Study of certain types of summability methods and Approximation theory

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SMT. RIMA GHOSH

Under the guidance of
Prof. Pratulananda Das
Dr. Sudipta Dutta

DEPARTMENT OF MATHEMATICS
JADAVPUR UNIVERSITY
KOLKATA-700032
WEST BENGAL, INDIA

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যাদবপুর বিশ্ববিদ্যালয়
JADAVPUR UNIVERSITY

CERTIFICATE FROM THE SUPERVISORS

This is to certify that the thesis entitled "**Study of certain types of summability methods and Approximation theory** " submitted by **Smt. Rima Ghosh** who got her name registered on 30.08.2018 **(Index No:171/18/Maths./26)** for the award of Ph. D. (Science) degree of Jadavpur University, is absolutely based upon her own work under the supervision of **Prof. Pratulananda Das** and **Dr. Sudipta Dutta** and that neither this thesis nor any part of it has been submitted for either any degree/ diploma or any other academic award anywhere before.

P. Das 25.08.2023

Prof. Pratulananda Das
Department of Mathematics
Jadavpur University
Kolkata-700032
West Bengal,India

Professor
DEPARTMENT OF MATHEMATICS
Jadavpur University
Kolkata – 700 032, West Bengal

Sudipta Dutta 25.08.2023

Dr. Sudipta Dutta
Assistant Professor
Department of Mathematics
Govt. General Degree College,
Manbazar-II
Purulia-723131
West Bengal,India

Asst. Professor (W.B.E.S.)
Dept. Of Mathematics
Govt. Gen. Degree College
Manbazar - II Purulia
Govt. of W.B

Dedicated to
my mother
Smt. Sumitra Ghosh
and
my father
Sri Sadhu Gopal Ghosh

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Glossary of Notations

$\mathcal{I}$	Ideal.
$\mathcal{F}(\mathcal{I})$	Filter associated with the ideal $\mathcal{I}$.
$\mathcal{I}\text{-}\lim\limits_{k} x_k = x$	$\{x_k\}_{k\in\mathbb{N}}$ is $\mathcal{I}$-convergent to x.
$x_k \xrightarrow{\mathcal{I}} x$	$\{x_k\}_{k\in\mathbb{N}}$ is $\mathcal{I}$-convergent to x.
$\mathcal{I}_{fin}$	The class of all finite subsets of $\mathbb{N}$.
$\mathcal{I}_d$	The class of all subsets of $\mathbb{N}$(or $\mathbb{N}\times\mathbb{N}$) with natural density zero.
$M_1(RS)$	The class of all non-negative regular summability matrix.
$M_2(RH-RS)$	The class of all non-negative RH-regular summability matrix.
$x_k \xrightarrow{A^{\mathcal{I}}-st} L$	$\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to L.
$A_2^{\mathcal{I}}\text{-}st\text{-}\lim\limits_{m,n} x_{mn} = L$	$\{x_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-statistically convergent to L.
$\mathcal{I}\text{-}\lim\limits_{n} \sum\limits_{k\in\mathbb{N}} a_{nk}x_k = L$	$\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-summable to L.
$\mathcal{I}_2\text{-}\lim\limits_{j,k} \sum\limits_{(m,n)\in\mathbb{N}^2} a_{jkmn}x_{mn} = L$	$\{x_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to L.
$f_n \xrightarrow{\mathcal{I}^*_\alpha-ue} f$	$\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*_\alpha$-uniformly equally convergent to f.
$f_n \xrightarrow{\mathcal{I}^*_\alpha-ud} f$	$\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*_\alpha$-uniformly discretely convergent to f.
$f_n \xrightarrow{\mathcal{I}^*_\alpha-sue} f$	$\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*_\alpha$-strongly uniformly equally convergent to f.
$f_n \xrightarrow{\mathcal{I}^*\alpha} f$	$\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^*\alpha$ convergent to f.
$f_n \xrightarrow{\mathcal{I}\alpha-e} f$	$\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}\alpha$ equally convergent to f.
$\Phi^{\mathcal{I}^*_\alpha-ue}$	The class of all $\mathcal{I}^*_\alpha$- uniform equal limit functions defined on X.
$\Phi^{\mathcal{I}^*_\alpha-ud}$	The class of all $\mathcal{I}^*_\alpha$- uniform discrete limit functions defined on X.
$\Phi^{\mathcal{I}^*_\alpha-sue}$	The class of all $\mathcal{I}^*_\alpha$-strongly uniform equal limit functions on X.
$\Phi^{\mathcal{I}\alpha-e}$	The class of all $\mathcal{I}\alpha$-equal limit functions defined on X.

Abstract

This thesis is a detailed work on generalization of Korovkin approximation process and related summability theory and some new types of convergence. As we know after the exploration of Korovkin approximation theorem numerous paths have opened for researcher to investigate the Korovkin set. Also summability theory is a very interesting part of mathematics as it helps to approximate the limit of divergent sequences. We have combined these two branches and derived some abundant theorems which have significant depth and utilities.

This Thesis contains four main chapters. In Chapter 2, we have established Korovkin type approximation theorems for positive linear operators on $UC_*[0,\infty)$, the Banach space of all real valued uniform continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0,\infty)$ using the notion of $A^{\mathcal{I}}$-statistical convergence and $A^{\mathcal{I}}$-summability method for real sequences and examined the Korovkin set on $UC_*\left([0,\infty)\times[0,\infty)\right)$. In Chapter 3, we have approximated a sequence of positive convolution operators on $C[a,b]$, the Banach space of all real valued continuous functions on $[a,b]$ endowed with the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C[a,b]$, based on the notion of $A^{\mathcal{I}}$-summability and $A^{\mathcal{I}}$-statistical convergence and calculated corresponding rates of convergence. In Chapter 4, we have visited on topological spaces and introduced statistical $A_{\mathcal{T}}$-strong convergence and $A^{\mathcal{I}}_{\mathcal{T}}$-strong convergence, both of which are generalizations of $A_{\mathcal{T}}$-strong convergence in Hausdorff spaces via a certain class of special functions. Similar to the classic scenario, we find some correlations between $A^{\mathcal{I}}$-statistical convergence and $A^{\mathcal{I}}_{\mathcal{T}}$-strong convergence. Additionally, we obtain a characterization of $A^{\mathcal{I}}$-statistical convergence. In Chapter 5, we have introduced new forms of convergence, namely, $\mathcal{I}^*_\alpha$-ue, $\mathcal{I}^*_\alpha$-ud, $\mathcal{I}^*_\alpha$-sue and $\mathcal{I}\alpha$-equal convergence and follow up some associated findings adding to the lattice features of the classes made up of all those real valued functions defined on a metric space (X,d).

Publications

1. S. Dutta, R. Ghosh, *Korovkin type approximation theorem on an infinite interval in $A^{\mathcal{I}}$-statistical sense*, **Acta Math. Univ. Comenianae**, Vol. LXXXIX (1) (2020), 131-142.

2. S. Dutta, R. Ghosh, *Korovkin type approximation on an infinite interval via generalized matrix summability method using ideal*, **Stud. Univ. Babeş-Bolyai Math.**, Vol. 65 (2) (2020), 243-254.

3. S. Dutta, R.Ghosh, *$A^{\mathcal{I}}$-statistical approximation of continuous functions by sequence of convolution operators*, **Kragujevac Journal of Mathematics**, Vol. 46 (3) (2022), 355-368.

4. R.Ghosh, S. Dutta, *Approximation of a continuous function by a sequence of convolution operators via a matrix summability method using ideals* **Sarajevo Journal of Mathematics**, Vol. 19 (1) (32) (2023), 13-27.

Contents

0 Preface

We go over the relevant history and background information for this thesis in this prelude. On the basis of a review of the literature on sequence, series, summability, and approximation theory, we have written this chapter. We use the symbols $\mathbb{N}$: set of natural numbers, and $\mathbb{R}$: set of real numbers throughout the thesis. The meaning of other symbols and notations used in this thesis is as usual. In putting together this thesis, we reference a few books [2, 3, 7, 58, 72], and we are grateful to their writers. We must first go back to the fundamental ideas behind this thesis, such as topological space, metric topology, neighbourhood basis, subbasis, metrizability, Hausdorff space etc.

Definition 0.0.1. *(Topology) Let X be a non-empty set. A collection τ of subsets of X is said to be a topology on X if τ satisfies the following properties:*
(i) *$X \in \tau$ and $\emptyset \in \tau$,*
(ii) *any arbitrary union of members of τ belongs to τ,*
(iii) *intersection of any finite number of members of τ is in τ.*

If τ is a topology on a set X then the pair (X, τ) is called a topological space. The members of τ are called the open sets of X.

Example 0.0.1. *Let X be a non-empty set. Then $\tau = \{\emptyset, X\}$ is a topology on X, called the indiscrete topology.*

Example 0.0.2. *Let X be a non-empty set. Then $\tau = \mathcal{P}(X)$, the power set of X, is a topology on X. Here every subset of X is an open set. This topology is called the discrete topology.*

If X is a topological space and $x \in X$, a neighbourhood of x is a set U which contains an open set V containing x. The collection $\mathcal{U}_x$ of all neighbourhoods of x is the neighbourhood system at x. A neighbourhood base at x in the topological space X is a subcollection $\mathcal{B}_x$ taken from the neighbourhood system $\mathcal{U}_x$, having the property that each $U \in \mathcal{U}_x$ contains some $V \in \mathcal{B}_x$.

Definition 0.0.2. *If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that*
(i) for each $x \in X$, there is at least one basis element B containing x and
(ii) if x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_τ, containing x such that $B_\tau \subset B_1 \cap B_2$.

Definition 0.0.3. *(Order Topology) Let X be a set containing more than one element with a simple order relation $<$. Let $\mathcal{B}$ be the collection of all sets of the following types:*
(i) All open intervals (a, b) in X.
(ii) All intervals of the form $[a, b)$, where a is the smallest element (if any) of X.
(iii) All intervals of the form $(a, b]$, where b is the largest element (if any) of X. The collection $\mathcal{B}$ is a basis for a topology on X, which is called the order topology (usual topology, lower limit topology, upper limit topology respectively).

Definition 0.0.4. *(Hausdorff Space) A topological space X is called a Hausdorff space if for each pair of distinct points u, v of X, there exist disjoint neighborhoods U and V containing preceding points respectively.*

Definition 0.0.5. *(Metric Space) Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is said to be a metric on X if it satisfies the following conditions:*
(i) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry),
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality).

A metric space is a non-empty set X equipped with a metric d on X and is denoted by the pair (X, d) or simply X.

Different metrics can be defined on a single non-empty set and this gives rise to distinct metric spaces. Given $\varepsilon > 0$, the set $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$ is called the ε-ball centered at x.

Example 0.0.3. *The function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by*

$$d(x, y) = |x - y| \quad \text{for all } x, y \in \mathbb{R}$$

is a metric on the set $\mathbb{R}$. It is called the usual metric on $\mathbb{R}$.

Example 0.0.4. *The set $C[0, 1]$ consisting of all real-valued continuous functions defined on $[0, 1]$ with the function d given by*

$$d(f, g) = \int_0^1 |f(x) - g(x)| dx, \quad \text{for all } f, g \in C[0, 1]$$

is a metric space.

Theorem 0.0.1. *In any metric space* (X, d)*,*
(i) *the union of an arbitrary family of open sets is open.*
(ii) *the intersection of a finite number of open sets is open.*

Definition 0.0.6. *(Convergence) Let* (X, d) *be any metric space. A sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of points of* X *is said to converge to a point* x *of* X*, if for each* $\varepsilon > 0$ *there exists a natural number* K *such that* $d(x_n, x) < \varepsilon$ *for all* $n \geq K$*.*

Definition 0.0.7. *Let* (X, d) *be a metric space and the collection of all* ε*-balls* $B(x, \varepsilon)$*, for* $x \in X$ *and* $\varepsilon > 0$*, is a basis for a topology on* X*. Then the topology is called the metric topology induced by* d*.*

Definition 0.0.8. *A topological space* X *is said to be metrizable if there exists a metric* d *on the set* X *that induces the topology of* X*. A metric space is a metrizable space* X *together with a specific metric* d *that gives the topology of* X*.*

The metric topology generated by the usual metric on $\mathbb{R}$ is called the usual topology on $\mathbb{R}$. Obviously, it is a metrizable space.

The lower limit topology τ_L is a topology defined on the set $\mathbb{R}$ generated by the basis of all half open intervals $[a, b)$, where a and b are real numbers. The space $\mathbb{R}$ with lower limit topology is called the Sorgenfrey line and is denoted by $(\mathbb{R}, \tau_L)$ or $\mathbb{R}_l$. Interestingly, it is a non metrizable Hausdorff space.

In approximation theory, positive approximation processes have numerous useful applications. It frequently occurs in issues involving the approximation of continuous functions, especially when additional requirements for qualitative characteristics like monotonicity, convexity, shape preservation, etc. are present. In order to determine if a given sequence $\{L_n(f)\}_{n\in\mathbb{N}}$ of positive linear operators on $C[0, 1]$, the space of real-valued continuous functions on [0, 1], is approximative, P.P. Korovkin [42] developed a powerful and straightforward criterion in 1953. As a matter of fact, it suffices to confirm that $L_n(f) \longrightarrow f$ uniformly on [0,1] only for the test functions $f \in \{1, x, x^2\}$.

Many academics have expanded Korovkin's theorem to include other function spaces. Numerous studies have been conducted to examine the test functions in relation to abstract spaces, Banach lattices, Banach algebras, Banach spaces, and many other spaces. This study introduces a new branch of mathematics known as Korovkin type approximation theory (see[23, 48]). Using the "modulus of continuity", numerous scholars simultaneously explored the rates of various forms of convergence. Here, we review this widely used jargon that is essential to our main chapters. The modulus of continuity is defined and denoted by

$$\omega(f, \alpha) = \sup_{|y-x|\leq\alpha} \mid f(y) - f(x) \mid .$$

It gives the maximum oscillation of f in any interval of length not exceeding $\alpha > 0$. That is generally known that

$$\lim_{\alpha \to 0} \omega(f, \alpha) = \omega(f, 0) = 0,$$

and that for any constants $\alpha > 0$ and $p > 0$,

$$\omega(f, p\alpha) \leq (1 + [p])\omega(f, \alpha),$$

where $[p]$ is the greatest integer not exceeding p.

On the other hand, divergent series were of non-debated part of mathematics. Divergent series, which are infinite series that do not converge, were seriously studied by L. Euler. A galaxy of extraordinarily talented mathematicians came after him. The foundation of summability theory is the study of divergent series and it is now widely used in analysis and practical mathematics.

The thought of statistical convergence for a real numbers sequence had been introduced independently by Fast [32], Steinhaus [67] in 1951 based on the well known natural density (or asymptotic density). In 1959, Schoenberg [63] reintroduced the same notion and studied related summability methods. Interestingly, statistical convergence generalized the concept of the ordinary convergence and expanded as a branch of mathematical analysis. Further, a lot of analysis and investigations had been made on this area by Šalát [59] and Fridy [33].

Let $K \subset \mathbb{N}$, K_n denotes the set $\{k \in K : k \leq n\}$ and $Card(K_n)$ stands for the cardinality of the set K_n.

(Upper natural density of K) $\bar{d}(K) = \limsup_n \dfrac{Card(K_n)}{n}$

(Lower natural density of K) $\underline{d}(K) = \liminf_n \dfrac{Card(K_n)}{n}$.

If $\bar{d}(K) = \underline{d}(K)$ then we say that the natural density (or asymptotic density) of K exists and it is simply denoted by $d\,(K)$. Clearly $d\,(K) = \lim_n \dfrac{Card(K_n)}{n}$. "A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is said to be statistically convergent to L if for arbitrary $\varepsilon > 0$ the set $K(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ has natural density zero" [63]
Define a sequence

$$x_n = \begin{cases} 1 & \text{when} \quad n = k^2 \\ 0 & \text{when} \quad n \neq k^2 \quad \text{where} \quad k = 1, 2, 3, \cdots . \end{cases}$$

This sequence is statistically convergent to 0 but not usually convergent to 0. Any finite subset of $\mathbb{N}$ has natural density zero. So, obviously statistical convergence is a generalization of usual convergence. This area was explored in the works of Šalát [59] and Fridy [33].

Example 0.0.5. *The set $\{n^2 : n \in \mathbb{N}\}$ has natural density zero.*

Note 0.0.1. *The set of all bounded statistically convergent sequences of real numbers is a linear subspace of the norm linear space of all bounded sequence of real numbers with supremum norm.*

After many years, $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence for real numbers sequence were introduced by Kostyrko et. al. [43] in 2001, generalizing the notion of statistical convergence based on the certain class of subsets $\mathcal{I}$ of $P(\mathbb{N})$, called ideal. Later, Lahiri and Das continued several developments on this branch (see [44, 45, 46, 70]). Subsequently, using the notion of $\mathcal{I}$-convergence, several noteworthy investigations had been done for sequences of functions with real value (see [5, 20, 41, 50]).

A family $\mathcal{I} \subset P(\mathbb{X})$ of subsets of a nonempty set $\mathbb{X}$ is known an ideal on $\mathbb{X}$ if it is closed under finite union and satisfy hereditary property (i.e., $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$). When $\mathcal{I}$ contains all singleton subsets of $\mathbb{X}$, is referred to an admissible ideal. The family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{X} : \exists\, C \in \mathcal{I} : M = \mathbb{X} \setminus C\}$ is called the filter associated with the ideal $\mathcal{I}$.

Definition 0.0.9. *[43]. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$.*

Definition 0.0.10. *[43] The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-convergent to $x \in \mathbb{R}$ if there is a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < ... < m_k < ...\}$ such that $\lim_{k\to\infty} x_{m_k} = x$.*

Definition 0.0.11. *An ideal $\mathcal{I} \subset P(\mathbb{N})$ is known as a P-ideal (or said to satisfy the (AP) condition) if it is admissible and for any countable family $\{C_1, C_2, C_3, ...\}$ of pairwise disjoint sets in $\mathcal{I}$ there is a countable family $\{D_1, D_2, D_3, ...\}$ of sets such that $C_i \Delta D_i$ $(i = 1, 2, \cdots)$ is finite and $D = \bigcup_{j=1}^{\infty} D_j \in \mathcal{I}$.*

To know more about P-ideals, we can see [43].

Definition 0.0.12. *[54] An admissible ideal $\mathcal{I} \subset P(\mathbb{N})$ is known a good ideal if for any countable family $\{C_1, C_2, C_3, ...\}$ of sets with $C_i \notin \mathcal{I}$ for each $i \in \mathbb{N}$ there exists a countable family $\{D_1, D_2, D_3, ...\}$ of pairwise disjoint sets such that $D_i \subset C_i$, $D_i \in \mathcal{I}$ and $\bigcup_{i=1}^{\infty} D_i \notin \mathcal{I}$.*

A real numbers sequence $\{y_n\}_{n\in\mathbb{N}}$ is called $\mathcal{I}$-divergent if for any $M > 0$, either $\{n \in \mathbb{N} : y_n \leq M\} \in \mathcal{I}$ or $\{n \in \mathbb{N} : y_n \geq -M\} \in \mathcal{I}$. B. K. Lahiri defined these as $\mathcal{I}$-convergent to $+\infty$ and $\mathcal{I}$-convergent to $-\infty$ respectively [44].

Definition 0.0.13. *[43] A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of the metric space (X, ρ) is said to be $\mathcal{I}$-convergent to $\xi \in X$ if and only if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \varepsilon\}$ belongs to $\mathcal{I}$. It is denoted by $\mathcal{I}\text{-}\lim_{n\to\infty} = \xi$.*

If $\mathcal{I}$ is an admissible ideal, then the usual convergence in X implies $\mathcal{I}$-convergence in X. Interestingly, the limit of any $\mathcal{I}$-convergent sequence is unique. The family $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible (or free) ideal of $\mathbb{N}$. For $\mathcal{I} = \mathcal{I}_d$, ideal convergence yields statistical convergence and for $\mathcal{I} = \mathcal{I}_{fin}$, the collection of all finite subsets of $\mathbb{N}$, ideal convergence coincides with usual convergence.

Definition 0.0.14. *[43] A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of the metric space (X, ρ) is said to be $\mathcal{I}^*$-convergent to $\xi \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < ... < m_k < ...\}$ such that $\lim_{k\to\infty} \rho(x_{m_k}, \xi) = 0$. It is denoted by $\mathcal{I}^*\text{-}\lim_{n\to\infty} = \xi$.*

Proposition 0.0.1. *[43] Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^*\text{-}\lim_{n\to\infty} = \xi$, then $\mathcal{I}\text{-}\lim_{n\to\infty} = \xi$.*

Theorem 0.0.2. *[43] Let $\mathcal{I} \subset P(\mathbb{N})$ be an admissible ideal. If the ideal $\mathcal{I}$ has the property (AP) then $\mathcal{I}\text{-}\lim_{n\to\infty} = \xi$ implies $\mathcal{I}^*\text{-}\lim_{n\to\infty} = \xi$.*

On the other hand, Kolk [38] expanded statistical convergence to include A-statistical convergence. Matrix summability and A-statistical convergence have since been the subject of numerous investigations (see [18, 22, 38]). The two methodologies indicated above were combined, and the broad concept of $A^{\mathcal{I}}$-statistical convergence was proposed and explored, in particular in [60, 61, 62].

If $x = \{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $A = (a_{nk})_{n,k=1}^{\infty}$ is an infinite matrix, then Ax is the sequence whose n-th term is given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

A matrix A is called regular if the limits of the convergent sequences are preserved, i.e., $\lim_{k\to\infty} A_k(x) = \lim_{k\to\infty} x_k$ for all convergent sequence $x = \{x_k\}_{k\in\mathbb{N}}$. It is well-known Silverman-Toeplitz theorem states that the necessary and sufficient conditions for A to be regular are

$$\begin{aligned} &1) \quad \sup_n \sum_k |a_{nk}| < \infty; \\ &2) \quad \lim_n a_{nk} = 0, \text{ for each } k; \\ &3) \quad \lim_n \sum_k a_{nk} = 1 (see [36]). \end{aligned}$$

When the sequence $\{A_n\}_{n\in\mathbb{N}}$ is convergent, the sequence $\{x_k\}_{k\in\mathbb{N}}$ is called A-summable. Following the notions of A-summability and $\mathcal{I}$-convergence, Das et. al. introduced the new notion of $A^{\mathcal{I}}$-summablity.

Recall the following definitions

Definition 0.0.15. *[61, 62] Let $A = (a_{nk})$ be a non-negative regular matrix. For an ideal $\mathcal{I}$ of $\mathbb{N}$, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta \right\} \in \mathcal{I}$$

where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. *In this case, we write* $A^{\mathcal{I}}\text{-}st\text{-}\lim_n x_n = L$.

Definition 0.0.16 ([62]). *Let* $A = (a_{nk})$ *be a non-negative regular matrix. Then a real sequence* $x = \{x_k\}_{k\in\mathbb{N}}$ *is said to be* $A^{\mathcal{I}}$*-summable to a number* L *if for every* $\varepsilon > 0$, $\{n \in \mathbb{N} : |A_n(x) - L| \geq \varepsilon\} \in \mathcal{I}$ *where* $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$.

Thus $x = \{x_k\}_{k\in\mathbb{N}}$ *is* $A^{\mathcal{I}}$*-summable to a number* L *if and only if* $\{A_n(x)\}_{n\in\mathbb{N}}$ *is* $\mathcal{I}$*-convergent to* L*. In this case, we write* $\mathcal{I}\text{-}\lim_n \sum_{k\in\mathbb{N}} a_{nk}x_k = L$.

We have worked with double sequences too and uses the notion of convergence in Pringsheim's sense [56].

Definition 0.0.17. *[49, 51] A real double sequence* $\{x_{mn}\}_{m,n\in\mathbb{N}}$ *is statistically convergent to* L *if for every* $\varepsilon > 0$, $\lim_{j,k} \frac{|\{m \leq j, n \leq k : |x_{mn} - L| \geq \varepsilon\}|}{jk} = 0$.

A non-trivial ideal $\mathcal{I}$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also. Let $\mathcal{I}_0 = \{A \subset \mathbb{N}\times\mathbb{N} : \text{there is } m(A) \in \mathbb{N} \text{ such that } i, j \geq m(A) \Longrightarrow (i, j) \notin A\}$. Then $\mathcal{I}_0$ is a non-trivial strongly admissible ideal [17]. Let $A = (a_{jkmn})$ be a four dimensional summability matrix. For a given double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$, the A-transform of x, denoted by $Ax := ((Ax)_{jk})$, is given by

$$(Ax)_{jk} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}x_{mn}$$

provided the double series converges in Pringsheim sense for every $(j, k) \in \mathbb{N}^2$. A four-dimensional analog of regularity was first proposed by Robison in 1926 [57] by taking the extra premise of boundedness into account. Because a convergent double sequence is not always bounded, this assumption was made.

