ON CERTAIN VERSIONS OF CONTINUITY AND RELATED PROPERTIES ASSOCIATED WITH CAUCHY AND QUASI-CAUCHY SEQUENCES

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LIST OF PUBLICATIONS

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7. **N. Adhikary** and S.K. Pal, On certain notions of precompactness, continuity and Lipschitz functions, communicated.

Abstract

The main objective of this thesis is to investigate various aspects of both metric space and uniform space concerning uniform continuity by using the notions of Cauchy regularity and ward continuity. In the initial three chapters, our main emphasis lies on exploring Cauchy regularity. There are two well-known notions related to uniform continuity, the UC space and a more general concept of straight space. A space X is called UC if every real valued continuous function on X is uniformly continuous and a space X is called straight if for any real-valued continuous function f on X and a closed cover $X = C_1 \cup C_2$ of X, the restrictions of f on both C_1, C_2 are uniformly continuous implies f is uniformly continuous. Hence straight space is a kind of generalization of UC space and one can see an additive type property of uniform continuity in a straight space. In Chapter 4, we define two types of straight spaces using Cauchy regularity namely pre-straight and pre($*$)-straight and we show that class of all straight spaces are basically the intersection of class of all pre-straight and class of all pre($*$)-straight spaces. The concept of straightness is closely linked with certain versions of connectedness. To investigate this direction in our context in Chapter 3, we introduce a new type of connectedness namely Cauchy connected space and present some relations of Cauchy connected space with various types of straight spaces. In Chapter 5, we analyze another perspective of Cauchy regularity specifically its preserving properties. We find several conditions under which a precompactness- and Cauchy connectedness- preserving function is Cauchy regular and discuss the role of a pre-straight space as generalization of complete space.

In Chapter 6, initially we work on several types of completeness in uniform space, which strictly lies in between compactness and completeness. We find some results on metrizability in line with the well-known results given by Čech, which state that a metrizable space X is completely metrizable iff $X = \bigcap_{n=1}^{\infty} G_n$, where each G_n is an open subspace of βX. At the end of this chapter we look into the notion of BqC sequence in metric space and connect it with quasi-Cauchy sequence. Finally, in the last chapter we discuss about some variants of Cauchy regular functions.

LIST OF SYMBOLS AND ABBREVIATIONS

Symbols

- $\mathbb{N}$: The set of all natural numbers.
- $\mathbb{Q}$: The set of all rational numbers.
- $\mathbb{R}$: The set of all real numbers.
- ℓ_∞ : The space of all bounded sequences.
- (e_n) : The sequence of unit vectors in ℓ_∞.
- $||.||_\infty$: Sup norm of ℓ_∞.
- $X \setminus A$ or A^c : The complement of A in X.
- $f|_A$: The restriction of the function f on A, where $f : X \to Y$ be a function and $\emptyset \neq A \subset X$.
- βX : The Stone Čech compactification of X.
- $C(X)$: Space of all real-valued continuous functions defined on X.
- $CC(X)$: Space of all Cauchy regular functions defined on X.
- $\overline{A}$: Closure of A in X.
- A° : Interior of A in X.

For the following notations, let (X, d) and (Y, ρ) be two metric spaces, $x \in X$ be a point and $\varepsilon > 0$ be given. Also, let $f : X \to Y$ be a function.

- $B_d(x, \varepsilon)$ or $B(x, \varepsilon)$: The open ball in (X, d) centered at x with radius ε.

- $\mathcal{B}_\varepsilon$: The cover of the open balls with a fixed radius $\varepsilon > 0$, and it equals to the set $\{B_d(x,\varepsilon) : x \in X\}$.
- μ_d : The uniformity induced by the metric d and the base of the uniformity is the family of all $\mathcal{B}_\varepsilon$.
- $(\widehat{X}, \widehat{d})$ or $\widehat{X}$: The completion of (X, d).
- $G(f)$: The graph of the function f.
- $\widehat{G(f)}$: The completion of the graph of f.
- $s_d X$: The Samuel compactification of (X, d).
- $B_d^n(x,\varepsilon)$ or $B^n(x,\varepsilon)$: nth ε-enlargement of x in X.
- $B_d^\infty(x,\varepsilon)$ or $B^\infty(x,\varepsilon)$: ε-chainable component of x in X.

For the following notations let (X,μ) be a uniform space, where μ be a distinguished family of coverings and $\mathcal{U}, \mathcal{V} \in \mu$.

- $\mathcal{U} < \mathcal{V}$: $\mathcal{U}$ is a refinement of $\mathcal{V}$.
- $\mathcal{U}^* < \mathcal{V}$: $\mathcal{U}$ is a star refinement of $\mathcal{V}$.
- $\mathbf{u}$: The fine uniformity.
- $s_f\mu$: The star finite modification of (X,μ).
- $s_f^\infty\mu$: The finite-component modification of (X,μ).

Abbreviations

- BqC filter (sequence): Bourbaki quasi-Cauchy filter (sequence).
- Bq-complete: Bourbaki quasi-complete.
- cBq-complete: cofinally Bourbaki quasi- complete.

Contents

1 Introduction

Mathematicians have always been captivated by continuity and its various forms. Not only are functions of great interest but the related sequences and filters also play a crucial role within the framework of metric space and uniform space. In 1981, Snipes [71] introduced a well-known notion called Cauchy regular or Cauchy continuous function, which lies between the classes of all continuous and uniformly continuous functions [71]. A function f between two uniform spaces is called Cauchy regular if it preserves Cauchy filters. From [51], it is evident that a function f between two metric spaces is Cauchy regular iff it preserves Cauchy sequences. In this thesis, we explore various notions, such as connectedness, variants of completeness using Cauchy conditions and investigate related properties. We also focus on different types of maps, which can be thought of as a generalization of Cauchy regularity.

In the realm of metric structures, there exists another type of sequence known as quasi-Cauchy sequence [23], which has been explored as a weaker version of the Cauchy sequence. However, it is worth noting that the investigation of quasi-Cauchy sequences has been primarily limited to metric spaces, and extending its definition to more general uniform spaces seems difficult. The function that preserves quasi-Cauchy sequence is called ward continuous [24]. Several forms of completeness and precompactness are well-known concepts in literature. But there are no such investigations in the case of quasi-Cauchy sequence and ward continuity, probably because of its wilder nature. For instance, a subsequence of a quasi-Cauchy sequence may not be quasi-Cauchy. In fact, in [23], it has been shown that in $\mathbb{R}$, every sequence can be subsequence of a quasi-Cauchy sequence, which makes this investigation different and much more challenging. In Chapter 6, we characterize an arbitrary subsequence of a quasi-Cauchy sequence and analyze the significance of quasi-Cauchy sequences in various domains.

In a compact metric space, continuity coincides with uniform continuity. But there are many non-compact spaces where every continuous function is uniformly continuous such as any infinite uniformly discrete space. Atsuji, in 1958, introduced a space commonly referred to as Atsuji space or UC space [4]. This space has since been extensively studied and has played a prominent role in numerous research works related to uniform continuity (see [2, 3, 10, 11, 13]) A metric space (X, d) is called UC if every real-valued continuous function on X is uniformly continuous. In 2005, Dikranjan et al [17] introduced straight space, which is a weaker notion of UC space. A metric space (X, d) is called straight if for a cover $X = C_1 \cup C_2$ with each C_i, $i \in \{1, 2\}$ is closed, any real-valued continuous function f defined on X is uniformly continuous iff the restricted functions $f|_{C_i}$, $i \in \{1, 2\}$ are uniformly continuous. There is a strong connection between straightness and various forms of connectedness. A totally disconnected space is straight iff it is UC. The second one is: if the space is locally connected, then it is straight iff uniformly locally connected. In this perspective, another concept that is closely linked to uniform continuity is the notion of uniform connectedness. It is natural to ask what would occur if we were to consider two weaker notions of uniform continuity, namely Cauchy regularity and ward continuity, in the above scenario. In chapters 3 and 4 we proceed in that direction.

In Chapter 3, we introduce a new type of connectedness using Cauchy condition. Typically, connectedness is defined through weak separation and the use of continuous functions, where a space is considered connected if every continuous function from the given space to a two-point set is constant. On the other hand, the concept of uniform connectedness was introduced in [60], which states that every uniformly continuous function from the given space to a two-point set is constant (further work can be found in [5, 6, 7, 8]). By replacing continuity (or uniform continuity) with Cauchy regularity, a new form of connectedness called Cauchy connectedness is obtained, which lies strictly between the classes of all connected spaces and uniform connected spaces. Primarily we examine several features of Cauchy connected space in the structure of uniform space and emphasize the points where it differs from usual connectedness. At the end of the chapter, we look back into uniform connectedness in metric space and observe that uniform connectedness is equivalent to the condition that every ward continuous function from the given space to a two-point set is constant.

In Chapter 4, we generalize the concept of straight space using Cauchy regular function and ward continuous function. Firstly, we deal with Cauchy regular function and show that two types of straight spaces, namely pre-straight space and pre($*$)straight space can be defined. The main results of this section are

- The class of all straight spaces is the intersection of the class of all pre-straight spaces and the class of all pre($*$)-straight spaces (Proposition 4.1.1).
- X is pre($*$)-straight iff its completion $\widehat{X}$ is straight (Theorem 4.1.2).

We establish several properties of pre-straight space, which are variations of complete-like properties. In the next section, we find some relations between Cauchy connected space and these types of straight spaces. We end this chapter by defining WC space in line with UC space and then define a new type of straight space, namely W-straight space using quasi-Cauchy sequences. We present some characterizations of both WC space and W-straight space, which will be again used in chapter 6.

In Chapter 5, we are interested in certain preserving properties of Cauchy regular function. Every continuous function is both compactness- and connectedness-preserving. Conversely, the question of determining the conditions under which these types of functions are continuous has been explored by many authors. J. Gerlits has provided further insights into the historical development of this question in [42]. One may naturally inquire about the consequences of considering a stronger concept of continuity, specifically Cauchy regularity, in the aforementioned investigation. In the first section, we endeavor to provide an answer in that particular aspect. It is known that every Cauchy regular function preserves precompactness and in Theorem 3.1.3, we have proved that Cauchy connectedness is preserved by every Cauchy regular function. We begin our pursuit of this analysis from these two points. In the work by E.R. McMillan [57], it was demonstrated that any function that preserves connectedness and compactness, defined on a locally connected space, is necessarily continuous (for additional details, refer to [52, 74]). However, these conditions are not sufficient for guaranteeing Cauchy regularity. To illustrate this point, consider the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \sin \frac{1}{x}$. It is evident that f preserves both precompactness and Cauchy connectedness, with a locally connected domain. Furthermore, the completion of the domain is also locally connected. However, despite these properties, f is not Cauchy regular. In the initial section, our focus is on exploring various conditions that lead to the Cauchy regularity of a function. We were intrigued by the significant role played by the graph of a function in this investigation. The main results of this section are

- Suppose that (X, d_X) and (Y, d_Y) are two metric spaces and $f : (X, d_X) \to (Y, d_Y)$ is both precompactness- and Cauchy connectedness-preserving function such that $\widehat{G(f)}$ is locally connected. Then f is Cauchy regular. (Theorem 5.1.1)

- Suppose that $D \subset \mathbb{R}$ is connected and $f : D \to \mathbb{R}$ is both boundedness- and

Cauchy connectedness-preserving function such that $\overline{G(f)}$ is path-connected. Then f is Cauchy regular. (Theorem 5.1.3)

In the subsequent part of this chapter, we define Cauchy separated fibers (CSF) in line with the notion of distance fibers (DF) [16], that gives Cauchy regularity of a continuous function defined on a pre-straight space with locally connected completion. Finally, we introduce the notion of Cauchy approachable (CA) functions inspired by the concept of uniformly approachable (UA) functions [15, 33]) and examine the connection of CA functions with the new type of fibers.

Chapter 6 of this thesis focuses primarily on exploring completeness properties within the framework of uniform structures. In recent times, various authors, including G. Beer, M.I. Garrido, A.S. Meroño, A. Hohti, H. Junnila have studied some variants of completeness strictly lie between compactness and completeness (see [11, 38, 39, 46, 47, 69, 58] for more references). This chapter builds upon the aforementioned papers and investigates these intermediate properties from multiple perspectives to gain a deeper understanding of their characteristics. In the first two sections, we specifically follow the way of M.I. Garrido and A.S. Meroño's works in [38, 58]. We start by defining Bourbaki quasi-Cauchy filters and cofinally Bourbaki quasi-Cauchy filters in line with Bourbaki Cauchy and cofinally Bourbaki Cauchy filters, respectively [39]. Following that, we address the fundamental question of when these filters converge, which prompts us to introduce two new types of completeness properties: Bourbaki quasi-completeness (Bq-completeness) and cofinally Bourbaki quasi-completeness (cBq-completeness), respectively. Firstly we intend to explore a new kind of modification of uniform spaces. A modification of a uniform space (X, μ), is defined by a uniformity of X such that the base or subbase consists of a subfamily of covers belonging to μ. In order to describe these new variants of completeness, we delve into the concept of finite-component covers [61] and then define the corresponding modification, namely, finite-component modification, ($s_f^\infty \mu$). This enables us to connect Bq-completeness and cBq-completeness of (X, μ) with completeness and cofinal completeness of $(X, s_f^\infty \mu)$, respectively. Furthermore, we also have a significant interest in the concept of paracompactness in this section. It has been established that various types of completeness properties are closely connected with paracompactness. To investigate this further, we revisit the concept of superparacompactness [20, 21, 61] and proceed to define uniformly star superparacompact spaces. We establish that a space is cBq-complete if and only if it is uniformly star superparacompact. Then we focus on some results of metrizability such as some variations (Theorem 6.2.1, Theorem 6.2.6) of the classical theorem by Čech (a metric space is completely metrizable iff it is a G_δ-set in its Stone-Čech compactification [25]).

In the last section of this chapter, we look into the concept of Bourbaki quasi-Cauchy sequence in metric space. Interestingly we find a relation between Bourbaki quasi-Cauchy sequence and quasi-Cauchy sequence. We prove that any arbitrary subsequence of a quasi-Cauchy sequence is Bourbaki quasi-Cauchy. Using this result, one can apply the concept of quasi-Cauchy sequences in many directions where Cauchy sequences usually work. Here we define a new type of Lipschitz function, which we call quasi-Cauchy Lipschitz function in line with the recently studied Cauchy Lipschitz function [14]. We investigate several coincidence results of this new Lipschitz function with the other types of Lipschitz functions. Finally, we prove that any real-valued ward continuous function can be uniformly approximated by a quasi-Cauchy Lipschitz function.

In the last chapter, we give attention to some variations of Cauchy regular map. We define Cauchy covering map which is a reversal of Cauchy regular map and also a generalization of well-known sequence covering map. In this chapter, we connect this investigation with a general form of convergence called Ideal convergence, which gives us more general types of mappings in this direction. Throughout this chapter, we investigate many conditions under which these mappings can be related to each other.

Our topological terminologies and notations are as in the book [35], from where the notions (undefined inside the thesis) can be found.

2 Preliminary

This chapter begins by discussing certain concepts in metric structures related to Cauchy and quasi-Cauchy sequences, followed by a brief description of uniform spaces. The first section primarily focuses on metric spaces and specifically describes various notions of continuity along with some conditions under which they coincide with each other. In the last section, we first discuss various equivalent definitions of uniform space and then consider certain variants of completeness in uniform structure which precisely lie in between compactness and completeness.

2.1 Some stronger versions of continuity

In this section, we concentrate on the notions of Cauchy regularity, ward continuity, and uniform continuity. Let (X, d) be a metric space. A sequence (x_n) is called Cauchy if for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n > n_0$ and a function is called Cauchy regular (It is also called Cauchy continuous) if it preserves Cauchy sequences. There is a weaker version of Cauchy sequence called quasi-Cauchy sequence [23] and the definition is as follows: a sequence (x_n) is said to be quasi-Cauchy if for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_{n+1}, x_n) < \varepsilon$ for all $n \geq n_0$. The sequence $(\sqrt{n})$ is an example of quasi-Cauchy which is not Cauchy (see [23, 24] for more examples). Consequently, another type of continuity can be obtained by using quasi-Cauchy sequences.

Definition 2.1.1 (see for example [24])**.** *A function $f : (X, d) \to (Y, \rho)$ is said to be ward continuous if for every quasi-Cauchy sequence (x_n) in X, $(f(x_n))$ is quasi-Cauchy in Y.*

The class of all ward continuous functions strictly lies in between the class of all Cauchy regular functions and the class of all uniform continuous functions. The following examples illustrate that the reverse inclusions are not generally true.

Example 2.1.1. *1. Let us consider $X = \mathbb{N} \cup \{n + \frac{1}{n} : n \in \mathbb{N}\}$ with usual metric of $\mathbb{R}$. Then the characteristic function of $\mathbb{N}$ from X to $\{0, 1\}$ is ward continuous but not uniformly continuous.*

2. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$. Then f is Cauchy regular but not ward continuous since $(\sqrt{n})$ is a quasi-Cauchy sequence but $(f(\sqrt{n}))$ is not quasi-Cauchy.

In a complete metric space, continuity coincides with Cauchy regularity. Conversely, we assume every real-valued continuous function defined on X is Cauchy regular. If possible let (x_n) be a Cauchy sequence in X, without any cluster point in X. Then $S = \{x_n : n \in \mathbb{N}\}$ is closed and the function $f : C \to \mathbb{R}$ defined by $f(x_n) = n$ for each n is continuous. Then using the Tietze extension theorem, f can be extended to a continuous function $\widetilde{f} : X \to \mathbb{R}$. Clearly f is not Cauchy regular, which contradicts our assumption. Hence X is complete. Therefore the conclusion is a metric space (X, d) is complete iff every real-valued continuous function on X is Cauchy regular.

In a compact metric space, every continuous function is uniformly continuous. But the converse is not generally true for example we can consider the set of all natural numbers $\mathbb{N}$. In this direction there is a larger class of spaces known as UC space or Atsuji space ([4], see also [2, 3, 10, 11]) in literature.

Definition 2.1.1. *A metric space (X,d) is called UC space if every real-valued continuous function on X is uniformly continuous.*

In [49], Hueber presented a characterization of UC space in terms of a geometric functional I, called as degree of isolation.

- Let (X,d) be a metric space. Then $I(x) = d(x, X \setminus \{x\})$, which is the degree of isolation of a point $x \in X$. $I(x) = 0$ iff x is a limit point of X [4].

Theorem 2.1.1. *[49] Let (X,d) be a metric space. Then X is a UC space iff every sequence (x_n) in X with $\lim_{n\to\infty} I(x_n) = 0$ has a cluster point.*

A metric space is called precompact (It is also called totally bounded) if every sequence has a Cauchy subsequence. In a precompact space, every Cauchy regular function is uniformly continuous. Conversely on $\mathbb{N}$ with the usual metric every Cauchy regular function is uniformly continuous but it is not precompact. In [10, 51], several types of equivalent conditions were given under which Cauchy regularity coincides with uniform continuity. In this direction, an interesting point is when a space (X,d) has Atsuji completion i.e, $(\widehat{X}, \widehat{d})$ is Atsuji space.

Theorem 2.1.2. *In a metric space (X,d) the following conditions are equivalent.*

1. *$(\widehat{X}, \widehat{d})$ is Atsuji space.*
2. *Every real-valued Cauchy regular function defined on X is uniformly continuous.*
3. *Every sequence (x_n) in X with $\lim_{n\to\infty} I(x_n) = 0$ has a Cauchy subsequence.*

In [10, 11, 14, 41], Beer, Garrido and Jaramillo studied different types of Lipschitz-like functions, and the definitions of these functions are summarized below.

Definition 2.1.2. *Let (X,d) and (Y,ρ) be two metric spaces. A function $f : (X,d) \to (Y,\rho)$ is said to be:*

1. *Lipschitz if there exists* $K > 0$ *such that* $\rho(f(x), f(y)) \leq Kd(x,y)\ \forall x, y \in X$.

2. *Lipschitz in the small if there exist* $\delta > 0$ *and* $K > 0$ *such that* $\rho(f(x), f(y)) \leq Kd(x,y)$ *whenever* $d(x,y) < \delta$.

3. *Uniformly locally Lipschitz if there exists* $\delta > 0$ *such that for every* $x \in X$, *there exists* $K_x > 0$ *with* $\rho(f(u), f(w)) \leq K_x d(u,w)$ *whenever* $u, w \in B_d(x, \delta)$.

4. *Cauchy-Lipschitz if* f *is Lipschitz when restricted to the range of each Cauchy sequence* (x_n) *in* X.

5. *Locally Lipschitz if for each* $x \in X$, *there exists* $\delta_x > 0$ *such that* f *restricted to* $B_d(x, \delta_x)$ *is Lipschitz.*

Clearly, every Lipschitz in the small function is uniformly locally Lipschitz. Moreover, it is shown in [12] that the collection of all Cauchy-Lipschitz functions is contained in the class of all locally Lipschitz functions which also contains the class of all uniformly locally Lipschitz functions. But the converse implications are not generally true.

Example 2.1.2. 1. *A function* $f : \mathbb{R} \to \mathbb{R}$, *defined by* $f(x) = x^2$ *is uniformly locally Lipschitz but not Lipschitz in the small.*

2. *A function* $f : (0,1) \to \mathbb{R}$ *is defined by* $f(x) = sin(\frac{1}{x})$. *Then* f *is locally Lipschitz but not Cauchy Lipschitz since* f *is not Lipschitz when restricted to* $(\frac{1}{n\pi + \frac{\pi}{2}})$.

3. *Let* $X = \mathbb{N} \cup \{(n + \frac{1}{n}) : n \in \mathbb{N}\}$ *with the usual metric of* $\mathbb{R}$. *A function* $f : X \to \mathbb{R}$ *is defined by* $f(n) = 0$ *and* $f(n + \frac{1}{n}) = 1$ *for every* $n \in \mathbb{N}$. *Then* f *is Cauchy Lipschitz but not uniformly locally Lipschitz.*

It is evident that the sets of all real-valued locally Lipschitz, Cauchy Lipschitz, and Lipschitz in the small functions are proper subsets of the sets of all real-valued continuous, Cauchy regular, and uniformly continuous functions, respectively. However, what makes this particularly intriguing is that these smaller sets are not only contained within the larger sets, but they are uniformly dense in them.

Theorem 2.1.3. *Let (X, d) be metric space. Then the following results hold:*

1. *Every real-valued continuous function defined on X can be uniformly approximated by real-valued locally Lipschitz functions.*
2. *Every real-valued Cauchy regular function defined on X can be uniformly approximated by real-valued Cauchy Lipschitz functions.*
3. *Every real-valued uniformly continuous function defined on X can be uniformly approximated by real-valued Lipschitz in the small functions.*

2.2 Some concepts in uniform spaces

2.2.1 Definitions of uniform spaces

Here we first give some brief descriptions of uniform spaces from [35, 44, 50]. There are three equivalent definitions for a uniform space using entourages, pseudometrics and uniform covers respectively. In this thesis, we mainly use the definition of a uniform space using entourages and uniform covers.

Definition with Entourage: Let X be a non-empty set. A set $U \subseteq X \times X$ is called entourage of the diagonal if $\Delta = \{(x, x) : x \in X\} \subseteq U$ and $U = U^{-1}$. Let Θ be a family of entourages of diagonal. The pair (X, Θ) is called a uniform space if for all entourages U, V the following conditions are satisfied:

(i) $U \in \Theta$ and $V \supset U \Longrightarrow V \in \Theta$;

(ii) $U, V \in \Theta \Longrightarrow U \cap V \in \Theta$;

(iii) $U \in \Theta \Longrightarrow (\exists V \in \Theta)\ V \circ V = \{(x, z) : (\exists y \in X)\ (x, y), (y, z) \in V\} \subseteq U$;

(iv) $\bigcap \Theta = \Delta$.

For every $x \in X$ and $U \in \Theta$, $U[x] = \{y \in X : (x,y) \in U\}$. The topology τ_Θ induced by the uniformity Θ is given by $O \in \tau_\Theta$ iff for every $x \in O$, there exists $U \in \Theta$ such that $U[x] \subseteq O$.

Now we will consider the following notations. Let X be a non-empty set, $A \subseteq X$ and $\mathcal{P}$, $\mathcal{Q}$ be covers of X. Then we write:

- $\mathcal{P} < \mathcal{Q}$ means $\mathcal{P}$ is a refinement of $\mathcal{Q}$ and $\mathcal{P} \wedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$.
- $St^0(A,\mathcal{P}) = A$, $St(A,\mathcal{P}) = \bigcup\{P \in \mathcal{P} : P \cap A \neq \emptyset\}$, $\mathcal{P}^* = \{St(P,\mathcal{P}) : P \in \mathcal{P}\}$, $St(x,\mathcal{P}) = St(\{x\},\mathcal{P})$, $St^m(A,\mathcal{P}) = St(St^{m-1}(A,\mathcal{P}),\mathcal{P})$ for $m \geq 1$ and $St^\infty(A,\mathcal{P}) = \bigcup\limits_{n\in\mathbb{N}} St^n(A,\mathcal{P})$.

Definition with Uniform cover: A uniform space (X,μ) is a non-empty set X together with a distinguished family of coverings μ, which is a subset of the set of all covering of X and satisfies the following conditions:

(i) $\{X\} \in \mu$.

(ii) $\mathcal{P}^* < \mathcal{Q}$ with $\mathcal{P} \in \mu$ and $\mathcal{Q}$ a cover of $X \implies \mathcal{Q} \in \mu$.

(iii) For every pair $\mathcal{P}, \mathcal{Q} \in \mu$, there exists $\mathcal{R} \in \mu$ such that $\mathcal{R}^* < \mathcal{P} \wedge \mathcal{Q}$.

In a uniform space (X,μ), each cover belonging to μ is called a uniform cover. Note that for every $\mathcal{P} \in \mu$, there exists an open cover $\mathcal{V} \in \mu$ such that $\mathcal{V} < \mathcal{P}$.

One can easily observed that a partition of X can be induced by a cover $\mathcal{P}$ in such a way that $X = \bigcup\{St^\infty(x_i,\mathcal{P}) : i \in I\}$, where $x_i \in X$ with $i \in I$ for some index set I and $St^\infty(x_i,\mathcal{P}) \cap St^\infty(x_j,\mathcal{P}) = \emptyset$ for $i \neq j$. Each $St^\infty(x_i,\mathcal{P})$ is called the chainable components of X induced by $\mathcal{P}$.

In a metric space (X,d), the family of covers of the open balls with a fixed radius $\varepsilon > 0$, that is, $\mathcal{B}_\varepsilon = \{B_d(x,\varepsilon) : x \in X\}$ forms a base for uniformity and the uniformity induced by the metric d will be denoted by μ_d. If A be a non-empty subset of X and $\varepsilon > 0$, we will denote the ε-enlargement of A by $A^\varepsilon = \bigcup\{B_d(x,\varepsilon) : x \in A\} = \{y : d(y,A) < \varepsilon\}$. Furthermore the ε-chainable component of $x \in X$ is defined by $B_d^\infty(x,\varepsilon) = \bigcup_{n\in\mathbb{N}} B_d^n(x,\varepsilon)$, where $B_d^1(x,\varepsilon) = B_d(x,\varepsilon)$ and for every $n \geq 2$, $B_d^n(x,\varepsilon) = (B_d^{n-1}(x,\varepsilon))^\varepsilon$. Clearly, $B_d^\infty(x,\varepsilon) = St^\infty(x,\mathcal{B}_\varepsilon)$.

When we consider a uniform space (X,Θ) in the sense of entourages, we can define a cover $\mathcal{P}$ to be uniform if there exists some entourage U in Θ such that for each $x \in X$ there is $P \in \mathcal{P}$ with $U[x] \subseteq P$. The family of all such uniform cover forms a uniform space as in the uniform cover definition. Conversely, for given a uniform space (X,μ) in the uniform cover sense, the supersets of $\bigcup\{A \times A : A \in \mathcal{P}\}$ as $\mathcal{P}$ ranges over μ are the entourages for a uniform space as described in the first definition.

Now we will present some concepts of uniform space in terms of uniform covers.

A topological space is uniformizable iff it is a Tychonoff space. The topology induced by (X,μ) is given by a set O is open iff for every $x \in X$, there exists $\mathcal{P} \in \mu$ such that $St(x,\mathcal{P}) \subseteq O$. A uniformity μ on a set X is said to be compatible with the topology of X if the topology induced by the uniformity is exactly the same as the original topology of X. Let us consider the family of all compatible uniformities over X with a relation $\leq$ defined by $\mu_1 \leq \mu_2$ iff $\mu_1 \subseteq \mu_2$. This relation is a partial order and there exists a supremum for the family, which is the finest uniformity, compatible with the topology X. This uniformity is known as the fine uniformity and is denoted by $\mathbf{u}$.

2.2.2 Certain versions of completeness in uniform space

In this section, we consider certain variants of completeness in uniform space, which precisely lie in between compactness and completeness. We are mainly interested in cofinally complete spaces, Bourbaki complete spaces and cofinally Bourbaki complete spaces. Throughout this section, we present all the definitions using uniform cover.

Definition 2.2.1. *[27, 58] Let (X, μ) be a uniform space and $\mathcal{F}$ be a filter of X.*

1. *$\mathcal{F}$ is called Cauchy if for every $\mathcal{U} \in \mu$ there exists $U \in \mathcal{U}$ such that $F \subseteq U$ for some $F \in \mathcal{F}$.*

   (X, μ) is called complete if every Cauchy filter has a cluster point in X.

2. *$\mathcal{F}$ is called Bourbaki-Cauchy if for every $\mathcal{U} \in \mu$ there exist $U \in \mathcal{U}$ and $n \in \mathbb{N}$ such that $F \subseteq St^n(U, \mathcal{U})$ for some $F \in \mathcal{F}$.*

   (X, μ) is called Bourbaki complete if every Bourbaki Cauchy filter has a cluster point in X.

3. *$\mathcal{F}$ is called cofinally Cauchy if for every $\mathcal{U} \in \mu$ there exists $U \in \mathcal{U}$ such that $F \cap U \neq \emptyset$ for every $F \in \mathcal{F}$.*

   (X, μ) is called cofinally complete if every cofinally Cauchy filter has a cluster point in X.

4. *$\mathcal{F}$ is called cofinally Bourbaki-Cauchy if for every $\mathcal{U} \in \mu$ there exist $U \in \mathcal{U}$ and $n \in \mathbb{N}$ such that $F \cap St^n(U, \mathcal{U}) \neq \emptyset$ for every $F \in \mathcal{F}$.*

   (X, μ) is called cofinally Bourbaki complete if every cofinally Bourbaki Cauchy filter has a cluster point in X.

The following implications are evident.

compact ⇒ cofinally Bourbaki-complete ⇒ cofinally complete ⇒ complete.

compact ⇒ cofinally Bourbaki-complete ⇒ Bourbaki-complete ⇒ complete.

One can naturally obtain the following definitions in metric structure.

Definition 2.2.2. *[38, 48] Let (X, d) be a metric space and (x_n) be sequence of X.*

1. *(x_n) is said to be cofinally Cauchy if for every $\varepsilon > 0$, there exists an infinite subset N_ε of $\mathbb{N}$ such that for every $m, n \in \mathbb{N}$, $d(x_m, x_n) < \varepsilon$.*
2. *(x_n) is said to be Bourbaki-Cauchy in X if for every $\varepsilon > 0$ there exist $m, n_0 \in \mathbb{N}$ such that for some $p \in X$ we have $x_n \in B_d^m(p, \varepsilon)$ for every $n \geq n_0$.*
3. *(x_n) is said to be cofinally Bourbaki-Cauchy in X if for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and an infinite subset N_ε of $\mathbb{N}$ such that for some $p \in X$ we have $x_n \in B_d^m(p, \varepsilon)$ for every $n \in N_\varepsilon$.*

In [11, 38, 58] it has shown that a metric space (X, d) is cofinally complete (Bourbaki complete, cofinally Bourbaki complete) iff every cofinally Cauchy (Bourbaki Cauchy, cofinally Bourbaki Cauchy) sequence has a cluster point in X.

Now we present some counterexamples related to these types of complete spaces.

Example 2.2.1. *1. Let us consider a partition of $\mathbb{N} \setminus \{1\}$ into a countable family of infinite subsets $\{M_n : n \in \mathbb{N}\}$, where each M_n can be written as: $M_n = \{n_k : k \in \mathbb{N}\}$. Now we define a sequence (x_n) such that $x_1 = e_1$ and $x_{n_k} = 10^n e_1 + \frac{1}{n} e_{n_k}$ for every $n, k \in \mathbb{N}$, where (e_n) is a sequence of unit vectors of ℓ^∞. Let us take $X = \{x_n : n \in \mathbb{N}\}$ with sup norm of ℓ^∞. Clearly X is complete. But X is not cofinally complete as (x_n) is a cofinally Cauchy sequence without having any cluster point. Note that here the sequence (x_n) is a cofinally Cauchy sequence, which has no Cauchy subsequence. Also, X is Bourbaki-complete, but not cofinally complete and so not cofinally Boubaki-complete also.*

2. *If we consider $X = \{re_n : n \in \mathbb{N}, r \in [0,1]\}$, where $\{e_n : n \in \mathbb{N}\}$ is the set of all unit vectors of ℓ^∞, endowed with sup norm of ℓ^∞, then X is cofinally complete but not Bourbaki complete. Hence the notions of cofinal completeness and Bourbaki-completeness are mutually independent.*

We end this section with some discussions on the metric structure.

Definition 2.2.3. *Let (X,d) be a metric space and $\varepsilon > 0$ be given. Then an ordered set of points $\{x_0, x_1, \dots, x_n\}$ in X satisfying $d(x_{i-1}, x_i) < \varepsilon$, where $i = 1, 2, \dots, n$ is said to be an ε-chain of length n from x_0 to x_n.*

Note that $y \in B_d^n(x, \varepsilon)$ iff x and y can be joined by an ε-chain of length n.

Definition 2.2.4. *[4]* (1) *A metric space (X, d) is called ε-chainable if any two points of X can be joined by an ε-chain, whereas X is called chainable if X is ε-chainable for every $\varepsilon > 0$.*

(2) *A subset B of a metric space (X, d) is said to be Bourbaki bounded (also known as finitely chainable subset of X [54]) if for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and a finite collection of points $p_1, p_2, \dots, p_k \in X$ such that $B \subseteq \bigcup_{i=1}^{k} B_d^m(p_i, \varepsilon)$.*

Every precompact set is Bourbaki bounded and every Bourbaki bounded set is bounded. The real line with the bounded metric $\widehat{d} = \min\{1, d\}$, where d is the usual Euclidean metric, is bounded but not Bourbaki bounded. On the other hand if we consider $X = \{re_n : n \in \mathbb{N}, r \in [0,1]\}$, where $\{e_n : n \in \mathbb{N}\}$ is the set of all unit vectors of ℓ^∞, endowed with sup norm of ℓ^∞, then X is Bourbaki bounded since for every $n \in \mathbb{N}$, every point can be joined with 0 by a $\frac{1}{n}$- chain of length $2n$. But X is not precompact.

Like precompact spaces, Bourbaki bounded spaces can be characterized in terms of Bourbaki-Cauchy and cofinally Bourbaki-Cauchy sequences.

Theorem 2.2.1. *[38] Let (X, d) be a metric space. Then the following conditions are equivalent.*

1. *X is Bourbaki bounded.*

2. *Every sequence has a Bourbaki-Cauchy subsequence in X.*

3. *Every sequence is cofinally Bourbaki-Cauchy in X.*

Theorem 2.2.2. *[38] A metric space (X, d) is compact iff it is Bourbaki-bounded and Bourbaki-complete.*

3

Connectedness Using Cauchy Condition

Connectedness (resp. uniform connectedness [60]) of uniform spaces can be defined in terms of continuous (resp. uniformly continuous) functions to a discrete space requiring that every continuous (resp. uniformly continuous) function to a discrete space has to be constant. Replacing uniformly continuous functions with the strictly weaker notion of Cauchy regular functions [71], we obtain a new notion of connectedness, namely Cauchy connectedness which happens to be an intermediate notion between connectedness and uniform connectedness. We primarily investigate several features of this new notion. Further, in metric spaces, we turn our attention to quasi-Cauchy sequences [23] and show that replacing Cauchy regular continuity with ward continuity one again gets back the notion of uniform connectedness.

The content of this chapter is based on the following research paper.

- P. Das, S.K. Pal and **N. Adhikary**, On Cauchy condition and related notion of connectedness, **Topology and its Applications**, 301 (2021), No. 107499. [30]

3.1 Cauchy connected spaces

In this chapter, we consider the definition of a uniform space in terms of entourages. Throughout this chapter, (X, Θ) (or sometimes only X) will stand for a uniform space.

Let $\mathcal{F}$ be a filter base in X and $\emptyset \neq C \subset X$. Then

• $\mathcal{F}$ is said to be eventually in C if there exists $F \in \mathcal{F}$ such that $F \subset C$.

• $\mathcal{F}$ is said to be frequently in C if for each $F \in \mathcal{F},\ F \cap C \neq \emptyset$.

• $\mathcal{F}$ is said to be a Cauchy filter base if for every $U \in \Theta$, there exists $F \in \mathcal{F}$ such that $F \times F \subset U$.

We now introduce our main definitions of this section.

Definition 3.1.1. *Suppose (X, Θ) is a uniform space and A, B are two non-empty subsets of X. Then (A, B) is called a Cauchy separation of X if $X = A \cup B$ and any Cauchy filter base in X, is either eventually in A or eventually in B but not eventually in both A and B.*

Definition 3.1.2. *A uniform space (X, Θ) is called Cauchy connected if X has no Cauchy separation. If X has a Cauchy separation, then (X, Θ) is called Cauchy disconnected.*

Several observations can be derived directly from the definition of Cauchy separation. Let (A, B) be a Cauchy separation of X.

- A and B are mutually disjoint. If not, then choose $x \in A \cap B$. Then clearly the filterbase $\mathcal{F} = \{\{x\}\}$ is eventually both in A and B, which is a contradiction.

- A and B both are open. Let $x \in A$ and $(x_k)_{k \in \mathcal{D}}$, with directed set $(\mathcal{D}, \geq)$ be a net converging to x. It is clear that the filter base $\mathcal{F} = \{F_i : i \in \mathcal{D}\}$ converges to x, where $F_i = \{x_k : k \geq i\}$ for $i \in D$. Define $\mathcal{F}_1 = \{F \cup \{x\} : F \in \mathcal{F}\}$. Since $\{\{x\}\}$ converges to x, $\mathcal{F}_1$ is Cauchy and so it must be eventually in A. Hence $(x_k)_{k \in \mathcal{D}}$ is eventually in A. This shows that A is open. Similarly, it can be shown that B is also open. Then A and B are also closed is now obvious.

