Study of certain selection principles in bornological spaces and its implications

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I would like to dedicate this thesis to their Lordships

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CHAPTER 0

PREFACE

0.1 CARDINALITY

We follow [57, 115] for the definitions and results of this chapter. A ***set, family*** or ***collection*** is an aggregate of things, called the elements or points of the set.

The *empty set* $\emptyset$ is the set having no elements. There is only one empty set and it is a subset of every other set. For a set A the collection of all subsets of A is called the *power set* of A and is denoted by $\mathcal{P}(A)$.

We list few well known sets and their conventional notations.

- $\mathbb{R}$: the set of all real numbers.
- $\mathbb{R}^n$: the Euclidean n-space.
- $\mathbb{N}$: the set of all positive integers.
- $\mathbb{Q}$: the set of all rational numbers in $\mathbb{R}$.
- $\mathbb{Z}$: the set of all integers.

Definition 0.1.1. *Let A and B be two sets. A is* ***equipotent*** *with B if and only if there is a one to one function from A onto B.*

Intuitively, equipotent sets have the same number of elements. We postulate the existence of a set, called *cardinal numbers*, so chosen that every set A is equipotent with precisely one cardinal number, called the ***cardinal number of*** A. It is denoted by $|A|$.

We have the following observations.

- The cardinality of the empty set is 0.
- The cardinality of $\mathbb{N}$ is $\aleph_0$.
- The cardinality of $\mathbb{R}$ is $\mathfrak{c}$ (continuum).
- A set A is *denumerable* if and only if A is equipotent with $\mathbb{N}$ and we write $|A| = \aleph_0$.
- A set A is said to have the *cardinal number of continuum* if and only if A is equipotent with $\mathbb{R}$ and we write $|A| = \mathfrak{c}$.
- A set is countable if it is denumerable or has the cardinal number n for some $n = 0, 1, 2, \ldots$; otherwise A is uncountable.
- The union of countably many countable sets is countable.
- The product of two countable sets is countable.
- $\mathbb{Q}$ is a countable set.
- The operations of addition and multiplication for cardinal numbers are defined as follows. The ***sum of cardinal numbers*** $\mathfrak{l}$ and $\mathfrak{m}$ is defined as the cardinality of $X \cup Y$, where $|X| = \mathfrak{l}$, $|Y| = \mathfrak{m}$ and $X \cap Y = \emptyset$. The ***product of cardinal numbers*** $\mathfrak{l}$ and $\mathfrak{m}$ is the cardinality of $X \times Y$, where $|X| = \mathfrak{l}$ and $|Y| = \mathfrak{m}$. The sum of cardinal numbers $\mathfrak{l}$ and $\mathfrak{m}$ is denoted by $\mathfrak{l} + \mathfrak{m}$, and the product of $\mathfrak{l}$ and $\mathfrak{m}$ is denoted by $\mathfrak{l} \cdot \mathfrak{m}$ or $\mathfrak{l}\mathfrak{m}$.
- For every cardinal number $\mathfrak{m}$, the number $2^{\mathfrak{m}}$, also denoted by $\exp \mathfrak{m}$, is defined as the cardinality of the power set of X, where $|X| = \mathfrak{m}$.
- For any cardinal number $\mathfrak{m}$, $\mathfrak{m} < 2^{\mathfrak{m}}$.
- For any positive integer n, $n < \aleph_0 < \mathfrak{c}$.
- We have $2^{\aleph_0} = \mathfrak{c}$.
- The continuum hypothesis states that there are no sets A for which $\aleph_0 < |A| < 2^{\aleph_0}$.

Definition 0.1.2. *Let $\mathfrak{l}$ and $\mathfrak{m}$ be two cardinal numbers of two sets X and Y respectively. We say that $\mathfrak{l}$ is* ***not larger than*** *$\mathfrak{m}$, or that $\mathfrak{m}$ is* ***not smaller than*** *$\mathfrak{l}$, if there is a one to one mapping of X to Y. We write this fact by $\mathfrak{l} \leq \mathfrak{m}$ or $\mathfrak{m} \geq \mathfrak{l}$ respectively.*

The fundamental fact about inequalities for cardinal numbers is the following.

Theorem 0.1.1 (Cantor Bernstein Theorem). *If $\mathfrak{l} \leq \mathfrak{m}$ and $\mathfrak{m} \leq \mathfrak{l}$, then $\mathfrak{l} = \mathfrak{m}$.*

The eventually dominating order $\leq^*$ on the Baire space $\mathbb{N}^{\mathbb{N}}$ is defined as follows. For $f, g \in \mathbb{N}^{\mathbb{N}}$, we say that $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n.

Definition 0.1.1. *Let A be a subset of $\mathbb{N}^{\mathbb{N}}$. The set A is* ***bounded*** *if there is a function $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for all $f \in A$. The symbol $\mathfrak{b}$ denotes the minimal cardinality of an unbounded subset of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.*

Definition 0.1.2. *A subset A of $\mathbb{N}^{\mathbb{N}}$ is* ***dominating*** *if for each function $g \in \mathbb{N}^{\mathbb{N}}$ there exists a function $f \in A$ such that $g \leq^* f$. The symbol $\mathfrak{d}$ denotes the minimal cardinality of a dominating subset of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.*

Definition 0.1.3. *Let $\mathcal{A}$ be a family of infinite subsets of $\mathbb{N}$. $P(\mathcal{A})$ denotes that there is a subset P of $\mathbb{N}$ such that for each $A \in \mathcal{A}$, $P \setminus A$ is finite. The symbol $\mathfrak{p}$ denotes the smallest cardinal number k for which the following statement is false: For each family $\mathcal{A}$ if any finite subfamily of $\mathcal{A}$ has infinite intersection and $|\mathcal{A}| \leq k$, then $P(\mathcal{A})$ holds.*

Definition 0.1.4. *The symbol $\mathsf{cov}(\mathcal{M})$ denotes the smallest cardinal number k such that a family of k first category subsets of the real line covers the real line. For any family $\mathcal{A}$ of subsets of $\mathbb{N}^{\mathbb{N}}$ with cardinality less than $\mathsf{cov}(\mathcal{M})$ implies that there is a $g \in \mathbb{N}^{\mathbb{N}}$ such that for every $f \in \mathcal{A}$ the set $\{n \in \mathbb{N} : f(n) = g(n)\}$ is infinite.*

Definition 0.1.5. *The symbol $\mathsf{add}(\mathcal{M})$ denotes the smallest cardinal number k for which there exists a family of k first category sets of real numbers whose union is no longer first category.*

The following relations between the cardinal numbers mentioned above are well known. $\mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d}$, $\mathfrak{p} \leq \mathsf{cov}(\mathcal{M})$, $\mathsf{add}(\mathcal{M}) \leq \mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$ and $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$.

Definition 0.1.6. *A collection $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ is said to be a* ***reaping family*** *if for each infinite set $A \subseteq \mathbb{N}$ there is a $C \in \mathcal{F}$ satisfying $C \subseteq^* A$ or $C \subseteq^* \mathbb{N} \setminus A$, where $C \subseteq^* A$ denotes that $C \setminus A$ is finite. The symbol $\mathfrak{r}$ denotes the minimum cardinality of a reaping family.*

0.2 TOPOLOGICAL SPACES

Definition 0.2.1. *A* ***topology*** *on a set X is a family τ of subsets of X satisfying the following conditions.*

(1) *Any union of elements of τ belongs to τ.*

(2) *Any finite intersection of elements of τ belongs to τ.*

(3) *$\emptyset$ and X belong to $\mathcal{T}$.*

We call the pair (X, τ), a ***topological space***, or sometimes *"X is a topological space"* when no confusion arise about τ.

Definition 0.2.1. *Let (X, τ) be a topological space and $U \subset X$. U is called an* ***open set*** *if U belongs to the collection τ.*

Definition 0.2.2. *Let τ_1 and τ_2 be two topologies on X. Then τ_1 is **weaker (smaller, coarser)** than τ_2 or τ_2 is **stronger (larger, finer)** than τ_1 if and only if $\tau_1 \subset \tau_2$.*

Example 0.2.1.

(1) *If X is any set, the collection of all subsets of X is a topology on X which is called the **discrete topology**. The collection consisting of X and $\emptyset$ only is also a topology on X. We call it the **indiscrete topology** or the **trivial topology**.*

(2) *Let $X = \{x_1, x_2, x_3\}$. There are many possible topologies on X, for example $\tau_1 = \{X, \emptyset\}$, $\tau_2 = \{X, \emptyset, \{x_1\}, \{x_1, x_2\}\}$ and $\tau_3 = \{X, \emptyset, \{x_1, x_2\}, \{x_2\}, \{x_2, x_3\}\}$ are topologies on X.*

(3) *For a set X let τ_f be the collection of all subsets U of X for which $X \setminus U$ is either finite or all of X. Then τ_f forms a topology on X and it is called the **finite complement topology** or **co-finite topology**.*

(4) *For a set X let τ_c be the collection of all subsets U of X for which $X \setminus U$ is either countable or all of X. Then τ_c forms a topology on X and it is called the **countable complement topology** or **co-countable topology**.*

0.2.1 CLOSED SETS, CLOSURES AND INTERIORS

Definition 0.2.3. *For a topological space X a subset E of X is said to be **closed** if and only if $X \setminus E$ is open.*

Example 0.2.2.

(1) *The subsets $[a, b]$ and $[a, \infty)$ of $\mathbb{R}$ are closed.*

(2) *The subset $[a, b)$ of $\mathbb{R}$ is neither open nor closed.*

(3) *In the plane $\mathbb{R}^2$, $\{(x, y) : x \geq 0$ and $y \geq 0\}$ is a closed set. Because it is the union of two sets $(-\infty, 0) \times \mathbb{R}$ and $\mathbb{R} \times (-\infty, 0)$, each of which is a product of open sets of $\mathbb{R}$ and it is open in $\mathbb{R}^2$.*

(4) *For a set X with the finite complement topology, the closed sets consist of X itself and all finite subsets of X only.*

(5) *In the discrete topology on the set X, every subset is open and consequently it follows that every subset is closed.*

(6) *In $\mathbb{R}$, consider the set $Y = [0, 1] \cup (2, 3)$. One can see that the sets $[0, 1]$ and $(2, 3)$ are both open and closed subsets of Y.*

Theorem 0.2.1. *Let X be a topological space and $\mathcal{A}$ be the collection of closed sets in X. The following statements hold.*

(1) *Any intersection of members of* $\mathcal{A}$ *belongs to* $\mathcal{A}$.

(2) *Any finite union of members of* $\mathcal{A}$ *belongs to* $\mathcal{A}$.

(3) X *and* $\emptyset$ *both belong to* $\mathcal{A}$.

Conversely, for any collection $\mathcal{A}$ *of subsets of* X *satisfying* (1), (2) *and* (3) *the collection of complements of members of* $\mathcal{A}$ *is a topology on* X *in which the collection of closed set is just* $\mathcal{A}$.

Definition 0.2.4. *For a topological space* X *and a subset* E *of* X *the* ***closure*** *of* E *in* X *is the set*

$$\overline{E} = CL(E) = \cap\{K \subset X : E \subset K \text{ and } K \text{ is closed}\}.$$

We write the closure of E in X by $CL_X(E)$. $\overline{E}$ is a closed set, the smallest one containing E, in the sense that it is contained in every closed set containing E.

Lemma 0.2.1. *If* $A \subset B$, *then* $\overline{A} \subset \overline{B}$.

Theorem 0.2.2. *Let* X *be a topological space and* $E \subset X$. *The following statements hold.*

(1) $E \subset \overline{E}$.

(2) $\overline{(\overline{E})} = \overline{E}$.

(3) $\overline{E \cup F} = \overline{E} \cup \overline{F}$ *for any* $F \subset X$.

(4) $\overline{\emptyset} = \emptyset$.

(5) E *is closed in* X *if and only if* $\overline{E} = E$.

Definition 0.2.5. *For a topological space* X *and a subset* E *of* X *the* ***interior*** *of* E *in* X *is the set*

$$E^\circ = Int(E) = \cup\{H \subset X : H \text{ is open and } H \subset E\}.$$

We write interior of E in X by $Int_X(E)$. E° is an open set and it is the largest open set contained in E, in the sense that it contains any other open set contained in E. The notions of closure and interior are dual to each other. We have the following observations of this duality.

- $\overline{X \setminus E} = X \setminus E^\circ$.
- $(X \setminus E)^\circ = X \setminus \overline{E}$.

Lemma 0.2.2. *If* $A \subset B$, *then* $A^\circ \subset B^\circ$.

Theorem 0.2.3. *Let* X *be a topological space. For* $E \subset X$ *the following statements hold.*

(1) $E^\circ \subset E$.

(2) $(E^\circ)^\circ = E^\circ$.

(3) $(E \cap F)^\circ = E^\circ \cap F^\circ$ *for* $F \subset X$.

(4) $X^\circ = X$.

(5) E *is open if and only if* $E^\circ = E$.

Example 0.2.3.

(1) *In* $\mathbb{R}$, *with the usual topology,* (a, b) *is the interior of the closed interval* $[a, b]$.

(2) *In* $\mathbb{R}^2$, *with the usual topology, the interior of the disk* $\{(x, y) : x^2 + y^2 \leq 1\}$ *is the disk* $\{(x, y) : x^2 + y^2 < 1\}$.

0.2.2 NEIGHBOURHOODS AND LIMIT POINTS

Definition 0.2.6. *For a topological space* X *a* ***neighbourhood*** *of* $x \in X$ *is a set* U *which contains an open set* V *containing* x.

Evidently, U is a neighbourhood of x if and only if $x \in U^\circ$. The family $\mathcal{U}_x$ of all neighbourhoods of x is called the ***neighbourhood system*** at x.

Theorem 0.2.4. *Let* X *be a topological space. For a* $x \in X$ *the neighbourhood system* $\mathcal{U}_x$ *at* x *has the following properties.*

(1) *If* $U \in \mathcal{U}_x$, *then* $x \in U$.

(2) *If* $U, U' \in \mathcal{U}_x$, *then* $U \cap U' \in \mathcal{U}_x$.

(3) *If* $U \in \mathcal{U}_x$, *then there is a* $U' \in \mathcal{U}_x$ *such that* $U \in \mathcal{U}_y$ *for each* $y \in U'$.

(4) *If* $U \in \mathcal{U}_x$ *and* $U \subset U'$, *then* $U' \in \mathcal{U}_x$.

(5) $G \subset X$ *is open if and only if* G *contains a neighbourhood for each of its points.*

Conversely, for $x \in X$ *if* $\mathcal{U}_x$ *is a collection of subsets of* X *so that it satisfies conditions* $(1) - (4)$ *and if we define 'open' using* (5) *the result is a topology on* X *in which the neighbourhood system at each* x *is precisely* $\mathcal{U}_x$.

Definition 0.2.7. *Let* X *be a topological space. For* $x \in X$ *a* ***neighbourhood base*** *at* x *is a subcollection* $\mathcal{B}_x$ *of the neighbourhood system* $\mathcal{U}_x$, *which has the property that for each* $U \in \mathcal{U}_x$ *there is a* $V \in \mathcal{B}_x$ *satisfying* $V \subset U$. $\mathcal{U}_x$ *is determined by* $\mathcal{B}_x$ *as follows.*

$$\mathcal{U}_x = \{U \subset X : V \subset U \text{ for some } V \in \mathcal{B}_x\}$$

When a neighbourhood base at x has been chosen, its elements are called *basic neighbourhoods*.

Example 0.2.4.

(1) *In a metrizable space X, generated by a metric ρ, each open set containing x contains some disk $S_\rho(x)$ about x, the disks $S_\rho(x)$ about x form a neighbourhood base at x.*

(2) *In $\mathbb{R}^2$, with the usual topology, the set of all squares with sides parallel to axes and centred at $x \in \mathbb{R}^2$ is a neighbourhood base at x.*

(3) *In a discrete space X each point $x \in X$ has an acceptable neighbourhood base consisting of a single set $\{x\}$.*

(4) *For a trivial space X, the only neighbourhood base at $x \in X$ is the collection consisting of the single set X.*

Theorem 0.2.5. *Let X be a topological space. For $x \in X$ if $\mathcal{B}_x$ is a neighbourhood base at x, then the following statements hold.*

(1) *If $V \in \mathcal{B}_x$, then $x \in V$.*

(2) *If $V_1, V_2 \in \mathcal{B}_x$, then there is some $V_3 \in \mathcal{B}_x$ such that $V_3 \subset V_1 \cap V_2$.*

(3) *If $V \in \mathcal{B}_x$, there is some $V_0 \in \mathcal{B}_x$ such that if $y \in V_0$, then there is some $W \in \mathcal{B}_y$ with $W \subset V$.*

(4) *$G \subset X$ is open if and only if G contains a basic neighbourhood of each of its points.*

Conversely, for $x \in X$ if $\mathcal{B}_x$ is a collection of subsets of X so that it satisfies conditions $(1) - (3)$ and if we define 'open' using (4) the result is a topology on X in which for each $x \in X$ $\mathcal{B}_x$ is a neighbourhood base at x.

Definition 0.2.2. *Let X be a topological space. For $A \subset X$ and $x \in X$, x is said to be a **limit point** (or **cluster point** or **point of accumulation**) of A if each neighborhood of X intersects A in some point other than x itself. In other words, x is a **limit point** of A if it belongs to the closure of $A \setminus \{x\}$.*

Example 0.2.5.

(1) *In $\mathbb{R}$, if $A = (0, 1)$, then the point 0 is a limit point of A.*

(2) *In $\mathbb{R}$, every point of the interval $[0, 1]$ is a limit point of A, but any point outside $[0, 1]$ is not a limit point of A.*

(3) *If $B = \{\frac{1}{n} : n \in \mathbb{Z}_+\}$, then 0 is the only limit point of B.*

Definition 0.2.3. *For a topological space X a point $x \in X$ is called an **isolated point** of X if the one point set $\{x\}$ is open in X.*

Definition 0.2.4. *For a topological space X a subset E of X is called **perfect** if E is closed and if each point of E is a limit point of E.*

Hence a perfect set in X is a set without isolated points. We have the following relation between the closure of a set and the limit points of a set.

Theorem 0.2.6. *Let X be a topological space. For a subset A of X if A' denotes the set of all limit points of A, then $\overline{A} = A \cup A'$.*

Corollary 0.2.1. *Let X be a topological space. $A \subset X$ is closed if and only if it contains all its limit points.*

0.2.3 BASES AND SUBBASES

Just as we can specify neighbourhood system at x by giving smaller collection of sets, called neighbourhood base. In the same way, the topology of X can be specified by a smaller family, called *base for the topology*.

Definition 0.2.8. *For a topological space (X, τ) a **base** for τ is a collection $\mathcal{B} \subset \tau$ for which*

$$\tau = \{\cup_{B \in \mathcal{B}_1} B : \mathcal{B}_1 \subset \mathcal{B}\}.$$

We can say that τ can be obtained from $\mathcal{B}$ by taking all possible unions of subcollections from $\mathcal{B}$.

Example 0.2.6.

(1) *Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane. Then $\mathcal{B}$ is a base for the usual topology on $\mathbb{R}^2$.*

(2) *Let $\mathcal{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes. Then $\mathcal{B}'$ is a base for the usual topology on $\mathbb{R}^2$.*

(3) *Let X be any set. Then the family of all one-point subsets of X is a basis for the discrete topology on X.*

Theorem 0.2.7. *Let X be a topological space. A collection $\mathcal{B}$ forms a base if and only if*

(1) $X = \bigcup_{B \in \mathcal{B}} B$.

(2) *Whenever $B_1, B_2 \in \mathcal{B}$ with $p \in B_1 \cap B_2$ there is some $B_3 \in \mathcal{B}$ with $p \in B_3 \subset B_1 \cap B_2$.*

Theorem 0.2.8. *Let X be a topological space. A collection $\mathcal{B}$ of open sets in X forms a base if and only if for each $x \in X$ the collection $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$ is a neighbourhood base at x.*

If topologies are described by bases, then one can determined whether one topology is finer than another by the following criterion.

Lemma 0.2.3. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies τ and τ' on X respectively. The following statements are equivalent.*

(1) *τ' is finer than τ.*

(2) *For every $x \in X$ and every basis element $B \in \mathcal{B}$ with $x \in B$, there is a $B' \in \mathcal{B}'$ satisfying $x \in B' \subset B$.*

Definition 0.2.9. *For a topological space (X, τ) a **subbase** for τ is a collection $\mathcal{B}' \subset \tau$ for which the collection of all finite intersections of members from $\mathcal{B}'$ forms a base for τ.*

Theorem 0.2.9. *Any collection of subsets of X is a subbase for some topology on X.*

0.2.4 SUBSPACES

Definition 0.2.5. *Let (X, τ) be a topological space and let Y be a subset of X. The collection $\tau_Y = \{Y \cap U : U \in \tau\}$ is a topology for Y, called the **relative topology** for Y. Endowed with this topology, Y is said to be a **subspace** of X.*

For a topological space X when a topology is used on a subset of X without explicitly being described, it is assumed to be the relative topology. We now present some examples of subspaces.

Example 0.2.7.

(1) *The real line, regarded as the x-axis in $\mathbb{R}^2$, inherits its usual topology from $\mathbb{R}^2$.*

(2) *$\mathbb{Z}$ inherits the discrete topology as a subspace of $\mathbb{R}$.*

(3) *Any subspace of a discrete space is discrete.*

Theorem 0.2.10. *Let X be a topological space. If Y is a subspace of X, then the following statements hold.*

(1) *Let $H \subset Y$. H is open in Y if and only if $H = G \cap Y$, where G is open in X.*

(2) *Let $F \subset Y$. F is closed in Y if and only if $F = K \cap Y$, where K is closed in X.*

(3) *Let $F \subset Y$. Then $Cl_Y(E) = Y \cap Cl_X(E)$.*

(4) *If $a \in Y$, then V is a neighbourhood of a in Y if and only if $V = U \cap Y$, where U is a neighbourhood of a in X.*

(5) *For $a \in Y$ when $\mathcal{B}_a$ is a neighbourhood base at a in X, the collection $\{B \cap Y : B \in \mathcal{B}_a\}$ forms a neighbourhood base at a in Y.*

(6) *When $\mathcal{B}$ is a base for X, the collection $\{B \cap Y : B \in \mathcal{B}\}$ forms a base for Y.*

0.2.5 CONTINUOUS FUNCTIONS

In this section, we define continuous function on a topological space and study various properties of it.

Definition 0.2.6. *Let X and Y be topological spaces. Let $f : X \to Y$ be a function. f is said to be* ***continuous*** *at $x_0 \in X$ if for each open subset V of $f(x_0)$ in Y, there is a neighbourhood U of x_0 in X for which $f(U) \subset V$. f is said to be continuous on X if f is continuous at each $x \in X$.*

Theorem 0.2.11. *Let X and Y be topological spaces. Let $f : X \to Y$ be a function. The following statements are equivalent.*

(1) *f is a continuous function.*

(2) *For each open set H in Y $f^{-1}(H)$ is open in X.*

(3) *For each closed set K in Y $f^{-1}(K)$ is closed in X.*

(4) *For each subset E of X $f(Cl_X(E)) = Cl_Y(f(E))$.*

Theorem 0.2.12. *Let X, Y and Z be three topological spaces and let $g : X \to Y$ and $h : Y \to Z$ be continuous functions. Then the composition $h \circ g : X \to Z$ is a continuous function.*

Definition 0.2.10. *Let $f : X \to Y$ be a function and $A \subset X$. The* ***restriction of** f **to*** *A is a map of A into Y denoted by $f|_A$ and defined by $f|_A(a) = f(a)$ for each $a \in A$.*

Theorem 0.2.13. *Let A be a subset of X and let $f : X \to Y$ be a continuous function. Then $f|_A : A \to Y$ is continuous.*

Theorem 0.2.14. *Let $X = A_1 \cup A_2$, where A_1 and A_2 are both open (or both closed) in X. Let $f : X \to Y$ be a continuous function for which both $f|_{A_1}$ and $f|_{A_2}$ are continuous. Then f is continuous.*

The map which preserve X set-theoretically and topologically are called homeomorphisms.

Definition 0.2.11. *Let X and Y be two topological spaces and let $f : X \to Y$ be a function. f is a* ***homeomorphism*** *if f is one-one, onto and continuous and f^{-1} is also continuous. In this case, X and Y are said to be homeomorphic.*

If f is everything but onto, then X is said to be *embedded* in Y by f. X is embedded in Y by f if and only if f is a homeomorphism between X and some subspace of Y. In the following we present examples of homeomorphisms.

Example 0.2.8.

(1) *$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 1$ is a homeomorphism.*

(2) $f : (-1, 1) \to \mathbb{R}$ *defined by* $f(x) = \frac{x}{1-x^2}$ *is a homeomorphism.*

Theorem 0.2.15. *Let X and Y be two topological spaces. Let $f : X \to Y$ be a one-one and onto function. The following statements are equivalent.*

(1) *f is a homeomorphism.*

(2) *For each subset G of X $f(G)$ is open in Y if and only if G is open in X.*

(3) *For each subset F of X $f(F)$ is closed in Y if and only if F is closed in X.*

(4) *For each subset E of X $f(Cl_X(E)) = Cl_Y(f(E))$.*

0.2.6 THE PRODUCT TOPOLOGY ON $X \times Y$

Definition 0.2.12. *Let X_α be a set for each $\alpha \in A$. The* ***Cartesian product*** *of the sets X_α is the set*

$$\Pi_{\alpha \in A} X_\alpha = \{x : A \to \cup_{\alpha \in A} X_\alpha : x(\alpha) \in X_\alpha \text{ for each } \alpha \in A\}.$$

We denote this by ΠX_α when no confusion arise about the indexing set.

Definition 0.2.13. *The map $\pi_\beta : \Pi X_\alpha \to X_\beta$ defined by $\pi_\beta(x) = x_\beta$ is called the* ***projection map*** *of ΠX_α on X_β.*

We also call it the βth projection map.

Definition 0.2.14. *The product topology on ΠX_α is obtained by taking as a base for the open sets, sets of the form ΠU_α, where*

(1) *$U_\alpha \in \mathcal{B}_\alpha$, where for each α, $\mathcal{B}_\alpha$ is a base for the topology of X_α.*

(2) *For all but finitely many co-ordinates, $U_\alpha = X_\alpha$.*

Definition 0.2.15. *Let X and Y be two topological spaces. Let $f : X \to Y$ be a function. f is open (closed) map if for every open (closed) set G in X, $f(G)$ is open (closed) in Y.*

Theorem 0.2.16. *The βth projection map $\pi_\beta : \Pi X_\alpha \to X_\beta$ is continuous and open, but need not be closed.*

Theorem 0.2.17. *The product topology on ΠX_α is the weakest topology such that each projection map π_β is continuous.*

Theorem 0.2.18. *$f : X \to \Pi X_\alpha$ is a continuous function if and only if $\pi_\alpha \circ f$ is a continuous function for each $\alpha \in A$.*

0.2.7 THE METRIC TOPOLOGY

Definition 0.2.7. *Let X be a set. A function $d : X \times X \to \mathbb{R}$ is said to be a* ***metric*** *on X if the following conditions are satisfied.*

(1) $d(x_1, x_2) \geq 0$ *for all $x_1, x_2 \in X$. Equality holds if $x_1 = x_2$.*

(2) $d(x_1, x_2) = d(x_2, x_1)$ *for all $x_1, x_2 \in X$.*

(3) *(Triangle inequality)* $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$ *for all x_1, x_2 and $x_3 \in X$.*

The pair (X, d) is called a metric space. The number $d(x, y)$ is often called the ***distance between** x **and** y* in the metric d.

Definition 0.2.8. *Given a metric space (X, d) and a $\delta > 0$, consider the collection $B_d(x, \delta) = \{y : d(x, y) < \delta\}$ of all points y whose distance from x is less than δ. It is called the* ***δ-ball centered at** x*.

When no confusion arises, we omit the metric d from the notation and write this ball simply as $B(x, \varepsilon)$.

One can impose a topology on set X in terms of metric d in the following way.

Definition 0.2.9. *Let (X, d) be a metric space. The collection of all δ-balls $B_d(x, \delta)$, for $x \in X$ and $\delta > 0$, is a basis for a topology on X, called the* ***metric topology*** *induced by d.*

The definition of the metric topology can be rephrased as follows.

Definition 0.2.10. *A set U is* ***open*** *in the metric topology induced by d if and only if for each $y \in U$, there exists a $\delta > 0$ satisfying $B_d(y, \delta) \subset U$.*

Example 0.2.9.

(1) *(**Discrete Metric**) Given a set X, define*

$$d(x_1, x_2) = \begin{cases} 1 & \text{if} \quad x_1 \neq x_2 \\ 0 & \text{if} \quad x_1 = x_2. \end{cases}$$

Then d is a discrete metric. The topology it induces is the discrete topology.

(2) *The standard metric on the real numbers $\mathbb{R}$ is defined by the equation $d(x_1, x_2) = |x_1 - x_2|$. The topology it induces is the standard topology.*

(3) *Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the norm of x is given by $||x|| = \left(x_1^2 + \cdots + x_n^2\right)^{\frac{1}{2}}$ and the* ***Euclidean metric** d **on** $\mathbb{R}^n$* *is given by the equation $d(x, y) = ||x - y|| = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{\frac{1}{2}}$. The* ***square metric*** *ρ is given by the equation $\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.*

Definition 0.2.11. *Let X be a topological space. X is called* ***metrizable*** *if there is a metric d on X that induces the topology of X.*

Metrizability of a space depends only on the topology of the space but properties that involve a specific metric for X in general do not. For instance, we have the following definition in a metric space.

Definition 0.2.12. *Let (X, d) be a metric space. A subset A of X is called* ***bounded*** *if there is some number M such that $d(a, b) \leq M$ for any $a, b \in A$. If A is bounded and nonempty, then the number $diam(A) = \sup\{d(a, b) : a, b \in A\}$ is defined as the* ***diameter*** *of A.*

Observe that boundedness of a set is not a topological property but it depends on the particular metric d used for X. If (X, d) is a metric space, then there exists a metric $\bar{d}$ that gives the topology of X relative to which every subset of X is bounded.

Theorem 0.2.19. *Let (X, d) be a metric space. Define $\bar{d} : X \times X \to \mathbb{R}$ by $\bar{d}(x_1, x_2) = \min\{d(x_1, x_2), 1\}$. Then $\bar{d}$ is a metric such that the topologies induced by d and $\bar{d}$ are same and $\bar{d}$ is called the standard bounded metric corresponding to d.*

Definition 0.2.13. *For a set X, a* ***sequence*** *(or an* ***infinite sequence****) in X is a function $x : \mathbb{N} \to X$. The value of x at $n \in \mathbb{N}$ is denoted by x_n rather than $x(n)$, and call it the n-th coordinate of x. We denote x itself by the symbol $\{x_1, x_2, \ldots\}$ or $\{x_n\}_{n \in \mathbb{N}}$.*

Definition 0.2.14. *The point $x \in X$ is the* ***limit of a sequence*** *of real numbers $\{x_n\}_{n \in \mathbb{N}}$ if for each $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ satisfying $|x_n - x| < \varepsilon$ whenever $n \geq n_0$. We denoted this fact by $\lim_{n \to \infty} x_n = x$.*

In an arbitrary topological space X, a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X **converges to the point** x of X if for every neighbourhood U of x, there exists a $n_0 \in \mathbb{N}$ satisfying $x_n \in U$ for all $n \geq n_0$. Observe that a sequence can converge to more than one point in an arbitrary topological space.

The convergence of a sequence in a metric space is defined as follows.

Definition 0.2.15. *Let (X, d) be a metric space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in X is said to* ***converge*** *to a point x in X if for each $\varepsilon > 0$ there is a $n_0 \in \mathbb{N}$ satisfying $d(x_n, x) < \varepsilon$ whenever $n \geq n_0$. When $\{x_n\}_{n \in \mathbb{N}}$ is convergent to x we often write it as $x_n \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$. If there is no such $x \in X$, then $\{x_n\}_{n \in \mathbb{N}}$ is said to* ***diverge****.*

0.2.8 COMPACTNESS

The notion of compactness plays an important role and extensively used in mathematics.

Definition 0.2.16. *A collection $\mathcal{A}$ of subsets of a topological space X is called a* ***cover*** *of X, or to be a* ***covering of*** *X if $X = \cup_{A \in \mathcal{A}} A$. It is called an* ***open covering of*** *X if its elements are open subsets of X.*

Definition 0.2.17. *A topological space X is called* ***compact*** *if each open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.*

Example 0.2.10.

(1) *Any topological space which contains finite number of points is compact.*

(2) *The interval $[0,1]$ is compact but the interval $(0,1]$ is not.*

(3) *The subspace $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$ is compact whereas the subspace $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not.*

(4) *The real line $\mathbb{R}$ is not compact.*

Definition 0.2.16. *A topological space X is called* ***countably compact*** *when each countable open cover has a finite subcover.*

Obviously, X is countably compact and Lindelöf if and only if X is compact.

Theorem 0.2.20. *X is countably compact if and only if each nested sequence $C_1 \supset C_2 \supset \cdots$ of closed nonempty subsets of X has a nonempty intersection.*

There is another criterion for a space to be compact which is formulated in terms of closed sets rather than open sets.

Definition 0.2.18. *A family $\mathcal{A}$ of subsets of X is said to have the* ***finite intersection property*** *if for every finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$, the intersection $A_1 \cap \ldots \cap A_n$ is nonempty.*

Theorem 0.2.21. *Let X be a topological space. Then X is compact if and only if for every family $\mathcal{A}$ of closed subsets of X having the finite intersection property, the intersection $\bigcap_{A \in \mathcal{A}} A$ of all the elements of $\mathcal{A}$ is nonempty.*

Let Y be a subset of X. A collection $\mathcal{A}$ of subsets of X is called a **cover** of Y if the union of its elements contains Y.

Lemma 0.2.4. *For a topological space X a subspace Y is compact if every covering of Y by sets open in X contains a finite subcollection covering Y.*

We now present some facts about subspaces.

Theorem 0.2.22.

(1) *For a compact space every closed subspace is compact.*

(2) *For a Hausdorff space every compact subspace is closed.*

(3) *For a Hausdorff space disjoint compact subsets can be separated by disjoint open sets.*

(4) *Let X be completely regular. Let $A, B \subset X$ be closed and $A \cap B = \emptyset$. If A is compact then there is a continuous function $f : X \to [0,1]$ satisfying $f(A) = \{0\}$ and $f(B) = \{1\}$.*

(5) *If $A \times B \subset X \times Y$ is a compact set contained in an open set W in $X \times Y$, then open sets U and V in X and Y, respectively, can be found for which $A \times B \subset U \times V \subset W$ hold.*

Theorem 0.2.23.

(1) *The image of a compact space under a continuous function is compact.*

(2) *A nonempty product space is compact if and only if each component space is compact.*

(3) *Let a function $f : X \to Y$ be bijective and continuous. When X is a compact space and Y is Hausdorff, then the function f is a homeomorphism.*

Compact subspace of $\mathbb{R}^n$, $n \in \mathbb{N}$, is characterized in the following theorem.

Theorem 0.2.24. *A subspace A of $\mathbb{R}^n$ is a compact space if and only if A is closed and bounded in the Euclidean metric d or in the square metric ρ.*

Definition 0.2.17. *A topological space X is called **locally compact** if each point of X has a neighbourhood base which consists of compact sets.*

Example 0.2.11.

(1) *The real line $\mathbb{R}$ is locally compact.*

(2) *The subspace $\mathbb{Q}$ of $\mathbb{R}$ is not locally compact.*

(3) *$\mathbb{R}^n$ is a locally compact space.*

Theorem 0.2.25.

(1) *Let X be a Hausdorff space. Then X is locally compact if and only if each point of X has a compact neighbourhood.*

(2) *Let X be locally compact and let $f : X \to Y$ be continuous and open map. Then Y is a locally compact.*

Definition 0.2.19. *A topological space X is called **σ-compact** (**σ-countably compact**) if it is the union of a countable set of compact (respectively, countably compact) subspaces.*

Definition 0.2.20. *A topological space X is called **hemicompact** if there is a countable collection $\mathcal{C}$ of compact subsets of X for which every compact subset of X is contained in some member of $\mathcal{C}$.*

Definition 0.2.21. *A topological space X is called* ***limit point compact*** *if every infinite subset of X has a limit point.*

Definition 0.2.22. *A topological space X is called* ***sequentially compact*** *if every sequence of points of X has a convergent subsequence.*

Theorem 0.2.26. *Let X be a metrizable space. The following statements are equivalent.*

(1) *X is compact.*

(2) *X is limit point compact.*

(3) *X is sequentially compact.*

0.3 SEPARABILITY

Definition 0.3.1. *A topological space X which contains a countable dense subset is called separable.*

Theorem 0.3.1. *A finite or countably infinite product of separable spaces is separable.*

Theorem 0.3.2. *Let X be a metric space and $Y \subseteq X$. If X is separable, then Y is also separable.*

Theorem 0.3.3. *Let X be a metric space. X is separable if and only if X has a countable open base.*

Definition 0.3.2. *A metric space is said to be totally bounded if for every $\varepsilon > 0$ it can be decomposed into a finite number of sets of diameter $< \varepsilon$.*

Theorem 0.3.4. *Let X be a metric space. If X is totally bounded, then X is separable.*

Theorem 0.3.5. *Let X be a metric space. If X is separable, then X is homeomorphic to a totally bounded space.*

Theorem 0.3.6. *Let X be a metric space. If X is separable, then X is the continuous and one-to-one image of a subset of the Cantor discontinuum as well as of a set of irrational numbers.*

Theorem 0.3.7 (Generalized Bolzano-Weierstrass theorem). *Every sequence of subsets of a separable space contains a convergent subsequence.*

0.4 FUNCTION SPACES

We define the notion of pointwise and uniform convergence of sequence of functions.

Definition 0.4.1. *Let $f_n : X \to Y$, $n \in \mathbb{N}$, be a function from a set X to the metric space Y. Let d be the metric for Y. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to* ***converge pointwise*** *to the function $f : X \to Y$ if given $x \in X$ and $\varepsilon > 0$ there is a $n_0 \in \mathbb{N}$ (depending on both x and ε) satisfying $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq n_0$ and we denote this by $f_n(x) \to f(x)$ on X.*

Definition 0.4.2. *Let $f_n : X \to Y$, $n \in \mathbb{N}$, be a function from a set X to the metric space Y. Let d be the metric for Y. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to* ***converge uniformly*** *to the function $f : X \to Y$ if given $\varepsilon > 0$ there is a $n_0 \in \mathbb{N}$ satisfying $d(f_n(x), f(x)) < \varepsilon$ for all $n \geq n_0$ and for all $x \in X$ and we denote this by $f_n \to f$* ***uniformly on*** *X.*

Example 0.4.1.

(1) ***(A Sequence of Continuous Functions with a Discontinuous Limit Function)****.*

Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$. Then f_n converges pointwise but not uniformly to f on $\mathbb{R}$, where

$$f(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

Observe that each f_n is continuous but f is not continuous.

We have the following theorem about uniformly convergent sequences.

Theorem 0.4.1 (Uniform Limit Theorem). *Let $f_n : X \to Y$, $n \in \mathbb{N}$, be a continuous function from the topological space X to the metric space Y. If the sequence $\{f_n\}$ converges uniformly to f, then f is continuous.*

Theorem 0.4.2 (The Cauchy Condition). *Let $f_n : X \to Y$, $n \in \mathbb{N}$, be a function between two metric spaces X and Y. Let d be the metric on Y. Then $\{f_n\}$ is said to converge uniformly if and only if the following condition (called the* ***Cauchy condition****) is satisfied. For every $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ satisfying $d(f_n(x), f_m(x)) < \varepsilon$ for every $m \geq n_0$, $n \geq n_0$ and for every x in X.*

Now we define the notion of uniform convergence of infinite series of real valued functions.

Definition 0.4.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued functions defined on a metric space X. Let $s_n : X \to \mathbb{R}$ be defined by $s_n(x) = \sum_{k=1}^{n} f_k(x)$ for each x in X and for each $n \in \mathbb{N}$. If there is a function $f : X \to \mathbb{R}$ for which $s_n \to f$ uniformly on X, then the series $\sum_{n=1}^{\infty} f_n(x)$* ***converges uniformly*** *on X and we write $\sum_{n=1}^{\infty} f_n(x) = f(x)$ (uniformly on X).*

Theorem 0.4.3 (Cauchy Condition for Uniform Convergence of Series). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued functions defined on a metric space X. The infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on X if and only if for each $\varepsilon > 0$ there is a $n_0 \in \mathbb{N}$ for which*

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| < \varepsilon,$$

for all $n \geq n_0$ and $p = 1, 2, \ldots$, and every x in X.

Theorem 0.4.4 (Weierstrass M-test). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued functions defined on a metric space X and $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers such that $0 \leq |f_n(x)| \leq M_n$ for $n = 1, 2, \ldots$ and for every x in X. Then f_n converges uniformly on X if M_n converges.*

Theorem 0.4.5. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous real valued functions defined on a metric space X for which the infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to f on X. Then f is also continuous on X.*

For two topological spaces X and Y let Y^X ($C(X,Y)$) denote the collection of all functions (continuous functions) from the space X into the space Y. Let $\mathcal{C}$ be a subcollection of Y^X.

Definition 0.4.1. *A subcollection $\mathcal{C}$ of Y^X has the* ***pointwise topology*** *[115] if it is endowed with the subspace topology induced by the product topology on Y^X.*

We denote this topology by τ_p.

Definition 0.4.2. *The* ***compact-open topology*** *[115] on a subcollection $\mathcal{C}$ of Y^X is the topology having subbase the sets $(K, U) = \{f \in \mathcal{C} : f(K) \subset U\}$ for K compact in X and U open in Y.*

We denote this topology by τ_k.

When $C(X,Y)$ is endowed with the topology of pointwise convergence (the compact-open topology), we denote this space by $C_p(X,Y)$ ($C_k(X,Y)$). When $Y = \mathbb{R}$, we denote the space $C_p(X,Y)$ ($C_k(X,Y)$) by $C_p(X)$ ($C_k(X)$).

Basic open sets of $C_p(X,Y)$ are of the form

$$W(x_1, \ldots, x_n; U_1, \ldots, U_n) = \{g \in C(X,Y) : g(x_i) \in U_i, i = 1, \ldots, n\},$$

where $x_1, \ldots, x_n \in X$ and $U_1, \ldots, U_n$, $n \in \mathbb{N}$, are open subsets of Y. For a function $g \in C_p(X)$, the basic neighbourhood of the point g is

$$W(g, F, \varepsilon) = \{h \in C_p(X) : |g(x) - h(x)| < \varepsilon \text{ for all } x \in F\},$$

where F is finite subset of X and $\varepsilon > 0$.

Basic open sets of $C_k(X,Y)$ are of the form

$$W(K_1, \ldots, K_n; U_1, \ldots, U_n) = \{g \in C(X,Y) : g(x_i) \in U_i, i = 1, \ldots, n\},$$

where $K_1, \ldots, K_n$ are compact subsets of X and $U_1, \ldots, U_n$, $n \in \mathbb{N}$, are open subsets of Y. For a function $g \in C_k(X)$, the basic neighbourhood of the point g is

$$W(g, K, \varepsilon) = \{h \in C_k(X) : |g(x) - h(x)| < \varepsilon \text{ for all } x \in K\},$$

where $K \subset X$ is compact and $\varepsilon > 0$.

For $x \in X$, we denote $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$ [62], and Σ_x stands for the collection of all sequences that converge to $x \in X$ [17]. We now list some topological properties of X.

Definition 0.4.3. *A topological space X is said to have **countable tightness** (CT for short) if for every $x \in X$ and every $A \in \Omega_x$, there is a countable subset B of A such that $B \in \Omega_x$ [4].*

Definition 0.4.4. *A topological space X is said to have **countable fan tightness** (**countable strong fan tightness**) at x if X satisfies $\mathsf{S}_{\mathrm{fin}}(\Omega_x, \Omega_x)$ (respectively, $\mathsf{S}_1(\Omega_x, \Omega_x)$) [4, 87]. For convenience we denote this property by CFT (respectively, CSFT).*

Definition 0.4.5. *A topological space X is said to have **countable T-tightness** [53] if for each uncountable regular cardinal ρ and each increasing sequence $\{A_\alpha : \alpha < \rho\}$ of closed subsets of X the set $\cup\{A_\alpha : \alpha < \rho\}$ is closed.*

Definition 0.4.6. *The **set-tightness** $t_s(X)$ of a topological space X is the smallest infinite cardinal ρ such that if $A \subseteq X$ and $x \in \overline{A} \setminus A$, then there exists a family $\{A_\alpha : \alpha < \rho\}$ of subsets of A satisfying $x \notin \overline{A_\alpha}$ for any $\alpha < \rho$ and $x \in \overline{\cup\{A_\alpha : \alpha < \rho\}}$ [3, 53].*

Definition 0.4.7. *A topological space X is said to be **Fréchet–Urysohn** (FU for short) if for each subset A of X and each $x \in \overline{A}$, there is a sequence in A converging to x [41].*

Definition 0.4.8. *A topological space X is said to be **strictly Fréchet-Urysohn** (SFU for short) if $\mathsf{S}_1(\Omega_x, \Sigma_x)$ holds for each $x \in X$ [41].*

Definition 0.4.9. *A topological space X is called **weakly Fréchet-Urysohn** [38] if for $x \in \overline{A} \setminus A$ and $A \subset X$ there exists a countable infinite disjoint family of finite subsets $\{A_n : n \in \mathbb{N}\}$ of A such that every neighbourhood of x intersects A_n for all but finitely many $n \in \mathbb{N}$. This property is also known as **Reznichenko property** of X.*

Definition 0.4.10. *For $x \in X$, a collection $\mathcal{A}$ of subsets of a topological space X is called a **π-network** at x if every neighbourhood of x contains some element of $\mathcal{A}$.*

Definition 0.4.11. *A topological space X is said to have **countable fan tightness for finite sets** ($\mathsf{CFT}_{\mathrm{fin}}$ for short) if for each point $x \in X$ and any sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of π-networks at x consisting of finite subsets of X, there is a finite subset $\mathcal{B}_n \subseteq \mathcal{A}_n$ for each n such that $\bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is a π-network at x [92].*

Definition 0.4.12. *A topological space X is said to have **countable strong fan tightness for finite sets** ($\mathsf{CSFT}_{\mathrm{fin}}$ for short) if for each point $x \in X$ and any sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of π-networks at x consisting of finite subsets of X, there is an $A_n \in \mathcal{A}_n$ for each n such that $\{A_n : n \in \mathbb{N}\}$ is a π-network at x [92].*

Definition 0.4.13. *A topological space X is said to be* ***Fréchet-Urysohn for finite sets*** (FU_{fin} *for short) if for each $x \in X$ and each $\mathcal{A}$, a π-network at x consisting of finite subsets of X, $\mathcal{A}$ contains a subfamily that converges to x [86] (see also [46]).*

Note that FU_{fin} implies FU. An equivalent formulation of FU_{fin} is that for every $x \in X$ and every sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of π-networks at x consisting of finite subsets of X there are $A_n \in \mathcal{A}_n$, $n \in \mathbb{N}$, such that $\{A_n : n \in \mathbb{N}\}$ converges to x [86, Proposition 6].

Let $\Gamma, \Omega, \mathcal{K}$ and Γ_k denote the classes of γ-cover, ω-cover, k-cover and γ_k-cover of X; for two families $\mathcal{A}, \mathcal{B}$ of sets $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ and $\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{B})$ denote the selection principles (see Chapter 1 for definitions). A space X is said to be *Lindelöf* (*ω-Lindelöf* or *k-Lindelöf*) if each open cover (ω-cover or k-cover) contains a countable set which is a cover (ω-cover or k-cover) of X [63].

Theorem 0.4.1. *For a topological space X the following statements hold.*

(1) *The space $C_p(X)$ has* CT *if and only if X is ω-Lindelöf [4].*

(2) *The space $C_p(X)$ has* CFT *if and only if X satisfies $\mathsf{S}_{\text{fin}}(\Omega, \Omega)$ [41].*

(3) *The space $C_p(X)$ has* CSFT *if and only if X satisfies $\mathsf{S}_1(\Omega, \Omega)$ [41].*

Theorem 0.4.2. *For a topological space X the following statements hold.*

(1) *The space $C_p(X)$ is* FU *if and only if X satisfies $\mathsf{S}_1(\Omega, \Gamma)$ [2].*

(2) *The space $C_p(X)$ is* SFU *if and only if X satisfies $\mathsf{S}_1(\Omega, \Gamma)$ [87].*

Let $\mathcal{U}$ be a family of cozero subsets of X and $\mathcal{Z}(\mathcal{U}) = \{Z(U) : U \in \mathcal{U}\}$ be a family of zero-sets of X.

Theorem 0.4.6 ([89])**.** *For a Tychonoff space X and an infinite cardinal ρ the following statements are equivalent.*
(1) *$t_s(C_p(X)) \leq \rho$.*
(2) *For each collection $\mathcal{U}$ open subsets of X, if $\mathcal{Z}(\mathcal{U})$ is an ω-cover of X, then there exists a family $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of subsets of $\mathcal{U}$ such that no $\mathcal{Z}(\mathcal{U}_\alpha)$ is an ω-cover of X whereas $\cup_{\alpha<\rho}\mathcal{U}_\alpha$ is an ω-cover of X.*

Theorem 0.4.3. *For a topological space X the following statements hold.*

(1) *The space $C_k(X)$ has* CT *if and only if X is k-Lindelöf [71].*

(2) *The space $C_k(X)$ has* CFT *if and only if X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K})$ [69].*

(3) *The space $C_k(X)$ has* CSFT *if and only if X satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$ [61].*

Theorem 0.4.4. *The space $C_k(X)$ is* SFU *if and only if X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ [69].*

Theorem 0.4.7 ([61]). *For a Tychonoff space X the following statements are equivalent.*
(1) $C_k(X)$ *has countable T-tightness.*
(2) *For each regular cardinal ρ and each increasing sequence $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of families of open subsets of X such that $\cup_{\alpha<\rho}\mathcal{U}_\alpha$ is a k-cover of X there is a $\beta < \rho$ so that $\mathcal{U}_\beta$ is a k-cover of X.*

CHAPTER 1

INTRODUCTION

We follow the notations and terminologies of [4, 36, 48, 72]. A bornology $\mathfrak{B}$ on a set X is a family of subsets of X satisfying the following three axioms:

1. $\mathfrak{B}$ is a cover of X;

2. If $B \in \mathfrak{B}$ and $B' \subseteq B$, then $B' \in \mathfrak{B}$;

3. $\mathfrak{B}$ is closed under taking finite unions.

A base $\mathfrak{B}_0$ for a bornology $\mathfrak{B}$ is a subfamily of $\mathfrak{B}$ such that for $B \in \mathfrak{B}$ there is a $B_0 \in \mathfrak{B}_0$ satisfying $B \subseteq B_0$. A base is called closed (compact) if all of its members are closed (compact). A bornology is said to have a *countable base* [48] if it has a base which consists of a sequence of bounded sets. In this case the sequence is always assumed to be increasing.

We list a few natural bornologies on a set X:

1. The family $\mathcal{F}$ of all finite subsets of X, the smallest bornology on X.

2. The family of all nonempty subsets of X, the largest bornology on X.

3. The family of all nonempty d-bounded subsets of X, where d is a metric on X.

4. The family K of nonempty subsets of X with compact closure.

The notion of bornology works well with the vector space structure. There is a well developed theory of bornological vector spaces in the literature (see [48]). For a vector space V

over $\mathbb{K}$ a bornology $\mathfrak{B}$ on V is said to be a *vector bornology* [48] on V if $\mathfrak{B}$ is stable under vector addition, homothetic transformation and formation of circled hulls, i.e., for any $A, B \in \mathfrak{B}$ $A + B \in \mathfrak{B}$, $\lambda A \in \mathfrak{B}$ for $\lambda \in \mathbb{K}$ and $\cup_{|\alpha| \leq 1} \alpha A \in \mathfrak{B}$.

The pair $(V, \mathfrak{B})$ consisting of a vector space V and a vector bornology $\mathfrak{B}$ is called *bornological vector space*. A vector bornology $\mathfrak{B}$ on V is said to be a *convex vector bornology* if it is stable under the formation of convex hulls. The pair $(V, \mathfrak{B})$, where $\mathfrak{B}$ is a convex vector bornology, is called a *convex bornological vector space*.

For two bornological sets X and Y a map $f : X \to Y$ is called a *bounded map* if image of every bounded subset of X under f is bounded in Y. If $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are two bornologies on X, then $\mathfrak{B}_1$ is said to be *finer bornology* than $\mathfrak{B}_2$ or $\mathfrak{B}_2$ is said to be *coarser bornology* than $\mathfrak{B}_1$ if the identity map $Id : (X, \mathfrak{B}_1) \to (X, \mathfrak{B}_2)$ is bounded.

The notion of convergence is associated with every bornological vector spaces. For a bornological vector space V a sequence $\{x_n\}$ in V is said to *converge bornologically* to 0 if there exists a circled bounded subset B of V and a sequence $\{\lambda_n\}$ of scalars converging to 0 such that $x_n \in \lambda_n B$ for every n.

The notion of bornological convergence is related to topological convergence in the following way. Every bornologically convergent sequence is topologically convergent. But the converse is in general false. Under certain conditions on V every topologically convergent sequence is bornologically convergent [48].

There are some standard methods by which one can construct new bornologies from given bornologies [48]. Let $\{X_n : n \in \mathbb{N}\}$ be a family of sets and $\mathfrak{B}_n$ is a bornology on X_n for each $n \in \mathbb{N}$. The *product bornology* on $\Pi_{n \in \mathbb{N}} X_n$ has a base consisting of sets of the form $B = \Pi_{n \in \mathbb{N}} B_n$, where $B_n \in \mathfrak{B}_n$ for all $n \in \mathbb{N}$. Let $\mathfrak{B}$ be a bornology on a set X and Y be a subset of X. The *induced bornology* on Y by $\mathfrak{B}$ has base consisting of sets of the form $B \cap Y$ for $B \in \mathfrak{B}$. If $\{\mathfrak{B}_i : i \in \mathcal{I}\}$ is a family of bornologies on a set X, where $\mathcal{I}$ is a nonempty index set, then $\{\cap_{i \in \mathcal{I}} B_i : B_i \in \mathfrak{B}_i\}$ forms a base for the *intersection bornology*. For a set X and a family $\mathcal{A}$ of subsets of X the *bornology generated by* $\mathcal{A}$ is the intersection of all bornologies containing $\mathcal{A}$. Let $\mathfrak{B}$ be a bornology on X and Y be another set and let $\phi : X \to Y$ be an onto map. Then $\phi(\mathfrak{B})$ is a base for the *image bornology* of $\mathfrak{B}$ under ϕ. If Y is the quotient of the set X by an equivalence relation, then Y endowed with $\phi(\mathfrak{B})$ is called the *bornological quotient* of X and the bornology $\phi(\mathfrak{B})$ is called the *quotient bornology* of $\mathfrak{B}$ by the given equivalence relation.

In 2009, Beer and Levi [14] during their bornological investigations introduced the notion of strong uniform continuity on a bornology. Let (X, d) and (Y, ρ) be two metric spaces.

> *A mapping $f : X \to Y$ is* ***strongly uniformly continuous*** *on a subset B of X if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x_1, x_2) < \delta$ and $\{x_1, x_2\} \cap B \neq \emptyset$ imply $\rho(f(x_1), f(x_2)) < \varepsilon$.*

For a bornology $\mathfrak{B}$ on X, f is called strongly uniformly continuous on $\mathfrak{B}$ if f is strongly uniformly continuous on B for each $B \in \mathfrak{B}$. Some facts on strong uniform continuity which easily follows are:

1. f is strongly uniformly continuous on $\{x\}$ if and only if f is continuous at x.

2. If f is continuous on a compact set $K \subseteq X$, then f is strongly uniformly continuous on K.

Let $\mathfrak{B}$ be a bornology with closed base on a metric space (X, d). For $B \in \mathfrak{B}$ and $\delta > 0$, let $B^\delta = \bigcup_{x \in B} S(x, \delta)$, where $S(x, \delta) = \{y \in X : d(x, y) < \delta\}$. For every $B \in \mathfrak{B}$ and $\delta > 0$, $\overline{B^\delta} \subseteq B^{2\delta}$.

They also introduced the notion of the topology of strong uniform convergence on a bornology for function spaces. Let $\mathfrak{B}$ be a bornology with a closed base on X.

*The topology of **strong uniform convergence** $\tau^s_{\mathfrak{B}}$ is determined by a uniformity on Y^X with a base consisting of all sets of the form*

$$[B, \varepsilon]^s = \{(f, g) : \exists \delta > 0 \text{ for every } x \in B^\delta, \rho(f(x), g(x)) < \varepsilon\},$$

for $B \in \mathfrak{B}, \varepsilon > 0$ [14].

The topology of strong uniform convergence $\tau^s_{\mathfrak{B}}$ is finer than the topology of pointwise convergence τ_p if $\mathfrak{B} = \mathcal{F}$. Also $\tau_p \leqslant \tau^s_{\mathfrak{B}} \leqslant \tau_k$ if $\mathfrak{B}$ has a compact base and $\tau_k = \tau^s_{\mathfrak{B}}$ if $\mathfrak{B} = \mathrm{K}$ [18].

The notion of the topology of strong uniform convergence on a bornology is of much significance. For a sequence of continuous functions, pointwise convergent to a limit function, the limit function is not necessarily continuous. Therefore the natural question always has been what can be added to a sequence of continuous functions, which is pointwise convergent, to preserve continuity of the limit function? In [5, 6], a necessary and sufficient condition was formulated by Arzelá for the continuity of a series of continuous functions defined in a compact interval of the real line. In 1926, Hobson [47] extended Arzelá's Theorem to closed and bounded sets of the real number space. In 1948, Alexandroff [1] investigated the question for a sequence of continuous functions from a topological space X to a metric space Y. Subsequently Bartle [9] extended Arzelá's Theorem to nets of real valued continuous functions on a topological space. Since then many authors have investigated in this direction (see [78, 79, 80]). In 1997, Ewert [35] defined a new type of convergence on spaces which is not uniformizable in terms of open covers of Y. This convergence has been shown to preserve continuity. In 2009, in the realm of metric spaces another necessary and sufficient condition was given by Beer and Levi [14] through the notion of strong uniform convergence on bornologies under the assumption that the bornology $\mathfrak{B} = \mathcal{F}$.

Let (X, d) and (Y, ρ) be metric spaces. Let $\{f_n\}$ be a sequence of continuous functions pointwise convergent to a function f. Then f is continuous if and only if the sequence $\{f_n\}$ is $\tau^s_{\mathcal{F}}$-convergent to f [14, Corollary 6.8].

On the other hand theory of Selection Principles is a branch of mathematics having a long history which goes back to the works by Borel, Menger, Hurewicz, Rothberger, Seirpiński in 1920-1930's. Since the initiation of systematic study of this field by Scheepers [95] (see also [54]), this beautiful area of mathematics have become more popular and attracted a big number of researchers in the last twenty-thirty years. Selection principles theory has since got much performances as it has been shown to have connections with several mathematical areas such as Set theory, General topology, Game theory, Ramsey theory, Functional analysis.

The classical selection principles are formulated in general form in [54, 95] as follows. For $\mathcal{A}$ and $\mathcal{B}$ be two families of sets we have the following selection principles.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$: *For every sequence* $\{A_n : n \in \mathbb{N}\}$ *of elements from* $\mathcal{A}$ *there exists a* $b_n \in A_n$ *for each* n *for which the sequence* $\{b_n : n \in \mathbb{N}\}$ *belongs to* $\mathcal{B}$.

$\mathsf{S}_{\text{fin}}(\mathcal{A}, \mathcal{B})$: *For every sequence* $\{A_n : n \in \mathbb{N}\}$ *of elements from* $\mathcal{A}$ *there exists a finite (possibly empty) set* $B_n \subseteq A_n$ *for each* n *for which the set* $\bigcup_{n\in\mathbb{N}} B_n$ *belongs to* $\mathcal{B}$.

We also consider the following selection principles.

$\mathsf{U}_{\text{fin}}(\mathcal{A}, \mathcal{B})$: *For every sequence* $\{A_n : n \in \mathbb{N}\}$ *of elements from* $\mathcal{A}$ *there exists a finite (possibly empty) set* $B_n \subseteq A_n$ *for each* n *for which either* $\{\cup B_n : n \in \mathbb{N}\} \in \mathcal{B}$ *or for some* n, $\cup B_n = X$ *([54, 95]).*

Let $\mathcal{O}$ denote the collection of all open covers of X. The selection principle $\mathsf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$ is defined as the Menger covering property [51, 67] and $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ is defined as the Rothberger covering property (or C'' property) [82].