Remember that if a four-dimensional matrix $A = (a_{jkmn})$ translates every bounded convergent double sequence into a convergent double sequence with the same limit, it is said to be a RH-regular. A four-dimensional matrix $A = (a_{jkmn})$ meets the Robison-Hamilton requirements and is deemed to be RH-regular if and only if

(i) $\lim_{j,k} a_{jkmn} = 0$ for each $(m,n) \in \mathbb{N}^2$,

(ii) $\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} = 1$,

(iii) $\lim_{j,k} \sum_{m\in\mathbb{N}} |a_{jkmn}| = 0$ for each $n \in \mathbb{N}$,

(iv) $\lim_{j,k} \sum_{n\in\mathbb{N}} |a_{jkmn}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{jkmn}|$ is convergent for each $(j,k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers M_0 and N_0 such that $\sum_{m,n>N_0} |a_{jkmn}| < M_0$

holds for every $(j,k) \in \mathbb{N}^2$.

If $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix and $K \subset \mathbb{N}^2$ then the A-density of K is given by

$$\delta_A^{(2)}\{K\} = \lim_{j,k} \sum_{(m,n)\in K} a_{jkmn}$$

provided the limit exists.

Definition 0.0.18. *[38] A real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be A-statistically convergent to a number L if for every $\varepsilon > 0$*

$$\delta_A^{(2)}\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\} = 0.$$

Recall the following definitions

Definition 0.0.19. *[28] A real double sequence $\{x_{m,n}\}_{m,n\in\mathbb{N}}$ is said to be $\mathcal{I}_2$-statistically convergent to L if for each $\varepsilon > 0$ and $\delta > 0$,*

$$\Big\{(j,k) \in \mathbb{N}^2 : \tfrac{1}{jk}|\{m \leq j, n \leq k : |x_{mn} - L| \geq \varepsilon\}| \geq \delta\Big\} \in \mathcal{I}.$$

Definition 0.0.20. *[28] Let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix. Then a real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-statistically convergent to a number L if for every $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\right\} \in \mathcal{I}$$

where $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : \ |x_{mn} - L| \geq \varepsilon\}$.
We write, $A_2^{\mathcal{I}}$-st-$\lim_{m,n} x_{mn} = L$.

Convergence is one of the important concepts of topology. In a topological space summability theory is now playing a relevant role. Summability theory is the theory of assigning limits to a scalar-valued or a linear space valued sequence, especially if the sequence is divergent [9]. Some authors have studied summability theory in the topological spaces by assuming the topological space to have a group structure or a linear structure. Also some are introducing certain summability methods those do not need a linear structure in the topological space as A-statistical convergence [11, 47, 69]. The idea of A-statistical convergence was introduced by Kolk [39] using a non-negative regular matrix A (which subsequently included the ideas of statistical, lacunary statistical or λ-statistical convergence as special cases). More recent works in this field can be found in [30, 40, 52] where many references are mentioned. We need some definitions to follow. According to Lahiri et. al. [45] "A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a topological space X is said to be $\mathcal{I}$-convergent to $\xi \in X$ if for any nonempty open set U containing ξ, the set $\{k \in \mathbb{N} : x_k \notin U\} \in \mathcal{I}$."

Now we recall some well-known notions from literature.

According to Maio et. al.[47] "Let $A = (b_{nk})$ be a matrix in $M_1(RS)$. If $\{x_k\}_{k\in\mathbb{N}}$ is a sequence in a Hausdorff space X such that for any open set U that contains ξ

$$\lim_n \sum_{k:x_k\notin U} b_{nk} = 0$$

then the sequence $\{x_k\}_{k\in\mathbb{N}}$ is called A-statistically convergent to $\xi \in X$."

In 1988-1989, J. S. Connor [12, 13] defined A-strongly convergence as "For a non-negative regular summability matrix $A = (b_{nk})$, a sequence $\{x_k\}_{k\in\mathbb{N}}$ in a metric space (X, d) is said to be A-strongly convergent to $\xi \in X$ if

$$\lim_n \sum_k d(x_k, \xi) b_{nk} = 0.''$$

The concepts of uniform equal convergence, uniform discrete convergence, and strong uniform equal convergence for sequences of real valued functions were established and investigated by Papanastassiou in [55]. In a later study, Das and Papanastassiou [21] looked at a number of lattice features of these classes of functions. Das et. al. [20] first proposed the idea of $\mathcal{I}^*$-uniform equal convergence of sequences of real valued functions. The lattice features of these new classes of functions were further explored by Das and Dutta [19].

The notion of continuous convergence for a sequence of real valued functions was introduced in the twentieth century (see [68]) and later it known as α-convergence in literature (see also [4]).

Definition 0.0.21. *[68] A sequence $\{f_n\}_{n\in\mathbb{N}}$ is known as* ***α-convergent*** *to f if for any element $x \in X$ and for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to x, $\{f_n(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$.*

Later R. Das [21] introduced the notions of α-equally and α-uniformly equally convergence and defined these in the following ways

Definition 0.0.22. *[21] A sequence $\{f_n\}_{n\in\mathbb{N}}$ is called **α-equally convergent** (or in short, α-e convergent) to the function f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_n \varepsilon_n = 0$ such that for each $x \in X$ and for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to x, there exists a natural number $k_0 \equiv k_0(x,(x_n))$ satisfying $|f_n(x_n) - f(x)| < \varepsilon_n$ for all $n \geq k_0$.*

Definition 0.0.23. *[21] A sequence $\{f_n\}_{n\in\mathbb{N}}$ is called **α-uniformly equally convergent** (or in short, α-ue convergent) to f if there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_n \varepsilon_n = 0$ and a natural number k_0 such that*

$$Card\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \varepsilon_n\} \leq k_0$$

for each $x \in X$ and for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to x.

In [20], Das et. al. introduced the idea of $\mathcal{I}^*$-uniformly equally convergence generalizing the idea of uniform eqal convergence.

Definition 0.0.24. *[20] A sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions is called **$\mathcal{I}^*$-uniformly equally convergent** (or in short, $\mathcal{I}^*$-ue convergent) to the function f if there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_n \varepsilon_n = 0$, a set $S := S(\{\varepsilon_n\})$ belonging to $\mathcal{F}(\mathcal{I})$ and a natural number $k \equiv k(\{\varepsilon_n\})$ such that*

$$Card\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\} \leq k \text{ for all } x \in X.$$

We write this as $f_n \xrightarrow{\mathcal{I}^*-ue} f$.

Interestingly, it is observed that $\mathcal{I}^*$-u convergence is stronger than $\mathcal{I}^*$-ue convergence which is again stronger than $\mathcal{I}^*$-e convergence (see Example 3.2 and Example 3.3 in [19]). It should be noted that, α-e convergence is intermediate between α-ue convergence and α convergence.

For each class of functions Φ on X, we must keep in mind the following widely recognized definitions of the various forms of lattice [16].

Definition 0.0.25. (a) *A class of functions, Φ, is referred to as a lattice if it contains all constants and for any $f, g \in \Phi$, $\max(f,g)$, $\min(f,g) \in \Phi$. A lattice Φ having translation property (i.e., for any real number c and $f \in \Phi$, $f + c \in \Phi$) is called a translation lattice.*

(b) *A translation lattice Φ is referred to as a congruence lattice if $f \in \Phi$ implies $-f \in \Phi$.*

(c) *A congruence lattice Φ is referred to as a weakly affine if there is an unbounded set $S \subset (0,\infty)$ such that $c \in S$ and $f \in \Phi$ implies $cf \in \Phi$.*

(d) A congruence lattice Φ is referred to as an affine lattice if $c \in \mathbb{R}$, $f \in \Phi$ implies $cf \in \Phi$.

(e) A lattice Φ is referred to as a subtractive lattice if $f - g \in \Phi$ for all $f, g \in \Phi$.

(f) A subtractive lattice Φ is referred to as an ordinary class if $f.g \in \Phi$ for all $f, g \in \Phi$ and for any $f \in \Phi$, s.t. $f(x) \neq 0$ for all $x \in X$, $1/f \in \Phi$.

We also need to think back to the Korovkin approximation theorem. Let X and Y be two normed spaces. The collection of all linear operators from X into Y will be denoted by $L(X, Y)$. Then $L(X, Y)$ is a linear(vector) space under the algebraic operations $(S+T)(x) = S(x) + T(x)$ and $(\alpha T)(x) = \alpha T(x)$. In fact, $L(X, Y)$ is a normed linear space (or normed spaces) under the operator norm $||T|| = \sup\{||T(x)|| : ||x|| = 1\}$. Moreover, if Y is a Banach space, then $L(X, Y)$ is also a Banach space.

Recall that a linear operator $f : X \to \mathbb{R}$, where X is a real vector space, is called a linear functional on X. An operator $T : X \to Y$ between two normed spaces is called positive if $T(x) \geq 0$ holds for all $x \geq 0$. We will close this chapter with a noteworthy convergence feature for sequences of positive linear operators on $C[0, 1]$ (thanks to P. P. Korovkin, see [42]). Korovkin's findings show how useful the order structures are.

Let $T : C[0, 1] \to C[0, 1]$ be a linear operator where $C[0, 1]$ is equipped with the sup norm $|| \cdot ||_\infty$. For instance, $\lim_n f_n = f$ in $C[0, 1]$ will mean $\lim_n ||f_n - f|| = 0$, i.e., $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to f. Also, 1, x and x^2 will denote the three functions of $C[0, 1]$ defined by $1(t) = 1$, $x(t) = t$, and $x^2(t) = t^2$ for each $t \in [0, 1]$.

Theorem 0.0.3. *[42] (Korovkin approximation theorem) Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of positive linear operators from $C[0, 1]$ into $C[0, 1]$. If $\lim_n T_n f = f$ holds when f equals 1, x and x^2 then $\lim_n T_n f = f$ holds for all $f \in C[0, 1]$.*

1 Introduction

All over the thesis, we use the abbreviation P.L.O. for "positive linear operators"; $M_1(RS)$ denotes the class of all non-negative regular summability matrices; $M_2(RH-RS)$ denotes the class of all non-negative RH-regular summability matrices.

In Chapter 2, our primary interest is to obtain a general Korovkin type approximation theorem for P.L.O. on the space $UC_*(D)$, the Banach space of all real valued uniform continuous functions on $D:=[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and finite, endowed with the supremum norm $\|f\|_* = \sup_{x\in D}|f(x)|$ for $f\in UC_*(D)$, using the concept of $A^{\mathcal{I}}$-statistical convergence for real sequences and test functions $1, e^{-x}, e^{-2x}$. We also construct an example which shows that our new result is stronger than its classical version. In section 2.3, we extend the Korovkin type approximation theorem for double sequence of P.L.O. on $UC_*([0,\infty)\times[0,\infty))$. In section 2.4, following the notion of $A^{\mathcal{I}}$-summability method for real sequences [61] we establish a Korovkin type approximation theorem for P.L.O. on $UC_*[0,\infty)$, the Banach space of all real valued uniform continuous functions on $[0,\infty)$ with the property that $\lim\limits_{x\to\infty} f(x)$ exists finitely for any $f\in UC_*[0,\infty)$. In the last section of the chapter, we extend the Korovkin type approximation theorem for P.L.O. on $UC_*([0,\infty)\times[0,\infty))$. We then construct an example which shows that our new result is stronger than its classical version.

Convolution is a mathematical operation on two functions (f and g) in functional analysis that results in a third function ($f*g$) that explains how the shape of one is changed by the other. Convolution describes both the method of computation and the resulting function. It is described as the integral of the two functions' product after one of them has been shifted and reflected around the y-axis. The function that is chosen to be reflected and moved before the integral has no effect on the outcome of the integral.

In Chapter 3, following the concept of $A^{\mathcal{I}}$-statistical convergence for real sequences introduced by Savas et. al.[61], we deal with Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a,b]$, the space of all real valued continuous functions on $[a, b]$, in the line of Duman[27]. In the section 3.3, we study the rate of $A^{\mathcal{I}}$-statistical convergence using the modulus of continuity. In section 3.4, for a sequence of positive convolution operators defined on $C[a, b]$, the Banach space of all real valued continuous functions on $[a, b]$ endowed with the supremum norm $||f|| = \sup_{x \in [a,b]} |f(x)|$ for $f \in C[a, b]$ we deal with Korovkin type approximation theory based on the notion of $A^{\mathcal{I}}$-summability. We construct an example to exhibit that the main result is more generalized than its statistical A-summable version. We also study the rate of $A^{\mathcal{I}}$-summability in section 3.5.

Recently study of strong convergence [71] has been introduced a class of functions with the help of pre-metrics on a topological space and they have some certain properties which are closely related to elements of topological base. We have studied strong convergence using $A^{\mathcal{I}}$ summability theory and shown that our results are stronger than classical strong convergence. The main motivation for using $A^{\mathcal{I}}$ summability theory (introduced by Savas et. al. for real sequences [62]) and $A^{\mathcal{I}}$-statistical convergence is always to make a non-convergent sequence to converge to a desirable limit.

Linear structure is required for study of strong ideal convergence in an arbitrary Hausdorff space. In section 4.3 of 4th chapter, we have introduced a convergence method that extends the thought of strong ideal convergence to topological spaces. We have also defined $A^{\mathcal{I}}$-statistical convergence in topological space. We have developed a relation between strong Ideal convergence and $A^{\mathcal{I}}$-statistical convergence and also characterized $A^{\mathcal{I}}$-statistical convergence in arbitrary Hausdorff space in section 4.4.

For sequences of functions with values in $\mathbb{R}$, the concepts of discrete and equal convergence were initially established in 1975 by Császár and M. Laczkovich [15]. Later, these notions were generalized in ideal setting namely, $\mathcal{I}$-discrete (or in short, $\mathcal{I}$-d) convergence, $\mathcal{I}$-equal (or in short, $\mathcal{I}$-e) convergence and $\mathcal{I}^*$-equal (or in short, $\mathcal{I}^*$-e) convergence [20]. In 2005, Gezer and Karakuş generalized the idea of pointwise and uniform convergence in terms of ideal, namely $\mathcal{I}$-pointwise convergence, $\mathcal{I}$-uniform (or in short, $\mathcal{I}$-u) convergence and $\mathcal{I}^*$-uniform (or in short, $\mathcal{I}^*$-u) convergence [34]. We can look at [5] for additional results.

Following this line of research, in Chapter 5, we introduce three new forms of convergence, namely, $\mathcal{I}^*_\alpha$-ue, $\mathcal{I}^*_\alpha$-ud and $\mathcal{I}^*_\alpha$-sue. We explore here a number of lattice features of the classes made up of all those real valued functions defined on a metric space (X, d), which are $\mathcal{I}^*_\alpha$-ue limits, $\mathcal{I}^*_\alpha$-ud limits and $\mathcal{I}^*_\alpha$-sue limits respectively for sequences of real valued functions belong to a particular class of functions. In addition, we define the term $\mathcal{I}\alpha$-equal convergence and follow up some associated findings.

2

Korovkin type approximation theorem on an infinite interval

2.1 Introduction

The necessary and sufficient conditions for the uniform convergence of $\{L_n\}_{n\in\mathbb{N}}$ to a function f were first established by Korovkin [42] using the test functions $e_1 = 1, e_2 = x, e_3 = x^2$ [2] for a sequence $\{L_n\}_{n\in\mathbb{N}}$ of P.L.O. on C(X), the space of real valued continuous functions on a compact subset X of real numbers. An established field of research with a lengthy history is the study of the Korovkin type approximation theory.

The idea of uniform statistical convergence has been used in recent years to show a number of statistical approximation results ([26]). A-statistical convergence was used by Erkuş and Duman [31] to study a Korovkin type approximation theorem in the space $H_w(I^2)$ where $I^2 = [0,\infty) \times [0,\infty)$ and was extended by Demirci and Dirik [23, 25] for double sequences of P.L.O. of two variables (see also [53]). By Dutta et al. [28, 29], it was further expanded for double sequences of P.L.O. of two variables in $A_2^{\mathcal{I}}$-statistical sense and in the sense of $A_2^{\mathcal{I}}$-summability method.

In this chapter, we have introduced general Korovkin type approximation theorem for P.L.O. on the space $UC_*(D)$, the Banach space of all real valued uniform continuous functions on $D := [0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and finite, endowed with the supremum norm $\|f\|_* = \sup_{x\in D} \mid f(x) \mid$ for $f \in UC_*(D)$, using the concepts of $A^{\mathcal{I}}$-statistical convergence and $A^{\mathcal{I}}$-summability method for real sequences and test functions 1, e^{-x}, e^{-y}. In section 2.3 and 2.5, for a double sequence of P.L.O. on $UC_*([0,\infty)\times[0,\infty))$, we expand the

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Korovkin-type approximation theorem. Additionally, we create instances that demonstrate how much stronger our new findings are than the corresponding classical version.

2.2 Korovkin-type approximation using $A^{\mathcal{I}}$-statistical convergence

Throughout the section $\mathcal{I}$ denotes the non-trivial admissible ideal in $\mathbb{N}$. If L be a positive linear operator then $L(f) \geq 0$ for any positive function f and we denote the value of $L(f)$ at x by $L(f; x)$.

We now establish the following Korovkin type approximation theorem for P.L.O. on $UC_*[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0,\infty)$, endowed with the supremum norm $\|f\|_* = \sup_{x\in[0,\infty)} |f(x)|$ for $f \in UC_*([0,\infty))$.

Theorem 2.2.1. *Let $\{L_n\}$ be a sequence of P.L.O. from $UC_*[0,\infty)$ into itself and let, $A = (a_{jn})$ be a member of $M_1(RS)$. Then for all $f \in UC_*[0,\infty)$*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\|_* = 0$$

if and only if the following statements hold

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(e^{-kt}) - e^{-kx}\|_* = 0, k = 0, 1, 2.$$

Proof. Since the necessity is clear, it is enough to proof sufficiency. Our objective is to show that given $\varepsilon > 0$ there exist constants C_0, C_1, C_2 (depending on $\varepsilon > 0$) such that

$$\|L_n(f) - f\|_* \leq \varepsilon + C_2\|L_n(e^{-2t}) - e^{-2x}\|_* + C_1\|L_n(e^{-t}) - e^{-x}\|_* + C_0\|L_n(1) - 1\|_*$$

If this is done then our hypotheses imply that for $\varepsilon > 0$, $\delta > 0$

$$\{n \in \mathbb{N} : \sum_{p\in P(\varepsilon)} a_{np} \geq \delta\} \in \mathcal{I}$$

where

$$P(\varepsilon) = \{p \in \mathbb{N} : \|L_p(f) - f\|_* \geq \varepsilon\}.$$

Let $f \in UC_*[0,\infty)$. Then there is a constant M such that $| f(x) | \leq M$ for each $x \in [0,\infty)$. Let ε be an arbitrary positive number. By hypothesis we may find $\delta := \delta(\varepsilon) > 0$ such that for every $t, x \in [0,\infty)$, $| e^{-t} - e^{-x} | < \delta$ implies $| f(t) - f(x) | < \varepsilon$. Further note that $| f(t) - f(x) | < 2M$ for all $t, x \in [0,\infty)$.

Also if $| e^{-t} - e^{-x} | \geq \delta$, then

$$| f(t) - f(x) | < \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2.$$

Then for all $t, x \in [0, \infty)$

$$| f(t) - f(x) | < \varepsilon + \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2.$$

Consequently for $n \in \mathbb{N}$ using the linearity and the positivity of the operators L_n,

$$\begin{aligned}
|L_n(f(t); x) - f(x)| &\leq L_n(| f(t) - f(x) |; x) + | f(x) || L_n(1; x) - 1 | \\
&\leq L_n(\varepsilon + \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2; x) + | f(x) || L_n(1; x) - 1 | \\
&\leq \varepsilon + (\varepsilon + M) | L_n(1; x) - 1 | + \frac{2M}{\delta^2} L_n((e^{-t} - e^{-x})^2; x) \\
&\leq \varepsilon + (\varepsilon + M) | L_n(1; x) - 1 | + \frac{2M}{\delta^2} | e^{-2x} || L_n(1; x) - 1 | \\
&\quad + \frac{2M}{\delta^2} | L_n(e^{-2t}; x) - e^{-2x} | + \frac{4M}{\delta^2} | e^{-x} || L_n(e^{-t}; x) - e^{-x} |
\end{aligned}$$

where $| e^{-kt} | \leq 1 \; \forall \; t \in [0, \infty)$ and $k \in \mathbb{N}$.
Now taking supremum over $x \in [0, \infty)$ we have

$$\|L_n(f) - f\|_* \leq \varepsilon + K\{\|L_n(1) - 1\|_* + \|L_n(e^{-t}) - e^{-x}\|_* + \|L_n(e^{-2t}) - e^{-2x}\|_*\} \tag{2.1}$$

where $K = \max\{\varepsilon + M + \frac{2M}{\delta^2}, \frac{2M}{\delta^2}, \frac{4M}{\delta^2}\}$. For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$ let us define the following sets

$$D = \{n \in \mathbb{N} : \|L_n(f) - f\|_* \geq r\}$$

$$D_1 = \{n \in \mathbb{N} : \|L_n(1) - 1\|_* \geq \frac{r - \varepsilon}{3K}\}$$

$$D_2 = \{n \in \mathbb{N} : \|L_n(e^{-t}) - e^{-x}\|_* \geq \frac{r - \varepsilon}{3K}\}$$

$$D_3 = \{n \in \mathbb{N} : \|L_n(e^{-2t}) - e^{-2x}\|_* \geq \frac{r - \varepsilon}{3K}\}.$$

It follows from (2.1) that $D \subset D_1 \cup D_2 \cup D_3$. Therefore, for each $n \in \mathbb{N}$ we may write

$$\sum_{p \in D} a_{np} \leq \sum_{p \in D_1} a_{np} + \sum_{p \in D_2} a_{np} + \sum_{p \in D_3} a_{np}$$

which implies that for any $\sigma > 0$ and $p \in D$

$$\{n \in \mathbb{N} : \sum_{p \in D} a_{np} \geq \sigma\} \subseteq \bigcup_{i=1}^{3} \{n \in \mathbb{N} : \sum_{p \in D_i} a_{np} \geq \frac{\sigma}{3}\}.$$

Since from hypotheses $\{n \in \mathbb{N} : \sum_{p \in D_i} a_{np} \geq \frac{\sigma}{3}\} \in \mathcal{I}$ for $i = 1, 2, 3$, therefore

$$\bigcup_{i=1}^{3} \{n \in \mathbb{N} : \sum_{p \in D_i} a_{np} \geq \frac{\sigma}{3}\} \in \mathcal{I}.$$

Hence

$$\{n \in \mathbb{N} : \sum_{p \in D} a_{np} \geq \sigma\} \in \mathcal{I}.$$

and this completes the proof. □

Remark 2.2.1. *We now demonstrate a sequence of P.L.O.* $\{L_n\}$ *s.t.* $A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f)-f\|_* = 0$ *but* $st_A\text{-}\lim_n \|L_n(f)-f\|_* \neq 0$.

Let $\mathcal{I}$ *be a non-trivial admissible ideal of* $\mathbb{N}$. *Choose an infinite subset* $\mathcal{C} = \{q_1 < q_2 < q_3 < \cdots\}$ *from* $\mathcal{I} \setminus \mathcal{I}_d$. *Let* $\{u_k\}_{k\in\mathbb{N}}$ *be given by*

$$u_k = \begin{cases} 0 & \text{when } k \text{ is odd} \\ 1 & \text{when } k \text{ is even.} \end{cases}$$

Let $A = (a_{nk})$ *be given by*

$$a_{nk} = \begin{cases} 1 & \text{when } n = q_i, k = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{when } n \neq q_i, \text{ for any } i, \ k = 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now for $0 < \varepsilon < 1$, $K(\varepsilon) = \{k \in \mathbb{N} : |u_k - 0| \geq \varepsilon\}$ *is the set of all even integers. Notice that*

$$\sum_{k\in K(\varepsilon)} a_{nk} = \begin{cases} 1 & \text{if } n = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } n \neq q_i, \text{ for any } i \in \mathbb{N}. \end{cases}$$

So for any $\delta > 0$, $\left\{ n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta \right\} = \mathcal{C} \in \mathcal{I}$ *which shows that* $\{u_k\}_{k\in\mathbb{N}}$ *is* $A^{\mathcal{I}}$*-statistically convergent to* 0 *though not* A*-statistically convergent.*

We now consider the following Baskakov operators $B_n : UC_*[0,\infty) \to UC_*[0,\infty)$ *defined by*

$$B_n f(x) = \sum_{k=0}^{\infty} \binom{n-1+k}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right).$$

Thus

$$B_n(1,x) = 1$$

$$B_n(e^{-u},x) = (1+x-xe^{-\frac{1}{n}})^{-n}$$

$$B_n(e^{-2u},x) = (1+x-xe^{-\frac{2}{n}})^{-n}$$

where $x \in [0,\infty)$.

Let us define $L_n(f,x) = (1+u_n)B_n(f,x)$ *for any* $f \in UC_*[0,\infty)$. *Then*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f_i)-f_i\|_* = 0, \ i = 0,1,2.$$

From previous theorem

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f)-f\|_* = 0.$$

But as $st_A\text{-}\lim_n u_n \neq 0$ *so* $st_A\text{-}\lim_n \|L_n(f)-f\|_* \neq 0$.

