QUALITATIVE ANALYSIS OF DYNAMICAL COMPLEXITIES ON APPLIED BIOECONOMIC MODELS IN PREDATOR-PREY FISHERY

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CERTIFICATE FROM THE SUPERVISORS

This is to certify that, the thesis entitled 'QUALITATIVE ANALYSIS OF DYNAMICAL COMPLEXITIES ON APPLIED BIOECONOMIC MODELS IN PREDATOR-PREY FISHERY' is submitted by Sudipta Sarkar whose name was registered vide INDEX NO: 210/16/Maths./25 dated 21.11.2016, for the award of Doctor of Philosophy (Science) degree of Jadavpur University, Kolkata-700032. This thesis is a record of bonafide research work carried out under our supervision and is worthy of consideration for award of Ph.D. (Sc.) degree. Neither this thesis nor any part of this work has been submitted before to any other University or Institute for the award of any degree/diploma or any other academic award.

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Dedicated to

my Father Late Prabir Kumar Sarkar

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Abstract

Management of ecological and biological resources like fish species, forests, etc. must be carefully monitored to prevent overexploitation and species extinction. Overexploitation of any species disrupts its ecology. Overexploitation of biological resources for commercial gain must be considered biologically and economically. To prevent environmental disasters and economic irresponsibility and income losses, biological and economic overexploitation must halt. Concerns and laws on ecological and biological resource exploitation must prevent biological and economic overexploitation. Quantitative modeling that considers biological and economic factors will help identify these issues and policies.

Non-linear mathematical models based on real occurrences are being studied in mathematical biology. These research help explain fish species' dynamical behaviour, including selective and non-selective harvesting, presence of toxic substances, Holling Type-II functional response, and more.

My study focuses on mathematical models for commercially harvested mixed species marine fisheries management. Developing concerns and policies to prevent biological and economic overexploitation of fish species is the main goal of such modeling.

In this thesis, fishery resource models were utilised to study predator-prey interactions and human activities like harvesting. Nonlinear dynamics features including local stability, persistence, and global stabil- ity are examined here. Local stability of equilibrium points can reveal the system's distinguishing traits. Here, we examin how well-regulated fish capture affects the ecology. The model is now an optimal control problem that considers harvesting socioeconomics. Applying Pontryagin's maximum principle to a modified catch rate function maximises net harvesting revenue while maintaining system stability. The study found that appropriate fish population harvesting stabilised systems.

Keywords: Fishery resource management, Functional response, Local and global stability, Lyapunov function, Optimal harvesting policy.

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1

INTRODUCTION

"Evolution is the control of development by ecology." - Leigh Van Valen.

Background and motivation

Management of exploitation of ecological and biological resources, such as fish species, forests, etc., must be investigated scientifically in order to reduce the likelihood of overexploitation and prevent the extinction of species. As a result of overexploitation of any species, the ecosystem in which that species resides becomes unstable. When such biological resources are exploited for commercial purposes, overexploitation must be analysed from both biological and economic perspectives. Biological overexploitation must be avoided to prevent environmental catastrophes, whereas economic overexploitation must cease to prevent economic negligence that results in revenue losses. The issues and policies surrounding the exploitation of such ecological and biological resources must be established in a way that prevents biological and economic overexploitation. Quantitative modeling that takes into account the pertinent biological and economic factors will help to clearly identify these issues and policies.

To evaluate the status of research, analysis, and development in this field, one must first examine its international standing. Mathematical studies and analyses based on resource management problems were first discussed by Hotelling, 1931, who dealt with the economics of exhaustible resources utilising the calculus of variations, which was not well-known to researchers at the time. Prior to 1973's energy crisis, Hotelling's article was not widely read (Solow, 1974). The energy crisis sparked an increase in the field. The importance of studies and analyses of depletable resources was greater than that of renewable resources. Schaefer, 1954 formulated the first mathematical model of a single-species fishery model. As a management aid for the Eastern Tropical Tuna Fishery, this model was utilised. This model was developed using the logistic law of growth first proposed by Verlhust, 1838 and the catch-per-unit-effort (CPUE) hypothesis proposed by Clark, 1976b. In the model proposed by Schaefer, 1954, the maximal value of the yield-effort curve is referred to as the Maximum Sustainable Yield (MSY). As fishing effort is increased, the curve declines after attaining its maximum value. As a result, at sufficiently high sustained effort levels, yield falls to zero. When sustainable yield declines below the maximum sustainable yield, biological overfishing is said to occur. Preventing overexploitation is the primary objective of the MSY concept. Utilising this MSY concept, the administration of biological resource exploitation has been maintained for decades.

Since the turn of the twentieth century, the problems associated with fishery resources, particularly commercial marine fisheries, have drawn the attention of scientists and researchers. The analyses and investigations conducted as a result of biological considerations were insufficient to resolve many unanswered questions regarding marine fishery resources. This required the participation of researchers and scientists from various disciplines to conduct integrated studies of fisheries resources, taking into account biological, economic, technological, environmental, and social factors. As a consequence, a rapid expansion of the fisheries literature from static bioeconomics to dynamic bioeconomics to stochastic bioeconomics took place. The analyses and studies conducted by the researchers were categorised from the following perspectives:

(i) single species versus multispecies models;
(ii) lumped parameter versus cohort models;
(iii) linear versus nonlinear models;
(iv) autonomous versus non-autonomous models;
(v) malleable capital versus non-malleable capital models;
(vi) deterministic versus stochastic models.

Both Munro, 1979 and Clark, 1980a investigated the exploitation of the fish species in a multicountry context. The models of seasonal fishing were analysed by Bradley, 1970, Horwood and Whittle, 1986, McCallum, 1988, Clark, 1980b, Deklark and Gatto, 1981, etc. The effects of schooling on the dynamical behaviour of exploited fish populations were studied by Clark, 1974a and Clark and Mangel, 1979. Clark et al., 1985 investigated the problem of calculating optimal fishing capacity for a developing fishery.

The fish species mate seasonally or annually, and each reproduction affects the recruitment of new fish into the population. In a classic expository article, Ricker, 1954 examines in depth a number of problems associated with stock and recruitment. For many species, recruitment to the reproductive population occurs many years after birth. This phenomenon of delayed recruitment to the reproductive population as it applies to populations of baleen whales was conceptualised by Clark, 1976a.

Overexploitation is an extremely common issue in commercial fisheries. Diverse marine fish species, such as the Peruvian anchoveta, California sardine, Hokkaido herring, North Sea herring, Japanese sardine, Antarctic fin whales, Antarctic blue whales, etc., have completely depreciated in this century as a result of exploitation. Later, H.S. Gordon was asked to provide an economic analysis of the persistent issue of low-income maritime fishermen in Canada. Gordon, 1954 modeled a static model based on common property (or open access) fisheries. Not only did it provide an explanation for the low income of fishermen, but it also provided an economic interpretation of the overexploitation issue. This model demonstrated how economic overfishing would be anticipated to occur in a fishery resource with unrestricted access, while biological overfishing would occur in the presence of a very high price or cost ratio. Gordon's theory predicts the existence of bionomic (combination of biological and economic) equilibrium between exploited and unexploited natural resources. Existence of bionomic equilibrium occurs at a stock level where revenue obtained equals the cost of exploitation precisely. When an open-access resource transforms into bionomic equilibrium, the economic rent (net economic revenue) is completely lost. This key characteristic of rent dissipation in open-access resources is used to introduce **the second fundamental theorem of resource economics**. Hotelling, 1931 is appended to the preceding result for privately owned resource stocks. The consideration of an exhaustible or renewable resource stock by its solitary owner as a **capital asset** and the expectation that it will generate dividends at the 'normal' rate of return were disapproved by Hotelling, 1931. If not, the owner would attempt to dispose of the asset. This remarkable resource stock phenomenon is referred to as **the first fundamental theorem of resource economics**.

Using the concept of Maximum Sustainable Yield (MSY) proposed by Clark, 1976b, the conservation and management of renewable resources such as fisheries, forests, and wildlife have typically been considered. The MSY concept is founded on a biological growth paradigm. It implies that there exists a surplus production that can be harvested permanently without affecting the resource stock level when any given population level falls below its carrying capacity. If the surplus production is not harvested, the population stock level rises until it reaches the carrying capacity. At the level of carrying capacity, excess production is reduced to zero. Maximum Sustainable Yield (MSY) is attained when the surplus production is at its optimum, as the surplus production at any population level equals the sustainable yield. This occurs at the population density where the growth rate of the species is maximised. However, the MSY concept has numerous limitations and drawbacks that are discussed below:

(i) When multiple ecologically interdependent species are harvested simultaneously, the biological concept of 'yield' may be ambiguous. Realistically, yield maximisation for each species population separately is impossible;

(ii) In the case of marine fisheries subject to large and unpredictable variations, the concept of 'sustainable yield' is erroneous;

(iii) All economic factors associated with fisheries resources are mismatched with the MSY concept; even the costs associated with the sustainable exploitation of fisheries resources are ignored. The MSY concept is therefore not optimal from an economic standpoint;

(iv) A minor, arbitrary modification to the MSY concept may result in overexploitation and, ultimately, the extinction of the species. Therefore, the MSY concept is not at all stable.

However, the MSY concept has a consequence that indicates the possibility of overexploitation (and subsequent extinction) is imminent whenever the stock level falls below the MSY level. This situation poses a significant hazard to the extinction of an entire species.

Few fishery scientists and managers later introduced the OSY (Optimum Sustainable Yield) to rectify the economic irrelevance of the MSY concept. The challenge of the OSY concept, however, is to clearly identify the objective of the concept to be optimised. Clark, 1976b elaborated on optimal resource management policies, including problems of recognition and analyses.

Now, the line between 'economic overfishing' and 'biological overfishing' may be permeable. Economic exploitation occurs when all economic revenue is lost (i.e., when total costs for harvesting equal total revenues obtained). When the fish stock level falls below the MSY level as a result of overexploitation, biological overfishing occurs.

The Maximum Economic Yield (MEY) is the yield of the resource stock level that maximises net economic revenue. Gordon, 1954 contradicts the fact that both the fisherman's and the social welfare optimum would occur at the MEY level. However, the MEY concept has certain limitations. It cannot be regarded as a beneficial management objective for reasons intangible to the static model proposed by Gordon. The Gordon-Schaefer dynamic model brought to the forefront the aforementioned viewpoint of Clark, 1985. Gordon, 1954 deliberated on a dynamic bioeconomic model for a population of a single species. The Gordon, 1954 model, which is the equilibrium solution of the model proposed by Schaefer, 1967, is still widely used in commercial fisheries despite its numerous drawbacks. This model is based on initial consideration of the logistic population model proposed by Verlhust, 1838. Clark, 1976b and Clark, 1985 conducted in-depth analyses of the Schaefer model.

Until the late 1960s, the static theory of Gordon, 1954 dominated the theoretical research in the field of fishery economics. As the static method is straightforward, it is still applicable for providing policymakers with an explanation of the outcomes of an unregulated fishery. The static approach provides the rationale for the necessity of incorporating tax and establishing individual catch quotas as optimal means of regulating the fishery resource. Brown, 1974, Plouride, 1970, and Quirk and Smith, 1970 analysed and corrected the static approach's limitations. In the 1970s, Clark, 1976b witnessed a rapid evolution of dynamic models as a consequence of advancements in the optimal control theory's application to economics. Recently, the cost-benefit criterion of maximising the present values of net economic revenues has become a standard instrument for optimal resource management. Utilising the maximum principle of Pontryagin et al., 1962, the present value of future economic rents are maximised. This concept is applicable to both private and public management decisions, despite the fact that the specification of costs and revenues is not necessarily identical. In private management, exact and internalised costs are considered, whereas in public management, social and external costs are taken into account. Frelek, Gatto, and Locatelli, 1982, Clark, Clarke, and Munro, 1979, Goh, 1980, Cliff and Vincent, 1973, Koen, 1988, Clark and Munro, 1982, etc., have made substantial contributions to dynamic models of fishery economics with a capital theoretic framework.

Various strategies to combat the common property externality in commercial fisheries have been periodically discussed. Allocation of fishermen's quotas by Christy, 1973, Mangel, 1982, Moloney and Pearse, 1979, Weitzmann, 1974, imposition of taxes on landed fish biomass by Clark, 1976b, Clark, 1985, Pearse, 1979, Chaudhuri and Johnson, 1990, licence limitation by Pearse, 1980, restricting fishing seasons by Kellog et al., 1988, etc., are included in these approaches. Anderson and Lee, 1986 studied and analysed a variety of issues pertaining to the selection and implementation of optimal governance instruments in dominant fisheries. Sutinen and Andersen, 1985 discussed the economic aspects of regulating marine fisheries through the application of laws.

The 'dynamic pool' model proposed by Beverton and Holt, 1985 for North Sea demersal fisheries is one of the most frequently utilised models in fishery management. This model is actually structured by age. Clark, 1976b, Clark, 1985, and Clark, Edwards, and Friedlaender, 1973 examined censorious analysis of the preceding model from a variety of vantage points.

The majority of papers on fisheries resources focus on deterministic fishery models. However, it is widely acknowledged that marine fishery resources and fish populations are subject to enormous and unpredictable fluctuations. Many scientists and researchers have taken an interest in this peculiar phenomenon of marine fishery resources in recent years. Walters, 1975, Reed, 1974, Reed, 1979, Beddington, 1977, Andersen, 1981, Andersen, 1982, Clark and Mangel, 1981, Abakuks and Prajneshu, 1981, Andersen and Sutinen, 1984, Clark and Kirkwood, 1986 carried out their researches in this area.

The majority of the aforementioned models focus on singular species. In addition to these models, there are numerous fascinating articles by Clark, 1973, Clark, 1979a, May, 1980, Smith, 1969, Rosenzweig, 1971, Turvey, 1964 etc. There are so many limitations imposed on models of populations of singular species. A singular ecological population is only one component of a complex ecosystem comprised of prey, predator, competitor, collaborator, disease-causing organisms, and other forms of life. A model of a particular species that disregards its biological interrelationships with other species in the ecosystem is not very realistic. Despite the sophistication of fishing gears and fleets, it is difficult to locate a vessel capable of exploiting a single fish species. In all marine fisheries, multiple fish species have been harvested, despite the fact that a portion of the capture is incidental in some cases. In multispecies trawl fisheries, the overexploitation of the more valuable fish species residing at higher trophic levels is a common occurrence, as described by Pauly, 1982. The incidents of capturing porpoises in tuna purse scenes, entanglement of dugongs in shark nets, assassination of sports-fish such as marlin and sailfish in long-liners, capturing of juveniles of large fin-fish species in shrimp trawlers, capturing of halibut in groundfish trawlers in the north-east pacific, etc., are examples of **incidental catches** or **by-catches**. These accidental captures may result in the unintended extinction of non-target species.

Deterministic fishery models are discussed in the majority of studies on fishery resources. However, it is widely acknowledged that fish populations and marine fisheries resources are vulnerable to significant and unpredictable changes. The odd phenomenon of marine fisheries resources has drawn the attention of numerous experts and researchers over the past few years. Researchers like Walters, 1975, Reed, 1974, Reed, 1979, Beddington, 1977, Abakuks and Prajneshu, 1981, Andersen and Sutinen, 1984, Clark and Kirkwood, 1986 conducted their studies in this field.

Species at various trophic levels, such as cod-capelin and krill whale populations, are fished using various boats and equipment. In these circumstances, a high degree of harvesting in the lower trophic levels suggested by Akenhead et al., 1982 has a significant impact on the production of the species at both trophic levels. A species that was eradicated due to fishing pressure may occasionally be replaced by one or more new species in the same biological location. An example of this kind of substitution mentioned by Murphy and Clupeoids, 1977 is the substitution of anchovies for the pacific sardine. The management of multi-species fisheries is getting more challenging due to the biological interdependence of the various oceanic species. The traditional MSY rule is no longer valid, because obtaining MSY for each species independently decided upon by May et al., 1979 is extremely challenging. Therefore, it is difficult to create a realistic model of a multi-species fishery that includes all of the biological relationships between the species. Even if the aforementioned model were taken into account, there is very little chance of doing the analytical investigation. One should implement his strategies by acknowledging that no amount of bioeconomic modeling will be able to address the issues with multi-species fisheries as mentioned by Clark, 1985 and Merrcer, 1982.

Larkin, 1963 discussed one of the early attempts at multi-species modeling. Here, using Lotka-Volterra equations, the effects of exploitation on either or both of a pair of competing species were examined. Another paper considered the predator-prey interaction model was proposed by Lotka-Volterra. However, in these analyses, economic concerns were completely disregarded.

Clark, 1976b examined the combined harvesting of two biodegradable separate population species that were subjected to logistic growth. The main findings of this study are:

(i) One population may be driven to extinction while the other population maintains the bionomic equilibrium for the fishery resource;

(ii) Populations with relatively low biotechnical productivity are subject to extinction;

(iii) Even with zero discounting, the extinction of one of the two species is an economically optimal outcome.

Therefore, maintaining every population that is a part of an ecosystem that is being exploited may not always be profitable. On the other hand, biologists want to preserve each species' place in an ecosystem. This clearly implies a significant divergence between the biological and economic points of view. With the aid of subsequent investigations and analyses, the optimal harvesting policies problem was mathematically constructed by Clark, 1976b; nevertheless, the remaining puzzle pieces were filled in by Mesterton-Gibbons, 1987.

Gause, 1935 built a model of two fish species that compete with one another and that each follow the logistic law of growth. Clark, 1976b examined the impact of harvesting only one species out of the two in the model proposed by Gause, 1935 . Before the effort reached its MSY level, the harvested species may have fallen. Clark, 1976b discovered this important feature. Using the aforementioned model, Culley, 1971 discussed how the pacific sardine fishery went extinct in the late 1940s and early 1950s. Sardine population replacement with anchovy population was maintained. This anchovy population currently predominates and prevents the sardine population from recovering. A rational plan for the rehabilitation of the sardine appears to be the evolution of an anchovy fishery. Due to the anchovy's importance as a food source for several sport fish species, sport fishermen campaign against its overfishing. This has indeed occurred in the instance of the California sardine fishery. Sport fisherman lobbied the California state legislature to forbid anchovy harvesting, and their efforts were successful. The management of fishing resources is severely hampered by this kind of uncontrollable condition, whether it be societal or otherwise. In the model proposed by Gause, 1935, combined harvesting of both species was examined by Chaudhuri, 1986. The aforementioned research went into great length about the bioeconomic equilibrium's existence, the dynamic nature of the exploited system, and the best harvesting strategy. Contrarily, analysis of Chaudhuri, 1988 of the prior issue revealed that the optimal control problem may be solved by treating effort as a dynamic variable. The discount rate was set at zero in this instance. When a non-zero finite discount rate was taken into account, the identical problem was examined by Mesterton-Gibbons, 1988.

While both species in the fishery resource model might be fished separately, Clark, 1976b also developed a model based on selective harvesting. An inshore-offshore fishing resource model was also taken into account, according to Clark, 1976b. Despite excluding economic considerations, Goh, Leitmann, and Vincent, 1974 examined the fishery model based on the Lotka-Volterra prey-predator system. Control theoretic models of multispecies fisheries were investigated in Quirk and Smith, 1970, Silver and Smith, 1977, etc. A statistical analysis of the dynamical model for the growth and harvesting of fishing resources was proposed by Chaudhuri, 1989.

A catch-rate function was used to develop a bioeconomic model of a single species fishery resource, which is more plausible than the CPUE hypothesis typically used in the literature on fisheries. In the model indicated above, imposing a reasonable tax on the biomass that was brought in was seen as a way to regulate the extinction of fish species. The ideal harvesting strategy was described in their model utilising control theory. The model proposed by Schaefer, 1954 was modified by Ganguly and Chaudhuri, 1991 by using the catch-rate function that was chosen by Chaudhuri and Johnson, 1990 because it was more realistic. Chaudhuri and Saha Ray, 1991 examined the bioeconomic exploitation of a two-species Lotka Volterra type predator-prey model. From a capital theoretic perspective, Ganguly and Chaudhuri, 1995 reexamined Chaudhuri Johnson's model. In a prey predator fishing resource system, Chaudhuri and Saha Ray, 1996 considered a model of certain preys seeking refuges someplace inside the dwelling area to prevent predation. The Gompertz rule of growth was used in place of the logistic growth law in the single species fishery resource model provided by Chaudhuri and Pradhan, 1998. A dynamic reaction model on the bioeconomic harvesting of a schooling fish species was developed by Chaudhuri and Pradhan, 1999b in his introduction. The aforementioned model was later expanded to include the example of a two-species fishery model by Chaudhuri and Pradhan, 1999a. This approach included capital theoretic analysis with taxation as a management objective. By include an infected prey, the model of non-selective harvesting of a predator-prey fishery was developed by Chattopadhyay and Pradhan, 1999. In an inshore offshore fishery, the subject of selective harvesting was examined by Pradhan and Chaudhuri, 1999b. The model of Chaudhuri and Pradhan, 1998 was expanded to include the scenario of combined harvesting of a prey predator fishery by Purohit and Chaudhuri, 2002. By using combined harvesting, a multispecies fishery model was thought through by Kar and Chaudhuri, 2002b. A model for the bioeconomic exploitation of a fishery resource made up of two prey species and one predator was put up by Kar and Chaudhuri, 2002a. According to Kar and Chaudhuri, 2003b, taxation can be used to control a prey predator fishery. A model of selective harvesting of two competing fish species in the presence of hazardous substances was also covered in Kar and Chaudhuri, 2003a. The author of Kar and Chaudhuri, 2003c developed a dynamic reaction model for the regulation of a prey predator fishery with the introduction of taxes on the catch. Pradhan and Chaudhuri, 2003 presented the model for harvesting a prey predator fishery with low predator population density. A bioeconomic model of an inshore-offshore fishery on selective harvesting, where the growth of the fish species is governed by Gompertz law of growth, was proposed by Purohit and Chaudhuri, 2004b. Kar and Chaudhuri, 2004a conducted their studies on the management of a single species fishery with stage structure. The model of selective harvesting of two competing fish species in the presence of environmental variation was explored by Kar and Chaudhuri, 2004b. A model for managing a prey-predator fisheries resource based on continuous fishing effort was presented by Kar, Pahari, and Chaudhuri, 2004. Purohit and Chaudhuri, 2004a developed a bioeconomic model for the joint harvesting of two fish species that are competitors. Two competing fish species are harvested nonselectively in a dynamic reaction model created by Purohit and Chaudhuri, 2007. Das, Mukherjee, and Chaudhuri, 2009c developed a mathematical model of combined harvesting of two competing fish species, obeying the logistic law of growth in the presence of hazardous chemicals. Later, the model adopts a suitable catch-rate function in place of the standard CPUE hypothesis. A Holling Tanner type prey-predator fishing model was presented by Das, Mukherjee, and Chaudhuri, 2009b using taxes as a control mechanism. It was discovered that a prey-predator fishery might be harvested bioeconomically by substituting an acceptable catch-rate function for CPUE, as described in Das, Mukherjee, and Chaudhuri, 2009a. Das, Mukherjee, and Chaudhuri, 2009d also created a model for prey-predator fishing resource harvesting in toxic environments. Here, toxicity directly affects the prey species, and consumption of the poisonous preys indirectly affects the predator species. The use of power law formalism in biological modeling was described by Chaudhuri, 2010. Sana and Chaudhuri, 2011 and Sana, Purohit, and Chaudhuri, 2012 have recently created two bioeconomic models on the joint project of harvesting prawn fishery and poultry of birds. We now provide succinct chapter-by-chapter overviews of the issues this thesis considers:

Chapter 1: Introduction introduces the thesis. Since it is the first chapter, it offers a succinct summary of fishery resource management. There are also definitions of several terms and fundamental principles of fishery resource management that are pertinent to the thesis. The problem statement and the thesis outline are presented at the chapter's conclusion.

Chapter 2: The dynamics of a fishery resource system in an aquatic environment subject to selective harvesting of only prey fish species are examined in this chapter using a non-linear model. We assume that all forms of predator fish harvesting are prohibited. Here, we suggest employing a modified catch rate function to harvest the prey fish species. The parameters of a system's biological and bioeconomic equilibrium are established. For the suggested model, the points of regional stability, global stability, and instability are derived. The maximum concept proposed by Pontryagin is used to determine the best harvesting strategy. To understand the theoretical outcomes of the model, numerical simulations are carried out.

Chapter 3: In a non-linear mathematical model where both species are poisoned by the harmful compounds generated by each species, the goal of this work is to investigate and analyse the dynamical behaviour of a prey-predator fishing resource in an aquatic ecosystem. Here, both species follow the logistic growth rate function and are harvested bioeconomically together. Here, using a modified catch rate function, the bioeconomic combined harvesting of prey-predator fish species in the presence of hazardous toxic chemicals released by them is examined.

Here, the proposed model's boundedness is looked at. The suggested model's biological and bionomic equilibria are derived, and this study illustrates the requirements for local stability, instability, and global stability. Pontryagin's maximum principle is used to determine the best harvesting strategy before numerical simulations are used to illustrate and validate the model's theoretical predictions.

Chapter 4: In this chapter, we use a nonlinear model to investigate the dynamics of a prey-predator fishing resource system in an aquatic environment where both species are harmed by the toxicants they emit when subjected to combined bioeconomic harvesting. Using a modified catch rate function, we investigate the bioeconomic harvesting of two competing fish species that produce toxic compounds that are detrimental to one another. By employing Pontryagin's maximal principle, equilibrium, stability, bionomic equilibrium, and the optimal harvesting policy have been established. We have determined that the characteristics of the system and the initial population densities of the interacting fish species will have a substantial effect on their dynamical behaviour. A few numerical simulations and corresponding solution contours are used to illustrate the theoretical outcomes of the proposed model. Finally, the existence of the limit cycle is demonstrated here.

Chapter 5: In this chapter, we have taken a predator-prey fishery resource system where, both the species are subjected to logistic growth, by incorporating the modified catch rate function. Here, the functional response of the predator species on the prey population is taken in a way that the functional response of the predators towards the prey population tends to a constant with the increasing prey population density. Boundedness of the proposed non-linear model is analyzed here. Steady states of the model along with their existence criteria for positivity are shown here. Further, local and global stability of the equilibria are found out here by using Eigenvalue analysis and constructing a suitable Lyapunov function respectively. Bionomic equilibria of the proposed non-linear model and their existence criteria have been carried out. The most salient feature of this current proposed non-linear predator-prey fishery resource model is the setting up of an optimal harvesting policy using Pontryagin's maximum principle, so that both the species can be prevented from annihilation though both are subjected to combined harvesting. Finally, some numerical simulations are done to validate the theoretical results of the proposed predator-prey model.