There are two more important classical covering properties in the theory of selection principles namely the Hurewicz property and the Gerlits-Nagy property. These are proved to be S_{fin}-type and S_1-type properties [62] respectively.

The ***Hurewicz property***: *For every sequence* $\{\mathcal{U}_n : n \in \mathbb{N}\}$ *of open covers of* X *we have a sequence* $\{\mathcal{V}_n : n \in \mathbb{N}\}$ *of finite sets with* $\mathcal{V}_n \subseteq \mathcal{U}_n$ *for each* n *such that for each* x *in* X $x \in \cup\mathcal{V}_n$ *for all but finitely many* n *[51, 52].*

The ***Gerlits-Nagy property*** *is a conjunction of the Hurewicz covering property and Rothberger covering property [41].*

There are some two person infinite games naturally associated to the above selection principles. In these games we are mainly interested in the following question: Does the player ONE or TWO have a winning strategy?

If the answers to the above questions are no, then the above game is said to be an *undetermined game*. For the study of selection principles *determined* and *undetermined games* both turn out to be very useful tools. The games which are naturally associated to the selection principles $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ and $\mathsf{S}_{\text{fin}}(\mathcal{A},\mathcal{B})$ respectively introduced above are given below.

> $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ *denotes the game for two players, ONE and TWO, who play a round for each* $n \in \mathbb{N}$. *In the n-th round ONE chooses a set* A_n *from* $\mathcal{A}$ *and in response TWO chooses an element* $b_n \in A_n$. *TWO wins the play* $\{A_1, b_1, \dots, A_n, b_n, \dots\}$ *if* $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$. *Otherwise ONE wins.*

> $\mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{B})$ *denotes the game where in the n-th round ONE chooses a set* A_n *from* $\mathcal{A}$ *and in response TWO chooses a finite (possibly empty) set* $B_n \subseteq A_n$. *TWO wins the play* $\{A_1, B_1, \dots, A_n, B_n, \dots\}$ *if* $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$. *Otherwise ONE wins.*

The game $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$ was explicitly defined by Telgársky in [107]. In [51, 52], Hurewicz observed that there is a natural connection between the Menger property and the game $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$. In fact, in [51], Hurewicz implicitly proved that the principle $\mathsf{S}_{\text{fin}}(\mathcal{O},\mathcal{O})$ is equivalent to the fact that ONE has no winning strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$.

Galvin explicitly defined the game $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ in [40] and Pawlikowski in [83] proved that ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ if and only if X satisfies $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. Considering certain classes of covers namely Γ, Ω, Γ_k and $\mathcal{K}$ (definitions are given below) many game theoretic results are proved. ONE has no winning strategy in the game $\mathsf{G}_1(\Omega,\Omega)$ ($\mathsf{G}_{\text{fin}}(\Omega,\Omega)$) if and only if X satisfies $\mathsf{S}_1(\Omega,\Omega)$ ($\mathsf{S}_{\text{fin}}(\Omega,\Omega)$) [96]. ONE has no winning strategy in the game $\mathsf{G}_1(\Omega,\Gamma)$ if and only if X satisfies $\mathsf{S}_1(\Omega,\Gamma)$ [41, 95]. Also ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{K},\Gamma)$ ($\mathsf{G}_1(\mathcal{K},\Gamma_k)$) if and only if X satisfies $\mathsf{S}_1(\mathcal{K},\Gamma)$ ($\mathsf{S}_1(\mathcal{K},\Gamma_k)$) [31, 64].

When the player TWO has a winning strategy Telgágarsky [105, 106] proved the following results. For a first countable space X TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ if and only if X is countable. For a metrizable space X TWO has a winning strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{O},\mathcal{O})$ if and only if X is σ-compact.

The Ramseyan partition relations play an important role in the theory of Selection Principles. Several selection principles are characterized using the Ramseyan partition relations. We now recall definitions of some symbols related to Ramseyan partition relations.

Let $\mathcal{U}$ *be an element in* $\mathcal{A}$. *A map* $f : [\mathcal{U}]^2 \to \{1,\dots,k\}$ *is called a coloring [95] if for each* $U \in \mathcal{U}$ *and every* $\mathcal{V} \in \mathcal{A}$ *with* $\mathcal{V} \subseteq \mathcal{U}$, *there is an* $i \in \{1,\dots,k\}$ *such that* $\{V \in \mathcal{V} : f(\{U,V\}) = i\}$ *belongs to* $\mathcal{A}$.

We say that X satisfies the partition relation $\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$ *for* $k \in \mathbb{N}$ *if for every* $\mathcal{U} \in \mathcal{A}$ *and any coloring* $f : [\mathcal{U}]^2 \to \{1, \dots, k\}$ *we have an* $i \in \{1, \dots, k\}$*, a set* $\mathcal{V} \in \mathcal{B}$ *with* $\mathcal{V} \subseteq \mathcal{U}$ *and a finite to one function* $\phi : \mathcal{V} \to \mathbb{N}$ *for which for every* $V, W \in \mathcal{V}$ *with* $\phi(V) \neq \phi(W)$*,* $f(\{V, W\}) = i$ *[8, 95].*

In this case $\mathcal{V}$ is said to be eventually homogeneous for f. This symbol is known as *Baumgartner–Taylor partition symbol* .

We say that X satisfies the partition relation $\mathcal{A} \to (\mathcal{B})^n_k$ *for* $n, k \in \mathbb{N}$ *if for every* $\mathcal{U} \in \mathcal{A}$ *and any coloring* $f : [\mathcal{U}]^n \to \{1, \dots, k\}$ *we have an* $i \in \{1, \dots, k\}$ *and a set* $\mathcal{V} \in \mathcal{B}$ *with* $\mathcal{V} \subseteq \mathcal{U}$ *for which for every* $V \in [\mathcal{V}]^n$*,* $f(V) = i$ *[81].*

Also in this case $\mathcal{V}$ is said to be homogeneous for f. This symbol is known as *ordinary partition symbol* .

The following selection principles are called the α_i properties which are defined in [65].

The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$ *for* $i = 1, 2, 3, 4$ *denotes that for every sequence* $\{A_n : n \in \mathbb{N}\}$ *of elements from* $\mathcal{A}$*, there exists a* $B \in \mathcal{B}$ *such that*

$\alpha_1(\mathcal{A}, \mathcal{B})$*: for each* $n \in \mathbb{N}$*, the set* $A_n \setminus B$ *is finite.*

$\alpha_2(\mathcal{A}, \mathcal{B})$*: for each* $n \in \mathbb{N}$*, the set* $A_n \cap B$ *is infinite.*

$\alpha_3(\mathcal{A}, \mathcal{B})$*: for infinitely many* $n \in \mathbb{N}$*, the set* $A_n \cap B$ *is infinite.*

$\alpha_4(\mathcal{A}, \mathcal{B})$*: for infinitely many* $n \in \mathbb{N}$*, the set* $A_n \cap B$ *is non empty.*

Scheepers [95] introduced the following notion:

> *An open cover* $\mathcal{U}$ *of X is said to be a* ***γ-cover*** *if it is infinite and each element of X belongs to all but finitely many elements of* $\mathcal{U}$*. The set of all* γ*-covers of X will be denoted by* Γ*.*

The following notion of open cover was first studied in [41] and since then extensively studied by S. García-Ferreira and A. Tamariz-Mascarúa in [42, 44].

> *An open cover* $\mathcal{U}$ *of X is an* ***ω-cover*** *if* $X \notin \mathcal{U}$ *and for every finite subset F of X there exists a* $U \in \mathcal{U}$ *satisfying* $F \subset U$ *and the collection of all* ω*-covers of X is denoted by* Ω*.*

The notion of large cover was introduced in [95].

> *A cover* $\mathcal{U}$ *is a* ***large cover*** *of X if X is infinite,* $X \notin \mathcal{U}$ *and for each* $x \in X$ *there exist infinitely many* $U \in \mathcal{U}$ *satisfying* $x \in U$ *and the collection of all large open covers of X is denoted by* Λ

The following inclusion relations among the above mentioned classes of covers holds. $\Gamma \subset \Omega \subset \Lambda$ [95]. The notion of k-cover was introduced in [71] by McCoy.

> *An open cover* $\mathcal{U}$ *of X is a* ***k-cover*** *if* $X \notin \mathcal{U}$ *and for every compact subset K of X there is a* $U \in \mathcal{U}$ *such that* $K \subset U$*. The collection of all k-covers of X is denoted by* $\mathcal{K}$

The notion of k-cover plays an important role in the function space $C(X)$ with compact-open topology. Kočinac [64] introduced another variation of the γ-cover namely γ_k-cover.

*An open cover $\mathcal{U}$ of X is called a γ_k-**cover** if each compact subset of X is contained in all but finitely many elements of U and X is not a member of $\mathcal{U}$. The collection of all γ_k-covers of X is denoted by Γ_k .*

In [91], the notion of γ_F-shrinkable cover was introduced by Sakai.

*A γ-cover $\mathcal{U}$ of co-zero sets of X is γ_F-**shrinkable** if there exists a γ-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subseteq U$ for every $U \in \mathcal{U}$. The symbol Γ_F denotes the collection of γ_F-shrinkable γ-covers of X.*

The notion of γ_k-shrinkable cover was introduced in [77].

*A γ_k-cover $\mathcal{U}$ of co-zero sets of X is γ_k-**shrinkable** if there exists a γ_k-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subseteq U$ for every $U \in \mathcal{U}$. The collection of γ_k-shrinkable γ_k-covers is denoted by Γ_k^{sh} .*

Let $\mathfrak{B}$ be a bornology on a metric space X with closed base.

*A cover $\mathcal{U}$ of X is said to be a **strong $\mathfrak{B}$-cover** ($\mathfrak{B}^s$-cover for short) of X [18] if X is not in $\mathcal{U}$ and for each $B \in \mathfrak{B}$ there exist $U \in \mathcal{U}$ and $\delta > 0$ such that $B^\delta \subseteq U$.*

A $\mathfrak{B}^s$-cover $\mathcal{U}$ is said to be an **open $\mathfrak{B}^s$-cover** if the members of $\mathcal{U}$ are open sets. The collection of all open $\mathfrak{B}^s$-covers of X is denoted by $\mathcal{O}_{\mathfrak{B}^s}$.

*X is said to be $\mathfrak{B}^s$-**Lindelöf** [19] if each $\mathfrak{B}^s$-cover of X contains a countable $\mathfrak{B}^s$-subcover.*

*An open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is said to be a $\gamma_{\mathfrak{B}^s}$-**cover** [19] (see also [18]) if it is infinite and for each $B \in \mathfrak{B}$ there is a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers such that $B^{\delta_n} \subseteq U_n$ for all $n \geq n_0$. The collection of all $\gamma_{\mathfrak{B}^s}$-covers of X is denoted by $\Gamma_{\mathfrak{B}^s}$.*

For $K \subseteq \mathbb{N}$, $K(n)$ denotes the set $\{k \in K : k \leq n\}$ and $|K(n)|$ is the cardinality of $K(n)$. The asymptotic density of K is defined by

$d(K) = \lim_{n\to\infty} \frac{|K(n)|}{n}$, provided the limit exists.

Though this notion has long been used in Number Theory, Ergodic Theory etc., one of its most interesting applications has been in Analysis where the notion of asymptotic density was used to define the idea of statistical convergence by Fast [37] (see also [58, 102, 103, 116]), generalizing the idea of usual convergence of real sequences.

A sequence $\{x_n : n \in \mathbb{N}\}$ in a topological space is said to converge statistically (s-converge for short) to x if for any neighbourhood U of x, $d(\{n \in \mathbb{N} : x_n \notin U\}) = 0$.

Statistical convergence has several applications in different fields of mathematics such as summability theory and trigonometric series. The statistical convergence can be viewed as a regular method of summability of sequences.

The notion of statistical γ-cover was introduced in [32].

A countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is said to be a statistical γ-cover (s-γ-cover for short) if for each $x \in X$, $d(\{n \in \mathbb{N} : x \notin U_n\}) = 0$. The collection of all s-γ-covers is denoted by s-Γ .

In [14], Beer and Levi introduced and developed the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence on $\mathfrak{B}$. It was shown that the topology $\tau^s_{\mathfrak{B}}$ on $C(X,Y)$ is metrizable or first countable if and only if the bornology $\mathfrak{B}$ has a countable base [14, Theorem 7.1]. In [18], the authors characterized several topological properties of function spaces which are mentioned below. The topology $\tau^s_{\mathfrak{B}}$ on $C(X,Y)$ is submetrizable if and only if $\mathfrak{B}$ has a countable subfamily whose union is dense in X [18, Theorem 3.3]. The space $(C(X), \tau^s_{\mathfrak{B}})$ is separable if and only if X has a weaker metrizable separable topology and $\mathfrak{B}$ has a base consisting of compact sets [18, Theorem 3.7]. The space $(C(X), \tau^s_{\mathfrak{B}})$ has countable netweight if and only if X is separable and $\mathfrak{B}$ has a base consisting of compact sets [18, Theorem 3.8]. Moreover, the countable tightness and Fréchetness of $(C(X), \tau^s_{\mathfrak{B}})$ are characterized in terms of bornological covering properties of X (see [18, Theorem 3.12, 3.14]). Next we recall some more investigations in this direction. In [49], the author characterized the complete metrizability of the topology $\tau^s_{\mathfrak{B}}$ on $C(X)$ using the notion of shields. Also it is shown that when $(C(X), \tau^s_{\mathfrak{B}})$ is Polish the topologies $\tau^s_{\mathfrak{B}}$ and τ_k coincide [49, Theorem 4.1]. In [50], the authors investigated the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence from the perspective of cardinal invariants namely cellularity, character, density, tightness etc. In [21], a new notion of convergence namely the "strong Whitney convergence" was introduced. This convergence lies between the Whitney and the strong uniform convergences. In [25], it is shown that whenever $\mathfrak{B} \subseteq$ K, the topology $\tau^{sw}_{\mathfrak{B}}$ [21] of strong Whitney convergence and the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence coincide.

Using the idea of strong uniform convergence on bornology, Caserta, Di Maio and Kočinac [19] studied open covers and selection principles in the realm of metric spaces (associated with a bornology) and function spaces (w.r.t. the topology of strong uniform convergence). It is shown that the space $(C(X), \tau^s_{\mathfrak{B}})$ has countable tightness if and only if X is $\mathfrak{B}^s$-Lindelöf [18, Theorem 3.12]. $(C(X), \tau^s_{\mathfrak{B}})$ has countable strong fan tightness (countable fan tightness) if and only if X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ ($\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$) [19, Theorem 2.3, 2.5]. Also $(C(X), \tau^s_{\mathfrak{B}})$ is strictly Fréchet-Urysohn if and only if X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ [19, Theorem 2.7]. We primarily continue in the line initiated in [19] and investigate the behaviour of various selection principles related to these classes of bornological covers.

By the symbol $\underline{0}$ we denote the zero function in the function space $(C(X), \tau^s_{\mathfrak{B}})$. The space $(C(X), \tau^s_{\mathfrak{B}})$ is homogeneous and so it is enough to concentrate at the point $\underline{0}$ when dealing with local properties of this function space. Throughout the investigations we use the convention that if $\mathfrak{B}$ is a bornology on a metric space X, then $X \in \mathfrak{B}$.

In **Chapter** 2, we obtain implications among selection principles related to bornological covers resulting in Scheepers' like diagrams. Some game theoretic observations are presented. We also introduce the notion of strong-$\mathfrak{B}$-Hurewicz property and investigate some of its consequences. In function space $C(X)$ with respect to the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence, important properties like *countable T-tightness*, *Reznichenko property* are characterized in terms of bornological covering properties of X.

In **Chapter** 3, we explore further ramifications, presenting characterizations of various selection principles related to certain classes of bornological covers using the Ramseyan partition relations, interactive results between the cardinalities of bornological bases and certain selection principles involving bornological covers which have not been studied before. Further, some new observations on the $\mathfrak{B}^s$-Hurewicz property introduced in Chapter 2 are presented. We also deals with product bornology $\mathfrak{B}^n$, $n \in \mathbb{N}$, and show that the $(\mathfrak{B}^n)^s$-Hurewicz property of X^n is equivalent to the $\mathfrak{B}^s$-Hurewicz property of X. Following the seminal work of [41], we introduce the $\mathfrak{B}^s$-Gerlits-Nagy property and characterize it using Ramseyan partition relations.

In **Chapter** 4, we introduce the notion of a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X and obtain various implications among selection principles related to the class of shrinkable $\gamma_{\mathfrak{B}^s}$-cover. We present characterizations of various local properties of $(C(X), \tau^s_{\mathfrak{B}})$ such as *countable fan tightness for finite sets*, *countable strong fan tightness for finite sets*, *Fréchet-Urysohn for finite sets* etc. in terms of selection principles related to bornological covers of X and these are represented by a diagram describing various implications among them. We also investigate topological properties of sequences of dense and sequentially dense subsets of $(C(X), \tau^s_{\mathfrak{B}})$ and present another diagram describing implications among those selective properties of $(C(X), \tau^s_{\mathfrak{B}})$.

In **Chapter** 5, we focus on the tightness property and some of its variations such as supertightness, Id-fan tightness and T-tightness in function spaces. We show that the tightness and the supertightness properties of $(C(X), \tau^s_{\mathfrak{B}})$ are interchangeable. The Id-fan tightness and the T-tightness of $(C(X), \tau^s_{\mathfrak{B}})$ are characterized in terms of bornological covering properties of X. We also study the concept of k-spaces from the bornological perspective. Two new notions of games namely the strong $\mathfrak{B}$- open game and the $\gamma_{\mathfrak{B}^s}$-open game on X are introduced and their consonances with other classical games on X are obtained. We investigate discretely selective property and associated games. Several interactions between topological games on $(C(X), \tau^s_{\mathfrak{B}})$ related to discretely selective property, the *Gruenhage game* on $(C(X), \tau^s_{\mathfrak{B}})$ and certain games on

X are also presented.

In **Chapter** 6, we primarily make a general approach to the study of open covers and related selection principles using the idea of statistical convergence in metric space. We introduce statistical analogues of certain types of open covers and investigate the behaviour of related selection principles using the idea of strong uniform convergence on a bornology. We introduce statistical versions of certain types of bornological open covers and observe the behaviour of related selection principles including the α_i-properties. We also introduce the notions of statistically-strong-$\mathfrak{B}$-Hurewicz property and statistically-strong-$\mathfrak{B}$-groupable cover and obtain some game theoretic results. In the function space $C(X)$ associated with the topology of strong uniform convergence on $\mathfrak{B}$ we investigate some properties like *statistically strictly Fréchet Urysohn*, *statistically Reznichenko* and *countable fan tightness*.

CHAPTER 2

ON BORNOLOGICAL APPROACH TO THE STUDY OF SELECTION PRINCIPLES

This Chapter is based on our following work:

D. Chandra, P. Das and S. Das `Applications of bornological covering properties in metric spaces`, **Indag. Math.**, 31 (2020), 43–63.

2.1 INTRODUCTION

In [54, 95]), Scheepers had started a systematic study of selection principles in topology and their relations to game theory. Study of open covers and selection principles and their interrelationship has a long and illustrious history and readers interested in selection principles and its recent developments can consult the papers [63, 99, 111] where many more references can be found.

In [14], Beer and Levi had introduced the notion of strong uniform continuity on a bornology and the topology of strong uniform convergence on a bornology for function spaces. Using the idea of strong uniform convergence [14, 18] on bornology, Caserta, Di Maio and Kočinac [19] studied open covers and selection principles in the realm of metric spaces (associated with a bornology) and function spaces (w.r.t. the topology of strong uniform convergence).

In this Chapter we continue in that direction and use this idea of strong uniform convergence on bornology to investigate the behaviour of various selection principles related to bornological classes of covers. The main objective of such investigations is to obtain "Scheepers' like diagrams" which we precisely attain (Figure 1 and Figure 2) as consequences of several results where implications between these selection principles are established. We introduce the notion of strong-$\mathfrak{B}$-Hurewicz property and strong-$\mathfrak{B}$-groupable cover and establish their relationships with certain selection principles and related games. Also we focus on the function space $C(X)$ endowed with the topology $\tau^s_{\mathfrak{B}}$ and consider several important properties like countable T-tightness, Reznichenko property and obtain their characterizations in terms of bornological covering properties of X.

2.2 SELECTION PRINCIPLES AND BORNOLOGICAL COVERING PROPERTIES

2.2.1 CERTAIN OBSERVATIONS ON COVERS

We now present some observations on open $\mathfrak{B}^s$-cover and $\gamma_{\mathfrak{B}^s}$-cover of X which will be used throughout our bornological investigation.

Proposition 2.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open cover of X. The following statements are equivalent.*
(1) $\mathcal{U}$ is an open $\mathfrak{B}^s$-cover of X.
(2) For each $B \in \mathfrak{B}$ there exist $U_n \in \mathcal{U}$ and $\delta_n > 0$ satisfying $B^{\delta_n} \subseteq U_n$ for infinitely many $n \in \mathbb{N}$.

Proof. (1) $\Rightarrow$ (2). Let $B \in \mathfrak{B}$. Since $X \notin \mathfrak{B}$, choose a $x_1 \in X \setminus B$. Clearly $B \cup \{x_1\} \in \mathfrak{B}$. Consequently, we can find $\delta_{n_1} > 0$ and $U_{n_1} \in \mathcal{U}$ satisfying $(B \cup \{x_1\})^{\delta_{n_1}} \subseteq U_{n_1}$ and so $B^{\delta_{n_1}} \subseteq U_{n_1}$. Choose $x_2 \in X \setminus (B \cup \{x_1\})$. Then $B \cup \{x_1, x_2\} \in \mathfrak{B}$. Again we can find $\delta_{n_2} > 0$ and $U_{n_2} \in \mathcal{U}$ satisfying $(B \cup \{x_1, x_2\})^{\delta_{n_2}} \subseteq U_{n_2}$ and so $B^{\delta_{n_2}} \subseteq U_{n_2}$. Since $\mathfrak{B}$ is closed under taking finite unions, by continuing the above process, we get a sequence $n_1 < n_2 < \cdots$ with positive real numbers δ_{n_k} and $U_{n_k} \in \mathcal{U}$ satisfying $B^{\delta_{n_k}} \subseteq U_{n_k}$ for all k. Hence (2) holds.

The implication (2) $\Rightarrow$ (1) easily follows. □

Lemma 2.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Consider a sequence of open $\mathfrak{B}^s$-covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of X, where $\mathcal{U}_n = \{U^n_j : j \in \mathbb{N}\}$ for each n. Let $\mathcal{V}_n = \{U^1_{k_1} \cap U^2_{k_2} \cap \ldots \cap U^n_{k_n} : U^i_{k_i} \in \mathcal{U}_i, k_i \in \mathbb{N}, i = 1, 2, \ldots, n\}$. Then $\mathcal{V}_n$ is an open $\mathfrak{B}^s$-cover of X for each $n \in \mathbb{N}$.*

Proof. Let $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$. Let $B \in \mathfrak{B}$. Then there are $U^i_{k_i} \in \mathcal{U}_i$ and $\delta^i_{k_i} > 0$ satisfying $B^{\delta^i_{k_i}} \subseteq U^i_{k_i}$ for each $i = 1, 2, \ldots, n$. Choose $\delta_k = \min\{\delta^i_{k_i} : i = 1, 2, \ldots, n\}$. Now we have $B^{\delta_k} \subseteq U^1_{k_1} \cap U^2_{k_2} \cap \ldots \cap U^n_{k_n} \in \mathcal{V}_n$. This shows that $\mathcal{V}_n$ is an open $\mathfrak{B}^s$-cover of X for each $n \in \mathbb{N}$. □

It can be easily seen that every $\gamma_{\mathfrak{B}^s}$-cover is a γ-cover. The following example shows that the class of $\gamma_{\mathfrak{B}^s}$-covers is properly contained in the class of γ-covers.

Example 2.2.1. *Let $X = \mathbb{R}$ with the Euclidean metric d and the bornology $\mathfrak{B}$ generated by $\{(-\infty, x) : x \in \mathbb{R}\}$. Consider $\mathcal{U} = \{(-m, m) : m \in \mathbb{N}\}$. Clearly $\mathcal{U}$ is a γ-cover of X. Let $B = (-\infty, x_0) \in \mathfrak{B}$*

and $\delta > 0$. Then $B^\delta = (-\infty, x_0 + \delta)$. Clearly for any $m \in \mathbb{N}$ and any $\delta > 0$ $B^\delta \nsubseteq (-m, m)$. Therefore $\mathcal{U}$ can not be a $\gamma_{\mathfrak{B}^s}$-cover of X.

Among the classes of covers $\mathcal{O}, \Gamma, \mathcal{O}_{\mathfrak{B}^s}$ and $\Gamma_{\mathfrak{B}^s}$ the following inclusion relations hold. $\Gamma_{\mathfrak{B}^s} \subsetneq \mathcal{O}_{\mathfrak{B}^s} \subsetneq \mathcal{O}$, $\Gamma_{\mathfrak{B}^s} \subsetneq \Gamma$.

Lemma 2.2.2 (cf. [19]). *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If a $\gamma_{\mathfrak{B}^s}$-cover of X is partitioned into finitely many pieces, then each of them is again a $\gamma_{\mathfrak{B}^s}$-cover.*

Lemma 2.2.3 (cf. [19]). *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Then the following statements hold.*
(1) *Infinite subset of a $\gamma_{\mathfrak{B}^s}$-cover is a $\gamma_{\mathfrak{B}^s}$-cover.*
(2) *Consider a sequence of $\gamma_{\mathfrak{B}^s}$-covers $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of X, where $\mathcal{U}_n = \{U^n_j : j \in \mathbb{N}\}$ for each n. Let $\mathcal{V}_n = \{U^1_k \cap U^2_k \cap \ldots \cap U^n_k : k \in \mathbb{N}, U^i_k \in \mathcal{U}_i, i = 1, 2, \ldots, n\}$. Then $\mathcal{V}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X.*

Lemma 2.2.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Consider an open $\mathfrak{B}^s$-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X. For $n \in \mathbb{N}$ let $V_n = \cup_{i=1}^n U_i$ and let $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$. Then $\mathcal{V}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. Let $B \in \mathfrak{B}$. Clearly there exist $U_{k_0} \in \mathcal{U}$ and $\delta > 0$ satisfying $B^\delta \subseteq U_{k_0}$. Then for all $k \geq k_0$ $B^\delta \subseteq \cup_{i=1}^k U_i$. Choose $\delta_n = \delta$ for all $n \geq k_0$. Then $B^{\delta_n} \subseteq V_n$ for all $n \geq k_0$. Therefore $\mathcal{V}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. □

2.2.2 IMPLICATIONS AMONG SELECTION PRINCIPLES AND SCHEEPERS' LIKE DIAGRAMS

Taking $\mathcal{A}, \mathcal{B} \in \{\mathcal{O}, \Gamma, \mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}\}$ and $\prod \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}, \mathsf{U}_{\text{fin}}\}$, we establish the equivalences among the selection principles $\prod(\mathcal{A}, \mathcal{B})$ in the next few results.

Theorem 2.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements hold.*
(1) $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}) = \mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$
(2) $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma) = \mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma)$
(3) $\mathsf{S}_1(\Gamma, \Gamma_{\mathfrak{B}^s}) = \mathsf{S}_{\text{fin}}(\Gamma, \Gamma_{\mathfrak{B}^s})$.

Proof. We only prove (1) and omit the remaining proofs as they are analogous.

It is easy to see that $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ implies $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Let X satisfy $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. We show that X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $\gamma_{\mathfrak{B}^s}$-covers of X. We bijectively enumerate $\mathcal{U}_n$ as $\mathcal{U}_n = \{U^n_k : n \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. Define $\mathcal{V}_n = \{U^1_k \cap U^2_k \cap \ldots \cap U^n_k : k \in \mathbb{N}\}$ for each n. By Lemma 2.2.3(2), $\mathcal{V}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Applying $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ to the sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$, we obtain a finite set

$\mathcal{W}_n \subseteq \mathcal{V}_n$ for each n for which the set $\cup_{n=1}^{\infty}\mathcal{W}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Now we choose a sequence of positive integers $n_1 < n_2 < \cdots$ so that $\mathcal{W}_{n_j} \setminus \cup_{i<j}\mathcal{W}_{n_i} \neq \emptyset$. This is possible because $\cup_{n\in\mathbb{N}}\mathcal{W}_n$, being a $\gamma_{\mathfrak{B}^s}$-cover, is infinite. Suppose that $n_1, n_2, \ldots, n_{j-1}$ have been chosen. If there is no such $n_j \in \mathbb{N}$, then $\mathcal{W}_n \setminus \cup_{i\leq j-1}\mathcal{W}_{n_i} = \emptyset$ for all $n > n_{j-1}$, i.e., $\cup_{n>n_{j-1}}\mathcal{W}_n \subseteq \cup_{i\leq j-1}\mathcal{W}_{n_i}$. This is a contradiction as the later set is finite. By Lemma 2.2.3, clearly $\cup_{k\in\mathbb{N}}\mathcal{W}_{n_k}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X.

Now for each j, choose m_j such that $V_{m_j}^{n_j}$ is an element of $\mathcal{W}_{n_j} \setminus \cup_{i<j}\mathcal{W}_{n_i}$. Then $\{V_{m_j}^{n_j} : j = 1,2,\ldots\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X by Lemma 2.2.3(1). Define $U_n = U_{m_{k+1}}^n$ for each $n \in (n_k, n_{k+1}]$. Consider the set $\{U_n : n = 1,2,\ldots\}$. For $B \in \mathfrak{B}$ there exist $k_0 \in \mathbb{N}$ and a sequence $\{\delta_{n_k} : k \geq k_0\}$ of positive reals such that $B^{\delta_{n_{k+1}}} \subseteq V_{m_{k+1}}^{n_{k+1}}$ for all $k \geq k_0$, i.e., $B^{\delta_{n_{k+1}}} \subseteq U_{m_{k+1}}^1 \cup U_{m_{k+1}}^2 \cup \cdots \cup U_{m_{k+1}}^{n_{k+1}}$. For $n \in (n_k, n_{k+1}]$, define $\delta_n = \delta_{n_{k+1}}$. Then $B^{\delta_n} \subseteq U_n$ for all $n \geq n_{k_0}$ and $k \geq k_0$. So $\{U_n : n \in \mathbb{N}\} \in \Gamma_{\mathfrak{B}^s}$. Hence X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. □

Theorem 2.2.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements hold.*
(1) $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}) = \mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$
(2) $\mathsf{S}_1(\mathcal{O}, \Gamma_{\mathfrak{B}^s}) = \mathsf{S}_{\text{fin}}(\mathcal{O}, \Gamma_{\mathfrak{B}^s})$
(3) $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma) = \mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma)$, *provided X is $\mathfrak{B}^s$-Lindelöf.*

Proof. We give proof of (1). Let X satisfy $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X, where $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$. Now define $\mathcal{V}_n = \{U_{k_1}^1 \cap U_{k_2}^2 \cap \ldots \cap U_{k_n}^n : U_{k_i}^i \in \mathcal{U}_i, k_i \in \mathbb{N}, i = 1,2,\ldots,n\}$. By Lemma 2.2.1, $\mathcal{V}_n$ is an open $\mathfrak{B}^s$-cover of X for each $n \in \mathbb{N}$. Applying $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ and proceeding as in the proof of Theorem 2.2.1(1), we can conclude that X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. □

Theorem 2.2.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and let X be a $\mathfrak{B}^s$-Lindelöf space. The following statements hold.*
(1) $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O}) = \mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O})$
(2) $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}) = \mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$
(3) $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}) = \mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$.

Proof. We give proof of (1). Let X satisfy $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$. Consider a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X. Let $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$. For each k, n let $V_k^n = U_1^n \cup U_2^n \cup \cdots \cup U_k^n$. Consider the collection $\mathcal{V}_n = \{V_k^n : k \in \mathbb{N}\}$. By Lemma 2.2.4, $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of $\gamma_{\mathfrak{B}^s}$-covers of X. Applying $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$, we obtain a finite set $\mathcal{W}_n \subseteq \mathcal{V}_n$ for each n for which the set $\cup_{n=1}^{\infty}\mathcal{W}_n$ is an open cover of X. Deconstructing the members of each $\mathcal{W}_n$, we find a finite subset $\mathcal{Z}_n \subseteq \mathcal{U}_n$ for each n such that $\{\cup\mathcal{Z}_n : n \in \mathbb{N}\}$ is an open cover of X. For $x \in X$ there exists a $V_{k_0}^{n_0} \in \cup_{n=1}^{\infty}\mathcal{W}_n$ such that $x \in V_{k_0}^{n_0} = U_1^{n_0} \cup \cdots \cup U_{k_0}^{n_0}$ and so $x \in U_i^{n_0}$ for some $i \in \{1,2,\ldots,k_0\}$. As $U_i^{n_0} \in \mathcal{Z}_{n_0}$, we have $x \in \cup\mathcal{Z}_{n_0}$. Hence $\{\cup\mathcal{Z}_{n_0} : n \in \mathbb{N}\} \in \mathcal{O}$. So X satisfies $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O})$.

Conversely, let X satisfy $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O})$. Consider a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of $\gamma_{\mathfrak{B}^s}$-covers of X. Applying $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O})$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, we obtain a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each n for which the set $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is an open cover of X. Now for $x \in X$ we have $x \in \cup\mathcal{V}_{k_0}$ for some $k_0 \in \mathbb{N}$. Clearly $x \in U$ for some $U \in \mathcal{V}_{k_0}$. Therefore $\cup_{n=1}^{\infty}\mathcal{V}_k \in \mathcal{O}$ and hence X satisfies $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$. □

Theorem 2.2.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and let X be a $\mathfrak{B}^s$-Lindelöf space. The following statements hold.*

(1) $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s}) = \mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.
(2) $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}) = \mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.
(3) $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}) = \mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$
(4) $\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathfrak{B}^s}) = \mathsf{U}_{\text{fin}}(\Gamma, \mathcal{O}_{\mathfrak{B}^s})$
(5) $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma) = \mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma)$
(6) $\mathsf{U}_{\text{fin}}(\mathcal{O}, \Gamma_{\mathfrak{B}^s}) = \mathsf{U}_{\text{fin}}(\Gamma, \Gamma_{\mathfrak{B}^s})$.

Proof. (1). Let X satisfies $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. It is easy to see that X also satisfies $\mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Now let X satisfy $\mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Consider a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X. We enumerate $\mathcal{U}_n$ as $\mathcal{U}_n = \{U_k^n : k = 1, 2, \dots\}$ for each n. Consider the collection $\mathcal{V}_n$ whose m-th member is $V_m = \cup_{i=1}^{m} U_i^n$. By Lemma 2.2.4, $\mathcal{V}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Clearly $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of $\gamma_{\mathfrak{B}^s}$-covers of X. Now applying $\mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$, we obtain a finite set $\mathcal{W}_n \subseteq \mathcal{V}_n$ for each n for which the set $\{\cup\mathcal{W}_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Deconstructing the members of each $\mathcal{W}_n$, we find a finite subset $\mathcal{T}_n$ of $\mathcal{U}_n$ for each n with $\cup\mathcal{W}_n = \cup\mathcal{T}_n$. Since $\{\cup\mathcal{W}_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X, $\{\cup\mathcal{T}_n : n \in \mathbb{N}\}$ is also an open $\mathfrak{B}^s$-cover of X. Hence X satisfies $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.
The remaining proofs are analogous and so are omitted. □

The equivalences among the selection principles from Theorem 2.2.1, Theorem 2.2.2, Theorem 2.2.3 and Theorem 2.2.4 are presented in the following implication diagram (Figure 1).

Moreover considering only the bornological classes of covers $\mathcal{O}_{\mathfrak{B}^s}$ and $\Gamma_{\mathfrak{B}^s}$, one can obtain the following diagrammatic representation (Figure 2).

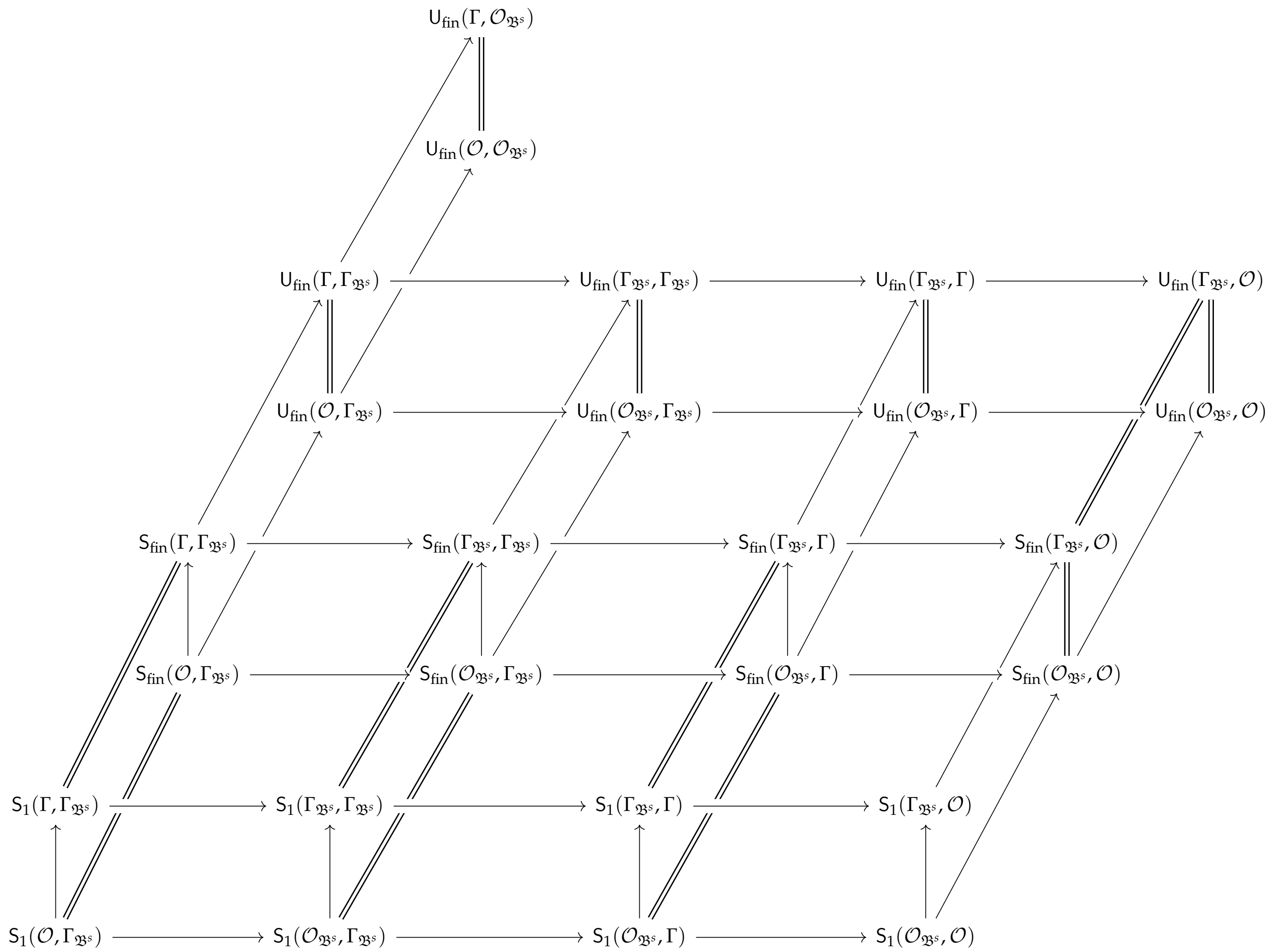

FIGURE 2.1

The Baire space is the set $\mathbb{N}^{\mathbb{N}}$ endowed with the Baire metric ρ (see [39, 56]) defined by

$$\rho(f,g)=\begin{cases}\dfrac{1}{\min\{n\in\mathbb{N}: f(n)\neq g(n)\}+1} & \text{if } f\neq g\\ 0 & \text{if } f=g.\end{cases}$$

The eventually dominating order $\leq^*$ on $\mathbb{N}^{\mathbb{N}}$ is defined as follows. For $f,g\in\mathbb{N}^{\mathbb{N}}$, we say that $f\leq^* g$ if $f(n)\leq g(n)$ for all but finitely many n. A set $A\subseteq\mathbb{N}^{\mathbb{N}}$ is said to be bounded if there is a function $g\in\mathbb{N}^{\mathbb{N}}$ for which $f\leq^* g$ for all $f\in A$. A set $X\subseteq\mathbb{R}$ is said to be an A_2-set [11] if each Borel image of X into the Baire space $\mathbb{N}^{\mathbb{N}}$ is bounded.

The next example shows that an A_2-set endowed with a certain bornology satisfies $\mathsf{S}_1(\Gamma,\Gamma_{\mathfrak{B}^s})$.

Example 2.2.2. *Suppose that X is an A_2-set endowed with the Euclidean metric d and $\mathfrak{B}$ is the bornology $\mathcal{F}$, the family of all finite subsets of X. We show that X satisfies $\mathsf{S}_1(\Gamma,\Gamma_{\mathfrak{B}^s})$. To see this, consider a sequence $\{\mathcal{U}_n : n\in\mathbb{N}\}$ of γ-covers of X. Let $\mathcal{U}_n=\{U^n_m : m\in\mathbb{N}\}$ for each n. Define $\varphi : X\to\mathbb{N}^{\mathbb{N}}$ by $\varphi(x)(n)=\min\{m : x\in U^n_k, \forall k\geq m\}$. Then φ is a Borel function (see [54]). Since X is*

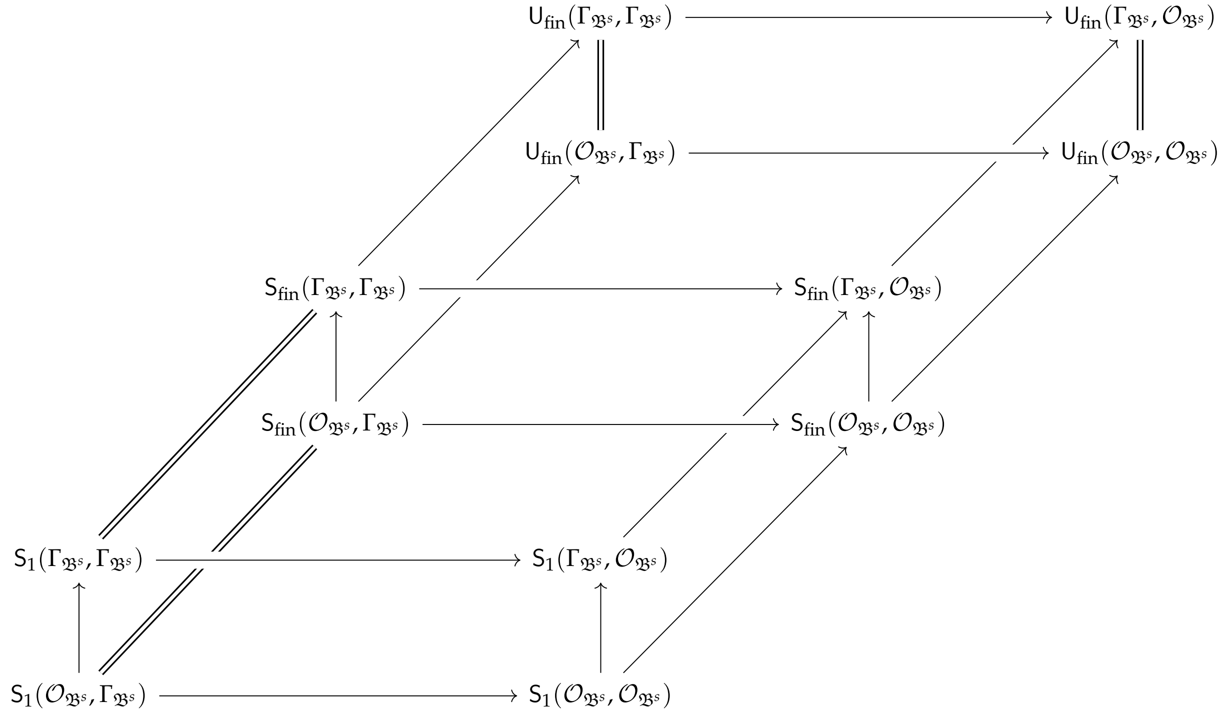

FIGURE 2.2

an A_2 set, the image $\varphi(X)$ is bounded. There is a $g \in \mathbb{N}^{\mathbb{N}}$ such that for every $x \in X$ there is a $n_x \in \mathbb{N}$ satisfying $\varphi(x)(n) \leq g(n)$ for all $n \geq n_x$. Clearly $x \in U^n_{g(n)}$ for all $n \geq n_x$. We prove that $\{U^n_{g(n)} : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Let $B \in \mathfrak{B}$ and $B = \{x_1, x_2, \ldots, x_k\}$. There is a $n_i \in \mathbb{N}$ for each $x_i \in B$ satisfying $x_i \in U^n_{g(n)}$ for all $n \geq n_i$. As $U^n_{g(n)}$ is open, there exists a $\delta^i_n > 0$ such that $S_d(x_i, \delta^i_n) \subseteq U^n_{g(n)}$ for all $n \geq n_i$ and $i = 1, 2, \ldots, k$. Choose $n_0 = \max\{n_i : i = 1, 2, \ldots, k\}$ and $\delta_n = \min\{\delta^i_n : i = 1, 2, \ldots, k\} > 0$. Then $B^{\delta_n} \subseteq U^n_{g(n)}$ for all $n \geq n_0$. Clearly $\{U^n_{g(n)} : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Hence X satisfies $\mathsf{S}_1(\Gamma, \Gamma_{\mathfrak{B}^s})$.

Remark 2.2.1. *Consider X and the bornology as in Example 2.2.2. Observe that X satisfies each of the following selection principles: $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, $\mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, $\mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma)$, $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$, $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma)$, $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$, $\mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma)$, $\mathsf{U}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \mathcal{O})$, $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma)$, $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O})$,$\mathsf{U}_{\text{fin}}(\Gamma, \mathcal{O}_{\mathfrak{B}^s})$,$\mathsf{U}_{\text{fin}}(\mathcal{O}, \mathcal{O}_{\mathfrak{B}^s})$ (see Figure 1 and 2).*

In the following example we consider the Baire space associated with certain bornology and show that it does not satisfy $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Example 2.2.3. *Suppose that $X = \mathbb{N}^{\mathbb{N}}$ endowed with the bornology $\mathfrak{B} = \mathcal{F}$, the family of all finite subsets of X. The set $U^n_m = \{f \in X : f(n) \leq m\}$ is open for each $m, n \in \mathbb{N}$ as for $f \in U^n_m$ $S_\rho(f, \frac{1}{n+1}) \subseteq U^n_m$. Also $U^n_m \subseteq U^n_j$ for $j \geq m$ for each n. For each n consider the open cover $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$ of X. We first show that $\mathcal{U}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. To see this, let $B \in \mathfrak{B}$ and let $B = \{f_1, f_2, \ldots, f_k\}$. Choose $m_0 \in \mathbb{N}$ such that $f_i \in U^n_m$ for each $m \geq m_0$ and $i = 1, 2, \ldots, k$.*

For $m \geq m_0$ choose $\delta_m = \frac{1}{n+1}$. Clearly $B^{\delta_m} \subseteq U^n_m$ for all $m \geq m_0$ and hence $\mathcal{U}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X.

We now show that X does not satisfy $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. We assume the contrary. Applying $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, we obtain a $g \in X$ for which the set $\{U^n_{g(n)} : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Choose a $h \in X$ satisfying $h(n) > g(n)$ for each n. Then $h \notin U^n_{g(n)}$ for any n. Otherwise for some n $h \in U^n_{g(n)}$ implies $h(n) \leq g(n)$, which is a contradiction. Let $B = \{h\}$. Clearly $B^{\delta_n} \nsubseteq U^n_{g(n)}$ for any n and $\delta > 0$. This contradicts that $\{U^n_{g(n)} : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Hence our assumption is false. Therefore X does not satisfy $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Remark 2.2.2. *Let X be the Baire space with the bornology $\mathcal{F}$ as in Example 2.2.3. By Figure 1 and 2, it follows that X does not satisfy any of the following selection principles: $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_1(\Gamma, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_{\text{fin}}(\Gamma, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_1(\mathcal{O}, \Gamma_{\mathfrak{B}^s})$, $\mathsf{S}_{\text{fin}}(\mathcal{O}, \Gamma_{\mathfrak{B}^s})$.*

2.2.3 GAME THEORETIC CHARACTERIZATIONS OF SOME SELECTION PRINCIPLES

The study of topological properties and their relations to Ramsey theory, function spaces, and other related topics can be described in terms of topological games. Rich surveys are available in [54, 83, 95, 104, 105, 107] and reference therein. Numerous properties in selection principles are interconnected with two player game and these properties are equivalent to the fact that the first player does not have a wining strategy in that game. From time to time these game theoretic observations become useful tools to derive results related to those properties in selection principles.

We first present the following game theoretic characterization of $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Theorem 2.2.5. *Let $\mathfrak{B}$ a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(2) *ONE has no wining strategy in the game $\mathsf{G}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. (1) $\Rightarrow$ (2). Suppose that ψ is a strategy for ONE in $\mathsf{G}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. The first move of ONE in the game $\mathsf{G}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is $\psi(\emptyset)$, a $\gamma_{\mathfrak{B}^s}$-cover of X, say $\psi(\emptyset) = \{U_{(n)} : n \in \mathbb{N}\}$. Let TWO choose $U_{(n_1)} \in \psi(\emptyset)$. The second move of ONE in the game $\mathsf{G}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is $\psi(U_{(n_1)}) \in \Gamma_{\mathfrak{B}^s}$, say $\psi(U_{(n_1)}) = \{U_{(n_1,m)} : m \in \mathbb{N}\}$. Let TWO choose $U_{(n_1,n_2)} \in \psi(U_{(n_1)}) \setminus \{U_{(n_1)}\}$ and so on. Hence for each finite sequence σ, the collection $\{U_{\sigma\frown(m)} : m \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Since X satisfies $S_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, for each σ there exists a $n_\sigma \in \mathbb{N}$ such that $\{U_{\sigma\frown(n_\sigma)} : \sigma \text{ is a finite sequence}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover. Define inductively $n_1, n_2, \dots$ so that $n_1 = n_\emptyset$ and $n_{k+1} = n_{(n_1,\dots,n_k)}$. Then the sequence $U_{(n_1)}, U_{(n_1,n_2)}, \dots$ of moves of TWO in the game $\mathsf{G}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is a $\gamma_{\mathfrak{B}^s}$-cover by Lemma 2.2.3. Thus ψ is not a winning strategy for ONE in $\mathsf{G}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Assume that X does not satisfy $S_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. Clearly we have a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of $\gamma_{\mathfrak{B}^s}$-covers of X for which $\{U_n : n \in \mathbb{N}\}$ is not a $\gamma_{\mathfrak{B}^s}$-cover of X for any choice of $U_n \in \mathcal{U}_n$. We now define a strategy ψ for ONE as follows. Define $\psi(\emptyset) = \mathcal{U}_1$. TWO responds by selecting $U_1 \in \psi(\emptyset)$. For each $n \geq 1$ define $\psi(U_1, \ldots, U_{n-1}) = \mathcal{U}_n$. TWO responds by choosing $U_n \in \psi(U_1, \ldots, U_{n-1})$. Since ψ is not a winning strategy, $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover which is a contradiction. Hence (1) is satisfied. □

Theorem 2.2.6. *Let $\mathfrak{B}$ a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *X satisfies $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(2) *ONE has no wining strategy in the game $\mathsf{G}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. Let ϕ be a strategy for ONE in the game $\mathsf{G}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. The first move of ONE in the game $\mathsf{G}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is $\phi(\emptyset)$, a $\gamma_{\mathfrak{B}^s}$-cover of X, say $\phi(\emptyset) = \{U_{(n)} : n \in \mathbb{N}\}$. Let TWO choose a $\mathcal{V}_{(n_1)}$, a finite subset of $\phi(\emptyset)$. The second move of ONE in the game $\mathsf{G}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is $\phi(\mathcal{V}_{(n_1)}) \in \Gamma_{\mathfrak{B}^s}$, say $\phi(\mathcal{V}_{(n_1)}) = \{U_{(n_1,m)} : m \in \mathbb{N}\}$. Let TWO choose a finite set $\mathcal{V}_{(n_1,n_2)} \subseteq \phi(U_{(n_1)}) \setminus \{\mathcal{V}_{(n_1)}\}$ and so on. Hence for each finite sequence σ, the collection $\{U_{\sigma\frown(m)} : m \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Since X satisfies $\mathsf{S}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, for each σ there exists a $n_\sigma \in \mathbb{N}$ such that $\cup\{\mathcal{V}_{\sigma\frown(n_\sigma)} : \sigma$ is a finite sequence$\}$ is a $\gamma_{\mathfrak{B}^s}$-cover. Define inductively $n_1, n_2, \ldots$ so that $n_1 = n_\emptyset$ and $n_{k+1} = n_{(n_1,\ldots,n_k)}$. Then $\mathcal{V}_{(n_1)}, \mathcal{V}_{(n_1,n_2)}, \cdots$ is a sequence of moves of TWO in the game $\mathsf{G}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ for which the set $\cup_{k\in\mathbb{N}}\mathcal{V}_{(n_1,\ldots,n_k)})$ is a $\gamma_{\mathfrak{B}^s}$-cover by Lemma 2.2.3. Thus ϕ is not a winning strategy for ONE in $\mathsf{G}_{\text{fin}}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

The converse part is easily followed. □

Theorem 2.2.7. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(2) *ONE has no wining strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. Let ψ be a strategy for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. The first move of ONE in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is $\psi(\emptyset)$, a open $\mathfrak{B}^s$-cover of X. Let $\psi(\emptyset) = \{U_{(n)} : n \in \mathbb{N}\}$. Let TWO choose $U_{(n_1)} \in \psi(X)$. The second move of ONE in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is $\psi(U_{(n_1)}) \in \mathcal{O}_{\mathfrak{B}^s}$. Let $\psi(U_{(n_1)}) = \{U_{(n_1,m)} : m \in \mathbb{N}\}$. Let TWO choose $U_{(n_1,n_2)} \in \psi(U_{(n_1)}) \setminus \{U_{(n_1)}\}$ and so on. Clearly for each finite sequence σ, the collection $\{U_{\sigma\frown(m)} : m \in \mathbb{N}\}$ is a open $\mathfrak{B}^s$-cover of X. As X satisfies $S_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, there exists a $n_\sigma \in \mathbb{N}$ for each σ for which $\{U_{\sigma\frown(n_\sigma)} : \sigma$ is a finite sequence$\}$ is a $\gamma_{\mathfrak{B}^s}$-cover. Define inductively $n_1, n_2, \ldots$ so that $n_1 = n_\emptyset$ and $n_{k+1} = n_{(n_1,\ldots,n_k)}$. Then the sequence $U_{(n_1)}, U_{(n_1,n_2)}, \cdots$ of moves of TWO in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is a $\gamma_{\mathfrak{B}^s}$-cover by Lemma 2.2.3. Thus ψ is not a winning strategy for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. □

2.3 THE STRONG $\mathfrak{B}$-HUREWICZ PROPERTY

We first introduce the definitions of the strong $\mathfrak{B}$-Hurewicz property and the strong $\mathfrak{B}$-Hurewicz game.

Definition 2.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. X is said to have the strong $\mathfrak{B}$-Hurewicz property ($\mathfrak{B}^s$-Hurewicz property for short) if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X there is a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ such that for every $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers satisfying $B^{\delta_n} \subseteq U$ for some $U \in \mathcal{V}_n$ for all $n \geq n_0$.*

Definition 2.3.2. *The strong $\mathfrak{B}$-Hurewicz game ($\mathfrak{B}^s$-Hurewicz game for short) on X is defined as follows. Two players named ONE and TWO play an infinite long game. In the n-th inning ONE selects an open $\mathfrak{B}^s$-cover $\mathcal{U}_n$ of X, TWO responds by choosing a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$. TWO wins the play: $\mathcal{U}_1, \mathcal{V}_1, \mathcal{U}_2, \mathcal{V}_2, \dots, \mathcal{U}_n, \mathcal{V}_n, \dots$ if for each $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers satisfying $B^{\delta_n} \subseteq U$ for some $U \in \mathcal{V}_n$ for all $n \geq n_0$. Otherwise ONE wins.*

We now present two examples where we show that the real line associated with a certain bornology has the $\mathfrak{B}^s$-Hurewicz property and on the other hand the Baire space with the bornology $\mathcal{F}$ does not have the $\mathfrak{B}^s$-Hurewicz property.

Example 2.3.1. *Let $X = \mathbb{R}$ with the Euclidean metric d and the bornology $\mathfrak{B}$ generated by the set $\{(-x, x) : x \in X\}$. We show that X has the $\mathfrak{B}^s$-Hurewicz property. Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \mathcal{O}_{\mathfrak{B}^s}$. Consider a sequence of positive integers $k_1 < k_2 < \cdots$. For each $n \in \mathbb{N}$, choose a $U_n \in \mathcal{U}_n$ satisfying $(-k_n, k_n) \subseteq U_n$ and define $\mathcal{W}_n = \{U_n\}$. We prove that $\{\mathcal{W}_n : n \in \mathbb{N}\}$ witnesses the $\mathfrak{B}^s$-Hurewicz property of X. Let $B \in \mathfrak{B}$. Clearly $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\} \in \Gamma_{\mathfrak{B}^s}$. We can find a $n_0 \in \mathbb{N}$ and a sequence of positive real numbers $\{\delta_n : n \geq n_0\}$ satisfying $B^{\delta_n} \subseteq (-n, n)$ for all $n \geq n_0$ and consequently $B^{\delta_n} \subseteq (-k_n, k_n) \subseteq U_n$ for $U_n \in \mathcal{W}_n$ for all $n \geq n_0$. Hence X has the $\mathfrak{B}^s$-Hurewicz property.*

Example 2.3.2. *Let $X = \mathbb{N}^{\mathbb{N}}$ endowed with the Bare metric ρ and the bornology $\mathfrak{B} = \mathcal{F}$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X, where $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$ and $U^n_m = \{f \in X : f(n) \leq m\}$. We prove that X does not have the $\mathfrak{B}^s$-Hurewicz property. Let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be any sequence, where $\mathcal{V}_n$ is finite subset of $\mathcal{U}_n$ for each n. Choose $h \in X$ in such a way that $h(n) > 2 \cdot \max\{m : U^n_m \in \mathcal{V}_n\}$. Again choose $f, g \in X$ for which $h = f + g$. Then $\max\{f(n), g(n)\} \geq \frac{1}{2}h(n)$. Clearly $\{f, g\} \nsubseteq U^n_m$ for any $U^n_m \in \mathcal{V}_n$ and $n \in \mathbb{N}$. Let $B = \{f, g\} \in \mathfrak{B}$. Then for any n and $\delta > 0$, $B^\delta \nsubseteq U^n_m$ for all $U^n_m \in \mathcal{V}_n$. Hence X does not have the $\mathfrak{B}^s$-Hurewicz property.*

We now present a characterization of the $\mathfrak{B}^s$-Hurewicz property by the $\mathfrak{B}^s$-Hurewicz game.

Theorem 2.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*

(1) *X has the $\mathfrak{B}^s$-Hurewicz property.*

(2) *ONE has no wining strategy in the $\mathfrak{B}^s$-Hurewicz game on X.*

Proof. (1) $\Rightarrow$ (2). Suppose that X has the $\mathfrak{B}^s$-Hurewicz property. Define a strategy ψ for ONE in the $\mathfrak{B}^s$-Hurewicz game on X. Let the first move of ONE in the $\mathfrak{B}^s$-Hurewicz game be $\psi(\emptyset) = \{U_{(n)} : n \in \mathbb{N}\}$. Let TWO choose a finite set $\mathcal{V}_{(n_1)} = \{U_{(n)} : n \leq n_1\}$. Suppose that for each finite sequence σ of positive integers of length at most m, U_σ has been defined. Now $\psi(\mathcal{V}_{(n_1)}, \dots, \mathcal{V}_{(n_1,\dots,n_{m-1})}) = \{U_{(n_1,\dots,n_{m-1},k)} : k \in \mathbb{N}\}$. Let TWO choose a finite set $\mathcal{V}_{(n_1,\dots,n_m)} \in F\left(\mathcal{V}_{(n_1)}, \dots, \mathcal{V}_{(n_1,\dots,n_{m-1})}\right) \setminus \{\mathcal{V}_{(n_1)} \cup \dots \cup \mathcal{V}_{(n_1,\dots,n_{m-1})}\}$. Clearly for each finite sequence σ of positive integers, $\mathcal{U}_\sigma = \{U_{\sigma\frown(n)} : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. By (1), we get for each σ, a $n_\sigma \in \mathbb{N}$ and a finite set $\mathcal{V}_\sigma = \{U_{\sigma\frown(n_\sigma)} : n \leq n_\sigma\}$ for which $\{\mathcal{V}_\sigma : \sigma \text{ is a finite sequence}\}$ witnesses the $\mathfrak{B}^s$-Hurewicz property of X.

