ \;\; b(a+a^*) \quad (\text{since } (a+a^*)a = a+a^*) \\ \Rightarrow & (a+a^*)ba \;\; \mathcal{H}^+ \;\; (a+a^*)b \quad (\text{Using } (A3)) \\ \Rightarrow & (a+a^*)ba \;\; \mathcal{H}^+ \;\; (a+a^*)ab \;\; (\text{as } (a+a^*)a = a+a^*). \end{aligned}$$

Since both $(a+a^*)ba$, $(a+a^*)ab$ are additive idempotents and every $\mathcal{H}^+$ -class is a near-ring, we have

$$\begin{aligned} (a+a^*)ba &= (a+a^*)ab \\ &= (a+a^*)b \quad (\text{using } (a+a^*)a = a+a^*). \end{aligned}$$

Again as $(a+a^*)a = a+a^*$ and $aba = a$, we have

$$\begin{aligned} (a+a^*)b &= (a+a^*)ba \\ &= ((a+a^*)a)ba \\ &= (a+a^*)a \\ &= a+a^* \in \mathcal{H}^+_a \end{aligned}$$

Hence $(a+a^*)b \;\; \mathcal{H}^+ \;\; a$. Now as $(a+a^*) \;\; \mathcal{H}^+ \;\; a$ and $\mathcal{H}^+$ is a right congruence, $(a+a^*)b \;\; \mathcal{H}^+ \;\; ab$, whence $ab \;\; \mathcal{H}^+ \;\; a$. Again in view of $(A3)$ and the fact that $(a+a^*)b \;\; \mathcal{H}^+ \;\; a$, we see that $ba \;\; \mathcal{H}^+ \;\; a$. Hence $bab \;\; \mathcal{H}^+ \;\; ab$ *i.e.*, $b \;\; \mathcal{H}^+ \;\; ab$ using $(A2)$. So $a \;\; \mathcal{H}^+ \;\; b$. Thus $\mathcal{H}^+_a$ becomes a regular near-ring following Definition 1.5.10.

(2) $\Rightarrow$ (3) trivially.

(3) $\Rightarrow$ (1): Let S be union (disjoint) of regular near-rings $\{N_\alpha : \alpha \in \Lambda\}$. In view of Theorem 3.1.15 and regularity of each $(N_\alpha, \cdot)$, it follows that S is a multiplicatively regular, GLCR seminearring. Let $a \in S$. Then $a \in N_\beta$ for some $\beta \in \Lambda$. As N_β is a

regular near-ring, there exists $b \in N_\beta$ such that $aba = a$, $bab = b$. Also $\mathcal{H}_a^+ = \mathcal{H}_b^+ = N_\beta$. Now in view of Lemma 3.1.14, there exists unique a^*, $b^* \in N_\beta$ such that $a + a^* = a^* + a = b^* + b = b + b^* = 0_{N_\beta}$ (where 0_{N_β} denotes the additive identity of N_β). Thus ba, $(a + a^*)b$ and $b(a + a^*)$, being members of N_β, are all related under the relation $\mathcal{H}^+$. □

The following result is the zero-symmetric counterpart of Theorem 4.1.7 whose proof follows from Theorem 4.1.4 and Theorem 4.1.7.

Theorem 4.1.8. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is both generalized left completely regular (GLCR) and generalized right completely regular (GRCR) in which for each $a \in S$ there exists $b \in S$ such that

(A1) $aba = a$,

(A2) $bab = b$, and

(A3) $(a + a^*)b$, $b(a + a^*) \in \mathcal{H}_{ba}^+$.

where according to Lemma 3.1.14, for each $a \in S$, a^* denotes the unique element in $\mathcal{H}_a^+$ such that $a + a^* + a = a$, $a + a^* = a^* + a$ and $(a + a^*)a = a + a^*$.

(2) Every $\mathcal{H}^+$ -class is a zero-symmetric regular near-ring;

(3) S is a union (disjoint) of zero-symmetric regular near-rings.

In the following result it is shown that by considering the multiplicative reduct of a seminearring S which is a union of some near-rings (*i.e.*, a GLCR seminearring) to be an inverse semigroup, then as the case of completely regular one, each component near-ring becomes multiplicatively inverse.

Theorem 4.1.9. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is generalized left completely regular (GLCR) as well as multiplicatively inverse seminearring;

(2) Every $\mathcal{H}^+$ -class is an inverse near-ring;

(3) S is a union (disjoint) of inverse near-rings.

Proof. (1) $\Rightarrow$ (2): Suppose (1) holds. Let $a \in S$. Then since in view of Theorem 3.1.15 every $\mathcal{H}^+$ -class is a near-ring, we have $a \;\; \mathcal{H}^+ \;\; a^2$. Also there exists a unique $b \in S$ satisfying the conditions $aba = a$ and $bab = b$. Now

$$\begin{aligned}
& \quad a \;\; \mathcal{H}^+ \;\; a^2 \\
& \Rightarrow ab^2a \;\; \mathcal{H}^+ \;\; a^2b^2a \;\; (\text{Since } \mathcal{H}^+ \text{ is a right congruence}) \\
& \Rightarrow ab^2a \;\; \mathcal{H}^+ \;\; a(ab)(ba) \\
& \Rightarrow ab^2a \;\; \mathcal{H}^+ \;\; a(ba)(ab) \;\; (\text{Since } ab \text{ and } ba \text{ both are idempotents of the inverse semigroup } (S,\cdot)) \\
& \Rightarrow ab^2a \;\; \mathcal{H}^+ \;\; (aba)(ab) \\
& \Rightarrow ab^2a \;\; \mathcal{H}^+ \;\; a^2b \;\; (\text{Since } aba = a).
\end{aligned}$$

Again in view of the facts that $a \;\; \mathcal{H}^+ \;\; a^2$ and $\mathcal{H}^+$ is a right congruence we have $a^2b \;\; \mathcal{H}^+ \;\; ab$. So $(ab, ab^2a) \in \mathcal{H}^+$. Also since

$$\begin{aligned}
& \quad b \;\; \mathcal{H}^+ \;\; b^2 \\
& \Rightarrow ba^2b \;\; \mathcal{H}^+ \;\; b^2a^2b \;\; (\text{Since } \mathcal{H}^+ \text{ is a right congruence}) \\
& \Rightarrow ba^2b \;\; \mathcal{H}^+ \;\; b(ba)(ab) \\
& \Rightarrow ba^2b \;\; \mathcal{H}^+ \;\; b(ab)(ba) \;\; (\text{Since } ab \text{ and } ba \text{ both are idempotents of the inverse semigroup } (S,\cdot)) \\
& \Rightarrow ba^2b \;\; \mathcal{H}^+ \;\; (bab)(ba) \\
& \Rightarrow ba^2b \;\; \mathcal{H}^+ \;\; b^2a \;\; (\text{Since } bab = b).
\end{aligned}$$

