EUCLIDEAN OPERATOR RADIUS INEQUALITIES OF HILBERT SPACE OPERATORS AND THEIR APPLICATIONS

Suvendu Jana
(Index No.: 77/21/Math./27)

THIS THESIS IS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF DOCTOR OF PHILOSOPHY IN SCIENCE

DEPARTMENT OF MATHEMATICS
JADAVPUR UNIVERSITY
KOLKATA-700 032
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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled "**EUCLIDEAN OPERATOR RADIUS INEQUALITIES OF HILBERT SPACE OPERATORS AND THEIR APPLICATIONS**" submitted by **Suvendu Jana** who got his name registered on 18/03/2021 (**Index No.: 77/21/Math./27**) for the award of Ph.D. (Science) degree of Jadavpur University, is absolutely based upon his own research work under the supervision of **Prof. Kallol Paul**, Department of Mathematics, Jadavpur University, Kolkata 700032, India and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

PROF. KALLOL PAUL
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Kallol Paul 28/08/2023
....

(Prof. Kallol Paul)
(Signature of the Supervisor
and date with official seal)

Dedicated to my mother ***Mrs. Kajal Jana***
and my father ***Mr. Shyamal Kumar Jana***

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Abstract

The numerical radius $w(T)$ of a bounded linear operator T defined on a complex Hilbert space $\mathcal{H}$, is defined as the radius of the smallest circular disc with centre at the origin that contains the numerical range, i.e, $w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$. Recall that the numerical range $W(T)$ of T is defined as the subset of complex plane $\mathbb{C}$ whose elements are the image of the unit circle of Hilbert space $\mathcal{H}$ under the continuous mapping $x \longmapsto \langle Tx, x\rangle$ from $\mathcal{H}$ to $\mathbb{C}$, i.e., $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$. There are several generalizations of numerical radius, one of them is Euclidean operator radius. Euclidean operator radius of any d-tuple operator $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ is defined as $w_e(\mathbf{T}) = \sup\left\{\left(\sum_{k=1}^{d} |\langle T_k x, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1\right\}$, where $\mathbb{B}^d(\mathcal{H})$ is the collection of all d-tuple of bounded linear operators defined on $\mathcal{H}$. The main focus of this thesis is to develop stronger lower and upper bounds of the numerical radius and Euclidean operator radius using various technique. Applying Euclidean operator radius inequalities we obtain various numerical radius inequalities which are finer than existing numerical radius inequalities. Among many inequalities, we obtain improvements and generalizations of the inequalities $\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|$. Then we obtain new bounds for the zeros of a complex monic polynomial $p(z)$ of higher degree by applying the bounds of numerical radius developed here. Next we study the Euclidean operator radius inequalities of a pair of bounded linear operators which improve existing ones. Then we present bounds for the numerical radius of bounded linear operators which generalize and improve on the well-known numerical radius bounds using Euclidean operator radius inequalities. We also study generalized Euclidean operator radius inequalities and their applications. Next we obtain power inequality for d-tuple operator and applying these power inequality we obtain the Euclidean operator radius bounds for product of two d-tuple operators. Finally we obtain bounds for Euclidean operator radius of d-tuple operator and of $n \times n$ operator matrix whose entries are d-tuple operators.

CONTENTS

CHAPTER 1

INTRODUCTION

The study of quadratic forms was the object of chief interest in the early studies of Hilbert space by renowned mathematicians such as Hilbert, Hellinger, Toeplitz, and others. Quadratic forms occupy a central place in various branches of Mathematics like number theory, linear algebra, group theory, differential geometry etc. In linear algebra, the concept of a quadratic form associated with a matrix and its applications are quite well known. Extension of the idea of quadratic form in both finite and infinite dimensional spaces comprises the theory of numerical range and numerical radius. The numerical range when considered on finite dimensional spaces is sometimes referred to as Field of values, a term commonly used in matrix theory. The first, that is, numerical range is mostly preferred by operator theorists. The concept of numerical range was initiated by Toeplitz and Hausdorff in 1918 for matrices, that is, in finite dimensional spaces, but this definition is equally applicable to operators on infinite dimensional Hilbert spaces. The study of numerical range, numerical radius and its generalizations and applications has great deal of research interest in many branches of pure and applied mathematics such as operator theory, functional analysis, Banach algebra, matrix norm, inequalities, numerical analysis, perturbation theory, quantum computing etc. The numerical radius and the distance of numerical range to the origin are used in studying perturbation, stability, convergence and approximation problems. In particular, as an example, very often, the numerical radius has been used as a reliable indicator for rate of convergence of iterative methods. It also plays a crucial role in the stability analysis of finite difference approximations of solutions to hyperbolic initial value problems. Furthermore, numerical radius has recently been associated with stability issues of Hermitian generalized eigen-problems and of higher order dynamical systems. The numerical range is used as a rough estimate of eigenvalues of an operator. There are several generalization of numerical range, such as joint numerical range. In connection with joint numerical range, one such natural multivariable generalization of the classical numerical radius of a bounded linear operator is Euclidean operator radius which is useful in various theoretical and applied subjects, in particular, in control system. Let us now turn attention to the word "inequalities". Inequality has been studied in various branches of Mathematics over the years. Two books about inequalities, the first one written by G. H. Hardy, J. Littlewood and J. Polya in 1934 and the second one written by E. Bechanbach and R. Bellman in 1961, turned the field of inequalities into a well organized field and provided motivations, ideas, techniques and applications for new research. Before proceeding further we now introduce notations and terminologies.

1.1 Introduction and preliminaries

First we introduce inner product space which is nothing but generalization of Euclidean space.

Definition 1.1. *Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}\,(=\mathbb{R}$ or $\mathbb{C})$. An inner product on $\mathbb{V}$ is a function that assigns to every ordered pair of vectors x and y in $\mathbb{V}$ to a scalar in $\mathbb{F}$, denoted $\langle x, y\rangle$, such that for all x, y and z in $\mathbb{V}$ and all $c \in \mathbb{F}$, the following hold:*

$$(i)\quad \langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle,$$
$$(ii)\quad \langle x, x\rangle \geq 0,$$
$$(iii)\quad \langle x, x\rangle = 0, \text{ if and only if } x = 0,$$
$$(iv)\quad \langle cx, y\rangle = c\langle x, y\rangle,$$
$$(v)\quad \overline{\langle x, y\rangle} = \langle y, x\rangle,$$

where $\overline{\langle x, y\rangle}$ denotes the complex conjugate of $\langle x, y\rangle$. Note that (v) reduce to $\langle x, y\rangle = \langle y, x\rangle$ if $\mathbb{F} = \mathbb{R}$. A vector space $\mathbb{V}$ equipped with such an inner product $\langle\cdot,\cdot\rangle$ is known as inner product space and is denoted as $(\mathbb{V}, \langle\cdot,\cdot\rangle)$.

If $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ is an inner product space then it is easy to see that the function $\|\cdot\| : \mathbb{V} \to \mathbb{R}$ defined by $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ for all $x \in \mathbb{V}$ satisfies the following:

- $\|x\| \geq 0$ for all $x \in \mathbb{V}$ (non-negativity), and $\|x\| = 0$ if and only if $x = 0$.
- $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{V}$ (triangle inequality).
- $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ and for all $x \in \mathbb{V}$ (homogeneity).

Therefore, the function $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle$ satisfies all the conditions of a norm and so $(\mathbb{V},\|\cdot\|)$ is a normed linear space. In general, a vector space $\mathbb{V}$ is said to be a normed linear space if there is a function $\|\cdot\|$ on $\mathbb{V}$ satisfying the above three properties.
Now we recall few basic properties on an inner product space.

1. (**Cauchy-Schwarz inequality**) Let $(\mathbb{V},\langle\cdot,\cdot\rangle)$ be an inner product space. Then

$$|\langle x,y\rangle| \leq \|x\|\|y\| \quad \text{for all} \quad x,y\in\mathbb{V}.$$

2. (**Parallelogram law**) Let $(\mathbb{V},\langle\cdot,\cdot\rangle)$ be an inner product space. Then

$$\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2) \quad \text{for all} \quad x,y\in\mathbb{V}.$$

3. (**Polarization identity**) Let $(\mathbb{V},\langle\cdot,\cdot\rangle)$ be an inner product space. Then

$$4\langle x,y\rangle = \|x+y\|^2-\|x-y\|^2+i\left(\|x+iy\|^2-\|x-iy\|^2\right) \quad \text{for all} \quad x,y\in\mathbb{V}.$$

A Hilbert space is an inner product space $(\mathbb{V},\langle\cdot,\cdot\rangle)$ such that the space is complete with respect to the metric $d(x,y)=\|x-y\|=\langle x-y,x-y\rangle^{\frac{1}{2}}$ for all $x,y\in\mathbb{V}$, induced from the inner product $\langle\cdot,\cdot\rangle$. Throughout, we restrict the symbol $\mathcal{H}$ for a complex Hilbert space with inner product $\langle .,.\rangle$. Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$. The norm induced by the inner product $\langle .,.\rangle$ is denoted by $\|\cdot\|$. The numerical range of a bounded linear operator $T\in\mathbb{B}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ is a subset of complex plane $\mathbb{C}$ is defined by

$$W(T)=\{\langle Tx,x\rangle, x\in\mathcal{H}, \|x\|=1\},$$

and the numerical radius of $T\in\mathbb{B}(\mathcal{H})$ is defined as

$$w(T)=\sup\{|\lambda| : \lambda\in W(T)\}.$$

Thus, the numerical range $W(T)$ is the image of unit circle in $\mathcal{H}$ under the quadratic form $f(x)=\langle Tx,x\rangle$ from $\mathcal{H}$ to $\mathbb{C}$ and the numerical radius of T is the smallest radius of a circular disc centred at the origin which contains the numerical range $W(T)$. It is clear from the definition of the numerical range and numerical radius that these two are nearly related. For $T\in\mathbb{B}(\mathcal{H})$, T^* denotes the adjoint of T and $|T|$ stands for the positive operator $(T^*T)^{1/2}$. The Cartesian decomposition of $T\in\mathbb{B}(\mathcal{H})$ is given by $T=\Re(T)+i\Im(T)$, where $\Re(T)$ and $\Im(T)$ denote the real part and the imaginary part of an operator T respectively, that is, $\Re(T)=\frac{1}{2}(T+T^*)$

and $\Im(T) = \frac{1}{2i}(T - T^*)$. The spectrum of an operator $T \in \mathbb{B}(\mathcal{H})$, denoted by $\sigma(T)$, is the complement of the resolvent set, i.e, $\sigma(T) = \mathbb{C} \setminus \rho(T)$, where $\rho(T)$ is the resolvent of T. Note that the resolvent set $\rho(T)$ is defined as the collection of all scalars λ for which $(T - \lambda I)^{-1}$ exists as a bounded linear operator on $\mathcal{H}$. The spectral radius $r(T)$ of a bounded linear operator T is the supremum of the absolute values of the elements of its spectrum, i.e,

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

The spectral radius of a bounded linear operator T is connected with the norms of its power by the formula $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$, where $\|T\|$ denotes the operator norm of T. Recall that $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$. If T is self-adjoint, i.e, $T = T^*$, then $\|T\|^2 = \|T^*T\| = \|T^2\|$ and so, by induction, $\|T\|^{2^n} = \|T^{2^n}\|$. Therefore, for the self-adjoint operator T, $r(T) = \lim_{n\to\infty} \|T^{2^n}\|^{1/2^n} = \|T\|$.

The properties of the numerical range $W(T)$ are given below :

(i) $W(\alpha I + \beta T) = \alpha + \beta W(T)$ for all $\alpha, \beta \in \mathbb{C}$.

(ii) $W(T^*) = \{\bar{\lambda} : \lambda \in W(T)\}$.

(iii) $W(U^*TU) = W(T)$ for every *unitary* operator $U \in \mathbb{B}(\mathcal{H})$.

(iv) (Ellipse lemma, [51, Lemma 1.1-1]) If T is an operator on a two-dimensional space $\mathcal{H}$, then $W(T)$ is an ellipse whose foci are the eigenvalues of T.

(V) (Toeplitz-Hausdorff theorem, [51, Th. 1.1-2]) The numerical range of an operator is convex.

(vi) The numerical range is a compact subset of $\mathbb{C}$, if the space $\mathcal{H}$ is finite-dimensional.

(vii) $W(T)$ is a real segment $[\alpha, \beta]$ if and only if T is a Hermitian matrix with its smallest and the largest eigenvalues being α and β, respectively.

Thus, geometric properties of numerical ranges helps us classify special types of operators, for example, the self adjoint, normal, unitary among others. Also given the numerical range of an operator, one is capable of making deductions on the properties of the operator, both algebraic and analytic .

Another fundamental theorem on numerical range is known as spectral inclusion theorem, which reads as follows.

Theorem 1.1. (*Spectral inclusion theorem, [51, p. 6]*) *Let* $T \in \mathbb{B}(\mathcal{H})$*. Then* $\sigma(T)$ *is contained in the closure of* $W(T)$*, that is,* $\sigma(T) \subseteq \overline{W(T)}$*.*

Thus, the numerical range is useful to locate the spectrum of an operator since the spectrum is known to be contained within closure of numerical range of the operator. The bounds for the numerical radius of a bounded linear operator have been studied by many mathematicians in recent years and a considerable improvement of the same has been obtained. From Spectral inclusion theorem it is easy to say that the spectral radius $r(T)$ of T always satisfies $r(T) \leq w(T)$. A basic property for the numerical radius is that it satisfies the power inequality, i.e., for $T \in \mathbb{B}(\mathcal{H})$, $w(T^n) \leq w^n(T)$ *for all* $n \in \mathbb{N}$. Here, $\mathbb{N}$ denotes the set of all natural numbers. Now we give a classical bound for numerical radius of a bounded linear operator $T \in \mathbb{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

Therefore, the numerical radius norm is equivalent to the operator norm on $\mathbb{B}(\mathcal{H})$. Let us note here that $w(\cdot)$ fails to be a norm if the Hilbert space is considered over the real field. The inequalities in (1.1) are sharp, $w(T) = \|T\|$ if T is normal (i.e., $T^*T = TT^*$) and $w(T) = \frac{1}{2}\|T\|$ if $T^2 = 0$. Therefore, The upper and lower norm bounds, dilations with simple structure, among others can also be obtained for some special class of operators. Here we note some important improvement of the inequalities for the numerical radius of a bounded linear operator T. In [64], Kittaneh improved the upper bound in (1.1) by establishing that for $T \in \mathbb{B}(\mathcal{H})$,

$$w(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{\|T^2\|}\right). \tag{1.2}$$

Again, Kittaneh [63] improved the inequalities in (1.1) by establishing that

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{1.3}$$

Observe that the upper bounds in (1.2),(1.3) are not comparable, in general. In [42], Dragomir obtained an another inequality, namely, for $T \in \mathbb{B}(\mathcal{H})$,

$$w^2(T) \leq \frac{1}{2}\left(\|T\| + w(T^2)\right), \tag{1.4}$$

which improve on the right hand inequality in (1.1). Further, Abu-Omar and Kittaneh in [2] obtained that for $T \in \mathbb{B}(\mathcal{H})$,

$$\frac{1}{2}c(T^2) + \frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{4}\|T^*T + TT^*\| + \frac{1}{2}w(T^2). \tag{1.5}$$

Clearly, the lower bound in (1.5) is stronger than the first inequality in (1.3). Also, the upper bound in (1.5) is stronger than the corresponding inequalities in (1.2), (1.3) and (1.4). Using the Aluthge transform, Yamazaki [86] proved that if $T \in \mathbb{B}(\mathcal{H})$, then

$$w(T) \leq \frac{1}{2}\left(\|T\| + w(\widetilde{T})\right), \tag{1.6}$$

where the Aluthge transform of T, denoted as $\widetilde{T}$, is defined as $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, while U is the partial isometry associated with the polar decomposition of T and so $kerT = kerU$. Since $w(\widetilde{T}) \leq \|\widetilde{T}\| \leq \sqrt{\|T^2\|}$, so the inequality in (1.6) is stronger than that in (1.2). After that, in [3], Abu-Omar and Kittaneh improved the inequality (1.6) by using t-Aluthge transformation establishing that

$$w(T) \leq \frac{1}{2}\left(\|T\| + \min_{0\leq t\leq 1} w(\widetilde{T}_t)\right), \tag{1.7}$$

where $\widetilde{T}_t$ is the t-Aluthge transformation of T, defined by

$$\widetilde{T}_t = |T|^t U|T|^{1-t}, t \in [0,1].$$

Here U is the partial isometry associated with the polar decomposition of T. Another improvement of the second bound in (1.1) is proved by Bhunia and Paul in [24], is given by

$$w(T) \leq \frac{1}{2}\left(\|T\| + \sqrt{r(|T||T^*|)}\right), \tag{1.8}$$

where $r(.)$ is the spectral radius of an operator. Another improvement of the first inequality in (1.3) is proved by Bhunia and Paul in [25] by established the inequality

$$w^2(T) \geq \frac{1}{4}\|T^*T + TT^*\| + \frac{1}{2}|\|\Re(T)\|^2 - \|\Im(T)\|^2|. \tag{1.9}$$

After that in [27], Bhunia and Paul proved the inequality

$$w^{2r}(T) \leq \frac{1}{4}\||T|^{2r} + |T^*|^{2r}\| + \frac{1}{2}w(|T|^r|T^*|^r), \tag{1.10}$$

for all $r \geq 1$, which is stronger than second inequality of (1.3). Another generalization of the second inequality in (1.5) is given by

$$w^{2r}(T) \leq \frac{1}{4}\|(T^*T)^r + (TT^*)^r\| + \frac{1}{2}w^r(T^2), \tag{1.11}$$

for all $r \geq 1$, proved by Bhunia, Bag and Paul in [30].

For further improvements of (1.1) - (1.11) we refer the interested readers to [2, 10, 18, 24, 25, 26, 27, 30, 64, 74, 75, 80, 81, 86]. For further readings on the numerical range and the numerical radius inequalities, see the books [17, 51, 85].

Now, let $\mathbb{B}^d(\mathcal{H}) = \mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H}) \times \ldots \times \mathbb{B}(\mathcal{H})$ (d times) and let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ be a d-tuple operator. The joint numerical range, Euclidean operator radius, joint Crawford number and joint operator norm of $\mathbf{T}$ are defined respectively as follows:

$$\begin{aligned}
JtW(\mathbf{T}) &= \{(\langle T_1x, x\rangle, \langle T_2x, x\rangle, \ldots, \langle T_dx, x\rangle) : x \in \mathcal{H}, \|x\| = 1\},\\
w_e(\mathbf{T}) &= \sup\left\{\left(\sum_{k=1}^{d} |\langle T_kx, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1\right\},\\
c_e(\mathbf{T}) &= \inf\left\{\left(\sum_{k=1}^{d} |\langle T_kx, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1\right\},\\
\|\mathbf{T}\| &= \sup\left\{\left(\sum_{k=1}^{d} \|T_kx\|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1\right\}.
\end{aligned}$$

It is well known that the joint numerical range $JtW(\mathbf{T})$ is not a convex subset of $\mathbb{C}^d$ if $d \geq 2$, (in general) (see [51]). Many eminent researchers have studied matrices with certain commutativity properties that have convex joint numerical ranges, see, [31, 32, 37, 39]. In particular, Dash [37, Proposition 2.4] proved that $W(T_1, T_2, \ldots, T_d)$ is always convex for any commuting family $\{T_1, T_2, \ldots, T_d\} \subseteq M_2$, where M_2 is the set of all 2×2 matrices with complex entries. From the definition of joint operator norm, it is easy to see that $\|\mathbf{T}\| = \left\|\sum_{k=1}^{d} T_k^*T_k\right\|^{\frac{1}{2}}$.

As defined in [78], there is a new norm and "spectral radius" on $\mathbb{B}^d(\mathcal{H})$, which are

$$\|(T_1, T_2, \ldots, T_d)\|_e = \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \|\lambda_1T_1 + \lambda_2T_2 + \ldots + \lambda_dT_d\|$$

and

$$r_e(T_1, T_2, \ldots, T_d) = \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} r\left(\lambda_1T_1 + \lambda_2T_2 + \ldots + \lambda_dT_d\right),$$

where $\mathbb{B}_d$ is the unit ball in $\mathbb{C}^d$ and $r(T)$ denotes the usual spectral radius of an operator $T \in \mathbb{B}(\mathcal{H})$.

Next, we summarize some of the basic properties of the Euclidean operator radius of a d-tuple of operators $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$.

Theorem 1.2. *The Euclidean operator radius $w_e : \mathbb{B}^d(\mathcal{H}) \to [0, \infty)$ for d-tuples of operators satisfies the following properties:*

(i) $w_e(T_1, T_2, \ldots, T_d) = 0$ *if and only if* $T_1 = T_2 = \ldots = T_d = 0$,

(ii) $w_e(\lambda T_1, \lambda T_2, \ldots, \lambda T_d) = |\lambda| w_e(T_1, T_2, \ldots, T_d)$ *for any* $\lambda \in \mathbb{C}$,

(iii) $w_e(T_1 + T_1', T_2 + T_2', \ldots, T_d + T_d') \leq w_e(T_1, T_2, \ldots, T_d) + w_e(T_1', T_2', \ldots, T_d')$,

(iv) $w_e(U^*T_1U, U^*T_2U, \ldots, U^*T_dU) = w_e(T_1, T_2, \ldots, T_d)$ *for any unitary operator* $U \in \mathbb{B}(\mathcal{H})$,

(v) $w_e(X^*T_1X, X^*T_2X, \ldots, X^*T_dX) \leq \|X\|^2 w_e(T_1, T_2, \ldots, T_d)$ *for any operator* $X \in \mathbb{B}(\mathcal{H})$,

(vi) $\frac{1}{2}\|(T_1, T_2, \ldots, T_d)\|_e \leq w_e(T_1, T_2, \ldots, T_d) \leq \|(T_1, T_2, \ldots, T_d)\|_e$,

(vii) $r_e(T_1, T_2, \ldots, T_d) \leq w_e(T_1, T_2, \ldots, T_d)$,

(ix) w_e *is a continuous map in the norm topology.*

Next, we state the following results:

Proposition 1.1. *[78, Corollary 2.3] If* $(T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$*, then*

$$w_e(T_1, T_2, \ldots, T_d) = \sup_{(\lambda_1, \ldots, \lambda_d) \in \mathbb{B}_d} w(\lambda_1 T_1 + \lambda_2 T_2 + \ldots + \lambda_d T_d).$$

As pointed out in [78], $w_e(\cdot)$ is a norm on $\mathbb{B}^d(\mathcal{H})$ and satisfies the following inequality:

$$\frac{1}{2\sqrt{d}}\|\sum_{k=1}^{d} T_k^* T_k\|^{\frac{1}{2}} \leq w_e(\mathbf{T}) \leq \|\sum_{k=1}^{d} T_k^* T_k\|^{\frac{1}{2}},$$
$$i.e, \qquad \frac{1}{2\sqrt{d}}\|\mathbf{T}\| \leq w_e(\mathbf{T}) \leq \|\mathbf{T}\|. \tag{1.12}$$

Here the constant $\frac{1}{2\sqrt{d}}$ and 1 are best possible.

In particular for $d = 2$, Dragomir [43, Th. 1] proved that if $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, then

$$\frac{1}{2} w(T_1^2 + T_2^2) \leq w_e^2(T_1, T_2) \leq \|T_1^* T_1 + T_2^* T_2\|, \tag{1.13}$$

where the constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a larger constant. For $d = 2$, lower bound in (1.13) is stronger than the first bound in (1.12) when both T_1, T_2 are self adjoint operators. Various other inequalities and their applications about Euclidean operator radius have been studied by many mathematicians [4, 5, 11, 31, 34, 35, 36, 41, 43, 45, 58, 73, 78, 79, 82].

In this thesis, we develop various new upper and lower bounds for the Euclidean operator radius which refine the bounds mentioned in (1.12), (1.13). Also, we develop various new upper and lower bounds for the numerical radius of bounded linear operators defined on $\mathcal{H}$ which improve the existing above bounds in (1.1) - (1.11). We next give a brief outline of the thesis.

1.2 Outline of the thesis

The thesis consists of seven chapters including the Introductory one. In the introductory chapter we provide a brief history about numerical range and its multivariable generalization, joint numerical range along with terminologies and preliminary notations to be used throughout the thesis.

In chapter 2, We develop various lower bounds for the numerical radius $w(T)$ of a bounded linear operator T defined on a complex Hilbert space, which improve the existing inequality $w^2(T) \geq \frac{1}{4}\|T^*T+TT^*\|$. In particular, for $r \geq 1$, we show that

$$\frac{1}{4}\|T^*T+TT^*\| \leq \frac{1}{2}\left(\frac{1}{2}\|\Re(T)+\Im(T)\|^{2r}+\frac{1}{2}\|\Re(T)-\Im(T)\|^{2r}\right)^{\frac{1}{r}} \leq w^2(T),$$

where $\Re(T)$ and $\Im(T)$ are the real and imaginary parts of T, respectively. Furthermore, we obtain upper bounds for $w^2(T)$ refining the well-known upper bound $w^2(T) \leq \frac{1}{2}\left(w(T^2)+\|T\|^2\right)$. Separate complete characterizations for $w(T) = \frac{\|T\|}{2}$ and $w(T) = \frac{1}{2}\sqrt{\|T^*T+TT^*\|}$ are also given.

In chapter 3, We present some new upper and lower bounds for the numerical radius of bounded linear operators on a complex Hilbert space and show that the bounds are stronger than the existing ones. In particular, we prove that if T is a bounded linear operator on a complex Hilbert space $\mathcal{H}$ and if $\Re(T)$, $\Im(T)$ are the real part, the imaginary part of T, respectively, then

$$w(T) \geq \frac{\|T\|}{2}+\frac{1}{2\sqrt{2}}\Big|\|\Re(T)+\Im(T)\|-\|\Re(T)-\Im(T)\|\Big|$$

and

$$w^2(T) \geq \frac{1}{4}\|T^*T+TT^*\|+\frac{1}{4}\Big|\|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2\Big|,$$

where $w(\cdot)$ and $\|\cdot\|$ denote the numerical radius and the operator norm, respectively. Further, we obtain refinements of the inequalities for the numerical radius of the product of two operators. Finally, as an application of the second inequality mentioned above, we obtain an improvement of upper bound for the numerical radius of the commutators of operators. Also, we develop inequalities involving numerical radius and spectral radius for the sum of the product operators, from which we derive the following inequalities

$$w^p(T) \leq \frac{1}{\sqrt{2}}w(|T|^p+\mathrm{i}|T^*|^p) \leq \|T\|^p,$$

for all $p \geq 1$. Further, we derive new bounds for the zeros of complex polynomials.

In Chapter 4, we present an improvement of the inequalities in (1.13), that is,

$$\frac{1}{2}w(T_1^2+T_2^2) \leq w_e^2(T_1,T_2) \leq \|T_1^*T_1+T_2^*T_2\|.$$

Further, applying those improve inequality we derive a numerical radius inequality of a bounded linear operator $T \in \mathbb{B}(\mathcal{H})$ which improves the bounds in (1.3), i.e,

$$\frac{1}{4}\|T^*T+TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T+TT^*\|.$$

Next, we derive some improvement and generalization of well known Euclidean operator radius inequality for 2-tuple operator and as its application we get known numerical radius inequality.

In Chapter 5, we present various lower and upper bounds for Euclidean numerical radius of 2-tuple bounded linear operator which generalize and improve the well-known bounds and applying these bound we derive some numerical radius inequality of a bounded linear operator stronger than the bounds both in (1.14) and (1.15), that is,

$$w(T) \geq \frac{1}{2}\|T\| + \frac{1}{2}\big|\|\Re(T)\| - \|\Im(T)\|\big| \tag{1.14}$$

and

$$w(T) \geq \sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{2}\big|\|\Re(T)\|^2 - \|\Im(T)\|^2\big|}, \tag{1.15}$$

proved by Bhunia and Paul in [25]. Further, we give necessary and sufficient condition for equality of the inequalities in (1.14) and (1.15).

In Chapter 6, we establish new inequalities for the Euclidean operator radius of a d-tuple bounded linear operator which is stronger than the first inequality in (1.12) i.e,

$$\frac{1}{2\sqrt{d}}\|\mathbf{T}\| \leq w_e(\mathbf{T}),$$

where $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$. Further, we develop power inequality for Euclidean operator radius of d-tuple operator $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$, namely $w_e(\mathbf{T}^n) \leq \sqrt{d}w_e^n(\mathbf{T})$. Next, using this power inequality we obtain Euclidean operator radius for the product of two d-tuple operator. Further, we study the Euclidean operator radius inequalities of 2×2 operator matrices whose entries are d-tuple operators. We also obtain an Euclidean operator norm inequality of 2×2 operator matrices.

In Chapter 7, We develop several Euclidean operator radius bounds for the product of two d-tuple operators using positivity criteria of a 2×2 block matrix whose entries are d-tuple operators. From these bounds, by using the polar decomposition of operators, we obtain Euclidean operator radius bounds for d-tuple operators. Among many other interesting bounds, it is shown that

$$w_e(\mathbf{T}) \leq \frac{1}{\sqrt{2}}\|\mathbf{T}\|^{1/2}\sqrt{\left\|\sum_{k=1}^{d}(|T_k| + |T_k^*|)\right\|},$$

where $w_e(\mathbf{T})$ and $\|\mathbf{T}\|$ are the Euclidean operator radius and the Euclidean operator norm, respectively, of a d-tuple operator $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$. Further, we develop an upper bound for the Euclidean operator radius of $n \times n$ operator matrix whose entries are d-tuple operators.

Before we end this section we would like to mention that in the beginning of each of the following chapter we provide a brief motivation along with the relevant notations and terminologies necessary to keep each chapter independent for the convenience of the reader.

CHAPTER 2

IMPROVED INEQUALITIES FOR NUMERICAL RADIUS VIA CARTESIAN DECOMPOSITION

2.1 Introduction

The purpose of the present chapter is to obtain improvements of the existing well-known upper and lower bounds for the numerical radius of bounded linear operators acting on Hilbert spaces in terms of their real and imaginary parts. Let us first introduce some notation and terminology.

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ induced by the inner product. Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ with the identity I. Let $T \in \mathbb{B}(\mathcal{H})$. We denote by $|T| = (T^*T)^{\frac{1}{2}}$ the positive square root of T^*T, and $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2\mathrm{i}}(T - T^*)$, respectively, stand for the real and imaginary parts of T. The numerical range of T, denoted as $W(T)$, is defined by $W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$. We denote the operator norm, the Crawford number and

Content of this chapter is based on the following paper:
P. Bhunia, **S. Jana**, M. S. Moslehian and K. Paul, Improved Inequalities for Numerical Radius via Cartesian Decomposition, Funct. Anal. Appl., 56 (2022), no. 3, 123–133.

the numerical radius of T by $\|T\|$, $c(T)$ and $w(T)$, respectively. Recall that

$$\|T\| = \sup_{\|x\|=1} \|Tx\|,$$

$$c(T) = \inf\left\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\right\}$$

and

$$w(T) = \sup\left\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\right\}.$$

2.2 Lower Bounds for the numerical radius of operators

We start with the observation that for every $T \in \mathbb{B}(\mathcal{H})$,

$$\begin{aligned} \frac{1}{4}\|T^*T + TT^*\| &= \frac{1}{2}\left\|(\Re(T))^2 + (\Im(T))^2\right\| \\ &= \frac{1}{2}\left\|\left(\frac{\Re(T)+\Im(T)}{\sqrt{2}}\right)^2 + \left(\frac{\Re(T)-\Im(T)}{\sqrt{2}}\right)^2\right\|. \end{aligned} \tag{2.1}$$

First by using the identity (2.1), we obtain the following improvement of the first inequality in (1.3).

Theorem 2.1. *If $T \in \mathbb{B}(\mathcal{H})$, then*

$$\begin{aligned} &\frac{1}{4}\|T^*T + TT^*\| \\ &\le \frac{1}{4}\|\Re(T)+\Im(T)\|^2 + \frac{1}{4}\|\Re(T)-\Im(T)\|^2 \\ &\le \frac{1}{4}\|\Re(T)+\Im(T)\|^2 + \frac{1}{4}\|\Re(T)-\Im(T)\|^2 + \frac{1}{4}c^2(\Re(T)+\Im(T)) + \frac{1}{4}c^2(\Re(T)-\Im(T)) \\ &\le w^2(T). \end{aligned}$$

Proof. It follows from (2.1) that

$$\begin{aligned} \frac{1}{4}\|T^*T + TT^*\| &= \frac{1}{4}\|(\Re(T)+\Im(T))^2 + (\Re(T)-\Im(T))^2\| \\ &\le \frac{1}{4}\|\Re(T)+\Im(T)\|^2 + \frac{1}{4}\|\Re(T)-\Im(T)\|^2. \end{aligned}$$

This is the first inequality, and the second follows trivially.

Now we prove the third inequality. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then from the Cartesian decomposition of T, we get

$$\begin{aligned} |\langle Tx,x\rangle|^2 &= \langle \Re(T)x,x\rangle^2 + \langle \Im(T)x,x\rangle^2 \\ &= \frac{1}{2}\left(\langle \Re(T)x,x\rangle + \langle \Im(T)x,x\rangle\right)^2 + \frac{1}{2}\left(\langle \Re(T)x,x\rangle - \langle \Im(T)x,x\rangle\right)^2 \\ &= \frac{1}{2}\langle (\Re(T)+\Im(T))x,x\rangle^2 + \frac{1}{2}\langle (\Re(T)-\Im(T))x,x\rangle^2. \end{aligned}$$

Therefore, we have the following two inequalities:

$$\frac{1}{2}c^2(\Re(T)+\Im(T)) + \frac{1}{2}\|\Re(T)-\Im(T)\|^2 \leq w^2(T) \tag{2.2}$$

and

$$\frac{1}{2}c^2(\Re(T)-\Im(T)) + \frac{1}{2}\|\Re(T)+\Im(T)\|^2 \leq w^2(T). \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\frac{1}{4}\|\Re(T)+\Im(T)\|^2 + \frac{1}{4}\|\Re(T)-\Im(T)\|^2 + \frac{1}{4}c^2(\Re(T)+\Im(T)) + \frac{1}{4}c^2(\Re(T)-\Im(T)) \leq w^2(T).$$

□

Clearly, Theorem 2.1 refines the first inequality in (1.3). Now, the following corollary is trivially inferred from Theorem 2.1.

Corollary 2.1. *If $T \in \mathbb{B}(\mathcal{H})$, then*

$$\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{4}c^2(\Re(T)+\Im(T)) + \frac{1}{4}c^2(\Re(T)-\Im(T)) \leq w^2(T).$$

Also, the next result follows easily from (2.2) and (2.3).

Corollary 2.2. *If $T \in \mathbb{B}(\mathcal{H})$, then $w^2(T) \geq \max\{\beta_1, \beta_2\}$, where*

$$\beta_1 = \frac{1}{2}c^2(\Re(T)+\Im(T)) + \frac{1}{2}\|\Re(T)-\Im(T)\|^2,$$

$$\beta_2 = \frac{1}{2}c^2(\Re(T)-\Im(T)) + \frac{1}{2}\|\Re(T)+\Im(T)\|^2.$$

Remark 2.2. *(i) We have*

$$
\begin{aligned}
&\max\{\beta_1,\beta_2\}\\
&=\frac{1}{2}\left\{\frac{c^2(\Re(T)+\Im(T))+\|\Re(T)-\Im(T)\|^2+c^2(\Re(T)-\Im(T))+\|\Re(T)+\Im(T)\|^2}{2}\right\}\\
&\quad+\frac{1}{2}\left\{\frac{|\ \|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2+c^2(\Re(T)-\Im(T))-c^2(\Re(T)+\Im(T))\ |}{2}\right\}\\
&\geq\frac{1}{4}\left\{c^2(\Re(T)+\Im(T))+c^2(\Re(T)-\Im(T))\right\}+\frac{1}{4}\|(\Re(T)-\Im(T))^2+(\Re(T)+\Im(T))^2\|\\
&\quad+\frac{1}{4}\left|\|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2+c^2(\Re(T)-\Im(T))-c^2(\Re(T)+\Im(T))\right|\\
&=\frac{1}{4}\|T^*T+TT^*\|+\frac{1}{4}c^2(\Re(T)+\Im(T))+\frac{1}{4}c^2(\Re(T)-\Im(T))\\
&\quad+\frac{1}{4}\ |\ \|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2+c^2(\Re(T)-\Im(T))-c^2(\Re(T)+\Im(T))\ |\ .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
w^2(T)\geq&\frac{1}{4}\|T^*T+TT^*\|+\frac{1}{4}c^2(\Re(T)+\Im(T))+\frac{1}{4}c^2(\Re(T)-\Im(T))\\
&+\frac{1}{4}\left|\|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2+c^2(\Re(T)-\Im(T))-c^2(\Re(T)+\Im(T))\right|.
\end{aligned}
$$

(ii) Also, we remark that Corollary 2.2 is stronger than the recently obtained inequality in [16, Th. 2.3].

To prove the next refinement of the first inequality in (1.3), we need the following lemma, which can be found in [28, Th. 2.17].

Lemma 2.1. *Let $S,T\in\mathbb{B}(\mathcal{H})$. Then*

$$\|S+T\|^2\leq\|S\|^2+\|T\|^2+\frac{1}{2}\|S^*S+T^*T\|+w(S^*T)$$

and

$$\|S+T\|^2\leq\|S\|^2+\|T\|^2+\frac{1}{2}\|SS^*+TT^*\|+w(ST^*).$$

Theorem 2.3. *If $T\in\mathbb{B}(\mathcal{H})$, then*

$$
\begin{aligned}
&\frac{1}{4}\|T^*T+TT^*\|\\
&\leq\frac{1}{4}\left\{\frac{3}{2}\|\Re(T)+\Im(T)\|^4+\frac{3}{2}\|\Re(T)-\Im(T)\|^4+\|\Re(T)+\Im(T)\|^2\|\Re(T)-\Im(T)\|^2\right\}^{\frac{1}{2}}\\
&\leq w^2(T).
\end{aligned}
$$

Proof. To prove this theorem we use the technique similar to [28, Th. 2.18]. It follows from (2.1) that

$$\begin{aligned}
&\frac{1}{16}\|T^*T+TT^*\|^2\\
&=\frac{1}{4}\left\|\left(\frac{\Re(T)+\Im(T)}{\sqrt{2}}\right)^2+\left(\frac{\Re(T)-\Im(T)}{\sqrt{2}}\right)^2\right\|^2\\
&=\frac{1}{16}\|(\Re(T)+\Im(T))^2+(\Re(T)-\Im(T))^2\|^2\\
&\leq\frac{1}{16}\left\{\|\Re(T)+\Im(T)\|^4+\|\Re(T)-\Im(T)\|^4+\frac{1}{2}\|(\Re(T)+\Im(T))^4+(\Re(T)-\Im(T))^4\|\right\}\\
&\quad+\frac{1}{16}w((\Re(T)+\Im(T))^2(\Re(T)-\Im(T))^2), \qquad \text{(using Lemma 2.1)}\\
&\leq\frac{1}{16}\left\{\|\Re(T)+\Im(T)\|^4+\|\Re(T)-\Im(T)\|^4+\frac{1}{2}\left(\|\Re(T)+\Im(T)\|^4+\|\Re(T)-\Im(T)\|^4\right)\right\}\\
&\quad+\frac{1}{16}\|\Re(T)+\Im(T)\|^2\|\Re(T)-\Im(T)\|^2\\
&=\frac{1}{16}\left\{\frac{3}{2}\|\Re(T)+\Im(T)\|^4+\frac{3}{2}\|\Re(T)-\Im(T)\|^4+\|\Re(T)+\Im(T)\|^2\|\Re(T)-\Im(T)\|^2\right\}\\
&\leq w^4(T),
\end{aligned}$$

where the last inequality is deduced from (2.4) and (2.5). □

Now, we state a lemma.

Lemma 2.2. *([22, Th. 2.2]) Let $S,T\in\mathbb{B}(\mathcal{H})$. Then*

$$\|S+T\|^2\leq 2\max\left\{\|S^*S+T^*T\|,\|SS^*+TT^*\|\right\}.$$

Based on the above lemma, we obtain the following refinement of the first inequality in (1.3).