Now we inductively define a sequence $n_1 = n_\emptyset$, $n_{k+1} = n_{(n_1,\dots,n_k)}$ for $k \geq 1$. The sequence $\{\mathcal{V}_{(n_1)}, \dots, \mathcal{V}_{(n_1,\dots,n_m)}, \dots\}$ witnesses the $\mathfrak{B}^s$-Hurewicz property of X. Since this is a sequence of moves of TWO in the $\mathfrak{B}^s$-Hurewicz game on X, ψ is not a winning strategy for ONE in the $\mathfrak{B}^s$-Hurewicz game in X.

(2) $\Rightarrow$ (1). We assume contrary. Then there exists a collection $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \mathcal{O}_{\mathfrak{B}^s}$ for which any sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite sets with $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each n fails to witness the $\mathfrak{B}^s$-Hurewicz property. A strategy ψ for ONE in the $\mathfrak{B}^s$-Hurewicz game on X is defined as follows. Let the 1st move of ONE be $\psi(\emptyset) = \mathcal{U}_1$. Let TWO choose a finite set $\mathcal{V}_1 \subseteq \mathcal{U}_1$. For $n > 1$ define $\psi(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_{n-1}) = \mathcal{U}_n \setminus (\mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots \cup \mathcal{V}_{n-1})$. By the given condition, ψ is not a wining strategy for ONE in the $\mathfrak{B}^s$-Hurewicz game. Therefore the sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of moves of TWO witnesses the $\mathfrak{B}^s$-Hurewicz property. This contradicts our assumption. $\square$

In the following we introduce the notion of a $\gamma_{\mathfrak{B}^s}$-set .

Definition 2.3.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. X is a $\gamma_{\mathfrak{B}^s}$-set if X satisfies the selection principle $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

From [19, Theorem 2.8], it is clear that X is a $\gamma_{\mathfrak{B}^s}$-set if and only if every open $\mathfrak{B}^s$-cover of X a set that is a $\gamma_{\mathfrak{B}^s}$-cover of X. Therefore we can say that X is a $\gamma_{\mathfrak{B}^s}$-set if and only if X satisfies $\binom{\mathcal{O}_{\mathfrak{B}^s}}{\Gamma_{\mathfrak{B}^s}}$.

It can be shown that the real line endowed with certain bornology is a $\gamma_{\mathfrak{B}^s}$-set and the Baire space with the bornology $\mathcal{F}$ is not a $\gamma_{\mathfrak{B}^s}$-set.

Example 2.3.3. *Consider Example 2.3.1. For $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \mathcal{O}_{\mathfrak{B}^s}$ we have obtained a sequence $\{\mathcal{W}_n : n \in \mathbb{N}\}$, where $\mathcal{W}_n = \{U_n\}$ and $U_n \in \mathcal{U}_n$, such that for each $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence of positive real numbers $\{\delta_n : n \geq n_0\}$ satisfying $B^{\delta_n} \subseteq U_n$ for all $n \geq n_0$. Clearly $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. This shows that X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ and hence X is a $\gamma_{\mathfrak{B}^s}$-set.*

Example 2.3.4. *Consider Example 2.2.3. We have seen that X fails to satisfy $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Clearly X does not satisfy $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ (by Figure 2) and hence X is not a $\gamma_{\mathfrak{B}^s}$-set.*

Lemma 2.3.1. *Every $\gamma_{\mathfrak{B}^s}$-set has the $\mathfrak{B}^s$-Hurewicz property.*

Now we introduce the following definition.

Definition 2.3.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. An open cover $\mathcal{U}$ of X is said to be $\mathfrak{B}^s$-groupable if there is a collection $\{\mathcal{U}_n : n \in \mathbb{N}\}$ with $\mathcal{U}_n \cap \mathcal{U}_m = \emptyset$ for $n \neq m$, $\mathcal{U}_n$'s are finite and $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{U}_n$ such that for each $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers with $B^{\delta_n} \subseteq U$ for some $U \in \mathcal{U}_n$ for all $n \geq n_0$.*

The family of all $\mathfrak{B}^s$-groupable covers of X is denoted by $\mathcal{O}^{gp}_{\mathfrak{B}^s}$.

First consider an example of a $\mathfrak{B}^s$-groupable open cover of $\mathbb{R}$.

Example 2.3.5. *Consider the real line $X = \mathbb{R}$ with the bornology as in Example 2.3.1. Clearly $\mathcal{U} = \{(-k, k) : k \in \mathbb{N}\}$ is an open-$\mathfrak{B}^s$-cover of X which is $\mathfrak{B}^s$-groupable. To see this, first choose $B \in \mathfrak{B}$. Then there is a $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $B^\delta \subseteq (-k, k)$ for $k \geq n_0$. Now choose a sequence of positive integers $k_1 < k_2 < \cdots$ and define $\mathcal{V}_1 = \{(-k, k) : k \leq k_1\}$, and for $n > 1$ $\mathcal{V}_n = \{(-k, k) : k_{n-1} < k \leq k_n\}$. Then $\mathcal{V}_n$' s are pairwise disjoint and finite subsets of $\mathcal{U}$ such that $\mathcal{U} = \cup_n\mathcal{V}_n$. Now choose $\delta_n = \delta$ for $n \geq n_0$. It follows that the collection $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the $\mathfrak{B}^s$-groupability of $\mathcal{U}$.*

Lemma 2.3.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If X has the $\mathfrak{B}^s$-Hurewicz property, then every countable open $\mathfrak{B}^s$-cover of X is $\mathfrak{B}^s$-groupable.*

Proof. Suppose that $\mathcal{U}$ is an open $\mathfrak{B}^s$-cover of X. Define a strategy σ for ONE in the $\mathfrak{B}^s$-Hurewicz game on X as follows. Define $\sigma(\emptyset) = \mathcal{U}$. Let TWO choose a finite set $\mathcal{V}_1 \subseteq \mathcal{U}$. Suppose that $\mathcal{V}_1, \ldots, \mathcal{V}_{n-1}$ are chosen. Now define $\sigma(\mathcal{V}_1, \ldots, \mathcal{V}_{n-1}) = \mathcal{U} \setminus (\mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_{n-1})$, which is an open $\mathfrak{B}^s$-cover. Let TWO choose a finite set $\mathcal{V}_n \in \sigma(\mathcal{V}_1, \ldots, \mathcal{V}_{n-1})$. By Theorem 2.3.1, the play $\sigma(\emptyset), \mathcal{V}_1, \ldots, \sigma(\mathcal{V}_1, \ldots, \mathcal{V}_{n-1}), \mathcal{V}_n, \ldots$ is lost by ONE. So for each $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ with $\delta_n > 0$ satisfying $B^{\delta_n} \subseteq U$ for $U \in \mathcal{V}_n$ for all $n \geq n_0$ and hence $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \mathcal{O}_{\mathfrak{B}^s}$. Now $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is so constructed that $\mathcal{V}_n$'s are pairwise disjoint and finite. So $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the groupability of $\cup_{n\in\mathbb{N}}\mathcal{V}_n$. Since $\mathcal{U}$ is countable, the elements of $\mathcal{U} \setminus \cup_{n\in\mathbb{N}}\mathcal{V}_n$ can be distributed among $\mathcal{V}_n$'s so that $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the $\mathfrak{B}^s$-groupability of $\mathcal{U}$. Hence $\mathcal{U}$ is $\mathfrak{B}^s$-groupable cover of X. □

The next result shows that an S_{fin}-type selection hypothesis suffices to classify the $\mathfrak{B}^s$-Hurewicz property.

Theorem 2.3.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) X has $\mathfrak{B}^s$-Hurewicz property.
(2) X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$.
(3) ONE has no wining strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$.

Proof. (2) $\Rightarrow$ (1). Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \mathcal{O}_{\mathfrak{B}^s}$. Enumerate $\mathcal{U}_n$ bijectively as $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$. Consider $\mathcal{V}_n = \{U^1_{m_1} \cap \ldots \cap U^n_{m_n} : n < m_1 < \cdots < m_n\}$. By Lemma 2.2.1, $\mathcal{V}_n \in \mathcal{O}_{\mathfrak{B}^s}$ for each n. Now apply $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ to obtain a sequence $\{\mathcal{W}_n : n \in \mathbb{N}\}$ of finite pairwise disjoint sets for which $\mathcal{W}_n \subseteq \mathcal{V}_n$ and $\cup_{n\in\mathbb{N}}\mathcal{W}_n$ is a $\mathfrak{B}^s$-groupable open cover of X. Let $\{\mathcal{Y}_n : n \in \mathbb{N}\}$ witness the $\mathfrak{B}^s$-groupability of $\cup_{n\in\mathbb{N}}\mathcal{W}_n$. Clearly $\mathcal{Y}_n$'s are pairwise disjoint sets satisfying $\cup_{n\in\mathbb{N}}\mathcal{W}_n = \cup_{n\in\mathbb{N}}\mathcal{Y}_n$.

Choose $n_1 > 1$ so large that $\mathcal{Y}_{n_1} \subseteq \cup_{j>1}\mathcal{W}_j$ and let $\mathcal{A}_1$ be the set of all U^1_m that appear as 1st component in the representation of elements of $\mathcal{Y}_{n_1}$. Choose $n_2 > n_1$ large enough so that $\mathcal{Y}_{n_2} \subseteq \cup_{j>2}\mathcal{W}_j$ and let $\mathcal{A}_2$ be set of all U^2_m that appear as 2nd component in the representation of elements of $\mathcal{Y}_{n_2}$. Continuing in this way at the k-th step we choose $\mathcal{A}_k \subseteq \mathcal{U}_k$. So we obtain a sequence $\{\mathcal{A}_k : k \in \mathbb{N}\}$ of finite sets such that $\mathcal{A}_k \subseteq \mathcal{U}_k$ for each k. For $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers such that $B^{\delta_n} \subseteq Y$ for all $n \geq n_0$ for some $Y \in \mathcal{Y}_n$. Choose a k_0 such that $k \geq k_0$ implies $n_{k_0} > n_0$. Clearly $B^{\delta_{n_k}} \subseteq Y$ for some $Y \in \mathcal{Y}_{n_k}$ for all $k \geq k_0$. Define $\delta_k = \delta_{n_k}$ for each k. By the definition of $\mathcal{A}_k$, we have $B^{\delta_k} \subseteq Y \subseteq U$ for some $U \in \mathcal{A}_k$ for all $k \geq k_0$. Hence $\{\mathcal{A}_k : k \in \mathbb{N}\}$ witnesses the $\mathfrak{B}^s$-Hurewicz property of X.

(1) $\Rightarrow$ (3). Let τ be a strategy for ONE in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$. A strategy σ for ONE in the $\mathfrak{B}^s$-Hurewicz game is defined as follows. Suppose that the first move of ONE is $\sigma(\emptyset) = \tau(\emptyset)$ ($= \mathcal{U}$ say). TWO responds with a finite set $\mathcal{V}_1 \subset \mathcal{U}$ (in the $\mathfrak{B}^s$-Hurewicz game). Then the second move of ONE is $\sigma(\mathcal{V}_1) = \tau(\mathcal{V}_1) \setminus \mathcal{V}_1$. Let TWO respond with a finite set $\mathcal{V}_2 \subseteq \sigma(\mathcal{V}_1)$. The n-th move of ONE is $\sigma(\mathcal{V}_1, \ldots, \mathcal{V}_{n-1}) = \tau(\mathcal{V}_1, \ldots, \mathcal{V}_{n-1}) \setminus (\mathcal{V}_1 \cup \ldots \mathcal{V}_{n-1})$ which is an open $\mathfrak{B}^s$-cover by Proposition 2.2.1. TWO responds with a finite set $\mathcal{V}_n \subseteq \sigma(\mathcal{V}_1, \ldots, \mathcal{V}_{n-1})$ and so on. Thus we obtain a legitimate strategy σ for ONE in the $\mathfrak{B}^s$-Hurewicz game on X. Since X has the $\mathfrak{B}^s$-Hurewicz property, the play $\sigma(\emptyset), \mathcal{V}_1, \sigma(\mathcal{V}_1, \mathcal{V}_2), \mathcal{V}_2, \ldots$ is lost by ONE. Thus for each $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$, $\delta_n > 0$, for which $B^{\delta_n} \subseteq V$ for all $n \geq n_0$ for some $V \in \mathcal{V}_n$. Clearly $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a $\mathfrak{B}^s$-groupable open cover of X as $\mathcal{V}_n$'s are pairwise disjoint finite sets. Now for the strategy τ, the play $\tau(\emptyset), \mathcal{V}_1, \tau(\mathcal{V}_1), \mathcal{V}_2, \tau(\mathcal{V}_1, \mathcal{V}_2), \mathcal{V}_3, \ldots$ is legitimate in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$. As $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of moves by TWO in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ and $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a $\mathfrak{B}^s$-groupable open cover of X. Hence τ is not a wining strategy for ONE in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$.

(3) $\Rightarrow$ (2). Assume that X does not satisfy $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$. Then there exists a sequence $\{\mathcal{U}_n :$

$n \in \mathbb{N}\}$ from $\mathcal{O}_{\mathfrak{B}^s}$ for which for any sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$, where each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, we have $\cup_{n\in\mathbb{N}}\mathcal{V}_n \notin \mathcal{O}^{gp}_{\mathfrak{B}^s}$. A strategy F for ONE in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ is defined as follows. Let the first move of ONE be $F(X) = \mathcal{U}_1$. TWO responds by choosing a finite set $\mathcal{V}_1 \subseteq \mathcal{U}_1$. In the n-th inning ONE's move is $F(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n-1}) = \mathcal{U}_n$ and TWO responds by choosing a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$. Since F is not a wining strategy for ONE, we must have $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \mathcal{O}^{gp}_{\mathfrak{B}^s}$, which contradicts our assumption. Hence X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$. □

2.4 RESULTS IN FUNCTION SPACES

In this section we investigate some properties of function space endowed with the topology of strong uniform convergence on a bornology. Let $\mathfrak{B}$ be a bornology on (X, d) with closed base and let (Y, ρ) be another metric space. For $f \in C(X, Y)$, the neighbourhood of f with respect to the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence is denoted by

$$[B, \varepsilon]^s(f) = \{g \in C(X, Y) : \exists\delta > 0, \rho(f(x), g(x)) < \varepsilon, \forall x \in B^\delta\},$$

for $B \in \mathfrak{B}$, $\varepsilon > 0$ ([14, 18]).

Lemma 2.4.1. *([19, Lemma 2.2]) Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements hold.*
(a) Let $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X. If $A = \{f \in C(X) : \exists U \in \mathcal{U}, f(x) = 1$ for all $x \in X \setminus U\}$. Then $\underline{0} \in \overline{A} \setminus A$ in $(C(X), \tau^s_{\mathfrak{B}})$.
(b) Let $A \subseteq (C(X), \tau^s_{\mathfrak{B}})$ and let $\mathcal{U} = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A\}$, where $n \in \mathbb{N}$. If $\underline{0} \in \overline{A}$ and $X \notin \mathcal{U}$, then $\mathcal{U} \in \mathcal{O}_{\mathfrak{B}^s}$.

2.4.1 CERTAIN APPLICATIONS OF $\mathfrak{B}^s$-COVERS

We start with a basic observation about $\mathfrak{B}^s$-covers.

Theorem 2.4.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\mathcal{U}$ be a collection of open subsets in X. The following statements are equivalent.*
(1) $\mathcal{U}$ is an open $\mathfrak{B}^s$-cover of X.
(2) For each $U \in \mathcal{U}$ there exists a closed set $C(U) \subseteq U$ for which $\{C(U) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X.

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{U} \in \mathcal{O}_{\mathfrak{B}^s}$. Let $A = \{f \in C(X) : \exists U \in \mathcal{U}, f(X \setminus U) = \{1\}\}$. By Lemma 2.4.1, $A \in \Omega_{\underline{0}}$. For $U \in \mathcal{U}$, if $f \in A$ such that $f(X \setminus U) = \{1\}$, then $f^{-1}([-\frac{1}{2}, \frac{1}{2}]) \subseteq U$. Now take the closed set $C(U)$ to be $f^{-1}[-\frac{1}{2}, \frac{1}{2}]$. Consider the collection $\mathcal{V} = \{C(U) : U \in \mathcal{U}\}$. We shall show that $\mathcal{V}$ is a $\mathfrak{B}^s$-cover of X. For $B \in \mathfrak{B}$, consider the neighbourhood $[B, \frac{1}{2}]^s(\underline{0})$. Clearly $f \in [B, \frac{1}{2}]^s(\underline{0}) \cap A \neq \emptyset$. There exists a $\delta > 0$ such that $|f(x)| < \frac{1}{2}$ for all $x \in B^\delta$ and so $B^\delta \subseteq f^{-1}(-\frac{1}{2}, \frac{1}{2}) \subseteq f^{-1}[-\frac{1}{2}, \frac{1}{2}]$. This shows that $\mathcal{V}$ is a $\mathfrak{B}^s$-cover of X.

The implication (2) $\Rightarrow$ (1) is straightforward. □

In the next few results we investigate how various properties of $C(X)$ can be characterized in terms of $\mathfrak{B}^s$-covers of X. We start with the property of countable T-tightness of $C(X)$ whose characterization is established in the next result.

Theorem 2.4.2. *Let $\mathfrak{B}$ a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *$(C(X), \tau^s_{\mathfrak{B}})$ has countable T-tightness.*
(2) *For each uncountable regular cardinal ρ and each increasing sequence $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of families of open subsets of X for which $\bigcup_{\alpha<\rho} \mathcal{U}_\alpha \in \mathcal{O}_{\mathfrak{B}^s}$, there is a $\beta < \rho$ with $\mathcal{U}_\beta$ being an open $\mathfrak{B}^s$-cover of X.*

Proof. (1) $\Rightarrow$ (2). Consider an increasing sequence of families of open subsets $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of X such that $\bigcup_{\alpha<\rho} \mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X. For $B \in \mathfrak{B}$ there exist a $\delta > 0$ and $U \in \bigcup_{\alpha<\rho} \mathcal{U}_\alpha$ such that $B^{2\delta} \subseteq U$. Define a continuous function $f_{B,U}$ on X such that $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. For each $\alpha < \rho$, consider $A_\alpha = \{f_{B,U} : U \in \mathcal{U}_\alpha, B \in \mathfrak{B}\}$. Clearly $\{\overline{A_\alpha} : \alpha < \rho\}$ is an increasing sequence of closed subsets of $C(X)$. By (1), the set $A = \bigcup_{\alpha<\rho} \overline{A_\alpha}$ is closed in $(C(X), \tau^s_{\mathfrak{B}})$. Since $\bigcup_{\alpha<\rho} \mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X, $\underline{0} \in \overline{\cup_{\alpha<\rho} A_\alpha} \setminus \cup_{\alpha<\rho} A_\alpha$ by Lemma 2.4.1(a). Clearly $\underline{0} \in A$ and so one can find a $\beta < \rho$ with $\underline{0} \in \overline{A_\beta}$. By Lemma 2.4.1($b$), $\{f^{-1}_{B,U}(-1,1) : f_{B,U} \in A_\beta\}$ is an open $\mathfrak{B}^s$-cover of X. Since $f^{-1}_{B,U}(-1,1) \subseteq U \in \mathcal{U}_\beta$, $\mathcal{U}_\beta$ is an open $\mathfrak{B}^s$-cover of X.
(2) $\Rightarrow$ (1). Consider an increasing sequence $\{A_\alpha : \alpha < \rho\}$ of closed subsets of $C(X)$. We shall show that $A = \cup_{\alpha<\rho} A_\alpha$ is closed in $C(X)$. Without any loss of generality assume $\underline{0} \in \overline{A}$. Consider the collection $\mathcal{U}_{\alpha,n} = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A_\alpha\}$ and $\mathcal{U}_n = \bigcup_{\alpha<\rho} \mathcal{U}_{\alpha,n}$. By Lemma 2.4.1($b$), $\mathcal{U}_n \in \mathcal{O}_{\mathfrak{B}^s}$ for each $n \in \mathbb{N}$. Now by our assumption for each $n \in \mathbb{N}$ there is an open $\mathfrak{B}^s$-cover $\mathcal{U}_{\beta_n,n} \subset \mathcal{U}_n$. Let $\beta_0 = \sup\{\beta_n : n \in \mathbb{N}\}$. It can be easily seen that for each n, $\mathcal{U}_{\beta_0,n} \in \mathcal{O}_{\mathfrak{B}^s}$. Now we show that $\underline{0} \in A_{\beta_0}$. Let $B \in \mathfrak{B}$ and $\varepsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ for all $n \geq n_0$. Since $\mathcal{U}_{\beta_0,n} = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A_{\beta_0}\}$ is an open $\mathfrak{B}^s$-cover of X, there exist a $f \in A_{\beta_0}$ and a $\delta > 0$ such that $B^\delta \subseteq f^{-1}(-\frac{1}{n}, \frac{1}{n}) \subseteq f^{-1}(-\varepsilon, \varepsilon)$ for $n \geq n_0$. Thus $f \in [B, \varepsilon]^s(\underline{0}) \cap A_{\beta_0} \neq \emptyset$ and so $\underline{0} \in \overline{A_{\beta_0}} = A_{\beta_0}$. Hence $\underline{0} \in A$. This proves that A is closed in X. □

Using techniques of the proof of [102, Theorem 12] and using Lemma 2.4.1 one can also characterize the weakly Fréchet-Urysohn property (Reznichenko property) as well.

Theorem 2.4.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Then the following statements are equivalent.*
(1) *$(C(X), \tau^s_{\mathfrak{B}})$ is weakly Fréchet-Urysohn.*
(2) *Every open $\mathfrak{B}^s$-cover of X is $\mathfrak{B}^s$-groupable.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{U} \in \mathcal{O}_{\mathfrak{B}^s}$. By Theorem 2.4.1, for each $U \in \mathcal{U}$ there is a closed set $C(U)$ with $C(U) \subseteq U$ such that $\{C(U) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X. Define a continuous function f_U on X such that $f_U(C(U)) = \{0\}$ and $f_U(X \setminus U) = \{1\}$. Clearly $C(U) \subseteq f^{-1}_U(\{0\})$ and so

$\{f_U^{-1}(\{0\}) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X. We can assume that f_U and $f_{U'}$ are distinct whenever U and U' are distinct. Choose $A = \{f_U : U \in \mathcal{U}\}$. Consider a neighbourhood $[B,\epsilon]^s(\underline{0})$ of $\underline{0}$, where $B \in \mathfrak{B}$ and $\epsilon > 0$. There exists a $\delta > 0$ such that $B^\delta \subseteq C(U)$. Clearly $f_U(B^\delta) = \{0\}$ and $f_U \in [B,\epsilon]^s(\underline{0}) \cap A$ and hence $\underline{0} \in \overline{A} \setminus A$. By (1), there is a sequence $\{A_n : n \in \mathbb{N}\}$ of pairwise disjoint finite subsets of A such that for $B \in \mathfrak{B}$ and $\epsilon > 0$ there is a $n_0 \in \mathbb{N}$ satisfying $[B,\epsilon]^s(\underline{0}) \cap A_n \neq \emptyset$ for all $n \geq n_0$. For each n consider $\mathcal{U}_n = \{U : f_U \in A_n\}$. Then $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is pairwise disjoint finite subsets of $\mathcal{U}$. If we choose $\epsilon = 1$, then $[B,1]^s(\underline{0}) \cap A_n \neq \emptyset$ for all $n \geq n_0$. Let $f_U \in [B,1]^s(\underline{0}) \cap A_n$. Then there is a $\delta > 0$ such that $|f_U(x)| < 1$ for all $x \in B^\delta$, i.e., $B^\delta \subseteq f_U^{-1}(-1,1) \subseteq U$ for some $U \in \mathcal{U}_n$ for all $n \geq n_0$. Consequently $\{\mathcal{U}_n : n \in \mathbb{N}\}$ witnesses the $\mathfrak{B}^s$-groupability for $\mathcal{U}$.

(2) $\Rightarrow$ (1). Let $A \subset C(X)$ with $\underline{0} \in \overline{A} \setminus A$. By Lemma 2.4.1$(b)$, $\mathcal{U}_1 = \{f^{-1}(-1,1) : f \in A\}$ is an open $\mathfrak{B}^s$-cover of X. Choose $B \in \mathfrak{B}$ and $\epsilon > 0$. By (2), there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $\mathcal{U}_1$, a $n_1 \in \mathbb{N}$ and a sequence of positive real numbers $\{\delta_n : n \geq n_1\}$ satisfying $B^{\delta_n} \subseteq f^{-1}(-1,1) \in \mathcal{V}_n$ for all $n \geq n_1$. Choose a sequence $\{H_n : n \in \mathbb{N}\}$ of pairwise disjoint finite subsets of A such that $\mathcal{V}_n = \{f^{-1}(-1,1) : f \in H_n\}$ and $[B,1]^s(\underline{0}) \cap H_n \neq \emptyset$ for all $n \geq n_1$. Let $H = \cup\{H_n : n \in \mathbb{N}\}$, $I_0 = \{H_{2n} : n \in \mathbb{N}\}$ and $I_1 = \{H_{2n+1} : n \in \mathbb{N}\}$. Clearly either $\underline{0} \in \overline{I_0 \cup (A \setminus H)}$ or $\underline{0} \in \overline{I_1 \cup (A \setminus H)}$. Without any loss of generality, let $\underline{0} \in \overline{I_1 \cup (A \setminus H)}$ and enumerate I_0 as $\{A_{1,n} : n \in \mathbb{N}\}$. Again by Lemma 2.4.1$(b)$, $\mathcal{U}_2 = \{f^{-1}(-\frac{1}{2}, \frac{1}{2}) : f \in I_1 \cup (A \setminus H)\}$ is an open $\mathfrak{B}^s$-cover of X. Proceeding similarly, we obtain a sequence $\{A_{2,n} : n \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $I_1 \cup (A \setminus H)$ and a $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$ $[B, \frac{1}{2}]^s(\underline{0}) \cap A_{2,n} \neq \emptyset$. Again $\underline{0} \in \overline{A \setminus \{A_{m,n} : n \in \mathbb{N}, m = 1,2\}}$. Repeating the same process, we obtain a family $\{A_{m,n} : n,m \in \mathbb{N}\}$ of pairwise disjoint finite subsets of A. For each m there exists a $n_m \in \mathbb{N}$ for which $[B, \frac{1}{m}]^s(\underline{0}) \cap A_{m,n} \neq \emptyset$ for all $n \geq n_m$. For each $k \in \mathbb{N}$ let $A_k = \cup\{A_{i,j} : i + j = k\}$. Then $\{A_k : k \in \mathbb{N}\}$ is family of pairwise disjoint finite subsets of A. Choose $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} < \epsilon$. Clearly there exists a $n_{m_0} \in \mathbb{N}$ for which $[B, \frac{1}{m_0}]^s(\underline{0}) \cap A_{m_0,n} \neq \emptyset$ for all $n \geq n_{m_0}$. Choose $k_0 = m_0 + n_{m_0}$. Consequently $[B, \frac{1}{m_0}]^s(\underline{0}) \cap A_k \neq \emptyset$ for all $k \geq k_0$ and so $[B,\epsilon]^s(\underline{0}) \cap A_k \neq \emptyset$ for all $k \geq k_0$. Hence $(C(X), \tau^s_{\mathfrak{B}})$ is weakly Fréchet-Urysohn. □

In the next result we obtain a characterization of a countable π-network at $\underline{0}$ of finite subsets of $A \subset (C(X), \tau^s_{\mathfrak{B}})$.

Theorem 2.4.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*

(1) *If A is a subset of $(C(X), \tau^s_{\mathfrak{B}})$ and if $\underline{0} \in \overline{A} \setminus A$, then there is a countable π-network at $\underline{0}$ of finite subsets of A.*

(2) *If $\mathcal{U}$ is an open $\mathfrak{B}^s$-cover of X, then there is a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of finite subfamilies of $\mathcal{U}$ for which $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{U} \in \mathcal{O}_{\mathfrak{B}^s}$. Let $A = \{f_U \in C(X) : \exists U \in \mathcal{U}, f_U(X \setminus U) = \{1\}\}$. By Lemma 2.4.1$(a)$, $A \in \Omega_{\underline{0}}$. By (1), choose a collection $\{A_n : n \in \mathbb{N}\}$ of finite subsets of A for which $\{A_n : n \in \mathbb{N}\}$ is a π-network at $\underline{0}$. Now choose a finite subset $\mathcal{U}_n$ of $\mathcal{U}$ for which $A_n = \{f_U : U \in \mathcal{U}_n\}$. We show that $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\} \in \mathcal{O}_{\mathfrak{B}^s}$. Let $B \in \mathfrak{B}$ and consider the neighbourhood $[B,1]^s(\underline{0})$ of $\underline{0}$. Then $A_n \subset [B,1]^s(\underline{0})$ for some $n \in \mathbb{N}$ as $\{A_n : n \in \mathbb{N}\}$ is a π-network at $\underline{0}$. Clearly $f_U \in [B,1]^s(\underline{0})$ for all $U \in \mathcal{U}_n$. If $\mathcal{U}_n = \{U_n^1, U_n^2, \cdots, U_n^{r_n}\}$, then for each $k = 1,2,\ldots,r_n$, there exists a $\delta_k > 0$ such that $|f_{U_n^k}(x)| < 1$ for all $x \in B^{\delta_k}$, i.e., $B^{\delta_k} \subseteq U_n^k$. Consequently there exists a $\delta > 0$ such that $B^\delta \subseteq \bigcap \mathcal{U}_n$. Hence $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\} \in \mathcal{O}_{\mathfrak{B}^s}$.

(2) $\Rightarrow$ (1). Let $A \subset C(X)$ be such that $\underline{0} \in \overline{A} \setminus A$. For each $n \in \mathbb{N}$, consider the collection $\mathcal{U}_n = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A\}$. Then by Lemma 2.4.1$(b)$, $\mathcal{U}_n$ is an open $\mathfrak{B}^s$-cover of X.
If there exists a sequence $n_1 < n_2 < \cdots$ of positive integers such that $f_k^{-1}(-\frac{1}{n_k}, \frac{1}{n_k}) = X$, then the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges to $\underline{0}$. Choose $A_1 = \{f_i : i < n_1\}$ and for each $k > 1$ choose $A_k = \{f_i : n_{k-1} \leq i < n_k\}$. Consider the family $\mathcal{A} = \{A_k : k \in \mathbb{N}\}$. For $B \in \mathfrak{B}$ and $\epsilon > 0$, consider the neighbourhood $[B,\epsilon]^s(\underline{0})$ of $\underline{0}$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{n_k} < \epsilon$ for all $k \geq k_0$. Now for any $\delta > 0$, $B^\delta \subseteq f_k^{-1}(-\frac{1}{n_k}, \frac{1}{n_k}) = X$ for all $k \in \mathbb{N}$ and hence $f_k \in [B,\epsilon]^s(\underline{0})$ for all $k \geq k_0$, i.e., $A_{k+1} \subset [B,\epsilon]^s(\underline{0})$ for all $k \geq k_0$. Hence $\mathcal{A} = \{A_k : k \in \mathbb{N}\}$ is a π-network at $\underline{0}$.

Otherwise, we assume that there is a $n_0 \in \mathbb{N}$ such that $X \notin \mathcal{U}_n$ for all $n \geq n_0$. Now by (2), for each $n \geq n_0$, there exists a sequence $\{\mathcal{V}_{n,m} : m \in \mathbb{N}\}$ of finite subfamilies of $\mathcal{U}_n$ for which $\{\bigcap \mathcal{V}_{n,m} : m \in \mathbb{N}\} \in \mathcal{O}_{\mathfrak{B}^s}$. Consequently there is a collection $\{A_{n,m} : n \geq n_0, m \in \mathbb{N}\}$ of finite subsets of A such that $\mathcal{V}_{n,m} = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A_{n,m}\}$ for each $n \geq n_0$ and $m \in \mathbb{N}$. Consider $\mathcal{A} = \{A_{n,m} : n \geq n_0, m \in \mathbb{N}\}$. Let $[B,\varepsilon]^s(\underline{0})$ be a neighbourhood of $(\underline{0})$, where $\epsilon > 0$. Choose $n_1 \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ for all $n \geq n_1$. Fix $n \geq \max\{n_0, n_1\}$. Then there exists a $\delta > 0$ such that $B^\delta \subseteq \cap \mathcal{V}_{n,m}$ for some $m \in \mathbb{N}$, i.e., $B^\delta \subseteq f^{-1}(-\frac{1}{n}, \frac{1}{n})$ for all $f \in A_{n,m}$ for some m. Consequently $f \in [B, \frac{1}{n}]^s(\underline{0})$ for all $f \in A_{n,m}$ for some m. So $A_{n,m} \subseteq [B, \frac{1}{n}]^s(\underline{0}) \subseteq [B,\varepsilon]^s(\underline{0})$ for some m and each $n \geq \max\{n_0, n_1\}$. Hence $\mathcal{A} = \{A_{n,m} : n \geq n_0, m \in \mathbb{N}\}$ is a π-network at $\underline{0}$. □

Similarly we can prove a sufficient condition for $C(X)$ to be Pytkeev. Recall that X is called a **Pytkeev space** ([84], see [68, 102] for more details) if $x \in \overline{A} \setminus A$ and $A \subset X$ imply the existence of a countable π-network at x of infinite subsets of A.

Theorem 2.4.5. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Consider the following statements.*
(1) If $\mathcal{U}$ is an open $\mathfrak{B}^s$-cover of X, then there is a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of countably infinite subfamilies of $\mathcal{U}$ for which $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ is Pytkeev.

Then (1) $\Rightarrow$ (2) holds.

Proof. Let $A \subset C(X)$ be such that $\underline{0} \in \overline{A} \setminus A$. For each $n \in \mathbb{N}$, consider the collection $\mathcal{U}_n =$

$\{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A\}$. Then by Lemma 2.4.1$(b)$, $\mathcal{U}_n$ is an open $\mathfrak{B}^s$-cover of X.
If there exists a sequence $n_1 < n_2 < \cdots$ of positive integers such that $f_k^{-1}(-\frac{1}{n_k}, \frac{1}{n_k}) = X$, then the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges to $\underline{0}$. For each $k \in \mathbb{N}$ choose $A_k = \{f_i : i \geq k\}$. Consider the family $\mathcal{A} = \{A_k : k \in \mathbb{N}\}$. For $B \in \mathfrak{B}$ and $\epsilon > 0$, consider the neighbourhood $[B, \epsilon]^s(\underline{0})$ of $\underline{0}$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{n_k} < \epsilon$ for all $k \geq k_0$. Now for any $\delta > 0$, $B^\delta \subseteq f_k^{-1}(-\frac{1}{n_k}, \frac{1}{n_k}) = X$ for all $k \in \mathbb{N}$ and hence $f_k \in [B, \epsilon]^s(\underline{0})$ for all $k \geq k_0$, i.e., $A_k \subset [B, \epsilon]^s(\underline{0})$ for all $k \geq k_0$. Hence $\mathcal{A} = \{A_k : k \in \mathbb{N}\}$ is a π-network at $\underline{0}$.

Otherwise, we assume that there is a $n_0 \in \mathbb{N}$ such that $X \notin \mathcal{U}_n$ for each $n \geq n_0$. Now by (1), for each $n \geq n_0$ we obtain a sequence $\{\mathcal{V}_{n,m} : m \in \mathbb{N}\}$ of countably infinite subfamilies of $\mathcal{U}_n$ for which $\{\bigcap \mathcal{V}_{n,m} : m \in \mathbb{N}\} \in \mathcal{O}_{\mathfrak{B}^s}$. Consequently, there is a collection $\{A_{n,m} : n \geq n_0, m \in \mathbb{N}\}$ of infinite subsets of A such that $\mathcal{V}_{n,m} = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A_{n,m}\}$ for each $n \geq n_0$ and $m \in \mathbb{N}$. Consider $\mathcal{A} = \{A_{n,m} : n \geq n_0, m \in \mathbb{N}\}$. Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $\epsilon > 0$. Choose $n_1 \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ for all $n \geq n_1$. Fix $n \geq \max\{n_0, n_1\}$. Then there exists a $\delta > 0$ such that $B^\delta \subseteq \cap \mathcal{V}_{n,m}$ for some $m \in \mathbb{N}$. Clearly $B^\delta \subseteq f^{-1}(-\frac{1}{n}, \frac{1}{n})$ for all $f \in A_{n,m}$ for some m and $f \in [B, \frac{1}{n}]^s(\underline{0})$ for all $f \in A_{n,m}$ for some m. Consequently $A_{n,m} \subseteq [B, \frac{1}{n}]^s(\underline{0}) \subseteq [B, \varepsilon]^s(\underline{0})$ for some m and each $n \geq \max\{n_0, n_1\}$. Therefore $\mathcal{A} = \{A_{n,m} : n \geq n_0, m \in \mathbb{N}\}$ is a π-network at $\underline{0}$ of infinite subsets of A and hence $C(X)$ is Pytkeev. □

2.4.2 GAME THEORETIC RESULTS IN FUNCTION SPACES

In the next two results we observe the connection between game theoretic relations on X and selection hypotheses on $C(X)$. First we state a result from [19]. It was shown [19, Theorem 2.3] that $(C(X), \tau^s_{\mathfrak{B}})$ *has countable strong fan tightness if and only if X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Theorem 2.4.6. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If ONE has no wining strategy in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ on X, then $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $S_1(\Omega_{\underline{0}}, \Omega^{gp}_{\underline{0}})$.*

Proof. Clearly if X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ then X also satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. By [19, Theorem 2.3], $(C(X), \tau^s_{\mathfrak{B}})$ has countable strong fan tightness, i.e., $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_1(\Omega_{\underline{0}}, \Omega_{\underline{0}})$. Now it suffices to show that each countable subset of $\Omega_{\underline{0}}$ is groupable. Let $A \in \Omega_{\underline{0}}$ be a countable subset of $C(X)$. By Lemma 2.4.1, $\mathcal{U}_1 = \{f^{-1}(-1, 1) : f \in A\} \in \mathcal{O}_{\mathfrak{B}^s}$. A strategy σ for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ is defined as follows. Let the first move of ONE be $\sigma(\emptyset) = \mathcal{U}_1$. TWO chooses $U_1 \in \mathcal{U}_1$. Take the corresponding function $f_1 \in A$ such that $U_1 = f_1^{-1}(-1, 1)$. Now consider $A_1 = A \setminus \{f_1\}$ with $\underline{0} \in \overline{A_1}$. Again by Lemma 2.4.1, $\mathcal{U}_2 = \{f^{-1}(-\frac{1}{2}, \frac{1}{2}) : f \in A_1\} \setminus \{U_1\} \in \mathcal{O}_{\mathfrak{B}^s}$. Let the second move of ONE be $\sigma(U_1) = \mathcal{U}_2$. Take the corresponding function $f_2 \in A_1$ such that $U_2 = f_2^{-1}(-\frac{1}{2}, \frac{1}{2})$. Continuing in this way we have the following.

(a) $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \mathcal{O}_{\mathfrak{B}^s}$, where $\mathcal{U}_n = \sigma(U_1, U_2, \ldots, U_{n-1})$.

(b) For each $n \in \mathbb{N}$, U_n is not in $\{U_1, U_2, \ldots, U_{n-1}\}$.

(c) For each $n \in \mathbb{N}$, $f_n \in A \setminus \{f_1, f_2, \dots, f_{n-1}\}$.

(d) For each $n \in \mathbb{N}$, $U_n = f_n^{-1}(-\frac{1}{n}, \frac{1}{n})$.

Since σ is not a wining strategy for ONE, the play $\mathcal{U}_1, U_1 \dots, \mathcal{U}_n, U_n, \dots$ is lost by ONE and consequently $\mathcal{V} = \{U_n : n \in \mathbb{N}\} \in \mathcal{O}^{gp}_{\mathfrak{B}^s}$. Therefore there is an increasing infinite sequence $n_1 < n_2 < \cdots$ such that the sets $\mathcal{H}_k = \{U_i : n_k \leq i \leq n_{k+1}\}$ (for $k = 1, 2, \dots$) are finite, pairwise disjoint and for every $B \in \mathfrak{B}$ there exist a $k_0 \in \mathbb{N}$ and a sequence $\{\delta_k : k \geq k_0\}$ of positive real numbers such that $B^{\delta_k} \subseteq U$ for $U \in \mathcal{H}_k$ for all $k \geq k_0$. Define $\mathcal{M}_k = \{f_i : n_k \leq i < n_{k+1}\}$. Then $\mathcal{M}_k$'s are finite, pairwise disjoint subsets of A. Clearly $A = \bigcup_{k \in \mathbb{N}} \mathcal{M}_k$. We claim that $\{\mathcal{M}_k : k \in \mathbb{N}\}$ witnesses the groupability of A.

To see this for $B \in \mathfrak{B}$, consider the neighbourhood $[B, \varepsilon]^s(\underline{0})$ of $\underline{0}$. Choose a $k_1 \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$ for all $k \geq k_1$. Again for that B there exist a $k_0 \in \mathbb{N}$ and a sequence $\{\delta_k : k \geq k_0\}$ of positive real numbers such that $B^{\delta_k} \subseteq U$ for $U \in \mathcal{H}_k$ for all $k \geq k_0$, i.e., $B^{\delta_k} \subseteq f^{-1}(-\frac{1}{k}, \frac{1}{k})$. Observe that $|f(x)| < \frac{1}{k} < \varepsilon$ for all $x \in B^{\delta_k}$ and $k \geq k_2$, where $k_2 \geq \max\{k_0, k_1\}$. Clearly $f \in \mathcal{M}_k \cap [B, \varepsilon]^s(\underline{0}) \neq \emptyset$ for all $k \geq k_2$. Hence every neighbourhood of $\underline{0}$ intersect all but finitely many $\mathcal{M}_k$. So $A \in \Omega^{gp}_{\underline{0}}$. Hence X satisfies $S_1(\Omega_{\underline{0}}, \Omega^{gp}_{\underline{0}})$. □

Theorem 2.4.7. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If ONE has no wining strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$, then $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_{\text{fin}}(\Omega_{\underline{0}}, \Omega^{gp}_{\underline{0}})$.*

Proof. The proof is similar to Theorem 2.4.6 and so is omitted. □

The countable fan tightness and Reznichenko's property of $C(X)$ can also be characterized in terms of the $\mathfrak{B}^s$-Hurewicz property of X. First we state the following results. In [62, Theorem 21] it was proved that *$C_p(X)$ has countable fan tightness and Reznichenko's property if and only if $C_p(X)$ has the property $\mathsf{S}_{\text{fin}}(\Omega_{\underline{0}}, \Omega^{gp}_{\underline{0}})$*. Also by [19, Theorem 2.5], *$(C(X), \tau_{\mathfrak{B}^s})$ has countable fan tightness if and only if X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

Theorem 2.4.8. *Let $\mathfrak{B}$ be bornology with closed base on a metric space X. The following statements are equivalent.*
(1) X has the $\mathfrak{B}^s$-Hurewicz property.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ has countable fan tightness and Reznichenko's property.

Proof. (1) ⇒ (2). The result directly follows from Theorem 2.3.2 and Theorem 2.4.7.
(2) ⇒ (1). Since $C(X)$ has countable fan tightness, by [19, Theorem 2.5] X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Also by Theorem 2.4.3, every open $\mathfrak{B}^s$-cover of X is $\mathfrak{B}^s$-groupable. Clearly X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$. By Theorem 2.3.2, X has the $\mathfrak{B}^s$-Hurewicz property. □

Theorem 2.4.9. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*

(1) *TWO has a wining strategy in* $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$ *on* $(C(X),\tau^s_{\mathfrak{B}})$.
(2) *TWO has a wining strategy in* $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$ *on* X.

Proof. (1) $\Rightarrow$ (2). Let ψ be a wining strategy for TWO in $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$. A strategy σ for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$ is defined as follows. Let the first move of ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$ be $\mathcal{U}_1\in\mathcal{O}_{\mathfrak{B}^s}$. For each $B\in\mathfrak{B}$, there exist a $\delta>0$ and $U\in\mathcal{U}_1$ such that $B^{2\delta}\subseteq U$. Take a continuous function $f_{B,U}$ on X such that $f_{B,U}(B^\delta)=\{0\}$ and $f_{B,U}(X\setminus U)=\{1\}$. Then the collection $A_1=\{f_{B,U}:B\in\mathfrak{B},U\in\mathcal{U}_1\}$ is in $\Omega_{\underline{0}}$ by Lemma 2.4.1. Let the first move of ONE in $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$ is A_1. TWO responds by choosing $\psi(A_1)$ $(=f_{B_1,U_1})$ (say). Now define $\sigma(\mathcal{U}_1)=U_1$. In the n-th inning, the move of ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$ is $\mathcal{U}_n$, then the corresponding move of ONE in $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$ is $A_n=\{f_{B,U}:B\in\mathfrak{B},U\in\mathcal{U}_n\}$ and $\psi(A_1,\dots,A_n)=f_{B_n,U_n}$. Now define $\sigma(\mathcal{U}_1,\dots,\mathcal{U}_n)=U_n$.

A play in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$ is

$$\mathcal{U}_1,\sigma(\mathcal{U}_1),\dots,\mathcal{U}_n,\sigma(\mathcal{U}_1,\dots,\mathcal{U}_n)\dots$$

and the corresponding play in the game $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$ is

$$A_1,\psi(A_1),\dots,A_n,\psi(A_1,\dots,A_n)\dots,$$

. Now by (1), ψ is a wining strategy for TWO in $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$. So $\{f_n:n\in\mathbb{N}\}\in\Sigma_{\underline{0}}$, where $f_n=f_{B_n,U_n}$. We want to show that $\{U_n:n\in\mathbb{N}\}\in\Gamma_{\mathfrak{B}^s}$. Let $B\in\mathfrak{B}$ and $\varepsilon=1$. For the neighbourhood $[B,1]^s(\underline{0})$ there exists a $n_0\in\mathbb{N}$ such that $f_n\in[B,1]^s(\underline{0})$ for all $n\geq n_0$, i.e., for each n there exists a $\delta_n>0$ such that $|f_n(x)|<1$ for all $x\in B^{\delta_n}$ and so $B^{\delta_n}\subseteq U_n$ for all $n\geq n_0$. So $\{U_n:n\in\mathbb{N}\}\in\Gamma_{\mathfrak{B}^s}$. Hence σ is a wining strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$.
(2) $\Rightarrow$ (1). Let σ be a wining strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$. For each n let $I_n=(-\frac{1}{n+1},\frac{1}{n+1})$. A strategy ψ for TWO in $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$ is defined as follows. In the n-th inning let the move of ONE in $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$ be $A_n\in\Omega_{\underline{0}}$. Then n-th move of ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$ is $\mathcal{U}(A_n)=\{f^{-1}(I_n):f\in A_n\}$, where $\mathcal{U}(A_n)$ is an open $\mathfrak{B}^s$-cover by Lemma 2.4.1. Now $\sigma(\mathcal{U}(A_1),\dots,\mathcal{U}(A_n))=U_n$, where $U_n=f_n^{-1}(I_n)$. Define $\psi(A_1,\dots,A_n)=f_n$.

Now a play in the game $\mathsf{G}_1(\Omega_{\underline{0}},\Sigma_{\underline{0}})$ is

$$A_1,\psi(A_1),\dots,A_n,\psi(A_1,\dots,A_n),\dots$$

and the corresponding play in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$ is

$$\mathcal{U}(A_1),\sigma(\mathcal{U}(A_1)),\dots,\mathcal{U}(A_n),\sigma(\mathcal{U}(A_1),\dots,\mathcal{U}(A_n)),\dots.$$

If for $A_n\in\Omega_{\underline{0}}$, $X\in\mathcal{U}(A_n)$ for infinitely many n, then the conclusion is trivial. So we assume that $X \notin \mathcal{U}(A_n)$ for all $n \geq n_0$, $n_0 \in \mathbb{N}$. Since σ is a wining strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, we have $\{U_n : n \in \mathbb{N}\} \in \Gamma_{\mathfrak{B}^s}$. We show that $\{f_n : n \in \mathbb{N}\} \in \Sigma_{\underline{0}}$. Let $B \in \mathfrak{B}$ and choose a neighbourhood $[B, \varepsilon]^s(\underline{0})$ of $\underline{0}$. Choose $n_1 \in \mathbb{N}$ such that $\frac{1}{n+1} < \varepsilon$ for $n \geq n_1$. Now there is a $n_2 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_2\}$ satisfying $B^{\delta_n} \subseteq U_n$ for all $n \geq n_2$. Since $U_n = f_n^{-1}(I_n) = f_n^{-1}(-\frac{1}{n+1}, \frac{1}{n+1})$, we have $|f_n(x)| < \frac{1}{n+1} < \varepsilon$ for all $n \geq n_1$ and $x \in U_n$. If $N = \max\{n_1, n_2\}$, then $|f_n(x)| < \varepsilon$ for all $x \in B^{\delta_n}$ and $n \geq N$. Consequently $f_n \in [B, \varepsilon]^s(\underline{0})$ for all $n \geq N$ and so $\{f_n : n \in \mathbb{N}\} \in \Sigma_{\underline{0}}$. Hence ψ is a wining strategy for TWO in $\mathsf{G}_1(\Omega_{\underline{0}}, \Sigma_{\underline{0}})$. □

CHAPTER 3

On selection principles related to bornological covers

This Chapter is based on our following work:

D. Chandra, P. Das and S. Das, `Certain observations on selection principles from (a) bornological viewpoint`, **Quaest. Math.**, 45(3) (2022), 423–442.

3.1 Introduction

This Chapter is a continuation of the study of bornological open covers and related selection principles in metric spaces done in Chapter 2 using the idea of strong uniform convergence [14] on bornology. In Chapter 2, the main focus was to obtain Schepeers' like diagrams and study the notion of strong-$\mathfrak{B}$-Hurewicz property (or $\mathfrak{B}^s$-Hurewicz property). In this Chapter we explore further ramifications, presenting characterizations of various selection principles related to certain classes of bornological covers using the Ramseyan partition relations, interactive results between the cardinalities of bornological bases and certain selection principles involving bornological covers which have not been studied before.

We first present some basic observations on bornological covers and related selection principles under continuous functions. Subsequently, we deal with a totally new perspective as we investigate the behaviour of certain selection principles involving bornological covers corresponding to the cardinality of a base $\mathfrak{B}_0$ and then obtain their characterizations in terms of Ramseyan partition relations. Bornological investigation of a selection principle introduced in [112] are also presented here. We then focus on the $\mathfrak{B}^s$-Hurewicz property. Considering the product bornology $\mathfrak{B}^n$ on X^n, the $(\mathfrak{B}^n)^s$-Hurewicz property of X^n is shown to be equivalent with the $\mathfrak{B}^s$-Hurewicz property of X and moreover it is characterized Ramsey-theoretically as well as game-theoretically. Following the seminal work of [41], we introduce the notion of the

$\mathfrak{B}^s$-Gerlits-Nagy property of X and proceed to establish equivalence with the $(\mathfrak{B}^n)^s$-Gerlits-Nagy property of X^n and further use Ramseyan partition relations to characterize it.

3.2 RESULTS RELATED TO CONTINUOUS IMAGE OF X

In this section we present some observations related to continuous image of X. First note that if $f : X \to Y$ is any map and if $\mathfrak{B}$ is a bornology on X, then the collection $f(\mathfrak{B}) = \{f(B) : B \in \mathfrak{B}\}$ is a bornology on $f(X)$. Moreover, if f is surjective, then $f(\mathfrak{B})$ is a bornology on Y. Note that if $\mathfrak{B}$ is a bornology on X with a compact base $\mathfrak{B}_0$ and $f : X \to Y$ is a continuous function, then $f(\mathfrak{B})$ is a bornology on $f(X)$ with compact base $f(\mathfrak{B}_0)$.

Lemma 3.2.1. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X and (Y, ρ) be a another metric space. Let $f : X \to Y$ be a continuous function. If $\mathcal{U}$ is an open $f(\mathfrak{B})^s$-cover ($\gamma_{f(\mathfrak{B})^s}$-cover) of $f(X)$, then $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open $\mathfrak{B}^s$-cover ($\gamma_{\mathfrak{B}^s}$-cover) of X.*

Proof. Let $B \in \mathfrak{B}_0$. Since $\mathcal{U}$ is an open $f(\mathfrak{B})^s$-cover of X so for $f(B) \in f(\mathfrak{B})$ there is a $\varepsilon > 0$ such that $f(B)^\varepsilon \subseteq U$ for some $U \in \mathcal{U}$. As f is continuous function on B and B is compact, f is strongly uniformly continuous on B [14], i.e., there is a $\delta > 0$ such that $f(B^\delta) \subseteq f(B)^\varepsilon$, i.e., $B^\delta \subseteq f^{-1}(U)$. Thus $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open $\mathfrak{B}^s$-cover of X. □

Proposition 3.2.1. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X and (Y, ρ) be a another metric space. Let $f : X \to Y$ be a continuous function. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}, \mathsf{U}_{\mathrm{fin}}\}$ and $\mathcal{P}, \mathcal{Q} \in \{\mathcal{O}, \Gamma\}$. If X satisfies $\Pi(\mathcal{P}_{\mathfrak{B}^s}, \mathcal{Q}_{\mathfrak{B}^s})$, then $f(X)$ satisfies $\Pi(\mathcal{P}_{f(\mathfrak{B})^s}, \mathcal{Q}_{f(\mathfrak{B})^s})$.*

Proof. We only show that if X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, then $f(X)$ satisfies $\mathsf{S}_1(\mathcal{O}_{f(\mathfrak{B})^s}, \Gamma_{f(\mathfrak{B})^s})$. Before proceeding with the proof note that if B is compact and U is open in X with $B \subseteq U$ then there is a $\delta > 0$ such that $B^\delta \subseteq U$.

Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $f(\mathfrak{B})^s$-covers of $f(X)$. By Lemma 3.2.1, $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}\} \in \mathcal{O}_{\mathfrak{B}^s}$ for each n. Apply $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ to $\{\mathcal{U}'_n : n \in \mathbb{N}\}$ to choose a $f^{-1}(U_n) \in \mathcal{U}'_n$ for each n for which $\{f^{-1}(U_n) : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. We now show that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{f(\mathfrak{B})^s}$-cover of $f(X)$. Let $B' \in f(\mathfrak{B}_0)$ and say $B' = f(B)$, where $B \in \mathfrak{B}_0$. Choose a n_0 and a sequence $\{\delta_n : n \geq n_0\}$ of positive reals such that $B^{\delta_n} \subseteq f^{-1}(U_n)$ for $n \geq n_0$, i.e., $f(B) \subseteq U_n$ for $n \geq n_0$. Since $f(B)$ is compact, there is a $\varepsilon_n > 0$ such that $f(B)^{\varepsilon_n} \subseteq U_n$ for $n \geq n_0$. Consequently $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{f(\mathfrak{B})^s}$-cover of $f(X)$ and hence $f(X)$ satisfies $\mathsf{S}_1(\mathcal{O}_{f(\mathfrak{B})^s}, \Gamma_{f(\mathfrak{B})^s})$. □

Proposition 3.2.2. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X. If X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, then every continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. Let ρ be the Baire metric on $\mathbb{N}^{\mathbb{N}}$ and $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ be continuous. By Proposition 3.2.1, $\varphi(X)$ satisfies $\mathsf{S}_1(\Gamma_{\varphi(\mathfrak{B})^s}, \Gamma_{\varphi(\mathfrak{B})^s})$. For $n, k \in \mathbb{N}$, let $U^n_k = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) \leq k\}$. Define

$$\begin{array}{ccc}
\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s},\mathcal{O}_{\mathfrak{B}^s}) & \longrightarrow & \mathsf{Split}(\Gamma_{\mathfrak{B}^s},\mathcal{O}_{\mathfrak{B}^s}) \\
\uparrow & & \| \\
\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s}) & \longrightarrow & \mathsf{Split}(\Gamma_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})
\end{array}$$

FIGURE 3.1

$\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$ for each n. Let $B \in \varphi(\mathfrak{B}_0)$. Since B is compact, choose a k_0 and a sequence $\{\delta_k : k \geq k_0\}$ of positive reals such that $B^{\delta_k} \subseteq U_k^n$ for all $k \geq k_0$. Therefore $\mathcal{U}_n$ is a $\gamma_{\varphi(\mathfrak{B})^s}$-cover of $\varphi(X)$. Now apply $\mathsf{S}_1(\Gamma_{\varphi(\mathfrak{B})^s},\Gamma_{\varphi(\mathfrak{B})^s})$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to choose a $U_{k_n}^n \in \mathcal{U}_n$ for each n for which $\{U_{k_n}^n : n \in \mathbb{N}\}$ is a $\gamma_{\varphi(\mathfrak{B})^s}$-cover of $\varphi(X)$. Define a function $h : \mathbb{N} \to \mathbb{N}$ by $h(n) = k_n$ for $n \in \mathbb{N}$. Let $f \in \varphi(X)$. Clearly there is a $n_0 \in \mathbb{N}$ such that $f \in U_{k_n}^n$ for all $n \geq n_0$. Consequently $f(n) \leq h(n)$ for all $n \geq n_0$ and so $f \leq^* h$. Hence $\varphi(X)$ is bounded in $\mathbb{N}^{\mathbb{N}}$. □

Proposition 3.2.3. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X. If X satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\mathfrak{B}^s},\mathcal{O}_{\mathfrak{B}^s})$, then for every continuous function $\varphi : X \to \mathbb{N}^{\mathbb{N}}$, $\varphi(X)$ is not dominating.*

Proof. By Proposition 3.2.1, $\varphi(X)$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\varphi(\mathfrak{B})^s},\mathcal{O}_{\varphi(\mathfrak{B})^s})$. Consider $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$, where $U_k^n = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) \leq k\}$ for $n, k \in \mathbb{N}$ which is a $\gamma_{\varphi(\mathfrak{B})^s}$-cover of $\varphi(X)$. Apply $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\varphi(\mathfrak{B})^s},\mathcal{O}_{\varphi(\mathfrak{B})^s})$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to choose a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each n for which $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is an open $\varphi(\mathfrak{B})^s$-cover of $\varphi(X)$. Define a function $h : \mathbb{N} \to \mathbb{N}$ by $h(n) = \max\{k \in \mathbb{N} : U_k^n \in \mathcal{V}_n\}$ for $n \in \mathbb{N}$. Now we show that for any $f \in \varphi(X)$, $f(n) \leq h(n)$ for infinitely many $n \in \mathbb{N}$. Let $f \in \varphi(X)$. Choose a $B_0 \in \varphi(\mathfrak{B})$ such that $f \in B_0$. Since $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \mathcal{O}_{\varphi(\mathfrak{B})^s}$, in view of [22, Proposition 3.1], there are $\delta_n > 0$ such that $B_0^{\delta_n} \subseteq U_k^n$ for some $U_k^n \in \mathcal{V}_n$ for infinitely many $n \in \mathbb{N}$, i.e., $f \in U_k^n$ for some $U_k^n \in \mathcal{V}_n$ for infinitely many $n \in \mathbb{N}$ and so $f(n) \leq h(n)$ for infinitely many $n \in \mathbb{N}$. Hence $\varphi(X)$ is not dominating. □

Recall that X is called a $\gamma_{\mathfrak{B}^s}$-set [22] if X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$. An equivalent condition is that every open $\mathfrak{B}^s$-cover of X contains a set that is a $\gamma_{\mathfrak{B}^s}$-cover of X (see [19]).

We recall the definition of splittability.

> $\mathcal{A}$ is called $\mathcal{B}$-splittable if for every element A of $\mathcal{A}$ there are two disjoint elements of $\mathcal{B}$, each a subset of $\mathcal{A}$ [95].

The symbol $\mathsf{Split}(\mathcal{A},\mathcal{B})$ denotes this phenomenon [95]. The splittability property can also be studied in bornological structures. The implications among the relations $\mathsf{Split}(\mathcal{A},\mathcal{B})$, where $\mathcal{A},\mathcal{B} \in \{\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s}\}$ are represented in Figure 3.1. Some of these properties are trivial, as for example, every space satisfies $\mathsf{Split}(\Gamma_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$. This fact can be used to obtain the following.