Also $b^2a \;\; \mathcal{H}^+ \;\; ba$. Thus $(ba^2b, ba) \in \mathcal{H}^+$. Now since ab and ba both are idempotents of the inverse semigroup $(S, \cdot)$,

$$\begin{aligned}
ab^2a &= (ab)(ba) \\
&= (ba)(ab) \\
&= ba^2b
\end{aligned}$$

whence $ab \, \mathcal{H}^+ \, ba$. Again $(a+a^*)b \in \mathcal{H}^+_{ab}$ and $(b+b^*)a \in \mathcal{H}^+_{ba} = \mathcal{H}^+_{ab}$, where according to Lemma 3.1.14, for each $a \in S$, a^* denotes the unique element in $\mathcal{H}^+_a$ such that $a+a^*+a = a$, $a + a^* = a^* + a$ and $(a + a^*)a = a + a^*$. Now,

$$\begin{aligned}
(a + a^*)b + (a + a^*)b &= (a + a^* + a + a^*)b \\
&= (a + a^*)b \in E^+(S).
\end{aligned}$$

Also $(b+b^*)a \in E^+(S)$. Since every $\mathcal{H}^+$ class contains a unique additive idempotent, therefore $(a+a^*)b = (b+b^*)a$. Now $(a+a^*)$, $(b+b^*)$, being zero elements of some $\mathcal{H}^+$ classes, are idempotents in the inverse semigroup $(S,\cdot)$ and so

$$\begin{aligned}
(a+a^*)(b+b^*) &= (b+b^*)(a+a^*)\\
\Rightarrow (a+a^*)(b+b^*)a &= (b+b^*)(a+a^*)a\\
\Rightarrow (a+a^*)(a+a^*)b &= (b+b^*)(a+a^*)a \quad (\text{as } (a+a^*)b = (b+b^*)a)\\
\Rightarrow (a+a^*)b &= (b+b^*)(a+a^*).
\end{aligned}$$

Again as $\mathcal{H}^+$ is a right congruence, we have $(b+b^*)(a+a^*)\ \mathcal{H}^+\ b(a+a^*)$. Hence $(a+a^*)b\ \mathcal{H}^+\ b(a+a^*)$. Then in view of the proof of (1) $\Rightarrow$ (2) of Theorem 4.1.7, we have (a, ab), $(ab, ba) \in \mathcal{H}^+$. Therefore, $b = bab\ \ \mathcal{H}^+\ \ a$. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1) holds in view of Theorem 4.1.3. $\square$

The following result is the zero-symmetric counterpart of Theorem 4.1.9 whose proof follows from Theorem 4.1.4 and Theorem 4.1.9.

Theorem 4.1.10. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is both generalized left completely regular (GLCR) and generalized right completely regular (GRCR) which is multiplicatively inverse seminearring as well;

(2) Every $\mathcal{H}^+$ -class is a zero-symmetric, inverse near-ring;

(3) S is a union (disjoint) of zero-symmetric, inverse near-rings.

Combining Theorems 4.1.3 and 4.1.9 we obtain the following result.

Theorem 4.1.11. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is generalized left completely regular (GLCR) as well as multiplicatively Clifford seminearring;

(2) Every $\mathcal{H}^+$ -class is a Clifford near-ring;

(3) S is a union (disjoint) of Clifford near-rings.

Combining Theorems 4.1.4 and 4.1.10 we obtain the following result.

Theorem 4.1.12. *Let S be a seminearring. Then the following statements are equivalent:*

(1) S is both generalized left completely regular (GLCR) and generalized right completely regular (GRCR) as well as multiplicatively Clifford seminearring;

(2) Every $\mathcal{H}^+$ -class is a Clifford near-ring as well as zero-symmetric;

(3) S is a union (disjoint) of zero-symmetric Clifford near-rings.

CHAPTER 5

ON COMPLETELY REGULAR SEMINEARRINGS

5

On Completely Regular Seminearrings

To extend the structure theory of completely regular semigroups in the setting of seminearrings, Mukherjee et al. formulated the notions of *left completely regular (LCR) seminearrings, right completely regular (RCR) seminearrings, left completely simple seminearrings, right completely simple seminearrings, left Clifford seminearrings and right Clifford seminearrings* in [90] and obtained some structure theorems. This study has been extended by obtaining some more structure theorems for left (right) Clifford seminearrings which has constituted **Chapter 2** and [89]. Mukherjee et al.'s study on LCR and RCR seminearings has further been extended to generalized left completely regular (GLCR) seminearrings and generalized right completely regular (GRCR) seminearrings which is included in **Chapter 3** and published in [88]. The above mentioned studies on seminearrings have their counterpart in semirings (Recall that if one of the distributive properties of a semiring is removed then the resulting structure is a seminearring) for which one can be referred to [113, 114]. But the striking difference between the study in seminearring setting and that in semiring setting is that left, right concepts coincide in semirings. In this context one natural question arises - what class of seminearrings can be obtained if the main axioms leading to left, right completely regular seminearrings are made to coincide? The present chapter is the outcome of an attempt to find an answer to the above question. The types of

---

This chapter is mainly based on the work published in the following paper:
Tuhin Manna et al., *On Completely Regular Seminearrings*, Communicated.

seminearrings we introduce and study in this chapter are called respectively *completely regular seminearring* and *completely simple seminearring.* Among other things, here the analogy between the decomposition of completely regular seminearrings and the decomposition of completely regular semigroups has been studied.

5.1 Completely Regular Seminearrings

Definition 5.1.1. A seminearring $(S, +, \cdot)$ is said to be a *completely regular seminearring* if

(i) for each $a \in S$ there exists $x_a \in S$ satisfying

(C1) $a + x_a + a = a$,

(C2) $a + x_a = x_a + a$, and

(C3) $a(a + x_a) + (a + x_a)a = (a + x_a)a + a(a + x_a) = a + x_a$

(ii) $xe \;\; \mathcal{J}^+ \;\; ex$ for any $x \in S$, for any $e \in E^+(S)$.

Example 5.1.2. A seminearring which is left as well as right completely regular seminearring is always a completely regular seminearring.

Example 5.1.3. A near-ring which is not zero-symmetric is always a left completely regular seminearring but not a completely regular one.