2.3 A Korovkin type approximation theorem for a sequence of positive linear operators of two variables

Throughout the section $\mathcal{I}$ denotes the non-trivial strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. It should be noted that, if we consider $A = C(1,1)$, the double Cesáro matrix defined as follows

$$a_{jkmn} = \begin{cases} \frac{1}{jk} & \text{for } m \leq j, n \leq k; \\ 0 & \text{otherwise} \end{cases}$$

then $A_2^{\mathcal{I}}$-statistical convergence coincides with the notion of $\mathcal{I}_2$-statistical convergence. Again if we replace the matrix A by the identity matrix for four dimensional matrices and $\mathcal{I} = \mathcal{I}_0$ then $A_2^{\mathcal{I}}$-statistical convergence reduces to the Pringsheim convergence for double sequences. For the ideal $\mathcal{I} = \mathcal{I}_0$, $A_2^{\mathcal{I}}$-statistical convergence implies A-statistical convergence for double sequences.

Now we establish the Korovkin type approximation theorem for a double sequence of P.L.O. on $UC_*([0,\infty)\times[0,\infty))$, the Banach space of all real valued uniformly continuous functions defined on $D := [0,\infty)\times[0,\infty)$ with the property that $\lim\limits_{(x,y)\to(\infty,\infty)} f(x,y)$ exists finitely for any $f \in UC_*(D)$, endowed with the supremum norm $\|f\|_* = \sup\limits_{(x,y)\in D} |f(x,y)|$ for $f \in UC_*(D)$, in $A_2^{\mathcal{I}}$-statistical sense.

Theorem 2.3.1. *Let* $\{L_{mn}\}_{m,n\in\mathbb{N}}$ *be a sequence of P.L.O. on* $UC_*([0,\infty)\times[0,\infty))$*, the Banach space of all real valued uniform continuous functions defined on* $[0,\infty)\times[0,\infty)$ *with the property that* $\lim\limits_{(x,y)\to(\infty,\infty)} f(x,y)$ *exists finitely for any* $f \in UC_*([0,\infty)\times[0,\infty))$ *and let* $A = (a_{jkmn}) \in M_2(RH-RS)$*. Then for any* $f \in UC_*([0,\infty)\times[0,\infty))$*,*

$$A_2^{\mathcal{I}}\text{-}st\text{-}\lim_{m,n} \|L_{mn}(f) - f\|_* = 0$$

is satisfied if the following hold

$$A_2^{\mathcal{I}}\text{-}st\text{-}\lim_{m,n} \|L_{mn}(f_i) - f_i\|_* = 0,\ i = 0,1,2,3. \tag{2.2}$$

where $f_0 = 1,\ f_1 = e^{-x},\ f_2 = e^{-y}, f_3 = e^{-2x} + e^{-2y}$.

Proof. Assume that (2.2) holds. Let $f \in UC_*([0,\infty)\times[0,\infty))$. Our objective is to show that for given $\varepsilon > 0$ there exist constants $C_0,\ C_1,\ C_2,\ C_3$ (depending on $\varepsilon > 0$) such that

$$\begin{aligned}\|L_{mn}f - f\|_* \leq \varepsilon &+ C_3\|L_{mn}f_3 - f_3\|_* + C_2\|L_{mn}f_2 - f_2\|_* \\ &+ C_1\|L_{mn}f_1 - f_1\|_* + C_0\|L_{mn}f_0 - f_0\|_*.\end{aligned}$$

If this is done then our hypotheses imply that for any $\varepsilon > 0,\ \delta > 0$,

$$\left\{(j,k)\in\mathbb{N}^2:\sum_{(m,n)\in K_2(\varepsilon)}a_{jkmn}\geq\delta\right\}\in\mathcal{I}$$

where $K_2(\varepsilon)=\{(m,n)\in\mathbb{N}^2:\ \|L_{mn}f-f\|_*\geq\varepsilon\}$.

To this end, start by observing that for each $(u,v)\in([0,\infty)\times[0,\infty))$ the function $0\leq g_{uv}\in UC_*\left([0,\infty)\times[0,\infty)\right)$ defined by

$$g_{uv}(s,t)=(e^{-s}-e^{-u})^2+(e^{-t}-e^{-v})^2$$

satisfies $g_{uv}=(e^{-x})^2+(e^{-y})^2-2e^{-u}e^{-x}-2e^{-v}e^{-y}+(e^{-u})^2+(e^{-v})^2$. Since each L_{mn} is a positive operator, $L_{mn}g_{uv}$ is a positive function. In particular, we have for each $(u,v)\in([0,\infty)\times[0,\infty))$,

$$\begin{aligned}
0 &\leq L_{mn}g_{uv}(u,v)\\
&= \left[L_{mn}\left(\left(e^{-x}\right)^2+\left(e^{-y}\right)^2-2e^{-u}e^{-x}-2e^{-v}e^{-y}+\left(e^{-u}\right)^2+\left(e^{-v}\right)^2;u,v\right)\right]\\
&= \left[L_{mn}\left(\left(e^{-x}\right)^2+\left(e^{-y}\right)^2;u,v\right)-\left(e^{-u}\right)^2-\left(e^{-v}\right)^2\right]\\
&\quad -2e^{-u}\left[L_{mn}\left(e^{-x};u,v\right)-e^{-u}\right]-2e^{-v}\left[L_{mn}\left(e^{-y};u,v\right)-e^{-v}\right]\\
&\quad +\left\{\left(e^{-u}\right)^2+\left(e^{-v}\right)^2\right\}\left[L_{mn}f_0-f_0\right]\\
&\leq ||L_{mn}f_3-f_3||_*+2e^{-u}||L_{mn}f_1-f_1||_*+2e^{-v}||L_{mn}f_2-f_2||_*\\
&\quad +\left\{\left(e^{-u}\right)^2+\left(e^{-v}\right)^2\right\}||L_{mn}f_0-f_0||_*.
\end{aligned}$$

Let $f\in UC_*\left([0,\infty)\times[0,\infty)\right)$. Then there exists a constant M such that $|f(x,y)|\leq M$ for each $(x,y)\in([0,\infty)\times[0,\infty))$. Let $\varepsilon>0$ be arbitrary. Then by the uniform continuity of f on $([0,\infty)\times[0,\infty))$ there exists a $\delta=\delta(\varepsilon)>0$ such that if $\mid e^{-x}-e^{-u}\mid<\delta$ and $\mid e^{-y}-e^{-v}\mid<\delta$ then

$$\mid f(x,y)-f(u,v)\mid<\varepsilon+\frac{2M}{\delta^2}\left[\left(e^{-x}-e^{-u}\right)^2+\left(e^{-y}-e^{-v}\right)^2\right]$$

for all $(x,y),(u,v)\in[0,\infty)\times[0,\infty)$. Since each L_{mn} is positive and linear it follows that

$$\begin{aligned}
-\varepsilon L_{mn}f_0-\frac{2M}{\delta^2}L_{mn}g_{uv} &\leq L_{mn}f-f(u,v)L_{mn}f_0\\
&\leq \varepsilon L_{mn}f_0+\frac{2M}{\delta^2}L_{mn}g_{uv}.
\end{aligned}$$

Therefore

$$\begin{aligned}
|L_{mn}(f;u,v)-f(u,v)L_{mn}(f_0;u,v)| &\leq \varepsilon+\varepsilon\left[L_{mn}(f_0;u,v)-f_0(u,v)\right]\\
&\quad +\frac{2M}{\delta^2}L_{mn}g_{uv}\\
&\leq \varepsilon+\varepsilon||L_{mn}f_0-f_0||_*+\frac{2M}{\delta^2}L_{mn}g_{uv}.
\end{aligned}$$

In particular, note that

$$\begin{aligned} |L_{mn}(f;u,v)-f(u,v)| &\leq |L_{mn}(f;u,v)-f(u,v)L_{mn}(f_0;u,v)| \\ &\quad +|f(u,v)|\,|L_{mn}(f_0;u,v)-f_0(u,v)| \\ &\leq \varepsilon+(M+\varepsilon)\|L_{mn}f_0-f_0\|_*+\frac{2M}{\delta^2}L_{mn}g_{uv} \end{aligned}$$

which implies

$$\begin{aligned} \|L_{mn}f-f\|_* \leq \varepsilon &+ C_3\|L_{mn}f_3-f_3\|_*+C_2\|L_{mn}f_2-f_2\|_* \\ &+C_1\|L_{mn}f_1-f_1\|_*+C_0\|L_{mn}f_0-f_0\|_* \end{aligned}$$

where there exist such A and B such that $C_0=\left[\frac{2M}{\delta^2}\{(e^{-A})^2+(e^{-B})^2\}+M+\varepsilon\right]$, $C_1=\frac{4M}{\delta^2}e^{-A}$, $C_2=\frac{4M}{\delta^2}e^{-B}$ and $C_3=\frac{2M}{\delta^2}$. i.e.

$$\|L_{mn}f-f|\|_* \leq \varepsilon+C\sum_{i=0}^{3}\|L_{mn}f_i-f_i\|_*,\ i=0,1,2,3$$

where $C=\max\{C_0,\ C_1,\ C_2,\ C_3\}$.

For a given $\gamma>0$, choose $\varepsilon>0$ such that $\varepsilon<\gamma$. Now let

$$U=\{(m,n):\|L_{mn}f-f\|_*\geq\gamma\}$$

and

$$U_i=\{(m,n):\|L_{mn}f_i-f_i\|_*\geq\tfrac{\gamma-\varepsilon}{4C}\},\ i=0,1,2,3.$$

It follows that $U\subset\bigcup_{i=0}^{3}U_i$ and consequently for all $(j,k)\in\mathbb{N}^2$

$$\sum_{(m,n)\in U}a_{jkmn}\leq\sum_{i=0}^{3}\sum_{(m,n)\in U_i}a_{jkmn}$$

which implies that for any $\sigma>0$ and $(m,n)\in U$,

$$\left\{(j,k)\in\mathbb{N}^2:\sum_{(m,n)\in U}a_{jkmn}\geq\sigma\right\}\subseteq\bigcup_{i=0}^{3}\left\{(j,k)\in\mathbb{N}^2:\sum_{(m,n)\in U_i}a_{jkmn}\geq\frac{\sigma}{3}\right\}.$$

Therefore from hypotheses, $\left\{(j,k)\in\mathbb{N}^2:\sum_{(m,n)\in U}a_{jkmn}\geq\sigma\right\}\in\mathcal{I}$ and this completes the proof. □

Remark 2.3.1. *We now show that our theorem is stronger than the A-statistical version [24] (and so the classical version). Let $\mathcal{I}$ be a non-trivial strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$. Choose an infinite subset $C=\{(p_i,q_i):i\in\mathbb{N}\}$, from $\mathcal{I}\setminus\mathcal{I}_d$ where $\mathcal{I}_d$ denotes the set of all subsets of $\mathbb{N}\times\mathbb{N}$ with natural density zero, such that $p_i\neq q_i$ for all i, $p_1<p_2<\ldots$ and*

$q_1 < q_2 < ...$

Let $\{u_{mn}\}_{m,n\in\mathbb{N}}$ *be given by*

$$u_{mn} = \begin{cases} 1 & m, n \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{jkmn})$ *be given by*

$$a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i, m = 2p_i, n = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i, m = 2j+1, n = 2k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now for $0 < \varepsilon < 1$, $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |u_{mn} - 0| \geq \varepsilon\} = \{(m,n) : m, n \text{ are even}\}$. *Observe that*

$$\sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i \in \mathbb{N}. \end{cases}$$

Thus for any $\delta > 0$,

$$\left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\right\} = C \in \mathcal{I}$$

which shows that $\{u_{mn}\}_{m,n\in\mathbb{N}}$ *is* $A_2^{\mathcal{I}}$*-statistically convergent to* 0. *Evidently this sequence is not* A*-statistically convergent to* 0.

Let $\mathcal{K} = [0,\infty)\times[0,\infty)$. *We consider the following Baskakov operators* $B_{mn} : UC_*(\mathcal{K}) \to UC_*(\mathcal{K})$ *defined by*

$$B_{mn}(f;x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f\left(\frac{j}{n},\frac{k}{n}\right)\binom{m-1+j}{j}\binom{n-1+k}{k}(1+x)^{-m-j}(1+y)^{-n-k}x^j y^k$$

Now we consider the double sequence $\{L_{mn}\}_{m,n\in\mathbb{N}}$ *of P.L.O. defined by* $L_{mn}(f;x,y) = (1+u_{mn})B_{mn}(f;x,y)$.

Then observe that

$$L_{mn}(f_0;x,y) = (1+u_{mn})f_0(x,y),$$
$$L_{mn}(f_1;x,y) = (1+u_{mn})\Big(1+x-xe^{-\frac{1}{m}}\Big)^{-m},$$
$$L_{mn}(f_2;x,y) = (1+u_{mn})\Big(1+y-ye^{-\frac{1}{n}}\Big)^{-n},$$
$$L_{mn}(f_3;x,y) = (1+u_{mn})\left[\Big(1+x-xe^{-\frac{1}{m}}\Big)^{-m} + \Big(1+y-ye^{-\frac{1}{n}}\Big)^{-n}\right]$$

Then

$$A_2^{\mathcal{I}}\text{-}st\text{-}\lim_{m,n} \|L_{mn}(f_i) - f_i\|_* = 0, \ i = 0,1,2,3.$$

Therefore by previous theorem, for any $f \in UC_*(\mathcal{K})$

$$A_2^{\mathcal{I}}\text{-}st\text{-}\lim_{m,n} \|L_{mn}(f) - f\|_* = 0$$

But since $\{u_{mn}\}_{m,n\in\mathbb{N}}$ *is not usual convergent and not* A*-statistical convergent, so we can say that the classical version and* A*-statistical version of the previous theorem do not work for the operator defined above.*

Here we introduce our new approach to Korovkin type approximation theory via generalized matrix summability methods using ideals.

2.4 Approximation process based on the notion of $A^{\mathcal{I}}$-summability

Throughout this section $\mathcal{I}$ denotes the non-trivial admissible ideal on $\mathbb{N}$. It should be noted that for $\mathcal{I} = \mathcal{I}_d$, the set of all subsets of $\mathbb{N}$ with natural density zero, $A^{\mathcal{I}}$-summability reduces to statistical A-summability [30].

We now establish a Korovkin type approximation theorem for P.L.O. on $UC_*[0,\infty)$, the Banach space of all real valued uniform continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0,\infty)$.

Theorem 2.4.1. *Let* $\{L_n\}$ *be a sequence of P.L.O. from* $UC_*[0,\infty)$ *into itself and let,* $A = (a_{jn})$ *be a member of* $M_1(RS)$ *then for all* $f \in UC_*[0,\infty)$

$$\mathcal{I}\text{-}\lim_n \|\sum_{k=1}^{\infty} a_{nk}L_k(f) - f\|_* = 0$$

if and only if the following statements hold

$$\mathcal{I}\text{-}\lim_n \|\sum_{k=1}^{\infty} a_{nk}L_k(e^{-pt}) - e^{-px}\|_* = 0, p = 0, 1, 2.$$

Proof. Since the necessity is clear, then it is enough to proof sufficiency. Our objective is to show that for given $\varepsilon > 0$ there exist constants C_0 , C_1 , C_2 (depending on $\varepsilon > 0$) such that

$$\|\sum_{k=1}^{\infty} a_{nk}L_k(f) - f\|_* \le \varepsilon + C_2\|\sum_{k=1}^{\infty} a_{nk}L_k(e^{-2t}) - e^{-2x}\|_* + C_1\|\sum_{k=1}^{\infty} a_{nk}L_k(e^{-t}) - e^{-x}\|_* + C_0\|\sum_{k=1}^{\infty} a_{nk}L_k(1) - 1\|_*.$$

If this is done then our hypotheses imply that for any $\varepsilon > 0$,

$$\{n \in \mathbb{N} : \|\sum_{k=1}^{\infty} a_{nk}L_k(f) - f\| \ge \varepsilon\} \in \mathcal{I}.$$

Let $f \in UC_*[0,\infty)$ then $\exists$ a constant M such that $\mid f(x) \mid \leq M$ for each $x \in [0,\infty)$. Let ε be an arbitrary positive number. By hypothesis we may find $\delta := \delta(\varepsilon) > 0$ such that for every $t, x \in [0,\infty)$, $\mid e^{-t} - e^{-x} \mid < \delta$ implies $\mid f(t) - f(x) \mid < \varepsilon$. We can write $\mid f(t) - f(x) \mid < 2M \ \forall \ t, x \in [0,\infty)$. Also if $\mid e^{-t} - e^{-x} \mid \geq \delta$ then

$$\mid f(t) - f(x) \mid < \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2.$$

Then for all $t, x \in [0,\infty)$,

$$\mid f(t) - f(x) \mid < \varepsilon + \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2.$$

Then for $n \in \mathbb{N}$, using the linearity and the positivity of the operators L_n,

$$\begin{aligned}
\mid \sum_{k=1}^{\infty} a_{nk} L_k(f(t);x) - f(x) \mid \ &\leq \ \sum_{k=1}^{\infty} a_{nk} L_k(\mid f(t) - f(x) \mid; x) \\
&\quad + \mid f(x) \mid \mid \sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1 \mid \\
&\leq \ \sum_{k=1}^{\infty} a_{nk} L_k(\varepsilon + \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2; x) \\
&\quad + \mid f(x) \mid \mid \sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1 \mid \\
&\leq \ \varepsilon + (\varepsilon + M) \mid \sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1 \mid \\
&\quad + \frac{2M}{\delta^2} \sum_{k=1}^{\infty} a_{nk} L_k((e^{-t} - e^{-x})^2; x) \\
&\leq \ \varepsilon + (\varepsilon + M) \mid \sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1 \mid \\
&\quad + \frac{2M}{\delta^2} \mid e^{-2x} \mid \mid \sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1 \mid \\
&\quad + \frac{2M}{\delta^2} \mid \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}; x) - e^{-2x} \mid \\
&\quad + \frac{4M}{\delta^2} \mid e^{-x} \mid \mid \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}; x) - e^{-x} \mid
\end{aligned}$$

where $\mid e^{-kt} \mid \leq 1 \ \forall \ t \in [0,\infty)$ and $k \in \mathbb{N}$.
Then taking supremum over $x \in [0,\infty)$ we have

$$\begin{aligned}
\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \|_* \leq \varepsilon + K\{ \| \sum_{k=1}^{\infty} a_{nk} L_k(1) - 1 \|_* + \| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}) - e^{-x} \|_* \\
+ \| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}) - e^{-2x} \|_* \}
\end{aligned}$$

where $K = \max\{\varepsilon + M + \frac{2M}{\delta^2}, \frac{2M}{\delta^2}, \frac{4M}{\delta^2}\}$. For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$ let us define the following sets

$$D = \{n \in \mathbb{N} : \| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \|_* \geq r\}$$

$$D_1 = \{n \in \mathbb{N} : \| \sum_{k=1}^{\infty} a_{nk} L_k(1) - 1 \|_* \geq \frac{r - \varepsilon}{3K}\}$$

$$D_2 = \{n \in \mathbb{N} : \| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}) - e^{-x} \|_* \geq \frac{r - \varepsilon}{3K}\}$$

$$D_3 = \{n \in \mathbb{N} : \| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}) - e^{-2x} \|_* \geq \frac{r - \varepsilon}{3K}\}.$$

It follows that $D \subset D_1 \cup D_2 \cup D_3$. Since from hypotheses $D_1,\ D_2,\ D_3$ are belong to $\mathcal{I}$ so $D \in \mathcal{I}$ i.e.

$$\{n \in \mathbb{N} : \| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \| \geq \varepsilon\} \in \mathcal{I}$$

and this completes the proof. □

2.5 Approximation for a sequence of positive linear operators of two variables via matrix summability methods using ideals

Throughout this section $\mathcal{I}$ denotes the non-trivial strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$.

Recall the following definition

Definition 2.5.1 ([29]). *Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix. Then a real double sequence $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-summable to a number L if for every $\varepsilon > 0$, $\{(j,k) \in \mathbb{N}^2 : |(Ax)_{j,k} - L| \geq \varepsilon\} \in \mathcal{I}$.*

Thus $x = \{x_{mn}\}_{m,n \in \mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to a number L if and only if $(Ax)_{j,k}$ is $\mathcal{I}$-convergent to L. In this case, we write $\mathcal{I}_2\text{-}\lim_{j,k} \sum_{(m,n) \in \mathbb{N}^2} a_{jkmn} x_{mn} = L$.

It should be noted that, if we take $\mathcal{I} = \mathcal{I}_d$, then $A_2^{\mathcal{I}}$-summability reduces to the notion of statistical A -summability for double sequence[8].

We now establish the Korovkin-type approximation theorem for a double sequence of P.L.O. on $UC_*\left([0,\infty) \times [0,\infty)\right)$, the Banach space of all real valued uniform continuous functions defined on $[0,\infty) \times [0,\infty)$ with the property that $\lim_{(x,y) \to (\infty,\infty)} f(x,y)$ exists finitely for any $f \in UC_*\left([0,\infty) \times [0,\infty)\right)$ endowed with the supremum norm $||f||_* = \sup_{x,y \in [0,\infty)} |f(x,y)|$, in $A_2^{\mathcal{I}}$-summability method. If L be a positive linear operator then $L(f) \geq 0$ for any positive function f. Also we denote the value of $L(f)$ at a point $(x,y) \in [0,\infty) \times [0,\infty)$ by $L(f;x,y)$.

Theorem 2.5.1. *Assume* $\mathcal{K} := [0,\infty) \times [0,\infty)$ *and let* $\{L_{mn}\}_{m,n\in\mathbb{N}}$ *be a sequence of P.L.O. on* $UC_*(\mathcal{K})$*, the Banach space of all real valued uniform continuous functions defined on* $\mathcal{K}$ *with the property that* $\lim\limits_{(x,y)\to(\infty,\infty)} f(x,y)$ *exists finitely for any* $f \in UC_*(\mathcal{K})$ *and let* $A = (a_{jkmn}) \in M_2(RH - RS)$*. Then for any* $f \in UC_*(\mathcal{K})$,

$$\mathcal{I}_2\text{-}\lim_{j,k} \| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \|_* = 0$$

is satisfied if the following hold

$$\mathcal{I}_2\text{-}\lim_{j,k} \| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \|_* = 0, \; i = 0,1,2,3 \tag{2.3}$$

where $f_0 = 1,\; f_1 = e^{-x},\; f_2 = e^{-y}, f_3 = e^{-2x} + e^{-2y}$.

Proof. Assume that (2.3) holds. Let $f \in UC_*(\mathcal{K})$. Our aim is to establish that for given $\varepsilon > 0$ there exist constants $C_0,\; C_1,\; C_2,\; C_3$ (depending on $\varepsilon > 0$) such that

$$|| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f ||_* \le \varepsilon + \sum_{i=0}^{3} C_i || \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i ||_*.$$

If this is accomplished then our hypotheses suggest that for any $\varepsilon > 0$,

$$\{(j,k) \in \mathbb{N}^2 : \| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \|_* \ge \varepsilon\} \in \mathcal{I}.$$

In order to achieve this, first note that for each $(u,v) \in \mathcal{K}$ the function $0 \le g_{uv} \in UC_*(\mathcal{K})$ defined by

$$g_{uv}(s,t) = (e^{-s} - e^{-u})^2 + (e^{-t} - e^{-v})^2$$

satisfies $g_{uv} = (e^{-x})^2 + (e^{-y})^2 - 2e^{-u}e^{-x} - 2e^{-v}e^{-y} + (e^{-u})^2 + (e^{-v})^2$. Here $L_{mn}g_{uv}$ is a positive function because each L_{mn} is a positive operator. In particular, we have for each $(u,v) \in \mathcal{K}$,

$$\begin{aligned}
0 &\le \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv})(u,v) \\
&= \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left((e^{-x})^2 + (e^{-y})^2 - 2e^{-u}e^{-x} - 2e^{-v}e^{-y} + (e^{-u})^2 + (e^{-v})^2 ; u,v \right) \right] \\
&= \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left((e^{-x})^2 + (e^{-y})^2 ; u,v \right) - (e^{-u})^2 - (e^{-v})^2 \right] \\
&\quad -2e^{-u} \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(e^{-x}; u,v\right) - e^{-u} \right] -2e^{-v} \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(e^{-y}; u,v\right) - e^{-v} \right] \\
&\quad + \left\{ (e^{-u})^2 + (e^{-v})^2 \right\} \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - f_0 \right]
\end{aligned}$$

$$\leq || \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_3) - f_3||_* + 2e^{-u}|| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_1) - f_1||_*$$
$$+ 2e^{-v}|| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_2) - f_2||_* + \left\{ \left(e^{-u}\right)^2 + \left(e^{-v}\right)^2 \right\} || \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - f_0||_*.$$

Let $f \in UC_*(\mathcal{K})$. Then, for each $(x, y) \in \mathcal{K}$, there exists a constant M such that $|f(x, y)| \leq M$. Let $\varepsilon > 0$ be any numbers. Then there exists a $\delta = \delta(\varepsilon) > 0$ due to uniform continuity of f on $\mathcal{K}$ such that if $| e^{-x} - e^{-u} |< \delta$ and $| e^{-y} - e^{-v} |< \delta$ then

$$| f(x, y) - f(u, v) |< \varepsilon + \frac{2M}{\delta^2} \left[\left(e^{-x} - e^{-u}\right)^2 + \left(e^{-y} - e^{-v}\right)^2 \right]$$

for all $(x, y), (u, v) \in \mathcal{K}$.
Now positivity and linearity of L_{mn} follows that

$$-\varepsilon \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv})$$
$$\leq \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f(u, v) L_{mn}(f_0)$$
$$\leq \varepsilon \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) + \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv}).$$

Therefore

$$| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f; u, v) - f(u, v) L_{mn}(f_0; u, v)|$$
$$\leq \varepsilon + \varepsilon[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0; u, v) - f_0(u, v)] + \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv})$$
$$\leq \varepsilon + \varepsilon\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - f_0\|_* + \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv}).$$

In particular, note that

$$| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f; u, v) - f(u, v)|$$
$$\leq | \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f; u, v) - f(u, v) \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0; u, v)|$$
$$+ | \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} f(u, v)|| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0; u, v) - f_0(u, v)|$$
$$\leq \varepsilon + (M + \varepsilon)|| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - f_0||_* + \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv})$$

which implies

$$\begin{aligned}\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f)-f\|_* \;\le\; & \varepsilon + C_3\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_3)-f_3\|_* \\ & +C_2\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_2)-f_2\|_* \\ & +C_1\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_1)-f_1\|_* \\ & +C_0\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0)-f_0\|_*\end{aligned}$$

where there exist such A and B such that $C_0 = \left[\frac{2M}{\delta^2}\{(e^{-A})^2+(e^{-B})^2\}+M+\varepsilon\right]$, $C_1 = \frac{4M}{\delta^2}e^{-A}$, $C_2 = \frac{4M}{\delta^2}e^{-B}$ and $C_3 = \frac{2M}{\delta^2}$. i.e.

$$||\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f)-f||_* \le \varepsilon + C\sum_{i=0}^{3}||\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_i)-f_i||_*,\ i=0,1,2,3$$

where $C=\max\{C_0,\ C_1,\ C_2,\ C_3\}$.