Proposition 3.2.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*

(1) *X satisfies* $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.
(2) *X is a* $\gamma_{\mathfrak{B}^s}$*-set.*

Proof. (1) $\Rightarrow$ (2). By (1), every open $\mathfrak{B}^s$-cover of X contains a $\gamma_{\mathfrak{B}^s}$-subcover, hence X is a $\gamma_{\mathfrak{B}^s}$-set.

(2) $\Rightarrow$ (1). Using the fact that every space satisfies $\mathsf{Split}(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ and by (2) every open $\mathfrak{B}^s$-cover of X contains a set that is a $\gamma_{\mathfrak{B}^s}$-cover of X, we can conclude that X satisfies $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. □

The Baire space $\mathbb{N}^{\mathbb{N}}$ with the bornology $\mathfrak{B} = \mathcal{F}$ does not satisfy $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ (see [22, Example 3.2]). Also by [19, Theorem 2.8], there is an open $\mathfrak{B}^s$-cover which does not contain any $\gamma_{\mathfrak{B}^s}$-subcover of this space. Thus the Baire space does not satisfy $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Proposition 3.2.5. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X and (Y, ρ) be a another metric space. Let $f : X \to Y$ be a continuous function. If X satisfies $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, then $f(X)$ satisfies $\mathsf{Split}(\mathcal{O}_{f(\mathfrak{B})^s}, \mathcal{O}_{f(\mathfrak{B})^s})$.*

Proof. Let $\mathcal{U}$ be an open $f(\mathfrak{B})^s$-cover of $f(X)$. By Lemma 3.2.1, $\mathcal{U}' = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open $\mathfrak{B}^s$-cover of X. By our assumption there are disjoint subsets $\mathcal{U}_1$, $\mathcal{U}_2$ of $\mathcal{U}$ which are open $\mathfrak{B}^s$-covers of X. Choose $\mathcal{V}_i = \{U \in \mathcal{U} : f^{-1}(U) \in \mathcal{U}_i\}$ for each $i = 1, 2$. Clearly $\mathcal{V}_1$, $\mathcal{V}_2$ are disjoint subsets of $\mathcal{U}$ which are open $f(\mathfrak{B})^s$-covers of $f(X)$. Hence $f(X)$ satisfies $\mathsf{Split}(\mathcal{O}_{f(\mathfrak{B})^s}, \mathcal{O}_{f(\mathfrak{B})^s})$.

□

3.3 RESULTS RELATED TO CARDINALITY AND RAMSEY THEORY

3.3.1 RESULTS CONCERNING CARDINALITY

Theorem 3.3.1. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ on a metric space X and let X be $\mathfrak{B}^s$-Lindelöf. If $|\mathfrak{B}_0| < \mathfrak{p}$, then X is a $\gamma_{\mathfrak{B}^s}$-set.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open $\mathfrak{B}^s$-cover of X. By Proposition 2.2.1, for $B \in \mathfrak{B}$ there are $\delta_n > 0$ and $U_n \in \mathcal{U}$ such that $B^{\delta_n} \subseteq U_n$ for infinitely many n. Define $A_B = \{n \in \mathbb{N} : B^{\delta_n} \subseteq U_n\}$ for $B \in \mathfrak{B}$. Clearly each A_B is an infinite subset of $\mathbb{N}$ and now consider the family $\mathcal{A} = \{A_B : B \in \mathfrak{B}_0\}$. Let $\{A_{B_1}, \dots, A_{B_k}\}$ be any finite subfamily of $\mathcal{A}$. Since $B_1 \cup \dots \cup B_k \in \mathfrak{B}$, there are $\delta_n > 0$ and $U_n \in \mathcal{U}$ such that $(B_1 \cup \dots \cup B_k)^{\delta_n} \subseteq U_n$ for infinitely many n. This means that $A_{B_1} \cap \ldots \cap A_{B_k}$ must be infinite and consequently we can conclude that any finite subfamily of $\mathcal{A}$ has infinite intersection. Now in view of our assumption that $|\mathfrak{B}_0| < \mathfrak{p}$, we can choose an infinite subset P of $\mathbb{N}$ such that for each $B \in \mathfrak{B}_0$, $P \setminus A_B$ is finite. Enumerating P as $\{n_k : k \in \mathbb{N}\}$, set $\mathcal{V} = \{U_{n_k} \in \mathcal{U} : k \in \mathbb{N}\}$. We show that $\mathcal{V}$ is a $\gamma_{\mathfrak{B}^s}$-cover. Let $B \in \mathfrak{B}_0$.

Using the fact that $P \setminus A_B$ is finite and the definition of A_B, there is a $k_0 \in \mathbb{N}$ and a sequence $\{\delta_{n_k} : k \geq k_0\}$ of positive reals such that $B^{\delta_{n_k}} \subseteq U_{n_k}$ for all $k \geq k_0$. This shows that $\mathcal{V} \in \Gamma_{\mathfrak{B}^s}$ and hence X is a $\gamma_{\mathfrak{B}^s}$-set. □

Theorem 3.3.2. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ on a metric space X and let X be $\mathfrak{B}^s$-Lindelöf. If $|\mathfrak{B}_0| < \mathsf{cov}(\mathcal{M})$, then X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \mathcal{O}_{\mathfrak{B}^s}$. Enumerate each $\mathcal{U}_n$ bijectively as $\{U^n_m : m \in \mathbb{N}\}$. For $B \in \mathfrak{B}_0$, choose a $\delta > 0$ and a $U^n_m \in \mathcal{U}_n$ such that $B^\delta \subseteq U^n_m$. Define a function $f_B : \mathbb{N} \to \mathbb{N}$ by $f_B(n) = \min\{m \in \mathbb{N} : B^\delta \subseteq U^n_m$ for some $\delta > 0\}$ for each $n \in \mathbb{N}$. Consider the set $\{f_B : B \in \mathfrak{B}_0\}$. Since $|\mathfrak{B}_0| < \mathsf{cov}(\mathcal{M})$, there exists a $f : \mathbb{N} \to \mathbb{N}$ for which for each $B \in \mathfrak{B}_0$, $\{n \in \mathbb{N} : f_B(n) = f(n)\}$ is infinite [39, Theorem 5] (for our consideration the fact that this particular set is non-empty is the most important fact). We show that $\{U^n_{f(n)} : n \in \mathbb{N}\}$ is the required open $\mathfrak{B}^s$-cover. Let $B \in \mathfrak{B}$. There is a $B_0 \in \mathfrak{B}_0$ with $B \subseteq B_0$. By definition of f_{B_0}, $B_0^\delta \subseteq U^n_{f_{B_0}(n)}$ for some $\delta > 0$. Since the set $\{n : f_{B_0}(n) = f(n)\} \neq \emptyset$, choosing an appropriate $k \in \mathbb{N}$ with $f_{B_0}(k) = f(k)$ we observe that $B^\delta \subseteq U^k_{f(k)}$ for some $\delta > 0$. So, as claimed, $\{U^n_{f(n)} : n \in \mathbb{N}\} \in \mathcal{O}_{\mathfrak{B}^s}$ showing that X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. □

Theorem 3.3.3. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ on a metric space X and let X be $\mathfrak{B}^s$-Lindelöf. If $|\mathfrak{B}_0| < \mathfrak{d}$, then X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \mathcal{O}_{\mathfrak{B}^s}$. Enumerate each $\mathcal{U}_n$ bijectively as $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$ for each n. For $B \in \mathfrak{B}_0$, there are $\delta > 0$ and $U^n_m \in \mathcal{U}_n$ such that $B^\delta \subseteq U^n_m$. Define a function $f_B : \mathbb{N} \to \mathbb{N}$ by $f_B(n) = \min\{m \in \mathbb{N} : B^\delta \subseteq U^n_m$ for some $\delta > 0\}$. Consider the set $\{f_B : B \in \mathfrak{B}_0\}$. Since $|\mathfrak{B}_0| < \mathfrak{d}$, the family $\{f_B : B \in \mathfrak{B}_0\}$ is not dominating. Therefore there is a function $g : \mathbb{N} \to \mathbb{N}$ such that for any $B \in \mathfrak{B}_0$, $f_B(n) < g(n)$ for infinitely many n. Define $\mathcal{V}_n = \{U^n_m : m \leq g(n)\}$ for each n. Clearly $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each n and $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \mathcal{O}_{\mathfrak{B}^s}$. Hence X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. □

Theorem 3.3.4. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ on a metric space X. If $|\mathfrak{B}_0| < \mathfrak{b}$, then X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \Gamma_{\mathfrak{B}^s}$, where $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$ for each n. For each $B \in \mathfrak{B}_0$, there exist a $m_0 \in \mathbb{N}$ and a sequence $\{\delta_m : m \geq m_0\}$ of positive reals such that $B^{\delta_m} \subseteq U^n_m$ for $m \geq m_0$. Define a function $f_B : \mathbb{N} \to \mathbb{N}$ by $f_B(n) = \min\{k \in \mathbb{N} : \text{for all } m \geq k, B^{\delta_m} \subseteq U^n_m\}$. Consider the set $\{f_B : B \in \mathfrak{B}_0\}$. Since $|\mathfrak{B}_0| < \mathfrak{b}$, there is a $g : \mathbb{N} \to \mathbb{N}$ such that $f_B \leq^* g$ for all $B \in \mathfrak{B}_0$. Now it can be easily verified that $\{U^n_{g(n)} : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Hence X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. □

Theorem 3.3.5. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ on a metric space X and let X be $\mathfrak{B}^s$-Lindelöf. If $|\mathfrak{B}_0| < \mathfrak{r}$, then X satisfies $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open $\mathfrak{B}^s$-cover of X. For $B \in \mathfrak{B}_0$ there are $\delta_n > 0$ and $U_n \in \mathcal{U}$ such that $B^{\delta_n} \subseteq U_n$ for infinitely many $n \in \mathbb{N}$. Let $A_B = \{n \in \mathbb{N} : B^{\delta_n} \subseteq U_n\}$. Clearly each A_B is an infinite subset of $\mathbb{N}$ for $B \in \mathfrak{B}_0$. Consider the family $\{A_B : B \in \mathfrak{B}_0\}$. Since $|\mathfrak{B}_0| < \mathfrak{r}$, choose an infinite subset A of $\mathbb{N}$ such that for every $B \in \mathfrak{B}_0$, $A_B \subseteq^* A$ and $A_B \subseteq^* \mathbb{N} \setminus A$ hold. Clearly $A_B \setminus A$ and $A_B \cap A$ are infinite subsets. Observe that $\{U_n : n \in A\}$ and $\{U_n : n \notin A\}$ are disjoint subsets of $\mathcal{U}$ as well as open $\mathfrak{B}^s$-covers of X. Hence X satisfies $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. □

3.3.2 RAMSEY THEORETIC RESULTS

We now formulate S_1 and S_{fin}-type selection principles using certain partition relations involving bornological covers. The following observation about open $\mathfrak{B}^s$-covers will be useful in this context.

Lemma 3.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on metric space X. If an open $\mathfrak{B}^s$-cover $\mathcal{U}$ can be expressed as a union of finite number of subfamilies of $\mathcal{U}$, then at least one of them must be an open $\mathfrak{B}^s$-cover of X.*

A family $\mathcal{G}$ of subsets of $\mathbb{N}$ is said to be an ultrafilter [114] on $\mathbb{N}$ if it is closed under taking supersets, does not contain the empty set as an element, is closed under finite intersection and for each $G \subseteq \mathbb{N}$ either $G \in \mathcal{G}$ or $\mathbb{N} \setminus G \in \mathcal{G}$. Also $\mathcal{G}$ is said to be a nonprincipal ultrafilter if it is not of the form $\{G \subseteq \mathbb{N} : n \in G\}$ for any $n \in \mathbb{N}$.

Proposition 3.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on metric space X. Suppose that an open $\mathfrak{B}^s$-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X can not be split into two open $\mathfrak{B}^s$-covers of X, then $\mathcal{G} = \{G \subseteq \mathbb{N} : \{U_n\}_{n \in G}$ is an open $\mathfrak{B}^s$-cover of $X\}$ is a nonprincipal ultrafilter on $\mathbb{N}$.*

Proof. Clearly $\mathcal{G}$ is closed under supersets. By Lemma 3.3.1, for any $G \subseteq \mathbb{N}$ either $G \in \mathcal{G}$ or $\mathbb{N} \setminus G \in \mathcal{G}$. We now show that $\mathcal{G}$ has the finite intersection property. Suppose that $G_1, \ldots, G_n \in \mathcal{G}$ and $G_1 \cap \ldots \cap G_n \notin \mathcal{G}$. Clearly $(\mathbb{N} \setminus G_1) \cup \cdots \cup (\mathbb{N} \setminus G_n) = \mathbb{N} \setminus (G_1 \cap \cdots \cap G_n) \in \mathcal{G}$. Again by Lemma 3.3.1, at least one of $(\mathbb{N} \setminus G_i)$'s say $(\mathbb{N} \setminus G_{i_0})$ is in $\mathcal{G}$, this contradicts that $G_{i_0} \in \mathcal{G}$. Therefore $\mathcal{G}$ has the finite intersection property. Since an open $\mathfrak{B}^s$-cover can not consist of a single open set, the conclusion now follows. □

We closely follow the approach of [95, Theorem 10] and also [54, Theorem 6.2] to obtain the following result.

Theorem 3.3.6. *Let $\mathfrak{B}$ be a bornology with closed base on metric space X. The following statements are equivalent.*
(1) *X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*
(2) *X satisfies $\mathcal{O}_{\mathfrak{B}^s} \to \lceil \mathcal{O}_{\mathfrak{B}^s} \rceil^2_2$, provided X is $\mathfrak{B}^s$-Lindelöf.*

Proof. (1) $\Rightarrow$ (2). Consider an open $\mathfrak{B}^s$-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X and a coloring $f : [\mathcal{U}]^2 \to \{1, 2\}$. Clearly $\mathcal{U}$ can be expressed as $\mathcal{T}_1 \cup \mathcal{T}_2$, where $\mathcal{T}_1 = \{V \in \mathcal{U} : f(\{U_1, V\}) = 1\}$ and $\mathcal{T}_2 = \{V \in \mathcal{U} : f(\{U_1, V\}) = 2\}$. By Lemma 3.3.1, either $\mathcal{T}_1 \in \mathcal{O}_{\mathfrak{B}^s}$ or $\mathcal{T}_2 \in \mathcal{O}_{\mathfrak{B}^s}$. Let $i_1 \in \{1, 2\}$ be such that $\mathcal{U}_1 = \mathcal{T}_{i_1}$ is an open $\mathfrak{B}^s$-cover. Inductively, choose $\{\mathcal{U}_n : n \in \mathbb{N}\}$ and $\{i_n : n \in \mathbb{N}\}$ for which $\mathcal{U}_n = \{V \in \mathcal{U}_{n-1} : f(U_n, V) = i_n\}$ is an open $\mathfrak{B}^s$-cover for $n > 1$. Now apply $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to obtain a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$, where $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each n, such that $\cup_{n\in\mathbb{N}}\mathcal{V}_n \in \mathcal{O}_{\mathfrak{B}^s}$. Further we can assume that $\mathcal{V}_n$'s are pairwise disjoint and there is an $i \in \{1, 2\}$ such that for each $U_m \in \cup_{n\in\mathbb{N}}\mathcal{V}_n$, $i_m = i$. Choose $k_1 \in \mathbb{N}$ large enough so that $i \le k_1$ whenever $U_i \in \mathcal{V}_1$. Further choose $k_2 > k_1$ so that $i \le k_2$ whenever $U_i \in \mathcal{V}_2$ and so on. Again choose a sequence $l_1 < l_2 < \cdots$ such that $\cup_{j>l_1}\mathcal{V}_j \subseteq \mathcal{U}_{k_1}$ and $\cup_{j>l_m}\mathcal{V}_j \subseteq \mathcal{U}_{k_{l_m}}$ for $m > 1$. Take $\mathcal{Z}_n = \cup_{l_n \le j < l_{n+1}}\mathcal{V}_j$ for each $n \in \mathbb{N}$. Clearly $\cup_{n\in\mathbb{N}}\mathcal{Z}_n$ is an open $\mathfrak{B}^s$-cover and $\{\mathcal{Z}_n : n \in \mathbb{N}\}$ is a partition of it into pairwise disjoint finite sets. Again by Lemma 3.3.1, at least one of $\cup_{n\in\mathbb{N}}\mathcal{Z}_{2n}$ and $\cup_{n\in\mathbb{N}}\mathcal{Z}_{2n-1}$ must be an open $\mathfrak{B}^s$-cover and without any loss of generality assume that $\cup_{n\in\mathbb{N}}\mathcal{Z}_{2n}$ is the one. Write $\mathcal{V} = \cup_{n\in\mathbb{N}}\mathcal{Z}_{2n}$. Define $\varphi : \mathcal{V} \to \mathbb{N}$ by $\varphi(V) = n$ if $V \in \mathcal{Z}_{2n}$. Then for all $V, W \in \mathcal{V}$ with $\varphi(V) \neq \varphi(W)$, $f(\{V, W\}) = i$.

(2) $\Rightarrow$ (1). Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subset \mathcal{O}_{\mathfrak{B}^s}$ and let $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. Consider the collection $\mathcal{V} = \{U^1_m \cap U^m_k : m, k \in \mathbb{N}\}$. To see that $\mathcal{V}$ is an open $\mathfrak{B}^s$-cover, let $B \in \mathfrak{B}$. There are $\delta_1, \delta_2 > 0$ and $U^1_m \in \mathcal{U}_1$, $U^m_k \in \mathcal{U}_m$ such that $B^{\delta_1} \subseteq U^1_m$ and $B^{\delta_2} \subseteq U^m_k$. Choosing $\delta = \min\{\delta_1, \delta_2\}$, we get that $B^\delta \subseteq U^1_m \cap U^m_k$. Define $f : [\mathcal{V}]^2 \to \{1, 2\}$ by

$$f(\{U^1_{m_1} \cap U^{m_1}_{k_1}, U^1_{m_2} \cap U^{m_2}_{k_2}\}) = \begin{cases} 1 & \text{if } m_1 = m_2 \\ 2 & \text{if } m_1 \neq m_2. \end{cases}$$

By (2), there are a $\mathfrak{B}^s$-subcover $\mathcal{W} = \{V_j : j \in \mathbb{N}\}$ of $\mathcal{V}$, where $V_j = U^1_{m_j} \cap U^{m_j}_{k_j}$, a finite-to-one function $\varphi : \mathcal{W} \to \mathbb{N}$ and an $i \in \{1, 2\}$ such that for all $V_j, V_p \in \mathcal{W}$ whenever $\varphi(V_j) \neq \varphi(V_p)$, $f(\{V_j, V_p\}) = i$.

If $i = 1$. Then for all $j, p \in \mathbb{N}$, $m_j = m_p$, i.e., $m_j = m_1$ for all $j \in \mathbb{N}$ which implies that every element of $\mathcal{W}$ refines $U^1_{m_1}$. But this contradicts that $\mathcal{W} \in \mathcal{O}_{\mathfrak{B}^s}$. Therefore we must have $i = 2$.

Let $\mathcal{Z} = \{U^{m_j}_{k_j} : j \in \mathbb{N}\}$. Clearly $\mathcal{Z}$ is an open $\mathfrak{B}^s$-cover as $\mathcal{W}$ refines $\mathcal{Z}$. Choose $\mathcal{Z}_n = \{U^{m_j}_{k_j} : m_j = n\}$. $\mathcal{Z}_n$'s are either empty or finite. Otherwise there will be infinitely many $j \in \mathbb{N}$ with $m_j = n$. Since φ is finite-to-one, choose j, p with $m_j = m_p = n$ and $\varphi(V_j) \neq \varphi(V_p)$, then $f(\{V_j, V_p\}) = 2$, i.e., $m_j \neq m_p$, which is a contradiction. Hence $\{\mathcal{Z}_n : n \in \mathbb{N}\}$ is a sequence of finite sets with $\mathcal{Z}_n \subseteq \mathcal{U}_n$ for each $n \in \mathbb{N}$ witnessing $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. □

Using the method of proof of Theorem 3.3.6, the induction argument on n and k and the usual method for proving the Ramsey theoretic statements for $n > 2$, $k > 2$ (see, for example, [62, Theorem 1] and [95, Theorem 25]), we obtain the following result.

Theorem 3.3.7. *Let $\mathfrak{B}$ be a bornology with closed base on metric space X. The following statements are equivalent.*
(1) *X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s},\mathcal{O}_{\mathfrak{B}^s})$.
(2) *X satisfies* $\mathcal{O}_{\mathfrak{B}^s} \to (\mathcal{O}_{\mathfrak{B}^s})^2_2$, *provided X is $\mathfrak{B}^s$-Lindelöf.*
(3) *For each $n,k \in \mathbb{N}$ X satisfies* $\mathcal{O}_{\mathfrak{B}^s} \to (\mathcal{O}_{\mathfrak{B}^s})^n_k$, *provided X is $\mathfrak{B}^s$-Lindelöf.*

Theorem 3.3.8. *Let $\mathfrak{B}$ be a bornology with closed base on metric space X. The following statements are equivalent.*
(1) *X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$.
(2) *ONE has no a winning strategy in* $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s},\Gamma_{\mathfrak{B}^s})$ *on X.*
(3) *For each $n,k \in \mathbb{N}$ X satisfies* $\mathcal{O}_{\mathfrak{B}^s} \to (\Gamma_{\mathfrak{B}^s})^n_k$, *provided X is $\mathfrak{B}^s$-Lindelöf.*
(4) *Every open $\mathfrak{B}^s$-cover contains a set that is a $\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is already proved in [22, Theorem 3.7] and the implication (4) $\Rightarrow$ (1) is due to [19, Theorem 2.8]. The implication (2) $\Rightarrow$ (3) follows from [95, Theorem 30] with minor modifications.

(3) $\Rightarrow$ (4). Let $\mathcal{U} \in \mathcal{O}_{\mathfrak{B}^s}$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a partition of $\mathcal{U}$ into nonempty finite sets. Define $f : [\mathcal{U}]^2 \to \{1,2\}$ by $f(\{U,V\}) = 1$ if $U,V \in \mathcal{U}_n$ for some $n \in \mathbb{N}$ and $f(\{U,V\}) = 2$ otherwise. Now using (3), we obtain a subset $\mathcal{V}$ of $\mathcal{U}$ which is a $\gamma_{\mathfrak{B}^s}$-cover of X. Hence (4) holds.

□

3.4 OBSERVATIONS ON THE SELECTION PRINCIPLE $\bigcap_\infty(\mathcal{A},\mathcal{B})$

For two nonempty classes of sets $\mathcal{A}$ and $\mathcal{B}$ of an infinite set S, the selection principle $\bigcap_\infty(\mathcal{A},\mathcal{B})$ was introduced and studied in [112]. In this section we make some observations on this selection principle involving bornological covers. Recall that

> $\bigcap_\infty(\mathcal{A},\mathcal{B})$: *For every sequence $\{A_n : n \in \mathbb{N}\}$ of elements from $\mathcal{A}$ there exists a B_n, an infinite subset of A_n for each $n \in \mathbb{N}$, for which the set $\{\cap B_n : n \in \mathbb{N}\}$ belongs $\mathcal{B}$.*

By the symbols $\boldsymbol{\mathcal{O}}_{\mathfrak{B}^s}$ and $\boldsymbol{\Gamma}_{\mathfrak{B}^s}$ we denote collection all $\mathfrak{B}^s$-covers [18] and all $\mathfrak{B}^s$-sequences [18] of X respectively.

Below we present two interpretative examples, one of a space satisfying $\bigcap_\infty(\mathcal{O}_{\mathfrak{B}^s},\boldsymbol{\mathcal{O}}_{\mathfrak{B}^s})$ and another which does not satisfy $\bigcap_\infty(\Gamma_{\mathfrak{B}^s},\boldsymbol{\Gamma}_{\mathfrak{B}^s})$.

Example 3.4.1. *Let X be the real line with Euclidean metric and $\mathfrak{B}$ be the bornology generated by $\{(-x,x) : x > 0\}$. We show that X satisfies $\bigcap_\infty(\mathcal{O}_{\mathfrak{B}^s},\boldsymbol{\mathcal{O}}_{\mathfrak{B}^s})$.*

Suppose that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of open $\mathfrak{B}^s$-covers of X. Consider a partition $\{\mathcal{P}_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into infinite subsets. Let $\mathcal{P}_n = \{n_1 < n_2 < \cdots < n_k < \cdots\}$ for $n \in \mathbb{N}$. Since

$\mathcal{U}_n \in \mathcal{O}_{\mathfrak{B}^s}$, for the interval $(-n_k, n_k)$ choose a $U_{n_k} \in \mathcal{U}_n$ such that $(-n_k, n_k) \subseteq U_{n_k}$. Set $\mathcal{V}_n = \{U_{n_k} \in \mathcal{U}_n : (-n_k, n_k) \subseteq U_{n_k}, n_k \in \mathcal{P}_n\}$. Clearly each $\mathcal{V}_n$ is an infinite subset of $\mathcal{U}_n$. We prove that $\{\cap\mathcal{V}_n : n \in \mathbb{N}\}$ is a $\mathfrak{B}^s$-cover of X. To see this, let $B \in \mathfrak{B}$ and $B = (-x, x)$ (say) for some positive real x. Choose $n_0 \in \mathbb{N}$ such that $x \leq n_0$. There is a $\mathcal{P}_n$ and a $\delta > 0$ such that $B^\delta \subseteq (-n_k, n_k)$ for all $n_k \in \mathcal{P}_n$. Consequently $B^\delta \subseteq U_{n_k}$ for all $n_k \in \mathcal{P}_n$ and so $B^\delta \subseteq \cap\mathcal{V}_n$. Hence $\{\cap\mathcal{V}_n : n \in \mathbb{N}\} \in \boldsymbol{\mathcal{O}}_{\mathfrak{B}^s}$ and X satisfies $\bigcap_\infty(\mathcal{O}_{\mathfrak{B}^s}, \boldsymbol{\mathcal{O}}_{\mathfrak{B}^s})$.

Example 3.4.2. *Consider the space $X = \mathbb{N}^\mathbb{N}$ with the Baire metric ρ and the bornology $\mathfrak{B} = \mathcal{F}$. We will show that X does not satisfy $\bigcap_\infty(\Gamma_{\mathfrak{B}^s}, \boldsymbol{\Gamma}_{\mathfrak{B}^s})$. Let $U^n_k = \{f \in X : f(n) \geq k\}$ and $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ for $n, k \in \mathbb{N}$. Clearly $\mathcal{U}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X for each n. Consider $\{\mathcal{U}_n : n \in \mathbb{N}\}$. If possible, suppose that for each $n \in \mathbb{N}$ there is an infinite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ for which $\{\cap\mathcal{V}_n : n \in \mathbb{N}\} \in \boldsymbol{\Gamma}_{\mathfrak{B}^s}$. Define a function $h : \mathbb{N} \to \mathbb{N}$ by $h(n) = \min\{k \in \mathbb{N} : U^n_k \in \mathcal{V}_n\}$. Now choose a $g \in \mathbb{N}^\mathbb{N}$ with $g(n) > h(n)$ for each n. Let $B = \{g\}$. By our assumption there is a n_0 and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers such that $B^{\delta_n} \subseteq \cap\mathcal{V}_n$ for all $n \geq n_0$ and so $g \in U^n_{h(n)}$ for all $n \geq n_0$. Clearly $g(n) \leq h(n)$ for all $n \geq n_0$ which is a contradiction. Hence X does not satisfy $\bigcap_\infty(\Gamma_{\mathfrak{B}^s}, \boldsymbol{\Gamma}_{\mathfrak{B}^s})$.*

The following results establish certain relations of the selection principle under consideration in this section with the α_i properties for $i = 1, 2, 3, 4$ in the bornological context and subsequently provides new perspectives for known results. The following theorem is one such instance (see [19, Theorem 3.1] and [22, Theorem 3.5]).

Theorem 3.4.1. *Let $\mathfrak{B}$ be a bornology with closed base on metric space X. The following statements are equivalent.*
(1) *X satisfies $\alpha_2(\Gamma_{\mathfrak{B}^s}, \boldsymbol{\Gamma}_{\mathfrak{B}^s})$.*
(2) *X satisfies $\alpha_3(\Gamma_{\mathfrak{B}^s}, \boldsymbol{\Gamma}_{\mathfrak{B}^s})$.*
(3) *X satisfies $\alpha_4(\Gamma_{\mathfrak{B}^s}, \boldsymbol{\Gamma}_{\mathfrak{B}^s})$.*
(4) *X satisfies $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \boldsymbol{\Gamma}_{\mathfrak{B}^s})$.*
(5) *ONE has no winning strategy in $\mathsf{G}_1(\Gamma_{\mathfrak{B}^s}, \boldsymbol{\Gamma}_{\mathfrak{B}^s})$.*
(6) *X satisfies $\bigcap_\infty(\Gamma_{\mathfrak{B}^s}, \boldsymbol{\Gamma}_{\mathfrak{B}^s})$.*

Proof. (1) ⇔ (4) follows from [19, Theorem 3.1] and (4) ⇔ (5) is already proved in [22, Theorem 3.5]. We only prove (1) ⇔ (6).

(1) ⇒ (6). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $\gamma_{\mathfrak{B}^s}$-covers of X. Since every infinite subset of a $\gamma_{\mathfrak{B}^s}$-cover of X is a $\gamma_{\mathfrak{B}^s}$-cover, we assume that $\mathcal{U}_n$'s are pairwise disjoint. By (1), there is a $\mathcal{V} \in \boldsymbol{\Gamma}_{\mathfrak{B}^s}$ such that $\mathcal{U}_n \cap \mathcal{V}$ is infinite for each n. Let $\mathcal{V}_n = \mathcal{U}_n \cap \mathcal{V}$ for each n. We intend to show that $\{\cap\mathcal{V}_n : n \in \mathbb{N}\} \in \boldsymbol{\Gamma}_{\mathfrak{B}^s}$. Assume the contrary. Then there is a $B_0 \in \mathfrak{B}$ such that for any sequence $\{\delta_n : n \in \mathbb{N}\}$ with $\delta_n > 0$ and for infinitely many n, $B_0^{\delta_n} \nsubseteq \cap\mathcal{V}_n$. Enumerate $\mathcal{V}$ as $\{V_n : n \in \mathbb{N}\}$. Choose a $n_0 \in \mathbb{N}$ and a sequence $\{\sigma_n : n \geq n_0\}$ with $\sigma_n > 0$ satisfying $B^{\sigma_n} \subseteq V_n$ for all $n \geq n_0$. But note that $B_0^{\sigma_n} \nsubseteq \cap\mathcal{V}_n$ for infinitely many $n \in \mathbb{N}$. Now for all

such n choose a $V_n \in \mathcal{V}_n \subseteq \mathcal{V}$ satisfying $B_0^{\sigma_n} \nsubseteq V_n$. This is not possible as $\mathcal{V} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$. Therefore $\{\cap\mathcal{V}_n : n \in \mathbb{N}\} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$ and hence (6) holds.

(6) $\Rightarrow$ (1). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence from $\Gamma_{\mathfrak{B}^s}$. Apply (6) to obtain a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ for which $\mathcal{V}_n$ is an infinite subset of $\mathcal{U}_n$ for each n and $\{\cap\mathcal{V}_n : n \in \mathbb{N}\} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$. Choose $\mathcal{V} = \cup_{n\in\mathbb{N}}\mathcal{V}_n$. Clearly $\mathcal{U}_n \cap \mathcal{V}$ is infinite for each n. It now remains to show that $\mathcal{V} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$.

We write $\mathcal{V}_n = \{V_k^n : k \in \mathbb{N}\}$, then $\mathcal{V} = \{V_k^n : n,k \in \mathbb{N}\}$. Let $B \in \mathfrak{B}$. Since $\{\cap\mathcal{V}_n : n \in \mathbb{N}\} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$, there are a n_0 and a sequence $\{\sigma_n : n \geq n_0\}$ with $\delta_n > 0$ such that $B^{\sigma_n} \subseteq \cap\mathcal{V}_n$ for all $n \geq n_0$ and so $B^{\sigma_n} \subseteq V_k^n$ for all $k \in \mathbb{N}$ and $n \geq n_0$. We first fix a n with $n \geq n_0$ and then for each $k \in \mathbb{N}$ choose $\delta_k^n = \sigma_n$. Clearly $B^{\delta_k^n} \subseteq V_k^n$ for all $k \in \mathbb{N}$. Now consider $\{\mathcal{V}_i : 1 \leq i < n_0\}$. Let $B \in \mathfrak{B}$. From the fact that each $\mathcal{V}_i$, $1 \leq i < n_0$ is a $\gamma_{\mathfrak{B}^s}$-cover of X, one is assured of a finite set $M_i \subseteq \mathbb{N}$ and a sequence $\{\delta_k^i : k \in \mathbb{N} \setminus M_i\}$ of positive real numbers such that $B^{\delta_k^i} \subseteq V_k^n$ for all $k \in \mathbb{N} \setminus M_i$. Choose $M = \cup_{1\leq i<n_0} M_i$. For the sequence $\{\delta_k^n : k \in \mathbb{N} \setminus M, n \in \mathbb{N}\}$, we have $B^{\delta_k^n} \subseteq V_k^n$ for all $k \in \mathbb{N} \setminus M$ and $n \in \mathbb{N}$. Therefore $\mathcal{V} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$. Hence (1) holds. □

Theorem 3.4.2. *Let $\mathfrak{B}$ be a bornology with closed base on metric space X. The following statements are equivalent.*

(1) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.*

(2) *X satisfies $\bigcap_\infty(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. By (1) and using [19, Theorem 2.8], one can find a $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each n, such that $\mathcal{V}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Also by Theorem 6.2.7, X satisfies $\bigcap_\infty(\Gamma_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$. Now applying $\bigcap_\infty(\Gamma_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ choose a $\mathcal{W}_n \subseteq \mathcal{V}_n$ for each n such that $\{\cap\mathcal{W}_n : n \in \mathbb{N}\} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$. Since each $\mathcal{W}_n$ is also a subset of $\mathcal{U}_n$, X satisfies $\bigcap_\infty(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. By (2), there is a $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each n for which $\{\cap\mathcal{V}_n : n \in \mathbb{N}\} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$. For each n choose a $U_n \in \mathcal{V}_n$. Clearly $\{U_n : n \in \mathbb{N}\} \in \mathbf{\Gamma}_{\mathfrak{B}^s}$. Hence X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$. □

Remark 3.4.1. *Moreover using [19] and [22, Theorem 3.7] we can add the following equivalent conditions to Theorem 3.4.2.*

(3) *X satisfies $\alpha_2(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.*

(4) *X satisfies $\alpha_3(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.*

(5) *X satisfies $\alpha_4(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.*

(6) *ONE has no winning strategy in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.*

We incorporate Theorem 3.4.1, Theorem 3.3.1 and Theorem 3.3.4 to obtain the following.

Corollary 3.4.1. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ on metric space X. The following statements hold.*

(1) *Let X be a $\mathfrak{B}^s$-Lindelöf. If $|\mathfrak{B}_0| < \mathfrak{p}$ then X satisfies $\bigcap_\infty(\mathcal{O}_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.*

(2) *If $|\mathfrak{B}_0| < \mathfrak{b}$ then X satisfies $\bigcap_\infty(\Gamma_{\mathfrak{B}^s}, \mathbf{\Gamma}_{\mathfrak{B}^s})$.*

3.5 THE $\mathfrak{B}^s$-HUREWICZ AND THE $\mathfrak{B}^s$-GERLITS-NAGY PROPERTIES

3.5.1 SOME OBSERVATIONS ON THE $\mathfrak{B}^s$-HUREWICZ PROPERTY

In Chapter 2, the notion of the strong $\mathfrak{B}$-Hurewicz property ($\mathfrak{B}^s$-Hurewicz property for short) was introduced and several of its basic properties were established. In this section first we again look back at this very important property and present some more new observations.

Recall the definition of the $\mathfrak{B}^s$-Hurewicz property of X.

X is said to have the strong $\mathfrak{B}$-Hurewicz property ($\mathfrak{B}^s$-Hurewicz property for short) if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X there is a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ such that for every $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers satisfying $B^{\delta_n} \subseteq U$ for some $U \in \mathcal{V}_n$ for all $n \geq n_0$.

Clearly every $\gamma_{\mathfrak{B}^s}$-set has the $\mathfrak{B}^s$-Hurewicz property and every space with the $\mathfrak{B}^s$-Hurewicz property satisfies $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

The following observation follows from Lemma 3.2.1.

Proposition 3.5.1. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X and (Y, ρ) be a another metric space. Let $f : X \to Y$ be a continuous function. If X has the $\mathfrak{B}^s$-Hurewicz property, then $f(X)$ has the $f(\mathfrak{B})^s$-Hurewicz property.*

Proof. Consider a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $f(\mathfrak{B})^s$-covers of $f(X)$. By Lemma 3.2.1, for each $n \in \mathbb{N}, \mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ is an open $\mathfrak{B}^s$-cover of X. In view of the $\mathfrak{B}^s$-Hurewicz property of X, we can find a sequence $\{\mathcal{V}'_n : n \in \mathbb{N}\}$ of finite sets with $\mathcal{V}'_n \subseteq \mathcal{U}'_n$ for each n, such that for each $B \in \mathfrak{B}$ there exist a n_0 and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers satisfying $B^{\delta_n} \subseteq f^{-1}(U)$ for some $f^{-1}(U) \in \mathcal{V}'_n$ for all $n \geq n_0$. For each n choose $\mathcal{V}_n = \{U \in \mathcal{U}_n : f^{-1}(U) \in \mathcal{V}'_n\}$. Clearly $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the $f(\mathfrak{B})^s$-Hurewicz property for $f(X)$. □

Proposition 3.5.2. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X. If X has the $\mathfrak{B}^s$-Hurewicz property, then every continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is bounded.*

Proof. Let ρ be the Baire metric on $\mathbb{N}^{\mathbb{N}}$ and $\varphi : X \to \mathbb{N}^{\mathbb{N}}$ be continuous. By Proposition 3.5.1, $\varphi(X)$ has the $\varphi(\mathfrak{B})^s$-Hurewicz property. Consider $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$, where $U^n_k = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) \leq k\}$ for $n, k \in \mathbb{N}$, which is an open $\varphi(\mathfrak{B})^s$-cover of $\varphi(X)$.

Apply the $\varphi(\mathfrak{B})^s$-Hurewicz property to $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to obtain $\{\mathcal{V}_n : n \in \mathbb{N}\}$, where each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, such that for each $B \in \varphi(\mathfrak{B}_0)$ there exist a n_0 and a sequence $\{\delta_n : n \geq n_0\}$ with $\delta_n > 0$ satisfying $B^{\delta_n} \subseteq U^n_k$ for some $U^n_k \in \mathcal{V}_n$ and for all $n \geq n_0$. Define a function $h : \mathbb{N} \to \mathbb{N}$ by $h(n) = \max\{k : U^n_k \in \mathcal{V}_n\}$. Now it can be easily shown that for any $f \in \varphi(X), f \leq^* h$ holds. Hence $\varphi(X)$ is bounded. □

The following result provides an important insight into the spaces with the $\mathfrak{B}^s$-Hurewicz property in terms of the cardinality of the base of the bornology and the significance of the result is due to the fact that several known bornolocical spaces have closed bases.

Theorem 3.5.1. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ on a metric space X. If $|\mathfrak{B}_0| < \mathfrak{b}$, then X has the $\mathfrak{B}^s$-Hurewicz property.*

Proof. Consider a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X. Let $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$ for $n \in \mathbb{N}$. For $B \in \mathfrak{B}_0$ there are $\delta > 0$ and $U^n_m \in \mathcal{U}_n$ such that $B^\delta \subseteq U^n_m$. For each $B \in \mathfrak{B}_0$, define a function $f_B : \mathbb{N} \to \mathbb{N}$ by $f_B(n) = \min\{m \in \mathbb{N} : B^\delta \subseteq U^n_m \text{ for some } \delta > 0\}$. Consider the set $\{f_B : B \in \mathfrak{B}_0\}$. Since $|\mathfrak{B}_0| < \mathfrak{b}$, there is a $g : \mathbb{N} \to \mathbb{N}$ such that $f_B \leq^* g$ for all $B \in \mathfrak{B}_0$. For each $n \in \mathbb{N}$, choose $\mathcal{V}_n = \{U^n_m : m \leq g(n)\}$. We claim that $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the $\mathfrak{B}^s$-Hurewicz property.

To see this we will show that for $B \in \mathfrak{B}$ there is a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers such that $B^{\delta_n} \subseteq U$ for some $U \in \mathcal{V}_n$ for all $n \geq n_0$. For $B \in \mathfrak{B}$ choose $B_0 \in \mathfrak{B}_0$ with $B \subseteq B_0$. Since $f_{B_0} \leq^* g$, there is a $n_0 \in \mathbb{N}$ such that $f_{B_0}(n) \leq g(n)$ for all $n \geq n_0$. Now by definition of f_{B_0}, for each $n \geq n_0$ there is a $\delta_n > 0$ such that $B_0^{\delta_n} \subseteq U^n_{f_{B_0}(n)}$. Clearly $U^n_{f_{B_0}(n)} \in \mathcal{V}_n$ for all $n \geq n_0$. So we have a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers such that $B^{\delta_n} \subseteq B_0^{\delta_n} \subseteq U$ for some $U \in \mathcal{V}_n$ for all $n \geq n_0$. Hence X has the $\mathfrak{B}^s$-Hurewicz property. □

Theorem 3.5.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If X has the $\mathfrak{B}^s$-Hurewicz property, then X satisfies* $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. Let $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X. Assume that $\mathcal{U}$ is countable. Since X has the $\mathfrak{B}^s$-Hurewicz property, $\mathcal{U}$ is $\mathfrak{B}^s$-groupable by Lemma 2.3.2. Therefore there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of pairwise disjoint subsets of $\mathcal{U}$ such that $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{V}_n$ and for $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ with $\delta_n > 0$ satisfying $B^{\delta_n} \subseteq U$ for some $U \in \mathcal{V}_n$ for all $n \geq n_0$. Set $\mathcal{U}_1 = \cup_{k\in\mathbb{N}}\mathcal{V}_{2k-1}$ and $\mathcal{U}_2 = \cup_{k\in\mathbb{N}}\mathcal{V}_{2k}$. Clearly $\mathcal{U}_1, \mathcal{U}_2$ are disjoint subsets of $\mathcal{U}$ which are open $\mathfrak{B}^s$-covers of X. This completes the proof. □

The space $\mathbb{R}$ with Euclidean metric d and a bornology $\mathfrak{B}$ generated by $\{(-x, x) : x > 0\}$ satisfies $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ as it has the $\mathfrak{B}^s$-Hurewicz property by Example 2.3.1.

Theorem 3.5.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*

(1) *X has the $\mathfrak{B}^s$-Hurewicz property.*

(2) *X satisfies* $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$.

(3) *ONE does not have a winning strategy in* $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$.

(4) *For each $k \in \mathbb{N}$ X satisfies $\mathcal{O}_{\mathfrak{B}^s} \to \lceil \mathcal{O}^{gp}_{\mathfrak{B}^s} \rceil^2_k$, provided X is $\mathfrak{B}^s$-Lindelöf.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are shown in [22, Theorem 4.2]. Also the implication (3) $\Rightarrow$ (4) can be obtained by using similar technique of proof of [62, Theorem 3].

(4) $\Rightarrow$ (2) Clearly X satisfies $\mathcal{O}_{\mathfrak{B}^s} \to \lceil \mathcal{O}_{\mathfrak{B}^s} \rceil^2_2$. By Theorem 3.3.6, X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Now we show that every countable open $\mathfrak{B}^s$-cover of X is $\mathfrak{B}^s$-groupable. Let $\mathcal{U}$ be a countable open $\mathfrak{B}^s$-cover of X. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a partition of $\mathcal{U}$ into nonempty finite sets. Define a coloring $f : [\mathcal{U}]^2 \to \{1,2\}$ by $f(\{U,V\}) = 1$ if $U, V \in \mathcal{U}_n$ for some $n \in \mathbb{N}$ and $f(\{U,V\}) = 2$ otherwise. By (4), there is a $\mathcal{V} \in \mathcal{O}^{gp}_{\mathfrak{B}^s}$ with $\mathcal{V} \subseteq \mathcal{U}$. So there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $\mathcal{V}$ witnessing the $\mathfrak{B}^s$-groupability of $\mathcal{V}$. Since $\mathcal{U}$ is countable, the elements of $\mathcal{U} \setminus \mathcal{V}$ can be distributed among V_n's so that $\{V_n : n \in \mathbb{N}\}$ witnesses the $\mathfrak{B}^s$-groupability of $\mathcal{U}$. Hence X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$.

□

Our next result concerns with the $\mathfrak{B}^s$-Hurewicz property for product spaces. For a metric space (X,d) one can consider the product space X^n endowed with the product metric d^n defined as

$$d^n((x_1,\dots,x_n),(y_1,\dots,y_n)) = \max\{d(x_1,y_1),\dots,d(x_n,y_n)\}.$$

Let $\mathfrak{B}$ be a bornology on X with closed base $\mathfrak{B}_0$. In [48], it has been shown that the collection $\{B^n : B \in \mathfrak{B}\}$ generates a bornology on X^n . We denote that bornology on X^n by $\mathfrak{B}^n$. Further it can be easily verified that for any $n \in \mathbb{N}$ and $\delta > 0$, $(B^\delta)^n = (B^n)^\delta$ which we will need repeatedly. We start with the following simple observation followed by a result on selection principles (Theorem 3.5.4) which though not related to the main topic of this section is interesting in its own right.

Lemma 3.5.1. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X. If $\mathcal{U}$ is an open $(\mathfrak{B}^n)^s$-cover of X^n, then there exists an open $\mathfrak{B}^s$-cover $\mathcal{V}$ of X such that $\{V^n : V \in \mathcal{V}\}$ is an open $(\mathfrak{B}^n)^s$-cover of X^n which refines $\mathcal{U}$.*

Proof. Consider an open $(\mathfrak{B}^n)^s$-cover $\mathcal{U}$ of X^n. Let $B \in \mathfrak{B}_0$. Then for $B^n \in \mathfrak{B}^n$ there exist a $\delta > 0$ and $U \in \mathcal{U}$ such $(B^n)^\delta \subseteq U$. Since B is a compact subset of X and U is open in X^n containing B^n, we use Wallace Theorem [36] to find an open set V_B in X such that $B^n \subseteq V_B^n \subseteq U$. Consider the collection $\mathcal{V} = \{V_B : B \in \mathfrak{B}_0\}$. We intend to show that $\mathcal{V}$ is an open $\mathfrak{B}^s$-cover of X. For $B \in \mathfrak{B}_0$, we have $B \subseteq V_B$. Since V_B is open and B is compact with $B \subseteq V_B$, there is a $\delta > 0$ such that $B^\delta \subseteq V_B$. Therefore $\mathcal{V} = \{V_B : B \in \mathfrak{B}_0\}$ is an open $\mathfrak{B}^s$-cover of X.

Again for $B \in \mathfrak{B}_0$, $(B^\delta)^n \subseteq V_B^n$. Clearly $(B^n)^\delta \subseteq V_B^n$ and $V_B^n \subseteq U$ for some $U \in \mathcal{U}$ implying that $\{V^n : V \in \mathcal{V}\}$ is an open $(\mathfrak{B}^n)^s$-cover of X^n as well as refines $\mathcal{U}$. □

Theorem 3.5.4. *Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$ and $\mathcal{P} = \mathcal{O}$, $\mathcal{Q} \in \{\mathcal{O}, \Gamma\}$. Let $\mathfrak{B}$ be a bornology with compact base on a metric space X. The following statements are equivalent.*

(1) *X satisfies* $\Pi(\mathcal{P}_{\mathfrak{B}^s}, \mathcal{Q}_{\mathfrak{B}^s})$.
(2) X^n *satisfies* $\Pi(\mathcal{P}_{(\mathfrak{B}^n)^s}, \mathcal{Q}_{(\mathfrak{B}^n)^s})$ *for each* $n \in \mathbb{N}$.

Proof. We only present the proof for the case $\Pi = \mathsf{S}_1$, $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \Gamma$ as other cases follow analogously.

(1) $\Rightarrow$ (2). Let $\{\mathcal{U}_m : m \in \mathbb{N}\}$ be a sequence of open $(\mathfrak{B}^n)^s$-covers of X^n. By Lemma 3.5.1, for each $\mathcal{U}_m$ there exists an open $\mathfrak{B}^s$-cover $\mathcal{V}_m$ of X for which $\{V^n : V \in \mathcal{V}_m\}$ refines $\mathcal{U}_m$. Apply $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ to the sequence $\{\mathcal{V}_m : m \in \mathbb{N}\}$ to choose a $V_m \in \mathcal{V}_m$ for each $m \in \mathbb{N}$ so that $\{V_m : m \in \mathbb{N}\}$ becomes a $\gamma_{\mathfrak{B}^s}$-cover of X. Now for each V_m we can choose a $U_m \in \mathcal{U}_m$ such that $V_m^n \subseteq U_m$. We will show that $\{U_m : m \in \mathbb{N}\}$ is a $\gamma_{(\mathfrak{B}^n)^s}$-cover of X^n. Let $B^n \in \mathfrak{B}^n$. For $B \in \mathfrak{B}$ there exist a $m_0 \in \mathbb{N}$ and a sequence $\{\delta_m : m \geq m_0\}$ of positive real numbers such that $B^{\delta_m} \subseteq V_m$ for all $m \geq m_0$. So $(B^{\delta_m})^n \subseteq V_m^n$ for all $m \geq m_0$, i.e., $(B^n)^{\delta_m} \subseteq V_m^n \subseteq U_m$ for all $m \geq m_0$. Therefore $\{U_m : m \in \mathbb{N}\}$ is a $\gamma_{(\mathfrak{B}^n)^s}$-cover of X^n. Hence X^n satisfies $\mathsf{S}_1(\mathcal{O}_{(\mathfrak{B}^n)^s}, \Gamma_{(\mathfrak{B}^n)^s})$.

(2) $\Rightarrow$ (1). Let $\{\mathcal{U}_m : m \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. For each $m \in \mathbb{N}$, let $\mathcal{U}'_m = \{U^n : U \in \mathcal{U}_m\}$. Clearly $\mathcal{U}'_m$'s are open $(\mathfrak{B}^n)^s$-covers of X^n. Apply $\mathsf{S}_1(\mathcal{O}_{(\mathfrak{B}^n)^s}, \Gamma_{(\mathfrak{B}^n)^s})$ to $\{\mathcal{U}'_m : m \in \mathbb{N}\}$ to choose a $U_m^n \in \mathcal{U}'_m$ for each $m \in \mathbb{N}$ such that $\{U_m^n : m \in \mathbb{N}\}$ is a $\gamma_{(\mathfrak{B}^n)^s}$-cover of X^n. It is easy to check that $\{U_m : m \in \mathbb{N}\}$ with $U_m \in \mathcal{U}_m$ for $m \in \mathbb{N}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. □

Theorem 3.5.5. *Let* $\mathfrak{B}$ *be a bornology with compact base on a metric space X. The following statements are equivalent.*
(1) *X has the* $\mathfrak{B}^s$*-Hurewicz property.*
(2) X^n *has the* $(\mathfrak{B}^n)^s$*-Hurewicz property for each* $n \in \mathbb{N}$.

Proof. We only present the proof of (1) $\Rightarrow$ (2).
(1) $\Rightarrow$ (2). Let $\{\mathcal{U}_k : k \in \mathbb{N}\}$ be a sequence of open $(\mathfrak{B}^n)^s$-covers of X^n. By Lemma 3.5.1, for each k there exists an open $\mathfrak{B}^s$-cover $\mathcal{V}_k$ of X such that $\{V^n : V \in \mathcal{V}_k\}$ refines $\mathcal{U}_k$. Consider the sequence $\{\mathcal{V}_k : k \in \mathbb{N}\}$. By (1), there is a sequence $\{\mathcal{W}_k : k \in \mathbb{N}\}$ of finite sets with $\mathcal{W}_k \subseteq \mathcal{V}_k$ for each k such that for $B \in \mathfrak{B}$ there exist a $m_0 \in \mathbb{N}$ and a sequence $\{\delta_m : m \geq m_0\}$ of positive real numbers satisfying $B^{\delta_m} \subseteq V$ for some $V \in \mathcal{W}_k$ for all $m \geq m_0$. Now for each k we can choose a finite subset $\mathcal{Z}_k$ of $\mathcal{U}_k$ such that for each $V \in \mathcal{W}_k$ there is a $U \in \mathcal{Z}_k$ with $V^n \subseteq U$. We will show that $\{\mathcal{Z}_k : k \in \mathbb{N}\}$ witnesses the $(\mathfrak{B}^n)^s$-Hurwicz property of X^n. Let $B^n \in \mathfrak{B}^n$. Note that for $B \in \mathfrak{B}$ there already exist a $p_0 \in \mathbb{N}$ and a sequence $\{\delta_p : p \geq p_0\}$ of positive real numbers satisfying $B^{\delta_p} \subseteq V$ for some $V \in \mathcal{W}_k$ for all $p \geq p_0$, i.e., $(B^n)^{\delta_p} \subseteq U$ for some $U \in \mathcal{Z}_k$ for all $p \geq p_0$. Hence X^n has the $(\mathfrak{B}^n)^s$-Hurewicz property.

□

Combining Theorem 3.5.3 and Theorem 3.5.5, we obtain the following.

Theorem 3.5.6. *Let* $\mathfrak{B}$ *be a bornology with compact base on a metric space X. The following statements are equivalent.*

(1) *X^n has the $(\mathfrak{B}^n)^s$-Hurewicz property for each $n \in \mathbb{N}$.*

(2) *X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$.*

(3) *ONE does not have a winning strategy in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ on X.*

(4) *X satisfies $\mathcal{O}_{\mathfrak{B}^s} \to \lceil \mathcal{O}^{gp}_{\mathfrak{B}^s} \rceil^2_k$ for each $k \in \mathbb{N}$, provided X is $\mathfrak{B}^s$-Lindelöf.*

3.5.2 THE STRONG $\mathfrak{B}$-GERLITS-NAGY PROPERTY AND SOME OBSERVATIONS

It is well known that the Gerlits-Nagy property was introduced in the seminal paper of the authors [41] as a property stronger than the classical Hurewicz property and had been extensively investigated since then (see [100, 113] for example). In particular in [62] certain new characterizations of this property were established using the notion of groupability of open covers. In this section we introduce the notion of strong $\mathfrak{B}$-Gerlits-Nagy property (which has not been investigated at all in bornological settings) and while defining the concept, follow the line of [62] which seems more effective for our purpose.

Definition 3.5.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. X is said to have the strong $\mathfrak{B}$-Gerlits-Nagy property ($\mathfrak{B}^s$-Gerlits-Nagy property for short) if X satisfies the selection principle $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$.*

It is clear that $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ implies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ which shows that every $\gamma_{\mathfrak{B}^s}$-set has the $\mathfrak{B}^s$-Gerlits-Nagy property. Below we prove certain results in line of [62] where the proofs are done with suitable modifications as is necessary for bornological structures.

The following example shows that the real line associated with a bornology has the $\mathfrak{B}^s$-Gerlits-Nagy property.

Example 3.5.1. *Consider the real line $X = \mathbb{R}$ with the Euclidean metric d and the bornology $\mathfrak{B}$ generated by $\{(-x, x) : x > 0\}$. We show that X has the $\mathfrak{B}^s$-Gerlits-Nagy property. To see this, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Then clearly for each $k \in \mathbb{N}$, there is a $U \in \mathcal{U}_n$ such that $(-k, k) \subseteq U$ (by choosing $(-k, k) \in \mathfrak{B}$). For $B = (-k, k) \in \mathfrak{B}$, where k is a positive integer, there exist a $\delta > 0$ and $U \in \mathcal{U}_n$ such that $B^\delta \subseteq U$ so $(-k, k) \subseteq U$.*

Let $B \in \mathfrak{B}$. Clearly $\mathcal{U} = \{(-n, n) : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Consequently we can find a $n_0 \in \mathbb{N}$ and a sequence of positive real numbers $\{\delta_n : n \geq n_0\}$ such that $B^{\delta_n} \subseteq (-n, n)$ for all $n \geq n_0$, i.e., $B^{\delta_n} \subseteq (-k_n, k_n) \subseteq U_n$ for $U_n \in \mathcal{V}_n$ for all $n \geq n_0$. Therefore $\{U_n : n \in \mathbb{N}\} \in \mathcal{O}^{gp}_{\mathfrak{B}^s}$ and so X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$. Hence X has the $\mathfrak{B}^s$-Gerlits-Nagy property.

Theorem 3.5.7. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*

(1) *X has the $\mathfrak{B}^s$-Gerlits-Nagy property.*

(2) *X has the $\mathfrak{B}^s$-Hurewicz property as well as it satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

Proof. (1) $\Rightarrow$ (2). By (1), X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ which implies that X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$ as well as $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Again by Theorem 2.3.2, X has the $\mathfrak{B}^s$-Hurewicz property. Hence (2) holds.

(2) $\Rightarrow$ (1). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Apply $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to choose a $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Using that X has the $\mathfrak{B}^s$-Hurewicz property and Lemma 2.3.2, $\{U_n : n \in \mathbb{N}\}$ is a $\mathfrak{B}^s$-groupable cover of X. Thus X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$. □

Combining Proposition 3.2.1 and Proposition 3.5.1, we have the following.

Corollary 3.5.1. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X and (Y, ρ) be a another metric space. Let $f : X \to Y$ be a continuous function. If X has the $\mathfrak{B}^s$-Gerlits-Nagy property, then $f(X)$ has the $f(\mathfrak{B})^s$-Gerlits-Nagy property.*

Proposition 3.5.3. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ on a metric space X. If $|\mathfrak{B}_0| < \mathsf{add}(\mathcal{M})$, then X has the $\mathfrak{B}^s$-Gerlits-Nagy property.*

Proof. Since $\mathsf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\}$ and $|\mathfrak{B}_0| < \mathsf{add}(\mathcal{M})$, so $|\mathfrak{B}_0| < \mathsf{cov}(\mathcal{M})$ as well as $|\mathfrak{B}_0| < \mathfrak{b}$. By Theorem 3.3.2, $|\mathfrak{B}_0| < \mathsf{cov}(\mathcal{M})$ implies that X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Again in view of $|\mathfrak{B}_0| < \mathfrak{b}$ and Theorem 3.5.1, we can conclude that X has the $\mathfrak{B}^s$-Hurewicz property. Hence X has the $\mathfrak{B}^s$-Gerlits-Nagy property. □

Theorem 3.5.8. *Let $\mathfrak{B}$ be bornology with closed base on a metric space X. The following statements are equivalent.*

(1) *X has the $\mathfrak{B}^s$-Gerlits-Nagy property.*

(2) *X satisfies $\mathcal{O}_{\mathfrak{B}^s} \to (\mathcal{O}^{gp}_{\mathfrak{B}^s})^n_k$ for each $n, k \in \mathbb{N}$, provided X is $\mathfrak{B}^s$-Lindelöf.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X and $f : [\mathcal{U}]^n \to \{1, \ldots, k\}$ be a coloring. Since X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, X also satisfies $\mathcal{O}_{\mathfrak{B}^s} \to (\mathcal{O}_{\mathfrak{B}^s})^n_k$ by Theorem 3.3.7. Consequently there are a $\mathcal{V} \subseteq \mathcal{U}$ with $\mathcal{V} \in \mathcal{O}_{\mathfrak{B}^s}$ and an $i \in \{1, \ldots, k\}$ such that for each $V \in [\mathcal{V}]^n$, $f(V) = i$. By (1), we can find a countable open $\mathfrak{B}^s$-cover $\mathcal{V}' \subseteq \mathcal{V}$ which is $\mathfrak{B}^s$-groupable. Thus we have a $\mathcal{V}' \in \mathcal{O}^{gp}_{\mathfrak{B}^s}$ and an $i \in \{1, \ldots, k\}$ such that for each $V \in [\mathcal{V}']^n$, $f(V) = i$ holds.