Example 5.1.4. Consider the semigroup $(S, +)$ defined as follows

$+$	e	a	f	b
e	e	a	f	b
a	a	e	b	f
f	f	b	f	b
b	b	f	b	f

Let $T = \{f_1, f_2, f_3\}$ be the subseminearring of the seminearring $M(S)$ of the self-maps of $(S, +)$ where

$$f_1(a) = b,\; f_2(a) = f,\; f_3(a) = e,$$
$$f_1(e) = e,\; f_2(e) = e,\; f_3(e) = e,$$
$$f_1(b) = e,\; f_2(b) = e,\; f_3(b) = e,$$
$$f_1(f) = f,\; f_2(f) = f,\; f_3(f) = f.$$

The Cayley tables are given below:

$+$	f_1	f_2	f_3
f_1	f_2	f_1	f_1
f_2	f_1	f_2	f_2
f_3	f_1	f_2	f_3

$\circ$	f_1	f_2	f_3
f_1	f_3	f_2	f_3
f_2	f_3	f_2	f_3
f_3	f_3	f_2	f_3

T is a completely regular seminearring which is right completely regular but not left completely regular.

Remark 5.1.5. We could not find examples of

(i) A completely regular seminearring which is neither left completely regular nor right completely regular, and

(ii) A right completely regular seminearring which is not completely regular.

Proposition 5.1.6. *Let S be a completely regular seminearring. Then $\mathcal{J}^+$ is a congruence on S.*

Proof. Since by definition of completely regular seminearring (*cf.* Definition 5.1.1)$(S,+)$ is a completely regular semigroup, $\mathcal{J}^+$ is compatible w. r. t. addition. Also using right ditributivity of the seminearring under consideration, it can be easily proved that $\mathcal{J}^+$ is right compatible w. r. t. the multiplication, *i. e.*, if $a \ \mathcal{J}^+ \ b$ then $ac \ \mathcal{J}^+ \ bc$ for any $c \in S$. Again let $s \ \mathcal{J}^+ \ t$. Then $s = z + t + y$ for some $z, y \in S$. Let $c \in S$. Then by repeated use of condition (ii) of Definition 5.1.1 and right distributive property of multiplication over addition we obtain

$$\begin{aligned}
cs &= c(z+t+y) \\
&= (c+x_c+c)(z+t+y) \\
&= (c+x_c)(z+t+y)+c(z+t+y) \\
&= r+(z+t+y)(c+x_c)+m+c(z+t+y) (\text{for some } r,m \in S) \\
&= r+z(c+x_c)+t(c+x_c)+y(c+x_c)+m+c(z+t+y) \\
&= r+z(c+x_c)+p+(c+x_c)t+q+y(c+x_c)+m+c(z+t+y) (\text{for some } p,q \in S) \\
&= r+z(c+x_c)+p+ct+x_ct+q+y(c+x_c)+m+c(z+t+y).
\end{aligned}$$

Similar calculation show that $ct = z' + cs + t'$ for some $z', t' \in S$ whence $cs \ \mathcal{J}^+ \ ct$. $\square$

Lemma 5.1.7. *Let S be a completely regular seminearring. Then each additive idempotent of S is a multiplicative idempotent, too.*

Proof. Let $e \in E^+(S)$. Then in view of Definition 5.1.1, there exists $x_e \in S$ satisfying $(C1)$ $e + x_e + e = e$, $(C2)$ $e + x_e = x_e + e$ and $(C3)$ $e(e + x_e) + (e + x_e)e = e + x_e$. From $(C1)$ and $(C2)$ we get $x_e + e + e = e$, *i. e.*, $x_e + e = e$. Hence from $(C3)$ we obtain $e^2 + e^2 = e$, *i. e.*, $(e + e)e = e^2 = e$. □

Proposition 5.1.8. *Let $(S, +, \cdot)$ be a completely regular seminearring which is also left completely regular. Then S is right completely regular seminearrings.*

Proof. Let $a \in S$ then there exists an $x_a \in S$ satisfying

(C1) $a + x_a + a = a$,

(C2) $a + x_a = x_a + a$,

(C3) $a(a + x_a) + (a + x_a)a = (a + x_a)a + a(a + x_a) = a + x_a$, and

(C4) $(a + x_a)a = a + x_a$

Now from $(C3)$ we get,

$$\begin{aligned} a + x_a &= a(a + x_a) + (a + x_a)a \\ &= a(a + x_a) + (a + x_a) \quad (\text{Using } (C4)) \\ &= a(a + x_a) + (a + x_a)^2 \quad (\text{Using } 5.1.7) \\ &= (a + a + x_a)(a + x_a) \\ &= (a + x_a + a)(a + x_a) \quad (\text{Using } (C2)) \\ &= a(a + x_a) \quad (\text{Using } (C1)). \end{aligned}$$

Thus S is right completely regular seminearring. □

Proposition 5.1.9. *Let S be a completely regular seminearring. Then $\mathcal{J}^+$ becomes a bi-semilattice congruence on S.*

Proof. Since $\mathcal{J}^+$ is a congruence on S (*cf.* Proposition 5.1.6), $S/\mathcal{J}^+ = \{\mathcal{J}^+_a | a \in S\}$ obviously becomes a seminearring w. r. t. the following addition and multiplication: $\mathcal{J}^+_a + \mathcal{J}^+_b = \mathcal{J}^+_{a+b}$ and $\mathcal{J}^+_a \cdot \mathcal{J}^+_b = \mathcal{J}^+_{a \cdot b}$. Again as $\mathcal{J}^+$ is a semilattice congruence on $(S,+)$ which is a completely regular semigroup, $(S/\mathcal{J}^+,+)$ becomes a semilattice. Now let $a,b \in S$. Then in view of Definition 5.1.1, there exists $e \in E^+(S)$ such that $a \;\; \mathcal{J}^+ \;\; e$. As $\mathcal{J}^+$ is a congruence on S, $a^2 \;\; \mathcal{J}^+ \;\; e^2$. Again in view of Lemma 5.1.7, $e^2 = e$. So $a^2 \;\; \mathcal{J}^+ \;\; e$. Thus $\mathcal{J}^+_a = \mathcal{J}^+_{a^2}$. Again as $a \;\; \mathcal{J}^+ \;\; e$ for some $e \in E^+(S)$ and $\mathcal{J}^+$ is a congruence on S, we see that $ab \;\; \mathcal{J}^+ \;\; eb$ and $ba \;\; \mathcal{J}^+ \;\; be$. Now in view of condition (ii) of Definition 5.1.1, $be \;\; \mathcal{J}^+ \;\; eb$ whence $\mathcal{J}^+_{ab} = \mathcal{J}^+_{ba}$. □

Lemma 5.1.10. *Let $(S,+,\cdot)$ be a completely regular seminearring. Then for each $s \in S$ there exists a unique element $s' \in S$ such that*

(C1) $s + s' + s = s$,

(C1′) $s' + s + s' = s'$,

(C2) $s + s' = s' + s$ *and*

(C3) $s(s+s') + (s+s')s = (s+s')s + s(s+s') = s+s'$.