For a given number $\gamma>0$, choose a number $\varepsilon>0$ s.t. $\varepsilon<\gamma$. Now let

$$U := \{(j,k)\in\mathbb{N}^2 : ||\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f)-f||_* \ge \gamma\}$$

and

$$U_i := \{(j,k)\in\mathbb{N}^2 : ||\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_i)-f_i||_* \ge \frac{\gamma-\varepsilon}{4C}\},\ i=0,1,2,3.$$

Thus, it follows $U\subset\bigcup_{i=0}^{3}U_i$. By hypotheses, each $U_i\in\mathcal{I}$, $i=0,1,2,3$ and accordingly $U\in\mathcal{I}$ i.e.

$$\{(j,k)\in\mathbb{N}^2 : \|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f)-f\|_* \ge \gamma\}\in\mathcal{I}.$$

The theorem's proof is now finished. □

Remark 2.5.1. *We now demonstrate the superiority of our theorem over the classical version and the statistical A-summable version [24]. Let $\mathcal{I}$ be a non-trivial strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$. Choose an infinite subset $\mathcal{C}=\{(p_i,q_i) : i\in\mathbb{N}$ where $p_i\neq q_i,\ p_1<p_2<...,$ and $q_1<q_2<...\}$ from $\mathcal{I}\setminus\mathcal{I}_d$. Let $\{u_{mn}\}_{m,n\in\mathbb{N}}$ be given by*

$$u_{mn}=\begin{cases}1 & \text{when } m,n \text{ both are even}\\ 0 & \text{otherwise.}\end{cases}$$

Let $A = (a_{jkmn})$ *be given by*

$$a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i, m = 2p_i, n = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i, m = 2j+1, n = 2k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$y_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} u_{mn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i \in \mathbb{N}. \end{cases}$$

Let $\varepsilon > 0$ *be given. Then* $\{(j,k) \in \mathbb{N}^2 : |y_{j,k} - 0| \geq \varepsilon\} = C \in \mathcal{I}$. *Then the sequence* $\{u_{mn}\}_{m,n\in\mathbb{N}}$ *is* $A_2^{\mathcal{I}}$*-summable to* 0. *Evidently, this sequence is not statistically A-summable to* 0.

Let $\mathcal{K} = [0,\infty) \times [0,\infty)$. *We consider the following Baskakov operators* $B_{mn} : UC_*(\mathcal{K}) \to UC_*(\mathcal{K})$ *defined by*

$$B_{mn}(f;x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f\left(\frac{j}{n}, \frac{k}{n}\right) \binom{m-1+j}{j} \binom{n-1+k}{k} (1+x)^{-m-j}(1+y)^{-n-k} x^j y^k.$$

We now consider the double sequence $\{L_{mn}\}_{m,n\in\,\mathbb{N}}$ *of P.L.O. defined by* $L_{mn}(f;x,y) = (1 + u_{mn})B_{mn}(f;x,y)$.
Then observe that

$$L_{mn}(f_0;x,y) = (1 + u_{mn})f_0(x,y),$$

$$L_{mn}(f_1;x,y) = (1 + u_{mn})\Big(1 + x - xe^{-\frac{1}{m}}\Big)^{-m},$$

$$L_{mn}(f_2;x,y) = (1 + u_{mn})\Big(1 + y - ye^{-\frac{1}{n}}\Big)^{-n},$$

$$L_{mn}(f_3;x,y) = (1 + u_{mn})\left[\Big(1 + x - xe^{-\frac{1}{m}}\Big)^{-m} + \Big(1 + y - ye^{-\frac{1}{n}}\Big)^{-n}\right].$$

Now as A is an element from $M_2(RH - RS)$ *and* $\{u_{mn}\}_{m,n\in\mathbb{N}}$ *is* $A_2^{\mathcal{I}}$*-summable to* 0 *then for any* $\varepsilon > 0$,

$$\left\{(j,k) \in \mathbb{N}^2 : || \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i||_* \geq \varepsilon\right\} \in \mathcal{I}, \ i = 0,1,2,3.$$

Therefore by previous theorem

$$\left\{(j,k) \in \mathbb{N}^2 : || \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f||_* \geq \varepsilon\right\} \in \mathcal{I}.$$

The classical version and statistical A-summable version of the prior theorem, however, do not apply to the operator stated above since $\{u_{mn}\}_{m,n\in\mathbb{N}}$ *is not typically convergent and statistically A-summable.*

2.6 Conclusion

We conclude this chapter by pointing out some important features of this study. The result that we have encountered, for a sequence $\{L_n\}_{n\in\mathbb{N}}$ of P.L.O. on $UC_*[0,\infty)$, established the necessary and sufficient conditions using the notions of $A^{\mathcal{I}}$-statistically convergence and $A^{\mathcal{I}}$-summability of $\{L_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $f_0 = 1,\ f_1 = e^{-x},\ f_2 = e^{-2x}$. The same type result is also established for a sequence of P.L.O. of two variables by using the test functions $f_0 = 1,\ f_1 = e^{-x},\ f_2 = e^{-y}, f_2 = e^{-2x} + e^{-2y}$. The examples show that our new results are stronger than its A-statistical version and consequently stronger than its classical version.

3

Approximation of continuous function by sequence of convolution operators

3.1 Introduction

In this chapter, following the concept of $A^{\mathcal{I}}$-statistical and $A^{\mathcal{I}}$-summability for real sequences introduced by Savas et. al.[61, 62] we deal with Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a,b]$, the space of all real valued continuous functions on $[a,b]$, in the line of Duman[27]. We also study the rate of convergence for both $A^{\mathcal{I}}$-statistical and $A^{\mathcal{I}}$-summablity.

3.2 $A^{\mathcal{I}}$-statistical approximation for a sequence of convolution operators

Following the idea of $A^{\mathcal{I}}$-statistical convergence we approach towards the succeeding theorems. Note that for $\mathcal{I} = \mathcal{I}_{fin}$, $A^{\mathcal{I}}$-statistical convergence becomes A-statistical convergence[39].

We consider the Banach space $C[a,b]$ endowed with the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C[a,b]$.

Theorem 3.2.1. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of P.L.O. from $C[a,b]$ into $C[a,b]$. If*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f_i) - f_i\| = 0$$

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with $f_i = t^i,\ i = 0,1,2$ *then for all* $f \in C[a,b]$ *then we have*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\| = 0.$$

Proof. Our objective is to show that for given $\varepsilon > 0$ there exist constants C_0,C_1,C_2, (depending on $\varepsilon > 0$) such that

$$\|L_n(f) - f\| \leq \varepsilon + C_2\|L_n(f_2) - f_2\| + C_1\|L_n(f_1) - f_1\| + C_0\|L_n(f_0) - f_0\|.$$

If this is done then our hypotheses imply that for $\varepsilon > 0$, $\delta > 0$

$$\{n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta\} \in \mathcal{I}$$

where $K(\varepsilon) = \{k \in N : \|L_k(f) - f\| \geq \varepsilon\}$.
To this end, start by observing that for each $x \in [a,b]$ the function $0 \leq \Psi \in C[a,b]$ defined by $\Psi(t) = (t-x)^2$. Since each L_n is positive, $L_n(\Psi; x)$ is a positive function. In particular, we have

$$\begin{aligned} 0 \leq L_n(\Psi; x) &= L_n(t^2; x) - 2xL_n(t; x) + x^2 L_n(1; x) \\ &= (L_n(t^2; x) - t^2(x)) - 2x(L_n(t; x) - t(x)) + x^2(L_n(1; x) - 1(x)) \\ &\leq \|L_n(t^2) - t^2\| + 2b\|L_n(t) - t\| + b^2\|L_n(1) - 1\| \end{aligned}$$

for each $x \in [a,b]$. Let $M = \|f\|$. Since f is bounded on the closed bounded interval $[a,b]$, we can write

$$|f(t) - f(x)| < 2M,\ t, x \in [a,b].$$

Also, since f is continuous on $[a,b]$, we have

$$|f(t) - f(x)| < \varepsilon$$

for all t, x satisfying $|t - x| \leq \delta$.
On the other hand, if $|t - x| \geq \delta$, then it follows that,

$$-\frac{2M}{\delta^2}(t-x)^2 \leq -2M \leq f(t) - f(x) \leq 2M \leq \frac{2M}{\delta^2}(t-x)^2.$$

Therefore for all $t \in (-\infty, \infty)$ and all $x \in [a,b]$ we get,

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(t-x)^2$$

where δ is a fixed real number. Since each L_n is positive, we have

$$-\varepsilon L_n(f_0; x) - \frac{2M}{\delta^2} L_n(\Psi; x) \leq L_n(f(t); x) - f(x) L_n(f_0; x) \leq \varepsilon L_n(f_0; x) + \frac{2M}{\delta^2} L_n(\Psi; x).$$

Next, let $K = \frac{2M}{\delta^2}$ and we get,

$$\begin{aligned}|L_n(f(t);x) - f(x)L_n(f_0;x)| &\leq \varepsilon L_n(f_0;x) + \frac{2M}{\delta^2}L_n(\Psi;x) \\ &= \varepsilon + \varepsilon[L_n(f_0;x) - f_0(x)] + KL_n(\Psi;x) \\ &\leq \varepsilon + \varepsilon|L_n(f_0;x) - f_0(x)| + KL_n(\Psi;x).\end{aligned}$$

In particular,

$$\begin{aligned}|L_n(f(t);x) - f(x)| &\leq |L_n(f(t);x) - f(x)L_n(f_0;x)| + |f(x)||L_n(f_0;x) - f_0(x)| \\ &\leq \varepsilon + KL_n(\Psi;x) + (M+\varepsilon)|L_n(f_0;x) - f_0(x)|\end{aligned}$$

which implies

$$\|L_n(f) - f\| \leq \varepsilon + C_2\|L_n(f_2) - f_2\| + C_1\|L_n(f_1) - f_1\| + C_0\|L_n(f_0) - f_0\|$$

where, $C_2 = K$, $C_1 = 2bK$ and $C_0 = (\varepsilon + b^2K + M)$ i.e. ,

$$\|L_n(f) - f\| \leq \varepsilon + C\sum_{i=0}^{2}\|L_n(f_i) - f_i\|, i = 0,1,2$$

where $C = \max\{C_0, C_1, C_2\}$. For a given $\varepsilon' > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$ and let us define the following sets

$$D = \{n : \|L_n(f) - f\| \geq \varepsilon'\};$$

$$D_1 = \{n : \|L_n(f_0) - f_0\| \geq \frac{\varepsilon' - \varepsilon}{3C}\};$$

$$D_2 = \{n : \|L_n(f_1) - f_1\| \geq \frac{\varepsilon' - \varepsilon}{3C}\};$$

$$D_3 = \{n : \|L_n(f_2) - f_2\| \geq \frac{\varepsilon' - \varepsilon}{3C}\}.$$

It follows that $D \subseteq D_1 \cup D_2 \cup D_3$ and consequently for all $n \in \mathbb{N}$

$$\sum_{k\in D} a_{nk} \leq \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk} + \sum_{k\in D_3} a_{nk}$$

which implies that, for any $\sigma > 0$

$$\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\} \subseteq \bigcup_{i=1}^{3}\{n \in \mathbb{N} : \sum_{k\in D_i} a_{nk} \geq \frac{\sigma}{3}\}.$$

Therefore from hypotheses,

$$\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\} \in \mathcal{I}.$$

Hence the proof is done. □

We will now have a look at the defined convolution operators on $C[a,b]$ below.

$$L_n(f;x) = \int_a^b f(y)K_n(y-x)dy,\ n \in \mathbb{N},\ x \in [a,b] \text{ and } f \in C[a,b] \tag{3.1}$$

where a and b are two real numbers such that $a < b$. Throughout the chapter, we assume that K_n is a continuous function on $[a-b, b-a]$ and also that $K_n(u) \geq 0$ for all $n \in \mathbb{N}$ and for every $u \in [a-b, b-a]$. Consider the function Ψ on $[a,b]$ defined by $\Psi(y) = (y-x)^2$ for each $x \in [a,b]$.

Theorem 3.2.2. *Let $A = (a_{ij})$ be a member of $M_1(RS)$ and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators from $C[a,b]$ into $C[a,b]$.*
If $A^{\mathcal{I}}$-st-$\lim_n \|L_n(f_0) - f_0\| = 0$ with $f_0(y) = 1$ and $A^{\mathcal{I}}$-st-$\lim_n \|L_n(\Psi)\| = 0$ then for all $f \in C[a,b]$ we have

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\| = 0.$$

Proof. Given, $\Psi(y) := (y-x)^2$ be a function on $[a,b]$ where $x \in [a,b]$ and $L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}; x \in [a,b]$ and $f \in C[a,b]$ where $a,\ b$ are two real numbers such that $a < b$. Since L_n is a positive linear operator then $L_n(\Psi;x) \geq 0$.

Let $M = \|f\|$ and $\varepsilon > 0$. By the uniform continuity of $f \in C[a,b]$ and $x \in [a,b]$ there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \varepsilon \text{ whenever } |y-x| \leq \delta$$

Let $I_\delta = [x-\delta, x+\delta] \cap [a,b]$. So

$$\begin{aligned} |f(y) - f(x)| &= |f(y)-f(x)|\Psi_{I_\delta}(y) + |f(y)-f(x)|\Psi_{[a,b]-I_\delta}(y) \\ &\leq \varepsilon + 2M\delta^{-2}(y-x)^2. \end{aligned}$$

Since L_n's are positive and linear so we have,

$$\begin{aligned} |L_n(f;x) - f(x)| &= |\int_a^b f(y)K_n(y-x)dy - f(x)| \\ &= |\int_a^b (f(y)-f(x))K_n(y-x)dy + f(x)\int_a^b K_n(y-x)dy - f(x)| \\ &\leq |\int_a^b (f(y)-f(x))K_n(y-x)dy| + |f(x)||\int_a^b K_n(y-x)dy - 1| \\ &\leq \int_a^b |f(y)-f(x)||K_n(y-x)dy| + |f(x)||L_n(f_0;x) - f_0(x)| \\ &\leq \int_a^b (\varepsilon + 2M\delta^{-2}(y-x)^2)K_n(y-x)dy + M|L_n(f_0;x) - f_0(x)| \\ &= \varepsilon + (\varepsilon + M)|L_n(f_0;x) - f_0(x)| + 2M\delta^{-2}|L_n(\Psi;x)| \\ &\leq \varepsilon + \alpha\{|L_n(f_0;x) - f_0(x)| + |L_n(\Psi;x)|\} \end{aligned}$$

where $\alpha = max\{\varepsilon + M, \frac{2M}{\delta^2}\}$.

Therefore

$$\|L_n(f) - f\| \leq \varepsilon + \alpha\{\|L_n(f_0) - f_0\| + \|L_n(\Psi)\|\}.$$

For any given $r > 0$, choose $\varepsilon > 0$ s.t. $0 < \varepsilon < r$ and define the following sets

$$D := \{n : \|L_n(f) - f\| \geq r\};$$

$$D_1 := \{n : \|L_n(f_0) - f_0\| \geq \frac{r-\varepsilon}{2\alpha}\};$$

$$D_2 := \{n : \|L_n(\Psi)\| \geq \frac{r-\varepsilon}{2\alpha}\}.$$

It follows that, $D \subseteq D_1 \cup D_2$ and consequently for all $n \in \mathbb{N}$

$$\sum_{k\in D} a_{nk} \leq \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk}$$

which implies that, for any $\sigma > 0$,

$$\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\} \subseteq \bigcup_{i=1}^{2}\{n \in \mathbb{N} : \sum_{k\in D_i} a_{nk} \geq \frac{\sigma}{2}\}.$$

Therefore from hypotheses

$$\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\} \in \mathcal{I}.$$

Hence the proof is conquered. □

Let δ be a positive real number so that $\delta < \frac{b-a}{2}$ and let $\|f\|_\delta = \sup_{a+\delta\leq x\leq b-\delta} |f(x)|,\ f \in C[a,b]$.

In order to give our main result we need the following lemmas.

Lemma 3.2.1. *Let $A = (a_{ij})$ be a member of $M_1(RS)$. Assume that δ is a fixed positive number such that $\delta < \dfrac{b-a}{2}$. If the conditions*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1 \tag{3.2}$$

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n (\sup_{|y|\geq\delta} K_n(y)) = 0 \tag{3.3}$$

hold, then for the operators L_n where $L_n(f;x) = \int_a^b f(y)K_n(y-x)dy,\ n \in \mathbb{N},\ x \in [a,b],\ f \in C[a,b]$ and a, b are real numbers $a < b$, we have

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0 \ \textit{with}\ f_0(y) = 1.$$

Proof. Let $0 < \delta < \frac{b-a}{2}$ and let $x \in [a+\delta, b-\delta]$. Then

$$\delta \leq x - a \leq b - a \Rightarrow -(b-a) \leq a - x \leq -\delta$$

and

$$\delta \leq b - x \leq b - a.$$

Now $L_n(f_0; x) = \int_a^b K_n(y-x)dy = \int_{a-x}^{b-x} K_n(y)dy$. Then we have,

$$\int_{-\delta}^{\delta} K_n(y)dy \leq L_n(f_0; x) \leq \int_{-(b-a)}^{b-a} K_n(y)dy.$$

Therefore $\|L_n(f_0) - f_0\|_\delta \leq u_n$ where $u_n := \max\{| \int_{-\delta}^{\delta} K_n(y)dy - 1 |, | \int_{-(b-a)}^{b-a} K_n(y)dy - 1 |\}$. Therefore $A^{\mathcal{I}}$-st-$\lim_n u_n = 0$ for all $\delta > 0$ such that $\delta < \dfrac{b-a}{2}$. Now for given $\varepsilon > 0$, define the following sets

$$D := \{n \in \mathbb{N} : \|L_n(f_0) - f_0\|_\delta \geq \varepsilon\};$$

$$D' := \{n \in \mathbb{N} : u_n \geq \varepsilon\}.$$

So $D \subseteq D'$. Then for all $n \in \mathbb{N}$ we have,

$$\sum_{k \in D} a_{nk} \leq \sum_{k \in D'} a_{nk}.$$

Then for any $\sigma > 0$,

$$\{n \in \mathbb{N} : \sum_{k \in D} a_{nk} \geq \sigma\} \subseteq \{n \in \mathbb{N} : \sum_{k \in D'} a_{nk} \geq \sigma\}.$$

From hypothesis,

$$\{n \in \mathbb{N} : \sum_{k \in D'} a_{nk} \geq \sigma\} \in \mathcal{I}.$$

Hence

$$\{n \in \mathbb{N} : \sum_{k \in D} a_{nk} \geq \sigma\} \in \mathcal{I}.$$

Hence the proof. □

Lemma 3.2.2. *Let $A = (a_{ij})$ be a member of $M_1(RS)$. If conditions 3.2 and 3.3 hold for a fixed $\delta > 0$ such that $\delta < \dfrac{b-a}{2}$, then for all convolution operators L_n defined by $L_n(f; x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}; x \in [a, b]$ and $f \in C[a, b]$ where a, b are two real numbers such that $a < b$, we have*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(\Psi)\|_\delta = 0 \text{ where } \Psi(y) = (y-x)^2.$$

Proof. For a fixed $0 < \delta < \frac{b-a}{2}$, let $x \in [a+\delta, b-\delta]$. Since $\Psi(y) = y^2 - 2xy + x^2$ then $\Psi \in C[a,b]$ for all $x \in [a+\delta, b-\delta]$. Now $L_n(\Psi; x) = L_n(f_2; x) - 2xL_n(f_1; x) + x^2 L_n(f_0; x)$ with $f_i(y) = y^i,\ i = 0,\ 1\ ,2$. Then for all $n \in N$

$$\begin{aligned} L_n(\Psi; x) &= \int_a^b (y-x)^2 K_n(y-x)dy \\ &= \int_{a-x}^{b-x} y^2 K_n(y)dy \\ &\leq \int_{-(b-a)}^{b-a} y^2 K_n(y)dy \end{aligned}$$

Since the f_2 is continuous at $y = 0$, given $\varepsilon > 0$ there exists a $\eta > 0$ such that $y^2 < \varepsilon$ for all y satisfying $\mid y \mid \leq \eta$. We have two cases such that $\eta \geq b - a \ or \ \eta < b - a$.

Case 1: Let $\eta \geq b - a$. Therefore $0 \leq L_n(\Psi; x) \leq \varepsilon \int_{-(b-a)}^{b-a} K_n(y)dy$. By condition 3.2, $0 \leq L_n(\Psi; x) \leq \varepsilon$ and $A^{\mathcal{I}}$-st-$\lim_n \|L_n(\Psi)\|_\delta = 0$ for $\eta \geq b - a$.

Case 2: Let $\eta < b - a$. Therefore $L_n(\Psi; x) \leq \int_{|y| \geq \eta} y^2 K_n(y)dy + \int_{|y| \leq \eta} y^2 K_n(y)dy$ and hence we obtain

$$\begin{aligned} \|L_n(\Psi; x)\|_\delta &\leq a_n \int_\eta^{b-a} y^2 dy + \varepsilon \int_{|y| \leq \eta} K_n(y)dy \\ &= a_n \frac{(b-a)^3 - \eta^3}{3} + \varepsilon b_n \end{aligned}$$

where $a_n = \sup_{|y| \geq \eta} K_n(y)$ and $b_n = \int_{|y| \leq \eta} K_n(y)dy$.
Also we have from hypotheses

$$A^{\mathcal{I}}\text{-st-}\lim_n a_n = 0$$

and

$$A^{\mathcal{I}}\text{-st-}\lim_n b_n = 1.$$

Taking, $M = \max\{\frac{(b-a)^3 - \eta^3}{3}, \varepsilon\}$ we have for all $n \in \mathbb{N}$ then

$$\|L_n(\Psi)\|_\delta \leq \varepsilon + M(a_n + \mid b_n - 1 \mid).$$

For given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$.
Let

$$D := \{n \in \mathbb{N} : \|L_n(\Psi)\|_\delta \geq r\};$$

$$D_1 := \{n \in \mathbb{N} : a_n \geq \frac{r - \varepsilon}{2M}\};$$

$$D_2 := \{n \in \mathbb{N} :| b_n - 1 |\geq \frac{r-\varepsilon}{2M}\}.$$

Therefore $D \subseteq D_1 \cup D_2$. Hence for all $n \in \mathbb{N}$ we have,

$$\sum_{k\in D} a_{nk} \leq \sum_{k\in D_1} a_{nk} + \sum_{k\in D_2} a_{nk}$$

which implies that for any $\sigma > 0$,

$$\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\} \subseteq \bigcup_{i=1}^{2} \{n \in \mathbb{N} : \sum_{k\in D_i} a_{nk} \geq \frac{\sigma}{2}\}.$$

Therefore from the hypothesis

$$\{n \in \mathbb{N} : \sum_{k\in D} a_{nk} \geq \sigma\} \in \mathcal{I}.$$

Hence the proof. □

Now the following main result follows from Theorem 3.2.2 and Lemma 3.2.1, 3.2.2.