(2) $\Rightarrow$ (1) Clearly X satisfies $\mathcal{O}_{\mathfrak{B}^s} \to (\mathcal{O}_{\mathfrak{B}^s})^2_2$. By Theorem 3.3.7, X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Now we show that every countable open $\mathfrak{B}^s$-cover of X is $\mathfrak{B}^s$-groupable. Let $\mathcal{U}$ be a countable open $\mathfrak{B}^s$-cover of X. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a partition of $\mathcal{U}$ into nonempty finite sets. Define a coloring $f : [\mathcal{U}]^2 \to \{1, 2\}$ by $f(\{U, V\}) = 1$ if $U, V \in \mathcal{U}_n$ for some $n \in \mathbb{N}$ and $f(\{U, V\}) = 2$ otherwise. By (2), there is a $\mathcal{V} \in \mathcal{O}^{gp}_{\mathfrak{B}^s}$ with $\mathcal{V} \subseteq \mathcal{U}$. Therefore $\mathcal{U}$ is $\mathfrak{B}^s$-groupable. Hence X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}^{gp}_{\mathfrak{B}^s})$. □

The following equivalent formulation of $\mathfrak{B}^s$-Gerlits-Nagy property in product spaces follows from Theorem 3.5.7 in combination with Theorem 3.5.4 and Theorem 3.5.5

Theorem 3.5.9. *Let $\mathfrak{B}$ be bornology with compact base on a metric space X. The following statements are equivalent.*
(1) X has the $\mathfrak{B}^s$-Gerlits-Nagy property.
(2) X^n has the $(\mathfrak{B}^n)^s$-Gerlits-Nagy property for each $n \in \mathbb{N}$.

3.5.3 OBSERVATIONS ON PRODUCT SPACE $X \times Y$

Finally we present two more observations related to both the properties in product spaces. Let (X, d) and (Y, ρ) be metric spaces. Let $\mathfrak{B}$ be a bornology on X. Then there is a natural bornology $\widehat{\mathfrak{B}}$ on $(X \times Y, d \times \rho)$ induced by $\mathfrak{B}$ and it is defined as $\widehat{\mathfrak{B}} = \{C \subseteq X \times Y : \pi_X(C) \in \mathfrak{B}\}$ [14], where $\pi_X : X \times Y \to X$ is the projection map. A base for $\widehat{\mathfrak{B}}$ is $\{B \times Y : B \in \mathfrak{B}\}$. The following observation is similar to Lemma 3.5.1.

Lemma 3.5.2. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X and (Y, ρ) be another compact metric space. If $\mathcal{U}$ is an open $(\widehat{\mathfrak{B}})^s$-cover of $X \times Y$, then there exists an open $\mathfrak{B}^s$-cover $\mathcal{V}$ of X such that $\{V \times Y : V \in \mathcal{V}\}$ refines $\mathcal{U}$.*

Theorem 3.5.10. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X and (Y, ρ) be another compact metric space. The following statements hold.*
(1) X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ if and only if $X \times Y$ satisfies $\mathsf{S}_1(\mathcal{O}_{(\widehat{\mathfrak{B}})^s}, \Gamma_{(\widehat{\mathfrak{B}})^s})$.
(2) X has the $\mathfrak{B}^s$-Hurewicz property if and only if $X \times Y$ has the $(\widehat{\mathfrak{B}})^s$-Hurewicz property.
(3) X has the $\mathfrak{B}^s$-Gerlits-Nagy property if and only if $X \times Y$ has the $(\widehat{\mathfrak{B}})^s$-Gerlits-Nagy property.

Proof. We give only proof of (1) and the other cases can be similarly verified.

Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $(\widehat{\mathfrak{B}})^s$-covers of $X \times Y$. By Lemma 3.5.2, for each $\mathcal{U}_n$ there is a $\mathcal{V}_n \in \mathcal{O}_{\mathfrak{B}^s}$ for which the set $\{V \times Y : V \in \mathcal{V}\}$ refines $\mathcal{U}_n$. Applying $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$, we can obtain a $\gamma_{\mathfrak{B}^s}$-cover $\{V_n : n \in \mathbb{N}\}$ of X, where $V_n \in \mathcal{V}_n$ for $n \in \mathbb{N}$. Now for each $V_n \times Y$ there is a $U_n \in \mathcal{U}_n$ satisfying $V_n \times Y \subseteq U_n$. It can be easily verified that $\{U_n : n \in \mathbb{N}\}$ witnesses $\mathsf{S}_1(\mathcal{O}_{(\widehat{\mathfrak{B}})^s}, \Gamma_{(\widehat{\mathfrak{B}})^s})$.

Conversely, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Then $\mathcal{U}'_n = \{U \times Y : U \in \mathcal{U}_n\}$ is an open $(\widehat{\mathfrak{B}})^s$-covers of $X \times Y$. Apply $\mathsf{S}_1(\mathcal{O}_{(\widehat{\mathfrak{B}})^s}, \Gamma_{(\widehat{\mathfrak{B}})^s})$ to $\{\mathcal{U}'_n : n \in \mathbb{N}\}$ to choose a $U_n \times Y$ for each n such that $\{U_n \times Y : n \in \mathbb{N}\} \in \Gamma_{(\widehat{\mathfrak{B}})^s}$. We show that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Let $B \in \mathfrak{B}_0$. For $B \times Y \in \widehat{\mathfrak{B}}$ there exist a n_0 and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers such that $(B \times Y)^{\delta_n} \subseteq U_n \times Y$ for all $n \geq n_0$, i.e., $B \subseteq U_n$ for all $n \geq n_0$. Since B is compact, for each $n \geq n_0$ there is a $\varepsilon_n > 0$ such that $B^{\varepsilon_n} \subseteq U_n$ for all $n \geq n_0$. Hence $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover and so X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

□

Let (Y, d_Y) be a subspace of (X, d), where d_Y is the induced metric on $Y \subseteq X$. Let $\mathfrak{B}$ be a bornology on X. It can be easily checked that $\mathfrak{B}_Y = \{B \cap Y : B \in \mathfrak{B}\}$ is a bornology on Y. If Y

is closed and $\mathfrak{B}$ has a compact base, then $\mathfrak{B}_Y$ has a compact base. The following result needs no further explanations.

Theorem 3.5.11. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X and Y be a closed subset of X. The following statements are true.*
(1) *If X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, then Y satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s_Y}, \Gamma_{\mathfrak{B}^s_Y})$.*
(2) *If X has the $\mathfrak{B}^s$-Hurewicz property, then Y has the $\mathfrak{B}^s_Y$-Hurewicz property.*
(3) *If X has the $\mathfrak{B}^s$-Gerlits-Nagy property, then Y has the $\mathfrak{B}^s_Y$-Gerlits-Nagy property.*

CHAPTER 4

ON BORNOLOGICAL COVERING PROPERTIES IN FUNCTION SPACES

This Chapter is based on our following work:

P. Das, D. Chandra and S. Das `Further applications of bornological covering properties in function spaces`, **Topology Appl.**, 310 (2022), 108005.

4.1 INTRODUCTION

Investigations on function spaces have always been one of the richest areas having supreme relevance in both topology and functional analysis. Since the initiation of a systematic study of selection principles in topology and their relations to game theory by Scheepers in [54, 95] (for the long and illustrious history of selection principles and its recent developments, one can consult [63, 99, 111]) a lot of activities have been going on, specifically in function spaces starting with the seminal papers [41, 96], which continued in [60, 75, 76, 77, 92, 97] and in other works.

As far as bornological perspectives of function spaces are concerned, it is a much newer area which started in [14] with a new topology on the set Y^X of all functions from X into Y, called the topology of strong uniform convergence, and since then various properties in function spaces were established in more recent articles, such as [18], [19] and [22], with a major emphasis given on characterizations of classical properties in respect of selection principles using certain kinds of bornological covers. Such investigations are important because of the fact that in general the topology of strong uniform convergence is not necessarily equal to the point open or compact open topologies.

This Chapter is a continuation of Chapter 2 focusing solely on the topological properties and related selection principles on the function space $C(X)$ with respect to the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence on $\mathfrak{B}$.

In this Chapter, we introduce the notion of a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X and present a diagram showing various implications among selection principles $\Pi(\mathcal{A}, \mathcal{B})$, where $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$ and $\mathcal{A}, \mathcal{B} \in \{\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}, \Gamma^{sh}_{\mathfrak{B}^s}\}$. Properties like countable fan tightness for finite sets, countable strong fan tightness for finite sets [92], Fréchet-Urysohn for finite sets [86] etc. of $(C(X), \tau^s_{\mathfrak{B}})$ are characterized by bornological covering properties of X, all of which are presented in a diagram describing various implications among them. The set tightness and the countable tightness of $(C(X), \tau^s_{\mathfrak{B}})$ are also interpreted similarly. A parallel study of selective versions for the class of dense and sequentially dense subsets of $(C(X), \tau^s_{\mathfrak{B}})$ and for that of certain bornological covers of X is presented. The relations among these selective properties of $(C(X), \tau^s_{\mathfrak{B}})$ are summarized into another implication diagram.

As there have been similar detailed investigations of the spaces $C_p(X)$ and $C_k(X)$ from which the motivation of this article came from, so an effort has been made present the $C_p(X)$ ($C_k(X)$) versions of the results obtained here for the bornological settings, alongside almost all the results with proper citations.

4.2 Observations on selective properties of $(C(X), \tau^s_{\mathfrak{B}})$

We first present few observations on closure properties of the function space $C(X)$.

Lemma 4.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $A = \{f_n : n \in \mathbb{N}\}$ be a sequence in $(C(X), \tau^s_{\mathfrak{B}})$. If A converges to the zero function, then for any neighbourhood U of 0 in the real line with $f_n^{-1}(U) \neq X$ for any n, the collection $\{f_n^{-1}(U) : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X.*

Proposition 4.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$.*
(1) If X satisfies $\Pi(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, then $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\Pi(\Sigma_{\underline{0}}, \Omega_{\underline{0}})$.
(2) If X satisfies $\Pi(\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$, then $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\Pi(\Sigma_{\underline{0}}, \Sigma_{\underline{0}})$.

Proof. Only the proof of (1) for the case $\Pi = \mathsf{S}_1$ is presented here, as the other cases can be proved using similar arguments. Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of elements from $\Sigma_{\underline{0}}$, where $A_n = \{f_{n,k} : k \in \mathbb{N}\}$ for $n \in \mathbb{N}$. Consider the set $\mathcal{U}_n = \{f^{-1}_{n,k}(-\frac{1}{n}, \frac{1}{n}) : k \in \mathbb{N}\}$. First assume that $X \in \mathcal{U}_n$ for infinitely many n. Choose M to be an infinite subset of $\mathbb{N}$ such that $X = f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n})$ for $n \in M$. Clearly $\underline{0} \in \overline{\{f_{n,k_n} : n \in M\}}$ and the conclusion follows. Now we assume that $X \notin \mathcal{U}_n$ for any n. By Lemma 4.2.1, each $\mathcal{U}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Now, applying $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, we choose an $f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n}) \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ so that $\{f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. We claim that $\underline{0} \in \overline{\{f_{n,k_n} : n \in \mathbb{N}\}}$. Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. Since $\{f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover, for $B \in \mathfrak{B}$ there are $\delta > 0$ and $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$ such that $B^\delta \subseteq f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n})$ (see [22, Proposition 3.1]). Clearly $f_{n,k_n} \in [B, \varepsilon]^s(\underline{0})$. Thus $\underline{0} \in \overline{\{f_{n,k_n} : n \in \mathbb{N}\}}$ and $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_1(\Sigma_{\underline{0}}, \Omega_{\underline{0}})$. ☐

4.2.1 SHRINKABLE $\gamma_{\mathfrak{B}^s}$-COVERS AND RELATED OBSERVATIONS

During the investigation of sequence selection property of $C_p(X)$, in 1999, Scheepers [98] proved that if $C_p(X)$ has Arhangel'skii's α_1-property, then X is a quasi-normal space (or alternatively, a QN-space) and conjectured that the converse is also true. Later in 2007, Sakai [91] proved the conjecture in the affirmative by introducing the notion of shrinkable γ-covers. In an independent study, Bukovský and Haleš [16] also proved Scheepers' conjecture in 2007, using the same notion. We follow here in their footsteps and introduce the notion of a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X.

Definition 4.2.1. *A $\gamma_{\mathfrak{B}^s}$-cover $\{U_n : n \in \mathbb{N}\}$ of X is shrinkable if there is a closed $\gamma_{\mathfrak{B}^s}$-cover $\{C(U_n) : n \in \mathbb{N}\}$ of X such that $C(U_n) \subseteq U_n$ for each $n \in \mathbb{N}$.*

Let $\Gamma^{sh}_{\mathfrak{B}^s}$ denote the family of of all shrinkable $\gamma_{\mathfrak{B}^s}$-covers of X. For the space $X = \mathbb{R}$ with Euclidean metric d and the bornology generated by $\mathfrak{B} = \{(-x, x) : x > 0\}$, the $\gamma_{\mathfrak{B}^s}$-cover $\{(-n, n) : n > 1, n \in \mathbb{N}\}$ is shrinkable as $\{[-n+1, n-1] : n > 1, n \in \mathbb{N}\}$ is a closed $\gamma_{\mathfrak{B}^s}$-cover of X. We now observe a few applications of shrinkable $\gamma_{\mathfrak{B}^s}$-covers of X. With the help of Theorem 4.2.1 (given below), we present an implication diagram (Figure 4.1) among the selection principles $\Pi(\mathcal{A}, \mathcal{B})$, where $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$ and $\mathcal{A}, \mathcal{B} \in \{\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}, \Gamma^{sh}_{\mathfrak{B}^s}\}$.

Lemma 4.2.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\{\mathcal{U}_n : n \in \mathbb{N}\} \subseteq \Gamma^{sh}_{\mathfrak{B}^s}$ and let $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$, $n \in \mathbb{N}$. If $\mathcal{V}_n = \{V^n_k : k \in \mathbb{N}\}$, where $V^n_k = U^1_k \cap \ldots \cap U^n_k$, then $\mathcal{V}_n$ is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X for each n.*

Proof. The proof is straightforward and so is omitted. □

Lemma 4.2.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If a sequence $\{f_k : k \in \mathbb{N}\}$ in $(C(X), \tau^s_{\mathfrak{B}})$ converges to h, then $\mathcal{U} = \{(f_k - h)^{-1}(-\frac{1}{n}, \frac{1}{n}) : k \in \mathbb{N}\}$, where $X \notin \mathcal{U}$, is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X for each $n \in \mathbb{N}$.*

Proof. As in Lemma 4.2.1, $\mathcal{U} = \{(f_k - h)^{-1}(-\frac{1}{n}, \frac{1}{n}) : k \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. To see that $\mathcal{U}$ is shrinkable, we show that $\{(f_k - h)^{-1}[-\frac{1}{n+1}, \frac{1}{n+1}] : k \in \mathbb{N}\}$ is a closed $\gamma_{\mathfrak{B}^s}$-cover of X. Let $B \in \mathfrak{B}$ and consider the neighbourhood $[B, \frac{1}{n+1}]^s(h)$. Choose a $k_0 \in \mathbb{N}$ such that $f_k \in [B, \frac{1}{n+1}]^s(h)$ for all $k \geq k_0$. Now choose a $\delta_k > 0$ such that $B^{\delta_k} \subseteq (f_k - h)^{-1}(-\frac{1}{n+1}, \frac{1}{n+1})$ for all $k \geq k_0$. Thus $B^{\delta_k} \subseteq (f_k - h)^{-1}[-\frac{1}{n+1}, \frac{1}{n+1}]$ for all $k \geq k_0$. This completes the proof. □

It is known that $\mathsf{S}_1(\Gamma^{sh}_k, \Gamma_k) = \mathsf{S}_{\text{fin}}(\Gamma^{sh}_k, \Gamma_k)$ (see [77, Lemma 3.2]) and $\mathsf{S}_1(\Gamma_F, \Gamma) = \mathsf{S}_{\text{fin}}(\Gamma_F, \Gamma)$ (see [75, Proposition 8.10]). Along this line we have the following result.

Theorem 4.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following relations hold.*

(1) $\mathsf{S}_1(\Gamma^{sh}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}) = \mathsf{S}_{\text{fin}}(\Gamma^{sh}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

(2) $\mathsf{S}_1(\Gamma^{\rm sh}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s}) = \mathsf{S}_{\rm fin}(\Gamma^{\rm sh}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s})$.
(3) $\mathsf{S}_1(\Gamma_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s}) = \mathsf{S}_{\rm fin}(\Gamma_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s})$.
(4) $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s}) = \mathsf{S}_{\rm fin}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s})$.

Proof. We only give the proof of (2). It is enough to show that $\mathsf{S}_{\rm fin}(\Gamma^{\rm sh}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s})$ implies $\mathsf{S}_1(\Gamma^{\rm sh}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s})$. Let X satisfy $\mathsf{S}_{\rm fin}(\Gamma^{\rm sh}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s})$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of shrinkable $\gamma_{\mathfrak{B}^s}$-covers of X, and let $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ for $n \in \mathbb{N}$. Let $\mathcal{V}_n = \{V^n_k : k \in \mathbb{N}\}$, where $V^n_k = U^1_k \cap \ldots \cap U^n_k$. By Lemma 4.2.2, $\mathcal{V}_n$ is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X for each n. Applying $\mathsf{S}_{\rm fin}(\Gamma^{\rm sh}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$, we can choose a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ for each n so that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X. Thus for each n, there is a finite collection of closed sets $C(\mathcal{W}_n)$ such that for each $V \in \mathcal{W}_n$ there is a $C(V) \in C(\mathcal{W}_n)$ with $C(V) \subseteq V$ and furthermore $\bigcup_{n\in\mathbb{N}} C(\mathcal{W}_n)$ is a closed $\gamma_{\mathfrak{B}^s}$-cover of X. Now choose a sequence $n_1 < n_2 < \cdots$ such that $C(\mathcal{W}_{n_j}) \setminus \cup_{i<j} C(\mathcal{W}_{n_i}) \neq \emptyset$. For each j, choose a $C(V^{n_j}_{k_j}) \in C(\mathcal{W}_{n_j}) \setminus \cup_{i<j} C(\mathcal{W}_{n_i})$. The sequence $\{C(V^{n_j}_{k_j}) : j \in \mathbb{N}\}$, being an infinite subset of a closed $\gamma_{\mathfrak{B}^s}$-cover, is again a closed $\gamma_{\mathfrak{B}^s}$-cover of X. We have $C(V^{n_j}_{k_j}) \subseteq V^{n_j}_{k_j}$ and $V^{n_j}_{k_j} = U^1_{k_j} \cap \ldots \cap U^{n_j}_{k_j}$ for $j \in \mathbb{N}$. For each $n \in \mathbb{N}$ with $n \in [1, n_1]$ choose $U_n = U^n_{k_1}$ and $C(U_n) = C(V^{n_1}_{k_1})$. For each $n, j \in \mathbb{N}$, $j > 1$ with $n \in (n_{j-1}, n_j]$ choose $U_n = U^n_{k_j}$ and $C(U_n) = C(V^{n_j}_{k_j})$. Clearly $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X and $\{C(U_n) : n \in \mathbb{N}\}$ is a closed $\gamma_{\mathfrak{B}^s}$-cover of X, where $C(U_n) \subseteq U_n$ for each n. Therefore $\{U_n : n \in \mathbb{N}\}$ is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X and hence X satisfies $\mathsf{S}_1(\Gamma^{\rm sh}_{\mathfrak{B}^s}, \Gamma^{\rm sh}_{\mathfrak{B}^s})$. □

From now on, for the results presenting selection principles $\Pi(\mathcal{A}, \mathcal{B})$ on either X or $C(X)$, where $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\rm fin}\}$, we would present only the proof for the case $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$, as all the other cases follow with minor modifications. One can use [15, Theorem 76], [75, Proposition 6.4] and [94, Theorem 3.2] to conclude that $C_p(X)$ satisfies $\Pi(\Sigma_{\underline{0}}, \Omega_{\underline{0}})$ if and only if X satisfies $\Pi(\Gamma_F, \Omega)$, provided X is a Tychonoff topological space.

Theorem 4.2.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\rm fin}\}$. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\Pi(\Sigma_{\underline{0}}, \Omega_{\underline{0}})$.
(2) X *satisfies* $\Pi(\Gamma^{\rm sh}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of shrinkable $\gamma_{\mathfrak{B}^s}$-covers of X and let $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ for each n. There is, for each n, a closed $\gamma_{\mathfrak{B}^s}$-cover $\{C(U^n_k) : k \in \mathbb{N}\}$ of X such that $C(U^n_k) \subseteq U^n_k$ for each k. Choose a $f_{n,k} \in C(X)$ satisfying $f_{n,k}(C(U^n_k)) = \{0\}$ and $f_{n,k}(X \setminus U^n_k) = \{1\}$. Consider $A_n = \{f_{n,k} : k \in \mathbb{N}\}$. We first show that $A_n \in \Sigma_{\underline{0}}$ for each n. Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. As $\{C(U^n_k) : k \in \mathbb{N}\}$ is a closed $\gamma_{\mathfrak{B}^s}$-cover of X, for $B \in \mathfrak{B}$ there exist a k_0 and a sequence $\{\delta^n_k : k \geq k_0\}$ of positive real numbers such that $B^{\delta^n_k} \subseteq C(U^n_k)$ for all $k \geq k_0$. Clearly $f_{n,k} \in [B, \varepsilon]^s(\underline{0})$ for all $k \geq k_0$ and hence $A_n \in \Sigma_{\underline{0}}$. Applying (1) to $\{A_n : n \in \mathbb{N}\}$, we choose an $f_{n,k_n} \in A_n$ such that $\underline{0} \in \overline{\{f_{n,k_n} : n \in \mathbb{N}\}}$. To

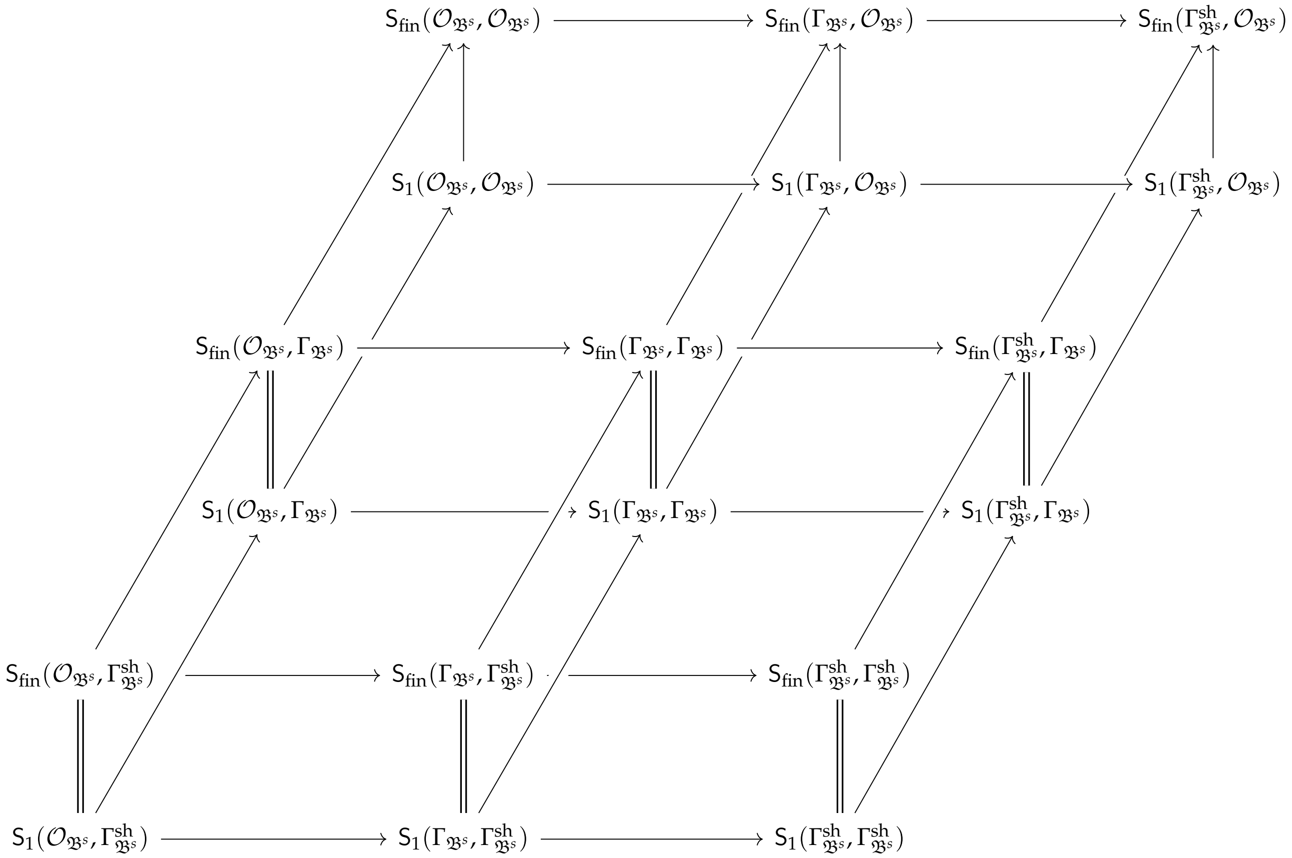

FIGURE 4.1: Diagram of the $\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}$-type selection principles in X for the classes $\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}, \Gamma^{\mathrm{sh}}_{\mathfrak{B}^s}$

each f_{n,k_n} associate a $U^n_{k_n} \in \mathcal{U}_n$ with $f_{n,k_n}(X \setminus U^n_{k_n}) = \{1\}$. Clearly $\{f^{-1}_{n,k_n}(-1,1) : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. As $f^{-1}_{n,k_n}(-1,1) \subseteq U^n_{k_n}$ for each n, $\{U^n_{k_n} : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Hence X satisfies $\mathsf{S}_1(\Gamma^{\mathrm{sh}}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of elements from $\Sigma_{\underline{0}}$, and let $A_n = \{f_{n,k} : k \in \mathbb{N}\}$. For each n let $\mathcal{U}_n = \{f^{-1}_{n,k}(-\frac{1}{n}, \frac{1}{n}) : k \in \mathbb{N}\}$. We assume that $X \notin \mathcal{U}_n$ for any n. By Lemma 4.2.3, $\mathcal{U}_n$ is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X. By (2), there is an $f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n})$ for each n so that $\{f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. It is easy to verify that $\underline{0} \in \overline{\{f_{n,k_n} : n \in \mathbb{N}\}}$, where $f_{n,k_n} \in A_n$. Hence (1) holds. □

Corollary 4.2.1 (see [76, Theorems 6.2, 7.1]). *Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$. The following statements are equivalent.*
(1) *$C_k(X)$ satisfies $\Pi(\Sigma_{\underline{0}}, \Omega_{\underline{0}})$.*
(2) *X satisfies $\Pi(\Gamma^{\mathrm{sh}}_k, \mathcal{K})$.*

Similarly we can obtain the following result. A similar result for $C_p(X)$ was obtained in [16, Theorem 10].

Theorem 4.2.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$. The following statements are equivalent.*

(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\Pi(\Sigma_{\underline{0}}, \Sigma_{\underline{0}})$.
(2) X *satisfies* $\Pi(\Gamma^{\text{sh}}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Corollary 4.2.2 (see [77, Theorems 3.3, 3.5])**.** *Let* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. *The following statements are equivalent.*
(1) $C_k(X)$ *satisfies* $\Pi(\Sigma_{\underline{0}}, \Sigma_{\underline{0}})$.
(2) X *satisfies* $\Pi(\Gamma^{\text{sh}}_k, \Gamma_k)$.

We will present some more applications of shrinkable $\gamma_{\mathfrak{B}^s}$-covers of X in the final section of this paper.

4.2.2 SELECTION PRINCIPLES AND SOME LOCAL PROPERTIES OF $(C(X), \tau^s_{\mathfrak{B}})$

One of the most interesting aspects of selection principles is that they can be used to characterize important classical properties. In Chapter 2, the function space $C(X)$ endowed with the topology $\tau^s_{\mathfrak{B}}$ was considered and certain important properties like countable T-tightness and the Reznichenko property were characterized in terms of bornological covering properties of X. Here we continue in that line and show that properties of $(C(X), \tau^s_{\mathfrak{B}})$ like $\mathsf{CSFT}_{\text{fin}}$, $\mathsf{CFT}_{\text{fin}}$ and FU_{fin} can be characterized by the selection properties $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ and $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ of X, respectively. One can naturally ask about the spaces $C_p(X)$ and $C_k(X)$, when X is a Tychonoff topological space. In [92, Lemma 2.8], Sakai obtained a characterization of $\mathsf{CFT}_{\text{fin}}$ for $C_p(X)$ in terms of the Menger property of X. Similarly the property $\mathsf{CSFT}_{\text{fin}}$ for $C_p(X)$ can be shown to be equivalent to the Rothberger property of X. It is also well known that $C_k(X)$ has CSFT (CFT) if and only if X satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$ (respectively, $\mathsf{S}_{\text{fin}}(\mathcal{K}, \mathcal{K})$) [61].

Theorem 4.2.4. *Let* $\mathfrak{B}$ *be a bornology with closed base on a metric space* X. *The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *has* CSFT.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *has* $\mathsf{CSFT}_{\text{fin}}$.
(3) X *satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. The equivalence $(1) \Leftrightarrow (3)$ follows from [19, Theorem 2.3]. We prove only $(2) \Leftrightarrow (3)$.

$(2) \Rightarrow (3)$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. For each n, let $A_n = \{f \in C(X) : \exists U \in \mathcal{U}_n \text{ such that } f(x) = 1 \text{ for all } x \in X \setminus U\}$. By Lemma 2.4.1, $\underline{0} \in \overline{A_n} \setminus A_n$. Define $\mathcal{A}_n = \{\{f\} : f \in A_n\}$ for $n \in \mathbb{N}$. Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$. Since $\underline{0} \in \overline{A_n} \setminus A_n$, we can choose an $f \in A_n$ such that $\{f\} \subseteq [B, \varepsilon]^s(\underline{0})$. Therefore $\mathcal{A}_n$ is a π-network at $\underline{0}$ consisting of finite subsets of $(C(X), \tau^s_{\mathfrak{B}})$. Using (2), choose a $\{f_n\} \in \mathcal{A}_n$, for each n, so that $\{\{f_n\} : n \in \mathbb{N}\}$ is a π-network at $\underline{0}$. For each f_n there is a $U_n \in \mathcal{U}_n$ such that $f_n(X \setminus U_n) = \{1\}$. Consider $\{U_n : n \in \mathbb{N}\}$. Let $B \in \mathfrak{B}$. Since $\{\{f_n\} : n \in \mathbb{N}\}$ is a π-network at $\underline{0}$, there is some n for which $\{f_n\} \subseteq [B, 1]^s(\underline{0})$, i.e., there exists a $\delta > 0$ such that $B^\delta \subseteq U_n$. Thus $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$ cover of X and (3) holds.

(3) $\Rightarrow$ (2). Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of π-networks at $\underline{0}$ consisting of finite subsets of $(C(X), \tau^s_{\mathfrak{B}})$. For each $A \in \mathcal{A}_n$, let $U(A) = \bigcap_{f \in A} f^{-1}(-\frac{1}{n}, \frac{1}{n})$. Clearly each such $U(A)$ is open, as A is finite. Let $\mathcal{U}_n = \{U(A) : A \in \mathcal{A}_n\}$, $n \in \mathbb{N}$. First assume that $X \in \mathcal{U}_n$ for infinitely many n. If $M \subseteq \mathbb{N}$ is an infinite set such that $X = U(A_n)$ for $n \in M$, then clearly $\{A_n : n \in M\}$ is a π-network at $\underline{0}$ and the conclusion follows. Therefore we proceed on the premise that $X \notin \mathcal{U}_n$ for each $n \in \mathbb{N}$. We claim that $\mathcal{U}_n$ is an open $\mathfrak{B}^s$-cover of X. Let $B \in \mathfrak{B}$. For the neighbourhood $[B, \frac{1}{n}]^s(\underline{0})$ of $\underline{0}$, there is an $A \in \mathcal{A}_n$ such that $A \subseteq [B, \frac{1}{n}]^s(\underline{0})$. It consequently follows that for each $f \in A$ there is a $\delta_f > 0$ satisfying $|f(x)| < \frac{1}{n}$ for all $x \in B^{\delta_f}$. This implies that $B^{\delta_f} \subseteq f^{-1}(-\frac{1}{n}, \frac{1}{n})$. Subsequently $B^{\delta} \subseteq U(A)$, where $\delta = \min\{\delta_f : f \in A\}$. Hence $\mathcal{U}_n$ is an open $\mathfrak{B}^s$-cover of X.

Applying (3) to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, we obtain, for each n, a $U(A_n) \in \mathcal{U}_n$ such that $\{U(A_n) : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. We now show that $\{A_n : n \in \mathbb{N}\}$ is a π-network at $\underline{0}$. Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. Since $\{U(A_n) : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X, in view of Proposition 2.2.1, there are a $\delta > 0$ and a $U(A_n)$ with $\frac{1}{n} < \varepsilon$ satisfying $B^{\delta} \subseteq U(A_n)$. Accordingly, $B^{\delta} \subseteq f^{-1}(-\frac{1}{n}, \frac{1}{n}) \subseteq f^{-1}(-\varepsilon, \varepsilon)$ for all $f \in A_n$. Hence $A_n \subseteq [B, \varepsilon]^s(\underline{0})$. Therefore $\{A_n : n \in \mathbb{N}\}$ is a π-network at $\underline{0}$ and (2) holds. □

Corollary 4.2.3. *The following statements are equivalent.*
(1) $C_k(X)$ *has* CSFT.
(2) $C_k(X)$ *has* $\mathsf{CSFT}_{\mathsf{fin}}$.
(3) X *satisfies* $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$.

We skip the proof of the following result, as the line of argument is similar to Theorem 4.2.4.

Theorem 4.2.5. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *has* CFT.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *has* $\mathsf{CFT}_{\mathsf{fin}}$.
(3) X *satisfies* $\mathsf{S}_{\mathsf{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Corollary 4.2.4. *The following statements are equivalent.*
(1) $C_k(X)$ *has* CFT.
(2) $C_k(X)$ *has* $\mathsf{CFT}_{\mathsf{fin}}$.
(3) X *satisfies* $\mathsf{S}_{\mathsf{fin}}(\mathcal{K}, \mathcal{K})$.

In [41], Gerlits and Nagy established that $C_p(X)$ is FU as well as SFU if and only if X is a γ-space. Also by [46, Page 3], $C_p(X)$ is $\mathsf{FU}_{\mathsf{fin}}$ if and only if $C_p(X)$ is FU. On the other hand, by [71, Theorem 1], $C_k(X)$ is FU if and only if X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$, while it is known that $C_k(X)$ is SFU if and only if X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ [69] (see also [17]). One should note that all the above mentioned results for both the spaces $C_p(X)$ and $C_k(X)$ were proved under the assumption that X is a Tychonoff topological space. For $(C(X), \tau^s_{\mathfrak{B}})$ we have the following result.

Theorem 4.2.6. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *is* FU.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *is* SFU.
(3) $(C(X), \tau^s_{\mathfrak{B}})$ *is* $\mathsf{FU}_{\mathsf{fin}}$.
(4) X *satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) follow from [19, Corollary 2.10]. It is enough to prove (4) $\Rightarrow$ (3).

(4) $\Rightarrow$ (3). Let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of π-networks at $\underline{0}$ consisting of finite subsets of $(C(X), \tau^s_{\mathfrak{B}})$. For each n, let $U(A) = \bigcap_{f\in A} f^{-1}(-\frac{1}{n}, \frac{1}{n})$, $A \in \mathcal{A}_n$. Let $\mathcal{U}_n = \{U(A) : A \in \mathcal{A}_n\}$. Assume that $X \notin \mathcal{U}_n$ for any n. Clearly $\mathcal{U}_n$'s are open $\mathfrak{B}^s$-covers of X. Applying (4) to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, we obtain a sequence $\{A_n : n \in \mathbb{N}\}$ such that $\{U(A_n) : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Consider a neighbourhood $[B, \varepsilon]^s(\underline{0})$ of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. For $B \in \mathfrak{B}$ there exist an $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < \varepsilon$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers satisfying $B^{\delta_n} \subseteq U(A_n)$ for all $n \geq n_0$. Thus we have $B^{\delta_n} \subseteq \bigcap_{f\in A_n} f^{-1}(-\varepsilon, \varepsilon)$ for all $n \geq n_0$, and so $A_n \subseteq [B, \varepsilon]^s(\underline{0})$ for all $n \geq n_0$. Therefore $\{A_n : n \in \mathbb{N}\}$ converges to $\underline{0}$. □

Corollary 4.2.5. *The following statements are equivalent.*
(1) $C_k(X)$ *is* FU.
(2) $C_k(X)$ *is* SFU.
(3) $C_k(X)$ *is* $\mathsf{FU}_{\mathsf{fin}}$.
(4) X *satisfies* $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$.

The proof of the following result is similar to that of [19, Theorem 2.7] with necessary modifications.

Theorem 4.2.7. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\mathsf{S}_{\mathsf{fin}}(\Omega_{\underline{0}}, \Sigma_{\underline{0}})$.
(2) X *satisfies* $\mathsf{S}_{\mathsf{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Corollary 4.2.6. *The following statements are equivalent.*
(1) $C_k(X)$ *satisfies* $\mathsf{S}_{\mathsf{fin}}(\Omega_{\underline{0}}, \Sigma_{\underline{0}})$.
(2) X *satisfies* $\mathsf{S}_{\mathsf{fin}}(\mathcal{K}, \Gamma_k)$.

Note that an analogous result for the space $C_p(X)$ also holds; details can be seen from [75, Theorem 5.5]. The selective properties of $(C(X), \tau^s_{\mathfrak{B}})$ discussed so far are represented by an implication diagram in Figure 4.2.

$$\begin{array}{ccccccc}
\mathsf{S}_{\mathrm{fin}}(\Sigma_{\underline{0}}, \Sigma_{\underline{0}}) & = & \mathsf{S}_1(\Sigma_{\underline{0}}, \Sigma_{\underline{0}}) & \to & \mathsf{S}_1(\Sigma_{\underline{0}}, \Omega_{\underline{0}}) & \to & \mathsf{S}_{\mathrm{fin}}(\Sigma_{\underline{0}}, \Omega_{\underline{0}}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathsf{S}_{\mathrm{fin}}(\Omega_{\underline{0}}, \Sigma_{\underline{0}}) & = & \mathsf{SFU} & \longrightarrow & \mathsf{CSFT} & \longrightarrow & \mathsf{CFT} \\
\| & & \| & & \| & & \| \\
\mathsf{FU} & = & \mathsf{FU}_{\mathrm{fin}} & \longrightarrow & \mathsf{CSFT}_{\mathrm{fin}} & \longrightarrow & \mathsf{CFT}_{\mathrm{fin}}
\end{array}$$

FIGURE 4.2: Diagram of the selective properties of $(C(X), \tau^s_{\mathfrak{B}})$

4.2.3 OBSERVATIONS RELATED TO VARIATIONS ON TIGHTNESS OF $(C(X), \tau^s_{\mathfrak{B}})$

Let $\mathcal{U}$ be a collection of open subsets of X, and let $\mathcal{Z}(\mathcal{U}) = \{Z(U) : U \in \mathcal{U}\}$ be a collection of closed sets, where $Z(U) \subseteq U$ for $U \in \mathcal{U}$. In [89, Theorem 3.1], Sakai characterized the set tightness of $C(X)$ with respect to the topology of pointwise convergence in terms of a covering property of X. In the language of the topology of strong uniform convergence on a bornology, we correlate Sakai's result with the following characterization.

Theorem 4.2.8. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $t_s((C(X), \tau^s_{\mathfrak{B}})) \leq \rho$.
(2) *For each collection $\mathcal{U}$ of open subsets of X, if $\mathcal{Z}(\mathcal{U})$ is a $\mathfrak{B}^s$-cover of X, then there exists a family $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of subsets of $\mathcal{U}$ such that no $\mathcal{Z}(\mathcal{U}_\alpha)$ is a $\mathfrak{B}^s$-cover of X, whereas $\bigcup_{\alpha<\rho} \mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{U}$ be a collection of open subsets of X and $\mathcal{Z}(\mathcal{U})$ be a $\mathfrak{B}^s$-cover of X. For each $U \in \mathcal{U}$ let $f_U \in C(X)$ be such that $f_U(Z(U)) = \{0\}$ and $f_U(X \setminus U) = \{1\}$. Consider the collection $A = \{f_U : U \in \mathcal{U}\}$. Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. Since $\mathcal{Z}(\mathcal{U})$ is a $\mathfrak{B}^s$-cover of X, choose a $\delta > 0$ and a $Z(U)$ such that $B^\delta \subseteq Z(U)$. Clearly $f_U \in [B, \varepsilon]^s(\underline{0})$ and so $[B, \varepsilon]^s(\underline{0}) \cap A \neq \emptyset$. Hence $\underline{0} \in \overline{A} \setminus A$. By (1), there is a family $\{A_\alpha : \alpha < \rho\}$ of subsets of A such that $\underline{0} \notin \overline{A_\alpha}$ for any $\alpha < \rho$ and $\underline{0} \in \overline{\cup_{\alpha<\rho} A_\alpha}$. Choose $\mathcal{U}_\alpha = \{U : f_U \in A_\alpha\}$. Now if $\mathcal{Z}(\mathcal{U}_\alpha)$ is a $\mathfrak{B}^s$-cover of X, then clearly $\underline{0} \in \overline{A_\alpha}$, a contradiction. Therefore no $\mathcal{Z}(\mathcal{U}_\alpha)$ is a $\mathfrak{B}^s$-cover of X. To show that $\bigcup_{\alpha<\rho} \mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover, let $B \in \mathfrak{B}$. We have $[B, 1]^s(\underline{0}) \cap (\bigcup_{\alpha<\rho} A_\alpha) \neq \emptyset$. There exits an $f_U \in A_\alpha$ for some $\alpha < \rho$ with $f_U \in [B, 1]^s(\underline{0})$, i.e., there is a $\delta > 0$ with $B^\delta \subseteq f_U^{-1}(-1, 1) \subseteq U$ for some $U \in \mathcal{U}_\alpha$. Hence $\bigcup_{\alpha<\rho} \mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X.

(2) $\Rightarrow$ (1). Let $A \subseteq (C(X), \tau^s_{\mathfrak{B}})$ with $\underline{0} \in \overline{A} \setminus A$. For each $n \in \mathbb{N}$ consider the collections $\mathcal{U}_n = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A\}$ and $\mathcal{Z}(\mathcal{U}_n) = \{f^{-1}[-\frac{1}{n+1}, \frac{1}{n+1}] : f \in A\}$. Assume that $X \notin \mathcal{U}_n$ for any n. Clearly $\mathcal{Z}(\mathcal{U}_n)$'s are $\mathfrak{B}^s$-covers of X. By (2), for each n there is a family $\{\mathcal{U}_{n,\alpha} : \alpha < \rho\}$ of subsets of $\mathcal{U}$ such that no $\mathcal{Z}(\mathcal{U}_{n,\alpha})$ is a $\mathfrak{B}^s$-cover of X and $\bigcup_{\alpha<\rho} \mathcal{U}_{n,\alpha}$ is an open $\mathfrak{B}^s$-cover

of X. Similarly, for each n and $\alpha < \rho$, choose an $A_{n,\alpha} \subseteq A$ such that $f \in A_{n,\alpha}$ whenever $f^{-1}(-\frac{1}{n}, \frac{1}{n}) \in \mathcal{U}_{n,\alpha}$. Consider the family $\{A_{n,\alpha} : n \in \mathbb{N}, \alpha < \rho\}$. Since $\mathcal{Z}(\mathcal{U}_{n,\alpha})$ is not a $\mathfrak{B}^s$-cover of X, there is a $B_0 \in \mathfrak{B}$ such that $B_0^\delta \nsubseteq f^{-1}[-\frac{1}{n+1}, \frac{1}{n+1}]$ for any $\delta > 0$ and any $f \in A_{n,\alpha}$, i.e., $f \notin [B_0, \frac{1}{n+1}]^s(\underline{0})$ for any $f \in A_{n,\alpha}$ and so $[B_0, \frac{1}{n+1}]^s(\underline{0}) \cap A_{n,\alpha} = \emptyset$. Therefore $\underline{0} \notin \overline{A_{n,\alpha}}$ for any n and $\alpha < \rho$. The proof will be finished if we show that $\underline{0} \in \overline{\cup\{A_{n,\alpha} : n \in \mathbb{N}, \alpha < \rho\}}$. Consider a neighbourhood $[B, \varepsilon]^s(\underline{0})$ of $\underline{0}$ and choose $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Since $\bigcup_{\alpha<\rho} \mathcal{U}_{n,\alpha}$ is an open $\mathfrak{B}^s$-cover of X, there are a $\delta > 0$ and $f^{-1}(-\frac{1}{n}, \frac{1}{n}) \in \mathcal{U}_{n,\alpha}$ for some $\alpha < \rho$ such that $B^\delta \subseteq f^{-1}(-\frac{1}{n}, \frac{1}{n})$, i.e., $f \in [B, \varepsilon]^s(\underline{0})$ for some $f \in A_{n,\alpha}$. Hence $[B, \varepsilon]^s(\underline{0}) \cap (\bigcup\{A_{n,\alpha} : n \in \mathbb{N}, \alpha < \rho\}) \neq \emptyset$ and we are done. □

If $\mathfrak{B}$ is a bornology on X, then it is easy to verify that for any $B \in \mathfrak{B}$ and $\delta > 0$, $(B^n)^\delta = (B^\delta)^n$ for any $n \in \mathbb{N}$. In [18, Theorem 3.12], the CT property of $(C(X), \tau^s_{\mathfrak{B}})$ was characterized by the $\mathfrak{B}^s$-Lindelöf property of X. In the following, we show that it can also be characterized by the $(\mathfrak{B}^n)^s$-Lindelöf property of X^n for each $n \in \mathbb{N}$. Here $\mathfrak{B}^n$, a bornology on X^n, is the n-th power of $\mathfrak{B}$. For a Tychonoff topological space X it is important to note that $C_p(X)$ has CT if and only if X^n is Lindelöf for each $n \in \mathbb{N}$ [72, Corollary 4.7.3] (see also [4]).

We need to use the following result from [26].

Lemma 4.2.4 (cf. [26, Lemma 5.1]). *For a bornology $\mathfrak{B}$ with compact base on a metric space X, if $\mathcal{U}$ is an open $(\mathfrak{B}^n)^s$-cover of X^n, then there exists an open $\mathfrak{B}^s$-cover $\mathcal{V}$ of X such that $\{V^n : V \in \mathcal{V}\}$ is an open $(\mathfrak{B}^n)^s$-cover of X^n which refines $\mathcal{U}$.*

Theorem 4.2.9. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ has CT.
(2) X^n is $(\mathfrak{B}^n)^s$-Lindelöf for each $n \in \mathbb{N}$.

Proof. (1) ⇒ (2). Let $\mathcal{U}$ be an open $(\mathfrak{B}^n)^s$-cover of X^n. There is an open $\mathfrak{B}^s$-cover $\mathcal{V}$ of X such that $\{V^n : V \in \mathcal{V}\}$ refines $\mathcal{U}$. Now for $B \in \mathfrak{B}$ there exist a $\delta > 0$ and a $V \in \mathcal{V}$ such that $B^{2\delta} \subseteq V$. Let $\mathcal{V}_B = \{V \in \mathcal{V} : B^{2\delta} \subseteq V\}$. For each $V \in \mathcal{V}_B$ choose a $f_{B,V} \in C(X)$ such that $f_{B,V}(B^\delta) = \{0\}$ and $f_{B,V}(X \setminus V) = \{1\}$. Consider the set $F = \{f_{B,V} : B \in \mathfrak{B}, V \in \mathcal{V}_B\}$. Clearly $\underline{0} \in \overline{F}$. By (1), there is a countable subset F' of F such that $\underline{0} \in \overline{F'}$. Let $\mathcal{W} = \{V : f_{B,V} \in F'\}$. Then $\mathcal{W}$ is a countable subset of $\mathcal{V}$. Now for $V \in \mathcal{W}$ choose a $U \in \mathcal{U}$ such that $V^n \subseteq U$. Consider the set $\mathcal{Z} = \{U \in \mathcal{U} : V^n \subseteq U \text{ for } V \in \mathcal{W}\}$. Clearly $\mathcal{Z}$ is a countable subset of $\mathcal{U}$. We show that $\mathcal{Z}$ is an open $(\mathfrak{B}^n)^s$-cover of X^n. Let $B^n \in \mathfrak{B}^n$. As $[B, 1]^s(\underline{0}) \cap F' \neq \emptyset$, there is a $f_{B_1,V_1} \in F'$ with $f_{B_1,V_1} \in [B, 1]^s(\underline{0})$. Choose a $\delta > 0$ such that $B^\delta \subseteq f^{-1}_{B_1,V_1}(-1, 1) \subseteq V_1$. Now $(B^\delta)^n \subseteq V_1^n \subseteq U_1$ for some $U_1 \in \mathcal{Z}$. This implies that $(B^n)^\delta \subseteq U_1$. So $\mathcal{Z}$ is a countable $(\mathfrak{B}^n)^s$-subcover of $\mathcal{U}$.

(2) ⇒ (1). Let $A \subseteq (C(X), \tau^s_{\mathfrak{B}^s})$ with $\underline{0} \in \overline{A}$. Let $\mathcal{U}_m = \{g^{-1}(-\frac{1}{m}, \frac{1}{m}) : g \in A\}$ for $m \in \mathbb{N}$. Assume that $X \notin \mathcal{U}_n$ for any n. By Lemma 2.4.1, $\mathcal{U}_m$ is an open $\mathfrak{B}^s$-cover of X for each $m \in \mathbb{N}$. Let $\mathcal{V}_m = \{U^n : U \in \mathcal{U}_m\}$. Clearly $\mathcal{V}_m$ is an open $(\mathfrak{B}^n)^s$-cover of X^n. By (2), for each $m \in \mathbb{N}$

there is a countable $(\mathfrak{B}^n)^s$-subcover $\mathcal{W}_m = \{U^n_{k,m} : k \in \mathbb{N}\}$ of $\mathcal{V}_m$, where $U_{k,m} = g^{-1}_{k,m}(-\frac{1}{m}, \frac{1}{m})$ for $k \in \mathbb{N}$. Choose $A' = \{g_{k,m} : k, m \in \mathbb{N}\}$, a countable subset of A. Clearly $\underline{0} \in \overline{A'}$. Hence (1) holds. □

4.3 ON SELECTORS FOR SEQUENCES OF DENSE SUBSETS OF $(C(X), \tau^s_{\mathfrak{B}})$

This section is devoted to selection principles in the function space $C(X)$ with respect to the topology $\tau^s_{\mathfrak{B}}$ involving dense and sequentially dense subsets in terms of certain bornological covers of X, which has not been investigated in the bornological universe before. The results obtained in this section are collected in the comprehensive diagram in Figure 4.3. Dense and sequentially dense subsets were used in the seventh article [97] in the series of papers by Scheepers and his co-authors while building the so called universe of selection principles theory and has subsequently been carried out for further investigations (see [75, 76, 77]). One must note that all the investigations done so far in this direction have been carried out in the function space $C(X)$ with respect to the topology of pointwise convergence τ_p and the compact-open topology τ_k.

4.3.1 ON DENSE SUBSETS OF $(C(X), \tau^s_{\mathfrak{B}})$

By the symbol $\mathcal{D}$ we denote the family of dense subsets of X. We will make extensive use of the following two basic observations about the open $\mathfrak{B}^s$-covers, which are easy to prove.

Lemma 4.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X. If P is dense in $(C(X), \tau^s_{\mathfrak{B}})$, then the set $D = \{f \in C(X) : f|_{B^\delta} = h$ and $f(X \setminus U) = \{1\}$, where $B^\delta \subseteq U$, for some $\delta > 0, B \in \mathfrak{B}, U \in \mathcal{U}, h \in P\}$ is also dense in $(C(X), \tau^s_{\mathfrak{B}})$.*

Lemma 4.3.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let D be dense in $(C(X), \tau^s_{\mathfrak{B}})$. For any $h \in C(X)$ and $n \in \mathbb{N}$ the collection $\mathcal{U} = \{(f-h)^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in D\}$, where $X \notin \mathcal{U}$, is an open $\mathfrak{B}^s$-cover of X.*

Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$ and consider the space $C_p(X)$, where X is a Tychonoff topological space. It follows from [12, Theorem 21, 57] that $C_p(X)$ satisfies $\Pi(\mathcal{D}, \Omega_{\underline{0}})$ if and only if X satisfies $\Pi(\Omega, \Omega)$. Also $C_p(X)$ satisfies $\Pi(\mathcal{D}, \Sigma_{\underline{0}})$ if and only if X satisfies $\Pi(\Omega, \Gamma)$, where $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$ [75, Theorem 5.6]. Considering the topology of strong uniform convergence $\tau^s_{\mathfrak{B}}$, we obtain the following results.

Theorem 4.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\Pi(\mathcal{D}, \Omega_{\underline{0}})$.
(2) X satisfies $\Pi(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X and D be dense in $(C(X), \tau^s_{\mathfrak{B}})$. For each positive integer n and $B \in \mathfrak{B}$ there are $\delta > 0$ and $U \in \mathcal{U}_n$ such that $B^{2\delta} \subseteq U$. Let $\mathcal{U}_{n,B} = \{U \in \mathcal{U}_n : B^{2\delta} \subseteq U \text{ for some } \delta > 0\}$. For $U \in \mathcal{U}_{n,B}$ and $h \in D$, choose an $f_{B,U,h} \in C(X)$ such that $f_{B,U,h}(x) = h(x)$ for all $x \in B^\delta$ and $f_{B,U,h}(x) = 1$ for all $x \in X \setminus U$. Let $D_n = \{f_{B,U,h} : B \in \mathfrak{B}, U \in \mathcal{U}_{n,B}, h \in D\}$. By Lemma 4.3.1, D_n's are dense in $(C(X), \tau^s_{\mathfrak{B}})$. Applying (1) to $\{D_n : n \in \mathbb{N}\}$, we choose, for each n, an $f_{B_n,U_n,h_n} \in D_n$ such that $\underline{0} \in \overline{\{f_{B_n,U_n,h_n} : n \in \mathbb{N}\}}$. Clearly $\{f^{-1}_{B_n,U_n,h_n}(-1,1) : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Since $f^{-1}_{B_n,U_n,h_n}(-1,1) \subseteq U_n$ for each n, $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Hence (2) holds.

(2) $\Rightarrow$ (1). Let $\{D_n : n \in \mathbb{N}\}$ be a sequence of elements from $\mathcal{D}$. For each n, let $\mathcal{U}_n = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in D_n\}$. For the case where $X \in \mathcal{U}_n$ for infinitely many n, the conclusion is immediate. We can proceed with the assumption that $X \notin \mathcal{U}_n$ for each n. By Lemma 4.3.2, $\mathcal{U}_n$'s are open $\mathfrak{B}^s$-covers of X. We apply (2) to $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to choose a $U_n \in \mathcal{U}_n$ for each n so that $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X, where $U_n = f_n^{-1}(-\frac{1}{n}, \frac{1}{n})$. Clearly $\underline{0} \in \overline{\{f_n : n \in \mathbb{N}\}}$ and the desired conclusion follows. □

Corollary 4.3.1 (see [76, Theorems 4.2, 5.1])**.** *Let* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$*. The following statements are equivalent.*
(1) $C_k(X)$ *satisfies* $\Pi(\mathcal{D}, \Omega_{\underline{0}})$.
(2) X *satisfies* $\Pi(\mathcal{K}, \mathcal{K})$.

Similarly we can obtain the following result.

Theorem 4.3.2. *Let* $\mathfrak{B}$ *be a bornology with closed base on a metric space* X*. Let* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$*. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\Pi(\mathcal{D}, \Sigma_{\underline{0}})$.
(2) X *satisfies* $\Pi(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Corollary 4.3.2 (see [76, Theorem 3.3])**.** *Let* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$*. The following statements are equivalent.*
(1) $C_k(X)$ *satisfies* $\Pi(\mathcal{D}, \Sigma_{\underline{0}})$.
(2) X *satisfies* $\Pi(\mathcal{K}, \Gamma_k)$.

Recall the following definition.

> *X is said to be* ***weakly Fréchet in the strict sense with respect to dense subspaces*** *if for every sequence* $\{D_n : n \in \mathbb{N}\}$ *of dense subspaces of* X *and for every* $x \in X$*, there is a sequence* $\{F_n : n \in \mathbb{N}\}$ *of finite sets such that* $F_n \subseteq D_n$ *for each* n *and for every neighbourhood* U *of* x*,* $U \cap F_n \neq \emptyset$ *for all but finitely many* $n \in \mathbb{N}$ *[12].*

In [12, Proposition 38] a criterion for a space $C_p(X)$ to be weakly Fréchet in the strict sense with respect to dense subspaces in terms of the Hurewicz property of X was obtained. For the bornological case we have the following.

Theorem 4.3.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *is weakly Fréchet in the strict sense with respect to dense subspaces.*
(2) *X has the $\mathfrak{B}^s$-Hurewicz property.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X and P be a dense subset of $(C(X), \tau^s_{\mathfrak{B}})$. By Lemma 4.3.1, for each n, $D_n = \{f_{B,U,h} : B \in \mathfrak{B}, U \in \mathcal{U}_{n,B}, h \in P\}$ is dense, where $\mathcal{U}_{n,B} = \{U \in \mathcal{U}_n : B^{2\delta} \subseteq U, \delta > 0\}$, $f_{B,U,h}(x) = h(x)$ for all $x \in B^{\delta}$ and $f_{B,U,h}(x) = 1$ for all $x \in X \setminus U$. By (1), for the sequence $\{D_n : n \in \mathbb{N}\}$ and the element $\underline{0}$, there is a sequence $\{F_n : n \in \mathbb{N}\}$ of finite sets such that $F_n \subseteq D_n$ for each n and every neighbourhood of $\underline{0}$ intersects F_n for all but finitely many n. Choose $\mathcal{V}_n = \{U \in \mathcal{U}_n : f_{B,U,h} \in F_n\}$. Clearly $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. We show that $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the $\mathfrak{B}^s$-Hurewicz property of X. Let $B' \in \mathfrak{B}$. For the neighbourhood $[B', 1]^s(\underline{0})$, there is an n_0 such that $F_n \cap [B', 1]^s(\underline{0}) \neq \emptyset$ for all $n \geq n_0$. Choose a $\delta_n > 0$ such that $B'^{\delta_n} \subseteq f^{-1}_{B,U,h}(-1, 1)$ for all $n \geq n_0$, where $f_{B,U,h} \in F_n$. Now for the sequence $\{\delta_n : n \geq n_0\}$, we have $B'^{\delta_n} \subseteq U$ for all $n \geq n_0$ for some $U \in \mathcal{V}_n$, and this shows that (2) holds.

(2) $\Rightarrow$ (1) Let $\{D_n : n \in \mathbb{N}\}$ be a sequence of dense sets, and let $g \in C(X)$. Consider $\mathcal{U}_n = \{(f - g)^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in D_n\}$. Assume that $X \notin \mathcal{U}_n$ for any n. Clearly $\mathcal{U}_n$'s are open $\mathfrak{B}^s$-covers of X. By (2), there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite sets such that $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each n witnessing $\mathfrak{B}^s$-Hurewicz property of X. Choose a finite subset F_n of D_n such that $f \in F_n$ whenever $f^{-1}(-\frac{1}{n}, \frac{1}{n}) \in \mathcal{V}_n$. It is easy to verify that $\{F_n : n \in \mathbb{N}\}$ is the desired sequence for (1) to hold. □

The rest of the results of this section are proved under the separability condition on $C(X)$. For this purpose the following characterization of separability of $(C(X), \tau^s_{\mathfrak{B}})$ in terms of a base of a bornology on X would be helpful.

Theorem 4.3.4 (cf. [18, Theorem 3.7])**.** *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(i) $(C(X), \tau^s_{\mathfrak{B}})$ *is separable.*
(ii) *X has a weaker metrizable separable topology and $\mathfrak{B}$ has a base of compacta.*

In [97, Theorems 13, 35], Scheepers showed that for a separable metric space X, $C_p(X)$ satisfies $\Pi(\mathcal{D}, \mathcal{D})$ if and only if X satisfies $\Pi(\Omega, \Omega)$, where $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. In our bornological framework the following characterization is obtained.