Proof. Let $s \in S$. Then there exists $x_s \in S$ such that $s + x_s + s = s$, $s + x_s = x_s + s$ and $s(s+x_s) + (s+x_s)s = (s+x_s)s + s(s+x_s) = s + x_s$. Consider $\overline{s} = x_s + s + x_s$. Then clearly $s + \overline{s} + s = s$, $s + \overline{s} = \overline{s} + s$, and $\overline{s} + s + \overline{s} = \overline{s}$. Also

$$\begin{aligned} s(s+\overline{s}) + (s+\overline{s})s &= s(s+x_s+s+x_s) + (s+x_s+s+x_s)s \\ &= s(s+x_s) + (s+x_s)s \\ &= s + x_s \\ &= s + x_s + s + x_s \\ &= s + \overline{s} \end{aligned}$$

and

$$\begin{aligned} (s+\overline{s})s + s(s+\overline{s}) &= (s+x_s+s+x_s)s + s(s+x_s+s+x_s) \\ &= (s+x_s)s + s(s+x_s) \\ &= s + x_s \\ &= s + x_s + s + x_s \\ &= s + \overline{s}. \end{aligned}$$

Hence $\overline{s}$ is the desired unique element since in view of Theorem 45 of [100], there exists a unique element $s' \in S$ such that $s + s' + s = s$, $s' + s + s' = s'$ and $s + s' = s' + s$.

□

Proposition 5.1.11. *Let S be a completely regular seminearring. Then $\mathcal{J}_a^+$ is a completely regular seminearring for each $a \in S$.*

Proof. Let $a \in S$ and consider $\mathcal{J}_a^+$. In view of the semigroup theoretic background of the completely regular seminearring we have $(\mathcal{J}_a^+, +)$ is a semigroup. Let $x, y \in \mathcal{J}_a^+$. Then $x \quad \mathcal{J}^+ \quad a$ and $y \quad \mathcal{J}^+ \quad a$. Therefore, in view of Proposition 5.1.6, $xy \quad \mathcal{J}^+ \quad a^2$. Since $\mathcal{J}^+$ is a bi-semilattice congruence on S (*cf.* Proposition 5.1.9), $a \quad \mathcal{J}^+ \quad a^2$ whence $xy \in \mathcal{J}_a^+$. Thus $(\mathcal{J}_a^+, +, \cdot)$ is a seminearring. Now let $s \in \mathcal{J}_a^+$. Then in view of Lemma 5.1.10, there exists unique element $s' \in S$ such that $s + s' + s = s$, $s' + s + s' = s'$, $s + s' = s' + s$ and $s(s+s') + (s+s')s = (s+s')s + s(s+s') = s+s'$. By construction, $s' \in \mathcal{J}_a^+$. Hence we are done. □

We now introduce the following notion for its use in a short while.

Definition 5.1.12. A seminearring $(S, +, \cdot)$ is said to be a *completely simple seminearring* if S is a completely regular seminearring in which any two elements are $\mathcal{J}^+$ related.

In view of Proposition 5.1.9, Proposition 5.1.11, Definition 5.1.12 and Definition 1.5.20 we obtain the following result.

Theorem 5.1.13. *A seminearring S is a completely regular seminearring if and only if it is a bi-semilattice of completely simple seminearrings.*

Theorem 5.1.14. *Let $(S, +, \cdot)$ be a seminearring. Then the following statements are equivalent:*

(i) S is a completely regular seminearring in which additive idempotent commute with each other.

(ii) S is a bi-semilattice of zero-symmetric near-rings.

(iii) S is right Clifford as well as left Clifford seminraring.

Proof. $(i) \Rightarrow (ii)$

Suppose (i) holds. Then $(S,+)$ is a Clifford semigroup and $\mathcal{J}^+ = \mathcal{H}^+$ (*cf.* Result 1.1.38). Hence in view of Proposition 5.1.9, $\mathcal{H}^+$ is a bi-semilattice congruence on S. In view of Theorem 1.1.34, $(\mathcal{H}^+_a, +)$ is a group. Let $b, c \in \mathcal{H}^+_a$. Then $b\ \mathcal{H}^+ a$ and $c\ \mathcal{H}^+ a$. As $\mathcal{H}^+$ is a bi-semilattice congruence on S, $bc\ \mathcal{H}^+ a^2$ and $a^2\ \mathcal{H}^+ a$. Hence $bc\ \mathcal{H}^+ a$. Therefore $(\mathcal{H}^+_a, \cdot)$ is a semigroup. So $(\mathcal{H}^+_a, +, \cdot)$ is a near-ring with its additive identity $0_{\mathcal{H}^+_a} = x_a + a = a + x_a$. As $(\mathcal{H}^+_a, +, \cdot)$ is a near-ring, $(a+x_a)a = 0_{\mathcal{H}^+_a} \cdot a = 0_{\mathcal{H}^+_a}$. Now

$$\begin{aligned} a(a+x_a) + (a+x_a)a &= a + x_a \\ \Rightarrow a \cdot 0_{\mathcal{H}^+_a} + 0_{\mathcal{H}^+_a} \cdot a &= 0_{\mathcal{H}^+_a} \\ \Rightarrow a \cdot 0_{\mathcal{H}^+_a} + 0_{\mathcal{H}^+_a} &= 0_{\mathcal{H}^+_a} \\ \Rightarrow a \cdot 0_{\mathcal{H}^+_a} = 0_{\mathcal{H}^+_a}. \end{aligned}$$

Hence $(\mathcal{H}^+_a, +, \cdot)$ is a zero-symmetric near-ring.

$(ii) \Rightarrow (iii)$

Suppose (ii) holds. Then in view of Corollaries 1.5.33 and 1.5.35, S is right Clifford as well as left Clifford seminraring.