Theorem 3.2.3. *Let $A = (a_{ij})$ be a member of $M_1(RS)$ and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by* (3.1). *If conditions* 3.2 *and* 3.3 *hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\|_\delta = 0.$$

If we take $\mathcal{I} = \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$, we get the following

Corollary 3.2.1 (Corollary 2.5,[27])**.** *Let $A = (a_{ij})$ be a non negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by*

$$L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$$

$n \in \mathbb{N}, x \in [a,b]$ and $f \in C[a,b]$ where a and b are two real numbers such that $a < b$. If conditions

$$st_A - \lim_n \int_{-\delta}^{\delta} K_n(y)dy = 1$$

and

$$st_A - \lim_n (\sup_{|y|\geq\delta} K_n(y)) = 0$$

hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have

$$st_A - \lim_n \|L_n(f) - f\|_\delta = 0.$$

Remark 3.2.1. *We now exhibit a sequence of positive convolution operators for which Corollary 3.2.1 does not apply but Theorem 3.2.3 does. Let*

$$u_n = \begin{cases} 1 & \text{for} \quad n \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{I}$ be a non-trivial admissible ideal of $\mathbb{N}$. Choose an infinite subset $C = \{q_1 < q_2 < q_3 ...\}$ from $\mathcal{I} \setminus \mathcal{I}_d$.

Let $A = (a_{nk})$ be given by

$$a_{nk} = \begin{cases} 1 & \text{if } n = q_i, k = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } n \neq q_i, \text{ for any } i, \ k = 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now for $0 < \varepsilon < 1$, $K(\varepsilon) = \{k \in \mathbb{N} : |u_k - 0| \geq \varepsilon\}$ is the set of all even positive integers. We can find that

$$\sum_{k \in K(\varepsilon)} a_{nk} = \begin{cases} 1 & \text{if } n = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } n \neq q_i, \text{ for any } i \in \mathbb{N}. \end{cases}$$

Thus for any $\delta > 0$, $\left\{ n \in \mathbb{N} : \sum_{k \in K(\varepsilon)} a_{nk} \geq \delta \right\} = C \in \mathcal{I} \setminus \mathcal{I}_d$. This shows that $\{u_k\}_{k \in \mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to 0 but not A-statistically convergent.

Now let the operators L_n on $C[a,b]$ be defined by

$$L_n(f;x) = \frac{n(1+u_n)}{\sqrt{\pi}} \int_a^b f(y) e^{-n^2(y-x)^2} dy.$$

If we choose $K_n(y) = \frac{n(1+u_n)}{\sqrt{\pi}} e^{-n^2 y^2}$ then

$$L_n(f;x) = \frac{n(1+u_n)}{\sqrt{\pi}} \int_a^b f(y) K_n(y-x) dy.$$

Now for every $\delta > 0$ such that $\delta < \frac{b-a}{2}$, we have

$$\begin{aligned} \int_{-\delta}^{\delta} K_n(y) dy &= \frac{n(1+u_n)}{\sqrt{\pi}} \left(\int_{-\infty}^{\infty} e^{-n^2 y^2} dy - \int_{|y| \geq \delta} e^{-n^2 y^2} dy \right) \\ &= \frac{2(1+u_n)}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-y^2} dy - \int_{\delta.n}^{\infty} e^{-y^2} dy \right). \end{aligned}$$

Since $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} < \infty$, it is clear that $\lim_n \int_{\delta.n}^{\infty} e^{-y^2} dy = 0$.
Also since $A^{\mathcal{I}}$-st-$\lim_n (1+u_n) = 1$, we immediately get

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \int_{-\delta}^{\delta} K_n(y) dy = 1.$$

On the other hand, we have

$$\begin{aligned}\sup_{|y|\geq\delta} K_n(y) &= \frac{n(1+u_n)}{\sqrt{\pi}} \sup_{|y|\geq\delta} e^{-n^2y^2} \\ &\leq \frac{n(1+u_n)}{e^{n^2\delta^2}}.\end{aligned}$$

Since $\lim_n \frac{n}{e^{n^2\delta^2}} = 0$ *and* $A^{\mathcal{I}}\text{-}st\text{-}\lim_n (1+u_n) = 1$, *we conclude that*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n (\sup_{|y|\geq\delta} K_n(y)) = 0.$$

Therefore from Theorem 3.2.3

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\|_\delta = 0 \text{ for all } f \in C[a,b].$$

However note that, as $\{u_k\}_{k\in\mathbb{N}}$ *is not A-statistically convergent to zero so* K_n *do not satisfy the hypotheses of Corollary 3.2.1.*

3.3 Rate of $A^{\mathcal{I}}$-Statistical Convergence

In this section we study the rates of $A^{\mathcal{I}}$-statistical convergence in Theorem 3.2.3 using the modulus of continuity already described.

Next we introduce the following definition

Definition 3.3.1. *Let* $A = (a_{jn})$ *be a member of* $M_1(RS)$ *and let* $\{c_n\}_{n\in\mathbb{N}}$ *be a positive non-increasing sequence of real numbers. Then a sequence* $x = \{x_n\}_{n\in\mathbb{N}}$ *is said to be* $A^{\mathcal{I}}$-*statistically convergent to a number L with the rate of* $o(c_n)$ *if for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\{j \in \mathbb{N} : \frac{1}{c_j} \sum_{\{n:|x_n-L|\geq\varepsilon\}} a_{jn} \geq \delta\} \in \mathcal{I}$.
In this case we write $A^{\mathcal{I}}\text{-}st\text{-}o(c_n)\text{-}\lim_n x_n = L$.

We establish the following Theorem

Theorem 3.3.1. *Let* $A = (a_{jn})$ *be a member of* $M_1(RS)$ *and let* $\{L_n\}_{n\in\mathbb{N}}$ *be a sequence of convolution operators given by* (3.1). *Assume further that* $\{c_n\}_{n\in\mathbb{N}}$ *and* $\{d_n\}_{n\in\mathbb{N}}$ *are two positive non-increasing sequences. If for a fixed* $\delta > 0$ *such that* $\delta < \frac{b-a}{2}$,

$$A^{\mathcal{I}}\text{-}st\text{-}o(c_n)\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0$$

and

$$A^{\mathcal{I}}\text{-}st\text{-}o(d_n)\text{-}\lim_n \omega(f, \alpha_n) = 0$$

where $\alpha_n := \sqrt{\|L_n(\Psi)\|_\delta}$, *then for all* $f \in C[a,b]$ *we have*

$$A^{\mathcal{I}}\text{-}st\text{-}o(p_n)\text{-}\lim_n \|L_n(f) - f\|_\delta = 0$$

where $p_n := \max\{c_n, d_n\}$.

Proof. Let $0 < \delta < \frac{b-a}{2}$, $f \in C[a,b]$ and $x \in [a+\delta, b-\delta]$. By positivity and linearity of the operators L_n and using the inequalities for any $\alpha > 0$ we get

$$\begin{aligned}
|L_n(f;x) - f(x)| &\leq L_n(|f(y) - f(x)|;x) + |f(x)||L_n(f_0;x) - f_0(x)| \\
&\leq L_n(\omega(f, \alpha\frac{|y-x|}{\alpha});x) + |f(x)||L_n(f_0;x) - f_0(x)| \\
&\leq \omega(f,\alpha)L_n(1 + [\frac{|y-x|}{\alpha}];x) + |f(x)||L_n(f_0;x) - f_0(x)| \\
&\leq \omega(f,\alpha)\{L_n(f_0;x) + \frac{1}{\alpha^2}L_n(\psi;x)\} + |f(x)||L_n(f_0;x) - f_0(x)|.
\end{aligned}$$

Therefore for all $n \in \mathbb{N}$

$$\|L_n(f) - f\|_\delta \leq \omega(f,\alpha)\{\|L_n(f_0)\|_\delta + \frac{1}{\alpha^2}\|L_n(\Psi)\|_\delta\} + M_1\|L_n(f_0) - f_0\|_\delta$$

where $M_1 := \|f\|_\delta$. Now let $\alpha := \alpha_n = \sqrt{\|L_n(\Psi)\|_\delta}$, then we have

$$\begin{aligned}
\|L_n(f) - f\|_\delta &\leq \omega(f,\alpha_n)\{\|L_n(f_0)\|_\delta + 1\} + M_1\|L_n(f_0) - f_0\|_\delta \\
&\leq 2\omega(f,\alpha_n) + \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta + M_1\|L_n(f_0) - f_0\|_\delta.
\end{aligned}$$

Let $M = \max\{2,\ M_1\}$. Then we can write for all $n \in \mathbb{N}$ that

$$\|L_n(f) - f\|_\delta \leq M\{\omega(f,\alpha_n) + \|L_n(f_0) - f_0\|_\delta\} + \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta.$$

Given $\varepsilon > 0$, define the following sets:

$$D := \{n : \|L_n(f) - f\|_\delta \geq \varepsilon\};$$

$$D_1 := \{n : \omega(f,\alpha_n) \geq \frac{\varepsilon}{3M}\};$$

$$D_2 := \{n : \omega(f,\alpha_n)\|L_n(f_0) - f_0\|_\delta \geq \frac{\varepsilon}{3}\};$$

$$D_3 := \{n : \|L_n(f_0) - f_0\|_\delta \geq \frac{\varepsilon}{3M}\}.$$

Then $D \subseteq D_1 \cup D_2 \cup D_3$. Also, we define

$$D_2^{'} = \{n : \omega(f,\alpha_n) \geq \sqrt{\frac{\varepsilon}{3}}\};$$

$$D_2^{''} = \{n : \|L_n(f_0) - f_0\|_\delta \geq \sqrt{\frac{\varepsilon}{3}}\}.$$

Therefore $D_2 \subseteq D_2^{'} \cup D_2^{''}$. Hence we get $D \subseteq D_1 \cup D_2^{'} \cup D_2^{''} \cup D_3$. Since $p_n = \max\{c_n,\ d_n\}$ we obtain for all $j \in \mathbb{N}$ that

$$\frac{1}{p_j} \sum_{n \in D} a_{jn} \leq \frac{1}{d_j} \sum_{n \in D_1} a_{jn} + \frac{1}{d_j} \sum_{n \in D_2^{'}} a_{jn} + \frac{1}{c_j} \sum_{n \in D_2^{''}} a_{jn} + \frac{1}{c_j} \sum_{n \in D_3} a_{jn}.$$

As

$$A^{\mathcal{I}}\text{-st-}o(c_n)\text{-}\lim_n \|L_n(f_0) - f_0\|_\delta = 0$$

and

$$A^{\mathcal{I}}\text{-st-}o(d_n)\text{-}\lim_n \omega(f, \alpha_n) = 0.$$

Therefore

$$\left\{ j \in \mathbb{N} : \frac{1}{p_j} \sum_{n \in D} a_{jn} \geq \delta \right\} \in \mathcal{I}$$

i.e.

$$A^{\mathcal{I}}\text{-st-}o(p_n)\text{-}\lim_n \|L_n(f) - f\|_\delta = 0 \text{ for all } f \in C[a,b],$$

where $p_n := \max\{c_n, d_n\}$. Hence the result. □

Now we enter the next section following the notion of $A^{\mathcal{I}}$-summability.

3.4 Approximation on the notion of $A^{\mathcal{I}}$-summability

We are concern that $A^{\mathcal{I}}$-summability instantiates to statistical A-summability for $\mathcal{I} = \mathcal{I}_d$ [30].

We consider the Banach space $C[a,b]$ endowed with the supremum norm $||f|| = \sup_{x \in [a,b]} |f(x)|$ for $f \in C[a,b]$.

Theorem 3.4.1. *Let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of P.L.O. from $C[a,b]$ into $C[a,b]$ and $A = (a_{jn})$ be a non-negative regular matrix. If*

$$\mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_i) - f_i \right\| = 0 \textit{ with } f_i(y) = y^i,\ i = 0, 1, 2$$

then for all $f \in C[a,b]$ we have

$$\mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\| = 0.$$

Proof. We start by observing that for each $x \in [a,b]$, the function $0 \leq \Psi \in C[a,b]$ defined by $\Psi(y) = (y-x)^2$. Since each L_n is positive, $L_n(\Psi; x)$ is a positive function. In particular, we have for each $x \in [a,b]$

$$\begin{aligned}
0 &\leq \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x) \\
&= \sum_{n=1}^{\infty} a_{jn} L_n(y^2;x) - 2x \sum_{n=1}^{\infty} a_{jn} L_n(y;x) + x^2 \sum_{n=1}^{\infty} a_{jn} L_n(1;x) \\
&= (\sum_{n=1}^{\infty} a_{jn} L_n(y^2;x) - y^2(x)) - 2x(\sum_{n=1}^{\infty} a_{jn} L_n(y;x) - y(x)) + x^2(\sum_{n=1}^{\infty} a_{jn} L_n(1;x) - 1(x)) \\
&\leq \left\| \sum_{n=1}^{\infty} a_{jn} L_n(y^2) - y^2 \right\| + 2b \left\| \sum_{n=1}^{\infty} a_{jn} L_n(y) - y \right\| + b^2 \left\| \sum_{n=1}^{\infty} a_{jn} L_n(1) - 1 \right\|.
\end{aligned}$$

Fix $f \in C[a,b]$. Let $M = \|f\|$. Then we can write $|f(y) - f(x)| < 2M$ for all $y, x \in [a,b]$. Also, since f is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$. Hence for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all y, x satisfying $| y - x |< \delta$. On the other hand, if $| y - x |\geq \delta$, then it follows that,

$$-\frac{2M}{\delta^2}(y-x)^2 \leq -2M \leq f(y) - f(x) \leq 2M \leq \frac{2M}{\delta^2}(y-x)^2.$$

Therefore for all $y, x \in [a,b]$ we get,

$$| f(y) - f(x) |< \varepsilon + \frac{2M}{\delta^2}(y-x)^2$$

where δ is a fixed real number. Since each L_n is positive, we have

$$\begin{aligned}
-\varepsilon \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - \frac{2M}{\delta^2} \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x) &\leq \sum_{n=1}^{\infty} a_{jn} L_n(f(y);x) - f(x) \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) \\
&\leq \varepsilon \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) + \frac{2M}{\delta^2} \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x).
\end{aligned}$$

Next, let $K = \frac{2M}{\delta^2}$ and we get

$$\begin{aligned}
\left| \sum_{n=1}^{\infty} a_{jn} L_n(f(y);x) - f(x) \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) \right| &\leq \varepsilon \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) + \frac{2M}{\delta^2} \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x) \\
&= \varepsilon + \varepsilon \left[\sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - f_0(x) \right] \\
&\quad + K \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x) \\
&\leq \varepsilon + \varepsilon \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\| \\
&\quad + K \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x).
\end{aligned}$$

In particular,

$$\begin{aligned}\left|\sum_{n=1}^{\infty} a_{jn}L_n(f(y);x) - f(x)\right| &\leq \left|\sum_{n=1}^{\infty} a_{jn}L_n(f(y);x) - f(x)\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x)\right| \\ &\quad + |f(x)|\left|\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right| \\ &\leq \varepsilon + K\sum_{n=1}^{\infty} a_{jn}L_n(\Psi;x) + (M+\varepsilon)\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right\|\end{aligned}$$

which implies

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f) - f\right\| &\leq \varepsilon + C_2\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_2) - f_2\right\| + C_1\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_1) - f_1\right\| \\ &\quad + C_0\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\|\end{aligned}$$

where, $C_2 = K$, $C_1 = 2bK$ and $C_0 = (\varepsilon + b^2K + M)$ i.e. ,

$$\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f) - f\right\| \leq \varepsilon + C\sum_{i=0}^{2}\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_i) - f_i\right\|,\ i = 0,1,2$$

where $C = \max\{C_0,\ C_1,\ C_2\}$. For a given $\varepsilon' > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$ and let us define the following sets

$$D = \left\{j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn}L_n(f) - f\right\| \geq \varepsilon'\right\};$$

$$D_1 = \left\{j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\| \geq \frac{\varepsilon' - \varepsilon}{3C}\right\};$$

$$D_2 = \left\{j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_1) - f_1\right\| \geq \frac{\varepsilon' - \varepsilon}{3C}\right\};$$

$$D_3 = \left\{j \in \mathbb{N} : \left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_2) - f_2\right\| \geq \frac{\varepsilon' - \varepsilon}{3C}\right\}.$$

It follows that $D \subseteq D_1 \cup D_2 \cup D_3$ and from hypotheses we have D_1, D_2, D_3 belong to $\mathcal{I}$. Therefore $D \in \mathcal{I}$. Hence the proof is completed. □

In [73], the authors investigated the classical versions of the following results in two variables and for sequence of infinite matrices. In particular, for Frechet ideal $\mathcal{I} = \mathcal{I}_{fin}$, the following results give the classical versions for single variable.

Theorem 3.4.2. *Let $A = (a_{jn})$ be a member of $M_1(RS)$ and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators from $C[a,b]$ into $C[a,b]$ as in* (3.1). *If*

$$\mathcal{I}\text{-}\lim_{j}\left\|\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\| = 0 \text{ with } f_0(y) = 1$$

and

$$\mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\| = 0$$

then for all $f \in C[a,b]$ *we have*

$$\mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\| = 0.$$

Proof. Fix $f \in C[a,b]$ and $x \in [a,b]$. Let $M = \|f\|$ and $\varepsilon > 0$. By the uniform continuity of $f \in C[a,b]$ and $x \in [a,b]$, there exists a $\delta > 0$ such that $| f(y) - f(x) | < \varepsilon$ whenever $| y - x | \leq \delta$. Let $I_\delta = [x-\delta, x+\delta] \cap [a,b]$. So

$$\begin{aligned} |f(y) - f(x)| &= | f(y) - f(x) | \Psi_{I_\delta}(y) + | f(y) - f(x) | \Psi_{[a,b]-I_\delta}(y) \\ &\leq \varepsilon + 2M\delta^{-2}(y-x)^2. \end{aligned}$$

Since L_n's are positive and linear so we have,

$$\begin{aligned} \left| \sum_{n=1}^{\infty} a_{jn} L_n(f;x) - f(x) \right| &= \left| \sum_{n=1}^{\infty} a_{jn} \int_a^b f(y) K_n(y-x) dy - f(x) \right| \\ &= | \sum_{n=1}^{\infty} a_{jn} \int_a^b (f(y) - f(x)) K_n(y-x) dy \\ &\quad + f(x) \sum_{n=1}^{\infty} a_{jn} \int_a^b K_n(y-x) dy - f(x) | \\ &\leq \left| \sum_{n=1}^{\infty} a_{jn} \int_a^b (f(y) - f(x)) K_n(y-x) dy \right| \\ &\quad + |f(x)| \left| \sum_{n=1}^{\infty} a_{jn} \int_a^b K_n(y-x) dy - 1 \right| \\ &\leq \sum_{n=1}^{\infty} a_{jn} \int_a^b |f(y) - f(x)| \, |K_n(y-x)| \, dy \\ &\quad + |f(x)| \left| \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - f_0(x) \right| \\ &\leq \sum_{n=1}^{\infty} a_{jn} \int_a^b (\varepsilon + 2M\delta^{-2}(y-x)^2) K_n(y-x) dy \\ &\quad + M \left| \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - f_0(x) \right| \\ &= \varepsilon + (\varepsilon + M) \left| \sum_{n=1}^{\infty} a_{jn} L_n(f_0;x) - f_0(x) \right| \\ &\quad + 2M\delta^{-2} \left| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x) \right| \end{aligned}$$

$$\leq \varepsilon + \alpha \left| \sum_{n=1}^{\infty} a_{jn} L_n(f_0; x) - f_0(x) \right| + \alpha \sum_{n=1}^{\infty} a_{jn} L_n(\Psi; x)$$

where $\alpha = \max\{\varepsilon + M, \ \frac{2M}{\delta^2}\}$.

Therefore

$$\left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\| \leq \varepsilon + \alpha \left\{ \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\| + \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\| \right\}.$$

For given $r > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < r$ and define the following sets

$$D = \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\| \geq r \right\};$$

$$D_1 = \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\| \geq \frac{r - \varepsilon}{2\alpha} \right\};$$

$$D_2 = \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\| \geq \frac{r - \varepsilon}{2\alpha} \right\}.$$

It follows that $D \subseteq D_1 \cup D_2$ and since D_1, D_2 belong to $\mathcal{I}$ so $D \in \mathcal{I}$. Hence this completes the proof. □

Let δ be a positive real number so that $\delta < \frac{b-a}{2}$ and let $\|f\|_\delta = \sup_{a+\delta \leq x \leq b-\delta} | f(x) |$, $f \in C[a, b]$.

We now study the main theorem of this section.

Theorem 3.4.3. *Let $A = (a_{jn})$ be a member of $M_1(RS)$ and let $\{L_n\}_{n \in \mathbb{N}}$ be a sequence of convolution operators on $C[a, b]$ given by (1). If conditions*

$$\mathcal{I}\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} \int_{-\delta}^{\delta} K_n(y) dy = 1 \tag{3.4}$$

$$\mathcal{I}\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} (\sup_{|y| \geq \delta} K_n(y)) = 0 \tag{3.5}$$

hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a, b]$ we have

$$\mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_\delta = 0.$$

In order to prove our main result we need the following lemma.

Lemma 3.4.1. *Let $A = (a_{jn})$ be a member of $M_1(RS)$. Assume that δ is a fixed positive number such that $\delta < \dfrac{b-a}{2}$. If the conditions 3.4 and 3.5 hold, then for the operators L_n where $L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$, $n \in \mathbb{N}$, $x \in [a,b]$, $f \in C[a,b]$ and a, b are real numbers $a < b$, we have*

$$(i)\ \mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_{\delta} = 0 \textit{ with } f_0(y) = 1$$

and

$$(ii)\ \mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_{\delta} = 0 \quad \textit{with } \Psi(y) = (y-x)^2.$$

Proof. (i) Let $0 < \delta < \frac{b-a}{2}$ and let $x \in [a+\delta, b-\delta]$. Then $\delta \leq x - a \leq b - a \Rightarrow -(b-a) \leq a - x \leq -\delta$ and $\delta \leq b - x \leq b - a$. Now $L_n(f_0;x) = \int_a^b K_n(y-x)dy = \int_{a-x}^{b-x} K_n(y)dy$. Then we have,

$$\int_{-\delta}^{\delta} K_n(y)dy \leq L_n(f_0;x) \leq \int_{-(b-a)}^{b-a} K_n(y)dy.$$

Therefore

$$\left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_{\delta} \leq u_j$$

where $u_j := \max \left\{ \left| \sum_{n=1}^{\infty} a_{jn} \int_{-\delta}^{\delta} K_n(y)dy - 1 \right|, \left| \sum_{n=1}^{\infty} a_{jn} \int_{-(b-a)}^{b-a} K_n(y)dy - 1 \right| \right\}$.

Therefore from given conditions, $\mathcal{I}\text{-}\lim_j u_j = 0$ for all $\delta > 0$ such that $\delta < \dfrac{b-a}{2}$. Now for given $\varepsilon > 0$

$$(\text{say})\ D := \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_{\delta} \geq \varepsilon \right\} \subseteq \{ j \in \mathbb{N} : u_j \geq \varepsilon \} = D' \ (\text{say}).$$

Since $D' \in \mathcal{I}$, so $D \in \mathcal{I}$. Hence this completes the proof of (i).

(ii) For a fixed $0 < \delta < \frac{b-a}{2}$, let $x \in [a+\delta, b-\delta]$. Since $\Psi(y) = y^2 - 2xy + x^2$ then $\Psi \in C[a,b]$ for all $x \in [a+\delta, b-\delta]$. Now $L_n(\Psi;x) = L_n(f_2;x) - 2xL_n(f_1;x) + x^2 L_n(f_0;x)$ with $f_i(y) = y^i$, $i = 0,1,2$. Then for all $n \in \mathbb{N}$

$$\begin{aligned} L_n(\Psi;x) &= \int_a^b (y-x)^2 K_n(y-x)dy \\ &= \int_{a-x}^{b-x} y^2 K_n(y)dy \\ &\leq \int_{-(b-a)}^{b-a} y^2 K_n(y)dy \end{aligned}$$

Since the function f_2 is continuous at $y = 0$, given $\varepsilon > 0$ there exists $\eta > 0$ such that for every y satisfying $| y | \leq \eta$, $y^2 < \varepsilon$ holds. We have two cases such that $\eta \geq b - a$ *or* $\eta < b - a$.

Case 1

Let $\eta \geq b-a$. Therefore $0 \leq L_n(\Psi;x) \leq \varepsilon \int_{-(b-a)}^{b-a} K_n(y)dy$.

By condition 3.4, $0 \leq \sum_{n=1}^{\infty} a_{jn} L_n(\Psi;x) \leq \varepsilon$ and $\mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_\delta = 0$ for $\eta \geq b-a$.

Case 2

Now let $\eta < b-a$. Therefore $L_n(\Psi;x) \leq \int_{|y|\geq\eta} y^2 K_n(y)dy + \int_{|y|\leq\eta} y^2 K_n(y)dy$

and hence we obtain for all $j \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_\delta &\leq \sum_{n=1}^{\infty} a_{jn} p_n \int_\eta^{b-a} y^2 dy + \varepsilon \sum_{n=1}^{\infty} a_{jn} \int_{|y|\leq\eta} K_n(y)dy \\ &= \frac{(b-a)^3 - \eta^3}{3} \sum_{n=1}^{\infty} a_{jn} p_n + \varepsilon \sum_{n=1}^{\infty} a_{jn} q_n \end{aligned}$$

where $p_n = \sup_{|y|\geq\eta} K_n(y)$ and $q_n = \int_{|y|\leq\eta} K_n(y)dy$.