Chapter 6: In this chapter, the Rosenzweig-MacArthur model of a predator-prey system is taken into account. Where, the predator population is absent, the logistic growth of the prey population is taken into consideration. The functional response of the predator is of Holling type-II for the prey population. The predator species is given access to a stock of an additional food source. Together with the positivity and existence of the solutions, the stability of the equilibrium points is examined. For the purpose of studying the system's overall stability, a Lyapunov function that is appropriate is established. The parameters used in the system are not precise in real life due to inadequate information. In order to make the model more realistic, the issue of the best taxation policy in a fuzzy environment is studied. Theorems are incomplete without numerical verifications of analytical results, so we analyse a few fictitious data sets and verify them numerically.

Chapter 7: This thesis uses dynamic fishery resource models to study the effects of predator-prey interactions and harvesting. Local stability, permanence, and global stability have been examined in nonlinear dynamics. Local equilibrium point stability reveals essential system properties. Well-regulated fish collecting affects the system. The model is now an optimum control problem that considers harvesting socioeconomics. A modified version of Pontryagin's maximal principle determines the optimal harvesting effort to maximise net harvesting revenue while maintaining system stability.

2

MODELING OF SINGLE SPECIES FISHERY RESOURCE HARVESTING WITH MODIFIED CATCH RATE FUNCTION

"The first law of ecology is that everything is related to everything else."

- Barry Commoner

2.1 Introduction

Fisheries, forestry, and wildlife management are all fields that regularly engage in the exploitation of biological resources. The goal of bioeconomic modeling is to help managers of renewable resources minimise economic costs and maximise profits. Humanity's insatiable appetite

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for more food and more energy has resulted in their depletion. Several studies on fisheries resources have been conducted in recent decades; some of the more notable of them are included below with their respective information: Bhattacharya and Begum, 1996, Clark, 1976b, Clark, 1979b, Clark, 1985, Clark, 1990, Goh, 1980, Meserton-Gibbons, 1988, Meserton-Gibbons, 1996. Kitabatake, 1982 suggested a predator-prey dynamical model for fishing resources. Using the work of Clark, 1976b as inspiration, Chaudhuri, 1986 examined the effects of harvesting two species of fish at once, finding evidence that the open access fishery may have a bionomic equilibrium in which one species is wiped out. The optimal strategy for maximising present value from the combined harvest of predator and prey was discovered by Meserton-Gibbons, 1996. The selective harvesting of a two-species fishery was also investigated by Mukhopadhyay, Chattopadhyay, and Tapaswi, 1997 who incorporated a discrete time delay for harvesting both species. A model for the expansion and exploitation of a schooling fish species was also proposed by Chaudhuri and Pradhan, 1999a. The best harvesting policy of a stage structured problem was analysed by Zhang, Chen, and Neumann, 2000, who also determined the conditions for species' survival and extinction using the model they developed. A dynamic model for a single-species fishery partially reliant on an increasing supply was investigated by Dubey, Chandra, and Sinha, 2002. Maximum sustainable yield (MSY) and equilibrium density of fish populations were observed to grow in tandem with resource biomass density. A model for fishing resources with a reserve area was also proposed by Dubey, Chandra, and Sinha, 2003. A bioeconomic model of a marine reserve and a single-species fishery was presented by Kar and Mastud, 2008. A model of selective harvesting in a time-delayed prey-predator fishery was also established by Kar, 2003. In order to examine optimal control of harvesting for a single population, Wang and Wang, 2004 studied a model. The idea of modeling a marine reserve was presented by Loisel

and Cartigny, 2009. The optimal harvesting policy in a two patch environment was analysed in a model published by Cai, Li, and Song, 2008. An example of a competition model that includes a buffer zone was explored by Du and Liang, 2008. Prey reserve in a prey-predator fishery was also the subject of a model proposed by Kar and Mishra, 2006. A model for a prey-predator fishery incorporating a marine reserve was given by Kar and Pahari, 2007. A model of a prey-predator type fishery with a reserve area was also described by Ghosh, 2010.
From the literature reviewed above and to the best of our knowledge, in this chapter we have provided a mathematical model to analyse the dynamics of a fishery resource system in an aquatic habitat subjected to selective harvesting of the prey fish population. A modified catch rate function is proposed for harvesting prey fish species in this model. The prey fish are assumed to be collected while the predator fish are off-limits. We derive the stability behaviour of biological and bionomic equilibrium and investigate the conditions under which they exist. Finally, we apply Maximum Principle of Pontryagin et al., 1962 to harvesting to determine the best course of action to take.

2.2 Mathematical model formulation

In order to model the interaction between prey and predator, we analyse the following dynamical system:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\alpha xy, \\ \frac{dy}{dt} &= -sy+\beta xy, \end{aligned} \tag{2.2.1}$$

where r is the highest specific growth rate of the prey population and x=x(t) is the biomass density of the prey population at time t, y=y(t) is the biomass density of the predator population at time t, and s is the relative rate at which the predators die off in the absence of prey.

According to the model, only the predators have an impact on prey reproduction, and the amount of prey they catch limits their ability to reproduce. Prey populations increase with a relative rate of r in the absence of predators, while predator populations decline exponentially with a relative rate of s in the absence of prey. The trophic function, or the predator's functional reaction to the prey population density, is given by αx, which is understood as the biomass of the prey devoured by the predator per unit time. A fraction $\frac{\beta}{\alpha}$ $(0 < \beta < \alpha \leq 1)$ of the energy consumed with this biomass goes into predator reproduction while the rest of the energy is used to sustain metabolism and hunting activity of predators. Here, k is the environmental carrying capacity of prey population.

The catch rate function usually taken in the beginning of fishery models are of the form $h = qEx$. This is based on the CPUE (catch-per-unit-effort) hypothesis proposed by Clark, 1976b. Later on, it is modified in the functional form of $h = \frac{qEx}{bE+lx}$.

Here, we suppose that, in order to successfully harvest the fish resource, the fisherman searches at random inside a specific area. We refer to this idea as a discovery-area search efficiency. The efficiency with which one catches fish while competing with other fishermen is directly proportional to the amount of effort expended. Based on these premises, we adjust the catch rate function $h = \frac{qEx}{b+E}$ to account for the varying sizes of the resource population (fish) being caught at any given amount of effort. Here, E denotes the harvesting effort, q (constant) the catchability coefficient and b is a positive constant.

Keeping these in view, the dynamics of prey-predator fish populations may be governed by the following autonomous system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \alpha xy - \frac{qEx}{b+E}, \\ \frac{dy}{dt} &= -sy + \beta xy. \end{aligned} \tag{2.2.2}$$

2.3 Steady states

The possible steady states of the dynamical system of equation (2.2.2) are $S_0(0,0)$, $S_1(\bar{x},0)$ where $\bar{x} = \frac{k}{r}\left(r - \frac{qE}{b+E}\right)$ and $S_2(x^*,y^*)$ where

$$\begin{aligned} r\left(1 - \frac{x^*}{k}\right) - \alpha y^* - \frac{qE}{b+E} &= 0, \\ -s + \beta x^* &= 0. \end{aligned} \tag{2.3.1}$$

We have $x^* = \frac{s}{\beta}$ which is always positive and $y^* = \frac{r\left(1 - \frac{s}{\beta k}\right) - \frac{qE}{b+E}}{\alpha}$ which is positive when $r > \frac{qE\beta k}{\left(b+E\right)\left(\beta k - s\right)}$ and $k > \frac{s}{\beta}$.

2.4 Local stability

For $S_0(0,0)$, the characteristic equation is the following: $\begin{vmatrix} r - \frac{qE}{b+E} - \lambda & 0 \\ 0 & -s - \lambda \end{vmatrix} = 0$. Here, λ denotes the eigenvalue. Roots of the equation are $\lambda_1 = r - \frac{qE}{b+E}$ and $\lambda_2 = -s$. Here, $\lambda_1 < 0$ if $\frac{r}{q} < \frac{E}{b+E}$ and $\lambda_2 < 0$. Thus, λ_1 and λ_2 are both real and negative when $\frac{r}{q} < \frac{E}{b+E}$. Therefore, the trivial steady state $S_0(0,0)$ is a stable node. When $\frac{r}{q} > \frac{E}{b+E}$, then $\lambda_1 > 0$ and $\lambda_2 < 0$ and hence, $S_0(0,0)$ is a saddle point. For $S_1(\bar{x},0)$, the characteristic equation is the following: $\begin{vmatrix} \frac{-rx}{k} - \lambda & -\alpha x \\ 0 & -s + \beta x - \lambda \end{vmatrix} = 0.$

Roots of this equation are $\lambda_1 = \frac{-rx}{k}$ and $\lambda_2 = -s + \beta x$. If $\lambda_2 < 0$ i.e.; $x < \frac{s}{\beta}$, the steady state $S_1(\bar{x},0)$ is a stable node. For the non-trivial steady state

$S_2(x^*, y^*)$, the characteristic equation is the following:

$$\begin{vmatrix} \frac{-rx^*}{k} - \lambda & -\alpha x^* \\ \\ \beta y^* & 0 - \lambda \end{vmatrix} = 0. \tag{2.4.1}$$

Here,

$$A_1 = \frac{rx^*}{k} \text{and } A_2 = \alpha\beta x^* y^*. \tag{2.4.2}$$

Here, sum of the roots$= -A_1 < 0$ and product of the roots $= A_2 > 0$. Therefore, the roots of the quadratic equation are either real and negative or complex conjugates with negative real parts. Hence, the steady state $S_2(x^*, y^*)$ is either locally stable node or a locally stable focus.

2.5 Global stability

In this section, we consider the global stability of the system of equation (2.2.2) by constructing a suitable Lyapunov function.

$$V(x, y) = \left[(x - x^*) - x^* \log \frac{x}{x^*} + h(y - y^*) - y^* \log(\frac{y}{y^*})\right]. \tag{2.5.1}$$

Here, h is a suitable constant to be determined in the subsequent steps. It can be easily verified that the function V is zero at the equilibrium point (x^*, y^*) and is positive for all other values of x, y. The time derivative of V along the trajectories of equation (2.2.2) is the following:

$$\frac{dv}{dt} = \frac{x - x^*}{x}\frac{dx}{dt} + h\frac{y - y^*}{y}\frac{dy}{dt} = (x - x^*)\left[r(1 - \frac{x}{k}) - \alpha y - \frac{qE}{b + E}\right] + h(y - y^*)\left[-s + \beta x\right]. \tag{2.5.2}$$

Also, we have the set of the equilibrium equations corresponding to the steady state $S_2(x^*, y^*)$

$$\begin{aligned} r\left(1 - \frac{x^*}{k}\right) - \alpha y^* - \frac{qE}{b + E} &= 0, \\ -s + \beta x^* &= 0. \end{aligned} \tag{2.5.3}$$

We can write equation (2.5.2) together with the above two equations in the form

$$\begin{aligned}\frac{dv}{dt} =&(x-x^*)\left[r(1-\frac{x}{k})-\alpha y-\frac{qE}{b+E}-r(1-\frac{x^*}{k})+\alpha y^*+\frac{qE}{b+E}\right]\\ &\qquad\qquad\qquad\qquad\qquad\qquad +h(y-y^*)\left[-s+\beta x+s-\beta x^*\right]\\ =&-\left[(x-x^*)^2(\frac{r}{k})+(x-x^*)(y-y^*)(\alpha-h\beta)\right]. \qquad (2.5.4)\end{aligned}$$

If we choose $h=\frac{\alpha}{\beta}$, $\frac{dv}{dt}=-(x-x^*)^2(\frac{r}{k})<0$. Now, since $\frac{dv}{dt}$ is negative semidefinite in some neighbourhood of (x^*,y^*), the interior equilibrium point (x^*,y^*) is globally asymptotically stable.

2.6 Bionomic equilibrium

The term bionomic equilibrium is an amalgamation of the concepts of biological equilibrium and economic equilibrium . As we already know, a biological equilibrium is given by $\dot{x}=0=\dot{y}$. In the fishery literature, the bionomic equilibrium is said to be achieved when the total revenue obtained by selling the harvested biomass equals the total cost utilized in harvesting. In such a case , the economic rent is completely dissipated. Let c be the fishing cost per unit effort and p the price per unit biomass of the prey species. The economic rent (net revenue) at any time is given by:

$$\pi(x,y,E)=\frac{pqEx}{b+E}-cE. \qquad (2.6.1)$$

The bionomic equilibrium is $P_\infty(x_\infty,y_\infty,E_\infty)$ where x_∞, y_∞, E_∞ are the positive solutions of the following equation:

$$\dot{x}=\dot{y}=\pi=0. \qquad (2.6.2)$$

It may be noted here that if $c>\frac{pqx}{b+E}$ i.e., if the fishing cost exceeds the revenue obtained by selling the fish resource, then the economic rent

obtained from the fishery becomes negative. Hence, the fishery will be closed and no bionomic equilibrium exists. Therefore, for the existence of bionomic equilibrium, it is natural to assume that $\frac{pqx}{b+E} > c$.
We then have,

$$x_\infty = \frac{s}{\beta}. \tag{2.6.3}$$

$$y_\infty = \frac{1}{\alpha}\left[r\left(1 - \frac{s}{\beta k}\right) - \left(\frac{pqs - bc\beta}{ps}\right)\right]. \tag{2.6.4}$$

The above one is positive when $r > \frac{\beta k\left(pqs-bc\beta\right)}{ps\left(\beta k-s\right)}$ and $p > c\left(\frac{b\beta}{qs}\right)$ and $k > \frac{s}{\beta}$.

$$E_\infty = \frac{pqx - bc}{c}, \tag{2.6.5}$$

which is positive if $x > \frac{bc}{pq}$.
So, the bionomic equilibrium $P_\infty(x_\infty, y_\infty, E_\infty)$ exists if the above inequalities hold.

2.7 Optimal harvesting policy

The present value $\mathfrak{J}$ of a continous time-stream of revenues is given by the following equation:

$$\mathfrak{J} = \int_0^\infty \pi(x, y, E, t)e^{-\delta t}dt. \tag{2.7.1}$$

Here, $\pi(x, y, E, t) = \frac{pqxE}{b+E} - cE$ and δ denotes the instantaneous annual rate of discount. Our problem is to maximize $\mathfrak{J}$ subject to the state equation (2.2.2) by invoking Maximum Principle of Pontryagin et al., 1962. The control variable $E(t)$ is subjected to the constraints $0 \leq E(t) \leq E_{max}$ so that $V_t = [0, E_{max}]$ is the control set where E_{max} is a feasible uper limit for the harvesting effort. The Hamiltonian for the problem is given by the following:

$$H=\left(\frac{pqx}{b+E}-c\right)Ee^{-\delta t}+\lambda_1\left[rx(1-\frac{x}{k})-\alpha xy-\frac{qEx}{b+E}\right]+\lambda_2\left[-sy+\beta xy\right]. \tag{2.7.2}$$

Here, $\lambda_1(t)$, $\lambda_2(t)$ are the adjoint variables. The adjoint equations are the folowing:

$$\begin{aligned}\frac{d\lambda_1}{dt}&=-\frac{\partial H}{\partial x_1}\\&=-\left[\frac{pq}{b+E}Ee^{-\delta t}+\lambda_1\left(r-\frac{2rx}{k}-\alpha y-\frac{qE}{b+E}\right)+\lambda_2\left(\beta y\right)\right].\end{aligned} \tag{2.7.3}$$

and

$$\begin{aligned}\frac{d\lambda_2}{dt}&=-\frac{\partial H}{\partial y}\\&=-\left[\lambda_1\left(-\alpha x\right)+\lambda_2\left(-s+\beta x\right)\right].\end{aligned} \tag{2.7.4}$$

Our aim is to find an optimal equilibrium solution of the problem so that we may take

$$E=\left(\frac{pqx-bc}{c}\right). \tag{2.7.5}$$

By using equation (2.7.5), equations (2.7.3) and (2.7.4) become respectively,

$$\frac{d\lambda_1}{dt}=\left[\left(\frac{pq}{b+E}\right)Ee^{-\delta t}+\lambda_1\left(\frac{-rx}{k}\right)+\lambda_2\left(\beta y\right)\right], \tag{2.7.6}$$

and

$$\frac{d\lambda_2}{dt}=-\left[\lambda_1\left(-\alpha x\right)\right]. \tag{2.7.7}$$

Eliminating λ_2 from the equations (2.7.6) and (2.7.7), we have

$$\begin{aligned}\frac{d^2\lambda_1}{dt^2}-\left(\frac{rx}{k}\right)\frac{d\lambda_1}{dt}+\alpha\beta xy\lambda_1&=\left(\frac{pq}{b+E}\right)E\delta e^{-\delta t}\\&=M_1e^{-\delta t}.\end{aligned} \tag{2.7.8}$$

Here, $M_1 = \left(\frac{pq}{b+E}\right)E\delta$.
The auxiliary equation for (2.7.8) is

$$\mu^2 - \left(\frac{rx}{k}\right)\mu + \alpha\beta xy = 0. \tag{2.7.9}$$

Here, sum of the roots=$\left(\frac{rx}{k}\right) > 0$ and product of the roots=$\left(\alpha\beta xy\right) > 0$. Therefore, the roots of the above equation are either both real and positive or complex conjugates with positive parts. The complete solution for equation (2.7.9) is of the form

$$\lambda_1(t) = B_1 e^{\mu_1(t)} + B_2 e^{\mu_2(t)} + \left(\frac{M_1}{N}\right)e^{-\delta t}, \tag{2.7.10}$$

where B_1, B_2 are arbitrary constants and

$$N = \delta^2 + \left(\frac{rx}{k}\right)\delta + \alpha\beta xy \neq 0. \tag{2.7.11}$$

Thus, $\lambda_1(t)$ remains bounded as $t \to \infty$ if and only if $B_1 = B_2 = 0$ and then

$$\lambda_1(t) = \left(\frac{M_1}{N}\right)e^{-\delta t} = constant. \tag{2.7.12}$$

Similarly, we get

$$\lambda_2(t) = \left(\frac{M_2}{N}\right)e^{-\delta t} = constant. \tag{2.7.13}$$

Here, $M_2 = \left(\frac{pqE\alpha x}{b+E}\right)$.
Thus, the shadow prices $\lambda_i(t)e^{\delta t}$, i=1, 2 remain constant over time in optimal equilibrium when they satisfy the transversality condition at ∞, i.e., when they remain bounded as $t \to \infty$. Again, the condition for

which the Hamiltonian H must be a maximum is the following:

$$\begin{aligned} \frac{\partial H}{\partial E} &= e^{-\delta t}\left[\frac{pqxb}{(b+E)^2} - c\right] + \lambda_1\left[\frac{-qbx}{(b+E)^2}\right] \\ &= 0 \end{aligned} \tag{2.7.14}$$

i.e., $\lambda_1\left[\frac{qbx}{(b+E)^2}\right] = e^{-\delta t}\left[\frac{pqxb}{(b+E)^2} - c\right]$,

i.e., $\lambda_1\left[\frac{qbx}{(b+E)^2}\right] = e^{-\delta t}\frac{\partial \pi}{\partial E}$.

The right hand side represents the discounted value of the future profit per unit effort at the steady state effort level.

Putting the values of $\lambda_1(t)$ and $\lambda_2(t)$ in equation (2.7.14), we get

$$\left(p - \frac{M_1}{N}\right)\frac{qbx}{(b+E)^2} = c. \tag{2.7.15}$$

Equation (2.7.15) together with equation (2.6.1) gives the optimal equilibrium populations $x_1 = x_{1\delta}, x_2 = x_{2\delta}$.

When $\delta \rightarrow \infty$, equation (2.7.15) leads to the result $\frac{pqbx}{(b+E)^2} = c$ which implies $\frac{\partial \pi}{\partial E}(x_{1\infty}, x_{2\infty}, E) = 0$.

Thus, the value of the future profit per unit effort at an infinite discount rate. Using equation (2.7.15), we have

$$\begin{aligned} \frac{\partial \pi}{\partial E} &= \frac{pqbx}{(b+E)^2} - c \\ &= \frac{M_1}{N}\frac{qbx}{(b+E)^2}. \end{aligned} \tag{2.7.16}$$

Since, M_1 is of $o(\delta)$ where N is of $o(\delta^2)$, we see that $\frac{\partial \pi}{\partial E}$ is of $o(\delta^{-1})$. Thus, $\frac{\partial \pi}{\partial E}$ is a decreasing function of $\delta(\geq 0)$.

We, therefore, conclude that $\delta = 0$ leads to maximization of $\frac{\partial \pi}{\partial E}$.

2.8 Numerical simulations

In this model, we assume that there is a complete ban on harvesting the predator fish species. Here, only the prey fish species are harvested by using catch-per-unit effort hypothesis. Numerical simulations have been carried out by making use of MATLAB-2016a and Maple-18. It is very difficult to validate the model results with realistic data so far harvesting is considered in the natural field. These results are all verified by means of numerical illustrations of which some chosen ones are shown in the figures. So, We take some hypothetical parameter values to illustrate the results we have established. In this section, we consider some numerical examples:

Example 1:- We take the parameter values as r=15, k=1000, $\alpha = 0.02$, q=1.6, E=11, b=0.3, s=0.6, $\beta = 0.005$, in appropriate units. For the above values, we find that

(i) $S_0(0, 0)$ is unstable,

(ii) $S_1(896.17, 0)$ is unstable,

(iii) $S_2(120, 582.12)$ is both locally and globally asymtotically stable node.

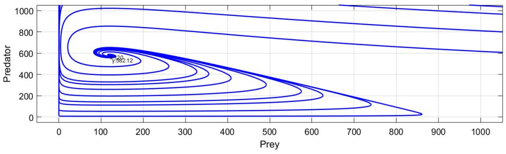

Figure 2.1: Phase plane trajectories of the prey-predator fishery with different initial values

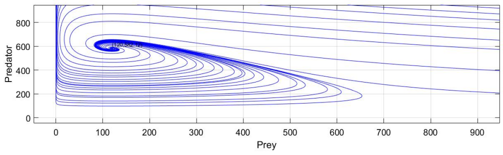

Figure 2.2: Phase plane trajectories of the prey-predator fishery with different initial values

From Figure 2.1 and Figure 2.2, it is clear that for the above set of parameter values, the system possesses an interior equilibrium point E_*=(120,582.12). It is also observed that the system (2.2.2) is globally asymptotically stable around the coexistence equilibrium E_*.

From Figure 2.3 and Figure 2.4, it is clear that the biomass density of prey species increases sharply with respect to time and then, decreases

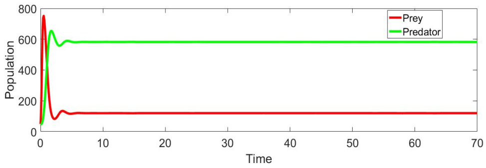

Figure 2.3: Solution curves of the prey-predator fishery for a period t=0 to 70 weeks

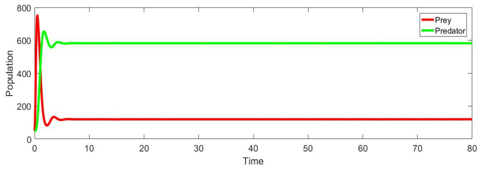

Figure 2.4: Solution curves of the prey-predator fishery for a period t=0 to 70 weeks

and settles down at its equilibrium level. The biomass density of predator species increases with respect to time and then, decreases slightly and settles down at its equilibrium level.

Example 2. Taking the same values of the parameters together with p=6 and c=20, from Figure 2.5, we find that the bionomic equilibrium exists at (23.54,670.42).

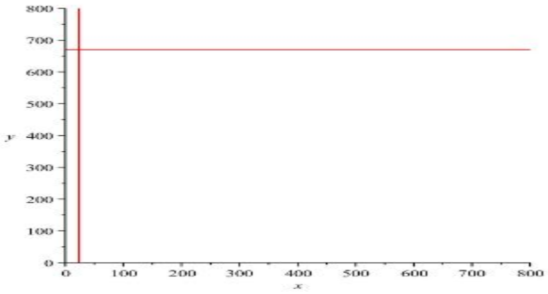

Figure 2.5: Bionomic equilibrium

2.9 Discussions

The dynamics of fishery resources have been studied in this research using a mathematical model that has been suggested and examined. It is considered that taking any of the predator species is totally prohibited. It is demonstrated that the inner equilibrium exists and is asymptotically stable. Additionally, it was discovered that the system under study lacks a limit cycle. It has also been noted that the fish population is kept at a suitable equilibrium level even when prey species are continuously harvested. This model establishes the existence of bionomic equilibria. A collection of parameter values is computed to determine the bionomic equilibrium. The Pontryagin's Maximum principle is then used in this model to analyse the best harvesting strategy. Economic rent has been shown to diminish as the discount rate rises, and it may even drift to zero as the discount rate approaches infinity. The maximisation of economic revenue occurs when the discount rate is zero, and the total evaporation of economic rent occurs when the discount rate is infinite. Therefore, it may be said that high interest rates will result in high inflation rates.

Finally, using MATLAB-2016a and Maple-18, the solution curves for the steady state and the bionomic equilibrium are shown. In this model, phase plane trajectories and growth curves are also examined.

3

A NON-LINEAR MODEL OF A FISHERY RESOURCE FOR ANALYZING THE EFFECTS OF TOXIC SUBSTANCES

"The definition of a good mathematical problem is the mathematics it generates rather than the problem itself." *- Andrew Wiles*

3.1 Introduction

Food and other essentials for human life are derived from renewable resources including forests, wild life, and fisheries, among others. These

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materials are essential to the survival and growth of biological communities. Our goal for managing these renewable resources is to maximise the benefits now derived from them and to ensure their correct maintenance in order to prevent their extinction. The impact of poisonous compounds is one of the most important issues in an aquatic ecological system. These hazardous compounds have a major impact on the growth of fish and other aquatic species. The management of fishery resources has been the subject of numerous research projects during the past few decades. Clark, 1976b, Clark, 1985, Mesterton-Gibbons, 1987, Mesterton-Gibbons, 1988 have pursued their insightful analyses based on mathematical modeling of fishery exploitation. There are numerous factors, including poisonous compounds, inter- and intra-species competition among the predator-prey fish resources, in addition to excessive and unrestrained fishing, which contribute to the extinction of fish species. By using non-linear mathematical modeling of predator-prey fishery resources, Mukhopadhyay, Chattopadhyay, and Tapaswi, 1998, Dubey and Hussain, 2000, Kar and Chaudhuri, 2003a, and Kar, Pahari, and Chaudhuri, 2004 conducted their mathematical analyses with toxicity as the primary focus. Our main goal is to determine the best control that can be used to preserve the extinction of both species while maximising the advantages derived from this predator-prey fisheries resource in the presence of hazardous agents. Additionally, Kar and Chaudhuri, 2003a expanded on the concept of a competing fishery resource system proposed by Maynard Smith, 1974 that is vulnerable to commercial exploitation and toxicity. A model was created by Ghosh, Chandra, and Sinha, 2002 to demonstrate how hazardous chemicals affect a fishery that is based on predator-prey relationships. A non-linear model was created by Haque and Sarwadi, 2016, Pal, Mahato, and Samanta, 2020, to show the effects of harmful compounds in an aquatic system. By modifying the catch rate function, Sarkar, Sarkar, and Chaudhuri, 2020 created a non-linear model.