From the above discussion, it is clear that a weak separation is also a Cauchy separation of X. Hence connectedness implies Cauchy connectedness. But the converse is not generally true, as can be seen from the following example.

Example 3.1.1. *Take X as the unit square $[0,1] \times [0,1]$ with the linear order topology coming from the lexicographic order on the unit square. X, a Tychonoff space, is consequently uniformizable and let Θ be the resulting uniformity on X (which is compatible with the order topology). Consider the subspace $Y = X \setminus \{(\frac{1}{2}, \frac{1}{2})\}$ endowed with the relative uniformity Θ_Y induced from Θ.*

If possible suppose that (Y, Θ_Y) is not Cauchy connected and let (A, B) be a Cauchy separation of (Y, Θ_Y). Note that the set $[(0,0), (\frac{1}{2}, \frac{1}{2}))$ is connected and so either $[(0,0), (\frac{1}{2}, \frac{1}{2})) \subset A$ or $[(0,0), (\frac{1}{2}, \frac{1}{2})) \subset B$. Similar observation also holds for the connected set $((\frac{1}{2}, \frac{1}{2}), (1,1)]$. Since A and B both are nonempty with $A \cap B = \emptyset$, without any loss of generality we can assume that $A = [(0,0), (\frac{1}{2}, \frac{1}{2}))$ and $B = ((\frac{1}{2}, \frac{1}{2}), (1,1)]$.

Now in view of the fact that X is also first countable and $(\frac{1}{2}, \frac{1}{2}) \in \overline{A} \cap \overline{B}$, we can find two sequences $(x_n)_n \in A$ and $(y_n)_n \in B$ such that $x_n \to (\frac{1}{2}, \frac{1}{2})$ as also $y_n \to (\frac{1}{2}, \frac{1}{2})$. Subsequently, considering $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ where $F_n = \{x_k : k \geq n\} \cup \{y_k : k \geq n\}$, we observe that $\mathcal{F}$ is a Cauchy filter base in Y which is frequently in both A and B. But this is a contradiction to the assumption that (A, B) is a Cauchy separation of (Y, Θ_Y). Hence Y must be Cauchy connected. Evidently Y is not connected, with $[(0,0), (\frac{1}{2}, \frac{1}{2}))$ and $((\frac{1}{2}, \frac{1}{2}), (1,1)]$ forming a weak separation.

Note that discrete uniform spaces having more than one element are always Cauchy disconnected as for any $x \in X$, $(\{x\}, X \setminus \{x\})$ forms a Cauchy separation of X.

The following notions will be helpful in understanding this new idea of connectedness better.

Definition 3.1.3. *Let X be a uniform space and $A \subset X$. Then A is called Cauchy clopen in X iff any Cauchy filter base in X is either eventually in A or in $X \setminus A$.*

Note that, for $A \subset X$, being Cauchy clopen implies it is a clopen set, as for any filter base $\mathcal{F}$ converging to x, the filter base $\mathcal{F}_1 = \{F \cup \{x\} : F \in \mathcal{F}\}$ is Cauchy and it is either eventually in A or in $X \setminus A$. But the converse may not be true as in Example 3.1.1, $[(0,0),(\frac{1}{2},\frac{1}{2}))$ is clopen in Y but not Cauchy clopen in Y.

Definition 3.1.4. *([71]) A function from a uniform space to a uniform space is called Cauchy-regular if it preserves Cauchy filter bases.*

In view of the above definitions, we immediately obtain the following characterization of Cauchy connectedness.

Theorem 3.1.1. *Let (X,Θ) be a uniform space. Then the following conditions are equivalent:*

1. *X is Cauchy connected.*

2. *Every Cauchy regular function from X to a two-point set with discrete topology is constant.*

3. *X has no nonempty proper Cauchy clopen subset.*

Proof. $(1) \Rightarrow (2)$ Suppose that $f : (X,\Theta) \to \{0,1\}$ is a nonconstant Cauchy regular function. Let us take $A = f^{-1}\{0\}$ and $B = f^{-1}\{1\}$. Clearly $X = A \cup B$ and for any Cauchy filter base $\mathcal{F}$ in X, $f(\mathcal{F})$ is Cauchy in $\{0,1\}$ and so it is eventually constant. Consequently, $\mathcal{F}$ is either eventually in A or in B but not eventually in both A and B. Therefore (A,B) is a Cauchy separation of X, which contradicts that X is Cauchy connected.

$(2) \Rightarrow (3)$ Suppose that (3) does not hold, then there is a nonempty proper Cauchy clopen subset A of X and the function $f : X \to \{0,1\}$, defined by $f(A) = 0$, $f(X \setminus A) = 1$, is a nonconstant Cauchy regular function, which is a contradiction.

$(3) \Rightarrow (1)$ If X has a nonempty proper Cauchy clopen subset A, then $(A, X \setminus A)$ forms a Cauchy separation of X. □

From the following observations, we can show that this new notion of connectedness has many of the usual features of the notion of connectedness, and they can be easily proved.

(1) Suppose that (X, Θ) is a uniform space and (A, B) is a Cauchy separation of X. For any Cauchy connected subset S of X either $S \subset A$ or $S \subset B$.

(2) X is Cauchy connected iff $X = A \cup B$ with $A, B \neq \emptyset$ implies that there is a Cauchy filter base $\mathcal{F}$ in X such that it is frequently both in A and B.

(3) Let A and B be two Cauchy connected spaces and let there be a Cauchy filter base in $A \cup B$, which is frequently both in A and B. Then $A \cup B$ is also Cauchy connected.

(4) Suppose that $A \subset X$ is Cauchy connected and $A \subset B \subset \overline{A}$. Then B is also Cauchy connected.

However, the notion of Cauchy connectedness is different from the usual notion of connectedness in several respects. The first major difference is in respect of closure. It is well known that for $E \subset X$, the connectedness of $\overline{E}$ does not necessarily imply that of E. A typical simple example is the set $E = \mathbb{Q} \cap [0, 1]$ in $\mathbb{R}$. For Cauchy connectedness, we have a stronger result in this respect.

Theorem 3.1.2. *Let (X, Θ) be a uniform space and $A \subset X$. Then A is Cauchy connected iff $\overline{A}$ is Cauchy connected.*

Proof. From the above observation (4), it is evident that A is Cauchy connected implies $\overline{A}$ is Cauchy connected.

Conversely, suppose that $\overline{A}$ is Cauchy connected. Let $A = C_1 \cup C_2$. Then $\overline{A} = \overline{C_1} \cup \overline{C_2}$. Since $\overline{A}$ is Cauchy connected, there is a Cauchy filter base $\mathcal{F}$ in $\overline{A}$ such that it is frequently both in $\overline{C_1}$ and $\overline{C_2}$. Let Θ_B be the base for the uniform structure Θ.

For every $U \in \Theta$ and for every point $a \in X$, we define $U[a] = \{x \in X : (a, x) \in U\}$. For every set $U \in \Theta$ and for every set $C \subset X$, we define $U[C] = \cup_{a \in C} U[a]$. It is easy to see that $\mathcal{F}_1 = \{U[F] : F \in \mathcal{F}, U \in \Theta_B\}$ is a Cauchy filter base in X. As $\mathcal{F}$ is a filter base in $\overline{A}$, sets of the form $U[F] \cap A$ where $F \in \mathcal{F}$ and $U \in \Theta_B$ are never empty. Consequently, $\mathcal{F}_1^A = \{U[F] \cap A : F \in \mathcal{F}, U \in \Theta_B\}$ is a Cauchy filter base in A. Since for each $F \in \mathcal{F},\ F \cap \overline{C_1}, F \cap \overline{C_2} \neq \emptyset$, for each $U \in \Theta_B,\ U[F] \cap WC_1 \neq \emptyset$ and $U[F] \cap C_2 \neq \emptyset$. Hence $\mathcal{F}_1^A$ is frequently both in C_1 and C_2 and it implies that A is Cauchy connected. □

Corollary 3.1.1. *Suppose that (X, Θ) is a uniform space. Then X is Cauchy connected iff any dense subset of X is Cauchy connected.*

Further, unlike connectedness, Cauchy connectedness is not a topological property.

Example 3.1.2. *Let $X = (0, 1) \cup (1, 2)$ and $Y = (0, 1) \cup (2, 3)$ endowed with the usual topology. Then X and Y are homeomorphic but note that Y is not Cauchy connected though X is so.*

However, we do have a positive result for the preservation of Cauchy connectedness if we choose appropriate functions.

Theorem 3.1.3. *A Cauchy-regular image of a Cauchy connected space is Cauchy connected.*

Proof. Suppose that $f : (X, \Theta_X) \to (Y, \Theta_Y)$ is a Cauchy regular function, and (A, B) is a Cauchy separation of $f(X)$. Let us consider $C_1 = f^{-1}(A), C_2 = f^{-1}(B)$. It is clear that $C_1, C_2 \neq \emptyset$ and $X = C_1 \cup C_2$. Then there exists a Cauchy filter base $\mathcal{F}$ in X, that is frequently both in C_1 and C_2. From Cauchy regularity, $f(\mathcal{F})$ is Cauchy in $f(X)$ and it is also frequently both in A and B, which is a contradiction. Hence $f(X)$ is Cauchy connected. □

One can easily understand that in a complete uniform space, the notions of Cauchy clopen sets, Cauchy regular continuity and of course, Cauchy connectedness coincide with the usual notions of clopen sets, continuity, and connectedness. So the next result seems in the expected line, though being interesting in its own right.

Theorem 3.1.4. *Let (X, Θ) be a uniform space. Then X is Cauchy connected iff its completion is connected.*

Proof. Suppose that X is Cauchy connected and $\widehat{X}$ is the completion of X. Then there is an isometry $f : X \to f(X)$ such that $\overline{f(X)} = \widehat{X}$. Since f is Cauchy regular, from Theorem 3.1.2 and Theorem 3.1.3, we can conclude that $\widehat{X}$ is Cauchy connected. Then being a complete uniform space, $\widehat{X}$ is connected.

Conversely, let $\widehat{X}$ be connected and $X = A \cup B$ where $A, B \neq \emptyset$. Then $\widehat{X} = \overline{f(A)} \cup \overline{f(B)}$. Evidently $\overline{f(A)} \cap \overline{f(B)} \neq \emptyset$. Choose $z \in \overline{f(A)} \cap \overline{f(B)}$. Then there are two filter bases $\mathcal{F}_A \subset f(A)$ and $\mathcal{F}_B \subset f(B)$ such that $\mathcal{F}_A$ and $\mathcal{F}_B$ both converge to z. Define $\mathcal{F} = \{F_A \cup F_B : F_A \in \mathcal{F}_A, F_B \in \mathcal{F}_B\}$. Clearly $\mathcal{F}$ is a Cauchy filter base and so, $f^{-1}(\mathcal{F})$ is again a Cauchy filter base that is frequently both in A and B. Hence X must be Cauchy connected. □

We had already seen an example of a Cauchy connected space that is not connected (Example 3.1.1). Another interesting but simple example is $\mathbb{Q}^+$ with the subspace topology induced from the usual topology of $\mathbb{R}$ (note that it is a multiplicative topological group with respect to the usual subspace topology). It follows from Theorem 3.1.4 that this space is Cauchy connected but obviously this space is totally disconnected.

Remark 3.1.1. *It is important to note that the property of Cauchy connectedness essentially depends on uniformity. For example, consider $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ first endowed with the subspace* *topology induced from the usual topology of* $\mathbb{R}$*. From Theorem 3.1.4,* $\mathbb{R}^*$ *is Cauchy connected. But when we consider* $\mathbb{R}^*$ *as a topological group (it is an abelian group with respect to multiplication, and again the usual subspace topology forms a group topology), then it inherits another uniform structure with respect to which it is not Cauchy connected, as* $\mathbb{R}^+$ *and* $\mathbb{R}^-$ *form a Cauchy separation.*

To observe this, let us consider the following left relations for $0 < \varepsilon < 1$*,*

$$R_\varepsilon = \{(x, y) : xy^{-1} \in (1 - \varepsilon, 1 + \varepsilon)\}.$$

It is easy to observe that $\mathcal{B} = \{R_\varepsilon : 0 < \varepsilon < 1\}$ *forms a filter base satisfying the conditions that (i) each* R_ε *contains the diagonal, (ii) for any* $U \in \mathcal{B}$ *there is a* $V \in \mathcal{B}$ *such that* $V \subset U^{-1}$*, (iii) for any* $U \in \mathcal{B}$ *there is a* $W \in \mathcal{B}$ *such that* $W \circ W \subset U$*. Let* $\mathcal{V}$ *be the uniformity on* $\mathbb{R}^*$ *generated by* $\mathcal{B}$*.*

Consider the separation $\mathbb{R} = \mathbb{R}^+ \cup \mathbb{R}^-$ *and note that for any* $x \in \mathbb{R}^+$ *and* $y \in \mathbb{R}^-$*,* $xy^{-1} \notin (1 - \frac{1}{2}, 1 + \frac{1}{2})$ *and so* $(x, y) \notin R_{\frac{1}{2}}$*. Now if there is a Cauchy filter base* $\mathcal{F}_0$ *which is frequently in both* $\mathbb{R}^+$ *and* $\mathbb{R}^-$ *then there is* $F \in \mathcal{F}_0$ *such that* $F \times F \subset R_{\frac{1}{2}}$*. Subsequently, noting that* $F \cap \mathbb{R}^+ \neq \emptyset \neq F \cap \mathbb{R}^-$ *we arrive at a contradiction.*

With similar reasoning, $GL_n(\mathbb{R})$*, the set of all non-singular matrices of order* $n \times n$ *is Cauchy connected when endowed with the uniformity induced from the Euclidean metric of* $\mathbb{R}^{n^2}$*. But it is not Cauchy connected when the uniformity is induced by the topological group structure as* $GL_n(\mathbb{R})$ *forms a topological group with respect to matrix multiplication and the induced topology from* $\mathbb{R}^{n^2}$*, as* $\{M : det(M) > 0\}$ *and* $\{M : det(M) < 0\}$ *form the required Cauchy separation.*

Remark 3.1.2. *Though completeness is a sufficient condition for the equality of the notions of Cauchy connectedness and connectedness, it is not necessary. For any Tychonoff space* X*, let, as usual,* $C(X)$ *be the class of all real-valued continuous functions on* X *and* $C^*(X)$ *stands for the class of all bounded real-valued continuous functions on* X*. Recall the following* *well-known result:*

Lemma 3.1.1. *(8.3.18, p.449 [35]) For a Tychonoff space X, the collection of all sets of the form $S(f,r) = \{(x,y) : |f(x) - f(y)| < r\}$, $r > 0$, $f \in C^*(X)$ forms a subbase for a precompact (i.e. totally bounded) uniformity Θ_0 (say) on X.*

Now if we take X to be non-compact, then Θ_0 cannot be complete (see Theorem 6.32 [53]). But in (X, Θ_0) every real-valued continuous function is uniformly continuous (and so is Cauchy regular). Then from Theorem 3.1.1, we can conclude that in such a space (X, Θ_0), Cauchy connectedness coincides with connectedness.

One of the nicest characterizations of connected sets comes in $\mathbb{R}$ with usual topology, or for that matter, in any orderable space D (a totally ordered set with order topology) which is complete (i.e., for any $A \subset D$, $\sup A$ and $\inf A$ exists in D). That characterization is that a set is connected iff it is an interval. The following result is in that line.

Definition 3.1.5. *Suppose X is an orderable space. $E \subset X$ is called a pre-interval if $a, b \in \overline{E}$ with $a < b$ implies that there exists $x \in E$ such that $a < x < b$.*

Clearly, all intervals are pre-intervals. But the converse is not true as one can see that $(0,1) \cup (1,2)$ and $(0,1) \cap \mathbb{Q}$ are pre-intervals but not intervals in $\mathbb{R}$ with the usual topology.

Lemma 3.1.2. *Let X be a complete orderable space and let $E \subset X$. E be a pre-interval iff $\overline{E}$ is an interval.*

Proof. Suppose that E is a pre-interval and let $\overline{E}$ be not an interval. Then there are $a, b \in \overline{E}$ with $a < b$ and there exists $x \in (a, b)$ such that $x \notin \overline{E}$. Let $U = [a, x] \cap \overline{E}$ and $V = [x, b] \cap \overline{E}$. From completeness of X, $sup(U)$ and $inf(V)$ exist. Clearly

$p = sup(U) \in U$ and $q = inf(V) \in V$. So $p, q \in \overline{E}$ but $(p, q) \cap E = \emptyset$, which is a contradiction. Hence $\overline{E}$ is an interval.

Conversely let $\overline{E}$ be an interval and $a, b \in \overline{E}$. If $(a, b) \cap E = \emptyset$ then $(a, b) \cap \overline{E} = \emptyset$, which is a contradiction. Hence E is a pre-interval. □

Corollary 3.1.2. *Suppose that X is a complete orderable space. Then A is a Cauchy connected subset of X iff A is a pre-interval.*

Proof. Let $A \subset X$. A is Cauchy connected iff $\overline{A}$ is connected i.e. iff $\overline{A}$ is an interval and this is true iff A is a pre-interval. □

Definition 3.1.6. *Let (X, Θ) be a uniform space and $A, B \subset X$. A, B are said to be close to each other if for each $U \in \Theta$ we have $(A \times B) \cap U \neq \emptyset$.*

Theorem 3.1.5. *Let X be a complete orderable space endowed with the uniformity generated by $C^*(X)$ (as in Remark 3.1.2). Then for any $A \subset X$, the following are equivalent.*

(a) A is Cauchy connected.

(b) $\emptyset \neq C_1, C_2 \subset A$ and $A = C_1 \cup C_2$ implies C_1, C_2 are close to each other.

Proof. Suppose that A is Cauchy connected and let $A = C_1 \cup C_2$ where $\emptyset \neq C_1, C_2 \subset A$. Then there is a Cauchy filter base $\mathcal{F}$ in A which is frequently both in C_1 and C_2. Hence C_1, C_2 are close to each other.

Now suppose (b) holds and on the contrary A is not Cauchy connected. Then A is not a pre-interval and there are $a, b \in \overline{A}$ such that $(a, b) \cap A = \emptyset$. Define $U = (-\infty, a] \cap A$ and $V = [b, \infty) \cap A$. Then $A = U \cup V$. Now by Tietze extension theorem there is a continuous function $f : X \to \mathbb{R}$ such that $f(\overline{U}) = 1$ and $f(\overline{V}) = 0$. Consider the entourage $D(f, \frac{1}{2})$ where $D(f, \frac{1}{2}) = \{(x, y) : |f(x) - f(y)| < \frac{1}{2}\}$. Then $(U \times V) \cap D(f, \frac{1}{2}) = \emptyset$. This contradicts the given condition in (b). Hence A is Cauchy connected. □

Note 3.1.1. *For an arbitrary uniform space* $(a) \Rightarrow (b)$ *is true. But* $(b) \Rightarrow (a)$ *is not generally true. If we consider* $A = \{(x, \frac{1}{x}) : x > 0\}, B = \{(x, 0) : x \geqslant 0\} \cup \{(0, y) : y \geqslant 0\}$ *and take* $X = A \cup B$*, then* A, B *satisfy the conditions in* (b) *but not Cauchy connected.*

Corollary 3.1.3. *(Generalized Intermediate Value Theorem) Let* $f : X \to Y$ *be a Cauchy regular function, where* Y *is a complete orderable space (having the uniformity of Theorem 3.1.5). Then the following are equivalent.*
(1) X *is Cauchy connected.*
(2) *For any* $p, q \in \overline{f(X)}$ *with* $p < q$ *there is* $c \in X$ *such that* $p < f(c) < q$*.*

Proof. $(1) \Rightarrow (2)$ Since $f(X)$ is a Cauchy connected subset of Y, $f(X)$ is a pre-interval. Then for any $p, q \in \overline{f(X)}$ with $p < q$ there is $k \in X$ such that $p < f(k) < q$.

$(2) \Rightarrow (1)$ If X is not Cauchy connected then from Theorem 3.1.1, there is a Cauchy regular function $f : X \to \mathbb{R}$ defined by $f(A) = 0$ and $f(X \setminus A) = 1$, where A is a nonempty proper subset of X. The function f does not satisfy the given condition. □

Definition 3.1.7. *Let* X *be a uniform space. A nonempty subset* U *of* X *is called a Cauchy component of* X *if* U *is Cauchy connected and there is no proper superset of* U *in* X *which is Cauchy connected.*

If we take $X = \{(0, 1) \cup (2, 3)\} \cap \mathbb{Q}$ with usual topology then $(0, 1) \cap \mathbb{Q}$ and $(2, 3) \cap \mathbb{Q}$ are two Cauchy components of X, but evidently they are not components of X (because X is totally disconnected).

In a uniform space X, the following statements hold, showing that Cauchy components have similar properties as components: (a) The Cauchy components of X are Cauchy separated. (b) The Cauchy components of X are closed in X. (c) Each Cauchy connected set is contained in a unique Cauchy component of X. (d) X is the union of its Cauchy connected components.

A uniform space in which the Cauchy components are all singleton sets is said to be totally Cauchy disconnected. Clearly, a totally Cauchy disconnected space is totally disconnected. But the converse is not true because $\mathbb{Q}$ with usual topology is totally disconnected though it is Cauchy connected. Any discrete space is evidently totally Cauchy disconnected. But the converse is not true as $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is not a discrete space but still is totally Cauchy disconnected. Also, the Cantor set $\mathcal{C}$ is totally Cauchy disconnected.

Definition 3.1.8. *A space X is said to be locally Cauchy connected at x if, for every neighborhood U of x, there is a Cauchy connected neighborhood V of x contained in U. If X is locally Cauchy connected at each of its points, it is said to be locally Cauchy connected.*

Note 3.1.2. *It is obvious that a locally connected space [5] is locally Cauchy connected. But the converse is not true as the unit square with only rational coordinates as a subspace of the unit square in $\mathbb{R}^2$ with usual topology is locally Cauchy connected but not locally connected. $(0,1) \cup (2,3)$ with usual topology is locally Cauchy connected but not Cauchy connected. The topologist's sine curve is Cauchy connected but not locally Cauchy connected.*

Next, we consider a uniformity, namely $\Theta_{\mathcal{C}}$ on X using Cauchy connectedness and show how the local Cauchy connectedness of the space is related to the new uniformity.

Definition 3.1.9. *In a uniform space (X, Θ) with the basis $\mathcal{B}$ of the uniformity, define $C_U = \{(x,y)$: there is a U-small Cauchy connected set containing x and $y\}$.*

Lemma 3.1.3. (1) *$\mathcal{C}_{\mathcal{B}} = \{C_U : U \in \mathcal{B}\}$ forms a basis of a uniformity on X.*
(2) *Let $\Theta_{\mathcal{C}}$ be the uniformity generated by $\mathcal{C}_{\mathcal{B}}$. Then $\Theta_{\mathcal{C}}$ is finer than Θ.*

Proof. (1) Since for each $x \in X$, $\{x\}$ is Cauchy connected, $\Delta \in C_U$ for each $U \in \mathcal{B}$. Evidently for each $U \in \Theta$, $C_U^{-1} = C_U$. Next let $C_{U_1}, C_{U_2} \in \mathcal{C}_{\mathcal{B}}$. Then there is $V \in \mathcal{B}$ such that $V \subset U_1 \cap U_2$. Clearly $C_V \subset C_{U_1} \cap C_{U_2}$. For each $U \in \mathcal{B}$ choose $W \in \mathcal{B}$ such

that $WoW \subset U$. We will show that $C_W o C_W \subset C_U$. Let $(x,y),\ (y,z) \in C_W$. So there are two W-small Cauchy connected sets $A,\ B$ such that $x, y \in A$ and $y, z \in B$. Now $A \cup B$ is a Cauchy connected set containing x and z. Let $p, q \in A \cup B$. Then (p, y) and $(y, q) \in W$. So $(p, q) \in WoW \subset U$. Hence $A \cup B$ is a U-small Cauchy connected set containing x and z. Therefore $C_W o C_W \subset C_U$.
(2) Follows from the definition. □

Let $\tau_\Theta, \tau_{\Theta_C}$ be the topologies on X generated by the uniformities Θ and Θ_C respectively.

Theorem 3.1.6. *$\tau_\Theta = \tau_{\Theta_C}$ iff X is locally Cauchy connected.*

Proof. Obviously $\tau_\Theta \subset \tau_{\Theta_C}$. Conversely, let $G \in \tau_{\Theta_C}$ and $x \in G$. Then there exists $U \in \mathcal{B}$ such that $C_U[x] \subset G$. There is $V \in \mathcal{B}$ such that $VoVoVoV \subset U$. From the local Cauchy connectedness of X there is $M \in \tau_\Theta$ such that $x \in M$ and M is Cauchy connected and $M \subset V[x]$. Now there is $W \in \mathcal{B}$ such that $W[x] \subset M$. For any $p, q \in M$, $(p, x) \in V$ and $(x, q) \in V$. So M is VoV-small. Take any point $y \in W[x]$. Then M is a VoV-small Cauchy connected set containing x, y. Hence $(x, y) \in C_{VoV} \subset C_U$. So $y \in C_U[x] \subset G$ from which it follows that $W[x] \subset G$. Hence G is open in τ_Θ.

For the converse part let $\tau_\Theta = \tau_{\Theta_C}$. Let $x \in X$ and $U \in \mathcal{B}$. Then there is $V \in \mathcal{B}$ such that $C_V[x] \subset U[x]$. We will show that $C_V[x]$ is Cauchy connected. Take any point $y \in C_V[x]$. Then there is a V-small Cauchy connected set A_y containing x, y. Note that for each $z \in A_y$, $z \in C_V[x]$ and so $y \in A_y \subset C_V[x]$. Hence $C_V[x] = \cup_{y \in C_V[x]} A_y$ where each A_y is Cauchy connected and having a common point x. So $C_V[x]$ is Cauchy connected. This completes the proof. □

Definition 3.1.10. *A uniform space (X, Θ) is said to be uniformly locally connected (Cauchy connected) if for each $U \in \Theta$ there exists a $V \in \Theta$ such that $V \subset U$ and $V[x]$ is connected (Cauchy connected) for each $x \in X$.*

Clearly, for a uniform space, uniformly locally connected implies uniformly locally Cauchy connected.

The following definition was given with the help of uniform connectedness.

Definition 3.1.11. *[5] A uniform space (X, Θ) has property S iff for each $U \in \Theta$, X can be covered by a finite family of connected U-small sets.*

Following the same line, one can consider a modification of property S using the notion of Cauchy connectedness.

Definition 3.1.12. *A uniform space (X, Θ) has property S^* iff for each $U \in \Theta$, X can be covered by a countable family of Cauchy connected U-small sets.*

Clearly, for a uniform space, property S implies property S^*. In precompact uniform spaces, weak property S implies property S^*. Further, it can be noted that property S^* is preserved by a uniformly continuous function.

Theorem 3.1.7. *Every Lindelöf uniformly locally Cauchy connected space has property S^*.*

Proof. Let $U \in \Theta$. There exists $V \in \Theta$ such that $VoV \subset U$. Since X is uniformly locally Cauchy connected, there is $W \in \Theta$ such that $W \subset V$ and each $W[x]$ is Cauchy connected. Then there is a sequence (x_n) such that $\{W[x_i] : i \in \mathbb{N}\}$ cover X. Now $W[x_i] \times W[x_i] \subset V[x_i] \times V[x_i] \subset VoV \subset U$. □

Theorem 3.1.8. *If X has property S^* then X is locally Cauchy connected.*

Proof. Let $x \in X$, $U \in \Theta$ and V be a closed entourage such that $V \subset U$. As X has the property S^*, there is a countable family $\{C_i\}_{i \in \mathbb{N}}$ of Cauchy connected V-small sets covering X. Let C be the union of those C_i's such that $x \in \overline{C_i}$. Thus C is Cauchy connected. It now readily follows that $C \subset \cup_{x \in \overline{C_i}} C_i \subset \cup_{x \in \overline{C_i}} \overline{C_i} \subset V[x] \subset U[x]$. Hence C is a neighbourhood of x in view of the fact that x neither belongs to nor is a limit point of $X \setminus C$. □

Finally, we state below certain observations in the line of the work done in [8], the proofs of which follow an analogous argument as the original results and so are omitted.

Theorem 3.1.9. *(cf. Th.1.4, [8]) Let $f : (X, \Theta) \to (Y, \mathcal{V})$ be a uniform quotient map. If (X, Θ) is uniformly locally Cauchy connected then so is $(Y, \mathcal{V})$.*

Theorem 3.1.10. *(cf. Th.1.6, [8]) $(\prod X_\alpha, \Theta)$ is uniformly locally Cauchy connected iff*
(1) *Each $(X_\alpha, \Theta_\alpha)$ is uniformly locally Cauchy connected.*
(2) *All but finitely many X_α are Cauchy connected.*

Theorem 3.1.11. *(cf. Th.1.5, [8]) $(\prod X_\alpha, \Theta)$ has property S^* iff*
(1) *Each $(X_\alpha, \Theta_\alpha)$ has property S^*.*
(2) *All but finitely many X_α are Cauchy connected.*

Though the notion of locally Cauchy connected spaces has been considered in uniform structure, but there are certain interesting facts of this notion, which are typically investigatable in metric structures only that we present now. To proceed in this direction, we first consider the definition of Cauchy connected space in the metric structure.

Definition 3.1.13. *(cf.[30]) Suppose that (X, d) is a metric space and A, B are two non-empty subsets of X. Then (A, B) is called a Cauchy separation of X if $X = A \cup B$ and any Cauchy sequence in X, is either eventually in A or eventually in B but not eventually in both sets simultaneously.*

A metric space (X, d) is said to be Cauchy connected if X has no Cauchy separation. X is said to be Cauchy disconnected if X has a Cauchy separation.

First, we define the following mapping in line of [11], which is non-trivial and much more relevant when the underlying space is locally Cauchy connected but not Cauchy connected.

Definition 3.1.14. *Suppose that $(X, d$ is a metric space. Let us define a function $l : X \to \mathbb{R}$ by $l(x) = \sup \{diam(C) : \ C$ is a Cauchy connected neighbourhood of $x\}$.*

Remark 3.1.3. *If there is a Cauchy connected neighbourhood of x which is unbounded then we define $l(x) = \infty$. Let $\mathcal{C}_x$ be the Cauchy connected component of x. Then $l(x) \leqslant diam(\mathcal{C}_x)$. $\mathcal{C}_x = \{x\}$ implies $l(x) = 0$. But the converse is not true. If we take $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup [-1, 0]$. Then $l(0) = 0$ though $diam(\mathcal{C}_x) = 1$. If X is locally Cauchy connected and x is not an isolated point of X, then $l(x) > 0$. But here also the converse is not necessarily true. For example, for the topologist sine curve, $l(0) > 0$ but it is not locally Cauchy connected. In other words if x is not an isolated point of X then $l(x) = 0$ implies X is not locally Cauchy connected at the point x.*

Theorem 3.1.12. *Let X be a metric space and a sequence (x_n) converges to x. Then $l(x) \leq \liminf l(x_n)$.*

Proof. If $\liminf l(x_n) = \infty$, then there is nothing to prove. So let $\liminf l(x_n) = \alpha < \infty$. We claim that $l(x)$ must be finite. If $l(x)$ is infinite, then x has an unbounded Cauchy connected neighborhood U. As $x_n \to x$, so $x_n \in U$ for all but finitely many n, say, for all $n \geq n_0$. Consequently each singleton (x_n) has an unbounded Cauchy connected neighbourhood for all $n \geq n_0$, which contradicts that $\liminf l(x_n) < \infty$. Next if possible let $l(x) - \alpha > 0$. Choose $\varepsilon > 0$, such that $l(x) - \alpha > \varepsilon$ and so $l(x) > \alpha + \varepsilon$. Then there is a Cauchy connected neighbourhood V of x such that $\alpha + \varepsilon < diam(V) \leq l(x)$. Now $x_n \in V$ for all $n \geq n_1$(say) and subsequently $\alpha + \varepsilon < diam(V) \leq l(x_n)$, for all $n \geq n_1$, which again contradicts that $\liminf l(x_n) = \alpha$. Hence $l(x) \leq \alpha$. □

Corollary 3.1.4. *Let X be a metric space and a sequence (x_n) converges to x. If $(l(x_n))$ is bounded then $l(x)$ is finite.*

In Theorem 3.1.12, the inequality may be strict.

Example 3.1.3. *Define* $A_n = (\frac{1}{n+1} - \delta_{n+1}, \frac{1}{n+1} + \delta_n)$*, where* $\delta_n = \frac{1}{3}(\frac{1}{n} - \frac{1}{n+1})$*. Let* $X = [\cup_{n=1}^{\infty} A_n \cup \{0\}] \times (-1, 1)$ *with the usual metric of* $\mathbb{R}^2$*. Now* $dist(A_n, A_{n+1}) = \delta_{n+1} > 0$*, so they can not contain any common Cauchy sequence. Then* $l(\frac{1}{n+1}, 0) = diam(A_n) = \sqrt{\delta_{n+1}^2 + 2^2}$ *and so* $l(\frac{1}{n}, 0) \to 2$*. But* $l(0, 0) = 0$*.*

The following example shows that the function l is not necessarily continuous.

Example 3.1.4. *Define* $A_n = (\frac{1}{n+1} - \delta_{n+1}, \frac{1}{n+1} + \delta_n) \times (-n, n)$*, where* $\delta_n = \frac{1}{3}(\frac{1}{n} - \frac{1}{n+1})$*. Let* $X = [\cup_{n=1}^{\infty} A_n \cup \{(0, 0)\}]$ *with the usual metric of* $\mathbb{R}^2$*. Now* $dist(A_n, A_{n+1}) = \delta_{n+1} > 0$*, so they can not contain any common Cauchy sequence. Then* $l(\frac{1}{n+1}, \frac{1}{n+1}) = diam(A_n) = \sqrt{\delta_{n+1}^2 + (2n)^2}$*. Here* $(l(\frac{1}{n}, \frac{1}{n}))$ *is not a convergent sequence though* $(\frac{1}{n}, \frac{1}{n}) \to (0, 0)$ *. So this function* l *is not continuous at* $(0, 0)$*.*

3.2 Uniformly connected spaces

We now look back into a stronger notion of connectedness, namely uniform connectedness which was introduced in [60] and show that our notion is strictly weaker than that notion.

Definition 3.2.1. *(cf. [60]) A uniform space* (X, Θ) *is called uniformly connected iff every uniformly continuous function from* (X, Θ) *to a discrete space is constant.*

Clearly, for any uniform space, Cauchy connectedness implies uniform connectedness. But the converse is not true, as can be seen from the next example below. For a compact uniform space, all three notions of connectedness coincide.

Example 3.2.1. *Let* $A = \{(x, \frac{1}{x}) : x > 0\}, B = \{(x, 0) : x \geqslant 0\} \cup \{(0, y) : y \geqslant 0\}$ *and take* $X = A \cup B$ *endowed with the usual subspace topology of* $\mathbb{R}^2$*. Let* $f : X \to \{0, 1\}$ *be a uniformly continuous function. Since* A, B *are connected,* $f(A), f(B)$ *must be singletons. Without any loss of generality, suppose that* $f(A) = 1$*. Since* $dist(A, B) = 0$*,* $f(B) = 1$ *by* *uniform continuity of f. So f is a constant function and hence X is uniformly connected. But clearly, X is complete, and X is not connected and so X cannot be Cauchy connected.*

Theorem 3.2.1. *Suppose that X is a complete orderable space and $A \subset X$. Then A is Cauchy connected iff A is uniformly connected.*

Proof. Necessity is trivial. Conversely, suppose that A is not Cauchy connected. Then from Theorem 3.1.5, there exist $\emptyset \neq C_1, C_2 \subset A$ such that C_1, C_2 are not closed to each other and $A = C_1 \cup C_2$. Then there exists a uniformly continuous function $g : A \to \{0, 1\}$ such that $g(C_1) = 0$ and $g(C_2) = 1$, which contradicts that A is uniformly connected. □

In the rest of this section, we continue our investigation into the stronger structure of metric spaces and obtain a new characterization of the much-investigated notion of uniform connectedness.

Recall that a metric space is called chainable if any two points can be joined by an ε-chain for every $\varepsilon > 0$.

Lemma 3.2.1. *[60] A metric space X is uniformly connected iff it is chainable.*

Lemma 3.2.2. *Suppose that X is chainable and $(a_i), (b_i)$ are two adjacent sequences (i.e $d(a_i, b_i) \to 0$). Then there exists a quasi-Cauchy sequence (x_i) with the property that for any integer $i \geq 1$ there exists $j \geq 1$ such that $a_i = x_j$ and $b_i = x_{j+1}$.*

Proof. As X is chainable, for every $k \geq 1$, b_k and a_{k+1} can be joined by a $\frac{1}{k}$-chain. Hence there is $b_k = y_0^k, y_1^k, ..., y_{n_k}^k = a_{k+1}$ in X such that $d(y_i^k, y_{i-1}^k) \leq \frac{1}{k}$ for $1 \leq i \leq n_k$. Observe that the sequence

$$a_1, b_1, y_0^1, y_1^1, ..., y_{n_1}^1, a_2, b_2, y_0^2, y_1^2, ..., y_{n_2}^2, a_3, b_3, ...$$

clearly has the required property. □

Lemma 3.2.3. *Let (X, d) be a uniformly connected space. Then a function $f : (X, d) \to (Y, \rho)$ is uniformly continuous iff it is ward continuous.*

Proof. Clearly, uniform continuity implies ward continuity. For the converse, let $f : (X, d) \to (Y, \rho)$ be a ward continuous function. If f is not uniformly continuous, then there exists $\varepsilon > 0$ such that there are two sequences $(x_n), (y_n) \in X$ with $d(x_n, y_n) < \frac{1}{n}$ but $\rho(f(x_n), f(y_n)) \geq \varepsilon$ for all $n \in \mathbb{N}$. Then by Lemma 3.2.2, there exists a quasi-Cauchy sequence (z_n) with the property that for any $n \geq 1$ there exists a $m \geq 1$ such that $x_n = z_m$ and $y_n = z_{m+1}$. This shows that f can not be ward continuous. □

Theorem 3.2.2. *For a metric space (X, d), the following are equivalent.*

1. *X is uniformly connected.*

2. *Any ward continuous function $f : X \to \{0, 1\}$ with discrete topology is constant.*

3. *If $X = A \cup B$, where A, B are non empty subsets of X, then there is a quasi-Cauchy sequence (x_n) in X such that A and B both contain an infinite subsequence of (x_n).*

Proof. From the above Lemma (1) $\Rightarrow$ (2) readily follows. For (2) $\Rightarrow$ (3), let $X = A \cup B$, where A, B are non empty subsets of X. If any quasi-Cauchy sequence in X is eventually either in A or in B but not eventually in both of them, then $A \cap B = \emptyset$, because otherwise, the constant sequence consisting of a point from $A \cap B$ is eventually both in A and B and consequently the function $f : X \to \{0, 1\}$ defined by $f(A) = 0, f(B) = 1$ is a nonconstant ward continuous function, which contradicts (1).