Theorem 2.4. *If $T\in\mathbb{B}(\mathcal{H})$, then*

$$\frac{1}{4}\|T^*T+TT^*\|\leq\frac{1}{2\sqrt{2}}\left(\|\Re(T)+\Im(T)\|^4+\|\Re(T)-\Im(T)\|^4\right)^{\frac{1}{2}}\leq w^2(T).$$

Proof. We prove this theorem by similar technique as in [28, Th. 2.13]. It follows from (2.1) that

$$\begin{aligned}\frac{1}{4}\|T^*T+TT^*\| &= \frac{1}{4}\|(\Re(T)+\Im(T))^2+(\Re(T)-\Im(T))^2\| \\ &\le \frac{1}{2\sqrt{2}}\|(\Re(T)+\Im(T))^4+(\Re(T)-\Im(T))^4\|^{\frac{1}{2}}, \qquad \text{(by Lemma 2.2)} \\ &\le \frac{1}{2\sqrt{2}}\left(\|\Re(T)+\Im(T)\|^4+\|\Re(T)-\Im(T)\|^4\right)^{\frac{1}{2}} \\ &\le w^2(T),\end{aligned}$$

where we deduce the last inequality from (2.4) and (2.5). □

We observe here that the convexity of the function $f(t)=t^2$ ensures that the first inequality in Theorem 2.4 is better than the first inequality in Theorem 2.1. Also, we observe that the second inequality in Theorem 2.4 is better than the second inequality in Theorem 2.3.

In the next theorem, we obtain another improvement of (1.3). First we note that (2.2) and (2.3) imply the following two inequalities, respectively:

$$\frac{1}{2}\|\Re(T)-\Im(T)\|^2 \le w^2(T) \tag{2.4}$$

and

$$\frac{1}{2}\|\Re(T)+\Im(T)\|^2 \le w^2(T). \tag{2.5}$$

Now, by employing the convexity property of the function $f(t)=t^r$, $r\ge 1$, in the first inequality in Theorem 2.1 and using inequalities (2.4) and (2.5), we get the following inequality.

Theorem 2.5. *If $T\in\mathbb{B}(\mathcal{H})$, then for $r\ge 1$,*

$$\frac{1}{4}\|T^*T+TT^*\| \le \frac{1}{2}\left(\frac{1}{2}\|\Re(T)+\Im(T)\|^{2r}+\frac{1}{2}\|\Re(T)-\Im(T)\|^{2r}\right)^{\frac{1}{r}} \le w^2(T).$$

Remark 2.6. *Clearly, Theorem 2.5 is a generalization of Theorem 2.4. We would like to remark that the second inequality in Theorem 2.5 gives more refinement as r increases.*

To prove our next result, we need the following lemma.

Lemma 2.3. ([38]) *Let $S,T\in\mathbb{B}(\mathcal{H})$ be positive. Then*

$$\|S+T\| \le \max\{\|S\|,\|T\|\}+\|ST\|^{\frac{1}{2}}.$$

Theorem 2.7. *If $T \in \mathbb{B}(\mathcal{H})$, then*

$$\begin{aligned}
&\frac{1}{4}\|T^*T + TT^*\| \\
&\leq \frac{1}{4}\left[\max\left\{\|\Re(T) + \Im(T)\|^2, \|\Re(T) - \Im(T)\|^2\right\} + \|\Re(T) + \Im(T)\|\|\Re(T) - \Im(T))\|\right] \\
&\leq w^2(T).
\end{aligned}$$

Proof. To prove this theorem we use the technique similar to [28, Th. 2.10]. It follows from (2.1) that

$$\begin{aligned}
&\frac{1}{4}\|T^*T + TT^*\| \\
&= \frac{1}{4}\|(\Re(T) + \Im(T))^2 + (\Re(T) - \Im(T))^2\| \\
&\leq \frac{1}{4}\left[\max\left\{\|\Re(T) + \Im(T)\|^2, \|\Re(T) - \Im(T)\|^2\right\} + \|\Re(T) + \Im(T))^2(\Re(T) - \Im(T))^2\|^{\frac{1}{2}}\right] \\
&\leq \frac{1}{4}\left[\max\left\{\|\Re(T) + \Im(T)\|^2, \|\Re(T) - \Im(T)\|^2\right\} + \|(\Re(T) + \Im(T))\|\|(\Re(T) - \Im(T))\|\right] \\
&\leq w^2(T),
\end{aligned}$$

in which we employ (2.4) and (2.5). □

We now concentrate our study on the equality of the first inequality in (1.3).

Corollary 2.3. *Let $T \in \mathbb{B}(\mathcal{H})$. If $w^2(T) = \frac{1}{4}\|T^*T + TT^*\|$, then the following assertions hold:*

(i) There exists a sequence $\{x_n\}$ in $\mathcal{H}$ with $\|x_n\| = 1$ such that

$$\lim_{n\to\infty} |\langle \Re(T)x_n, x_n\rangle| = \lim_{n\to\infty} |\langle \Im(T)x_n, x_n\rangle|.$$

(ii) $\|\Re(T) + \Im(T)\|^2 = \|\Re(T) - \Im(T)\|^2 = \frac{1}{2}\|T^*T + TT^*\|$.

Proof. Let $w^2(T) = \frac{1}{4}\|T^*T + TT^*\|$. It follows from Theorem 2.1 that $c(\Re(T) + \Im(T)) = c(\Re(T) - \Im(T)) = 0$. This implies that there exist sequences $\{y_n\}$ and $\{z_n\}$ in $\mathcal{H}$ with $\|y_n\| = \|z_n\| = 1$ such that $\lim_{n\to\infty}\langle(\Re(T) + \Im(T))y_n, y_n\rangle = 0$ and $\lim_{n\to\infty}\langle(\Re(T) - \Im(T))z_n, z_n\rangle = 0$. Thus (i) holds.

Also, from (i) of Remark 2.2, we have $\|\Re(T) + \Im(T)\|^2 = \|\Re(T) - \Im(T)\|^2$. In addition, we conclude from Theorem 2.5 that $\|\Re(T) + \Im(T)\|^2 = \frac{1}{2}\|T^*T + TT^*\|$, which yields (ii). □

Considering the matrix $T = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$, we conclude that the converse of Corollary 2.3 is not true.

Remark 2.8. *Considering the following two examples, we observe that the bounds obtained in Theorems 2.3 and 2.7 (also, Theorems 2.4 and 2.7) are not comparable, in general.*

(i) Let $T = \begin{pmatrix} 2+2i & 0 \\ 0 & 0 \end{pmatrix}$. *Then* $\Re(T) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\Im(T) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. *Clearly,* $\|\Re(T) + \Im(T)\| = 4$ *and* $\|\Re(T) - \Im(T)\| = 0$. *By simple calculations, we have*

$$\frac{1}{2\sqrt{2}}\left(\|\Re(T)+\Im(T)\|^4 + \|\Re(T)-\Im(T)\|^4\right)^{\frac{1}{2}} = 4\sqrt{2} \approx 5.65685424949,$$

$$\frac{1}{4}\left\{\frac{3}{2}\|\Re(T)+\Im(T)\|^4 + \frac{3}{2}\|\Re(T)-\Im(T)\|^4 + \|\Re(T)+\Im(T)\|^2\|\Re(T)-\Im(T)\|^2\right\}^{\frac{1}{2}}$$
$$= 2\sqrt{6} \approx 4.89897948557,$$
$$\frac{1}{4}\left[\max\left\{\|\Re(T)+\Im(T)\|^2, \|\Re(T)-\Im(T)\|^2\right\} + \|\Re(T)+\Im(T)\|\|\Re(T)-\Im(T)\|\right] = 4.$$

(ii) Let $T = \begin{pmatrix} 3+2i & 0 \\ 0 & 4i \end{pmatrix}$. *Then* $\Re(T) = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\Im(T) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. *Therefore,* $\|\Re(T) + \Im(T)\| = 5$ *and* $\|\Re(T) - \Im(T)\| = 4$. *By simple calculations, we get*

$$\frac{1}{2\sqrt{2}}\left(\|\Re(T)+\Im(T)\|^4 + \|\Re(T)-\Im(T)\|^4\right)^{\frac{1}{2}} = \frac{1}{2\sqrt{2}}\sqrt{881} \approx 10.4940459309,$$

$$\frac{1}{4}\left\{\frac{3}{2}\|\Re(T)+\Im(T)\|^4 + \frac{3}{2}\|\Re(T)-\Im(T)\|^4 + \|\Re(T)+\Im(T)\|^2\|\Re(T)-\Im(T)\|^2\right\}^{\frac{1}{2}}$$
$$= \frac{1}{4}\sqrt{\frac{17215}{10}} \approx 10.37274071,$$
$$\frac{1}{4}\left[\max\left\{\|\Re(T)+\Im(T)\|^2, \|\Re(T)-\Im(T)\|^2\right\} + \|\Re(T)+\Im(T)\|\|\Re(T)-\Im(T)\|\right]$$
$$= \frac{45}{4} = 11.25.$$

2.3 Upper Bounds for the numerical radius of operators

Now, we obtain an upper bound for the numerical radius of bounded linear operators. For this, we need the following two lemmas. The first one is known as Buzano's inequality, and the second one is known as the weighted arithmetic-geometric mean inequality.

Lemma 2.4. *([33]) Let* $x, y, e \in \mathcal{H}$ *with* $\|e\| = 1$. *Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\|\right).$$

Lemma 2.5. *([53]) If $a, b \geq 0$ and $0 \leq \alpha \leq 1$, then*

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b.$$

Theorem 2.9. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$w^2(T) \leq \frac{1}{2}\left[\|T\|^2\left(\min_{t\in[0,1]} \|tT^*T + (1-t)TT^*\|\right) + w^2(T^2) + w(T^2)\|T^*T + TT^*\|\right]^{\frac{1}{2}}.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then

$$\begin{aligned}
&|\langle Tx, x\rangle|^2 \\
&= |\langle Tx, x\rangle\langle x, T^*x\rangle| \\
&\leq \frac{1}{2}\left(|\langle Tx, T^*x\rangle| + \|Tx\|\|T^*x\|\right) \qquad \text{(using Lemma 2.4)} \\
&= \frac{1}{2}\left\{|\langle Tx, T^*x\rangle|^2 + \|Tx\|^2\|T^*x\|^2 + 2|\langle Tx, T^*x\rangle|\|Tx\|\|T^*x\|\right\}^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left\{|\langle T^2x, x\rangle|^2 + \langle T^*Tx, x\rangle\langle TT^*x, x\rangle + |\langle T^2x, x\rangle|\langle (TT^* + T^*T)x, x\rangle\right\}^{\frac{1}{2}} \\
&= \frac{1}{2}\Bigg\{|\langle T^2x, x\rangle|^2 + \langle T^*Tx, x\rangle^t\langle TT^*x, x\rangle^{1-t}\langle T^*Tx, x\rangle^{1-t}\langle TT^*x, x\rangle^t \\
&\qquad + |\langle T^2x, x\rangle|\langle (TT^* + T^*T)x, x\rangle\Bigg\}^{\frac{1}{2}} \\
&\leq \frac{1}{2}\Bigg\{|\langle T^2x, x\rangle|^2 + \langle (tT^*T + (1-t)TT^*)x, x\rangle\langle ((1-t)T^*T + tTT^*)x, x\rangle \\
&\qquad + |\langle T^2x, x\rangle|\langle (TT^* + T^*T)x, x\rangle\Bigg\}^{\frac{1}{2}} \qquad \text{(using Lemma 2.5)} \\
&\leq \frac{1}{2}\Bigg\{|\langle T^2x, x\rangle|^2 + \|tT^*T + (1-t)TT^*\|\|(1-t)T^*T + tTT^*\| \\
&+ |\langle T^2x, x\rangle|\langle (TT^* + T^*T)x, x\rangle\Bigg\}^{\frac{1}{2}} \\
&\leq \frac{1}{2}\Bigg\{|\langle T^2x, x\rangle|^2 + \|tT^*T + (1-t)TT^*\|\|T\|^2 + |\langle T^2x, x\rangle|\langle (TT^* + T^*T)x, x\rangle\Bigg\}^{\frac{1}{2}} \\
&\leq \frac{1}{2}\Bigg\{w^2(T^2) + \|tT^*T + (1-t)TT^*\|\|T\|^2 + w(T^2)\|TT^* + T^*T\|\Bigg\}^{\frac{1}{2}}.
\end{aligned}$$

Taking supremum over $\|x\| = 1$, we get

$$w^2(T) \le \frac{1}{2}\left[\|T\|^2\left(\|tT^*T + (1-t)TT^*\|\right) + w^2(T^2) + w(T^2)\|T^*T + TT^*\|\right]^{\frac{1}{2}}.$$

This holds for all $t \in [0,1]$, so considering minimum over $t \in [0,1]$, we have

$$w^2(T) \le \frac{1}{2}\left[\|T\|^2\left(\min_{t\in[0,1]}\|tT^*T + (1-t)TT^*\|\right) + w^2(T^2) + w(T^2)\|T^*T + TT^*\|\right]^{\frac{1}{2}},$$

as required. □

Remark 2.10. *(i) Dragomir [42, Th. 1] proved that for $T \in \mathbb{B}(\mathcal{H})$,*

$$w^2(T) \le \frac{1}{2}\Big(\|T\|^2 + w(T^2)\Big). \tag{2.6}$$

We would like to remark that the inequality in Theorem 2.9 is sharper than that in Dragomir's result [42, Th. 1].

(ii) Now, we consider $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. *Then, by simple calculations, we arrive at*

$$\frac{1}{2}\left[\|T\|^2\left(\min_{t\in[0,1]}\|tT^*T + (1-t)TT^*\|\right) + w^2(T^2) + w(T^2)\|T^*T + TT^*\|\right]^{\frac{1}{2}} = \frac{3}{4}$$

and

$$\frac{1}{2}\|T^*T + TT^*\| = 1.$$

Again, considering another operator $T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$, *we have*

$$\frac{1}{2}\left[\|T\|^2\left(\min_{t\in[0,1]}\|tT^*T + (1-t)TT^*\|\right) + w^2(T^2) + w(T^2)\|T^*T + TT^*\|\right]^{\frac{1}{2}} = \sqrt{5}$$

and

$$\frac{1}{2}\|T^*T + TT^*\| = 2.$$

Thus, we conclude that the second inequality in (1.3) and our obtained inequality in Theorem 2.9 are not comparable, in general.

In the following theorem, we present another refinement of (2.6).

Theorem 2.11. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$w^4(T) \leq \frac{1}{4}\left[w^2(T^2) + \frac{1}{4}\left\|(T^*T)^2 + (TT^*)^2\right\| + \frac{1}{2}w(T^*T^2T^*) + w(T^2)\|T^*T + TT^*\|\right].$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. It follows from Lemma 2.4 that

$$\begin{aligned}
\langle T^*Tx, x\rangle\langle TT^*x, x\rangle &= \langle T^*Tx, x\rangle\langle x, TT^*x\rangle \\
&\leq \frac{\|T^*Tx\|\|TT^*x\| + |\langle TT^*x, T^*Tx\rangle|}{2} \\
&\leq \frac{1}{4}\left(\|T^*Tx\|^2 + \|TT^*x\|^2\right) + \frac{1}{2}|\langle T^*T^2T^*x, x\rangle| \\
&= \frac{1}{4}\left\langle\left((T^*T)^2 + (TT^*)^2\right)x, x\right\rangle + \frac{1}{2}|\langle T^*T^2T^*x, x\rangle| \\
&\leq \frac{1}{4}\left\|(T^*T)^2 + (TT^*)^2\right\| + \frac{1}{2}w(T^*T^2T^*).
\end{aligned}$$

Following the proof of Theorem 2.9, we infer that

$$\begin{aligned}
&|\langle Tx, x\rangle|^4 \\
&\leq \frac{1}{4}\left\{|\langle T^2x, x\rangle|^2 + \langle T^*Tx, x\rangle\langle TT^*x, x\rangle + |\langle T^2x, x\rangle|\langle (TT^* + T^*T)x, x\rangle\right\} \\
&\leq \frac{1}{4}\left\{w^2(T^2) + \langle T^*Tx, x\rangle\langle TT^*x, x\rangle + w(T^2)\|TT^* + T^*T\|\right\} \\
&\leq \frac{1}{4}\left\{w^2(T^2) + \frac{1}{4}\left\|(T^*T)^2 + (TT^*)^2\right\| + \frac{1}{2}w(T^*T^2T^*) + w(T^2)\|T^*T + TT^*\|\right\}.
\end{aligned}$$

Considering supremum over $\|x\| = 1$, we arrive at the desired inequality. □

Remark 2.12. *Clearly, for $T \in \mathbb{B}(\mathcal{H})$, we have*

$$\begin{aligned}
&\frac{1}{4}\left[w^2(T^2) + \frac{1}{4}\left\|(T^*T)^2 + (TT^*)^2\right\| + \frac{1}{2}w(T^*T^2T^*) + w(T^2)\|T^*T + TT^*\|\right] \\
&\leq \frac{1}{4}\left[w^2(T^2) + \frac{1}{2}\|T\|^4 + \frac{1}{2}\left\|T^*T^2T^*\right\| + 2w(T^2)\|T\|^2\right] \\
&\leq \frac{1}{4}\left[w^2(T^2) + \frac{1}{2}\|T\|^4 + \frac{1}{2}\|T\|^4 + 2w(T^2)\|T\|^2\right] \\
&= \left[\frac{\|T\|^2 + w(T^2)}{2}\right]^2.
\end{aligned}$$

Therefore, Theorem 2.11 refines inequality (2.6).

In our next theorem, we obtain an inequality involving norm and numerical radius of a bounded linear operator. First we recall the following well-known identity from [86, p. 85]:

$$w(T) = \sup_{\theta\in\mathbb{R}} \left\|\Re(e^{\mathrm{i}\theta}T)\right\|. \tag{2.7}$$

Theorem 2.13. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$w^3(T) \leq \frac{1}{4}\Big[w(T^3) + \|T\|\|T^2\| + w(T)\|T^*T + TT^*\|\Big].$$

Proof. By a short calculation, we get

$$\Re^3(T) = \frac{1}{4}\Re(T^3) + \frac{1}{8}(T^2T^* + T^{*2}T) + \frac{1}{4}(T^*T + TT^*)\Re(T).$$

Since $\Re(T)$ is selfadjoint, we have

$$\begin{aligned}
\|\Re(T)\|^3 &= \left\|\frac{1}{4}\Re(T^3) + \frac{1}{8}(T^2T^* + T^{*2}T) + \frac{1}{4}(T^*T + TT^*)\Re(T)\right\| \\
&\leq \frac{1}{4}\left\|\Re(T^3)\right\| + \frac{1}{8}\left\|T^2T^* + T^{*2}T\right\| + \frac{1}{4}\|T^*T + TT^*\|\|\Re(T)\| \\
&\leq \frac{1}{4}\left\|\Re(T^3)\right\| + \frac{1}{4}\left\|T^2\right\|\|T\| + \frac{1}{4}\|T^*T + TT^*\|\|\Re(T)\|.
\end{aligned}$$

Now let $\theta \in \mathbb{R}$. Replacing T with $e^{\mathrm{i}\theta}T$ in the last inequality yields that

$$\|\Re(e^{\mathrm{i}\theta}T)\|^3 \leq \frac{1}{4}\left\|\Re(e^{3\mathrm{i}\theta}T^3)\right\| + \frac{1}{4}\left\|T^2\right\|\|T\| + \frac{1}{4}\|T^*T + TT^*\|\|\Re(e^{\mathrm{i}\theta}T)\|.$$

Taking supremum over all $\theta \in \mathbb{R}$ and using identity (2.7), we derive that

$$w^3(T) \leq \frac{1}{4}\Big[w(T^3) + \|T\|\|T^2\| + w(T)\|T^*T + TT^*\|\Big],$$

as desired. ∎

Remark 2.14. *Let $T \in \mathbb{B}(\mathcal{H})$ with $T \neq 0$ and $T^2 = 0$. It follows from Theorem 2.13 that*

$$w(T) \leq \frac{1}{2}\sqrt{\|T^*T + TT^*\|}.$$

This inequality combined with the first inequality in (1.3) ensures that

$$w(T) = \frac{1}{2}\sqrt{\|T^*T + TT^*\|}.$$

It should be mentioned that the reverse part is not true, in general. To see this, consider the matrix $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then one can easily verify that $w(T) = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{\|T^*T + TT^*\|}$, but $T^2 \neq 0$.

At the end of the article, we give separate complete characterizations for $w(T) = \frac{1}{2}\|T\|$ and $w(T) = \frac{1}{2}\sqrt{\|T^*T + TT^*\|}$. First we need the following lemma.

Lemma 2.6. *([25, Th. 2.14]) Let $T \in \mathbb{B}(\mathcal{H})$. Then*
(i) $w(T) = \frac{1}{2}\|T\|$ if and only if $\overline{W(T)}$ is a circular disk with center at the origin and radius $\frac{1}{2}\|T\|$;
(ii) $w(T) = \frac{1}{2}\sqrt{\|T^*T + TT^*\|}$ if and only if $\overline{W(T)}$ is a circular disk with center at the origin and radius $\frac{1}{2}\sqrt{\|T^*T + TT^*\|}$.

Theorem 2.15. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

(i) $w(T) = \frac{1}{2}\|T\|$ if and only if $w(T + \lambda I) = \frac{1}{2}\|T\| + |\lambda|$ for all $\lambda \in \mathbb{C}$;

(ii) $w(T) = \frac{1}{2}\sqrt{\|T^*T + TT^*\|}$ if and only if $w(T+\lambda I) = \frac{1}{2}\sqrt{\|T^*T + TT^*\|} + |\lambda|$ for all $\lambda \in \mathbb{C}$.

Proof. (i) The sufficient part is trivial, so we only prove the necessary part. Let $w(T) = \frac{1}{2}\|T\|$. Clearly, $\overline{W(T + \lambda I)} = \overline{W(T)} + \lambda$ for all $\lambda \in \mathbb{C}$. Therefore, it follows from Lemma 2.6(i) that $\overline{W(T + \lambda I)}$ is a circular disk with center at λ and radius $\frac{1}{2}\|T\|$. This implies that $w(T + \lambda I) = \frac{1}{2}\|T\| + |\lambda|$.
(ii) The proof follows as in (i). □

Remark 2.16. *Let $T \in \mathbb{B}(\mathcal{H})$. We would like to remark that if $\overline{W(T)}$ is a circular disk with center at the origin, then $w(T + \lambda I) = w(T) + |\lambda|$ for all $\lambda \in \mathbb{C}$. Hence, it follows from [25, Lemma 2.13] that if $\left\|\Re(e^{i\theta}T)\right\| = k$ (a constant) for all $\theta \in \mathbb{R}$, then $w(T + \lambda I) = w(T) + |\lambda|$ for all $\lambda \in \mathbb{C}$. This shows that if $\left\|\Re(e^{i\theta}T)\right\| = k$ (a constant) for all $\theta \in \mathbb{R}$, then $w(T + \lambda I) \geq w(T)$ for all $\lambda \in \mathbb{C}$, that is, T is Birkhoff–James numerical radius orthogonal to I (for the details of Birkhoff–James numerical radius orthogonality, we refer to [70, 87]). Finally, we remark that if either $w(T) = \frac{1}{2}\|T\|$ or $w(T) = \frac{1}{2}\sqrt{\|T^*T + TT^*\|}$, then T is Birkhoff–James numerical radius orthogonal to I.*

CHAPTER 3

REFINED INEQUALITIES FOR THE NUMERICAL RADIUS AND ITS APPLICATIONS

3.1 Introduction

In this chapter, we present some new upper and lower bounds for the numerical radius of bounded linear operators on a complex Hilbert space and show that the bounds are stronger than the existing ones. Further, we obtain refinements of the inequalities for the numerical radius of the product of two operators. As an application of some inequality proved in this chapter, we obtain an improvement of upper bound for the numerical radius of the commutators of operators. In last section, using bounds of numerical radius we derive the bounds of zeros of complex monic polynomials.

Let $\mathbb{B}(\mathcal{H})$ denote the $\mathcal{C}^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$ and the corresponding norm $\|.\|$ induced by the inner product $\langle ., . \rangle$. For

Content of this chapter is based on the following two papers:
P. Bhunia, **S. Jana** and K. Paul, Refined inequalities for the numerical radius of Hilbert space operators, Rocky Mountain J. Math., To appear (2023).
P. Bhunia, **S. Jana** and K. Paul, Numerical radius inequalities and estimation of zeros of polynomials, Georgian Math. J., To appear (2023).

$T \in \mathbb{B}(\mathcal{H})$, let T^* be the adjoint of T and $|T| = (T^*T)^{\frac{1}{2}}$. Let $\Re(T)$ and $\Im(T)$ denote the real part and the imaginary part of T, respectively, i.e., $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2\mathrm{i}}(T - T^*)$. Let $\|T\|$ and $w(T)$ denote the operator norm and the numerical radius of T, respectively. Recall that

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

The Aluthge transform of T, denoted as $\widetilde{T}$, is defined as $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where U is the partial isometry appearing in the polar decomposition $T = U|T|$ of T with $\ker T = \ker U$. It follows from the definition of $\widetilde{T}$ that $\|\widetilde{T}\| \leq \|T\|$. Also, $w(\widetilde{T}) \leq w(T)$.

3.2 Numerical radius inequalities of operators

We start our work with the following improvement of the first inequality in (1.1).

Theorem 3.1. *If $T \in \mathbb{B}(\mathcal{H})$, then*

$$w(T) \geq \frac{1}{2}\|T\| + \frac{1}{2\sqrt{2}}\ \big|\ \|\Re(T) + \Im(T)\| - \|\Re(T) - \Im(T)\|\ \big|\ .$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then by the Cartesian decomposition of T, we have

$$\begin{aligned} |\ \langle Tx, x\rangle\ | &= \sqrt{\langle \Re(T)x, x\rangle^2 + \langle \Im(T)x, x\rangle^2} \\ &\geq \frac{1}{\sqrt{2}}(|\ \langle \Re(T)x, x\rangle\ | + |\ \langle \Im(T)x, x\rangle\ |) \\ &\geq \frac{1}{\sqrt{2}}\ |\ \langle (\Re(T) \pm \Im(T))x, x\rangle\ |\ . \end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$, $\|x\| = 1$, we get

$$w(T) \geq \frac{1}{\sqrt{2}}\|\Re(T) \pm \Im(T)\|.$$

Thus,

$$w(T) \geq \frac{1}{\sqrt{2}}\max\{\|\Re(T) + \Im(T)\|, \|\Re(T) - \Im(T)\|\}.$$

Now,

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\max\{\|\Re(T)+\Im(T)\|,\|\Re(T)-\Im(T)\|\}\\
=\;&\frac{1}{\sqrt{2}}\left\{\frac{\|\Re(T)+\Im(T)\|+\|\Re(T)-\Im(T)\|}{2}+\frac{|\,\|\Re(T)+\Im(T)\|-\|\Re(T)-\Im(T)\|\,|}{2}\right\}\\
\geq\;&\frac{1}{\sqrt{2}}\left\{\frac{\|(\Re(T)+\Im(T))+\mathrm{i}(\Re(\mathrm{T})-\Im(\mathrm{T}))\|}{2}+\frac{|\,\|\Re(T)+\Im(T)\|-\|\Re(T)-\Im(T)\|\,|}{2}\right\}\\
=\;&\frac{1}{\sqrt{2}}\left\{\frac{\|(1+\mathrm{i})\mathrm{T}^*\|}{2}+\frac{|\,\|\Re(T)+\Im(T)\|-\|\Re(T)-\Im(T)\|\,|}{2}\right\}\\
=\;&\frac{\|T\|}{2}+\frac{|\,\|\Re(T)+\Im(T)\|-\|\Re(T)-\Im(T)\|\,|}{2\sqrt{2}}.
\end{aligned}$$

This completes the proof. □

Remark 3.2. *(i) Clearly, the inequality in Theorem 3.1 refines the first inequality in (1.1).*
(ii) If $w(T)=\frac{\|T\|}{2}$, *then* $\|\Re(T)+\Im(T)\|=\|\Re(T)-\Im(T)\|$, *but the converse is not necessarily true. For example, if* T *is (non-zero) self-adjoint, then* $\|\Re(T)+\Im(T)\|=\|\Re(T)-\Im(T)\|$ *and* $w(T)\neq\frac{\|T\|}{2}$.
(iii) In [25, Th. 2.1], Bhunia and Paul proved that

$$w(T)\;\;\geq\;\;\frac{1}{2}\|T\|+\frac{1}{2}\,|\,\|\Re(T)\|-\|\Im(T)\|\,|\,. \tag{3.1}$$

Clearly, if $T\in\mathbb{B}(\mathcal{H})$ *is (non-zero) self-adjoint, then*

$$\frac{1}{2}\|T\|+\frac{1}{2}\,|\,\|\Re(T)\|-\|\Im(T)\|\,|>\frac{\|T\|}{2}+\frac{|\,\|\Re(T)+\Im(T)\|-\|\Re(T)-\Im(T)\|\,|}{2\sqrt{2}}$$

and if $T=\begin{pmatrix}1+\mathrm{i} & 0\\ 0 & 0\end{pmatrix}$, *then*

$$\frac{1}{2}\|T\|+\frac{1}{2}\,|\,\|\Re(T)\|-\|\Im(T)\|\,|<\frac{\|T\|}{2}+\frac{|\,\|\Re(T)+\Im(T)\|-\|\Re(T)-\Im(T)\|\,|}{2\sqrt{2}}.$$

Thus, we conclude that the inequality in Theorem 3.1 and the inequality (3.1) are not comparable, in general.

Next we prove the following inequality which improve on the first inequality in (1.3).

Theorem 3.3. *If* $T\in\mathbb{B}(\mathcal{H})$, *then*

$$w^2(T)\geq\frac{1}{4}\|T^*T+TT^*\|+\frac{1}{4}\,|\,\|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2\,|\,.$$

Proof. As the proof of Theorem 3.1, we have

$$w^2(T) \geq \frac{1}{2}\max\{\|\Re(T)+\Im(T)\|^2, \|\Re(T)-\Im(T)\|^2\}.$$

Now,

$$\begin{aligned}
& \frac{1}{2}\max\{\|\Re(T)+\Im(T)\|^2, \|\Re(T)-\Im(T)\|^2\} \\
= & \frac{1}{2}\left\{\frac{\|\Re(T)+\Im(T)\|^2+\|\Re(T)-\Im(T)\|^2}{2}+\frac{|\,\|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2\,|}{2}\right\} \\
\geq & \frac{1}{2}\left\{\frac{\|(\Re(T)+\Im(T))^2+(\Re(T)-\Im(T))^2\|}{2}+\frac{|\,\|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2\,|}{2}\right\} \\
= & \frac{\|T^*T+TT^*\|}{4}+\frac{|\,\|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2\,|}{4}.
\end{aligned}$$

This completes the proof. □

Remark 3.4. *(i) Clearly, the inequality in Theorem 3.3 refines the first inequality in (1.3).*
(ii) If $w^2(T)=\frac{\|T^*T+TT^*\|}{4}$*, then* $\|\Re(T)+\Im(T)\|=\|\Re(T)-\Im(T)\|$*, but the converse is not necessarily true.*
(iii) In [25, Th. 2.9], Bhunia and Paul proved that if $T\in\mathbb{B}(\mathcal{H})$*, then*

$$w^2(T) \geq \frac{1}{4}\|T^*T+TT^*\|+\frac{1}{2}\,|\,\|\Re(T)\|^2-\|\Im(T)\|^2\,|\,. \tag{3.2}$$

Considering the same examples as in Remark 3.2 (iii), we conclude that the inequality in Theorem 3.3 and the inequality (3.2) are not comparable, in general.

Next we obtain an upper bound for the numerical radius which improve on (1.6). For this purpose, first we need the following lemma, based on polarization principle.

Lemma 3.1. *Let* $T\in\mathbb{B}(\mathcal{H})$*, and let* $x,y\in\mathcal{H}$*. Then*

$$\begin{aligned}
\langle Tx,x\rangle = & \frac{\langle T(x+y),x+y\rangle-\langle T(x-y),x-y\rangle}{4} \\
& +\mathrm{i}\frac{\langle T(x+\mathrm{i}y),x+\mathrm{i}y\rangle-\langle T(x-\mathrm{i}y),x-\mathrm{i}y\rangle}{4}.
\end{aligned}$$

Theorem 3.5. *If* $T\in\mathbb{B}(\mathcal{H})$*, then*

$$w(T) \leq \frac{1}{2}\left(\|T\|^2+w^2(\widetilde{T})+w(|T|\widetilde{T}+\widetilde{T}|T|)\right)^{\frac{1}{2}}.$$

Proof. First we note that

$$w(T) = \sup_{\theta \in \mathbb{R}} \|\Re(e^{i\theta}T)\| = \sup_{\theta \in \mathbb{R}} w(\Re(e^{i\theta}T)).$$

Let $T = U|T|$ be the polar decomposition of T. Then, by Lemma 3.1, we have

$$\begin{aligned} & \langle e^{i\theta}Tx, x\rangle = \langle e^{i\theta}|T|x, U^*x\rangle \\ = \; & \frac{1}{4}\Big(\langle |T|(e^{i\theta}x + U^*x), e^{i\theta}x + U^*x\rangle - \langle |T|(e^{i\theta}x - U^*x), e^{i\theta}x - U^*x\rangle\Big) \\ + \; & \frac{i}{4}\Big(\langle |T|(e^{i\theta}x + iU^*x), e^{i\theta}x + iU^*x\rangle - \langle |T|(e^{i\theta}x - iU^*x), e^{i\theta}x - iU^*x\rangle\Big). \end{aligned}$$

Therefore, we have

$$\begin{aligned} Re\langle e^{i\theta}Tx, x\rangle &= \frac{1}{4}(\langle |T|(e^{i\theta}x + U^*x), e^{i\theta}x + U^*x\rangle - \langle |T|(e^{i\theta}x - U^*x), e^{i\theta}x - U^*x\rangle) \\ &\leq \frac{1}{4}\langle |T|(e^{i\theta}x + U^*x), e^{i\theta}x + U^*x\rangle \\ &= \frac{1}{4}\langle (e^{-i\theta} + U)|T|(e^{i\theta} + U^*)x, x\rangle \\ &\leq \frac{1}{4}\|(e^{-i\theta} + U)|T|(e^{i\theta} + U^*)\| \\ &= \frac{1}{4}\||T|^{\frac{1}{2}}(e^{i\theta} + U^*)(e^{-i\theta} + U)|T|^{\frac{1}{2}}\| \qquad (\|T^*T\| = \|TT^*\|) \\ &= \frac{1}{4}\|2|T| + e^{i\theta}\widetilde{T} + e^{-i\theta}\widetilde{T}^*\| \\ &= \frac{1}{2}\||T| + \Re(e^{i\theta}\widetilde{T})\| \\ &= \frac{1}{2}\left\|\Big(|T| + \Re(e^{i\theta}\widetilde{T})\Big)^2\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\left\||T|^2 + (\Re(e^{i\theta}\widetilde{T}))^2 + |T|\Re(e^{i\theta}\widetilde{T}) + \Re(e^{i\theta}\widetilde{T})|T|\right\|^{\frac{1}{2}} \\ &= \frac{1}{2}\left\||T|^2 + (\Re(e^{i\theta}\widetilde{T}))^2 + \Re(e^{i\theta}(|T|\widetilde{T} + \widetilde{T}|T|))\right\|^{\frac{1}{2}} \\ &\leq \frac{1}{2}\Big(\|T\|^2 + \|\Re(e^{i\theta}\widetilde{T})\|^2 + \|\Re(e^{i\theta}(|T|\widetilde{T} + \widetilde{T}|T|))\|\Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\Big(\|T\|^2 + w^2(\widetilde{T}) + w(|T|\widetilde{T} + \widetilde{T}|T|)\Big)^{\frac{1}{2}}. \end{aligned}$$

Since $Re\langle e^{i\theta}Tx, x\rangle = \langle \Re(e^{i\theta}T)x, x\rangle$, so taking supremum over $\theta \in \mathbb{R}$, we get

$$w(T) \leq \frac{1}{2}\Big(\|T\|^2 + w^2(\widetilde{T}) + w(|T|\widetilde{T} + \widetilde{T}|T|)\Big)^{\frac{1}{2}},$$

as desired. □

Remark 3.6. *From [48] we have if $T, X \in \mathbb{B}(\mathcal{H})$, then $w(T^*X + XT) \leq 2\|T\|w(X)$ and so*

$$\begin{aligned} \|T\|^2 + w^2(\widetilde{T}) + w(|T|\widetilde{T} + \widetilde{T}|T|) &\leq \|T\|^2 + w^2(\widetilde{T}) + 2\||T|\|w(\widetilde{T}) \\ &= \|T\|^2 + w^2(\widetilde{T}) + 2\|T\|w(\widetilde{T}) \\ &= (\|T\| + w(\widetilde{T}))^2. \end{aligned}$$

Therefore,

$$w(T) \leq \frac{1}{2}\left(\|T\|^2 + w^2(\widetilde{T}) + w(|T|\widetilde{T} + \widetilde{T}|T|)\right)^{\frac{1}{2}} \leq \frac{1}{2}(\|T\| + w(\widetilde{T})).$$

Thus the inequality in Theorem 3.5 is stronger than the inequality (1.6).

Next we need the following two lemmas, first one is known as Heinz inequality and second one is known as Buzano's inequality.

Lemma 3.2. *([60]) Let $T \in \mathbb{B}(\mathcal{H})$. Then, for all $x, y \in \mathcal{H}$,*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}y, y\rangle,$$

for all $\alpha \in [0, 1]$.

Lemma 3.3. *([33]) Let $a, b, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle a, e\rangle\langle e, b\rangle| \leq \frac{1}{2}(|\langle a, b\rangle| + \|a\|\|b\|).$$

Now we are in a position to prove an upper bound which improves on both the upper bounds in (1.2) and (1.3).

Theorem 3.7. *If $T \in \mathbb{B}(\mathcal{H})$, then*

$$w^2(T) \leq \frac{1}{4}\left\||T|^{4\alpha} + |T^*|^{4(1-\alpha)}\right\| + \frac{1}{2}w\left(|T^*|^{2(1-\alpha)}|T|^{2\alpha}\right),$$

for all $\alpha \in [0, 1]$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$, then by Lemma 3.2, we have

$$\begin{aligned} |\langle Tx, x\rangle|^2 &\leq \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}x, x\rangle \\ &= \langle |T|^{2\alpha}x, x\rangle\langle x, |T^*|^{2(1-\alpha)}x\rangle \\ &\leq \frac{1}{2}\||T|^{2\alpha}x\|\||T^*|^{2(1-\alpha)}x\| + \frac{1}{2}|\langle |T|^{2\alpha}x, |T^*|^{2(1-\alpha)}x\rangle| \text{ (by Lemma 3.3)} \\ &= \frac{1}{2}\langle |T|^{4\alpha}x, x\rangle^{\frac{1}{2}}\langle |T^*|^{4(1-\alpha)}x, x\rangle^{\frac{1}{2}} + \frac{1}{2}|\langle |T|^{2\alpha}x, |T^*|^{2(1-\alpha)}x\rangle| \\ &\leq \frac{1}{4}\left(\langle |T|^{4\alpha}x, x\rangle + \langle |T^*|^{4(1-\alpha)}x, x\rangle\right) + \frac{1}{2}|\langle |T|^{2\alpha}x, |T^*|^{2(1-\alpha)}x\rangle| \\ &= \frac{1}{4}\langle\left(|T|^{4\alpha} + |T^*|^{4(1-\alpha)}\right)x, x\rangle + \frac{1}{2}|\langle |T^*|^{2(1-\alpha)}|T|^{2\alpha}x, x\rangle| \\ &\leq \frac{1}{4}\||T|^{4\alpha} + |T^*|^{4(1-\alpha)}\| + \frac{1}{2}w\left(|T^*|^{2(1-\alpha)}|T|^{2\alpha}\right). \end{aligned}$$

Therefore, taking supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we get

$$w^2(T) \leq \frac{1}{4}\||T|^{4\alpha} + |T^*|^{4(1-\alpha)}\| + \frac{1}{2}w\left(|T^*|^{2(1-\alpha)}|T|^{2\alpha}\right),$$

as desired. □

In particular, considering $\alpha = \frac{1}{2}$ in Theorem 3.7, we get the following inequality [27, Cor. 2.6]

$$w^2(T) \leq \frac{1}{4}\||T|^2 + |T^*|^2\| + \frac{1}{2}w\left(|T^*||T|\right). \tag{3.3}$$

Next, considering the minimum over $\alpha \in [0, 1]$ in Theorem 3.7, we get the following corollary.

Corollary 3.1. *If $T \in \mathbb{B}(\mathcal{H})$, then*

$$w^2(T) \leq \min_{0\leq\alpha\leq1}\left\{\frac{1}{4}\left\||T|^{4\alpha} + |T^*|^{4(1-\alpha)}\right\| + \frac{1}{2}w\left(|T^*|^{2(1-\alpha)}|T|^{2\alpha}\right)\right\}.$$

Clearly, we have

$$\min_{0\leq\alpha\leq1}\left\{\frac{1}{4}\left\||T|^{4\alpha} + |T^*|^{4(1-\alpha)}\right\| + \frac{1}{2}w\left(|T^*|^{2(1-\alpha)}|T|^{2\alpha}\right)\right\} \leq \frac{1}{4}\||T|^2 + |T^*|^2\| + \frac{1}{2}w\left(|T^*||T|\right).$$

In order to appreciate the inequality in Corollary 3.1 we give the following example.