The chapter is organized as follows. In Section 3.2, the discretized model is formulated, followed by the investigation of the existence and local and global stability of the equilibrium points in Section 3.3, and Section 3.4, respectively. In Section 3.5, some discussions on the optimal harvesting problem have been carried out. In Section 3.6, numerical simulations are performed to verify the analytical results. Lastly, in Section 3.7, a brief conclusion on the biological interpretations and significance of our findings is provided.

3.2 Model formulation

The dynamical system of prey-predator competing fishery resource is proposed by the following autonomous system of differential equations after incorporation of toxic affects:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\alpha xy-\gamma_1 x^3 y-\frac{q_1 E x}{b_1+E},\\ \frac{dy}{dt} &= sy\left(1-\frac{y}{L}\right)-\beta xy-\gamma_2 xy^2-\frac{q_2 E y}{b_2+E},\end{aligned} \tag{3.2.1}$$

where the population densities of fish that are predators and prey at time t, respectively, are represented by $y = y(t)$ and $x = x(t)$, respectively. Prey and predator fish species' maximum growth rates, competition coefficients, and environmental carrying capacities are denoted by the letters r, s, α, β, K, and L, respectively.

γ_1 and γ_2 here stand for the toxicity coefficients of prey and predator fish species, respectively. In this paradigm, every parameter is taken to be a positive constant. The term $\gamma_1 x^3$ is viewed as one type of functional reaction of the predator fish to the density of the prey population, and it results from the predator species' manufacture of harmful compounds to dissuade the prey population from sharing the common resources. The expressions $\frac{d(\gamma_1 x^3)}{dx} = 3\gamma_1 x^2 > 0$ and $\frac{d^2(\gamma_1 x^3)}{dx} = 6\gamma_1 x > 0$ are positive in this case. As a result, poisonous components are seen to proliferate more quickly as the biomass density of prey-predator species rises. Here, q_1 and q_2 stand for the catchability coefficients of the two fish species, respectively, while b_1 and b_2 are positive constants.

In the beginning of fishery models, the catch rate function is often of the type $h = qEx$. The CPUE (catch-per-unit-effort) theory put forth by Clark, 1976b serves as the foundation for this. It is subsequently changed to have the functional form $h = \frac{qEx}{bE+lx}$.

Here, we make the assumption that a fisherman will effectively search at random in a specific area based on how hard he or she will work to get the fish resource. This idea is renamed as a searching effectiveness for the field of discovery. The effectiveness (efficiency) of the effort level is employed in the presence of other fishermen to determine the rate of fish resource collection. In accordance with the aforementioned hypothesis, we modify the catch rate function in the form $h = \frac{qEx}{b+E}$, where E stands for harvesting effort, q (constant) for catchability coefficient, and b is a positive constant, to account for the fish resource population being captured for different effort levels.

3.3 Equilibria and their feasibility

There are four equilibrium points of the dynamical system (3.2.1) which are $E_0(0,0)$, $E_1(x_1,0)$, $E_2(0,y_2)$ and $E^*(x^*,y^*)$, where $x_1 = \frac{K}{r}\left(r - \frac{q_1E}{b_1+E}\right)$, $y_2 = \frac{L}{s}\left(s - \frac{q_2E}{b_2+E}\right)$, $y^* = \frac{L(s-\beta x^* - \frac{q_2E}{b_2+E})}{s+L\gamma_2 x^*}$ and we get the value of x^* from the following cubic equation:

$$A_1{x_1}^{*3} + A_2{x_1}^{*2} + A_3{x_1}^{*} + A_4 = 0, \tag{3.3.1}$$

where $A_1 = \gamma_1 L\beta$, $A_2 = \frac{-r}{K}L\gamma_2 - \gamma_1 Ls + \frac{L\gamma_1 q_2E}{b_2+E}$, $A_3 = \frac{-rs}{K} + rL\gamma_2 + \alpha L\beta - \frac{q_1EL\gamma_2}{b_1+E}$ and $A_4 = rs - \alpha Ls + \frac{q_2E\alpha L}{b_2+E} - \frac{q_1Es}{b_1+E}$.

It is observed that the steady state E_0 is always feasible, E_1 is feasible if $r > \frac{q_1E}{b_1+E}$ and E_2 exists if $s > \frac{q_2E}{b_2+E}$.

If $D < 0$, then the equation (3.4.1) has at least one positive root say

x^*. So, the existence condition of $E^*(x^*, y^*)$ is $s - \beta x^* - \frac{q_2 E}{b_2+E} > 0$ and $s > \frac{\alpha L q_2 E}{\alpha L(b_2+E) - r(b_2+E) + \frac{q_1 E(b_2+E)}{(b_1+E)}}$.

3.4 Qualitative analysis

3.4.1 Boundedness of the model

In the system (3.2.1), the solutions are uniformly bounded.

3.4.1.1 Proof

To prove the boundedness of the system let us choose a function $\omega = x + y$.

$$\begin{aligned} \frac{d\omega}{dt} + \zeta\omega &= rx\left(1 - \frac{x}{K}\right) + sy\left(1 - \frac{y}{L}\right) - (\alpha + \beta)\, xy - xy(\gamma_1 x^2 + \gamma_2 y) \\ &\quad - E\left(\frac{q_1 x}{b_1 + E} + \frac{q_2 y}{b_2 + E}\right) + \zeta\,(x + y) \\ &< \frac{K}{4r}\left(r + \zeta - \frac{Eq_1}{b_1 + E}\right)^2 + \frac{L}{4s}\left(s + \zeta - \frac{Eq_2}{b_2 + E}\right)^2 = \eta. \end{aligned}$$

Thus, we get $0 < \omega(x, y) < \frac{\eta}{\zeta}(1 - \exp^{-\zeta t}) + \omega(0)\exp^{-\zeta t} < \max\left\{\frac{\eta}{\zeta}, \omega(0)\right\}$. Therefore, the solutions are uniformly bounded in

$$\mathbf{R}_{xy} = \left\{(x, y) \in \Re^2_+ : \omega(x, y) \le \frac{\eta}{\zeta} + \varepsilon \text{ for any } \varepsilon > 0\right\}.$$

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3.4.2 Local behaviour of the equilibria

Let J_n be the variational matrix of the steady states E_n where $n = 0, 1, 2$. The eigenvalues of J_0 are $r - \frac{q_1 E}{b_1+E}$ and $s - \frac{q_2 E}{b_2+E}$. So, $E_0(0, 0)$ is stable node if $E > \max\left(\frac{b_1 r}{q_1 - r}, \frac{b_2 s}{q_2 - s}\right)$ and unstable if $E < \min\left(\frac{b_1 r}{q_1 - r}, \frac{b_2 s}{q_2 - s}\right)$.

The eigen values of J_1 are $\lambda_1 = \frac{-rx_1}{K}$ and $\lambda_2 = s - \beta x_1 - \frac{q_2 E}{b_2+E}$. Obviously, $\lambda_1 < 0$ and hence $E_1(x_1, 0)$ is a stable node if $\lambda_2 < 0$ which imply that $E > \frac{b_2(s-\beta x_1)}{q_2-s+\beta x_1}$. It can be shown that $E_1(x_1, 0)$ is saddle if $E < \frac{b_2(s-\beta x_1)}{q_2-s+\beta x_1}$.

The eigen values of J_2 are $\lambda_1 = r - \alpha y_2 - \frac{q_1 E}{b_1+E}$ and $\lambda_2 = \frac{-sy_2}{L}$. It is observed that $\lambda_2 < 0$. Hence $E_2(0, y_2)$ is a stable node when $\lambda_1 < 0$ i.e., if $E > \frac{b_1(r-\alpha y_2)}{q_1-r+\alpha y_2}$ and saddle point if $E < \frac{b_1(r-\alpha y_2)}{q_1-r+\alpha y_2}$.

The variational matrix of the system of equations (3.2.1) around E^* is

$$J^* = \begin{pmatrix} \frac{-r_1 x^*}{K} - 2\gamma_1 x^{*2} y^* & -\alpha x^* - \gamma_1 x^{*3} \\ -\beta y^* - \gamma_2 y^{*2} & \frac{-sy^*}{L} - \gamma_2 x^* y^* \end{pmatrix}$$

Here, $\mathrm{trace}(J^*) < 0$. So, for the local stability of the system of equations (3.2.1) around E^*, we have to prove that $\det(J^*) > 0$.

$$\det(J^*) = x^* y^* \left[\frac{rs}{KL} - \alpha\beta + \gamma_2 \left(\frac{q_1 E}{b_1 + E} - r \right) - \beta\gamma_1 x^{*2} + 2x^* \left(\gamma_1\gamma_2 x^* y^* + \frac{s\gamma_1 y^*}{L} + \frac{r\gamma_2}{K} \right) \right]. \tag{3.4.1}$$

So, $\det(J^*) > 0$ if $x^{*2} < \frac{1}{\beta\gamma_1} \left[\frac{rs}{KL} + \gamma_2 \left(\frac{q_1 E}{b_1+E} - r \right) - \alpha\beta \right]$.

3.4.3 Non-existence of periodic solution

Let us write the system of Equations (3.2.1) in the form $\dot{Y} = G(Y)$, where $Y = (x, y)$ and $G = (G_1, G_2)$. Here, $G_1, G_2 \in C^\infty(\mathbf{R})$, $G_1 = rx\left(1 - \frac{x}{K}\right) - \alpha xy - \gamma_1 x^3 y - \frac{q_1 E x}{b_1+E}$ and $G_2 = y\left(1 - \frac{y}{L}\right) - \beta xy - \gamma_2 x y^2 - \frac{q_2 E y}{b_2+E}$. Let us consider a function $F(x, y) = \frac{1}{xy}$. Then $F(x, y) > 0$ for $(x, y) \in \mathbf{R}_{xy}$.

$$\nabla.(F\vec{G}) = -\left[\frac{r}{Ky} + 2\gamma_1 x + \frac{s}{Lx} + \frac{\gamma_2 y}{x} \right] < 0 \; \forall (x, y) \in \mathbf{R}_{xy}. \tag{3.4.2}$$

It follows that $\nabla.(P\vec{G}) < 0$ always. So, by Bendixson-Dulac's criterion, there does not exist any periodic orbit for the proposed model.

3.4.4 Global Stability Analysis

To discuss the global stability of the dynamical system (3.2.1), a suitable Lyapunov function has been constructed as follows:

$$V(x,y)=\left[\left(x-x^*\log\frac{x}{x^*}\right)+\left(y-y^*\log\frac{y}{y^*}\right)\right]. \tag{3.4.3}$$

It is observed that V is positive definite $\forall (x,y)\in\Re^2_+$. Now, we have

$$\frac{dV}{dt}=-\left[P\left(x-x^*\right)^2+R\left(x-x^*\right)\left(y-y^*\right)+Q\left(y-y^*\right)^2\right], \tag{3.4.4}$$

where $P=\frac{r}{K}+\gamma_1 y\left(x+x^*\right)>0$, $Q=\frac{s}{L}+\gamma_2 x>0$, $R=\alpha+\beta+\gamma_1 x^{*2}+\gamma_2 y^*>0$.
Now, $4PQ-R^2=4\left\{\frac{r}{K}+\gamma_1 y(x+x^*)\right\}\left\{\frac{s}{L}+\gamma_2 x\right\}-\left\{\alpha+\beta+\gamma_1 x^{*2}+\gamma_2 y^*\right\}^2>0$, if $4rs>KL\{\alpha+\beta+\gamma_1 x^*+\gamma_2\}^2$.
So, $\frac{dV}{dt}=0$ at $E^*(x^*,y^*)$ and $\frac{dV}{dt}<0$ at all the points other than $E^*(x^*,y^*)$. Using Lyapunov- LaSalle's invariance principle, we can conclude that the interior equilibrium point is globally asymptotically stable under certain conditions.

3.4.5 Bionomic equilibrium

In the fishery literature, the bionomic equilibrium is said to be obtained when the total revenue achieved (TR) by selling the collected biomass is equal to the total cost (TC) for the harvesting. In that case, the economic rent is completely dissipated. Then the net revenue at any time is as follows:

$$\pi(x,y,E,t)=\left(\frac{p_1q_1x}{b_1+E}+\frac{p_2q_2y}{b_2+E}-C\right)E, \tag{3.4.5}$$

where C=constant cost for fishing per unit effort, p_1=constant price per unit biomass of the prey, p_2= constant price per unit biomass of the predator. To make the calculation simple, the harvesting cost per unit effort is considered as constant.
Now, from the equation $\dot{x}=0$, we get $E=\frac{\alpha y b_1+\gamma_1 x^2 y b_1-r\left(1-\frac{x}{K}\right)b_1}{r\left(1-\frac{x}{K}\right)-\alpha y-\gamma_1 x^2 y-q_1}$.

Thus, E is positive, when $\alpha y + \gamma_1 x^2 y < r\left(1-\frac{x}{K}\right) < \alpha y + \gamma_1 x^2 y + q_1$.
Similarly, from the equation $\dot{y} = 0$ we get $E = \frac{\beta b_2 x + \gamma_2 x y b_2 - s\left(1-\frac{y}{L}\right) b_2}{s\left(1-\frac{y}{L}\right) - \beta x - \gamma_2 x y - q_2}$.
So, E is positive when $\beta x + \gamma_2 x y < s\left(1-\frac{y}{L}\right) < \beta x + \gamma_2 x y + q_2$. Therefore, the interior equilibrium point exists on the curve

$$\frac{\alpha y b_1 + \gamma_1 x^2 y b_1 - r\left(1-\frac{x}{K}\right) b_1}{r\left(1-\frac{x}{K}\right) - \alpha y - \gamma_1 x^2 y - q_1} = \frac{\beta b_2 x + \gamma_2 x y - s\left(1-\frac{y}{L}\right) b_2}{s\left(1-\frac{y}{L}\right) - \beta x - \gamma_2 x y - q_2}, \tag{3.4.6}$$

where $0 \leq x \leq K$ and $0 \leq y \leq L$.
We can find the bionomic equilibrium of the open access fishery model using the equation (3.4.6) and the condition $\pi = TR - TC = 0$, which gives the result $\left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right) = 0$.

3.5 Optimal harvesting policy

In this part, we have found the optimal harvesting policy for our model which is assumed to be a commercial fishery model. Our salient task is to fix the optimal adjustment between the current and future harvests. The current value I of revenues is given by:

$$I = \int_0^\infty \pi(x, y, E, t) e^{-\delta t} dt, \tag{3.5.1}$$

where $\pi(x, y, E, t) = \frac{p_1 q_1 x E}{b_1 + E} + \frac{p_2 q_2 y E}{b_2 + E} - CE$ and δ denotes the instantaneous annual rate of discount, C is the cost of fishing per unit effort, p_1, p_2 are the prices per unit biomass of x and y species respectively. Our main focus is to maximize I along with the equation (3.2.1) by using Maximum Principle proposed by Pontryagin et al., 1962. Here, $0 \leq E(t) \leq E_{max}$ and control set $V_t = [0, E_{max}]$ where E_{max} is the possible maximum value of the effort of harvesting.

For our problem, the Hamiltonian is given by:

$$\begin{aligned} H &= \left(\frac{p_1q_1x}{b_1+E}+\frac{p_2q_2y}{b_2+E}-C\right)Ee^{-\delta t}+\lambda_1\left[rx\left(1-\frac{x}{K}\right)-\alpha xy-\gamma_1x^3y-\frac{q_1Ex}{b_1+E}\right] \\ &\quad +\lambda_2\left[sy\left(1-\frac{y}{L}\right)-\beta xy-\gamma_2xy^2-\frac{q_2Ey}{b_2+E}\right]. \end{aligned} \tag{3.5.2}$$

Here, $\lambda_1(t)$, $\lambda_2(t)$ are two adjoint variables. For finding an optimal solution, we are taking

$$E=\frac{\alpha yb_1+\gamma_1x^2yb_1-r\left(1-\frac{x}{K}\right)b_1}{r\left(1-\frac{x}{K}\right)-\alpha y-\gamma_1x^2y-q_1}=\frac{\beta b_2x+\gamma_2xyb_2-s\left(1-\frac{y}{L}\right)b_2}{s\left(1-\frac{y}{L}\right)-\beta x-\gamma_2xy-q_2}. \tag{3.5.3}$$

Now, from the two adjoint equations ($\frac{d\lambda_1}{dt}=-\frac{\partial H}{\partial x}$ and $\frac{d\lambda_2}{dt}=-\frac{\partial H}{\partial y}$), we get

$$\begin{aligned} &\frac{d^2\lambda_1}{dt^2}-\left(\frac{rx}{K}+2\gamma_1x^2y+\frac{sy}{L}+xy\gamma_2\right)\frac{d\lambda_1}{dt} \\ &+\left[\left(\frac{rx}{K}+2\gamma_1x^2y\right)\left(\frac{sy}{L}+xy\gamma_2\right)-\left(\beta y+\gamma_2y^2\right)\left(\alpha x+\gamma_1x^3\right)\right]\lambda_1=M_1e^{-\delta t}, \end{aligned} \tag{3.5.4}$$

where $M_1=\left(\frac{p_1q_1E\delta}{b_1+E}\right)+\left(\frac{sy}{L}+xy\gamma_2\right)\frac{p_1q_1E}{b_1+E}-\left(\beta y+\gamma_2y^2\right)\frac{p_2q_2E}{b_2+E}$.
The auxiliary equation for the equation (3.5.4) is as follows:

$$\begin{aligned} &\mu^2-\left(\frac{rx}{K}+2\gamma_1x^2y+\frac{sy}{L}+xy\gamma_2\right)\mu \\ &+\left[\left(\frac{rx}{K}+2\gamma_1x^2y\right)\left(\frac{sy}{L}+xy\gamma_2\right)-\left(\beta y+\gamma_2y^2\right)\left(\alpha x+\gamma_1x^3\right)\right]=0. \end{aligned} \tag{3.5.5}$$

Then, complete solution of the equation (3.5.5) is as follows:

$$\lambda_1(t)=A_1e^{\mu_1(t)}+A_2e^{\mu_2(t)}+\left(\frac{M_1}{N_1}\right)e^{-\delta t}, \tag{3.5.6}$$

where $N_1=\left[\delta^2-\delta\left(\frac{rx}{K}+2\gamma_1x^2y+\frac{sy}{L}+xy\gamma_2\right)+\left(\frac{rx}{K}+2\gamma_1x^2y\right)\left(\frac{sy}{L}+xy\gamma_2\right)-\left(\beta y+\gamma_2y^2\right)\left(\alpha x+\gamma_1x^3\right)\right]\neq 0$. It is true that λ_1 is bounded if $A_1=A_2=0$.

Then, we have $\lambda_1(t) = \left(\frac{M_1}{N_1}\right)e^{-\delta t}$ and $\lambda_2(t) = \left(\frac{M_2}{N_1}\right)e^{-\delta t}$, where $M_2 = \frac{p_2q_2E\delta}{b_2+E} + \left(\frac{rx}{K} + 2\gamma_1x^2y\right)\frac{p_2q_2E}{b_2+E} - \left(\alpha x + \gamma_1x^3\right)\frac{p_1q_1E}{b_1+E}$.

At $t \to \infty$, the shadow prices $\lambda_i(t)e^{\delta t}$, $i = 1, 2$ become constant if they satisfy the transversality condition.

Now, the equation $\frac{\partial H}{\partial E} = 0$ along with the values of $\lambda_1(t)$ and $\lambda_2(t)$ gives the following:

$$\left(p_1 - \frac{M_1}{N_1}\right)\frac{q_1b_1x}{(b_1+E)^2} + \left(p_2 - \frac{M_2}{N_1}\right)\frac{q_2b_2y}{(b_2+E)^2} = C. \tag{3.5.7}$$

Using the equations (3.5.3) and (3.5.7), we get the optimal equilibrium of the populations i.e., $x = x_\delta, y = y_\delta$.

At $\delta \to \infty$, equation (3.5.7) gives the result $\frac{p_1q_1b_1x}{(b_1+E)^2} + \frac{p_2q_2b_2y}{(b_2+E)^2} = C$ which implies $\frac{\partial \pi}{\partial E}(x_\infty, y_\infty, E) = 0$.

Thus, the economic rent is totally vanished and hence the fishery remains undeveloped. Then, from the equations (3.5.3) and (3.5.7), we get the following:

$$\frac{\partial \pi}{\partial E} = \frac{M_1}{N_1}\frac{q_1b_1x}{(b_1+E)^2} + \frac{M_2}{N_1}\frac{q_2b_2y}{(b_2+E)^2}. \tag{3.5.8}$$

Since M_1 and M_2 are of $o(\delta)$ where N_1 is of $o(\delta^2)$, it is found that $\frac{\partial \pi}{\partial E}$ is of $o(\delta^{-1})$. So, $\frac{\partial \pi}{\partial E}$ is gradually decreasing with $\delta(\geq 0)$.

Hence, we conclude that $\frac{\partial \pi}{\partial E}$ attains it's maximum value at $\delta = 0$.

3.6 Numerical results

In this section, we verify the analytical results numerically using MATLAB-2016a and Maple-18. For this purpose, we have taken a set of parameter values: r=3.5, k=300, $\alpha = 0.02$, $q_1 = 0.9$, $E = 10$, $b_1 = 0.5$, $\gamma_1 = 0.0002$, s=1.1, l=100, $\beta = 0.001$, $\gamma_2 = 0.0001$, $q_2 = 0.7$, $b_2 = 0.3$. For the above set of parameter values, it is shown that: (i) $E_0(0, 0)$ is

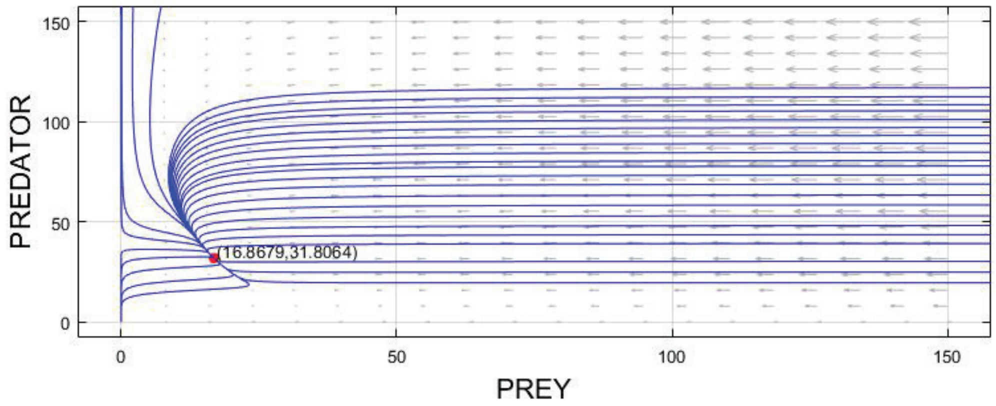

Figure 3.1: Globally stable steady state of the fishery model with different initial values.

unstable, (ii) $E_1(226.53, 0)$ is stable node, (iii) $E_2(0, 38.22)$ is stable node, (iv) $E^*(16.8679, 31.8064)$ is locally as well as globally asymptotically stable node.

From Figure 3.1 and 3.2, it is seen that the system possesses an interior equilibrium point E^*=(16.8679,31.8064) and around that point, the system (3.2.1) is globally asymptotically stable.

It is found that the bionomic equilibrium i.e., $(x_\infty, y_\infty) = (16.5346, 33.2731)$ (Figure3.3) occurs for the same parameter values $p_1 = 10$, $p_2 = 10$, C=50 and $E = \frac{\alpha y b_1+\gamma_1 x^2 y b_1 - r\left(1-\frac{x}{k}\right)b_1}{r\left(1-\frac{x}{k}\right)-\alpha y-\gamma_1 x^2 y-q_1}$. Without the effect of toxicity i.e., $(\gamma_1 = \gamma_2 = 0)$, we found $(x_\infty, y_\infty) = (190.84, 21.24)$ (Figure3.4). From the figures, it is concluded that the population density of prey $(x_\infty = 190.84)$ at which the bionomic equilibrium occurs without toxicity is higher than the population density of prey $(x_\infty = 16.5346)$ with toxicity. On the other hand, the population density of predator $(y_\infty = 21.24)$ at which the bionomic equilibrium occurs without toxicity is lower than the population density of predator $(y_\infty = 33.2731)$ with toxicity.

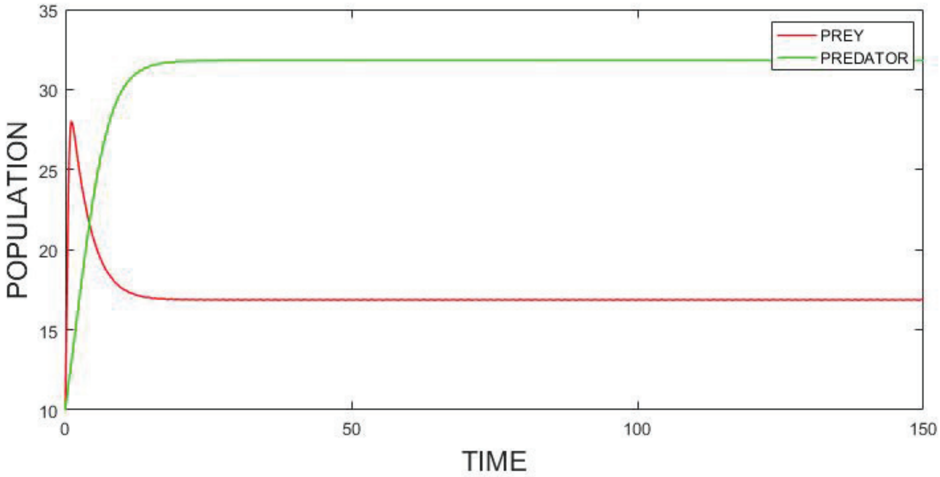

Figure 3.2: Globally stable steady state of the fishery model with different initial values.