$(ii) \Rightarrow (i)$

Suppose (ii) holds. Let S be a bi-semilattice B of zero-symmetric near-rings N_i $(i \in B)$. Then S admits of a bi-semilattice congruence β such that $B = S/\beta$ with each N_i $(i \in B)$ a β-class as well as a zero-symmetric near-ring. Then $(S,+)$ admits of a semilattice congruence β such that $B = S/\beta$ with each N_i $(i \in B)$ a β-class and a group as well. Hence $(S,+)$ is a Clifford semigroup (*cf.* Definition 1.1.33 and Theorem 1.1.40). Let $a \in S$ then $a \in N_i$ for some $i \in B$. Since each N_i is a zero-symmetric near-ring, there exists an unique $x_a \in N_i$ such that $a + x_a = x_a + a = 0_{N_i}$ where 0_{N_i} denotes the additive identity of N_i. Also $a + x_a + a = a + 0_{N_i} = a$. As N_i is a zero-symmetric near-ring, $(a+x_a)a = 0_{N_i} \cdot a = 0_{N_i} = a \cdot 0_{N_i} = a(a+x_a)$. Hence

$$\begin{aligned} a(a+x_a) + (a+x_a)a &= 0_{N_i} + 0_{N_i} \\ &= 0_{N_i} \\ &= a + x_a \end{aligned}$$

and

$$\begin{aligned}(a+x_a)a+a(a+x_a) &= 0_{N_i}+0_{N_i}\\ &= 0_{N_i}\\ &= a+x_a.\end{aligned}$$

Again for $e \in E^+(S)$, $(S/\beta, +, \cdot)$ being a bi-semilattice, ae and ea both belong to some zero-symmetric near-ring N_j. Thus $ae\ \mathcal{J}^+\ ea$. Consequently, S is a completely regular seminearring.

$\square$

Some Remarks and Scope of Further Study

We list below some remarks and observations which are mainly related with some possible extension of the research work undertaken in this thesis.

1. In **Chapter 2**, we have established that a left (right) Clifford seminearring $(S, +, \cdot)$ can be decomposed as a strong bi-semilattice of near-rings if (i) S is E^+-unitary seminearring and (ii) $a(b+e) = ab + a^0e$ for all $a, b \in S$ and for all $e \in E^+(S)$ holds. In Examples 2.1.12 and 2.1.18, we have shown that property (i) and (ii) are not inherent in left (right) Clifford seminearrings. But we could not find the following two examples:

   (a) A left (right) Clifford seminearring which is E^+-unitary but does not satisfy property (ii), and

   (b) A left (right) Clifford seminearring which satisfies property (ii) but not E^+-unitary seminearring.

   So we could not verify that whether these properties become dependent on a left (right) Clifford seminearring or not.

2. In view of the nice work done on the sub-direct product of *left Clifford* and *right Clifford* semigroups by Sen et al. [112], the study of sub-direct product of left completely regular (*cf.* Definition 1.5.23) and right completely regular seminearrings (*cf.* Definition 1.5.24) may lead to some interesting results as it is easily verifiable that the corresponding direct product is neither a left completely regular nor a right completely regular seminearring.

3. In semigroup theory, the notions of Clifford semigroups, semilattice of groups and strong semilattice of groups coincide. Mukherjee et al. [90] extended the study of structure theory of Clifford semigroups in the setting of seminearrings by obtaining the semigroup theoretic analogue of “A semigroup is Clifford if and only if it is a semilattice of groups” (*cf.* Corollaries 1.5.33 and 1.5.35) and we continued that work in the Chapter 2 by obtaining semigroup theoretic analogue of “A semigroup is Clifford if and only if it is strong semilattice of groups” (*cf.* Theorems 2.1.16, 2.1.21, 2.2.4 and Corollaries 2.3.2, 2.3.3). Recently in [79], S. K. Maity et al. have studied the topological Clifford semigroups and characterized some restricted type of topological Clifford semigroups as a semilattice of topological groups as well as a strong semilattice of topological groups. They have obtained the following theorem:

   Let (S, τ) **be a topological semigroup. Then the following conditions are equivalent:**

   (i) (S, τ) **is a topological Clifford semigroup satisfying the property that for each** $G \in \tau$ **and every** $x \in G$**, there exists an element** $U \in \tau$ **such that** $x \in U \subseteq G \cap J_x$**,**

   (ii) (S, τ) **is a strong semilattice of topological groups,**

   (iii) (S, τ) **is a semilattice of topological groups.**

   In view the work of Mukherjee et al. [90] and the word done in **Chapter 2** of the thesis, one may attempt to obtain an analogue of Maity et al.'s result in the seminearring setting.

Bibliography

[1] Adhikari, M. R., Sen, M. K., Weinert, H. J.: *On k-regular semirings,* Bull. Calcutta Math. Soc., 88 (2), 141-144 (1996).

[2] Ahsan, J.: *Seminear-rings characterized by their $\mathcal{S}$-ideals. I*, Proc. Japan Acad. Ser. A Math. Sci., 71 (5), 101-103 (1995).

[3] Ahsan, J.: *Seminear-rings characterized by their $\mathcal{S}$-ideals. II,* Proc. Japan Acad. Ser. A Math. Sci., 71 (6), 111-113 (1995).

[4] Ahsan, J., Zhongkui, L.: *Strongly idempotent seminearrings and their prime ideal spaces,* Nearrings, Nearfields and $\overline{K}$-Loops (Hamburg, 1995), Math. Appl., Vol. 26, Kluwer Acad. Publ., Dordrecht, 151-166 (1997).

[5] Armstrong, A., Struth, G., Weber, T.: *Programming and automating mathematics in the Tarski-Kleene hierarchy*, J. Log. Algebr. Methods Program., 83 (2), 87-102 (2014).

[6] Ayaragarnchanakul, J., Mitchell, S.: *Division seminear-rings,* Kyungpook Math. J., 34 (1), 67-72 (1994).

[7] Baeten, J. C. M.: *A brief history of process algebra*, Theor. Comput. Sci., 335, 131-146 (2005).

[8] Baeten, J. C. M., Weijland, W. P.: *Process algebra*, Cambridge Tracts in Theoretical Computer Science. 18, Cambridge University Press (1990).

[9] Balakrishnan, R., Perumal, R.: *Left duo seminear-rings*, Scientia Magna, 8 (3), 115-120 (2012).

[10] Bandelt, H. J., Petrich, M.: *Subdirect products of rings and distributive lattics*, Proc. Edinb. Math. Soc., 25, 155-171 (1982).

[11] Blackett, D. W.: *Connecting seminearrings to probability generating functions*, Near-Rings and Near-Fields, 75-81 (1995).

[12] Blair, R. L.: *Ideal lattices and the structure of rings*, Trans. Amer. Math. Soc., 75 (1), 136-153 (1953).

[13] Boykett, T.: *Seminearrings of Polynomials Over Semifields: A Note on Blackett's Fredericton Paper*, Nearrings, Nearfields and K-Loops, Vol 426, 225-236 (1997).