Example 3.8. *Let*

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then by simple calculation, we have

$$|T|^{4\alpha}+|T^*|^{4(1-\alpha)}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1+16^{1-\alpha} & 0\\ 0 & 0 & 16^{\alpha}\end{pmatrix} \textit{ and } |T^*|^{2(1-\alpha)}|T|^{2\alpha}=\begin{pmatrix}1 & 0 & 0\\ 0 & 4^{1-\alpha} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Therefore, we have

$$\||T|^{4\alpha}+|T^*|^{4(1-\alpha)}\| = \begin{cases}1+16^{1-\alpha} & \textit{if } 0\le\alpha\le r_0\\ 16^{\alpha} & \textit{if } r_0\le\alpha\le 1,\end{cases}$$

where $16^{r_0}=\frac{1+\sqrt{65}}{2}$. *Also,*

$$w(|T^*|^{2(1-\alpha)}|T|^{2\alpha})=\begin{cases}4^{1-\alpha} & \textit{if } 0\le\alpha<1\\ 1 & \textit{if } \alpha=1.\end{cases}$$

Now,

$$\min_{0\le\alpha\le r_0}\left\{\frac{1+16^{1-\alpha}}{4}+\frac{4^{1-\alpha}}{2}\right\} = \left\{\frac{33+\sqrt{65}}{4(1+\sqrt{65})}+\frac{4\sqrt{2}}{2\sqrt{1+\sqrt{65}}}\right\}\approx 2.0724.$$

$$\min_{r_0\le\alpha\le 1}\left\{\frac{16^{\alpha}}{4}+\frac{4^{1-\alpha}}{2}\right\} = \left\{\frac{1+\sqrt{65}}{8}+\frac{4\sqrt{2}}{2\sqrt{1+\sqrt{65}}}\right\}\approx 2.0724.$$

Thus,

$$\min_{0\le\alpha\le 1}\left\{\frac{1}{4}\||T|^{4\alpha}+|T^*|^{4(1-\alpha)}\|+\frac{1}{2}w\left(|T^*|^{2(1-\alpha)}|T|^{2\alpha}\right)\right\}\approx 2.0724.$$

Also, we have

$$\frac{1}{4}\||T|^2+|T^*|^2\|+\frac{1}{2}w\left(|T^*||T|\right)=\frac{9}{4}=2.25.$$

Hence, for the matrix T,

$$\min_{0\le\alpha\le 1}\left\{\frac{1}{4}\||T|^{4\alpha}+|T^*|^{4(1-\alpha)}\|+\frac{1}{2}w\left(|T^*|^{2(1-\alpha)}|T|^{2\alpha}\right)\right\}<\frac{1}{4}\||T|^2+|T^*|^2\|+\frac{1}{2}w\left(|T^*||T|\right).$$

Thus the inequality in Corollary 3.1 is a refinement of the inequality (3.3) and hence it also refines the inequalities (1.2) and (1.3).

3.3 Numerical radius inequalities of product of operators

We begin this section with the following two lemmas, first one can be found in [83, p. 20] and second one is obvious.

Lemma 3.4. *Let $T \in \mathbb{B}(\mathcal{H})$ be posittive and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then for all $r \geq 1$, we have*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle.$$

Lemma 3.5. *For all $a, b \in \mathbb{R}$, we have $|a + b| \leq \sqrt{2}|a + \mathrm{i}b|$.*

We now prove an improvement of the numerical radius bounds product of two bounded linear operator namely in [40], if $S, T \in \mathbb{B}(\mathcal{H})$ and $r \geq 1$, then

$$w^r(T^*S) \leq \frac{1}{2}(\||S|^{2r} + |T|^{2r}\|). \tag{3.4}$$

.

Theorem 3.9. *Let $S, T \in \mathbb{B}(\mathcal{H})$. Then, for all $r \geq 1$,*

$$w^{2r}(T^*S) \leq \frac{1}{2}w^2\left(|S|^{2r} + \mathrm{i}|T|^{2r}\right).$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then we have,

$$\begin{aligned}
|\langle T^*Sx, x\rangle|^{2r} &= |\langle Sx, Tx\rangle|^{2r} \\
&\leq \|Sx\|^{2r}\|Tx\|^{2r} \\
&\leq \langle |S|^{2r}x, x\rangle\langle |T|^{2r}x, x\rangle \ (\textit{by Lemma 3.4}) \\
&\leq \frac{1}{4}\left(\langle |S|^{2r}x, x\rangle + \langle |T|^{2r}x, x\rangle\right)^2 \\
&\leq \frac{1}{2}\left|\langle |S|^{2r}x, x\rangle + \mathrm{i}\langle |T|^{2r}x, x\rangle\right|^2 \ (\textit{by Lemma 3.5}) \\
&= \frac{1}{2}\left|\langle\left(|S|^{2r} + \mathrm{i}|T|^{2r}\right)x, x\rangle\right|^2 \\
&\leq \frac{1}{2}w^2\left(|S|^{2r} + \mathrm{i}|T|^{2r}\right).
\end{aligned}$$

Taking supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we get

$$w^{2r}(S^*T) \leq \frac{1}{2}w^2(|S|^{2r} + \mathrm{i}|T|^{2r}),$$

as required. ☐

It is easy to verify that, $w^2(|S|^{2r}+\mathrm{i}|T|^{2r}) \leq \||S|^{4r}+|T|^{4r}\|$. Thus, the inequality in Theorem 3.9 refines the inequality (3.4) for $r \geq 2$.

For next result we need the following lemma, that can be found in [9].

Lemma 3.6. *Let f be a non-negative increasing convex function on $[0,\infty)$ and let $S,T \in \mathbb{B}(\mathcal{H})$ be positive. Then*

$$\left\| f\left(\frac{S+T}{2}\right)\right\| \leq \left\|\frac{f(S)+f(T)}{2}\right\|.$$

Theorem 3.10. *If $S,T \in \mathbb{B}(\mathcal{H})$, then for $r \geq 1$,*

$$w^{2r}(T^*S) \leq \frac{1}{2}\left(\frac{\||T|^2|S|^2+|S|^2|T|^2\|}{2}\right)^r + \frac{1}{4}\left\||T|^{4r}+|S|^{4r}\right\|.$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|=1$. Then, we have

$$\begin{aligned}
|\langle T^*Sx,x\rangle|^2 &= |\langle Sx,Tx\rangle|^2 \\
&\leq \|Sx\|^2\|Tx\|^2 \\
&\leq \langle |S|^2x,x\rangle\langle |T|^2x,x\rangle \\
&\leq \frac{1}{4}\left(\langle |S|^2x,x\rangle + \langle |T|^2x,x\rangle\right)^2 \\
&= \frac{1}{4}\left(\langle\left(|S|^2+|T|^2\right)x,x\rangle\right)^2 \\
&\leq \frac{1}{4}\langle\left(|S|^2+|T|^2\right)^2 x,x\rangle \\
&= \frac{1}{4}\langle\left(|S|^4+|T|^4+|S|^2|T|^2+|T|^2|S|^2\right)x,x\rangle \\
&\leq \frac{1}{4}\|\left(|S|^4+|T|^4+|S|^2|T|^2+|T|^2|S|^2\right)\| \\
&\leq \frac{1}{4}\||S|^4+|T|^4\| + \frac{1}{4}\||S|^2|T|^2+|T|^2|S|^2\|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|\langle T^*Sx,x\rangle|^{2r} &\leq \left(\frac{1}{4}\||S|^4+|T|^4\| + \frac{1}{4}\||S|^2|T|^2+|T|^2|S|^2\|\right)^r \\
&\leq \frac{1}{2}\left(\frac{\||S|^4+|T|^4\|}{2}\right)^r + \frac{1}{2}\left(\frac{\||S|^2|T|^2+|T|^2|S|^2\|}{2}\right)^r \\
&\leq \frac{1}{4}\||S|^{4r}+|T|^{4r}\| + \frac{1}{2}\left(\frac{\||S|^2|T|^2+|T|^2|S|^2\|}{2}\right)^r \ (\textit{by Lemma 3.6}).
\end{aligned}$$

Hence, taking supremum over $x \in \mathcal{H}$, $\|x\| = 1$, we get

$$w^{2r}(T^*S) \leq \frac{1}{2}\left(\frac{\||T|^2|S|^2 + |S|^2|T|^2\|}{2}\right)^r + \frac{1}{4}\||T|^{4r} + |S|^{4r}\|.$$

□

Remark 3.11. *By using convexity property of $f(t) = t^r, r \geq 1$, we have*

$$\begin{aligned}
& \frac{1}{2}\left(\frac{\||T|^2|S|^2 + |S|^2|T|^2\|}{2}\right)^r + \frac{1}{4}\||T|^{4r} + |S|^{4r}\| \\
= \ & \frac{1}{2}w^r\left(\frac{|T|^2|S|^2 + |S|^2|T|^2}{2}\right) + \frac{1}{4}\||T|^{4r} + |S|^{4r}\| \\
\leq \ & \frac{1}{2}\left(\frac{1}{2}w(|T|^2|S|^2) + \frac{1}{2}w(|S|^2|T|^2)\right)^r + \frac{1}{4}\||T|^{4r} + |S|^{4r}\| \\
\leq \ & \frac{1}{4}\left(w^r(|T|^2|S|^2) + w^r(|S|^2|T|^2)\right) + \frac{1}{4}\||T|^{4r} + |S|^{4r}\| \\
= \ & \frac{1}{2}w^r(|T|^2|S|^2) + \frac{1}{4}\||T|^{4r} + |S|^{4r}\|.
\end{aligned}$$

Thus, for all $r \geq 1$, we have

$$\begin{aligned}
w^{2r}(T^*S) \ & \leq \ \frac{1}{2}\left(\frac{\||T|^2|S|^2 + |S|^2|T|^2\|}{2}\right)^r + \frac{1}{4}\||T|^{4r} + |S|^{4r}\| \\
& \leq \ \frac{1}{2}w^r(|T|^2|S|^2) + \frac{1}{4}\||T|^{4r} + |S|^{4r}\|.
\end{aligned}$$

Therefore, the inequality in Theorem 3.10 is a refinement of $w^{2r}(T^*S) \leq \frac{1}{2}w^r(|T|^2|S|^2) + \frac{1}{4}\||T|^{4r} + |S|^{4r}\|$, that in [54].

Finally, as a direct application of Theorem 3.3, we obtain the following inequality for the generalized commutators of operators.

Theorem 3.12. *Let $S, T, X, Y \in \mathbb{B}(\mathcal{H})$. Then*

$$\begin{aligned}
& w(SXT \pm TYS) \\
\leq \ & 2\sqrt{2}\|T\| \max\{\|X\|, \|Y\|\} \sqrt{w^2(S) - \frac{|\ \|\Re(S) + \Im(S)\|^2 - \|\Re(S) - \Im(S)\|^2\ |}{4}}.
\end{aligned}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. If $\|X\| \leq 1$ and $\|Y\| \leq 1$, then by Cauchy Schwarz inequality, we get

$$\begin{aligned}|\langle (SX \pm YS)x, x\rangle| &\leq |\langle Xx, S^*x\rangle| + |\langle Sx, Y^*x\rangle| \\ &\leq \|S^*x\| + \|Sx\| \\ &\leq \sqrt{2}(\|S^*x\|^2 + \|Sx\|^2)^{\frac{1}{2}} \\ &\leq \sqrt{2}\|SS^* + S^*S\|^{\frac{1}{2}}.\end{aligned}$$

Taking supremum over $\|x\| = 1$, we get

$$w(SX \pm YS) \leq \sqrt{2}\|SS^* + S^*S\|^{\frac{1}{2}}, \tag{3.5}$$

when $\|X\| \leq 1$ and $\|Y\| \leq 1$. Now we consider the general case, that is, $X, Y \in \mathbb{B}(\mathcal{H})$ are arbitrary. If $X = Y = 0$, then (3.5) holds trivially. If $\max\{\|X\|, \|Y\|\} \neq 0$, then $\left\|\frac{X}{\max\{\|X\|,\|Y\|\}}\right\| \leq 1$ and $\left\|\frac{Y}{\max\{\|X\|,\|Y\|\}}\right\| \leq 1$, and so it follows from the inequality (3.5) that

$$w(SX \pm YS) \leq \sqrt{2}\max\{\|X\|, \|Y\|\}\, \|SS^* + S^*S\|^{\frac{1}{2}}. \tag{3.6}$$

Replacing X by XT and Y by TY in the inequality (3.6) we get,

$$w(SXT \pm TYS) \leq \sqrt{2}\max\{\|XT\|, \|TY\|\}\, \|SS^* + S^*S\|^{\frac{1}{2}}.$$

This implies that

$$w(SXT \pm TYS) \leq \sqrt{2}\|T\|\max\{\|X\|, \|Y\|\}\, \|SS^* + S^*S\|^{\frac{1}{2}}.$$

Therefore, by using the inequality in Theorem 3.3, we have

$$\begin{aligned}&w(SXT \pm TYS) \\ \leq\ & 2\sqrt{2}\|T\|\max\{\|X\|, \|Y\|\}\sqrt{w^2(S) - \frac{|\ \|\Re(S) + \Im(S)\|^2 - \|\Re(S) - \Im(S)\|^2\ |}{4}},\end{aligned}$$

as desired. □

Considering $X = Y = I$ (the identity operator) in Theorem 3.12 we get the following corollary.

Corollary 3.2. *If $S, T \in \mathbb{B}(\mathcal{H})$, then*

$$w(ST \pm TS) \leq 2\sqrt{2}\|T\|\sqrt{w^2(S) - \frac{|\ \|\Re(S) + \Im(S)\|^2 - \|\Re(S) - \Im(S)\|^2\ |}{4}}.$$

Remark 3.13. *Let $S, T \in \mathbb{B}(\mathcal{H})$. (i) Fong and Holbrook [48] proved that*

$$w(ST + TS) \le 2\sqrt{2}\|T\| w(S). \tag{3.7}$$

Clearly, the inequality in Corollary 3.2 is stronger than the inequality (3.7).
(ii) Hirzallah and Kittaneh [55] improved on the inequality (3.7) to prove that

$$w(ST \pm TS) \quad \le \quad 2\sqrt{2}\|T\|\sqrt{w^2(S) - \frac{\big|\ \|\Re(S)\|^2 - \|\Im(T)\|^2\ \big|}{2}}. \tag{3.8}$$

Considering the same examples as in Remark 3.2(iii), we see that the inequalities in Corollary 3.2 and (3.8) are not comparable, in general.
(iii) It follows from the Corollary 3.2 that if $w(ST \pm TS) = 2\sqrt{2}\|T\|w(T)$, then

$$\|\Re(S) + \Im(S)\| = \|\Re(S) - \Im(S)\|.$$

At the end of this section, we give a sufficient condition for the equality of $w(T) = \frac{1}{2}\|T^*T + TT^*\|^{1/2}$. For this purpose first we note the following known lemma.

Lemma 3.7. *[66] Let $S, T \in \mathbb{B}(\mathcal{H})$ be positive. Then, $\|S + T\| = \|S\| + \|T\|$ if and only if $\|ST\| = \|S\|\|T\|$.*

Theorem 3.14. *Let $T \in \mathbb{B}(\mathcal{H})$. Then $\|T\|^4 = \|\Re^2(T)\Im^2(T)\|$ implies*

$$w^2(T) = \frac{1}{4}\|T^*T + TT^*\|.$$

Proof. We have

$$\begin{aligned} \|T\|^4 &= \|\Re^2(T)\Im^2(T)\| \le \|\Re^2(T)\|\|\Im^2(T)\| = \|\Re(T)\|^2\|\Im(T)\|^2 \\ &\le \frac{1}{2}\left(\|\Re(T)\|^4 + \|\Im(T)\|^4\right) \le \max\left(\|\Re(T)\|^4, \|\Im(T)\|^4\right) \\ &\le w^4(T) \le \|T\|^4. \end{aligned}$$

This implies that

$$\|\Re^2(T)\Im^2(T)\| = \|\Re(T)\|^2\|\Im(T)\|^2. \tag{3.9}$$

Also, we have

$$\frac{1}{2}\left(\|\Re(T)\|^4 + \|\Im(T)\|^4\right) = \max\left(\|\Re(T)\|^4, \|\Im(T)\|^4\right) = w^4(T). \tag{3.10}$$

This implies that

$$\|\Re(T)\| = \|\Im(T)\| = w(T). \tag{3.11}$$

Now, by using Lemma 3.7, it follows from the identity (3.9) that

$$\begin{aligned}\frac{1}{2}\|\Re^2(T)+\Im^2(T)\| &= \frac{1}{2}\left(\|\Re^2(T)\|+\|\Im^2(T)\|\right)\\ &= \frac{1}{2}\left(\|\Re(T)\|^2+\|\Im(T)\|^2\right)\\ &= \|\Re(T)\|^2 = w^2(T)\ (\textit{using}\ (3.11)).\end{aligned}$$

This completes the proof.

□

It should be mentioned here that the converse of Theorem 3.14 is not true, in general. For example, we consider $T = \begin{pmatrix} 0 & 3 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$. Then, $w^2(T) = \frac{1}{4}\|T^*T + TT^*\| = \frac{9}{4}$, however $\|T\|^4 \neq \|\Re^2(T)\Im^2(T)\|$.

3.4 Application : Bounds for zeros of polynomials

Suppose $p(z) = z^n + a_n z^{n-1} + \ldots + a_2 z + a_1$ is a complex monic polynomial of degree $n \geq 2$ and $a_1 \neq 0$. Location of the zeros of $p(z)$ have been obtained by applying numerical radius inequalities to Frobenius companion matrix of the polynomial $p(z)$. The Frobenius companion matrix of the polynomial $p(z)$ is given by

$$C_p = \begin{pmatrix} -a_n & -a_{n-1} & \ldots. & -a_2 & -a_1\\ 1 & 0 & \ldots & 0 & 0\\ 0 & 1 & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \ldots. & 1 & 0\end{pmatrix}.$$

The characteristic polynomial of C_p is the polynomial $p(z)$. Thus, the zeros of $p(z)$ are exactly the eigenvalues of C_p, see [57, p. 316]. The square of C_p is given by

$$C_p^2 = \begin{pmatrix} b_n & b_{n-1} & \cdots\cdots & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{pmatrix},$$

where $b_j = a_n a_j - a_{j-1}$ for $j = 1, 2, \ldots, n$, with $a_0 = 0$.
Also,

$$C_p^3 = \begin{pmatrix} c_n & c_{n-1} & \cdots\cdots & c_4 & c_3 & c_2 & c_1 \\ b_n & b_{n-1} & \cdots\cdots & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_4 & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \end{pmatrix},$$

where $b_j = a_n a_j - a_{j-1}$ and $c_j = -a_n b_j + a_{n-1} a_j - a_{j-2}$ for $j = 1, 2, \ldots, n$, with $a_0 = a_{-1} = 0$, and

$$C_p^4 = \begin{pmatrix} d_n & d_{n-1} & \cdots\cdots & d_5 & d_4 & d_3 & d_2 & d_1 \\ c_n & c_{n-1} & \cdots\cdots & c_5 & c_4 & c_3 & c_2 & c_1 \\ b_n & b_{n-1} & \cdots\cdots & b_5 & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_5 & -a_4 & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $b_j = a_n a_j - a_{j-1}$, $c_j = -a_n b_j + a_{n-1} a_j - a_{j-2}$, and $d_j = -a_n c_j - a_{n-1} b_{j-1} + a_{n-2} a_j - a_{j-3}$ for $j = 1, 2, \ldots, n$, with $a_0 = a_{-1} = a_{-2} = 0$.

The exact value of $\|C_p\|$ is well known (see in [67]), it is given by

$$\|C_p\| = \sqrt{\frac{\alpha + 1 + \sqrt{(\alpha+1)^2 - 4|a_1|^2}}{2}}, \tag{3.12}$$

where $\alpha = \sum_{j=1}^{n} |a_j|^2$.

An estimation of $\|C_p^2\|$ obtained in [64] is as follows

$$\|C_p^2\| \leq \sqrt{\frac{\delta + 1 + \sqrt{(\delta-1)^2 + 4\delta'}}{2}}, \tag{3.13}$$

where $\delta = \frac{1}{2}\left(\alpha + \beta + \sqrt{(\alpha-\beta)^2 + 4|\gamma|^2}\right)$ and $\delta' = \frac{1}{2}\left(\alpha' + \beta' + \sqrt{(\alpha'-\beta')^2 + 4|\gamma'|^2}\right)$, $\alpha = \sum_{j=1}^{n} |a_j|^2$, $\beta = \sum_{j=1}^{n} |b_j|^2$, $\alpha' = \sum_{j=3}^{n} |a_j|^2$, $\beta' = \sum_{j=3}^{n} |b_j|^2$, $\gamma = -\sum_{j=1}^{n} \bar{a}_j b_j$, $\gamma' = -\sum_{j=3}^{n} \bar{a}_j b_j$.

We note that

$$\|C_p^2\|^{\frac{1}{2}} \leq \left(\sqrt{\frac{\delta + 1 + \sqrt{(\delta-1)^2 + 4\delta'}}{2}}\right)^{1/2} \leq \sqrt{\frac{\alpha + 1 + \sqrt{(\alpha+1)^2 - 4|a_1|^2}}{2}} = \|C_p\|.$$

Motivated by the above estimation, here we will obtain an estimation of $\|C_p^4\|^{1/4}$. For this purpose first we note the following norm inequality for the sum of two positive operators.

Lemma 3.8. *[66] If $S, T \in \mathbb{B}(\mathcal{H})$ are positive, then*

$$\|S+T\| \leq \frac{1}{2}\left(\|S\| + \|T\| + \sqrt{(\|S\| - \|T\|)^2 + 4\left\|S^{\frac{1}{2}}T^{\frac{1}{2}}\right\|^2}\right).$$

Now, we are in a position to obtain an estimation of $\|C_p^4\|^{1/4}$. Let $C_p^4 = R + S + T$, where

$$R = \begin{pmatrix} d_n & d_{n-1} & \ldots\ldots & d_5 & d_4 & d_3 & d_2 & d_1 \\ c_n & c_{n-1} & \ldots\ldots & c_5 & c_4 & c_3 & c_2 & c_1 \\ 0 & 0 & \ldots\ldots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots. & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 0 & \ldots\ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots\ldots & 0 & 0 & 0 & 0 & 0 \\ b_n & b_{n-1} & \ldots\ldots & b_5 & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \ldots. & -a_5 & -a_4 & -a_3 & -a_2 & -a_1 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots. & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T = \begin{pmatrix} 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & ..... & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .... & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & ... & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & ... & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & .... & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now,

$$\begin{aligned} \|C_p^4\|^2 &= \|R+S+T\|^2 \\ &= \|(R+S+T)^*(R+S+T)\| \\ &= \|R^*R+S^*S+T^*T\| \,(\textit{since } R^*S = R^*T = S^*R = S^*T = T^*R = T^*S = 0) \\ &\leq \|R^*R+S^*S\| + \|T^*T\| \\ &\leq \frac{1}{2}\left(\|R\|^2+\|S\|^2+\sqrt{(\|R\|^2-\|S\|^2)^2+4\|RS^*\|^2}\right)+1 \text{ (by Lemma 3.8)}. \end{aligned}$$

By simple calculations, we have

$$\begin{aligned} \|R\|^2 &= \|R^*R\| = \|RR^*\| \\ &= \frac{1}{2}\left(\alpha_1+\beta_1+\sqrt{(\alpha_1-\beta_1)^2+4|\gamma_1|^2}\right) = \delta_1, \end{aligned}$$

where $\alpha_1 = \sum_{j=1}^n |d_j|^2$, $\beta_1 = \sum_{j=1}^n |c_j|^2$, $\gamma_1 = \sum_{j=1}^n d_j\bar{c_j}$,

$$\begin{aligned} \|S\|^2 &= \|S^*S\| = \|SS^*\| \\ &= \frac{1}{2}\left(\alpha+\beta+\sqrt{(\alpha-\beta)^2+4|\gamma|^2}\right) = \delta, \end{aligned}$$

where $\alpha = \sum_{j=1}^n |a_j|^2$, $\beta = \sum_{j=1}^n |b_j|^2$, $\gamma = -\sum_{j=1}^n b_j\bar{a_j}$,

$$\begin{aligned} &\|RS^*\|^2 \\ =\ & \frac{1}{2}\left(|\gamma_2|^2+|\gamma_3|^2+|\gamma_4|^2+|\gamma_5|^2+\sqrt{((|\gamma_2|^2+|\gamma_3|^2)-(|\gamma_4|^2+|\gamma_5|^2))^2+4|\gamma_2\bar{\gamma_4}+\gamma_3\bar{\gamma_5}|^2}\right) \\ =\ & \delta_2, \end{aligned}$$

where $\gamma_2 = \sum_{j=1}^n d_j\bar{b_j}$, $\gamma_3 = \sum_{j=1}^n d_j\bar{a_j}$, $\gamma_4 = \sum_{j=1}^n c_j\bar{b_j}$, $\gamma_5 = \sum_{j=1}^n c_j\bar{a_j}$.

Therefore,

$$\|C_p^4\| \leq \sqrt{\frac{1}{2}\left(\delta_1+\delta+\sqrt{(\delta_1-\delta)^2+4\delta_2}\right)+1}. \tag{3.14}$$

We observe that the estimation of $\|C_p^4\|^{1/4}$ in (3.14) is incomparable with the existing estimation of $\|C_p^2\|^{1/2}$ in (3.13). In the following theorem we derive an upper bound for the spectral radius of the Frobenius companion matrix C_p, by using the estimations in (3.13) and (3.14).

Theorem 3.15. *The following inequality holds:*

$$r(C_p) \leq \left\{\frac{1}{4}\left(\frac{\delta+1+\sqrt{(\delta-1)^2+4\delta'}}{2}\right)+\frac{3}{4}\left(\frac{1}{2}\left(\delta_1+\delta+\sqrt{(\delta_1-\delta)^2+4\delta_2}\right)+1\right)^{\frac{1}{2}}\right\}^{\frac{1}{4}},$$

where $\delta' = \frac{1}{2}\left(\alpha'+\beta'+\sqrt{(\alpha'-\beta')^2+4|\gamma'|^2}\right)$,
$\delta = \frac{1}{2}\left(\alpha+\beta+\sqrt{(\alpha-\beta)^2+4|\gamma|^2}\right)$,
$\delta_1 = \frac{1}{2}\left(\alpha_1+\beta_1+\sqrt{(\alpha_1-\beta_1)^2+4|\gamma_1|^2}\right)$,
$\delta_2 = \frac{1}{2}\left(|\gamma_2|^2+|\gamma_3|^2+|\gamma_4|^2+|\gamma_5|^2+\sqrt{\left((|\gamma_2|^2+|\gamma_3|^2)-(|\gamma_4|^2+|\gamma_5|^2)\right)^2+4|\gamma_2\bar{\gamma_4}+\gamma_3\bar{\gamma_5}|^2}\right)$,
$\alpha' = \sum_{j=3}^n |a_j|^2$, $\beta' = \sum_{j=3}^n |b_j|^2$, $\gamma' = -\sum_{j=3}^n \bar{a_j}b_j$,
$\alpha = \sum_{j=1}^n |a_j|^2$, $\beta = \sum_{j=1}^n |b_j|^2$, $\gamma = -\sum_{j=1}^n b_j\bar{a_j}$,
$\alpha_1 = \sum_{j=1}^n |d_j|^2$, $\beta_1 = \sum_{j=1}^n |c_j|^2$, $\gamma_1 = \sum_{j=1}^n d_j\bar{c_j}$,
$\gamma_2 = \sum_{j=1}^n d_j\bar{b_j}$, $\gamma_3 = \sum_{j=1}^n d_j\bar{a_j}$, $\gamma_4 = \sum_{j=1}^n c_j\bar{b_j}$, $\gamma_5 = \sum_{j=1}^n c_j\bar{a_j}$.

Proof. Let $T \in \mathbb{B}(\mathcal{H})$. Putting $T = T^2$ in the inequality $w^2(T) \leq \frac{1}{4}\|T^*T+TT^*\|+\frac{1}{2}w(T^2)$ (see [2, Th. 2.4]), we get

$$w^2(T^2) \quad \leq \quad \frac{1}{4}\left\||T^2|^2+|(T^*)^2|^2\right\|+\frac{1}{2}w(T^4).$$

It follows that

$$r^2(T) = r(T^2) \leq w(T^2) \leq \left\{\frac{1}{4}\left\||T^2|^2+|(T^*)^2|^2\right\|+\frac{1}{2}w(T^4)\right\}^{\frac{1}{2}},$$

i.e.,

$$r(T) \leq \left\{\frac{1}{4}\left\||T^2|^2+|(T^*)^2|^2\right\|+\frac{1}{2}w(T^4)\right\}^{\frac{1}{4}}. \tag{3.15}$$

Now, it follows from (3.15) and the inequality $\|C_p^*C_p+C_pC_p^*\| \leq \|C_p\|^2+\|C_p^2\|$ (see [24, Remark

3.9]) that

$$\begin{aligned} r(C_p) &\leq \left\{\frac{1}{4}\left\||C_p^2|^2+|(C_p^*)^2|^2\right\|+\frac{1}{2}w(C_p^4)\right\}^{\frac{1}{4}} \\ &\leq \left\{\frac{1}{4}(\|C_p^2\|^2+\|C_p^4\|)+\frac{1}{2}\|C_p^4\|\right\}^{\frac{1}{4}} \\ &\leq \left\{\frac{1}{4}\left\|C_p^2\right\|^2+\frac{3}{4}\left\|C_p^4\right\|\right\}^{\frac{1}{4}}. \end{aligned}$$

Therefore, the required inequality follows by using the estimations in (3.13) and (3.14). □

By using the fact $|\lambda_j(C_p)| \leq r(C_p)$, where $\lambda_j(C_p)$ is the j-th eigenvalue of C_p, we infer the following estimation for the zeros of the polynomial $p(z)$.

Theorem 3.16. *If z is any zero of $p(z)$, then*

$$|z| \leq \left\{\frac{1}{4}\left(\frac{\delta+1+\sqrt{(\delta-1)^2+4\delta'}}{2}\right)+\frac{3}{4}\left(\frac{1}{2}\left(\delta_1+\delta+\sqrt{(\delta_1-\delta)^2+4\delta_2}\right)+1\right)^{\frac{1}{2}}\right\}^{\frac{1}{4}},$$

where δ, δ_1, δ_2 and δ' are same as in Theorem 3.15.

Applying the spectral mapping theorem, we conclude that if z is any zero of $p(z)$ then $|z| \leq \|C_p^4\|^{\frac{1}{4}}$. Thus, by using the inequality (3.14) we achieve another new estimation for the zeros of $p(z)$.

Theorem 3.17. *If z is any zero of $p(z)$, then*

$$|z| \leq \left\{\frac{1}{2}\left(\delta_1+\delta+\sqrt{(\delta_1-\delta)^2+4\delta_2}\right)+1\right\}^{\frac{1}{8}},$$

where δ, δ_1 and δ_2 are given in Theorem 3.15.

Again, putting $T = T^2$ in the inequality $w(T) \leq \frac{1}{2}\left(\|T\|+\|T^2\|^{\frac{1}{2}}\right)$ (see [64, Th. 1]), and proceeding as (3.15), we get

$$r(T) \leq \left\{\frac{1}{2}\|T^2\|+\frac{1}{2}\|T^4\|^{\frac{1}{2}}\right\}^{\frac{1}{2}}. \tag{3.16}$$

Proceeding similarly as in Theorem 3.15 we obtain the following estimation by using the inequalities in (3.16), (3.13) and (3.14).

Theorem 3.18. *If z is any zero of $p(z)$, then*

$$|z| \leq \left\{ \frac{1}{2}\sqrt{\frac{\delta+1+\sqrt{(\delta-1)^2+4\delta'}}{2}} + \frac{1}{2}\left(\frac{1}{2}\left(\delta_1+\delta+\sqrt{(\delta_1-\delta)^2+4\delta_2}\right)+1\right)^{\frac{1}{4}} \right\}^{\frac{1}{2}},$$

where δ, δ_1, δ_1 and δ' are given in Theorem 3.15.

Finally, we compare the bounds obtained here for the zeros of $p(z)$ with the existing ones. First we note some well known existing bounds. Let z be any zero of $p(z)$. Then

Linden [69] obtained that

$$|z| \leq \frac{|a_n|}{n} + \left(\frac{n-1}{n}\left(n-1+\sum_{j=1}^{n}|a_j|^2-\frac{|a_n|^2}{n}\right)\right)^{\frac{1}{2}}.$$

Montel [50, Th. 3] obtained that

$$|z| \leq \max\{1, |a_1|+\cdots+|a_n|\}.$$

Cauchy [57] obtained that

$$|z| \leq 1+\max\{|a_1|,\cdots,|a_n|\}.$$

Kittaneh [65] proved that

$$|z| \leq \frac{1}{2}\left(|a_n|+1+\sqrt{(|a_n|-1)^2+4\sqrt{\sum_{j=1}^{n-1}|a_j|^2}}\right).$$

Fujii and Kubo [49] obtained that

$$|z| \leq \cos\frac{\pi}{n+1}+\frac{1}{2}\left(|a_n|+\sqrt{\sum_{j=1}^{n}|a_j|^2}\right).$$

Bhunia and Paul [19, Th. 2.6] proved that

$$|z|^2 \leq \cos^2\frac{\pi}{n+1}+|a_{n-1}|+\frac{1}{4}\left(|a_n|+\sqrt{\alpha}\right)^2+\frac{1}{2}\sqrt{\alpha-|a_n|^2}+\frac{1}{2}\sqrt{\alpha},$$

where $\alpha=\sum_{j=1}^{n}|a_j|^2$.

We consider a polynomial $p(z)=z^3+z^2+\frac{1}{2}z+1$. Different upper bounds for the modulus of the zeros of this polynomial, mentioned above, are as shown in the following table.

Linden [69]	1.9492
Montel[50]	2.5
Cauchy[57]	2
Kittaneh[65]	2.0547
Fujii and Kubo[49]	1.9571
Bhunia and Paul[19]	1.96761

However, Theorem 3.16 gives $|z| \leq 1.38047091798$, Theorem 3.17 gives $|z| \leq 1.3798438819$ and Theorem 3.18 gives $|z| \leq 1.381095966$, which are better than the above mentioned bounds.

CHAPTER 4

EUCLIDEAN OPERATOR RADIUS AND INEQUALITIES OF A PAIR OF BOUNDED LINEAR OPERATORS

4.1 Introduction

The main objective of this chapter is to obtain several sharp lower and upper bounds for the Euclidean operator radius of a pair of bounded linear operators defined on a complex Hilbert space. As applications of these bounds we deduce a chain of new bounds for the classical numerical radius of a bounded linear operator which improve on the existing ones.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ induced by the inner product. Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For $T \in \mathbb{B}(\mathcal{H})$, T^* denotes the adjoint of T and $|T| = (T^*T)^{\frac{1}{2}}$ denotes the positive square root of T. The real part and imaginary part of T, denoted by $\Re(T)$ and $\Im(T)$, are defined as $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2\mathrm{i}}(T - T^*)$ respectively. The numerical range of T, denoted by $W(T)$, is defined as $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$. We denote the operator norm, the

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Crawford number and the numerical radius of T by $\|T\|$, $c(T)$ and $w(T)$, respectively. Note that

$$c(T) = \inf\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$$

and

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, the Euclidean operator radius of T_1 and T_2, denoted by $w_e(T_1, T_2)$, is defined as

$$w_e(T_1, T_2) = \sup\left\{\sqrt{|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2} : x \in \mathcal{H}, \|x\| = 1\right\}.$$

Following [78], $w_e(.\,,.) : \mathbb{B}^2(\mathcal{H}) \to [0, \infty)$ is a norm that satisfies the inequality

$$\frac{\sqrt{2}}{4}\|T_1^*T_1 + T_2^*T_2\|^{\frac{1}{2}} \le w_e(T_1, T_2) \le \|T_1^*T_1 + T_2^*T_2\|^{\frac{1}{2}}. \tag{4.1}$$

The constants $\frac{\sqrt{2}}{4}$ and 1 are best possible in (4.1). If T_1 and T_2 are self-adjoint operators, then (4.1) becomes

$$\frac{\sqrt{2}}{4}\|T_1^2 + T_2^2\|^{\frac{1}{2}} \le w_e(T_1, T_2) \le \|T_1^2 + T_2^2\|^{\frac{1}{2}}. \tag{4.2}$$

We note that for self-adjoint operators T_1 and T_2, $w_e(T_1, T_2) = w(T_1 + iT_2)$, its proof follows easily from the definition of $w_e(T_1, T_2)$. In [43, Th. 1], Dragomir proved that if $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, then

$$\frac{1}{2}w(T_1^2 + T_2^2) \le w_e^2(T_1, T_2) \le \|T_1^*T_1 + T_2^*T_2\|, \tag{4.3}$$

4.2 Euclidean operator radius inequalities

To prove our first theorem we need the following lemma.

Lemma 4.1. *[60](Cauchy-Schwarz inequality) If $T \in \mathbb{B}(\mathcal{H})$ and $0 \le \alpha \le 1$, then*

$$|\langle Tx, y\rangle|^2 \le \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}y, y\rangle$$

for all $x, y \in \mathcal{H}$.

Theorem 4.1. *Let* $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, *then*

$$\begin{aligned}&\frac{1}{2}w(T_1^2+T_2^2)+\frac{1}{2}\max\{w(T_1),w(T_2)\}\big|w(T_1+T_2)-w(T_1-T_2)\big|\\ &\leq w_e^2(T_1,T_2)\leq \min\left\{w(|T_1|+i|T_2|)w(|T_1^*|+i|T_2^*|),w(|T_1|+i|T_2^*|)w(|T_1^*|+i|T_2|)\right\}.\end{aligned}$$

Proof. Let x be an unit vector in $\mathcal{H}$. Then we have

$$\begin{aligned}|\langle T_1x,x\rangle|^2+|\langle T_2x,x\rangle|^2 &\geq \frac{1}{2}\left(|\langle T_1x,x\rangle|+|\langle T_2x,x\rangle|\right)^2\\ &\geq \frac{1}{2}\left(|\langle T_1x,x\rangle\pm\langle T_2x,x\rangle|\right)^2\\ &= \frac{1}{2}|\langle (T_1\pm T_2)x,x\rangle|^2.\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$, $\|x\|=1$, we get

$$w_e^2(T_1,T_2)\geq \frac{1}{2}w^2(T_1\pm T_2). \tag{4.4}$$

Therefore, it follows from the inequalities in (4.4) that

$$\begin{aligned}w_e^2(T_1,T_2) &\geq \frac{1}{2}\max\{w^2(T_1+T_2),w^2(T_1-T_2)\}\\ &= \frac{w^2(T_1+T_2)+w^2(T_1-T_2)}{4}+\frac{\big|w^2(T_1+T_2)-w^2(T_1-T_2)\big|}{4}\\ &\geq \frac{w((T_1+T_2)^2)+w((T_1-T_2)^2)}{4}\\ &\quad +(w(T_1+T_2)+w(T_1-T_2))\frac{\big|w(T_1+T_2)-w(T_1-T_2)\big|}{4}\\ &\geq \frac{w((T_1+T_2)^2+(T_1-T_2)^2)}{4}\\ &\quad +(w(T_1+T_2)+(T_1-T_2))\frac{\big|w(T_1+T_2)-w(T_1-T_2)\big|}{4}.\end{aligned}$$

Therefore,

$$w_e^2(T_1,T_2) \geq \frac{w(T_1^2+T_2^2)}{2}+\frac{w(T_1)}{2}\big|w(T_1+T_2)-w(T_1-T_2)\big|. \tag{4.5}$$

Interchanging T_1 and T_2 in (5.18), we have that

$$w_e^2(T_1,T_2) \geq \frac{w(T_1^2+T_2^2)}{2}+\frac{w(T_2)}{2}\big|w(T_1+T_2)-w(T_1-T_2)\big|. \tag{4.6}$$

Therefore, the desired first inequality follows by combining the inequalities in (4.5) and (4.6).