Theorem 4.3.5. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $(C(X), \tau^s_{\mathfrak{B}})$ be separable. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\Pi(\mathcal{D}, \mathcal{D})$.
(2) *X satisfies* $\Pi(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2). Since on $(C(X), \tau^s_{\mathfrak{B}})$, $S_1(\mathcal{D}, \mathcal{D})$ implies $S_1(\mathcal{D}, \Omega_{\underline{0}})$, it follows from Theorem 4.3.1 that X satisfies $S_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Let $\{D_n : n \in \mathbb{N}\}$ be a sequence of elements from $\mathcal{D}$. Let $\{\mathcal{P}_m : m \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets, and let $D = \{h_m : m \in \mathbb{N}\}$ be a countable dense subset of $(C(X), \tau^s_{\mathfrak{B}})$. Fix $m \in \mathbb{N}$. Now for each $n \in \mathcal{P}_m$, let $\mathcal{U}_n = \{(f - h_m)^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in D_n\}$. Without any loss of generality, we assume that $X \notin \mathcal{U}_n$ for each $n \in \mathcal{P}_m$. By Lemma 4.3.2, $\mathcal{U}_n$'s are open $\mathfrak{B}^s$-covers of X. Applying (2) to $\{\mathcal{U}_n : n \in \mathcal{P}_m\}$, we choose a $U_n \in \mathcal{U}_n$ for each $n \in \mathcal{P}_m$ so that $\{U_n : n \in \mathcal{P}_m\}$ is an open $\mathfrak{B}^s$-cover of X, where $U_n = (f_n - h_m)^{-1}(-\frac{1}{n}, \frac{1}{n})$ and $f_n \in D_n$. Thus we obtain a collection $\{f_n : n \in \mathcal{P}_m\}$. We now show that $\bigcup_{m \in \mathbb{N}}\{f_n : n \in \mathcal{P}_m\}$ is dense in $(C(X), \tau^s_{\mathfrak{B}})$. Let $[B, \varepsilon]^s(g)$ be a neighbourhood of g, where $g \in C(X)$, $B \in \mathfrak{B}$ and $\varepsilon > 0$. Using the density of D, choose an $h_{m_1} \in D$ such that $h_{m_1} \in [B, \frac{\varepsilon}{2}]^s(g)$, which means that there exists a $\sigma > 0$ such that $|h_{m_1}(x) - g(x)| < \frac{\varepsilon}{2}$ for all $x \in B^\sigma$. Since $\{U_n : n \in \mathcal{P}_{m_1}\}$ is an open $\mathfrak{B}^s$-cover of X, for $B \in \mathfrak{B}$, there are $\delta > 0$ and U_n with $\frac{1}{n} < \frac{\varepsilon}{2}$ (in view of [22, Proposition 3.1]) such that $B^\delta \subseteq U_n$, i.e., $B^\delta \subseteq (f_n - h_{m_1})^{-1}(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Thus $|f_n(x) - h_{m_1}(x)| < \frac{\varepsilon}{2}$ for all $x \in B^\delta$. For $\delta_1 = \min\{\sigma, \delta\}$ we have $|f_n(x) - g(x)| < \varepsilon$ for all $x \in B^{\delta_1}$ and so $f_n \in [B, \varepsilon]^s(g)$. Clearly $\bigcup_{m \in \mathbb{N}}\{f_n : n \in \mathcal{P}_m\}$ is dense in $(C(X), \tau^s_{\mathfrak{B}})$ and the proof is now complete. □

Recall that the *i*-weight $iw(X)$ of a space X is the smallest infinite cardinal number $\mathfrak{m}$ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than $\mathfrak{m}$. It is known that the space $C_k(X)$ is separable if and only if $iw(X) = \aleph_0$ [74].

Corollary 4.3.3 (see [76, Theorems 4.2, 5.1]). *Let $iw(X) = \aleph_0$ and $\Pi \in \{S_1, S_{fin}\}$. The following statements are equivalent.*
(1) *$C_k(X)$ satisfies $\Pi(\mathcal{D}, \mathcal{D})$.*
(2) *X satisfies $\Pi(\mathcal{K}, \mathcal{K})$.*

Corollary 4.3.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $(C(X), \tau^s_{\mathfrak{B}})$ be separable. Let $\Pi \in \{S_1, S_{fin}\}$. The following statements are equivalent.*
(1) *$(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\Pi(\mathcal{D}, \mathcal{D})$.*
(2) *$(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\Pi(\mathcal{D}, \Omega_{\underline{0}})$.*
(3) *X satisfies $\Pi(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

4.3.2 ON SEQUENTIALLY DENSE SUBSETS OF $(C(X), \tau^s_{\mathfrak{B}})$

The symbol $[A]_{seq}$ denotes the set of all limits of sequences from A. A set $D \subseteq X$ is said to be *sequentially dense* in X if $X = [D]_{seq}$. The symbol $\mathcal{S}$ denotes the family of sequentially dense subsets of X.

> *X is said to be* ***strongly sequentially dense in itself*** *if every dense subset of X is sequentially dense.*

In this section we investigate certain topological properties of sequentially dense subsets of $(C(X), \tau^s_{\mathfrak{B}})$. Osipov had proved that the space $C_p(X)$ is strongly sequentially dense in itself if and only if X satisfies $\mathsf{S}_1(\Omega, \Gamma)$ [75, Theorem 5.6], where X is a Tychonoff topological space. In our bornological settings we have the following.

Theorem 4.3.6. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *is strongly sequentially dense in itself.*
(2) *Every open $\mathfrak{B}^s$-cover of X contains a set which is a $\gamma_{\mathfrak{B}^s}$-cover of X.*
(3) *X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(4) *X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. The equivalences (2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (4) follow from [19, Theorem 2.8] and [22, Theorem 3.2(1)] respectively. So we only prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Let $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X and P be dense in $(C(X), \tau^s_{\mathfrak{B}})$. Consider the set $D = \{f_{B,U,h} : B \in \mathfrak{B}, U \in \mathcal{U}_B, h \in P\}$, where $\mathcal{U}_B = \{U \in \mathcal{U} : B^{2\delta} \subseteq U, \delta > 0\}$ and $f_{B,U,h} \in C(X)$ is any function such that $f_{B,U,h}(x) = h(x)$ for all $x \in B^\delta$ and $f_{B,U,h}(x) = 1$ for all $x \in X \setminus U$.

By Lemma 4.3.1, D is dense in $(C(X), \tau^s_{\mathfrak{B}})$ and hence D is sequentially dense. Choose a sequence $\{f_{B_n,U_n,h_n} : n \in \mathbb{N}\}$ in D that converges to $\underline{0}$. We now show that $\{U_n : n \in \mathbb{N}\} \subseteq \mathcal{U}$ is the required $\gamma_{\mathfrak{B}^s}$-cover of X. Let $B \in \mathfrak{B}$. For the neighbourhood $[B, 1]^s(\underline{0})$, there is an n_0 such that $f_{B_n,U_n,h_n} \in [B, 1]^s(\underline{0})$ for all $n \geq n_0$. Choose a $\delta_n > 0$ such that $B^{\delta_n} \subseteq f^{-1}_{B_n,U_n,h_n}(-1, 1)$ for all $n \geq n_0$, i.e., $B^{\delta_n} \subseteq U_n$ for all $n \geq n_0$. This shows that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X.

(3) $\Rightarrow$ (1). Let D be a dense subset of $(C(X), \tau^s_{\mathfrak{B}})$ and let $g \in C(X)$. To show that D is sequentially dense, we need to find a sequence in D converging to g. For each n, let $\mathcal{U}_n = \{(f - g)^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in D\}$. First suppose that $X \in \mathcal{U}_n$ for infinitely many n and choose $M = \{n \in \mathbb{N} : X \in \mathcal{U}_n\}$. For each $n \in M$ there is an $f_n \in D$ such that $X = (f_n - g)^{-1}(-\frac{1}{n}, \frac{1}{n})$. Clearly $\{f_n : n \in M\}$ converges to g. Therefore we can assume that $X \notin \mathcal{U}_n$ for each $n \in \mathbb{N}$. By Lemma 4.3.2, $\mathcal{U}_n$'s are open $\mathfrak{B}^s$-covers of X. Applying $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, we choose a $U_n \in \mathcal{U}_n$ for each n so that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X, where $U_n = (f_n - g)^{-1}(-\frac{1}{n}, \frac{1}{n})$. Clearly $\{f_n : n \in \mathbb{N}\}$ is the desired sequence converging to g. $\square$

Corollary 4.3.5 (see [76, Theorem 3.3])**.** *The following statements are equivalent.*
(1) *$C_k(X)$ is strongly sequentially dense in itself.*
(2) *Every k-cover of X contains a set which is a γ_k-cover of X.*
(3) *X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$.*
(4) *X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{K}, \Gamma_k)$.*

Theorem 4.3.7. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $(C(X), \tau^s_{\mathfrak{B}})$ be separable. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. The following statements are equivalent.*

(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\Pi(\mathcal{D}, \mathcal{S})$.
(2) X *satisfies* $\Pi(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Proof. The result follows from Theorem 4.3.5 and Theorem 4.3.6. □

Corollary 4.3.6 (see [76, Theorem 3.3]). *Let $iw(X) = \aleph_0$ and $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. The following statements are equivalent.*
(1) $C_k(X)$ *satisfies* $\Pi(\mathcal{D}, \mathcal{S})$.
(2) X *satisfies* $\Pi(\mathcal{K}, \Gamma_k)$.

Corollary 4.3.7. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $(C(X), \tau^s_{\mathfrak{B}})$ be separable. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\mathsf{S}_1(\mathcal{D}, \mathcal{S})$.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\mathsf{S}_1(\mathcal{D}, \Sigma_{\underline{0}})$.
(3) X *satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.
(4) X *satisfies* $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.
(5) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\mathsf{S}_{\text{fin}}(\mathcal{D}, \Sigma_{\underline{0}})$.
(6) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\mathsf{S}_{\text{fin}}(\mathcal{D}, \mathcal{S})$.

Coming back to shrinkable $\gamma_{\mathfrak{B}^s}$-covers, the following observation is useful.

Lemma 4.3.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If a $\gamma_{\mathfrak{B}^s}$-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of X is shrinkable, then the set $S = \{f \in C(X) : f(X \setminus U_n) = \{1\}$ for some $n\}$ is sequentially dense in $(C(X), \tau^s_{\mathfrak{B}})$.*

Proof. Let $g \in C(X)$. We show that there is a sequence in S which converges to g. First choose a closed $\gamma_{\mathfrak{B}^s}$-cover $\{C(U_n) : n \in \mathbb{N}\}$ of X such that $C(U_n) \subseteq U_n$ for each n. Then choose an $f_n \in C(X)$ for which $f_n(x) = g(x)$ for all $x \in C(U_n)$ and $f_n(x) = 1$ for all $x \in X \setminus U_n$. Clearly $f_n \in S$ for each n. We claim that the sequence $\{f_n : n \in \mathbb{N}\}$ converges to g. Let $[B, \varepsilon]^s(g)$ be a neighbourhood of g, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. For $B \in \mathfrak{B}$ there exist an n_0 and a sequence $\{\delta_n : n \geq n_0\}$ of positive reals such that $B^{\delta_n} \subseteq C(U_n)$ for all $n \geq n_0$. Clearly $f_n(x) = g(x)$ for all $x \in B^{\delta_n}$ and $n \geq n_0$. Hence $f_n \in [B, \varepsilon]^s(g)$ for all $n \geq n_0$. Therefore $\{f_n : n \in \mathbb{N}\}$ converges to g. □

Next we consider selectors of $(C(X), \tau^s_{\mathfrak{B}})$ of type $\Pi(\mathcal{S}, \mathcal{Q})$, where $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$ and $\mathcal{Q} \in \{\Omega_{\underline{0}}, \Sigma_{\underline{0}}, \mathcal{D}, \mathcal{S}\}$, and establish that they can be characterized in terms of selection principles on X involving the class of shrinkable $\gamma_{\mathfrak{B}^s}$-covers of X. For similar characterizations in the context of $C_p(X)$, one should see [75, Theorems 6.6, 7.2, 8.8, 8.11].

Theorem 4.3.8. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. The following statements are equivalent.*

(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\Pi(\mathcal{S}, \Omega_{\underline{0}})$.
(2) X *satisfies* $\Pi(\Gamma^{\text{sh}}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of shrinkable $\gamma_{\mathfrak{B}^s}$-covers of X and $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ for $n \in \mathbb{N}$. For each n define $S_n = \{f \in C(X) : f(X \setminus U^n_k) = \{1\}$ for some $k \in \mathbb{N}\}$. By Lemma 4.3.3, S_n's are sequentially dense in $(C(X), \tau^s_{\mathfrak{B}})$. Applying (1) to $\{S_n : n \in \mathbb{N}\}$, we choose an $f_n \in S_n$ for each n so that $\underline{0} \in \overline{\{f_n : n \in \mathbb{N}\}}$. To each f_n, associate a $U^n_{k_n}$ such that $f_n(X \setminus U^n_{k_n}) = \{1\}$. Clearly $\{U^n_{k_n} : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Hence (2) holds.

(2) $\Rightarrow$ (1). Let $\{S_n : n \in \mathbb{N}\}$ be a sequence of elements from $\mathcal{S}$. Since S_n is sequentially dense, there is a sequence $\{f_{n,k} : k \in \mathbb{N}\}$ in S_n converging to $\underline{0}$. Let $\mathcal{U}_n = \{f^{-1}_{n,k}(-\frac{1}{n}, \frac{1}{n}) : k \in \mathbb{N}\}$. We assume that $X \notin \mathcal{U}_n$ for any n. By Lemma 4.2.3, $\mathcal{U}_n$ is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X. Now, applying (2) to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, we choose an $f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n}) \in \mathcal{U}_n$ for each n so that $\{f^{-1}_{n,k_n}(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Clearly $\underline{0} \in \overline{\{f_{n,k_n} : n \in \mathbb{N}\}}$ and hence $\mathsf{S}_1(\mathcal{S}, \Omega_{\underline{0}})$ holds. □

Corollary 4.3.8 (see [76, Theorems 6.4, 7.2]). *Let* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. *The following statements are equivalent.*
(1) $C_k(X)$ *satisfies* $\Pi(\mathcal{S}, \Omega_{\underline{0}})$.
(3) X *satisfies* $\Pi(\Gamma^{\text{sh}}_k, \mathcal{K})$.

The following result is obtained similarly.

Theorem 4.3.9. *Let* $\mathfrak{B}$ *be a bornology with closed base on a metric space* X. *Let* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. *The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\Pi(\mathcal{S}, \Sigma_{\underline{0}})$.
(2) X *satisfies* $\Pi(\Gamma^{\text{sh}}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Corollary 4.3.9 (see [77, Theorem 3.5]). *Let* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. *The following statements are equivalent.*
(1) $C_k(X)$ *satisfies* $\Pi(\mathcal{S}, \Sigma_{\underline{0}})$.
(3) X *satisfies* $\Pi(\Gamma^{\text{sh}}_k, \Gamma_k)$.

Applying techniques similar to that of the proof of Theorem 4.3.5 along with Lemma 4.2.3 and Theorem 4.3.8, one can obtain the following result.

Theorem 4.3.10. *Let* $\mathfrak{B}$ *be a bornology with closed base on a metric space* X *and* $(C(X), \tau^s_{\mathfrak{B}})$ *be separable. Let* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. *The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *satisfies* $\Pi(\mathcal{S}, \mathcal{D})$.
(2) X *satisfies* $\Pi(\Gamma^{sh}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Corollary 4.3.10 (see [76, Theorems 6.4, 7.2]). *Let* $iw(X) = \aleph_0$ *and* $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. *The following statements are equivalent.*
(1) $C_k(X)$ *satisfies* $\Pi(\mathcal{S}, \mathcal{D})$.
(3) X *satisfies* $\Pi(\Gamma^{\text{sh}}_k, \mathcal{K})$.

Theorem 4.3.11. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $(C(X), \tau^s_{\mathfrak{B}})$ be sequentially separable. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\Pi(\mathcal{S}, \mathcal{S})$.
(2) X satisfies $\Pi(\Gamma^{\text{sh}}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of shrinkable $\gamma_{\mathfrak{B}^s}$-covers of X and $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ for $n \in \mathbb{N}$. Let $\mathcal{V}_n = \{V^n_k : k \in \mathbb{N}\}$, where $V^n_k = U^1_k \cap \ldots \cap U^n_k$. By Lemma 4.2.2, $\mathcal{V}_n$ is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X for each n. We set $S_n = \{f \in C(X) : f(X \setminus V^n_k) = \{1\}$ for some $k \in \mathbb{N}\}$. By Lemma 4.3.3, S_n's are sequentially dense in $(C(X), \tau^s_{\mathfrak{B}})$. Applying (1) to $\{S_n : n \in \mathbb{N}\}$, we choose an $f_n \in S_n$ for each n so that $\{f_n : n \in \mathbb{N}\}$ is sequentially dense. Thus there is a sequence $1 < n_1 < n_2 < \cdots$ such that $\{f_{n_m} : m \in \mathbb{N}\}$ converges to $\underline{0}$. To each f_{n_m} we associate a $V^{n_m}_{k_{n_m}}$ such that $f_{n_m}(X \setminus V^{n_m}_{k_{n_m}}) = \{1\}$. Clearly $V^{n_m}_{k_{n_m}} = U^1_{k_{n_m}} \cap \ldots \cap U^{n_m}_{k_{n_m}}$ for each m and $\{V^{n_m}_{k_{n_m}} : m \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. For each $i \in \mathbb{N}$ with $1 \leq i \leq n_1$, choose the ith component $U^i_{k_{n_1}}$ in the representation of $V^{n_1}_{k_{n_1}}$ and for each $i, m \in \mathbb{N}$ $(m > 1)$ with $n_{m-1} < i \leq n_m$, choose the ith component $U^i_{k_{n_m}}$ in the representation of $V^{n_m}_{k_{n_m}}$. The family $\{U^i_{k_{n_m}} : i \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X witnessing $\mathsf{S}_1(\Gamma^{sh}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Let $\{S_n : n \in \mathbb{N}\}$ be a sequence of elements from $\mathcal{S}$ and $S = \{h_n : n \in \mathbb{N}\}$ be a countable sequentially dense subset of $(C(X), \tau^s_{\mathfrak{B}})$. Since S_n is sequentially dense, for h_n there is a sequence $\{f_{n,k} : k \in \mathbb{N}\}$ in S_n converging to h_n. Let $\mathcal{U}_n = \{(f_{n,k} - h_n)^{-1}(-\frac{1}{n}, \frac{1}{n}) : k \in \mathbb{N}\}$ for $n \in \mathbb{N}$. We assume that $X \notin \mathcal{U}_n$ for any $n \in \mathbb{N}$. By Lemma 4.2.3, $\mathcal{U}_n$ is a shrinkable $\gamma_{\mathfrak{B}^s}$-cover of X for each n. Applying (2) to $\{\mathcal{U}_n : n \in \mathbb{N}\}$, we choose a $U_n \in \mathcal{U}_n$ for each n so that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X, where $U_n = (f_{n,k_n} - h_n)^{-1}(-\frac{1}{n}, \frac{1}{n})$. It now remains to show that $\{f_{n,k_n} : n \in \mathbb{N}\} \in \mathcal{S}$.

Let $g \in C(X)$. Since S is sequentially dense, there is a sequence $\{h_{n_m} : m \in \mathbb{N}\}$ in S that converges to g. We claim that $\{f_{n_m,k_{n_m}} : m \in \mathbb{N}\}$ is the expected sequence converging to g. Consider a neighbourhood $[B, \varepsilon]^s(g)$ of g, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. There is an m_0 such that $h_{n_m} \in [B, \frac{\varepsilon}{2}]^s(g)$ for all $m \geq m_0$. Again $\{U_{n_m} : m \in \mathbb{N}\}$, being an infinite subset of the $\gamma_{\mathfrak{B}^s}$-cover $\{U_n : n \in \mathbb{N}\}$, is also a $\gamma_{\mathfrak{B}^s}$-cover of X. For $B \in \mathfrak{B}$ there is a $m_1 \in \mathbb{N}$ with $\frac{1}{n_{m_1}} < \frac{\varepsilon}{2}$ and a sequence $\{\delta_m : m \geq m_1\}$ of positive real numbers satisfying $B^{\delta_m} \subseteq U_{n_m}$ for all $m \geq m_1$. This means that $B^{\delta_m} \subseteq (f_{n_m,k_{n_m}} - h_{n_m})^{-1}(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ for all $m \geq m_1$, i.e., $f_{n_m,k_{n_m}} \in [B, \frac{\varepsilon}{2}]^s(h_{n_m})$ for all $m \geq m_1$. Choose $m_2 = \max\{m_0, m_1\}$. Clearly $f_{n_m,k_{n_m}} \in [B, \varepsilon]^s(g)$ for all $m \geq m_2$. Therefore $\{f_{n_m,k_{n_m}} : m \in \mathbb{N}\}$ converges to g. Hence $\{f_{n,k_n} : n \in \mathbb{N}\}$ is sequentially dense in $(C(X), \tau^s_{\mathfrak{B}})$. ☐

Corollary 4.3.11 (see [77, Theorem 3.5]). *Let $C_k(X)$ be sequentially separable and $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. The following statements are equivalent.*
(1) $C_k(X)$ satisfies $\Pi(\mathcal{S}, \mathcal{S})$.
(3) X satisfies $\Pi(\Gamma^{\text{sh}}_k, \Gamma_k)$.

Corollary 4.3.12. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $(C(X), \tau^s_{\mathfrak{B}})$ be separable. Let $\Pi \in \{\mathsf{S}_1, \mathsf{S}_{\text{fin}}\}$. The following statements are equivalent.*
(1) *$(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\Pi(\mathcal{S}, \mathcal{D})$.*
(2) *$(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\Pi(\mathcal{S}, \Omega_{\underline{0}})$.*
(3) *X satisfies $\Pi(\Gamma^{sh}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.*

Corollary 4.3.13. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $(C(X), \tau^s_{\mathfrak{B}})$ be sequentially separable. The following statements are equivalent.*
(1) *$(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_1(\mathcal{S}, \mathcal{S})$.*
(2) *$(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_1(\mathcal{S}, \Sigma_{\underline{0}})$.*
(3) *X satisfies $\mathsf{S}_1(\Gamma^{sh}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(4) *X satisfies $\mathsf{S}_{\text{fin}}(\Gamma^{sh}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(5) *$(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_{\text{fin}}(\mathcal{S}, \Sigma_{\underline{0}})$.*
(6) *$(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_{\text{fin}}(\mathcal{S}, \mathcal{S})$.*

The relationship between selective properties of dense and sequentially dense subsets of $(C(X), \tau^s_{\mathfrak{B}})$ found above in terms of bornological covering properties of X under suitable assumptions on $C(X)$ are summarized in the diagram in Figure 4.3.

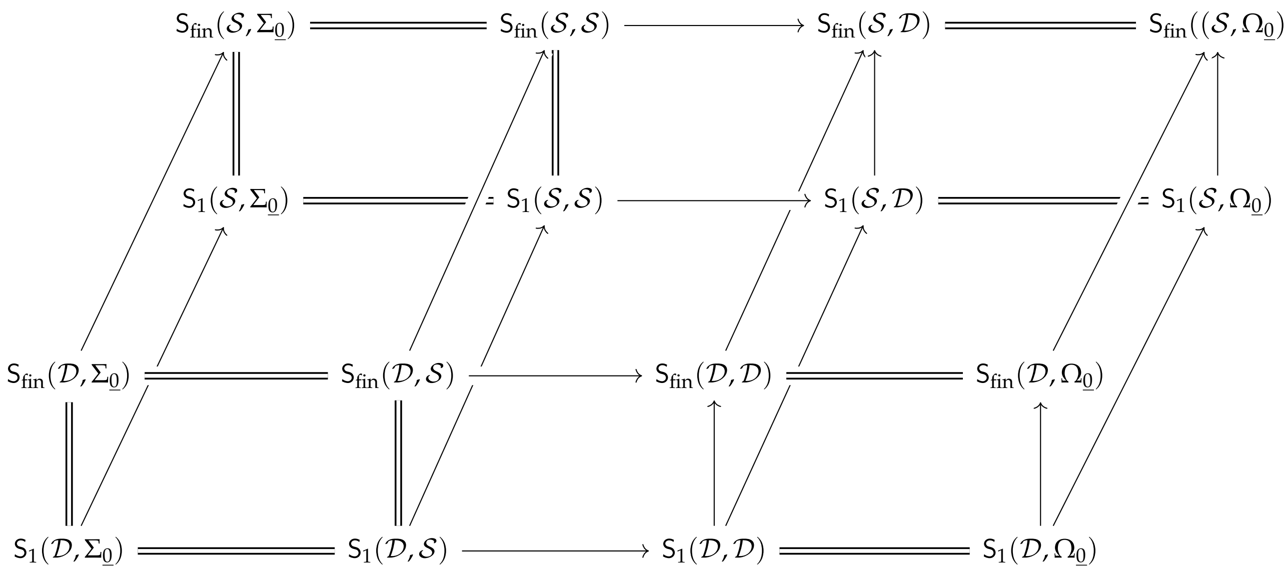

FIGURE 4.3: Diagram of selectors for sequences of dense subsets of $(C(X), \tau^s_{\mathfrak{B}})$

CHAPTER 5

On tightness and topological games in bornology

This Chapter is based on our following work:

D. Chandra, P. Das and S. Das `Certain observations on tightness and topological games in bornology`, **submitted**.

5.1 Introduction

This Chapter is a continuation of our investigations in the function space $C(X)$ with respect to the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence on $\mathfrak{B}$ in line of Chapter 2 and Chapter 4 using the idea of strong uniform convergence [14] on a bornology.

The study of selection principles and related games in function spaces endowed with a suitable topology is one of the most fascinating research areas where focus had been given on characterizations of classical properties in function space in respect of selection principles using open covers. Such investigations had been done with respect to the point-open and the compact-open topologies respectively. For more information on selection principles and its recent expansion, see [63, 99, 111].

From the bornological point of view, with respect to the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence on $\mathfrak{B}$, the study of selection principles in function spaces had been initiated in [19]. Further advancement in this direction has been carried out in [22, 26, 29], where selection principles and some classical properties have been elaborately investigated using the topology $\tau^s_{\mathfrak{B}}$ of strong uniform convergence on $\mathfrak{B}$. In this Chapter, we focus on the tightness property and some of its variations such as supertightness, Id-fan tightness and T-tightness in function spaces. It is shown that the tightness and the supertightness properties of $(C(X), \tau^s_{\mathfrak{B}})$ are interchangeable. The Id-fan tightness and the T-tightness of $(C(X), \tau^s_{\mathfrak{B}})$ are characterized in terms of bornological covering properties of X. We also study the concept of k-spaces from the bornological perspective. We show that whenever $(C(X), \tau^s_{\mathfrak{B}})$ is a k-space it is equivalent to a selection principle related to bornological covers of X. We introduce the notions of strong $\mathfrak{B}$-open game and $\gamma_{\mathfrak{B}^s}$-open game on X and obtain their consonances with other classical games on X. We investigate discretely selective property and associated games. Under certain condition on $\mathfrak{B}$, it is shown that $(C(X), \tau^s_{\mathfrak{B}})$ is discretely selective. Several interactions between topological games on $(C(X), \tau^s_{\mathfrak{B}})$ related to discretely selective property, the Gruenhage game on $(C(X), \tau^s_{\mathfrak{B}})$ and certain games on X are also presented.

5.2 VARIATIONS ON TIGHTNESS OF $(C(X), \tau^s_{\mathfrak{B}})$

5.2.1 CERTAIN OBSERVATIONS ON TIGHTNESS, SUPERTIGHTNESS AND ID-FAN TIGHTNESS

In this section we consider the tightness property of $(C(X), \tau^s_{\mathfrak{B}})$ and some of its variations such as the supertightness and the Id-fan tightness. We now recall the following definitions

> ***Tightness*** *$t(X)$ of X is the smallest infinite cardinal $\mathfrak{m}$ such that if $A \subseteq X$ and $x \in \overline{A}$, then there exists a $B \subseteq A$ satisfying $x \in \overline{B}$ and $|B| \leq \mathfrak{m}$.*

When $t(X) = \omega$, it is called countable tightness (CT) . The tightness of $(C(X), \tau^s_{\mathfrak{B}})$ is denoted by $t(\tau^s_{\mathfrak{B}})$ [50].

> ***Lindelöf number*** *$l(X)$ of X is the smallest infinite cardinal $\mathfrak{m}$ such that every open cover of X has a subcover with cardinality less than or equal to $\mathfrak{m}$.*

The Lindelöf number of $(C(X), \tau^s_{\mathfrak{B}})$ is denoted by $l(\tau^s_{\mathfrak{B}})$. For a bornology $\mathfrak{B}$, $L^s(X, \mathfrak{B})$ is the smallest cardinal $\mathfrak{m}$ for which every open $\mathfrak{B}^s$-cover of X has an open $\mathfrak{B}^s$-subcover with cardinality less than or equal to $\mathfrak{m}$ [50].

> *The **supertightness** $st(x, X)$ of x is the smallest cardinal $\mathfrak{m}$ for which every π-network $\mathcal{A}$ at x consisting of finite subsets of X has a subfamily $\mathcal{G}$ of cardinality less than or equal to $\mathfrak{m}$ which is a π-network at x.*

$st(X) = \omega \cdot \sup\{st(x, X) : x \in X\}$ [66] (see also [88]) is defined as the supertightness $st(X)$ of X. When $st(X) = \omega$, it is called *countable supertightness*. For simplicity we denote the supertightness of $(C(X), \tau^s_{\mathfrak{B}})$ by $st(\tau^s_{\mathfrak{B}})$ and countable supertightness by CST .

It is known that for a bornology $\mathfrak{B}$ with closed base on a metric space X $t(\tau^s_{\mathfrak{B}}) = L^s(X, \mathfrak{B})$ [50, Theorem 3.5] holds. Our first result shows that in the product space X^n endowed with the product bornology $\mathfrak{B}^n$ with a compact base $t(\tau^s_{\mathfrak{B}}) = L^s(X^n, \mathfrak{B}^n)$ holds for all $n \in \mathbb{N}$. For a similar result for $C_p(X)$ see [72, Corollary 4.7.3].

Theorem 5.2.1. *If $\mathfrak{B}$ is a bornology with compact base on a metric space X, then for all $n \in \mathbb{N}$ $t(\tau^s_{\mathfrak{B}}) = L^s(X^n, \mathfrak{B}^n)$.*

Proof. Let $t(\tau^s_{\mathfrak{B}}) = \mathfrak{m}$. Let $\mathcal{U}$ be an open $(\mathfrak{B}^n)^s$-cover of X^n. There is an open $\mathfrak{B}^s$-cover $\mathcal{V}$ of X such that $\{V^n : V \in \mathcal{V}\}$ refines $\mathcal{U}$ by Lemma 3.5.1. For $B \in \mathfrak{B}$ choose a $V_B \in \mathcal{V}$ and a $\delta > 0$ such that $B^{2\delta} \subseteq V_B$. Choose a $f_{B,V_B} \in C(X)$ satisfying $f_{B,V_B}(B^\delta) = \{0\}$ and $f_{B,V_B}(X \setminus V_B) = \{1\}$. Clearly $\underline{0} \in \overline{\{f_{B,V_B} : B \in \mathfrak{B}\}}$. As $t(\tau^s_{\mathfrak{B}}) = \mathfrak{m}$, there is a $\mathfrak{B}' \subseteq \mathfrak{B}$ with $|\mathfrak{B}'| \leq \mathfrak{m}$ such that $\underline{0} \in \overline{\{f_{B,V_B} : B \in \mathfrak{B}'\}}$. Let $\mathcal{W} = \{V_B : B \in \mathfrak{B}'\}$. For each V_B choose a $U_B \in \mathcal{U}$ such that $V_B^n \subseteq U_B$. We claim that $\{U_B : B \in \mathfrak{B}'\}$ is an open $(\mathfrak{B}^n)^s$-cover of X^n. Let $B^n \in \mathfrak{B}^n$ for $B \in \mathfrak{B}$. Choose a $f_{B_1,V_{B_1}} \in [B,1]^s(\underline{0}) \cap \{f_{B,V_B} : B \in \mathfrak{B}'\}$. Now $f_{B_1,V_{B_1}} \in [B,1]^s(\underline{0})$ implies $B^\delta \subseteq f^{-1}_{B_1,V_{B_1}}(-1,1)$ for some $\delta > 0$ and hence $B^\delta \subseteq V_{B_1}$. Clearly $(B^\delta)^n \subseteq V^n_{B_1}$ and $(B^n)^\delta \subseteq U_{B_1}$. This proves our claim. Hence $L^s(X^n, \mathfrak{B}^n) \leq t(\tau^s_{\mathfrak{B}})$.

Conversely, let $L^s(X^n, \mathfrak{B}^n) = \mathfrak{m}$. Let $A \subseteq C(X)$ with $\underline{0} \in \overline{A}$. By Lemma 2.4.1, $\mathcal{U}_m = \{f^{-1}(-\frac{1}{m}, \frac{1}{m}) : f \in A\} \setminus \{X\}$ is an open $\mathfrak{B}^s$-cover of X. Let $\mathcal{V}_m = \{f^{-1}(-\frac{1}{m}, \frac{1}{m})^n : f \in A\}$. Clearly $\mathcal{V}_m$ is an open $(\mathfrak{B}^n)^s$-cover of X^n. As $L^s(X^n, \mathfrak{B}^n) = \mathfrak{m}$, there is an $A_m \subseteq A$ with $|A_m| \leq \mathfrak{m}$ such that $\{f^{-1}(-\frac{1}{m}, \frac{1}{m})^n : f \in A_m\}$ is an open $(\mathfrak{B}^n)^s$-cover. If $A' = \cup_{m \in \mathbb{N}} A_m$, then it follows that $\underline{0} \in \overline{A'}$ with $|A'| \leq \mathfrak{m}$. Hence $t(\tau^s_{\mathfrak{B}}) \leq L^s(X^n, \mathfrak{B}^n)$. □

For $C_p(X)$ Sakai showed that $st(C_p(X)) = t(C_p(X))$ [88, Theorem 2.3] and $l(C_p(X)) \geq st(X^n)$ for each $n \in \mathbb{N}$ [88, Theorem 2.1]. We now present similar observations in the bornological setting.

Theorem 5.2.2. *If $\mathfrak{B}$ is a bornology with closed base on a metric space X, then $st(\tau^s_{\mathfrak{B}}) = t(\tau^s_{\mathfrak{B}})$.*

Proof. Let $t(\tau^s_{\mathfrak{B}}) = \mathfrak{m}$. Let $\mathcal{A}$ be a π-network at $\underline{0}$ consisting of finite subsets of $C(X)$. For each n and $A \in \mathcal{A}$, let $U_n(A) = \cap_{f \in A} f^{-1}(-\frac{1}{n}, \frac{1}{n})$ and let $F_n = \{f \in C(X) : f(X \setminus U_n(A)) = \{1\}$ for some $A \in \mathcal{A}\}$. We first show that $\underline{0} \in \overline{F_n}$. Let $[B,\varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. Choose a $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. There is an $A \in \mathcal{A}$ with $A \subseteq [B, \frac{1}{n}]^s(\underline{0})$. Clearly for every $f \in A$ there is a $\delta_f > 0$ such that $B^{\delta_f} \subseteq f^{-1}(-\frac{1}{n}, \frac{1}{n})$. Choose $\delta = \min\{\delta_f : f \in A\}$. We now have $B^\delta \subseteq U_n(A)$. Choose a $f \in C(X)$ satisfying $f(B^{\frac{\delta}{2}}) = \{0\}$ and $f(X \setminus U_n(A)) = \{1\}$. Clearly $f \in F_n \cap [B,\varepsilon]^s(\underline{0})$ and hence $\underline{0} \in \overline{F_n}$. As $t(\tau^s_{\mathfrak{B}}) = \mathfrak{m}$, there is a $G_n \subseteq F_n$ with $|G_n| \leq \mathfrak{m}$ such that $\underline{0} \in \overline{G_n}$. For each $g \in G_n$ there exists an $A_g \in \mathcal{A}$ satisfying $g(X \setminus U_n(A_g)) = \{1\}$. Consider $\mathcal{A}_n = \{A_g : g \in G_n\}$. Choose $\mathcal{G} = \cup_{n \in \mathbb{N}} \mathcal{A}_n$. We now show that $\mathcal{G}$ is a π-network at $\underline{0}$. Let $[B,\varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. Choose a $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. As $\underline{0} \in \overline{G_n}$, choose a $g \in [B, \frac{1}{n}]^s(\underline{0}) \cap G_n$. Now $B^\delta \subseteq g^{-1}(-\frac{1}{n}, \frac{1}{n})$ for some $\delta > 0$. Clearly $B^\delta \subseteq U_n(A_g)$ and $B^\delta \subseteq \cap_{f \in A_g} f^{-1}(-\frac{1}{n}, \frac{1}{n})$. Subsequently $f \in [B, \frac{1}{n}]^s(\underline{0})$ for all $f \in A_g$ and $A_g \subseteq [B,\varepsilon]^s(\underline{0})$. Therefore $\mathcal{G}$ is a π-network at $\underline{0}$ of cardinality less than or equal to $\mathfrak{m}$. Hence $st(\tau^s_{\mathfrak{B}}) \leq t(\tau^s_{\mathfrak{B}})$.

Conversely, let $st(\tau^s_{\mathfrak{B}}) = \mathfrak{m}$. Let $A \subseteq C(X)$ with $\underline{0} \in \overline{A}$. Let $\mathcal{A} = \{\{f\} : f \in A\}$. Clearly $\mathcal{A}$ is a π-network at $\underline{0}$. There is a subfamily $\mathcal{G}$ of $\mathcal{A}$ with $|\mathcal{G}| \leq \mathfrak{m}$ such that $\mathcal{G}$ is a π-network at $\underline{0}$. Choose $B = \{f : \{f\} \in \mathcal{G}\}$. Then $\underline{0} \in \overline{B}$ with $|B| \leq \mathfrak{m}$. Hence $t(\tau^s_{\mathfrak{B}}) \leq st(\tau^s_{\mathfrak{B}})$. □

Note that $(C(X), \tau^s_{\mathfrak{B}})$ has CT if and only if X is $\mathfrak{B}^s$-Lindelöf [19, Theorem 2.1].

Corollary 5.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Then the following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *is* CST.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *is* CT.
(3) *X is $\mathfrak{B}^s$-Lindelöf.*

Corollary 5.2.2. *For a Tychonoff space X $st(C_k(X)) = t(C_k(X))$ holds.*

Theorem 5.2.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements hold.*
(1) $l(\tau^s_{\mathfrak{B}}) \geq st(X^n)$ *for all $n \in \mathbb{N}$.*
(2) $l(\tau^s_{\mathfrak{B}}) \geq t(X^n)$ *for all $n \in \mathbb{N}$.*

Proof. We prove only (1). Let $l(\tau^s_{\mathfrak{B}}) = \mathfrak{m}$. Let $(x_1, \ldots, x_n) \in X^n$ and $\mathcal{A}$ be a π-network at $(x_1, \ldots, x_n)$ consisting of finite subsets of X^n. We need to show that there is a subfamily $\mathcal{G}$ of $\mathcal{A}$ with $|\mathcal{G}| \leq \mathfrak{m}$ which is a π-network at $(x_1, \ldots, x_n)$. Assume the contrary.

Let $F = \{x_1, \ldots, x_n\}$ and $S = \{f \in C(X) : f(F) = \{0\}\}$. Since S is closed in $(C(X), \tau^s_{\mathfrak{B}})$, $l(S) \leq \mathfrak{m}$. For each $A \in \mathcal{A}$ choose a finite set $B_A \in \mathfrak{B}$ such that $A = \{(y_1, \ldots, y_n) : y_1, \ldots, y_n \in B_A\}$. We claim that $S \subseteq \cup\{[B_A, 1]^s(\underline{0}) : A \in \mathcal{A}\}$. Let $f \in S$. As $f(F) = \{0\}$, $f^{-1}(-1,1)^n$ is a neighbourhood of $(x_1, \ldots, x_n)$. There is an $A \in \mathcal{A}$ such that $A \subseteq f^{-1}(-1,1)^n$ and hence $B_A \subseteq f^{-1}(-1,1)$. As B_A is finite, choose a $\delta > 0$ such that $B_A^\delta \subseteq f^{-1}(-1,1)$. Hence $f \in [B_A, 1]^s(\underline{0})$. This proves our claim. Since $l(S) = \mathfrak{m}$, there is a subfamily $\mathcal{G}$ of $\mathcal{A}$ with $|\mathcal{G}| \leq \mathfrak{m}$ such that $S \subseteq \cup\{[B_A, 1]^s(\underline{0}) : A \in \mathcal{G}\}$. By our assumption, $\mathcal{G}$ is not a π-network at $(x_1, \ldots, x_n)$. Therefore there is a neighbourhood $U_1 \times \cdots \times U_n$ (say) of $(x_1, \ldots, x_n)$ such that $A \nsubseteq U_1 \times \cdots \times U_n$ for any $A \in \mathcal{G}$. Let $V = U_1 \cap \cdots \cap U_n$. Choose an $g \in C(X)$ satisfying $g(F) = \{0\}$ and $g(X \setminus V) = \{1\}$. Clearly $g \in S$. Now $A \nsubseteq U_1 \times \cdots \times U_n$ for any $A \in \mathcal{G}$ implies that $B_A \nsubseteq V$ for any $A \in \mathcal{G}$. Therefore $g(x) = 1$ for some $x \in B_A$. Thus $g \notin [B_A, 1]^s(\underline{0})$ for any $A \in \mathcal{G}$. This contradicts that $S \subseteq \cup\{[B_A, 1]^s(\underline{0}) : A \in \mathcal{G}\}$. Hence our assumption is false. □

Corollary 5.2.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements hold.*
(1) *X^n has* CST *for all $n \in \mathbb{N}$ if $(C(X), \tau^s_{\mathfrak{B}})$ is Lindelöf.*
(2) *X^n has* CT *for all $n \in \mathbb{N}$ if $(C(X), \tau^s_{\mathfrak{B}})$ is Lindelöf.*

Corollary 5.2.4. *For a Tychonoff space X the following statements holds.*
(1) $l(C_k(X)) \geq st(X^n)$ *for all* $n \in \mathbb{N}$.
(2) $l(C_k(X)) \geq t(X^n)$ *for all* $n \in \mathbb{N}$.

We now recall the following definition.

> *X has* ***Id-fan tightness*** *if for every sequence* $\{A_n : n \in \mathbb{N}\}$ *and for every* $x \in X$, *where* A_n *is a subsets of X with* $x \in \overline{A_n}$, *there is an* $F_n \subseteq A_n$ *for each n with* $|F_n| \leq n$ *satisfying* $x \in \overline{\cup\{F_n : n \in \mathbb{N}\}}$ *[43].*

For convenience we denote this property by IdFT . The following implications immediately follow. $\mathsf{CSFT_{fin}} \Rightarrow \mathsf{CSFT} \Rightarrow \mathsf{IdFT} \Rightarrow \mathsf{CFT}$. In [43, Theorem 3.13], it is shown that the IdFT of $C_p(X)$ is equivalent to the fact that every finite power of X has the C'' property. We now present a characterization of IdFT of $(C(X), \tau^s_{\mathfrak{B}})$ in terms of the selection principle $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ of X. We first need the following lemma.

Lemma 5.2.1. *Let* $\mathfrak{B}$ *be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.
(2) *X satisfies* $\mathsf{S}_{\mathsf{id}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.
(3) *There is an* $h \in \mathbb{N}^{\mathbb{N}}$ *such that X satisfies* $\mathsf{S}_{\mathsf{h}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is follows easily.

(3) $\Rightarrow$ (1). Assume $h \in \mathbb{N}^{\mathbb{N}}$ to be strictly increasing for which (3) holds. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Define $\mathcal{V}_1 = \mathcal{U}_1 \wedge \cdots \wedge \mathcal{U}_{h(1)}$ and $\mathcal{V}_n = \mathcal{U}_{h(n-1)+1} \wedge \cdots \wedge \mathcal{U}_{h(n)}$ for $n > 1$. By Lemma 2.2.1, $\mathcal{V}_n$ is an open $\mathfrak{B}^s$-cover of X. Also there is a $\mathcal{W}_n \subseteq \mathcal{V}_n$ with $|\mathcal{W}_n| \leq h(n)$ such that $\cup\{\mathcal{W}_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Let $\mathcal{W}_1 = \{V_i : i \leq h(1)\}$ and $\mathcal{W}_n = \{V_{h(n-1)+1}, \ldots, V_{h(n)}\}$ for $n > 1$. Now for each $i \leq h(1)$ there is a $U_i \in \mathcal{U}_i$ satisfying $V_i \subseteq U_i$. Also for each $n, i \in \mathbb{N}$ with $n > 1$ and $i \leq h(n) - h(n-1)$ there exists a $U_{h(n-1)+i} \in \mathcal{U}_{h(n-1)+i}$ satisfying $V_{h(n-1)+i} \subseteq U_{h(n-1)+i}$. Thus $\{U_i : i \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X and (1) holds. □

Theorem 5.2.4. *Let* $\mathfrak{B}$ *be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *has* CSFT.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *has* $\mathsf{CSFT_{fin}}$.
(3) $(C(X), \tau^s_{\mathfrak{B}})$ *has* IdFT.
(4) *X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) are from [29, Theorem 3.4]. The implication (1) $\Rightarrow$ (3) is easily followed.

(3) $\Rightarrow$ (4). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. For each $n \in \mathbb{N}$ and $B \in \mathfrak{B}$ there exist a $\delta > 0$ and a $U \in \mathcal{U}_n$ such that $B^{2\delta} \subseteq U$. Let $\mathcal{U}_{n,B} = \{U \in \mathcal{U}_n : B^{2\delta} \subseteq U\}$. Choose a $f_{B,U} \in C(X)$ satisfying $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. Let $A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{n,B}\}$. Clearly $\underline{0} \in \overline{A_n}$. By (3), there is a $C_n \subseteq A_n$ with $|C_n| \leq n$ satisfying $\underline{0} \in \overline{\cup\{C_n : n \in \mathbb{N}\}}$. Choose $\mathcal{V}_n = \{U \in \mathcal{U}_n : f_{B,U} \in C_n\}$. Observe that $|\mathcal{V}_n| \leq n$. Let $B \in \mathfrak{B}$. For the neighbourhood $[B,1]^s(\underline{0})$ of $\underline{0}$, we have $[B,1]^s(\underline{0}) \cap C_n \neq \emptyset$ for some n and $B^\delta \subseteq f^{-1}_{B,U}(-1,1)$ for some $f_{B,U} \in C_n$ and $\delta > 0$. Clearly $B^\delta \subseteq U$ for some $U \in \mathcal{V}_n$. Therefore $\cup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. By Lemma 5.2.1, X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

□

It is known that $C_k(X)$ has CSFT (respectively, $\mathsf{CSFT}_{\mathsf{fin}}$) if and only if X satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$ [61, Theorem 2.2], [29, Corollary 3.3]. We now have the following.

Corollary 5.2.5. *For a Tychonoff space X the following statements are equivalent.*
(1) $C_k(X)$ *has* CSFT.
(2) $C_k(X)$ *has* $\mathsf{CSFT}_{\mathsf{fin}}$.
(3) $C_k(X)$ *has* IdFT.
(4) X *satisfies* $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$.

5.2.2 ON T-TIGHTNESS OF $(C(X), \tau^s_{\mathfrak{B}})$

The T-tightness property is defined as follows.

> *The* ***T-tightness*** $T(X)$ *is the smallest infinite cardinal* $\mathfrak{m}$ *such that if* $\{F_\alpha : \alpha < \kappa\}$ *is an increasing family of closed subsets and* κ *is a regular cardinal greater than* $\mathfrak{m}$, *then* $\cup\{F_\alpha : \alpha < \kappa\}$ *is closed in X [53, 89].*

For a Tychonoff topological space X, Sakai characterized the T-tightness of $C_p(X)$ in terms of a covering property of X^n, $n \in \mathbb{N}$ (see [89, Theorem 2.3]). In this section we present a characterization of the T-tightness of $(C(X), \tau^s_{\mathfrak{B}})$ in terms of a bornological covering property of X^n endowed with the product bornology $\mathfrak{B}^n$, $n \in \mathbb{N}$. For two infinite cardinals $\mathfrak{a}, \mathfrak{b}$ with $\mathfrak{a} \leq \mathfrak{b}$ by $[\mathfrak{a}, \mathfrak{b}]$ we denote the collection of all cardinals $\mathfrak{m}$ with $\mathfrak{a} \leq \mathfrak{m} \leq \mathfrak{b}$. The symbol $\mathfrak{m}^+$ denotes the successor cardinal of $\mathfrak{m}$. We first introduce the following notion.

Definition 5.2.1. A space X is called a $[\mathfrak{a}, \mathfrak{b}]_r$-$\mathfrak{B}^s$ space if every open $\mathfrak{B}^s$-cover of X whose cardinality is a regular cardinal $\mathfrak{m} \in [\mathfrak{a}, \mathfrak{b}]$ has an open $\mathfrak{B}^s$-subcover of cardinality less than $\mathfrak{m}$.

The superscript r denotes the restriction to regular cardinals. A space X is called a $[\mathfrak{a}, \infty]_r$-$\mathfrak{B}^s$ space if for all $\mathfrak{b} \geq \mathfrak{a}$ X is a $[\mathfrak{a}, \mathfrak{b}]_r$-$\mathfrak{B}^s$ space. When $L^s[X, \mathfrak{B}] \leq \mathfrak{m}$, X is $[\mathfrak{m}^+, \infty]_r$-$\mathfrak{B}^s$ space. In line of the definition of $T(\tau)$ [89], we introduce the following property.

Definition 5.2.2. Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. X satisfies the property $T(\mathfrak{m}, \mathfrak{B})$ if for a regular cardinal $\kappa > \mathfrak{m}$ $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ is an increasing family of open sets so that $\cup_{\alpha<\kappa}\mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X, then there exists a $\alpha < \kappa$ such that $\mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X.

Theorem 5.2.5. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X. The following statements are equivalent.*
(1) *X is a $[\mathfrak{a}, \mathfrak{b}]_r$-$\mathfrak{B}^s$ space.*
(2) *Every decreasing sequence $\{F_\alpha : \alpha < \mathfrak{m}\}$ of nonempty closed subsets of X, where $\mathfrak{m} \in [\mathfrak{a}, \mathfrak{b}]$ is regular, has nonempty intersection.*

Proof. (1) $\Rightarrow$ (2). Let $\mathfrak{m} \in [\mathfrak{a}, \mathfrak{b}]$ be regular. Let $\{F_\alpha : \alpha < \mathfrak{m}\}$ be a decreasing sequence of nonempty closed subsets of X. Define $U_\alpha = X \setminus F_\alpha$. Clearly $U_\alpha \neq X$ for all $\alpha < \mathfrak{m}$. We show that $\cap_{\alpha<\mathfrak{m}}F_\alpha \neq \emptyset$. Assume that $\cap_{\alpha<\mathfrak{m}}F_\alpha = \emptyset$. Observe that $\cap_{\alpha<\mathfrak{m}}(X \setminus U_\alpha) = \emptyset$ and $X = \cup_{\alpha<\mathfrak{m}}U_\alpha$. Let $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{m}\}$ and choose $B \in \mathfrak{B}_0$. Since B is compact and $B \subseteq \cup_{\alpha<\mathfrak{m}}U_\alpha$ and U_α's are increasing, there is a U_α such that $B \subseteq U_\alpha$. Moreover, there is a $\delta > 0$ such that $B^\delta \subseteq U_\alpha$. Hence $\mathcal{U}$ is an open $\mathfrak{B}^s$-cover of X. By (1), there exists a $\tau < \mathfrak{m}$ for which $\{U_\alpha : \alpha < \tau\}$ is an open $\mathfrak{B}^s$-cover of X. As U_α's are increasing and $\cup_{\alpha<\tau}U_\alpha = X$, we have $U_\tau = X$. This is a contradiction as $U_\tau \neq X$. Hence our assumption is false.

(2) $\Rightarrow$ (1). Let $\mathfrak{m} \in [\mathfrak{a}, \mathfrak{b}]$ be regular. Let $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{m}\}$ be an open $\mathfrak{B}^s$-cover of X which has no $\mathfrak{B}^s$-subcover of cardinality less than $\mathfrak{m}$. Let $\mathcal{U}_\beta = \{U_\alpha : \alpha < \beta\}$. Clearly $\mathcal{U}_\beta \subset \mathcal{U}_\tau$ whenever $\beta < \tau$. Define $F_\beta = X \setminus \cup\mathcal{U}_\beta$. Clearly the sequence $\{F_\beta : \beta < \mathfrak{m}\}$ is decreasing and each F_β is nonempty closed subset of X. We have $\cup_{\beta<\mathfrak{m}}(\cup\mathcal{U}_\beta) = X$ and $\cap_{\beta<\mathfrak{m}}(X \setminus \cup\mathcal{U}_\beta) = \emptyset$. Thus $\cap_{\beta<\mathfrak{m}}F_\beta = \emptyset$. This contradicts (2). Hence there is a $\beta < \mathfrak{m}$ such that $\mathcal{U}_\beta$ is an open $\mathfrak{B}^s$-subcover of X. □

Theorem 5.2.6. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X. The following statements are equivalent.*
(1) *X satisfies $T(\mathfrak{m}, \mathfrak{B})$.*
(2) *X^n satisfies $T(\mathfrak{m}, \mathfrak{B}^n)$ for any $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2). Let $\kappa > \mathfrak{m}$ be regular. Let $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ be a sequence of families of open sets with $\mathcal{U}_\alpha \subset \mathcal{U}_\beta$ for $\alpha < \beta$ so that $\mathcal{U} = \cup_{\alpha<\kappa}\mathcal{U}_\alpha$ is an open $(\mathfrak{B}^n)^s$-cover of X^n. By Lemma 3.5.1, there is an open $\mathfrak{B}^s$-cover $\mathcal{V}$ of X such that $\{V^n : V \in \mathcal{V}\}$ refines $\mathcal{U}$. Let $\mathcal{V}_\alpha = \{V \in \mathcal{V} : V^n \subseteq U$ for some $U \in \mathcal{U}_\alpha\}$. Clearly $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$ whenever $\alpha < \beta$. By (1), there is a $\alpha < \kappa$ such that $\mathcal{V}_\alpha$ is an open $\mathfrak{B}^s$-cover of X. We now show that $\mathcal{U}_\alpha$ is an open $(\mathfrak{B}^n)^s$-cover of X. Let $B^n \in \mathfrak{B}^n$ for some $B \in \mathfrak{B}$. Choose a $V \in \mathcal{V}_\alpha$ and a $\delta > 0$ such that $B^\delta \subseteq V$. Clearly $(B^\delta)^n \subseteq V^n$ and so $(B^n)^\delta \subseteq U$ for some $U \in \mathcal{U}_\alpha$ with $V^n \subseteq U$. Hence $\mathcal{U}_\alpha$ is an open $(\mathfrak{B}^n)^s$-cover of X^n and (2) holds.

(2) $\Rightarrow$ (1). Let $\mathcal{U} = \cup_{\alpha<\kappa}\mathcal{U}_\alpha$ be an open $\mathfrak{B}^s$-cover of X and $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ whenever $\alpha < \beta$. Let $\mathcal{V}_\alpha = \{U^n : U \in \mathcal{U}_\alpha\}$. Now $\mathcal{V} = \cup_{\alpha<\kappa}\mathcal{V}_\alpha$ is an open $(\mathfrak{B}^n)^s$-cover of X^n. By (2), there is a $\alpha < \kappa$ such that $\mathcal{V}_\alpha$ is an open $(\mathfrak{B}^n)^s$-cover of X^n. Thus $\mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X. □

Theorem 5.2.7. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X. The following statements are equivalent.*
(1) *X satisfies $T(\mathfrak{m}, \mathfrak{B})$.*
(2) *X^n is a $[\mathfrak{m}^+, \infty]_r$-$(\mathfrak{B}^n)^s$ space for any $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2). We first show that X is a $[\mathfrak{m}^+, \infty]_r$-$\mathfrak{B}^s$ space. Let $\kappa > \mathfrak{m}$ be regular and $\{F_\alpha : \alpha < \kappa\}$ be a decreasing sequence of nonempty closed subsets of X. Let $U_\alpha = X \setminus F_\alpha$ and $\mathcal{U}_\beta = \{U_\alpha : \alpha < \beta\}$. Clearly $\mathcal{U}_\beta$'s are increasing. Assume that $\cap_{\alpha<\kappa}F_\alpha = \emptyset$. Clearly $\cap_{\alpha<\kappa}(X \setminus U_\alpha) = \emptyset$ and $X = \cup_{\alpha<\kappa}U_\alpha$. Let $\mathcal{U} = \cup_{\beta<\kappa}\mathcal{U}_\beta$. Now $\mathcal{U}$ is an open $\mathfrak{B}^s$-covers of X. By (1), there is a $\beta < \kappa$ such that $\mathcal{U}_\beta$ is an open $\mathfrak{B}^s$-cover of X. Since U_α's are increasing, we have $U_\beta = X$ and $F_\beta = \emptyset$. This contradicts that $F_\beta \neq \emptyset$. Therefore $\cap_{\alpha<\kappa}F_\alpha \neq \emptyset$. Hence X is a $[\mathfrak{m}^+, \infty]_r$-$\mathfrak{B}^s$ space by Theorem 5.2.5. Also by Theorem 5.2.6, X^n satisfies $T(\mathfrak{m}, \mathfrak{B}^n)$ for any $n \in \mathbb{N}$. Repeating the above procedure, it is easy to see that X^n is a $[\mathfrak{m}^+, \infty]_r$-$(\mathfrak{B}^n)^s$ space for $n > 1$.

(2) $\Rightarrow$ (1). Let $\kappa > \mathfrak{m}$ be regular. Let $\mathcal{U} = \cup_{\alpha<\kappa}\mathcal{U}_\alpha$ be an open $\mathfrak{B}^s$-cover of X, where $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ if $\alpha < \beta$. Fix a $n \in \mathbb{N}$. Let $\mathcal{V}_\alpha = \{U^n : U \in \mathcal{U}_\alpha\}$. The collection $\mathcal{V} = \cup_{\alpha<\kappa}\mathcal{V}_\alpha$ is an open $(\mathfrak{B}^n)^s$-cover of X^n such that $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$ if $\alpha < \beta$. By (2), there is a $(\mathfrak{B}^n)^s$-subcover of $\mathcal{V}$ of cardinality less than κ. As $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$ whenever $\alpha < \beta$, let $\mathcal{V}_\tau$ be that $(\mathfrak{B}^n)^s$-subcover of $\mathcal{V}$, where $\tau < \kappa$. Clearly $\mathcal{U}_\tau$, $\tau < \kappa$, is an open $\mathfrak{B}^s$-cover of X. Hence (1) holds. □

Theorem 5.2.8. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X. The following statements are equivalent.*
(1) *$T(C(X), \tau^s_{\mathfrak{B}}) \leq \mathfrak{m}$.*
(2) *X^n is a $[\mathfrak{m}^+, \infty]_r$-$(\mathfrak{B}^n)^s$ space for any $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2). Let $T(C(X), \tau^s_{\mathfrak{B}}) \leq \mathfrak{m}$. We show that X satisfies $T(\mathfrak{m}, \mathfrak{B})$. To see this, let $\kappa > \mathfrak{m}$ be regular and $\cup_{\alpha<\kappa}\mathcal{U}_\alpha$ be an open $\mathfrak{B}^s$-cover of X such that $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ whenever $\alpha < \beta$. For $B \in \mathfrak{B}$ choose a $\delta > 0$ and a $U \in \cup_{\alpha<\kappa}\mathcal{U}_\alpha$ such that $B^{2\delta} \subseteq U$. Choose a $f_{B,U} \in C(X)$ satisfying $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. Consider $A_\alpha = \{f_{B,U} : U \in \mathcal{U}_\alpha, B \in \mathfrak{B}\}$ for each $\alpha < \kappa$. Clearly $\{\overline{A_\alpha} : \alpha < \kappa\}$ is an increasing sequence of closed subsets of $C(X)$. By (1), the set $\cup_{\alpha<\kappa}\overline{A_\alpha}$ is closed in $(C(X), \tau^s_{\mathfrak{B}})$. Using [19, Lemma 2.2] and the fact that $\cup_{\alpha<\kappa}\mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X, we obtain $\underline{0} \in \overline{\cup_{\alpha<\kappa}A_\alpha} \setminus \cup_{\alpha<\kappa}A_\alpha$. It is easy to see that $\underline{0} \in \cup_{\alpha<\kappa}\overline{A_\alpha}$. Thus $\underline{0} \in \overline{A_\alpha}$ for some $\alpha < \kappa$. Again by [19, Lemma 2.2], $\{f^{-1}_{B,U}(-1,1) : f_{B,U} \in A_\alpha\}$ is an open $\mathfrak{B}^s$-cover of X. Since $f^{-1}_{B,U}(-1,1) \subseteq U$, $\mathcal{U}_\alpha$ is an open $\mathfrak{B}^s$-cover of X. Hence X satisfies $T(\mathfrak{m}, \mathfrak{B})$. The result now follows from Theorem 5.2.7.