[14] Boykett, T.: *Seminearring models of reversible computation I.* preprint, University of Linz (Published electronically at *citeseer.ist.psu.edu/368766.html*) (1997).

[15] Chakraborty, K., Pal, P., Sardar, S. K.: *Near-ring congruences on additively regular seminearrings*, Semigroup Forum, 101, 285-302 (2020).

[16] Chakraborty, K., Mukherjee, R., Sardar, S. K.: *Near-ring congruences on seminearrings*, Semigroup Forum, 104, 584-593 (2022).

[17] Chakraborty, K., Mukherjee, R.: *Lattice of Near-ring congruences on seminearrings*, Bull. Calcutta Math. Soc., 115 (1), 75-94 (2023).

[18] Chakraborty, K.: *A study of congruences on seminearrings,* Ph.D. Dissertation (Department of Mathematics, Jadavpur University), 2022.

[19] Changphas, T., Denecke, K.: *Complexity of hypersubstitutions and lattices of varieties*, Discuss. Math. Gen. Algebra Appl., 23 (1), 31-43 (2003).

[20] Changphas, T., Denecke, K.: *Green's relations on the seminearring of full hypersubstitutions of type* (n)*,* Algebra Discrete Math., 2, 6-19 (2003).

[21] Clay, J. R.: *Circular block designs from planar near-rings*, Ann. of Discrete Math., 37, 95-105 (1988).

[22] Clay, J. R.: *Generating balanced incomplete block designs from planar near rings*, J. Algebra, 22, 319-331 (1972).

[23] Clifford, A. H., Preston, G. B.: *The algebraic theory of semigroups*, American Mathematical Society (1961).

[24] Davey, B. A., Priestley H. A.: *Introduction to lattice and order*, Cambridge University Press (2002).

[25] Desharnais, J., Struth, G.: *Domain axioms for a family of near-semirings*, AMAST, 330-345 (2008).

[26] Dickson, L. E.: *Definitions of a group and a field by independent postulates*, Trans. Amer. Math. Soc., 6 (2), 198-204 (1905).

[27] Droste, M., Stüber, T., Vogler, H.: *Weighted finite automata over strong bimonoids*, Inform. Sci., 180 (1), 156-166 (2010).

[28] Edwards, P. M.: *On joins with group congruences*, Proc. Edinb. Math. Soc., 40 (1), 63-67 (1997).

[29] Fuchs, L.: *Uber die ideale arithmetischer ringe*, Comment. Math. Helv., 23, 334-341 (1949).

[30] Ghosh, S.: *Characterization of semirings which are subdirect products of a distributive lattice and a ring*, Semigroup Forum, 59, 106-120 (1999).

[31] Gigoń, R. S.: *Completely simple congruences on E-inversive semigroups*, J. Algebra Appl., 15 (6), 1650052 (2016).

[32] Gigoń, R. S.: *Congruences and group congruences on a semigroup*, Semigroup Forum, 86, 431-450 (2013).

[33] Gilbert, N. D., Samman, M.: *Clifford semigroups and seminear-rings of endomorphisms,* Int. Electron. J. Algebra, 7, 110-119 (2010).

[34] Gilbert, N. D., Samman, M.: *Endomorphism seminear-rings of Brandt semigroups,* Comm. Algebra, 38 (11), 4028-4041 (2010).

[35] Glazek, K.: *A Guide to the literature of semirings and their applicatiobs in mathematics and information sciences*, Kluwer Academic Publishers, London (2001).

[36] Golan, J. S.: *Semirings and their applications*, Kluwer Academic Publishers (1999) .

[37] Gomes, G. M. S.: *A characterization of the group congruences on a semigroup*, Semigroup Forum, 46, 48-53 (1993).

[38] Grillet, M. P.: *Semirings with a completely simple additive semigroup,* J. Aust. Math. Soc., 20 (Ser A), 257-267 (1975) .

[39] Hebisch, U., Weinert, H. J.: *Euclidean seminearrings and nearrings,* Near-rings and near-fields (Tübingen, 1985), North Holland Math. Stud., Vol. 137, North Holland, Amsterdam, 105-121, 1987.

[40] Hebisch, U., Weinert, H. J.: *Semirings and semifields*, Handbook of Algebra (editor: M. Hazewinkel), Elsevier, Amsterdam, 425-462, 1996.

[41] Hebisch, U., Weinert, H. J.: *Semirings: algebraic theory and applications in computer science,* Series in Algebra, Vol. 5, World Scientific, 1998.

[42] Holcombe, W. M. L.: *The syntactic near-ring of a linear sequential machine,* Proc. Edinb. Math. Soc., 26 (1), 15-24 (1983).

[43] Holcombe, W. M. L.: *A radical for linear sequential machines,* Proc. Roy. Irish Acad. Sect. A, 84 A (1), 27-35 (1984).

[44] Hoogewijs, A.: *Semi-nearring embeddings,* Med. Konink. Vlaamse Acad. Wetensch. Lett. Schone Kunst. België Kl. Wetensch., 32 (2), 3-11 (1970).

[45] Hoogewijs, A.: *$\mathcal{I}$-Congruences on seminearrings,* An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.), 22 (1), 3-9 (1976).

[46] van Hoorn, G., van Rootselaar, B.: *Fundamental notions in the theory of seminearrings,* Compos. Math., 18, 65-78 (1967).

[47] van Hoorn, W. G.: *Some generalisations of the Jacobson radical for seminearrings and semirings,* Math. Z., 118, 69-82 (1970).

[48] Howie, J. M.: *An introduction to semigroup theory,* Academic Press, London-New York, 1976.

[49] Howie, J. M.: *Fundamentals of semigroup theory,* Clarendon Press, Oxford, 1995.

[50] Huq, S. A.: *Embedding problems, module theory, semisimplicity of seminearrings,* Ann. Soc. Sci. Bruxelles Sér. I, 103 (1), 49-62 (1989).

[51] Hussain, F., Tahir, M., Abdullah, S., Sadiq, N.: *Quotient seminearrings*, Indian J. Sci. Tech., 9 (38), 1-7 (2016).

[52] Jenila, C., Dheena, P.: *Ideal theory in near-semirings and its applications to automata*, Adv. Math. Sci. J., 9, 4293-4302 (2020).

[53] Kalman, J. A.: *Subdirect decomposition of distributive quasilattices*, Fund. Math., 71, 161-163 (1971).

[54] Khachorncharoenkul, P., Laipaporn, K., Wananiyakul, S.: *Left almost seminearrings*, Lobachevskii J. Math., 41 (3), 349-361 (2020).