Now we prove the second inequality. Let $x \in \mathcal{H}$ with $\|x\| = 1$.

$$\begin{aligned}
& |\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2 \\
\leq\ & \langle |T_1|x, x\rangle \langle |T_1^*|x, x\rangle + \langle |T_2|x, x\rangle \langle |T_2^*|x, x\rangle \quad (\textit{using Lemma 4.1}) \\
\leq\ & \Big\{(\langle |T_1|x, x\rangle^2 + \langle |T_2|x, x\rangle^2)(\langle |T_1^*|x, x\rangle^2 + \langle |T_2^*|x, x\rangle^2)\Big\}^{\frac{1}{2}} \\
& \quad (\textit{by the inequality } (ab+cd)^2 \leq (a^2+c^2)(b^2+d^2) \textit{ for } a,b,c,d \in \mathbb{R}) \\
=\ & \big\{|\langle |T_1|x, x\rangle + i\langle |T_2|x, x\rangle|^2 |\langle |T_1^*|x, x\rangle + i\langle |T_2^*|x, x\rangle|^2\big\}^{\frac{1}{2}} \\
=\ & \big\{|\langle (|T_1| + i|T_2|)x, x\rangle|^2 |\langle (|T_1^*| + i|T_2^*|)x, x\rangle|^2\big\}^{\frac{1}{2}} \\
\leq\ & w(|T_1| + i|T_2|)w(|T_1^*| + i|T_2^*|).
\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\| = 1$, we get

$$w_e^2(T_1, T_2) \quad \leq \quad w(|T_1| + i|T_2|)w(|T_1^*| + i|T_2^*|). \tag{4.7}$$

Replacing T_2 by T_2^* in (5.20), we have

$$w_e^2(T_1, T_2) \quad \leq \quad w(|T_1| + i|T_2^*|)w(|T_1^*| + i|T_2|). \tag{4.8}$$

Therefore, combining the inequalities in (4.7) and (4.8) we obtain the desired second inequality. □

We would like to remark that the lower bound of $w_e(T_1, T_2)$ obtained in Theorem 4.1 is stronger than the lower bound in (4.3). Also, it is not difficult to verify that $w^2(|T_1| + i|T_2|) \leq \|T_1^*T_1 + T_2^*T_2\|$ and $w^2(|T_1^*| + i|T_2^*|) \leq \|T_1T_1^* + T_2T_2^*\|$. Therefore,

$$w(|T_1| + i|T_2|)w(|T_1^*| + i|T_2^*|) \quad \leq \quad \|T_1^*T_1 + T_2^*T_2\|^{\frac{1}{2}}\|T_1T_1^* + T_2T_2^*\|^{\frac{1}{2}}.$$

Similarly,

$$w(|T_1| + i|T_2^*|)w(|T_1^*| + i|T_2|) \quad \leq \quad \|T_1^*T_1 + T_2T_2^*\|^{\frac{1}{2}}\|T_1T_1^* + T_2^*T_2\|^{\frac{1}{2}}.$$

Therefore, it follows from the second inequality in Theorem 4.1 that

$$w_e^2(T_1, T_2) \quad \leq \quad \min\left\{\|T_1^*T_1 + T_2^*T_2\|^{\frac{1}{2}}\|T_1T_1^* + T_2T_2^*\|^{\frac{1}{2}}, \|T_1^*T_1 + T_2T_2^*\|^{\frac{1}{2}}\|T_1T_1^* + T_2^*T_2\|^{\frac{1}{2}}\right\}.$$

The above bound for $w_e(T_1, T_2)$ is better than the upper bound in (4.1) if $\|T_1T_1^* + T_2T_2^*\| \leq \|T_1^*T_1 + T_2^*T_2\|$.

Also, it follows from Theorem 4.1 that $w_e^2(T_1, T_2) = \frac{1}{2}w(T_1^2 + T_2^2)$ implies $w(T_1 + T_2) = w(T_1 - T_2)$. But, by considering $T_2 = 0$, we conclude that the converse part does not always hold.

The following corollary is an immediate consequence of Theorem 4.1 assuming T_1 and T_2 to be self-adjoint operators.

Corollary 4.1. *Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$ be self-adjoint, then*

$$\frac{1}{2}\|T_1^2 + T_2^2\| + \frac{1}{2}\max\{\|T_1\|, \|T_2\|\}\big|\|T_1 + T_2\| - \|T_1 - T_2\|\big| \\ \leq w_e^2(T_1, T_2) \leq w^2(|T_1| + i|T_2|). \tag{4.9}$$

We remark that the second inequality in (4.9) gives better bound than that in (4.2).

Next lower bound for $w_e(T_1, T_2)$ reads as follows.

Theorem 4.2. *Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, then*

$$\frac{1}{2}\max\left\{w^2(T_1 + T_2) + c^2(T_1 - T_2), w^2(T_1 - T_2) + c^2(T_1 + T_2)\right\} \leq w_e^2(T_1, T_2).$$

Proof. Let x be an unit vector in $\mathcal{H}$. Then we have

$$|\langle T_1x, x\rangle + \langle T_2x, x\rangle|^2 + |\langle T_1x, x\rangle - \langle T_2x, x\rangle|^2 = 2(|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2).$$

Therefore,

$$\begin{aligned} |\langle (T_1 + T_2)x, x\rangle|^2 + |\langle (T_1 - T_2)x, x\rangle|^2 &= 2(|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2) \\ &\leq 2w_e^2(T_1, T_2). \end{aligned}$$

Thus,

$$\begin{aligned} |\langle (T_1 + T_2)x, x\rangle|^2 &\leq 2w_e^2(T_1, T_2) - |\langle (T_1 - T_2)x, x\rangle|^2 \\ &\leq 2w_e^2(T_1, T_2) - c^2(T_1 - T_2). \end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\| = 1$, we get

$$w^2(T_1 + T_2) \leq 2w_e^2(T_1, T_2) - c^2(T_1 - T_2),$$

that is,

$$w^2(T_1+T_2)+c^2(T_1-T_2)\leq 2w_e^2(T_1,T_2). \tag{4.10}$$

Similarly, we can prove that

$$w^2(T_1-T_2)+c^2(T_1+T_2)\leq 2w_e^2(T_1,T_2). \tag{4.11}$$

Combining the inequalities (4.10) and (4.11) we get,

$$\frac{1}{2}\max\left\{w^2(T_1+T_2)+c^2(T_1-T_2),w^2(T_1-T_2)+c^2(T_1+T_2)\right\}\leq w_e^2(T_1,T_2),$$

as desired. □

Remark 4.3. *(i) Clearly, the bound in Theorem 4.2 is stronger then the first bound in [43, Th. 2], that is,* $\frac{1}{2}\max\left\{w^2(T_1+T_2),w^2(T_1-T_2\right\}\leq w_e^2(T_1,T_2)$.
(ii) For self-adjoint operators $T_1,T_2\in\mathbb{B}(\mathcal{H})$*, the bound in Theorem 4.2 is of the form*

$$\frac{1}{2}\max\left\{\|T_1+T_2\|^2+c^2(T_1-T_2),\|T_1-T_2\|^2+c^2(T_1+T_2)\right\}\leq w_e^2(T_1,T_2).$$

We next obtain the following inequality.

Theorem 4.4. *Let* $T_1,T_2\in\mathbb{B}(\mathcal{H})$*, then*

$$\max\left\{w^2(T_1)+c^2(T_2),w^2(T_2)+c^2(T_1)\right\}\leq w_e^2(T_1,T_2).$$

Proof. Let $x\in\mathcal{H}$ with $\|x\|=1$. Then we have

$$|\langle T_1x,x\rangle+\langle T_2x,x\rangle|^2+|\langle T_1x,x\rangle-\langle T_2x,x\rangle|^2=2(|\langle T_1x,x\rangle|^2+|\langle T_2x,x\rangle|^2),$$

that is,

$$|\langle (T_1+T_2)x,x\rangle|^2+|\langle (T_1-T_2)x,x\rangle|^2=2(|\langle T_1x,x\rangle|^2+|\langle T_2x,x\rangle|^2).$$

This implies that

$$w_e^2(T_1+T_2,T_1-T_2)=2w_e^2(T_1,T_2). \tag{4.12}$$

Now replacing T_1 by $T_1 + T_2$ and T_2 by $T_1 - T_2$ in Theorem 4.2 we obtain that

$$2 \max \left\{ w^2(T_1) + c^2(T_2), w^2(T_2) + c^2(T_1) \right\} \leq w_e^2(T_1 + T_2, T_1 - T_2). \tag{4.13}$$

Therefore, the required inequality follows from (4.13) by using the fact (4.12).

□

Remark 4.5. *(i) Following Theorem 4.4, we have for $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, $w_e(T_1, T_2) = w(T_1)$ implies if $\lim_{n\to\infty} |\langle T_1 x_n, x_n\rangle| = w(T_1)$ then $\lim_{n\to\infty} |\langle T_2 x_n, x_n\rangle| = 0$. It should be mentioned here that the converse part is not necessarily true.*
(ii) For normal operators $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, the inequality in Theorem 4.4 turns into the form

$$\max \left\{ \|T_1\|^2 + c^2(T_2), \|T_2\|^2 + c^2(T_1) \right\} \leq w_e^2(T_1, T_2).$$

To prove the next result we need the following inequality for vectors in $\mathcal{H}$, knows as Buzano's inequality.

Lemma 4.2. *([33]) Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$, then*

$$|\ \langle x, e\rangle\langle e, y\rangle\ | \leq \frac{1}{2} \left(|\ \langle x, y\rangle\ | + \|x\|\|y\|\right).$$

Theorem 4.6. *Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, then*

$$w_e^2(T_1, T_2) \leq \min \left\{ w^2(T_1 + T_2), w^2(T_1 - T_2) \right\} + \frac{1}{2}\|T_2^* T_2 + T_1 T_1^*\| + w(T_1 T_2).$$

Proof. Let x be an unit vector in $\mathcal{H}$. Then we have

$$\begin{aligned} |\ \langle T_2 x, x\rangle\ |^2 - 2\Re[\langle T_2 x, x\rangle\ \overline{\langle T_1 x, x\rangle}] + |\ \langle T_1 x, x\rangle\ |^2 &= |\ \langle T_2 x, x\rangle - \langle T_1 x, x\rangle\ |^2 \\ &= |\ \langle (T_2 - T_1) x, x\rangle\ |^2 \\ &\leq w^2(T_2 - T_1). \end{aligned}$$

Thus,

$$\begin{aligned}
|\langle T_2x,x\rangle|^2+|\langle T_1x,x\rangle|^2 &\le w^2(T_2-T_1)+2\Re[\langle T_2x,x\rangle\,\overline{\langle T_1x,x\rangle}]\\
&\le w^2(T_2-T_1)+2|\langle T_2x,x\rangle\,\langle T_1x,x\rangle|\\
&\le w^2(T_2-T_1)+\|T_2x\|\|T_1^*x\|+|\langle T_2x,T_1^*x\rangle|\,(\textit{by Lemma 4.2})\\
&\le w^2(T_2-T_1)+\frac{1}{2}(\|T_2x\|^2+\|T_1^*x\|^2)+w(T_1T_2)\\
&\le w^2(T_2-T_1)+\frac{1}{2}\|T_2^*T_2+T_1T_1^*\|+w(T_1T_2).
\end{aligned}$$

Therefore, taking supremum over all x in $\mathcal{H}$ with $\|x\|=1$, we get

$$w_e^2(T_1,T_2)\le w^2(T_1-T_2)+\frac{1}{2}\|T_2^*T_2+T_1T_1^*\|+w(T_1T_2). \tag{4.14}$$

Replacing T_2 by $-T_2$ in the above inequality, we obtain that

$$w_e^2(T_1,T_2)\le w^2(T_1+T_2)+\frac{1}{2}\|T_2^*T_2+T_1T_1^*\|+w(T_1T_2). \tag{4.15}$$

Hence, the desired bound follows from (4.14) and (4.15). □

In particular, taking $T_1=T_2=T$ in (4.14), we get the following upper bound (see [2]) for the numerical radius of a bounded linear operator T on $\mathcal{H}$:

$$w^2(T)\le\frac{1}{4}\|T^*T+TT^*\|+\frac{1}{2}w(T^2).$$

In [43, Th. 1], it is proved that if T_1,T_2 are bounded linear operators on $\mathcal{H}$, then

$$w\left(\frac{1}{2}T_1^2+\frac{1}{2}T_2^2\right)\le w_e^2(T_1,T_2). \tag{4.16}$$

In the next result we establish a generalization of the above lower bound for the Euclidean operator radius for a pair (T_1,T_2) of two bounded linear operators on $\mathcal{H}$.

Theorem 4.7. *Let $T_1,T_2\in\mathbb{B}(\mathcal{H})$, then*

$$\max_{0\le\alpha\le1} w\left(\alpha T_1^2+(1-\alpha)T_2^2\right)\le w_e^2(T_1,T_2). \tag{4.17}$$

Proof. Let x be an unit vector in $\mathcal{H}$. Then applying the inequality $(ab+cd)^2\le(a^2+c^2)(b^2+d^2)$

for real numbers a, b, c, d, we have

$$\begin{aligned} \sqrt{\alpha}|\langle T_1x, x\rangle| + \sqrt{1-\alpha}|\langle T_2x, x\rangle| &\leq (|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2)^{\frac{1}{2}}((\sqrt{\alpha})^2 + (\sqrt{1-\alpha})^2)^{\frac{1}{2}} \\ &= (|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2)^{\frac{1}{2}}, \end{aligned}$$

for any $\alpha \in [0, 1]$.

Therefore,

$$\begin{aligned} (|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2)^{\frac{1}{2}} &\geq \sqrt{\alpha}|\langle T_1x, x\rangle| + \sqrt{1-\alpha}|\langle T_2x, x\rangle| \\ &= |\langle \sqrt{\alpha}T_1x, x\rangle| + |\langle \sqrt{1-\alpha}T_2x, x\rangle| \\ &\geq |\langle \sqrt{\alpha}T_1x, x\rangle \pm \langle \sqrt{1-\alpha}T_2x, x\rangle| \\ &= |\langle \left(\sqrt{\alpha}T_1 \pm \sqrt{1-\alpha}T_2\right) x, x\rangle|. \end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\| = 1$, we get

$$w_e(T_1, T_2) \geq w(\sqrt{\alpha}T_1 \pm \sqrt{1-\alpha}T_2).$$

Therefore,

$$\begin{aligned} 2w_e^2(T_1, T_2) &\geq w^2(\sqrt{\alpha}T_1 + \sqrt{1-\alpha}T_2) + w^2(\sqrt{\alpha}T_1 - \sqrt{1-\alpha}T_2) \\ &\geq w((\sqrt{\alpha}T_1 + \sqrt{1-\alpha}T_2)^2) + w((\sqrt{\alpha}T_1 - \sqrt{1-\alpha}T_2)^2) \\ &\geq w((\sqrt{\alpha}T_1 + \sqrt{1-\alpha}T_2)^2 + (\sqrt{\alpha}T_1 - \sqrt{1-\alpha}T_2)^2) \\ &= 2w(\alpha T_1^2 + (1-\alpha)T_2^2). \end{aligned}$$

This implies $w_e^2(T_1, T_2) \geq w(\alpha T_1^2 + (1-\alpha)T_2^2)$. This holds for all $\alpha \in [0, 1]$ and so we get the desired inequality. □

Now, we would like to remark that the inequality (4.17) is a refinement of the inequality (4.16) obtained in [43, Th. 1]. To see that the refinement is proper, consider $T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. Then we get,

$$w\left(\frac{1}{2}T_1^2 + \frac{1}{2}T_2^2\right) = 2 < 4 = \max_{0\leq\alpha\leq 1} w(\alpha T_1^2 + (1-\alpha)T_2^2).$$

Next, we prove the following theorem.

Theorem 4.8. *Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, then*

$$w_e^2(T_1, T_2) \le w^2(\sqrt{\alpha}T_1 + \sqrt{1-\alpha}T_2) + w^2(\sqrt{1-\alpha}T_1 + \sqrt{\alpha}T_2),$$

for all $\alpha \in [0, 1]$.

Proof. Let x be an unit vector in $\mathcal{H}$. Then we have

$$\begin{aligned} |\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2 &= |\langle \sqrt{\alpha}T_1x, x\rangle + \langle \sqrt{1-\alpha}T_2x, x\rangle|^2 + |\langle \sqrt{1-\alpha}T_1x, x\rangle - \langle \sqrt{\alpha}T_2x, x\rangle|^2 \\ &= |\langle (\sqrt{\alpha}T_1 + \sqrt{1-\alpha}T_2)x, x\rangle|^2 + |\langle (\sqrt{1-\alpha}T_1 - \sqrt{\alpha}T_2)x, x\rangle|^2 \\ &\le w^2(\sqrt{\alpha}T_1 + \sqrt{1-\alpha}T_2) + w^2(\sqrt{1-\alpha}T_1 - \sqrt{\alpha}T_2). \end{aligned}$$

Therefore, taking supremum over all x in $\mathcal{H}$ with $\|x\| = 1$, we get

$$w_e^2(T_1, T_2) \le w^2(\sqrt{\alpha}T_1 + \sqrt{1-\alpha}T_2) + w^2(\sqrt{1-\alpha}T_1 - \sqrt{\alpha}T_2),$$

as desired. □

Remark 4.9. *(i) In particular, if we consider $\alpha = \frac{1}{2}$ in Theorem 4.8, then we get the following upper bound (see [43, Th. 2]) for Euclidean operator radius*

$$w_e^2(T_1, T_2) \le \frac{1}{2}\left(w^2(T_1 + T_2) + w^2(T_1 - T_2)\right).$$

(ii) Putting $T_1 = \Re(T)$ and $T_2 = \Im(T)$ in Theorem 4.8 we obtain the following upper bound for the numerical radius of a bounded linear operator T on $\mathcal{H}$,

$$w^2(T) \le \|\sqrt{\alpha}\Re(T) + \sqrt{1-\alpha}\Im(T)\|^2 + \|\sqrt{1-\alpha}\Re(T) - \sqrt{\alpha}\Im(T)\|^2,$$

for all $\alpha \in [0, 1]$.

To prove the next upper bound we need the following lemma known as Jensen's inequality, obtained from more general result for superquadratic functions [1].

Lemma 4.3. *The following inequality*

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \le \frac{1}{n}\sum_{k=1}^{n} a_k^p - \frac{1}{n}\sum_{k=1}^{n} \left|a_k - \frac{1}{n}\sum_{j=1}^{n} a_j\right|^p,$$

holds for $p \ge 2$ and for every finite positive sequence of real numbers $a_1, a_2, \ldots, a_n$.

Theorem 4.10. *Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, then*

$$w_e^{2r}(T_1, T_2) \leq \frac{1}{2}w^{2r}(T_1+T_2) + \frac{1}{2}w^{2r}(T_1-T_2) - 2^r \inf_{\|x\|=1} |\Re(\langle T_1x, x\rangle \overline{\langle T_2x, x\rangle})|$$

holds for every $r \geq 2$.

(Here $\Re(\lambda)$ denotes the real part of the complex number λ, i.e., $\Re(\lambda) = (\lambda + \overline{\lambda})/2$.)

Proof. Let x be an unit vector in $\mathcal{H}$. Then

$$\begin{aligned}
& \left(|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2\right)^r \\
= & \left(\frac{1}{2}|\,\langle T_1x, x\rangle + \langle T_2x, x\rangle\,|^2 + \frac{1}{2}|\,\langle T_1x, x\rangle - \langle T_2x, x\rangle\,|^2\right)^r \\
= & \left(\frac{1}{2}|\,\langle (T_1+T_2)x, x\rangle\,|^2 + \frac{1}{2}|\,\langle (T_1-T_2)x, x\rangle\,|^2\right)^r \\
\leq & \frac{1}{2}|\,\langle (T_1+T_2)x, x\rangle\,|^{2r} + \frac{1}{2}|\,\langle (T_1-T_2)x, x\rangle\,|^{2r} \\
& -\frac{1}{2}\Big|\frac{1}{2}\,|\langle (T_1+T_2)x, x\rangle\,|^2 - \frac{1}{2}|\,\langle (T_1-T_2)x, x\rangle\,|^2\Big|^r \\
& -\frac{1}{2}\Big|\frac{1}{2}|\,\langle (T_1-T_2)x, x\rangle\,|^2 - \frac{1}{2}|\,\langle (T_1+T_2)x, x\rangle\,|^2\Big|^r \text{ (using Lemma 4.3)} \\
= & \frac{1}{2}|\,\langle (T_1+T_2)x, x\rangle\,|^{2r} + \frac{1}{2}|\,\langle (T_1-T_2)x, x\rangle\,|^{2r} \\
& -\frac{1}{2^r}\Big||\,\langle (T_1+T_2)x, x\rangle\,|^2 - |\,\langle (T_1-T_2)x, x\rangle\,|^2\Big|^r \\
= & \frac{1}{2}|\,\langle (T_1+T_2)x, x\rangle\,|^{2r} + \frac{1}{2}|\,\langle (T_1-T_2)x, x\rangle\,|^{2r} - \frac{2^{2r}}{2^r}|\Re(\langle T_1x, x\rangle \overline{\langle T_2x, x\rangle})| \\
\leq & \frac{1}{2}w^{2r}(T_1+T_2) + \frac{1}{2}w^{2r}(T_1-T_2) - 2^r \inf_{\|x\|=1} |\Re(\langle T_1x, x\rangle \overline{\langle T_2x, x\rangle})|.
\end{aligned}$$

Therefore, taking supremum over all x in $\mathcal{H}$ with $\|x\| = 1$, we get the desire result. □

Next we need the following lemma, called generalized Cauchy-Schwarz inequality [68].

Lemma 4.4. *If f and g be two non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$, then*

$$|\langle Tx, y\rangle| \leq \|f(|T|)x\|\|g(|T^*|y\|,$$

for all $T \in \mathbb{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$.

Finally, we obtain the following inequality involving non-negative continuous functions.

Theorem 4.11. *Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$ and let f, g be two non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$, then*

$$\frac{1}{2}\|T_1+T_2\|^2 \leq w_e(f^2(|T_1|), f^2(|T_2|))w_e(g^2(|T_1^*|), g^2(|T_2^*|)).$$

In particular,

$$\frac{1}{2}\|T_1+T_2\|^2 \le w_e(|T_1|,|T_2|)w_e(|T_1^*|,|T_2^*|).$$

Proof. Let x, y be two unit vectors in $\mathcal{H}$. Then

$$\begin{aligned}
& |\langle (T_1+T_2)x,y\rangle|^2 \\
= \; & |\langle T_1x,y\rangle + \langle T_2x,y\rangle|^2 \\
\le \; & 2(|\langle T_1x,y\rangle|^2 + |\langle T_2x,y\rangle|^2) \\
\le \; & 2(\|f(|T_1|)x\|^2\|g(|T_1^*|)y\|^2 + \|f(|T_2|)x\|^2\|g(|T_2^*|)y\|^2) \text{ (using Lemma 4.4)} \\
= \; & 2(\langle f^2(|T_1|)x,x\rangle\langle g^2(|T_1^*|)y,y\rangle + \langle f^2(|T_2|)x,x\rangle\langle g^2(|T_2^*|)y,y\rangle) \\
\le \; & 2\left(\langle f^2(|T_1|)x,x\rangle^2 + \langle f^2(|T_2|)x,x\rangle^2\right)^{\frac{1}{2}}\left(\langle g^2(|T_1^*|)x,x\rangle^2 + \langle g^2(|T_2^*|)x,x\rangle^2\right)^{\frac{1}{2}} \\
\le \; & 2w_e(f^2(|T_1|), f^2(|T_2|))w_e(g^2(|T_1^*|), g^2(|T_2^*|)).
\end{aligned}$$

Taking supremum over $\|x\| = \|y\| = 1$, we get

$$\frac{1}{2}\|T_1+T_2\|^2 \le w_e(f^2(|T_1|), f^2(|T_2|))w_e(g^2(|T_1^*|), g^2(|T_2^*|)).$$

□

In particular, if we take $f(t) = g(t) = t^{\frac{1}{2}}$, then

$$\frac{1}{2}\|T_1+T_2\|^2 \le w_e(|T_1|,|T_2|)w_e(|T_1^*|,|T_2^*|),$$

as desired. As an application to the inequalities obtain here we develop numerical radius inequalities of a bounded linear operator T.

4.3 Application to numerical radius inequalities

Considering $T_1 = \Re(T)$ and $T_2 = \Im(T)$ in (4.9) we obtain the following new upper and lower bounds for the numerical radius of a bounded linear operator T.

Corollary 4.2. *Let* $T \in \mathbb{B}(\mathcal{H})$*, then*

$$\begin{aligned}
\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{2}\max\{\|\Re(T)\|,\|\Im(T)\|\}\big|\|\Re(T)+\Im(T)\| - \|\Re(T)-\Im(T)\|\big| \\
\le w^2(T) \le w^2(|\Re(T)| + i|\Im(T)|).
\end{aligned}$$

Remark 4.12. *It is easy to show that* $w^2(|\Re(T)| + i|\Im(T)|) \leq \frac{1}{2}\|T^*T + TT^*\|$. *Therefore, the inequality in Corollary 4.2 is stronger then the inequality (1.3). Also,* $\frac{1}{2}\|T^*T + TT^*\| \leq \|\Re(T)\|^2 + \|\Im(T)\|^2$. *So, the upper bound for* $w(T)$ *in Corollary 4.2 is stronger than the well-known bound* $w(T) \leq \sqrt{\|\Re(T)\|^2 + \|\Im(T)\|^2}$.

Now, if we consider $T_1 = T$ and $T_2 = T^*$ in Theorem 4.1, we get the following inequality.

Corollary 4.3. *Let* $T \in \mathbb{B}(\mathcal{H})$, *then*

$$\frac{1}{2}\|\Re(T^2)\| + \frac{1}{2}w(T)\big|\|\Re(T)\| - \|\Im(T)\|\big| \leq w^2(T) \leq \frac{1}{2}w(|T| + i|T^*|)w(|T^*| + i|T|).$$

Remark 4.13. *Clearly, the first inequality in Corollary 4.3 is sharper than the existing inequality* $\frac{1}{2}\|\Re(T^2)\| \leq w^2(T)$, *given in [43, Remark 2]. Observe that* $\frac{1}{2}w(|T| + i|T^*|)w(|T^*| + i|T|) \leq \frac{1}{2}\|T^*T + TT^*\|$, *and so the second inequality in Corollary 4.3 is stronger then the second inequality in (1.3).*

Remark 4.14. *Replacing* T_1 *by* $\Re(T)$ *and* T_2 *by* $\Im(T)$ *in Theorem 4.2 we get the following lower bound for the numerical radius of* $T \in \mathbb{B}(\mathcal{H})$:

$$w^2(T) \geq \frac{1}{2}\max\left\{\|\Re(T) + \Im(T)\|^2 + c^2(\Re(T) - \Im(T)), \|\Re(T) - \Im(T)\|^2 + c^2(\Re(T) + \Im(T))\right\}.$$

This bound appeared recently in [18, Cor. 2.3].

Remark 4.15. *If we replace* T_1 *by* $\Re(T)$ *and* T_2 *by* $\Im(T)$ *in the inequality in Theorem 4.4, then we get the following lower bound (see [30, Th. 3.3]) for the numerical radius of a bounded linear operator* T *on* $\mathcal{H}$:

$$\max\left\{\|\Re(T)\|^2 + c^2(\Im(T)), \|\Im(T)\|^2 + c^2(\Re(T))\right\} \leq w^2(T).$$

Further, replacing T_1 *by* $\Re(T)$ *and* T_2 *by* $\Im(T)$ *in Theorem 4.7, we obtain that for* $T \in \mathbb{B}(\mathcal{H})$,

$$\left\|\alpha(\Re(T))^2 + (1-\alpha)(\Im(T))^2\right\| \leq w^2(T), \tag{4.18}$$

for all $\alpha \in [0, 1]$. *In particular, for* $\alpha = \frac{1}{2}$, *we get the well-known lower bound*

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T).$$

CHAPTER 5

EUCLIDEAN OPERATOR RADIUS AND NUMERICAL RADIUS INEQUALITIES

5.1 Introduction

Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. We obtain various lower and upper bounds for the numerical radius of T by developing the Euclidean operator radius bounds of a pair of operators, which are stronger than the existing ones. In particular, we develop an inequality that improves on the inequality

$$w(T) \geq \frac{1}{2}\|T\| + \frac{1}{4}\left|\|\Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|\Im(T)\| - \frac{1}{2}\|T\|\right|.$$

Various equality conditions of the existing numerical radius inequalities are also provided. Further, we study the numerical radius inequalities of 2×2 off-diagonal operator matrices. Applying the numerical radius bounds of operator matrices, we develop the upper bounds of $w(T)$ by using

Content of this chapter is based on the following papers:
S. Jana , P. Bhunia and K. Paul, Euclidean operator radius and numerical radius inequalities. arXiv:2308.09252
S. Jana, P. Bhunia and K. Paul, Refinements of generalized Euclidean operator radius inequalities of 2-tuple operators. arXiv:2308.09261

t-Aluthge transform. In particular, we improve the well known inequality

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}),$$

where $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ is the Aluthge transform of T and $T = U|T|$ is the polar decomposition of T.

Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle .,. \rangle)$. For $T \in \mathbb{B}(\mathcal{H})$, T^* denotes the adjoint of T and $|T| = (T^*T)^{\frac{1}{2}}$. The Cartesian decomposition of T is $T = \Re(T) + i\Im(T)$, where $\Re(T) = \frac{1}{2}(T + T^*)$ and $\Im(T) = \frac{1}{2i}(T - T^*)$. For $0 \leq t \leq 1$, the t-Aluthge transform of $T \in \mathbb{B}(\mathcal{H})$ is defined as $\widetilde{T_t} = |T|^t U|T|^{1-t}$, where $T = U|T|$ is the polar decomposition of T and U is the partial isometry. In particular, for $t = \frac{1}{2}$, let $\widetilde{T} = \widetilde{T_{\frac{1}{2}}} = |T|^{1/2}U|T|^{1/2}$ be the Aluthge transform of T. The numerical radius of T, denoted by $w(T)$, is defined as $w(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}$.

For $T_1, T_2 \in \mathbb{B}(\mathcal{H})$, the Euclidean operator radius of T_1 and T_2, denoted by $w_e(T_1, T_2)$, is defined as

$$w_e(T_1, T_2) = \sup\left\{\sqrt{|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2} : x \in \mathcal{H}, \|x\| = 1\right\}.$$

The Euclidean operator radius $w_e(.,.)$, defines a norm on $\mathbb{B}^2(\mathcal{H})(= \mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H}))$, which satisfies the inequality (see [78])

$$\frac{1}{8}\|T_1^*T_1 + T_2^*T_2\| \leq w_e^2(T_1, T_2) \leq \|T_1^*T_1 + T_2^*T_2\|. \tag{5.1}$$

Here the constants $\frac{1}{8}$ and 1 are best possible. In [43, Th. 1], Dragomir proved that

$$\frac{1}{2}w\left(T_1^2 + T_2^2\right) \leq w_e^2(T_1, T_2) \tag{5.2}$$

and the constant $\frac{1}{2}$ is best possible.

5.2 Euclidean operator radius inequalities

We begin with the following proposition that gives lower bounds for the Euclidean operator radius $w_e(T_1, T_2)$.

Proposition 5.1. *Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$. Then the following inequalities hold:*
(i) $w_e(T_1, T_2) \geq \max\{w(T_1), w(T_2)\}$.
(ii) $w_e(T_1, T_2) \geq \frac{1}{\sqrt{2}}w\left(T_1 + e^{i\theta}T_2\right)$ *for all* $\theta \in \mathbb{R}$.
(iii) $w_e(T_1, T_2) \geq \sqrt{\frac{1}{2}w\left(T_1^2 + e^{i\theta}T_2^2\right) + \frac{1}{2}\left|w^2(T_1) - w^2(T_2)\right|}$ *for all* $\theta \in \mathbb{R}$.

(iv) $w_e(T_1, T_2) \geq \sqrt{\frac{1}{2} w\left(T_1T_2 + T_2T_1\right)}$.

Proof. (i) Follows trivially from the definition.

(ii) We have

$$\begin{aligned} w_e(T_1, T_2) &= \sup_{\|x\|=1} \sqrt{|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2} \\ &\geq \sup_{\|x\|=1} \sqrt{\frac{1}{2}\left(|\langle T_1x, x\rangle| + |\langle T_2x, x\rangle|\right)^2} \\ &\geq \sup_{\|x\|=1} \sqrt{\frac{1}{2}\left(|\langle T_1x, x\rangle + e^{i\theta}\langle T_2x, x\rangle|\right)^2} \\ &= \frac{1}{\sqrt{2}} w\left(T_1 + e^{i\theta}T_2\right). \end{aligned}$$

(iii) From (i), we have

$$\begin{aligned} w_e^2(T_1, T_2) &\geq \max\left\{w^2(T_1), w^2(T_2)\right\} \\ &= \frac{1}{2}(w^2(T_1) + w^2(T_2)) + \frac{1}{2}|w^2(T_1) - w^2(T_2)| \\ &\geq \frac{1}{2}\left(w(T_1^2) + w(T_2^2)\right) + \frac{1}{2}|w^2(T_1) - w^2(T_2)| \\ &\geq \frac{1}{2}w(T_1^2 + e^{i\theta}T_2^2) + \frac{1}{2}|w^2(T_1) - w^2(T_2)|. \end{aligned}$$

(iv) From (ii), we have $w_e(T_1, T_2) \geq \frac{1}{\sqrt{2}}w\left(T_1 + T_2\right)$ and $w_e(T_1, T_2) \geq \frac{1}{\sqrt{2}}w\left(T_1 - T_2\right)$. Thus,

$$\begin{aligned} 2w_e^2(T_1, T_2) &\geq \frac{1}{2}w^2\left(T_1 + T_2\right) + \frac{1}{2}w^2\left(T_1 - T_2\right) \\ &\geq \frac{1}{2}w\left((T_1 + T_2)^2\right) + \frac{1}{2}w\left((T_1 - T_2)^2\right) \\ &\geq \frac{1}{2}w\left((T_1 + T_2)^2 - (T_1 - T_2)^2\right) \\ &= w\left(T_1T_2 + T_2T_1\right). \end{aligned}$$

This completes the proof. □

Clearly, Proposition 5.1 (iii) generalizes and improves the inequality $w_e(T_1, T_2) \geq \sqrt{\frac{1}{2}w\left(T_1^2 + T_2^2\right)}$, proved in [43, Th. 1]. Now, by using Proposition 5.1 we prove the following theorem.

Theorem 5.1. *Let* $T_1, T_2 \in \mathbb{B}(\mathcal{H})$*. Then*

$$\sqrt{\frac{1}{4}w(T_1^2 + T_2^2) + \frac{1}{4}\left(w^2(T_1) + w^2(T_2)\right) + \frac{1}{2}\left|w^2(T_1) - w^2(T_2)\right|} \leq w_e(T_1, T_2).$$

Proof. Take $t_1 = \max\left\{w^2(T_1), \frac{1}{2}w\left(T_1^2 + T_2^2\right)\right\}$, $t_2 = \max\left\{w^2(T_2), \frac{1}{2}w\left(T_1^2 + T_2^2\right)\right\}$,

$m_1 = \left|w^2(T_1) - \frac{1}{2}w\left(T_1^2+T_2^2\right)\right|$ and $m_2 = \left|w^2(T_2) - \frac{1}{2}w\left(T_1^2+T_2^2\right)\right|$. From the inequalities (i) and (iii) of Proposition 5.1, we have

$$\begin{aligned} w_e^2(T_1,T_2) &\geq \max\{t_1,t_2\} \\ &= \frac{1}{2}(t_1+t_2) + \frac{1}{2}\left|t_1-t_2\right| \\ &= \frac{1}{4}\left(w^2(T_1)+w^2(T_2)\right) + \frac{1}{4}w(T_1^2+T_2^2) + \frac{1}{4}(m_1+m_2) + \frac{1}{2}\left|t_1-t_2\right| \\ &\geq \frac{1}{4}\left(w(T_1^2)+w(T_2^2)\right) + \frac{1}{4}w(T_1^2+T_2^2) + \frac{1}{4}(m_1+m_2) + \frac{1}{2}\left|t_1-t_2\right| \\ &\geq \frac{1}{4}w(T_1^2+T_2^2) + \frac{1}{4}w(T_1^2+T_2^2) + \frac{1}{4}(m_1+m_2) + \frac{1}{2}\left|t_1-t_2\right| \\ &= \frac{1}{2}w(T_1^2+T_2^2) + \frac{1}{4}(m_1+m_2) + \frac{1}{2}\left|t_1-t_2\right| \\ &= \frac{1}{4}w(T_1^2+T_2^2) + \frac{1}{4}\left(w^2(T_1)+w^2(T_1)\right) + \frac{1}{2}\left|w^2(T_1)-w^2(T_2)\right|, \end{aligned}$$

as desired. □

Remark 5.2. *(i) Clearly, the inequality obtained in Theorem 5.1 is a refinement of the inequality* $\sqrt{\frac{1}{2}w\left(T_1^2+T_2^2\right)} \leq w_e(T_1,T_2)$, *given in [43, Th. 1].*
(ii) If $w_e(T_1,T_2) = \sqrt{\frac{1}{2}w\left(T_1^2+T_2^2\right)}$, *then from Theorem 5.1 it follows that* $w(T_1) = w(T_2) = \sqrt{\frac{1}{2}w\left(T_1^2+T_2^2\right)}$. *However, the converse, in general, may not hold. As for example, considering a non-zero normal operator* $T_1 = T_2$, *we get* $w(T_1) = w(T_2) = \sqrt{\frac{1}{2}w\left(T_1^2+T_2^2\right)}$, *but* $w_e(T_1,T_2) = \sqrt{2}w(T_1) \neq w(T_1) = \sqrt{\frac{1}{2}w\left(T_1^2+T_2^2\right)}$.
(iii) If $w_e(T_1,T_2) = \sqrt{\frac{1}{2}w\left(T_1^2+T_2^2\right) + \frac{1}{2}\left|w^2(T_1)-w^2(T_2)\right|}$ *then from Theorem 5.1 it follows that* $w\left(T_1^2+T_2^2\right) = w^2(T_1)+w^2(T_2)$ *and* $w_e(T_1,T_2) = \max\{w(T_1),w(T_2)\}$. *The converse of the result is also valid.*

As an immediate consequence of Theorem 5.1 we have the following result.

Corollary 5.1. *Let* $T_1, T_2 \in \mathbb{B}(\mathcal{H})$ *be normal, then*

$$\begin{aligned} w_e(T_1,T_2) &\geq \sqrt{\frac{1}{4}\|T_1^2+T_2^2\| + \frac{1}{4}\left(\|T_1\|^2+\|T_2\|^2\right) + \frac{1}{2}\left|\|T_1\|^2-\|T_2\|^2\right|} \\ &= \sqrt{\frac{1}{2}\left\|T_1^2+T_2^2\right\| + \frac{1}{4}(p_1+p_2) + \frac{1}{2}\left|s_1-s_2\right|}, \end{aligned}$$

where $s_1 = \max\left\{\|T_1\|^2, \frac{1}{2}\|T_1^2+T_2^2\|\right\}$, $s_2 = \max\left\{\|T_2\|^2, \frac{1}{2}\|T_1^2+T_2^2\|\right\}$, $p_1 = \left|\|T_1\|^2 - \frac{1}{2}\|T_1^2+T_2^2\|\right|$ *and* $p_2 = \left|\|T_2\|^2 - \frac{1}{2}\|T_1^2+T_2^2\|\right|$.

Here we note that Corollary 5.1 is better than the first inequality in (5.1) when T_1 and T_2 are self-adjoint operators.

Next, we obtain an upper bound for the Euclidean operator radius $w_e(T_1, T_2)$.

Theorem 5.3. *If $T_1, T_2 \in \mathbb{B}(\mathcal{H})$ then for all $t \in [0,1]$,*

$$\begin{aligned} & w_e(T_1, T_2) \\ \leq \ & \left\|t^2 T_1^* T_1 + (1-t)^2 T_2^* T_2\right\|^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \left\{w^2((1-t)T_1 + tT_2) + w^2((1-t)T_1 - tT_2)\right\}^{\frac{1}{2}}. \end{aligned}$$

In particular, for $t = \frac{1}{2}$

$$w_e(T_1, T_2) \ \leq \ \frac{1}{2}\|T_1^* T_1 + T_2^* T_2\|^{\frac{1}{2}} + \frac{1}{2\sqrt{2}} \left\{w^2(T_1 + T_2) + w^2(T_1 - T_2)\right\}^{\frac{1}{2}}. \tag{5.3}$$

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned} & \left(|\langle T_1 x, x\rangle|^2 + |\langle T_2 x, x\rangle|^2\right)^{\frac{1}{2}} \\ = \ & \left(|t\langle T_1 x, x\rangle + (1-t)\langle T_1 x, x\rangle|^2 + |(1-t)\langle T_2 x, x\rangle + t\langle T_2 x, x\rangle|^2\right)^{\frac{1}{2}} \\ \leq \ & \left(t^2|\langle T_1 x, x\rangle|^2 + (1-t)^2|\langle T_2 x, x\rangle|^2\right)^{\frac{1}{2}} + \left((1-t)^2|\langle T_1 x, x\rangle|^2 + t^2|\langle T_2 x, x\rangle|^2\right)^{\frac{1}{2}} \\ & \qquad\qquad\qquad\qquad\qquad\qquad (\textit{by Minkowski inequality}) \\ \leq \ & \left(t^2\|T_1 x\|^2 + (1-t)^2\|T_2 x\|^2\right)^{\frac{1}{2}} \\ & + \left(\frac{1}{2}|\langle((1-t)T_1 + tT_2)\, x, x\rangle|^2 + \frac{1}{2}|\langle((1-t)T_1 - tT_2)\, x, x\rangle|^2\right)^{\frac{1}{2}} \\ \leq \ & \left\|t^2 T_1^* T_1 + (1-t)^2 T_2^* T_2\right\|^{\frac{1}{2}} + \left\{\frac{1}{2}w^2((1-t)T_1 + tT_2) + \frac{1}{2}w^2((1-t)T_1 - tT_2)\right\}^{\frac{1}{2}}. \end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get the first inequality. In particular, considering $t = \frac{1}{2}$ we get the second inequality. □

To present our next result need the following Hermite-Hadamard inequality, see [76, p. 137]. For a convex function $f : J \to \mathbb{R}$ and $a, b \in J$, we have

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f(ta + (1-t)b)dt \leq \frac{f(a) + f(b)}{2}. \tag{5.4}$$

Also, we need the following lemmas.