(2) $\Rightarrow$ (1). Let $\kappa > \mathfrak{m}$ be regular and let $\{A_\alpha : \alpha < \kappa\}$ be a sequence of closed subsets of $C(X)$ which are increasing. We show that $A = \cup_{\alpha<\kappa} A_\alpha$ is closed in $C(X)$. For this let $\underline{0} \in \overline{A}$. Consider the collection $\mathcal{U}_{\alpha,n} = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A_\alpha\}$ and $\mathcal{U}_n = \cup_{\alpha<\kappa}\mathcal{U}_{\alpha,n}$, $n \in \mathbb{N}$. Clearly $\mathcal{U}_n$ is an open $\mathfrak{B}^s$-cover of X. By Theorem 5.2.7, X satisfies $T(\mathfrak{m}, \mathfrak{B})$. Hence there exists a $\alpha_n < \kappa$ for each n such that $\mathcal{U}_{\alpha_n,n}$ is an open $\mathfrak{B}^s$-cover. Let $\beta = \sup\{\alpha_n : n \in \mathbb{N}\}$. Now it can be easily checked that $\mathcal{U}_{\beta,n}$ is an open $\mathfrak{B}^s$-cover of X for each n. We prove that $\underline{0} \in \overline{A_\beta}$. Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. Choose a $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} < \varepsilon$. Since $\mathcal{U}_{\beta,n_1}$ is an open $\mathfrak{B}^s$-cover of X, there exist a $f \in A_\beta$ and a $\delta > 0$ such that $B^\delta \subseteq f^{-1}(-\frac{1}{n_1}, \frac{1}{n_1})$. Thus $f \in [B, \varepsilon]^s(\underline{0}) \cap A_\beta$ and $\underline{0} \in \overline{A_\beta} = A_\beta$. Hence $\underline{0} \in A$. This completes the proof. $\square$

5.3 SOME OBSERVATIONS ON k-SPACES

The concept of k-space was studied in [70, 71] by R. A. McCoy in function spaces endowed with the point-open topology and the compact-open topology. For a Tychonoff space X it was known that $C_p(X)$ is a k-space if and only if X satisfies $\mathsf{S}_1(\Omega, \Gamma)$ (see [70, Theorem 1]). In this section we present some observations on k-spaces in the context of the topology of strong uniform convergence on $\mathfrak{B}$. Recall that

> *X is said to be a* ***k-space*** *if the closed subsets of X are precisely those subset A such that for each compact subset K of X, $A \cap K$ is closed in K [70].*

We first present the following lemma.

Lemma 5.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If $K \subseteq (C(X), \tau^s_{\mathfrak{B}})$ is compact then the set $\{f(x) : f \in K\}$ is bounded in $\mathbb{R}$ for every $x \in X$.*

Proof. Fix a $x_0 \in X$. Let $B = \{x_0\}$. Clearly $K \subseteq \cup\{[B, n]^s(\underline{0}) : n \in \mathbb{N}\}$. Since K is compact, there are $n_1, \ldots, n_k$ such that $K \subseteq \cup\{[B, n_i]^s(\underline{0}) : i = 1, \ldots, k\}$. Choose $m = \max\{n_1, \ldots, n_k\}$. Now for each $f \in K$, $|f(x_0)| < m$. Hence $\{f(x_0) : f \in K\}$ is bounded. $\square$

Recall that with respect to the topology of strong uniform convergence on a bornology $\mathfrak{B}$, $C(X)$ is FU if and only if X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ ([19, Corollary 2.10]). Thus we have the following.

Theorem 5.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *is a* k-*space.*
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *is* FU.
(3) *X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (3). We prove this by contrapositive argument. Suppose that X does not satisfy $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. Then we get a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X for which $\{U_n : n \in \mathbb{N}\}$ is not a $\gamma_{\mathfrak{B}^s}$-cover of X for any $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$. Construct a new sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X as follows. Define $\mathcal{V}_1 = \mathcal{U}_1$ and for $n > 1$ $\mathcal{V}_n$ is a refinement of both $\mathcal{V}_{n-1}$ and $\mathcal{U}_n$.

For each $B \in \mathfrak{B}$ and each $n \in \mathbb{N}$ there are a $V \in \mathcal{V}_n$ and a $\delta > 0$ satisfying $B^{2\delta} \subseteq V$. Let $\mathcal{V}_{n,B} = \{V \in \mathcal{V}_n : B^{2\delta} \subseteq V\}$. For every $V \in \mathcal{V}_{n,B}$ choose a continuous function $f_{B,V}$ from X to $[\frac{1}{n}, n]$ such that $f_{B,V}(B^\delta) = \{\frac{1}{n}\}$ and $f_{B,V}(X \setminus V) = \{n\}$. Let $F_n = \{f_{B,V} : B \in \mathfrak{B}, V \in \mathcal{V}_{n,B}\}$. Choose $F = \cup_{n\in\mathbb{N}} F_n$ and $F^* = F \setminus \{\underline{0}\}$. It can be easily seen that $\underline{0} \in \overline{F}$. Clearly F^* is not closed. We now show that $(C(X), \tau^s_{\mathfrak{B}})$ is not a k-space by showing that $F^* \cap K$ closed in K for each compact subset K of $(C(X), \tau^s_{\mathfrak{B}})$. Let K be a compact subset of $(C(X), \tau^s_{\mathfrak{B}})$. By Lemma 5.3.1, for every $x \in X$ the set $\{f(x) : f \in K\}$ is bounded. Define $M(x) = \sup\{f(x) : f \in K\}$. Consider $X_m = \{x \in X : M(x) \leq m\}$. Clearly $X_m \subseteq X_{m+1}$ for $m \in \mathbb{N}$ and the collection $\{X_m : m \in \mathbb{N}\}$ is a $\mathfrak{B}^s$-cover of X.

Suppose that for each $m, n \in \mathbb{N}$, there exist a $k \geq n$ and a $V \in \mathcal{V}_k$ such that $X_m \subseteq V$. We now use induction to choose $\{U_n : n \in \mathbb{N}\}$, where $U_n \in \mathcal{U}_n$. Choose a $k_1 \geq 1$ and a $V_1 \in \mathcal{V}_{k_1}$ such that $X_1 \subseteq V_1$. By construction of $\mathcal{V}_{k_1}$, for each $i \in \{1, \ldots, k_1\}$ choose a $U_i \in \mathcal{U}_i$ such that $V_1 \subseteq U_i$. Suppose that k_m and $U_1, \ldots, U_{k_m}$ have been chosen. Now choose a $k_{m+1} \geq k_m + 1$ and a $V_{m+1} \in \mathcal{V}_{k_{m+1}}$ such that $X_{m+1} \subseteq V_{m+1}$. For each $i \in \{k_m + 1, \ldots, k_{m+1}\}$ choose a $U_i \in \mathcal{U}_i$ such that $V_{m+1} \subseteq U_i$. Proceeding in this way we obtain a sequence $\{U_n : n \in \mathbb{N}\}$, where $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$, which is not a $\gamma_{\mathfrak{B}^s}$-cover of X by our assumption. Let $B \in \mathfrak{B}$. There are X_m and $\delta > 0$ such that $B^\delta \subseteq X_m$. For $n \geq k_m$ choose a $j \geq m$ with $k_{j-1} + 1 \leq n \leq k_j$ and define $\delta_n = \delta$. Clearly $B^{\delta_n} \subseteq X_m \subseteq X_j \subseteq V_j \subset U_n$. Thus $B^{\delta_n} \subseteq U_n$ for all $n \geq k_m$. This shows that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover which is a contradiction. Hence there exist $m_0, n_0 \in \mathbb{N}$ such that for any $k \geq n_0$ and any $V \in \mathcal{V}_k$, $X_{m_0} \nsubseteq V$.

Choose $N = \max\{m_0, n_0\}$. Consider the neighbourhood $[B, \frac{1}{N}]^s(\underline{0})$. Let $f \in [B, \frac{1}{N}]^s(\underline{0}) \cap F$. Clearly $f \in F_k$ for some $k \in \mathbb{N}$ and $f = f_{B,V}$, where $V \in \mathcal{V}_{k,B}$. Now $\frac{1}{k} \leq f_{B,V}(x) \leq \frac{1}{N}$ for all $x \in B^\delta$, where $\delta > 0$. Hence $k > N \geq n_0$ and $X_{m_0} \nsubseteq V$ for any $V \in \mathcal{V}_k$. Let $x \in X_{m_0} \setminus V$. Observe that $f_{B,V}(x) = k$. Also since $k > N \geq m_0$ and $x \in X_{m_0}$, we have $k \geq M(x)$. Therefore $f_{B,V} \notin K$ and $[B, \frac{1}{N}]^s(\underline{0}) \cap F \cap K = \emptyset$. This shows that $\underline{0}$ is not a limit point of $F^* \cap K$ and so $F^* \cap K$ is closed in K. Thus $(C(X), \tau^s_{\mathfrak{B}})$ is not a k-space. $\square$

In combination with Theorem 2.2.2, Proposition 3.2.4, Theorem 4.2.6 and [19, Corollary 2.10], the following corollary is obtained.

Corollary 5.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Then the following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ *is* FU.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ *is* SFU.

(3) $(C(X), \tau^s_{\mathfrak{B}})$ *is* $\mathsf{FU}_{\mathsf{fin}}$.
(4) $(C(X), \tau^s_{\mathfrak{B}})$ *is a* k*-space.*
(5) X *satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.
(6) X *satisfies* $\mathsf{S}_{\mathsf{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.
(7) X *satisfies* $\binom{\mathcal{O}_{\mathfrak{B}^s}}{\Gamma_{\mathfrak{B}^s}}$.
(8) X *satisfies* $\mathsf{Split}(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Corollary 5.3.2 (see [71, Theorem 1])**.** *Let X be a Tychonoff space. The following statements are equivalent.*
(1) $C_k(X)$ *is a* k*-space.*
(2) $C_k(X)$ *is* FU.
(3) X *satisfies* $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$.

As an immediate application of the preceding result, we present the following examples.

Example 5.3.1. Let $X = \mathbb{R}$ and $\mathfrak{B}$ be a bornology generated by $\{(-x, x) : x > 0\}$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. For each $n \in \mathbb{N}$, choose a $U_n \in \mathcal{U}_n$ such that $(-n, n) \subseteq U_n$. It is easy to see that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. By Theorem 5.3.1, $(C(X), \tau^s_{\mathfrak{B}})$ is a k-space.

Example 5.3.2. Let $X = \mathbb{N}^{\mathbb{N}}$ with the Baire metric ρ and the bornology $\mathfrak{B} = \mathcal{F}$. We first show that X does not satisfy $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. To see this, consider $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$, where $U^n_m = \{f \in X : f(n) \leq m\}$. Let there exist a $U^n_{m_n} \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ such that $\{U^n_{m_n} : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Choose an $h \in X$ such that $h(n) = m_n + 1$ for each $n \in \mathbb{N}$ and a $B \in \mathfrak{B}$ with $h \in B$. Since $\{U^n_{m_n} : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X, for $B \in \mathfrak{B}$ there exist a $n_0 \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq n_0\}$ of positive real numbers satisfying $B^{\delta_n} \subseteq U^n_{m_n}$ for all $n \geq n_0$. Consequently, $h(n) \leq m_n$ for all $n \geq n_0$, which is a contradiction. Hence X does not satisfy $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. By Theorem 5.3.1, $(C(X), \tau^s_{\mathfrak{B}})$ is not a k-space.

Proposition 5.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If $\mathfrak{B}$ has a countable base, then $(C(X), \tau^s_{\mathfrak{B}})$ is a* k*-space.*

Proof. Consider a base $\mathfrak{B}_0 = \{B_n : n \in \mathbb{N}\}$ for $\mathfrak{B}$ and assume that $B_n \subseteq B_{n+1}$ for each n. Let $A \subseteq C(X)$ be such that for every compact subset K of $C(X)$ $A \cap K$ is closed in K . We show that A is closed. Let $\underline{0} \in \overline{A}$. Choose a $f_n \in A \cap [B_n, \frac{1}{n}]^s(\underline{0})$ for each $n \in \mathbb{N}$. We claim that the sequence $\{f_n : n \in \mathbb{N}\}$ converges to $\underline{0}$. Let $[B, \varepsilon]^s(\underline{0})$ be a neighbourhood of $\underline{0}$, where $B \in \mathfrak{B}$ and $\varepsilon > 0$. Choose a $n_0 \in \mathbb{N}$ such that $B \subseteq B_n$ and $\frac{1}{n} < \varepsilon$ for all $n \geq n_0$. Clearly $f_n \in [B, \varepsilon]^s(\underline{0})$ for all $n \geq n_0$. Therefore $\{f_n : n \in \mathbb{N}\}$ converges to $\underline{0}$. Let $K = \{f_n : n \in \mathbb{N}\} \cup \{\underline{0}\}$. Clearly $\underline{0} \in \overline{A \cap K}$. Since $A \cap K$ is closed, $\underline{0} \in A$. Hence A is closed. □

Proposition 5.3.2. *Let $\mathfrak{B}$ be a bornology with closed base $\mathfrak{B}_0$ on a metric space X and let X be $\mathfrak{B}^s$-Lindelöf. If $|\mathfrak{B}_0| < \mathfrak{p}$, then $(C(X), \tau^s_{\mathfrak{B}})$ is a* k*-space.*

Proof. By [26, Theorem 3.1], X is a $\gamma_{\mathfrak{B}^s}$-set. Therefore X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. By Theorem 5.3.1, $(C(X), \tau^s_{\mathfrak{B}})$ is a k-space. □

A bornology $\mathfrak{B}$ is *local* if $\mathfrak{B}$ contains as a member a neighbourhood of each $x \in X$ [14, 52].

Proposition 5.3.3. *Let $\mathfrak{B}$ be a bornology with a closed base $\mathfrak{B}_0$ which contains the nonempty compact subsets of a metric space X. If $(C(X), \tau^s_{\mathfrak{B}})$ is a* k*-space, then $\mathfrak{B}$ is local.*

Proof. Suppose that $\mathfrak{B}$ is not local. Choose a $x_0 \in X$ such that $S_\delta(x_0) \setminus B \neq \emptyset$ for any $B \in \mathfrak{B}$ and any $\delta > 0$. Let $\{S_{\frac{1}{n}}(x_0) : n \in \mathbb{N}\}$ be a sequence of neighbourhoods of x_0. Let $x_{n,B} \in S_{\frac{1}{n}}(x_0) \setminus B$ for each $B \in \mathfrak{B}_0$ and each $n \in \mathbb{N}$. Since B is closed, $d(x_{n,B}, B) > 0$. Choose $\delta_{n,B} = \frac{1}{2} d(x_{n,B}, B)$. Clearly $S_{\frac{1}{n}}(x_0) \setminus B^{\delta_{n,B}} \neq \emptyset$. For each $n \in \mathbb{N}$, consider the collection $\mathcal{U}_n = \{B^{\delta_{n,B}} : B \in \mathfrak{B}_0\}$, which is an open $\mathfrak{B}^s$-cover of X. Now there exists a $U_n \in \mathcal{U}_n$ for each n such that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Let $y_n \in S_{\frac{1}{n}}(x_0) \setminus U_n$. Clearly the sequence $\{y_n : n \in \mathbb{N}\}$ converges to x_0. Now $A = \{y_n : n \in \mathbb{N}\} \cup \{x_0\}$ is compact and so $A \in \mathfrak{B}_0$. Clearly $A \nsubseteq U_n$ for any $n \in \mathbb{N}$. This contradicts the fact that $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Hence $\mathfrak{B}$ is local. □

Proposition 5.3.4. *Let $\mathfrak{B}$ be a bornology with closed base on metric space X. If $(C(X), \tau^s_{\mathfrak{B}})$ is a* k*-space, then it has* CT*.*

Proof. By the given condition X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. Also by [19, Theorem 2.8], we obtain that every open $\mathfrak{B}^s$-cover of X has a $\gamma_{\mathfrak{B}^s}$-cover of X. Hence X is $\mathfrak{B}^s$-Lindelöf. Now by [19, Theorem 2.1], $(C(X), \tau^s_{\mathfrak{B}})$ has countable tightness. □

Using Theorem 5.3.1, Theorem 3.5.11 and Proposition 3.2.1 we have the following.

Proposition 5.3.5. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X. The following statements hold.*
(1) Let $(C(X), \tau^s_{\mathfrak{B}})$ be a k*-space and Y be a closed subset X. Then $(C(Y), \tau^s_{\mathfrak{B}_Y})$ is a* k*-space, where $\mathfrak{B}_Y = \{B \cap Y : B \in \mathfrak{B}\}$.*
(2) If $(C(X), \tau^s_{\mathfrak{B}})$ is a k*-space, then $(C(Y), \tau^s_{f(\mathfrak{B})})$ is a* k*-space, where $f : X \to Y$ is a continuous function.*

5.4 TOPOLOGICAL GAMES AND DISCRETE SELECTIVITY

5.4.1 CERTAIN OBSERVATIONS ON STRONG $\mathfrak{B}$-OPEN GAME

We first introduce two games.

Definition 5.4.1. *The* ***strong** $\mathfrak{B}$**-open*** *($\mathfrak{B}^s$-open for short) game on X is defined as follows. Suppose that an infinitely long game is played by ONE and TWO. In the nth inning, ONE chooses B_n, where $B_n \in \mathfrak{B}$, and TWO responds by choosing an open set U_n satisfying $B_n^{\delta_n} \subseteq U_n$ for some $\delta_n > 0$. ONE wins the play if $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Otherwise TWO wins.*

We denote the $\mathfrak{B}^s$-open game on X by $\mathcal{O}_{\mathfrak{B}^s}(X)$.

Definition 5.4.2. *The $\gamma_{\mathfrak{B}^s}$**-open** game on X is defined as follows. Suppose that an infinitely long game is played by ONE and TWO. In the nth inning, ONE chooses B_n, where $B_n \in \mathfrak{B}$, and TWO responds by choosing an open set U_n satisfying $B_n^{\delta_n} \subseteq U_n$ for some $\delta_n > 0$. ONE wins the play if $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Otherwise TWO wins.*

We denote the $\gamma_{\mathfrak{B}^s}$-open game on X by $\Gamma_{\mathfrak{B}^s}(X)$.

To prove the next result we use the technique of [73, Lemma 4.1] and the fact that a sequence of open sets $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X if and only if every infinite subsequence of $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X.

Theorem 5.4.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *ONE has a winning strategy in the game $\mathcal{O}_{\mathfrak{B}^s}(X)$.*
(2) *ONE has a winning strategy in the game $\Gamma_{\mathfrak{B}^s}(X)$.*

Proof. (1) $\Rightarrow$ (2). Let ψ be a winning strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$. We define a strategy σ for ONE in $\Gamma_{\mathfrak{B}^s}(X)$ as follows. Define $\sigma(\emptyset) = \psi(\emptyset)$, where $\psi(\emptyset) \in \mathfrak{B}$. Let TWO respond by choosing an open set U_1 such that $\sigma(\emptyset)^{\delta_1} \subseteq U_1$ for some $\delta_1 > 0$. Suppose that $U_1, \dots, U_{n-1}$ have already been chosen. We now define $\sigma(U_1, \dots, U_{n-1})$ in such a way that $\psi(\emptyset) \cup [\cup\{\psi(U_{i_1}, \dots U_{i_k}) : 1 \leq i_1 \leq \cdots \leq i_k \leq n-1\}] \subseteq \sigma(U_1, \dots, U_{n-1})$. Let TWO respond by choosing an open set U_n such that $\sigma(U_1, \dots, U_{n-1})^{\delta_n} \subseteq U_n$ for some $\delta_n > 0$. This defines the strategy σ for ONE in $\Gamma_{\mathfrak{B}^s}(X)$. We show that the sequence $\{U_n : n \in \mathbb{N}\}$ of moves of TWO in $\Gamma_{\mathfrak{B}^s}(X)$ forms a $\gamma_{\mathfrak{B}^s}$-cover of X. For this we prove that every infinite subsequence of $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Let $\{U_{n_k} : k \in \mathbb{N}\}$ be an infinite subsequence of $\{U_n : n \in \mathbb{N}\}$. Clearly $\psi(\emptyset) \subseteq \sigma(U_1, \dots, U_{n_1 - 1})$. Since $\sigma(U_1, \dots, U_{n_1-1})^{\delta_{n_1}} \subseteq U_{n_1}$ for some $\delta_{n_1} > 0$, $\psi(\emptyset)^{\delta_{n_1}} \subseteq U_{n_1}$. Again $\psi(U_{n_1}, \dots, U_{n_k}) \subseteq \sigma(U_1, \dots, U_{n_{k+1}-1})$ and since $\sigma(U_1, \dots, U_{n_{k+1}-1})^{\delta_{n_{k+1}}} \subseteq U_{n_{k+1}}$ for some $\delta_{n_{k+1}} > 0$, $\psi(U_{n_1}, \dots, U_{n_k})^{\delta_{n_{k+1}}} \subseteq U_{n_{k+1}}$ and so on.

Now $\psi(\emptyset), U_{n_1}, \dots, \psi(U_{n_1}, \dots, U_{n_k}), U_{n_{k+1}}, \dots$ is a legitimate play in $\mathcal{O}_{\mathfrak{B}^s}(X)$. Now $\{U_{n_k} : k \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X as ψ is a winning strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$. Therefore $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Hence σ is a winning strategy for ONE in $\Gamma_{\mathfrak{B}^s}(X)$. □

For the next two results we use the method of Theorems 15, 17 given in [24]

Theorem 5.4.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *ONE has a winning strategy in the game $\mathcal{O}_{\mathfrak{B}^s}(X)$.*
(2) *ONE has a winning strategy in the game $\Gamma_{\mathfrak{B}^s}(X)$.*
(3) *TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ on X.*
(4) *TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ on X.*

Proof. We prove (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (3). Let ψ be a winning strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$. Define a strategy σ for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ as follows. Let the first move of ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ be $\mathcal{U}_1$. For $\psi(\emptyset) \in \mathfrak{B}$ there are a $U_1 \in \mathcal{U}_1$ and a $\delta_1 > 0$ satisfying $\psi(\emptyset)^{\delta_1} \subseteq U_1$ for some $\delta_1 > 0$. Define $\sigma(\mathcal{U}_1) = U_1$. Suppose that $U_1, \ldots, U_{n-1}$ have been chosen. In the nth round, let the move of ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ be $\mathcal{U}_n$. For $\psi(U_1, \ldots, U_{n-1}) \in \mathfrak{B}$ there are a $U_n \in \mathcal{U}_n$ and a $\delta_n > 0$ satisfying $\psi(U_1, \ldots U_{n-1})^{\delta_n} \subseteq U_n$. Define $\sigma(\mathcal{U}_1, \ldots, \mathcal{U}_n) = U_n$.

For a play $\mathcal{U}_1, \sigma(\mathcal{U}_1), \ldots, \mathcal{U}_n, \sigma(\mathcal{U}_1, \ldots, \mathcal{U}_n), \ldots$ in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, the corresponding play in $\mathcal{O}_{\mathfrak{B}^s}(X)$ is $\psi(\emptyset), U_1, \ldots, \psi(U_1, \ldots, U_{n-1}), U_n, \ldots$. Since ψ is a winning strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$, $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Hence σ is a winning strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

(3) $\Rightarrow$ (1). Let σ be a winning strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Define a strategy ψ for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$ as follows. Let $\{U_1, \ldots, U_{n-1}\}$ be a finite sequence of open subsets of X. Assume that $\psi(U_1, \ldots, U_k)$ has been defined for all $k < n-1$ and $\mathcal{U}_k$, an open $\mathfrak{B}^s$-cover of X, is defined for all $k \leq n-1$.

Suppose that for each $B \in \mathfrak{B}$ there exist an open set U_B and a $\delta > 0$ satisfying $B^\delta \subseteq U_B$ such that for every open $\mathfrak{B}^s$-cover $\mathcal{U}$, $\sigma(\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}, \mathcal{U}) \neq U_B$. Clearly $\mathcal{U}' = \{U_B : B \in \mathfrak{B}\}$ is an open $\mathfrak{B}^s$-cover of X. Now $\sigma(\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}, \mathcal{U}') = U_B$ for some $B \in \mathfrak{B}$, which is a contradiction. Hence there exists a $B_n \in \mathfrak{B}$ such that for every open set U with $B_n^\delta \subseteq U$ for some $\delta > 0$ there exists an open $\mathfrak{B}^s$-cover $\mathcal{U}$ satisfying $\sigma(\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}, \mathcal{U}) = U$. We define $\psi(U_1, \ldots, U_{n-1}) = B_n$. Thus ψ is a strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$.

In response to the move $\psi(U_1, \ldots, U_{n-1})$ of ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$, let TWO choose U_n, where $B_n^{\delta_n} \subseteq U_n$ for some $\delta_n > 0$. Now for U_n there exists an open $\mathfrak{B}^s$-cover $\mathcal{U}_n$ of X such that $\sigma(\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}, \mathcal{U}_n) = U_n$. For a play in $\mathcal{O}_{\mathfrak{B}^s}(X)$

$$\psi(\emptyset), U_1, \ldots, \psi(U_1, \ldots, U_{n-1}), U_n, \ldots,$$

the corresponding play in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ is

$$\mathcal{U}_1, \sigma(\mathcal{U}_1), \ldots, \mathcal{U}_n, \sigma(\mathcal{U}_1, \ldots, \mathcal{U}_n), \ldots.$$

Since σ is a winning strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Hence ψ is a winning strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$. $\square$

A *Markov strategy* for TWO in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ is a function σ satisfying $\sigma(A, n) = a$ for some $a \in A$ for $A \in \mathcal{A}$ and $n \in \mathbb{N}$. We say this Markov strategy is winning if whenever ONE chooses $A_n \in \mathcal{A}$ during each round $n \in \mathbb{N}$, TWO wins the play by choosing $\sigma(A_n, n)$ during each round $n \in \mathbb{N}$ [24].

A *predetermined strategy* for ONE is a strategy which depends only on the round of the game. If this strategy for ONE is winning then ONE will be able to win a game irrespective of what TWO is playing [24].

Theorem 5.4.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *ONE has a predetermined winning strategy in the game $\mathcal{O}_{\mathfrak{B}^s}(X)$.*
(2) *ONE has a predetermined winning strategy in the game $\Gamma_{\mathfrak{B}^s}(X)$.*
(3) *TWO has a winning Markov strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ on X.*
(4) *TWO has a winning Markov strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ on X.*

Proof. (1) $\Rightarrow$ (3). Let ψ be a predetermined winning strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$. We define a Markov strategy σ for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ as follows. Let $n \in \mathbb{N}$ and $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X. For $\psi(n)$ there exist a $\delta > 0$ and a $U \in \mathcal{U}$ such that $\psi(n)^\delta \subseteq U$. Define $\sigma(\mathcal{U}, n) = U$. Thus σ is a Markov strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. For a play $\mathcal{U}_1, \sigma(\mathcal{U}_1, 1), \ldots, \mathcal{U}_n, \sigma(\mathcal{U}_n, n) \ldots$ in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, the corresponding play in $\mathcal{O}_{\mathfrak{B}^s}(X)$ is $\psi(1), U_1, \ldots, \psi(n), U_n \ldots$, where $\sigma(\mathcal{U}_n, n) = U_n$ for $n \in \mathbb{N}$. Since ψ is a winning strategy for ONE, $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Hence σ is a winning strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

(3) $\Rightarrow$ (1). Let σ be a winning Markov strategy for TWO in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. We define a predetermined strategy ψ for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$ as follows. Fix a $n \in \mathbb{N}$. Suppose that for each $B \in \mathfrak{B}$ there exists an open set U_B with $B^\delta \subseteq U_B$ such that for every open $\mathfrak{B}^s$-cover $\mathcal{U}$, $\sigma(\mathcal{U}, n) \neq U_B$. Now $\sigma(\{U_B : B \in \mathfrak{B}\}, n) \notin \{U_B : B \in \mathfrak{B}\}$, a contradiction. Hence there exists a B_n such that for every open set U with $B_n^\delta \subseteq U$, there exists an open $\mathfrak{B}^s$-cover $\mathcal{U}_n$ satisfying $\sigma(\mathcal{U}_n, n) = U$. Define $\psi(n) = B_n$. Thus ψ is a predetermined strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$. For $n \in \mathbb{N}$ and $\psi(n)$, let TWO's response in $\mathcal{O}_{\mathfrak{B}^s}(X)$ is U_n. Now choose an open $\mathfrak{B}^s$-cover $\mathcal{U}_n$ satisfying $\sigma(\mathcal{U}_n, n) = U_n$. For a play $\psi(1), U_1, \ldots, \psi(n), U_n, \ldots$ in $\mathcal{O}_{\mathfrak{B}^s}(X)$, the corresponding play in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ is $\mathcal{U}_1, \sigma(\mathcal{U}_1, 1), \ldots, \mathcal{U}_n, \sigma(\mathcal{U}_n, n), \ldots$. Since σ is a winning strategy for TWO, $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Hence ψ is a winning strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$.

Other implications can be similarly verified. □

We skip the proof of the next result.

Proposition 5.4.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If TWO has a winning strategy in the game $\mathcal{O}_{\mathfrak{B}^s}(X)$, then TWO has a winning strategy in the game $\Gamma_{\mathfrak{B}^s}(X)$.*

Theorem 5.4.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *TWO has a winning strategy in the game $\mathcal{O}_{\mathfrak{B}^s}(X)$.*
(2) *ONE has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ on X.*

Proof. (1) $\Rightarrow$ (2). Let ψ be a winning strategy for TWO in $\mathcal{O}_{\mathfrak{B}^s}(X)$. We define a strategy σ for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ as follows. For a finite sequence t of members from $\mathfrak{B}$ let $\mathcal{U}_t = \{\psi(t \frown < B >) : B \in \mathfrak{B}\}$. Define $\sigma(\emptyset) = \mathcal{U}_\emptyset$. Let TWO respond with $U_1 \in \sigma(\emptyset)$, where $U_1 = \psi(B_1)$. Suppose that $U_1, \ldots, U_{n-1}$ have been chosen. Now define $\sigma(U_1, \ldots, U_{n-1}) = \mathcal{U}_{<B_1,\ldots,B_{n-1}>}$. Let TWO respond with U_n, where $U_n = \psi(B_1, \ldots, B_n)$. A play in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ is

$$\sigma(\emptyset), U_1, \ldots, \sigma(U_1, \ldots, U_{n-1}), U_n, \ldots.$$

The corresponding play in $\mathcal{O}_{\mathfrak{B}^s}(X)$ is

$$B_1, \psi(B_1), \ldots, B_n, \psi(B_1, \ldots, B_n) \ldots.$$

Since ψ is a winning strategy for TWO in $\mathcal{O}_{\mathfrak{B}^s}(X)$, $\{U_n : n \in \mathbb{N}\}$ is not an open $\mathfrak{B}^s$-cover of X. Hence σ is a winning strategy for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Let σ be a winning strategy for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. Define a strategy ψ for TWO in $\mathcal{O}_{\mathfrak{B}^s}(X)$ as follows. Let B_1 be the first move of ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$. For $B_1 \in \mathfrak{B}$ there exist a $U_1 \in \sigma(\emptyset)$ and a $\delta_1 > 0$ such that $B_1^{\delta_1} \subseteq U_1$. Define $\psi(B_1) = U_1$. In $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, let TWO respond with U_1. Suppose that $U_1, \ldots, U_{n-1}$ have been chosen. Now define $\psi(B_1, \ldots, B_n) = U_n$, where $B_n^{\delta_n} \subseteq U_n$ for some $\delta > 0$ and $U_n \in \sigma(U_1, \ldots, U_{n-1})$. In $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, let TWO respond with U_n and so on. A play in $\mathcal{O}_{\mathfrak{B}^s}(X)$ is

$$B_1, \psi(B_1), \ldots, B_n, \psi(B_1, \ldots, B_n), \ldots$$

The corresponding play in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ is

$$\sigma(\emptyset), U_1 \ldots, \sigma(U_1, \ldots, U_{n-1}), U_n, \ldots$$

Since σ is a winning strategy for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, $\{U_n : n \in \mathbb{N}\}$ is not an open $\mathfrak{B}^s$-cover of X. Hence ψ is a winning strategy for TWO in $\mathcal{O}_{\mathfrak{B}^s}(X)$. □

Theorem 5.4.5. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *TWO has a winning strategy in the game $\Gamma_{\mathfrak{B}^s}(X)$.*
(2) *ONE has a winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ on X.*

Theorem 5.4.6. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *TWO has a winning Markov strategy in the game $\mathcal{O}_{\mathfrak{B}^s}(X)$.*
(2) *ONE has a predetermined winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ on X.*

Proof. (1) $\Rightarrow$ (2). Let ψ be a winning Markov strategy for TWO in $\mathcal{O}_{\mathfrak{B}^s}(X)$. We define a predetermined strategy σ for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ as follows. Fix a $n \in \mathbb{N}$. For $B \in \mathfrak{B}$ $\psi(B,n)$ is an open set satisfying $B^\delta \subseteq \psi(B,n)$ for some $\delta > 0$. Thus $\{\psi(B,n) : B \in \mathfrak{B}\}$ is an open $\mathfrak{B}^s$-cover of X. Define $\sigma(n) = \{\psi(B,n) : B \in \mathfrak{B}\}$. Hence σ is a predetermined strategy for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. It is easy to see that σ is a winning strategy.

(2) $\Rightarrow$ (1). Let σ be a predetermined winning strategy for ONE in $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$. We define a Markov strategy ψ for TWO in $\mathcal{O}_{\mathfrak{B}^s}(X)$ as follows. For $B \in \mathfrak{B}$ and $n \in \mathbb{N}$ there exists a $U \in \sigma(n)$ and a $\delta > 0$ such that $B^\delta \subseteq U$. Define $\psi(B,n) = U$. Thus ψ is a Markov strategy for TWO in $\mathcal{O}_{\mathfrak{B}^s}(X)$. It is easy to see that ψ is a winning strategy. □

Theorem 5.4.7. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) *TWO has a winning Markov strategy in the game $\Gamma_{\mathfrak{B}^s}(X)$.*
(2) *ONE has a predetermined winning strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ on X.*

5.4.2 RESULTS ON DISCRETE SELECTIVITY AND RELATED GAMES

The concept of discrete selectivity and related games were introduced and studied in [108, 109] by V.V. Tkachuk in function spaces with respect to the point-open topology. We say that

> *X is **discretely selective** if for any sequence $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X there exists a $x_n \in U_n$ for each n for which the set $\{x_n : n \in \mathbb{N}\}$ is a closed and discrete [108].*

$C_p(X)$ is discretely selective if and only if X is uncountable, provided X is a Tychonoff space [108, Proposition 3.3]. In this section we study discretely selective property and related games in function spaces with respect to the topology of strong uniform convergence on a bornology.

Theorem 5.4.8. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X. The following statements are equivalent.*
(1) *$\mathfrak{B}_0$ is uncountable.*
(2) *$(C(X), \tau^s_{\mathfrak{B}})$ is discretely selective.*

Proof. (1) $\Rightarrow$ (2). Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of nonempty open subsets of $(C(X), \tau^s_{\mathfrak{B}})$. Without loss of generality we assume that $U_n = [B_n, \frac{1}{n}]^s(g_n)$ for some $B_n \in \mathfrak{B}_0$ and $g_n \in C(X)$. Since $\mathfrak{B}_0$ is uncountable, there exists a $B \in \mathfrak{B}_0 \setminus \{B_n : n \in \mathbb{N}\}$. For each n let $x_n \in B \setminus B_n$. Since B_n is closed, there exists a $\delta_n > 0$ such that $S_{2\delta_n}(x_n) \cap B_n = \emptyset$. Choose a $f_n \in C(X)$ satisfying $f_n(x) = g_n(x)$ for all $x \in \overline{B^{\delta_n}}$ and $f_n(x_n) = n$. Clearly $f_n \in [B_n, \frac{1}{n}]^s(g_n)$. We now show that $\{f_n : n \in \mathbb{N}\}$ $(= D)$ is closed and discrete. Let $f \in C(X)$. As B is compact, $f(B)$ is bounded and let $m = \sup\{2|f(x)| : x \in B\}$. Consider the neighbourhood $[B, \frac{m}{2}]^s(f)$. If $f_n \in [B, \frac{m}{2}]^s(f)$, then $|f_n(x)| < \frac{m}{2} + |f(x)|$ for all $x \in B^\delta$, where $\delta > 0$. Thus $|f_n(x_n)| < m$ and so $n < m$. Therefore $[B, \frac{m}{2}]^s(f) \cap D \subseteq \{f_1, \dots, f_{m-1}\}$. Hence D is closed and discrete.

(2) $\Rightarrow$ (1). If $\mathfrak{B}_0$ is countable, then $(C(X), \tau^s_{\mathfrak{B}})$ is first countable by [18, Theorem 3.1]. Again by (2) and [108, Proposition 3.2(b)], $(C(X), \tau^s_{\mathfrak{B}})$ is discrete, which is a contradiction. Hence $\mathfrak{B}_0$ is uncountable. □

We consider the following games for our subsequent results.

*For $x \in X$ the **Gruenhage game** $\mathsf{G}(X, x)$ on X is defined as follows. Suppose that an infinitely long game is played by two players ONE and TWO. In the nth round, ONE chooses an open neighbourhood U_n of x. In response to that TWO chooses a point x_n from U_n. ONE wins the play if $\{x_n : n \in \mathbb{N}\}$ converges to x. Otherwise TWO wins [45].*

For simplicity we denote the Gruenhage game $\mathsf{G}((C(X), \tau^s_{\mathfrak{B}}), \underline{0})$ on $(C(X), \tau^s_{\mathfrak{B}})$ by $\mathsf{G}(\tau^s_{\mathfrak{B}})$.

The game $\mathsf{CD}(X)$ on X is defined as follows. Suppose that an infinitely long game is played by two players ONE and TWO. In the nth round, ONE chooses a nonempty open subset U_n of X. In response to that TWO chooses a point x_n from U_n. TWO wins the play if the set $\{x_n : n \in \mathbb{N}\}$ is closed and discrete. Otherwise ONE wins [109].

For simplicity we denote the game $\mathsf{CD}(C(X), \tau^s_{\mathfrak{B}})$ on $(C(X), \tau^s_{\mathfrak{B}})$ by $\mathsf{CD}(\tau^s_{\mathfrak{B}})$.

For $x \in X$ the game $\mathsf{CL}(X, x)$ on X is defined as follows. Suppose that an infinitely long game is played by two players ONE and TWO. In the nth round, ONE chooses a nonempty open subset U_n of X. In response to that TWO chooses a point x_n from U_n. ONE wins the play if $x \in \overline{\{x_n : n \in \mathbb{N}\}}$. Otherwise TWO wins [109].

For simplicity we denote the game $\mathsf{CL}((C(X), \tau^s_{\mathfrak{B}}), \underline{0})$ on $(C(X), \tau^s_{\mathfrak{B}})$ by $\mathsf{CL}(\tau^s_{\mathfrak{B}})$. For a Tychonoff space X it is known that the player ONE has a winning strategy in $\mathsf{G}(C_p(X), u)$, $\mathsf{CL}(C_p(X), u)$ for some $u \in C_p(X)$ and $\mathsf{CD}(C_p(X))$ respectively are equivalent to the fact that the player ONE has a winning strategy in the point-open game on X [109, Theorem 3.8]. A similar result holds for the space $(C(X), \tau^s_{\mathfrak{B}})$.

Theorem 5.4.9. *Let $\mathfrak{B}$ be a bornology with a compact base $\mathfrak{B}_0$ on a metric space X. The following statements are equivalent.*
(1) *ONE has a winning strategy in the game $\mathsf{G}(\tau^s_{\mathfrak{B}})$.*
(2) *ONE has a winning strategy in the game $\mathsf{CL}(\tau^s_{\mathfrak{B}})$.*
(3) *ONE has a winning strategy in the game $\mathsf{CD}(\tau^s_{\mathfrak{B}})$.*
(4) *ONE has a winning strategy in the game $\mathcal{O}_{\mathfrak{B}^s}(X)$.*

Proof. We prove only the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3) Let σ be a winning strategy for ONE in $\mathsf{CL}(\tau^s_{\mathfrak{B}})$. We define another strategy ψ for ONE in $\mathsf{CL}(\tau^s_{\mathfrak{B}})$ as follows. Define $\psi(\emptyset) = \sigma(\emptyset) \setminus \{\underline{0}\}$. Suppose that $f_1, \ldots, f_{n-1}$ are already

chosen. Now define $\psi(f_1,\ldots,f_{n-1}) = \sigma(f_1,\ldots f_{n-1}) \setminus \{\underline{0}\}$. TWO responds by choosing a $f_n \in \psi(f_1,\ldots,f_{n-1})$. Since σ is a winning strategy for ONE, $\underline{0} \in \overline{\{f_n : n \in \mathbb{N}\}}$. Therefore ψ is also a winning strategy for ONE in $\mathsf{CL}(\tau^s_{\mathfrak{B}})$. Now $\underline{0} \in \overline{\{f_n : n \in \mathbb{N}\}} \setminus \{f_n : n \in \mathbb{N}\}$. This shows that $\{f_n : n \in \mathbb{N}\}$ is not closed and hence ψ is a winning strategy for ONE in $\mathsf{CD}(\tau^s_{\mathfrak{B}})$.

(3) $\Rightarrow$ (4). Let ψ be a winning strategy for ONE in $\mathsf{CD}(\tau^s_{\mathfrak{B}})$. We define a strategy τ for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$ as follows. Let $[B_1,\varepsilon_1]^s(g_1) \subseteq \psi(\emptyset)$ for some $B_1 \in \mathfrak{B}$, $\varepsilon_1 > 0$ and $g_1 \in \psi(\emptyset)$. Define $\tau(\emptyset) = B_1$. TWO responds by choosing an open set U_1 such that $B_1^{2\delta_1} \subseteq U_1$ for some $\delta_1 > 0$. Choose a $f_1 \in C(X)$ satisfying $f_1(x) = g_1(x)$ for all $x \in \overline{B_1^{\delta_1}}$ and $f_1(x) = 1$ for all $x \in X \setminus U_1$. Clearly $f_1 \in \psi(\emptyset)$. In $\mathsf{CD}(\tau^s_{\mathfrak{B}})$, let TWO respond by choosing f_1. Suppose that $f_1,\ldots,f_{n-1}$ and $U_1,\ldots,U_{n-1}$ are already chosen. Let $[B_n,\varepsilon_n]^s(g_n) \subseteq \psi(f_1,\ldots,f_{n-1})$ for some $B_n \in \mathfrak{B}$, $\varepsilon_n > 0$ and $g_n \in \psi(f_1,\ldots,f_{n-1})$. Define $\tau(U_1,\ldots,U_{n-1}) = B_n$. TWO responds by choosing an open set U_n such that $B_n^{2\delta_n} \subseteq U_n$ for some $\delta_n > 0$. Choose a $f_n \in C(X)$ satisfying $f_n(x) = g_n(x)$ for all $x \in \overline{B_n^{\delta_n}}$ and $f_n(x) = n$ for all $x \in X \setminus U_n$. In $\mathsf{CD}(\tau^s_{\mathfrak{B}})$, let TWO respond by choosing f_n and so on. A play in $\mathsf{CD}(\tau^s_{\mathfrak{B}})$ is

$\psi(\emptyset), f_1, \ldots, \psi(f_1,\ldots,f_{n-1}), f_n, \ldots$

The corresponding play in $\mathcal{O}_{\mathfrak{B}^s}(X)$ is

$\tau(\emptyset), U_1, \ldots, \tau(U_1,\ldots,U_{n-1}), U_n, \ldots$

We now prove that $\{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover of X. Assume the contrary. There exists a $B \in \mathfrak{B}_0$ such that $B^\delta \nsubseteq U_n$ for any $\delta > 0$ and any $n \in \mathbb{N}$. It follows that $B \nsubseteq U_n$ for any n, as B is compact. Let $f \in C(X)$ and $m = \sup\{2|f(x)| : x \in B\}$. Consider the neighbourhood $[B, \frac{m}{2}]^s(f)$. It can be easily seen that $[B, \frac{m}{2}]^s(f) \cap \{f_n : n \in \mathbb{N}\} \subseteq \{f_1,\ldots,f_{m-1}\}$ and hence the collection $\{f_n : n \in \mathbb{N}\}$ is discrete and closed. This contradicts the fact that ψ is a winning strategy for ONE in $\mathsf{CD}(\tau^s_{\mathfrak{B}})$. Therefore our assumption is wrong. Hence τ is a winning strategy for ONE in $\mathcal{O}_{\mathfrak{B}^s}(X)$.

(4) $\Rightarrow$ (1). By Theorem 5.4.1, let σ be a winning strategy for ONE in $\Gamma_{\mathfrak{B}^s}(X)$. We define a strategy ψ for ONE in $\mathsf{G}(\tau^s_{\mathfrak{B}})$ as follows. Define $\psi(f_1,\ldots,f_{n-1}) = [B_n, \frac{1}{n}]^s(\underline{0})$, where $\sigma(U_1,\ldots,U_{n-1}) = B_n$. TWO responds by choosing a $f_n \in [B_n, \frac{1}{n}]^s(\underline{0})$. There is a $\delta_n > 0$ satisfying $B_n^{\delta_n} \subseteq f_n^{-1}(-\frac{1}{n}, \frac{1}{n})$. Consider $U_n = f_n^{-1}(-\frac{1}{n}, \frac{1}{n})$. Assume that $U_n \neq X$. (If $f_n^{-1}(-\frac{1}{n}, \frac{1}{n}) = X$, then consider $U_n = B_n^{2\delta_n}$). In $\Gamma_{\mathfrak{B}^s}(X)$, TWO chooses U_n.

A play in $\Gamma_{\mathfrak{B}^s}(X)$ is

$\sigma(\emptyset), U_1, \ldots, \sigma(U_1,\ldots,U_{n-1}), U_n, \ldots$

The corresponding play in $\mathsf{G}(\tau^s_{\mathfrak{B}})$ is

$\psi(\emptyset), f_1, \ldots, \psi(f_1,\ldots,f_{n-1}), f_n, \ldots$

Since σ is a winning strategy for ONE in $\Gamma_{\mathfrak{B}^s}(X)$, $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. It is easy to see that $\{f_n : n \in \mathbb{N}\}$ converges to $\underline{0}$. Hence ψ is a winning strategy for ONE in $\mathsf{G}(\tau^s_{\mathfrak{B}})$. □

Finally, we consider a predetermined strategy for ONE. For a similar result for the space

$C_p(X)$ we refer readers to [24, Theorem 17].

Theorem 5.4.10. *Let $\mathfrak{B}$ be a bornology with compact base on a metric space X. The following statements are equivalent.*
(1) *$\mathfrak{B}$ has a countable base.*
(2) *ONE has a predetermined winning strategy in the game $\mathsf{G}(\tau^s_{\mathfrak{B}})$.*
(3) *ONE has a predetermined winning strategy in the game $\mathsf{CL}(\tau^s_{\mathfrak{B}})$.*
(4) *ONE has a predetermined winning strategy in the game $\mathsf{CD}(\tau^s_{\mathfrak{B}})$.*
(5) *ONE has a predetermined winning strategy in the game $\mathcal{O}_{\mathfrak{B}^s}(X)$.*

Proof. We prove only $(1) \Leftrightarrow (2)$.

$(1) \Rightarrow (2)$. Let $\{B_n : n \in \mathbb{N}\}$ be a base for $\mathfrak{B}$. We assume that $B_n \subseteq B_{n+1}$ for each n. We define a strategy ψ for ONE in $\mathsf{G}(\tau^s_{\mathfrak{B}})$ as follows. For each n, define $\psi(n) = [B_n, \frac{1}{n}]^s(\underline{0})$. Clearly the strategy ψ depends only on n. Now whenever $f_n \in \psi(n), n \in \mathbb{N}$, the sequence $\{f_n : n \in \mathbb{N}\}$ converges to $\underline{0}$. Hence ψ is a predetermined winning strategy for ONE in $\mathsf{G}(\tau^s_{\mathfrak{B}})$.

$(2) \Rightarrow (1)$. Let ψ be a predetermined winning strategy for ONE in $G(\tau^s_{\mathfrak{B}})$. Now $\{\psi(n) : n \in \mathbb{N}\}$ is a sequence of open neighbourhoods of $\underline{0}$. Since $\mathfrak{B}$ is closed under finite unions and has closed base, we assume that $\psi(n) = [B_n, \frac{1}{n}]^s(\underline{0})$, where B_n is closed for every $n \in \mathbb{N}$ and $B_n \subseteq B_{n+1}$ for every $n \in \mathbb{N}$. We first show that $X = \cup_{n\in\mathbb{N}} B_n$. Let $x_0 \in X \setminus \cup_{n\in\mathbb{N}} B_n$. For each n there exists a $\delta_n > 0$ such that $S_{2\delta_n}(x_0) \cap B_n = \emptyset$. Choose a $f_n \in C(X)$ satisfying $f_n(x) = 0$ for all $x \in \overline{B_n^{\delta_n}}$ and $f_n(x_0) = n$. Clearly for each n, $f_n \in [B_n, \frac{1}{n}]^s(\underline{0})$. If we choose a $B \in \mathfrak{B}$ with $x_0 \in B$, then $f_n \notin [B, 1]^s(\underline{0})$ for any n. Now $\{f_n : n \in \mathbb{N}\}$ is a sequence of moves by TWO which does not converge to $\underline{0}$. This contradicts that ψ is a winning strategy for ONE in $G(\tau^s_{\mathfrak{B}})$. Therefore $X = \cup_{n\in\mathbb{N}} B_n$. To complete the proof we next show that $\{B_n : n \in \mathbb{N}\}$ is cofinal. Assume the contrary. Therefore there is a $B \in \mathfrak{B}$ satisfying $B \setminus B_n \neq \emptyset$ for each n. Now for each n choose a $x_n \in B \setminus B_n$ and a $\delta_n > 0$ with $S_{2\delta_n}(x_n) \cap B_n = \emptyset$. Again choose a $f_n \in C(X)$ satisfying $f_n(x) = 0$ for all $x \in \overline{B_n^{\delta_n}}$ and $f_n(x_n) = 1$. Clearly $f_n \in [B_n, \frac{1}{n}]^s(\underline{0})$ but $f_n \notin [B, 1]^s(\underline{0})$ for each n. Therefore $\{f_n : n \in \mathbb{N}\}$ is a sequence of moves by TWO which does not converge to $\underline{0}$. Again it contradicts that ψ is a winning strategy for ONE in $G(\tau^s_{\mathfrak{B}})$. Hence $\{B_n : n \in \mathbb{N}\}$ is cofinal. □

CHAPTER 6

ON STATISTICAL VARIATIONS OF BORNOLOGICAL COVERS

This Chapter is based on our following work:

S.Das and D. Chandra `Certain observations on statistical variations of bornological covers`, **Filomat.** 35(7) (2021), 2303--2315.

6.1 INTRODUCTION

We primarily make a general approach to the study of open covers and related selection principles using the idea of statistical convergence in metric space. In [32], the authors had studied selection principles, function spaces and hyperspaces using the notion of statistical convergence in topological and uniform spaces. For more details of the study of statistical convergence in topological and function spaces related to selection principles see also [20, 33, 34] and references therein.

In [19], the authors had studied open covers and related selection principles in function space with respect to the topology of strong uniform convergence on a bornology. Very recently in [22], a further advancement has been made in this direction (see also [7]). Motivated by [32], in this Chapter, we introduce statistical analogues of certain types of open covers and investigate the behaviour of related selection principles using the idea of strong uniform convergence on bornologies. Our main objective is to study some results of [19, 22] in a more general setup using the idea of statistical convergence. We introduce statistical versions of certain types of bornological open covers and observe the behaviour of related selection principles including the α_i-properties. We also introduce the notions of statistically-strong-$\mathfrak{B}$-Hurewicz property and statistically-strong-$\mathfrak{B}$-groupable cover and obtain some game theoretic results. In the function space $C(X)$ associated with the topology of strong uniform convergence on $\mathfrak{B}$ we investigate some properties like statistically strictly Frèchet Urysohn, statistically Reznichenko and countable fan tightness.

6.2 STATISTICAL VARIATIONS OF CERTAIN BORNOLOGICAL NOTIONS

6.2.1 THE s-$\gamma_{\mathfrak{B}^s}$-COVER AND RELATED SELECTION PRINCIPLES

First we introduce the following definition which plays a central role in our paper.

Definition 6.2.1. *A countable open cover $\mathcal{U}$ is said to be a statistical-$\gamma_{\mathfrak{B}^s}$-cover (a s-$\gamma_{\mathfrak{B}^s}$-cover for short) if there is an enumeration of $\mathcal{U}$, say $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ such that for $B \in \mathfrak{B}$ there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers such that $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U_n\}) = 0$.*

In contrast to the classical definition, this definition depends on the enumeration of pieces. A s-$\gamma_{\mathfrak{B}^s}$-cover may not still be a s-$\gamma_{\mathfrak{B}^s}$-cover under a changed enumeration (see Example 6.2.2 below). Throughout we follow the convention that whenever we consider a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, we always consider a fixed enumeration.
The collection of all s-$\gamma_{\mathfrak{B}^s}$-covers of X is denoted by s-$\Gamma_{\mathfrak{B}^s}$. It is clear from the context that every $\gamma_{\mathfrak{B}^s}$-cover is a s-$\gamma_{\mathfrak{B}^s}$-cover, i.e., $\Gamma_{\mathfrak{B}^s} \subset s$-$\Gamma_{\mathfrak{B}^s}$. The following example shows that the inclusion is proper.

Example 6.2.1. *Consider $X = \mathbb{R}$ and a bornology $\mathfrak{B}$ on X generated by $\{(-x, x) : x \in \mathbb{R}\}$. Now consider an open $\mathfrak{B}^s$-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, where $U_n = (0, n)$ when $n = k^2$ and $U_n = (-n, n)$ when $n \neq k^2$ for each $k \in \mathbb{N}$. We show that $\mathcal{U}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover. Let $B \in \mathfrak{B}$. Say, $B = (-x_0, x_0)$. Now for a $\delta > 0$ there is a $n_0 \in \mathbb{N}$ such that $B^\delta \subseteq U_n$ for all $n \geq n_0$ and $n \neq k^2$ for any $k \in \mathbb{N}$. Define $\delta_n = \delta$ for each n, then for this sequence $\{\delta_n : n \in \mathbb{N}\}$ we have $\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U_n\} \subseteq \{n \in \mathbb{N} : n = k^2$ for $k \in \mathbb{N}\} \cup \{1, 2, \ldots, n_0 - 1\}$. Clearly $\mathcal{U}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover, as $d(\{n \in \mathbb{N} : n = k^2$ for $k \in \mathbb{N}\}) = 0$. It is also clear that for any $\delta > 0$, $B^\delta \nsubseteq U_n$ for infinitely many n. Thus $\mathcal{U}$ can not be a $\gamma_{\mathfrak{B}^s}$-cover of X.*

Example 6.2.2. *Under a changed enumeration the s-$\gamma_{\mathfrak{B}^s}$-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of Example 6.2.1 may not remain a s-$\gamma_{\mathfrak{B}^s}$-cover of X.*
First consider a partition $\{\mathcal{P}_j : j \in \mathbb{N}\}$ of $A = \{1^2, 2^2, \ldots\}$, where $\mathcal{P}_1 = \{1^2, 2^2\}$ and $\mathcal{P}_j = \{(j^2 - j + 1)^2, (j^2 - j + 2)^2, \ldots, (j^2 + j)^2\}$ for $j > 1$.

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be the bijection given by

$$\sigma(n) = \begin{cases} k + i & \text{if } n = k^2 \text{ and } n \in \mathcal{P}_i \text{ for some } k, i \in \mathbb{N} \\ (n - k)^2 & \text{if } k^2 < n < (k+1)^2 \text{ for some } k \in \mathbb{N} \end{cases}$$

and consider the enumeration $\{U_{\sigma(n)} : n \in \mathbb{N}\}$ of $\mathcal{U}$. Clearly $U_{\sigma(n)} = (-\sigma(n), \sigma(n))$ if $n = k^2$ for $k \in \mathbb{N}$ and $U_{\sigma(n)} = (0, \sigma(n))$ if $n \neq k^2$. Let $B = (-1, 1)$. It is clear that for any sequence $\{\delta_n : n \in$

$\mathbb{N}\}$ *of positive real numbers* $\mathbb{N} \setminus A \subseteq \{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U_{\sigma(n)}\}$. *Also* $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U_{\sigma(n)}\}) \neq 0$ *as* $d(\mathbb{N} \setminus A) = 1$. *Thus* $\{U_{\sigma(n)} : n \in \mathbb{N}\}$ *is not a* s-$\gamma_{\mathfrak{B}^s}$-*cover of* X.

It is also interesting to observe in Example 6.2.1 that $\{U_{k^2} : k \in \mathbb{N}\}$ is an infinite subset of $\mathcal{U}$ which is not a s-$\gamma_{\mathfrak{B}^s}$-cover (not even a cover) of X. Generally an infinite subset of a s-$\gamma_{\mathfrak{B}^s}$-cover is not necessarily a s-$\gamma_{\mathfrak{B}^s}$-cover. However, on the positive side, the result holds if we consider any statistically dense subset of this cover.

A subset $\mathcal{V}$ *of a cover* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *of* X *is called* ***statistically dense*** *[32] in* $\mathcal{U}$ *if the set of indices of elements from* $\mathcal{V}$ *has asymptotic density* 1.

Lemma 6.2.1. *A statistically dense subset of a* s-$\gamma_{\mathfrak{B}^s}$-*cover of* X *is again a* s-$\gamma_{\mathfrak{B}^s}$-*cover of* X.

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Let $\{U_{n_k} : k \in \mathbb{N}\}$ be a statistically dense subset of $\mathcal{U}$. We aim to show that this is again a s-$\gamma_{\mathfrak{B}^s}$-cover. Assume on the contrary that there is a $B \in \mathfrak{B}$ such that for any sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers, $d(\{k \in \mathbb{N} : B^{\delta_{n_k}} \nsubseteq U_{n_k}\}) \neq 0$. Since $d(\{k : B^{\delta_{n_k}} \nsubseteq U_{n_k}\}) \leq d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U_n\})$, it follows that $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U_n\}) \neq 0$. Which in turn contradicts that $\mathcal{U}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. □

The next two observations about the s-$\gamma_{\mathfrak{B}^s}$-cover will be useful in what follows.

Lemma 6.2.2. *Let* $\mathfrak{B}$ *be a bornology with closed base on a metric space* X *and let* $\{\mathcal{U}_n : n \in \mathbb{N}\}$ *be a sequence of* s-$\gamma_{\mathfrak{B}^s}$-*covers of* X, *where* $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$. *Then for each* n *the collection* $\mathcal{V}_n = \{U_k^1 \cap U_k^2 \ldots \cap U_k^n : U_k^i \in \mathcal{U}_i, 1 \leq i \leq n, k \in \mathbb{N}\}$ *is also a* s-$\gamma_{\mathfrak{B}^s}$-*cover of* X.

Proof. Let $B \in \mathfrak{B}$ and fix a positive integer n. For each $i = 1, 2, \ldots, n$ choose a sequence $\{\delta_k^i : k \in \mathbb{N}\}$ of positive real numbers such that $d(T_i) = 0$, where $T_i = \{k \in \mathbb{N} : B^{\delta_k^i} \nsubseteq U_k^i\})$. Choose $V_k^n = U_k^1 \cap U_k^2 \ldots \cap U_k^n$ and take $\delta_k = \min\{\delta_k^i : i = 1, 2, \ldots, n\}$. We show that $d(S) = 0$, where $S = \{k \in \mathbb{N} : B^{\delta_k} \nsubseteq V_k^n\}$. If $k \in S$, then $B^{\delta_k} \nsubseteq V_k^n$, i.e., $B^{\delta_k^i} \nsubseteq U_k^i$ for some $i \in \{1, 2, \ldots, n\}$. Clearly $S \subseteq \cup_{i=1}^n T_i$ and $d(S) = 0$. Hence $\mathcal{V}_n$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. □

Since every $\gamma_{\mathfrak{B}^s}$-cover is a s-$\gamma_{\mathfrak{B}^s}$-cover, the next result follows from Lemma 2.2.4.

Lemma 6.2.3. *Let* $\mathfrak{B}$ *be a bornology with closed base on a metric space* X *and* $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ *be an open* $\mathfrak{B}^s$-*cover of* X. *If* $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$, *where* $V_n = \cup_{i=1}^n U_i$, *then* $\mathcal{V}$ *is a* s-$\gamma_{\mathfrak{B}^s}$-*cover of* X.

The statistical version of the α_i, $i = 1, 2, 3, 4$ properties are introduced and studied in [32]. In particular, we consider the s-$\alpha_4(\mathcal{A}, \mathcal{B})$ property.