Also we have from conditions 3.4 and 3.5

$$\mathcal{I}\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} p_n = 0$$

and

$$\mathcal{I}\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} q_n = 1.$$

Taking, $M = \max\{\frac{(b-a)^3-\eta^3}{3}, \varepsilon\}$ we have for all $j \in \mathbb{N}$ then

$$\left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_\delta \leq \varepsilon + M \left(\sum_{n=1}^{\infty} a_{jn} p_n + \left| \sum_{n=1}^{\infty} a_{jn} q_n - 1 \right| \right).$$

For given $r > 0$, choose $\varepsilon > 0$ such that $\varepsilon < r$.

Let

$$D = \left\{ j \in \mathbb{N} : \left\| \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_\delta \geq r \right\};$$

$$D_1 = \left\{ j \in \mathbb{N} : \sum_{n=1}^{\infty} a_{jn} p_n \geq \frac{r-\varepsilon}{2M} \right\};$$

$$D_2 = \left\{ j \in \mathbb{N} : \left| \sum_{n=1}^{\infty} a_{jn} q_n - 1 \right| \geq \frac{r-\varepsilon}{2M} \right\}.$$

Therefore $D \subseteq D_1 \cup D_2$. Since from hypotheses, D_1 and D_2 belong to $\mathcal{I}$, so $D \in \mathcal{I}$. Hence this completes the proof. □

Proof. Proof of Theorem 3.4.3

The main result (Theorem 3.4.3) follows from Theorem 3.4.2, Lemma 3.4.1. □

If we take $\mathcal{I} = \mathcal{I}_d$, we get the following

Corollary 3.4.1. *Let $A = (a_{jn})$ be a member of $M_1(RS)$ and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators on $C[a,b]$ given by*

$$L_n(f;x) = \int_a^b f(y)K_n(y-x)dy$$

$n \in \mathbb{N}, x \in [a,b]$ and $f \in C[a,b]$ where a and b are two real numbers such that $a < b$. If conditions

$$st\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} \int_{-\delta}^{\delta} K_n(y)dy = 1$$

and

$$st\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn}(\sup_{|y|\geq\delta} K_n(y)) = 0$$

hold for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$, then for all $f \in C[a,b]$ we have

$$st\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_{\delta} = 0.$$

The above corollary can be proved independently in a straightforward way and it is the statistical A-summable version of the Theorem 2.4. in [27].

Remark 3.4.1. *We now exhibit a sequence of positive convolution operators for which Corollary 3.4.1 does not apply but Theorem 3.4.3 does. Let*

$$u_n = \begin{cases} 1 & \textit{for} \quad n \textit{ even} \\ 0 & \textit{otherwise.} \end{cases}$$

Let $\mathcal{I}$ be a non-trivial admissible ideal of $\mathbb{N}$ such that $\mathcal{I} \neq \mathcal{I}_{fin}$(Frechet ideal) and $\mathcal{I} \neq \mathcal{I}_d$. Choose an infinite subset $C = \{p_1 < p_2 < p_3 ...\}$ from $\mathcal{I} \setminus \mathcal{I}_d$.

Let $A = (a_{jn})$ be given by

$$a_{jn} = \begin{cases} 1 & \textit{if } j = p_i, n = 2p_i \textit{ for some } i \in \mathbb{N} \\ 1 & \textit{if } j \neq p_i, \textit{ for any } i, \ n = 2j+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Observe that

$$y_j = \sum_{n=1}^{\infty} a_{jn} u_n = \begin{cases} 1 & \textit{if } j = p_i \textit{ for some } i \in \mathbb{N} \\ 0 & \textit{if } j \neq p_i, \textit{ for any } i \in \mathbb{N}. \end{cases}$$

Let $\varepsilon > 0$ be given and $\{j \in \mathbb{N} : |y_j - 0| \geq \varepsilon\} = C \in \mathcal{I} \setminus \mathcal{I}_d$. Thus $\{u_n\}_{n\in\mathbb{N}}$ is $A^{\mathcal{I}}$-summable to 0 but not statistically A-summable.

Now let the operators L_n *on* $C[a,b]$ *be defined by*

$$L_n(f;x) = \frac{n(1+y_n)}{\sqrt{\pi}} \int_a^b f(y) e^{-n^2(y-x)^2} dy.$$

If we choose $K_n(y) = \frac{n(1+y_n)}{\sqrt{\pi}} e^{-n^2y^2}$ *then*

$$L_n(f;x) = \frac{n(1+y_n)}{\sqrt{\pi}} \int_a^b f(y) K_n(y-x) dy.$$

Now for every $\delta > 0$ *such that* $\delta < \frac{b-a}{2}$, *we have*

$$\begin{aligned} \int_{-\delta}^{\delta} K_n(y)dy &= \frac{n(1+y_n)}{\sqrt{\pi}} \left(\int_{-\infty}^{\infty} e^{-n^2y^2} dy - \int_{|y|\geq\delta} e^{-n^2y^2} dy \right) \\ &= \frac{2(1+y_n)}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-y^2} dy - \int_{\delta.n}^{\infty} e^{-y^2} dy \right). \end{aligned}$$

Since $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2} < \infty$, *it is clear that* $\lim_n \int_{\delta.n}^{\infty} e^{-y^2} dy = 0$.
Also since $\mathcal{I}\text{-}\lim_j ||1 + y_j|| = 1$, *we immediately get*

$$\mathcal{I}\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} \int_{-\delta}^{\delta} K_n(y)dy = 1.$$

On the other hand, we have

$$\begin{aligned} \sup_{|y|\geq\delta} K_n(y) &= \frac{n(1+y_n)}{\sqrt{\pi}} \sup_{|y|\geq\delta} e^{-n^2y^2} \\ &\leq \frac{n(1+u_n)}{e^{n^2\delta^2}}. \end{aligned}$$

Since $\lim_n \frac{n}{e^{n^2\delta^2}} = 0$ *we conclude that*

$$\mathcal{I}\text{-}\lim_j \sum_{n=1}^{\infty} a_{jn} (\sup_{|y|\geq\delta} K_n(y)) = 0.$$

Therefore from Theorem 3.4.3

$$\mathcal{I}\text{-}\lim_j \left\| \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_{\delta} = 0 \textit{ for all } f \in C[a,b].$$

However note that, as $\{u_n\}_{n\in\mathbb{N}}$ *is not statistical A-summable to zero so Corollary 3.4.1. do not work for the operator defined above.*

We now recall the following note from [14] and make a remark in support of the existence of the set C in the above remark.

Remark 3.4.2. *The simple density ideal $\mathcal{Z}_g$ for which $\frac{n}{g(n)}$ is bounded does not necessarily coincide with $\mathcal{Z}$, where in particular, $\mathcal{Z}$ is the simple density ideal generated by $g(n) = n$ and in fact $\mathcal{Z} = \mathcal{I}_d$. Consider the set $S_k = [(2k)!, (2k+1)!]$ for all $k \in \mathbb{N}$ and $S := \cup_k S_{2k}$. If we consider the simple density ideal $\mathcal{Z}_g$ where $g : \mathbb{N} \to [0, \infty)$ be defined by*

$$g(n) = \begin{cases} n^2 & \text{if} \quad n \in S \\ n & \text{if} \quad n \notin S \end{cases}$$

then $S \in \mathcal{Z}_g \setminus \mathcal{Z}$ [14].

3.5 Rate of $A^{\mathcal{I}}$-Summability

In this section we investigate the rates of $A^{\mathcal{I}}$-summability in Theorem 3.4.3 using the modulus of continuity.

Next we introduce the following definition

Definition 3.5.1. *Let $A = (a_{jn})$ be a member of $M_1(RS)$ and let $\{c_n\}_{n\in\mathbb{N}}$ be a positive non-increasing sequence of real numbers. Then a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-summable to a number L with the rate of $o(c_n)$ if for every $\varepsilon > 0$ such that*

$$\left\{ j \in \mathbb{N} : \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} x_n - L \right\| \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case we write $A^{\mathcal{I}}$-sum-$o(c_n)$-$\lim_n x_n = L$.

In particular, for the non-increasing sequence $\{c_n\}_{n\in\mathbb{N}}$, where $c_n = 1,\ n \in \mathbb{N}$, Definition 3.5.1 implies $A^{\mathcal{I}}$-summability to a number L.

We establish the following Theorem

Theorem 3.5.1. *Let $A = (a_{jn})$ be a member of $M_1(RS)$ and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of convolution operators given by (3.1). Assume further that $\{c_n\}_{n\in\mathbb{N}}$is a positive non-increasing sequence. If for a fixed $\delta > 0$ such that $\delta < \frac{b-a}{2}$,*

$$A^{\mathcal{I}}\text{-}sum\text{-}o(c_n)\text{-}\lim_n L_n(f_0) = f_0$$

and

$$\mathcal{I}\text{-}\lim_j \omega(f, \alpha_j) = 0$$

where $\alpha_j := \sqrt{\left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(\Psi) \right\|_{\delta}}$, then for all $f \in C[a, b]$ we have

$$\mathcal{I}\text{-}\lim_j \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_{\delta} = 0.$$

Proof. Let $0 < \delta < \frac{b-a}{2}$, $f \in C[a,b]$ and $x \in [a+\delta, b-\delta]$. By positivity and linearity of the operators L_n and using the inequalities for any $\alpha > 0$ we get

$$\begin{aligned}
\left|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f;x) - f(x)\right| &\leq \frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(\mid f(y) - f(x) \mid; x) \\
&\quad + \mid f(x) \mid \left|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right| \\
&\leq \frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n\left(\omega(f, \alpha\frac{\mid y-x \mid}{\alpha}\right); x) \\
&\quad + \mid f(x) \mid \left|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right| \\
&\leq \omega(f,\alpha)\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n\left(1 + \left[\frac{\mid y-x \mid}{\alpha}\right]; x\right) \\
&\quad + \mid f(x) \mid \left|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right| \\
&\leq \omega(f,\alpha)\left\{\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) + \frac{1}{\alpha^2}\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(\Psi;x)\right\} \\
&\quad + \mid f(x) \mid \left|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0;x) - f_0(x)\right|.
\end{aligned}$$

Therefore for all $n \in \mathbb{N}$

$$\begin{aligned}
\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f) - f\right\|_\delta &\leq \omega(f,\alpha)\left\{\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)\right\|_\delta + \frac{1}{\alpha^2}\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(\Psi)\right\|_\delta\right\} \\
&\quad + M_1\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\|_\delta
\end{aligned}$$

where $M_1 := \|f\|_\delta$. Now let $\alpha := \alpha_j = \sqrt{\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(\Psi)\right\|_\delta}$, then we have

$$\begin{aligned}
\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f) - f\right\|_\delta &\leq \omega(f,\alpha_j)\left\{\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0)\right\|_\delta + 1\right\} \\
&\quad + M_1\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\|_\delta \\
&\leq 2\omega(f,\alpha_j) + \omega(f,\alpha_j)\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\|_\delta \\
&\quad + M_1\left\|\frac{1}{c_j}\sum_{n=1}^{\infty} a_{jn}L_n(f_0) - f_0\right\|_\delta.
\end{aligned}$$

Let $M = \max\{2,\ M_1\}$. Then we can write for all $n \in \mathbb{N}$ that

$$\left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_\delta \leq M \left\{ \omega(f, \alpha_j) + \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_\delta \right\} + \omega(f, \alpha_j) \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_\delta .$$

Given $\varepsilon > 0$, define the following sets:

$$D := \left\{ j \in \mathbb{N} : \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f) - f \right\|_\delta \geq \varepsilon \right\};$$

$$D_1 := \left\{ j \in \mathbb{N} : \omega(f, \alpha_j) \geq \frac{\varepsilon}{3M} \right\};$$

$$D_2 := \left\{ j \in \mathbb{N} : \omega(f, \alpha_j) \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_\delta \geq \frac{\varepsilon}{3} \right\};$$

$$D_3 := \left\{ j \in \mathbb{N} : \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_\delta \geq \frac{\varepsilon}{3M} \right\}.$$

Then $D \subseteq D_1 \cup D_2 \cup D_3$. Also, we define

$$D_2^{'} = \left\{ j \in \mathbb{N} : \omega(f, \alpha_j) \geq \sqrt{\frac{\varepsilon}{3}} \right\};$$

$$D_2^{''} = \left\{ j \in \mathbb{N} : \left\| \frac{1}{c_j} \sum_{n=1}^{\infty} a_{jn} L_n(f_0) - f_0 \right\|_\delta \geq \sqrt{\frac{\varepsilon}{3}} \right\}.$$

Therefore $D_2 \subseteq D_2^{'} \cup D_2^{''}$. Hence we get $D \subseteq D_1 \cup D_2^{'} \cup D_2^{''} \cup D_3$. Since $D_1,\ D_2^{'},\ D_2^{''},\ D_3$ belong to $\mathcal{I}$ so $D \in \mathcal{I}$. This completes the proof. □

3.6 Conclusion

We generalize Korovkin type approximation theory for a sequence of positive convolution operators defined on $C[a, b]$ in some sense of generalized matrix summability method, namely, $A^{\mathcal{I}}$-summability method and $A^{\mathcal{I}}$-statistical sense for real sequences. We construct examples in support of this generalizations. We are very much interested whether the results of this chapter are valid for the function f with two variables. Again we are interested whether the results are relevant on infinite interval.

3.7 Open problem

We now leave an open problem that the results of this chapter may be extended to a larger class of matrices, namely $(\mathcal{I}, \mathcal{J})$-regular matrices, the one which maps $\mathcal{I}$-convergent sequences into $\mathcal{J}$-convergent sequences and preserves the ideal limits, for some choice of ideals $\mathcal{I}$ and $\mathcal{J}$ [14].

4

Strong Ideal convergence in topological spaces

4.1 Introduction

Summability theory is currently playing an intriguing role in a topological space. A scalar-valued or linear space valued sequence can be given limits using summability theory, especially if the sequence is non-convergent [9]. Summability theory in topological spaces had been addressed by certain authors under the assumption that topological space has a group structure or a linear structure. Additionally, there are some summability techniques that do not require a linear structure in the topological space, such as statistical convergence [32, 33] as well as A-statistical convergence [39] (see [11, 47, 69, 64, 65]). The thought of $\mathcal{I}$-convergence was given by Kostyrko et. al. in 2000-2001 [43] on metric spaces and the same notion was explored by Lahiri et. al. in 2005 [45] on topological spaces.
A class of pre-metrics on arbitrary Hausdorff spaces, having several characteristics that are quite similar to the topological base, had been used to introduce the study of strong convergence, namely, $A_{\mathcal{T}}$-strongly convergence, in [71]. In this article, in the line of [71], we have investigated strong convergence using the theory of $\mathcal{I}$-convergence, and we have demonstrated that our findings are more robust than those of classical strong convergence.

4.2 $A^{\mathcal{I}}_{\mathcal{T}}$-Strong Convergence in a Topological Space

The idea of strong convergence can not be examined in arbitrary topological spaces because of the dependence of the strong convergence on the metric functions. The well-known con-

This chapter is based on a paper communicated to an international journal.

nection between strong convergence and statistical convergence can not, therefore, be easily extended to topological spaces. In this section, we define $A^{\mathcal{I}}$-statistical convergence in a topological space and introduce the notion of $A^{\mathcal{I}}_{\mathcal{T}}$ -strong convergence using a particular function defined on the topological space. We determine how these convergences relate to one another. Throughout $\mathcal{I}$ denotes the non-trivial proper (i.e., $\mathbb{N} \notin \mathcal{I}$, $\mathcal{I} \neq \{\emptyset\}$) admissible ideal on $\mathbb{N}$.

Definition 4.2.1. *Let $A = (b_{nk})$ be a member of $M_1(RS)$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a Hausdorff space X is said to be $A^{\mathcal{I}}$-statistically convergent to $\xi \in X$ if for any $\delta > 0$ and for any open set U containing ξ*

$$\left\{n \in \mathbb{N} : \sum_{k:x_k \notin U} b_{nk} \geq \delta \right\} \in \mathcal{I}.$$

It is simple to observe that, in place of open sets, the members of the topology's base can be used to define $A^{\mathcal{I}}$-statistically convergence.

Using the concept of pre-metric [1] and the following properties in topological space we define $A^{\mathcal{I}}_{\mathcal{T}}$-strong convergence in topological space. Let (X, τ) be a Hausdorff space and $\xi \in X$. Throughout $\mathcal{B}_\xi$ denotes the family of elements of the base of τ that contains ξ and $B_T(\xi, \varepsilon) := \{\beta \in X : T(\beta, \xi) < \varepsilon\}$. Also we denote $\mathcal{L}(X)$ as the set of functions $T : X \times X \to [0, \infty)$ that satisfy the condition "For any $\varepsilon > 0$ and for any $\xi \in X$ there exists $U_\varepsilon \in \mathcal{B}_\xi$ such that $U_\varepsilon \subset B_T(\xi, \varepsilon)$" [71]. Though any function from $\mathcal{L}(X)$ is a pre-metric (see, for instance, [1]) on X, the topology need not be pre-metrizable. These pre-metrics are here partly compatible with the topology because they satisfy that certain condition.

Definition 4.2.2. *Let (X, τ) be a Hausdorff space, $A = (b_{nk})$ be a member of $M_1(RS)$ and $\mathcal{T} \subset \mathcal{L}(X)$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in X is said to be statistical $A_{\mathcal{T}}$-strongly convergent to $\xi \in X$ if for any $T \in \mathcal{T}$, the sequence $\left\{\sum_k T(x_k, \xi) b_{nk}\right\}_{n\in\mathbb{N}}$ is statistically convergent to zero, provided the series is convergent for each n.*

Definition 4.2.3. *Let (X, τ) be a Hausdorff space, $A = (b_{nk})$ be a member of $M_1(RS)$ and $\mathcal{T} \subset \mathcal{L}(X)$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in X is said to be $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent to $\xi \in X$ if for any $\varepsilon > 0$ and for any $T \in \mathcal{T}$*

$$\left\{n \in \mathbb{N} : \sum_k T(x_k, \xi) b_{nk} \geq \varepsilon \right\} \in \mathcal{I},$$

provided the series is convergent for each n.

For $\mathcal{I} = \mathcal{I}_d$, the above convergences coincide. Here we are giving an example of $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent sequence in a non-metrizable Hausdorff space.

Example 4.2.1. *Consider the topological space $(\mathbb{R}, \tau_L)$, where τ_L be the lower limit topology. It is a non-metrizable Hausdorff space [66]. Let us take the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by*

$$x_k = \begin{cases} 0, & \text{if } k \text{ is even} \\ 1, & \text{otherwise} \end{cases}$$

and let us consider the family of functions $\mathcal{T} = \{T_r\}_{r\geq 0}$ defined for any $r \geq 0$ that

$$T_r(x, y) = \begin{cases} x - y, & x \geq y \\ r, & x < y. \end{cases}$$

Then $T_r \in \mathcal{T}$. Let $\mathcal{I}$ be a non-trivial admissible ideal of $\mathbb{N}$ such that $\mathcal{I} \neq \mathcal{I}_{fin}$. Consider an infinite set $\mathcal{D} = \{q_1 < q_2 < q_3 < \cdots\}$ from $\mathcal{I}$ and take the infinite matrix $A = (b_{nk})$ be given by,

$$b_{nk} = \begin{cases} 1 & \text{if } \ n = q_i, k = 2q_i \ \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } \ n \neq q_i, \text{for any } i, k = 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

For any $\varepsilon > 0$ and $\xi \in X$ we consider $U_\varepsilon = [\xi, \xi + \varepsilon) \in \mathcal{B}_\xi$ then for all $\beta \in U_\varepsilon$ we have $T(\beta, \xi) = \beta - \xi < \varepsilon$. So $\mathcal{T} \subset \mathcal{L}(X)$ and $\{x_k\}_{k\in\mathbb{N}}$ is not convergent in the corresponding topology. But we have

$$T_r(x_k, 1) = \begin{cases} r, & k \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

Now for any $r \geq 0$ and $\delta > 0$ we get that

$$\sum_k T(x_k, 1) b_{nk} = \sum_{k\in 2\mathbb{N}} b_{nk} = \sum_{k\in 2\mathbb{N}} r \geq \delta$$

for $n = q_i$. Thus for any $\delta > 0$, $\left\{ n \in \mathbb{N} : \sum_k T(x_k, 1) b_{nk} \geq \delta \right\} = \mathcal{D} \in \mathcal{I}$ showing that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent to 1 but not $A_{\mathcal{T}}$-strong convergent to 1. So $A^{\mathcal{I}}_{\mathcal{T}}$ strong convergence is stronger than $A_{\mathcal{T}}$-strong convergence.

In particular, if we choose $\mathcal{I} \neq \mathcal{I}_{fin}$ and $\mathcal{I} \neq \mathcal{I}_d$ and the infinite set $\mathcal{D} \in \mathcal{I} \setminus \mathcal{I}_d$ then the sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent to 1 but not statistical $A_{\mathcal{T}}$-strongly convergent to 1.

Connor [12], Khan and Orhan [37] established the relation between A-statistical convergence and A-strong convergence. Also in the field of $A^{\mathcal{I}}$-statistical convergence Savas, Das and Dutta [61, 62] generalized its relation with $A^{\mathcal{I}}$-summability.
We denote $\mathcal{Q}(X)$ as the set of functions $T : X \times X \to [0, \infty)$ that satisfy the condition: " For all $\xi \in X$ and for all $B \in \mathcal{B}_\xi$ there exists $M > 0$ such that for all $\beta \notin B$, $T(\beta, \xi) > M$" [71].

Theorem 4.2.1. *Let X be a Hausdorff space and $A = (b_{nk})$ be a member of $M_1(RS)$ and let $\mathcal{T} \subset \mathcal{L}(X)$. Then*
(i) $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergence with $\mathcal{T} \cap \mathcal{Q}(X) \neq \varnothing$ implies $A^{\mathcal{I}}$-statistically convergence.
(ii) $A^{\mathcal{I}}$-statistically convergence with the condition

$$\sup_{T\in\mathcal{T}} \sup_k T(x_k, \xi) < \infty \tag{4.1}$$

follows $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergence.

Proof. (i) Let $\{x_k\}_{k\in\mathbb{N}}$ be $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent to ξ in X. Let $B \in \mathcal{B}_\xi$ and $T \in \mathcal{T} \cap \mathcal{Q}(X)$. Then there exists $M > 0$ such that for all $\beta \notin B$, $T(\beta, \xi) > M$. From non-negativity of T we get

$$\begin{aligned}
\sum_k T(x_k, \xi) b_{nk} &= \sum_{k:x_k\in B} T(x_k, \xi) b_{nk} + \sum_{k:x_k\notin B} T(x_k, \xi) b_{nk} \\
&\geq \sum_{k:x_k\notin B} T(x_k, \xi) b_{nk} \\
&\geq M \sum_{k:x_k\notin B} b_{nk}.
\end{aligned}$$

Let $\delta > 0$ be given. Therefore

$$\left\{ n \in \mathbb{N} : \sum_{k:x_k\notin B} b_{nk} \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \sum_k T(x_k, \xi) b_{nk} \geq \delta \cdot M \right\}.$$

As $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent to ξ in X then the right side set belongs to $\mathcal{I}$ and this implies

$$\left\{ n \in \mathbb{N} : \sum_{k:x_k\notin B} b_{nk} \geq \delta \right\} \in \mathcal{I}.$$

As B is arbitrary, $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to ξ. Hence the proof of (i) is completed.

(ii) Let us suppose $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to $\xi \in X$ and holds (4.1) and let $T \in \mathcal{T}$. Then for any $\varepsilon > 0$ there exists $B \in \mathcal{B}_\xi$ such that for all $\beta \in B$, $T(\beta, \xi) < \varepsilon$. Hence non-negativity of T gives

$$\begin{aligned}
\sum_k T(x_k, \xi) b_{nk} &= \sum_{k:x_k\in B} T(x_k, \xi) b_{nk} + \sum_{k:x_k\notin B} T(x_k, \xi) b_{nk} \\
&\leq \varepsilon \sum_{k:x_k\in B} b_{nk} + \sup_{T\in\mathcal{T}} \sup_k T(x_k, \xi) \sum_{k:x_k\notin B} b_{nk} \\
&\leq \varepsilon + M \sum_{k:x_k\notin B} b_{nk}
\end{aligned}$$

where $M = \sup_{T\in\mathcal{T}} \sup_k T(x_k, \xi)$. For a given $\delta > 0$ choose $\varepsilon > 0$ such that $\varepsilon < \delta$. Then

$$\left\{n \in \mathbb{N} : \sum_k T(x_k, \xi) b_{nk} \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \sum_{k: x_k \notin B} b_{nk} \geq \frac{\delta - \varepsilon}{M}\right\}.$$

Since $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to ξ in X, the r.h.s. set is in $\mathcal{I}$ and consequently this implies that $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent to ξ. □

Remark 4.2.1. *In Example 4.2.1 we have given an example of an $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent sequence and we can see that the family $\mathcal{T}$ is a subfamily of $\mathcal{Q}(X)$. For $T_r \in \mathcal{T}$, $\xi \in \mathbb{R}$ and $U \in \mathcal{B}_\xi$ we have U in the most general form of $[\xi, \gamma)$. Then for any $\beta \notin U$, $T_r(\beta, \xi) = \beta - \xi$ for $\beta > \xi$; otherwise $T_r(\beta, \xi) = r$. Also $T(\gamma, \xi) = \gamma - \xi$ and $\beta > \xi \Rightarrow \beta > \gamma$ so, $T_r(\beta, \xi) > T(\gamma, \xi)$. Hence $T_r(\beta, \xi) > \min\{\frac{r}{2}, T(\gamma, \xi)\}$. Therefore, by Theorem 4.2.1 the given sequence in Example 4.2.1 converges $A^{\mathcal{I}}$-statistically to 1 with respect to the given ideal.*

On the other hand in the preceding example as the given sequence is $A^{\mathcal{I}}$-statistically convergent to 1 and we can define a family of functions $\mathcal{T} = \{T_r\}_{r\geq 0}$ for any $r \geq 0$ as

$$T_r(x, y) = \begin{cases} x - y, & x \geq y \\ \frac{1}{r+1}, & x < y \end{cases}.$$

that satisfies (4.1) i.e., $\sup_{T\in\mathcal{T}} \sup_k T(x_k, 1) = 1 < \infty$ and also $\mathcal{T} \subset \mathcal{L}(X) \cap \mathcal{Q}(X)$. Hence it justifies part (ii) of Theorem 4.2.1.