(3) $\Rightarrow$ (1) Let $f : X \to \{0, 1\}$ be a uniformly continuous function. If f is non-constant, then there is a nonempty proper subset A of X such that $f(A) = 0$ and $f(X \setminus A) = 1$. But then by (3), there is a quasi-Cauchy sequence (x_n) in X

such that A and $X \setminus A$ both contain infinite subsequences of (x_n). Passing onto an appropriate subsequence, we can then find a subsequence (x_{r_n}) of (x_n) such that $x_{r_n} \in A$ and $(x_{r_n+1}) \in X \setminus A$, for otherwise (x_n) is either eventually in A or in $X \setminus A$. Now $d(x_{r_n}, x_{r_n+1}) \to 0$ implies $(f(x_{r_n+1}))$ is eventually 0, which in turn leads to a contradiction. □

Note that Theorem 3.2.2 presents new characterizations of uniform connectedness in terms of quasi-Cauchy sequences and ward continuous functions. Also from condition (3) of the above theorem, we can conclude that in metric space, uniform connectedness can be defined using a kind of separation.

4 Variations of Straightness

Straight spaces are metric spaces X having the property that for a cover $X = A \cup B$ by two closed sets, any continuous function $f : X \to \mathbb{R}$ is uniformly continuous provided it is so on each of the sets A and B (it is actually called "2-straight" but is easier to deal with [17]). In this chapter, we consider this nice idea and instead of uniformly continuous functions, we consider Cauchy regular functions [71] and ward continuous functions [24], as these classes of functions strictly lie between the classes of continuous and uniformly continuous functions. In the process, we obtain two natural variations of straightness which we name pre-straight and W-straight spaces respectively. We primarily investigate these notions along with another notion called pre($*$)-straight which actually helps us to obtain a better understanding of the relationship between the notions of straight and pre-straightness.

The entire investigation is done in metric space setting and the content of this chapter is based on the research papers listed below.

- P. Das, S.K. Pal, **N. Adhikary**, On certain versions of straightness, **Topology and its Applications**, 284 (2020), No. 107369. [29]

- S. K. Pal and **N. Adhikary**, Characterization of Cauchy regular functions, **Topology and its Applications**, 315 (2022) 108148. [66]

4.1 Variations of straight spaces using Cauchy regular functions

We start this section by recalling the definition of straight space together with its certain properties.

Definition 4.1.1. *[18] A metric space (X, d) is said to be straight if whenever X is the union of two closed sets, then $f \in C(X)$ is uniformly continuous iff its restriction to each of the closed sets is uniformly continuous.*

It is known that a locally connected space is straight iff it is uniformly locally connected [17] (A metric space X is uniformly locally connected if, for every $\varepsilon > 0$, there is $\delta > 0$ such that any two points at distance $< \delta$ lie in a connected set of diameter $< \varepsilon$ [7]).

Obviously, every compact metric space is straight. Similarly, every UC space is straight. But the converse implication is not true as for example, the closed unit disk minus a point with the usual metric of $\mathbb{R}^2$ is straight but it is not compact as well as not a UC space. From this example, we can say that straight spaces may not be complete.

Definition 4.1.2. *[17] Let (X, d) be a metric space. A pair C^+, C^- of closed sets of X is said to be u-placed if $d(C^+_\varepsilon, C^-_\varepsilon) > 0$ holds for every $\varepsilon > 0$, where $C^+_\varepsilon = \{x \in C^+ : d(x, C^+ \cap C^-) \geq \varepsilon\}$ and $C^-_\varepsilon = \{x \in C^- : d(x, C^+ \cap C^-) \geq \varepsilon\}$.*

One can prove the following equivalent condition for straightness.

Theorem 4.1.1. *[17] A metric space (X, d) is straight iff every pair of closed subsets, which form a cover of X, is u-placed.*

Since the concept of Cauchy-regularity lies between the concepts of continuity and uniform continuity (as discussed in [71]), it becomes natural to inquire about the consequences of replacing uniform continuity with Cauchy-regularity and then also continuity with Cauchy regularity in the definition of straight space. In this direction, we can introduce two variations of straightness. The first one, referred to as pre-straightness seems a more relatable property and exhibits several analogous features to straightness. The second variation is called pre($*$)-straightness (Definition 4.1.5), which completes the relation between straightness and pre-straightness.

Definition 4.1.3. *A space X is said to be pre-straight if whenever X is the union of two closed sets, then $f \in C(X)$ is Cauchy regular iff its restriction to each of the closed sets is Cauchy regular.*

We have already mentioned in the Preliminary chapter that a metric space (X, d) is complete iff every real-valued continuous function defined on X is Cauchy regular. In view of this fact, it is now evident that the concept of pre-straightness can be thought of as a generalization of completeness. From Proposition 4.1.1, it is clear that the closed unit disk minus a point is a pre-straight space, which is not complete. The following example differentiates the concept of pre-straightness from straightness.

Example 4.1.1. *Let us consider $A_1 = \{(x,0) : x \in \mathbb{R}\} \cup \{(0,y) : y \in \mathbb{R}\}$, $A_2 = \{(x,1) : 0 \leq x \leq 1\}$, $A_3 = \{(x, \frac{1}{x}) : x \geq 1\}$ and $A_4 = A_2 \cup A_3$. Then take $X = A_1 \cup A_4$. Being a complete space X is pre-straight. Now a function $f : X \to \mathbb{R}$ is defined by $f(x,1) = x \, \forall x \in [0,1]$ and $f(A_3) = 1, f(A_1) = 0$. Here f is continuous and $f|_{A_1}, f|_{A_4}$ are uniformly continuous but f is not uniformly continuous, since if we take $\varepsilon = \frac{1}{2}$, then for any $\delta > 0$, there exists $n \in \mathbb{N}$ such that $d((n,0),(n,\frac{1}{n})) < \frac{1}{n} < \delta$, but $|f(n,0) - f(n,\frac{1}{n})| = 1 > \varepsilon$. Hence X is not straight.*

Recall that two sets A and B ($A, B \subset X$) do not have any "common Cauchy sequence“ if for each Cauchy sequence (x_n) in X the sets, $\{x_n : n \in \mathbb{N}\} \cap A$ and $\{x_n : n \in \mathbb{N}\} \cap B$ can not be infinite simultaneously. In order to obtain a characterization of pre-straight spaces in line with Theorem 4.1.1, we introduce now the following variation of the notion of u-placed.

Definition 4.1.4. *A pair C^+, C^- of closed sets of X is said to be c-placed if $C^+_\varepsilon, C^-_\varepsilon$ have no common Cauchy sequence for every $\varepsilon > 0$.*

Remark 4.1.1. *Note that $C^+_\varepsilon = C^+$ and $C^-_\varepsilon = C^-$ when $C^+ \cap C^- = \phi$. Hence a partition $X = C^+ \cup C^-$ of X is c-placed iff C^+, C^- are Cauchy clopen.*

We begin with the following result, which is a variation of Lemma 2.6 [17].

Lemma 4.1.1. *Let (X, d) be a metric space and C^+, C^- be a pair of closed subsets of X. Then the following conditions are equivalent:*

(1) The pair C^+, C^- is c-placed.

(2) If (Y, ρ) is a metric space and $f : C^+ \cup C^- \to (Y, \rho)$ is continuous, then f is Cauchy regular whenever $f|_{C^+}$ and $f|_{C^-}$ are Cauchy regular.

(3) If $f : C^+ \cup C^- \to \mathbb{R}$ is continuous, then f is Cauchy regular whenever $f|_{C^+}$ and $f|_{C^-}$ are Cauchy regular.

Proof. (1) $\Rightarrow$ (3) Let $f : C^+ \cup C^- \to (Y, \rho)$ be a continuous function with $f|_{C^+}$ and $f|_{C^-}$ be Cauchy regular. If either $C^+ = C^+ \cup C^-$ or $C^- = C^+ \cup C^-$ then we have done, so assume that $C^+ \neq C^+ \cup C^- \neq C^-$. Now If $C^+ \cap C^- = \phi$, then from Remark 4.1.1, C^+ and C^- are Cauchy clopen i.e C^+ and C^- do not have any common Cauchy sequence. Hence (3) evidently holds.

Next, we assume that $C^+ \cap C^- \neq \phi$. Let (x_n) be a Cauchy sequence in $C^+ \cup C^-$. If (x_n) is eventually either in C^+ or in C^-, by Cauchy regularity of $f|_{C^+}$ and $f|_{C^-}$ it is clear that $(f(x_n))$ is Cauchy. On the other hand suppose that (x_n) has a subsequence (x_{r_n}) in C^+ and (x_{p_n}) in C^-. If $(x_{r_n}) \in C^+_{\frac{1}{k}}$ and $(x_{p_n}) \in C^-_{\frac{1}{k}}$, then (x_n) will be the common Cauchy sequence. So without loss of generality, assume $(x_{r_n}) \notin C^+_{\frac{1}{k}}$. Now since (x_n) is Cauchy $(x_{p_n}) \notin C^-_{\frac{1}{k}}$ (by definition of $C^+_{\frac{1}{k}}$ and $C^-_{\frac{1}{k}}$

). So for each $k \in \mathbb{N}$ there are $y_k, z_k \in C^+ \cap C^-$ and $x_{r_{n_k}} \in (x_{r_n})$ and $x_{p_{n_k}} \in (x_{p_n})$ such that $d(x_{r_{n_k}}, y_k) < \frac{1}{k}$ and $d(x_{p_{n_k}}, z_k) < \frac{1}{k}$. For each $\varepsilon > 0$ choose $k_1, k_2 \in \mathbb{N}$ such that $\frac{1}{k_1} < \frac{\varepsilon}{4}$ and $d(x_{r_{n_l}}, x_{r_{n_k}}) < \frac{\varepsilon}{2}$ for all $l, k \geq k_2$. Take $k_0 = max\{k_1, k_2\}$. Now $d(y_k, y_l) \leq d(y_l, x_{r_{n_l}}) + d(x_{r_{n_l}}, x_{r_{n_k}}) + d(x_{r_{n_k}}, y_k) < \frac{1}{l} + \frac{\varepsilon}{2} + \frac{1}{k} < \varepsilon$ whenever $l, k \geq k_0$. This shows that (y_k) is a Cauchy sequence. Similarly, (z_k) is Cauchy. Now define three sequences (y_n'), (z_n') and (t_n') as follows: $y'_{2k} = y_k$ and $y'_{2k-1} = x_{r_{n_k}}$ and $z'_{2k} = z_k, z'_{2k-1} = x_{p_{n_k}}$ and $t_{2k} = y_k, t_{2k-1} = z_k$. Clearly $(y'_k) \subset C^+$ and $(z'_k) \subset C^-$ and $(t_k) \subset C^+ \cap C^-$. It is easy to check that $(y'_k), (z'_k)$ and (t_k) are Cauchy. Hence $(f(y'_k)), (f(z'_k))$ and $(f(t_k))$ are also Cauchy.

Next define $f(q_{2k}) = f(x_{r_{n_k}}), f(q_{2k-1}) = f(x_{p_{n_k}})$. Let $\varepsilon > 0$ be given. Then there exist $n_1, n_2, n_3, n_4 \in \mathbb{N}$ such that $\rho(f(y'_m), f(y'_n)) < \frac{\varepsilon}{4}$ for all $m, n \geq n_1$ and $\rho(f(y_m), f(y_n)) < \frac{\varepsilon}{4}$ for all $m, n \geq n_2$ and $\rho(f(z'_m), f(z'_n)) < \frac{\varepsilon}{4}$ for all $m, n \geq n_3$ and $\rho(f(y_m), f(z_n)) < \frac{\varepsilon}{4}$ for all $m, n \geq n_4$. Now take $n_0 = max\{n_1, n_2, n_3, n_4\}$. Then $\rho(f(x_{r_{n_k}}), f(x_{p_{n_l}})) \leq \rho(f(x_{r_{n_k}}), f(y_k)) + \rho(f(y_k), f(y_l)) + \rho(f(y_l), f(z_l)) + \rho(f(z_l), f(x_{p_{n_l}}))$ $< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$ for all $l, k \geq n_0$. So $(f(q_k))$ is Cauchy. Note that $(f(x_{r_n})), (f(x_{p_n}))$ are both Cauchy and they contain a common Cauchy sequence $(f(q_k))$. Hence $(f(x_n))$ must be a Cauchy sequence in (Y, ρ) and so f is Cauchy regular.

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Assume that (2) holds. If $C^+ \cap C^- = \phi$, then from Remark 4.1.1, it is enough to show that the sets C^+, C^- are Cauchy clopen. If we consider the characteristic function $\chi_{C^+} : C^+ \cup C^- \to \mathbb{R}$, it is continuous and by (2) χ_{C^+} is Cauchy regular also. Hence C^+ is Cauchy clopen. Similarly, one can prove that C^- is Cauchy clopen.

Now assume that $C^+ \cap C^- \neq \phi$ and consider the function $f : C^+ \cup C^- \to \mathbb{R}$ defined by $f(x) = d(x, C^+ \cap C^-)$, for $x \in C^+$ and $f(x) = -d(x, C^+ \cap C^-)$, for $x \in C^-$. Obviously f is continuous and $f|_{C^+}, f|_{C^-}$ are Cauchy regular. So by (2) f is Cauchy regular. This implies C^+_ε and C^-_ε do not have any common Cauchy sequence for any $\varepsilon > 0$. For otherwise there is some $\varepsilon > 0$ such that C_ε^+ and C_ε^- have a common Cauchy sequence (x_n) with two subsequences $(x_{r_n}) \subset C_\varepsilon^+$ and $(x_{p_n}) \subset C_\varepsilon^-$. Consequently, $f(x_{r_n}) \geq \varepsilon$ whereas $f(x_{p_n}) \leq -\varepsilon \; \forall n \in \mathbb{N}$. Therefore $(f(x_n))$ can not be Cauchy, which contradicts the Cauchy regularity of f. Hence the pair C^+, C^- is c-placed. $\square$

Corollary 4.1.1. *A metric space (X, d) is pre-straight iff every pair of closed subsets, which form a cover of X, is c-placed.*

We now introduce the second version of straightness.

Definition 4.1.5. *A space X is said to be pre$(*)$-straight if whenever X is the union of two closed sets, then $f \in CC(X)$ is uniformly continuous iff its restriction to each of the closed sets is uniformly continuous.*

Every precompact space is pre$(*)$-straight since in a precompact space Cauchy regularity coincides with uniform continuity. The converse is not true as the set of all natural numbers $\mathbb{N}$, with usual metric is pre$(*)$-straight but not precompact.

The following two examples show that the notions of pre-straight and pre$(*)$-straight are independent of each other with this new notion also being different from the notion of straightness.

Example 4.1.2. *Let $X = (0,1) \cup (1,2)$, with the usual metric of $\mathbb{R}$. Then X is pre$(*)$-straight as X is precompact. But X is not straight as the characteristic function $g = \chi_{(0,1)} : X \to \mathbb{R}$ of $(0,1)$ is continuous. Note that $g|_{(0,1)}$ and $g|_{(1,2)}$ are uniformly continuous but g itself is not uniformly continuous as well as not Cauchy regular. So X is also not pre-straight.*

Example 4.1.3. *Consider $X = \bigcup_{n=2}^{\infty} [n + \frac{1}{n}, n+1]$, with the usual metric of $\mathbb{R}$. Clearly X is pre-straight. Let us take $A = \bigcup_{k=1}^{\infty} [2k + \frac{1}{2k}, 2k+1]$ and $B = \bigcup_{k=2}^{\infty} [2k - 1 + \frac{1}{2k-1}, 2k]$.*

Then $X = A \cup B$ and A, B are closed subsets of X. Consider the characteristic function $g = \chi_A : X \to \mathbb{R}$ of A. Then g is Cauchy regular as A, B are both Cauchy clopen. Again $g|_A, g|_B$ are uniformly continuous but χ_A is not uniformly continuous, since for $\varepsilon = \frac{1}{2}$ and for any $\delta > 0$ there is $n \in \mathbb{N}$ such that $|2n - (2n + \frac{1}{2n})| < \frac{1}{2n} < \delta$, but $|\chi(2n) - \chi(2n + \frac{1}{2n})| = 1 > \varepsilon$. Hence X is not pre$(*)$-straight.

Recall that two sequences $(x_n), (y_n)$ in X with $d(x_n, y_n) \to 0$ are called adjacent sequences. If (x_n) (or equivalently (y_n)) form a closed discrete set then $(x_n), (y_n)$ are called discrete adjacent sequences. In this line, we can consider the following notion. Furthermore in above, if (x_n) (or equivalently (y_n)) is Cauchy then these two sequences $(x_n), (y_n)$ are called Cauchy adjacent sequences.

Definition 4.1.6. *Let (X, d) be a metric space. A pair C^+, C^- of closed sets of X is said to be $\bar{c}-$placed if the pair $C^+_\varepsilon, C^-_\varepsilon$ has no non Cauchy adjacent sequences $(x_n), (y_n)$ with $(x_n) \subset C^+_\varepsilon$ and $(y_n) \subset C^-_\varepsilon$ for every $\varepsilon > 0$.*

We use the above definition to present a sufficient condition for a space to be pre$(*)$-straight.

Lemma 4.1.2. *Suppose that (X, d) is metric space and every pair of closed subsets, which form a cover of X, is $\bar{c}$-placed then X is pre$(*)$-straight.*

Proof. Let $X = C^+ \cup C^-$ where C^+, C^- are a pair of $\bar{c}$-placed closed subsets of X. Suppose that $f : X \to \mathbb{R}$ is a Cauchy regular function where $f|_{C^+}$ and $f|_{C^-}$ are uniformly continuous. We will show that f is uniformly continuous. For this it is sufficient to take two adjacent sequences $(x_n), (y_n)$ and show that $|f(x_n) - f(y_n)| \to 0$. If (x_n) and (y_n) are both in C^+ or C^-, then the proof is finished. Next if (x_n) and (y_n) are Cauchy adjacent then define $z_{2k} = x_k$ and $z_{2k-1} = y_k$. (z_k) is a Cauchy sequence and by Cauchy regularity of f , $(f(z_k))$ must be Cauchy and the proof is over.

Now let us assume that (x_n) and (y_n) are non-Cauchy adjacent sequences (having no Cauchy subsequences) and $(x_n) \subset C^+$ while $(y_n) \subset C^-$. As for each $\varepsilon > 0$, C^+_ε and C^-_ε do not contain any non Cauchy adjacent sequences (x_n) and (y_n) with $(x_n) \subset C^+_\varepsilon$ and $(y_n) \subset C^-_\varepsilon$, so for each $k \in \mathbb{N}$, $x_n \notin C^+_{\frac{1}{k}}$ and $y_n \notin C^-_{\frac{1}{k}}$ for all but finitely many n. Thus for each $n \in \mathbb{N}$ we can obtain $x'_n, y'_n \in C^+ \cap C^-$ such that $d(x_n, x'_n) < \frac{1}{n}$ and $d(y_n, y'_n) < \frac{1}{n}$. Let $\varepsilon > 0$ be given. Then there exist n_1, n_2 such that $\frac{1}{n_1} < \frac{\varepsilon}{4}$ and $d(x_n, y_n) < \frac{\varepsilon}{2}$ for all $n \geq n_2$. Take $n_0 = max\{n_1, n_2\}$. Evidently $d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$, for all $n \geq n_0$. This implies that (x'_n) and (y'_n) are adjacent sequences in $C^+ \cap C^-$ and so $|f(x'_n) - f(y'_n)| \to 0$. Next (x_n) and (x'_n) are adjacent in C^+ and $(y_n), (y'_n)$ are adjacent in C^-. Then by uniform continuity of $f|_{C^+}$ and $f|_{C^-}$ we can conclude that $|f(x_n) - f(x'_n)| \to 0$ and $|f(y_n) - f(y'_n)| \to 0$. Now $|f(x_n) - f(y_n)| \leq |f(x_n) - f(x'_n)| + |f(x'_n) - f(y'_n)| + |f(y'_n) - f(y_n)|$ which readily implies that $|f(x_n) - f(y_n)| \to 0$. Hence f is uniformly continuous and so X is pre$(*)$-straight. □

The following example shows that the converse of Lemma 4.1.2 is not generally true.

Example 4.1.4. *Put* $A_k = \{(x, k) : |x| \leq \frac{1}{k}\}$. *Consider the set* $X = \bigcup_{k=1}^{\infty} A_k \setminus \{(0, k) : k \in \mathbb{N}\}$. *Then* $\widehat{X}$ *is UC and so straight. Therefore from Theorem 4.1.2, we can conclude that* X *is pre*$(*)$*-straight. Let us consider* $C^+ = \{(x, k) : 0 < x \leq \frac{1}{k}\}$ *and* $C^- = \{(x, k) : \frac{-1}{k} \leq x < 0\}$. *Then* C^+ *and* C^- *are closed with* $C^+ \cap C^- = \phi$. *Consider the sequences* (x_n) *and* (y_n) *where* $x_n = (\frac{1}{n}, n)$ *and* $y_n = (\frac{-1}{n}, n)$. *Clearly* $d(x_n, y_n) \to 0$. *Also* (x_n) *and* (y_n) *are non-Cauchy with* $x_n \in C^+$ *and* $y_n \in C^-$. *Therefore* C^+ *and* C^- *contain non-Cauchy adjacent sequences.*

Now let us denote by $\mathcal{S}_1$ the set of all straight spaces, by $\mathcal{S}_2$ the set of all pre-straight spaces and by $\mathcal{S}_3$ the set of all pre$(*)$-straight spaces.

The following result, though simple, presents the relation between the classes $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and at the same time, provides a new characterization of straight spaces.

Proposition 4.1.1. $\mathcal{S}_1 = \mathcal{S}_2 \cap \mathcal{S}_3$.

Proof. Suppose X is straight. Let $X = C^+ \cup C^-$ where C^+, C^- are closed in X. C^+ and C^- are u-placed implies they are c-placed. Hence by Corollary 4.1.1, X is pre-straight. Again as every Cauchy regular function is continuous, so straight implies pre($*$)-straight. So $\mathcal{S}_1 \subset \mathcal{S}_2 \cap \mathcal{S}_3$.

For the converse, let X be both pre-straight and pre($*$)-straight. Let $X = C^+ \cup C^-$ and f be continuous where $f|_{C^+}, f|_{C^-}$ are uniformly continuous. Then by pre-straightness, f is Cauchy regular and subsequently, by pre($*$)-straightness f is uniformly continuous. So X is straight and $\mathcal{S}_1 = \mathcal{S}_2 \cap \mathcal{S}_3$. □

The next result presents an interesting relation between pre($*$)-straight and straight spaces, demonstrating how completeness of the space plays a key role.

Theorem 4.1.2. *Let X be a metric space. X is pre($*$)-straight iff its completion $\widehat{X}$ is straight.*

Proof. Let X be pre($*$)-straight and $f : (X, d) \to (\widehat{X}, \widehat{d})$ be the isometry such that $\overline{f(X)} = \widehat{X}$. Suppose $\widehat{X} = C^+ \cup C^-$ with C^+, C^- being closed subsets of $\widehat{X}$ and $g : \widehat{X} \to \mathbb{R}$ is a continuous function where $g|_{C^+}$ and $g|_{C^-}$ are uniformly continuous. Then $gof : X \to \mathbb{R}$ is a Cauchy regular function on X and $X = f^{-1}(C^+) \cup f^{-1}(C^-)$. For any $x_n, y_n \in f^{-1}(C^+)$ with $d(x_n, y_n) \to 0$ we have $\widehat{d}(f(x_n), f(y_n)) \to 0$. Then by uniform continuity of $g|_{C^+}$, $|g(f(x_n)) - g(f(y_n))| \to 0$. Hence $(gof)|_{f^{-1}(C^+)}$ is uniformly continuous. Similarly $(gof)|_{f^{-1}(C^-)}$ is also uniformly continuous. As X is pre($*$)-straight, gof becomes uniformly continuous on X. Now let $x_n, y_n \in \widehat{X}$ with $\widehat{d}(x_n, y_n) \to 0$. Now for each $n \in \mathbb{N}$ there exist p_n and $q_n \in X$ such that $\widehat{d}(f(p_n), x_n) < \frac{1}{n}$ with $|g(x_n) - (gof)(p_n)| < \frac{1}{n}$ and $\widehat{d}(f(q_n), y_n) < \frac{1}{n}$ with

$|g(y_n) - (gof)(q_n)| < \frac{1}{n}$. Let $\varepsilon > 0$ be given. Choose $n_0, n_1 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{4}$ and $\widehat{d}(x_n, y_n) < \frac{\varepsilon}{2}$ for all $n \geq n_1$. Take $n_2 = max\{n_0, n_1\}$. Then $\widehat{d}(f(p_n), f(q_n)) \leq \widehat{d}(f(p_n), x_n) + \widehat{d}(x_n, y_n) + \widehat{d}(y_n, f(q_n)) < \frac{1}{n} + \frac{\varepsilon}{2} + \frac{1}{n} < \varepsilon$. So $\widehat{d}(f(p_n), f(q_n)) \to 0$, which implies $d(p_n, q_n) \to 0$ and subsequently $|(gof)(p_n) - (gof)(q_n)| \to 0$. Now $|g(x_n) - g(y_n)| \leq |g(x_n) - (gof)(p_n)| + |(gof)(p_n) - (gof)(q_n)| + |(gof)(q_n) - g(y_n)|$ and hence $|g(x_n) - g(y_n)| \to 0$. This shows that g is uniformly continuous on $\widehat{X}$ and and so it is straight.

Conversely suppose that $\widehat{X}$ is straight and $X = C^+ \cup C^-$ with C^+, C^- being closed subsets of X and $g : X \to \mathbb{R}$ is a Cauchy regular function where $g|_{C^+}$ and $g|_{C^-}$ are uniformly continuous. Let $f : X \to \widehat{X}$ be the isometry such that $\overline{f(X)} = \widehat{X}$. Then $\widehat{X} = \overline{f(C^+)} \cup \overline{f(C^-)}$ is a closed cover of $\widehat{X}$. Clearly $gof^{-1} : f(X) \to \mathbb{R}$ is a Cauchy regular function and it can be extended to a Cauchy regular function $\widehat{gof^{-1}} : \widehat{X} \to \mathbb{R}$. Note that $\widehat{gof^{-1}}|_{f(C^+)}$ and $\widehat{gof^{-1}}|_{f(C^-)}$ are uniformly continuous as $g|_{C^+}$ and $g|_{C^-}$ are uniformly continuous respectively. Now we will show that $\widehat{gof^{-1}}|_{\overline{f(C^+)}}$ and $\widehat{gof^{-1}}|_{\overline{f(C^+)}}$ are uniformly continuous. Let $\varepsilon > 0$ be given. Then there is $\delta > 0$ such that for all $p, q \in f(C^+)$ with $\widehat{d}(p, q) < 3\delta$ implies $|gof^{-1}(p) - gof^{-1}(q)| < \frac{\varepsilon}{3}$. Now for each $x, y \in \overline{f(C^+)}$ there exist two sequences $x_n, y_n \in f(C^+)$ such that $x_n \to x$ and $y_n \to y$ respectively. So for $\delta > 0$ one can find a positive integer n_0 such that $\widehat{d}(x_n, x) < \delta$ and $\widehat{d}(y, y_n) < \delta$ and by continuity $|\widehat{gof^{-1}}(x) - \widehat{gof^{-1}}(x_n)| < \frac{\varepsilon}{3}$ and $|\widehat{gof^{-1}}(y) - \widehat{gof^{-1}}(y_n)| < \frac{\varepsilon}{3}$ for all $n \geq n_0$. Now if $\widehat{d}(x, y) < \delta$ then $\widehat{d}(x_n, y_n) \leq \widehat{d}(x_n, x) + \widehat{d}(x, y) + \widehat{d}(y, y_n) < 3\delta$ for all $n \geq n_0$. Hence for each $x, y \in \overline{f(C^+)}$ with $\widehat{d}(x, y) < \delta$ we have $|\widehat{gof^{-1}}(x) - \widehat{gof^{-1}}(y)| \leq |\widehat{gof^{-1}}(x) - \widehat{gof^{-1}}(x_{n_0})| + |\widehat{gof^{-1}}(x_{n_0}) - \widehat{gof^{-1}}(y_{n_0})| + |\widehat{gof^{-1}}(y_{n_0}) - \widehat{gof^{-1}}|(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Hence $\widehat{gof^{-1}}|_{\overline{f(C^+)}}$ is uniformly continuous. Similarly we can show that $\widehat{gof^{-1}}|_{\overline{f(C^-)}}$ is also uniformly continuous. As $\widehat{X}$ is straight, we can conclude that $\widehat{gof^{-1}}$ is uniformly continuous on $f(X)$. Hence g is uniformly continuous on X implying that X is pre$(*)$-straight. □

4.2 Certain properties of pre-straight spaces

We start this section with some complete-like properties of pre-straight spaces as we have already shown that pre-straightness is a generalization of completeness.

Theorem 4.2.1. *Suppose X is pre-straight. Then a subspace is clopen iff it is Cauchy clopen.*

Proof. Suppose that A is a clopen subset of X. Then there exists a continuous function $f : X \to \{0,1\}$ such that $f(A) = 0$ and $f(X \setminus A) = 1$. Since X is pre-straight, f is Cauchy regular, which implies that A is Cauchy clopen. □

Theorem 4.2.2. *Suppose that X is a pre-straight space. If C is a closed subspace of X such that $X \setminus C$ is complete, then C is also pre-straight.*

Proof. Suppose that X is a pre-straight space and C is a closed subspace of X such that $X \setminus C$ is complete. Let $f : C \to \mathbb{R}$ be a continuous function and $C = C^+ \cup C^-$ be a closed cover of C where $f|_{C^+}, f|_{C^-}$ are Cauchy regular. Clearly $X = [C^+ \cup (X \setminus C)] \cup C^-$. By Tietze extension theorem, f can be extended to a continuous function $\widetilde{f} : X \to \mathbb{R}$. We will show that the restriction of $\widetilde{f}$ on $C^+ \cup (X \setminus C)$ is Cauchy regular. Note that $\widetilde{f}|_{C^+}$ and $\widetilde{f}|_{(X \setminus C)}$ are Cauchy regular. Let (x_n) be a Cauchy sequence in $C^+ \cup (X \setminus C)$. Then it must be eventually either in C^+ or in $X \setminus C$. Otherwise if (x_n) has two subsequence $(x_{r_n}) \subset C^+$ and $(x_{p_n}) \subset (X \setminus C)$, then by completeness of $X \setminus C$, (x_{p_n}) converges to some point $x \in X \setminus C$. Consequently $x_n \to x$ and $x \in C^+$, which is a contradiction. So $(\widetilde{f}(x_n))$ is Cauchy which implies that $\widetilde{f}$ is Cauchy regular and hence f is also so. □

Theorem 4.2.3. *Suppose that X is a straight space. If C is a closed subspace of X such that $X \setminus C$ is UC then C is also straight.*

Proof. The proof is analogous to the proof of Theorem 4.2.2 and so is omitted. □

Remark 4.2.1. *Theorem 4.2.3 provides a possible answer to the open problem Problem 6.1 posed in [18] where the description of those closed subspaces of a straight space was asked which are again straight.*

The next result presents a necessary and sufficient condition for a space to be complete in terms of pre-straightness.

Theorem 4.2.4. *(cf. Proposition 5.1 [17]) Suppose X is a metric space. Then X is complete iff each of its closed subspaces is pre-straight.*

Proof. If X is complete then the necessity of the condition is obvious. For the converse, let there be a Cauchy sequence (x_n) in X such that (x_n) is not convergent in X (without loss of generality we can assume that each x_n is distinct). Clearly $C = \{x_n : n \in \mathbb{N}\}$ is closed. Define a function $f : C \to \mathbb{R}$ by $f(x_{2k}) = 1$ and $f(x_{2k-1}) = 0$ for each $k \in \mathbb{N}$. Take $C^+ = \{x_{2k} : k \in \mathbb{N}\}$ and $C^- = \{x_{2k-1} : k \in \mathbb{N}\}$. Clearly f is continuous, $f|_{C^+}$ and $f|_{C^-}$ are Cauchy regular, but f is not Cauchy regular. Hence C is not pre-straight. So (x_n) must be convergent in X which implies that X is complete. □

Theorem 4.2.5. *Every closed subspace of a metric space X is pre-straight iff every pair of closed subsets is c-placed.*

Proof. Follows from Theorem 4.2.4. □

Theorem 4.2.1. *Let X be a pre-straight space with the completion $\widehat{X}$ being locally connected. Then X is locally connected.*

Proof. If possible suppose that X is not locally connected. Then there exist $x \in X$ and an open subset U of X containing x such that U does not contain any open, connected set containing x. Since $\widehat{X}$ is locally connected, one can easily observe that there exists an open, connected subset C of $\widehat{X}$ containing x with $C_X = C \cap X \subset U$. Then from

Corollary 5.1.1, it is clear that C_X is a Cauchy connected subset of X containing x but not connected in X. Now there exists $\varepsilon > 0$ such that the open ball $B \subset \widehat{X}$ of radius ε with center x is contained in C. Let $B_1 \subset \widehat{X}$ be the open ball of radius $\frac{\varepsilon}{2}$ with center x. Then we can obtain an open Cauchy connected but not connected subset C'_X of X such that $x \in C'_X \subset B_1 \cap X \subset B \cap X \subset C_X$. Consequently, we can choose two disjoint open subsets P and Q of X such that $C_X = P \cup Q$ and $P \cap C'_X \neq \phi$, $Q \cap C'_X \neq \phi$. Now consider $C^+ = P \cup (X \setminus C_X)$ and $C^- = Q \cup (X \setminus C_X)$. It is evident that C^+, C^- are two closed subsets of X such that $d(C'_X, C^+ \cap C^-) \geq \frac{\varepsilon}{2}$. Since C'_X is Cauchy connected, there is a Cauchy sequence (x_n) in C'_X, which is frequently both in $P \cap C'_X$ and $Q \cap C'_X$. Hence C^+, C^- are not c-placed, which contradicts the fact that X is pre-straight. Therefore X is locally connected. □

Now we establish a nice characterization of pre-straight space in terms of a metric introduced in [17].

Definition 4.2.1. *[17] Given a metric space (X, d) and $x, y \in X$ define $d^*(x, y) = min\{1, inf\{\varepsilon|$ there is a connected set of diameter $\leq \varepsilon$ containing x and $y\}\}$*

(So $d^*(x, y) = 1$ if there is no connected set containing x and y).

The function d^* induced by (X, d) is a metric on X and if d is bounded by 1 then $d^* \geq d$. Furthermore, the topologies of X induced by d and d^* coincide iff (X, d) is locally connected. Equivalently one can say that the identity function $I_d : (X, d) \to (X, d^*)$ is continuous iff (X, d) is locally connected. In Lemma 4.2.1, we prove that Cauchy regularity of I_d implies local connectedness of $\widehat{X}$. On general metric spaces, the converse is not true. For example, if we consider $X = (-1, 0) \cup (0, 1)$ with the usual metric of $\mathbb{R}$, then $\widehat{X}$ is locally connected but on X, I_d is not Cauchy regular as $d^*(-\frac{1}{n}, \frac{1}{n}) = 1$ for all $n \in \mathbb{N}$. In this line, pre-straightness gives the appropriate equivalency.

Lemma 4.2.1. *Suppose that (X, d) is a metric space such that the identity function $I_d : (X, d) \to (X, d^*)$ is Cauchy regular. Then the completion $(\widehat{X}, \widehat{d})$ is locally connected.*

Proof. We will show that $\widehat{X}$ is weakly locally connected. Let $x \in \widehat{X}$ and $\varepsilon > 0$ be given. Consider the set $S = \{(x_n) : (x_n) \subset X \text{ and } (x_n) \text{ converges to } x\}$. Clearly S is non-empty. Choose $(x_n) \in S$. Since I_d is Cauchy regular, (x_n) is Cauchy in (X, d^*). Then there exists $n_0 \in \mathbb{N}$ such that $d^*(x_n, x_{n_0}) < \frac{\varepsilon}{4}$ for all $n \geq n_0$. Let $z = (z_n)$ be any sequence in X that converges to x. So there exists $n_z \in \mathbb{N}$ such that for some $m > n_0$, $d^*(z_n, x_m) < \frac{\varepsilon}{4}$ for all $n \geq n_z$ and also one can conclude that $d^*(z_n, x_{n_0}) \leq d^*(z_n, x_m) + d^*(x_m, x_{n_0}) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$ for all $n \geq n_z$. Consequently, there exists a connected subset C_n^z of X with a diameter $\leq \frac{\varepsilon}{2}$ containing x_{n_0} and z_n. Hence $C = \bigcup_{n \geq n_z, z \in S} C_n^z$ is a connected set of diameter $\leq \varepsilon$. Then $\widehat{C}$ is a connected subset of $\widehat{X}$ containing x. Now we claim that x is an interior point of $\widehat{C}$. Let (y_n) be any sequence in $\widehat{X}$ which converges to x. If (y_n) is not eventually in $\widehat{C}$, then without any loss of generality we can assume that $y_n \notin \widehat{C}$ for all $n \in \mathbb{N}$. So $d_n = dist(y_n, \widehat{C}) > 0$ for all $n \in \mathbb{N}$ and $d_n \to 0$. We can choose $z_n \in X$ with $\widehat{d}(y_n, z_n) < \frac{d_n}{2}$. Hence (z_n) converges to x but $z_n \notin C$, which is a contradiction. Therefore $\widehat{C}$ is a connected subset of $\widehat{X}$ of diameter $\leq \varepsilon$ with x is an interior point of $\widehat{C}$. Then from Lemma 5.1.2, $\widehat{X}$ is locally connected. □

Lemma 4.2.2. *(Lemma 3.4 of [17]) Let (X, ρ) be a metric space, and let (x_n), (y_n) be two sequences in X with $\rho(x_n, y_n) > \varepsilon$ for all n. Then there is an infinite set $J \subset \mathbb{N}$ such that $\rho(x_n, y_m) > \frac{\varepsilon}{4}$ for all $n, m \in J$.*

Theorem 4.2.6. *Suppose that (X, d) is a metric space. Then the following statements are equivalent:*

(1) *The identity function $I_d : (X, d) \to (X, d^*)$ is Cauchy regular.*

(2) *(X, d) is a pre-straight space with the completion $\widehat{X}$ is locally connected.*

Proof. (1) $\Rightarrow$ (2) Suppose that the identity function $I_d : (X, d) \to (X, d^*)$ is Cauchy regular. Then from Lemma 4.2.1, $\widehat{X}$ is locally connected. Now we only need to prove that X is pre-straight. Suppose that $X = C^+ \cup C^-$ is a cover of X by closed sets. Let $f \in C(X)$ and assume that $f|_{C^+}, f|_{C^-}$ are Cauchy regular. If f is not Cauchy regular, there is a Cauchy sequence (x_n) in X such that $(f(x_n))$ is not a Cauchy sequence. Consequently, there exist $\varepsilon > 0$ and two subsequences (x_{m_k}) and (x_{n_k}) of (x_n) such that $|f(x_{m_k}) - f(x_{n_k})| > \varepsilon$ for each k. (Note that $(x_{n_k}), (x_{m_k})$ cannot have accumulation points in X). Since (x_n) is a Cauchy sequence in (X, d^*), for k large enough, there are connected sets I_k joining x_{m_k} and x_{n_k} whose diameters tend to 0. On the other hand, the diameter of $f(I_k)$ is greater than ε. Now $f(I_k) = f(I_k \cap C^+) \cup f(I_k \cap C^-)$ is connected. Then either $\limsup_n diam(f(I_k \cap C^+)) > \frac{\varepsilon}{2}$ or $\limsup_n diam(f(I_k \cap C^-)) > \frac{\varepsilon}{2}$, which is impossible as $f|_{C^+}, f|_{C^-}$ are Cauchy regular.