Lemma 5.1. *[14, (4.24)] Let $T \in \mathbb{B}(\mathcal{H})$ be self-adjoint with spectrum contained in the interval J and let $x \in \mathcal{H}$ with $\|x\| = 1$. If f is a convex function on J, then*

$$f(\langle Tx, x\rangle) \leq \langle f(T)x, x\rangle.$$

Lemma 5.2. *[60](Generalized Cauchy-Schwarz inequality) If $T \in \mathbb{B}(\mathcal{H})$, then*

$$|\langle Tx, y\rangle|^2 \leq \langle |T|^{2\alpha}x, x\rangle\langle |T^*|^{2(1-\alpha)}y, y\rangle,$$

for all $x, y \in \mathcal{H}$ and for all $\alpha \in [0, 1]$.

Now we are in a position to prove our result.

Theorem 5.4. *Let $T_1, T_2 \in \mathbb{B}(\mathcal{H})$. If $f : [0, \infty) \to [0, \infty)$ is an increasing operator convex function, then*

$$\begin{aligned} f\left(w_e^2(T_1, T_2)\right) &\leq \left\| \int_0^1 f\left(t(T_1^*T_1 + T_2^*T_2) + (1-t)(T_1T_1^* + T_2T_2^*)\right) dt \right\| \\ &\leq \frac{1}{2}\left\| f(T_1^*T_1 + T_2^*T_2) + f(T_1T_1^* + T_2T_2^*)\right\|. \end{aligned}$$

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned} &f(|\langle T_1x, x\rangle|^2 + |\langle T_2x, x\rangle|^2) \\ &\leq f(\langle |T_1|x, x\rangle\langle |T_1^*|x, x\rangle + \langle |T_2|x, x\rangle\langle |T_2^*|x, x\rangle) \text{ (by Lemma 5.2)} \\ &\leq f\left(\{\langle |T_1|x, x\rangle^2 + \langle |T_2|x, x\rangle^2\}^{\frac{1}{2}}\{\langle |T_1^*|x, x\rangle^2 + \langle |T_2^*|x, x\rangle^2\}^{\frac{1}{2}}\right) \\ &\leq f\left(\frac{1}{2}\langle (T_1^*T_1 + T_2^*T_2))x, x\rangle + \frac{1}{2}\langle (T_1T_1^* + T_2T_2^*)x, x\rangle\right) \\ &\leq \int_0^1 f\left(\left\langle \left(t(T_1^*T_1 + T_2^*T_2)) + (1-t)(T_1T_1^* + T_2T_2^*)\right)x, x\right\rangle\right) dt \text{ (by (5.4)).} \end{aligned}$$

Now,

$$\begin{aligned} &f\left(\left\langle \left(t(T_1^*T_1 + T_2^*T_2) + (1-t)(T_1T_1^* + T_2T_2^*)\right)x, x\right\rangle\right) \\ &\leq \left\langle f\left(t(T_1^*T_1 + T_2^*T_2) + (1-t)(T_1T_1^* + T_2T_2^*)\right)x, x\right\rangle \text{ (by Lemma 5.1)} \\ &\leq t\langle f(T_1^*T_1 + T_2^*T_2)x, x\rangle + (1-t)\langle f(T_1T_1^* + T_2T_2^*)x, x\rangle, \end{aligned}$$

where the last inequality follows form operator convexity of f. Therefore,

$$\begin{aligned} &\int_0^1 f\left(\left\langle \left(t(T_1^*T_1 + T_2^*T_2) + (1-t)(T_1T_1^* + T_2T_2^*)\right)x, x\right\rangle\right) dt \\ &\leq \langle \int_0^1 f\left(t(T_1^*T_1 + T_2^*T_2) + (1-t)(T_1T_1^* + T_2T_2^*)\right) dt\, x, x\rangle \\ &\leq \frac{1}{2}\left(\langle f(T_1^*T_1 + T_2^*T_2)x, x\rangle + \langle f(T_1T_1^* + T_2T_2^*)x, x\rangle\right). \end{aligned}$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\begin{aligned} f\left(w_e^2(T_1, T_2)\right) &\leq \left\| \int_0^1 f\big(t(T_1^*T_1 + T_2^*T_2) + (1-t)(T_1T_1^* + T_2T_2^*)\big)dt \right\| \\ &\leq \frac{1}{2}\left\| f(T_1^*T_1 + T_2^*T_2) + f(T_1T_1^* + T_2T_2^*) \right\|. \end{aligned}$$

Thus, we complete the proof. □

Since for $1 \leq r \leq 2$ the function $f(x) = x^r$, $x \geq 0$ is an increasing operator convex function, we have

$$w_e^{2r}(T_1, T_2) \leq \left\| \int_0^1 \left(t(T_1^*T_1 + T_2^*T_2) + (1-t)(T_1T_1^* + T_2T_2^*)\right)^r dt \right\| \tag{5.5}$$

$$\leq \frac{1}{2}\|(T_1^*T_1 + T_2^*T_2)^r + (T_1T_1^* + T_2T_2^*)^r\|. \tag{5.6}$$

In particular, for $r = 1$,

$$\begin{aligned} w_e^2(T_1, T_2) &\leq \left\| \int_0^1 \left(tT_1^*T_1 + T_2^*T_2) + (1-t)(T_1T_1^* + T_2T_2^*)\right) dt \right\| \\ &\leq \frac{1}{2}\|(T_1^*T_1 + T_2^*T_2) + (T_1T_1^* + T_2T_2^*)\|. \end{aligned} \tag{5.7}$$

The above inequality can also be derived from

$$w_e^2(T_1, T_2) \leq \left\| \alpha(|T_1|^2 + |T_2|^2) + (1-\alpha)(|T_1^*|^2 + |T_2^*|^2) \right\|, \; 0 \leq \alpha \leq 1,$$

proved by Moslehian et al. [73, Prop. 3.9]. Now, if we take $T_1 = T_2 = T$ in (5.5) we obtain the following numerical radius inequality.

Corollary 5.2. *Let $T \in \mathbb{B}(\mathcal{H})$, then*

$$w^2(T) \leq \left\| \int_0^1 \left(tT^*T + (1-t)TT^*\right)^r dt \right\|^{1/r} \leq \left\| \frac{(T^*T)^r + (TT^*)^r}{2} \right\|^{1/r},$$

for $1 \leq r \leq 2$.

Next, in the following theorem we develop a lower bound for the numerical radius of a bounded linear operator T.

Theorem 5.5. *Let $T \in \mathbb{B}(\mathcal{H})$, then*

$$\frac{1}{4}\|T\| + \frac{1}{4}\left(\|\Re(T)\| + \|\Im(T)\|\right) + \frac{1}{2}\left|\|\Re(T)\| - \|\Im(T)\|\right| \leq w(T).$$

Proof. From the Cartesian decomposition of T, it is easy to verify that $\|\Re(T)\| \leq w(T)$, $\|\Im(T)\| \leq w(T)$ and $\frac{1}{2}\|T\| \leq w(T)$. Take $r_1 = \left|\|\Re(T)\| - \frac{1}{2}\|T\|\right|$, $r_2 = \left|\|\Im(T)\| - \frac{1}{2}\|T\|\right|$, $q_1 = \max\left\{\|\Re(T)\|, \frac{1}{2}\|T\|\right\}$ and $q_2 = \max\left\{\|\Im(T)\|, \frac{1}{2}\|T\|\right\}$. We have

$$\begin{aligned}
w(T) &\geq \max\{q_1, q_2\} \\
&= \frac{1}{2}(q_1+q_2) + \frac{1}{2}\left|q_1 - q_2\right| \\
&= \frac{1}{4}\|T\| + \frac{1}{4}(\|\Re(T)\| + \|\Im(T)\|) + \frac{1}{4}(r_1+r_2) + \frac{1}{2}\left|q_1 - q_2\right| \\
&\geq \frac{1}{4}\|T\| + \frac{1}{4}\|\Re(T) + i\Im(T)\| + \frac{1}{4}(r_1+r_2) + \frac{1}{2}\left|q_1 - q_2\right| \\
&= \frac{1}{2}\|T\| + \frac{1}{4}(r_1+r_2) + \frac{1}{2}\left|q_1 - q_2\right| \\
&= \frac{1}{2}\|T\| + \frac{1}{4}\left|\|\Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|\Im(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{2}\left|q_1 - q_2\right| \\
&= \frac{1}{4}\|T\| + \frac{1}{4}\left(\|\Re(T)\| + \|\Im(T)\|\right) + \frac{1}{2}\left|\|\Re(T)\| - \|\Im(T)\|\right|,
\end{aligned}$$

as desired. □

Remark 5.6. *(i) It follows from [56] that*

$$\frac{1}{2}\|T\| + \frac{1}{4}\left|\|\Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|\Im(T)\| - \frac{1}{2}\|T\|\right| \leq w(T). \tag{5.8}$$

Clearly, the inequality in Theorem 5.5 refines the inequality (5.8).
(ii) It follows from Theorem 5.5 that if

$$\frac{1}{2}\|T\| + \frac{1}{4}\left|\|\Re(T)\| - \frac{1}{2}\|T\|\right| + \frac{1}{4}\left|\|\Im(T)\| - \frac{1}{2}\|T\|\right| = w(T)$$

then $\max\left\{\|\Re(T)\|, \frac{1}{2}\|T\|\right\} = \max\left\{\|\Im(T)\|, \frac{1}{2}\|T\|\right\}$. *However, the converse may not be true.*
(iii) For $T \in \mathbb{B}(\mathcal{H})$, *Bhunia and Paul [25, Th. 2.1] proved that*

$$\frac{1}{2}\|T\| + \frac{1}{2}\left|\|\Re(T)\| - \|\Im(T)\|\right| \leq w(T). \tag{5.9}$$

Clearly, the inequality in Theorem 5.5 refines (5.9).
(iv) It follows from Theorem 5.5 that if $w(T) = \frac{1}{2}\|T\| + \frac{1}{2}\left|\|\Re(T)\| - \|\Im(T)\|\right|$, *then* $\|T\| = \|\Re(T)\| + \|\Im(T)\|$ *and* $w(T) = \max\{\|\Re(T)\|, \|\Im(T)\|\}$. *The converse is also true.*

As an application to the bounds develop here we develop numerical radius bounds of bounded linear operator T.

5.3 Application to numerical radius inequalities

From Corollary 5.1 we obtain the following numerical radius bound of a bounded linear operator T.

Corollary 5.3. *Let $T \in \mathbb{B}(\mathcal{H})$. Then*

$$\sqrt{\frac{1}{8}\|T^*T+TT^*\| + \frac{1}{4}\left(\|\Re(T)\|^2 + \|\Im(T)\|^2\right) + \frac{1}{2}\left|\|\Re(T)\|^2 - \|\Im(T)\|^2\right|} \le w(T),$$

Proof. Considering $T_1 = \Re(T)$ and $T_2 = \Im(T)$ in Corollary 5.1 we obtain that

$$\begin{aligned} w(T) &\ge \sqrt{\frac{1}{8}\|T^*T+TT^*\| + \frac{1}{4}\left(\|\Re(T)\|^2 + \|\Im(T)\|^2\right) + \frac{1}{2}\left|\|\Re(T)\|^2 - \|\Im(T)\|^2\right|} \\ &= \sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{4}(\alpha+\beta) + \frac{1}{2}\big|\gamma-\delta\big|}, \end{aligned}$$

where $\alpha = \left|\|\Re(T)\|^2 - \frac{1}{4}\|T^*T+TT^*\|\right|$, $\beta = \left|\|\Im(T)\|^2 - \frac{1}{4}\|T^*T+TT^*\|\right|$,
$\gamma = \max\left\{\|\Re(T)\|^2, \frac{1}{4}\|TT^*+T^*T\|\right\}$ and $\delta = \max\left\{\|\Im(T)\|^2, \frac{1}{4}\|TT^*+T^*T\|\right\}$. □

Remark 5.7. *(i) Clearly, the bound obtained in Corollary 5.8 is stronger than the bound obtained in [25, Th. 2.9], which is,*

$$\sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{2}\Big|\|\Re(T)\|^2 - \|\Im(T)\|^2\Big|} \le w(T). \tag{5.10}$$

(ii) From Corollary 5.3 it follows that, if $\sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{2}\Big|\|\Re(T)\|^2 - \|\Im(T)\|^2\Big|} = w(T)$, then $\frac{1}{2}\|T^*T+TT^*\| = \|\Re(T)\|^2 + \|\Im(T)\|^2$ and $w(T) = \max\{\|\Re(T)\|, \|\Im(T)\|\}$. The converse also holds.

Using Corollary 5.1 we also obtain the following lower bound for the numerical radius.

Corollary 5.4. *Let $T \in \mathbb{B}(\mathcal{H})$, then*

$$\sqrt{\frac{1}{8}\|T^*T+TT^*\| + \frac{1}{8}\left(\|\Re(T)+\Im(T)\|^2 + \|\Re(T)-\Im(T)\|^2\right) + \frac{\beta}{4}} \quad \le \quad w(T),$$

where $\beta = \left|\|\Re(T)+\Im(T)\|^2 - \|\Re(T)-\Im(T)\|^2\right|$.

Proof. Considering $T_1 = \frac{\Re(T)+\Im(T)}{\sqrt{2}}$ and $T_2 = \frac{\Re(T)-\Im(T)}{\sqrt{2}}$ in Corollary 5.24, we have

$$\begin{aligned} w(T) &\geq \sqrt{\frac{1}{8}\|T^*T+TT^*\| + \frac{1}{8}\left(\|\Re(T)+\Im(T)\|^2 + \|\Re(T)-\Im(T)\|^2\right) + \frac{\beta}{4}} \\ &= \sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{4}(\gamma+\delta) + \frac{1}{2}\big|\xi-\eta\big|}, \end{aligned}$$

where

$$\begin{aligned} \gamma &= \left|\frac{\|\Re(T)+\Im(T)\|^2}{2} - \frac{1}{4}\|T^*T+TT^*\|\right|, \\ \delta &= \left|\frac{\|\Re(T)-\Im(T)\|^2}{2} - \frac{1}{4}\|T^*T+TT^*\|\right|, \\ \xi &= \max\left\{\frac{\|\Re(T)+\Im(T)\|^2}{2}, \frac{1}{4}\|TT^*+T^*T\|\right\}, \\ \eta &= \max\left\{\frac{\|\Re(T)-\Im(T)\|^2}{2}, \frac{1}{4}\|TT^*+T^*T\|\right\}. \end{aligned}$$

□

Remark 5.8. *(i) In [16, Th. 2.3] authors developed the inequality*

$$\sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{4}\left|\|\Re(T)+\Im(T)\|^2 - \|\Re(T)-\Im(T)\|^2\right|} \leq w(T).$$

It is easy to conclude that the inequality obtained in Corollary 5.4 is a refinement of the above inequalty.
(ii) From the inequality developed in Corollary 5.4, it follows that if

$$\sqrt{\frac{1}{4}\|T^*T+TT^*\| + \frac{1}{4}\left|\|\Re(T)+\Im(T)\|^2 - \|\Re(T)-\Im(T)\|^2\right|} = w(T)$$

then $\|T^*T+TT^*\| = \|\Re(T)+\Im(T)\|^2 + \|\Re(T)-\Im(T)\|^2$ *and*

$$w(T) = \max\left\{\frac{\|\Re(T)+\Im(T)\|}{\sqrt{2}}, \frac{\|\Re(T)-\Im(T)\|}{\sqrt{2}}\right\}.$$

The converse also holds.

Our next result reads as follows.

Theorem 5.9. *Let* $T \in \mathbb{B}(\mathcal{H})$*, then the following inequalities hold:*

$$(i)\ \sqrt{\frac{1}{4}\|T^*T+TT^*\| + \alpha} \leq w(T) \leq \sqrt{\frac{1}{4}\|T^*T+TT^*\| + \beta},$$

$$(ii)\ \sqrt{\frac{1}{4}\|T^*T+TT^*\|+\gamma} \le w(T) \le \sqrt{\frac{1}{4}\|T^*T+TT^*\|+\delta},$$

where

$$\begin{aligned}\gamma &= \frac{1}{4}\left|\|\Re(T)+\Im(T)\|^2-\|\Re(T)-\Im(T)\|^2\right|,\\ \delta &= \frac{1}{4}\left(\|\Re(T)+\Im(T)\|^2+\|\Re(T)-\Im(T)\|^2\right).\end{aligned}$$

Proof. (i) First inequality follows from [25, Th. 2.9] and the second inequality follows from the inequality (5.3) by considering $T_1 = T$ and $T_2 = T^*$.
(ii) First inequality follows from [16, Th. 2.3] and the second inequality follows from the inequality (5.3) by considering $T_2 = \Re(T)$ and $T_2 = \Im(T)$. □

5.4 Numerical radius bounds of 2×2 operator matrices

Using the numerical radius inequalities obtained in Section 2, here we develop the numerical radius bounds of 2×2 off-diagonal operator matrices. Suppose $\mathcal{H}\oplus\mathcal{H}$ is the direct sum of two copies of $\mathcal{H}$, and $\begin{pmatrix} B & X \\ Y & C \end{pmatrix} \in \mathbb{B}(\mathcal{H}\oplus\mathcal{H})$ is a 2×2 operator matrix, defined by $\begin{pmatrix} B & X \\ Y & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Bx+Xy \\ Yx+Cy \end{pmatrix}$, $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}\oplus\mathcal{H}$. Considering $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in \mathbb{B}(\mathcal{H}\oplus\mathcal{H})$ in Theorem 5.5, Corollary 5.8, Corollary 5.4 and Theorem 5.9 respectively, we get the following bounds for the numerical radius of the 2×2 off-diagonal operator matrix $\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$.

Theorem 5.10. *Let* $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in \mathbb{B}(\mathcal{H}\oplus\mathcal{H})$, *then the following inequalities hold:*

$$\begin{aligned}
(i)\ w\left(T\right) &\geq \max\left\{\frac{\|X\|}{4},\frac{\|Y\|}{4}\right\} \\
&+\frac{1}{4}\left(\frac{\|X+Y^*\|}{2}+\frac{\|X-Y^*\|}{2}\right)+\frac{1}{2}\left|\frac{\|X+Y^*\|}{2}-\frac{\|X-Y^*\|}{2}\right|. \\
(ii)\ w^2\left(T\right) &\geq \max\left\{\frac{\|X^*X+YY^*\|}{8},\frac{\|XX^*+Y^*Y\|}{8}\right\} \\
&+\frac{1}{4}\left(\frac{\|X+Y^*\|^2}{4}+\frac{\|X-Y^*\|^2}{4}\right)+\frac{1}{2}\left|\frac{\|X+Y^*\|^2}{4}-\frac{\|X-Y^*\|^2}{4}\right|. \\
(iii)\ w^2(T) &\geq \max\left\{\frac{\|X^*X+YY^*\|}{8},\frac{\|XX^*+Y^*Y\|}{8}\right\} \\
&+\frac{1}{8}\left(\frac{\|(1-i)X+(1+i)Y^*\|^2}{4}+\frac{\|(1+i)X-(1-i)Y^*\|^2}{4}\right) \\
&+\frac{1}{4}\left|\frac{\|(1-i)X+(1+i)Y^*\|^2}{4}-\frac{\|(1+i)X-(1-i)Y^*\|^2}{4}\right|. \\
(iv)\ w^2(T) &\leq \max\left\{\frac{\|X^*X+YY^*\|}{4},\frac{\|XX^*+Y^*Y\|}{4}\right\}+\frac{1}{2}\left|\frac{\|X+Y^*\|^2}{4}+\frac{\|X-Y^*\|^2}{4}\right|. \\
(v)\ w^2(T) &\leq \max\left\{\frac{\|X^*X+YY^*\|}{4},\frac{\|XX^*+Y^*Y\|}{4}\right\} \\
&+\frac{1}{4}\left|\frac{\|(1-i)X+(1+i)Y^*\|^2}{4}+\frac{\|(1+i)X-(1-i)Y^*\|^2}{4}\right|.
\end{aligned}$$

Remark 5.11. *(i) We remark that the bound in Theorem 5.10 (i) is stronger than the same in [15, Th. 2.7], namely,*

$$w\left(T\right) \geq \max\left\{\frac{\|X\|}{2},\frac{\|Y\|}{2}\right\}+\frac{1}{2}\left|\frac{\|X+Y^*\|}{2}-\frac{\|X-Y^*\|}{2}\right|.$$

(ii) It is easy to verify that the bound in Theorem 5.10 (ii) is stronger than the same in [15, Th. 2.12], namely,

$$w^2\left(T\right) \geq \max\left\{\frac{\|X^*X+YY^*\|}{4},\frac{\|XX^*+Y^*Y\|}{4}\right\}+\frac{1}{2}\left|\frac{\|X+Y^*\|^2}{4}-\frac{\|X-Y^*\|^2}{4}\right|.$$

Now, by applying the operator matrix technique we develop upper bounds for the numerical radius of a bounded linear operator T by using the t-Aluthge transform. First we give the following upper bound for the numerical radius $w\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, where $X, Y \in \mathbb{B}(\mathcal{H})$.

Theorem 5.12. *[23, Th. 2.5 and Cor. 2.6] Let $X, Y \in \mathbb{B}(\mathcal{H})$ and $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \in \mathbb{B}(\mathcal{H}\oplus\mathcal{H})$.*

If $S = |X|^2 + |Y^*|^2$ *and* $P = |X^*|^2 + |Y|^2$*, then*

$$w^2 \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} = w^2(T) \leq \sqrt{\min\{\beta, \gamma\}},$$

where

$$\beta = \frac{1}{16}\|S\|^2 + \frac{1}{4}w^2(YX) + \frac{1}{8}w(YXS + SYX),$$
$$\gamma = \frac{1}{16}\|P\|^2 + \frac{1}{4}w^2(XY) + \frac{1}{8}w(XYP + PXY).$$

For $X, Y \in \mathbb{B}(\mathcal{H})$, we have the following inequalities:

$$w(XY) \leq w\begin{pmatrix} XY & 0 \\ 0 & YX \end{pmatrix} = w\left(\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}^2\right) \leq w^2\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}. \tag{5.11}$$

Now, by using (5.11) and Theorem 5.12, we prove the following result.

Corollary 5.5. *Let* $T \in \mathbb{B}(\mathcal{H})$*. If* $P_t = |T|^{2(1-t)} + |T|^{2t}$, $0 \leq t \leq 1$*, then*

$$\begin{aligned} w(T) &\leq \sqrt{\frac{1}{16}\|P_t\|^2 + \frac{1}{4}w^2(\widetilde{T}_t) + \frac{1}{8}w(\widetilde{T}_t P_t + P_t\widetilde{T}_t)} \\ &\leq \frac{1}{4}\left\||T|^{2(1-t)} + |T|^{2t}\right\| + \frac{1}{2}w(\widetilde{T}_t). \end{aligned} \tag{5.12}$$

In particular, for $t = \frac{1}{2}$

$$\begin{aligned} w(T) &\leq \sqrt{\frac{1}{4}\|T\|^2 + \frac{1}{4}w^2(\widetilde{T}) + \frac{1}{4}w(\widetilde{T}|T| + |T|\widetilde{T})} \\ &\leq \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}). \end{aligned} \tag{5.13}$$

Proof. Taking $X = U|T|^{1-t}$ and $Y = |T|^t$ in Theorem 5.12 (in the expression β) we obtain the inequality (5.12), and the next inequality follows from the inequality (see [48]) $w(XY + Y^*X) \leq 2\|Y\|w(X)$ for all $X, Y \in \mathbb{B}(\mathcal{H})$. The rest of the inequalities follows by considering $t = \frac{1}{2}$. □

Remark 5.13. *(i) Let* $T \in \mathbb{B}(\mathcal{H})$*. Then, clearly the inequality* (5.12) *refines the bound* $w(T) \leq \frac{1}{4}\left\||T|^{2(1-t)} + |T|^{2t}\right\| + \frac{1}{2}w(\widetilde{T}_t)$*, obtained by Kittaneh et al. [62, Cor. 2.2].*
(ii) We would like to remark that the inequality (5.13) *is stronger than the inequality* $w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T})$ $\left(\leq \frac{1}{2}\|T\| + \frac{1}{2}\left\|T^2\right\|^{1/2}\right)$*, proved by Yamazaki [86, Th. 2.1].*

Note that the inequality (5.13) is already proved in [16, Th. 2.6] but the approach is different and simple. Finally, we prove the following result.

Corollary 5.6. *Let* $T \in \mathbb{B}(\mathcal{H})$. *If* $Q_t = |T^*|^{2(1-t)} + |T|^{2t}$, $0 \le t \le 1$, *then*

$$\begin{aligned} w(T) &\le \sqrt{\frac{1}{16}\|Q_t\|^2 + \frac{1}{4}w^2(T) + \frac{1}{8}w(TQ_t + Q_tT)} \qquad (5.14)\\ &\le \frac{1}{4}\||T^*|^{2(1-t)} + |T|^{2t}\| + \frac{1}{2}w(T)\\ &\le \frac{1}{2}\left\||T^*|^{2(1-t)} + |T|^{2t}\right\|. \end{aligned}$$

Proof. Taking $X = U|T|^{1-t}$, $Y = |T|^t$ in Theorem 5.12 (in the expression γ) we obtain the inequality (5.14). The second inequality follows from $w(XY + Y^*X) \le 2\|Y\|w(X)$ for all $X, Y \in \mathbb{B}(\mathcal{H})$ (see [48]). The last inequality follows trivially. $\square$

Remark 5.14. *Let* $T \in \mathbb{B}(\mathcal{H})$. *Then, clearly the bound in* (5.14) *is sharper than the bound* $w(T) \le \frac{1}{4}\left\||T^*|^{2(1-t)} + |T|^{2t}\right\| + \frac{1}{2}w(T)$, *proved by Kittaneh et al. [62].*

5.5 Generalized Euclidean operator radius inequalities

Every positive operator A in $\mathbb{B}(\mathcal{H})$ defines the following positive semi-definite sesquilinear form:

$$\langle .,.\rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad (x, y) \to \langle x, y\rangle_A = \langle Ax, y\rangle.$$

Seminorm $\|\cdot\|_A$ induced by the semi-inner product $\langle .,.\rangle_A$, is given by $\|x\|_A = \langle Ax, x\rangle^{1/2} = \|A^{1/2}x\|$. This makes $\mathcal{H}$ into a semi-Hilbertian space. It is easy to verify that the seminorm induces a norm if and only if A is injective. Also, $(\mathcal{H}, \|\cdot\|_A)$ is complete if and only if the range space of operator A, denoted by $\mathcal{R}(A)$, is closed subspace of $\mathcal{H}$. Henceforth, we reserve the symbol A for a non-zero positive operator in $\mathbb{B}(\mathcal{H})$. We denote the A-unit sphere and A-unit ball of the semi-Hilbertian space $(\mathcal{H}, \|\cdot\|_A)$ by $\mathbb{S}_{\|\cdot\|_A}$ and $\mathbb{B}_{\|\cdot\|_A}$, respectively, i.e.,

$$\mathbb{S}_{\|\cdot\|_A} = \{x \in \mathcal{H} : \|x\|_A = 1\}, \ \mathbb{B}_{\|\cdot\|_A} = \{x \in \mathcal{H} : \|x\|_A \le 1\}.$$

For $T \in \mathbb{B}(\mathcal{H})$, let $c_A(T)$ and $w_A(T)$ denote the A-Crawford number and the A-numerical radius of T, respectively and are defined as

$$c_A(T) = \inf\left\{|\langle Tx, x\rangle_A| : x \in \mathbb{S}_{\|\cdot\|_A}\right\}, \ w_A(T) = \sup\left\{|\langle Tx, x\rangle_A| : x \in \mathbb{S}_{\|\cdot\|_A}\right\}.$$

Note that $w_A(T)$ is not necessarily finite, see [29]. An operator $S \in \mathbb{B}(\mathcal{H})$ is called an A-adjoint of $T \in \mathbb{B}(\mathcal{H})$ if for every $x, y \in \mathcal{H}$, $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$ holds, i.e., S is a solution of the operator equation $AX = T^*A$. There are operators T for which A-adjoint may fail to exist, when it do exist then there may be more than one A-adjoint. The set of all operators in $\mathbb{B}(\mathcal{H})$ which possess A-adjoint is denoted by $\mathbb{B}_A(\mathcal{H})$. By Douglas theorem [39], we have

$$\begin{aligned}\mathbb{B}_A(\mathcal{H}) &= \{T \in \mathbb{B}(\mathcal{H}) : \ \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\} \\ &= \{T \in \mathbb{B}(\mathcal{H}) : \ \exists\, \lambda > 0 \ \textit{such that} \ \|ATx\| \leq \lambda \|Ax\|, \ \forall x \in \mathcal{H}\}.\end{aligned}$$

If $T \in \mathbb{B}_A(\mathcal{H})$, then there exists a unique solution of $AX = T^*A$, is denoted by $T^{\sharp_A}$, satisfying $\mathcal{R}(T^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}$, where $\overline{\mathcal{R}(A)}$ is the norm closure of $\mathcal{R}(A)$. For simplicity we will write $T^\sharp$ instead of $T^{\sharp_A}$. If $T \in \mathbb{B}_A(\mathcal{H})$, then $T^\sharp \in \mathbb{B}_A(\mathcal{H})$. Moreover, $\left[T^\sharp\right]^\sharp = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and $\left[\left[T^\sharp\right]^\sharp\right]^\sharp = T^\sharp$, where $P_{\overline{\mathcal{R}(A)}}$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$. For more about $T^\sharp$, the reader can see [7, 8]. Again, clearly we have

$$\begin{aligned}\mathbb{B}_{A^{1/2}}(\mathcal{H}) &= \left\{T \in \mathbb{B}(\mathcal{H}) : \ \mathcal{R}(T^*A^{1/2}) \subseteq \mathcal{R}(A^{1/2})\right\} \\ &= \{T \in \mathbb{B}(\mathcal{H}) : \ \exists\, \lambda > 0 \ \textit{such that} \ \|Tx\|_A \leq \lambda \|x\|_A, \ \forall x \in \mathcal{H}\}.\end{aligned}$$

An operator in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ is called A-bounded operator. The inclusion $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H})$ always holds. Both of them are subalgebras of $\mathbb{B}(\mathcal{H})$ which are neither closed and nor dense in $\mathbb{B}(\mathcal{H})$. The semi-inner product $\langle ., .\rangle_A$ induces the A-operator seminorm on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ defined as follows:

$$\|T\|_A = \sup_{\substack{x \in \overline{\mathcal{R}(A)} \\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup\left\{\|Tx\|_A : \ x \in \mathbb{S}_{\|\cdot\|_A}\right\} < \infty.$$

Also, it is easy to verify that

$$\|T\|_A = \sup\left\{|\langle Tx, y\rangle_A| : \ x, y \in \mathbb{S}_{\|\cdot\|_A}\right\}.$$

By Cauchy-Schwarz inequality, it follows that $|\langle Tx, x\rangle_A| \leq \|Tx\|_A \|x\|_A$ for all $x \in \mathcal{H}$, and so $w_A(T) \leq \|T\|_A$ for all $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. For A-selfadjoint operator T (i.e., $AT = T^*A$), we have $w_A(T) = \|T\|_A$, see in [88]. An operator $T \in \mathbb{B}_A(\mathcal{H})$ can be expressed as $T = \Re_A(T) + i\Im_A(T)$, where $\Re_A(T) = \frac{1}{2}(T + T^{\sharp_A})$ and $\Im_A(T) = \frac{1}{2i}(T - T^{\sharp_A})$. This decomposition is called A-Cartesian decomposition, using this we have $|\langle \Re_A(T)x, x\rangle_A|^2 + |\langle \Im_A(T)x, x\rangle_A|^2 = |\langle Tx, x\rangle_A|^2$ for all $x \in \mathcal{H}$. This implies $\|\Re_A(T)\|_A \leq w_A(T)$ and $\|\Im_A(T)\|_A \leq w_A(T)$, since $\Re_A(T)$ and $\Im_A(T)$ both are A-selfadjoint. Therefore, $\|T\|_A \leq \|\Re_A(T) + i\Im_A(T)\|_A \leq 2w_A(T)$. Thus, for every $T \in \mathbb{B}_A(\mathcal{H})$, we get $w_A(T) \leq \|T\|_A \leq 2w_A(T)$. One can also easily verify that the above inequality holds for every $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, and $w_A(T^n) \leq [w_A(T)]^n$ holds for every positive integer n, see [12]. The A-numerical radius inequalities have been studied by many mathematicians [20, 21, 72] over the years. The A-Euclidean operator radius of d-tuple operators $\mathbf{T} = (T_1, T_2, \ldots.., T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ is defined as

$$w_{A,e}(\mathbf{T}) = \sup\left\{ \left(\sum_{k=1}^{d} |\langle T_k x, x\rangle_A|^2 \right)^{1/2} : x \in \mathbb{S}_{\|\cdot\|_A} \right\}.$$

This is also known as A-joint numerical radius of $\mathbf{T}$. The A-Euclidean operator seminorm of d-tuple operators $\mathbf{T} = (T_1, T_2, \ldots.., T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$ is defined as

$$\|\mathbf{T}\|_A = \sup\left\{ \left(\sum_{k=1}^{d} \|T_k x\|_A^2 \right)^{1/2} : x \in \mathbb{S}_{\|\cdot\|_A} \right\}.$$

Clearly, the A-Euclidean operator radius and A-Euclidean operator seminorm of d-tuple operators are generalizations of A-numerical radius and A-operator seminorm of an operator in $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. Observe that for $A = I$, $\|\cdot\|_A = \|\cdot\|$, $w_A(\cdot) = w(\cdot)$, $c_A(\cdot) = c(\cdot)$, $w_{A,e}(\cdot) = w_e(\cdot)$ and $\|\cdot\|_{A,e} = \|\cdot\|_e$ are the usual operator norm, numerical radius, Crawford number, Euclidean operator radius and Euclidean operator norm, respectively.

In this section, we obtain several inequalities involving A-Euclidean operator radius and A-Euclidean operator seminorm of 2-tuple operators, and we show that these inequalities improve on the earlier related inequalities.

We end this introductory section with a brief description of the space $\mathbf{R}(A^{1/2})$ (see [6]) as follows: The semi-inner product $\langle ., .\rangle_A$ induces an inner product on the quotient space $\mathcal{H}/\mathcal{N}(A)$, defined by $[\overline{x}, \overline{y}] = \langle Ax, y\rangle$, $\forall\, \overline{x}, \overline{y} \in \mathcal{H}/\mathcal{N}(A)$. The space $(\mathcal{H}/\mathcal{N}(A), [., .])$ is, in general, not a complete space. The completion of $(\mathcal{H}/\mathcal{N}(A), [., .])$ is isometrically isomorphic to the Hilbert space $R(A^{1/2})$ via the canonical construction mentioned in [32], where $R(A^{1/2})$ is equipped with the inner product

$$(A^{1/2}x, A^{1/2}y) = \langle P_{\overline{\mathcal{R}(A)}}x, P_{\overline{\mathcal{R}(A)}}y\rangle, \ \forall x, y \in \mathcal{H}.$$

In the sequel, the Hilbert space $(\mathcal{R}(A^{1/2}), (., .))$ will be denoted by $\mathbf{R}(A^{1/2})$ and we use the symbol $\|\cdot\|_{\mathbf{R}(A^{1/2})}$ to represent the norm induced by the inner product $(., .)$. Note that, the fact $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2})$ implies that $(Ax, Ay) = \langle x, y\rangle_A$, $\forall x, y \in \mathcal{H}$. This gives $\|Ax\|_{\mathcal{R}(A^{1/2})} = \|x\|_A$, $\forall x \in \mathcal{H}$. Now, we give a nice connection of an operator $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ with an operator $\widetilde{T} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$, in the form of the following proposition, see [6].

Proposition 5.2. *Let $T \in \mathbb{B}(\mathcal{H})$. Then $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exist a unique $\widetilde{T} \in \mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_A T = \widetilde{T} Z_A$, where $Z_A : \mathcal{H} \to \mathbf{R}(A^{1/2})$ is defined by $Z_A x = Ax$.*

We begin with the following sequence of known lemmas.
First lemma is known as Holder-McCarthy inequality.

Lemma 5.3. *[71] If $T \in \mathcal{B}(\mathcal{H})$ is positive, then the following inequalities hold: For any $x \in \mathcal{H}$,*

$$\langle T^r x, x\rangle \geq ||x||^{2(1-r)}\langle Tx, x\rangle^r, \quad \textit{for } r \geq 1$$

and

$$\langle T^r x, x\rangle \leq ||x||^{2(1-r)}\langle Tx, x\rangle^r, \quad \textit{for } 0 \leq r \leq 1.$$

Second lemma is related to A-selfadjoint operators.

Lemma 5.4. *[46] Let $T \in \mathcal{B}(\mathcal{H})$ be A-selfadjoint. Then $T^\sharp$ is also A-selfadjoint and $[T^\sharp]^\sharp = T^\sharp$.*

Fourth lemma is related to semi-Hilbertian space operator T and Hilbert space operator $\widetilde{T}$.

Lemma 5.5. *[6, 47] Let $T \in \mathcal{B}_A(\mathcal{H})$. Then*
(i) $\widetilde{T^\sharp} = \left(\widetilde{T}\right)^*$ and $\widetilde{(T^{\sharp_A})^{\sharp_A}} = \widetilde{T}$.
(ii) $\|T\|_A = \|\widetilde{T}\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$, $w_A(T) = w(\widetilde{T})$ and $c_A(T) = c(\widetilde{T})$.
(Here $\|\widetilde{T}\|_{\mathcal{B}(\mathbf{R}(A^{1/2}))}$ denotes the usual operator norm of $\widetilde{T}$).

Now, we prove the following result related to A-Euclidean operator radius and Euclidean operator radius.

Theorem 5.15. *Let $\mathbf{T} = (T_1, T_2, \dots, T_d) \in \mathbb{B}_{A^{1/2}}(\mathcal{H})^d$. Then*

$$w_{A,e}(\mathbf{T}) = w_{A,e}(T_1, T_2, ..., T_d) = w_e(\widetilde{T_1}, \widetilde{T_2}, ..., \widetilde{T_d}) = w_e(\widetilde{\mathbf{T}}),$$

where $\widetilde{\mathbf{T}} = (\widetilde{T_1}, \widetilde{T_2}, ..., \widetilde{T_d}) \in \mathbb{B}(\mathbf{R}(A^{1/2}))^d$.

Proof. First we prove $w_{A,e}(\mathbf{T}) \leq w_e(\widetilde{\mathbf{T}})$. We recall that

$$\begin{aligned} w_{A,e}(\mathbf{T}) &= \sup\left\{\left(\sum_{i=1}^d |\langle T_i x, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|x\|_A = 1\right\} \\ &= \sup\left\{\left(\sum_{i=1}^d |(AT_i x, Ax)|^2\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|Ax\|_{\mathbf{R}(A^{1/2})} = 1\right\} \\ &= \sup\left\{\left(\sum_{i=1}^d |(\widetilde{T}_i Ax, Ax)|^2\right)^{\frac{1}{2}} : x \in \mathcal{H},\ \|Ax\|_{\mathbf{R}(A^{1/2})} = 1\right\} \\ &\qquad\qquad (\textit{using Proposition } 5.2). \end{aligned}$$

From the decomposition $\mathcal{H} = \mathcal{N}(A^{1/2}) \oplus \overline{\mathcal{R}(A^{1/2})}$, we obtain that

$$w_{A,e}(\mathbf{T}) \;=\; \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T}_i Ax, Ax)|^2\right)^{\frac{1}{2}} : \; x \in \overline{\mathcal{R}(A^{1/2})},\; \|Ax\|_{\mathbf{R}(A^{1/2})} = 1\right\}. \tag{5.15}$$

Now,

$$\begin{aligned}
& w_e(\widetilde{\mathbf{T}}) \\
= \;& \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T}_i y, y)|^2\right)^{\frac{1}{2}} : \; y \in \mathcal{R}(A^{1/2}),\; \|y\|_{\mathbf{R}(A^{1/2})} = 1\right\} \\
= \;& \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T}_i A^{1/2}x, A^{1/2}x)|^2\right)^{\frac{1}{2}} : \; x \in \mathcal{H},\; \|A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 1\right\} \\
= \;& \sup\left\{\left(\sum_{i=1}^{d} |(\widetilde{T}_i A^{1/2}x, A^{1/2}x)|^2\right)^{\frac{1}{2}} : x \in \overline{\mathcal{R}(A^{1/2})}, \|A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 1\right\}. \quad (5.16)
\end{aligned}$$

Since $\mathcal{R}(A) \subseteq \mathcal{R}(A^{1/2})$, (5.15) together with (5.16) implies $w_{A,e}(\mathbf{T}) \le w_e(\widetilde{\mathbf{T}})$.