The symbol s-$\alpha_4(\mathcal{A}, \mathcal{B})$ *denotes that for each sequence* $\{A_n : n \in \mathbb{N}\}$ *of elements of* $\mathcal{A}$, *there is a* $B \in \mathcal{B}$ *and a set* $K \subseteq \mathbb{N}$ *with* $d(K) = 1$ *such that for each* $k \in K$ *the set* $A_k \cap B$ *is non empty.*

In the next result we show that the s-α_4 property lies between S_1- and S_{fin}-type selection properties for some suitable classes of covers. We will further investigate these types of statistical selection properties in the final section.

Proposition 6.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Consider the following statements.*
(1) *X satisfies $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.*
(2) *X satisfies s-$\alpha_4(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.*
(3) *X satisfies $\mathsf{S}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.*
Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) holds.

Proof. We only give proof of (2) $\Rightarrow$ (3). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X and let $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. By Lemma 6.2.1 and using (2), there is a subset $K = \{n_1 < n_2 < \cdots\}$ of $\mathbb{N}$ with $d(K) = 1$ and a s-$\gamma_{\mathfrak{B}^s}$ cover $\{U^j_{m_j} : j \in K\}$ such that $U^{n_i}_{m_{n_i}} \in \mathcal{U}_{n_i}$, for each $n_i \in K$ (see [32, Theorem 6.1]). If $n = n_i$, choose $\mathcal{W}_n = \{U^{n_i}_{m_{n_i}}\}$ and choose $\mathcal{W}_n = \emptyset$ otherwise. Clearly $\cup_{n\in\mathbb{N}}\mathcal{W}_n$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X and $\mathcal{W}_n$ is finite subset of $\mathcal{U}_n$ for each n. Hence (3) holds. □

We now present certain implications among the selection principles in the next few results.

Theorem 6.2.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements hold.*
(1) $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s}) = \mathsf{S}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$
(2) $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma) = \mathsf{S}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$
(3) $\mathsf{S}_1(s\text{-}\Gamma, \Gamma_{\mathfrak{B}^s}) = \mathsf{S}_{\text{fin}}(s\text{-}\Gamma, \Gamma_{\mathfrak{B}^s})$.

Proof. We only give proof of (1) as the other proofs are analogous. Suppose that X satisfies $\mathsf{S}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X. Let $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. For each n, consider the collection $\mathcal{V}_n = \{V^n_k : k \in \mathbb{N}\}$, where $V^n_k = U^1_k \cap U^2_k \cap \ldots \cap U^n_k$ and $U^i_k \in \mathcal{U}_i$, $i = 1, 2, \ldots, n$. By Lemma 6.2.2, $\mathcal{V}_n$'s are s-$\gamma_{\mathfrak{B}^s}$-covers of X. Now applying $\mathsf{S}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ to choose a finite subset $\mathcal{W}_n \subseteq \mathcal{V}_n$ for each n such that $\cup_{n\in\mathbb{N}}\mathcal{W}_n$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. Choose a sequence of positive integers $n_1 < n_2 < \cdots$ such that $\mathcal{W}_{n_j} \setminus \cup_{i<j}\mathcal{W}_{n_i} \neq \emptyset$ for $j \in \mathbb{N}$.

Now for each j, choose a $V^{n_j}_{k_j} \in \mathcal{W}_{n_j} \setminus \cup_{i<j}\mathcal{W}_{n_i}$. As infinite subset of a $\gamma_{\mathfrak{B}^s}$-cover of X is a $\gamma_{\mathfrak{B}^s}$-cover, $\{V^{n_j}_{k_j} : j \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X. For $1 \leqslant n \leqslant n_1$, define $U_n = U^n_{k_1}$, where $V^{n_1}_{k_1} = U^1_{k_1} \cap U^2_{k_1} \cap \cdots \cap U^{n_1}_{k_1}$ and for each $n \in (n_j, n_{j+1}]$, define $U_n = U^n_{k_{j+1}}$, where $V^{n_{j+1}}_{k_{j+1}} = U^1_{k_{j+1}} \cap U^2_{k_{j+1}} \cap \cdots \cap U^{n_{j+1}}_{k_{j+1}}$. We show that $\{U_n : n \in \mathbb{N}\} \in \Gamma_{\mathfrak{B}^s}$. Let $B \in \mathfrak{B}$. Since $\{V^{n_j}_{k_j} : j \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X, there exist a $j_0 \in \mathbb{N}$ and a sequence $\{\delta_j : j \geq j_0\}$ of positive real numbers such that $B^{\delta_{j+1}} \subseteq V^{n_{j+1}}_{k_{j+1}}$ for all $j \geq j_0$, i.e., $B^{\delta_{j+1}} \subseteq U^1_{k_{j+1}} \cap U^2_{k_{j+1}} \cap \cdots \cap U^{n_{j+1}}_{k_{j+1}}$. For each $n \in (n_j, n_{j+1}]$, define $\delta_n = \delta_{j+1}$. Thus we have $B^{\delta_n} \subseteq U_n$ for all $n \geq n_{j_0}$. Consequently $\{U_n : n \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X and hence X satisfies $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. The other direction is straightforward. □

Theorem 6.2.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If X is $\mathfrak{B}^s$-Lindelöf, then the following statements hold.*
(1) $\mathsf{S}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda) = \mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda)$
(2) $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda) = \mathsf{S}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$
(3) $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda) = \mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$.

Proof. We prove only (3). Let X satisfy $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda)$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X. Apply $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \Lambda)$ to $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to choose a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ for each n such that $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a large cover of X. Choose a sequence $1 = k_1 < k_2 < \cdots$ of positive integers and enumerate $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ as $\{V_i : i \in \mathbb{N}\}$, where $\mathcal{V}_n = \{V_i : k_n \leqslant i < k_{n+1}\}$. Since each x belongs to infinitely many V_i's, it follows that each x belongs to $\cup\mathcal{V}_n$ for infinitely many n. Clearly $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is a large cover of X and also X satisfies $\mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$.

In the other direction, assume that X satisfies $\mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Enumerate each $\mathcal{U}_n$ bijectively as $\{U^n_k : k \in \mathbb{N}\}$ and for each n consider the collection $\mathcal{V}_n = \{V^n_k : k \in \mathbb{N}\}$, where $V^n_k = U^n_1 \cup \cdots \cup U^n_k$. By Lemma 6.2.3, each $\mathcal{V}_n$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Apply $\mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \Lambda)$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ to choose a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ for each n such that $\{\cup\mathcal{W}_n : n \in \mathbb{N}\}$ is a large cover of X. By deconstructing members of $\mathcal{W}_n$, we can find a finite subset $\mathcal{Z}_n$ of $\mathcal{U}_n$ for each n. The proof will be complete if we show that $\cup_{n\in\mathbb{N}}\mathcal{Z}_n$ is a large cover of X. Let $x \in X$. Now $x \in \cup\mathcal{W}_n$ for infinitely many n, i.e., for infinitely many n there is a $V^n_k \in \mathcal{W}_n$ such that $x \in V^n_k = U^n_1 \cup \cdots \cup U^n_k$. Thus there is a $U^n_j \in \mathcal{Z}_n$ such that $x \in U^n_j$ for infinitely many n and consequently $\cup_{n\in\mathbb{N}}\mathcal{Z}_n$ is a large cover of X. □

Theorem 6.2.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If X is $\mathfrak{B}^s$-Lindelöf, then the following statements hold.*
(1) $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s}) = \mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$.
(2) $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s}) = \mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.
(3) $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}) = \mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, \mathcal{O})$
(4) $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma) = \mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma)$
(5) $\mathsf{U}_{\text{fin}}(\mathcal{O}, s\text{-}\Gamma_{\mathfrak{B}^s}) = \mathsf{U}_{\text{fin}}(s\text{-}\Gamma, s\text{-}\Gamma_{\mathfrak{B}^s})$.

Proof. We prove only (2). Suppose that X satisfies $\mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X, where $\mathcal{U}_n = \{U^n_k : k \in \mathbb{N}\}$ for each n. Now for each $n \in \mathbb{N}$ consider the collection $\mathcal{V}_n = \{V^n_k : k \in \mathbb{N}\}$, where $V^n_k = U^n_1 \cup \cdots \cup U^n_k$. By Lemma 6.2.3, $\mathcal{V}_n$'s are s-$\gamma_{\mathfrak{B}^s}$-covers of X. Apply $\mathsf{U}_{\text{fin}}(s\text{-}\Gamma_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ to $\{\mathcal{V}_n : n \in \mathbb{N}\}$ to find a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ for each n such that $\{\cup\mathcal{W}_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. By deconstructing members of $\mathcal{W}_n$, we find a finite subset $\mathcal{Z}_n$ of $\mathcal{U}_n$ for each n. Clearly $\cup\mathcal{W}_n = \cup\mathcal{Z}_n$ for each n. We show that $\{\cup\mathcal{Z}_n : n \in \mathbb{N}\} \in s\text{-}\Gamma_{\mathfrak{B}^s}$. Let $B \in \mathfrak{B}$. Since $\{\cup\mathcal{W}_n : n \in \mathbb{N}\} \in s\text{-}\Gamma_{\mathfrak{B}^s}$, there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers such that $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq \cup\mathcal{W}_n\}) = 0$, i.e., $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq \cup\mathcal{Z}_n\}) = 0$. Consequently $\{\cup\mathcal{Z}_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X and hence X

satisfies $\mathsf{U}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$. The other direction is straightforward. $\square$

Extending Theorem 2.2.5, we obtain the following game theoretic characterization of $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.

Theorem 6.2.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following conditions are equivalent.*
(1) *X satisfies $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(2) *ONE has no wining strategy in the game $\mathsf{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. It is enough to prove (1) $\Rightarrow$ (2). Let F be a strategy for ONE in the game $\mathsf{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. Let the first move of ONE be $F(X)$, a $s\text{-}\gamma_{\mathfrak{B}^s}$-cover of X enumerated bijectively as $\{U_{(n)} : n \in \mathbb{N}\}$. Let for each finite sequence τ of natural numbers of length at most m, U_τ have been already defined. Now define $\{U_{(n_1,\ldots,n_k,m)} : m \in \mathbb{N}\}$ to be $F(U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_k)}) \setminus \{U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_k)}\}$, where the enumeration $\{U_{(n_1,\ldots,n_k,m)} : m \in \mathbb{N}\}$ is bijective. It is clear that for each finite sequence τ of natural numbers, $\{U_{\tau\frown(m)} : m \in \mathbb{N}\}$ is a $s\text{-}\gamma_{\mathfrak{B}^s}$-cover of X. Now using (1) and proceeding as in Theorem 2.2.5, we can conclude that F is not a wining strategy for ONE. $\square$

Likewise the following characterization can be obtained.

Theorem 6.2.5. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following conditions are equivalent.*
(1) *X satisfies $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.*
(2) *ONE has no wining strategy in the game $\mathsf{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.*

Combining with Theorem 6.2.4, we obtain the following characterization related to the α_i-properties.

Theorem 6.2.6. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following conditions are equivalent.*
(1) *X satisfies $\alpha_2(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(2) *X satisfies $\alpha_3(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(3) *X satisfies $\alpha_4(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(4) *X satisfies $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*
(5) *ONE does not have a wining strategy in $\mathsf{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$.*

Proof. We give only proof of the following implications. The other implications follow from the standard argument.
(3) $\Rightarrow$ (4). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of $s\text{-}\gamma_{\mathfrak{B}^s}$-covers of X and $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$, $n \in \mathbb{N}$. By Lemma 6.2.2, $\mathcal{V}_n = \{V_m^n : m \in \mathbb{N}\} \in s\text{-}\Gamma_{\mathfrak{B}^s}$ for each n, where $V_m^n = U_m^1 \cap U_m^2 \cap \ldots \cap U_m^n$. Apply (3) to obtain a sequence $1 = n_0 < n_1 < n_2 < \cdots$ of positive integers such that

$\mathcal{V} = \{V^{n_i}_{m_i} : i \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover, where $V^{n_i}_{m_i} \in \mathcal{V}_{n_i}$ for each i. Now for each $i \geqslant 0$, each j with $n_i < j \leq n_{i+1}$, consider $V^{n_{i+1}}_{m_{i+1}} = U^1_{m_{i+1}} \cap \ldots \cap U^{n_{i+1}}_{m_{i+1}}$ and let $U_j = U^j_{m_{i+1}}$. Clearly $\{U_j : j \in \mathbb{N}\}$ is a $\gamma_{\mathfrak{B}^s}$-cover of X.
(5) $\Rightarrow$ (1). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of s-$\gamma_{\mathfrak{B}^s}$-covers of X. Let $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. We define a strategy σ for ONE in the game $\mathsf{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ as follows. Let the first move of ONE be $\sigma(\emptyset) = \mathcal{U}_1$. TWO chooses $U^1_{m_{i_1}} \in \mathcal{U}_1$. Now by Lemma 6.2.2, $\{U^1_m \cap U^2_m : m \geq m_{i_1}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Let the second move of ONE be $\sigma(U^1_{m_{i_1}}) = \{U^1_m \cap U^2_m : m \geq m_{i_1}\}$. TWO chooses $U^1_{m_{i_2}} \cap U^2_{m_{i_2}}$ and so on. Since the play $\sigma(\emptyset)$, $U^1_{m_{i_1}}$, $\sigma(U^1_{m_{i_1}})$, $U^1_{m_{i_2}} \cap U^2_{m_{i_2}}$ $\ldots$ in $\mathsf{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$ is lost by ONE, the collection $\{U^1_{m_{i_1}}, U^1_{m_{i_2}}, U^2_{m_{i_2}} \ldots\}$ is a $\gamma_{\mathfrak{B}^s}$-cover, which contains infinitely many elements of $\mathcal{U}_n$ for each n. Hence X satisfies $\alpha_2(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma_{\mathfrak{B}^s})$. □

Quite similarly the following characterization can be obtained.

Theorem 6.2.7. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following conditions are equivalent.*
(1) *X satisfies $\alpha_2(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.*
(2) *X satisfies $\alpha_3(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.*
(3) *X satisfies $\alpha_4(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.*
(4) *X satisfies $\mathsf{S}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.*
(5) *ONE does not have a wining strategy in $\mathsf{G}_1(s\text{-}\Gamma_{\mathfrak{B}^s}, \Gamma)$.*

6.2.2 THE s-$\mathfrak{B}^s$-HUREWICZ PROPERTY

We now define the statistically-strong-$\mathfrak{B}$-Hurewicz property and the corresponding game.

Definition 6.2.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. X is said to have the statistically-strong-$\mathfrak{B}$-Hurewicz property (s-$\mathfrak{B}^s$-Hurewicz property for short) if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$, where $\mathcal{V}_n$ is finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ such that for every $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers satisfying $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U \text{ for any } U \in \mathcal{V}_n\}) = 0$.*

Definition 6.2.3. *The statistically-strong-$\mathfrak{B}$-Hurewicz game (s-$\mathfrak{B}^s$-Hurewicz game for short) on X is defined as follows. Two players named ONE and TWO play an infinite long game. In the n-th inning ONE selects an open $\mathfrak{B}^s$-cover $\mathcal{U}_n$ of X, TWO responds by choosing a finite set $\mathcal{V}_n \subseteq \mathcal{U}_n$. TWO wins the play: $\mathcal{U}_1, \mathcal{V}_1, \mathcal{U}_2, \mathcal{V}_2, \ldots, \mathcal{U}_n, \mathcal{V}_n, \ldots$ if for each $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers satisfying $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U \text{ for any } U \in \mathcal{V}_n\}) = 0$. Otherwise ONE wins.*

We now present an example of a space having the s-$\mathfrak{B}^s$-Hurewicz property.

Example 6.2.3. *The space $X = \mathbb{R}^2$ together with the Euclidean metric d and the bornology $\mathfrak{B}$ generated by $\{S(0,r) : r > 0\}$, where $S(0,r)$ is an open ball centred at 0 with radius r, has the s-$\mathfrak{B}^s$-Hurewicz* *property. To see this, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Then clearly for each $k \in \mathbb{N}$, there is a $U \in \mathcal{U}_n$ such that $S(0,k) \subseteq U$, where $S(0,k) \in \mathfrak{B}$.*

Consider a sequence of positive integers $k_1 < k_2 < \cdots$. For each $n \in \mathbb{N}$, choose a $U_n \in \mathcal{U}_n$ such that $S(0,k_n) \subseteq U_n$ and define $\mathcal{W}_n = \{U_n\}$. We show that $\{\mathcal{W}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-Hurewicz property. Let $B \in \mathfrak{B}$. Since $\mathcal{U} = \{S(0,n) : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover, for $B \in \mathfrak{B}$ there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers satisfying $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq S(0,n)\}) = 0$. We show that $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U$ for any $U \in \mathcal{W}_n\}) = 0$. Observe that if for $n \in \mathbb{N}$ $B^{\delta_n} \nsubseteq U$ for any $U \in \mathcal{W}_n$, then $B^{\delta_n} \nsubseteq S(0,k_n)$, i.e., $B^{\delta_n} \nsubseteq S(0,n)$ as $n \leq k_n$. Clearly $\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U$ for any $U \in \mathcal{W}_n\} \subseteq \{n \in \mathbb{N} : B^{\delta_n} \nsubseteq S(0,n)\}$ and $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U$ for any $U \in \mathcal{W}_n\}) = 0$. The conclusion now follows.

We now give an example of a space without the s-$\mathfrak{B}^s$-Hurewicz property (see Example 2.3.2).

Example 6.2.4. *Let X be the Baire space with the bornology $\mathfrak{B} = \mathcal{F}$. Choose a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open $\mathfrak{B}^s$-covers of X, where $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$ and $U^n_m = \{f \in X : f(n) \leq m\}$. In order to show that the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ fails to witness the s-$\mathfrak{B}^s$-Hurewicz property, let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be any sequence, where $\mathcal{V}_n$ is finite subset of $\mathcal{U}_n$ for each n. Let $h \in X$ be such that $h(n) > 2 \cdot \max\{m : U^n_m \in \mathcal{V}_n\}$. Choose $f, g \in X$ in such a way that $h = f + g$. Clearly $\max\{f(n), g(n)\} \geq \frac{1}{2}h(n)$ and hence $\{f,g\} \nsubseteq U^n_m$ for each $U^n_m \in \mathcal{V}_n$ and each n. Let $B = \{f,g\} \in \mathfrak{B}$. Thus for any sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers $\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U^n_m$ for any $U^n_m \in \mathcal{V}_n\} = \mathbb{N}$. Since $d(\mathbb{N}) = 1$, it follows that X does not have the s-$\mathfrak{B}^s$-Hurewicz property.*

Next we introduce the notion of statistically-strong-$\mathfrak{B}$-groupable cover.

Definition 6.2.4. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. An open cover $\mathcal{U}$ of X is said to be statistically-strong-$\mathfrak{B}$-groupable (s-$\mathfrak{B}^s$-groupable for short) if it can be expressed as a union of countably many finite pairwise disjoint sets $\mathcal{U}_n$ such that for each $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers satisfying $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U$ for any $U \in \mathcal{U}_n\}) = 0$.*

The collection of all s-$\mathfrak{B}^s$-groupable covers is denoted by s-$\mathcal{O}^{gp}_{\mathfrak{B}^s}$. Clearly every $\mathfrak{B}^s$-groupable cover is s-$\mathfrak{B}^s$-groupable. Using the techniques of [27, Note 2.2] and [32, Theorem 3.5], we show that under certain condition the s-$\mathfrak{B}^s$-Hurewicz property is equivalent to the selection hypothesis $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$ (see also [28]). We recall the following definition.

> $\mathsf{CDR}_{\text{sub}}(\mathcal{A},\mathcal{B})$: *For each sequence $\{A_n : n \in \mathbb{N}\}$ of elements of $\mathcal{A}$ there is a sequence $\{B_n : n \in \mathbb{N}\}$ of pairwise disjoint elements of $\mathcal{B}$ such that for each n, $B_n \subseteq A_n$ [95].*

Theorem 6.2.8. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X and $\mathsf{CDR}_{\text{sub}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ hold. The following statements are equivalent.*

(1) *X has the s-$\mathfrak{B}^s$-Hurewicz property.*

(2) *X satisfies $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$.*

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. Since X satisfies $\mathsf{CDR}_{\mathsf{sub}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$, we can assume that $\mathcal{U}_n$'s are pairwise disjoint. Since X has the s-$\mathfrak{B}^s$-Hurewicz property, there is a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite sets with $\mathcal{V}_n \subseteq \mathcal{U}_n$ for each n such that for $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers with $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U \text{ for any } U \in \mathcal{V}_n\}) = 0$. Since $\mathcal{V}_n$'s are pairwise disjoint, $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-groupability of $\cup_{n\in\mathbb{N}}\mathcal{V}_n$. Hence X satisfies $\mathsf{S}_{\mathsf{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X and $\mathcal{U}_n = \{U^n_l : l \in \mathbb{N}\}$ for each n. Consider $\mathcal{V}_n = \{U^1_{l_1} \cap U^2_{l_2} \cap \ldots \cap U^n_{l_n} : n < l_1 < \cdots < l_n\}$ for each n. By Lemma 2.2.1 and Proposition 2.2.1, $\mathcal{V}_n$ is an open $\mathfrak{B}^s$-cover of X for each n. Now we apply $\mathsf{S}_{\mathsf{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$ to the sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ and obtain a sequence $\{\mathcal{F}_n : n \in \mathbb{N}\}$ of pairwise disjoint finite sets such that $\mathcal{F}_n \subseteq \mathcal{V}_n$ for each n and $\cup_{n\in\mathbb{N}}\mathcal{F}_n$ is a s-$\mathfrak{B}^s$-groupable cover of X. So there is a sequence $\{\mathcal{H}_k : k \in \mathbb{N}\}$ of pairwise disjoint finite subsets of $\cup_{n\in\mathbb{N}}\mathcal{F}_n$ such that for $B \in \mathfrak{B}$ there exists a sequence $\{\delta_k : k \in \mathbb{N}\}$ of positive real numbers for which $d(\{k \in \mathbb{N} : B^{\delta_k} \nsubseteq H \text{ for any } H \in \mathcal{H}_k\}) = 0$.

Define $A_i = \{k \in \mathbb{N} : \mathcal{H}_k \subseteq \cup_{j>i}\mathcal{F}_j\}$ for each $i \in \mathbb{N}$. Since each $\mathcal{F}_j$'s are finite and $\mathcal{H}_k$'s are pairwise disjoint sets, each A_i is cofinite and so $d(A_i) = 1$. Also $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Choose a $m_1 \in A_1$ such that $m_1 > 1$. Choose a $m_2 \in A_2$ with $m_2 > m_1$ in such a way that for all $n \geqslant m_2$, $\frac{|A_1(n)|}{n} > \frac{1}{2}$ and so on. Thus we obtain a sequence $m_1 < m_2 < \cdots$ of positive integers such that $m_i \in A_i$ and $\frac{|A_i(n)|}{n} > \frac{i-1}{i}$ for every $n \geqslant m_i$. We now define a subset K of positive integers as follows. $k \in K$ if $k \in (1, m_1] \cap A_1$. Also for $i > 1$, $k \in K$ if and only if $k \in (m_i, m_{i+1}] \cap A_i$. We write $K = \{k_1 < k_2 < \cdots\}$. It is easy to verify that $d(K) = 1$. For the finite number of elements k of K coming from A_1, we take the set of all U^1_l, the first components in the representation of elements of $\mathcal{H}_k$ and denote that collection by $\mathcal{G}_1$. Again for the next finite numbers of element k of K coming from A_2, we take the set of all U^2_l, the second component in the representation of the elements of $\mathcal{H}_k$ as $\mathcal{H}_k \subseteq \cup_{j>2}\mathcal{F}_j$. We denote that collection by $\mathcal{G}_2$. Continuing in this way we obtain a sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$, where $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ for each n.

We observe that $d(\{k_n \in K : B^{\delta_{k_n}} \nsubseteq H \text{ for any } H \in \mathcal{H}_{k_n}\}) = 0$ and hence $d(\{k_n \in K : B^{\delta_{k_n}} \subseteq H \text{ for some } H \in \mathcal{H}_{k_n}\}) = 1$ as $d(K) = 1$. Also it follows that $d(\{k_n \in K : B^{\delta_{k_n}} \subseteq H \text{ for some } H \in \mathcal{H}_{k_n}\}) \leqslant d(\{n \in \mathbb{N} : B^{\delta_{k_n}} \subseteq H \text{ for some } H \in \mathcal{H}_{k_n}\})$, i.e., $d(\{n \in \mathbb{N} : B^{\delta_{k_n}} \subseteq H \text{ for some } H \in \mathcal{H}_{k_n}\}) = 1$.

We now choose $\sigma_n = \delta_{k_n}$ for each n. Clearly $d(\{n \in \mathbb{N} : B^{\sigma_n} \subseteq G \text{ for some } G \in \mathcal{G}_n\}) = 1$ and hence $d(\{n \in \mathbb{N} : B^{\sigma_n} \nsubseteq G \text{ for any } G \in \mathcal{G}_n\}) = 0$. Consequently $\{\mathcal{G}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-Hurewicz property of X for $\{\mathcal{U}_n : n \in \mathbb{N}\}$. Thus (1) holds. □

Remark 6.2.1. *We do not require the assumption* $\mathsf{CDR}_{\mathsf{sub}}(\mathcal{O}_{\mathfrak{B}^s}, \mathcal{O}_{\mathfrak{B}^s})$ *to prove the implication* (2) $\Rightarrow$ (1). *We do not know whether the other direction can be obtained without this assumption.*

The intent of the next result is to show that under certain condition, a countable open $\mathfrak{B}^s$-

cover becomes s-$\mathfrak{B}^s$-groupable. In the following result we use the fact that if $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open $\mathfrak{B}^s$-cover, then for each $B \in \mathfrak{B}$ there are positive real numbers $\delta_n > 0$ and $U_n \in \mathcal{U}$ such that $B^{\delta_n} \subseteq U_n$ for infinitely many n and conversely (see Proposition 2.2.1). Thus for any finite subset $\mathcal{V}$ of $\mathcal{U}$, $\mathcal{U} \setminus \mathcal{V}$ is also an open $\mathfrak{B}^s$-cover of X.

Proposition 6.2.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If ONE has no wining strategy in the s-$\mathfrak{B}^s$-Hurwicz game in X, then every countable open $\mathfrak{B}^s$-cover of X is s-$\mathfrak{B}^s$-groupable.*

Proof. Let $\mathcal{U}$ be a countable open $\mathfrak{B}^s$-cover of X. Let σ be the strategy in the s-$\mathfrak{B}^s$-Hurewicz game on X. Let the first move of ONE be $\sigma(\emptyset) = \mathcal{U}$. TWO responds with a finite set $\mathcal{V}_1 \subseteq \mathcal{U}$. The second move of ONE is $\sigma(\mathcal{V}_1) = \mathcal{U} \setminus \mathcal{V}_1$. TWO responds with a finite set $\mathcal{V}_2 \subseteq \sigma(\mathcal{V}_1)$ and so on. By our assumption, there is a play $\sigma(\emptyset), \mathcal{V}_1, \sigma(\mathcal{V}_1), \mathcal{V}_2, \dots$ which is lost by ONE. So for each $B \in \mathfrak{B}$ there exists a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers such that $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U \text{ for any } U \in \mathcal{V}_n\}) = 0$. By construction, $\mathcal{V}_n$'s are pairwise disjoint finite sets and hence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-groupability of $\cup_{n\in\mathbb{N}}\mathcal{V}_n$. Since $\mathcal{U}$ is countable, we can show that $\mathcal{U}$ is a s-$\mathfrak{B}^s$-groupable cover of X. □

We end this section with the following game theoretic implication.

Proposition 6.2.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If ONE has no wining strategy in the s-$\mathfrak{B}^s$-Hurewicz game, then ONE has no wining strategy in the game $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$.*

Proof. Let τ be a strategy for ONE in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$. We define a strategy σ for ONE in the s-$\mathfrak{B}^s$-Hurewicz game as follows. Let the first move of ONE in the s-$\mathfrak{B}^s$-Hurewicz game be $\sigma(\emptyset) = \tau(\emptyset)$. TWO responds with a finite set $\mathcal{V}_1 \subset \sigma(\emptyset)$. Let the second move of ONE be $\sigma(\mathcal{V}_1) = \tau(\mathcal{V}_1) \setminus \mathcal{V}_1$. TWO responds with a finite set $\mathcal{V}_2 \subseteq \sigma(\mathcal{V}_1)$ and so on. By our assumption, there is a play $\sigma(\emptyset), \mathcal{V}_1, \sigma(\mathcal{V}_1), \mathcal{V}_2, \dots$ which is lost by ONE. Thus for each $B \in \mathfrak{B}$ there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers such that $d(\{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U \text{ for any } U \in \mathcal{V}_n\}) = 0$. Clearly $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a s-$\mathfrak{B}^s$-groupable open cover of X. Now for the strategy τ, consider the play $\tau(\emptyset), \mathcal{V}_1, \sigma(\mathcal{V}_1), \mathcal{V}_2, \dots$ in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$. As $\cup_{n\in\mathbb{N}}\mathcal{V}_n$ is a s-$\mathfrak{B}^s$-groupable open cover of X, τ is not a wining strategy for ONE in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\mathcal{O}^{gp}_{\mathfrak{B}^s})$.

□

6.3 RESULTS IN FUNCTION SPACES

We first recall that

> *A sequence $\{f_n : n \in \mathbb{N}\}$ of functions in $(C(X), \tau^s_{\mathfrak{B}})$ **s-converges** to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$ if for any neighbourhood $[B, \varepsilon]^s(\underline{0})$, ($B \in \mathfrak{B}$, $\varepsilon > 0$), $d(\{n \in \mathbb{N} : f_n \notin [B, \varepsilon]^s(\underline{0})\}) = 0$.*

The following lemmas are very useful in our subsequent results.

Lemma 6.3.1. *(cf. [32, Lemma 2.3]) Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. A sequence of functions in $(C(X), \tau^s_{\mathfrak{B}})$ is s-convergent if and only if any of its statistically dense subsequence is s-convergent.*

Lemma 6.3.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence in $(C(X), \tau^s_{\mathfrak{B}})$ that s-converge to $\underline{0}$. If for each n there is an open set U_n in X such that $f_n(X \setminus U_n) = \{1\}$, then $\{U_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. For the neighbourhood $[B,1]^s(\underline{0})$, we have $d(T) = 0$, where $T = \{n \in \mathbb{N} : f_n \notin [B,1]^s(\underline{0})\}$. We need to show that for $B \in \mathfrak{B}$, there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers for which $d(S) = 0$, where $S = \{n \in \mathbb{N} : B^{\delta_n} \nsubseteq U_n\}$. First note that $f_n^{-1}(-1,1) \subseteq U_n$ for each n. If $n \notin T$, then $f_n \in [B,1]^s(\underline{0})$, i.e., there is a $\eta_n > 0$ with $B^{\eta_n} \subseteq f_n^{-1}(-1,1) \subseteq U_n$. Fix a $\delta > 0$. Define $\delta_n = \delta$ if $n \in T$ and $\delta_n = \eta_n$ otherwise. Then $n \notin T$ implies $n \notin S$. Thus we obtain a sequence $\{\delta_n : n \in \mathbb{N}\}$ for which $S = T$ and $d(S) = 0$. Hence $\{U_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. □

Lemma 6.3.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of functions in $(C(X), \tau^s_{\mathfrak{B}})$. If $\{f_n^{-1}(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X, then $\{f_n : n \in \mathbb{N}\}$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$.*

Proof. For $B \in \mathfrak{B}$ there is a sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive real numbers for which $d(S) = 0$, where $S = \{n \in \mathbb{N} : B^{\delta_n} \nsubseteq f_n(-\frac{1}{n}, \frac{1}{n})\}$. We now show that for any neighbourhood $[B,\varepsilon]^s(\underline{0})$, $d(T) = 0$, where $T = \{n \in \mathbb{N} : f_n \notin [B,\varepsilon]^s(\underline{0})\}$. For $\varepsilon > 0$, choose a n_1 such that $\frac{1}{n_1} < \varepsilon$. Now if $n \in T \setminus \{1, 2, \ldots, n_1 - 1\}$, then $f_n \notin [B,\varepsilon]^s(\underline{0})$, i.e., for any $\zeta > 0$, $B^\zeta \nsubseteq f_n^{-1}(-\varepsilon, \varepsilon)$, i.e., $B^\zeta \nsubseteq f_n^{-1}(-\frac{1}{n}, \frac{1}{n})$. Choose $\zeta = \delta_n$, then $B^{\delta_n} \nsubseteq f_n^{-1}(-\frac{1}{n}, \frac{1}{n})$. Clearly $n \in S$ and hence $T \setminus \{1, 2, \ldots, n_1 - 1\} \subseteq S$. It is now evident that $d(T) = 0$. Hence $\{f_n : n \in \mathbb{N}\}$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$. □

Under certain assumption on a subset of $C(X)$, in the following result we show that every open $\mathfrak{B}^s$-cover of X has a s-$\gamma_{\mathfrak{B}^s}$-subcover.

Proposition 6.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. For $A \subseteq (C(X), \tau^s_{\mathfrak{B}})$ with $\underline{0} \in \overline{A}$, if there is a sequence in A which s-converges to $\underline{0}$, then every open $\mathfrak{B}^s$-cover of X contains a s-$\gamma_{\mathfrak{B}^s}$-cover of X.*

Proof. Let $\mathcal{U}$ be an open $\mathfrak{B}^s$-cover of X. For $B \in \mathfrak{B}$ there exist a $U_B \in \mathcal{U}$ and a $\delta > 0$ such that $B^{2\delta} \subseteq U_B$. Choose a $f_B \in C(X)$ such that $f_B(B^\delta) = \{0\}$ and $f_B(X \setminus U_B) = \{1\}$. Consider the set $A = \{f_B : B \in \mathfrak{B}\}$. Clearly $\underline{0} \in \overline{A}$. By our assumption, there is a sequence $\{f_{B_n} : n \in \mathbb{N}\}$ s-converging to $\underline{0}$. Clearly $\{U_{B_n} : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover by Lemma 6.3.2. □

The statistically strictly Frèchet Urysohn property of $C(X)$ can be characterized by a S_1-type selective property of X.

A space X is said to be ***statistically strictly Frèchet Urysohn*** *(s-SFU for short) [32] if* $\mathsf{S}_1(\Omega_x, s\text{-}\Sigma_x)$ *holds for each* $x \in X$.

Theorem 6.3.1. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. The following statements are equivalent.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ is s-SFU.
(2) X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.

Proof. (1) $\Rightarrow$ (2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of open $\mathfrak{B}^s$-covers of X. For $B \in \mathfrak{B}$ and each $n \in \mathbb{N}$, there are $U_{B,n} \in \mathcal{U}_n$ and $\delta > 0$ such that $B^{2\delta} \subseteq U_{B,n}$. Consider the collection $\mathcal{U}_{B,n} = \{U \in \mathcal{U}_n : B^{2\delta} \subseteq U\}$. For each $U \in \mathcal{U}_{B,n}$ choose a $f_{B,U} \in C(X)$ such that $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. Now define $A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_{B,n}\}$ for each $n \in \mathbb{N}$. Clearly $\underline{0} \in \overline{A_n} \setminus A_n$. Apply (1) to the sequence $\{A_n : n \in \mathbb{N}\}$ to find a $f_{B_n,U_n} \in A_n$ for each n such that $\{f_{B_n,U_n} : n \in \mathbb{N}\}$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$, where $U_n \in \mathcal{U}_n$ for each n. Clearly $\{U_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover by Lemma 6.3.2. Hence X satisfies $\mathsf{S}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of subsets of $C(X)$ such that $\underline{0} \in \overline{A_n} \setminus A_n$ for each $n \in \mathbb{N}$. By Lemma 2.4.1, $\mathcal{U}_n = \{f^{-1}(-\frac{1}{n}, \frac{1}{n}) : f \in A_n\}$ is an open $\mathfrak{B}^s$-cover of X for each $n \in \mathbb{N}$. Apply (2) to the sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ to obtain a $U_n \in \mathcal{U}_n$ for each n such that $\{U_n : n \in \mathbb{N}\}$ is a s-$\gamma_{\mathfrak{B}^s}$-cover of X. Now $U_n = f_n^{-1}(-\frac{1}{n}, \frac{1}{n})$. By Lemma 6.3.3, it follows that the sequence $\{f_n : n \in \mathbb{N}\}$ s-converges to $\underline{0}$ with respect to $\tau^s_{\mathfrak{B}}$ and hence $(C(X), \tau^s_{\mathfrak{B}})$ is s-SFU. □

Similar to Proposition 6.2.1, we can compare the following selective properties in $C(X)$.

Proposition 6.3.2. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. Consider the following statements.*
(1) $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_1(s\text{-}\Sigma_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$.
(2) $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $s\text{-}\alpha_4(s\text{-}\Sigma_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$.
(3) $(C(X), \tau^s_{\mathfrak{B}})$ satisfies $\mathsf{S}_{\text{fin}}(s\text{-}\Sigma_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$.
Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) holds.

Proof. The proof of (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (3). Let $\{S_n : n \in \mathbb{N}\}$ be a sequence of elements in s-$\Sigma_{\underline{0}}$. For each n take $S_n = \{f_{n,m} : m \in \mathbb{N}\}$. By (2), there is a $T \in s\text{-}\Sigma_{\underline{0}}$ and a (statistically dense) subset $K = \{n_1 < n_2 < \cdots\}$ of $\mathbb{N}$ with $d(K) = 1$ such that the set $S_{n_i} \cap T$ is non empty for each $n_i \in K$. Let $f_{n_i,m_i} \in S_{n_i} \cap T$ for each $n_i \in K$. By Lemma 6.3.1, the subsequence $\{f_{n_i,m_i} : i \in \mathbb{N}\}$ is s-convergent to $\underline{0}$. Define $F_n = \{f_{n_i,m_i}\}$ if $n = n_i$ and $F_n = \emptyset$ otherwise. The conclusion now follows from the fact that $\cup_{n\in\mathbb{N}} F_n$ is s-convergent to $\underline{0}$ and each F_n is a finite subset of S_n. □

We now give another sufficient condition for a countable open $\mathfrak{B}^s$-cover of X to be s-$\mathfrak{B}^s$-groupable (compare with Proposition 6.2.2) with the help of s-Reznichenko property of $C(X)$.

*X is said to have the s-**Reznichenko property** at $x \in X$ [32] if each countable set A in Ω_x can be represented as a countable union of finite and pairwise disjoint subsets of A such that for each neighbourhood W of x, $d(\{n \in \mathbb{N} : W \cap A_n = \emptyset\}) = 0$.*

In the following result we use the fact that a collection $\mathcal{U}$ of open subsets of X is an open $\mathfrak{B}^s$-cover if and only if for each $U \in \mathcal{U}$ there is a closed set (may be empty) $C(U) \subseteq U$ such that $\{C(U) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X (see Theorem 2.4.1).

Proposition 6.3.3. *Let $\mathfrak{B}$ be a bornology with closed base on a metric space X. If $(C(X), \tau^s_{\mathfrak{B}})$ has the s-Reznichenko property, then every countable open $\mathfrak{B}^s$-cover of X is s-$\mathfrak{B}^s$-groupable.*

Proof. Let $\mathcal{U}$ be a countable open $\mathfrak{B}^s$-cover of X. For each $U \in \mathcal{U}$ there is a closed set $C(U)$ with $C(U) \subseteq U$ such that $\{C(U) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X. Define a continuous function f_U on X such that $f_U(C(U)) = \{0\}$ and $f_U(X \setminus U) = \{1\}$. Clearly $\{f_U^{-1}(\{0\}) : U \in \mathcal{U}\}$ is a $\mathfrak{B}^s$-cover of X. We can assume that f_U and f'_U are distinct whenever U and U' are distinct. Consider the set $A = \{f_U : U \in \mathcal{U}\}$. Evidently $\underline{0} \in \overline{A} \setminus A$. By our assumption, there is a sequence $\{A_n : n \in \mathbb{N}\}$ of pairwise disjoint finite subsets of A such that for the neighbourhood $[B,1]^s(\underline{0})$, $d(T) = 0$, where $T = \{n \in \mathbb{N} : [B,1]^s(\underline{0}) \cap A_n = \emptyset\}$. Let $\mathcal{U}_n = \{U : f_U \in A_n\}$ for each n. Clearly $\mathcal{U}_n$'s are pairwise disjoint and finite. Using the fact that $d(T) = 0$, we can show that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ witnesses the s-$\mathfrak{B}^s$-groupability of $\mathcal{U}$. This completes the proof.

□

Theorem 6.3.2. *Let $\mathfrak{B}$ be a bornology on X with closed base. The following statements are equivalent.*
(1) TWO has a wining strategy in the game $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ on $(C(X), \tau^s_{\mathfrak{B}})$.
(2) TWO has a wining strategy in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ on X.

Proof. Let ψ be a wining strategy for TWO in the game $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$. We define a strategy σ for TWO in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ as follows. In the n-th inning, let the move of ONE in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ be $\mathcal{U}_n$. For each $B \in \mathfrak{B}$, there exist a $\delta > 0$ and a $U \in \mathcal{U}_n$ such that $B^{2\delta} \subseteq U$. Choose a $f_{B,U} \in C(X)$ such that $f_{B,U}(B^\delta) = \{0\}$ and $f_{B,U}(X \setminus U) = \{1\}$. Then the collection $A_n = \{f_{B,U} : B \in \mathfrak{B}, U \in \mathcal{U}_n\}$ is in $\Omega_{\underline{0}}$. In the game $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$, let the move of ONE be A_n. TWO responds with $\psi(A_1, \dots, A_n) = f_{B_n,U_n}$. Now we define $\sigma(\mathcal{U}_1, \dots, \mathcal{U}_n) = U_n$. This defines a strategy for TWO in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$. A play in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$ is $\mathcal{U}_1, \sigma(\mathcal{U}_1), \dots, \mathcal{U}_n, \sigma(\mathcal{U}_1, \dots, \mathcal{U}_n), \dots$ and the corresponding play in the game $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ is $A_1, \psi(A_1), \dots, A_n, \psi(A_1, \dots, A_n), \dots$ Since ψ is a wining strategy for TWO in the game $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$, $\{f_n : n \in \mathbb{N}\} \in s\text{-}\Sigma_{\underline{0}}$, where $f_n = f_{B_n,U_n}$. Now by Lemma 6.3.2, σ a wining strategy for TWO in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$.

(2) $\Rightarrow$ (1). Let σ be a wining strategy for TWO in the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$. For each n let $I_n = (-\frac{1}{n}, \frac{1}{n})$. We define a strategy ψ for TWO in the game $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ as follows. In the n-th inning

let the move of ONE in the game $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$ is $A_n \in \Omega_{\underline{0}}$. By Lemma 2.4.1 the set $\mathcal{U}(A_n) = \{f^{-1}(I_n) : f \in A_n\}$ is an open $\mathfrak{B}^s$-cover. In the game $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}^s}, s\text{-}\Gamma_{\mathfrak{B}^s})$, let the n-th move of ONE be $\mathcal{U}(A_n)$. TWO responds with $\sigma(\mathcal{U}(A_1), \ldots, \mathcal{U}(A_n)) = U_n$. Now define $\psi(A_1, \ldots, A_n) = f_n$, where $U_n = f_n^{-1}(I_n)$. This defines a strategy for TWO in the game $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$. Using the fact that σ is a winning strategy for TWO and Lemma 6.3.3, we can show that ψ is a wining strategy for TWO in $\mathsf{G}_1(\Omega_{\underline{0}}, s\text{-}\Sigma_{\underline{0}})$. □

References

[1] P.S. Alexandroff, Einführung in die Mengenlehre und die Theorie der reellen Funktionem, Deutsch Verlag Wissenschaft, 1956, translated from the 1948 Russian edition.

[2] A.V. Arhangel'skii, Hurewicz spaces, analytic sets and fan tightness of function spaces, Soviet Math. Dokl., 33 (1986), 396–399.

[3] A.V. Arhangel'skii, R. Isler and G. Tironi, On pseudo-radial spaces, Comment. Math. Univ. Carolin., 27 (1986), 137–154.

[4] A.V. Arhangel'skii, Topological Function Spaces, Kluwer, (1992).

[5] C. Arzelá, Intorno alla continuitá della somma di infinitá di funzioni continue, Rend. R. Accad. Sci. Istit., Bologna, (1883/1884), 79–84.

[6] C. Arzelá, Sulle serie di funzioni, Mem. R. Accad. Sci. Ist., Bologna, 5 (8) (1899/1900), 131–186, 701–744.

[7] L.F. Aurichi and R.M. Mezabarba, Bornologies and filters applied to selection principles and function spaces, Topology Appl., 258 (2019), 187–201.

[8] J.E. Baumgartner and A.D. Taylor, Partition theorems and ultrafilters, Trans. Amer. Math. Soc., 241 (1978), 283–309.

[9] R.G. Bartle, On compactness in functional analysis, Trans. Amer. Math. Soc., 79 (1955), 35–57.

[10] L. Babinkostova, Lj.D.R. Kočinac and M. Scheepers, Combinatorics of open covers (VIII), Topology Appl., 140 (2004), 15–32.

[11] T. Bartoszynski and M. Scheepers, A-sets, Real Anal. Exchange, 19(2) (1993-94), 521–528.

[12] A. Bella, M. Bonanzinga and M. Matveev, Variations of selective separability, Topology Appl., 156 (2009), 1241–1252.

[13] G. Beer and S. Levi, Pseudometrizable bornological convergence is Attouch–Wets convergence, J. Convex Anal., 15 (2008), 439–453.

[14] G. Beer and S. Levi, Strong uniform continuity, J. Math. Anal. Appl., 350 (2009), 568–589.

[15] M. Bonanzinga, F. Cammaroto and M. Matveev, Projective versions of selection principles, Topology Appl., 157 (2010), 874–893.

[16] L. Bukovský and J. Haleš, QN-spaces, wQN-spaces and covering properties, Topology Appl., 154 (2007), 848–858.

[17] A. Caserta, G. Di Maio, Lj.D.R. Kočinac and E. Meccariello, Applications of k-covers II, Topology Appl., 153 (2006), 3277–3293.

[18] A. Caserta, G. Di Maio and Ĺ. Holá, Arzelà's theorem and strong uniform convergence on bornologies, J. Math. Anal. Appl., 371 (2010), 384–392.

[19] A. Caserta, G. Di Maio and Lj.D.R. Kočinac, Bornologies, selection principles and function spaces, Topology Appl., 159 (7) (2012), 1847–1852.

[20] A. Caserta, Lj.D.R. Kočinac, On statistical exhaustiveness, Appl. Math. Lett., 25 (2012), 1447–1451.

[21] A. Caserta, Strong Whitney convergence, Filomat, 26 (1) (2012), 81–91.

[22] D. Chandra, P. Das and S. Das, Applications of bornological covering properties in metric spaces, Indag. Math., 31 (2020), 43–63.

[23] D. Chandra, P. Das and S. Das, Certain observations on tightness and topological games in bornology, submitted.

[24] S. Clontz and J. Holshouser, Limited Information Strategies and Discrete Selectivity, Topology Appl., 265 (2019), 106815.

[25] T.K. Chauhan and Varun Jindal, Strong Whitney and strong uniform convergences on a bornology, J. Math. Anal. Appl., 505 (2022), 125634.

[26] P. Das, D. Chandra and S. Das, Certain observations on selection principles from (a) bornological viewpoint, Quaest. Math., 45(3) (2022), 423–442.

[27] P. Das, Certain types of open covers and selection principles using ideals, Houston J. Math., 39 (2) (2013), 637-650.

[28] P. Das, Lj.D.R. Kočinac and D. Chandra, Some remarks on open covers and selection principles using ideals, Topology Appl., 202 (2016), 183–193.

[29] P. Das, D. Chandra and S. Das, Further applications of bornological covering properties in function spaces Topology Appl., 310 (2022), 108005.

[30] S. Das and D. Chandra, Certain observations on statistical variations of bornological covers, Filomat, 35(7) (2021), 2303–2315.

[31] G. Di Maio, Lj.D.R. Kočinac and E. Meccariello, Applications of k-covers, Acta Math. Sinica, English Series 22 (4) (2006).

[32] G. Di Maio and Lj.D.R. Kočinac, Statistical convergence in topology, Topology Appl., 156 (2008), 28–45.

[33] G. Di Maio, D. Djurčić, Lj.D.R. Kočinac and M.R. Žižović, Statistical convergence, selection principles and asymptotic analysis, Chaos Solitons Fractals, 42 (2009), 2815–2821.

[34] G. Di Maio and Lj.D.R. Kočinac, Statistical convergence in function spaces, Abstr. Appl. Anal., 2011 (2011), Article ID 420419.

[35] J. Ewert, On strong form of Arzelà convergence, Int. J. Math. Math. Sci., 20 (1997), 417–422.

[36] R. Engelking, General Topology, Sigma Ser. Pure Math., Heldermann, Berlin, (1989).

[37] H. Fast, Sur la convergence statistique, Colloq. Math., 2 (1951), 241–244.

[38] A. Fedel and A. Le Donne, Pytkeev spaces and sequential extensions, Topology Appl., 117 (2002), 345–348.

[39] D.H. Fremlin and A.W. Miller, On some properties of Hurewicz, Menger and Rothberger, Fund. Math., 129 (1988), 17–33.

[40] F. Galvin, Indeterminacy of point-open games, Bulletin de L'Academie Polonaise des Sciences 26 (1978), 445–448.

[41] J. Gerlits and Zs. Nagy, Some properties of $C(X)$, I, Topology Appl., 14 (1982), 151–161.

[42] S. García-Ferreira and A. Tamariz-Mascarúa, *p-Fréchet-Uryshon property of function spaces*, Topology Appl., 58 (1994), 157–172.

[43] S. García-Ferreira and A. Tamariz-Mascarúa, Some Generalizations of rapid ultrafilters in topology and Id-fan tightness, Tsukuba J. Math., 19 (1995), 173–185.

[44] S. García-Ferreira and A. Tamariz-Mascarúa, *The γ_p-property and the reals*, Proc. Amer. Math. Soc., 126 (1998), 1791–1798.

[45] G. Gruenhage, Infinite games and generalizations of first countable spaces, Gen. Top. and Appl., 6 (1976), 339–352.

[46] G. Gruenhage and P.J. Szeptycki, Fréchet-Urysohn for finite sets, Topology Appl., 151 (2005), 238–259.

[47] E.W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier Series, vols. 1 and 2, second edition, Cambridge University Press, 1926.

[48] H. Hogbe-Nlend, Bornologies and Functional Analysis, North-Holland, Amsterdam, (1977).

[49] Ĺ. Holá, Complete metrizability of topologies of strong uniform convergence on bornologies, J. Math. Anal. Appl., 387 (2012), 770–775.

[50] Ĺ. Holá and B. Novotný, Cardinal functions, bornologies and function spaces, Ann. Mat. Pura Appl., 193 (2014), 1319–1327.

[51] W. Hurewicz, Über die Verallgemeinerung des Borelschen Theorems, Mathematische Zeitschrift 24 (1925), 401–425.

[52] W. Hurewicz, Über Folgen stetiger Funktionen, Fund. Math., 9 (1927), 193–204.

[53] I. Juhász, Variations on tightness, Studia Sci. Math. Hungar., 24 (1989), 179–186.

[54] W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki, The combinatorics of open covers (II), Topology Appl., 73 (1996), 241–266.

[55] W. Just and A. Tanner, Splitting ω-covers, Comment. Math. Univ. Carolin., 38 (1997), 375–378.

[56] K. Kuratowski, Introduction to set theory and topology, Pergamon Press, (1961).

[57] K. Kuratowski, Topology, volume 1, Academic Press (1966).

[58] P. Kostyrko, M. Mačaj, T. Šalát and O. Strauch, On statistical limit points, Proc. Amer. Math. Soc., 129 (2000), 2647–2654.

[59] Lj.D.R. Kočinac, The Pixley-Roy topology and selection principles, Quest. Ans. Gen. Top., 19 (2001), 219–225.

[60] Lj.D.R. Kočinac and M. Scheepers, Function spaces and a property of Reznichenko, Topology Appl., 123 (2002), 135–143.

[61] Lj.D.R. Kočinac, Closure properties of function spaces, Appl. Gen. Topol., 4 (2) (2003), 255–261.

[62] Lj.D.R. Kočinac and M. Scheepers, Combinatorics of open covers (VII), Fund. Math., 179 (2003), 131–155.

[63] Lj.D.R. Kočinac, Selected results on selection principles, in: Proc. Third Seminar on Geometry and Topology, Tabriz, Iran, July 15–17, (2004), 71–104.

[64] Lj.D.R. Kočinac, γ-sets, γ_k-sets and hyperspaces, Math. Balkanica., 19 (2005), 109–118.

[65] Lj.D.R. Kočinac, Selection principles related to α_i -properties, Taiwanese J. Math., 12 (2008), 561–571.

[66] J. van Mill and C.F. Mill, On the character of supercompact spaces, Topology Proc., 3 (1978), 227–236.

[67] K. Menger, Einige Überdeckungssätze der Punktmengenlehre, Sitzungsberischte Abt. 2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie, Wien) 133 (1924), 421–444.

[68] V.I. Malykhin and G. Tironi, Weakly Fréchet–Urysohn and Pytkeev spaces, Topology Appl., 104 (2000), 181–190.

[69] S. Lin, C. Liu and H. Teng, Fan tightness and strong Fréchet property of $C_k(X)$, Adv. Math., (China) 23 (3) (1994), 234–237 (in Chinese); MR. 95e:54007, Zbl. 808.54012.

[70] R.A. McCoy, k-space function spaces, Intern. J. Math. Math. Sci., 3 (1980), 701–711.

[71] R.A. McCoy, Function spaces which are k-spaces, Topology Proc., 5 (1980), 139–146.

[72] R.A. McCoy and I. Ntantu, Topological properties of spaces of continuous functions, Lecture Notes in Math., Vol. 1315, Springer-Verlag, Berlin, (1988).

[73] R.A. McCoy and I. Ntantu, Countability properties of function spaces with set open topologies, Topology Proc., 10 (1985), 329–345.

[74] N. Noble, The density character of functions spaces, Proc. Amer. Math. Soc., 42 (1974), 228–233.

[75] A.V. Osipov, Classification of selectors for sequences of dense sets of $C_p(X)$, Topology Appl., 242 (2018), 20–32.

[76] A.V. Osipov, Selection principles in function spaces with the compact-open topology, Filomat, 15 (2018), 5403–5413.

[77] A.V. Osipov, On selective sequential separability of function spaces with the compact-open topology, Hacet. J. Math. Stat., 48 (6) (2019), 1761–1766.

[78] H. Poppe, Compactness in function spaces with a generalized uniform structure II, Bull. Acc. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 18 (1970), 567–573.

[79] M. Predoi, Sur la convergence quasi-uniforme, Period. Math. Hungar., 10 (1979), 31–40.

[80] M. Predoi, Sur la convergence quasi-uniforme topologique, An. Univ. Craiova., 11 (1983), 15–20.

[81] F. P. Ramsey, On a problem of formal logic, Proc. London Math. Soc., 30 (1930), 264–286.

[82] F. Rothberger, Eine Verschärfung der Eigenschaft C, Fund. Math., 30 (1938), 50–55.

[83] J. Pawlikowski, Undetermined sets of point-open games, Fund. Math., 144 (1994), 279–285.

[84] E.G. Pytkeev, On maximally resolvable spaces, Proc. Steklov Inst. Math., 154 (1984), 225–230.

[85] E.G. Reznichenko, The talk of 25.04.96 at Arhangel'skii's Seminar at Moscow University.

[86] E. Reznichenko and O. Sipacheva, Fréchet-Urysohn type properties in topological spaces, groups and locally convex vector spaces, Moscow Univ. Math. Bull., 54 (3) (1999), 33–38.

[87] M. Sakai, Property C'' and function spaces, Proc. Amer. Math. Soc., 104 (1988), 917–919.

[88] M. Sakai, On supertightness and function spaces, Comment. Math. Univ. Carolin., 29 (1988), 249–251.

[89] M. Sakai, Variations on tightness in function spaces, Topology Appl., 101 (2000), 273–280.

[90] M. Sakai, The Pytkeev property and the Reznichenko property in function spaces, Note Mat., 22 (2003), 43–52.

[91] M. Sakai, The sequence selection properties of $C_p(X)$, Topology Appl., 154 (2007), 552–560.

[92] M. Sakai, Selective separability of Pixley Roy hyperspaces, Topology Appl., 159 (2012), 1591–1598.

[93] M. Sakai and M. Scheepers, The combinatorics of open covers, Chapter in Recent Progress in General Topology III, Atlantis Press, (2014), 751–799.

[94] M. Sakai, The projective Menger property and an embedding of S_ω into function spaces, Topology Appl., 220 (2017), 118–130.

[95] M. Scheepers, Combinatorics of open covers (I): Ramsey theory, Topology Appl., 69 (1996), 31–62.

[96] M. Scheepers, Combinatorics of open covers (III): $C_p(X)$ and games, Fund. Math., 152 (1997), 231–254.

[97] M. Scheepers, Combinatorics of open covers (VI): Selectors for sequences of dense sets, Quaest. Math., 22 (1999), 109–130.

[98] M. Scheepers, Sequential convergence in $C_p(X)$ and a covering property, East-West J. Math., 1 (2) (1999), 207–214.

[99] M. Scheepers, Selection principles and covering properties in topology, Note Mat., 22 (2) (2003/2004), 3–41.

[100] M. Scheepers, Gerlits and function spaces, Studia Sci. Math. Hungar., 47 (4) (2010), 529–557.

[101] M. Schepeers, Remarks on countable tightness, Topology Appl., 161 (2014), 407–432.

[102] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math., 2 (1951), 73–74.

[103] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66 (1959), 361–375.

[104] R. Telgársky, On topological properties defined by games, Topics in Topology (Proc. Colloq. Keszthely 1972), Colloq. Math. Soc. János Bolyai, Vol. 8, North-Holland, Amsterdam, (1974), 617–624.

[105] R. Telgársky, Spaces defined by topological games, Fund. Math., 88 (1975), 193–223.

[106] R. Telgársky, Spaces defined by topological games II, Fund. Math., 116 (1984), 189–207.

[107] R. Telgársky, On games of Topsøe, Math. Scand., 54 (1984), 170–176.

[108] V.V. Tkachuk, Closed discrete selections for sequences of open sets in functions spaces, Acta. Math. Hungar., 154 (2018), 56–68.

[109] V.V. Tkachuk, Two point-picking games derived from a property of function spaces, Quaest. Math., (2017), 1–15.

[110] B. Tsaban, The combinatorics of splittability, Ann. Pure Appl. Logic, 129 (2004), 107–130.

[111] B. Tsaban, Some new directions in infinite combinatorial topology, in: J. Bagaria, S. Todorcevic (Eds), Set Theory, in Trends Math., Birkhauser, 2006, pp. 225–255.

[112] B. Tsaban, A new selection principle, Topology Proc., 31 (1) (2007), 319–329.

[113] B. Tsaban and L. Zdomskyy, Additivity of the Gerlits-Nagy property and concentrated sets, Proc. Amer. Math. Soc., 142 (2014), 2881–2890.

[114] J.E. Vaughan, Small uncountable cardinals and topology, in: Open Problems in Topology, J. van Mill and G.M. Reed, eds., North Holland, (1990), 195–218.

[115] S. Willard, General Topology, Addison-Wesley, Reading, MA, 1970.

[116] A. Zygmund, Trigonometric Series, second ed., Cambridge University Press, Cambridge, 1979.

Index

Symbols

PUBLICATIONS

The thesis is based on the following research works:

1. *Debraj Chandra, Pratulananda Das and Subhankar Das*
   APPLICATIONS OF BORNOLOGICAL COVERING PROPERTIES IN METRIC SPACES
   Indagationes Mathematicae, 31 (2020), 43–63.

2. *Subhankar Das and Debraj Chandra*
   CERTAIN OBSERVATIONS ON STATISTICAL VARIATIONS OF BORNOLOGICAL COVERS
   Filomat, 35(7) (2021), 2303–2315.

3. *Debraj Chandra, Pratulananda Das and Subhankar Das*
   CERTAIN OBSERVATIONS ON SELECTION PRINCIPLES FROM (A) BORNOLOGICAL VIEW-POINT
   Quaestiones Mathematicae, 45(3) (2022), 423–442.

4. *Pratulananda Das, Debraj Chandra and Subhankar Das*
   FURTHER APPLICATIONS OF BORNOLOGICAL COVERING PROPERTIES IN FUNCTION SPACES
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5. *Debraj Chandra, Pratulananda Das and Subhankar Das*
   CERTAIN OBSERVATIONS ON TIGHTNESS AND TOPOLOGICAL GAMES IN BORNOLOGY
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