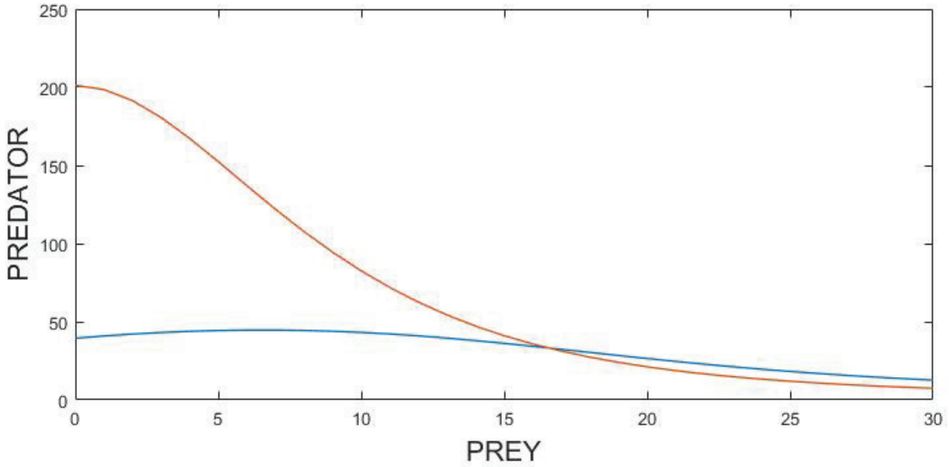

Figure 3.3: Bionomic equilibrium of the fishery model when toxicity$\neq 0$.

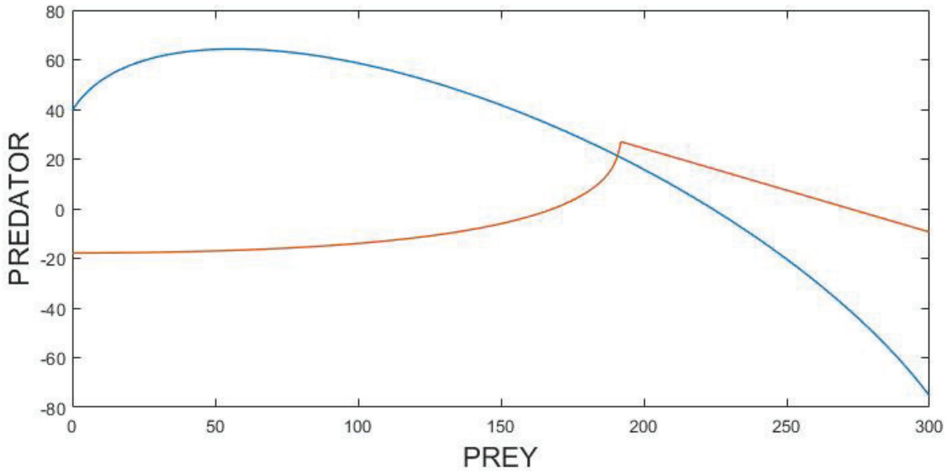

Figure 3.4: Bionomic equilibrium of the fishery model when toxicity= 0.

3.7 Discussions

This work has attempted to analyse the consequences of toxicants emitted by both species in a competing prey-predator fisheries model, where both species are harvested using a modified catch rate function. The stability of both the local and global levels has been investigated. Along with the problem of describing the condition of the bionomic equilibrium in the absence of toxicity, an effort has been made in this article to define the existence of bionomic equilibrium in the presence of toxicity. The numerical examples used for this purpose reveal that the two species' toxin release rates may modify the qualitative nature of the suggested model system, elucidating the possibility that any one of the species could become extinct as a result of toxicity. The population level of the bionomic equilibrium has been seen to rapidly decline for the first species in the presence of toxicity, but it slowly increases for the second species. To summarise, it may be claimed that the progressive increase in poisonous compounds produced by both species will have a significant negative impact on one another and eventually result in extinction. Another important aspect of the study that has been examined is the optimal harvesting policy. The policy that has been used, employing the Pontryagin maximum principle, has shown that, subject to the state equation and control limitations, the net revenue is maximised with the assistance of the Pontryagin maximum principle. Due to the fact that optimal equations satisfy the transversality criterion at infinity, the inquiry has also determined that the shadow price stays constant over time. According to the model, zero discounting results in the highest net revenue, whereas an infinite discount rate causes all net revenues to disappear. This work has attempted to analyse the consequences of toxicants emitted by both species in a competing prey-predator fisheries model, where both species are harvested using a modified catch rate function. The stability of both the local and global levels has been investigated. Along with the problem of describing the condition of the bionomic equilibrium in the absence of toxicity, an effort has been made in this article to define the existence of bionomic equilibrium in the presence of toxicity. The numerical examples used for this purpose reveal that the two species' toxin release rates may modify the qualitative nature of the suggested model system, elucidating the possibility that any one of the species could become extinct as a result of toxicity. The population level of the bionomic equilibrium has been seen to rapidly decline for the first species in the presence of toxicity, but it slowly increases for the second species. To summarise, it may be claimed that the progressive increase in poisonous compounds produced by both species will have a significant negative impact on one another and eventually result in extinction. Another important aspect of the study that has been examined is the optimal harvesting policy. The policy that has been used, employing the Pontryagin maximum principle, has shown that, subject to the state equation and control limitations, the net revenue is maximised with the assistance of the Pontryagin maximum principle. Due to the fact that optimal equations satisfy the transversality criterion at infinity, the inquiry has also determined that the shadow price stays constant over time. According to the model, zero discounting results in the highest net revenue, whereas an infinite discount rate causes all net revenues to disappear.

4

MODELING OF HARVESTING OF PREY-PREDATOR FISHERY IN THE PRESENCE OF TOXICITY WITH A MODIFIED CATCH RATE FUNCTION

"Mathematics is the art of giving the same name to different things."

- Henri Poincare

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4.1 Introduction

The scientific management of the exploitation of renewable resources, such as fisheries, forests, and wildlife management, is the focus of bioeconomic modeling. The need for more food and energy by an expanding population has resulted in more exploitation of these resources. In addition to these, there are a variety of human needs, industrial uses, and other purposes for which renewable resources are exploited. Our goal in bioeconomic modeling is to determine ways to control the utilisation and depletion of renewable resources. Researchers have recently become interested in this issue's multispecies fisheries harvesting. In addition to harmful compounds, intra- and inter-species competition, and other factors, overfishing also contributes to the extinction of fish populations. The animals living in that marina habitat will be severely impacted by these toxicants, though, as a result of the expanding human requirements companies are also manufacturing large amounts of toxicants, some of which are collected in marine water. Our primary goal is to investigate the effects of fish toxicity among different species that are discharged by different sources, such as industries, agricultural fields, etc.

In recent decades, the impact of toxicants on biological populations has grown to be a serious environmental issue. With the investigations of Hallam and Clark, 1982, Hallam, Clark, and Jordan, 1983, Hallam and De Luna, 1984, De Luna and Hallam, 1987, Freedman and Shukla, 1990, and others, mathematical modelling connected to such ecotoxicological difficulties was initiated. Other mathematicians who have studied this subject include Chattopadhyay, 1997, Shukla and Dubey, 1996, Mukhopadhyay, Chattopadhyay, and Tapaswi, 1998, Dubey and Hussain, 2000 and Shukla et al., 2001. The majority of these models focus on generic, single or dual-species biological communities without giving particular attention to the terrestrial or aquatic environment.

However, the effects of toxins on different fish species have grown to be a significant environmental issue. With the research of Kar and Chaudhuri, 2002b, Kar, Pahari, and Chaudhuri, 2004 etc., mathematical modeling in dealing with such situations was initiated. Without placing a particular emphasis on aquatic conditions, the majority of the models deal with generic single species or two species fishing models. The ecotoxicological consequences of the toxicants released by the marine biological species themselves have recently attracted the attention of researchers. The toxin emitted by one species has an impact on not just that species but may also have an impact on the development of other species. The idea of Maynard Smith, 1974 was expanded further by Kar and Chaudhuri, 2002b to a two species competing fish species model which are commercially exploited. A model was created by Haque and Sarwadi, 2016 to examine the impact of toxicity on a harvested fishing model.

Since fish is one of the most significant renewable resources in the ecological system, the fishery model is the main topic of discussion in this chapter. To the best of our knowledge and based on the literature discussed above, we have proposed a mathematical model in this chapter to analyse the dynamics of a fishery resource system in an aquatic habitat subject to fish population harvesting, where the growth of both species is influenced by some toxicants released by the other species. In this model, we suggest that the modified catch rate function put out by Sarkar, Sarkar, and Chaudhuri, 2020 be used to harvest both fish species. The structure of the current chapter is as follows: In Section 4.2, a model formulation is suggested. In Section 4.3, the equilibria and their viability are covered. In Section 4.4 and 4.5, the system's local and global stability analyses are covered. In Sections 4.6, respectively, we formulate the requirements for the existence of biological and bionomic equilibrium and examine their stable behaviour. The numerical data are presented in Section 4.7, and Section 4.8 presents a conclusion and an ecological interpretation of the current chapter.

4.2 Mathematical model formulation

We consider the following dynamical system as a simple prey-predator interaction model:

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\alpha xy,\\ \frac{dy}{dt} &= -sy+\beta xy, \end{aligned} \tag{4.2.1}$$

Where r is the highest specific growth rate of the prey population; x=x(t) is the biomass density of the prey population at time t; y=y(t) is the biomass density of the predator population at time t; and s is the relative rate at which the predators die out in the absence of prey.

According to the model, only the predators have an impact on prey reproduction, and the amount of prey they catch limits their ability to reproduce. Prey populations increase with a relative rate of r in the absence of predators, while predator populations decline exponentially with a relative rate of s in the absence of prey. The trophic function, or the predator's functional reaction to the prey population density, is given by αx, which is understood as the biomass of the prey devoured by the predator per unit time. Predator reproduction uses a portion of the energy required by this biomass, $\frac{\beta}{\alpha}$ $(\ 0<\beta<\alpha\leq 1)$, while predator metabolism and hunting activity utilise the remaining energy.Here, k represents the population of prey that the environment can sustain.

In the Lotka-Voltera prey-predator paradigm, in the absence of predator species, the prey species increases exponentially up to infinity. We change the Lotka-Voltera prey-predator model's initial term to $rx\left(1-\frac{x}{k}\right)$, where $k =$ the prey species' environmental carrying capacity. Additionally, both the predator and the prey are subjected to a joint harvesting effort here E.

In the beginning of fishery models, the catch rate function is often of the type $h = qEx$. The CPUE (catch-per-unit-effort) hypothesis [Clark, 1976b] serves as the foundation for this. Later, it is updated to have the functional form $h = \frac{qEx}{bE+lx}$.

Here, we make the assumption that a fisherman will effectively search at random in a specific area based on how hard he or she will work to get the fish resource. This idea is renamed as a searching effectiveness for the field of discovery. The effectiveness (efficiency) of the effort level is employed in the presence of other fishermen to determine the rate of fish resource collection. In accordance with the aforementioned hypothesis, the catch rate function is modified to take the form $h = \frac{qEx}{b+E}$, where E stands for harvesting effort, q(constant) for catchability, and b is a positive constant, depending on the amount of effort required to capture a given population of fish.

Considering these, the following autonomous system of differential equations may control the dynamics of prey-predator fish populations:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \alpha xy - \frac{q_1 Ex}{b_1+E}, \\ \frac{dy}{dt} &= -sy + \beta xy - \frac{q_2 Ey}{b_2+E}, \end{aligned} \tag{4.2.2}$$

where q_1, q_2 represent the catchability coefficient of the two species and b_1, b_2 are suitable constants.

The above mentioned model is now extended to the following one after incorporation of toxic effect:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - \alpha xy - \gamma_1 x^3 - \frac{q_1 Ex}{b_1+E}, \\ \frac{dy}{dt} &= -sy + \beta xy - \gamma_2 y^2 - \frac{q_2 Ey}{b_2+E}. \end{aligned} \tag{4.2.3}$$

The new parameters γ_1 and γ_2 are the toxicity coefficients in this case. All of the model's parameters, including r, s, α, β, and γ_1, k, L, and γ_2 are all positive constants. The phrase $\gamma_1 x^3$ is intimately related to the infection of prey species by some hazardous substances from the outside, such as industrial pollution. In this case, we see that $\frac{d(\gamma_1 x^3)}{dx} = 3\gamma_1 x^2 > 0$ and $\frac{d^2(\gamma_1 x^3)}{dx} = 6\gamma_1 x > 0$. As more and more members of the species ingest the contaminated food, the production of the toxicants to the density of the prey species increases. The coefficient of toxicity to the prey species in this case is γ_1. Similar circumstances apply to predator species, with the exception that because their toxicity is smaller than that of their prey, it is assumed that $\gamma_2 y^2$. The coefficient of toxicity for the predator species is γ_2 in this instance.

4.3 Equilibria and their feasibility

The possible steady states of the dynamical system of equation (4.2.3) are $S_0(0,0)$, $S_1(x_1,0)$ and $S^*(x^*,y^*)$, where $x_1 = \frac{\frac{-r}{k}+\sqrt{\left(\frac{r}{k}\right)^2+4\gamma_1\left(r-\frac{q_1E}{b_1+E}\right)}}{2\gamma_1}$ which exists if $\frac{r}{q_1} > \frac{E}{b_1+E}$ and $S^*(x^*,y^*)$ where

$$\begin{aligned} \frac{dx}{dt} &= r\left(1-\frac{x^*}{k}\right) - \alpha y^* - \gamma_1 x^{*2} - \frac{q_1E}{b_1+E}, \\ \frac{dy}{dt} &= -s + \beta x^* - \gamma_2 y^* - \frac{q_2E}{b_2+E}. \end{aligned} \tag{4.3.1}$$

Now, $x^* = \frac{-\left(\frac{r}{k}+\frac{\alpha\beta}{\gamma_2}\right)+\sqrt{\left(\frac{r}{k}+\frac{\alpha\beta}{\gamma_2}\right)^2+4\gamma_1\left(r+\frac{s\alpha}{\gamma_2}+\frac{\alpha q_2E}{\gamma_2(b2+E)}-\frac{q_1E}{b_1+E}\right)}}{2\gamma_1}$ which is positive when $\frac{\alpha q_2}{(b_2+E)} > \frac{q_1\gamma_2}{(b_1+E)}$ and $y^* = \frac{\beta x^*-s-\frac{q_2E}{b_2+E}}{\gamma_2}$
which is positive when $x^* > 0$ and $\beta x^* > s + \frac{q_2E}{b_2+E}$.

4.4 Local stability analysis

We shall now investigate the local behaviour of the model (4.2.3) around each of the above mentioned steady states. The variational matrix of the system of equations (4.2.3) is defined as follows:

$$\begin{vmatrix} r-\frac{2rx}{k}-\alpha y-3\gamma_1x^2-\frac{q_1E}{b_1+E} & -\alpha x \\ \\ \beta y & -s+\beta x-2\gamma_2y-\frac{q_2E}{b_2+E} \end{vmatrix}$$

For $S_0(0,0)$, the characteristic equation is the following:

$$\begin{vmatrix} r-\frac{q_1E}{b_1+E}-\lambda & 0 \\ \\ 0 & -s-\frac{q_2E}{b_2+E}-\lambda \end{vmatrix} = 0.$$

The eigenvalue of the variational matrix are $r-\frac{q_1E}{b_1+E}$ and $-s-\frac{q_2E}{b_2+E}$. Roots of this equation $\lambda_1 = r-\frac{q_1E}{b_1+E}$ and $\lambda_2 = -s-\frac{q_2E}{b_2+E}$. Here $\lambda_1 < 0$ if $\frac{r}{q_1} < \frac{E}{b_1+E}$ and $\lambda_2 < 0$. So λ_1 and λ_2 are both real and negative when $E > \frac{b_1r}{q_1-r}$.
So, the trivial steady state $S_0(0,0)$ is a stable node. When $E < \frac{b_1r}{q_1-r}$, then $\lambda_1 > 0$ and $\lambda_2 < 0$ and $S_0(0,0)$ is a saddle point.
For $S_1(x_1,0)$, the characteristic equation is the following:

$$\begin{vmatrix} \frac{-rx}{k}-\gamma_1x_1^2-\lambda & -\alpha x_1 \\ \\ 0 & -s+\beta x_1+\frac{q_2E}{b_2+E}-\lambda \end{vmatrix} = 0.$$

Here $\lambda_1 = -\left(\frac{rx_1}{k}+\gamma_1x_1^2\right) < 0$ and $\lambda_2 = -\left(s-\beta x_1+\frac{q_2E}{b_2+E}\right)$. Here $\lambda_1 < 0$ and hence the steady state $S_1(x_1,0)$ is a stable node when $\lambda_2 < 0 \implies \beta x_1 < s+\frac{q_2E}{b_2+E}$ The interior equilibrium (x^*,y^*) is the solution of the following system of equations:

$$\begin{aligned} r\left(1-\frac{x^*}{k}\right)-\alpha y^*-\gamma_1x^{*2}-\frac{q_1E}{b_1+E} &= 0, \\ -s+\beta x^*-\gamma_2y^*-\frac{q_2E}{b_2+E} &= 0. \end{aligned} \tag{4.4.1}$$

The variational matrix of the system of equations (4.2.3) around S^* is

$$J^* = \begin{vmatrix} \frac{-rx^*}{k}-2\gamma_1x^{*2} & -\alpha x^* \\ \\ \beta y^* & -\gamma_2y^*. \end{vmatrix}.$$

So, the charateristic equation of the above matrix J^* is

$$\begin{vmatrix} \frac{-rx^*}{k} - 2\gamma_1 x^{*2} - \lambda & -\alpha x^* \\ \\ \beta y^* & -\gamma_2 y^* - \lambda \end{vmatrix} = 0$$

$$\text{or}, \lambda^2 + \left(\frac{rx^*}{k} + 2\gamma_1 x^{*2} + \gamma_2 y^* \right) \lambda + \left(\frac{rx^*}{k} + 2\gamma_1 x^{*2} \right) \gamma_2 y^* + \alpha\beta x^* y^* \;=\; 0. \tag{4.4.2}$$

Here, the sum of the roots $= -b = -\left(\frac{rx^*}{k} + 2\gamma_1 x^{*2} + \gamma_2 y^*\right) < 0$ and the product of the roots $= c = \left(\frac{rx^*}{k} + 2\gamma_1 x^{*2}\right)\gamma_2 y^* + \alpha\beta x^* y^* > 0$. In absence of toxicity, we have $\gamma_1 = \gamma_2 = 0$ and then $c = \alpha\beta x^* y^*$ which is positive for all positive values of α, β, x^*, y^*.

So either real and negative or complex conjugates with negative real components are the roots of the quadratic equation. As a result, the steady state $S*$ is either a locally stable node or focus. Toxic conditions cause $b, c > 0$. As a result, the equation's two roots, (4.4.2), are either real and negative or complex conjugates with real parts that are negative. Therefore, depending on whether toxicity is present or not, the non-trivial steady state $S^*(x^*, y^*)$ is either a locally stable node or a locally stable focus.

Since x^* and y^* are both positive and must meet the connection $\frac{\alpha q_2}{(b_2+E)} > \frac{q_1\gamma_2}{(b_1+E)}$, the local stability of the system is not directly reliant on the intensities of the toxicants. Therefore, both species' population densities must be positive for the dynamical system to be stable, which means that as the effects of toxicity increase over time, both species' densities will gradually decrease. This will have an impact on the system's stability, and both species will eventually tend towards extinction.

4.5 Global stability

In this section, we consider the global stability of the system of equations (4.2.3) by constructing a suitable Lyapunov function which one is stated following:

$$V(x,y) = \left[(x-x^*) - x^* \log\frac{x}{x^*} + h(y-y^*) - y^* \log(\frac{y}{y^*})\right], \tag{4.5.1}$$

where h is a suitable constant to be determined in the subsequent steps. It can be easily verified that the function V is zero at the equilibrium point (x^*, y^*) and is positive for all other values of x,y. The time derivative of V along the trajectories of equation (4.2.3) is the following:

$$\begin{aligned} \frac{dv}{dt} = \frac{x-x^*}{x}\frac{dx}{dt} + h\frac{y-y^*}{y}\frac{dy}{dt} = (x-x^*)\left[r(1-\frac{x}{k}) - \alpha y - \gamma_1 x^2 - \frac{q_1E}{b_1+E}\right] \\ + h(y-y^*)\left[-s+\beta x - \gamma_2 y - \frac{q_2E}{b_2+E}\right]. \end{aligned} \tag{4.5.2}$$

Also, we have the set of the equilibrium equations

$$\begin{aligned} r\left(1-\frac{x^*}{k}\right) - \alpha y^* - \gamma_1 x^{*2} - \frac{q_1E}{b_1+E} &= 0, \\ -s + \beta x^* - \gamma_2 y^* - \frac{q_2E}{b_2+E} &= 0, \end{aligned} \tag{4.5.3}$$

corresponding to the steady state $S_2(x^*, y^*)$.

We can write equation (4.5.1) together with the above two equations in the form:

$$
\begin{aligned}
\frac{dv}{dt} &= (x-x^*)\left[r(1-\frac{x}{k})-\alpha y-\gamma_1 x^2-\frac{q_1E}{b_1+E}-r(1-\frac{x^*}{k})+\alpha y^*+\gamma_1 x^{*2}+\frac{q_1E}{b_1+E}\right] \\
&\quad + h(y-y^*)\left[-s+\beta x-\gamma_2 y-\frac{q_2E}{b_2+E}+s-\beta x^*+\gamma_2 y^*+\frac{q_2E}{b_2+E}\right] \\
&= -\left[(x-x^*)^2\left(\frac{r}{k}+\gamma_1(x+x^*)\right)+(x-x^*)(y-y^*)(\alpha-h\beta)+(y-y^*)^2h\gamma_2\right].
\end{aligned}
\tag{4.5.4}
$$

If we choose $h=\frac{\alpha}{\beta}$,
$\frac{dv}{dt}=-\left[(x-x^*)^2\left(\frac{r}{k}+\gamma_1(x+x^*)\right)+(y-y^*)^2\frac{\alpha}{\beta}\gamma_2\right]<0$
Now since $\frac{dv}{dt}$ is negative semidefinite in some neighbourhood of (x^*,y^*) , the interior equilibrium point (x^*,y^*) is globally asymptotically stable.

4.6 Bionomic equilibrium

The concepts of biological equilibrium and economic equilibrium are combined to form the phrase bionomic equilibrium. A biological equilibrium is expressed by the equation $\dot{x}=0=\dot{y}$. The bionomic equilibrium is considered to be reached in the fishing literature when the total revenue from selling the harvested biomass (TR) equals the total cost for the effort put into harvesting (TC). The economic rent fully disappears in such a situation. Let C represent the constant fishing cost per unit of effort, p_1 the constant price per unit of prey species biomass, and p_2 the constant price per unit of predator species biomass.
The economic rent (net revenue) at any time is given by the following equation:

$$\pi(x,y,E,t)=\left(\frac{p_1q_1x}{b_1+E}+\frac{p_2q_2y}{b_2+E}-C\right)E. \tag{4.6.1}$$

Although, the harvesting cost per unit effort is not a constant, we take it to be a constant for the sake of simplicity. Now,

$$\dot{x}=0\Rightarrow r\left(1-\frac{x}{k}\right)-\alpha y-\gamma_1 x^2=\frac{q_1E}{b_1+E}. \tag{4.6.2}$$

From the equation (4.6.2), we get

$$E = \frac{\alpha y b_1 + \gamma_1 x^2 b_1 - r\left(1 - \frac{x}{k}\right) b_1}{r\left(1 - \frac{x}{k}\right) - \alpha y - \gamma_1 x^2 - q_1}. \tag{4.6.3}$$

Thus, E is positive when $\alpha y + \gamma_1 x^2 < r\left(1 - \frac{x}{k}\right) < \alpha y + \gamma_1 x^2 + q_1$.

Again,

$$\dot{y} = 0 \Rightarrow -s + \beta x - \gamma_2 y = \frac{q_2 E}{b_2 + E}. \tag{4.6.4}$$

From the equation (4.6.4), we get

$$E = \frac{b_2\left(s + \gamma_2 y - \beta x\right)}{\beta x - q_2 - s - \gamma_2 y}. \tag{4.6.5}$$

So, E is positive when $\beta x - q_2 < s + \gamma_2 y < \beta x$.

Hence, the non-trivial equilibrium solution occurs at the point on the curve

$$\frac{\alpha y b_1 + \gamma_1 x^2 b_1 - r\left(1 - \frac{x}{k}\right) b_1}{r\left(1 - \frac{x}{k}\right) - \alpha y - \gamma_1 x^2 - q_1} = \frac{b_2\left(s + \gamma_2 y - \beta x\right)}{\beta x - q_2 - s - \gamma_2 y}, \tag{4.6.6}$$

where $0 \leq x \leq k$.