(2) $\Rightarrow$ (1) Suppose on the contrary that I_d is not Cauchy regular. Then there exists a Cauchy sequence (x_n) in (X, d) such that (x_n) is not a Cauchy sequence in (X, d^*). Hence there exist $\varepsilon > 0$ and two subsequences (x_{n_k}) and (x_{p_k}) of (x_n) with $d^*(x_{n_k}, x_{p_k}) > \varepsilon$ for each $k \in \mathbb{N}$. Since d^* is a metric taking a subsequence if necessary we can assume by Lemma 4.2.2, that $d^*(x_{n_i}, x_{p_j}) > \frac{\varepsilon}{4}$ for all i, j. As (x_n) is Cauchy in (X, d), there is a closed ball C_1 of diameter $\frac{\varepsilon}{8}$ containing x_n for all but finitely many n, say for all $n \geq n_0$. Let $C_1 \subset C$ be a ball of diameter $\frac{\varepsilon}{4}$ with the center as C_1 and let $H = X \setminus C^\circ$. For $n \geq n_0$, $dist(x_n, H) \geq \frac{\varepsilon}{8}$. We claim that interior of C can be partitioned in two relatively closed sets F and G, one containing all the x_{n_k} with $k \geq n_0$ and the other containing all the x_{p_k} with $k \geq n_0$. To prove this, first we note that C has been chosen so small that it cannot contain a connected set joining a point in $\{x_{n_k} : k \in \mathbb{N}\}$ to a point in $\{x_{p_k} : k \in \mathbb{N}\}$. Let $A_1 \subset C$ be the closure of the union of connected components of the points x_{n_k} belonging to C and let $A_2 \subset C$ be defined with respect to the points x_{p_k}. Finally, $A_3 \subset C$ is the closure of the union of all components of C which are disjoint from $\{x_{n_k} : k \in \mathbb{N}\}$ and $\{x_{p_k} : k \in \mathbb{N}\}$.

From Theorem 4.2.1, X is locally connected. Consequently, a point in the interior of C belongs to exactly one of the sets A_1, A_2 and A_3. Then we can prove the claim by setting $F = A_1 \cap C^\circ$ and $G = (A_2 \cup A_3) \cap C^\circ$. Therefore X can be written as the union of the two closed sets $H \cup F$ and $H \cup G$ intersecting in H. But they are not c-placed, which is a contradiction. Hence I_d is Cauchy regular. □

Remark 4.2.2. *Suppose that* $A = \{(x, \frac{1}{x}) : x > 0\}, B = \{(x, 0) : x \geqslant 0\} \cup \{(0, y) : y \geqslant 0\}$. *Consider* $X = A \cup B$ *with usual metric of* $\mathbb{R}^2$. *Then on* X *the identity function* I_d *is Cauchy regular, but not uniformly continuous. Here* X *is a pre-straight space, but not a straight space. Further, a closed disk minus a point is a non-complete locally connected pre-straight space with locally connected completion where we can have a continuous function that is not Cauchy regular.*

4.3 Relation between Cauchy connectedness and straightness

In [17], it has been shown that the notion of straight spaces is related to some versions of connectedness. In our context, Cauchy connectedness will play a significant role in the study of pre-straight and pre$(*)$-straight spaces. We consider a metric, namely d^c on X using Cauchy connectedness in line with Definition 4.2.1 and show how the local Cauchy connectedness depends on the nature of that metric.

Definition 4.3.1. *Given a metric space* (X, d) *and* $x, y \in X$ *define* $d^c(x, y) = min\{1, inf\{\varepsilon|$: *there is a Cauchy connected set of diameter* $\leq \varepsilon$ *containing* x *and* $y\}\}$ *and* $d^c(x, y) = 1$ *if there is no Cauchy connected set containing* x *and* y.

Lemma 4.3.1. (1) *The map* $d^c : X \times X \to \mathbb{R}$ *is a metric on* X.
(2) *If* d *is bounded by* 1 *then* $d \leq d^c \leq d^*$.

Proof. (1) Clearly from the definition $d^c(x,y) \geq 0$ and $d^c(x,y) = 0 \Leftrightarrow x = y$ and $d^c(x,y) = d^c(y,x)$. Let $x, y, z \in X$. If either $d^c(x,z)$ or $d^c(z,y) = 1$ then clearly $d^c(x,y) \leq d^c(x,z) + d^c(z,y)$. Thus assume that $d^c(x,z) < 1$ and $d^c(z,y) < 1$. Then there exist Cauchy connected sets A containing x, z and B containing z, y. Now $A \cup B$ is a Cauchy connected set containing x, y. So $d^c(x,y) \leq diam(A \cup B) \leq diam(A) + diam(B)$ and this implies that $d^c(x,y) \leq d^c(x,z) + d^c(z,y)$. Hence (X, d^c) is a metric space.

(2) Take any $x, y \in X$. If there is no Cauchy connected set containing x, y, then $d^c(x,y) = d^*(x,y) = 1$ and so $d(x,y) \leq d^c(x,y) \leq d^*(x,y)$. Now if there is a Cauchy connected set containing x, y but there is no connected set containing x, y, then again $d^*(x,y) = 1$. Clearly $d(x,y) \leq diam(A)$ for every such Cauchy connected set A. Hence $d(x,y) \leq d^c(x,y) \leq d^*(x,y)$. Next if there is a connected set containing x, y then $d(x,y) \leq d^c(x,y) \leq diam(A)$ for every such connected set A. So $d(x,y) \leq d^c(x,y) \leq d^*(x,y)$. □

The following example shows that in general $d \neq d^c \neq d^*$.

Example 4.3.1. *Let A be the semicircle with center at origin and radius $\frac{1}{7}$ minus its diameter and $X = A \cap (\mathbb{Q} \times \mathbb{Q})$. Let x, y be the two endpoints. Then $d(x,y) = \frac{2}{7}$, $d^c(x,y) = \frac{22}{49}$ and $d^*(x,y) = 1$*

Our next result is in line with Lemma 3.8 [17].

Theorem 4.3.1. *The metric d^c is equivalent to d iff (X, d) is locally Cauchy connected.*

Proof. Assume that (X, d) is locally Cauchy connected. Clearly, $\tau_d \subset \tau_{d^c}$, where the notations stand for the respective topologies. For the reverse inclusion, take any open ball $B_{d^c}(x, r)$. Since (X, d) is locally Cauchy connected, there exists $U \in \tau_d$ such that $x \in U \subset B_d(x, \frac{r}{4})$ and U is Cauchy connected. Now there is a $t > 0$ such that $B_d(x, t) \subset U$. Take any point $y \in B_d(x, t)$. Obviously U is a Cauchy connected set containing x, y. So $d^c(x,y) \leq diam(U) \leq diam(B_d(x, \frac{r}{4})) < r$, which implies $y \in B_{d^c}(x,r)$. Hence $B_d(x,t) \subset B_{d^c}(x,r)$, which implies $\tau_d^c \subset \tau_d$.

Conversely, assume that d^c is equivalent to d. Take any $U \in \tau_d$ and $x \in U$. Then there exists $r > 0$ $(r < 1)$ such that $B_{d^c}(x,r) \subset U$. We will show that $B_{d^c}(x,r)$ is Cauchy connected. Take any point $y \in B_{d^c}(x,r)$. Then there is a Cauchy connected set C_y containing x, y with $diam(C_y) < r$. Let $z \in C_y$. Evidently C_y is also a Cauchy connected set containing x, z. So $d^c(x,z) \leq diam(C_y) < r$. Hence $C_y \subset B_{d^c}(x,r)$. Observe that $B_{d^c}(x,r)$ is the union of all such Cauchy connected set C_y for each $y \in B_{d^c}(x,r)$ having x as a common point. Hence $B_{d^c}(x,r)$ is Cauchy connected and this proves that (X,d) is locally Cauchy connected. □

Corollary 4.3.1. *The identity map $I_d : (X,d) \to (X,d^c)$ is a homeomorphism iff (X,d) is locally Cauchy connected.*

Theorem 4.3.2. *Let (X,d) be locally Cauchy connected and pre-straight. Then $I_d : (X,d) \to (X,d^c)$ is Cauchy regular.*

Proof. The proof is similar to the proof of (2) $\implies$ (1) of Theorem 4.2.6, so is omitted. □

Example 4.3.2. *In the above Theorem, the given conditions are essential. The topologist sine curve is not locally Cauchy connected but pre-straight. Here I_d is not Cauchy regular. Again $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ is locally Cauchy connected but not pre-straight. Here also I_d is not Cauchy regular as $d^c(\frac{1}{n}, \frac{1}{m}) = 1$.*

Corollary 4.3.2. *Let (X,d) be a locally Cauchy connected and pre-straight. Then A is Cauchy connected in (X,d) iff A is Cauchy connected in (X,d^c).*

It is known that a locally connected space is straight iff it is uniformly locally connected (Theorem 3.9 of [17]).

Definition 4.3.2. *(cf. Remark 3.2 [17]) A metric space X is said to be weakly uniformly locally Cauchy connected, iff every pair of non-Cauchy d- adjacent sequence is d^c-adjacent i.e., for any two non-Cauchy sequences (x_n) and (y_n) with $d(x_n, y_n) \to 0$ we have $d^c(x_n, y_n) \to 0$.*

Theorem 4.3.3. *(cf. Lemma 3.1 [17]) If (X, d) is weakly uniformly locally Cauchy connected then (X, d) is pre$(*)$-straight.*

Proof. The proof is similar to $(1) \implies (2)$ of Theorem 4.2.6, so is omitted. □

4.4 Notion of straightness via ward continuity

In this section, we consider a space in line with the much investigated UC space or Atsuji space [4].

Definition 4.4.1. *A metric space X is called a WC space iff every real-valued continuous function on X is ward continuous.*

Clearly, every *UC* space is a *WC* space, but the converse is not generally true.

Example 4.4.1. *Take $X = \mathbb{N} \cup \{n + \frac{1}{n} : n \in \mathbb{N}\}$ with usual metric. Any quasi-Cauchy sequence in X is eventually constant. Hence X is evidently a WC space. But the characteristic function of $\mathbb{N}$ is a function from X to $\{0, 1\}$, which is continuous but not uniformly continuous.*

Example 4.4.2. *Define $X_n = \{[-1, -\frac{1}{n}] \cup [\frac{1}{n}, 1]\} \times \{n\}$. Then take $X = \bigcup_{n=1}^{\infty} X_n$ endowed with the usual metric of $\mathbb{R}^2$. Let f be a real valued continuous function on X and (x_n) be a quasi-Cauchy sequence in X. Then (x_n) is eventually in some X_{n_0}, but has no subsequence in any other X_n with $n \neq n_0$. Since each X_n is compact so $f|_{X_{n_0}}$ is uniformly continuous. Hence $(f(x_n))$ is a quasi-Cauchy sequence in $\mathbb{R}$. So f is a ward continuous function and hence X is a WC space. Now define $f : X \to \mathbb{R}$ by $f(x) = 0$, if $x \in [-1, -\frac{1}{n}] \times \{n\}$ and*

$f(x) = 1$, if $x \in [\frac{1}{n}, 1] \times \{n\}$, for each $n \in \mathbb{N}$. Then f is a continuous function but not a uniformly continuous function. Therefore X is not a UC space.

From Lemma 3.2.3, it is easy to observe that a uniformly connected space is UC iff it is WC.

Theorem 4.4.1. *Every WC space is complete.*

Proof. Let X be a WC space. If X is not complete, then there is a Cauchy sequence (x_n) in X, which is not convergent. By Tietze extension theorem, there is a continuous function $f : X \to \mathbb{R}$ such that $f(x_n) = n$, which is not ward continuous. □

However, complete metric spaces need not be WC. For example take $X = \{\sqrt{n} : n \in \mathbb{N}\}$ with the usual metric. X is trivially complete but is not WC because $f : X \to \mathbb{R}$ defined by $f(\sqrt{n}) = n$ is continuous but not ward continuous. Hence WC spaces properly lie between the UC spaces and complete spaces.

Definition 4.4.2. *Let (X, d) be a metric space and A, B be two non-empty subsets of X. Then*

1. *A, B are said to be connected through a quasi-Cauchy sequence if there exists a quasi-Cauchy sequence (x_n) in X such that $x_{n_k} \in A$ and $x_{n_k+1} \in B$ for some subsequence (x_{n_k}) of (x_n).*

2. *A, B do not have any common quasi-Cauchy sequence if for each quasi-Cauchy sequence (x_n) in X, the sets $\{x_n : n \in \mathbb{N}\} \cap A$ and $\{x_n : n \in \mathbb{N}\} \cap B$ cannot be infinite simultaneously.*

Clearly, if two sets A, B do not have any common quasi-Cauchy sequence, then they cannot be connected through a quasi-Cauchy sequence. But in general, the converse part is not true. For example let us consider X to be a real valued quasi-Cauchy sequence (x_n) where $x_n = \{1 + \frac{1}{2} + ... + \frac{1}{n} : n \in \mathbb{N}\}$. Take $A = \{x_{k^2} : k \in \mathbb{N}\}$

and $B = \{x_{[\frac{k^2+(k+1)^2}{2}]} : k \in \mathbb{N}\}$. Here A, B have a common quasi-Cauchy sequence, but they cannot be connected through any quasi-Cauchy sequence.

If A and B form a cover of X then these two concepts coincide. Also, in the case of Cauchy sequences, these two concepts are same. These two concepts play an important role in characterizing WC space as well as W-straight spaces (Definition 4.4.3).

Now we present two characterizations of WC spaces.

Theorem 4.4.2. *Suppose that (X, d) and (Y, ρ) are two metric spaces. Then the following conditions are equivalent.*

(1) *Any continuous function $f : (X, d) \to (Y, \rho)$ is ward continuous.*

(2) *X is a WC space.*

(3) *Any two non empty disjoint closed subsets of X cannot be connected through quasi-Cauchy sequences.*

(4) *Every subsequence of a quasi-Cauchy sequence has a cluster point.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) On the contrary, let us take two disjoint non empty closed subsets A, B of X which are connected through quasi-Cauchy sequences. Then there is a quasi-Cauchy sequence (x_n) in X such that $x_{n_k} \in A$ and $x_{n_k+1} \in B$ for some subsequence (x_{n_k}) of (x_n). By Tietze extension Theorem there is a continuous function $f : X \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) = 1$. But evidently f cannot be ward continuous, which contradicts with (2).

(3) $\Rightarrow$ (4) Let there be a quasi-Cauchy sequence (x_n) (without any loss of generality we can assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$.) having a subsequence (x_{n_k}) which has no cluster point. Take $A = \{x_{n_k} : k \in \mathbb{N}\}$ and $B = \{x_{n_k+1} : k \in \mathbb{N}\}$. Clearly A is closed. B should also be closed because, if B has a cluster point then A will have the same cluster point. So A, B are two disjoint closed subsets of X, which are connected

through the quasi-Cauchy sequence (x_n). This contradicts (3).

$(4) \Rightarrow (1)$ Suppose $f : (X, d) \to (Y, \rho)$ is a continuous function which is not ward continuous. Then there is a quasi-Cauchy sequence (x_n) in X such that $(f(x_n))$ is not quasi-Cauchy. So there exists $\varepsilon > 0$ and a subsequence (x_{n_k}) such that for each $k \in \mathbb{N}$, $\rho(f(x_{n_k}), f(x_{n_k+1})) \geq \varepsilon$. Now by the given condition (x_{n_k}) has a cluster point p. Without any loss of generality passing onto a subsequence we can assume that $x_{n_{k_i}} \to p$. Then obviously $x_{n_{k_i}+1} \to p$. Define $z_{2i} = x_{n_{k_i}}$ and $z_{2i-1} = x_{n_{k_i}+1}$. Clearly the sequence (z_n) is convergent (to p) but the sequence $(f(z_n))$ cannot be convergent. This fact contradicts with the assumption that f is continuous. Hence f must be ward continuous. □

If every quasi-Cauchy sequence in a metric space is convergent, then it is clear that the space is WC. But the converse is again not true. To produce an example the following Lemma would be needed.

Lemma 4.4.1. *Suppose that (X, d) is a uniformly connected metric space having at least two points. Then there is a nonconvergent quasi-Cauchy sequence in X.*

Proof. Choose $x, y \in X$ such that $x \neq y$. Since X is uniformly connected, from Lemma 3.2.1, it is chainable. Then for each $n \in \mathbb{N}$, there is a $\frac{1}{n}$-chain joining x, y. In other words, there is a finite sequence of points $x = x_1^n, x_2^n, ..., x_{n_i}^n = y$ such that $d(x_l^n, x_{l-1}^n) < \frac{1}{n}$, for $1 \leq l \leq n_i$. Now construct the sequence

$$x, x_2^1, .., y, x_{2_i-1}^2, x_{2_i-2}^2 ..., x, x_2^3, x_3^3, .., y, ..., x, x_2^n, ..., y, ...$$

Clearly, this sequence is quasi-Cauchy but not convergent, as it has two different cluster points. □

From Lemma 4.4.1, we can conclude that $[0, 1]$ with the usual metric is WC space but again in this space, a nonconvergent quasi-Cauchy sequence can be constructed following the above formulation.

Now we introduce the following notion of straightness which can be thought of as a generalization of WC space.

Definition 4.4.3. *A metric space (X, d) is said to be W-straight if whenever X is the union of two closed sets, then $f \in C(X)$ is ward continuous iff its restriction to each of the closed sets is ward continuous.*

In general, the concepts of straight and W-straightness are different which will be illustrated in Example 4.4.3.

Lemma 4.4.2. *A precompact pre-straight space is W-straight.*

Clearly WC spaces are W-straight. But the converse is not generally true. For example $(0, 1]$ with usual metric is W-straight by Lemma 4.4.2 but not WC as it is not complete.

We now introduce the following definition in line with the notion c-placed defined before.

Definition 4.4.4. *Let (X, d) be a metric space. A pair C^+, C^- of closed sets of X is said to be qc-placed if $C^+_\varepsilon, C^-_\varepsilon$ do not have any common quasi-Cauchy sequence for every $\varepsilon > 0$.*

Theorem 4.4.3. *Suppose that (X, d) is a metric space and every pair of closed subsets C^+, C^- of X, which form a cover of X, is qc-placed. Then X is W-straight.*

Proof. Let $X = C^+ \cup C^-$ be a closed cover and $f : C^+ \cup C^- \to \mathbb{R}$ be a continuous function such that $f|_{C^+}$ and $f|_{C^-}$ are ward continuous. If either $C^+ = C^+ \cup C^-$ or $C^- = C^+ \cup C^-$ then the proof is finished, so assume that $C^+ \neq C^+ \cup C^- \neq C^-$. If $C^+ \cap C^- = \phi$ then C^+ and C^- can not have any common quasi-Cauchy sequence. Hence additionally assume $C^+ \cap C^- \neq \phi$.

Let (x_n) be a quasi-Cauchy sequence in $C^+ \cup C^-$. If (x_n) is eventually either in C^+ or in C^- then by ward continuity of $f|_{C^+}$ and $f|_{C^-}$, $(f(x_n))$ is quasi-Cauchy.

Suppose on the other hand that (x_n) is frequently both in C^+ and C^-. Then there is a subsequence (x_{n_k}) such that

$A = \{x_1, x_2, ..., x_{n_1}, x_{n_2+1}, ..., x_{n_3}, x_{n_4+1} ....., x_{n_k+1}, x_{n_k+2}, ...x_{n_{k+1}}, ...\} \subset C^+$.

and $B = \{x_{n_1+1}, x_{n_1+2}, ..., x_{n_2}, x_{n_3+1}, ..., x_{n_4}, ..., x_{n_{k+1}+1}, ..., x_{n_{k+2}}, ...\} \subset C^-$.

Then for each $\varepsilon > 0$, A, B intersects $C^+_\varepsilon, C^-_\varepsilon$ respectively at most finitely many points. Otherwise $C^+_\varepsilon, C^-_\varepsilon$ would have a common quasi-Cauchy sequence. So for each $n \in \mathbb{N}$, there exists $y_n \in C^+ \cap C^-$ such that $d(x_n, y_n) < \frac{1}{n}$. Consequently we get three sequences as follows.

$P = \{x_1, .., x_{n_1}, y_{n_1+1}, .., y_{n_2}, ..., x_{n_k+1}, .., x_{n_{k+1}}, y_{n_{k+1}+1}, .., y_{n_{k+2}} ...\} \subset C^+$

$Q = \{y_1, .., y_{n_1}, x_{n_1+1}, .., x_{n_2}, ..., y_{n_k+1}, .., y_{n_{k+1}}, x_{n_{k+1}+1}, .., x_{n_{k+2}} ...\} \subset C^-$

$R = \{y_1, .., y_{n_1}, y_{n_1+1}, .., y_{n_2}, ..., y_{n_k+1}, .., y_{n_{k+1}}, x_{n_{k+1}+1}, .., x_{n_{k+2}} ...\} \subset C^+ \cap C^-$

Now for $\varepsilon > 0$ choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{3}$ and $d(x_n, x_{n+1}) < \frac{\varepsilon}{3}$ for all $n \geq n_0$. Note that $d(y_n, y_{n+1}) \leq d(y_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ for all $n \geq n_0$. This shows that R is a quasi-Cauchy sequence in $C^+ \cap C^-$. Next observe that $d(x_k, y_{k+1}) \leq d(x_k, y_k) + d(y_k, y_{k+1}) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$ for all $n \geq n_0$. Hence P, Q are again quasi-Cauchy sequence in C^+ and C^- respectively. So $f(P), f(Q), f(R)$ are all quasi-Cauchy. We will show that $(f(x_n))$ is quasi-Cauchy. For that if x_n, x_{n+1} are both in C^+, then as $f(P)$ is quasi-Cauchy, so $d(f(x_n), f(x_{n+1})) \to 0$. Similarly for if x_n, x_{n+1} both are in C^- then $d(f(x_n), f(x_{n+1})) \to 0$. Finally if $x_n \in C^+$ and $x_{n+1} \in C^-$ then $d(f(x_n), f(y_{n+1})) \to 0$, $d(f(y_{n+1}), f(y_n)) \to 0$, $d(f(y_n), f(x_{n+1})) \to 0$ as $f(P), f(R), f(Q)$ are all quasi-Cauchy.

Hence $d(f(x_n), f(x_{n+1})) \to 0$. So $(f(x_n))$ is quasi-Cauchy and X is W-straight. □

The next example shows that the condition of *qc*-placedness in Theorem 4.4.3 is not a necessary condition, unlike the roles played by similar notions as can be seen from Corollary 4.1.1 or Lemma 2.6 [17].

Example 4.4.3. *Consider* $X = \bigcup_{n=1}^{\infty} \{(x, \frac{1}{n}) : |x| \leq 1\} \cup \{(x, 0) : |x| \leq 1\} \cup \{(x, y) : x =$

1 or -1 and $0 \leq y \leq 1\}$ with Euclidean metric. Clearly X is compact and so W-straight. Take $C^{+} = \bigcup_{n=1}^{\infty} \{(x, \frac{1}{n}) : 0 \leq x \leq 1\} \cup \{(x,0) : 0 \leq x \leq 1\} \cup \{(1,y) : 0 \leq y \leq 1\}$ *and* $C^{-} = \bigcup_{n=1}^{\infty} \{(x, \frac{1}{n}) : -1 \leq x \leq 0\} \cup \{(x,0) : -1 \leq x \leq 0\} \cup \{(-1,y) : 0 \leq y \leq 1\}$. *Clearly C^+, C^- form a closed cover of X and $C^+ \cap C^- = \{(0, \frac{1}{n}) : n \in \mathbb{N}\} \cup \{(0,0)\}$. Now consider $x_n = (-1, \frac{1}{n})$ and $y_n = (1, \frac{1}{n})$. Clearly $x_n \in C_1^-$ and $y_n \in C_1^+$ for each $n \in \mathbb{N}$. Since X is uniformly connected, so x_n and y_n can be joined by a $\frac{1}{n}-$chain and y_n, x_{n+1} also can be joined by $\frac{1}{n}-$chain. Hence we get $x_n = p_{n,1}, p_{n,2}..., p_{n,n_k} = y_n$ and $y_n = q_{n,1}, q_{n,2}..., q_{n,n_k} = x_{n+1}$ such that two consecutive point having distance less than $\frac{1}{n}$. Thus we get a quasi-Cauchy sequence $\{x_1, p_{1,2}, ..., y_1, q_{1,2}, ..., x_2, ...\}$ in X such that (x_n) and (y_n) are two subsequences of it. So (C^+, C^-) is not qc-placed.*

W-straightness does not imply qc-placedness. But there is another property of the closed cover (C^+, C^-), which has a slight variation from qc-placedness, is implied by W-straightness.

Theorem 4.4.4. *Suppose X is W-straight. Then for any closed cover (C^+, C^-) of X and for any $\varepsilon > 0$, C_ε^+ and C_ε^- can not be connected through a quasi-Cauchy sequence.*

Proof. If $C^+ \cap C^- = \phi$, then consider the characteristic function $\chi : C^+ \cup C^- \to \mathbb{R}$ of the set C^+. It is continuous and its restriction to C^+ and C^- are ward continuous and so by the given condition χ must be ward continuous and hence C^+, C^- satisfies the required property. Now assume $C^+ \cap C^- \neq \phi$ and consider the function $f : C^+ \cup C^- \to \mathbb{R}$ defined by $f(x) = d(x, C^+ \cap C^-)$, for $x \in C^+$ and $f(x) = -d(x, C^+ \cap C^-)$, for $x \in C^-$. Obviously f is continuous and $f|_{C^+}, f|_{C^-}$ are ward continuous. So f is ward continuous. This implies C_ε^+ and C_ε^- have the given property for every $\varepsilon > 0$. If not, then for some $\varepsilon > 0$, there is a quasi-Cauchy sequence (x_n) in X such that $x_{n_k} \in C_\varepsilon^+$ and $x_{n_k+1} \in C_\varepsilon^-$ for some subsequence (x_{n_k}) of (x_n). Then $f(x_{n_k}) \geq \varepsilon$ and

$f(x_{n_k+1}) \leq -\varepsilon$ for each $k \in \mathbb{N}$. So $(f(x_n))$ can not be quasi-Cauchy, which contradicts the ward continuity of f. □

The next example shows that this property of C^+ and C^- can not yield W-straightness of a space, unlike the roles played by similar notions as can be seen from Corollary 4.1.1.

Example 4.4.4. *Let A_{2k-1} be the line segment with end points $(1, 2k-1)$ and $(0, 2k)$ whereas A_{2k} is the line segment with end points $(0, 2k)$ and $(1, 2k+1)$. Consider $X = \cup_{n=1}^{\infty} A_n$ with Euclidean metric. Clearly, X is uniformly locally connected and so is straight. Then for any closed cover (C^+, C^-) of X and for any $\varepsilon > 0$ $dist(C_\varepsilon^+, C_\varepsilon^-) > 0$. So $C_\varepsilon^+, C_\varepsilon^-$ can not be connected through a quasi-Cauchy sequence. But X is not W-straight. For this, consider $C^+ = \cup_{k=1}^{\infty} A_{2k}$ and $C^- = \cup_{k=1}^{\infty} A_{2k-1}$. Clearly C^+ and C^- are closed and $C^+ \cap C^- = \{(1, 2k-1) : k \in \mathbb{N}\} \cup \{(0, 2k) : k \in \mathbb{N}\}$. Take $x_k = (1, 2k-1)$ and $y_k = (0, 2k)$. Since X is uniformly connected, a quasi Cauchy sequence (z_n) can be constructed having subsequence (x_k) and (y_k). Now $A_{2k-1} \cap \{z_n : n \in \mathbb{N}\} = \{x_k, p_{k,2}, ...y_k\}$ and $A_{2k} \cap \{z_n : n \in \mathbb{N}\} = \{y_k, q_{k,2}, ...x_{k+1}\}$, which are finite. Define a function f_k on A_{2k-1} such that $f_k(x_k) = f_k(p_{k,2}) = ... = f_k(p_{k,n_k-1}) = k$ and $f_k(y_k) = k+1$ (we actually define the function on $A_{2k-1} \cap \{z_n : n \in \mathbb{N}\} = \{x_k, p_{k,2}, ...y_k\}$ which is continuous and then use Tietze extension theorem to extend it to the space A_{2k-1}). Similarly there is a continuous function g_k on A_{2k} such that $g_k(A_{2k}) = k+1$. Since each A_n is compact, so each f_k and g_k are uniformly continuous. Note that both $dist(A_{2k-1}, A_{2k+1}) > 1$ and $dist(A_{2k}, A_{2k+2}) > 1$. Define $f : X \to \mathbb{R}$ such that restriction of f on A_{2k-1} is f_k and restriction of f on A_{2k} is g_k. Clearly f is continuous and $f|_{C^+}$ is ward continuous (though not uniform continuous) as any quasi-Cauchy sequence in C^+ eventually in some A_{2k}. Similarly, $f|_{C^-}$ is ward continuous (though not uniform continuous). But f is not ward continuous as $(f(z_n))$ has a subsequence $(f(z_{n_k}))$ such that $|f(z_{n_k}) - f(z_{n_k+1})| = 1$.*

Remark 4.4.1. *For a W-straight space any closed cover (C^+, C^-) is c-placed because if not* *then there is* $\varepsilon > 0$ *such that* C_ε^+ *and* C_ε^- *has a subsequence* x_{n_k} *and* x_{m_k} *of a Cauchy sequence* x_n*. Define* $z_{2k-1} = x_{n_k}$ *and* $z_{2k} = x_{m_k}$ *for each* $k \in \mathbb{N}$*. Clearly* z_n *is quasi-Cauchy sequence. Then by the Theorem 4.4.4, this contradicts that* X *is W-straight. Hence* X *is pre-straight.*

A pre-straight space need not be W-straight. For example let $x_n = \sqrt{n}$ *and consider* $X = \{x_n : n \in \mathbb{N}\}$ *with the usual metric. Then* X *is pre-straight as it is complete. But considering the characteristic function of* $\{x_{2k} : k \in \mathbb{N}\}$ *we can conclude that* X *is not W-straight.*

Finally, we consider the following characterization of WC space in terms of W-straightness.

Theorem 4.4.5. *Let* (X, d) *be a metric space. Then the followings are equivalent.*
(1) *X is WC.*
(2) *Every closed subspace of X is W-straight, i.e., X is hereditarily W-straight.*
(3) *Every pair of closed subsets of X cannot be connected through a quasi-Cauchy sequence.*

Proof. By Theorem 4.4.2, every closed subspace of a WC space is WC. So $(1) \Rightarrow (2)$ is obvious. We only need to prove $(2) \Rightarrow (1)$. Assume on the contrary that X is not WC. Then there is a quasi-Cauchy sequence (x_n) in X such that no subsequence of (x_n) has any cluster point in X. Consider $A_i = \{j : x_j = x_i\}$. Clearly each A_i must be finite as otherwise (x_n) has a convergent subsequence. So we can assume (x_n) as a sequence of distinct terms. Take $C^+ = \{x_{2k} : k \in \mathbb{N}\}$ and $C^- = \{x_{2k-1} : k \in \mathbb{N}\}$. Then $C^+ \cap C^- = \phi$. Clearly the closed subspace $Y = C^+ \cup C^-$ is not W-straight by taking the characteristic function of C^+. The equivalence with (3) is straightforward. □

5

Preserving Properties of Cauchy Regular Functions

In this chapter, we focus on some preserving properties of Cauchy regular function. Particularly we are interested in the reverse implications. Every Cauchy regular function preserves precompactness and Cauchy connectedness. Firstly, we find some conditions under which a precompactness- and Cauchy connectedness preserving function is Cauchy regular. The interesting fact is that the graph of a function plays a significant role in this analysis. Additionally, We define Cauchy separated fibers, which yields Cauchy regularity of a continuous map defined on a pre-straight space. Finally, we define CA functions in line with the UA functions and establish a relation of Cauchy separated fibers with CA functions.

The entire investigation is done in the context of metric spaces and the content of the chapter is based on the following research paper.

- S. K. Pal and **N. Adhikary**, Characterization of Cauchy regular functions, **Topology and its Applications**, 315 (2022) 108148. [66]

5.1 Precompactness- and Cauchy connectedness-preserving function

We have already mentioned in the Preliminary chapter that every Cauchy regular function preserves precompactness. In Chapter 3, we have introduced the notion of Cauchy connectedness in terms of Cauchy separation and observed that every Cauchy regular function preserves Cauchy connectedness (Theorem 3.1.3), but these two preserving properties on a map cannot get back to Cauchy regularity. We are interested in bringing up the necessary conditions under which a precompactness- and Cauchy connectedness-preserving function is Cauchy regular. Taking a cue from the following result of McMillan [57]: "a function on a locally connected space is continuous iff it preserves connectedness and compactness" we obtain similar results for Cauchy regular functions where the role of connectedness and compactness are taken by Cauchy connectedness and precompactness, respectively. The hypothesis of local connectedness of the domain is replaced by the assumption that the completion of graph of the function is locally connected. We start our pursuit of this new idea with the following Lemma.

Lemma 5.1.1. *Suppose that (X, d) is a metric space and C is an open, connected subspace of X. Then, for any dense subset D of X, $C \cap D$ is a Cauchy connected subspace of X.*

Proof. It is easy to check that $C \subset \overline{C \cap D} \subset \overline{C}$ and which implies that $\overline{C \cap D}$ is a connected subspace of X. Then from Theorem 3.1.2, we can conclude that $C \cap D$ is a Cauchy connected subspace of X. □

Corollary 5.1.1. *Let $\widehat{X}$ be the completion of the metric space X. Then, for any open, connected subspace C of $\widehat{X}$, $C \cap X$ is Cauchy connected in X.*

Proof. The proof follows from Lemma 5.1.1. □

We are now in a position to present the main results of this section.

Theorem 5.1.1. *Suppose that (X, d_X) and (Y, d_Y) are two metric spaces and $f : (X, d_X) \to (Y, d_Y)$ is both precompactness- and Cauchy connectedness-preserving function satisfies the property that the completion of $G(f)$ (written as $\widehat{G(f)}$ and here $G(f)$ is the graph of f) is locally connected. Then f is Cauchy regular.*

Proof. Suppose, on the contrary, that f is not Cauchy regular. So f cannot be extended to a continuous function from $\widehat{X}$ to $\widehat{Y}$. Then there exist $x \in \widehat{X}$ and a sequence (x_n) in X such that (x_n) converges to x but $(f(x_n))$ does not converge in $\widehat{Y}$. Since f is a precompactness-preserving function, every subsequence of $(f(x_n))$ has a Cauchy subsequence. Now it is evident that $(f(x_n))$ has at least two distinct cluster points y_1 and y_2 in $\widehat{Y}$. Then there are two open sets U and V in $\widehat{Y}$ containing y_1 and y_2 respectively such that $\overline{U} \cap \overline{V} = \phi$. Since $\widehat{G(f)}$ is locally connected, for some open set W containing x in $\widehat{X}$, there are two open, connected sets C_1 and C_2 in $\widehat{G(f)}$ such that $(x, y_1) \in C_1 \subset W \times U$ and $(x, y_2) \in C_2 \subset W \times V$. Therefore from Corollary 5.1.1, it is clear that $C_1^g = C_1 \cap G(f)$ and $C_2^g = C_2 \cap G(f)$ are Cauchy connected subspaces of $G(f)$. Let π be the projection map from $G(f)$ to X. Then from Theorem 3.1.3, it is evident that $\pi(C_1^g)$ and $\pi(C_2^g)$ are Cauchy connected subspaces of X. Furthermore, the Cauchy sequence (x_n) is frequently both in $\pi(C_1^g)$ and $\pi(C_2^g)$. Hence $\pi(C_1^g) \cup \pi(C_2^g)$ is a Cauchy connected subspace of X, but $f(\pi(C_1^g) \cup \pi(C_2^g))$ is not Cauchy connected as (U, V) is a Cauchy separation of $f(\pi(C_1^g) \cup \pi(C_2^g))$. Indeed $f(z) \in f(\pi(C_1^g) \cup \pi(C_2^g))$ implies that $(z, f(z)) \in C_1^g \subset W \times U$ or $(z, f(z)) \in C_2^g \subset W \times V$ and consequently $f(\pi(C_1^g) \cup \pi(C_2^g)) \subset U \cup V$. Moreover it is clear that $f(\pi(C_1^g) \cup \pi(C_2^g)) \cap U \neq \phi$ and $f(\pi(C_1^g) \cup \pi(C_2^g)) \cap V \neq \phi$. □

The converse of this result is true provided $\widehat{X}$ is locally connected. In order to serve our purpose, we recall the definition of weakly locally connected space.