Let (X, d) be a metric space and $d \in \mathcal{L}(X) \cap \mathcal{Q}(X)$. If $\mathcal{T} = \{d\}$ then any bounded sequence satisfies (4.1) and we get the following corollary for $\mathcal{I} = \mathcal{I}_{fin}$.

Corollary 4.2.1. *[12] "Let A=(b_{nk}) be a member of $M_1(RS)$. A sequence in X which is A-strongly convergent to $\xi \in X$, is also A-statistically convergent to ξ. For a bounded sequence in X, A-statistically convergence and A-strongly convergence are equivalent."*

Theorem 4.2.2. *Let $A = (b_{nk})$ be a member of $M_1(RS)$ and let $\mathcal{T} \subset \mathcal{L}(X)$ where X be a Hausdorff space. If a sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to ξ in X and satisfies*
(i) if for any $\varepsilon > 0$ and $T \in \mathcal{T}$ there exists a compact subset $F \subset X$ such that

$$\sup_n \sum_{x_k \notin K} T(x_k, \xi) b_{nk} < \varepsilon;$$

(ii) for any compact set $C \subset X$ and $T \in \mathcal{T}$ there exists $M > 0$ such that $\sup_{x_k\in C} T(x_k, \xi) < M$

then $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}_{\mathcal{T}}$-strongly convergent to ξ.

Proof. Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $A^{\mathcal{I}}$-statistically convergent to ξ. Let $\varepsilon > 0$ and $T \in \mathcal{T}$ Also, from the hypothesis there exists a compact set $F \subset X$ such that

$$\sup_n \sum_{k: x_k \notin F} T(x_k, \xi) b_{nk} < \frac{\varepsilon}{2}$$

and there exists $M > 0$ such that

$$\sup_{k:x_k \in F} T(x_k, \xi) < M.$$

As $T \in \mathcal{L}(X)$ there exists $U_\varepsilon \in \mathcal{B}_\xi$ such that $U_\varepsilon \subset B_T(\xi, \frac{\varepsilon}{2})$. Now for any positive integer n

$$\begin{aligned}
\sum_k T(x_k,\xi)b_{nk} &= \sum_{k:x_k\in F} T(x_k,\xi)b_{nk} + \sum_{k:x_k\notin F} T(x_k,\xi)b_{nk} \\
&= \sum_{k:x_k\in F\cap U_\varepsilon} T(x_k,\xi)b_{nk} + \sum_{k:x_k\in F\setminus U_\varepsilon} T(x_k,\xi)b_{nk} + \sum_{k:x_k\notin F} T(x_k,\xi)b_{nk} \\
&\leq \frac{\varepsilon}{2}\sum_k b_{nk} + \sup_{k:x_k\in F} T(x_k,\xi) \sum_{k:x_k\notin U_\varepsilon} b_{nk} + \sup_n \sum_{k:x_k\notin F} T(x_k,\xi)b_{nk} \\
&\leq \frac{\varepsilon}{2}\sum_k b_{nk} + M \sum_{k:x_k\notin U_\varepsilon} b_{nk} + \frac{\varepsilon}{2}.
\end{aligned}$$

As $A = (a_{nk})$ is a matrix from $M_1(RS)$ then for any given $\delta > 0$ choosing $\varepsilon < \delta$

$$\left\{n \in \mathbb{N} : \sum_k T(x_k,\xi)b_{nk} \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \sum_{k:x_k\notin U_\varepsilon} b_{nk} \geq \frac{\delta - \varepsilon}{M}\right\}.$$

Since the set on r.h.s. belongs to $\mathcal{I}$, $\left\{n \in \mathbb{N} : \sum_k T(x_k,\xi)b_{nk} \geq \delta\right\} \in \mathcal{I}$ for any $\delta > 0$. Hence the proof is done. □

4.3 Characterization of $A^{\mathcal{I}}$-statistical convergence

The last theorem determines a characterization of $A^{\mathcal{I}}$-statistical convergence.

Theorem 4.3.1. *Let X be a Hausdorff space. Let $A = (b_{nk})$ be a member of $M_1(RS)$ and let $\mathcal{T} \subset \mathcal{L}(X) \cap \mathcal{Q}(X)$. Then a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X is $A^{\mathcal{I}}$-statistically convergent to $\xi \in X$ if and only if*

$$\mathcal{I} - \lim_n \sum_k \frac{T(x_k,\xi)}{1 + T(x_k,\xi)} b_{nk} = 0. \tag{4.2}$$

Proof. First, assume that $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to $\xi \in X$ and let $\varepsilon > 0$. As $T \in \mathcal{L}(X)$ there exists $U_\varepsilon \in \mathcal{B}_\xi$ such that $U_\varepsilon \subset B_T(\xi, \varepsilon)$. Then we have

$$\begin{aligned}
\sum_k \frac{T(x_k,\xi)}{1+T(x_k,\xi)} b_{nk} &= \sum_{x_k\notin U_\varepsilon} \frac{T(x_k,\xi)}{1+T(x_k,\xi)} b_{nk} + \sum_{x_k\in U_\varepsilon} \frac{T(x_k,\xi)}{1+T(x_k,\xi)} b_{nk} \\
&\leq \sum_{x_k\notin U_\varepsilon} b_{nk} + \varepsilon \sum_{x_k\in U_\varepsilon} b_{nk} \\
&\leq \sum_{x_k\notin U_\varepsilon} b_{nk} + \varepsilon.
\end{aligned}$$

For any given $\delta > 0$ choose $\varepsilon > 0$ such that $\varepsilon < \delta$. Then

$$\left\{n \in \mathbb{N} : \sum_k \frac{T(x_k, \xi)}{1 + T(x_k, \xi)} b_{nk} \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \sum_{x_k \notin U_\varepsilon} b_{nk} \geq \delta - \varepsilon\right\}.$$

Since $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to $\xi \in X$, the set on r.h.s. is a member of $\mathcal{I}$. Thus (4.2) holds.

Conversely, assume that (4.2) holds and let $B \in \mathcal{B}_\xi$. Since $T \in \mathcal{Q}(X)$ there exists $M > 0$ such that for all $\beta \notin B, T(\beta, \xi) > M$. Then we have

$$\begin{aligned}\sum_{x_k \notin B} b_{nk} &\leq \frac{1+M}{M} \sum_{x_k \notin B} \frac{T(x_k, \xi)}{1 + T(x_k, \xi)} b_{nk} \\ &\leq \frac{1+M}{M} \sum_k \frac{T(x_k, \xi)}{1 + T(x_k, \xi)} b_{nk}\end{aligned}$$

Therefore for any $\delta > 0$ choose $\varepsilon = \frac{\delta \cdot M}{1+M}$ and then

$$\left\{n \in \mathbb{N} : \sum_{x_k \notin B} b_{nk} \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \sum_k \frac{T(x_k, \xi)}{1 + T(x_k, \xi)} b_{nk} \geq \varepsilon\right\}.$$

Now (4.2) gives that the r.h.s. set belongs to $\mathcal{I}$ and hence $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to $\xi \in X$. □

The above Theorem readily follows the characterization of $A^{\mathcal{I}}$-statistical convergence of real sequences.

Corollary 4.3.1. *Let $A = (b_{nk})$ be a member of $M_1(RS)$ and let $\{x_k\}_{k\in\mathbb{N}}$ be a real sequence in $\mathbb{R}$ endowed with usual topology. Then $\{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to a real number L if and only if*

$$\mathcal{I} - \lim_n \sum_k \frac{| x_k - L |}{1 + | x_k - L |} b_{nk} = 0.$$

In particular for $\mathcal{I} = \mathcal{I}_{fin}$, the above corollary follows the Corollary 3. in [71].

4.4 Conclusion

A new form of strong convergence method, namely, $A^{\mathcal{I}}_T$-strong convergence, is introduced here utilizing a class of pre-metrics having characteristics similar to base on arbitrary Hausdorff spaces to overcome the lacking of linearity. Considering this new type of strong convergence, some interconnections with $A^{\mathcal{I}}$-statistical convergence have been investigated. Finally, we characterize $A^{\mathcal{I}}$-statistical convergence to some extent.

5

A note on generalized continuous convergence in ideal setting

5.1 Introduction

In [15], Császár and Laczkovich introduced equal (known also as Quasi-normal convergence [10] in literature) and discrete convergences for real valued functions sequence (see also [16]); later these notions were generalized to uniform discrete, uniform equal and strong uniform equal convergence by Papanastassiou [55] and following year Das and Papanastassiou [21] studied several lattice features for the classes consisting of the limit functions. After a long period of time, these notions were generalized in terms of ideal convergence by Das et. al. in [20] (see also [19]).

The notion of continuous convergence for a sequence of functions with real value was introduced in the twentieth century (see [68]) and later it was known as α-convergence in literature (see also [4]). Very recently, Banerjee et. al. introduced the notions of "$\mathcal{I}^*\alpha$-uniform equal convergence" and "$\mathcal{I}^*\alpha$-strong uniform equal convergence" in [6] available in arxiv:2201.10660v1[math.GN] 22 April, 2022.

In this chapter, our objective is to introduce the new ideas of convergence, namely, $\mathcal{I}^*_\alpha$-uniform equal, $\mathcal{I}^*_\alpha$-uniform discrete and $\mathcal{I}^*_\alpha$-strong uniform equal. We study the relationship between these new forms of convergence and then explore several lattice properties for these new classes of limit functions. We also discuss some results relating to the concept of $\mathcal{I}\alpha$-equal convergence

Throughout, let (X, d) be a metric space and $f, f_n : X \to \mathbb{R},\ n = 1, 2, 3, ...$ Here all the

This chapter is based on a research paper communicated in an international journal functions are real valued.

5.2 Key Outcomes

In this section, we review the Definition 4.1. from the work of Das et. al. [20]. Throughout $Card\ S$ means the cardinality of the set S. It is interesting to note that $\mathcal{I}^*$-u convergence is stronger than $\mathcal{I}^*$-ue convergence, which is stronger still than $\mathcal{I}^*$-e convergence (see Example 3.2 and Example 3.3 in [19]).

In addition, before introducing new kinds of convergence in this chapter, we would like to review the definitions of "α-convergence" (also known as "continuous convergence"), "α-equal convergence" and "α-uniform equal convergence". At the same time, we must remember the concepts of "α-equally and "α-uniformly equally" convergence established by R. Das [21]. It should be noted that, α-e convergence is intermediate between α-ue convergence and α convergence.

Here we define the notions of $\mathcal{I}^*_\alpha$-uniform equal, $\mathcal{I}^*_\alpha$-uniform discrete, $\mathcal{I}^*_\alpha$-strongly uniform equal convergence and investigate some of its lattice properties. Throughout $\mathcal{I}$ denotes a non-trivial admissible ideal on $\mathbb{N}$.

Definition 5.2.1. *A sequence $\{f_n\}_{n\in\mathbb{N}}$ is called $\mathcal{I}^*_\alpha$**-uniformly equally convergent** (or in short, $\mathcal{I}^*_\alpha$-ue convergent) to f if there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_n \varepsilon_n = 0$, a set $S := S(\{\varepsilon_n\})$ belonging to $\mathcal{F}(\mathcal{I})$ and a natural number $k_0 \equiv k_0(\{\varepsilon_n\})$ s.t.*

$$Card\{n \in S : |f_n(y_n) - f(y)| \geq \varepsilon_n\} \leq k_0$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. In symbols $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$.

The definition makes it apparent that $\mathcal{I}^*_\alpha$-ue convergence implies $\mathcal{I}^*$-ue convergence; but the converse implication fails.

Example 5.2.1. *Consider an ideal $\mathcal{I}$ different from $\mathcal{I}_{fin}$. Undoubtedly, there exists an infinite set A belonging to $\mathcal{I}$. Let us take a pairwise disjoint family $\{A_n\}_{n\in\mathbb{N}\setminus A}$, where each $A_n \neq \emptyset$, $A_n \subset \mathbb{R}$. Consider the sequence $\{f_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}$ with usual metric defined by*

$$\begin{aligned} f_n &= \chi_{A_n} \text{ when } n \in \mathbb{N} \setminus A \\ &= 1 \text{ when } n \in A. \end{aligned}$$

This sequence is $\mathcal{I}^*$-ue convergent to $f \equiv 0$. Next, we consider a set $D := \{n \in \mathbb{N} \setminus A :$ for some $p_n \in A_n$ for finitely many $n\}$ with finite cardinality and $y_0 \in \mathbb{R}$ and take the sequence

$\{y_n\}_{n\in\mathbb{N}}$ *defined by*

$$y_n = \begin{cases} p_n & \text{if } n \in D \\ y_0 & \text{if } n \in (\mathbb{N}\setminus A)\setminus D \\ y_0 & \text{if } n \in A. \end{cases}$$

Then $y_n \xrightarrow{\mathcal{I}} y_0$*. If for all* $n \in \mathbb{N}$, $\varepsilon_n < 1$, *then* $Card\{n \in \mathbb{N}\setminus A : |f_n(y_n) - f(y_0)| \geq \varepsilon_n\} = Card\{n \in \mathbb{N}\setminus A : f_n(y_n) \geq \varepsilon_n\} = Card\ D$*. Hence* $\{f_n\}_{n\in\mathbb{N}}$ *is not* $\mathcal{I}^*_\alpha$*-ue convergent to* 0.

Next we establish the equivalent condition of $\mathcal{I}^*_\alpha$-ue convergence.

Theorem 5.2.1. $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$ *iff there exist an* $\mathcal{I}$*-divergent sequence* $\{\rho_n\}_{n\in\mathbb{N}}$ *of positive integers, a sequence* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ *with* $\lim_n \varepsilon_n = 0$, *a set* $S := S(\{\varepsilon_n\}) \in \mathcal{F}(\mathcal{I})$ *and a natural number* $k_0 \equiv k_0(\{\varepsilon_n\})$ *such that*

$$Card\{n \in S : \rho_n|f_n(y_n) - f(y)| \geq \sqrt{\varepsilon_n}\} \leq k_0$$

for each $y \in X$ *and for any sequence* $\{y_n\}_{n\in\mathbb{N}} \subset X$ *with* $y_n \xrightarrow{\mathcal{I}} y$.

Proof. Suppose that $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$. Then from Definition 5.2.1 there is a positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\lim_n \varepsilon_n = 0$, a set $S := S(\{\varepsilon_n\})$ belonging to $\mathcal{F}(\mathcal{I})$ and a natural number $k_0 \equiv k_0(\{\varepsilon_n\})$ such that

$$Card\{n \in S : |f_n(y_n) - f(y)| \geq \varepsilon_n\} \leq k_0 \tag{5.1}$$

for each element $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Now, take a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive integers defined as

$$\begin{aligned} \rho_n &= \left[\frac{1}{\sqrt{\varepsilon_n}}\right] \quad &&\text{, if } n \in S \\ &= 1 &&\text{, if } n \notin S. \end{aligned}$$

Clearly, $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\mathcal{I}$-divergent sequence. Thus from (5.1)

$$Card\{n \in S : \rho_n|f_n(y_n) - f(y)| \geq \sqrt{\varepsilon_n}\} \leq k_0$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} x$.

Conversely, let there exist an $\mathcal{I}$-divergent sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive integers, a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_n \varepsilon_n = 0$, a set $S := S(\{\varepsilon_n\}) \in \mathcal{F}(\mathcal{I})$ and a number $k_0 \equiv k_0(\{\varepsilon_n\}) \in \mathbb{N}$ such that

$$Card\{n \in S : \rho_n|f_n(y_n) - f(y)| \geq \sqrt{\varepsilon_n}\} \leq k_0 \tag{5.2}$$

for each element $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Define a sequence $\{\theta_n\}_{n\in\mathbb{N}}$ by

$$\begin{aligned}\theta_n &= \frac{\sqrt{\varepsilon_n}}{\rho_n} \quad ,\text{if } n \in S\\ &= \frac{1}{n} \quad ,\text{if } n \notin S.\end{aligned}$$

So $\lim_n \theta_n = 0$ and $Card\{n \in S : |f_n(y_n) - f(y)| \geq \theta_n\} \leq k_0$ for each element $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Hence $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$. □

Lemma 5.2.1. *If* $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} 0$ *then* $f_n^2 \xrightarrow{\mathcal{I}^*_\alpha - ue} 0$.

Proof. Since $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} 0$, by definition, a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_n \varepsilon_n = 0$, a set $S := S(\{\varepsilon_n\}) \in \mathcal{F}(\mathcal{I})$ and $k \equiv k(\{\varepsilon_n\}) \in \mathbb{N}$ exist s.t.

$$Card\{n \in S : |f_n(y_n)| \geq \varepsilon_n\} \leq k$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} x$. Then

$$Card\{n \in S : |f_n^2(y_n)| \geq \varepsilon_n^2\} \leq k$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} x$. This implies the result. □

Lemma 5.2.2. *Let* f $(\neq 0)$ *be a constant function and* $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$ *then* $f_n.f \xrightarrow{\mathcal{I}^*_\alpha - ue} f^2$.

Proof. Suppose $f(x) = c(\neq 0)$ for all $x \in X$, a nonzero constant. Since $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$, there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_n \varepsilon_n = 0$, a number $k = k(\{\varepsilon_n\}) \in \mathbb{N}$ and a set $S := S(\{\varepsilon_n\}) \in \mathcal{F}(\mathcal{I})$ such that

$$Card\{n \in S : |f_n(y_n) - f(y)| \geq \varepsilon_n\} \leq k$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Since $|(f_n.f)(y_n) - f^2(y)| \leq |c||f_n(y_n) - f(y)|$, we have

$$\{n \in S : |(f_n.f)(y_n) - f^2(y)| \geq \varepsilon_n.|c|\} \subseteq \{n \in S : |f_n(y_n) - f(y)| \geq \varepsilon_n\}$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Thus $Card\{n \in S : |(f_n.f)(y_n) - f^2(y)| \geq \varepsilon_n.|c|\} \leq k$ for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. The proof is now complete. □

Lemma 5.2.3. *Let* $f, g, f_n, g_n : X \to \mathbb{R}$, $n = 1, 2, 3, ...$ *and* f, g *be bounded functions. If* $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$ *and* $g_n \xrightarrow{\mathcal{I}^*_\alpha - ue} g$ *then the product sequence* $f_n.g_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f.g$.

Proof. Since $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$ and $g_n \xrightarrow{\mathcal{I}^*_\alpha - ue} g$ then $f_n + g_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f + g$ and $f_n - g_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f - g$. Now Lemma 5.2.1, 5.2.2 and expression $f_n.g_n = \frac{(f_n+g_n)^2 - (f_n-g_n)^2}{4}$ deduce that $f_n.g_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f.g$. □

Lemma 5.2.4. *If $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$ then $|f_n| \xrightarrow{\mathcal{I}^*_\alpha - ue} |f|$.*

Proof. Let $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$. So there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_n \varepsilon_n = 0$, a set $S := S(\{\varepsilon_n\}) \in \mathcal{F}(\mathcal{I})$ and a natural number $k \equiv k(\{\varepsilon_n\})$ such that $Card\{n \in S : |f_n(y_n) - f(y)| \geq \varepsilon_n\} \leq k$ for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Now $||f_n|(y_n) - |f|(y)| \leq |f_n(y_n) - f(y)|$. Therefore $Card\{n \in S : ||f_n|(y_n) - |f|(y)| \geq \varepsilon_n\} \leq k$ for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$, i.e. $|f_n| \xrightarrow{\mathcal{I}^*_\alpha - ue} |f|$. □

Assume that Φ is any class of functions defined on the metric space (X, d). We denote by $\Phi^{\mathcal{I}^*_\alpha - ue}$, the class of functions $\{f : X \to \mathbb{R} : \exists$ a sequence $\{f_n\}_{n\in\mathbb{N}}$ in Φ s.t. $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f\}$.

Theorem 5.2.2. *Let Φ be a class of real valued functions on X. Then $\Phi^{\mathcal{I}^*_\alpha - ue}$ is a lattice if Φ is a lattice.*

Proof. Assume that Φ be a lattice and since Φ contains the constant functions, $\Phi^{\mathcal{I}^*_\alpha - ue}$ contains the constant functions. By Lemma 5.2.4, if $f \in \Phi^{\mathcal{I}^*_\alpha - ue}$ then $|f| \in \Phi^{\mathcal{I}^*_\alpha - ue}$. Next we show that if $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$, $g_n \xrightarrow{\mathcal{I}^*_\alpha - ue} g$ and $p, q \in \mathbb{R}$ then $pf_n + qg_n \xrightarrow{\mathcal{I}^*_\alpha - ue} pf + qg$. Indeed, there exist the sets $S_f, S_g \in \mathcal{F}(\mathcal{I})$, positive sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\lim_n \varepsilon_n = 0$, $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lim_n \lambda_n = 0$, and $n_f \equiv n_f(\{\varepsilon_n\})$, $n_g \equiv n_g(\{\lambda_n\}) \in \mathbb{N}$ such that

$$Card\{n \in S_f : |pf_n(y_n) - pf(y)| \geq |p|\varepsilon_n\} \leq n_f$$

and

$$Card\{n \in S_g : |qg_n(y_n) - qg(y)| \geq |q|\lambda_n\} \leq n_g.$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Now assume that $\theta_n = |p|\varepsilon_n + |q|\lambda_n$ and $k_0 = n_f + n_g$. Hence we observe that

$$Card\{n \in S_f \cap S_g : |(pf_n + qg_n)(y_n) - (pf + qg)(y) \geq \theta_n\}| \leq k_0$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$, where $S_f \cap S_g \in \mathcal{F}(\mathcal{I})$ and $\lim_n \theta_n = 0$. Hence $pf_n + qg_n \xrightarrow{\mathcal{I}^*_\alpha - ue} pf + qg$. Therefore if $f, g \in \Phi^{\mathcal{I}^*_\alpha - ue}$, $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} f$ and $g_n \xrightarrow{\mathcal{I}^*_\alpha - ue} g$ then $\frac{f_n+g_n}{2} + \frac{|f_n-g_n|}{2} \xrightarrow{\mathcal{I}^*_\alpha - ue} \max(f, g)$ which implies that $\max(f, g) \in \Phi^{\mathcal{I}^*_\alpha - ue}$. Similarly it can be shown that $\min(f, g) \in \Phi^{\mathcal{I}^*_\alpha - ue}$. Thus $\Phi^{\mathcal{I}^*_\alpha - ue}$ is a lattice. □

Note 5.2.1. $\Phi^{\mathcal{I}^*_\alpha - ue}$ *preserves the lattice characteristics of* Φ.