Next we show the reverse inequality, i.e, $w_A(\widetilde{\mathbf{T}}) \le w_{A,e}(\mathbf{T})$. Suppose that

$$\beta \in \left\{\left(\sum_{i=1}^{d} |(\widetilde{T}_i A^{1/2}x, A^{1/2}x)|^2\right)^{\frac{1}{2}} : \; x \in \overline{\mathcal{R}(A^{1/2})},\; \|A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 1\right\} = W_e(\widetilde{\mathbf{T}}),\ (say).$$

So, there exists $x \in \overline{\mathcal{R}(A^{1/2})}$ with $\|A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 1$ such that

$$\beta = \left(\sum_{i=1}^{d} |(\widetilde{T}_i A^{1/2}x, A^{1/2}x)|^2\right)^{\frac{1}{2}}.$$

Since $A^{1/2}x \in \mathbf{R}(A^{1/2})$ and $\mathcal{R}(A)$ is dense in $\mathbf{R}(A^{1/2})$, there exist a sequence $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} \|Ax_n - A^{1/2}x\|_{\mathbf{R}(A^{1/2})} = 0$. Hence $\beta = \lim_{n\to\infty} \left(\sum_{i=1}^{d} |(\widetilde{T}_i Ax_n, Ax_n)|^2\right)^{\frac{1}{2}}$ and $\lim_{n\to\infty} \|Ax_n\|_{\mathbf{R}(A^{1/2})} = 1$. Now, let $y_n = \frac{x_n}{\|Ax_n\|_{\mathbf{R}(A^{1/2})}}$. Then clearly we have,
$\beta = \lim_{n\to\infty} \left(\sum_{i=1}^{d} |(\widetilde{T}_i Ay_n, Ay_n)|^2\right)^{\frac{1}{2}}$ and $\|Ay_n\|_{\mathbf{R}(A^{1/2})} = 1$. Therefore,

$$\beta \in \overline{\left\{\left(\sum_{i=1}^{n} |(\widetilde{T}_i Ax, Ax)|^2\right)^{\frac{1}{2}} : \; x \in \overline{\mathcal{R}(A^{1/2})},\; \|Ax\|_{\mathbf{R}(A^{1/2})} = 1\right\}} = \overline{W_{A,e}(\mathbf{T})},\ (say).$$

Hence, $W_e(\widetilde{\mathbf{T}}) \subseteq \overline{W_{A,e}(\mathbf{T})}$. This implies $w_e(\widetilde{\mathbf{T}}) \le w_{A,e}(\mathbf{T})$, and this completes the proof. □

Now, we are in a position to prove the bounds of A-Euclidean operator radius. In the following theorem we obtain upper and lower bound for the A-Euclidean operator radius of 2-tuple operators in $\mathbb{B}_A(\mathcal{H})$ involving A-numerical radius.

Theorem 5.16. *Let $T_1, T_2 \in \mathbb{B}_A(\mathcal{H})$, then*

$$\begin{aligned}
&\frac{1}{2}w_A(T_1^2+T_2^2)+\frac{1}{2}\max\{w_A(T_1),w_A(T_2)\}\big|w_A(T_1+T_2)-w_A(T_1-T_2)\big| \\
&\le w_{A,e}^2(T_1,T_2) \\
&\le \frac{1}{\sqrt{2}}w_A((T_1^\sharp T_1+T_2^\sharp T_2)+i(T_1T_1^\sharp+T_2T_2^\sharp)).
\end{aligned}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have,

$$\begin{aligned}
|\langle T_1x,x\rangle_A|^2+|\langle T_2x,x\rangle_A|^2 &\ge \frac{1}{2}\left(|\langle T_1x,x\rangle_A|+|\langle T_2x,x\rangle_A|\right)^2 \\
&\ge \frac{1}{2}\left(|\langle T_1x,x\rangle_A \pm \langle T_2x,x\rangle_A|\right)^2 \\
&= \frac{1}{2}|\langle (T_1\pm T_2)x,x\rangle_A|^2.
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$, $\|x\|_A = 1$, we get

$$w_{A,e}^2(T_1,T_2) \ge \frac{1}{2}w_A^2(T_1\pm T_2). \tag{5.17}$$

Therefore, it follows from the inequalities in (5.17) that

$$\begin{aligned}
w_{A,e}^2(T_1,T_2) &\ge \frac{1}{2}\max\{w_A^2(T_1+T_2),w_A^2(T_1-T_2)\} \\
&= \frac{w_A^2(T_1+T_2)+w_A^2(T_1-T_2)}{4}+\frac{\big|w_A^2(T_1+T_2)-w_A^2(T_1-T_2)\big|}{4} \\
&\ge \frac{w_A((T_1+T_2)^2)+w_A((T_1-T_2)^2)}{4} \\
&\quad +(w_A(T_1+T_2)+w_A(T_1-T_2))\frac{\big|w_A(T_1+T_2)-w_A(T_1-T_2)\big|}{4} \\
&\ge \frac{w_A((T_1+T_2)^2+(T_1-T_2)^2)}{4} \\
&\quad +w_A((T_1+T_2)+(T_1-T_2))\frac{\big|w_A(T_1+T_2)-w_A(T_1-T_2)\big|}{4}.
\end{aligned}$$

Therefore,

$$w_{A,e}^2(T_1,T_2) \ge \frac{w_A(T_1^2+T_2^2)}{2}+\frac{w_A(T_1)}{2}\big|w_A(T_1+T_2)-w_A(T_1-T_2)\big|. \tag{5.18}$$

Interchanging T_1 and T_2 in (5.18), we arrive

$$w_{A,e}^2(T_1,T_2) \quad \geq \quad \frac{w_A(T_1^2+T_2^2)}{2}+\frac{w_A(T_2)}{2}\big|w_A(T_1+T_2)-w_A(T_1-T_2)\big|. \tag{5.19}$$

The inequality (5.18) together with (5.19), gives the first inequality.

Next, we prove the second inequality. Let $x \in \mathcal{H}$ with $\|x\|=1$. Then we have,

$$\begin{aligned}
&(|\langle T_1x,x\rangle|^2+|\langle T_2x,x\rangle|^2)^2\\
&\leq (\langle |T_1|x,x\rangle\langle |T_1^*|x,x\rangle+\langle |T_2|x,x\rangle\langle |T_2^*|x,x\rangle)^2 \ \ (\textit{using Lemma 4.1})\\
&\leq (\langle |T_1|x,x\rangle^2+\langle |T_2|x,x\rangle^2)(\langle |T_1^*|x,x\rangle^2+\langle |T_2^*|x,x\rangle^2)\\
&\qquad\qquad (\textit{since } (ab+cd)^2\leq (a^2+c^2)(b^2+d^2) \textit{ for all } a,b,c,d\in\mathbb{R})\\
&\leq (\langle |T_1|^2x,x\rangle+\langle |T_2|^2x,x\rangle)(\langle |T_1^*|^2x,x\rangle+\langle |T_2^*|^2x,x\rangle) \ \ (\textit{using Lemma 5.3})\\
&= \langle (T_1^*T_1+T_2^*T_2)x,x\rangle\langle (T_1T_1^*+T_2T_2^*)x,x\rangle\\
&\leq \frac{1}{2}\left\{\langle (T_1^*T_1+T_2^*T_2)x,x\rangle^2+\langle (T_1T_1^*+T_2T_2^*)x,x\rangle^2\right\}\\
&= \frac{1}{2}|\langle (T_1^*T_1+T_2^*T_2)x,x\rangle+i\langle (T_1T_1^*+T_2T_2^*)x,x\rangle|^2\\
&= \frac{1}{2}|\langle ((T_1^*T_1+T_2^*T_2)+i(T_1T_1^*+T_2T_2^*))x,x\rangle|^2\\
&\leq \frac{1}{2}w^2((T_1^*T_1+T_2^*T_2)+i(T_1T_1^*+T_2T_2^*)).
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\|=1$, we get

$$w_e^2(T_1,T_2) \quad \leq \quad \frac{1}{\sqrt{2}}w((T_1^*T_1+T_2^*T_2)+i(T_1T_1^*+T_2T_2^*)). \tag{5.20}$$

As $T_1,T_2 \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, following Proposition 5.2, there exist unique $\widetilde{T_1}$ and $\widetilde{T_2}$ in $\mathbb{B}(\mathbf{R}(A^{1/2}))$ such that $Z_AT_1=\widetilde{T_1}Z_A$ and $Z_AT_2=\widetilde{T_2}Z_A$. The inequality (5.20) implies that

$$w_e^2(\widetilde{T_1},\widetilde{T_2}) \leq \frac{1}{\sqrt{2}}w((\widetilde{T_1}^*\widetilde{T_1}+\widetilde{T_2}^*\widetilde{T_2})+i(\widetilde{T_1}\widetilde{T_1}^*+\widetilde{T_2}\widetilde{T_2}^*)). \tag{5.21}$$

Since $(\widetilde{T_1})^*=\widetilde{T_1^\sharp}$, the inequality (5.21) becomes

$$w_e^2(\widetilde{T_1},\widetilde{T_2}) \leq \frac{1}{\sqrt{2}}w((\widetilde{T_1^\sharp}\widetilde{T_1}+\widetilde{T_2^\sharp}\widetilde{T_2})+i(\widetilde{T_1}\widetilde{T_1^\sharp}+\widetilde{T_2}\widetilde{T_2^\sharp})). \tag{5.22}$$

For any $S,T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, it is easy to see that $\widetilde{ST}=\widetilde{S}\widetilde{T}$ and $\widetilde{S+\lambda T}=\widetilde{S}+\lambda\widetilde{T}$ for all $\lambda\in\mathbb{C}$.

So, the inequality (5.22) is of the following form

$$w_e^2(\widetilde{T_1}, \widetilde{T_2}) \leq \frac{1}{\sqrt{2}} w(\widetilde{(T_1^\sharp T_1 + T_2^\sharp T_2) + i(T_1 T_1^\sharp + T_2 T_2^\sharp)}). \tag{5.23}$$

Now, by applying Theorem 5.15 and Lemma 5.5, we have

$$w_{A,e}^2(T_1, T_2) \leq \frac{1}{\sqrt{2}} w_A((T_1^\sharp T_1 + T_2^\sharp T_2) + i(T_1 T_1^\sharp + T_2 T_2^\sharp)).$$

This completes the proof. □

Remark 5.17. *(i) The lower bound of* $w_e(T_1, T_2)$ *in Theorem 5.16 is stronger than the lower bound in [44, Th. 2.8], namely,* $\frac{1}{2} w_A(T_1^2 + T_2^2) \leq w_{A,e}^2(T_1, T_2)$. *Also, it is not difficult to verify that*

$$\frac{1}{\sqrt{2}} w_A((T_1^\sharp T_1 + T_2^\sharp T_2) + i(T_1 T_1^\sharp + T_2 T_2^\sharp)) \leq \frac{1}{\sqrt{2}} \left\{ \|T_1^\sharp T_1 + T_2^\sharp T_2\|_A^2 + \|T_1 T_1^\sharp + T_2 T_2^\sharp\|_A^2 \right\}^{\frac{1}{2}}.$$

Therefore, the upper bound of $w_{A,e}(T_1, T_2)$ *in Theorem 7.1 is better than the upper bound in [44, Th. 2.8], namely,* $w_{A,e}^2(T_1, T_2) \leq \|T_1 T_1^\sharp + T_2 T_2^\sharp\|_A$ *if* $\|T_1 T_1^\sharp + T_2 T_2^\sharp\|_A \leq \|T_1^\sharp T_1 + T_2^\sharp T_2\|_A$.

(ii) Following Theorem 5.16, $w_{A,e}^2(T_1, T_2) = \frac{1}{2} w_A(T_1^2 + T_2^2)$ *implies* $w_A(T_1 + T_2) = w_A(T_1 - T_2)$. *However, the converse is not true, in general.*

The following corollary is an immediate consequence of Theorem 5.16.

Corollary 5.7. *If* $T_1, T_2 \in \mathbb{B}_A(\mathcal{H})$ *are A-selfadjoint, then*

$$\frac{1}{2}\|T_1^2 + T_2^2\|_A + \frac{1}{2} \max\{\|T_1\|_A, \|T_2\|_A\} \big| \|T_1 + T_2\|_A - \|T_1 - T_2\|_A \big| \leq w_{A,e}^2(T_1, T_2).$$

Next we obtain an upper bound for the A-Euclidean operator radius of 2-tuple operators admitting A-adjoint. First we need the following proposition.

Proposition 5.3. *Let* $x \in \mathcal{H}$ *with* $\|x\|_A = 1$. *Suppose that* $T = x \otimes_A x$, *is defined as* $Tz = (x \otimes x)z = \langle z, x \rangle_A x$, $\forall z \in \mathcal{H}$. *Then we have*

$$|\alpha - 1| \leq \|\alpha T - I\|_A \leq \max\{1, |\alpha - 1|\},$$

for all $\alpha \in \mathbb{C}$. *Moreover, if* $|\alpha - 1| \geq 1$, *then* $\|\alpha T - I\|_A = |\alpha - 1|$.

Proof. For any $z \in \mathcal{H}$, we have

$$\begin{aligned} \|(\alpha T - I)z\|_A^2 &= \langle (\alpha T - I)z, (\alpha T - I)z \rangle_A \\ &= |\alpha|^2 \|Tz\|_A^2 - \alpha \langle Tz, z\rangle_A - \bar{\alpha}\langle z, Tz\rangle_A + \|z\|_A^2 \\ &= |\langle z, x\rangle_A|^2 (|\alpha|^2 - \alpha - \bar{\alpha}) + \|z\|_A^2 \\ &= |\langle z, x\rangle_A|^2 (|\alpha - 1|^2 - 1) + \|z\|_A^2 \qquad (5.24) \\ &\leq \max\{1, |\alpha - 1|^2\} \|z\|_A^2. \qquad (5.25) \end{aligned}$$

Taking supremum over $\|z\|_A = 1$, we have

$$\|\alpha T - I\|_A \leq \max\{1, |\alpha - 1|\}.$$

Again, from the equation (5.24) we have,

$$\|(\alpha T - I)z\|_A^2 + |\langle z, x\rangle_A|^2 = |\langle z, x\rangle_A|^2 |\alpha - 1|^2 + \|z\|_A^2.$$

This implies that

$$\|(\alpha T - I)z\|_A \geq |\langle z, x\rangle_A| |\alpha - 1|.$$

Taking supremum over $\|z\|_A = 1$, we get

$$\|\alpha T - I\| \geq \sup_{\|z\|_A = 1} |\langle z, x\rangle_A| |\alpha - 1| \geq |\alpha - 1|.$$

This completes the proof. □

By using the above proposition we obtain a generalization of Buzano's inequality ([33]), in the setting of a semi-Hilbertian space.

Lemma 5.6. *If $x, y, e \in \mathcal{H}$ with $\|e\|_A = 1$, then*

$$| \langle x, e\rangle_A \langle e, y\rangle_A | \leq \frac{| \langle x, y\rangle_A | + \max\{1, |\alpha - 1|\} \|x\|_A \|y\|_A}{|\alpha|},$$

for all non-zero scalar α.

Proof. Suppose that $T = e \otimes_A e$. Then we have,

$$\begin{aligned} |\ \alpha\langle x, e\rangle_A\langle e, y\rangle_A - \langle x, y\rangle_A\ | &= |\ \alpha\langle Tx, y\rangle_A - \langle x, y\rangle_A\ | \\ &= |\ \langle(\alpha T - I)x, y\rangle_A\ | \\ &\leq \|\alpha T - I\|_A\|x\|_A\|y\|_A \\ &\leq \max\{1, |\alpha - 1|\}\|x\|_A\|y\|_A \ \textit{(by Proposition 5.3)}. \end{aligned}$$

This gives that

$$|\ \alpha\langle x, e\rangle_A\langle e, y\rangle_A\ | \leq \{1, |\alpha - 1|\}\|x\|_A\|y\|_A + |\langle x, y\rangle_A|.$$

This completes the proof. □

Note that the inequality in Lemma 5.6 was studied (for the case $A = I$) in [61, Cor. 2.5], using different approaches. In particular, for $\alpha = 2$ in Lemma 5.6, we have

$$|\ \langle x, e\rangle_A\langle e, y\rangle_A\ | \leq \frac{\|x\|_A\|y\|_A + |\langle x, y\rangle_A|}{2}, \tag{5.26}$$

which was also obtained in [13].

Now, by using Lemma 5.6 we obtain the following upper bound for A-Euclidean operator radius.

Theorem 5.18. *If $T_1, T_2 \in \mathbb{B}_A(\mathcal{H})$, then*

$$w_{A,e}^2(T_1, T_2) \leq \frac{\max\{1, |1 - \alpha|\}\|(T_1, T_2)\|_{A,e}\|(T_1^\sharp, T_2^\sharp)\|_{A,e} + w_A(T_1^2) + w_A(T_2^2)}{|\alpha|},$$

for any non-zero scalar α.

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have,

$$\begin{aligned}
& |\langle T_1x, x\rangle_A|^2 + |\langle T_2x, x\rangle_A|^2 \\
= & |\langle T_1x, x\rangle_A \langle x, T_1^\sharp x\rangle_A| + |\langle T_2x, x\rangle_A \langle x, T_2^\sharp x\rangle_A| \\
\leq & \frac{\max\{1, |\alpha-1|\}\|T_1x\|_A\|T_1^\sharp x\|_A + |\langle T_1x, T_1^\sharp x\rangle_A|}{|\alpha|} \\
& + \frac{\max\{1, |\alpha-1|\}\|T_2x\|_A\|T_2^\sharp x\|_A + |\langle T_2x, T_2^\sharp x\rangle_A|}{|\alpha|} \text{ (using Lemma 5.6)} \\
= & \frac{\max\{1, |\alpha-1|\}(\|T_1x\|_A\mathbb{T}_1^\sharp x\|_A + \|T_2x\|_A\|T_2^\sharp x\|_A)}{|\alpha|} \\
& + \frac{|\langle T_1x, T_1^\sharp x\rangle_A| + |\langle T_2x, T_2^\sharp x\rangle_A|}{|\alpha|} \\
\leq & \frac{\max\{1, |\alpha-1|\}(\|T_1x\|_A^2 + \|T_2x\|_A^2)^{\frac{1}{2}}(\|T_1^\sharp x\|_A^2 + \|T_2^\sharp x\|_A^2)^{\frac{1}{2}}}{|\alpha|} \\
& + \frac{|\langle T_1^2x, x\rangle_A| + |\langle T_2^2x, x\rangle_A|}{|\alpha|} \\
\leq & \frac{\max\{1, |\alpha-1|\}\|(T_1, T_2)\|_{A,e}\|(T_1^\sharp, T_2^\sharp)\|_{A,e}}{|\alpha|} + \frac{w_A(T_1^2) + w_A(T_2^2)}{|\alpha|}.
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$, we get the desired inequality. $\square$

Next bound reads as follows:

Theorem 5.19. *If $T_1, T_2 \in \mathbb{B}_A(\mathcal{H})$, then*

$$\begin{aligned}
w_{A,e}^2(T_1, T_2) \leq & \min\{w_A^2(T_1 - T_2), w_A^2(T_1 + T_2)\} \\
& + \frac{\max\{1, |1-\alpha|\}\|T_2^\sharp T_2 + T_1T_1^\sharp\|_A + 2w_A(T_1T_2)}{|\alpha|},
\end{aligned}$$

for any non-zero scalar α.

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then we have,

$$\begin{aligned}
|\langle T_2x, x\rangle_A|^2 - 2\Re[\langle T_2x, x\rangle_A \overline{\langle T_1x, x\rangle_A}] + |\langle T_1x, x\rangle_A|^2 &= |\langle T_2x, x\rangle_A - \langle T_1x, x\rangle_A|^2 \\
&= |\langle (T_2 - T_1)x, x\rangle_A|^2 \\
&\leq w_A^2(T_2 - T_1).
\end{aligned}$$

Thus,

$$\begin{aligned}
& |\langle T_2x,x\rangle_A|^2+|\langle T_1x,x\rangle_A|^2\\
\leq\ & w_A^2(T_2-T_1)+2\Re[\langle T_2x,x\rangle_A\overline{\langle T_1x,x\rangle_A}]\\
\leq\ & w_A^2(T_2-T_1)+2|\langle T_2x,x\rangle_A\langle T_1x,x\rangle_A|\\
\leq\ & w_A^2(T_2-T_1)+\frac{2\max\{1,|\alpha-1|\}\|T_2x\|_A\|T_1^\sharp x\|_A+2|\langle T_2x,T_1^\sharp x\rangle_A|}{|\alpha|}\ (\textit{by Lemma 5.6})\\
\leq\ & w_A^2(T_2-T_1)+\frac{\max\{1,|1-\alpha|\}(\|T_2x\|_A^2+\|T_1^\sharp x\|_A^2)+2w_A(T_1T_2)}{|\alpha|}\\
\leq\ & w_A^2(T_2-T_1)+\frac{\max\{1,|1-\alpha|\}\|T_2^\sharp T_2+T_1T_1^\sharp\|_A+2w_A(T_1T_2)}{|\alpha|}.
\end{aligned}$$

Taking supremum over all $x\in\mathcal{H}$ with $\|x\|_A=1$, we get

$$w_{A,e}^2(T_1,T_2)\leq w_A^2(T_1-T_2)+\frac{\max\{1,|1-\alpha|\}\|T_2^\sharp T_2+T_1T_1^\sharp\|_A+2w_A(T_1T_2)}{|\alpha|}. \tag{5.27}$$

Replacing T_2 by $-T_2$, we obtain that

$$w_{A,e}^2(T_1,T_2)\leq w_A^2(T_1+T_2)+\frac{\max\{1,|1-\alpha|\}\|T_2^\sharp T_2+T_1T_1^\sharp\|_A+2w_A(T_1T_2)}{|\alpha|}. \tag{5.28}$$

Following the inequality (5.28) together with (5.27), we get the desired inequality. □

In particular, considering $\alpha=2$ in Theorem 5.19, we get

$$w_{A,e}^2(T_1,T_2)\leq \min\{w_A^2(T_1-T_2),w_A^2(T_1+T_2)\}+\frac{\|T_2^\sharp T_2+T_1T_1^\sharp\|_A+2w_A(T_1T_2)}{2}. \tag{5.29}$$

Finally, we obtain the following upper and lower bounds for A-Euclidean operator radius involving A-numerical radius.

Theorem 5.20. *Let $T_1,T_2\in\mathbb{B}(\mathcal{H})$, then*

$$w_A^2(\sqrt{\alpha}T_1\pm\sqrt{1-\alpha}T_2)\leq w_{A,e}^2(T_1,T_2)\leq w_A^2(\sqrt{\alpha}T_1+\sqrt{1-\alpha}T_2)+w_A^2(\sqrt{1-\alpha}T_1+\sqrt{\alpha}T_2),$$

for all $\alpha\in[0,1]$.

Proof. Let $x\in\mathcal{H}$ with $\|x\|_A=1$. Then we have,

$$\begin{aligned}
&\sqrt{\alpha}|\langle T_1x,x\rangle_A|+\sqrt{1-\alpha}|\langle T_2x,x\rangle_A|\\
&\leq(|\langle T_1x,x\rangle_A|^2+|\langle T_2x,x\rangle_A|^2)^{\frac12}((\sqrt{\alpha})^2+(\sqrt{1-\alpha})^2)^{\frac12}\\
&=(|\langle T_1x,x\rangle_A|^2+|\langle T_2x,x\rangle_A|^2)^{\frac12}.
\end{aligned}$$

Therefore,

$$\begin{aligned}(|\langle T_1x,x\rangle_A|^2+|\langle T_2x,x\rangle_A|^2)^{\frac{1}{2}} &\geq |\langle\sqrt{\alpha}T_1x,x\rangle_A|+|\langle\sqrt{1-\alpha}T_2x,x\rangle_A|\\ &\geq |\langle\sqrt{\alpha}T_1x,x\rangle_A\pm\langle\sqrt{1-\alpha}T_2x,x\rangle_A|\\ &= |\langle\left(\sqrt{\alpha}T_1\pm\sqrt{1-\alpha}T_2\right)x,x\rangle_A|.\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\|_A=1$, we get the first inequality, i.e.,

$$w_{A,e}(T_1,T_2)\geq w_A(\sqrt{\alpha}T_1\pm\sqrt{1-\alpha}T_2).$$

Next, we prove the second inequality. By simple calculation, we get

$$\begin{aligned}&|\langle T_1x,x\rangle_A|^2+|\langle T_2x,x\rangle_A|^2\\ =\ &|\langle\sqrt{\alpha}T_1x,x\rangle_A+\langle\sqrt{1-\alpha}T_2x,x\rangle_A|^2+|\langle\sqrt{1-\alpha}T_1x,x\rangle_A-\langle\sqrt{\alpha}T_2x,x\rangle_A|^2\\ =\ &|\langle(\sqrt{\alpha}T_1+\sqrt{1-\alpha}T_2)x,x\rangle_A|^2+|\langle(\sqrt{1-\alpha}T_1-\sqrt{\alpha}T_2)x,x\rangle_A|^2\\ \leq\ &w_A^2(\sqrt{\alpha}T_1+\sqrt{1-\alpha}T_2)+w_A^2(\sqrt{1-\alpha}T_1-\sqrt{\alpha}T_2).\end{aligned}$$

Taking supremum over all x in $\mathcal{H}$ with $\|x\|_A=1$, we get

$$w_{A,e}^2(T_1,T_2)\leq w_A^2(\sqrt{\alpha}T_1+\sqrt{1-\alpha}T_2)+w_A^2(\sqrt{1-\alpha}T_1-\sqrt{\alpha}T_2),$$

as desired. □

Remark 5.21. *(i) It is easy to verify that*

$$\begin{aligned}w_{A,e}^2(T_1,T_2) &\geq \max_{0\leq\alpha\leq 1} w_A^2(\sqrt{\alpha}T_1\pm\sqrt{1-\alpha}T_2)\\ &\geq \frac{1}{2}\max w_A^2(T_1\pm T_2)\\ &\geq \frac{1}{2}w_A(T_1^2+T_2^2).\end{aligned}$$

(ii) Putting $T_1=\Re_A(T)$ and $T_2=\Im_A(T)$ in (i) we obtain that

$$\begin{aligned}w_A^2(T) &\geq \frac{1}{2}\max\|\Re_A(T)\pm\Im_A(T)\|_A^2\\ &\geq \frac{1}{4}\|T^\sharp T+TT^\sharp\|_A.\end{aligned}$$

As an application to the inequalities obtain here we develop A-numerical radius inequalities of bounded linear operator T.

5.6 Application to A-numerical radius inequalities

Considering $T_1 = [\Re_A(T)]^\sharp$ and $T_2 = [\Im_A(T)]^\sharp$ in Theorem 5.16, and the using the Lemma 5.4. we obtain the following new upper and lower bounds for the A-numerical radius of a bounded linear operator $T \in \mathbb{B}_A(\mathcal{H})$.

Corollary 5.8. *If $T \in \mathbb{B}_A(\mathcal{H})$, then*

$$\frac{1}{4}\|T^\sharp T + TT^\sharp\|_A + \frac{\alpha}{2}\max\{\|\Re_A(T)\|_A, \|\Im_A(T)\|_A\} \leq w_A^2(T) \leq \frac{1}{2}\|TT^\sharp + T^\sharp T\|_A,$$

where $\alpha = \big|\|\Re_A(T) + \Im_A(T)\|_A - \|\Re_A(T) - \Im_A(T)\|_A\big|$.

Again, considering $T_1 = T$ and $T_2 = T^\sharp$ in Theorem 5.16, we get the following new lower bound for the A-numerical radius of $T \in \mathbb{B}_A(\mathcal{H})$.

Corollary 5.9. *Let $T \in \mathbb{B}_A(\mathcal{H})$, then*

$$\frac{1}{2}\|\Re_A(T^2)\|_A + \frac{1}{2}w_A(T)\big|\|\Re_A(T)\|_A - \|\Im_A(T)\|_A\big| \leq w_A^2(T).$$

In particular, considering $T_1 = T_2 = T$ in Theorem 5.18, we obtain the following corollary.

Corollary 5.10. *If $T \in \mathbb{B}_A(\mathcal{H})$, then*

$$w_A^2(T) \leq \frac{\max\{1, |1-\alpha|\}\|T\|_A^2 + w_A(T^2)}{|\alpha|},$$

for any non-zero scalar α.

For $\alpha = 2$,

$$w_A^2(T) \leq \frac{1}{2}\left(\|T\|_A^2 + w_A(T^2)\right),$$

which was also obtained in [44, Cor. 2.5].

Again, considering $T_1 = T_2 = T$ in Theorem 5.19, we get the following upper bound for the A-numerical radius of $T \in \mathbb{B}_A(\mathcal{H})$:

$$w_A^2(T) \leq \frac{\frac{1}{2}\max\{1, |1-\alpha|\}\|T^\sharp T + TT^\sharp\|_A + w_A(T^2)}{|\alpha|}. \tag{5.30}$$

Putting $\alpha = 2$ in (5.30), we get

$$w_A^2(T) \leq \frac{1}{4}\|T^\sharp T + TT^\sharp\|_A + \frac{1}{2}w_A(T^2),$$

which was also obtained in [88, Th. 2.11].

CHAPTER 6

EUCLIDEAN OPERATOR RADIUS INEQUALITIES OF D-TUPLE OPERATORS

6.1 Introduction

In this chapter, we study Euclidean operator radius inequalities of d-tuple operators, as well as the sum and the product of d-tuple operators. A power inequality for the Euclidean operator radius of d-tuple operators is also studied. Further, we study the Euclidean operator radius inequalities of 2×2 operator matrices whose entries are d-tuple operators.
The numerical radius has various applications in sciences, more precisely, perturbation problem, convergence problem, approximation problem and iterative method as well as recently developed quantum information system. Because of the importance of the numerical radius, various generalizations of it has been studied over the years. The Euclidean operator radius of d-tuple operators is one such generalization and it helps to study various problems in multivariable operator theory.

Content of this chapter is based on the following paper:
S. Jana, P. Bhunia and K. Paul, Euclidean operator radius inequalities of d-tuple operators and operator matrices. arXiv:2304.08033v1

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\| \cdot \|$ be the norm induced by the inner product. Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ and let $T \in \mathbb{B}(\mathcal{H})$. The numerical range of T is given by $W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$. The numerical radius of T, denoted by $w(T)$, is defined as $w(T) = \sup\ \{|\lambda| : \lambda \in W(T)\}$. It is well known that the numerical radius $w(\cdot)$ defines a norm on $\mathbb{B}(\mathcal{H})$ and it satisfies $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$. Now, let $\mathbb{B}^d(\mathcal{H}) = \mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H}) \times \ldots \times \mathbb{B}(\mathcal{H})$ (d times) and let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ be a d-tuple operator. The joint numerical range of $\mathbf{T}$ is defined as $JtW(\mathbf{T}) = \{(\langle T_1x, x\rangle, \langle T_2x, x\rangle, \ldots, \langle T_dx, x\rangle) : x \in \mathcal{H}, \|x\| = 1\}$. Following [78], the Euclidean operator radius, the Euclidean operator norm of $\mathbf{T}$ are defined respectively as follows:

$$w_e(\mathbf{T}) = \sup\left\{ \left(\sum_{k=1}^{d} |\langle T_kx, x\rangle|^2 \right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1 \right\},$$

$$\|\mathbf{T}\| = \sup\left\{ \left(\sum_{k=1}^{d} \|T_kx\|^2 \right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1 \right\}.$$

As pointed out in [78], the Euclidean operator radius $w_e(\cdot)$ is a norm on $\mathbb{B}^d(\mathcal{H})$ and it satisfies the following lower and upper bounds:

$$\frac{1}{2\sqrt{d}} \left\| \sum_{k=1}^{d} T_k^*T_k \right\|^{\frac{1}{2}} \leq w_e(\mathbf{T}) \leq \left\| \sum_{k=1}^{d} T_k^*T_k \right\|^{\frac{1}{2}}. \tag{6.1}$$

Here the constant $\frac{1}{2\sqrt{d}}$ and 1 are best possible.

Definition 6.1. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ be a d-tuple operator. Then $\mathbf{T}$ is said to be commuting if $T_iT_j = T_jT_i$ for all $i, j = 1, 2, \ldots, d$.*

Definition 6.2. *[36] Let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ be a d-tuple operator. Then $\mathbf{T}$ is said to be joint normal (or simply normal) if $\mathbf{T}$ is commuting and each T_i is normal.*

For d-tuple operators $\mathbf{S} = (S_1, S_2, \ldots, S_d)$, $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$, we write $\mathbf{ST} = (S_1T_1, S_2T_2, \ldots, S_dT_d)$, $\mathbf{S}+\mathbf{T} = (S_1+T_1, S_2+T_2, \ldots, S_d+T_d)$ and $\alpha\mathbf{T} = (\alpha T_1, \alpha T_2, \ldots, \alpha T_d)$ for any scalar $\alpha \in \mathbb{C}$. Also, for $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$, $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_d)$, $\mathbf{W} = (W_1, W_2, \ldots, W_d) \in \mathbb{B}^d(\mathcal{H})$, the 2×2 operator matrix, whose entries are d-tuple operators $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}$, is

$$\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix} = \left(\begin{bmatrix} X_1 & Y_1 \\ Z_1 & W_1 \end{bmatrix}, \begin{bmatrix} X_2 & Y_2 \\ Z_2 & W_2 \end{bmatrix}, \ldots, \begin{bmatrix} X_d & Y_d \\ Z_d & W_d \end{bmatrix} \right) \in \mathbb{B}^d(\mathcal{H} \oplus \mathcal{H}).$$

The inner product of the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ is $\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$ for all (x_1, x_2) and $(y_1, y_2) \in \mathcal{H} \oplus \mathcal{H}$.

6.2 Euclidean operator radius of d-tuple operators

We begin this section with the following proposition, proof of which follows from the definition of the Euclidean operator norm.

Proposition 6.1. *[45] If* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$, *then*

$$\|\mathbf{T}\| = \sqrt{\|T_1^*T_1 + T_2^*T_2 + \ldots + T_d^*T_d\|}.$$

Combining Proposition 6.1 and (6.1), we obtain

$$\frac{1}{2\sqrt{d}}\|\mathbf{T}\| \leq w_e(\mathbf{T}) \leq \|\mathbf{T}\|. \tag{6.2}$$

Now, we prove the following inequality involving the Euclidean operator radius.

Theorem 6.1. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$. *Then*

$$\sum_{k=1}^{d} \|T_k x\|^2 + \sum_{k=1}^{d} |\langle T_k^2 x, x\rangle| \leq 2\sqrt{d}\, w_e(\mathbf{T}) \left(\sum_{k=1}^{d} \|T_k x\|^2\right)^{\frac{1}{2}} \|x\|,$$

for all $x \in \mathcal{H}$.

Proof. Let λ_k and θ_k $(k = 1, 2, \ldots, d)$ be real numbers with $\lambda_k \neq 0$. Then, we have

$$\begin{aligned}
&\sum_{k=1}^{d} \|T_k x\|^2 + \sum_{k=1}^{d} e^{2i\theta_k} \langle T_k^2 x, x\rangle \\
&= \sum_{k=1}^{d} \Big\{ \frac{1}{2} \langle \lambda_k e^{2i\theta_k} T_k^2 x + \lambda_k^{-1} e^{i\theta_k} T_k x, \lambda_k e^{i\theta_k} T_k x + \lambda_k^{-1} x\rangle \\
&\quad - \frac{1}{2} \langle \lambda_k e^{2i\theta_k} T_k^2 x - \lambda_k^{-1} e^{i\theta_k} T_k x, \lambda_k e^{i\theta_k} T_k x - \lambda_k^{-1} x\rangle \Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big|\sum_{k=1}^{d}\|T_kx\|^2+\sum_{k=1}^{d}e^{2i\theta_k}\langle T_k^2x,x\rangle\Big|\\
&\le\sum_{k=1}^{d}\frac{1}{2}\big|\langle\lambda_k e^{2i\theta_k}T_k^2x+\lambda_k^{-1}e^{i\theta_k}T_kx,\lambda_k e^{i\theta_k}T_kx+\lambda_k^{-1}x\rangle\big|\\
&\quad+\sum_{k=1}^{d}\frac{1}{2}\big|\langle\lambda_k e^{2i\theta_k}T_k^2x-\lambda_k^{-1}e^{i\theta_k}T_kx,\lambda_k e^{i\theta_k}T_kx-\lambda_k^{-1}x\rangle\big|\\
&\le\sum_{k=1}^{d}\frac{1}{2}w(T_k)\|\lambda_k e^{i\theta_k}T_kx+\lambda_k^{-1}x\|^2+\sum_{k=1}^{d}\frac{1}{2}w(T_k)\|\lambda_k e^{i\theta_k}T_kx-\lambda_k^{-1}x\|^2.
\end{aligned}$$

Since $|\langle T_kx,x\rangle|\le\left(\sum_{k=1}^{d}|\langle T_kx,x\rangle|^2\right)^{\frac{1}{2}}$ for all $x\in\mathcal{H}$, $w(T_k)\le w_e(\mathbf{T})$. Thus,

$$\begin{aligned}
&\Big|\sum_{k=1}^{d}\|T_kx\|^2+\sum_{k=1}^{d}e^{2i\theta_k}\langle T_k^2x,x\rangle\Big|\\
&\le\sum_{k=1}^{d}\frac{1}{2}w_e(\mathbf{T})\|\lambda_k e^{i\theta_k}T_kx+\lambda_k^{-1}x\|^2+\sum_{k=1}^{d}\frac{1}{2}w_e(\mathbf{T})\|\lambda_k e^{i\theta_k}T_kx-\lambda_k^{-1}x\|^2\\
&=w_e(\mathbf{T})\sum_{k=1}^{d}\left\{\frac{1}{2}\|\lambda_k e^{i\theta_k}T_kx+\lambda_k^{-1}x\|^2+\frac{1}{2}\|\lambda_k e^{i\theta_k}T_kx-\lambda_k^{-1}x\|^2\right\}\\
&=w_e(\mathbf{T})\sum_{k=1}^{d}\left\{\lambda_k^2\|T_kx\|^2+\lambda_k^{-2}\|x\|^2\right\}.
\end{aligned}$$

Suppose $T_kx\neq 0$ for all $k=1,2,\ldots,d$ and we choose θ_k in such a way that $e^{2i\theta_k}\langle T_k^2x,x\rangle=|\langle T_k^2x,x\rangle|$ and $\lambda_k=\sqrt{\frac{\|x\|}{\|T_kx\|}}$ for all $k=1,2,\ldots,d$. Then, we have

$$\sum_{k=1}^{d}\|T_kx\|^2+\sum_{k=1}^{d}|\langle T_k^2x,x\rangle| \quad\le\quad 2w_e(\mathbf{T})\sum_{k=1}^{d}\|T_kx\|\|x\|.$$

Therefore, the Cauchy-Schwarz inequality implies that

$$\sum_{k=1}^{d}\|T_kx\|^2+\sum_{k=1}^{d}|\langle T_k^2x,x\rangle| \quad\le\quad 2\sqrt{d}w_e(\mathbf{T})\left(\sum_{k=1}^{d}\|T_kx\|^2\right)^{\frac{1}{2}}\|x\|.$$

Also, this inequality holds when $\|T_kx\|=0$ for all or some $k\in\{1,2,\ldots,d\}$. This completes the proof. □

Applying Theorem 6.1, we obtain a refinement of the first inequality in (6.2).

Corollary 6.1. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ *and* $\|\mathbf{T}\| \neq 0$. *Then*

$$\frac{1}{2\sqrt{d}}\left(\|\mathbf{T}\| + \frac{c_e(\mathbf{T^2})}{\|\mathbf{T}\|}\right) \leq w_e(\mathbf{T}),$$

where $c_e(\mathbf{T}) = \inf\left\{\left(\sum_{k=1}^d |\langle T_k x, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1\right\}$.

