Ref. No.: Ex/ME(M2)/BS/B/MATH/T/111/2024

B.E. MECHANICAL ENGINEERING EXAMINATIONS - 2024

First Year First Semester

Mathematics-I

Time : Three hours Full Marks:100

(*Notations and symbols have their usual meanings.*
Use separate answerscript for each group.)

GROUP- A

Answer any five questions from the following.

1. (a) Define a convergent sequence of real numbers. Show that, $x_n = \left\{\dfrac{\sqrt{n}}{n+1}\right\}$ is a convergent sequence. 4

(b)Examine the convergence of the series:

(i) $\sum \sin\dfrac{1}{n}$ (ii) $\dfrac{1}{2} + \dfrac{1}{2\cdot 2^2} + \dfrac{1}{3\cdot 2^3} + \dfrac{1}{4\cdot 2^4} + \cdots$ 3 + 3

2. (a) If

$$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2,$$

find values of a, b, c. 5

(b) Find the range of x for which $y = \left(x^4 - 6x^3 + 12x^2 + 5x + 7\right)$ is concave upwards and concave downwards. Find also its point of inflection. 5

3. (a) State the Lagrange's *Mean Value Theorem.* Use this theorem to prove $\dfrac{x}{1+x^2} < \tan^{-1} x < x$, if $x > 0$. 1+3

(b) Define a monotone sequence and a bounded sequence. Show that the sequence $\{x_n\}$ defined by $x_1 = \dfrac{3}{2}$ and $x_{n+1} = 2 - \dfrac{1}{x_n}$ for $n \geq 1$ is monotonic and bounded. Find the limit of the sequence. 6

4. (a) Find all the asymptotes of the curve

$$y^3 - 2xy^2 - x^2y + 2x^3 + 2x^2 - 3xy + x - 2y + 1 = 0. \quad 7$$

[Turn over

(b) Prove that radius of curvature at any point of the curve

$$x = a(\theta + \sin\vartheta), \quad y = a(1 - \cos\vartheta)$$

is $4a\cos\frac{\theta}{2}$. 3

5. (a) If $y = \sin ax + \cos ax$, then prove that

$$y_n = a^n\sqrt{1 + (-1)^n \sin 2ax}$$

5

(b) Show that the function

$$f(x) = \begin{cases} \dfrac{x^3 + y^3}{x - y} & \text{when } x \neq y, \\ 0, & \text{when } x = y \end{cases}$$

is not continuous at $(0, 0)$. 3

(c) State the Euler's theorem for a homogeneous function defined on $\mathbb{R}^2$. 2

6. (a) Find all local maxima and minima of the function $f(x, y) = x^2 + xy + y^2 - 3x$. 5

(b) Use Lagrange's method of undetermined multipliers to find the maximum and minimum of the function $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$. 5

7. (a) If $v = r^3$ and $r^2 = x^2 + y^2 + z^2$ then show that

$$\frac{1}{yz}v_{yz} + \frac{1}{zx}v_{zx} + \frac{1}{xy}v_{xy} = \frac{9}{r}$$

4

(b) If $u = \sin^{-1}\left[\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}\right]^{\frac{1}{2}}$ then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left(13 + \tan^2 u\right)$$

6

8. (a) Using Lagrange's method of multiplier, find the shortest distance of the point $(2, 1, -3)$ from the plane $2x + y = 2z + 4$. 5

(b) Define *directional derivative* for a real valued function defined on $\mathbb{R}^2$ at some direction. Show that the function

$$f(x) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{when } (x,y) \neq (0,0), \\ 0 & \text{when } x = y = 0 \end{cases}$$

has directional derivative at $(0,0)$ at any direction. 5

GROUP- B

Answer any five questions from the following.

1. (a) Prove that the necessary and sufficient condition that a bounded real valued function f(x) be integrable on the closed interval [a,b] is that for each $\epsilon > 0$, however small, there exists a partition P of [a,b] such that $0 \leq U(P,f) - L(P,f) \leq \epsilon$. 6

(b) Given

$$f(x) = \begin{cases} 0 & \text{x is rational} \\ 1, & \text{x is irrational} \end{cases}$$

Prove from definition that f is not Riemann integrable on [a,b] for any $a < b$. 4

2. (a) Prove that the improper integral $\int_1^\infty \frac{dx}{x^p}$, p is a real number, converges if $p > 1$ and diverges if $p \leq 1$. 4

(b) Examine the convergence of following improper integrals

(i) $\int_0^\infty \frac{x^{\frac{3}{2}}}{3x^2+5}dx$. (ii) $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ 3 + 3

3. (a)Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy$ over the positive quadrant of the circle $x^2+y^2=1$. 6

(b) Prove that $\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta cos^{2n-1}\theta d\theta, m,n > 0$. Hence deduce that $\beta(\frac{1}{2},\frac{1}{2}) = \pi$. 4

4. (a) Evaluate $\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{6}}}$ 5

(b) Evaluate $\iint r cos\theta dr d\theta$ over the region R bounded by the initial line and the two circles $r = 3cos\theta$ and $r = 5cos\theta$. 5

[Turn over

5. (a) Find the entire area enclosed by the curve $r = acos2\theta$. 4

(b) Find the approximate value of $\int_0^3 \sqrt{1+x^3}dx$ by Simpson's $\frac{1}{3}$ Rule taking six intervals. 6

6. (a) Find the area of the loop formed by the curve $4y^2 = x^2(4-x)$. 5

(b) Find the length of the arc of the parabola $y^2 = 16x$ measured from the vertex to an extremity of the latus rectum. 5

7. (a) Find the volume of the solid generated by revolving the cardiode $r = a(1 - cos\theta)$ about the initial line. 5

(b) Find the surface of the solid generated by revolution of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis. 5