Definition 5.1.1. *[26] A space X is said to be weakly locally connected at a point $x \in X$ if for every open set V containing x, there exists a connected set $N \subset V$ such that x lies in the interior of N. X is said to be weakly locally connected if it is weakly locally connected at every point x of X.*

Lemma 5.1.2. *[26] A weakly locally connected space is locally connected.*

Theorem 5.1.2. *Suppose that $f : (X, d_X) \to (Y, d_Y)$ is a Cauchy regular function. Then $\widehat{X}$ is locally connected iff $\widehat{G(f)}$ is locally connected.*

Proof. Being a Cauchy regular function f can be extended to a continuous function $\widetilde{f} : \widehat{X} \to \widehat{Y}$ such that $\widetilde{f}(x) = f(x)$ for all $x \in X$. It is obvious that $\widehat{G(f)} = G(\widetilde{f})$. Let $(x, \widetilde{f}(x)) \in G(\widetilde{f})$ and $(U \times V) \cap G(\widetilde{f})$ be a basic open set containing $(x, \widetilde{f}(x))$ where U and V be the open sets containing x and $\widetilde{f}(x)$ respectively. Then $W = U \cap \widetilde{f}^{-1}(V)$ is an open set containing x in $\widehat{X}$. Assume $\widehat{X}$ is locally connected. So there is an open, connected set W_1 such that $x \in W_1 \subset W$. Now from continuity of $\widetilde{f}$ we can conclude that the set $N = \{(a, \widetilde{f}(a)) : a \in W_1\}$ is a connected set containing $(x, \widetilde{f}(x))$ such that $N \subset (U \times V) \cap G(\widetilde{f})$. One can observe that for every sequence $((x_n, \widetilde{f}(x_n)))$ which converges to $(x, \widetilde{f}(x))$, (x_n) is eventually in W_1. This implies that $((x_n, \widetilde{f}(x_n)))$ is eventually in N. Hence $(x, \widetilde{f}(x))$ is an interior point of N. As $(x, \widetilde{f}(x))$ is chosen arbitrarily, $G(\widetilde{f})$ is weakly locally connected. Therefore from Lemma 5.1.2, $\widehat{G(f)}$ is a locally connected space.

To prove the converse part, we assume that $\widehat{G(f)}$ is locally connected. We have already explained that $\widehat{G(f)} = G(\widetilde{f})$, where $\widetilde{f} : \widehat{X} \to \widehat{Y}$ is a continuous function with $\widetilde{f}(x) = f(x)$ for all $x \in X$. Now the projection map $\pi : G(\widetilde{f}) \to \widehat{X}$ is open and continuous. Let $x \in \widehat{X}$ and V be an open set containing x. Then there exits an open connected subset C of $G(\widetilde{f})$ such that $(x, \widetilde{f}(x)) \in C \subset \pi^{-1}(V)$. Consequently $\pi(C)$ is an open connected subset of V containing x. Therefore $\widehat{X}$ is locally connected. □

In the following two theorems, we give special attention to a real-valued function defined on a connected subset of $\mathbb{R}$. Generally, for a real-valued function f, local connectedness and path-connectedness of $\widehat{G(f)}$ are not related. In fact, if we consider a real-valued function f defined on an arbitrary metric space X along with the property that f preserves precompactness and Cauchy connectedness, then generally local connectedness and path-connectedness of $\widehat{G(f)}$ are not equivalent. This fact is illustrated in the following examples. Interestingly if we consider the domain as a connected subset of $\mathbb{R}$ with the assumption that the function f preserves precompactness as well as Cauchy connectedness then local connectedness and path-connectedness of $\widehat{G(f)}$ are equivalent to the condition that f is Cauchy regular.

Example 5.1.1. (1) *Consider* $X = \bigcup_{k=1}^{\infty}\{(x, \dfrac{x}{k}) : 0 \le x \le 1\} \subset \mathbb{R}^2$ *endowed with the Euclidean metric of* $\mathbb{R}^2$*. A function* $f : X \to \mathbb{R}$ *is defined by* $f((x, \frac{x}{2k})) = x$ *and* $f((x, \frac{x}{2k-1})) = -x$ *for each* $k \in \mathbb{N}$*. It is evident that* f *preserves boundedness as well as Cauchy connectedness. One can observe that* $\overline{G(f)} = [\bigcup_{k=1}^{\infty}[\{(x, \dfrac{x}{2k}, x) : 0 \le x \le 1\} \cup \{(x, \dfrac{x}{2k-1}, -x) : 0 \le x \le 1\}]] \cup \{(x, 0, x) : 0 \le x \le 1\} \cup \{(x, 0, -x) : 0 \le x \le 1\}$*. Then clearly* $\overline{G(f)}$ *is path-connected but not locally connected. Note that* f *is not a Cauchy regular function as* $(f((1, \frac{1}{n})))$ *is not a Cauchy sequence.*

(2) *Let us define a function* $f : [0, 1] \to \mathbb{R}$ *by* $f(x) = x$ *when* $x \in \mathbb{Q}$ *and* $f(x) = 1$*, otherwise. Obviously,* f *is bounded but does not preserve Cauchy connectedness. One can easily observe that* $\overline{G(f)}$ *is path-connected as well as locally connected, but* f *is not a Cauchy regular function.*

(3) *Let us define a function* $f : (0, 1) \to \mathbb{R}$ *by* $f(x) = \sin\frac{1}{x}$ *when* $x \in \mathbb{Q}$ *and* $f(x) = 0$*, otherwise. Obviously,* f *is bounded but does not preserve Cauchy connectedness. One can easily observe that* $\overline{G(f)}$ *is path-connected but not locally connected. Also,* f *is not a Cauchy regular function.*

(4) *Consider a function* $f : (0, 1) \to \mathbb{R}$ *such that* $f(x) = \frac{1}{x}$*. It is very easy to check*

that f is a Cauchy connectedness-preserving function and $\overline{G(f)}$ is path-connected as well as locally connected. Still, f is neither a boundedness-preserving function nor a Cauchy regular function.

(5) *Consider $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ with the usual metric. A function $f : X \to \mathbb{R}$ is defined by $f(\frac{1}{n}) = n$. Then f is a Cauchy connectedness-preserving function and $\overline{G(f)}$ is locally connected but not path-connected. Note that f is neither bounded nor a Cauchy regular function.*

Theorem 5.1.3. *Let D be a connected subset of $\mathbb{R}$. Suppose that $f : D \to \mathbb{R}$ is both boundedness- and Cauchy connectedness-preserving function along with the property that $\overline{G(f)}$ is a path-connected space. Then f is a Cauchy regular function.*

Proof. If possible, suppose that f is not Cauchy regular. Then there exist $x \in \overline{D}$ and a sequence (x_n) in D such that (x_n) converges to x but $(f(x_n))$ does not converge in $\mathbb{R}$. Since f preserves precompactness, every subsequence of $(f(x_n))$ has a Cauchy subsequence. Now it is evident that $(f(x_n))$ has at least two distinct cluster points y_1 and y_2 in $\mathbb{R}$. We will proceed with the proof through certain steps.

First of all we claim that $\overline{G(f)}$ intersects the line $\{x\} \times \mathbb{R}$ at a point other than (x, y_1) and (x, y_2). On the contrary suppose that (x, y_1) and (x, y_2) are the only intersection points of the line $\{x\} \times \mathbb{R}$ and $\overline{G(f)}$. Take $\varepsilon = \frac{1}{4}|y_1 - y_2|$. Then there is $\delta > 0$ such that $f((x-\delta, x+\delta)) \subset (y_1 - \varepsilon, y_1 + \varepsilon) \cup (y_2 - \varepsilon, y_2 + \varepsilon)$, otherwise there is a sequence (z_n) which converges to x but $\{f(z_n)\}$ converges to a point other than y_1 and y_2. Next as $\overline{G(f)}$ is path-connected, there is a continuous function $\gamma : [0,1] \to \overline{G(f)}$ such that $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ with $\gamma(0) = (x, y_1)$ and $\gamma(1) = (x, y_2)$. Note that either $\gamma_1(t) > x$ or $\gamma_1(t) < x$ for all $t \in (0,1)$, otherwise there is some $t \in (0,1)$ such that $\gamma_1(t) = x$, which again contradicts the fact that (x, y_1) and (x, y_2) are the only intersection points of the line $\{x\} \times \mathbb{R}$ and $\overline{G(f)}$. Without any loss of generality we can assume that $\gamma_1(t) > x$ for all $t \in (0,1)$. Now as f is a

Cauchy connectedness-preserving function so either $f([x, x+\delta)) \subset (y_1-\varepsilon, y_1+\varepsilon)$ or $f([x, x+\delta)) \subset (y_2-\varepsilon, y_2+\varepsilon)$, but not contained in both sets simultaneously. Let $f([x, x+\delta)) \cap (y_1-\varepsilon, y_1+\varepsilon) = \phi$. For each $k \in \mathbb{N}$, choose $r_k \in (0, \frac{1}{k})$. It is clear that $\gamma_1(r_k) > x$ and $\gamma(r_k) \to (x, y_1)$. Since $\gamma(r_k) \in \overline{G(f)}$, $[(x, x+\delta) \times (y_1-\varepsilon, y_1+\varepsilon)] \cap G(f) \neq \phi$. But this contradicts the fact that $f([x, x+\delta)) \cap (y_1-\varepsilon, y_1+\varepsilon) = \phi$. If $f([x, x+\delta)) \cap (y_2-\varepsilon, y_2+\varepsilon) = \phi$, then choose $r_k \in (1-\frac{1}{k}, 1)$. Hence we can conclude that $\overline{G(f)}$ intersects the line $\{x\} \times \mathbb{R}$ at least three distinct points. As a result, now without any loss of generality, we can choose two distinct points say (x, y_1), $(x, y_2) \in \overline{G(f)}$ and two sequences (x_{m_k}) and (x_{n_k}) in D with $x_{m_k}, x_{n_k} \geq x$ for all $k \in \mathbb{N}$ such that $((x_{m_k}, f(x_{m_k}))) \to (x, y_1)$ and $((x_{n_k}, f(x_{n_k}))) \to (x, y_2)$. Since $\overline{G(f)}$ is path-connected, there is a path $\rho : [0,1] \to \overline{G(f)}$ from the point $(x, y_1) \in (\{x\} \times \mathbb{R}) \cap \overline{G(f)}$ to $(x_{m_1}, f(x_{m_1})$ with $\rho(t) = (\rho_1(t), \rho_2(t))$. Take $B = [\{x\} \times \mathbb{R}] \cap \overline{G(f)}$. It is clear that $\rho^{-1}(B)$ is closed and $0 \in \rho^{-1}(B)$. Let t_0 be the least upper bound of $\rho^{-1}(B)$. Clearly $t_0 \in \rho^{-1}(B)$ and $t_0 < 1$. Then we can write $\rho(t_0) = (x, y_0)$ for some $y_0 \in \mathbb{R}$. Now it is evident that the image of $(t_0, 1]$ under ρ is disjoint from $\{x\} \times \mathbb{R}$. Then by continuity of ρ_1, we can conclude that $\rho_1((t_0, 1]) > x$.

Case-1: Suppose that $y_0 \neq y_1$. Then we can choose two open sets U and V in $\mathbb{R}$ containing y_0 and y_1 respectively with $\overline{U} \cap \overline{V} = \phi$. Now we claim that there is $\mu > 0$ such that the set $A_\mu = \{a : a \in (x, x+\mu), f(a) \in U\}$ is dense in $(x, x+\mu)$. If this does not hold, then for each $k \in \mathbb{N}$, there is $a_k \in (x, x+\frac{1}{k})$ with $a_k \notin \overline{A_{\frac{1}{k}}}$. Clearly (a_k) converges to x and for each $k \in \mathbb{N}$ with $t_0 + \frac{1}{k} \leq 1$, $\rho_1([t_0, t_0+\frac{1}{k}))$ is a non-singleton connected set containing x. Then without any loss of generality, we can assume that $a_k \in \rho_1((t_0, t_0+\frac{1}{k})$ for each $k \in \mathbb{N}$. Consequently, there is $t_k \in (t_0, t_0+\frac{1}{k})$ such that $\rho_1(t_k) = a_k$ for each k. Now one can observe that $(\rho_2(t_k))$ converges to y_0. Therefore we can choose some $k \in \mathbb{N}$ such that $\rho(t_k) = (a_k, \rho_2(t_k)) \in (x, x+\frac{1}{k}) \times U$. As $\rho(t_k) \in \overline{G(f)}$, there is a sequence (c_n^k) that converges to a_k with $c_n^k \in (x, x+\frac{1}{k})$ and $f(c_n^k) \in U$ for each $n \in \mathbb{N}$, but this contradicts the fact that $a_k \notin \overline{A_{\frac{1}{k}}}$. Hence there is some $\mu > 0$ such that A_μ is dense in $(x, x+\mu)$. Moreover, there is some $k_0 \in \mathbb{N}$ for which the set $C = A_\mu \cup \{x_{m_k} : k \geq k_0, k \in \mathbb{N}\}$ is a dense subset of $[x, x+\mu)$. Hence from Theorem 3.1.2, C is Cauchy connected, but $f(C)$ is not a Cauchy connected subset of $\mathbb{R}$ as (U, V) is a Cauchy separation of $f(C)$). This contradicts that f preserves Cauchy connectedness, Hence we can conclude that $y_0 = y_1$.

Case-2: On the other hand, if we assume $y_0 \neq y_2$, the proof is the same as the above. In that case, take $C = A_\mu \cup \{x_{n_k} : k \geq k_0, k \in \mathbb{N}\}$. Therefore combining these two cases we have $y_1 = y_0 = y_2$, which again contradicts our assumption that $y_1 \neq y_2$. □

Theorem 5.1.4. *Suppose that D is a connected subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ is a boundedness-preserving as well as a Cauchy connectedness-preserving function. Then the following conditions are equivalent:*

(a) *f is Cauchy regular.*

(b) *$\overline{G(f)}$ is a locally connected space.*

(c) *$\overline{G(f)}$ is a path-connected space.*

Proof. $(a) \Rightarrow (c)$ is clear from the facts that f can be extended to a continuous function $\widetilde{f} : \overline{D} \to \mathbb{R}$ and path-connectedness is preserved by continuity. Other implications follow from Theorems 5.1.1,5.1.2, and 5.1.3. □

Now we present another condition under which a precompactness-preserving function is Cauchy regular. In this endeavour, the preservation of Cauchy connectedness is not required. Recall that two non-void subsets A and B of X are said to be Cauchy separated if any Cauchy sequence $(x_n) \subset A \cup B$ is either eventually in A or eventually in B but not eventually in both sets simultaneously.

Theorem 5.1.5. *Suppose that (X, d_X) and (Y, d_Y) are two metric spaces and $f : (X, d_X) \to (Y, d_Y)$ is a precompactness-preserving function satisfying the property that for any two* *precompact subsets K, M of X with $dist(K, M) > 0$, $f^{-1}(K)$ and $f^{-1}(M)$ are Cauchy separated. Then f is Cauchy regular.*

Proof. If possible suppose that there is a Cauchy sequence (x_n) such that $(f(x_n))$ is not a Cauchy sequence. Since f preserves precompactness, every subsequence of $(f(x_n))$ has a Cauchy subsequence. Then it is evident that $(f(x_n))$ has at least two cluster points y_1 and y_2 in $\widehat{Y}$, otherwise it is convergent in $\widehat{Y}$. So we can find two subsequences (x_{m_k}) and (x_{n_k}) of (x_n) such that $(f(x_{m_k}))$ and $(f(x_{n_k}))$ converge to y_1 and y_2 respectively. Then there is $k_0 \in \mathbb{N}$ such that $K = \{f(x_{m_k}) : k \geq k_0\}$ and $M = \{f(x_{n_k}) : k \geq k_0\}$ are two precompact sets with $dist(K, M) > 0$. But $f^{-1}(K)$ and $f^{-1}(M)$ have a common Cauchy sequence, which contradicts our given condition. □

Note 5.1.1. *Every Cauchy regular function satisfies the property mentioned in the above theorem, but if we remove the condition that f preserves precompactness, then Theorem 5.1.5 badly fails. To illustrate this fact, we consider a function $f : \{\frac{1}{n} : n \in \mathbb{N}\} \to \mathbb{R}$ defined by $f(\frac{1}{n}) = n$ for each $n \in \mathbb{N}$. Clearly f satisfies this property, but f is not a Cauchy regular function. Further, if we define $f(\frac{1}{2n}) = 1$ and $f(\frac{1}{2n-1}) = 0$ for each $n \in \mathbb{N}$, then f is bounded, but neither f satisfies this property nor a Cauchy regular function. A precompactness-preserving function along with this property yields Cauchy regularity.*

5.2 Function with Cauchy separated fibers

In complete metric spaces, Cauchy regularity coincides with continuity, but in general metric spaces, it is not true. In this section, we obtain a necessary and sufficient condition with the help of fibers under which, on a pre-straight space, Cauchy regularity coincides with continuity. Now we introduce the notion of Cauchy separated fibers (CSF), which plays an important role in yielding Cauchy regularity of a real-valued continuous function defined on a pre-straight space with locally connected completion. Also, it helps to characterize a generalization of Cauchy regular functions, which we define at the end of the section. We write $C(X, Y)$ for the set of all continuous functions $f : X \to Y$.

Definition 5.2.1. *A function $f : X \to Y$ has Cauchy separated fibers (CSF) if any two distinct fibers $f^{-1}(x)$ and $f^{-1}(y)$ are Cauchy separated i.e., for any two distinct points $x, y \in Y$ the following property holds: for each Cauchy sequence (x_n) in X the sets $\{n : f(x_n) = x\}$ and $\{n : f(x_n) = y\}$ cannot be infinite simultaneously.*

Clearly, every Cauchy regular function has CSF, but the converse is not generally true. If we consider a function $f : \{\frac{1}{n} : n \in \mathbb{N}\} \to \mathbb{R}$ defined by $f(\frac{1}{2n}) = \frac{1}{n}$ and $f(\frac{1}{2n-1}) = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$ then clearly f has CSF, but f is not Cauchy regular.

Theorem 5.2.1. *Suppose that X is a pre-straight space with $\widehat{X}$ is locally connected. Let $f : X \to \mathbb{R}$ be a connectedness- and precompactness-preserving function with Cauchy separated fibers (CSF). Then f is Cauchy regular.*

Proof. If possible, suppose that f is not Cauchy regular. Then there is a Cauchy sequence (x_n) such that $(f(x_n))$ is not a Cauchy sequence. Since f is a precompactness-preserving function, one can obtain two subsequences (x_{m_k}) and (x_{n_k}) of (x_n) such that $(f(x_{m_k})) \to y_1$ and $(f(x_{n_k})) \to y_2$ with $y_1 < y_2$. Then choose $u, v \in \mathbb{R}$ with $y_1 < u < v < y_2$. Taking subsequences we can assume that $f(x_{m_k}) < u < v < f(x_{n_k})$ for every k. Then from Theorem 4.2.6, we can conclude that there is a connected set I_k joining x_{m_k} and x_{n_k} for each k, such that $diam(I_k) \to 0$. On the connected set I_k, the function f takes a value greater than v (at x_{n_k}) and a value smaller than u (at x_{m_k}), so it must take the values u and v. Hence there is a Cauchy sequence, which is frequently both in $f^{-1}(u)$ and $f^{-1}(v)$, which contradicts the fact that f has CSF. □

Corollary 5.2.1. *A continuous precompactness-preserving real-valued function on a pre-straight space with locally connected completion is Cauchy regular iff it has CSF.*

Corollary 5.2.2. *A bounded continuous real-valued function on a pre-straight space with locally connected completion is Cauchy regular iff it has CSF.*

The failure of the above result for pre($*$)straight space can be illustrated as follows:

Example 5.2.1. *Let $\{p_k : k \in \mathbb{N}\}$ be a countable set of distinct prime numbers. For every $k \in \mathbb{N}$, we consider the set $D_k = \{\frac{m}{(p_k)^n} : m, n \in \mathbb{Z}\}$. Clearly, every D_k is a countable dense subset of $\mathbb{R}$. Now take $A_k = (\frac{1}{k+1}, \frac{1}{k}) \cap D_k$ and consider $X = \bigcup_{k=1}^{\infty} A_k$ with usual metric of $\mathbb{R}$. Next, choose $B_k = (0,1) \cap D_k$. Then obviously, $B_i \cap B_j = \phi$ for $i \neq j$ and each B_k is a countable dense subset of $(0,1)$. Take a strictly increasing function f_{2k-1} from A_{2k-1} onto B_{2k-1} and a strictly decreasing function f_{2k} from A_{2k} onto B_{2k}. Now define $f : X \to \mathbb{R}$ by $f(x) = f_k(x)$, when $x \in A_k$ for each k. Then it is clear that f is precompactness-, connectedness- and also Cauchy connectedness-preserving function. Furthermore, $\widehat{X}$ is locally connected and as $\widehat{X}$ is straight so from Theorem 4.1.2, X is pre($*$)straight. Note that X is not pre-straight as if we consider $C^+ = \bigcup_{k=2}^{\infty} A_k$ and $C^- = A_1$, then the characteristic function of C^+ is not Cauchy regular but its restriction to each of C^+ and C^- are Cauchy regular. If we take $x_{2k-1} \in f^{-1}[(0, \frac{1}{4}) \cap B_k]$ and $x_{2k} \in f^{-1}[(\frac{3}{4}, 1) \cap B_k]$, then clearly (x_k) is a Cauchy sequence but $(f(x_k))$ is not a Cauchy sequence.*

.

Theorem 5.2.2. *(cf. Th.3.10, [16]) Let (X, d) be a connected and locally connected metric space. Suppose $f, g \in C(X, [0,1])$ have the same fibers (We say that f and g have the same fibers if for every x we have $f^{-1}(f(x)) = g^{-1}(g(x))$) and f is Cauchy regular. Then also g is Cauchy regular.*

Remark 5.2.1. *A compact space with a compatible total order can be substituted for $[0,1]$ as the range space in the above theorem. If we consider the circle as a range space, then the above* *theorem does not hold. Our next example illustrates this fact and this provides a possible answer to the open question [Question 3.12] posed in [16], where the description of those spaces was asked, which can be substituted in Theorem 3.10 of [16] and also asked about the role of a circle as a range space in this theorem.*

Example 5.2.2. *Let $X = \{e^{i\theta} : \theta \in (0, 2\pi)\}$. We define two functions from X to a circle. Firstly define f as a identity function and define g by $g(e^{i\theta}) = e^{i(\frac{\pi}{2}+\frac{\theta}{2})}$. Clearly X is connected and locally connected and both f and g are injective. Hence they have the same fibers. Here f is Cauchy regular, but g is not.*

In the rest of this section, we consider a function in line of the much investigated UA function ([33], see also [15, 16]).

Definition 5.2.2. *We say that $f \in C(X, Y)$ is CA (Cauchy approachable), if for every point $x \in X$ and every set $M \subset X$, there is a Cauchy regular function $g \in C(X, Y)$ such that $f(x) = g(x)$ and $g(M) \subset f(M)$. We then say that g is a $(x, M)-$approximation of f.*

Note 5.2.1. *It is clear that every Cauchy regular function is CA since we can take $g = f$, but the converse is not true in general. We consider a function $f : X = \{\frac{1}{n} : n \in \mathbb{N}\} \to \mathbb{R}$ defined by $f(\frac{1}{2n}) = 0$ and $f(\frac{1}{2n-1}) = 1$ for each $n \in \mathbb{N}$. Here for each $x \in X$ and $M \subset X$, g is taken constant in $X \setminus \{x\}$. Our following example presents a non-CA continuous function.*

Example 5.2.3. *Let $A_k = \{(x, \frac{x}{k}) : 0 < x \leq 1\}$ and $X = \bigcup_{k=1}^{\infty} A_k \cup \{(x, 0) : 0 < x \leq 1\}$ endowed with the usual metric of $\mathbb{R}^2$. A function $f : X \to \mathbb{R}$ is defined by $f((x, \frac{x}{k})) = 1 + \frac{1}{k}$ and $f(x, 0) = 1$, for each $k \in \mathbb{N}$ and for each $x \in (0, 1]$. It is very easy to check that f is continuous. Now we consider $M = \bigcup_{k=1}^{\infty} f^{-1}(1 + \dfrac{1}{k})$ and $z = (1, 0)$. Note that $z \in \overline{M}$. We claim that f has no (z, M)-approximation. On the contrary suppose that there is a Cauchy regular function $g \in C(X)$ with $g(M) \subset f(M)$ and $g(z) = f(z) = 1$. First of all $g(M)$ is countable. Then g is constant on each of the connected sets A_k. Since A_i and A_j have a* *common Cauchy sequence for* $i \neq j$ *and* g *is Cauchy regular so* g *must then be constant on their union* M. *By continuity as* g *is constant on* M *and* $z \in \overline{M}$ *so* g *must be equal to* 1 *on* M. *But this contradicts* $g(M) \subset f(M)$ *since the latter set does not contain* 1.

Before establishing a relation between CSF and CA functions, we recall a definition from [16].

Definition 5.2.3. *[16] Let* $f \in C(X)$ *and* $a, b \in \mathbb{R}$ *with* $a \leq b$. *The* (a, b)*-truncation of* f *is the bounded function* $f_{(a,b)}$ *which coincides with* f *on the* f*-counter image of* $[a, b]$, *has value* a *whenever* f *has value* $\leq a$, *and has value* b *whenever* f *has value* $\geq b$.

Theorem 5.2.3. *Let* X *be a pre-straight space with locally connected completion and let* $f \in C(X)$ *have* CSF. *Then for every* $a < b$ *in* $\mathbb{R}$, *the* (a, b)*-truncation of* f *has* CSF *(and so is Cauchy regular being a bounded function).*

Proof. Let $g := f_{(a,b)}$. Suppose on the contrary that there are $u < v$ in $\mathbb{R}$ and a Cauchy sequence (x_n) in X with $g(x_{m_k}) = u < v = g(x_{n_k})$ for each $k \in \mathbb{N}$. Since X is locally connected pre-straight space so from Theorem 4.2.6, for k large enough x_{m_k} and x_{n_k} are contained in a connected set I_k with $diam(I_k) \to 0$. Now $g(I_k)$ is connected, so it contains the whole interval $[u, v]$. Hence by replacing x_{m_k} and x_{n_k} with other two points x'_{m_k} and x'_{n_k} inside I_k (so as to assure that $(x'_{m_k}) \cup (x'_{n_k})$ still a Cauchy sequence) we can arrange so that u and v are different from a and b. Then $g = f$ on the new sequence, which contradicts that f has CSF. Therefore g has CSF. □

Theorem 5.2.4. $CSF \to CA$ *for functions* $f \in C(X)$ *on a pre-straight space* X *with locally connected completion.*

Proof. Using Theorem 5.2.3, the proof is similar to Theorem 3.15 of [16]. □

6

New Types of Completeness in Uniform Space

This chapter aligns with previous research work presented in [38, 39, 40, 46, 47, 58] and other similar studies. Our first objective is to present and examine two new types of complete spaces called Bourbaki quasi-complete and cofinally Bourbaki quasi-complete spaces in the structure of uniform space (Instead of utilizing finite chains, we take the help of infinite chains in our approach). These new completeness properties exist as intermediary notions between compactness and completeness. In this direction firstly we define a new type of modification in uniform space using finite-component covers, which is significantly used in the first two sections. Additionally, we address a significant and inherent problem related metrizability of a uniform space using a Bq-complete and a cBq-complete metric. In the first two sections, we consider Hausdorff uniform spaces, defined by uniform covers. Results and techniques in [38, 58] are especially relevant to the work presented in the first two sections and will be our guide throughout. In the first two sections, we work in uniform space using the uniform cover definition.

Finally, in the last section we relook the notion of Bourbaki quasi-Cauchy sequences in metric structure and connect it to quasi-Cauchy sequence. We define quasi-Cauchy Lipschitz function and prove that the class of all real-valued quasi-Cauchy Lipschitz functions is uniformly dense in the class of all real-valued ward continuous functions.

The content of this chapter is based on the research papers listed below.

- P. Das, **N. Adhikary** and S. K. Pal, On certain new types of completeness properties using infinite chainability and associated metrization problems in Uniform spaces, communicated. [31]
- **N. Adhikary** and S.K. Pal, On certain notions of precompactness, continuity and Lipschitz functions, communicated. [1]

6.1 Finite-component modification of uniform space and the role of superparacompactness

We begin this section by introducing some versions of Cauchy filters along with the corresponding notions of completeness.

Definition 6.1.1. *Let (X, μ) be a uniform space and $\mathcal{F}$ be a filter of X. Then*

1. *$\mathcal{F}$ is said to be a Bourbaki quasi-Cauchy filter (short in BqC filter) if for every $\mathcal{U} \in \mu$, $\exists x \in X$ such that $F \subset St^{\infty}(x, \mathcal{U})$ for some $F \in \mathcal{F}$.*
2. *$\mathcal{F}$ is said to be a cofinally Bourbaki quasi-Cauchy filter (short in cofinally BqC filter) if for every $\mathcal{U} \in \mu$, $\exists x \in X$ such that $F \cap St^{\infty}(x, \mathcal{U}) \neq \emptyset$ for every $F \in \mathcal{F}$.*

Definition 6.1.2. *A uniform space (X, μ) is said to be*

1. *Bourbaki quasi-complete (short in Bq-complete) if every BqC filter has a cluster point.*
2. *cofinally Bourbaki quasi-complete (short in cBq-complete) if every cofinally BqC filter has a cluster point.*

One can easily obtain the following impications:

compact $\Rightarrow$ cBq-complete $\Rightarrow$ cofinally Bourbaki-complete $\Rightarrow$ cofinally complete $\Rightarrow$ complete.

compact $\Rightarrow$ cBq-complete $\Rightarrow$ Bq-complete $\Rightarrow$ Bourbaki-complete $\Rightarrow$ complete.

Now we provide several counter-examples that illustrate the reversals of the aforementioned implications.

Example 6.1.1.

1. *Any infinite uniformly discrete space is a non-compact cBq complete space.*
2. $\mathbb{R}$ *with the usual metric is Bourbaki-complete (also cofinally Bourbaki-complete) but not Bq-complete.*
3. *Let us take a partition* $\{M_n : n \in \mathbb{N}\}$ *of* $\mathbb{N}$ *such that each* M_n *is infinite. Let* $M_n = \{n_k : k \in \mathbb{N}\}$*. Now we define a sequence* (x_n) *such that* $x_{n_k} = (1 + \frac{k-1}{n})e_n$ *for every* $n, k \in \mathbb{N}$*, where* (e_n) *is a sequence of unit vectors of* ℓ^∞*. Let us take* $X = \{x_n : n \in \mathbb{N}\}$ *with sup norm of* ℓ^∞*. Clearly* X *is Bq-complete. But* X *is not cBq-complete as* (x_n) *is a cofinally BqC sequence without having any cluster point.*

Inspired by the work presented in [58], we introduce a new type of modification of a uniform space using finite-component covers (for easy reference, see [20, 21, 61]) to connect the notions of Bq- and cBq-complete spaces with completeness and cofinal completeness of this new modification. A modification of a uniform space (X, μ) is defined by a uniformity of X such that the base or subbase consists of a subfamily of covers belonging to μ. There are several types of modifications of a uniform space (X, μ) in literature named as the finite modification (precompact modification) $p\mu$, the point-finite modification $p_f\mu$, the star-finite modification $s_f\mu$ and the countable modification $e\mu$. These modifications are compatible with the topology of X induced by μ and the corresponding bases are the family of all finite, point-finite, star-finite and countable open covers belonging to μ, respectively (see page 23 Theorem 31, Page 69 Proposition 28 of [50],[44, 58, 63, 68]). There is another type of modification in literature, named as uniformly 0-dimensional modification ($r\mu$), generated by the collection of all uniform partitions from μ (A partition of X having a refinement belonging to μ is called the uniform partition of (X, μ)). This modification may not preserve topology and in fact, it may not be Hausdorff also. Now we present some basic properties of Bq-complete and cBq-complete spaces. The proofs are easy so we omit them.

Suppose that (X, μ) is a uniform space and $r\mu$ is the uniformly 0-dimensional modification of (X, μ). Then the following properties hold:

1. BqC and cofinally BqC filters are preserved by uniformly continuous functions. But Bq-complete and cBq-completeness may not be preserved by uniformly continuous functions.

2. Every closed subspace of a Bq-complete (cBq-complete) space is also Bq-complete (cBq-complete).

3. Any non-empty product of uniform spaces is Bq-complete iff each factor is Bq-complete. But countable products of cBq-complete uniform spaces may not be cBq-complete. As an example, we can consider the countable product of infinite discrete spaces $\prod_{n\in\mathbb{N}} D_n$ endowed with metric ρ, defined by $\rho((x_n),(y_n)) = 0$ if $x_n = y_n$ for every $n \in \mathbb{N}$ and $\rho((x_n),(y_n)) = \frac{1}{k}$ if $x_n = y_n$ for every $n = 1, ..., k-1$ and $x_k \neq y_k$.

4. Every filter on X larger than a BqC filter is BqC. But it may not be true for cofinally BqC filters. Thus we can say that X is Bq-complete iff every BqC ultrafilter is convergent.

5. Suppose that a uniformity ν on X satisfies $r\mu < \nu < \mu$. Then a filter is BqC (cofinally BqC) in (X, ν) iff it is Cauchy (cofinally Cauchy) in $(X, r\mu)$.

Proof. The one implication follows from the fact that for every $\mathcal{P} \in r\mu$, $\exists \mathcal{V} \in \nu$ with $\mathcal{V} < \mathcal{P}$ and so $\{St^{\infty}(x, \mathcal{V}) : x \in X\} < \mathcal{P}$. The converse part is evident as for every $V \in \mathcal{V}$, the cover $\{St^{\infty}(x, \mathcal{V}) : x \in X\} \in r\mu$. □

6. A filter is BqC (cofinally BqC) in (X, μ) iff it is Cauchy (cofinally Cauchy) in $(X, r\mu)$.

Now we recall the definition of finite-component covers [20, 21, 61], which will play a key role in our investigation.

Definition 6.1.3. *A cover $\mathcal{U}$ of X is called a finite-component cover if for every $U \in \mathcal{U}$, $St^{\infty}(U, \mathcal{U})$ intersects at most finitely many members of $\mathcal{U}$.*

Suppose that (X, μ) is a uniform space. Now we are going to prove that the family of all finite-component open covers from μ forms a base of uniformity on X, which is compatible with the topology of X induced by μ. Firstly, we need to establish the subsequent lemmas.

Lemma 6.1.1. *Suppose that X is a uniform space and H, K are two subsets of X with $H \subset K$ and $\mathcal{U}, \mathcal{V}$ are two covers of X with $\mathcal{U} < \mathcal{V}$. Then $St^{\infty}(H, \mathcal{U}) \subset St^{\infty}(K, \mathcal{V})$.*

Proof. It is clear that $St(H, \mathcal{U}) \subset St(K, \mathcal{V})$. Then by induction, the result can be easily proved. □

Lemma 6.1.2. *Suppose that (X, μ) is a uniform space and $\mathcal{U}$ is a finite-component open cover of X with $\mathcal{U} \in \mu$. Then there exists a finite-component open cover $\mathcal{W} \in \mu$ such that $\mathcal{W}^* < \mathcal{U}$.*

Proof. Suppose that (X, μ) is a uniform space and $\mathcal{U}$ is a finite-component open cover of X with $\mathcal{U} \in \mu$. Then there exists an open cover $\mathcal{V} \in \mu$ such that $\mathcal{V}^* < \mathcal{U}$. Now for every $V \in \mathcal{V}$ we define the following subsets of $\mathcal{U}$: $I(V) = \{U \in \mathcal{U} : St(V, \mathcal{V}) \subset U\}$, $J(V) = \{U \in \mathcal{U} : V \subset U\}$ and $K(V) = \{U \in \mathcal{U} : St^{\infty}(V, \mathcal{V}) \cap U \neq \emptyset\}$. Clearly for each $V \in \mathcal{V}$, $I(V) \subset J(V) \subset K(V)$ and they are non-empty as $\mathcal{V}^* < \mathcal{U}$. Moreover, for every $V \in \mathcal{V}$, $k(V)$ is a finite subset of $\mathcal{U}$ as $\mathcal{U}$ is finite-component and there is some $U \in \mathcal{U}$ such that $St^\infty(V,\mathcal{V}) \subset St^\infty(U,\mathcal{U})$. Consequently, both $I(V)$ and $J(V)$ must be finite. Now for any non-empty finite subsets I, J and K of $\mathcal{U}$ with $I \subset J \subset K$, we define $W_{IJK} = \bigcup\{V \in \mathcal{V} : I(V) = I, J(V) = J, K(V) = K\}$ and consider a new cover $\mathcal{W}$ of X consisting of all such W_{IJK}. Obviously, $\mathcal{W}$ is an open cover of X with $\mathcal{V} < \mathcal{W}$, which implies that $\mathcal{W} \in \mu$. Now we will show that $\mathcal{W}$ is a finite-component cover. To prove this let $W_{IJK} \in \mathcal{W}$. Note that $W_{IJK} \cap W_{I_1,J_1,K_1} \neq \emptyset$ implies that there exist $V, V_1 \in \mathcal{V}$ with $V \subset W_{IJK}$ and $V_1 \subset W_{I_1J_1K_1}$ such that $V \cap V_1 \neq \emptyset$, and hence $St^\infty(V_1,\mathcal{V}) = St^\infty(V,\mathcal{V})$. Then from the construction, we can conclude that $K = K_1$ and so at most finitely many choices of the sets I_1, J_1, K_1 are possible for which $W_{I_1J_1K_1} \subset St^\infty(W_{IJK},\mathcal{W})$. Hence $\mathcal{W}$ is finite-component. Finally, to show that $\mathcal{W}^* < \mathcal{U}$, take $W_{IJK} \in \mathcal{W}$. Then $W_{IJK} \cap W_{I_1J_1K_1} \neq \emptyset$ implies that there exist $V, V_1 \in \mathcal{V}$ with $V \subset W_{IJK}$ and $V_1 \subset W_{I_1J_1K_1}$ such that $V_1 \subset St(V,\mathcal{V})$. Furthermore, $V_1 \subset St(V,\mathcal{V}) \subset U$ for some $U \in I(V)$. Then from the construction of $W_{I_1J_1K_1}$, we can conclude that $W_{I_1J_1K_1} \subset U$, which implies that $St(W_{IJK},\mathcal{W}) \subset U$. □

Theorem 6.1.1. *Suppose that (X,μ) is a uniform space. Then the family of all finite-component open covers belonging to μ forms a base of a uniformity, compatible with the topology of X induced by μ.*

Proof. Suppose that (X,μ) is a uniform space and $\mathcal{U},\mathcal{V}$ are two open and finite-component cover with $\mathcal{U},\mathcal{V} \in \mu$. Then for every $U \in \mathcal{U}$ and $V \in \mathcal{V}$, we have $St^\infty(U \cap V,\mathcal{U} \wedge \mathcal{V}) \subset St^\infty(U,\mathcal{U}) \cap St^\infty(V,\mathcal{V})$. Hence $\mathcal{U} \wedge \mathcal{V}$ is also finite-component. Moreover, every finite open cover is finite-component. So the uniformity generated by the given family lies between $p\mu$ and μ. Evidently, this uniformity is compatible with the topology of X induced by μ. Hence from Lemma 6.3.1, the proof is clear. □

Now we can proceed to define the desired modification of a uniform space (X,μ).