Proof. From Theorem 6.1 and together with $\sum_{k=1}^d |\langle T_k^2 x, x\rangle|^2 \leq \left(\sum_{k=1}^d |\langle T_k^2 x, x\rangle|\right)^2$, we have

$$\sum_{k=1}^d \|T_k x\|^2 + \left(\sum_{k=1}^d |\langle T_k^2 x, x\rangle|^2\right)^{\frac{1}{2}} \leq 2\sqrt{d} w_e(\mathbf{T}) \left(\sum_{k=1}^d \|T_k x\|^2\right)^{\frac{1}{2}} \|x\|.$$

Taking $\|x\| = 1$,

$$\begin{aligned}\sum_{k=1}^d \|T_k x\|^2 + \left(\sum_{k=1}^d |\langle T_k^2 x, x\rangle|^2\right)^{\frac{1}{2}} &\leq 2\sqrt{d} w_e(\mathbf{T}) \left(\sum_{k=1}^d \|T_k x\|^2\right)^{\frac{1}{2}} \\ &\leq 2\sqrt{d} w_e(\mathbf{T})\|\mathbf{T}\|.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k=1}^d \|T_k x\|^2 &\leq 2\sqrt{d} w_e(\mathbf{T})\|\mathbf{T}\| - \left(\sum_{k=1}^d |\langle T_k^2 x, x\rangle|^2\right)^{\frac{1}{2}} \\ &\leq 2\sqrt{d} w_e(\mathbf{T})\|\mathbf{T}\| - c_e(\mathbf{T^2}).\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$ with $\|x\| = 1$, we get

$$\|\mathbf{T}\|^2 + c_e(\mathbf{T^2}) \leq 2\sqrt{d} w_e(\mathbf{T})\|\mathbf{T}\|,$$

as desired. □

Considering the following example we show that the inequality in Corollary 6.1 is a proper improvement of the first inequality in (6.2).

Consider $\mathbf{T} = (T_1, T_2) \in \mathbb{B}^2(\mathcal{H})$ is a 2-tuple operator, where $T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $T_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then the inequality (6.2) gives $\frac{1}{2\sqrt{2}} \leq w_e(\mathbf{T})$, whereas Corollary 6.1 gives $\frac{1}{2\sqrt{2}}\left(1 + \frac{1}{\sqrt{2}}\right) \leq w_e(\mathbf{T})$.

Next result for d-tuple normal operators.

Theorem 6.2. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ *be a d-tuple normal operator. Then*

$$\|\mathbf{T^2}\| = \|(T_1^*T_1, T_2^*T_2, \ldots, T_d^*T_d)\| \leq \|\mathbf{T}\|^2 = \|\mathbf{T}^*\|^2 \leq \sqrt{d}\|\mathbf{T^2}\|.$$

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
\|\mathbf{T^2}\| &= \|(T_1^2, T_2^2, \ldots, T_d^2)\| = \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|T_k^2 x\|^2 \right)^{\frac{1}{2}} \\
&= \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \langle T_k^2 x, T_k^2 x \rangle \right)^{\frac{1}{2}} = \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \langle T_k T_k x, T_k T_k x \rangle \right)^{\frac{1}{2}} \\
&= \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \langle T_k^* T_k x, T_k^* T_k x \rangle \right)^{\frac{1}{2}} \quad \textit{(since each } T_k \textit{ is normal)} \\
&= \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|T_k^* T_k x\|^2 \right)^{\frac{1}{2}} = \|(T_1^*T_1, T_2^*T_2, \ldots, T_d^*T_d)\|.
\end{aligned}$$

Now,

$$\begin{aligned}
\|(T_1^*T_1, T_2^*T_2, \ldots, T_d^*T_d)\| &= \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|T_k^* T_k x\|^2 \right)^{\frac{1}{2}} \\
&\leq \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|T_k^*\|^2 \|T_k x\|^2 \right)^{\frac{1}{2}} \\
&\leq \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|\mathbf{T}\|^2 \|T_k x\|^2 \right)^{\frac{1}{2}} \\
&\qquad \left(\textit{since } \|T_k x\| \leq \left(\sum_{k=1}^{d} \|T_k x\|^2 \right)^{\frac{1}{2}}, \ \|T_k\| \leq \|\mathbf{T}\| \textit{ for each } k\right) \\
&= \|\mathbf{T}\| \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|T_k x\|^2 \right)^{\frac{1}{2}} = \|\mathbf{T}\|^2.
\end{aligned}$$

Also, we have

$$\|\mathbf{T}\| = \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|T_k x\|^2 \right)^{1/2} = \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|T_k^* x\|^2 \right)^{1/2} = \|\mathbf{T}^*\|.$$

Again,

$$\begin{aligned}
\|\mathbf{T}\|^2 &= \sup_{\|x\|=1}\left(\sum_{k=1}^{d}\|T_kx\|^2\right) = \sup_{\|x\|=1}\left(\sum_{k=1}^{d}\langle T_kx, T_kx\rangle\right) \\
&= \sup_{\|x\|=1}\left(\sum_{k=1}^{d}\langle T_k^*T_kx, x\rangle\right) \leq \sup_{\|x\|=1}\left(\sum_{k=1}^{d}\|T_k^*T_kx\|\|x\|\right) \\
&\leq \sqrt{d}\sup_{\|x\|=1}\left(\sum_{k=1}^{d}\|T_k^*T_kx\|^2\right)^{\frac{1}{2}} \quad (\textit{by Cauchy-Schwarz inequality}) \\
&= \sqrt{d}\sup_{\|x\|=1}\left(\sum_{k=1}^{d}\|T_k^2x\|^2\right)^{\frac{1}{2}} \quad (\textit{since each } T_k \textit{ is normal}) \\
&= \sqrt{d}\|\mathbf{T^2}\|.
\end{aligned}$$

This completes the proof of the theorem. □

Remark 6.3. *If we take T_k $(k = 1, 2, \ldots, d)$ is a $d \times d$ matrix whose only (k, k) diagonal entries is 1 and others are zero, then the first inequality in Theorem 6.2 becomes equality. Also if we take $T_k = \sqrt{d}I$ (I is the $d \times d$ identity matrix), then the second inequality in Theorem 6.2 becomes equality. Thus, we would like to remark that the inequalities in Theorem 6.2 are sharp.*

Next we develop a power inequality for the Euclidean operator radius.

Theorem 6.4. *If $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$, then*

$$w_e(\mathbf{T^n}) \leq \sqrt{d}w_e^n(\mathbf{T}).$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. The inequality $|\langle T_kx, x\rangle| \leq \left(\sum_{k=1}^{d}|\langle T_kx, x\rangle|^2\right)^{\frac{1}{2}}$ implies $w(T_k) \leq w_e(\mathbf{T})$ for each $k = 1, 2, \ldots, d$. Thus, if $w_e(\mathbf{T}) \leq 1$, then $w(T_k) \leq 1$ for each $k = 1, 2, \ldots, d$. The power inequality [77] implies that $w(T_k^n) \leq 1$ for each $k = 1, 2, \ldots, d$, whenever $w(T_k) \leq 1$. Therefore, if $w(T_k) \leq 1$, then

$$\begin{aligned}
w_e(\mathbf{T^n}) &= \sup_{\|x\|=1}\left(\sum_{k=1}^{d}|\langle T_k^nx, x\rangle|^2\right)^{\frac{1}{2}} \leq \left(\sum_{k=1}^{d}\sup_{\|x\|=1}|\langle T_k^nx, x\rangle|^2\right)^{\frac{1}{2}} \leq \left(\sum_{k=1}^{d}w^2(T_k^n)\right)^{\frac{1}{2}} \\
&\leq \sqrt{d}.
\end{aligned}$$

Now, if we take $T_k^{'} = \frac{T_k}{w_e(\mathbf{T})}$ for all $k = 1, 2, \ldots, d$, then $w_e(\mathbf{T}^{'}) = 1$, where $\mathbf{T}^{'} = (T_1^{'}, T_2^{'}, \ldots, T_d^{'})$. So, $w(T_k^{'}) \leq 1$. Thus, $w_e((\mathbf{T}^{'})^{\mathbf{n}}) \leq \sqrt{d}$, which gives $w_e(\mathbf{T^n}) \leq \sqrt{d}w_e^n(\mathbf{T})$. □

Applying Theorem 6.4, we get the following result.

Corollary 6.2. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$*. If* $w_e(\mathbf{T}) \leq 1$*, then*

$$\|\mathbf{T^n}\| \leq 2d.$$

Proof. From (6.2) and Theorem 6.4, we get

$$\frac{\|\mathbf{T^n}\|}{2\sqrt{d}} \leq w_e(\mathbf{T^n}) \leq \sqrt{d} w_e^n(\mathbf{T}) \leq \sqrt{d}.$$

□

In the following theorem we obtain the Euclidean operator radius bounds for the product of d-tuple operators. For this, we need the following lemma in which we study the Euclidean operator norm is submultiplicative and the Euclidean operator radius is subadditive.

Lemma 6.1. *Let* $\mathbf{S} = (S_1, S_2, \ldots, S_d)$, $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$*. Then*
(a) $\|\mathbf{ST}\| \leq \|\mathbf{S}\|\|\mathbf{T}\|$.
(b) $w_e(\mathbf{S}+\mathbf{T}) \leq w_e(\mathbf{S}) + w_e(\mathbf{T})$.

Proof. Take $x \in \mathcal{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
\|\mathbf{ST}\| &= \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|S_k T_k x\|^2 \right)^{\frac{1}{2}} \leq \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|S_k\|^2 \|T_k x\|^2 \right)^{\frac{1}{2}} \\
&\leq \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|\mathbf{S}\|^2 \|T_k x\|^2 \right)^{\frac{1}{2}} \left(\textit{since } \|S_k x\| \leq \left(\sum_{k=1}^{d} \|S_k x\|^2 \right)^{\frac{1}{2}}, \|S_k\| \leq \|\mathbf{S}\| \right) \\
&= \|\mathbf{S}\| \sup_{\|x\|=1} \left(\sum_{k=1}^{d} \|T_k x\|^2 \right)^{\frac{1}{2}} = \|\mathbf{S}\|\|\mathbf{T}\|.
\end{aligned}$$

The subadditive property of the Euclidean operator radius is known, it follows from the norm of $w_e(\cdot)$.

□

Theorem 6.5. *If* $\mathbf{S} = (S_1, S_2, \ldots, S_d)$, $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$*, then*

$$w_e(\mathbf{ST}) \leq 4d w_e(\mathbf{S}) w_e(\mathbf{T}).$$

Proof. From Lemma 6.1 (a) and (6.2), we have

$$w_e(\mathbf{ST}) \leq \|\mathbf{ST}\| \leq \|\mathbf{S}\|\|\mathbf{T}\| \leq 4d w_e(\mathbf{S}) w_e(\mathbf{T}).$$

□

Further, we develop a bound of $w_e(\mathbf{ST})$ when $\mathbf{ST} = \mathbf{TS}$.

Theorem 6.6. *Let* $\mathbf{S} = (S_1, S_2, \ldots, S_d)$, $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$. *If* $\mathbf{ST} = \mathbf{TS}$ *(i.e, if* $S_kT_k = T_kS_k$ *for all* $k = 1, 2, \ldots, d$*), then*

$$w_e(\mathbf{ST}) \leq 2\sqrt{d}w_e(\mathbf{S})w_e(\mathbf{T}).$$

Proof. Suppose $w_e(\mathbf{S}) = w_e(\mathbf{T}) = 1$. Then, we have

$$\begin{aligned}
w_e(\mathbf{ST}) &= w_e\left(\frac{1}{4}(\mathbf{S}+\mathbf{T})^2 - \frac{1}{4}(\mathbf{S}-\mathbf{T})^2\right) \\
&\leq \frac{1}{4}w_e\left((\mathbf{S}+\mathbf{T})^2\right) + \frac{1}{4}w_e\left((\mathbf{S}-\mathbf{T})^2\right) \\
&\quad (\textit{using Lemma 6.1 (b) and the fact } w_e(c\mathbf{T}) = |c|w_e(\mathbf{T})) \\
&\leq \frac{\sqrt{d}}{4}w_e^2(\mathbf{S}+\mathbf{T}) + \frac{\sqrt{d}}{4}w_e^2(\mathbf{S}-\mathbf{T}) \ (\textit{using Theorem 6.4}) \\
&\leq \frac{\sqrt{d}}{4}\left(w_e(\mathbf{S}) + w_e(\mathbf{T})\right)^2 + \frac{\sqrt{d}}{4}\left(w_e(\mathbf{S}) + w_e(\mathbf{T})\right)^2 \ (\textit{using Lemma 6.1 (b)}) \\
&= 2\sqrt{d}.
\end{aligned}$$

This completes the proof of theorem. □

Next we develop a bound for $w_e(\mathbf{ST})$ when $\mathbf{S}, \mathbf{T}$ are joint normal operators.

Theorem 6.7. *Let* $\mathbf{S} = (S_1, S_2, \ldots, S_d)$, $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$. *If* $\mathbf{S}$, $\mathbf{T}$ *are joint normal, then*

$$w_e(\mathbf{ST}) \leq w_e(\mathbf{S})w_e(\mathbf{T}).$$

Proof. We have $w_e(\mathbf{ST}) \leq \|\mathbf{ST}\| \leq \|\mathbf{S}\|\|\mathbf{T}\| = w_e(\mathbf{S})w_e(\mathbf{T})$, where the last equality follows from $\|\mathbf{T}\| = w_e(\mathbf{T})$ and $\|\mathbf{S}\| = w_e(\mathbf{S})$ (see [35]). □

6.3 Euclidean operator radius of 2×2 operator matrices

We begin this section with proving the following lemma.

Lemma 6.2. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_d), \mathbf{Y} = (Y_1, Y_2, \ldots, Y_d) \in \mathbb{B}^d(\mathcal{H})$. *Then the following results hold:*

(a) $w_e\left(\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right) = \max\{w_e(\mathbf{X}), w_e(\mathbf{Y})\}.$

(b) $\left\|\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right\| = \max\{\|\mathbf{X}\|, \|\mathbf{Y}\|\}.$

(c) $w_e\left(\begin{bmatrix}0 & \mathbf{X}\\ \mathbf{Y} & 0\end{bmatrix}\right) = w_e\left(\begin{bmatrix}0 & \mathbf{Y}\\ \mathbf{X} & 0\end{bmatrix}\right).$

(d) $w_e\left(\begin{bmatrix}0 & \mathbf{X}\\ \mathbf{Y} & 0\end{bmatrix}\right) = w_e\left(\begin{bmatrix}0 & \mathbf{X}\\ \mathbf{e}^{\mathbf{i}\theta}\mathbf{Y} & 0\end{bmatrix}\right)$ *for all* $\theta \in \mathbb{R}$.

(e) $w_e\left(\begin{bmatrix}\mathbf{X} & \mathbf{Y}\\ \mathbf{Y} & \mathbf{X}\end{bmatrix}\right) = \max\{w_e(\mathbf{X}-\mathbf{Y}), w_e(\mathbf{X}+\mathbf{Y})\}.$

In particular $w_e\left(\begin{bmatrix}0 & \mathbf{Y}\\ \mathbf{Y} & 0\end{bmatrix}\right) = w_e(\mathbf{Y}).$

Proof. (a) Let $u = (x, y) \in \mathcal{H} \oplus \mathcal{H}$ with $\|u\| = 1$, i.e., $\|x\|^2 + \|y\|^2 = 1$. Then,

$$
\begin{aligned}
&\left(\sum_{k=1}^{d} |\langle \begin{bmatrix}X_k & 0\\ 0 & Y_k\end{bmatrix} u, u\rangle|^2\right)^{\frac{1}{2}}\\
= \quad & \left(\sum_{k=1}^{d} |\langle X_k x, x\rangle + \langle Y_k y, y\rangle|^2\right)^{\frac{1}{2}}\\
\leq \quad & \left(\sum_{k=1}^{d} |\langle X_k x, x\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{d} |\langle Y_k y, y\rangle|^2\right)^{\frac{1}{2}} \quad \textit{(using Minkowski inequality)}\\
\leq \quad & w_e(\mathbf{X})\|x\|^2 + w_e(\mathbf{Y})\|y\|^2\\
\leq \quad & \max\{w_e(\mathbf{X}), w_e(\mathbf{Y})\}\left(\|x\|^2 + \|y\|^2\right) = \max\{w_e(\mathbf{X}), w_e(\mathbf{Y})\}.
\end{aligned}
$$

Taking supremum over $\|u\| = 1$, we get

$$
w_e\left(\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right) \leq \max\{w_e(\mathbf{X}), w_e(\mathbf{Y})\}.
$$

Suppose $u = (x, 0) \in \mathcal{H} \oplus \mathcal{H}$ where $\|x\| = 1$, then

$$
\left(\sum_{k=1}^{d} |\langle \begin{bmatrix}X_k & 0\\ 0 & Y_k\end{bmatrix} u, u\rangle|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{d} |\langle X_k x, x\rangle|^2\right)^{\frac{1}{2}}.
$$

Taking supremum over $\|x\|=1$, we get

$$\sup_{\|x\|=1}\left(\sum_{k=1}^{d}|\langle\begin{bmatrix}X_k & 0\\ 0 & Y_k\end{bmatrix}u,u\rangle|^2\right)^{\frac{1}{2}}=w_e(\mathbf{X}).$$

This implies that $w_e\left(\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right)\geq w_e(\mathbf{X})$. Similarly, $w_e\left(\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right)\geq w_e(\mathbf{Y})$. Hence, $w_e\left(\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right)\geq \max\left\{w_e(\mathbf{X}),w_e(\mathbf{Y})\right\}$. This completes the proof (a).

(b) Let $u=(x,y)\in\mathcal{H}\oplus\mathcal{H}$ with $\|u\|=1$, i.e., $\|x\|^2+\|y\|^2=1$. Then, we have

$$\begin{aligned}\sum_{k=1}^{d}\left\|\begin{bmatrix}X_k & 0\\ 0 & Y_k\end{bmatrix}u\right\|^2 &= \sum_{k=1}^{d}\|(X_kx,Y_ky)\|^2\\ &= \sum_{k=1}^{d}\|X_kx\|^2+\|Y_ky\|^2\\ &\leq \|\mathbf{X}\|^2\|x\|^2+\|\mathbf{Y}\|^2\|y\|^2\\ &\leq \max\left\{\|\mathbf{X}\|^2,\|\mathbf{Y}\|^2\right\}\left(\|x\|^2+\|y\|^2\right)=\max\left\{\|\mathbf{X}\|^2,\|\mathbf{Y}\|^2\right\}.\end{aligned}$$

Taking supremum over $\|u\|=1$, we get

$$\left\|\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right\|\leq\max\left\{\|\mathbf{X}\|,\|\mathbf{Y}\|\right\}.$$

Now, let $u=(x,0)\in\mathcal{H}\oplus\mathcal{H}$ with $\|x\|=1$, then

$$\sum_{k=1}^{d}\left\|\begin{bmatrix}X_k & 0\\ 0 & Y_k\end{bmatrix}u\right\|^2=\sum_{k=1}^{d}\|X_kx\|^2.$$

Taking supremum over $\|x\|=1$, we get that $\sup\limits_{\|x\|=1}\sum_{k=1}^{d}\left\|\begin{bmatrix}X_k & 0\\ 0 & Y_k\end{bmatrix}u\right\|^2=\|\mathbf{X}\|^2$. This implies that $\left\|\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right\|\geq\|\mathbf{X}\|$. Similarly, $\left\|\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right\|\geq\|\mathbf{Y}\|$. Therefore, $\left\|\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{Y}\end{bmatrix}\right\|\geq\max\left\{\|\mathbf{X}\|,\|\mathbf{Y}\|\right\}$. This completes the proof (b).

(c) It is easy to verify (see also [78, Section 2]) that

$$w_e(T_1,T_2,\ldots,T_d)=w_e(U^*T_1U,U^*T_2U,\ldots,U^*T_dU)\tag{6.3}$$

for every unitary operator U. The proof (c) follows from (6.3) by taking $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

(d) The proof (d) follows from (6.3) by taking $U = \begin{bmatrix} I & 0 \\ 0 & e^{\frac{i\theta}{2}} I \end{bmatrix}$.

(e) Let $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix}$ and $T_k = \begin{bmatrix} X_k & Y_k \\ Y_k & X_k \end{bmatrix}$. Then $U^* T_k U = \begin{bmatrix} X_k - Y_k & 0 \\ 0 & Y_k + X_k \end{bmatrix}$. Using (a) and (6.3), we get $w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y} & \mathbf{X} \end{bmatrix}\right) = \max\left\{w_e(\mathbf{X} - \mathbf{Y}), w_e(\mathbf{X} + \mathbf{Y})\right\}$.

In particular, if we take $\mathbf{X} = 0$, then $w_e\left(\begin{bmatrix} 0 & \mathbf{Y} \\ \mathbf{Y} & 0 \end{bmatrix}\right) = w_e(\mathbf{Y})$. This completes the proof (e). □

Next, we develop an upper bound for the Euclidean operator radius of general 2×2 operator matrices whose entries are d-tuple operators.

Theorem 6.8. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$, $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_d)$, $\mathbf{W} = (W_1, W_2, \ldots, W_d) \in \mathbb{B}^d(\mathcal{H})$. Then*

$$w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \leq w\left(\begin{bmatrix} w_e(\mathbf{X}) & \|\mathbf{Y}\| \\ \|\mathbf{Z}\| & w_e(\mathbf{W}) \end{bmatrix}\right).$$

Proof. Let $u=(x,y)\in\mathcal{H}\oplus\mathcal{H}$ with $\|u\|=1$, i.e., $\|x\|^2+\|y\|^2=1$. Now,

$$
\begin{aligned}
&\left(\sum_{k=1}^{d}\left|\langle\begin{bmatrix}X_k & Y_k\\ Z_k & W_k\end{bmatrix}u,u\rangle\right|^2\right)^{\frac12}\\
=&\left(\sum_{k=1}^{d}\left|\langle(X_kx+Y_ky,Z_kx+W_ky),(x,y)\rangle\right|^2\right)^{\frac12}\\
\leq&\left(\sum_{k=1}^{d}\left|\langle X_kx,x\rangle+\langle W_ky,y\rangle\right|^2\right)^{\frac12}+\left(\sum_{k=1}^{d}\left|\langle Y_ky,x\rangle+\langle Z_kx,y\rangle\right|^2\right)^{\frac12}\\
&\quad(\textit{using Minkowski inequality})\\
\leq&\left(\sum_{k=1}^{d}\left|\langle X_kx,x\rangle\right|^2\right)^{\frac12}+\left(\sum_{k=1}^{d}\left|\langle W_ky,y\rangle\right|^2\right)^{\frac12}+\left(\sum_{k=1}^{d}\left|\langle Z_kx,y\rangle\right|^2\right)^{\frac12}+\left(\sum_{k=1}^{d}\left|\langle Y_ky,x\rangle\right|^2\right)^{\frac12}\\
&\quad(\textit{using Minkowski inequality})\\
\leq& w_e(\mathbf{X})\|x\|^2+w_e(\mathbf{W})\|y\|^2+\left(\sum_{k=1}^{d}\|Z_kx\|^2\|y\|^2\right)^{\frac12}+\left(\sum_{k=1}^{d}\|Y_ky\|^2\|x\|^2\right)^{\frac12}\\
\leq& w_e(\mathbf{X})\|x\|^2+w_e(\mathbf{W})\|y\|^2+\|\mathbf{Z}\|\|x\|\|y\|+\|\mathbf{Y}\|\|y\|\|x\|\\
=&\langle\begin{bmatrix}w_e(\mathbf{X}) & \|\mathbf{Y}\|\\ \|\mathbf{Z}\| & w_e(\mathbf{W})\end{bmatrix}\widetilde{x},\widetilde{x}\rangle,\ \textit{where } \widetilde{x}=(\|x\|,\|y\|)\in\mathbb{C}^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
w_e\left(\begin{bmatrix}\mathbf{X} & \mathbf{Y}\\ \mathbf{Z} & \mathbf{W}\end{bmatrix}\right) &= \sup\left\{\left(\sum_{k=1}^{d}\left|\langle\begin{bmatrix}X_k & Y_k\\ Z_k & W_k\end{bmatrix}u,u\rangle\right|^2\right)^{\frac12}: u\in\mathcal{H}\oplus\mathcal{H}, \|u\|=1\right\}\\
&\leq w\left(\begin{bmatrix}w_e(\mathbf{X}) & \|\mathbf{Y}\|\\ \|\mathbf{Z}\| & w_e(\mathbf{W})\end{bmatrix}\right).
\end{aligned}
$$

□

Since $w_e(\mathbf{X})\leq\|\mathbf{X}\|, w_e(\mathbf{W})\leq\|\mathbf{W}\|$ and $w\left([a_{ij}]_{2\times2}\right)\leq w\left([b_{ij}]_{2\times2}\right)$, for $0\leq a_{ij}\leq b_{ij}$ for all i,j, the following corollary is immediate from Theorem 6.8.

Corollary 6.3. *Let* $\mathbf{X}=(X_1,X_2,\ldots,X_d)$, $\mathbf{Y}=(Y_1,Y_2,\ldots,Y_d)$, $\mathbf{Z}=(Z_1,Z_2,\ldots,Z_d)$, $\mathbf{W}=(W_1,W_2,\ldots,W_d)\in\mathbb{B}^d(\mathcal{H})$. *Then*

$$
w_e\left(\begin{bmatrix}\mathbf{X} & \mathbf{Y}\\ \mathbf{Z} & \mathbf{W}\end{bmatrix}\right)\leq w\left(\begin{bmatrix}\|\mathbf{X}\| & \|\mathbf{Y}\|\\ \|\mathbf{Z}\| & \|\mathbf{W}\|\end{bmatrix}\right).
$$

Now, we need the following lemma.

Lemma 6.3. *[52, p. 44] Let $B = [b_{ij}]$ be an $n \times n$ matrix such that $b_{ij} \geq 0$ for all $i, j = 1, 2, ..., n$. Then*

$$w(B) = r\left(\left[\tfrac{b_{ij}+b_{ji}}{2}\right]\right),$$

where $r(\cdot)$ denotes the spectral radius.

Applying Theorem 6.8 and using Lemma 6.3, we obtain the following result.

Corollary 6.4. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$, $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_d)$, $\mathbf{W} = (W_1, W_2, \ldots, W_d) \in \mathbb{B}^d(\mathcal{H})$. Then*

$$w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \leq \frac{1}{2}\left(w_e(\mathbf{X}) + w_e(\mathbf{W}) + \sqrt{(w_e(\mathbf{X}) - w_e(\mathbf{W}))^2 + (\|\mathbf{Y}\| + \|\mathbf{Z}\|)^2}\right).$$

By using the power inequality obtained in Theorem 6.4, we develop a lower bound for the Euclidean operator radius of 2×2 off-diagonal operator matrices.

Theorem 6.9. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d) \in \mathbb{B}^d(\mathcal{H})$. Then*

$$\sqrt[2n]{\frac{1}{\sqrt{d}} \max\{w_e((\mathbf{XY})^n), w_e((\mathbf{YX})^n)\}} \leq w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right).$$

Proof. Let $\mathbf{T} = \begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}$. Then $\mathbf{T}^{2n} = \begin{bmatrix} (\mathbf{XY})^n & 0 \\ 0 & (\mathbf{YX})^n \end{bmatrix}$ for all $n = 1, 2, 3, \ldots$. Using Lemma 6.2 (a) and Theorem 6.4, we get

$$\max\{w_e((\mathbf{XY})^n), w_e((\mathbf{YX})^n)\} = w_e(\mathbf{T}^{2n}) \leq \sqrt{d} w_e^{2n}(\mathbf{T}).$$

□

Next bounds reads as follows.

Theorem 6.10. *Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d) \in \mathbb{B}^d(\mathcal{H})$. Then*

$$\frac{1}{2} \max\left\{w_e(\mathbf{X} + \mathbf{Y}), w_e(\mathbf{X} - \mathbf{Y})\right\} \leq w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) \leq \frac{1}{2}(w_e(\mathbf{X} + \mathbf{Y}) + w_e(\mathbf{X} - \mathbf{Y})).$$

Proof. It follows from Lemma 6.2 (e) that

$$\begin{aligned} w_e(\mathbf{X}+\mathbf{Y}) &= w_e\left(\begin{bmatrix} 0 & \mathbf{X}+\mathbf{Y} \\ \mathbf{X}+\mathbf{Y} & 0 \end{bmatrix}\right) \\ &= w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{Y} \\ \mathbf{X} & 0 \end{bmatrix}\right) \\ &\leq w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) + w_e\left(\begin{bmatrix} 0 & \mathbf{Y} \\ \mathbf{X} & 0 \end{bmatrix}\right) \quad (\textit{using Lemma 6.1}) \\ &= 2w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) \quad (\textit{using Lemma 6.2 (c)}). \end{aligned} \tag{6.4}$$

Replacing $\mathbf{Y}$ by $-\mathbf{Y}$, we have

$$w_e(\mathbf{X}-\mathbf{Y}) \leq 2w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ -\mathbf{Y} & 0 \end{bmatrix}\right) = 2w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right). \tag{6.5}$$

Therefore, the first inequality follows from (6.4) and (6.5). To prove the second inequality, consider an unitary operator $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. Then we have,

$$\begin{aligned} w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) &= w_e\left(U^*\begin{bmatrix} 0 & X_1 \\ Y_1 & 0 \end{bmatrix}U, U^*\begin{bmatrix} 0 & X_2 \\ Y_2 & 0 \end{bmatrix}U, \ldots, U^*\begin{bmatrix} 0 & X_d \\ Y_d & 0 \end{bmatrix}U\right) \\ &= \frac{1}{2}w_e\left(\begin{bmatrix} \mathbf{X}+\mathbf{Y} & \mathbf{X}-\mathbf{Y} \\ -(\mathbf{X}-\mathbf{Y}) & -(\mathbf{X}+\mathbf{Y}) \end{bmatrix}\right) \\ &= \frac{1}{2}w_e\left(\begin{bmatrix} \mathbf{X}+\mathbf{Y} & 0 \\ 0 & -(\mathbf{X}+\mathbf{Y}) \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{X}-\mathbf{Y} \\ -(\mathbf{X}-\mathbf{Y}) & 0 \end{bmatrix}\right) \\ &\leq \frac{1}{2}w_e\left(\begin{bmatrix} \mathbf{X}+\mathbf{Y} & 0 \\ 0 & -(\mathbf{X}+\mathbf{Y}) \end{bmatrix}\right) + \frac{1}{2}w_e\left(\begin{bmatrix} 0 & \mathbf{X}-\mathbf{Y} \\ -(\mathbf{X}-\mathbf{Y}) & 0 \end{bmatrix}\right) \\ &\qquad (\textit{using Lemma 6.1}) \\ &= \frac{w_e(\mathbf{X}+\mathbf{Y}) + w_e(\mathbf{X}-\mathbf{Y})}{2} \quad (\textit{using Lemma 6.2}). \end{aligned}$$

□

As an application of Theorem 6.10, we derive the following inequalities.

Corollary 6.5. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ and let $T_k = X_k + iY_k$ be the Cartesian* *decomposition of* T_k, $k = 1, 2, \dots, d$. *If* $\mathbf{X} = (X_1, X_2, \dots, X_d)$ *and* $\mathbf{Y} = (Y_1, Y_2, \dots, Y_d)$, *then*

$$\frac{w_e(\mathbf{T})}{2} \leq w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ e^{i\theta}\mathbf{Y} & 0 \end{bmatrix}\right) \leq w_e(\mathbf{T}),$$

for all $\theta \in \mathbb{R}$.

Proof. Replacing $\mathbf{Y}$ by $i\mathbf{Y}$ in Theorem 6.10 and then using Lemma 6.2, we have

$$\frac{\max\{w_e(\mathbf{X}+i\mathbf{Y}), w_e(\mathbf{X}-i\mathbf{Y})\}}{2} \leq w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ e^{i\theta}\mathbf{Y} & 0 \end{bmatrix}\right) \leq \frac{w_e(\mathbf{X}+i\mathbf{Y}) + w_e(\mathbf{X}-i\mathbf{Y})}{2},$$

as desired. □

Now, we prove the following pinching inequalities.

Lemma 6.4. *Let* $\mathbf{X} = (X_1, X_2, \dots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \dots, Y_d)$, $\mathbf{Z} = (Z_1, Z_2, \dots, Z_d)$, $\mathbf{W} = (W_1, W_2, \dots, W_d) \in \mathbb{B}^d(\mathcal{H})$. *Then*

$$w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \geq w_e\left(\begin{bmatrix} \mathbf{X} & 0 \\ 0 & \mathbf{W} \end{bmatrix}\right)$$

and

$$w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \geq w_e\left(\begin{bmatrix} 0 & \mathbf{Y} \\ \mathbf{Z} & 0 \end{bmatrix}\right).$$

Proof. Let $u = (x, 0) \in \mathcal{H} \oplus \mathcal{H}$ with $\|u\| = 1$, i.e., $\|x\| = 1$. Now, we have

$$\begin{aligned} \left(\sum_{k=1}^{d} \left|\left\langle \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix} u, u\right\rangle\right|^2\right)^{\frac{1}{2}} &= \left(\sum_{k=1}^{d} |\langle (X_k x, Z_k x), (x, 0)\rangle|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^{d} |\langle X_k x, x\rangle|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Taking supremum over $\|u\| = 1$, we get

$$\sup_{\|u\|=1} \left(\sum_{k=1}^{d} \left|\left\langle \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix} u, u\right\rangle\right|^2\right)^{\frac{1}{2}} = \sup_{\|x\|=1} \left(\sum_{k=1}^{d} |\langle X_k x, x\rangle|^2\right)^{\frac{1}{2}} = w_e(\mathbf{X}).$$

This gives,

$$w_e(\mathbf{X}) \leq w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right). \tag{6.6}$$

Similarly,

$$w_e(\mathbf{Y}) \leq w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right). \tag{6.7}$$

Therefore, the desired first inequality follows from (6.6) and (6.7) together with Lemma 6.2 (a). To prove the second inequality, we write $\begin{bmatrix} 0 & \mathbf{Y} \\ \mathbf{Z} & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & -\mathbf{W} \end{bmatrix}$. It follows from Lemma 6.1 that $w_e\left(\begin{bmatrix} 0 & \mathbf{Y} \\ \mathbf{Z} & 0 \end{bmatrix}\right) \leq \frac{1}{2}w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) + \frac{1}{2}w_e\left(\begin{bmatrix} -\mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & -\mathbf{W} \end{bmatrix}\right)$. By considering the unitary operator $U = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$, we have $\left(U^*\begin{bmatrix} -X_1 & Y_1 \\ Z_1 & -W_1 \end{bmatrix}U, U^*\begin{bmatrix} -X_2 & Y_2 \\ Z_2 & -W_2 \end{bmatrix}U, \ldots, U^*\begin{bmatrix} -X_d & Y_d \\ Z_d & -W_d \end{bmatrix}U\right) = \begin{bmatrix} -\mathbf{W} & -\mathbf{Z} \\ -\mathbf{Y} & -\mathbf{X} \end{bmatrix}$. So, from the weakly unitary invariant property of the Euclidean operator radius we infer that

$$w_e\left(\begin{bmatrix} 0 & \mathbf{Y} \\ \mathbf{Z} & 0 \end{bmatrix}\right) \leq \frac{1}{2}w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) + \frac{1}{2}w_e\left(\begin{bmatrix} -\mathbf{W} & -\mathbf{Z} \\ -\mathbf{Y} & -\mathbf{X} \end{bmatrix}\right). \tag{6.8}$$

Again, considering the unitary operator $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, we have $\left(U^*\begin{bmatrix} -W_1 & -Z_1 \\ -Y_1 & -X_1 \end{bmatrix}U, U^*\begin{bmatrix} -W_2 & -Z_2 \\ -Y_2 & -X_2 \end{bmatrix}U, \ldots, U^*\begin{bmatrix} -W_d & -Z_d \\ -Y_d & -X_d \end{bmatrix}U\right) = \begin{bmatrix} -\mathbf{X} & -\mathbf{Y} \\ -\mathbf{Z} & -\mathbf{W} \end{bmatrix}$ and so $w_e\left(\begin{bmatrix} -\mathbf{W} & -\mathbf{Z} \\ -\mathbf{Y} & -\mathbf{X} \end{bmatrix}\right) = w_e\left(\begin{bmatrix} -\mathbf{X} & -\mathbf{Y} \\ -\mathbf{Z} & -\mathbf{W} \end{bmatrix}\right) = w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right)$. This completes the proof of the lemma.

□

Using the above pinching inequality we obtain the following result.

Theorem 6.11. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_d)$*,* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d) \in \mathbb{B}^d(\mathcal{H})$*. Then*

$$\max\{w_e(\mathbf{X}), w_e(\mathbf{Y})\} \leq w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & -\mathbf{X} \end{bmatrix}\right) \leq w_e(\mathbf{X}) + w_e(\mathbf{Y}).$$

Proof. The first inequality follows from Lemma 6.4 together with Lemma 6.2. For the other part,

$$\begin{aligned} w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & -\mathbf{X} \end{bmatrix}\right) &= w_e\left(\begin{bmatrix} \mathbf{X} & 0 \\ 0 & -\mathbf{X} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{Y} \\ -\mathbf{Y} & o \end{bmatrix}\right) \\ &\leq w_e\left(\begin{bmatrix} \mathbf{X} & 0 \\ 0 & -\mathbf{X} \end{bmatrix}\right) + w_e\left(\begin{bmatrix} 0 & \mathbf{Y} \\ -\mathbf{Y} & o \end{bmatrix}\right) \quad (\textit{using Lemma } 6.1) \\ &= w_e(\mathbf{X}) + w_e(\mathbf{Y}). \end{aligned}$$

□

Taking $\mathbf{Y} = \mathbf{X}$ in Theorem 6.11, we get the following result.

Corollary 6.6. *If* $\mathbf{X} \in \mathbb{B}^d(\mathcal{H})$*, then*

$$w_e(\mathbf{X}) \leq w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{X} \\ -\mathbf{X} & -\mathbf{X} \end{bmatrix}\right) \leq 2w_e(\mathbf{X}).$$

Using Theorem 6.10, we obtain the following lower and upper bounds for the Euclidean operator radius of general 2×2 operator matrices.