The bionomic equilibrium of the open access fishery is found by equation (4.6.6) along with the condition

$$\begin{aligned} \pi &= TR - TC \\ &= \left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right) E \\ &\Rightarrow \left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right) = 0. \end{aligned} \tag{4.6.7}$$

4.7 Optimal harvesting policy

The fundamental problem for finding out an optimal policy in a economic (commercial fishery) is to determine the optimal trade off between the current and future harvests. The present value $\mathfrak{I}$ of a continous time-stream of revenues is given by the following:

$$\mathfrak{I} = \int_0^\infty \pi(x, y, E, t)e^{-\delta t}dt, \tag{4.7.1}$$

where $\pi(x, y, E, t) = \frac{p_1q_1xE}{b_1+E} + \frac{p_2q_2yE}{b_2+E} - CE$ and δ denotes the instantaneous annual rate of discount, C is the cost of fishing per unit effort, p_1, p_2 are the price per unit biomass of x and y species respectively. Our problem is to maximize $\mathfrak{I}$ subject to the state equation (4.2.3) by invoking Maximum Principle of Pontryagin et al., 1962. The control variable $E(t)$ is subjected to the constraints $0 \leq E(t) \leq E_{max}$ so that $V_t = [0, E_{max}]$ is the control set where E_{max} is a feasible upper limit for the harvesting effort. The Hamiltonian for the problem is given by the following:

$$\begin{aligned} H = \left(\frac{p_1q_1x}{b_1+E} + \frac{p_2q_2y}{b_2+E} - C\right)Ee^{-\delta t} + \lambda_1\left[rx\left(1-\frac{x}{k}\right) - \alpha xy - \gamma_1 x^3 - \frac{q_1Ex}{b_1+E}\right] \\ + \lambda_2\left[-sy + \beta xy - \gamma_2 y^2 - \frac{q_2Ey}{b_2+E}\right], \end{aligned} \tag{4.7.2}$$

where $\lambda_1(t)$, $\lambda_2(t)$ are the adjoint variables. The adjoint equations are

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial x} \\ &= -\left[\frac{p_1q_1}{b_1+E}Ee^{-\delta t} + \lambda_1\left(r - \frac{2rx}{k} - \alpha y - 3\gamma_1 x^2 - \frac{q_1E}{b_1+E}\right) + \lambda_2\Big(\beta y\Big)\right]. \end{aligned} \tag{4.7.3}$$

and

$$\begin{aligned}\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial y}\\ &= -\left[\frac{p_2 q_2}{b_2+E}Ee^{-\delta t} + \lambda_1\Big(-\alpha x\Big) + \lambda_2\Big(-s+\beta x-2\gamma_2 y-\frac{q_2 E}{b_2+E}\Big)\right].\end{aligned} \tag{4.7.4}$$

Our aim is to find an optimal equilibrium solution of the problem so that we may take

$$E = \frac{\alpha y b_1 + \gamma_1 x^2 b_1 - r\left(1-\frac{x}{k}\right)b_1}{r\left(1-\frac{x}{k}\right) - \alpha y - \gamma_1 x^2 - q_1} = \frac{b_2\left(s+\gamma_2 y-\beta x\right)}{\beta x - q_2 - s - \gamma_2 y}. \tag{4.7.5}$$

By using equation (4.7.5), equations (4.7.3) and (4.7.4) become respectively,

$$\frac{d\lambda_1}{dt} = \lambda_1\Big(\frac{rx}{k}+2\gamma_1 x^2\Big) + \lambda_2\Big(-\beta y\Big) - \left[\frac{p_1 q_1}{b_1+E}Ee^{-\delta t}\right]. \tag{4.7.6}$$

and

$$\frac{d\lambda_2}{dt} = \lambda_1\Big(\alpha x\Big) + \lambda_2\Big(\gamma_2 y\Big) - \left[\frac{p_2 q_2}{b_2+E}Ee^{-\delta t}\right]. \tag{4.7.7}$$

Eliminating λ_2 from the equations (4.7.6) and (4.7.7), we have

$$\begin{aligned}&\frac{d^2\lambda_1}{dt^2} - \Big(\frac{rx}{k}+2\gamma_1 x^2+\gamma_2 y\Big)\frac{d\lambda_1}{dt}\\ &+\left[\Big(\frac{rx}{k}+2\gamma_1 x^2\Big)\gamma_2 y + \alpha\beta x y\right]\lambda_1 = M_1 e^{-\delta t},\end{aligned} \tag{4.7.8}$$

where $D = \frac{d}{dt}$ and $M_1 = \frac{p_1 q_1 E\delta}{b_1+E} + \frac{p_2 q_2 \beta y E}{b_2+E} + \frac{p_1 q_1 E \gamma_2 y}{b_1+E}$.

The auxiliary equation for (4.7.8) is the following:

$$\mu^2 - \Big(\frac{rx}{k}+2\gamma_1 x^2+\gamma_2 y\Big)\mu + \left[\Big(\frac{rx}{k}+2\gamma_1 x^2\Big)\gamma_2 y + \alpha\beta x y\right] = 0. \tag{4.7.9}$$

This is a qudratic equation in μ where sum of the roots$=\Big(\frac{rx}{k}+2\gamma_1 x^2 + \gamma_2 y\Big) > 0$ and

product of the roots=$\left[\left(\frac{rx}{k}+2\gamma_1 x^2\right)\gamma_2 y+\alpha\beta xy\right]>0$.
Therefore, the roots μ_1 and μ_2 of the above equation are either both real and positive or complex conjugates with positive parts. The complete solution for equation (4.7.9) is of the form

$$\lambda_1(t)=A_1e^{\mu_1(t)}+A_2e^{\mu_2(t)}+\left(\frac{M_1}{N}\right)e^{-\delta t}, \tag{4.7.10}$$

where $N=\left[\delta^2-\delta\left(\frac{rx}{K}+2\gamma_1x^2+\gamma_2y\right)+\left(\frac{rx}{k}+2\gamma_1x^2\right)\gamma_2y+\alpha\beta xy\right]\neq 0$. It is true that λ_1 is bounded if $A_1=A_2=0$. Then we have

$$\lambda_1(t)=\left(\frac{M_1}{N}\right)e^{-\delta t}=constant. \tag{4.7.11}$$

Similarly, we get

$$\lambda_2(t)=\left(\frac{M_2}{N}\right)e^{-\delta t}=constant, \tag{4.7.12}$$

where $M_2=\frac{p_2q_2E\delta}{b_2+E}+\left(\frac{rx}{K}+2\gamma_1x^2\right)\frac{p_2q_2E}{b_2+E}-\alpha x\frac{p_1q_1E}{b_1+E}$.
Hence, the shadow prices $\lambda_i(t)e^{\delta t}$, i=1, 2 remain constant over time in optimal equilibrium when they satisfy the transversality condition at ∞, i.e., when they remain bounded as $t\to\infty$. Again, the condition that the Hamiltonian H must be a maximum gives the condition

$$\begin{aligned}\frac{\partial H}{\partial E}&=e^{-\delta t}\left[\frac{p_1q_1b_1x}{(b_1+E)^2}+\frac{p_2q_2b_2y}{(b_2+E)^2}-C\right]+\lambda_1\left[\frac{-q_1b_1x}{(b_1+E)^2}\right]+\lambda_2\left[\frac{-q_2b_2y}{(b_2+E)^2}\right]\\&=0.\end{aligned} \tag{4.7.13}$$

The right hand side represents the discounted value of the future profit per unit effort at the steady state effort level.
Putting the values of $\lambda_1(t)$ and $\lambda_2(t)$ in (4.7.13) we get,

$$\left(p_1-\frac{M_1}{N}\right)\frac{q_1b_1x}{(b_1+E)^2}+\left(p_2-\frac{M_2}{N}\right)\frac{q_2b_2y}{(b_2+E)^2}=C \tag{4.7.14}$$

Equation (4.7.14) together with equation (4.7.5) gives the optimal equilibrium populations $x = x_\delta, y = y_\delta$.

When $\delta \to \infty$, we get the result as $\frac{p_1 q_1 b_1 x}{(b_1+E)^2} + \frac{p_2 q_2 b_2 y}{(b_2+E)^2} = C$ which implies $\frac{\partial \pi}{\partial E}(x_\infty, y_\infty, E) = 0$

Thus, the economic rent is completely dissipated and hence the fishery remains unexploited. When the discount rate is infinite, using equation (4.7.15) we get the folloiwing:

$$\begin{aligned}\frac{\partial \pi}{\partial E} &= \frac{p_1 q_1 b_1 x}{(b_1+E)^2} + \frac{p_2 q_2 b_2 y}{(b_2+E)^2} - C \\ &= \frac{M_1}{N}\frac{q_1 b_1 x}{(b_1+E)^2} + \frac{M_2}{N}\frac{q_2 b_2 y}{(b_2+E)^2}.\end{aligned} \tag{4.7.15}$$

Since M_1 and M_2 is of $o(\delta)$ where N is of $o(\delta^2)$, we see that $\frac{\partial \pi}{\partial E}$ is of $o(\delta^{-1})$.

Thus, $\frac{\partial \pi}{\partial E}$ is a decreasing function of $\delta (\geq 0)$.

We, therefore, conclude that $\delta = 0$ leads to maximization of $\frac{\partial \pi}{\partial E}$.

4.8 Numerical results

Maple-18 and MATLAB-2016a have been used to conduct numerical simulations. As long as harmful effects and harvesting are taken into consideration in the natural environment, it is exceedingly difficult to confirm the model results with realistic data. All of these findings are supported by numerical examples, some of which are displayed in the figures. In order to show the findings from the current model, we take a set of fictitious parameter values: r=15, k=1000, $\alpha = 0.02$, $q_1 = 1.6$, $E = 11$, $b_1 = 0.3$, $\gamma_1 = 0.000005$, s=0.6, $\beta = 0.005$, $\gamma_2 = 0.000008$, $q_2 = 1.01$, $b_2 = 0.2$.

Example 1:- We take the parameter values as r=15, k=1000, $\alpha = 0.02$, $q_1 = 1.6$, $E = 11$, $b_1 = 0.3$, $\gamma_1 = 0.000005$, s=0.6, $\beta = 0.005$, $\gamma_2 = 0.000008$, $q_2 = 1.01$, $b_2 = 0.2$ in appropriate units. For the above values, we find that

(i) $S_0(0,0)$ is unstable.

(ii) $S_1(722.27, 0)$ is unstable.

(iii) $S_2(320.15, 408.26)$ is both locally and globally asymtotically stable node.

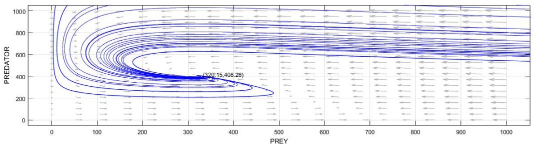

Figure 4.1: Phase plane trajectories of the prey -predator fishery with different initial values

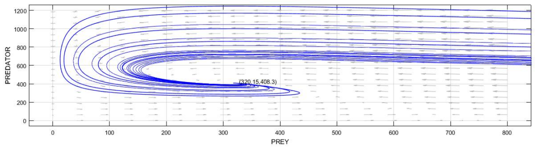

Figure 4.2: Phase plane trajectories of the prey -predator fishery with different initial values

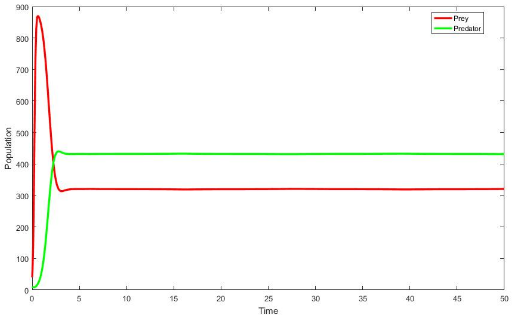

Figure 4.3: Solution curves of the prey-predator fishery for a period t=0 to 50 weeks

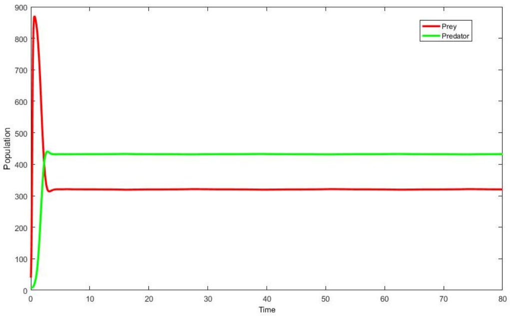

Figure 4.4: Solution curves of the prey-predator fishery for a period t=0 to 80 weeks

From Figure 4.1 and Figure 4.2, it is clear that for the above set of parameter values, the system possesses an interior equilibrium point E_*=(320.15,408.26). It is also observed that the system (4.2.2) is globally asymptotically stable around the coexistence equilibrium E_*. From Figure 4.3 and Figure 4.4, it is clear that the biomass density of prey species increases sharply with respect to time and then decreases and settles down at its equilibrium level. The biomass density of predator species increases with respect to time and then decreases slightly and settles down at its equilibrium level.

Example 2:- Taking the same values of the parameters together with $p_1 = 5$,$p_2 = 6$, C=50 and $E = \frac{\alpha y b_1+\gamma_1 x^2 b_1-r\left(1-\frac{x}{k}\right)b_1}{r\left(1-\frac{x}{k}\right)-\alpha y-\gamma_1 x^2-q_1}$ from Figure 4.5 we find that the bionomic equilibrium exists $(x_\infty, y_\infty) = (201.51, 1044.59)$. In the

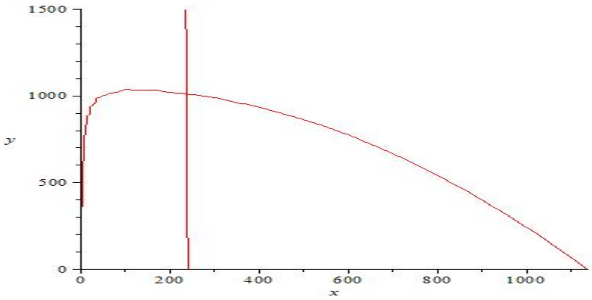

Figure 4.5: Bionomic equilibrium

absence of toxicity $(\gamma_1 = \gamma_2 = 0)$, $(x_\infty, y_\infty) = (158.21, 1078.42)$ was determined. If $\delta = 0.1$ and the same parameter values are used, the optimal equilibrium is found to be $(x_\delta, y_\delta) = (174.31, 1069.59)$. In the absence of toxicity $(\gamma_1 = \gamma_2 = 0)$, $(x_\delta, y_\delta) = (692.09, 1078.29)$ was determined.

From numerical example, we can deduce the following:

(i) The optimal equilibrium level for the prey species is $x_\delta = 174.31$, which is less than the steady state level $x^* = 320.15$. However, the optimal equilibrium population level for predator fish species $y_\delta = 1069.59$ is significantly greater than the corresponding steady state level $y^* = 408.26$. In the absence of toxicity, the optimal equilibrium level $(x_\delta, y_\delta) = (692.09, 1078.29)$ occurs at a higher population level for the predator species and a lower population level for the first species compared to $(x_\delta, y_\delta) = (174.31, 1069.59)$ in the presence of toxicity.

(ii) In the absence of toxicity, bionomic equilibrium exists at a higher population level $(x_\infty, y_\infty) = (318.21, 1078.42)$ than $(x_\infty, y_\infty) = (201.51, 1044.59)$ (in the presence of toxicity) for the first population, but at a marginally lower population density for the second species.

(iii) Both bionomic equilibrium and optimal equilibrium are critically dependent on the parameter values of $p_1, p_2, C, q_1, q_2, r, s, b_1, b_2, k$.

4.9 Conclusions

The impacts of toxicants released by various different resources in the aquatic environment on a model of a prey-predator fishery, where both species are harvested using a modified catch rate function, are covered in this work. In order to limit the growth of the prey species to a predetermined level in the absence of a predator, we adapt the Lotka-Volterra system by taking environmental conditions into account. For harvesting, we are using a modified catch rate function. Here, both local and global stabilities are seen.

Examined is the possibility of bionomic equilibrium. These are the biological equilibrium line and the zero profit line. The biochemical equilibrium is calculated for a given set of parameter values.

The best harvesting strategy is then described. Using Pontryagin's maximum principle, subject to the state equations and the control restrictions, the present value of the revenue is maximised. Here are illustrated the numerous instances of optimal equilibrium. When the shadow prices satisfy the transversality criterion, it is discovered that they remain constrained throughout time in optimal equilibrium. Additionally, it is demonstrated that a discount rate of zero leads to the maximisation of economic revenue and an infinite discount rate results in the total evaporation of economic rent. For a set of data, the ideal steady state solution is discovered.

Using the Matlab and Maple packages, solution curves corresponding to the steady state and bionomic equilibrium are displayed. We explore phase plane trajectories and growth curves. Finally, the topic of limit cycles is covered here.

5

OPTIMAL HARVESTING OF PREDATOR-PREY FISHERY MODEL WITH MODIFIED PREDATION TERM

"God used beautiful mathematics in creating the world." *- Paul Dirac*

5.1 Introduction

Management of exploitation of ecological as well as biological living resources like fish species, forests etc., has become a matter of serious concern of late all over the world, as these resources are substantial

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Part of this chapter is communicated for publication as a research article titled 'Optimal harvesting of predator-prey fishery model with modified predation term' authored by Sudipta Sarkar et al. in *Punjab University Journal of Mathematics*, (Communicated) sources of food, medicine and other necessary commodities useful in human life. Many researchers, scientists and mathematicians have deliberated their researches, studies and analyses on conservation of these ecological resources so that they can be prevented from over-exploitation and annihilation. Clark, 1976b, Clark, 1985 proposed the researches and analyses associated with the bio-economic exploitation of these renewable resources. Further, many researches and analyses have been focused on multi-species fisheries, though analyses on them are not sufficient. The formulation of a multi-species fishery resource model is not easy at all. Clark, 1976b has introduced a non-linear model based on bioeconomic non-selective harvesting of multi-species resource system. Later, Chaudhuri, 1986 Chaudhuri, 1988, Mesterton-Gibbons, 1987, Kar and Chaudhuri, 2002b, Kar and Chaudhuri, 2003a, Chaudhuri and Saha Ray, 1996, Kar, Pahari, and Chaudhuri, 2004, Das, Mukherjee, and Chaudhuri, 2009a have deliberated their researches and analyses based on non-linear multispecies fishery models based on non-selective harvesting to prevent abolition of the species. In this current article, a non-linear model of the exploitation of a predator-prey fishery resource system, subjected to non-selective or combined harvesting where both the species follow the logistic growth rate is proposed. Growth patterns widely differing from the logistic law of growth may be revealed by a case study of real marine fishery, due to the exceedingly complex nature of the marine ecosystem, consisting of several species that are interdependent on one another. Here, the functional response of the predator population to the prey population density is taken in such a manner that the individual response of each predator to the prey population density tends to a constant as the prey density increases. In this proposed non-linear model, we are considering non-selective harvesting of both the species by using modified catch rate function proposed by Sarkar, Sarkar, and Chaudhuri, 2020, Sarkar et al., 2022. Boundedness of the proposed model as well as equilibria of the dynamical system along with their conditions for positivity are thoroughly discussed. Local stability and global stability of the steady states are examined by Eigenvalue analysis and constructing a suitable Lyapunov function respectively. Actually, bionomic equilibria are the amalgamation of biological and economic equilibrium. The existence condition of a bionomic equilibrium is shown here. All the equilibria i.e., local, global and bionomic are illustrated numerically and graphically for verifying the theoretical results.

Maximization of the total discounted net revenue obtained from the exploited fishery resource is the objective of a fishery owners. The owner's attempt is always to set up the harvesting rate in such a way that total discounted net revenue obtained from the fishery resource is maximum depending on the population level. In our proposed model, optimal harvesting policy is discussed by using maximum principle proposed by Pontryagin et al., 1962. Lastly, some remarks are made on the nature of our proposed model.

5.2 Model formulation

We consider a non-linear system of equations:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k_1}\right)-\frac{mxy}{a+x}, \\ \frac{dy}{dt} &= sy\left(1-\frac{y}{k_2}\right)+\frac{m\beta xy}{a+x}, \end{aligned} \tag{5.2.1}$$

where x=x(t) and y=y(t) denote the biomass of prey and predator population respectively. k_1 and k_2 denote environmental carrying capacity of the prey and predator species respectively. Here, β is taken as the conversion factor. Throughout the model, we assume that $0 < \beta < 1$. Here, the parameter m denotes the maximal relative increase of predation and a is Michaelis-Menten constant. The parameters r and s are the growth rates of the prey and predator population respectively. Here, the catch rate function was considered as $h = qEx$. In our current model, it is taken in modified form as $h = \frac{qEx}{b+E}$ proposed by Sarkar, Sarkar, and Chaudhuri, 2020, Sarkar et al., 2022.

Incorporating the modified catch rate function, the dynamical system of non-linear predator-prey fishery model becomes:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k_1}\right) - \frac{mxy}{a+x} - \frac{q_1Ex}{b_1+E}, \\ \frac{dy}{dt} &= sy\left(1-\frac{y}{k_2}\right) + \frac{m\beta xy}{a+x} - \frac{q_2Ey}{b_2+E}. \end{aligned} \tag{5.2.2}$$

Here, E stands for the effort of harvesting. The constants q_1 and q_2 denote the catchability coefficients for the prey and predator species respectively. Here, b_1 and b_2 are constants.

Usually in the prey-predator model, the predation term can be taken in the form of mxy. In our current proposed model, mx is interpreted as the functional term of the predator species to the prey density in broader sense. Here, we are not considering the predation term in this form as it personifies the unrealistic fact that predation becomes to infinity as the prey density tends to infinity for a finite and definite predator population. Considering this matter, the predation term has been taken as $\frac{mxy}{a+x}$.

5.3 Equilibria existence criteria

The possible equilibria of the above mentioned dynamical system of non-linear differential equations are the following:

(i) $P_0(0,0)$,

(ii) $P_1(\bar{x},0)$,

where $\bar{x} = \frac{k_1}{r}\left(r - \frac{q_1E}{b_1+E}\right)$. $P_1(\bar{x},0)$ exists if $r > \frac{q_1E}{b_1+E}$.

(iii) $P_2(0,\bar{y})$,

where $\bar{y} = \frac{k_2}{s}\left(s - \frac{q_2E}{b_2+E}\right)$ which exists if $s > \frac{q_2E}{b_2+E}$.

(iv) $P_3(x^*,y^*)$,

where $\overline{y} = \frac{k_2}{s}\left(s + \frac{m\beta\overline{x}}{a+\overline{x}} - \frac{q_2E}{b_2+E}\right)$. Now, $\overline{y} > 0$ if $\left(s + \frac{m\beta\overline{x}}{a+\overline{x}} > \frac{q_2E}{b_2+E}\right)$ holds and $\overline{x}$ is the root of the following cubic equation:

$$A\overline{x}^3 + B\overline{x}^2 + C\overline{x} + D \quad = \quad 0, \tag{5.3.1}$$

where $A = rs\left(b_1+E\right)\left(b_2+E\right)$, $B = q_1Esk_1\left(b_2+E\right) - rs\left(b_1+E\right)\left(b_2+E\right)\left(k_1-2a\right)$, $C = mk_1k_2s\left(b_1+E\right)\left(b_2+E\right) + \beta m^2k_1k_2\left(b_1+E\right)\left(b_2+E\right) + 2aq_1Esk_1\left(b_2+E\right) - rs\left(b_1+E\right)\left(b_2+E\right)\left(2ak-a^2\right) - mk_1k_2q_2E\left(b_1+E\right)$ and

$D = mk_1k_2sa\left(b_1+E\right)\left(b_2+E\right) + q_1Esk_1\left(b_2+E\right)a^2 - rs\left(b_1+E\right)\left(b_2+E\right)k_1a^2 - mk_1k_2q_2Ea\left(b_1+E\right)$.

Here, $\lambda_1 > 0$. By Descarte's rule of sign, there exists at least one positive root in the above equation if the last term i.e., $D < 0$,

i.e., $rs > \frac{mk_2s}{a} + \frac{q_1Es}{(b_1+E)} - \frac{mk_2q_2E}{(b_2+E)}$.

So, the existence conditions for the interior equilibrium $P_3(\overline{x},\overline{y})$ are $\left(s + \frac{m\beta\overline{x}}{a+\overline{x}} > \frac{q_2E}{b_2+E}\right)$ and $\left(rs > \frac{mk_2s}{a} + \frac{q_1Es}{(b_1+E)} - \frac{mk_2q_2E}{(b_2+E)}\right)$.

5.4 Local stability analysis

Here, the local behaviour of the proposed dynamical non-linear system of equations around each of the equilibria has been analysed. The variational matrix of the above mentioned non-linear system of equations (5.2.2) is the following:

$$V(x,y)) \quad = \quad \begin{vmatrix} r - \frac{2rx}{k_1} - \frac{may}{(a+x)^2} - \frac{q_1E}{b_1+E} & \frac{-mx}{a+x} \\ \\ \frac{\beta my}{a+x} & s - \frac{2sy}{k_2} + \frac{\beta mx}{a+x} - \frac{q_2E}{b_2+E} \end{vmatrix}. \tag{5.4.1}$$

From the above equation (5.4.1) we get,

$$V(0,0)) = \begin{vmatrix} r - \frac{q_1E}{b_1+E} & 0 \\ \\ 0 & s - \frac{q_2E}{b_2+E} \end{vmatrix}.$$

Eigenvalues corresponding to $P_0(0,0)$ are

$r - \frac{q_1E}{b_1+E}$ and $s - \frac{q_2E}{b_2+E}$. Both eigenvalues are negative if the conditions $r > \frac{q_1E}{b_1+E}$ and $s < \frac{q_2E}{b_2+E}$ hold respectively for which $P_0(0,0)$ is a stable node.

Again,

$$V(\overline{x},0)) = \begin{vmatrix} r - \frac{2r\overline{x}}{k_1} - \frac{q_1E}{b_1+E} & -\frac{m\overline{x}}{a+x} \\ & \\ 0 & s + \frac{m\beta\overline{x}}{a+\overline{x}} - \frac{q_2E}{b_2+E} \end{vmatrix}.$$

Eigenvalues corresponding to the equilibrium $P_1(\overline{x},0)$ are $r - \frac{2r\overline{x}}{k_1} - \frac{q_1E}{b_1+E}$ and $s + \frac{m\beta\overline{x}}{a+\overline{x}} - \frac{q_2E}{b_2+E}$ which will be negative for the conditions $r < \frac{2r\overline{x}}{k_1} - \frac{q_1E}{b_1+E}$ and $s < -\frac{m\beta\overline{x}}{a+\overline{x}} + \frac{q_2E}{b_2+E}$ for being a stable node.

Now,

$$V(0,\overline{y})) = \begin{vmatrix} r - \frac{m\overline{y}}{a} - \frac{q_1E}{b_1+E} & 0 \\ & \\ 0 & s - \frac{2s\overline{y}}{k_2} - \frac{q_2E}{b_2+E} \end{vmatrix}.$$

Eigenvalues corresponding to the equilibrium point $P_2(0,\overline{y})$ are $\left(r - \frac{m\overline{y}}{a} - \frac{q_1E}{b_1+E}\right)$ and $\left(s - \frac{2s\overline{y}}{k_2} - \frac{q_2E}{b_2+E}\right)$. Hence, $P_2(0,\overline{y})$ is a stable node if the conditions $r < \frac{m\overline{y}}{a} - \frac{q_1E}{b_1+E}$ and $s < \frac{2s\overline{y}}{k_2} - \frac{q_2E}{b_2+E}$ hold simultaneously.