[55] Khan, W. A., Arif, W., Taouti, A.: *Group seminearrings*, J. Math. Computer Sci., 24, 338-344 (2022).

[56] Khan, W. A., Davvaz, B.: *Soft intersection nearsemirings and its algebraic applications*, Lobachevskii J. Math., 41(3), 362-372 (2020).

[57] Khan, W. A., Rehman, A., Taouti, A.: *Soft near-semirings*, AIMS Mathematics, 5(6), 6464-6478 (2020).

[58] Kim, H. S., Kim, E. S., Lee, D. S., Park, Y. S.: *Some structures of seminear-rings,* Pure Math. Appl., 9 (3-4), 341-350 (1998).

[59] Koppula, K., Kedukodi, B. S., Kuncham, S. P.: *On perfect ideals of seminear-rings*, Beiträge zur Algebra und Geometrie, 1-20 (2020).

[60] Koppula, K., Srinivas, K. B., Prasad, K. S.: *On prime strong ideals of a seminearring*, Mat. Vesnik, 72, 243-256 (2020).

[61] Kornthorng, N., Iampan, A.: *A note on right full k-ideals of seminearrings*, Journal of Informatics and Mathematical Sciences, 4 (3), 255-261 (2012).

[62] Krishna, K. V.: *Near-semirings theory and application,* Ph.D. Dissertation (Department of Mathematics, Indian Institution of Technology Delhi), 2005.

[63] Krishna, K. V., Chatterjee, N.: *A computational model for real-world planning and scheduling problems,* in M. C. Puri (ed.), 6th International Conference of the APORS 2003 on Operational Research: Emerging Paradigms for Information Technology, APORS, New Delhi, India, 58 (2003).

[64] Krishna, K. V., Chatterjee, N.: *A necessary condition to test the minimality of generalized linear sequential machines using the theory of near-semirings,* Algebra Discrete Math., 3, 30-45 (2005).

[65] Krishna, K. V., Chatterjee, N.: *Reprensentation of near-semirings and approximation of their Categories,* Southeast Asian Bull. Math., 31 (5), 903-914 (2007).

[66] Krishna, K. V., Chatterjee, N.: *Holonomy decomposition of seminearrings,* Southeast Asian Bull. Math., 31 (6), 1113-1122 (2007).

[67] Krishna, K. V., Chatterjee, N.: *Reprensentation of near-semirings and approximation of their Categories,* Southeast Asian Bull. Math., 31 (5), 903-914 (2007).

[68] Krishna, K. V., Chatterjee, N.: *A necessary condition to test the minimality of generalized linear sequential machines using the theory of near-semirings*, Algebra Discrete Math., 3, 30-45 (2005).

[69] Kumar, J.: *Affine near-semirings over Brandt semigroups*, Ph.D. Dissertation (Department of Mathematics, Indian Institution of Technology Guwahati) (2014).

[70] Kumar, J., Krishna, K. V.: *Syntactic semigroup problem for the semigroup reducts of affine near-semirings over Brandt semigroups*, Asian-Eur. J. Math., 9 (1), 1650021 (2016).

[71] Kumar, J., Krishna, K. V.: *Rank properties of the semigroup reducts of affine near-semirings over Brandt semigroups*, Semigroup Forum, 93 (3), 516-534 (2016).

[72] Kumar, J., Krishna, K. V.: *Radicals and ideals of affine near-semirings over Brandt semigroups*, Semigroups, Algebras and Operator Theory, 123-133 (2015).

[73] Kumar, J., Krishna, K. V.: *Affine near-semirings over Brandt semigroups*, Comm. Algebra, 42 (12), 5152-5169 (2014).

[74] LaTorre, D. R.: *Group congruences on regular semigroups*, Semigroup Forum, 24 (1), 327-340 (1982).

[75] Ljapin, E. S.: *Semigroups,* Translations of Mathematical Monographs, Amer. Math. Soc., Vol. 3, 1963.

[76] Luo, Y., Fan, X., Li, X.: *Regular congruences on an E-inversive semigroup*, 76 (1), 107-123 (2008).

[77] Magill, K. D. Jr.: *Semigroups and near-rings of continuous functions, general topology and its relations to modern analysis and algebra,* Proceedings of the Third Prague Topological Symposium (1971), Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 283-288 (1972).

[78] Maity, S. K.: *On completely regular semiring,* Ph.D. Dissertation (Department of Pure Mathematics, University of Calcutta), 2006 .

[79] Maity, S. K.: *Semilattice of topological groups*, Comm. Algebra, 49 (9), 3905-3925 (2021).

[80] Manikandan, G., Perumal, R.: *Mid-units in duo seminearrings*, J. Phys. Conf. Ser., 1850 (1), (2021).

[81] Manikandan, G., Perumal, R.: *Mate and mutual mate functions in a seminearring*, AIMS Mathematics, 5 (5), 4974-4982 (2020).

[82] Manikandan, G., Perumal., R., Arulprakasam, R.: *Strong K (r, s) seminear-rings*, AIP Conference Proceedings, 2112 (1), (2019).

[83] Maxson, C. J., Oswald, A.: *The centralizer of the general linear group*, Proc. Edinb. Math. Soc., 27, 73-89 (1984).

[84] Maxson, C. J., Smith, K. C.: *Centralizer near-rings that are endomorphism rings*, Proc. Amer. Math. Soc., 80 (2), 189–95 (1980).

[85] Meldrum, J. D. P., Samman, M.: *On free d.g. seminear-rings,* Riv. Mat. Univ. Parma, 6, 93-102 (1997).

[86] Mikhalev, A. V., Pilz, G. F.: *The concise handbook of algebra,* Kluwer Academic Publishers, 2002.

[87] Mukherjee, R., Chakraborty, K., Sardar, S. K.: *On full subdirect products of a bi-semilattice and a (zero-symmetric) near-ring*, Asian-Eur. J. Math., 16 (9), 2350157 (2023).

[88] Mukherjee, R., Manna, T., Pal, P.: *A note on additively completely regular seminearrings*, Semigroup Forum, 100 (1), 339-347 (2020).

[89] Mukherjee, R., Pal, P., Manna, T., Sardar, S. K.: *On additively completely regular seminearrings: II,* Comm. Algebra, 47 (5), 1954-1963 (2019).

[90] Mukherjee (Pal), R., Pal, P., Sardar, S. K.: *On additively completely regular seminearrings,* Comm. Algebra, 45 (12), 5111-5122 (2017).

[91] Mukherjee, R.: *A study of congruences and decomposition of seminearrings,* Ph.D. Dissertation (Department of Mathematics, Jadavpur University), 2015 .