Definition 6.1.4. *Let (X,μ) be a uniform space. Then the finite-component modification $s_f^\infty\mu$ of μ is the uniformity generated by the family of all finite-component open covers from μ.*

Clearly, $p\mu \leq s_f^\infty\mu \leq s_f\mu \leq \mu$ and $r\mu \leq s_f^\infty\mu$. But the converse implications are not generally true and this is illustrated in the following examples.

Example 6.1.2. 1. *Let us consider $X_{2n} = \{e_{2n}(1+\frac{k}{n+1}) : k \in \mathbb{N}\cup\{0\}\}$ and $X_{2n-1} = \{10^n e_{2n-1} + \frac{1}{n+1}e_k : k \in \mathbb{N}\}$ and then take $X = \bigcup_{n\in\mathbb{N}} X_n$ with the metric d_∞ induced by sup norm of ℓ^∞. Let μ_{d_∞} be the uniformity induced by d_∞. It is evident that the open cover $\mathcal{C} = \{X_n : n \in \mathbb{N}\}$ has no finite refinement but $\mathcal{C} \in s_f^\infty\mu_{d_\infty}$. Now let us consider a cover $\mathcal{G} = \{X_{2n-1} : n \in \mathbb{N}\} \cup \{B_{d_\infty}(x,1) : x \in X_{2n}, n \in \mathbb{N}\}$. Clearly $\mathcal{G} \in s_f\mu_{d_\infty}$ but $\mathcal{G} \notin s_f^\infty\mu_{d_\infty}$. Because if $\mathcal{G} \in s_f^\infty\mu_{d_\infty}$, then there must exist a $\delta > 0$ such that for each $x \in X$, we can choose finitely many elements $G_1,...,G_k$ of $\mathcal{G}$ satisfying $B^\infty_{d_\infty}(x,\delta) \subset \bigcup_{i=1}^{k} G_i$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. In particular, we can choose $x_1,...,x_k \in X_{2N}$ such that $X_{2N} = B^\infty_{d_\infty}(e_{2N},\delta) \subset \bigcup_{i=1}^{k} B_{d_\infty}(x_i,1)$, which contradicts that X_{2N} is unbounded. Finally, as X is not cofinally complete, from Theorem 1.2.28 [58], we can conclude that $\mu_{d_\infty} \neq s_f\mu_{d_\infty}$. Hence in X $p\mu_{d_\infty} \neq s_f^\infty\mu_{d_\infty} \neq s_f\mu_{d_\infty} \neq \mu_{d_\infty}$.*

2. *If we consider $X = [0,1]$ with the usual metric d then $r\mu_d = \{[0,1]\}$ whereas every open cover belongs to $s_f^\infty\mu_d$ as each open cover has a finite subcover.*

After defining various suitable forms related to our approach in the rest of this section the results as well as the proofs are generally routine modifications of arguments in [58], but included some details for the reader's convenience. Firstly, we will characterize Bq-completeness and cBq-completeness of (X,μ) in terms of completeness and cofinally completeness of $(X,s_f^\infty\mu)$ respectively

Lemma 6.1.3. *Let (X,μ) be a uniform space. Then the following statements are true:*

1. *An ultafilter $\mathcal{F}$ of X is Cauchy in $(X,s_f^\infty\mu)$ iff it is BqC in (X,μ).*

2. *A filter $\mathcal{F}$ of X is cofinally Cauchy in $(X, s_f^{\infty}\mu)$ iff it is cofinally BqC in (X, μ).*

Proof. 1. The proof of one implication is evident as for every $\mathcal{U} \in \mu$, the cover $\{St^{\infty}(x,\mathcal{U}) : x \in X\} \in s_f^{\infty}\mu$. For the converse part, let $\mathcal{F}$ be a BqC ultrafilter in (X, μ) and $\mathcal{C} \in s_f^{\infty}\mu$. Then there exist finitely many $C_i \in \mathcal{C}$ with $i \in \{1, \ldots, m\}$ such that $F \subset \bigcup_{i=1}^{m} C_i$, for some $F \in \mathcal{F}$, which implies that there exists some i with $F \subset C_i$. Hence $\mathcal{F}$ is Cauchy $(X, s_f^{\infty}\mu)$.

2. The proof is similar to Lemma 1.2.7 of [58].

□

Corollary 6.1.1. *In a uniform space (X, μ), the followings hold:*

(1) (X, μ) *is Bq-complete iff* $(X, s_f^{\infty}\mu)$ *is complete.*

(2) (X, μ) *is cBq-complete iff* $(X, s_f^{\infty}\mu)$ *is cofinally complete.*

Recall that (X, μ) is said to be uniformly paracompact [46, 67] if every open cover of X has a uniformly locally finite open refinement (A cover $\mathcal{A}$ is said to be uniformly locally finite if $\exists \mathcal{U} \in \mu$ such that every $U \in \mathcal{U}$ intersects at most finitely many $A \in \mathcal{A}$). In [48, 70], it was shown that a uniform space is cofinally complete iff it is uniformly paracompact. Moreover, a Tychonoff space X is paracompact iff $(X, \mathbf{u})$ is cofinally complete (see [27]). Now we will establish an analogous result for cBq-complete spaces in terms of the well-known notion of superparacompactness [61].

Definition 6.1.5. *[61] A space X is said to be superparacompact if every open cover has an open finite-component refinement.*

A connected space is superparacompact iff it is compact. $\{\sqrt{n} : n \in \mathbb{N}\}$ with the usual metric of $\mathbb{R}$ is superparacompact and complete but not a Bq-complete space. One can naturally consider the following stronger version of uniformly paracompact space.

Definition 6.1.6. *A uniform space (X, μ) is said to be uniformly star superparacompact if every open cover $\mathcal{G}$ has an open refinement $\mathcal{A}$, satisfying the following property that $\exists \mathcal{U} \in \mu$ such that for every $x \in X$, $St^{\infty}(x, \mathcal{U})$ intersects at most finitely many elements of $\mathcal{A}$.*

In [62], another uniform extension of superparacompact space is defined and called it as R-superparacompact space. A uniform space (X, μ) is said to be R-superparacompact [62] if every open cover has a finite-component uniformly locally finite open refinement. R-superparacompactness is uniformly weaker than uniformly star superparacompactness. Indeed since a uniformly star superparacompact space (X, μ) is both superparacompact (see Theorem 6.1.3) and uniformly paracompact, every open cover $\mathcal{G}$ has an open finite-component refinement $\mathcal{Q}$ and $\mathcal{Q}$ has a uniformly locally finite open refinement. Then from Lemma 1 of [62], we can conclude that $\mathcal{Q}$ is also uniformly locally finite. But the converse is not generally true. For example, if we consider $X = \{\sqrt{n} : n \in \mathbb{N}\}$ with the usual metric of $\mathbb{R}$ is R-superparacompact as $\{\{\sqrt{n}\} : n \in \mathbb{N}\}$ is a finite-component uniformly locally finite open cover but X is not cBq-complete. Then from Theorem 6.1.2 we can conclude that X is not uniformly star superparacompact. Proceeding as [58], we can prove an equivalency between cBq complete space and uniformly star superparacompact space.

Lemma 6.1.4. *(cf. Th.1.2.27, [58]) In a uniform space (X, μ), $\mu = s_f^{\infty}\mu$ iff for every $\mathcal{U} \in \mu$, $\exists \mathcal{V} \in \mu$ such that for every $x \in X$ we have finitely many $U_1, ..., U_k \in \mathcal{U}$ with $St^{\infty}(x, \mathcal{V}) \subset \bigcup_{i=1}^{k} U_i$.*

Proof. The one implication is evident. For the converse part, let $\mathcal{U} \in \mu$ and choose an open cover $\mathcal{V} \in \mu$ such that $\mathcal{V}^* < \mathcal{U}$. By the given condition, there exists an open $\mathcal{W} \in \mu$ such that for every $x \in X$, there exists a finite subset $\mathcal{V}_x$ of $\mathcal{V}$ such that $St^{\infty}(x, \mathcal{W}) \subset \bigcup\{V : V \in \mathcal{V}_x\}$. Let us consider $\mathcal{C} = \{St^{\infty}(x_i, \mathcal{W}) : i \in I\}$, the family of all chainable components of X generated by $\mathcal{W}$. Then the cover $\mathcal{G} =$

$\{St^{\infty}(x_i, \mathcal{W}) \cap St(V, \mathcal{V}) : V \in \mathcal{V}_{x_i}, i \in I\}$ is finite-component. Since $\mathcal{C} \wedge \mathcal{V} < \mathcal{G}$, we have $\mathcal{G} \in \mu$ and also $\mathcal{G} < \mathcal{V}^* < \mathcal{U}$, which complete the proof. □

Theorem 6.1.2. *(cf. Th.1.2.28, [58]) Let (X, μ) be a uniform space. Then the following statements are equivalent:*

(1) (X, μ) *is uniformly star superparacompact.*

(2) (X, μ) *is uniformly paracompact and* $\mu = s_f^{\infty}\mu$.

(3) (X, μ) *is cofinally complete and* $\mu = s_f^{\infty}\mu$.

(4) (X, μ) *is cBq-complete.*

Proof. Using Lemma 6.1.4, the proof of $(1) \implies (2)$ can be followed by the arguments applying in $(1) \Rightarrow (2)$ of Th.1.2.28 [58] with a much simpler modification. The rest of the proof is analogous to Th.1.2.28 of [58]. □

Next, we consider the above theorem in the case of the fine uniformity **u**, which gives a characterization of superparacompact space. The proof of the following theorem is clear from the fact that in a paracompact space, the family of all open covers forms a base for the fine uniformity **u** [59, 67].

Theorem 6.1.3. *(cf. Corollary 1.2.30, [58]) Let X be a Tychonoff space. Then the following statements are equivalent:*

(1) X *is superparacompact.*

(2) X *is paracompact and* $\mathbf{u} = s_f^{\infty}\mathbf{u}$.

(3) $(X, \mathbf{u})$ *is uniformly paracompact (equivalently cofinally complete) and* $\mathbf{u} = s_f^{\infty}\mathbf{u}$.

(4) $(X, \mathbf{u})$ *is uniformly star superparacompact (equivalently cBq-complete).*

6.2 Results related to metrizability

6.2.1 Bourbaki quasi-completely metrizability

In this subsection, we will deal with the (topological) problem of metrization of a uniform space through the Bq-complete metric. A uniform space X is called Bq-completely metrizable if there exists a Bq-complete metric on X, which is compatible with its topology. Here our main interest is to obtain a result in line with the well-known results given by Čech and Garrido [25, 38], respectively, which state that a metrizable space X is completely metrizable (Bourbaki completely metrizable) iff $X = \bigcap_{n=1}^{\infty} G_n$, where each G_n is an open (open paracompact) subspace of βX. To proceed in this direction, we first consider the definition of BqC sequences in a metric space.

Definition 6.2.1. *[1] Let (X, d) be a metric space. A sequence (x_n) is said to be BqC if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for some $p \in X$, $x_n \in B_d^{\infty}(p, \varepsilon)$ for every $n \geq n_0$.*

Proceeding as Th.1.3.8 of [58], one can easily prove that a metric space (X, d) is Bq-complete iff every BqC sequence of X has a cluster point. Now we present the desired result of Bq-completely metrization.

Theorem 6.2.1. *(cf. Th.23, [38]) Let X be a metrizable space. Then the following statements are equivalent:*

(1) *X is Bq-completely metrizable.*

(2) *$X = \bigcap_{n=1}^{\infty} G_n$, where each G_n is an open superparacompact subspace of βX.*

(3) *There exists some compactification cX of X such that $X = \bigcap_{n=1}^{\infty} G_n$, where each G_n is an open superparacompact subspace of cX.*

Proof. (1) $\Rightarrow$ (2) Let (X, d) be a Bq-complete metric space and $\varepsilon > 0$ be given. First of all, we consider the family of all ε-chainable components of X: $C_\varepsilon = \{B_d^\infty(x_i, \varepsilon) : i \in I_\varepsilon\}$, where $B_d^\infty(x_i, \varepsilon) \cap B_d^\infty(x_j, \varepsilon) = \emptyset$ whenever $i \neq j$. Let $n \in \mathbb{N}$. Observe that for every $i \in I_{\frac{1}{n}}$, $B_d^\infty(x_i, \frac{1}{n})$ is clopen in X. So $\overline{B_d^\infty(x_i, \frac{1}{n})}^{\beta X}$ is clopen in βX and also $\overline{B_d^\infty(x_i, \frac{1}{n})}^{\beta X} \cap \overline{B_d^\infty(x_j, \frac{1}{n})}^{\beta X} = \emptyset$ whenever $i \neq j$. Let us define $G_n = \bigcup_{i \in I_{\frac{1}{n}}} \overline{B_d^\infty(x_i, \frac{1}{n})}^{\beta X}$, where $B_d^\infty(x_i, \frac{1}{n}) = \overline{B_d^\infty(x_i, \frac{1}{n})}^{\beta X} \cap X$ for every $i \in I_{\frac{1}{n}}$. Clearly, G_n is open and $X \subset G_n$. Now we will show that G_n is a superparacompact subspace of βX. To prove that, let $\mathcal{U}$ be an open cover of G_n. Then for each $i \in I_{\frac{1}{n}}$, consider $\mathcal{U}_i = \{U \cap \overline{B_d^\infty(x_i, \frac{1}{n})}^{\beta X} : U \in \mathcal{U}\}$. It is clear that $\mathcal{U}_i$ is an open cover of $\overline{B_d^\infty(x_i, \frac{1}{n})}^{\beta X}$. By the compactness of $\overline{B_d^\infty(x_i, \frac{1}{n})}^{\beta X}$, there exists a finite subcover $\mathcal{V}_i$ of $\mathcal{U}_i$. Moreover, each element of $\mathcal{V}_i$ is also open in G_n and $[\bigcup \mathcal{V}_i] \cap [\bigcup \mathcal{V}_j] = \emptyset$ for $i \neq j$. Then take $\mathcal{V} = \bigcup_{i \in I_{\frac{1}{n}}} \mathcal{V}_i$. It is evident that $\mathcal{V}$ is an open finite-component refinement of $\mathcal{U}$. Now we only need to show that $\bigcap_{n=1}^{\infty} G_n \subset X$. Choose $z \in G_n$ for every $n \in \mathbb{N}$. Take $\mathcal{F} = \{N \cap X : N$ is an open neighbourhood of z in $\beta X\}$. As X is dense in βX, $\mathcal{F}$ is a filter base of X. It is easy to observe that for each $n \in \mathbb{N}$, there is some $i_n \in I_{\frac{1}{n}}$ such that $\overline{B_d^\infty(x_{i_n}, \frac{1}{n})}^{\beta X}$ is an open neighbourhood of z in βX. Consequently, $B_d^\infty(x_{i_n}, \frac{1}{n}) \in \mathcal{F}$ for every $n \in \mathbb{N}$. Hence $\mathcal{F}$ is BqC in (X, d) and so $\mathcal{F}$ has a cluster point x in X. Then x is also a cluster point of the filter base $\{N : N$ is an open neighbourhood of z in $\beta X\}$ in βX. As βX is Hausdorff, we can conclude that $x = z$.

(2) $\Rightarrow$ (3) is obvious and (3) $\implies$ (1) can be proved in the same way as in the proof of (3) $\implies$ (1) of Theorem 23 [38] with slight modification at the end. □

Remark 6.2.1. *It is easy to observe that in a connected Tychonoff space X, every filter is BqC with respect to any compatible uniformity. Then we can obtain the following observation: a connected Tychonoff space is Bq-completely metrizable iff it is compact. Hence we can conclude* *that $\mathbb{R}$ with the usual topology is Bourbaki complete but not Bq-completely metrizable.*

6.2.2 Cofinally Bourbaki quasi-completely metrizablity

This subsection solely concerns with the notion of cBq-complete metric spaces and one of our prime interests is the investigation of when a metrizable space admits an equivalent cBq-complete metric, that is, when it is cBq-completely metrizable. In [69], Romaguera solved the analogous problem for cofinally completely metrizable spaces and then Meroño and Garrido in [38], proved a similar result in the case of cofinally Bourbaki-completely metrizable spaces. Clearly, every cBq-completely metrizable space is cofinally Bourbaki-completely metrizable. But the converse is not necessarily true as one can easily observe that a connected metrizable space is cBq-completely metrizable iff it is compact. Further, we present an example of a Bq-complete space, which is not cBq-completely metrizable in Example 6.2.1.

In particular, in a metric space (X, d), a sequence (x_n) is said to be cofinally BqC if for every $\varepsilon > 0$, there exists an infinite subset N_ε of $\mathbb{N}$ such that for some $p \in X$, $x_n \in B_d^\infty(p, \varepsilon)$ for every $n \in N_\varepsilon$. Proceeding as Th.1.3.27 of [58], one can easily prove that a metric space (X, d) is cBq-complete iff every cofinally BqC sequence of X has a cluster point.

Before proceeding to the problem of metrizability, we present another feature of cBq-complete spaces. In [67], (see also [11, 38]) it was shown that a uniform space (X, μ) is uniformly locally compact (Recall that (X, μ) is uniformly locally compact if there exists $\mathcal{U} \in \mu$ such that for each $x \in X$, $\overline{St(x, \mathcal{U})}$ is compact) iff it is locally compact and cofinally complete. $\mathbb{R}$ with the usual metric is uniformly locally compact but not cBq-complete. One can think of a stronger version of uniform local compactness in order to tackle the situation when locally compact cBq-complete spaces come into the picture.

Definition 6.2.2. (1) *A uniform space* (X, μ) *is said to be strongly locally compact if for each* $x \in X$*, there exists* $\mathcal{U}_x \in \mu$ *such that* $St^\infty(x, \mathcal{U}_x)$ *is compact.*

(2) *A uniform space* (X, μ) *is said to be strongly uniformly locally compact if there exists* $\mathcal{U} \in \mu$ *such that for each* $x \in X$*,* $St^\infty(x, \mathcal{U})$ *is compact.*

Clearly, strongly local compactness (strongly uniformly local compactness) implies local compactness (uniformly local compactness). But converses are not generally true as $\mathbb{R}$ with the usual metric is uniformly locally compact but not strongly uniformly locally compact. Now we will present a spatial characterization of locally compact cBq-complete uniform spaces in line with Theorem 14 [38].

Theorem 6.2.2. *Let* (X, μ) *be a uniform space. Then the following statements are equivalent:*

(1) (X, μ) *is strongly uniformly locally compact.*

(2) (X, μ) *is strongly locally compact and cBq-complete.*

(3) X *is locally compact and* (X, μ) *is cBq-complete.*

Proof. (1) $\Rightarrow$ (2) One part is obvious and to prove the other part, let $\mathcal{F}$ be a cofinally BqC filter in (X, μ). From strong uniform local compactness, one can find $\mathcal{U} \in \mu$ such that for each $x \in X$, $St^\infty(x, \mathcal{U})$ is compact. Furthermore, there exists $U \in \mathcal{U}$ such that $F \cap St^\infty(U, \mathcal{U}) \neq \emptyset$ for every $F \in \mathcal{F}$. Observe that for each $x \in U$, $St^\infty(x, \mathcal{U}) = St^\infty(U, \mathcal{U})$. Clearly $\mathcal{F}' = \{F \cap St^\infty(U, \mathcal{U}) : F \in \mathcal{F}\}$ is a filter base in $St^\infty(U, \mathcal{U})$. Therefore from compactness of $St^\infty(U, \mathcal{U})$, we can conclude that $\mathcal{F}'$ has a cluster point. Hence (X, μ) is cBq-complete.

(2) $\Rightarrow$ (3) is clear from the fact that $\overline{St(x, \mathcal{U}_x)}$ is a closed subspace of $St^\infty(x, \mathcal{U}_x)$.

(3) $\Rightarrow$ (1) Suppose that (X, μ) is locally compact and cBq-complete. Then from [38], (X, μ) is uniformly locally compact and so there exists $\mathcal{V} \in \mu$ such that $\overline{St(x, \mathcal{V})}$ is compact for every $x \in X$. Moreover, from Theorem 6.1.2, $\mu = s_f^\infty \mu$ and this ensures the existence of an open finite-component refinement $\mathcal{U}$ of $\mathcal{V}$ with $\mathcal{U} \in \mu$. Now it is evident that for each $x \in X$ we can choose finitely many $V_1, ..., V_k \in \mathcal{V}$ satisfying that

$St^{\infty}(x,\mathcal{U}) \subset \bigcup_{i=1}^{k} V_i \subset \bigcup_{i=1}^{k} \overline{V_i}$. Observe that for each $i \in \{1,..,k\}$, $\overline{V_i} \subset \overline{St(x,\mathcal{V})}$ for some $x \in V_i$. Consequently, from compactness of $\overline{St(x,\mathcal{V})}$, we can conclude that $St^{\infty}(x,\mathcal{U})$ is compact. □

Results for Metric Spaces

In [11], Beer defined a functional ν on X to characterize cofinal completeness in the framework of metric spaces. In our context, we define two types of functionals on X, namely f_c and f_p, which measure the compactness and precompactness, respectively of chainable components at each point to establish analogous results of [11] for cBq-complete spaces. Recall that if $x \in X$ has a compact neighbourhood we define the functional ν by $\nu(x) = \sup\{\varepsilon > 0 : \overline{B_d(x,\varepsilon)}$ is compact$\}$, otherwise $\nu(x) = 0$. The set $\{x \in X : \nu(x) = 0\}$ is the set of points of non-local compactness of X, which was denoted by $nlc(X)$ in [11]. Now we define the following functionals on a metric space (X,d):

- The functional f_c on X is defined by $f_c(x) = \sup\{\varepsilon > 0 : B_d^{\infty}(x,\varepsilon)$ is compact$\}$ if there exists some $\varepsilon > 0$ for which $B_d^{\infty}(x,\varepsilon)$ is compact, $f_c(x) = 0$, otherwise.
- The functional f_p on X is defined by $f_p(x) = \sup\{\varepsilon > 0 : B_d^{\infty}(x,\varepsilon)$ is precompact$\}$ if there exists some $\varepsilon > 0$ for which $B_d^{\infty}(x,\varepsilon)$ is precompact, $f_p(x) = 0$, otherwise.

Notice that if $f_c(x_0) = \infty$ for some x_0, then $f_c(x) = \infty$ for all $x \in X$ and the same holds for f_p as well. It is easy to check that if f_c (f_p) is bounded, then f_c (f_p) is uniformly continuous. Also, $f_c(x) \leq \nu(x)$ and $f_c(x) \leq f_p(x)$ for all $x \in X$. We define $nslc(X) = \{x : f_c(x) = 0\}$. Clearly, (X,d) is strongly locally compact iff $f_c(x) > 0$ for all $x \in X$ and (X,d) is strongly uniformly locally compact iff $\inf\{f_c(x) : x \in X\} > 0$.

Below we will present certain nice characterizations of cBq-complete metric spaces in terms of these functionals. For this purpose, we recall the following lemma from [11].

Lemma 6.2.1. *(Proposition 3.1, [11]) Let (A_n) be a decreasing sequence of nonempty closed subsets of (X, d) with intersection A. Then the following conditions are equivalent:*

(1) Whenever (a_n) is a sequence in X with $a_n \in A_n$ for each n, then (a_n) has a cluster point.

(2) A is a non-empty compact set and for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $A_n \subset A^{\varepsilon}$.

Theorem 6.2.3. *In a metric space (X, d) the following statements are equivalent:*

1. *(X, d) is cBq-complete.*
2. *Every sequence (x_n) with $\lim_{n\to\infty} f_c(x_n) = 0$, has a cluster point.*
3. *Either X is strongly uniformly locally compact or $nslc(X)$ is non-empty and compact and for every $\varepsilon > 0$ $(nslc(X)^{\varepsilon})^c$ is strongly uniformly locally compact in its relative topology.*
4. *(X, d) is complete and every sequence (x_n) with $\lim_{n\to\infty} f_p(x_n) = 0$, has a Cauchy subsequence.*

Proof. (1) $\Rightarrow$ (2) Suppose that condition (2) fails. Then there is a sequence (x_n) with $\lim_{n\to\infty} f_c(x_n) = 0$, but (x_n) has no cluster point. Without any loss of generality, we can assume by passing to a subsequence that $f_c(x_n) < \frac{1}{n}$. Then we can obtain a sequence (y_j^n) in $B_d^{\infty}(x_n, \frac{1}{n})$ having no cluster point. Partition $\mathbb{N}$ into a countable family of infinite subsets $\{M_n : n \in \mathbb{N}\}$. Then the sequence (z_j), defined by $z_j = y_j^n$ if $j \in M_n$ is cofinally BqC. Hence there is some subsequence $(y_{j_k}^{n_k})_k$ of (z_j), which is convergent. Consequently, the corresponding sequence (x_{n_k}) is BqC. Now from cBq-completeness of X, we can conclude that (x_{n_k}) has a cluster point, which contradicts our assumption.

(2) $\Rightarrow$ (3) If $nslc(X) = \emptyset$, then from continuity of f_c, we can conclude that X is strongly uniformly locally compact. Otherwise, from the given condition and continuity of f_c it is clear that $nslc(X)$ is a non-empty compact subset of X. To prove the another part, let $\varepsilon > 0$ be given and consider $F_n = \{x : f_c(x) \leq \frac{1}{n}\}$, which is closed by the continuity of f_c. Since $nslc(X)$ is non-empty, each F_n is non-empty. By (2) whenever $x_n \in F_n$ for each $n \in \mathbb{N}$, (x_n) has a cluster point and by continuity of f_c, it must lie in $nslc(X)$. Therefore from Lemma 6.2.1, there exists $n \in \mathbb{N}$ such that $F_n \subset nslc(X)^\varepsilon$. As a result, if $x \in (nslc(X)^\varepsilon)^c$, then $B_d^\infty(x, \frac{1}{n})$ is compact and again from closedness of $(nslc(X)^\varepsilon)^c$, we can conclude that $B_d^\infty(x, \frac{1}{n}) \cap (nslc(X)^\varepsilon)^c$ is also compact. Hence $(nslc(X)^\varepsilon)^c$ is strongly uniformly locally compact in its relative topology.

(3) $\Rightarrow$ (1) If X is strongly uniformly locally compact then from Theorem 6.2.2, X is cBq-complete. Otherwise $nslc(X)$ is non-empty and compact. Let (x_n) be a cofinally BqC sequence without a constant subsequence. Then for each $n \in \mathbb{N}$, there exist an infinite subset K_n of $\mathbb{N}$ and $p_n \in X$ such that $x_j \in B_d^\infty(p_n, \frac{1}{n})$ for all $j \in K_n$. If for some $\varepsilon > 0$ and for infinitely many $n \in \mathbb{N}$ $\{x_j : j \in K_n\} \cap (nslc(X)^\varepsilon)^c$ is infinite, then (x_n) has a cluster point by (relative) strong uniform local compactness of $(nslc(X)^\varepsilon)^c$. Otherwise, for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\{x_j : j \in K_n\} \cap (nslc(X)^\varepsilon)^c$ is a finite set. In particular, for each $\varepsilon > 0$, $nslc(X)^\varepsilon$ contains an infinite number of terms of (x_n), and by compactness (x_n) must have a cluster point in $nslc(X)$. Hence X is sequentially cBq-complete.

(1) $\Rightarrow$ (4) The proof is similar to (1) $\Rightarrow$ (2). So it is omitted.

(4) $\Rightarrow$ (1) On the contrary, suppose that there is cofinally BqC sequence (x_n), which has no cluster point. Since X is complete, (x_n) has no Cauchy subsequence.

Observe that for each $n \in \mathbb{N}$, there exist an infinite subset K_n of $\mathbb{N}$ and $p_n \in X$ such that $x_j \in B_d^{\infty}(p_n, \frac{1}{n})$ for all $j \in K_n$. Then one can conclude that there is a subsequence (x_{k_n}) of (x_n) with $k_n \in K_n$ and $f_p(x_{k_n}) < \frac{1}{n}$, where (x_{k_n}) has no Cauchy subsequence. This contradicts the given condition. □

Now we present a nice characterization of cofinally Bourbaki complete spaces in terms of Samuel compactification. From the following theorem, we can conclude that the Stone-Čech compactification βX in the condition (2) of Theorem 34 [38] can be replaced by the Samuel compactification $s_d X$.

Theorem 6.2.4. *Let (X, d) be a cofinally Bourbaki-complete metric space. Then there is a countable family $\{G_n : n \in \mathbb{N}\}$ of open paracompact subspaces of the Samuel compactification $s_d X$ containing X such that if G is an open subset of $s_d X$ containing X, then $X \subset G_k \subset G$ for some $k \in \mathbb{N}$.*

Proof. Since (X, d) is cofinally complete, there exists a countable family $\{\widehat{G_n} : n \in \mathbb{N}\}$ of open subspaces of $s_d X$ such that for any open subset G of $s_d X$ containing X, $X \subset \widehat{G_k} \subset G$ for some $k \in \mathbb{N}$. From the regularity of $s_d X$, for each $x \in X$, there exists an open neighbourhood U_x^n of x in $s_d X$ with $x \in U_x^n \subset \overline{U_x^n} \subset \widehat{G_n}$. Let us take the open cover $\mathcal{U}_n = \{U_x^n \cap X : x \in X\}$ of X. Since X is cofinally Bourbaki-complete, X is uniformly strongly paracompact (see Theorem 32 [38]). So there exist an open refinement $\mathcal{A}_n$ of $\mathcal{U}_n$ and $\mathcal{V}_n \in \mu_d$ such that for each $A \in \mathcal{A}_n$, $St(A, \mathcal{V}_n)$ meets at most finitely many elements of $\mathcal{A}_n$. Consider the family $\{St^{\infty}(x_i, \mathcal{V}_n) : i \in I_n\}$ of all chainable components of X with respect to $\mathcal{V}_n$. Observe that for each $m \in \mathbb{N}$, there is a cozero subset $V_{x_i,n}^m$ of $s_d X$ such that $V_{x_i,n}^m \cap X = St^m(x_i, \mathcal{V}_n)$ and $V_{x_i,n}^m \subset \overline{St^m(x_i, \mathcal{V}_n)}^{s_d X}$. Consequently, we can show that $St^{\infty}(x_i, \mathcal{V}_n) \subset \bigcup_{m=1}^{\infty} V_{x_i,n}^m$. Now take $G_n = \bigcup_{i \in I_n} [\bigcup_{m=1}^{\infty} V_{x_i,n}^m]$. Clearly, G_n is an open subset of $s_d X$ containing X. Being a cozero set in normal Hausdorff space, $\bigcup_{m=1}^{\infty} V^m_{x_i,n}$ is an open F_σ set for each $i \in I_n$. Also, we have $\bigcup_{m=1}^{\infty} V^m_{x_i,n} \subset \bigcup_{m=1}^{\infty} \overline{St^m(x_i,\mathcal{V}_n)}^{s_dX} \subset \overline{St^\infty(x_i,\mathcal{V}_n)}^{s_dX}$ and $\overline{St^\infty(x_i,\mathcal{V}_n)}^{s_dX} \cap \overline{St^\infty(x_j,\mathcal{V}_n)}^{s_dX} = \emptyset$ whenever $i \neq j$. Therefore G_n is a free union of open F_σ subsets of s_dX that is a free union of open Lindelöf subspaces of s_dX. Consequently, G_n is paracompact. Now we will show that $G_n \subset \widehat{G_n}$. Since for each $m \in \mathbb{N}$ and $i \in I_n$, $St^m(x_i,\mathcal{V}_n)$ meets at most finitely many elements $\mathcal{A}_n$ and $\mathcal{A}_n \leq \mathcal{U}_n$, we can choose finitely many $y_1, ..., y_k \in X$ such that $V^m_{x_i,n} \subset \overline{St^m(x_i,\mathcal{V}_n)}^{s_dX} \subset \bigcup_{j=1}^{k} \overline{U^n_{y_j}}^{s_dX} \subset \widehat{G_n}$. This completes the proof. □

Results for Uniform Spaces

In the following, we present our first result about cBq-completely metrization for metrizable spaces using the set $nlc(X)$ of points that have no compact neighborhood, in line with the investigation given in [38, 69] for cofinal complete and cofinal Bourbaki-complete metrization, respectively. In the following theorem, we will use the well-known metrization theorem by Dugundji [34]: "If X is a metrizable space and $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a family of open covers of X, then X can be made metrizable by a metric d such that the family of open balls of radius $\frac{1}{n}$, $\{B_d(x, \frac{1}{n}) : x \in X\}$ refines $\mathcal{U}_n$ for all $n \in \mathbb{N}$". The last part of the proof of the next theorem follows the similar arguments using in Theorem 33 of [38]. For the sake of completeness, we give the detailed proof.

Theorem 6.2.5. *(cf. Th.33, [38]) Let X be metrizable space. Then the following statements are equivalent:*

(1) *X is cBq-completely metrizable.*

(2) *The set $nlc(X)$ is compact and X is superparacompact.*

Proof. (1) ⇒ (2) It is clear that X is cofinally completely metrizable and subsequently, $nlc(X)$ is compact [11]. Moreover, from Theorem 6.1.2, X is uniformly star superparacompact and so is superparacompact.

(2) $\Rightarrow$ (1)

- Suppose that $nlc(X)$ is empty. Then X is locally compact and hence for every $x \in X$, there exists an open neighbourhood V^x with compact closure. Let us consider the open cover $\mathcal{V} = \{V^x : x \in X\}$. As X is superparacompact, there exists an open finite-component refinement $\mathcal{U}$ of $\mathcal{V}$. Now by Dugundji's metrization theorem [34], there exists a compatible metric d such that the cover of open balls $\{B_d(x,1) : x \in X\}$ refines $\mathcal{U}$. It is easy to observe that for each $x \in X$ there is some $U \in \mathcal{U}$ such that $B_d^\infty(x,1) \subset St^\infty(U,\mathcal{U})$ and by finite-componentness of $\mathcal{U}$, we can choose finitely many $x_1, ..., x_k \in X$ such that $B_d^\infty(x,1) \subset St^\infty(U,\mathcal{U}) \subset \overline{\bigcup_{i=1}^{k} V^{x_i}} \subset \bigcup_{i=1}^{k} \overline{V^{x_i}}$. Therefore $B_d^\infty(x,1)$ is compact for every $x \in X$ and then from Theorem 6.2.2, we can conclude that (X,d) is cBq-complete.

- Now we assume that $nlc(X)$ is non-empty. Then from compactness of $nlc(X)$, we have a countable family of open sets $\{U_1, ..., U_k, ...\}$ in X, which satisfies that for every open subset $A \subset X$ containing $nlc(X)$, $nlc(X) \subset U_k \subset A$ for some $k \in \mathbb{N}$. Now for every $x \in (nlc(X))^c$ we take an open neighbourhood V^x of x with compact closure. Then for every $k \in \mathbb{N}$ let us consider the open cover $\mathcal{W}_k = \{V^x : x \notin nlc(X)\} \cup \{U_k\}$. Moreover, by superparacompactness of X there is an open finite-component refinement $\mathcal{V}_k$ of $\mathcal{W}_k$. Now by Dugundji's metrization theorem [34], there exists a compatible metric d such that the cover of open balls $\{B_d(x, \frac{1}{k}) : x \in X\}$ refines $\mathcal{V}_k$ for every $k \in \mathbb{N}$.

  To prove the cBq-completeness of (X,d) we consider a cofinally BqC sequence (x_n) in (X,d). Then for every $k \in \mathbb{N}$ there exists some infinite subset $N_k \subset \mathbb{N}$ such that for some $p_k \in X$, $x_n \in B_d^\infty(p_k, \frac{1}{k})$ for every $n \in N_k$. It is easy to check that a union of finitely many members of $\mathcal{W}_k$ contains $B_d^\infty(p_k, \frac{1}{k})$, which implies that $(x_n)_{n\in N_k}$ must be cofinally contained in some member of $\mathcal{W}_k$. If there is some V^x, that contains a subsequence of $(x_n)_{n\in N_k}$, then we have done. On the other hand suppose that $(x_n)_{n\in N_k}$ is eventually in U_k, but (x_n) does not cluster. Then $\bigcap_{n\in\mathbb{N}} H_n = \emptyset$, where $H_n = \overline{\{x_j : j \geq n\}}$ for every $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$, let us take the open cover $\mathcal{H} = \{X \setminus H_n : n \in \mathbb{N}\}$ of X. Consequently, there exist finitely many $H_{n_1}, ..., H_{n_r}$ and some $k \in \mathbb{N}$ such that $nlc(X) \subset U_k \subset \bigcup_{i=1}^{r}(X \setminus H_{n_i})$. Then (x_n) must have a subsequence contained in some $(X \setminus H_{n_i})$, $i \in \{1, ..., r\}$, which is a contradiction. Therefore (x_n) has a cluster point.

□

We illustrate an example of a Bq-complete metric space, which is not cBq-completely metrizable with the help of the above theorem.

Example 6.2.1. *First of all for each n, partition $\mathbb{N} \setminus \{n\}$ into a countable family of infinite subsets $\{M_k^n : k \in \mathbb{N}\}$. Then consider $X_n = \{10^n e_n + \frac{1}{k}e_i : i \in M_k^n$ and $k \in \mathbb{N}\} \cup \{10^n e_n\}$. Take $X = \bigcup_{n\in\mathbb{N}} X_n$ with sup norm of ℓ_∞. One can observe that $(10^n e_n)$ is a sequence in $nlc(X)$. So $nlc(X)$ is not compact. Therefore from the above theorem, X is not cBq-completely metrizable. Here X is Bq-complete as every X_n is Bq-complete. To prove that, let (x_n) be a BqC sequence in X_n without a constant subsequence. We can write $X_n = \bigcup_{k=1}^{\infty} Y_k^n$, where $Y_k^n = \{10^n e_n + \frac{1}{k}e_i : i \in M_k^n\}$. Observe that for each $y \in Y_k^n$, $dist(y, \{y\}^c) = \frac{1}{k}$. Hence we can obtain a subsequence (x_{r_k}) of (x_n), which converges to $10^n e_n$.*

We know that for a metrizable space X, X is cofinally completely metrizable iff there is a countable family $\{G_n : n \in \mathbb{N}\}$ of open subspaces of βX containing X such that if G is an open subset of βX containing X then $X \subset G_k \subset G$ for some $k \in \mathbb{N}$

(see [22, 75]). In fact, this result is also true for every compactification cX. Finally, we give attention to the result of cBq-complete metrization in line of Theorem 6.2.1, which indicates the place of cBq-completely metrizable spaces into their Stone-Čech compactification.