We can get the interior equilibrium $P_3(x^*,y^*)$ from the following system of equations:

$$\begin{aligned} rx^*\left(1 - \frac{x^*}{k_1}\right) - \frac{mx^*y^*}{a+x^*} - \frac{q_1Ex^*}{b_1+E} &= 0 \quad , \\ sy^*\left(1 - \frac{y^*}{k_2}\right) + \frac{m\beta x^*y^*}{a+x^*} - \frac{q_2Ey^*}{b_2+E} &= 0 \quad . \end{aligned} \tag{5.4.2}$$

Therefore, we have:

$$V(x^*,y^*)) = \begin{vmatrix} \frac{-rx^*}{k_1} + \frac{mx^*y^*}{(a+x^*)^2} & \frac{-mx^*}{a+x^*} \\ & \\ \frac{a\beta my^*}{(a+x^*)^2} & \frac{-sy^*}{k_2} \end{vmatrix}. \tag{5.4.3}$$

So, the characteristic equation of the variational matrix (5.4.3) is as follows:

$$\lambda^2 - \lambda\left(\frac{-rx^*}{k_1} + \frac{mx^*y^*}{(a+x^*)^2} - \frac{sy^*}{k_2}\right) + \left(\frac{mx^*}{a+x^*}\right)\left(\frac{a\beta my^*}{(a+x^*)^2}\right) + \left(\frac{-rx^*}{k_1} + \frac{mx^*y^*}{(a+x^*)^2}\right)\left(\frac{-sy^*}{k_2}\right) = 0. \qquad (5.4.4)$$

From the above equation, we get
$\lambda_1 + \lambda_2 = -x^*\left(\frac{r}{k_1} - \frac{my^*}{(a+x^*)^2}\right) - y^*\left(\frac{s}{k_2}\right)$ and
$\lambda_1\lambda_2 = \left(\frac{-rx^*}{k_1} + \frac{mx^*y^*}{(a+x^*)^2}\right)\left(\frac{-sy^*}{k_2}\right) + \frac{a\beta m^2x^*y^*}{(a+x^*)^3}$.
From the above equation, it is obvious that $P_3(x^*, y^*)$ is a stable node if $\lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 > 0$.

So, the conditions for the equilibrium point $P_3(x^*, y^*)$ to be a stable node are the following:
$\frac{rx^*}{k_1} + \frac{sy^*}{k_2} > \frac{mx^*y^*}{(a+x^*)^2}$ and
$\left(\frac{rx^*}{k_1} - \frac{mx^*y^*}{(a+x^*)^2}\right)\left(\frac{sy^*}{k_2}\right) + \frac{a\beta m^2x^*y^*}{(a+x^*)^3} > 0$, which must hold simultaneously.

5.5 Global stability analysis

Here, we have constructed a suitable Lyapunov function $G(x, y)$ to discuss the global stability of the system of equations (5.2.2).

$$G(x,y) = \left[(x - x^*) - x^*\log\frac{x}{x^*} + h(y - y^*) - y^*\log(\frac{y}{y^*})\right], \qquad (5.5.1)$$

where $h =$ a suitable constant. Now, we have

$$\frac{dG}{dt} = \frac{x - x^*}{x}\frac{dx}{dt} + h\frac{y - y^*}{y}\frac{dy}{dt} = (x - x^*)\left[r(1 - \frac{x}{k_1}) - \frac{my}{a+x} - \frac{q_1E}{b_1+E}\right] + h(y - y^*)\left[s\left(1 - \frac{y}{k_2}\right) + \frac{m\beta x}{a+x} - \frac{q_2E}{b_2+E}\right], \qquad (5.5.2)$$

where $h = \frac{1}{\beta}$.

From the equations (5.2.2) and (5.5.2) we get,

$$\frac{dG}{dt} = -\left\{ (x-x^*)^2 \left[\frac{r}{k_1} - \frac{my^*}{(a+x)(a+x^*)}\right] + h(y-y^*)^2\left[\frac{s}{k_2}\right] + (x-x^*)(y-y^*)\left[\frac{m}{a+x} - \frac{mah\beta}{(a+x)(a+x^*)}\right]\right\}$$

$$= -X^T P X, \tag{5.5.3}$$

where $X^T = \left[(x-x^*), (y-y^*)\right]$ and

$$P = \begin{bmatrix} \frac{r}{k_1} - \frac{my^*}{(a+x)(a+x^*)} & \frac{m}{2}\left(\frac{1}{a+x} - \frac{a\beta h}{(a+x)(a+x^*)}\right) \\ \frac{m}{2} & h\left(\frac{s}{k_2}\right) \end{bmatrix}. \tag{5.5.4}$$

Now, $\frac{dG}{dt} < 0$, if the following conditions hold:

(i)$\frac{r}{k_1} > \frac{my^*}{(a+x)(a+x^*)}$ and

(ii)$h\left(\frac{r}{k_1} - \frac{my^*}{(a+x)(a+x^*)}\right)\left(\frac{s}{k_2}\right) > \frac{m^2}{4}\left(\frac{1}{a+x} - \frac{a\beta h}{(a+x)(a+x^*)}\right)^2$.

So, for the above conditions $\frac{dG}{dt} < 0$ for which $P_3(x^*, y^*)$ is globally asymptotically stable.

5.6 Bionomic Equilibrium Analysis

From the amalgamation of the biological equilibrium and the economic equilibrium, we can get the bionomic equilibrium. Biological equilibrium is obtained from $\dot{x} = 0 = \dot{y}$. In the fishery literature, when the total revenue equals the total cost, then the bionomic equilibrium comes. For this type of case, the economic rent is completely dissipated . We assume here that:

$C =$ the fishing cost per unit effort,

$p_1 =$ the price per unit biomass of the prey, and

$p_2 =$ the price per unit biomass of the predator fish.

The economic rent (net revenue) is as follows:

$$\pi(x, y, E) = \left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right)E. \tag{5.6.1}$$

Here, $P_\infty(x_\infty, y_\infty)$ is the bionomic equilibrium. Now, x_∞, y_∞ can be obtained from the following equation:

$$\dot{x} = \dot{y} = \pi = 0. \tag{5.6.2}$$

Now, $\dot{x} = 0$ gives

$$E = \frac{b_1\left[\frac{my}{a+x} - r\left(1 - \frac{x}{k_1}\right)\right]}{r\left(1 - \frac{x}{k_1}\right) - \frac{my}{a+x} - q_1}. \tag{5.6.3}$$

So, E is positive when $\frac{my}{a+x} < r\left(1 - \frac{x}{k_1}\right) < \frac{my}{a+x} + q_1$ holds. Again, $\dot{y} = 0$ gives

$$E = \frac{b_2\left[s\left(1 - \frac{y}{k_2}\right) + \frac{mx\beta}{a+x}\right]}{\left[q_2 - s\left(1 - \frac{y}{k_2}\right) - \frac{mx\beta}{a+x}\right]}. \tag{5.6.4}$$

So, E is positive when $q_2 > s\left(1 - \frac{y}{k_2}\right) + \frac{mx\beta}{a+x}$ holds.

Hence, the non-trivial equilibrium exists at a point on the following curve:

$$\frac{b_1\left[\frac{my}{a+x} - r\left(1 - \frac{x}{k_1}\right)\right]}{r\left(1 - \frac{x}{k_1}\right) - \frac{my}{a+x} - q_1} = \frac{b_2\left[s\left(1 - \frac{y}{k_2}\right) + \frac{mx\beta}{a+x}\right]}{\left[q_2 - s\left(1 - \frac{y}{k_2}\right) - \frac{mx\beta}{a+x}\right]}. \tag{5.6.5}$$

where $0 \le x \le k_1$ and $0 \le y \le k_2$.

The bionomic equilibrium is evaluated by the equation (5.6.4) along with the following condition:

$$\pi(x, y, E) = \left(\frac{p_1 q_1 x}{b_1 + E} + \frac{p_2 q_2 y}{b_2 + E} - C\right)E. \tag{5.6.6}$$

5.7 Optimal harvesting policy

For a continuous time-stream of revenues, the current value $\mathfrak{I}$ is expressed by the following expression:

$$\mathfrak{I} = \int_0^{\infty} \pi(x, y, E, t) e^{-\delta t} dt. \tag{5.7.1}$$

Here, $\pi(x, y, E, t) = \left(\frac{p_1 q_1 x}{b_1+E} + \frac{p_2 q_2 y}{b_2+E} - C\right) E$ and the instantaneous annual rate of discount is denoted by δ. By incorporating the Maximum Principle proposed by Pontryagin et al., 1962, our purpose is to maximize $\mathfrak{I}$ subject to the state equation (5.2.2). The control variable $E(t)$ is subjected to the constraints $0 \le E(t) \le E_{max}$ where $V_t = [0, E_{max}]$ is the control set and E_{max} is a feasible upper limit for the harvesting effort. For our proposed model, the Hamiltonian is given by:

$$H = \left(\frac{p_1 q_1 x}{b_1+E} + \frac{p_2 q_2 y}{b_2+E} - C\right) E e^{-\delta t} + \lambda_1 \left[rs\left(1 - \frac{x}{k_1}\right) - \frac{mxy}{a+x} - \frac{q_1 E x}{b_1+E}\right] + \lambda_2 \left[sy\left(1 - \frac{y}{k_2}\right) + \frac{m\beta xy}{a+x} - \frac{q_2 E y}{b_2+E}\right]. \tag{5.7.2}$$

Here, the adjoint variables are defined by $\lambda_1(t)$, $\lambda_2(t)$. The adjoint equations for our proposed model are the following:

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial x_1} \\ &= -\left[\frac{p_1 q_1}{b_1+E} E e^{-\delta t} + \lambda_1 \left(r - \frac{2rx}{k_1} - \frac{may}{(a+x)^2} - \frac{q_1 E}{b_1+E}\right) + \lambda_2 \left(\frac{ma\beta y}{(a+x)^2}\right)\right]. \end{aligned} \tag{5.7.3}$$

and

$$\begin{aligned} \frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial y} \\ &= -\left[\frac{p_2 q_2}{b_2+E} E e^{-\delta t} + \lambda_1 \left(\frac{-mx}{a+x}\right) + \lambda_2 \left(s - \frac{2sy}{k_2} + \frac{m\beta x}{a+x} - \frac{q_2 E}{b_2+E}\right)\right]. \end{aligned} \tag{5.7.4}$$

Our goal is to detect an optimal equilibrium solution of the proposed problem so that we may take

$$E = \frac{b_1\left[\frac{my}{a+x} - r\left(1-\frac{x}{k_1}\right)\right]}{r\left(1-\frac{x}{k_1}\right) - \frac{my}{a+x} - q_1} = \frac{b_2\left[s\left(1-\frac{y}{k_2}\right) + \frac{mx\beta}{a+x}\right]}{\left[q_2 - s\left(1-\frac{y}{k_2}\right) - \frac{mx\beta}{a+x}\right]}. \tag{5.7.5}$$

By using equation (5.7.5), equations (5.7.3) and (5.7.4) become respectively,

$$\frac{d\lambda_1}{dt} = \left[\left(-\frac{p_1c_1}{b_1+E}\right)Ee^{-\delta t} + \lambda_1\left(\frac{rx}{k_1} - \frac{mxy}{(a+x)^2}\right) + \lambda_2\left(\frac{ma\beta y}{(a+x)^2}\right)\right]. \tag{5.7.6}$$

and

$$\frac{d\lambda_2}{dt} = \left[\left(-\frac{p_2c_2}{b_2+E}\right)Ee^{-\delta t} + \lambda_1\left(\frac{mx}{a+x}\right) + \lambda_2\left(\frac{sy}{k_2}\right)\right]. \tag{5.7.7}$$

Eliminating λ_2 from the equations (5.7.6) and (5.7.7), we have

$$\begin{aligned}
&\frac{d^2\lambda_1}{dt^2} + \left(\frac{rx}{k_1} + \frac{sy}{k_2} - \frac{mxy}{(a+x)^2}\right)\frac{d\lambda_1}{dt} + \left[\left(\frac{rx}{k_1} - \frac{mxy}{(a+x)^2}\right)\left(\frac{sy}{k_2}\right) + \frac{m^2a\beta xy}{(a+x)^3}\right]\lambda_1 \\
&= \frac{\delta p_1c_1E}{b_1+E}e^{-\delta t} + \frac{p_1c_1Esy}{(b_1+E)\,k_2}e^{-\delta t} + \left(\frac{ma\beta y}{(a+x)^2}\right)\left(\frac{p_2c_2Ee^{-\delta t}}{(b_2+E)}\right) \\
&= P_1e^{-\delta t}.
\end{aligned} \tag{5.7.8}$$

Here, $P_1 = \frac{\delta p_1c_1E}{b_1+E}e^{-\delta t} + \frac{p_1c_1Esy}{(b_1+E)k_2}e^{-\delta t} + \left(\frac{ma\beta y}{(a+x)^2}\right)\left(\frac{p_2c_2Ee^{-\delta t}}{(b_2+E)}\right)$. The auxiliary equation for (5.7.8) is the following:

$$\mu^2 + \left(\frac{rx}{k_1} + \frac{sy}{k_2} - \frac{mxy}{(a+x)^2}\right)\mu + \left[\left(\frac{rx}{k_1} - \frac{mxy}{(a+x)^2}\right)\left(\frac{sy}{k_2}\right) + \frac{m^2a\beta xy}{(a+x)^3}\right] = 0. \tag{5.7.9}$$

Here, the sum of the roots=$\left(\frac{rx}{k_1} + \frac{sy}{k_2} - \frac{mxy}{(a+x)^2}\right) > 0$

and the product of the roots=$\left[\left(\frac{rx}{k_1} - \frac{mxy}{(a+x)^2}\right)\left(\frac{sy}{k_2}\right) + \frac{m^2a\beta xy}{(a+x)^3}\right] > 0$. Hence, the roots of the above equation are either both real and positive or complex conjugates with positive parts. For equation (5.7.9), the total solution is of the form:

$$\lambda_1(t) = A_1 e^{\mu_1(t)} + A_2 e^{\mu_2(t)} + \left(\frac{P_1}{Q}\right) e^{-\delta t}. \tag{5.7.10}$$

Here, A_1, A_2 are arbitrary constants and

$$Q = \delta^2 - \left[\frac{rx}{k_1} + \frac{sy}{k_2} - \frac{mxy}{(a+x)^2}\right]\delta + \left[\frac{sy}{k_2}\right]\left[\frac{rx}{k_1} - \frac{mxy}{(a+x)^2}\right] + \frac{m^2\beta xy}{(a+x)^3} \neq 0. \tag{5.7.11}$$

Thus, $\lambda_1(t)$ remains bounded as $t \to \infty$ if and only if $A_1 = A_2 = 0$ and then

$$\lambda_1(t) = \left(\frac{P_1}{Q}\right) e^{-\delta t} = constant. \tag{5.7.12}$$

Again,

$$\begin{aligned}
&\frac{d^2\lambda_2}{dt^2} - \left(\frac{rx}{k_1} + \frac{sy}{k_2} - \frac{mxy}{(a+x)^2}\right)\frac{d\lambda_2}{dt} + \left[\left(\frac{rx}{k_1} - \frac{mxy}{(a+x)^2}\right)\left(\frac{sy}{k_2}\right)\right]\lambda_2 \\
&= \frac{-p_1c_1Ee^{-\delta t}}{b_1+E}\left(\frac{mx}{a+x}\right) + \frac{p_2c_2E\delta e^{-\delta t}}{b_2+E} + \left(\frac{rx}{k_1}\right)\left(\frac{p_2c_2Ee^{-\delta t}}{b_2+E}\right) - \left(\frac{mxy}{(a+x)^2}\right)\left(\frac{p_2c_2Ee^{-\delta t}}{b_2+E}\right) \\
&= P_2 e^{-\delta t}.
\end{aligned} \tag{5.7.13}$$

So,

$$\lambda_2(t) = \left(\frac{P_2}{Q}\right) e^{-\delta t} = constant. \tag{5.7.14}$$

Here, $P_2 = \frac{-p_1c_1Ee^{-\delta t}}{b_1+E}\left(\frac{mx}{a+x}\right) + \frac{p_2c_2E\delta e^{-\delta t}}{b_2+E} + \left(\frac{rx}{k_1}\right)\left(\frac{p_2c_2Ee^{-\delta t}}{b_2+E}\right) - \left(\frac{mxy}{(a+x)^2}\right)\left(\frac{p_2c_2Ee^{-\delta t}}{b_2+E}\right)$. So, when the shadow prices $\lambda_i(t)e^{\delta t}$, i=1, 2 satisfy the transversality condition at ∞, i.e., when they remain bounded as $t \to \infty$, they achieve constant over time in optimal equilibrium. So, the Hamiltonian H is maximum for the following condition:

$$\begin{aligned}
\frac{\partial H}{\partial E} &= e^{-\delta t}\left[\frac{p_1c_1b_1x}{(b_1+E)^2} + \frac{p_2c_2b_2y}{(b_2+E)^2} - C\right] - \lambda_1\frac{b_1q_1x}{(b_1+E)^2} - \lambda_2\frac{b_2q_2y}{(b_2+E)^2} \\
&= 0.
\end{aligned} \tag{5.7.15}$$

This implies that

$$\lambda_1 \frac{b_1 q_1 x}{(b_1+E)^2} + \lambda_2 \frac{b_2 q_2 y}{(b_2+E)^2} \\ = e^{-\delta t} \frac{\partial H}{\partial E}. \tag{5.7.16}$$

The discounted value of the future profit line per unit effort at the steady state effort level is expressed in the right hand side of the above equation. Substituting the values of $\lambda_1(t)$ and $\lambda_2(t)$ in the equation (5.7.14), we obtain

$$\left\{p_1 - \frac{P_1}{Q}\right\} \frac{q_1 b_1 x}{(b_1+E)^2} + \left\{p_2 - \frac{P_2}{Q}\right\} \frac{q_2 b_2 x}{(b_2+E)^2} = C. \tag{5.7.17}$$

So, the optimal equilibrium solution $x_1 = x_{1\delta}, x_2 = x_{2\delta}$ is found out from equation (5.7.15) along with the equation (5.6.1). When $\delta \to \infty$, equation (5.7.17) becomes $\frac{p_1 q_1 b_1 x}{(b_1+E)^2} + \frac{p_2 q_2 b_2 y}{(b_2+E)^2} = C$ which implies $\frac{\partial \pi}{\partial E}(x_\infty, y_\infty, E) = 0$. So, at the infinite discount rate the value of the future profit line vanishes. Using equation (5.7.17), we get

$$\begin{aligned} \frac{\partial \pi}{\partial E} &= \frac{p_1 q_1 b_1 x}{(b_1+E)^2} + \frac{p_2 q_2 b_2 y}{(b_2+E)^2} - C \\ &= \frac{P_1}{Q} \frac{q_1 b_1 x}{(b_1+E)^2} + \frac{P_2}{Q} \frac{q_2 b_2 y}{(b_2+E)^2}. \end{aligned} \tag{5.7.18}$$

We find that $\frac{\partial \pi}{\partial E}$ is of $o(\delta^{-1})$ since P_1 and P_2 are of $o(\delta)$ and Q is of $o(\delta^2)$. So, we can conclude that $\frac{\partial \pi}{\partial E}$ is a decreasing function of $\delta(\geq 0)$. Hence, we conclude that the maximization of $\frac{\partial \pi}{\partial E}$ is led by $\delta = 0$.

5.8 Numerical results

This section of the current paper exhibits our attempts to validate the mathematical results obtained from the various sections of this paper. Numerical simulations have been done by using MATLAB-2016a.

Example 1:- We first consider the case using the parameter values as r=3.5, $k_1 = 230$, m=0.4, a=1.5, $q_1 = 0.7$, E=10, $b_1 = 0.5$, s=3.0, $k_2 = 275$, $\beta = 0.001$, $q_2 = 0.8$, $b_2 = 0.6$. For the above set of parameter values, we get that

(i) $P_0(0, 0)$ is unstable,

(ii)$P_1(186.1905, 0)$ is unstable,

(iii)$P_2(0, 205.8176)$ is stable,

(iv)$P_3(150.6197, 205.8539)$ is stable.

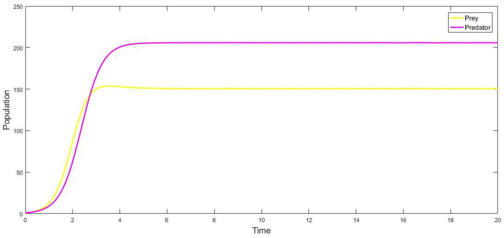

Figure 5.1: Solution curves of the predator-prey fishery with same initial values

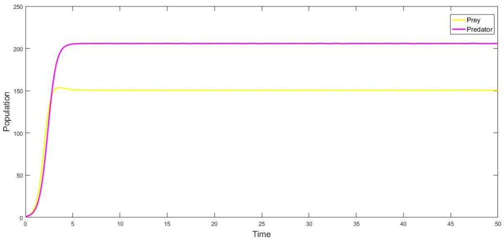

Figure 5.2: Solution curves of the predator-prey fishery with same initial values

From Figure 5.1 and Figure 5.2, it has been shown that the solution curves of the both populations initially increase and then finally, settle down. In this proposed model, it has been shown that the value of the biomass of the predator population is marginally greater than the prey population. The corresponding phase portraits of the proposed model are shown in the Figure 5.3 and Figure 5.4.

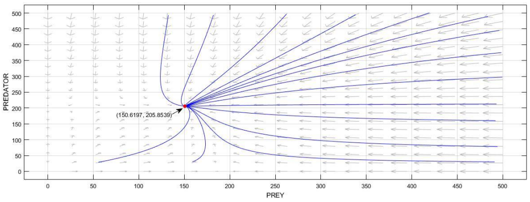

Figure 5.3: Phase plane trajectories of the predator-prey fishery for different initial values

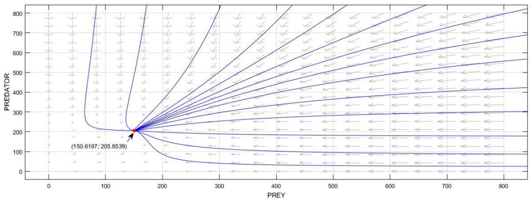

Figure 5.4: Phase plane trajectories of the predator-prey fishery for different initial values

5.9 Discussions

This proposed model takes into account a non-linear mathematical model of a predator-prey fishery with a modified catch-rate-function and non-selective harvesting. In this case, the functional response to the prey population has been chosen so that it approaches a constant as the prey population grows.

In this paper, the local and global stabilities of the proposed dynamical system are examined under non-selective harvesting. Here, we have derived the conditions under which all the possible steady states P_0, P_1, P_2 and P_3 are stable nodes. Numerically, we have found only one trivial state $P_2(0, 205.8176)$ and one non-trivial steady state $P_3(150.6197, 205.8539)$ which is locally asymptotically stable. Here, $P_3(150.6197, 205.8539)$ is also examined as globally stable. All steady states have been shown in Figure 5.3 and Figure 5.4.

Next, the existence criteria of bionomic (biological as well as economic) equilibria of the proposed model have been found out. Actually, bionomic equilibria are the points of intersection of the zero profit line and biological equilibrium line.

Finally, we have completed our analysis of the optimal harvesting policy according to maximal principle proposed by Pontryagin et al., 1962. With the control restrictions and state equations in mind, we may maximise the present value of a steady stream of income over time. When the ideal transversality requirement is met, shadow prices are stable throughout time. It has been found that the zero discount rate maximises economic revenues while the infinite discount rate completely dissipates economic rent.

6

OPTIMAL HARVESTING OF A PREDATOR-PREY SYSTEM WITH ADDITIONAL FOOD FOR PREDATOR IN FUZZY ENVIRONMENT

"With me everything turns into mathematics." - *Rene Descartes*

Part of this chapter is communicated for publication as a research article titled 'Optimal harvesting of predator-prey system with additional food for predator in fuzzy environment' authored by Sudipta Sarkar et al. in Springer Proceedings of the OLAB 2022, (Communicated).

6.1 Introduction

In recent years, the pattern of bioeconomy on the mishandling and overuse of biological supplies like fisheries and forestry has become more popular due to the overexploitation of resources to suit the needs of the global population. The pattern of how the prey grows logistically in the absence of the predator has always piqued the interest of scientists worldwide by Holt, 1984; Lafferty and Kuris, 1996; Powell, 2003; Rauwald and Ives, 2001; Ludwig et al., 2018; Sullivan and Klenner, 1993; Wäckers, Fadamiro, et al., 2005; Wade et al., 2008. Many researchers have considered Holling type-II predator functional response on the prey population which was further modified with the inclusion of an extra and added food term delivered to the predator species by Das and Samanta, 2018. Additionally, the predator-prey reserve model proposed by Zhang, Sun, and Yang, 2007; Louartassi et al., 2019 is viewed as a reaction model which is dyaniamic and represents the interaction between the effort put forth in the open-access model and the perceived rent. In an open-access dynamic prey-predator model the level of harvesting effort exerted to the prey and the predator species increases or decreases depending on the perceived rent. Assuming that the overall rate of capital investment applied to the predator-prey resource is proportionate to the perceived rent, a regulatory body manages the misuse and improper handling of the predator-prey reserve by imposing a tax or charge on each biomass unit of landed species. The novelty of the chapter lies in building a fuzzy inflated optimal control problem with the control variables being the harvesting effort and discount so that the fuzzy objective functional is maximized deliberated by Pal, Mahato, and Samanta, 2020; Ömer and Ömer, 2012.

For the present manuscript, one of the aims is to optimize the taxation policy so that best possible benefit is obtained when the prey and the predator population are harvested while preventing their extinction.

Although several similar types of models have recently surfaced in literature, the study of a predator-prey model that follows Rosenzweig-MacArthur pattern is the key distinguishing factor in the suggested pattern. Here, the prey increases in the absence of predators logistically. The discount rate considered is fuzzy in nature. Holling type-II functional response on prey population has been considered and both the species are harvested accordingly. Additionally, the impact of the prey population and additional food on the functional response of the predator is included. The value of money in terms of time which is represented by inflation along with discount rates are undetermined and hence fuzzy in character. While calculating the rate of discount which is annual and instantaneous, the difference between inflation and discount rates is considered, and since both the parameters are fuzzy in nature, the result is also a fuzzy quantity. The model becomes more environmentally accurate than other ones already in use as a result of this additional effect.

The construction and model assumptions are thoroughly addressed in this study. The dynamical system's local and global stabilities are then assessed for the mode. δ, which signifies the instantaneous yearly rate of discount is the difference between two arbitrary quantities which symbolise the time worth of money, the quantities being discount rate and inflation rate. Due to the trapezoidal fuzzy nature by Grzegorzewski and Pasternak-Winiarska, 2014; Mateos and Jiménez, 2009, which is intuitive, user-friendly, computationally simple, and helpful in fostering representation, we consider inflation along with discount rates to be fuzzy in nature proposed by Pal, Mahato, and Samanta, 2020. The optimal control issue is then presented for most return on investment and solved for best species harvesting utilising maximal principle of Pontryagin. The model is finally validated with numerical examples.

6.2 Preliminaries

Zadeh, 1965 first introduced the concept of fuzzy sets to standardized the concept of classification in class membership with human knowledge representations. Fuzzy set is used to describe and solve the complex systems where data are often incomplete or imprecise suggested by Klir and Yuan, 1996; Tudu, Mondal, and Alam, 2020.

Fuzzy Set

Let X be a given set and let $A \subset X$ be a set. A fuzzy set A^* in X is characterized by its membership function $\mu_{A^*} : X \to [0,1]$ such that $0 <= \mu_{A^*} <= 1$. wsz

Fuzzy Number

A fuzzy set $\tilde{A}$ on $\mathfrak{R}$ is a fuzzy number if

(i) $\tilde{A}$ is normal i.e. $x_0 \in \mathfrak{R}$ exists such that.