[92] Mukherjee, R., Sardar, S. K., Pal, P.: *On additively commutative near-ring congruences on seminearrings*, Semigroup Forum, 91(3), 573-583 (2015).

[93] Neuerburg, K. M.: *Semi-nearrings of bivariate polynomials over a field*, Nearrings and Nearfields. Springer, Dordrecht, 253-262 (2005).

[94] Pal, P., Mukherjee, R., Ghosh, M.: *Rees matrix seminearrings*, Asian-Eur. J. Math., 15 (9), 2250172 (2022).

[95] Perumal, R., Arulprakasam, R., Radhakrishnan, M.: *A note on ideals in seminear-rings*, J. Phys. Conf. Ser., 1000 (1), (2018).

[96] Perumal, R., Arulprakasam, R., Radhakrishnan, M.: *A note on normal seminear-rings*, Asia math., 1 (2), 15-21 (2017).

[97] Perumal, R., Balakrishnan, R.: *Left bipotent seminear-rings*, Internat. J. Algebra, 6 (26), 1289-1295 (2012).

[98] Perumal, R., Chinnaraj, P.: *Medial left bipotent seminear-rings*, Mathematics and Computing. Springer Proceedings in Mathematics and Statistics, 139, 451-457 (2015).

[99] Petrich, M.: *Inverse semigroups,* John Wiley and Sons, 1984.

[100] Petrich, M., Reilly, N. R.: *Completely regular semigroups,* Wiley, New York, 1999.

[101] Pilz, G.: *Near-Rings,* North Holland, 1977.

[102] Rivas, E., Jaskelioff, M., Schrijvers, T.: *From monoids to near-semirings: the essence of monadplus and alternative*, In Proceedings of the 17th International Symposium on Principles and Practice of Declarative Programming, 196-207 (2015).

[103] Romanowska, A. B., Smith, J.: *Bisemilattices of subsemilattices,* J. Algebra, 70 (1), 78-88 (1981).

[104] van Rootselaar, B.: *Algebraische kennzeichnung freier wortarithmetiken,* Compositio Math., 15, 156-168 (1963).

[105] Samman, M.: *Topics in seminear-ring theory* Doctoral Dissertation, University of Edinburgh (1998).

[106] Samman, M.: *On decompositions of d. g. seminear-rings,* Math. Pannon., 14 (1), 129-134 (2003).

[107] Samman, M.: *On the representation of d. g. seminear-rings,* Algebra Colloq., 14 (1), 79-84 (2007).

[108] Samman, M.: *Inverse semigroups and seminear-rings,* J. Algebra Appl., 8 (5), 713-721 (2009).

[109] Sardar, S. K., Mukherjee, R.: *On additively regular seminearrings,* Semigroup Forum, 88 (3), 541-554 (2014).

[110] Sardar, S. K., Mukherjee, R.: *Erratum to: On additively regular seminearring*, Semigroup Forum, 93 (3), 629-631 (2016).

[111] Scott, S. D.: *The automorphism group of a near-ring*, Proc. Amer. Math. Soc, 80 (4), 560-62 (1980).

[112] Sen, M. K., Ghosh, S., Pal, S.: *On a class of subdirect products of left and right Clifford semigroups,* Comm. Algebra, 32 (7), 2609-2615 (2004).

[113] Sen, M. K., Maity, S. K., Weinert, H. J.: *Completely simple semirings,* Bull. Calcutta Math. Soc., 97 (2), 163-172 (2005).

[114] Sen, M. K., Maity, S. K., Shum, K. P.: *Clifford semirings and generalized Clifford semirings,* Taiwanese J. Math., 9 (3), 433-444 (2005).

[115] Senthil, S., Perumal, R.: *Minimal prime ideals in seminearrings*, J. Phys. Conf. Ser., 1850 (1), (2021).

[116] Senthil, S., Perumal, R., Arulprakasam, R.: *On structures of right duo seminearring*, Advances in Mathematics: Scientific Journal, 9 (11), 9231-9236 (2020).

[117] Shabir, M., Ahmed, I.: *Weakly regular seminearrings,* Int. Electron. J. Algebra, 2, 114-126 (2007).

[118] Vasantha Kandasamy, W. B.: *The units of semigroup semi-near rings,* Opuscula Math., 15, 113-114 (1995).

[119] Weinert, H. J.: *Related representation theorems for rings, semirings, near-rings and semi-near-rings by partial transformations and partial endomorphisms,* Proc. Edinb. Math. Soc. (2), 20 (4), 307-315 (1976-77).

[120] Weinert, H. J.: *Seminearrings, seminearfields and their semigroup-theoretical background,* Semigroup Forum, 24, 231-254 (1982).

[121] Weinert, H. J.: *Extensions of seminearrings by semigroups of right quotients,* Recent developments in the algebraic, analytical and topological theory of semigroups (Oberwolfach, 1981), Lecture Notes in Math., Vol. 998, Spriger, Berlin, 412-486 (1983).

[122] Weinert, H. J.: *Partially and fully ordered seminearrings and nearrings,* Nearrings and near-fields (Tübingen, 1985), North-Holland Math. Stud., Vol. 137, North-Holland, Amsterdam, 277-294 (1987).

[123] Zeleznekow, J.: *Regular semirings*, Semigroup Forum, 23, 119-136 (1981).

[124] Zhu, X., Zhou, Y., Xiong, W.: *Characterizations of right ring congruences on a semiring,* Facta Univ. Ser. Math. Inform., 25, 1-5 (2010).

[125] Zulfiqar, M.: *A note on the upper radicals of seminearrings,* Proyecciones, 29 (1), 49-56 (2010).

List of Publications Based on the Thesis

A list of publications resulted from the work of this thesis has been appended below.

(1) **Rajlaxmi Mukherjee, Pavel Pal, Tuhin Manna, Sujit Kumar Sardar, On Additively Completely Regular Seminearrings: II, *Communications in Algebra,* 47 (5), 1954-1963 (2019)**.

(2) **Rajlaxmi Mukherjee, Tuhin Manna, Pavel Pal, A Note on Additively Completely Regular Seminearrings**, ***Semigroup Forum,* 100 (1), 339-347 (2020)**.

(3) **Tuhin Manna, Kamalika Chakraborty, Rajlaxmi Mukherjee, Sujit Kumar Sardar, On Union of Regular Near-rings**, **Communicated.**

(4) **Rajlaxmi Mukherjee, Tuhin Manna, Kamalika Chakraborty, Sujit Kumar Sardar, On Completely Regular Seminearrings**, **Communicated.**

Subject Index