Theorem 6.2.6. *For a metrizable space X the following statements are equivalent:*

(1) *X is a cBq-completly metrizable space.*

(2) *There is a countable family $\{G_n : n \in \mathbb{N}\}$ of open superparacompact subspaces of βX containing X such that if G is an open subset of βX containing X then $X \subset G_k \subset G$ for some $k \in \mathbb{N}$.*

(3) *There exist some compactification cX of X and a countable family $\{G_n : n \in \mathbb{N}\}$ of open superparacompact subspaces of cX containing X such that if G is an open subset of cX containing X then $X \subset G_k \subset G$ for some $k \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that X is cBq-completely metrizable. Then clearly X is cofinally completely metrizable. Then there is a countable family $\{\widehat{G_n} : n \in \mathbb{N}\}$ of open subspaces of βX containing X that is a basis for those open sets in βX which contain X (see [75]). On the other hand, there exists a compatible metric d on X such that (X, d) is cBq-complete. So from Theorem 6.1.2, we can conclude that (X, d) is uniformly star superparacompact. Let $n \in \mathbb{N}$. From the regularity of βX, for each $x \in X$ there exists an open neighbourhood V_x^n of x in βX such that $x \in V_x^n \subset \overline{V_x^n}^{\beta X} \subset \widehat{G_n}$. Consider the open cover of X, $\mathcal{U}_n = \{V_x^n \cap X : x \in X\}$. As (X, d) is uniformly star superparacompact, there exists an open refinement $\mathcal{A}_n$ of $\mathcal{U}_n$ and $\mathcal{V}_n \in \mu_d$ satisfying that for each $A \in \mathcal{A}_n$, $St^\infty(A, \mathcal{V}_n)$ meets atmost finite number of elements of $\mathcal{A}_n$. Let us consider the family $\{St^\infty(x_i, \mathcal{V}_n) : i \in I_n\}$ of all chainable components of X induced by $\mathcal{V}_n$. As for each $i \in I_n$, $St^\infty(x_i, \mathcal{V}_n)$ is clopen in X, $\overline{St^\infty(x_i, \mathcal{V}_n)}^{\beta X}$ is also clopen in βX. Consequently, for each $i \in I_n$, $\overline{St^\infty(x_i, \mathcal{V}_n)}^{\beta X}$ is a compact subspace of βX. Consider $G_n = \bigcup\{\overline{St^\infty(x_i, \mathcal{V}_n)}^{\beta X} : i \in I_n\}$. It is evident that G_n is open in βX. Now we will show that G_n is superparacompact. Let $\mathcal{G}$ be an open cover of G_n. It is clear that $\mathcal{G}_i = \{G \cap \overline{St^\infty(x_i, \mathcal{V}_n)}^{\beta X} : G \in \mathcal{G}\}$ is an open cover of $\overline{St^\infty(x_i, \mathcal{V}_n)}^{\beta X}$. Then by compactness of $\overline{St^\infty(x_i, \mathcal{V}_n)}^{\beta X}$, $\mathcal{G}_i$ has a finite subcover $\mathcal{H}_i$ and moreover, $\overline{St^\infty(x_i, \mathcal{V}_n)}^{\beta X} \cap \overline{St^\infty(x_j, \mathcal{V}_n)}^{\beta X} = \emptyset$ whenever $i \neq j$. Now one can easily observe that $\bigcup\{\mathcal{H}_i : i \in I_n\}$ is an open finite-component refinement of $\mathcal{G}$. Finally, we will show that for each $n \in \mathbb{N}$, $G_n \subset \widehat{G_n}$. Observe that as $\mathcal{A}_n \leq \mathcal{U}_n$, for each $i \in I_n$ we can choose $y_1, ..., y_k \in X$ such that $St^\infty(x_i, \mathcal{V}_n) \subset \bigcup\limits_{j=1}^{k} V^n_{y_j}$, which implies that $\overline{St^\infty(x_i, \mathcal{V}_n)}^{\beta X} \subset \overline{\bigcup\limits_{j=1}^{k} V^n_{y_j}}^{\beta X} \subset \bigcup\limits_{j=1}^{k} \overline{V^n_{y_j}}^{\beta X} \subset \widehat{G_n}$. Hence $X \subset G_n \subset \widehat{G_n}$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) From the given condition, it is clear that X is cofinally completely metrizable, which implies that $nlc(X)$ is compact. Then in view of Theorem 6.2.5, we only need to prove that X is superparacompact. Let $\mathcal{V}$ be an open cover of X. Observe that for each $V \in \mathcal{V}$, there exists an open set V_c in cX such that $V_c \cap X = V$. Let us take $H = \bigcup\{V_c : V \in \mathcal{V}\}$. Clearly, H is an open subspace of cX containing X. Now from the given condition, there exists $k \in \mathbb{N}$ satisfying that $X \subset G_k \subset H$. Consequently, $\mathcal{U} = \{V_c \cap G_k : V \in \mathcal{V}\}$ is an open cover of G_k. As G_k is superparacompact, there exists an open finite-component refinement $\mathcal{W}$ of $\mathcal{U}$. Consider $\mathcal{W}_X = \{W \cap X : W \in \mathcal{W}\}$. It is evident that $\mathcal{W}_X$ is an open finite-component refinement of $\mathcal{V}$. □

Remark 6.2.2. *We do not know whether the implications* (1) $\Rightarrow$ (2) *of Theorem 6.2.1 and Theorem 6.2.6 are true in every compactification X. If we carefully review the proofs of these theorems, we can see that the following property of βX is required: "Every uniformly continuous function $f : (X, d) \to [0, 1]$ can be extended to a uniformly continuous function $\widetilde{f} : \beta X \to [0, 1]$". It is known that the Samuel compactification $s_d X$ also satisfies this property. Hence we can obtain the following observations:*

(1) *A metric space* (X, d) *is Bq-complete iff* $X = \bigcap_{n=1}^{\infty} G_n$*, where each* G_n *is an open superparacompact subspace of* $s_d X$*.*

(2) *A metric space* (X, d) *is a cBq-complete iff there is a countable family* $\{G_n : n \in \mathbb{N}\}$ *of open superparacompact subspaces of* $s_d X$ *containing* X *such that if* G *is an open subset of* $s_d X$ *containing* X*, then* $X \subset G_k \subset G$ *for some* $k \in \mathbb{N}$*.*

The above theorems are also true for every compactification cX *with* $cX \geq s_d X$*. In Theorem 34 of [38], an analogous result was proved for cofinally Bourbaki-completely metrizable spaces, but whether the result is true for arbitrary compactification remains open like our case.*

6.3 Some results related to quasi-Cauchy sequences

In this section, we relook on the notion of BqC sequence in metric structure and investigate another direction of this concept. First, we consider the notion of α-boundedness [72, 73] and show that it is one kind of precompactness in the case of quasi-Cauchy sequences. Then we define a new type of Lipschitz function using quasi-Cauchy sequence in line with Cauchy Lipschitz function. In dealing with this type of problem the first objective is to characterize an arbitrary subsequence of a quasi-Cauchy sequence. Interestingly there is a nice connection between BqC sequence and any arbitrary subsequence of quasi-Cauchy sequence. Throughout this section, we will show that the notion of BqC sequences plays a key role in any investigation related to quasi-Cauchy sequence.

6.3.1 Notion of precompactness with quasi-Cauchy sequences

As has already been mentioned, precompact or totally bounded metric spaces are those where every sequence has a Cauchy subsequence. When quasi-Cauchy sequences come into the picture, it is tempting to define a notion of precompactness in the same way, only replacing Cauchy sequences by quasi-Cauchy sequences but it does not seem to be that much useful notion because of the wilder nature of quasi-Cauchy sequences. However, we consider the notion of α-boundedness, which was introduced and studied by Tashjain in the context of metric spaces in [72] and for uniform spaces in [73]. Then we investigate a nice connection in between the notion of α-boundedness and quasi-Cauchy sequences, which helps to open a new direction of this concept. Further, we show that α-bounded spaces play an important role for quasi-Cauchy sequences as precompact spaces play for Cauchy sequences.

Now we consider the definition of α-boundedness, a weaker version of precompactness, which we name as “Bq-precompactness" as it would be clearer later (see Theorem 6.3.2) the important role played by quasi-Cauchy sequences in this notion.

Definition 6.3.1. *Let (X, d) be a metric space. Then $B \subset X$ is said to be a Bq-precompact subset of X (or sometimes it is called Bq-precompact in X) if for every $\varepsilon > 0$ there exists a finite collection of points $p_1, p_2, \ldots, p_k \in X$ such that $B \subset \bigcup_{i=1}^{k} B_d^{\infty}(p_i, \varepsilon)$.*

It is clear that precompact $\Rightarrow$ Bourbaki bounded$\Rightarrow$ Bq-precompact. But the converse implications are not generally true. Every chainable metric space is Bq-precompact, but it may not be Bourbaki bounded. On the other hand, it is clear that Bq-precompactness is uniform property and the family of Bq-precompact subsets forms a (closed) bornology. One must also keep in mind the subtle difference between “Bq-precompact in X" and “Bq-precompact in itself" depending on whether the points forming the chains are coming from the concerned subset itself or not. For example note that every subset of a chainable metric space X is Bq-precompact in X but an infinite uniformly discrete subset of X cannot be Bq-precompact in itself. Evidently every quasi-Cauchy sequence is Bq-precompact in itself. The notion of Bq-precompactness is independent with the notion of boundedness. The following examples show that even a bounded Bq-precompact set may not be Bourbaki bounded.

Example 6.3.1. *The real line with the bounded metric $\widehat{d} = \min\{1, d\}$, where d is the usual Euclidean metric, is bounded Bq-precompact, but not Bourbaki bounded.*

In the following examples, we consider bounded Bq-precompact subset of ℓ^∞, which are not Bourbaki bounded.

Example 6.3.2. *Consider $X_n = \{(1 - \frac{k}{n+1})e_n + \frac{k}{n+1}e_{n+1} : k \in \mathbb{Z}, 0 \leq k \leq n+1\}$, where (e_n) is sequence of unit vectors of ℓ^∞. Take $X = \bigcup_{n \in \mathbb{N}} X_n$ with sup norm of ℓ^∞. By suitably arranging the terms of X one can observe that X is a quasi-Cauchy sequence and so Bq-precompact in itself. Evidently, X is bounded. Note that $d(X_i, X_j) = \frac{1}{2}$ for all $i, j \in \mathbb{N}$ with $|i - j| \geq 2$. So any $\frac{1}{4}$-chain joining e_{2n} and e_{2m} for $m > n$ must meet $X_{2n+1}, X_{2n+2}, \ldots, X_{2m-1}$. Consequently the length of this chain must be at least $2(m - n) - 1$. Hence X cannot be Bourbaki bounded.*

Now we are going to present a sequential characterization of Bq-precompact subsets of a space. For that, we recall the notion of BqC sequences in metric space.

Definition 6.3.2. *Let (X, d) be a metric space. A sequence (x_n) is said to be Bourbaki quasi-Cauchy or BqC in X if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for some $p \in X$ we have $x_n \in B_d^\infty(p, \varepsilon)$ for every $n \geq n_0$.*

Clearly, every subsequence of a BqC sequence is BqC in the underlying space. We will say that a sequence (x_n) has a BqC subsequence in X if (x_n) has a subsequence which is BqC in X. The reason behind the inclusion of the term "quasi-Cauchy" in Definition 6.3.2 can be understood from the following result.

Theorem 6.3.1. *Let (X, d) be a metric space. A sequence (x_n) is BqC in X iff (x_n) is a subsequence of some quasi-Cauchy sequence of X.*

Proof. Suppose that (x_n) is a subsequence of a quasi-Cauchy sequence (z_k). Let $\varepsilon > 0$ be given. Then there exists $k_0 \in \mathbb{N}$ such that $d(z_{k+1}, z_k) < \varepsilon\ \forall k \geq k_0$ which implies that $z_k \in B_d^\infty(z_{k_0}, \varepsilon)$ for all $k \geq k_0$. Consequently $x_n \in B_d^\infty(z_{k_0}, \varepsilon)$ for all but finitely many n.

For the converse, let (x_n) be a BqC sequence in X. Now for each $k \in \mathbb{N}$ there exists n_k such that $x_i \in B_d^\infty(x_{n_k}, \frac{1}{k})$ for all $i \geq n_k$. Equivalently we can say that x_{i+1} and x_i can be joined by a $\frac{1}{k}$-chain for each $i \geq n_k$ i.e. there exists a finite collection of points $x_i = p_0^{i,k}, p_1^{i,k}, \ldots, p_{r_i}^{i,k} = x_{i+1}$ with the property that $d(p_{j+1}^{i,k}, p_j^{i,k}) < \frac{1}{k}$ for all $j = 0, 1, \ldots, r_i$. Now consider the sequence

$$\{x_1, \ldots, x_{n_1}, p_1^{n_1,1}, p_2^{n_1,1}, \ldots, x_{n_1+1}, \ldots, x_{n_2}, p_1^{n_2,\frac{1}{2}}, \ldots, x_{n_k}, p_1^{n_k,\frac{1}{k}}, ...\}.$$

Clearly, the above sequence is the required quasi-Cauchy sequence. □

Corollary 6.3.1. *Every subsequence of a quasi-Cauchy sequence (x_n) is BqC in (x_n).*

In the following, we present a sequential characterization of Bq-precompact sets in line of the classical result that a space is precompact iff every sequence has a Cauchy subsequence.

Theorem 6.3.2. *Let (X, d) be a metric space. Then a non-void subset A of X is Bq-precompact in X iff every sequence in A has a BqC subsequence in X.*

Proof. Suppose that $\emptyset \neq A \subset X$ is Bq-precompact in X and (x_n) is a sequence in A. Without any loss of generality let us assume that all x_n's are distinct. By Bq-precompactness, the set A can be covered by finitely many 1-enlargements $B_d^\infty(y, 1)$ where $y \in X$. Thus, there exists $y_1 \in X$ such that $B_d^\infty(y_1, 1)$ must contain infinitely many terms of (x_n). Take $I_1 = \{n : x_n \in B_d^\infty(y_1, 1)\}$. Similarly there exists $y_2 \in X$ such that $B_d^\infty(y_2, \frac{1}{2})$ contains infinite number of terms of $(x_n)_{n \in I_1}$, then take $I_2 = I_1 \cap \{n : x_n \in B_d^\infty(y_2, \frac{1}{2})\}$ and so on. Continuing this process we obtain a decreasing sequence (I_k) of infinite subsets of $\mathbb{N}$ where $I_{k+1} = I_k \cap \{n : x_n \in B_d^\infty(y_{k+1}, \frac{1}{k+1})\}$ for each $k \in \mathbb{N}$. Choose an increasing sequence (n_k) of natural numbers with $n_k \in I_k$. We claim that (x_{n_k}) is a BqC sequence in X. This is true because given $\varepsilon > 0$, one can first choose $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \varepsilon$ and then from the construction of I_k it can be concluded that $x_{n_k} \in B_d^{\infty}(y_{k_0}, \frac{1}{k_0})$ for all $k \geq k_0$.

Conversely, suppose on the contrary that the given condition holds, but A is not Bq-precompact in X. Then there exists an $\varepsilon > 0$ for which we can choose $x_1, x_2 \in A$ such that $x_2 \notin B_d^{\infty}(x_1, \varepsilon)$. Again we can choose $x_3 \in A$ such that $x_3 \notin B_d^{\infty}(x_1, \varepsilon) \cup B_d^{\infty}(x_2, \varepsilon)$. Continuing in this process a sequence (x_k) in A is obtained which has the property that $x_{k+1} \notin \bigcup_{i=1}^{k} B_d^{\infty}(x_i, \varepsilon)$. Obviously (x_n) has no BqC subsequence in X. This contradicts the given condition. Hence A is Bq-precompact in X. □

Corollary 6.3.2. *A metric space X is Bq-precompact iff every sequence has a BqC subsequence.*

Theorem 6.3.3. *A metric space X is compact iff it is Bq-precompact and Bq-complete.*

Proof. The necessity of the conditions is clear. For the converse part, let (x_n) be a sequence in X. From Theorem 6.3.2, (x_n) has a BqC subsequence and so it has a cluster point by Bq-completeness. Hence X is compact. □

For a subset of $\mathbb{R}$, we can actually characterize compactness with the help of Bq-precompactness along with a weaker condition, known as weakly G-completeness [45].

Theorem 6.3.4. *Let $X \subset \mathbb{R}$ be endowed with the usual metric of $\mathbb{R}$. Then X is compact iff it is Bq-precompact in itself and every quasi-Cauchy sequence in X has a cluster point in X.*

Proof. One part is obvious. Conversely, suppose that X is Bq-precompact in itself and every quasi-Cauchy sequence in X has a cluster point in X. As closedness of X in $\mathbb{R}$ is quite obvious, one only needs to prove that X is bounded. Suppose on the contrary that X is not bounded. Then we can choose a sequence (x_n) which satisfies the property that $x_1 > 1$ and $x_{n+1} > x_n + 1$ for all $n \in \mathbb{N}$. Now as X is a Bq-precompact space, from Theorem 6.3.2 and by passing to a subsequence we can conclude that (x_n) is BqC in X. So according to Theorem 6.3.1, there is a quasi-Cauchy sequence (z_n) in X such that $z_{r_n} = x_n$ for some subsequence (z_{r_n}) of (z_n). Now consider $G_n = \{z_i : r_n < i < r_{n+1} \text{ and } z_i \in (z_{r_n}, z_{r_{n+1}})\}$ for each $n \in \mathbb{N}$. Clearly each G_n is non-empty for all but finitely many n. We can write G_n as $\{z_1^n < z_2^n < .... < z_{p_n}^n\}$. Now consider the following increasing sequence

$$\{z_{r_1}, z_1^1, z_2^1, \ldots, z_{p_1}^1, z_{r_2}, z_1^2, \ldots, z_{r_3}, \ldots, z_{r_n}, z_1^n, \ldots, z_{p_n}^n, z_{r_{n+1}}, ...\}.$$

Observe that it still remains a quasi-Cauchy sequence in X, but it has no cluster point. This contradicts the given condition. Hence X must be bounded and consequently, X must be compact. □

However, the following example shows that the above characterization is not generally true for an arbitrary metric space. Also, it shows that Bq-completeness is strictly stronger than weak G-completeness.

Example 6.3.3. *Consider* $X = \{re_n : n \in \mathbb{N}, r \in [0,1]\}$, *where* $\{e_n : n \in \mathbb{N}\}$ *is the set of all unit vectors of* ℓ^∞, *endowed with sup norm of* ℓ^∞. X *is a Bq-precompact space as it is chainable. Let* (x_n) *be a quasi-Cauchy sequence in* X. *Without any loss of generality we can assume that* x_n*'s are distinct. Choose* $X_k = \{re_k : r \in [0,1]\}$. *If* (x_n) *intersects only finitely many of* X_k, *then from the compactness of* X_k*'s it follows that* (x_n) *has a cluster point. Now if* (x_n) *intersects infinitely many of* X_k, *then we can choose a subsequence* (x_{n_p}) *of* (x_n) *and a sub collection* $\{X_{k_p} : p \in \mathbb{N}\}$ *of* $\{X_k : k \in \mathbb{N}\}$ *such that* $x_{n_p} \in X_{k_p}$ *but* $x_{n_p+1} \notin X_{k_p}$. *Let* r_p *and* s_p *be the non-zero terms of* x_{n_p} *and* x_{n_p+1} *respectively. Then we have* $||x_{n_p+1} - x_{n_p}||_\infty = \max\{s_p, r_p\} \to 0$. *Hence* $x_{n_p} \to 0$. *This shows that* X *is weakly G-complete. But* X *is not compact, so by Theorem 6.3.3,* X *is not Bq-complete. Note that* (e_n) *is BqC in* X *without any cluster point, in* X.

6.3.2 Quasi-Cauchy Lipschitz functions

In analysis there is a well-known group of continuous functions that is even stronger than uniformly continuous functions, namely Lipschitz functions. In [12, 13, 14, 41], Beer, Garrido and Jaramillo considered various kinds of Lipschitz-type functions. As our main objective in this section is to ascertain the role of quasi-Cauchy sequences in different spheres, naturally one can ask what would happen if the notion of Cauchy Lipschitz functions can be modified in terms of quasi-Cauchy sequences and with precisely this in mind, we introduce the following notion.

Definition 6.3.3. *A function $f : (X, d) \to (Y, \rho)$ is said to be quasi-Cauchy Lipschitz if for any quasi-Cauchy sequence (x_n) in X there exists $\lambda > 0$ such that $\rho(f(x_k), f(x_{k+1})) \leq \lambda d(x_k, x_{k+1})$ for all $k \in \mathbb{N}$.*

Clearly, every quasi-Cauchy Lipschitz function is ward continuous. Our main objective in this section is to study circumstances under which this new type of Lipschitz function coincides with any of the existing notions and secondly to investigate the density position of these functions in the space of ward continuous functions. We start with Lipschitz in the small functions and the following sequential characterization of Lipschitz in the small functions will come in handy for our said purpose.

Lemma 6.3.1. *A function $f : (X, d) \to (Y, \rho)$ is Lipschitz in the small iff for any two sequences (x_n) and (y_n) with $d(x_n, y_n) \to 0$ there exists $\lambda > 0$ such that $\rho(f(x_n), f(y_n)) \leq \lambda d(x_n, y_n)$ for all $n \in \mathbb{N}$.*

Proof. Let $f : (X, d) \to (Y, \rho)$ be a Lipschitz in the small function. Then there exist $\delta > 0$ and $\lambda > 0$ such that $d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) \leq \lambda d(x, y)$. Let $(x_n), (y_n)$ be two sequences with $d(x_n, y_n) \to 0$. Without any loss of generality we can assume that $x_n \neq y_n$ for all $n \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that $d(x_n, y_n) < \delta$

for all $n \geq n_0$ and consequently $\rho(f(x_n), f(y_n)) \leq \lambda d(x_n, y_n)$ for all $n \geq n_0$. Take $\lambda_0 = \max\{\lambda, \sup_{n<n_0} \dfrac{\rho(f(x_n), f(y_n))}{d(x_n, y_n)}\}$. Clearly $\rho(f(x_n), f(y_n)) \leq \lambda_0 d(x_n, y_n)$ for all $n \in \mathbb{N}$.

Conversely, suppose that f is not Lipschitz in the small. This means that for each $n \in \mathbb{N}$, $\{\frac{\rho(f(x),f(y))}{d(x,y)} : 0 < d(x,y) < \frac{1}{n}\}$ is not bounded. But this implies the existence of two sequences (x_n) and (y_n) with $d(x_n, y_n) \to 0$ satisfying $\frac{\rho(f(x_n),f(y_n))}{d(x_n,y_n)} > n$, which contradicts the given assumption. □

Theorem 6.3.5. *Let (X, d) and (Y, ρ) be two metric spaces. Then*

(1) *Each Lipschitz in the small function from X to Y is quasi-Cauchy Lipschitz function.*

(2) *Each quasi-Cauchy Lipschitz function from X to Y is Cauchy-Lipschitz.*

Proof. (1) is an immediate consequence of Lemma 4.1. For (2), suppose that $f : (X, d) \to (Y, \rho)$ is a quasi-Cauchy Lipschitz function but not a Cauchy-Lipschitz function. Then there exist a Cauchy sequence (x_n) and two subsequences (y_k) and (z_k) of (x_n) such that $\frac{\rho(f(y_k),f(z_k))}{d(y_k,z_k)} > k$ for all $k \in \mathbb{N}$. Define a new sequence (w_k) by taking $w_{2k} = y_k$ and $w_{2k-1} = z_k$ for all $k \in \mathbb{N}$. Clearly (w_k) is quasi-Cauchy and so f is not quasi-Cauchy Lipschitz. □

The following examples illustrate the relations of quasi-Cauchy Lipschitz functions with other types of Lipschitz functions.

Example 6.3.4.

1. *Let $X = \mathbb{N} \cup \{n + \frac{1}{n} : n \in \mathbb{N}\}$ endowed with usual metric of $\mathbb{R}$. The characteristic function $\chi_{\mathbb{N}}$ of $\mathbb{N}$ on X is quasi-Cauchy Lipschitz but not Lipschitz in the small as $\frac{|\chi_{\mathbb{N}}(n+\frac{1}{n})-\chi_{\mathbb{N}}(n)|}{\frac{1}{n}} = n \to \infty$. Note that $\chi_{\mathbb{N}}$ is uniformly locally Lipschitz.*
2. *Now let $X = \{\sqrt{n} : n \in \mathbb{N}\}$ with usual metric of $\mathbb{R}$. Observe that the characteristic function $\chi_{\sqrt{2n}}$ of $\{\sqrt{2n} : n \in \mathbb{N}\}$ on X is Cauchy-Lipschitz but not quasi-Cauchy Lipschitz. Here again $\chi_{\sqrt{2n}}$ is uniformly locally Lipschitz as for each $n \in \mathbb{N}$, $B(\sqrt{n}, 1)$ contains at most finitely many points.*

3. *Consider* $X = \{1 + \frac{1}{2} + ... + \frac{1}{n} : n \in \mathbb{N}\}$ *with usual metric of* $\mathbb{R}$*. Define* $f : X \to \mathbb{R}$ *by* $f(1 + ... + \frac{1}{n}) = \sqrt{n}$*. Clearly* f *is ward Continuous But* f *is not quasi-Cauchy Lipschitz as* $\frac{\sqrt{n+1}-\sqrt{n}}{\frac{1}{n+1}} = \frac{n+1}{\sqrt{n+1}+\sqrt{n}}$*, which produces an unbounded set of numbers as* n *ranges over* $\mathbb{N}$*.*

4. *Finally let* $X = \{ne_1 + \frac{1}{n}e_k : k, n \in \mathbb{N}\}$ *equipped with sup norm of* ℓ^∞*, where* (e_k) *is the usual basis of* ℓ^∞*. Define* $f : X \to \mathbb{R}$ *by* $f(ne_1 + \frac{1}{n}e_k) = n^k$*. Note that* f *is quasi-Cauchy Lipschitz as every quasi-Cauchy sequence is eventually constant. But* f *is not uniformly locally Lipschitz as* f *is unbounded on the sets* $\{ne_1 + \frac{1}{n}e_k : k \in \mathbb{N}\}$ *of diameter* $\frac{1}{n}$*.*

Remark 6.3.1. *From the above examples we can conclude that set of all quasi-Cauchy Lipschitz functions and the set of all uniformly locally Lipschitz functions both properly contain the set of all Lipschitz in the small functions and on the other side are themselves contained in the set of all Cauchy-Lipschitz functions. Moreover, these two classes neither coincide, nor their intersection coincides with the class of Lipschitz in the small functions. Further, there exists a Cauchy-Lipschitz function which is neither quasi-Cauchy Lipschitz nor uniformly locally Lipschitz. Consider* $X = \{\sqrt{n}e_1 + \frac{1}{n}e_k : k, n \in \mathbb{N}\}$ *equipped with sup norm of* ℓ^∞*. Define* $f : X \to \mathbb{R}$ *by* $f(\sqrt{n}e_1 + \frac{1}{n}e_k) = n^k$*. It can be easily seen that* f *is the required example.*

Now we will discuss about certain conditions under which ward continuity coincides with other types of continuity and quasi-Cauchy Lipschitz function coincides with other types of Lipschitz functions. As we will see from the next few results, some of these conditions are expectedly analogous to existing ones [51], but there are also certain conditions that were never brought up in the literature. Before going to the results we recall a lemma from [14].

Lemma 6.3.2. *[14] Let* (X, d) *and* (Y, ρ) *be two metric spaces. Then* $f : X \to Y$ *is locally Lipschitz iff the restriction of* f *to the range of each convergent sequence in* X *is Lipschitz.*

The first result in this line concerns with the coincidence of quasi-Cauchy Lipschitz (ward continuous) functions with locally Lipschitz (continuous) functions.

Theorem 6.3.6. *Let (X, d) and (Y, ρ) be two metric spaces. Then the following statements are equivalent.*

1. *Every real-valued locally Lipschitz function defined on X is quasi-Cauchy Lipschitz.*

2. *X is Bq-complete.*

3. *Every subsequence of a quasi-Cauchy sequence in X has a cluster point in X.*

4. *Every real-valued continuous function defined on X is ward continuous.*

Proof. (2) $\iff$ (3) is clear from Theorem 6.3.1 and (3) $\iff$ (4) follows from Theorem 4.4.2.

(1) $\Rightarrow$ (2) On the contrary, suppose that there is a BqC sequence (x_k) which has no cluster point in X. By Theorem 6.3.1 one can find a quasi-Cauchy sequence (z_n) such that $z_{n_k} = x_k$ for some subsequence (z_{n_k}) of (z_n). Evidently (z_{n_k+1}) has no cluster point. Now we define a new sequence (y_k) by taking $y_{2k-1} = z_{n_k}$ and $y_{2k} = z_{n_k+1}$ for each k. Clearly (y_k) has no cluster point. This fact allows us to choose a sequence of positive real numbers (δ_k), where $\delta_k = \frac{1}{4}d(y_k, \{y_n : n \neq k\})$. From this construction it immediately follows that the distance between two open balls $B(y_i, \delta_i)$ and $B(y_j, \delta_j)$ is positive whenever $i \neq j$. As a result, for each $x \in X$ there exists $\delta_x > 0$ such that $B(x, \delta_x)$ intersects at most one member from the family $\{B(y_k, \delta_k) : k \in \mathbb{N}\}$. Define $f : X \to \mathbb{R}$ by $f(x) = k - \frac{k}{\delta_k}d(x, y_k)$ if there exists some k with $d(x, y_k) < \delta_k$ and $f(x) = 0$, otherwise. Note that for each $x \in X$, f is Lipschitz on $B(x, \delta_x)$ which implies that f is locally Lipschitz. But f cannot be quasi-Cauchy Lipschitz as $|f(z_{n_k+1}) - f(z_{n_k})| = |2k - 2k + 1| = 1$, where (z_n) is a quasi-Cauchy sequence.

$(3) \Rightarrow (1)$ If possible suppose that $f : (X,d) \to (Y,\rho)$ is a locally Lipschitz function which is not quasi-Cauchy Lipschitz. Then there exist a quasi-Cauchy sequence (x_n) and a subsequence (x_{n_k}) of (x_n) such that $\frac{\rho(f(x_{n_k+1}),f(x_{n_k}))}{d(x_{n_k+1},x_{n_k})} \geq k$ for each $k \in \mathbb{N}$. Now being a BqC sequence, (x_{n_k}) must have a cluster point in X. By passing to a subsequence we can assume that (x_{n_k}) is convergent and consequently (x_{n_k+1}) also converges to the same limit. But note that the restriction of f cannot be Lipschitz to the range of the convergent sequence $(x_{n_k}) \cup (x_{n_k+1})$ which contradicts that f is locally Lipschitz.

□

Our next result concerns with the coincidence of quasi-Cauchy Lipschitz (ward continuous) functions with Cauchy Lipschitz (Cauchy regular) functions.

Theorem 6.3.7. *Let (X,d) be a metric space and $(\widehat{X},\widehat{d})$ be denote the completion of X. Then the following conditions are equivalent.*

1. *Every real-valued Cauchy Lipschitz function defined on X is quasi-Cauchy Lipschitz.*
2. *Every BqC sequence in X has a Cauchy subsequence.*
3. *$(\widehat{X},\widehat{d})$ is Bq-complete.*
4. *Every real-valued Cauchy regular function defined on X is ward continuous.*
5. *Any two Cauchy separated sets cannot be connected through a quasi-Cauchy sequence.*
6. *For a complete subset A and a closed subset B of X with $A, B \neq \emptyset$ and $A \cap B = \emptyset$, A and B cannot be connected through a quasi-Cauchy sequence.*

Proof. $(1) \Rightarrow (2)$ On the contrary, suppose that there is a BqC sequence (x_k) which has no Cauchy subsequence. Therefore one can choose a $\delta > 0$ and passing to a subsequence, we will have $d(x_i, x_j) \geq \delta$ whenever $i \neq j$. Note that by Theorem 6.3.1

one can find a quasi-Cauchy sequence (z_n) such that $z_{n_k} = x_k$ for some subsequence (z_{n_k}) of (z_n). Now define $f : X \to \mathbb{R}$ by $f(x) = k - \frac{4k}{\delta} d(x_k, x)$ if $\exists k \in \mathbb{N}$ such that $x \in B(x_k, \frac{\delta}{4})$, and $f(x) = 0$, otherwise. Clearly, f is Cauchy-Lipschitz as any Cauchy sequence can share an infinite subsequence with at most one open ball from the family $\{B(x_k, \frac{\delta}{4}) : k \in \mathbb{N}\}$ and can only meet finitely many such balls. But f fails to be quasi-Cauchy Lipschitz as $\frac{|f(z_{n_k}) - f(z_{n_k+1})|}{d(z_{n_k}, z_{n_k+1})} = \frac{4k}{\delta}$ for all but finitely many k.

(2) $\iff$ (3) is clear from the fact that a sequence (x_n) is BqC in $\widehat{X}$ implies that there exists a BqC sequence (y_n) in X with $\widehat{d}(x_n, y_n) \to 0$.

(2) $\Rightarrow$ (4) Let $f : (X, d) \to (Y, \rho)$ be Cauchy regular. If possible suppose that f is not ward continuous. Then there exists a quasi-Cauchy sequence (x_n) in X but $(f(x_n))$ is not quasi-Cauchy in Y. Consequently one can find a suitable $\varepsilon > 0$ and a subsequence (x_{n_k}) of (x_n) such that $\rho(f(x_{n_k+1}), f(x_{n_k})) \geq \varepsilon$ for all $k \in \mathbb{N}$. From Cauchy regularity of f we can then conclude that (x_{n_k}) has no Cauchy subsequence, which contradicts the given condition.

(4) $\Rightarrow$ (5) On the contrary suppose that A and B are two Cauchy separated subsets of X. and they can be connected through a quasi-Cauchy sequence. Then there exist a quasi-Cauchy sequence (x_k) in X and a subsequence (x_{k_p}) of (x_k) such that $x_{k_p} \in A$ and $x_{k_p+1} \in B$ for all $p \in \mathbb{N}$. Clearly (x_{k_p}) as well as (x_{k_p+1}) has no Cauchy subsequence. Then we define a function $f : S = \{x_{k_p}, x_{k_p+1} : p \in \mathbb{N}\} \to \mathbb{R}$ such that $f(x_{k_p}) = 0$ and $f(x_{k_p+1}) = 1$ for all $p \in \mathbb{N}$. Since S is closed in $\widehat{X}$, by Tietz extension theorem f can be extended to a continuous function $\widetilde{f}$ on $\widehat{X}$. Then the restriction of $\widetilde{f}$ on X is Cauchy regular but not a ward continuous function.

(5) $\Rightarrow$ (6) Let A and B be two non-empty subsets of X where A is complete and B is closed with $A \cap B = \emptyset$. It is obvious that A and B are Cauchy separated and hence from (5), we can conclude that they cannot be connected through a quasi-Cauchy sequence.

(6) $\Rightarrow$ (1) Let $f : (X, d) \to (Y, \rho)$ be a Cauchy-Lipschitz function. Suppose that

f is not quasi-Cauchy Lipschitz. Then there exists a quasi-Cauchy sequence (x_n) such that $\frac{\rho(f(x_{n_k+1}),f(x_{n_k}))}{d(x_{n_k+1},x_{n_k})} \geq k$ for some subsequence (x_{n_k}) of (x_n). Now (x_{n_k}) cannot have any Cauchy subsequence as f cannot be Lipschitz to the range of the sequence $(x_{n_k}) \cup (x_{n_k+1})$. Now taking $A = \{x_{n_k+1} : k \in \mathbb{N}\}$ and $B = \{x_{n_k} : k \in \mathbb{N}\}$ we obtain two non-empty complete subsets of X, which are connected through a quasi-Cauchy sequence. This contradicts the given assumption and hence f must be quasi-Cauchy Lipschitz.

□

Next, we focus on the coincidence of quasi-Cauchy Lipschitz (ward continuous) functions with Lipschitz in the small (uniform continuous) functions.

Theorem 6.3.8. *Let (X, d) be a metric spaces. Then the following conditions are equivalent.*

1. *Every real-valued quasi-Cauchy Lipschitz function f defined on X is Lipschitz in the small.*
2. *Every sequence (x_n) with $\lim_{n\to\infty} I(x_n) = 0$ has a BqC subsequence in X.*
3. *Every real-valued ward continuous function defined on X is uniformly continuous.*
4. *Any two nonempty subsets $A, B \subset X$ with $d(A, B) = 0$ are connected through a quasi-Cauchy sequence.*
5. *Every W-straight subspace of X is straight.*

Proof. (1) ⇒ (2) On the contrary suppose that there is a sequence (x_n) with $\lim_{n\to\infty} I(x_n) = 0$, which has no BqC subsequence in X. Without any loss of generality by passing through a subsequence we can assume that there exists a sequence (y_n) such that $d(x_n, y_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Then from Theorem 6.3.2, there exists an $\varepsilon > 0$ and passing through a subsequence we have $x_i \notin B^\infty(x_j, \varepsilon)$ whenever $i \neq j$. Let us Choose a sequence of positive real numbers (δ_n) where $\delta_n = \frac{1}{2}\min\{\varepsilon, d(x_n, y_n)\}$. We define a function $f : X \to \mathbb{R}$ by $f(x) = \frac{\delta_n - d(x_n,x)}{\delta_n}$ if there exists $n \in \mathbb{N}$ such that $x \in B(x_n, \delta_n)$ and $f(x) = 0$, otherwise. Note that any quasi-Cauchy sequence in X can share an infinite subsequence with at most one open ball from the family $\{B(x_n, \delta_n) : n \in \mathbb{N}\}$ and at the same time can only meet finitely many balls of this family. Because otherwise there would exist x_i and x_j $(i \neq j)$ which can be joined by an ε-chain, which contradicts our construction. So for each quasi-Cauchy sequence (z_n) in X there exists $k \in \mathbb{N}$ such that $|f(z_{n+1}) - f(z_n)| \leq \frac{1}{\delta_k} d(z_{n+1}, z_n)$ for all but finitely many n. Therefore f is quasi-Cauchy Lipschitz but f cannot be Lipschitz in the small as $f(x_n) = 1$ and $f(y_n) = 0$ for all $n \in \mathbb{N}$, which contradicts (1).