Theorem 5.2.3. *Let Φ be an ordinary class of functions defined on X. Let $f \in \Phi^{\mathcal{I}^*_\alpha - ue}$ be bounded s.t. $f(x) \neq 0$ for all $x \in X$. Then $\frac{1}{f} \in \Phi^{\mathcal{I}^*_\alpha - ue}$, provided $\frac{1}{f}$ is bounded on X.*

Proof. Suppose $\frac{1}{f}$ is bounded on X and then there exists $\mu > 0$ s.t. $f^2(x) > \mu$ for each $x \in X$. Since $f \in \Phi^{\mathcal{I}^*_\alpha - ue}$ as well as it is bounded then $f^2 \in \Phi^{\mathcal{I}^*_\alpha - ue}$. Thus there must be a sequence $\{f_n\}_{n\in\mathbb{N}}$ in Φ, a set $S \in \mathcal{F}(\mathcal{I})$ and $k \in \mathbb{N}$ s.t. $Card\{n \in S : |f_n(y_n) - f^2(y)| \geq \frac{1}{n^3}\} \leq k$ for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Let $g_n(y) = \max\{f_n(y), \frac{1}{n}\}$ for $y \in X$. Then $g_n \in \Phi$ for each $n \in \mathbb{N}$. Therefore

$$Card\{n \in S : g_n(y_n) = f_n(y_n), |g_n(y_n) - f^2(y)| \geq \tfrac{1}{n^3}\} \leq k$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Again

$$\begin{aligned}
&\left\{n \in S : g_n(y_n) = \frac{1}{n},\ |g_n(y_n) - f^2(y)| \geq \frac{1}{n^3}\right\} \\
= &\left\{n \in S : g_n(y_n) = \frac{1}{n},\ g_n(y_n) - f^2(y) \geq \frac{1}{n^3}\right\} \\
&\cup \left\{n \in S : g_n(y_n) = \frac{1}{n},\ -g_n(y_n) + f^2(y) \geq \frac{1}{n^3}\right\} \\
\subseteq &\left\{n \in S : f^2(y) \leq \frac{1}{n} - \frac{1}{n^3}\right\} \cup \left\{n \in S : f^2(y) \geq f_n(y_n) + \frac{1}{n^3}\right\} \\
\subseteq &\left\{n \in S : f^2(y) < \frac{1}{n}\right\} \cup \left\{n \in S : f^2(y) \geq f_n(y_n) + \frac{1}{n^3}\right\}.
\end{aligned}$$

Therefore $Card\left\{n \in S : g_n(y_n) = \frac{1}{n},\ |g_n(y_n) - f^2(y)| \geq \frac{1}{n^3}\right\} \leq k' + k = k_1$(say), where $k' = [\frac{1}{\mu}] + 1$, for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Now

$$\begin{aligned}
&\left\{n \in S : |g_n(y_n) - f^2(y)| \geq \frac{1}{n^3}\right\} \\
= &\left\{n \in S : g_n(y_n) = f_n(y_n), |g_n(y_n) - f^2(y)| \geq \frac{1}{n^3}\right\} \\
&\cup \left\{n \in S : g_n(y_n) = \frac{1}{n}, |g_n(y_n) - f^2(y)| \geq \frac{1}{n^3}\right\}.
\end{aligned}$$

This implies that $Card\left\{n \in S : |g_n(y_n) - f^2(y)| \geq \frac{1}{n^3}\right\} \leq k_1 + k = k_2$ (say) for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Therefore

$$\begin{aligned}
&Card\left\{n \in S : |\frac{1}{g_n(y_n)} - \frac{1}{f^2(y)}| \geq \frac{1}{n^3}.n.\frac{1}{\mu}\right\} \\
= \ &Card\left\{n \in S : \frac{|f^2(y) - g_n(y_n)|}{|g_n(y_n)||f^2(y)|} \geq \frac{1}{n^3}.n.\frac{1}{\mu}\right\} \\
\leq \ &Card\left\{n \in S : |g_n(y_n) - f^2(y)| \geq \frac{1}{n^3}\right\} \\
\leq \ &k_2
\end{aligned}$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Therefore $f^{-2} \in \Phi^{\mathcal{I}^*_\alpha - ue}$ and consequently $f.f^{-2} = f^{-1} \in \Phi^{\mathcal{I}^*_\alpha - ue}$. □

We now offer the following concept which is analogous to the notion of $\mathcal{I}^*$-uniformly discretely convergence [19].

Definition 5.2.2. *A sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is said to be* $\mathcal{I}^*_\alpha$***-uniformly discretely convergent*** *(or in short* $\mathcal{I}^*_\alpha$*-ud convergent) to the function* f *if there exist a set* M *belonging to* $\mathcal{F}(\mathcal{I})$ *and a number* $k \in \mathbb{N}$ *such that*

$$Card\{n \in M : |f_n(y_n) - f(y)| > 0\} \leq k$$

for each element $y \in X$ *and for any sequence* $\{y_n\}_{n\in\mathbb{N}} \subset X$ *with* $y_n \xrightarrow{\mathcal{I}} y$*. Specifically, we write* $f_n \xrightarrow{\mathcal{I}^*_\alpha - ud} f$*.*

For a class Φ, we use the notation $\Phi^{\mathcal{I}^*_\alpha - ud}$ to denote the class of functions $\{f : X \to \mathbb{R} : \exists$ a sequence $\{f_n\}_{n\in\mathbb{N}}$ in Φ s.t. $f_n \xrightarrow{\mathcal{I}^*_\alpha - ud} f\}$.

The following lattice properties of the class $\Phi^{\mathcal{I}^*_\alpha - ud}$ readily follows from Definition 5.2.2.

Theorem 5.2.4. $\Phi^{\mathcal{I}^*_\alpha - ud}$ *has the similar lattice characteristics as* Φ*.*

Theorem 5.2.5. *Let* Φ *be an ordinary class. Then*
(i) $f, g \in \Phi^{\mathcal{I}^*_\alpha - ud}$ *implies* $f.g \in \Phi^{\mathcal{I}^*_\alpha - ud}$*.*
(ii) If $f \in \Phi^{\mathcal{I}^*_\alpha - ud}$ *s.t.* $f(x) \neq 0$ *for each* $x \in X$ *and* $\frac{1}{f}$ *is bounded on* X *then* $\frac{1}{f}$ *belongs to* $\Phi^{\mathcal{I}^*_\alpha - ud}$*.*

Proof. Let $f, g \in \Phi^{\mathcal{I}^*_\alpha - ud}$. Then there exist sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ in Φ such that $f_n \xrightarrow{\mathcal{I}^*_\alpha - ud} f$ and $g_n \xrightarrow{\mathcal{I}^*_\alpha - ud} g$. Then from definition, it follows that $f_n.g_n \xrightarrow{\mathcal{I}^*_\alpha - ud} f.g$.

Let $f \in \Phi^{\mathcal{I}^*_\alpha - ud}$ and $f(x) \neq 0$ for each x in X and $\frac{1}{f}$ is bounded on X. Now choose $\lambda > 0$ such that $f^2(x) > \lambda > 0$ for each $x \in X$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \Phi$ s.t. $f_n \xrightarrow{\mathcal{I}^*_\alpha - ud} f$. Given that Φ is an ordinary class, then for each $n \in \mathbb{N}$, $f_n^2 \in \Phi$. Assume that $\{\sigma_n\}_{n\in\mathbb{N}}$ be a positive sequence that converges to zero and $g_n = \max\{f_n^2, \sigma_n\}$. Then $g_n \in \Phi$. Since $f_n \xrightarrow{\mathcal{I}^*_\alpha - ud} f$, then by Definition 5.2.2

$$Card\,\{n \in M : f_n(y_n) \neq f(y)\} \leq k$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$, which implies that

$$Card\,\{n \in M : g_n(y_n) \neq \max\{f^2(y), \sigma_n\}\} \leq k$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} x$ i.e.,

$$Card\left\{n \in M : \frac{1}{g_n(y_n)} \neq \frac{1}{\max\{f^2(y), \sigma_n\}}\right\} \leq k \tag{5.3}$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Now since $\lim_n \sigma_n = 0$, there exists a $k_0 \in \mathbb{N}$ such that $\sigma_n < \lambda$ for all $n \in M$ s.t. $n \geq k_0$. Therefore (5.3) becomes

$$Card\left\{n \in M : \tfrac{1}{g_n(y_n)} \neq \tfrac{1}{f^2(y)}\right\} \leq k + k_0$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Hence $f^{-2} \in \Phi^{\mathcal{I}^*_\alpha - ud}$ and consequently $f.f^{-2} = f^{-1} \in \Phi^{\mathcal{I}^*_\alpha - ud}$. □

Next we introduce the new form of convergence analogous to the notion of "$\mathcal{I}^*$-strongly uniformly equally convergence" introduced by Das and Dutta [19].

Definition 5.2.3. *A sequence $\{f_n\}_{n\in\mathbb{N}}$ is called* ***$\mathcal{I}^*_\alpha$-strongly uniformly equally convergent*** *(or in short $\mathcal{I}^*_\alpha$-sue convergent) to the function f if there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, a set $S := S(\{\varepsilon_n\}) \in \mathcal{F}(\mathcal{I})$ and $k \equiv k(\{\varepsilon_n\}) \in \mathbb{N}$ such that*

$$Card\{n \in S : |f_n(y_n) - f(y)| \geq \varepsilon_n\} \leq k$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. We write this as $f_n \xrightarrow{\mathcal{I}^*_\alpha - sue} f$.

Assume that $\Phi^{\mathcal{I}^*_\alpha - sue}$ denotes, the class of all $\mathcal{I}^*_\alpha$-sue limits of sequences of functions defined on X belonging to Φ , i.e.,

$$\Phi^{\mathcal{I}^*_\alpha - sue} = \{f : X \to \mathbb{R} : \ \exists \text{ a sequence } \{f_n\}_{n\in\mathbb{N}} \text{ in } \Phi \text{ s.t. } f_n \xrightarrow{\mathcal{I}^*_\alpha - sue} f\}$$

.

Example 5.2.2. *Let $S \in \mathcal{F}(\mathcal{I})$ and $\{f_n\}_{n\in\mathbb{N}}$ be the sequence defined on $\mathbb{R}$ with usual metric defined by*

$$\begin{aligned} f_n(y) &= \frac{1}{n}, \ \textit{if } n \in S \\ &= 0, \ \textit{if } n \notin S \end{aligned}$$

for all $y \in X$. Then $f_n \xrightarrow{\mathcal{I}^*_\alpha - ue} 0$ but $f_n \overset{\mathcal{I}^*_\alpha - sue}{\not\to} 0$.

The definition and preceding illustration follows that $\mathcal{I}^*_\alpha$-sue convergence is stronger than $\mathcal{I}^*_\alpha$-ue convergence. The results below are simple to confirm, much like in the case of $\mathcal{I}^*_\alpha$-ue convergence.

Lemma 5.2.5. *If $f_n \xrightarrow{\mathcal{I}^*_\alpha - sue} 0$ then $f_n^2 \xrightarrow{\mathcal{I}^*_\alpha - sue} 0$.*

Lemma 5.2.6. *If f is bounded and $f_n \xrightarrow{\mathcal{I}^*_\alpha - sue} f$ then $f_n.f \xrightarrow{\mathcal{I}^*_\alpha - sue} f^2$.*

Theorem 5.2.6. *Let $f_n, g_n, f, g : X \to \mathbb{R}$, $n \in \mathbb{N}$ and f, g be bounded. If $f_n \xrightarrow{\mathcal{I}^*_\alpha - sue} f$ and $g_n \xrightarrow{\mathcal{I}^*_\alpha - sue} g$ then $f_n.g_n \xrightarrow{\mathcal{I}^*_\alpha - sue} f.g$.*

Theorem 5.2.7. *If Φ is a lattice then $\Phi^{\mathcal{I}^*_\alpha - sue}$ is also a lattice.*

Proof. Due to the fact that Φ is a lattice and as such contains constant functions, $\Phi^{\mathcal{I}^*_\alpha - sue}$ does too. Let $f_n \xrightarrow{\mathcal{I}^*_\alpha - sue} f$. Then a positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, a set $S := S(\{\varepsilon_n\}) \in \mathcal{F}(\mathcal{I})$ and a number $k \equiv k(\{\varepsilon_n\}) \in \mathbb{N}$ exist such that $Card\{n \in S : |f_n(y_n) - f(y)| \geq \varepsilon_n\} \leq k$ for each element $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Now $||f_n|(y_n) - |f|(y)| \leq |f_n(y_n) - f(y)|$. Therefore $Card\{n \in S : ||f_n|(y_n) - |f|(y)| \geq \varepsilon_n\} \leq k$ for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$, i.e., $|f_n| \xrightarrow{\mathcal{I}^*_\alpha - sue} |f|$.

Now we show that if $f_n \xrightarrow{\mathcal{I}^*_\alpha - sue} f$, $g_n \xrightarrow{\mathcal{I}^*_\alpha - sue} g$ and $s, t \in \mathbb{R}$ then $sf_n + tg_n \xrightarrow{\mathcal{I}^*_\alpha - sue} sf + tg$. To see this, by Definition 5.2.3, there exist $M_f, M_g \in \mathcal{F}(\mathcal{I})$, $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and $n_f \equiv n_f(\{\varepsilon_n\})$, $n_g \equiv n_g(\{\lambda_n\}) \in \mathbb{N}$ such that

$$Card\{n \in M_f : |sf_n(y_n) - sf(y)| \geq |s|\varepsilon_n\} \leq n_f$$

and

$$Card\{n \in M_g : |tg_n(y_n) - tg(y)| \geq |t|\lambda_n\} \leq n_g$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Let us choose $\theta_n = |s|\varepsilon_n + |t|\lambda_n$ and $k_0 = n_f + n_g$. Then we have

$$Card\{n \in M_f \cap M_g : |(sf_n + tg_n)(y_n) - (sf + tg)(y) \geq \theta_n\} \leq k_0$$

for each $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$ where

$$\sum_{n=1}^{\infty}\theta_n = \sum_{n=1}^{\infty}|s|\varepsilon_n + |t|\lambda_n < \infty$$

and $M_f \cap M_g \in \mathcal{F}(\mathcal{I})$. Hence $sf_n + tg_n \xrightarrow{\mathcal{I}^*_\alpha - sue} sf + tg$. Therefore $f, g \in \Phi^{\mathcal{I}^*_\alpha - sue}$, $f_n \xrightarrow{\mathcal{I}^*_\alpha - sue} f$ and $g_n \xrightarrow{\mathcal{I}^*_\alpha - sue} g$ imply that $\frac{f_n + g_n}{2} + \frac{|f_n - g_n|}{2} \xrightarrow{\mathcal{I}^*_\alpha - sue} \max(f, g)$ i.e., $\max(f, g) \in \Phi^{\mathcal{I}^*_\alpha - sue}$. Similarly, $\min(f, g) \in \Phi^{\mathcal{I}^*_\alpha - sue}$. Thus $\Phi^{\mathcal{I}^*_\alpha - sue}$ is a lattice. □

Note 5.2.2. $\Phi^{\mathcal{I}^*_\alpha - sue}$ *preserves the lattice features of* Φ.

5.3 $\mathcal{I}\alpha$-equal Convergence

In 2010, Papachristodoulos et. al. [54] generalized the idea of α-convergence in ideal setting, namely, $\mathcal{I}\alpha$-convergence. "A sequence $\{f_n\}_{n\in\mathbb{N}}$ of real valued functions defined on X is said to be $\mathcal{I}\alpha$-**convergent** to f if for each $x \in X$ and sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \xrightarrow{\mathcal{I}} x$, $f_n(x_n) \xrightarrow{\mathcal{I}} f(x)$." Specifically, we write $f_n \xrightarrow{\mathcal{I}\alpha} f$.

Very recently, in [35], A. Ghosh proposed the notion of $\mathcal{I}^*\alpha$-convergence. We independently, here define the same notion in different aspects.

Definition 5.3.1. *A sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is said to be* $\mathcal{I}^*\alpha$***-convergent*** *to the function* f *if there exists a set* $P = \{p_1 < p_2 < p_3 < ...\} \in \mathcal{F}(\mathcal{I})$ *such that* $\{f_{p_k}\}_{k\in\mathbb{N}}$ *is* α*-convergent to* f*. We write* $f_n \xrightarrow{\mathcal{I}^*\alpha} f$*.*

Definition 5.3.2. *A sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is said to be* $\mathcal{I}\alpha$***-equally convergent*** *(or in short* $\mathcal{I}\alpha$*-e convergent) to the function* f *if there exists a sequence* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ *with* $\mathcal{I} - \lim_n \varepsilon_n = 0$ *such that for each* $y \in X$ *and for any sequence* $\{y_n\}_{n\in\mathbb{N}}$ *of* X *with* $\mathcal{I} - \lim_n y_n = y$ *there exists a set* $A \equiv A(y, \{y_n\}) \in \mathcal{I}$ *satisfying* $|f_n(y_n) - f(y)| < \varepsilon_n$ *for all* $n \in A^c$*. We write as* $f_n \xrightarrow{\mathcal{I}\alpha-e} f$*.*

The definitions make it obvious that $\mathcal{I}\alpha$-equal convergence implies $\mathcal{I}$-equal convergence but the reverse is not true. The following sequence is $\mathcal{I}$-equal convergent but not $\mathcal{I}\alpha$-equal convergent.

Example 5.3.1. *Let* $\mathcal{I}$ *be an admissible non-trivial ideal and* $C \in \mathcal{I}$*. Let* $\{\varepsilon_n\}_{n\in\mathbb{N}}$ *be a positive sequence such that* $\varepsilon_n \xrightarrow{\mathcal{I}} 0$*. Consider a sequence* $\{f_n\}_{n\in\mathbb{N}}$ *on* $[0, 1]$ *defined by*

$$f_n(y) = \begin{cases} 1 & \text{if } n \in C \\ y^n & \text{if } n \notin C \end{cases}$$

for each $n \in \mathbb{N}$ *and the function* f *defined by*

$$f(y) = \begin{cases} 0 & \text{if } y \in [0, 1) \\ 1 & \text{if } y = 1 \end{cases}$$

Here, $\{f_n\}_{n\in\mathbb{N}}$ *is* $\mathcal{I}$*-equal convergent to* f *but not* $\mathcal{I}\alpha$*-equal convergent to* f *at* $y = 1$*. Indeed, take the sequence* $\{y_n\}_{n\in\mathbb{N}} = \{1 - \frac{1}{n}\}_{n\in\mathbb{N}}$*. So* $y_n \xrightarrow{\mathcal{I}} 1$*. But*

$$f_n(y_n) = \begin{cases} 1 & \text{if } n \in C \\ (1 - \frac{1}{n})^n & \text{if } n \notin C \end{cases}$$

for each $n \in \mathbb{N}$*. So* $\mathcal{I}\text{-}\lim_{n\to\infty} f_n(x_n) = e^{-1} \neq 1 = f(1)$*.*

Theorem 5.3.1. *If* $f_n \xrightarrow{\mathcal{I}\alpha-e} f$ *then* $f_n \xrightarrow{\mathcal{I}^*\alpha} f$*.*

Proof. Let $f_n \xrightarrow{\mathcal{I}\alpha-e} f$. So there exists a positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\mathcal{I} - \lim_n \varepsilon_n = 0$ such that for each $y \in X$ and sequence $\{y_n\}_{n\in\mathbb{N}}$ of X s.t. $\mathcal{I} - \lim_n y_n = y$ there exists a set $A \equiv A(y, \{y_n\}) \in \mathcal{I}$ satisfying $|f_n(y_n) - f(y)| < \varepsilon_n$ for all $n \in A^c$, i.e., $\{n \in \mathbb{N} : |f_n(y_n) - f(y)| \leq \varepsilon_n\} \in \mathcal{F}(\mathcal{I})$.

Now let $\varepsilon > 0$ be given and since $\mathcal{I} - \lim_n \varepsilon_n = 0$ then $\{n \in \mathbb{N} : \varepsilon_n < \varepsilon\} = M(\text{say}) \in \mathcal{F}(\mathcal{I})$. Since $\{n \in \mathbb{N} : |f_n(y_n) - f(y)| < \varepsilon_n\} \cap M \subseteq \{n \in \mathbb{N} : |f_n(y_n) - f(y)| < \varepsilon\} \cap M$ and the left hand set belongs to $\mathcal{F}(\mathcal{I})$ for each $y \in X$ and $y_n \xrightarrow{\mathcal{I}} y$, hence $\{n \in \mathbb{N} : |f_n(y_n) - f(y)| < \varepsilon\} \cap M \in \mathcal{F}(\mathcal{I})$ for each $y \in X$ and $y_n \xrightarrow{\mathcal{I}} y$. Hence $f_n \xrightarrow{\mathcal{I}^*\alpha} f$. ∎

Theorem 5.3.2. *Let $f_n, f : X \to \mathbb{R}$, $n \in \mathbb{N}$. Then $f_n \xrightarrow{\mathcal{I}\alpha-e} f$ iff there exist an $\mathcal{I}$-divergent sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive integers and a positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\mathcal{I} - \lim_n \varepsilon_n = 0$ such that*

$$\{n \in \mathbb{N} : \rho_n.|f_n(y_n) - f(y)| \geq \sqrt{\varepsilon_n}\} \in \mathcal{I}$$

for any element $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$.

Proof. Suppose that $f_n \xrightarrow{\mathcal{I}\alpha-e} f$. Then there exist a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\mathcal{I}$-$\lim_n \varepsilon_n = 0$ s.t. for any $y \in X$ and $\{y_n\}_{n\in\mathbb{N}}$ in X with $\mathcal{I}$-$\lim_n y_n = y$

$$\{n \in \mathbb{N} : |f_n(y_n) - f(y)| \geq \varepsilon_n\} \in \mathcal{I}.$$

Consider a sequence $\{\rho_n\}_{n\in\mathbb{N}} = \left\{\left[\frac{1}{\sqrt{\varepsilon_n}}\right]\right\}_{n\in\mathbb{N}}$ which is an $\mathcal{I}$-divergent sequence of positive integers. Hence

$$\{n \in \mathbb{N} : \rho_n.|f_n(y_n) - f(y)| \geq \sqrt{\varepsilon_n}\} \in \mathcal{I}$$

for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$.

Conversely, let there exist an $\mathcal{I}$-divergent sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive integers and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I} - \lim_n \varepsilon_n = 0$ s.t.

$$\{n \in \mathbb{N} : \rho_n.|f_n(y_n) - f(y)| \geq \sqrt{\varepsilon_n}\} \in \mathcal{I}$$

for any $y \in X$ and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y_n \xrightarrow{\mathcal{I}} y$. Take a sequence $\{\theta_n\}_{n\in\mathbb{N}}$, where $\theta_n = \frac{\sqrt{\varepsilon_n}}{\rho_n}$. Then $\mathcal{I}$-$\lim_n \theta_n = 0$ and $\{n \in \mathbb{N} : |f_n(y_n) - f(y)| \geq \theta_n\} \in \mathcal{I}$ for each element y and for any sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ with $y \in X$ and $y_n \xrightarrow{\mathcal{I}} y$. Hence $f_n \xrightarrow{\mathcal{I}\alpha-e} f$. □

Note 5.3.1. *$f_n \xrightarrow{\mathcal{I}^*\alpha} f$ implies $f_n \xrightarrow{\mathcal{I}\alpha} f$.*

Note 5.3.2. *[35] "$f_n \xrightarrow{\mathcal{I}\alpha} f$ implies $f_n \xrightarrow{\mathcal{I}^*\alpha} f$, provided $\mathcal{I}$ be a good and P-ideal."*

Lemma 5.3.1. *If $\{f_n\}_{n\in\mathbb{N}}$ $\mathcal{I}\alpha$-equally converges to zero then so is $\{f_n^2\}_{n\in\mathbb{N}}$.*

Lemma 5.3.2. *Let f be a nonzero constant function. Then $f_n \xrightarrow{\mathcal{I}\alpha-e} f$ implies $f_n.f \xrightarrow{\mathcal{I}\alpha-e} f^2$.*

Lemma 5.3.3. *Let $f, g, f_n, g_n : X \to \mathbb{R}$, $(n \in \mathbb{N})$ and f, g be bounded functions. Then $f_n \xrightarrow{\mathcal{I}\alpha-e} f$ and $g_n \xrightarrow{\mathcal{I}\alpha-e} g$ imply $f_n.g_n \xrightarrow{\mathcal{I}\alpha-e} f.g$.*

The class of all functions defined on X that are $\mathcal{I}\alpha$-equal limits of sequences of functions belonging to Φ is denoted by the symbol $\Phi^{\mathcal{I}\alpha-e}$, i.e.,

$$\Phi^{\mathcal{I}\alpha-e} = \{f : X \to \mathbb{R} : \exists \text{ a sequence } \{f_n\}_{n\in\mathbb{N}} \text{ in } \Phi \text{ s.t. } f_n \xrightarrow{\mathcal{I}\alpha-e} f\}.$$

Theorem 5.3.3. *$\Phi^{\mathcal{I}\alpha-e}$ preserves the lattice properties of Φ.*

5.4 Conclusion

For sequences of real valued functions defined on a metric space (X, d), we define a few novel types of convergence in the current study and, to some extent, observe their relationships. In this chapter, we attempt to investigate the lattice features of some specific classes of real valued functions defined on the metric space. Through the ensuing figures, we accumulate the key relationships between the convergences.

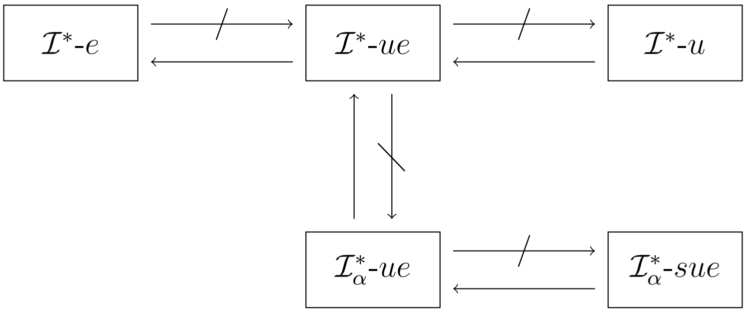

Figure 5.1: First Diagram

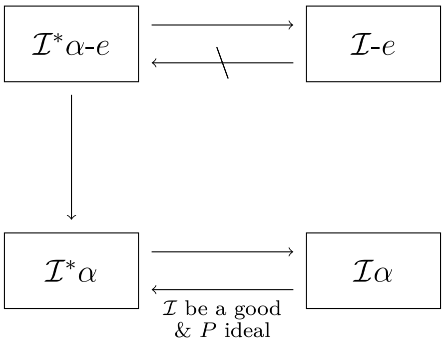

Figure 5.2: Second Diagram

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Index