(ii) $\forall \alpha \in (0,1], \tilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\}$ is a closed interval.

Trapezoidal Fuzzy Number (TrFN) Kaufmann and Gupta, 1988

A Trapezoidal fuzzy number $\widetilde{A}$ is specified by (a, b, c, d) (cf. Fig-6.1) and is defined by its continuous membership function $\mu_{\widetilde{A}}(x)$ as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{for } a \leq x \leq b, \\ 1 & \text{for } b \leq x \leq c, \\ \dfrac{d-x}{d-c} & \text{for } c \leq x \leq d, \\ 0 & \text{otherwise .} \end{cases}$$

zadeh1972fuzzy Let $\tilde{A}$ be a fuzzy number. The α-cut (cf. Fig-6.2) of $\tilde{A}$ in X is defined by the following set :

$$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha, \alpha \in [0,1]\}$$

$A_\alpha = [A_L(\alpha), A_R(\alpha)]$ is a non-empty bounded closed interval contained in

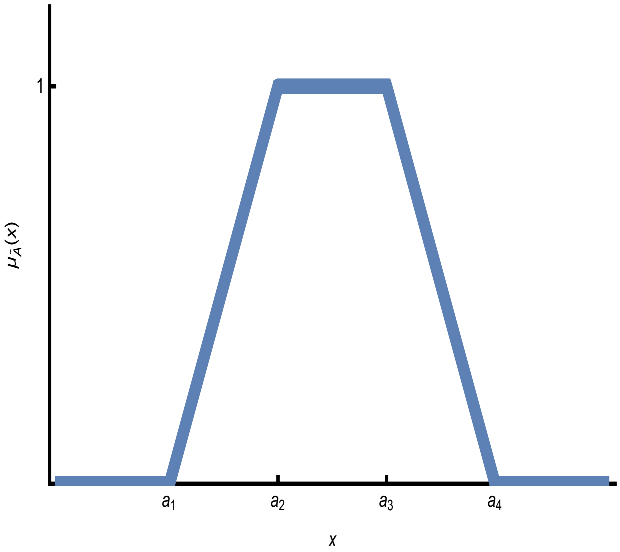

Figure 6.1: Trapezoidal Fuzzy Number (TrFN)

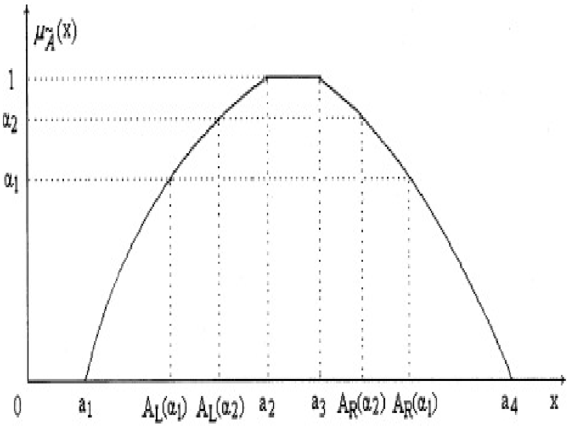

Figure 6.2: α-cut of a generalized fuzzy number

X, where $A_L(\alpha)$ and $A_R(\alpha)$ are the lower and upper bounds of the fuzzy set $\tilde{A}$.

Note: α-cut of TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$ is defined as follows:

$$A_\alpha = \{x : \mu_{\tilde{A}(x)} \geq \alpha\} = [A_L(\alpha), A_R(\alpha)] = [a_1 + \alpha(a_2 - a_1), a_4 - \alpha(a_4 - a_3)]$$

6.3 Formulation of the problem

Assuming the increase of prey logistically in the absence of predators, the Holling type II or Rosenzweig-MacArthur model is taken into account. Furthermore, the aspect of the predator's functional response to the prey population along with additional food are taken into consideration proposed by Rauwald and Ives, 2001. When the predator is given more food, a coupled differential equation reflecting the dynamics of prey and predator is created which is as follows:

$$\frac{dx}{dt} = rx(1 - \frac{x}{k}) - \frac{axy}{d + x + \beta\alpha A},$$
$$\frac{dy}{dt} = \frac{b(x + \alpha A)y}{d + x + \beta\alpha A} - ly.$$

An amount of tax $\tau > 0$ is charged which is calculated as per unit biomass of the population and a regulatory body manages it's exploitation. The subsidy which is provided could be translated as a value of τ which is negative. Now

$$E\{q_1(p_1 - \tau)x + q_2(p_2 - \tau)y - C\},$$

represents the perceived rent which implies the net economic revenue.

The constant price per unit biomass of the first species and second species are are p_1 and p_2 respectively. The constant cost per unit of harvesting effort is C. The catchability coefficients are denoted by q_1 and q_2 which are of the prey and predator population in order. E is a dynamic or time dependent variable which depicts the effort devoted to the combined harvesting of both species. The catch per unit effort hypothesis is adhered, which states that for the purpose of harvesting the predator population, catch per unit effort is proportionate to stock level. The tax τ is considered as the control technique which is imposed by the regulatory agency for the harvesting of the prey species, i.e., τ is the tax per unit of the biomass. The perceived rent and the gross rate of capital investment are taken to be proportional suggested by Pradhan and Chaudhuri, 1999a; Chakraborty, Chakraborty, and Kar, 2011. With the above mentioned assumptions we arrive at:

$$I = \gamma\{q_1(p_1 - \tau)x + q_2(p_2 - \tau)y - C\}E.$$

$I(t)$ being the gross investment rate that is stated at a time t in physical terms. The stiffness parameter, or constant γ, gauges how strongly an effort will respond to a perceived rent. According to the aforementioned equation, the maximum investment rate is equal to the perceived rent (for $\gamma = 1$) at any given time. Only in cases of negative perceived rent and perfectly non-malleable capital stocks, the case $\gamma = 0$ can be applied. Sanctioning a continual disinvestment of the capital assets, provided the policy operates at a loss and making the capital stocks completely malleable implying the permission of disinvestment of capital, the individual owner of the business earns a benefit. Since negative investment means disinvestment, we have $I < 0$ and $\gamma > 0$. In view of the above context we get,

$$\frac{dE}{dt} = [\gamma\{q_1(p_1 - \tau)x + q_2(p_2 - \tau)y - C\} - \mu]E.$$

The constant rate of depreciation of the capital is denoted by μ.

Thus, we arrive at the following system of equations:

$$\begin{aligned} \frac{dx}{dt} &= rx(1 - \frac{x}{k}) - \frac{axy}{d + x + \beta\alpha A} - q_1 Ex, \\ \frac{dy}{dt} &= \frac{b(x + \alpha A)y}{d + x + \beta\alpha A} - ly - q_2 Ey, \\ \frac{dE}{dt} &= [\gamma\{q_1(p_1 - \tau)x + q_2(p_2 - \tau)y - C\} - \mu]E, \end{aligned} \tag{6.3.1}$$

with initial conditions

$$x(t) \geq 0, y(t) \geq 0, E \geq 0. \tag{6.3.2}$$

The above mentioned parameters $r, k, a, d, \alpha, \beta, A, q_1, b, l, q_2, \gamma, p_1, p_2, C, \tau, \mu$ are all positive constants. $x(t)$, $y(t)$ are the density of the prey and predator respectively at time t. $E(t)$ denotes the effort devoted to the harvesting of both the species. The term $\frac{axy}{d + x + \beta\alpha A}$ represents the predator's functional response where the maximum rate of predation is a. In the absence of additional food, the predators' half-saturation value is d and the quality of the additional food in relation to the prey is β. αA is the effectual food level, r is the intrinsic growth rate of the prey species and l is the death rate of the predator species.

6.4 Existence of solution

Theorem 6.4.1 With initial condition (6.3.2), every element of the solution to equation (6.3.1) is positive and bounded.

Proof. We obtain from the first equation of (6.3.1)

$$x(t) = x(0)\exp[\textstyle\int_0^t\{r(1 - \frac{x(s)}{k}) - \frac{ay(s)}{d + x(s) + \beta\alpha A} - q_1E\}ds] \geq 0,$$

$$y(t) = y(0)\exp[\textstyle\int_0^t\{\frac{b(x(s) + \alpha A)}{d + x(s) + \beta\alpha A} - l - q_2E\}] \geq 0,$$

$$E(t) = z(0)\exp[[\textstyle\int_0^t\{\gamma\{q_1(p_1 - \tau)x + q_2(p_2 - \tau) - C\} - \mu\}] \geq 0.$$

Therefore, $x(t)$, $y(t)$, $E(t) \geq 0$ for all $t \geq 0$.

6.5 Stability and existence of equilibrium points

This section will analyze the system (6.3.1) for the existence and stability at each of it's equilibrium point. The equilibrium points of (6.3.1) being:

1. Trivial equilibrium: $P_0(0, 0, 0)$.
2. Axial equilibrium: $P_1\Big(x, 0, 0\Big)$ where $x = k$.
3. Planar equilibrium

   (i) $P_2(\bar{x}, \bar{y}, 0)$ where $\bar{x} = \frac{l(d + \beta\alpha A) - b\alpha A}{b - l}$ and $\bar{y} = \frac{r}{ka}(k - \bar{x})(d + \bar{x} + \beta\alpha A)$
   $\implies \bar{y} = \frac{r}{(b - l)^2ka}[kb - kl - ld - l\beta\alpha A - b\alpha A][(b - l)(d + \beta\alpha A) + ld + l\beta\alpha A - b\alpha A]$ which exists if $\bar{x} < k$, $l(d + \beta\alpha A) > b\alpha A$ and $b > l$.

   (ii) $P_3(\hat{x}, 0, \hat{E})$ where $\hat{x} = \frac{\mu + \gamma C}{\gamma q_1(p_1 - \tau)}$ and $\hat{E} = \frac{r}{q_1k}[k - \hat{x}]$, i.e., $\hat{E} = \frac{r}{q_1k}[k - \frac{\mu + \gamma C}{\gamma q_1(p_1 - \tau)}]$ which exists if $p_1 > \tau$ and $\hat{x} < k$.

   (iii) $P_4(0, y', E')$ where $y' = \frac{\mu + C\gamma}{\gamma q_2(p_2 - \tau)}$ and $E' = \frac{b\alpha A}{d + \beta\alpha A} - l$.
   The equilibrium point exists provided $p_2 > \tau$ and $b\alpha A > ld + l\beta\alpha A$.
4. Interior equilibrium: $P_5(x^*, y^*, E^*)$ with $y^* = \frac{(d + x^* + \beta\alpha A)}{a}[r(1 - \frac{x^*}{k}) - q_1E^*]$
   which exists if $r > \frac{rx^*}{k} + q_1E^*$

and $E^* = \dfrac{b(x^*+\alpha A)}{q_2(d+x^*+\beta\alpha A)} - \dfrac{l}{q_2}$ which exists if $b(x^*+\alpha A) > l(d+x^*+\beta\alpha A)$.

Further,

$x^* = \dfrac{-B+\sqrt{B^2-4AC}}{2A}$ where $A = q_2r, B = q_2rk - q_2rd - \beta\alpha A - q_1bk - \dfrac{q_2ak}{\gamma q_2(p_2-\tau)}\gamma q_1(p_1-\tau) + lq_1k$ and $C = q_2rkd + q_2rk\beta\alpha A - q_1bk\alpha A + \dfrac{q_2ak}{\gamma q_2(p_2-\tau)}[\gamma C+\mu] + lq_1(kd+\beta\alpha A)$ which exists if $-B+\sqrt{B^2-4AC} > 0$.

6.5.1 Local Stability Analysis

6.5.1.1 Stability of the trivial euilibrium P_0

The system (6.3.1)'s variational matrix changes as follows at the trivial equilibrium point, or $P_0(0,0,0)$

$$B(P_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & \dfrac{b\alpha A}{d+\beta\alpha A} - l & 0 \\ 0 & 0 & -\gamma C - \mu \end{bmatrix}$$

Therefore, the eigenvalues of the characteristic equation of $V(P_0)$ are $\lambda_1 = r, \lambda_2 = \dfrac{b\alpha A}{d+\beta\alpha A} - l, \lambda_3 = -\gamma C - \mu$. Since, λ_1 is positive, P_0 unstable. Therefore, for the system (6.3.1), we can say that the trivial equilibrium is unstable although it exists.

6.5.1.2 Stability of P_1, the Axial equilibrium

The variational matrix of the system (6.3.1) at $P_1(k,0,0)$ is given by the following:

$$B(P_1) = \begin{bmatrix} -r & -\dfrac{ak}{d+k+\beta\alpha A} & -q_1k \\ 0 & \dfrac{b(k+\alpha A)}{d+k+\beta\alpha A} - l & 0 \\ 0 & 0 & \gamma\{q_1(p_1-\tau)k - C\} - \mu \end{bmatrix}$$

So, the eigenvalues of the characteristic equation of $V(P_1)$ are $\lambda_1 = -r, \lambda_2 = \dfrac{b\{k+\alpha A\}}{d+k+\beta\alpha A} - l$ and $\lambda_3 = \gamma\{q_1(p_1-\tau)k - C\} - \mu$. Now, λ_2 is negative if $\dfrac{b(k+\alpha A)}{d+k+\beta\alpha A} < l$ and λ_3 is negative if $q_1(p_1-\tau) < \dfrac{\mu}{\gamma} + C$.
Thus we can say that the axial equilibrium of the system (6.3.1) is stable if $\dfrac{b(k+\alpha A)}{d+k+\beta\alpha A} < l$ and $q_1(p_1-\tau)k < \dfrac{\mu}{\gamma} + C$.

6.5.1.3 Stability of planar equilibrium points P_2, P_3 and P_4

(i) The variational matrix of the system(6.3.1) at $P_2(\bar{x}, \bar{y}, 0)$ is given by

$$B(P_2) = \begin{bmatrix} r - \dfrac{2r\bar{x}}{k} - \dfrac{(d+\beta\alpha A)a\bar{y}}{(d+\bar{x}+\beta\alpha A)^2} & -\dfrac{a\bar{x}}{d+\bar{x}+\beta\alpha A} & -q_1\bar{x} \\ \dfrac{(d+\beta\alpha A-\alpha A)b\bar{y}}{(d+\bar{x}+\beta\alpha A)^2} & \dfrac{b(\bar{x}+\alpha A)}{d+\bar{x}+\beta\alpha A} - l & -q_2\bar{y} \\ 0 & 0 & \gamma\{q_1(p_1-\tau)\bar{x} + q_2(p_2-\tau)\bar{y} - C\} - \mu \end{bmatrix}$$

One of the eigenvalues of the characteristic equation of $V(P_2)$ is $\gamma\{q_1(p_1-\tau)\bar{x} + q_2(p_2-\tau)\bar{y} - C\} - \mu$. and the other two eigenvalues can be found on solving $\lambda^2 + B_1\lambda + B_2 = 0$ where $B_1 = -\Big[\dfrac{b(\bar{x}+\alpha A)}{d+\bar{x}+\beta\alpha A} - l + r - \dfrac{2r\bar{x}}{k} - \dfrac{(d+\beta\alpha A)a\bar{y}}{(d+\bar{x}+\beta\alpha A)^2}\Big]$.

and $B_2 = \Big[r - \dfrac{2r\bar{x}}{k} - \dfrac{(d+\beta\alpha A)a\bar{y}}{(d+\bar{x}+\beta\alpha A)^2}\Big] * \Big[\dfrac{b(\bar{x}+\alpha A)}{d+\bar{x}+\beta\alpha A} - l\Big] + \Big[\dfrac{a\bar{x}}{d+\bar{x}+\beta\alpha A} * \dfrac{(d+\beta\alpha A-\alpha A)b\bar{y}}{(d+\bar{x}+\beta\alpha A)^2}\Big]$ which has negative real parts if

$$B_1 > 0, B_2 > 0. \tag{6.5.1}$$

So, we can say that if (6.5.1) is satisfied along with $q_1p_1\bar{x} + q_2p_2\bar{y} < \dfrac{\mu}{\gamma} + C + \tau(q_1\bar{x} + q_2\bar{y})$, the planar equilibrium point P_2 of equation (6.3.1) becomes stable.

(ii) Now the variational matrix of the system (6.3.1) at $P_3(\hat{x}, 0, \hat{E})$ is given by:

$$B(P_3) = \begin{bmatrix} r - \dfrac{2r\hat{x}}{k} - q_1\hat{E} & -\dfrac{a\bar{x}}{d + \bar{x} + \beta\alpha A} & -q_1\hat{x} \\ \\ 0 & \dfrac{b(\hat{x} + \alpha A)}{d + \hat{x} + \beta\alpha A} - l - q_2\hat{E} & 0 \\ \\ \hat{E}\gamma q_1(p_1 - \tau) & \hat{E}\gamma q_2(p_2 - \tau) & \gamma\{q_1(p_1 - \tau)\hat{x} - C\} - \mu \end{bmatrix}$$

One eigenvalue being $\dfrac{b(\hat{x} + \alpha A)}{d + \hat{x} + \beta\alpha A} - l - q_2\hat{E}$, and for the other two we have $\lambda^2 + C_1\lambda + C_2 = 0$ where

$C_1 = -\Big[\gamma\{q_1(p_1 - \tau)\hat{x} - C\} - \mu + r - \dfrac{2r\hat{x}}{k} - q_1\hat{E}\Big]$ and $C_2 = \Big[r - \dfrac{2r\hat{x}}{k} - q_1\hat{E}\Big] * \Big[\gamma\{q_1(p_1 - \tau)\hat{x} - C\} - \mu\Big] + \bar{x}\hat{E}\gamma {q_1}^2(p_1 - \tau)\hat{E}$.

According to the Routh-Hurwitz criterion, the above characteristic equation's eigenvalues have negative real portions if and only if

$$C_1 > 0, \;\; C_2 > 0. \tag{6.5.2}$$

Hence, we can say that the planar equilibrium P_3 of the system (6.3.1) exists and is locally asymptotically stable if condition (6.5.2) is satisfied along with $\dfrac{b(\hat{x} + \alpha A)}{d + \hat{x}\beta\alpha A} < l + q_2\hat{E}$.

(iii) Now the variational matrix of the system (6.3.1) at $P_4(0, y', E')$ is given by:

$$B(P_4) = \begin{bmatrix} r - \dfrac{(d + \beta\alpha A)ay'}{(d + \beta\alpha A)^2} - q_1E' & 0 & 0 \\ \\ \dfrac{(d + \beta\alpha A - \alpha A)by'}{(d + \beta\alpha A)^2} & \dfrac{b\alpha A}{d + \beta\alpha A} - l - q_2E' & -q_2y' \\ \\ E'\gamma q_1(p_1 - \tau) & E'\gamma q_2(p_2 - \tau) & \gamma\{q_2(p_2 - \tau)y' - C\} - \mu \end{bmatrix}$$

One eigenvalue is $r - \dfrac{(d + \beta\alpha A)ay'}{(d + \beta\alpha A)^2} - q_1E'$

and the roots of the characteristic equation of $V(P_4)$ is given by:,

$$\lambda^2 + D_1\lambda + D_2 = 0, \tag{6.5.3}$$

which correspond to the other two eigenvalues According to the Routh-Hurwitz criterion, all eigenvalues of the characteristic equation (6.5.3) have negative real portions if and only if

$$D_1 > 0,\ D_2 > 0, \tag{6.5.4}$$

where $D_1 = \dfrac{b\alpha A}{d+\beta\alpha A} - l - q_2E' + \gamma\{q_2(p_2-\tau)y' - C\} - \mu$ and $D_2 = \left[\dfrac{b\alpha A}{d+\beta\alpha A} - l - q_2E'\right] * \left[\gamma\{q_2(p_2-\tau)y' - C\} - \mu\right] + E'^2\gamma q_2 y'(p_2-\tau)$. So, we say that the planar equilibrium P_4 of the system (6.3.1) exists and is locally asymptotically stable if condition (6.5.4) is satisfied and $r < \dfrac{(d+\beta\alpha A)ay'}{(d+\beta\alpha A)^2} + q_1E'$.

6.5.1.4 Stability of the interior equilibrium P_5

The variational matrix of (6.3.1) at $P_5(x^*, y^*, E^*)$is given by

$$B(P_5) = \begin{bmatrix} -r - \dfrac{2rx^*}{k} - \dfrac{(d+\beta\alpha A)ay^*}{(d+x^*+\beta\alpha A)^2} - q_1E^* & -\dfrac{ax^*}{d+x^*+\beta\alpha A} & -q_1x^* \\ \dfrac{(d+\beta\alpha A - b\alpha A)by^*}{(d+x^*+\beta\alpha A)^2} & \dfrac{b(x^*+\alpha A)}{d+x^*+\beta\alpha A} - l - q_2E^* & -q_2y^* \\ E^*\gamma q_1(p_1-\tau) & E^*\gamma q_2(p_2-\tau) & \gamma\{q_1(p_1-\tau)x^* + q_2(p_2-\tau)y^* - C\} - \mu \end{bmatrix}$$

Therefore, the characteristic equation of $V(P_5)$ is given by,

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{6.5.5}$$

where a_1, a_2 and a_3 are functions of (x^*, y^*, E^*) and all the parameters.

By Routh-Hurwitz criterion, it follows that all eigenvalues of the characteristic equation (6.5.3) have negative real parts if and only if

$$a_1 > 0,\ a_3 > 0,\ a_1a_2 > a_3. \tag{6.5.6}$$

Therefore, the interior equilibrium P_5 of the model (6.3.1) is locally asymptotically stable when conditions (6.5.6) are satisfied.

6.6 Global stability analysis

By creating a suitable Lyapunov function, we discuss the system of equations (6.3.1)'s global stability in this section.

$$G(x,y,E) = \Bigg[(x-x^*)-x^*\log\frac{x}{x^*}+k_1(y-y^*)-y^*\log(\frac{y}{y^*})+k_2(E-E^*)-E^*\log(\frac{E}{E^*})\Bigg]. \tag{6.6.1}$$

Here, k_1 and k_2 being the appropriate constants that will be calculated in the next steps. It is obvious that at the interior equilibrium point (x^*, y^*, E^*), G is zero and positive definite for all other values of x, y and E. At the interior equilibrium point (x^*, y^*, E^*), the derivative of G with respect to time is the following:

$$\begin{aligned}
\frac{dG}{dt} &= \frac{x-x^*}{x}\frac{dx}{dt}+k_1\frac{y-y^*}{y}\frac{dy}{dt}+k_2\frac{E-E^*}{E}\frac{dE}{dt}\\
&= \Bigg[-\frac{r}{k}(x-x^*)^2+\frac{ay^*(x-x^*)^2}{(d+x+\beta\alpha a)(d+x^*+\beta\alpha a)}\\
&-\frac{a(d+x^*+\beta\alpha a)(x-x^*)(y-y^*)}{(d+x+\beta\alpha a)(d+x^*+\beta\alpha a)}-q_1(x-x^*)(E-E^*)\Bigg]\\
&+k_1\Bigg[-\frac{b\alpha a(x-x^*)(y-y^*)}{(d+x+\beta\alpha a)(d+x^*+\beta\alpha a)}+\frac{b(d+\beta\alpha a)(x-x^*)(y-y^*)}{(d+x+\beta\alpha a)(d+x^*+\beta\alpha a)}\\
&-q_2(y-y^*)(E-E^*)\Bigg]+k_2\gamma\Bigg[q_1(p_1-\tau)(x-x^*)(E-E^*)\\
&+q_2(p_2-\tau)(y-y^*)(E-E^*)\Bigg].
\end{aligned} \tag{6.6.2}$$

Choosing $k_1=\frac{1}{b}$ and $k_2=\frac{1}{q_1\gamma(p_1-\tau)}$, we get

$$\Big(d+x+\beta\alpha a\Big) > \frac{kay^*}{r\Big(d+x^*+\beta\alpha a\Big)}, \tag{6.6.3}$$

which is the sufficient condition for $\frac{dG}{dt}$ to be negative definite.

Thus, we find that the interior equilibrium $P_5(x^*, y^*, E^*)$ is globally asymptotically stable in some neighbourhood of $P_5(x^*, y^*, E^*)$ if the above condition (6.6.3) holds.

6.7 Fuzzy Environment Optimal Harvesting Policy

Assuming a fuzzy environment, let $\tilde{F}$ and $\tilde{R}$ be the inflation and discount rates that represent the time worth of money proposed by Sadhukhan et al., 2010; Pal and Mahapatra, 2014.

The present value J of continuous time-stream of revenues is given by the following:

$$\tilde{J} = \int_0^{\infty} e^{-\tilde{\delta}t}\{p_1q_1x + p_2q_2y - C\}E dt. \tag{6.7.1}$$

We have considered $\tilde{\delta} = \tilde{R} - \tilde{F}$ as trapeziodal fuzzy number, i.e. $\tilde{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4)$. The control set is defined as $V_t = [0, E_{max}]$ when the control variable $E(t)$ is subjected to the constraints $0 \le E(t) \le E_{max}$.