(2) $\Rightarrow$ (3) Let f be a real valued ward continuous function on X. If possible, assume that f is not uniformly continuous. Then there exist $\varepsilon > 0$ and two sequences $(x_n), (y_n)$ such that $d(x_n, y_n) \to 0$ but $|f(x_n) - f(y_n)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Consequently from the given condition, we can construct a quasi-Cauchy sequence (z_k) with the property that $z_{k_p} = x_{n_p}$ and $z_{k_p+1} = y_{n_p}$ for some subsequences $(z_{k_p}), (x_{n_p})$ and (y_{n_p}) of $(z_k), (x_n)$ and (y_n) respectively. But this contradicts the ward continuity of f. Hence f is uniformly continuous.

(3) $\implies$ (4) Suppose that A and B are two nonempty subsets of X with $d(A, B) = 0$, but they cannot be connected through a quasi-Cauchy sequence. Then there exist two sequences $(x_n) \subset A$ and $(y_n) \subset B$ with $d(x_n, y_n) \to 0$. Here (x_n) as well as (y_n) has no BqC subsequence in X. Otherwise passing to a subsequence we would obtain a quasi-Cauchy sequence (z_k) such that $z_{k_n} = x_n$ and $z_{k_n+1} = y_n$ for some subsequence (z_{k_n}) of (z_k). This would contradict our assumption that A, B cannot be connected through a quasi-Cauchy sequence. Then proceeding as (1) $\implies$ (2) we can complete the proof.

(4) $\Rightarrow$ (5) Let $A \subset X$ be a W-straight space and let (C^+, C^-) be a closed cover of A. If possible, suppose that there exists an $\varepsilon > 0$ for which we have $d(C^+_\varepsilon, C^-_\varepsilon) = 0$.

Then from the given condition, C_ε^+ and C_ε^- are connected through quasi-Cauchy sequence, which contradicts the W-straightness of A by Theorem 4.4.4.

(5) $\implies$ (1) Let us assume that $f : X \to Y$ be a quasi-Cauchy Lipschitz function but not a Lipschitz in small function. Then there exist two sequences (x_n) and (y_n) such that $d(x_n, y_n) \to 0$ and $\frac{|f(x_n)-f(y_n)|}{d(x_n,y_n)} \geq n$ for all $n \in \mathbb{N}$. Since f is quasi-Cauchy Lipschitz, it is clear that (x_n) as well as (y_n) has no BqC subsequence in X. Let us take $S = \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$. Now from Theorem 4.4.2, we can conclude that S is WC space and so W-straight. But S is not straight as is evident by taking $C^+ = \{x_n : n \in \mathbb{N}\}$ and $C^- = \{y_n : n \in \mathbb{N}\}$, and noting that $d(C^+, C^-) = 0$, which contradicts (5).

□

In [14, 41] it was shown that the set of all real-valued locally Lipschitz (Cauchy Lipschitz and Lipschitz in the small) functions on an arbitrary metric space (X, d) are uniformly dense in the set of all real-valued continuous (Cauchy regular and uniformly continuous respectively) functions. Interestingly this same pattern follows in the case of the real-valued ward continuous functions also. In our final result, we will establish the density of the set of all real-valued quasi-Cauchy Lipschitz functions on X in the set of all real-valued ward continuous functions.

Theorem 6.3.9. *Let (X, d) be a metric space. Then every real-valued ward continuous function defined on X can be uniformly approximated by real-valued quasi-Cauchy Lipschitz functions.*

Proof. Let f be a real-valued ward continuous function defined on X and let $\varepsilon > 0$ be given. First of all for each $n \in \mathbb{Z}$, let us consider the set $C_n = \{x : (n-1)\varepsilon < f(x) < (n+1)\varepsilon\}$ and let us define a function $g_n(x) = \inf\{1, d(x, X \setminus C_n)\}$. Note that by continuity of f, the family $\{C_n\}_{n \in \mathbb{Z}}$ satisfies the property that for each $x \in X$, there exists $\delta_x > 0$ such that the ball of radius δ_x with centre x is contained in some

C_m. Moreover $C_n \cap C_m = \emptyset$, whenever $|n-m| > 1$. So the function $g(x) = \sum\limits_{n\in\mathbb{Z}} g_n(x)$ is well defined and it satisfies that $\delta_x \leq g(x) \leq 2$ for all $x \in X$.

We define $h : X \to \mathbb{R}$ by $h(x) = \frac{1}{g(x)}(\sum\limits_{n\in\mathbb{Z}} n g_n(x))$. Now we are going to prove that h is quasi-Cauchy Lipschitz. Take a quasi-Cauchy sequence (x_n) in X. We claim that there exists $\delta > 0$ such that $f(B(x_n, \delta)) \subset (f(x_n) - \frac{\varepsilon}{4}, f(x_n) + \frac{\varepsilon}{4})$ for all $n \in \mathbb{N}$. If not, then for each $k \in \mathbb{N}$ there would exist $y_{n_k} \in B(x_{n_k}, \frac{1}{k})$ such that $|f(x_{n_k}) - f(y_{n_k})| \geq \frac{\varepsilon}{4}$. Now let us define a new sequence (z_n) by taking $z_{n_k+1} = y_{n_k}$ for each $k \in \mathbb{N}$ and $z_n = x_n$, otherwise. Clearly, (z_n) is quasi-Cauchy and consequently, this contradicts the fact that f is ward continuous. Therefore for each $n \in \mathbb{N}$ we can choose $m \in \mathbb{Z}$ such that $B(x_n, \delta)$ is contained in C_m and consequently $\delta \leq g(x_n) \leq 2$ for all $n \in \mathbb{N}$. Moreover there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \delta$ for all $n \geq n_0$. Now we will estimate $|h(x_n) - h(x_{n+1})|$ for every $n \geq n_0$. Clearly for each $n \geq n_0$ there exists $m \in \mathbb{Z}$ such that x_n, x_{n+1} both are in C_m and so we can obtain that

$$|g(x_n) - g(x_{n+1})| = |(g_{m-1} + g_m + g_{m+1})(x_n) - (g_{m-1} + g_m + g_{m+1})(x_{n+1})|$$

$$\leq \sum_{i=m-1}^{m+1} |g_i(x_n) - g_i(x_{n+1})| \leq 3d(x_n, x_{n+1}).$$

Now from Theorem 1 of [41], we can conclude that $|h(x_n) - h(x_{n+1})| \leq \frac{10}{\delta^2} d(x_n, x_{n+1})$. Let $\lambda = \max\{\frac{10}{\delta^2}, \sup\{\frac{|h(x_n)-h(x_{n+1})|}{d(x_n,x_{n+1})} : n < n_0, x_{n+1} \neq x_n\}\}$ and so $|h(x_n) - h(x_{n+1})| \leq \lambda d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Finally from Theorem 1 of [41], $|\varepsilon h(x) - f(x)| < \varepsilon$ for every $x \in X$. Hence εh is the required quasi-Cauchy Lipschitz function. □

7

Some Versions of Cauchy Regular Map

In this chapter, we define a reversal of Cauchy regularity, namely Cauchy covering maps and study their related properties. The concept of Cauchy covering maps can be thought also as a generalization of so-called sequence covering maps, which is a reversal of continuity. We also give an almost necessary and sufficient condition for a metric space to be complete in terms of Cauchy covering maps. Then we define some generalizations of these types maps using statistical convergence and also in terms of the more general context of ideal convergence. Finally, we investigate several conditions under which all the maps can relate to each other and discuss various counterexamples. The entire investigation is done in metric space setting.

The content of this chapter is based on the research papers listed below.

- S.K. Pal, **N. Adhikary** and U. Samanta, On ideal sequence covering maps, **Applied General Topology**, 20, no. 2 (2019), 363-377. [64]

- S. K. Pal and **N. Adhikary**, On Cauchy covering maps and complete metric spaces, **Topology Proceedings**, 57 (2021), 1-13. [65]

7.1 Preliminaries

In this section, we first recall some basic facts related to ideal convergence from [9, 55]. A family $\mathcal{I} \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$. Further an admissible ideal $\mathcal{I}$ of Y satisfies $\{x\} \in Y$ for each $x \in Y$. Such ideals are called free ideals. An admissible ideal $\mathcal{I}$ is said to satisfy the condition (AP) (or is called a P-ideal or sometimes an AP-ideal) if, for every countable family of mutually disjoint sets $(A_1, A_2, ...)$ of $\mathcal{I}$, there exists a countable family of sets $(B_1, B_2, ...)$ such that $A_j \triangle B_j$ is finite for each $j \in \mathbb{N}$ and $\bigcup B_k \in \mathcal{I}$. Whereas $\mathcal{I}$ is called maximal if there does not exist any non-trivial proper ideal, properly containing $\mathcal{I}$. An ideal $\mathcal{I}$ is called tall ideal if each infinite subset of $\mathbb{N}$ contains an infinite element of $\mathcal{I}$. If $\mathcal{I}$ is a proper non-trivial ideal in Y (i.e. $Y \notin \mathcal{I}$, $\mathcal{I} \neq \emptyset$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I} : M = Y \setminus A\}$ is called the filter associated with the ideal $\mathcal{I}$. The set $\mathcal{I}_{fin} = \{A \subset \mathbb{N} : A \ is \ finite\}$ is an ideal. On the other hand density of a subset A of $\mathbb{N}$ is defined by $d(A) = \lim\limits_{n\to\infty} \frac{1}{n} |\{k : k \in A, k \leq n\}|$, provided the limit exists and the set $\mathcal{I}_d = \{A : A \subset \mathbb{N}, d(A) = 0\}$ forms an ideal. Throughout this chapter $\mathcal{I}$ be a proper admissible ideal of $\mathbb{N}$.

Definition 7.1.1. *[55] Let (X, d_X) be a metric space and $\mathcal{I}$ be an admissible ideal.*

1. *A sequence (x_n) in X is said to be $\mathcal{I}$-convergent to $x \in X$ (i.e $\mathcal{I}$-$\lim\limits_{n\to\infty} x_n = x$) if for every open neighbourhood U of x, $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$.*

   A sequence (x_n) is said to be statistically convergent to x [36] if it is $\mathcal{I}_d$-convergent to x i.e for every U of x, density of the set $\{n \in \mathbb{N} : x_n \notin U\}$ equals to zero. It is clear that (x_n) convergent to x iff it is $\mathcal{I}_{fin}$-convergent.

2. *A sequence (x_n) in X is called $\mathcal{I}^*$-convergent to x if there exists a set $M = \{m_k : k \in \mathbb{N}\} \subset \mathbb{N}$, $M \subset \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) converges to x..*

Definition 7.1.2. *[32] Let (X, d_X) be a metric space and $\mathcal{I}$ be an admissible ideal.*

1. *A sequence (x_n) in X is said to be $\mathcal{I}$-Cauchy if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\{n : d_X(x_N, x_n) \geq \varepsilon\} \in \mathcal{I}$.*

   (x_n) is called statistical Cauchy if it is $\mathcal{I}_d$-Cauchy.

2. *A sequence (x_n) in X is called $\mathcal{I}^*$-Cauchy if there exists a set $M = \{m_k : k \in \mathbb{N}\} \subset \mathbb{N}$, $M \subset \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_k}) is an ordinary Cauchy sequence.*

Definition 7.1.3. *[56] Let X, Y be two metric space and $f : X \to Y$ be a mapping.*

1. *f is a sequence covering map if for every convergent sequence (y_n) in Y, there is a convergent sequence (x_n) in X with each $x_n \in f^{-1}(y_n)$.*

2. *f is called a sequentially quotient map if for each convergent sequence (y_n) in Y there is a convergent sequence (x_k) in X with $f(x_k) = y_{n_k}$ for each k, where (y_{n_k}) is a subsequence of (y_n)*

3. *f is called compact covering map if every compact subset of Y is the image of some compact subset of X.*

7.2 Main Results

7.2.1 Cauchy covering maps

In this section, we consider the notion of Cauchy covering maps and investigate its relation with the sequence covering maps.

Definition 7.2.1. *Let $f : (X, d_X) \to (Y, d_Y)$ be a mapping.*

1. *f is called a Cauchy covering map if for each Cauchy sequence (y_n) in Y, there is a Cauchy sequence (x_n) in X with each $x_n \in f^{-1}(y_n)$.*

2. *f is called a cofinally Cauchy covering map if whenever (y_n) is a cofinally Cauchy sequence in Y then there is a cofinally Cauchy sequence (x_n) in X with each $x_n \in f^{-1}(y_n)$.*

3. *f is called precompact covering map if every precompact subset of Y is the image of some precompact subset of X.*

Definition 7.2.2. *Let $f : (X, d_X) \to (Y, d_Y)$ be a mapping.*

1. *f is called an $\mathcal{I}$-sequence covering map [64] if for each $\mathcal{I}$-convergent sequence (y_n) in Y, there is an $\mathcal{I}$-convergent sequence (x_n) in X with each $x_n \in f^{-1}(y_n)$.*

2. *f is called an $\mathcal{I}$-Cauchy covering map if for each $\mathcal{I}$-Cauchy sequence (y_n) in Y, there is an $\mathcal{I}$-Cauchy sequence (x_n) in X with each $x_n \in f^{-1}(y_n)$.*

In general, the concepts of Cauchy covering maps and sequence covering maps are independent. The following examples are in this direction.

Example 7.2.1. *Let $f : (0, 1] \to Y$ be a mapping where Y be a convergent sequence (y_n) with its limit y. Let $(0, 1]$ have the usual metric and Y be homeomorphic to the metric space $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with usual metric. f is defined by $f(\frac{1}{2n}) = y_n$ for each $n \in \mathbb{N}$ and $f(x) = y$, otherwise. Then f is a Cauchy covering as well as precompact covering map but not sequence covering and also not compact covering.*

The above example also shows that precompact covering map does not imply the compact covering map.

Example 7.2.2. *Let $f : [1, \infty) \to (0, 1]$ be defined by $f(x) = \frac{1}{x}$. Clearly f is a sequence covering map and compact covering map. But for the Cauchy sequence $(\frac{1}{n})$ the preimage (n) is not Cauchy sequence. So f is not Cauchy covering map and not precompact covering .*

Definitions of Cauchy covering maps and precompact covering maps are analogous but they are mutually independent.

Example 7.2.3. *Consider,* $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 + \frac{1}{n} : n \in \mathbb{N}\}$ *and* $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$ *where* $y_n \to y$. *Define* $f : X \to Y$ *as* $f(\frac{1}{n}) = y_{2n}, f(1 + \frac{1}{n}) = y_{2n-1}$. *Then* f *is precompact covering map but not Cauchy covering map.*

Example 7.2.4. *Let* Y *be any metric space and* X *be the disjoint union of all Cauchy sequences of* Y. *Also let* f *be the natural map from* X *onto* Y. *Then* f *is Cauchy covering but not precompact covering.*

Note 7.2.1. *Consider the metric space* X *in Example 7.2.4. Let* x *be any point in* X *then* x *has a precompact neighbourhood. From this construction, we can say that each metric space is Cauchy covering image of a metric space which has the property that each point has precompact neighbourhood.*

Now we investigate how the Cauchy covering maps are related to other maps.

Theorem 7.2.1. *Suppose* $f : (X, d_X) \to (Y, d_Y)$ *is an one to one Cauchy covering map. Then* f *is a precompact covering map.*

Proof. Let V be a precompact subset of Y and (x_n) be a sequence in $f^{-1}(V)$. If (x_n) has no Cauchy subsequence then so $(f(x_n))$, which contradicts the fact that V is precompact. So $f^{-1}(V)$ is precompact. □

Theorem 7.2.2. *Suppose* $f : (X, d_X) \to (Y, d_Y)$ *be an one to one Cauchy covering map. Then* f *is a sequence covering map.*

Proof. Suppose (y_n) converges to y in Y. Then there exists a Cauchy sequence (x_n) in X with $f(x_n) = y_n$. Consider the sequence (z_k) defined by $z_{2n-1} = y_n$ and $z_{2n} = y$. Then $f(r_k) = z_k$, where $r_{2n-1} = x_n$ and $r_{2n} = x$. So (x_n) converges to $x \in f^{-1}(y)$. Hence f is a sequence covering map. □

Theorem 7.2.3. *Suppose $f : (X, d_X) \to (Y, d_Y)$ is an $\mathcal{I}$-sequence covering map and $\mathcal{I}$ is an admissible ideal and $\mathcal{I}$ is not a tall ideal. Then f is a sequence covering map.*

Proof. Suppose f is an $\mathcal{I}$-sequence covering map and (y_n) is a convergent sequence converges to y in Y. Since $\mathcal{I}$ is not a tall ideal so there is an infinite subset $A = \{n_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ with each $n_k < n_{k+1}$ such that any infinite subset of A does not belongs to $\mathcal{I}$. Now define a new sequence $z_{n_k} = y_k$ and $z_n = y$, otherwise. Clearly (z_n) converges to y. So there exists a sequence (x_n) with each $x_n \in f^{-1}(z_n)$ and it $\mathcal{I}$-converges to a point $x \in f^{-1}(y)$. So for $\varepsilon > 0$ the set $\{n \in \mathbb{N} : d_X(x_n, x) \geq \varepsilon\} \in \mathcal{I}$, which shows that $B = \{n_k : d_X(x_{n_k}, x) \geq \varepsilon\} \in \mathcal{I}$. But $B \subset A$. So B is finite. Hence (x_{n_k}) converges to $x \in f^{-1}(y)$ with each $x_{n_k} \in f^{-1}(y_k)$. Thus f is a sequence covering map. □

For the next Theorem we recall the following Lemma.

Lemma 7.2.1. *[28] If $\mathcal{I}$ is an admissible ideal with property (AP) then in a metric space the concepts $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence coincide.*

Theorem 7.2.4. *Suppose $f : (X, d_X) \to (Y, d_Y)$ is a Cauchy covering map and $\mathcal{I}$ is an AP ideal. Then f is an $\mathcal{I}$-Cauchy covering map.*

Proof. Let (y_n) be an $\mathcal{I}-$Cauchy sequence. Then by Lemma 7.2.1 there is a subset $M = \{m_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ with $M \in \mathcal{F}(\mathcal{I})$ such that (y_{m_k}) is Cauchy. So there is a Cauchy sequence (x_{m_k}) in X with each $x_n \in f^{-1}(y_n)$ and (x_n) is $\mathcal{I}$-Cauchy. Hence f is an $\mathcal{I}$-Cauchy covering map. □

Now we formulate an almost equivalent condition for a metric space to be complete in terms of Cauchy covering maps.

Theorem 7.2.5. *Let X be a metric space. If each Cauchy covering map on X is sequence covering then X is complete.*

Proof. Suppose X is not complete. Then there is a Cauchy sequence (x_n) in X which is not convergent in X. Let $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$ where (y_n) is a convergent sequence of distinct points with its limit y. Define a function $f : X \to Y$ as $f(x_{2n}) = y_n$ for each $n \in \mathbb{N}$ and $f(x) = y$, otherwise. Now any Cauchy sequence (z_n) in Y is a subsequence of (y_n). So choose $r_n \in f^{-1}(z_n)$ is a subsequence of (x_n). Therefore f is a Cauchy covering map but not a sequence covering map, which is a contradiction. Hence X is complete. □

The converse of the theorem will be true if we impose the continuity on f. Hence our next result has been described below with the necessity of its condition.

Theorem 7.2.6. *Suppose (X, d_X) is a complete metric space and $f : (X, d_X) \to (Y, d_Y)$ is a continuous Cauchy covering map. Then f is a sequence covering map.*

Proof. Suppose (y_n) converges to y in Y. Then there is a Cauchy sequence (x_n) with each $x_n \in f^{-1}(y_n)$ in X. Since X is complete so x_n is convergent to some point x in X. Now from continuity of f we get $f(x_n) \to f(x)$. So $f(x) = y$. Hence f is a sequence covering map. □

The following example asserts that continuity is necessary for the above theorem.

Example 7.2.5. *Suppose $X = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ where each point of X is distinct and $x_n \to x$ and $Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0, 2\}$. Now $f : X \to Y$ is defined by $f(x) = 2$, $f(x_{2n}) = \frac{1}{n}$ and $f(x_{2n-1}) = 0$ for each $n \in \mathbb{N}$. Clearly X is complete. Any Cauchy sequence (z_n) in Y is either eventually constant or contain a subsequence of $\{\frac{1}{n}\}$ and 0. So in the second case $r_n \in f^{-1}(z_n)$ is a subsequence of (x_n). Hence f is a Cauchy covering map. But f is not continuous and not sequence covering map.*

Further, completeness of X is necessary for the above theorem. The following example is in this direction.

Example 7.2.6. *Let $\bigwedge = \{\alpha : \alpha$ is an increasing function from $\mathbb{N}$ into $\mathbb{N}\}$ and $X_\alpha = \{\frac{1}{n} : n \in \mathbb{N}\} \times \{\alpha\}$. Also let (X_α, d_α) be a metric space where $d_\alpha((\frac{1}{m}, \alpha), (\frac{1}{n}, \alpha)) = |\frac{1}{m} - \frac{1}{n}|$. Put $X = \bigoplus_{\alpha \in \bigwedge} X_\alpha$ and the metric D on X is defined by $D(x,y) = min\{|x-y|, 1\}$ if $x, y \in X_\alpha$ and $D(x,y) = 1$ if $x \in X_\alpha, y \in X_\beta$ for $\alpha \neq \beta$. Take a real valued convergent sequence (y_n) converges to y and consider $Y = \{y_n : n \in \mathbb{N}\} \cup \{y\}$ with the usual metric and each point of Y distinct. Clearly, X is not complete. Now $f : X \to Y$ is defined by $f(\frac{1}{2k}, \alpha) = y_{\alpha(k)}$ and $f(\frac{1}{2k-1}, \alpha) = y$ for each $\alpha \in \bigwedge$. Any Cauchy sequence (z_n) in Y contains a subsequence $(y_{\alpha(k)})$ and y. So choose $r_n \in f^{-1}(z_n)$ which is a subsequence of $((\frac{1}{n}, \alpha))$. Hence (r_n) is Cauchy and hence f is Cauchy covering. Clearly f is continuous but not a sequence covering map. Hence the completeness of X is necessary for the above theorem.*

The following example shows that in general, cofinally Cauchy sequence may not have Cauchy subsequence.

Example 7.2.7. *Let $\mathbb{N} = \bigcup_{k=1}^{\infty} A_k$ where each A_k is infinite and $A_i \cap A_j = \phi$ for $i \neq j$. Also let $A_i = \{a_{i1}, a_{i2}, ...\}$. If $n \in A_i$, $n \neq a_{i1}$ put $x_{n,a_{i1}} = 2$, $x_{n,n} = \frac{1}{i}$ and $x_{n,k} = 0$ for other $k \in \mathbb{N}$. If $n = a_{i1}$ put $x_{n,a_{i1}} = 10^i$ and $x_{n,k} = 0$ for other $k \in \mathbb{N}$. Let $z_n = (x_{n,k})$. Now for $\varepsilon > 0$ there exists n_0 such that $\frac{1}{n_0} < \varepsilon$ and then for all $m, n \in A_{n_0}$, $||z_m - z_n|| = \frac{1}{n_0} < \varepsilon$. So (z_n) is a cofinally Cauchy sequence. But (z_n) has no Cauchy subsequence.*

Generally, the concepts of cofinally Cauchy covering maps and Cauchy covering maps are independent.

Example 7.2.8. *$f : \mathbb{R} \to \{z_n : n \in \mathbb{N}\}$ is defined by $f(n) = z_n$ for all $n \in \mathbb{N}$ and $f(x) = z_1$ otherwise, where (z_n) is a sequence described in the above example. Clearly, f is a Cauchy regular and Cauchy covering map but not a cofinally Cauchy covering map.*

Example 7.2.9. *Let $\mathcal{F}$ be the set of all one to one mapping from $\mathbb{N}$ to $\mathbb{N}$. For each $f \in \mathcal{F}$ consider S_f be a cofinally Cauchy sequence but not Cauchy sequence. $S_f = \{x_{f,n} : n \in \mathbb{N}\}$ where for all $n \in \mathbb{N}$ $x_{f,2n} = r_f$, $r_f \in \mathbb{R}$ and $x_{f,2n+1} = n$ for each $n \in \mathbb{N}$. $X = \bigoplus_{f \in \mathcal{F}} S_f$*

and $Y = \{\frac{1}{n} : n \in \mathbb{N}\}$. f is defined by $f(x_{f,n}) = \frac{1}{n}$. So f is Cauchy regular and cofinally Cauchy covering but not Cauchy covering.

Theorem 7.2.7. *Let (X, d_X) be a cofinally complete metric space. If $f : (X, d_X) \to (Y, d_Y)$ is a continuous and cofinally Cauchy covering map. Then f is a sequentially quotient and hence quotient map.*

Proof. Suppose $y_n \to y$ in Y. Then there exist a cofinally Cauchy sequence (x_n) in X with each $x_n \in f^{-1}(y_n)$. As X is cofinally complete, so there exist a subsequence (x_{n_k}) converges to some point x in X. By continuity of f, $f(x_{n_k}) \to f(x)$ which implies $x \in f^{-1}(y)$. So f is sequentially quotient and hence quotient map. □

Now we discuss the necessity of the condition of the above theorem. The necessity of continuity is proved in Example 7.2.5. Here f is not sequentially quotient map (Because if we take the sequence $(\frac{1}{n})$ then the only pre-image $(x_{2n}) \to x$. But $x \notin f^{-1}(0)$). The following Example will prove the necessity of cofinally completeness.

Example 7.2.10. *Let (x_n) be the sequence described in Example 7.2.7 and X and Y be same as the Example 7.2.6 just replaced the sequence $(\frac{1}{n})$ by the above sequence (x_n). Clearly X is not cofinally complete. $f : X \to Y$ is defined by $f(x_1, \alpha) = y, f(x_k, \alpha) = y_{\alpha(k)}$ for $k \geq 1$. For cofinally Cauchy sequence (z_n) in Y contains a subsequence $(y_{\alpha(k)})$. We choose $r_n \in f^{-1}(z_n)$ such that (r_n) contain $((x_k, \alpha))_{k \in \mathbb{N}}$. Hence f is cofinally Cauchy covering map. But f is not sequentially quotient.*

Note 7.2.2. *If we consider a continuous map $f : (X, d_X) \to (Y, d_Y)$ such that X, Y complete. Then sequence covering and Cauchy covering are equivalent. Also, precompact and compact covering are equivalent.*

7.2.2 Cauchy regular functions

In this section, we consider some generalized Cauchy regular functions and investigate how Cauchy regular function are related to these types of functions. First, we consider the following definitions:

Definition 7.2.3. *[28] Let (X, d_X) and (Y, d_Y) be two metric spaces and $\mathcal{I}$ be an admissible ideal.*

1. *A function $f : (X, d_X) \to (Y, d_Y)$ is called bounded continuous iff for each bounded sequence (x_n) in X, $(f(x_n))$ is a bounded sequence in Y.*
2. *A sequence (x_n) in X is called $\mathcal{I}$-bounded in X if there exist an element $x \in X$ and a positive real number r such that $\{n : d_X(x_n, x) \geq r\} \in \mathcal{I}$.*
3. *A function $f : (X, d_X) \to (Y, d_Y)$ is called $\mathcal{I}$-bounded continuous (Statistical bounded continuous) iff for each $\mathcal{I}$-bounded (Statistically bounded) sequence (x_n) in X, $(f(x_n))$ is a $\mathcal{I}$-bounded (Statistically bounded) sequence in Y.*

Definition 7.2.4. *Let $\mathcal{I}$ be an admissible ideal. A function $f : (X, d_X) \to (Y, d_Y)$ is called an $\mathcal{I}$-Cauchy regular (Statistical Cauchy regular) iff for each $\mathcal{I}$-Cauchy (Statistical Cauchy) sequence (x_n) in X, $(f(x_n))$ is $\mathcal{I}$-Cauchy (Statistical Cauchy) sequence in Y. (It is known as $\mathcal{I}$-uniformly continuous in [43])*

Now we establish the relation of $\mathcal{I}$-Cauchy regular map with Cauchy regularity as well as continuity. In general $\mathcal{I}$-Cauchy regular does not imply the Cauchy regularity.

Example 7.2.11. *Suppose $\mathcal{I}$ be an admissible maximal ideal. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and $Y = \{0, 2\}$ with the discrete metric. Let $f : (X, d_X) \to (Y, d_Y)$ be defined by $f(\frac{1}{2n}) = 2$ and $f(x) = 0$ otherwise. Then f is $\mathcal{I}$-Cauchy regular as every sequence in Y is $\mathcal{I}$-Cauchy (from Note7.2.3, which discuss in later). But f is not continuous and hence not Cauchy regular.*

Theorem 7.2.8. *Let $f : (X, d_X) \to (Y, d_Y)$ be a mapping. f is statistical Cauchy regular iff Cauchy regular.*

Proof. Suppose f is Cauchy regular. Let (x_n) be a statistical Cauchy sequence in X. Then by Lemma 3.9 there exists $M \subset \mathbb{N}$ with density of M, $d(M) = 1$ and $(x_n)_{n \in M}$ is Cauchy sequence. So $(f(x_n))_{n \in M}$ is Cauchy sequence. Hence $(f(x_n))_{n \in \mathbb{N}}$ is statistical Cauchy sequence.

Conversely, suppose that f is a statistical Cauchy regular map and (x_n) is a Cauchy sequence in X. Since every subsequence (x_{n_k}) of (x_n) is Cauchy so corresponding $(f(x_{n_k}))$ is statistical Cauchy. So every subsequence of $(f(x_n))$ is statistical Cauchy. Now we will show that $(f(x_n))$ is a Cauchy sequence.

Suppose $(f(x_n))$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there exist m_k and n_k such that $d_X(f(x_{m_k}), f(x_{n_k})) \geq \varepsilon$.

Now define a new sequence $y_{2k-1} = f(x_{m_k})$ and $y_{2k} = f(x_{n_k})$ for each $k \in \mathbb{N}$. Then we have $d_X(y_{2k-1}, y_{2k}) \geq \varepsilon$ for each $k \in \mathbb{N}$. Next we will show that (y_k) is not statistical Cauchy. Suppose (y_k) is statistical Cauchy. So there is $P \subset \mathbb{N}$ with $d(P) = 1$, $(y_k)_{k \in P}$ is Cauchy. Now there is infinitely many k such that P contains both $2k - 1$ and $2k$, otherwise density of P, $d(P) \leq \frac{1}{2}$. Therefore for infinitely many k with $2k - 1, 2k \in P$ and $d_X(y_{2k-1}, y_{2k}) \geq \varepsilon$, which contradicts the fact that $(y_k)_{k \in P}$ is Cauchy. Hence (y_k) is not statistical Cauchy, Which again contradicts that every subsequence of $(f(x_n))$ is statistical Cauchy. Thus $(f(x_n))$ is a Cauchy sequence. Hence f is Cauchy regular. □

Theorem 7.2.9. *Let $f : (X, d_X) \to (Y, d_Y)$ be a mapping and $\mathcal{I}$ be an admissible ideal. Then f is $\mathcal{I}$-bounded continuous iff bounded continuous.*

Proof. Let f be a bounded continuous function and (x_n) be an $\mathcal{I}$-bounded sequence in X. Then there exists $M \subset \mathbb{N}$ with $M \in \mathcal{F}(\mathcal{I})$ and $(x_n)_{n \in M}$ is a bounded sequence. So $(f(x_n))_{n \in M}$ is bounded sequence. Hence $(f(x_n))_{n \in \mathbb{N}}$ is $\mathcal{I}$-bounded sequence.

Conversely, let f be $\mathcal{I}$-bounded continuous and (x_n) be a bounded sequence in X. Since every subsequence (x_{n_k}) of (x_n) is bounded so corresponding $(f(x_{n_k}))$ is $\mathcal{I}$-bounded. So every subsequence of $(f(x_n))$ is $\mathcal{I}$-bounded. Suppose $(f(x_n))$ is not bounded. Then there is an element x in X and for each $k \in \mathbb{N}$ there is $f(x_{n_k})$ such that $d_X(f(x_{n_k}), x) \geq k$. Now for any positive real number a there is $k \in \mathbb{N}$ such that $k > a$. So we have $d_X(f(x_{n_i}), x) \geq k > a$, for all $i \geq k$. Hence $\{k : d_X(f(x_{n_k}), x) \geq a\} \notin \mathcal{I}$. Thus $(f(x_{n_k}))$ is not $\mathcal{I}$-bouded in X, which contradicts that every subsequence of $(f(x_n))$ is $\mathcal{I}$-bounded. Therefore $(f(x_n))$ is a bounded sequence. Hence f is bounded continuous. □

Lemma 7.2.2. *[28] Let $\mathcal{I}_0$ be an admissible ideal. Then $\mathcal{I}_0$ is a maximal ideal if and only $A \in \mathcal{I}_0 \vee \mathbb{N} \setminus A \in \mathcal{I}_0$ for each $A \subset \mathbb{N}$.*

Note 7.2.3. *One can observe the following facts.*

- *Let (X, d_X) be a metric space and (x_n) be a sequence in X. If each subsequence of (x_n) is statistical Cauchy then (x_n) is a Cauchy sequence. But in general this is not true for any admissible ideal. Suppose $\mathcal{I}$ be a maximal ideal. A sequence (x_n) is defined by $x_{2k-1} = 0$ and $x_{2k} = 2$ for each $k \in \mathbb{N}$. Clearly (x_n) is not a Cauchy sequence. Let (x_{n_k}) be a subsequence. Then there exists a subset A of $\mathbb{N}$ such that $x_{n_k} = 0$, for $k \in A$ and $x_{n_k} = 2$, otherwise. From Lemma 7.2.2 either $A \in \mathcal{I}$ or $A^c \in \mathcal{I}$. If $A \in \mathcal{I}$ then $(x_{n_k})-_{k \in A^c}$ is a constant sequence where $A^c \in \mathcal{F}(I)$. So (x_{n_k}) is an $\mathcal{I}-$Cauchy sequence. The other case is similar. So every subsequence of (x_n) is $\mathcal{I}$-Cauchy.*

- *If we consider the case of $\mathcal{I}$-boundedness the result also holds good. Let (X, d_X) be a metric space and (x_n) be a sequence in X. If each subsequence of (x_n) is $\mathcal{I}$-bounded then (x_n) is bounded.(Suppose (x_n) is not bounded. Then there is an element x in X and for each $k \in \mathbb{N}$ there is (x_{n_k}) such that $d_X(x_{n_k}, x) \geq k$. Now for any positive real number a there is $k \in \mathbb{N}$ such that $k > a$. So we have $d_X(x_{n_i}, x) \geq k > a$ for all $i \geq k$.*

So $\{k : d_X(x_{n_k}, x) \geq a\} \notin \mathcal{I}$. So (x_{n_k}) is not $\mathcal{I}$-bounded in X, which contradicts that every subsequence of (x_n) is $\mathcal{I}$-bounded. So (x_n) is a bounded sequence.

- *Further if we consider the case of $\mathcal{I}$-convergent then the result is also true. Let (X, d_X) be a metric space and (x_n) be a sequence in X. If each subsequence of (x_n) is $\mathcal{I}$-Convergent to the same limit then (x_n) is convergent to that limit. (Suppose (x_k) is $\mathcal{I}$-convergent to x in X but not convergent to x in X. Then there exists $\varepsilon > 0$ such that there is a subsequence (x_{n_k}) with $d_X(x_{n_k}, x) \geq \varepsilon$, for each $k \in \mathbb{N}$. Put $y_k = x_{n_k}$ for each $k \in \mathbb{N}$. So (y_k) is a subsequence of (x_n) that can not be $\mathcal{I}$-convergent to x. So (x_n) is convergent to x.)*

Theorem 7.2.10. *Let $f : (X, d_X) \to (Y, d_Y)$ be a mapping and $\mathcal{I}$ be an admissible ideal, satisfy condition AP. Then f is Cauchy regular implies f is $\mathcal{I}$-Cauchy regular.*

Proof. Let f be a Cauchy regular map and (x_n) is an $\mathcal{I}$-Cauchy sequence. Then by 7.2.1 there is $M = \{n_k : k \in \mathbb{N}\} \subset \mathbb{N}$ with $M \in \mathcal{F}(\mathcal{I})$ such that (x_{n_k}) is a Cauchy sequence. Therefore $(f(x_{n_k}))$ is a Cauchy sequence and which implies that $(f(x_n))$ is an $\mathcal{I}$-Cauchy sequence. This completes the proof. □

The following theorem classified all those ideals for which $\mathcal{I}$-Cauchy regularity of a function is equivalent to the continuity of f.

Theorem 7.2.11. *Let $f : (X, d_X) \to (Y, d_Y)$ be a mapping and $\mathcal{I}$ be an admissible ideal and there exists disjoint set A_1, A_2 with for each $i = 1, 2$ $A_i \subset \mathbb{N}$, also $A_i \notin \mathcal{I}$ and $\mathbb{N} = A_1 \cup A_2$. Then f is $\mathcal{I}$-Cauchy regular implies f is continuous.*

Proof. Suppose f is $\mathcal{I}$-Cauchy regular but f is not continuous. Then there is a convergent sequence (x_n) converges to x such that $(f(x_n))$ is not converges to $f(x)$. So there exist $\varepsilon > 0$ and a subsequence $(f(x_{n_k}))$ such that $d_Y(f(x_{n_k}), f(x)) > \varepsilon$ for each $k \in \mathbb{N}$. Define a new sequence $z_k = x_{n_k}$ if $k \in A_1$ and $z_k = x$ if $k \in A_2$. Clearly

(z_k) is $\mathcal{I}$-Cauchy sequence in X. Now for each $k \in \mathbb{N}$ we have two cases either $k \in A_1$ or $k \in A_2$. If $k \in A_1$ then $f(z_k) = f(x_{n_k})$ and so $A_2 \subset \{n : d_Y(f(z_n), f(z_k)) > \varepsilon\} \notin \mathcal{I}$. Next if $k \in A_2$ then $f(z_k) = f(x)$ and so $A_1 \subset \{n : d_Y(f(z_n), f(z_k)) > \varepsilon\} \notin \mathcal{I}$. So $(f(z_k))$ cannot be $\mathcal{I}$-Cauchy, Which contradicts that f is $\mathcal{I}$-Cauchy regular. Hence f is continuous. □

Note 7.2.4. *For an AP ideal satisfying the property mentioned in the above theorem, we have Cauchy regular* $\Rightarrow$ $\mathcal{I}$*-Cauchy regular* $\Rightarrow$ *continuous. But continuous does not imply* $\mathcal{I}$*-Cauchy regular for any admissible ideal.*

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