As proposed by Sadhukhan et al., 2010; Pal and Mahapatra, 2014 and Pal, Mahato, and Samanta, 2020, the integral given in equation (6.7.1) which is to be maximized can be expressed as

$$Max[J_L, J_R] = \int_0^{\infty} e^{-[\delta_L,\delta_R]t}\{p_1q_1x + p_2q_2y - C\}E(t)dt, \tag{6.7.2}$$

where

$$J_L = \int_0^{\infty} e^{-\delta_L t}\{p_1q_1x + p_2q_2y - C\}E(t)dt, \tag{6.7.3}$$

$$J_R = \int_0^{\infty} e^{-\delta_R t}\{p_1q_1x + p_2q_2y - C\}E(t)dt, \tag{6.7.4}$$

$$\delta_L = \delta_1 + \alpha(\delta_2 - \delta_1), \delta_R = \delta_4 - \alpha(\delta_4 - \delta_3), \tag{6.7.5}$$

subject to the constraints (6.3.1). We can write

$$\mathrm{Max}J = \mathrm{Max}[J_L, J_R] = w_1 J_L + w_2 J_R, \tag{6.7.6}$$

where w_1, w_2 are two weights such that $w_1 + w_2 = 1, w_1, w_2 \geq 0$.
The Hamiltonian is given by:

$$\begin{aligned} H &= (w_1 e^{-\delta_L t} + w_2 e^{-\delta_R t})(p_1 q_1 x + p_2 q_2 y - C)E(t) \\ &+ \mu_1 \left(rx(1 - \frac{x}{k}) - \frac{axy}{d + x + \beta\alpha A} - q_1 E x \right) \\ &+ \mu_2 \left(\frac{b(x + \alpha A)y}{d + x + \beta\alpha A} - ly - q_2 E y \right) \\ &+ \mu_3 \left([\gamma\{q_1(p_1 - \tau)x + q_2(p_2 - \tau)y - C\} - \mu]E \right), \end{aligned} \tag{6.7.7}$$

where $\mu_i = \mu_i(t)(i = 1, 2, 3)$ are adjoint variables. Hamiltonian (6.7.7) must be maximized for $\tau \in [\tau_{min}, \tau_{max}]$. Assuming that the control constraints are not binding , we have singular control by Clark, 1974b given by

$$\frac{\partial H}{\partial \tau} = -\mu_3 \gamma (q_1 x + q_2 y)E = 0 \text{ which implies } \mu_3 = 0. \tag{6.7.8}$$

The adjoint equations are

$$\frac{d\mu_1}{dt} = -\frac{\partial H}{\partial x}, \frac{d\mu_2}{dt} = -\frac{\partial H}{\partial y}, \frac{d\mu_3}{dt} = -\frac{\partial H}{\partial E}. \tag{6.7.9}$$

Therefore

$$\begin{aligned} \frac{d\mu_1}{dt} &= -\Bigg\{ (w_1 e^{-\delta_L t} + w_2 e^{-\delta_R t}) p_1 q_1 E + \mu_1 \left(r - \frac{2rx}{k} \right) \\ &+ \frac{(d + \beta\alpha A)(\mu_2 b - \mu_1 a)y - \mu_2 b\alpha A y}{(d + x + \beta\alpha A)^2} - \mu_1 q_1 E + \mu_3 \gamma q_1 (p_1 - \tau)E \Bigg\}, \end{aligned} \tag{6.7.10}$$

$$\frac{d\mu_2}{dt} = -\Bigg\{(w_1e^{-\delta_L t} + w_2e^{-\delta_R t})p_2q_2E - \mu_1\frac{ax}{(d+x+\beta\alpha A)} + \mu_2\left(\frac{b(x+\alpha A)}{d+x+\beta\alpha A} - l - q_2E\right) + \mu_3q_2\gamma(p_2-\tau)E\Bigg\}, \tag{6.7.11}$$

$$\frac{d\mu_3}{dt} = -\Bigg\{(w_1e^{-\delta_L t} + w_2e^{-\delta_R t})(p_1q_1x + p_2q_2y - C) - \mu_1q_1x - \mu_2q_2y + \mu_3[\gamma\{q_1(p_1-\tau)x + q_2(p_2-\tau)y - C\} - \mu]\Bigg\}. \tag{6.7.12}$$

At the equilibria,

$$r(1-\frac{x}{k}) - q_1E = \frac{ay}{d+x+\beta\alpha A}, \tag{6.7.13}$$

$$\frac{b(x+\alpha A)}{d+x+\beta\alpha A} - l - q_2E = 0, \tag{6.7.14}$$

$$\gamma\{q_1(p_1-\tau)x + q_2(p_2-\tau)y - C\} - \mu = 0. \tag{6.7.15}$$

From (6.7.13) and (6.7.14), we have

$$y = Ux^2 + Vx + W, \tag{6.7.16}$$

where

$$U = -\frac{r}{ak}, V = \frac{r}{a} - \frac{r(d+\beta\alpha A)}{ak} - \frac{q_1b}{aq_2} \ \ \& \ \ W = -\frac{r}{a}(d+\beta\alpha A) + \frac{q_1b\alpha A}{aq_2} + \frac{q_1l\alpha A}{aq_2}. \tag{6.7.17}$$

By using (6.7.8), (6.7.10), (6.7.11) and (6.7.12), at the equilibria, we get

$$\mu_1 = A_Le^{-\delta_L t} + A_Re^{-\delta_R t}, \tag{6.7.18}$$

$$\mu_2 = B_Le^{-\delta_L t} + B_Re^{-\delta_R t}, \tag{6.7.19}$$

where

$$\begin{aligned}
A_L &= \frac{w_1}{\delta_L + A_1}\left\{\frac{(bd + b\beta\alpha - b\alpha A y)(p_1q_1x + p_2q_2y - c)}{q_2y(d + x + \beta\alpha A)^2} - p_1q_1E\right\},\\
A_R &= \frac{w_2}{\delta_R + A_1}\left\{\frac{(bd + b\beta\alpha A - b\alpha A y)(p_1q_1x + p_2q_2y - c)}{q_2y(d + x + \beta\alpha A)^2} - p_1q_1E\right\},\\
A_1 &= r_1(1 - \frac{2x}{k}) - \frac{(d + \beta\alpha A)a}{(d + x + \beta\alpha A)^2} - \frac{(bd + b\beta\alpha A - b\alpha A y)q_1x}{q_2y(d + x + \beta\alpha A)^2} - q_1E,\\
B_L &= \frac{w_1}{B_1 - \delta_L}\left\{\frac{a}{d + x + \beta\alpha A}\frac{p_1q_1x + p_2q_2y - c}{q_1} - p_2q_2E\right\},\\
B_R &= \frac{w_2}{B_1 - \delta_R}\left\{\frac{a}{d + x + \beta\alpha A}\frac{p_1q_1x + p_2q_2y - c}{q_1} - p_2q_2E\right\},\\
B_1 &= \frac{aq_2y}{q_1(d + x + \beta\alpha A)}.
\end{aligned}$$

From (6.7.12), we have

$$(w_1e^{-\delta_L t} + w_2e^{-\delta_R t})(p_1q_1x + p_2q_2y - C) = \mu_1q_1x + \mu_2q_2y. \qquad (6.7.20)$$

Now from (6.7.13) and (6.7.14) and (6.7.16), we have

$$\begin{aligned}
E &= \frac{1}{q_1}\left\{r(1 - \frac{x}{k}) - \frac{a(Ux^2 + Vx + W)}{d + x + \beta\alpha A}\right\}, && (6.7.21)\\
E &= \frac{1}{q_2}\left\{\frac{b(x + \alpha A)}{d + x + \beta\alpha A} - l\right\}. && (6.7.22)
\end{aligned}$$

From (6.7.16), (6.7.20), (6.7.21) and (6.7.22) we get an equation of x by Pradhan and Chaudhuri, 1999a. Assuming $x_{[\delta_L,\delta_R]}$ as the positive root of this equation, if it exists.

Using this value of $x_{[\delta_L,\delta_R]}$ in (6.7.16) and (6.7.21), we have $y_{[\delta_L,\delta_R]}$ and $E_{[\delta_L,\delta_R]}$, respectively. Thus, the optimal equilibrium point is $(x_{[\delta_L,\delta_R]}, y_{[\delta_L,\delta_R]}, E_{[\delta_L,\delta_R]})$. The optimal value of τ for the optimal equilibrium values, is given by the following:

$$\tau = \frac{\gamma(q_1p_1x_{[\delta_L,\delta_R]} + q_2p_2y_{[\delta_L,\delta_R]}) - (\mu + \gamma C)}{\gamma(q_1x_{[\delta_L,\delta_R]} + q_2y_{[\delta_L,\delta_R]})}. \qquad (6.7.23)$$

6.8 Numerical Simulations

This section demonstrates our attempt to verify the mathematical findings from the earlier sections. We first consider the case when tax (τ)=6 and use the remaining parameter values as provided in Table 1. In this case, four equilibrium points of our system (3.1) exist: (i) Trivial equilibrium $P_0(0, 0, 0)$, (ii) Axial equilibrium $P_1(100, 0, 0)$, (iii) Planner equilibrium $P_2(0.97, 11.9048, 0)$, and (iv) Interior equilibrium $P_5(85.16, 56.1623, 69.7873)$, among which only interior equilibrium is stable having eigenvalues $(-6.3346, -0.00111298+0.0708877i, -0.00111298-0.0708877i)$ and the other equilibria are unstable. Fig 6.2 (a) shows the phase space trajectory corresponding to the tax(τ)=6 using the initial values $(18, 90, 64)$, which indicates the interior equilibrium $P_5(85.16, 56.1623, 69.7873)$ is asymptotically stable. Fig 6.2(b) shows the solution curves corresponding to the values used for Fig 6.1(a). The fluctuation of prey, predator, and harvesting effort against time for different tax levels is shown in Fig. 6.3(a)-(c) where (tax= $1, 6$ and 8) with initial values $(85, 56, 69)$ and other parameter values given in Table 1. These plots show that predator population rises as tax rate rises, although prey population and harvesting effort decrease..

Table 6.1

Parameter	Values
r	8.5
k	100
q_1	0.005
q_2	0.007
p_1	16
p_2	20
C	5
μ	0.01
γ	0.0021
a	1.45
β	0.5
d	1
α	0.02
A	1
b	1
l	0.5

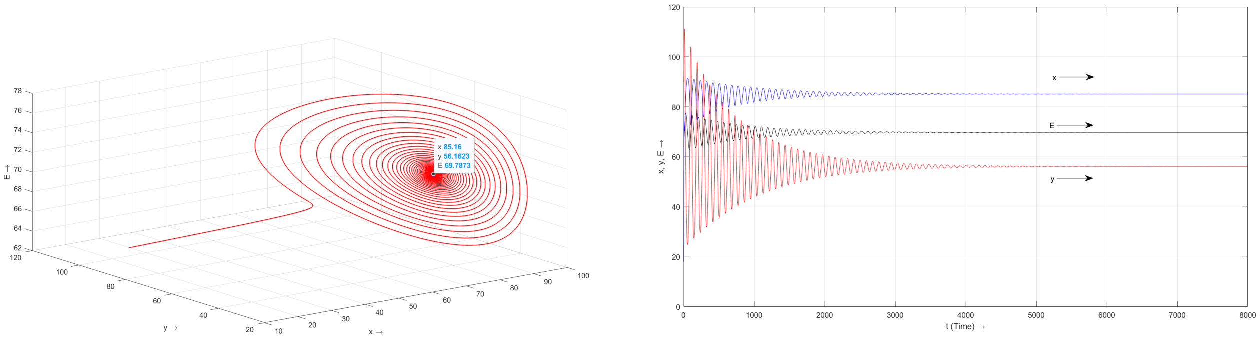

Fig.6.2(a) **Fig 6.2(b)**

Fig.6.2. *(a) Phase space trajectory, (b) solution curves of prey population* (x), *predator population* (y) *and harvesting effort* (E) *corresponding to tax* $\tau = 6$ *with initial values* $(18, 90, 64)$ *along with* **Table 6.1** *which shows other parameter values.*

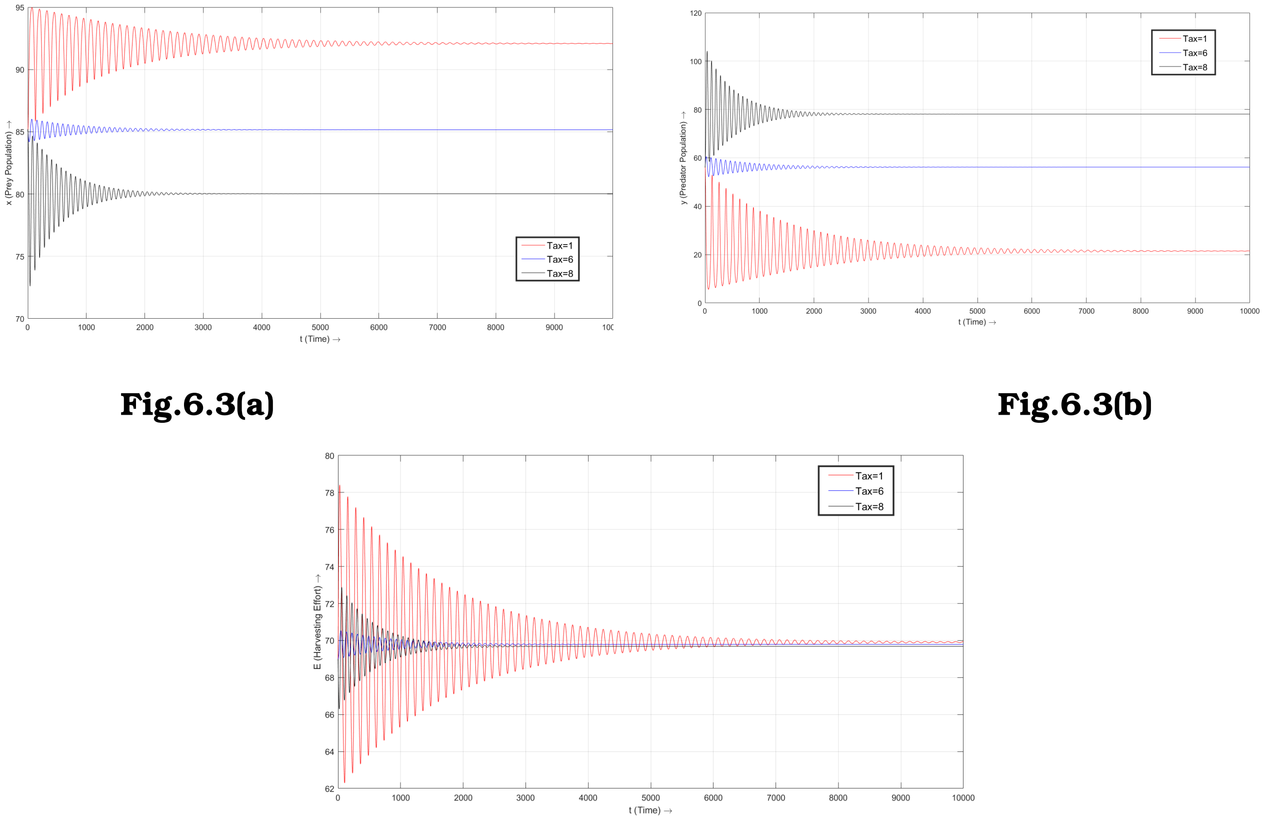

Fig.6.3(a) **Fig.6.3(b)**

Fig.6.3(c)

Fig.6.3. *Alteration of (a)* (x), *the prey population, (b)* (y), *the predator population and (c)* (E), *the harvesting effort with respect to time for different tax levels (*$\tau = 1, 6, 8$*) with initial values* $(85, 56, 69)$ *along with* **Table 6.1** *which shows other parameter values.*

6.9 Discussions and conclusions

In this paper, we have considered a Rosenzweig-MacArthur two species predator-prey harvesting model with harvesting effort as a variable of time. The prey population follows the logistic growth in the absence of predator whereas Holling type-II predator functional response on the prey population has been considered with the inclusion of an additional food term provided to the predator species. The predator-prey reserve model is treated as a dynamic reaction model which consists of the interaction between the perceived rent and the effort applied to the open-access model. In an open-access prey-predator dynamical model the level of harvesting effort applied to the prey and the predator species increases or decreases depending on the perceived rent. Assuming that the overall rate of capital investment applied to the predator-prey resource is proportionate to the perceived rent, a regulatory body controls the exploitation of the predator-prey reserve by imposing a tax per unit biomass of landed species. The most important part of this essay focuses on setting up an optimum control problem with fuzzy inflation and discount together with harvesting effort as the control variable in order to maximise the fuzzy objective functional.

The non-selective exploitation of a mixed species (predator-prey) is the topic of the current research, which emphasises the need of employing taxes as an effective regulatory tool to limit exploitation. This paper's goal is to maximise taxation policy in order to maximise advantage from the harvesting of the prey and predator populations while averting their extinction. The instantaneous yearly rate of discount, which is the difference of two fuzzy quantities—the inflation rate and the discount rate—and which represents the time worth of money, is also uncertain and fuzzy in nature. The model becomes more ecologically accurate than the current models as a result of this added effect.

The dynamical system's local and global stabilities for the model are assessed in this paper. The difference between two arbitrary quantities inflation and discount rates, which symbolise the time worth of money, is the instantaneous yearly rate of discount δ. We regard δ to be fuzzy in nature due to it's triangular fuzzy nature, which is innate, convenient, computationally simple, and beneficial in various representations. The optimal control issue is then presented for most return on investment and solved for best species harvesting utilising Pontryagin's maximal principle.

In this work the MATLAB software has been used to verify all the analytical findings. No case study is used in our model. Choosing parameter values from quantitative estimation is very challenging. We have used hypothetical sets of parameter values to verify our analytical results. With these parameter values, our analysis and simulations reach a good agreement and validate our analytical findings.

Finally, we draw the conclusion that the dynamics of our predator-prey harvesting model are quite intriguing. Due to the assumptions made and the challenges in estimating the model parameters, the mathematical model described in this study should be viewed with caution. The process through which predators consume their prey and obtain energy from the biomass is not immediate. There must be a gestation delay, sometimes known as a time lag. Therefore, as part of ongoing work to enhance the model, we can add the gestation delay to make it more accurate. Also, a time lag should be given in the system to provide enough time to fish populations to grow as a form of another delay term in the model which is very important in both the ecological and economical point of view. This can be incorporated in our model to improve our future work. The present idea can be further considered under other imprecise environments like intuitionistic fuzzy, fuzzy rough, neutrosophic etc.

7

CONCLUSION

"Mathematics is the art of explanation." *- Paul Lockhart*

7.1 Summary of key findings

In this thesis, continuous-time fishery resource models have been used to examine the consequences of various changes in predator-prey interactions and human activity, such as harvesting. Here, several nonlinear dynamics properties have been investigated, including local stability, permanence and global stability. The effects of the key system characteristics are perceived through the examination of the local stability of the equilibrium points. Here, we look into how the system is affected by well regulated fish collection. The model has been converted into an optimal control problem that considers the socioeconomic aspects of the harvesting. The ideal effort for harvesting is determined using a modified catch rate function applying Pontryagin's maximal principle in order to maximise net harvesting revenue while preserving system stability. In the study, it was found that optimum fish population harvesting brought about system stability.

In **Chapter 1**, the scope and summary of this thesis are presented which also offers a general introduction to fishery resource management.

Chapter 2 examines the dynamics of a fishery resource system in an aquatic environment subject to selective harvesting of only prey fish species using a nonlinear model. We presume that all methods of harvesting predatory fish are prohibited. To harvest the prey fish species, we suggest using a modified capture rate function. The parameters of the biological and bioeconomic equilibrium of a system are determined. Regional stability, global stability, and instability are derived for the suggested model. The Pontryagin maximum concept is used to determine the optimal harvesting strategy. To comprehend the model's theoretical outcomes, numerical simulations are performed.

In **Chapter 3**, the purpose of this study is to investigate and analyse the dynamical behaviour of a prey-predator fishery resource in an aquatic ecosystem using a non-linear mathematical model in which both species are poisoned by the harmful compounds produced by each species. Here, both species exhibit a logistic growth rate function and are harvested together for bioeconomic reasons. Using a modified catch rate function, this article examines the bioeconomics of the combined harvesting of prey-predator fish species in the presence of the toxic chemicals they emit. This section examines the boundedness of the proposed model. The biological and bionomic equilibria of the proposed model are derived, and the requirements for local stability, instability, and global stability are illustrated. Prior to using numerical simulations to elucidate and validate the model's theoretical predictions, the optimal harvesting strategy is determined using Pontryagin's maximum principle.

In **Chapter 4**, we use a nonlinear model to investigate the dynamics of a prey-predator fishing resource system in an aquatic environment where both species are harmed by the toxicants they emit when subjected to combined bioeconomic harvesting. Using a modified capture rate function, we investigate the bioeconomic harvesting of two competing fish species that produce toxic compounds that are detrimental to one another. By employing Pontryagin's maximal principle, equilibrium, stability, bionomic equilibrium, and the optimal harvesting policy have been established. We have determined that the characteristics of the system and the initial population densities of the interacting fish species will have a substantial effect on their dynamical behaviour. A few numerical simulations and corresponding solution contours are used to illustrate the theoretical outcomes of the proposed model. Finally, the existence of the limit cycle is demonstrated here.

Chapter 5 examines a predator-prey fishery resource system where both species are subject to logistic growth incorporating the modified capture rate function. Here, the functional response of predator species towards prey population density tends to be constant. Here, we analyse the boundedness of the proposed nonlinear model. Here are displayed the model's steady states and their existence criteria for positivity. In addition, the local and global stability of the equilibria are determined through Eigenvalue analysis and the construction of a suitable Lyapunov function, respectively. The bionomic equilibria of the proposed nonlinear model and the criteria for their existence have been determined. The most notable aspect of this proposed non-linear predator-prey fishery resource model is the establishment of an optimal harvesting policy based on Pontryagin's maximum principle, so that both species can be protected from extinction despite being subject to combined harvesting. Finally, numerical simulations are performed to validate the proposed predator-prey model's theoretical results.

In **Chapter 6**, the Rosenzweig-MacArthur predator-prey system paradigm is considered. In the absence of a predator population, the logistic growth of the prey population is considered. The functional response of the predator to the prey population is of Holling type II. The predator species is provided with access to an additional food source. In addition to the existence and positivity of solutions, the stability of equilibrium points is examined. Appropriate Lyapunov functions are established for the purpose of analysing the overall stability of the system. In reality, the system's parameters are not precise due to insufficient information. To increase the realism of the model, the optimal taxation policy in an ambiguous environment is investigated. Theorems are inadequate without numerical verifications of analytical results, so we analyse and numerically verify a few fictitious data sets.

7.2 Future scope of research

The research done in this thesis can be expanded into a more extensive investigation employing an improved model, new tools, and methodologies. Some potential scopes for future study can be listed as follows.

- For more realistic results, the parameters taken into account in a deterministic system can be thought of as stochastic in nature. It makes sense to think about and is more comparable to real-world phenomena to consider stochastic perturbations in a system.

- The right parameter values must be chosen in order to accurately simulate any system. So it is possible to get a more accurate image by using parameter estimation from actual data.

- Consideration of an impulsive harvesting technique may be a more realistic course of action in this situation because harvesting is not a continuous operation and does not occur at every time step.

- Certain communities frequently move through ecological systems in quest of better food supplies. For more accurate results, the model might be expanded to include the effects of prey species migration.

8

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[1] **Sarkar S**, Sarkar A K, Chaudhuri K S. "Modeling of single species fishery resource harvesting with modified catch rate function" ***Bull. Cal. Math.***, (2020) 112(6):511-524.

[2] **Sarkar S**, Murmu T, Sarkar A K, Chaudhuri K S. "A non-linear model of a fishery resource for analyzing the effects of toxic substances" ***Springer Proceedings Complexity of the ICNDA 2022***, (2022) 837-847.

[3] **Sarkar S**, Sarkar A K, Chaudhuri K S. "Modeling of harvesting of prey-predator fishery in the presence of toxicity with a modified catch rate function" ***Advances and Applications in Mathematical Sciences***, (2022) 22(9):2023-2043.

[4] Sarkar M, Sharma S, **Sarkar S**, Chakraborty D. "Optimal harvesting of a predator-prey system with additional food for predator in fuzzy environment" (2022) **(Communicated)**.

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MODELING OF SINGLE SPECIES FISHERY RESOURCE HARVESTING WITH MODIFIED CATCH RATE FUNCTION

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Abstract. In this paper we consider a non-linear model to analyse the dynamics of a fishery resource system in an aquatic environment subjected to selective harvesting of only prey fish species. We assume that there is a complete ban on harvesting the predator fish species. Here we propose that the prey fish species are harvested using modified catch rate function. The criteria of biological and bionomic equilibrium of system are established. The points of local stability, global stability and instability are derived for the proposed model. The optimal harvesting policy is established by using Pontryagin's maximal principle. Numerical simulations are done to explain the theoretical results of the model.

Keywords : Fishery resource, Single species harvesting, Stability, Steady state, Optimal equilibrium, Net revenue.

1. Introduction. The exploitation of biological resources are commonly practiced in fisheries, forestry and wild life management. Bio economic modeling is concerned with scientific management of exploitation of renewable resources taking into account economic loss or gain. The growing human needs for more food and more energy have led to the depletion of these resources. During the last few decades several investigations regarding fishery resource have been done (Bhattacharyya and Begum, 1996, Clark, 1976, 1979, 1985, 1990, Goh, 1980 and Meserton-Gibbons, 1988, 1996). Kitabatake (1982) proposed a dynamical model for fishery resources with predator-prey relationship. Based on the work of Clark (1976), Chaudhuri (1986) developed a model to study the combined harvesting of two competing fish species and established that the open access fishery may have a bionomic equilibrium and one of the species may be driven to annihilation. Meserton-Gibbons (1996) also found out an optimal

A Non-linear Model of a Fishery Resource for Analyzing the Effects of Toxic Substances

Sudipta Sarkar, Tanushree Murmu, Ashis Kumar Sarkar, and Kripasindhu Chaudhuri

Abstract The goal of the proposed model is to investigate and analyze the qualitative behaviour of predator-prey fishery resource in an aquatic ecosystem by a non-linear mathematical model in which prey and predator species are contaminated by the toxic substances released by each of the species. In this model the species are subjected to bio-economic combined harvesting and obey the logistic growth rate function. Bio-economic harvesting of prey-predator species in presence of harmful toxic substances released by them is analyzed here by using modified catch rate function. Boundedness of the proposed model is examined here. Biological and bionomic steady states of the proposed model are derived. The conditions for local behaviour, instability and global behaviour of the steady states are exhibited in this paper. Optimal harvesting policy with the help of Pontryagin's maximal principle and finally, numerical exmples are illustrated to verify theoretical observations obtained from proposed model.

Keywords Fishery resource · Bioeconomic combined harvesting · Stability · Steady state · Optimal equilibrium · Toxicity · Net revenue

1 Introduction

Sustainable resources fishery, forestry, wild life etc. are vital origins of food and other necessary commodities in human life. These resources play a salient role for existence and advancement of biological populations. For these renewable resources management, our aim is to maximize the current value of advantages obtained from

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MODELING OF HARVESTING OF PREY-PREDATOR FISHERY IN THE PRESENCE OF TOXICITY WITH A MODIFIED CATCH RATE FUNCTION

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Abstract

In this paper we consider a non-linear model to analyse the dynamics of a prey predator fishery resource system in an aquatic environment in which both the species are infected by the toxicants released by some other species subjected to bio-economic combined harvesting. Bio-economic harvesting of prey predator fish species where each of species is affected by the toxicants released from other resources is discussed here by using modified catch rate function. The equilibria, stability, bionomic equilibrium and optimal harvesting policy by using Pontryagin's maximal principle have been established. We have derived that the dynamical behavior of the fish species will be much sensitive to the system parameters and their initial population densities. Some numerical simulations and the corresponding solution curves are cited to illustrate the theoretical results of the proposed model. Finally, the existence of limit cycle is shown here.

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Keywords: fishery resource, bio-economic combined harvesting, stability, optimal equilibrium, toxicity.
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