Theorem 6.12. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_d)$*,* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$*,* $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_d)$*,* $\mathbf{W} = (W_1, W_2, \ldots, W_d) \in \mathbb{B}^d(\mathcal{H})$*. Then*

$$w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \geq \max\left\{w_e(\mathbf{X}), w_e(\mathbf{W}), w_e\left(\frac{\mathbf{Y}+\mathbf{Z}}{2}\right), w_e\left(\frac{\mathbf{Y}-\mathbf{Z}}{2}\right)\right\}$$

and

$$w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \leq \max\{w_e(\mathbf{X}), w_e(\mathbf{W})\} + w_e\left(\frac{\mathbf{Y}+\mathbf{Z}}{2}\right) + w_e\left(\frac{\mathbf{Y}-\mathbf{Z}}{2}\right).$$

Proof. It follows from Lemma 6.4 that

$$\begin{aligned}
& w_e\left(\begin{bmatrix}\mathbf{X} & \mathbf{Y}\\ \mathbf{Z} & \mathbf{W}\end{bmatrix}\right)\\
\geq\ & \max\left\{w_e\left(\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{W}\end{bmatrix}\right), w_e\left(\begin{bmatrix}0 & \mathbf{Y}\\ \mathbf{Z} & 0\end{bmatrix}\right)\right\}\\
=\ & \max\left\{w_e(\mathbf{X}), w_e(\mathbf{X}), w_e\left(\begin{bmatrix}0 & \mathbf{Y}\\ \mathbf{Z} & 0\end{bmatrix}\right)\right\} \quad (\textit{using Lemma 6.2(a)})\\
\geq\ & \max\left\{w_e(\mathbf{X}), w_e(\mathbf{W}), w_e\left(\frac{\mathbf{Y}+\mathbf{Z}}{2}\right), w_e\left(\frac{\mathbf{Y}-\mathbf{Z}}{2}\right)\right\} \quad (\textit{using Theorem 6.10})\ .
\end{aligned}$$

Again, it follows from Lemma 6.2(a) and Theorem 6.10 that

$$\begin{aligned}
w_e\left(\begin{bmatrix}\mathbf{X} & \mathbf{Y}\\ \mathbf{Z} & \mathbf{W}\end{bmatrix}\right) &= w_e\left(\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{W}\end{bmatrix} + \begin{bmatrix}0 & \mathbf{Y}\\ \mathbf{Z} & 0\end{bmatrix}\right)\\
&\leq w_e\left(\begin{bmatrix}\mathbf{X} & 0\\ 0 & \mathbf{W}\end{bmatrix}\right) + w_e\left(\begin{bmatrix}0 & \mathbf{Y}\\ \mathbf{Z} & 0\end{bmatrix}\right) \quad (\textit{using Lemma 6.1})\\
&\leq \max\{w_e(\mathbf{X}), w_e(\mathbf{W})\} + w_e\left(\frac{\mathbf{Y}+\mathbf{Z}}{2}\right) + w_e\left(\frac{\mathbf{Y}-\mathbf{Z}}{2}\right).
\end{aligned}$$

□

Next inequality reads as follows:

Theorem 6.13. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d) \in \mathbb{B}^d(\mathcal{H})$. *Then*

$$w_e\left(\begin{bmatrix}0 & \mathbf{X}\\ \mathbf{Y} & 0\end{bmatrix}\right) + \frac{|w_e(\mathbf{X}+\mathbf{Y}) - w_e(\mathbf{X}-\mathbf{Y})|}{2} \leq w_e(\mathbf{X}) + w_e(\mathbf{Y}).$$

Proof. From Theorem 6.10, we have

$$\begin{aligned}
w_e\left(\begin{bmatrix}0 & \mathbf{X}\\ \mathbf{Y} & 0\end{bmatrix}\right) &\leq \frac{w_e(\mathbf{X}+\mathbf{Y}) + w_e(\mathbf{X}-\mathbf{Y})}{2}\\
&= \max\{w_e(\mathbf{X}+\mathbf{Y}), w_e(\mathbf{X}-\mathbf{Y})\} - \frac{|w_e(\mathbf{X}+\mathbf{Y}) - w_e(\mathbf{X}-\mathbf{Y})|}{2}\\
&\leq w_e(\mathbf{X}) + w_e(\mathbf{Y}) - \frac{|w_e(\mathbf{X}+\mathbf{Y}) - w_e(\mathbf{X}-\mathbf{Y})|}{2}.
\end{aligned}$$

□

Next theorem is as follows:

Theorem 6.14. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$, $\mathbf{A} = (A_1, A_2, \ldots, A_d)$, $\mathbf{B} = (B_1, B_2, \ldots, B_d) \in \mathbb{B}^d(\mathcal{H})$. *Then*

$$w_e(\mathbf{A}^*\mathbf{X}\mathbf{B} + \mathbf{B}^*\mathbf{Y}\mathbf{A}) \leq 2\|\mathbf{A}\|\|\mathbf{B}\| w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right).$$

In particular, for $\mathbf{X} = \mathbf{Y}$

$$w_e(\mathbf{A}^*\mathbf{X}\mathbf{B} + \mathbf{B}^*\mathbf{X}\mathbf{A}) \leq 2\|\mathbf{A}\|\|\mathbf{B}\| w_e(\mathbf{X}).$$

Proof. Let $x, y \in \mathcal{H}$ be non-zero and let $z = \frac{1}{\sqrt{\|x\|^2+\|y\|^2}}(x, y)$. Then z is an unit vector in $\mathcal{H} \oplus \mathcal{H}$ and so

$$w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) \geq \left(\sum_{k=1}^{d} \left|\left\langle \begin{bmatrix} 0 & X_k \\ Y_k & 0 \end{bmatrix} z, z\right\rangle\right|^2\right)^{\frac{1}{2}} = \frac{\left(\sum_{k=1}^{d} |\langle X_k y, x\rangle + \langle Y_k x, y\rangle|^2\right)^{\frac{1}{2}}}{(\|x\|^2 + \|y\|^2)}.$$

Therefore, $\left(\|x\|^2 + \|y\|^2\right) w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) \geq \left(\sum_{k=1}^{d} |\langle X_k y, x\rangle + \langle Y_k x, y\rangle|^2\right)^{\frac{1}{2}}$ for all $x, y \in \mathcal{H}$. This implies that $\left(\|x\|^2 + \|y\|^2\right) w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) \geq |\langle X_k y, x\rangle + \langle Y_k x, y\rangle|$ holds for each $k = 1, 2, \ldots, d$. Now, replacing x and y by $A_k x$ and $B_k x$, respectively and then summing, we get

$$\begin{aligned} \sum_{k=1}^{d} \left|\langle X_k B_k x, A_k x\rangle + \langle Y_k A_k x, B_k x\rangle\right| &\leq \sum_{k=1}^{d} \left(\left(\|A_k x\|^2 + \|B_k x\|^2\right) w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right)\right) \\ &= w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) \sum_{k=1}^{d} \left(\|A_k x\|^2 + \|B_k x\|^2\right) \\ &\leq w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) \left(\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2\right) \|x\|^2. \end{aligned}$$

So,

$$\begin{aligned} \left(\sum_{k=1}^{d} \left|\langle X_k B_k x, A_k x\rangle + \langle Y_k A_k x, B_k x\rangle\right|^2\right)^{\frac{1}{2}} &\leq \sum_{k=1}^{d} \left|\langle X_k B_k x, A_k x\rangle + \langle Y_k A_k x, B_k x\rangle\right| \\ &\leq w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right) \left(\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2\right) \|x\|^2. \end{aligned}$$

Taking supremum over $\|x\| = 1$, we have

$$w_e(\mathbf{A}^*\mathbf{X}\mathbf{B} + \mathbf{B}^*\mathbf{Y}\mathbf{A}) \leq \left(\|\mathbf{A}\|^2 + \|\mathbf{B}\|^2\right) w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right). \tag{6.9}$$

Therefore, the desired inequality follows from (6.9) by replacing $\mathbf{A}$ and $\mathbf{B}$ by t$\mathbf{A}$ and $\frac{1}{t}$ $\mathbf{B}$ respectively, where $t = \sqrt{\frac{\|\mathbf{B}\|}{\|\mathbf{A}\|}}$. □

Next, we obtain the following inequality which involves the Euclidean operator norm and the classical operator norm.

Theorem 6.15. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_d)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_d)$, $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_d)$, $\mathbf{W} = (W_1, W_2, \ldots, W_d) \in \mathbb{B}^d(\mathcal{H})$. *Then*

$$\left\| \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \|\mathbf{X}\| & \|\mathbf{Y}\| \\ \|\mathbf{Z}\| & \|\mathbf{W}\| \end{bmatrix} \right\|.$$

Proof. Let $u = (x, y) \in \mathcal{H} \oplus \mathcal{H}$ with $\|u\| = 1$, i.e., $\|x\|^2 + \|y\|^2 = 1$. Then we have,

$$
\begin{aligned}
& \sum_{k=1}^{d} \left\| \begin{bmatrix} X_k & Y_k \\ Z_k & W_k \end{bmatrix} u \right\|^2 \\
= & \sum_{k=1}^{d} \langle \begin{bmatrix} X_k & Y_k \\ Z_k & W_k \end{bmatrix} u, \begin{bmatrix} X_k & Y_k \\ Z_k & W_k \end{bmatrix} u \rangle \\
= & \sum_{k=1}^{d} \left(\|X_k x + Y_k y\|^2 + \|Z_k x + W_k y\|^2 \right) \\
= & \sum_{k=1}^{d} (\|X_k x\|^2 + \|Y_k y\|^2 + \|Z_k x\|^2 + \|W_k y\|^2 + 2Re\langle X_k x, Y_k y\rangle + 2Re\langle W_k y, Z_k x\rangle) \\
& \qquad \qquad \textit{(where } Re\langle x, y\rangle \textit{ means real part of } \langle x, y\rangle) \\
\leq & \sum_{k=1}^{d} (\|X_k x\|^2 + \|Y_k y\|^2 + \|Z_k x\|^2 + \|W_k y\|^2 + 2|\langle X_k x, Y_k y\rangle| + 2|\langle W_k y, Z_k x\rangle|) \\
\leq & \sum_{k=1}^{d} \|X_k x\|^2 + \sum_{k=1}^{d} \|Y_k y\|^2 + \sum_{k=1}^{d} \|Z_k x\|^2 + \sum_{k=1}^{d} \|W_k y\|^2 \\
& + 2 \sum_{k=1}^{d} \|X_k x\| \|Y_k y\| + 2 \sum_{k=1}^{d} \|Z_k x\| \|W_k y\| \\
\leq & \|\mathbf{X}\|^2 \|x\|^2 + \|\mathbf{Y}\|^2 \|y\|^2 + \|\mathbf{Z}\|^2 \|x\|^2 + \|\mathbf{W}\|^2 \|y\|^2 \\
& \quad + 2 \left(\sum_{k=1}^{d} \|X_k x\|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{d} \|Y_k y\|^2 \right)^{\frac{1}{2}} + 2 \left(\sum_{k=1}^{d} \|Z_k x\|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{d} \|W_k y\|^2 \right)^{\frac{1}{2}}, \\
& \qquad \qquad \textit{(by Cauchy-Schwarz inequality)} \\
\leq & \left(\|\mathbf{X}\|^2 + \|\mathbf{Z}\|^2 \right) \|x\|^2 + \left(\|\mathbf{Y}\|^2 + \|\mathbf{W}\|^2 \right) \|y\|^2 + 2\|\mathbf{X}\|\|\mathbf{Y}\|\|x\|\|y\| + 2\|\mathbf{Z}\|\|\mathbf{W}\|\|x\|\|y\| \\
= & \langle \begin{bmatrix} \|\mathbf{X}\| & \|\mathbf{Y}\| \\ \|\mathbf{Z}\| & \|\mathbf{W}\| \end{bmatrix}^* \begin{bmatrix} \|\mathbf{X}\| & \|\mathbf{Y}\| \\ \|\mathbf{Z}\| & \|\mathbf{W}\| \end{bmatrix} \widetilde{x}, \widetilde{x} \rangle \ \textit{(where } \widetilde{x} = (\|x\|, \|y\|) \in \mathbb{C}^2) \\
\leq & \left\| \begin{bmatrix} \|\mathbf{X}\| & \|\mathbf{Y}\| \\ \|\mathbf{Z}\| & \|\mathbf{W}\| \end{bmatrix}^* \begin{bmatrix} \|\mathbf{X}\| & \|\mathbf{Y}\| \\ \|\mathbf{Z}\| & \|\mathbf{W}\| \end{bmatrix} \right\| = \left\| \begin{bmatrix} \|\mathbf{X}\| & \|\mathbf{Y}\| \\ \|\mathbf{Z}\| & \|\mathbf{W}\| \end{bmatrix} \right\|^2.
\end{aligned}
$$

Therefore,

$$
\left\| \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix} \right\| = \sup_{\|u\|=1} \left(\sum_{k=1}^{d} \left\| \begin{bmatrix} X_k & Y_k \\ Z_k & W_k \end{bmatrix} u \right\|^2 \right)^{1/2} \leq \left\| \begin{bmatrix} \|\mathbf{X}\| & \|\mathbf{Y}\| \\ \|\mathbf{Z}\| & \|\mathbf{W}\| \end{bmatrix} \right\|.
$$

□

Finally we provide an example presenting different bounds obtained from different inequalities studied here. Consider $\mathbf{X} = (X_1, X_2)$, $\mathbf{Y} = (Y_1, Y_2)$, $\mathbf{Z} = (Z_1, Z_2)$, $\mathbf{W} = (W_1, W_2) \in \mathbb{B}^2(\mathcal{H})$, where $X_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $Y_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $Z_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $Z_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $W_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then

(i) Theorem 6.8 gives $w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \leq 2$.

(ii) Corollary 6.3 gives $w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \leq 2$.

(iii) Theorem 6.9 gives $\frac{1}{\sqrt{2}} \leq w_e\left(\begin{bmatrix} 0 & \mathbf{X} \\ \mathbf{Y} & 0 \end{bmatrix}\right)$ (for $n = 1$).

(iv) Theorem 6.10 gives $\frac{1}{\sqrt{2}} \leq w_e\left(\begin{bmatrix} 0 & \mathbf{Y} \\ \mathbf{Z} & 0 \end{bmatrix}\right) \leq \sqrt{2}$.

(v) Theorem 6.12 gives $1 \leq w_e\left(\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}\right) \leq 1 + \sqrt{2}$.

CHAPTER 7

ESTIMATIONS OF EUCLIDEAN OPERATOR RADIUS

7.1 Introduction

In this chapter, we develop several Euclidean operator radius bounds for the product of two d-tuple operators using positivity criteria of a 2×2 block matrix whose entries are d-tuple operators. From these bounds, by using the polar decomposition of operators, we obtain Euclidean operator radius bounds for d-tuple operators. Let $\mathbb{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the corresponding norm. Let $T \in \mathbb{B}(\mathcal{H})$ and let $|T| = (T^*T)^{1/2}$, where T^* is the adjoint of T. The numerical radius of T is defined as

$$w(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

It is well known that $w(\cdot) : \mathbb{B}(\mathcal{H}) \to \mathbb{R}$ defines a norm and satisfies the following relation

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|.$$

Content of this chapter is based on the following paper:
P. Bhunia, **S. Jana** and K. Paul, Estimations of Euclidean operator radius. arXiv:2308.09258

The Euclidean operator radius of d-tuple operators is one such generalization. Let $\mathbf{T} = (T_1, T_2, \ldots, T_d)$ be a d-tuple operator in $\mathbb{B}^d(\mathcal{H}) = \mathbb{B}(\mathcal{H}) \times \mathbb{B}(\mathcal{H}) \times \ldots \times \mathbb{B}(\mathcal{H})$ (d times) and let $\langle \mathbf{T}x, y\rangle = (\langle T_1x, y\rangle, \langle T_2x, y\rangle, \ldots, \langle T_dx, y\rangle) \in \mathbb{C}^d$ for all $x, y \in \mathcal{H}$. The Euclidean operator radius and the Euclidean operator norm of $\mathbf{T}$ are defined respectively as

$$w_e(\mathbf{T}) \;=\; \sup\left\{\left(\sum_{k=1}^{d} |\langle T_kx, x\rangle|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1\right\}$$

and

$$\|\mathbf{T}\| \;=\; \sup\left\{\left(\sum_{k=1}^{d} \|T_kx\|^2\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1\right\}.$$

A d-tuple operator $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ is said to be positive if each T_k is positive for all $k = 1, 2, \ldots, d$. We write $|\mathbf{T}|^t = (|T_1|^t, |T_2|^t, \ldots, |T_d|^t)$ for $t > 0$, and $\alpha\mathbf{T} = (\alpha T_1, \alpha T_2, \ldots, \alpha T_d)$ for any scalar $\alpha \in \mathbb{C}$. For $\mathbf{S} = (S_1, S_2, \ldots, S_d) \in \mathbb{B}^d(\mathcal{H})$, we write $\mathbf{TS} = (T_1S_1, T_2S_2, \ldots, T_dS_d)$, $\mathbf{T} + \mathbf{S} = (T_1 + S_1, T_2 + S_2, \ldots, T_d + S_d)$. Let $\mathbf{T_{ij}} = (T^1_{ij}, T^2_{ij}, \ldots, T^d_{ij}) \in \mathbb{B}^d(\mathcal{H})$, $1 \le i, j \le n$. Then the $n \times n$ operator matrix, whose entries are d-tuple operators $\mathbf{T_{ij}}$, is defined as

$$\left[\mathbf{T_{ij}}\right]_{n\times n} = \left(\left[T^1_{ij}\right]_{n\times n}, \left[T^2_{ij}\right]_{n\times n}, \ldots, \left[T^d_{ij}\right]_{n\times n}\right) \in \mathbb{B}^d\left(\oplus_{i=1}^{d}\mathcal{H}\right).$$

It is well known that the Euclidean operator radius defines a norm on $\mathbb{B}^d(\mathcal{H})$ and satisfies the following relation

$$\frac{1}{2\sqrt{d}}\|\mathbf{T}\| \le w_e(\mathbf{T}) \le \|\mathbf{T}\|. \tag{7.1}$$

which can be found in [78]. Note that the constants $\frac{1}{2\sqrt{d}}$ and 1 are best possible.

This chapter is organized as follows: In Section 2, by using positivity of a 2×2 block matrix, whose entries are d-tuple operators, we obtain several upper bounds of the Euclidean operator radius of the product of two d-tuple operators. From these estimations and by using the polar decomposition, we develop several upper bounds for the Euclidean operator radius of d-tuple operators. In Section 3, we develop an upper bound for the Euclidean operator radius of $n \times n$ operator matrix whose entries are d-tuple operators. As an application of these bounds we derive upper bounds for the Euclidean operator radius of d-tuple operators.

7.2 Euclidean operator radius of d-tuple operators

We begin this section with the following lemmas. First lemma is known as McCarthy inequality.

Lemma 7.1. *[71] Let $T \in \mathbb{B}(\mathcal{H})$ be positive, and $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$\langle Tx, x\rangle^p \leq \langle T^p x, x\rangle,$$

for all $p \geq 1$.

Second lemma is known as Buzano's inequality, which is an extension of Schwarz's inequality.

Lemma 7.2. *[33] Let $x, y, z \in \mathcal{H}$ be such that $\|z\| = 1$. Then*

$$|\langle x, z\rangle\langle z, y\rangle| \leq \frac{\|x\|\|y\| + |\langle x, y\rangle|}{2}.$$

Third lemma is on non-negative real numbers and is known as Bohr's inequality.

Lemma 7.3. *[84] Let $a_k \geq 0$ for $k = 1, 2, \ldots, n$. Then*

$$\left(\sum_{k=1}^{n} a_k\right)^p \leq n^{p-1}\sum_{k=1}^{n} a_k^p,$$

for all $p \geq 1$.

The next lemma involves 2×2 operator matrix, whose entries are d-tuple operators.

Lemma 7.4. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_d), \mathbf{S} = (S_1, S_2, \ldots, S_d), \mathbf{X} = (X_1, X_2, \ldots, X_d) \in \mathbb{B}^d(\mathcal{H})$, where $\mathbf{T}$ and $\mathbf{S}$ are positive (i.e., T_k and S_k are positive for each $k = 1, 2, \ldots, d$). If $\begin{bmatrix} \mathbf{T} & \mathbf{X}^* \\ \mathbf{X} & \mathbf{S} \end{bmatrix}$ is positive, then*

$$\|\langle \mathbf{X}x, y\rangle\|^2 \leq \sum_{k=1}^{d} \langle T_k x, x\rangle\langle S_k y, y\rangle, \quad \textit{for all } x, y \in \mathcal{H}.$$

Proof. Take $x, y \in \mathcal{H}$. We have

$$\begin{aligned}
\|\langle \mathbf{X}x, y\rangle\|^2 &= \sum_{k=1}^{d} |\langle X_k x, y\rangle|^2 \\
&= \sum_{k=1}^{d} \left| \left\langle \begin{bmatrix} T_k & X_k^* \\ X_k & S_k \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\rangle \right|^2 \\
&\leq \sum_{k=1}^{d} \left\langle \begin{bmatrix} T_k & X_k^* \\ X_k & S_k \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\rangle \left\langle \begin{bmatrix} T_k & X_k^* \\ X_k & S_k \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\rangle \\
&\quad (\textit{by Cauchy-Schwarz inequality for positive operators}) \\
&= \sum_{k=1}^{d} \langle T_k x, x\rangle \langle S_k y, y\rangle.
\end{aligned}$$

□

We are now in a position to prove our first theorem.

Theorem 7.1. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d), \mathbf{S} = (S_1, S_2, \ldots, S_d), \mathbf{X} = (X_1, X_2, \ldots, X_d) \in \mathbb{B}^d(\mathcal{H})$, *where* $\mathbf{T}$ *and* $\mathbf{S}$ *are positive (i.e.,* T_k *and* S_k *are positive for each* $k = 1, 2, \ldots, d$*). If* $\begin{bmatrix} \mathbf{T} & \mathbf{X}^* \\ \mathbf{X} & \mathbf{S} \end{bmatrix}$ *is positive, then*

(i) $w_e(\mathbf{X}) \leq \sqrt{\frac{1}{2} \left\| \sum_{k=1}^{d} \left(T_k^2 + S_k^2\right) \right\|}$.

(ii) $w_e(\mathbf{X}) \leq \sqrt{\frac{1}{2} \|\mathbf{T}\| \|\mathbf{S}\| + \frac{\sqrt{d}}{2} w_e(\mathbf{TS})}$.

(iii) $w_e(\mathbf{X}) \leq \sqrt{\frac{1}{4} \left\| \sum_{k=1}^{d} \left(T_k^2 + S_k^2\right) \right\| + \frac{\sqrt{d}}{2} w_e(\mathbf{TS})}$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$.

(i) From Lemma 7.4, we have

$$\begin{aligned}
\|\langle \mathbf{X}x, x\rangle\|^2 &= \sum_{k=1}^{d} \left|\langle X_k x, x\rangle\right|^2 \leq \sum_{k=1}^{d} \langle T_k x, x\rangle \langle S_k x, x\rangle \\
&\leq \sum_{k=1}^{d} \frac{1}{2}\left(\langle T_k x, x\rangle^2 + \langle S_k x, x\rangle^2\right) \\
&\leq \sum_{k=1}^{d} \frac{1}{2}\left(\langle T_k^2 x, x\rangle + \langle S_k^2 x, x\rangle\right) \quad (\textit{using Lemma 7.1}) \\
&= \sum_{k=1}^{d} \frac{1}{2}\langle (T_k^2 + S_k^2)x, x\rangle \\
&= \frac{1}{2}\left\langle \sum_{k=1}^{d} (T_k^2 + S_k^2)x, x \right\rangle \\
&\leq \frac{1}{2}\left\| \sum_{k=1}^{d} (T_k^2 + S_k^2) \right\|.
\end{aligned}$$

This holds for all $x \in \mathcal{H}$ with $\|x\| = 1$ and so taking supremum we get the desired inequality.

(ii) Again from Lemma 7.4, we get

$$\begin{aligned}
\|\langle \mathbf{X}x, x\rangle\|^2 &= \sum_{k=1}^{d} \left|\langle X_k x, x\rangle\right|^2 \leq \sum_{k=1}^{d} \langle T_k x, x\rangle \langle S_k x, x\rangle \\
&\leq \sum_{k=1}^{d} \frac{1}{2}\left(\|T_k x\|\|S_k x\| + |\langle T_k x, S_k x\rangle|\right) \quad (\textit{using Lemma 7.2}) \\
&\leq \frac{1}{2}\left(\sum_{k=1}^{d} \|T_k x\|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{d} \|S_k x\|^2\right)^{\frac{1}{2}} + \frac{1}{2}\sum_{k=1}^{d} |\langle T_k x, S_k x\rangle| \\
&\qquad (\textit{using Cauchy-Schwarz inequality}) \\
&\leq \frac{1}{2}\|\mathbf{T}\|\|\mathbf{S}\| + \frac{\sqrt{d}}{2}\left(\sum_{k=1}^{d} |\langle x, T_k S_k x\rangle|^2\right)^{\frac{1}{2}} \qquad (\textit{using Lemma 7.3}) \\
&\leq \frac{1}{2}\|\mathbf{T}\|\|\mathbf{S}\| + \frac{\sqrt{d}}{2} w_e(\mathbf{TS}).
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$, $\|x\| = 1$, we get the desired result.

(iii) From Lemma 7.4, we have

$$\begin{aligned}
\|\langle \mathbf{X}x, x\rangle\|^2 &= \sum_{k=1}^{d} \left|\langle X_k x, x\rangle\right|^2 \leq \sum_{k=1}^{d} \langle T_k x, x\rangle\langle S_k x, x\rangle \\
&\leq \sum_{k=1}^{d} \frac{1}{2}\left(\|T_k x\|\|S_k x\| + |\langle T_k x, S_k x\rangle|\right) \quad (\textit{using Lemma 7.2}) \\
&\leq \frac{1}{4}\sum_{k=1}^{d}(\|T_k x\|^2 + \|S_k x\|^2) + \frac{1}{2}\sum_{i=1}^{d}|\langle T_k x, S_k x\rangle| \\
&= \frac{1}{4}\sum_{k=1}^{d}\left(\langle T_k^2 x, x\rangle + \langle S_k^2 x, x\rangle\right) + \frac{1}{2}\sum_{i=1}^{d}|\langle T_k x, S_k x\rangle| \\
&= \frac{1}{4}\sum_{k=1}^{d}\langle (T_k^2 + S_k^2)x, x\rangle + \frac{1}{2}\sum_{k=1}^{d}|\langle T_k x, S_k x\rangle| \\
&\leq \frac{1}{4}\left\langle \sum_{k=1}^{d}\left(T_k^2 + S_k^2\right)x, x\right\rangle + \frac{\sqrt{d}}{2}\left(\sum_{k=1}^{d}|\langle x, T_k S_k x\rangle|^2\right)^{\frac{1}{2}} (\textit{using Lemma 7.3}) \\
&\leq \frac{1}{4}\left\|\sum_{k=1}^{d}\left(T_k^2 + S_k^2\right)\right\| + \frac{\sqrt{d}}{2}w_e(\mathbf{TS})
\end{aligned}$$

Taking supremum over all $x \in \mathcal{H}$, $\|x\| = 1$ we obtain the desired bound. □

As an application of Theorem 7.1, we obtain the following bounds for the Euclidean operator radius of the product of two d-tuple operators.

Corollary 7.1. *Let* $\mathbf{S} = (S_1, S_2, \ldots, S_d), \mathbf{P} = (P_1, P_2, \ldots, P_d) \in \mathbb{B}^d(\mathcal{H})$, *then*
(i) $w_e(\mathbf{SP}) \leq \sqrt{\frac{1}{2}\left\|\sum_{k=1}^{d}\left(|S_k^*|^4 + |P_k|^4\right)\right\|}$.
(ii) $w_e(\mathbf{SP}) \leq \sqrt{\frac{1}{2}\|\mathbf{SS}^*\|\|\mathbf{P}^*\mathbf{P}\| + \frac{\sqrt{d}}{2}w_e(\mathbf{S}(\mathbf{PS})^*\mathbf{P})}$.
(iii) $w_e(\mathbf{SP}) \leq \sqrt{\frac{1}{4}\|\sum_{k=1}^{d}|S_k^*|^4 + |P_k|^4\| + \frac{\sqrt{d}}{2}w_e(\mathbf{S}(\mathbf{PS})^*\mathbf{P})}$.

Proof. Observe that $\begin{bmatrix} \mathbf{SS}^* & \mathbf{SP} \\ \mathbf{P}^*\mathbf{S}^* & \mathbf{P}^*\mathbf{P} \end{bmatrix}$ is positive. Using this positive operator matrix in Theorem 7.1, we obtain the desired inequalities. □

In the next results, we obtain an upper bound for the Euclidean operator radius of d-tuple operators.

Corollary 7.2. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$, *then*

$$w_e(\mathbf{T}) \leq \sqrt{\frac{1}{2}\left\|\sum_{k=1}^{d}\left(|T_k^*|^{4(1-t)} + |T_k|^{4t}\right)\right\|},$$

for all t, $0 \le t \le 1$. *In particular,*

$$w_e(\mathbf{T}) \le \sqrt{\frac{1}{2}\left\|\sum_{k=1}^{d}\left(|T_k^*|^2 + |T_k|^2\right)\right\|}.$$

Proof. Let $T_k = U_k|T_k|$ be the polar decomposition of T_k, for each $k = 1, 2, \ldots, d$. Considering $S_k = U_k|T_k|^{1-t}$ and $P_k = |T_k|^t$ in Corollary 7.1 (i), we get

$$w_e(\mathbf{T}) \le \sqrt{\frac{1}{2}\left\|\sum_{k=1}^{d}\left((U_k|T_k|^{2(1-t)}U_k^*)^2 + |A_k|^{4t}\right)\right\|}.$$

Since $U_k|T_k|^{2(1-t)}U_k^* = |T_k^*|^{2(1-t)}$, we obtain the first inequality. And the second inequality follows by considering $t = \frac{1}{2}$. □

Corollary 7.3. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$, *then*

$$\begin{aligned} w_e(\mathbf{T}) &\le \sqrt{\frac{1}{2}\left\||\mathbf{T}^*|^{2(1-t)}\right\| \||\mathbf{T}|^{2t}\| + \frac{\sqrt{d}}{2} w_e\left(|\mathbf{T}|^{2t}|\mathbf{T}^*|^{2(1-t)}\right)} \\ &= \sqrt{\frac{1}{2}\sqrt{\left\|\sum_{k=1}^{d}|T_k^*|^{4(1-t)}\right\|}\sqrt{\left\|\sum_{k=1}^{d}|T_k|^{4t}\right\|} + \frac{\sqrt{d}}{2} w_e\left(|\mathbf{T}|^{2t}|\mathbf{T}^*|^{2(1-t)}\right)}, \end{aligned}$$

for all t, $0 \le t \le 1$. *In particular,*

$$\begin{aligned} w_e(\mathbf{T}) &\le \sqrt{\frac{1}{2}\|\mathbf{T}^*\|\|\mathbf{T}\| + \frac{\sqrt{d}}{2} w_e(|\mathbf{T}||\mathbf{T}^*|)} \\ &= \sqrt{\frac{1}{2}\sqrt{\left\|\sum_{k=1}^{d}|T_k^*|^2\right\|}\sqrt{\left\|\sum_{k=1}^{d}|T_k|^2\right\|} + \frac{\sqrt{d}}{2} w_e(|\mathbf{T}||\mathbf{T}^*|)}. \end{aligned}$$

Proof. Let $T_k = U_k|T_k|$ be the polar decomposition of T_k for each $k = 1, 2, \ldots, d$. Considering $S_k = U_k|T_k|^{1-t}$ and $P_k = |T_k|^t$ in Corollary 7.1 (ii) and using similar arguments as Corollary 7.2, we obtain the desired bounds. □

Corollary 7.4. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$, *then*

$$w_e(\mathbf{T}) \le \sqrt{\frac{1}{4}\left\|\sum_{k=1}^{d}\left(|T_k^*|^{4(1-t)} + |T_k|^{4t}\right)\right\| + \frac{\sqrt{d}}{2} w_e\left(|\mathbf{T}|^{2t}|\mathbf{T}^*|^{2(1-t)}\right)},$$

for all t, $0 \leq t \leq 1$. *In particular,*

$$w_e(\mathbf{T}) \leq \sqrt{\frac{1}{4}\left\|\sum_{k=1}^{d}\left(|T_k^*|^2 + |T_k|^2\right)\right\| + \frac{\sqrt{d}}{2} w_e(|\mathbf{T}||\mathbf{T}^*|)}.$$

Proof. Let $T_k = U_k|T_k|$ be the polar decomposition of T_k for each $k = 1, 2, \ldots, d$. Considering $S_k = U_k|T_k|^{1-t}$ and $P_k = |T_k|^t$ in Corollary 7.1 (iii) and using the similar argument as Corollary 7.2, we get the desired results. □

To obtain our next result we need the following lemma.

Lemma 7.5. *[68] Let $S, T \in \mathbb{B}(\mathcal{H})$ be such that $|S|T = T^*|S|$. Let f and g be two non-negative continuous functions on $[0, \infty)$ such that $f(\lambda)g(\lambda) = \lambda$, for all $\lambda \in [0, \infty)$. Then*

$$|\langle STx, y\rangle| \leq r(T)\|f(|S|)x\|\|g(|S^*|)y\|,$$

for all $x, y \in \mathcal{H}$.

The following result provides an upper bound for the Euclidean operator radius of the product of two d-tuple operators.

Theorem 7.2. *Let $\mathbf{S} = (S_1, S_2, \ldots, S_d)$, $\mathbf{P} = (P_1, P_2, \ldots, P_d) \in \mathbb{B}^d(\mathcal{H})$ be such that $|\mathbf{S}|\mathbf{P} = \mathbf{P}^*|\mathbf{S}|$ (i.e., $|S_k|P_k = P_k^*|S_k|$ for all $k = 1, 2, \ldots, d$). If $f, g : [0, \infty) \to [0, \infty)$ are continuous function with $f(\lambda)g(\lambda) = \lambda$ for all $\lambda \geq 0$, then*

$$\begin{aligned} w_e(\mathbf{ST}) &\leq \frac{1}{\sqrt{2}} \max_k \{r(P_k)\}\, w_e\left(f^2(|\mathbf{S}|) + ig^2(|\mathbf{S}^*|)\right) \\ &\leq \frac{1}{\sqrt{2}} \max_k \{r(P_k)\} \sqrt{\left\|\sum_{k=1}^{d}\left(f^4(|S_k|) + g^4(|S_k^*|)\right)\right\|}. \end{aligned}$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then we get,

$$\begin{aligned}
\|\langle \mathbf{SP}x, x\rangle\|^2 &= \sum_{k=1}^{d} |\langle S_k P_k x, x\rangle|^2 \\
&\leq \sum_{k=1}^{d} r^2(P_k)\|f(|S_k|)x\|^2 \|g(|S_k^*|)x\|^2 \quad (\textit{using Lemma 7.5}) \\
&= \sum_{k=1}^{d} r^2(P_k)\langle f^2(|S_k|)x, x\rangle \langle g^2(|S_k^*|)x, x\rangle \\
&\leq \frac{1}{2}\sum_{k=1}^{d} r^2(P_k)\left(\langle f^2(|S_k|)x, x\rangle^2 + \langle g^2(|S_k^*|)x, x\rangle^2\right) \\
&= \frac{1}{2}\sum_{k=1}^{d} r^2(P_k)|\langle f^2(|S_k|)x, x\rangle + i\langle g^2(|S_k^*|)x, x\rangle|^2 \\
&\leq \frac{1}{2}\max_k\{r^2(P_k)\}\sum_{k=1}^{d} |\langle (f^2(|S_k|) + ig^2(|S_k^*|))x, x\rangle|^2 \\
&\leq \frac{1}{2}\max_k\{r^2(P_k)\} w_e^2(f^2(|\mathbf{S}|) + ig^2(|\mathbf{S}^*|)).
\end{aligned}$$

Therefore, taking supremum over all $x \in \mathcal{H}$, $\|x\| = 1$, we obtain the first inequality. Next, we see that

$$\begin{aligned}
& w_e^2(f^2(|\mathbf{S}|) + ig^2(|\mathbf{S}^*|)) \\
= & \sup_{\|x\|=1}\sum_{k=1}^{d}\left(\langle f^2(|S_k|)x, x\rangle^2 + \langle g^2(|S_k^*|)x, x\rangle^2\right) \\
\leq & \sup_{\|x\|=1}\sum_{k=1}^{d}\left(\langle f^4(|S_k|)x, x\rangle + \langle g^4(|S_k^*|)x, x\rangle\right) \quad (\textit{using Lemma 7.1}) \\
= & \sup_{\|x\|=1}\sum_{k=1}^{d}\langle (f^4(|S_k|) + g^4(|S_k^*|))x, x\rangle \\
= & \sup_{\|x\|=1}\left\langle \sum_{k=1}^{d}(f^4(|S_k|) + g^4(|S_k^*|))x, x\right\rangle \\
= & \left\|\sum_{k=1}^{d}(f^4(|S_k|) + g^4(|S_k^*|))\right\|,
\end{aligned}$$

which gives the second inequality. □

The inequalities in Theorem 7.2 include several Euclidean operator radius inequalities for d-tuple operators. Some of these are demonstrated in the following corollaries.

Corollary 7.5. *Let* $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$ *and let* $f, g : [0, \infty) \to [0, \infty)$ *be continuous* *functions, where* $f(\lambda)g(\lambda)=\lambda$ *for all* $\lambda\geq 0$. *Then*

$$\begin{aligned} w_e(\mathbf{T}) &\leq \frac{1}{\sqrt{2}}\max_k\left\{\|T_k\|^t\right\} w_e\left(f^2(|\mathbf{T}|^{1-t})+ig^2(|\mathbf{T}^*|^{1-t})\right)\\ &\leq \frac{1}{\sqrt{2}}\max_k\left\{\|T_k\|^t\right\}\sqrt{\left\|\sum_{k=1}^d f^4(|T_k|^{1-t})+g^4(|T_k^*|^{1-t})\right\|}\\ &\leq \frac{1}{\sqrt{2}}\|\mathbf{T}\|^t\sqrt{\left\|\sum_{k=1}^d f^4(|T_k|^{1-t})+g^4(|T_k^*|^{1-t})\right\|}, \end{aligned}$$

for all t, $0\leq t\leq 1$.

Proof. Let $T_k=U_k|T_k|$ be the polar decomposition of T_k for each $k=1,2,\ldots,d$. Considering $S_k=U_k|T_k|^{1-t}$ and $P_k=|T_k|^t$ in Theorem 7.2, we obtain

$$\begin{aligned} w_e(\mathbf{T}) &\leq \frac{1}{\sqrt{2}}\max_k\{r(|T_k^t|)\}w_e\left(f^2\left(|\mathbf{T}|^{1-t}\right)+ig^2\left(|\mathbf{T}^*|^{1-t}\right)\right)\\ &= \frac{1}{\sqrt{2}}\max_k\{\|T_k\|^t\}w_e\left(f^2\left(|\mathbf{T}|^{1-t}\right)+ig^2\left(|\mathbf{T}^*|^{1-t}\right)\right)\\ &\leq \frac{1}{\sqrt{2}}\max_k\{\|T_k\|^t\}\sqrt{\left\|\sum_{k=1}^d f^4\left(|T_k|^{1-t}\right)+g^4\left(|T_k^*|^{1-t}\right)\right\|}\\ &\leq \frac{1}{\sqrt{2}}\|\mathbf{T}\|^t\sqrt{\left\|\sum_{i=1}^d f^4\left(|T_k|^{1-t}\right)+g^4\left(|T_k^*|^{1-t}\right)\right\|}. \end{aligned}$$

Since $\|T_k\|\leq\|\mathbf{T}\|$ for all $k=1,2,\ldots,d$, the last inequality follows easily. This completes the proof. □

Remark 7.3. *Suppose* $\mathbf{T}=(T_1,T_2,\ldots,T_d)\in\mathbb{B}^d(\mathcal{H})$.
(i) Considering $f(\lambda)=\lambda^\alpha$ *and* $g(\lambda)=\lambda^{1-\alpha}$, $0\leq\alpha\leq 1$, *in Corollary 7.5, we get*

$$\begin{aligned} w_e(\mathbf{T}) &\leq \frac{1}{\sqrt{2}}\max_k\{\|T_k\|^t\}w_e\left(|\mathbf{T}|^{2\alpha(1-t)}+i|\mathbf{T}^*|^{2(1-\alpha)(1-t)}\right)\\ &\leq \frac{1}{\sqrt{2}}\max_k\{\|T_k\|^t\}\sqrt{\left\|\sum_{k=1}^d\left(|T_k|^{4\alpha(1-t)}+|T_k^*|^{4(1-\alpha)(1-t)}\right)\right\|}\\ &\leq \frac{1}{\sqrt{2}}\|\mathbf{T}\|^t\sqrt{\left\|\sum_{k=1}^d\left(|T_k|^{4\alpha(1-t)}+|T_k^*|^{4(1-\alpha)(1-t)}\right)\right\|}, \end{aligned}$$

for all α,t, $0\leq\alpha,t\leq 1$.

(ii) In particular, for $t = \frac{1}{2}$,

$$\begin{aligned} w_e(\mathbf{T}) &\leq \frac{1}{\sqrt{2}} \max_k \left\{ \|T_k\|^{1/2} \right\} w_e \left(|\mathbf{T}|^{\alpha} + i|\mathbf{T}^*|^{(1-\alpha)} \right) \\ &\leq \frac{1}{\sqrt{2}} \max_k \{ \|T_k\|^{1/2} \} \sqrt{\left\| \sum_{k=1}^{d} \left(|T_k|^{2\alpha} + |T_k^*|^{2(1-\alpha)} \right) \right\|} \\ &\leq \frac{1}{\sqrt{2}} \|\mathbf{T}\|^{1/2} \sqrt{\left\| \sum_{k=1}^{d} \left(|T_k|^{2\alpha} + |T_k^*|^{2(1-\alpha)} \right) \right\|}, \end{aligned}$$

for all α, $0 \leq \alpha \leq 1$.

(iii) In particular, for $\alpha = \frac{1}{2}$,

$$\begin{aligned} w_e(\mathbf{T}) &\leq \frac{1}{\sqrt{2}} \max_k \left\{ \|T_k\|^{1/2} \right\} w_e(|\mathbf{T}|^{\frac{1}{2}} + i|\mathbf{T}^*|^{\frac{1}{2}}) \\ &\leq \frac{1}{\sqrt{2}} \max_k \left\{ \|T_k\|^{1/2} \right\} \sqrt{\left\| \sum_{k=1}^{d} (|T_k| + |T_k^*|) \right\|} \\ &\leq \frac{1}{\sqrt{2}} \|\mathbf{T}\|^{1/2} \sqrt{\left\| \sum_{k=1}^{d} (|T_k| + |T_k^*|) \right\|}. \end{aligned}$$

Next, we obtain an upper bound for the Euclidean operator radius of product of two d-tuple operators.

Theorem 7.4. *Let* $\mathbf{S} = (S_1, S_2, \ldots, S_d)$, $\mathbf{P} = (P_1, P_2, \ldots, P_d) \in \mathbb{B}^d(\mathcal{H})$. *Then*

$$w_e(\mathbf{SP}) \leq \frac{1}{\sqrt{2}} w_e(|\mathbf{P}|^2 + i|\mathbf{S}^*|^2).$$

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Then we have

$$\begin{aligned}
\|\langle \mathbf{SP}x, x\rangle\|^2 &= \sum_{k=1}^{d} |\langle S_k P_k x, x\rangle|^2 = \sum_{k=1}^{d} |\langle P_k x, S_k^* x\rangle|^2 \\
&\leq \sum_{k=1}^{d} \|P_k x\|^2 \|S_k^* x\|^2 = \sum_{k=1}^{d} \langle |P_k|^2 x, x\rangle \langle |S_k^*|^2 x, x\rangle \\
&\leq \frac{1}{2}\sum_{k=1}^{d} \left(\langle |P_k|^2 x, x\rangle^2 + \langle |S_k^*|^2 x, x\rangle^2\right) \\
&= \frac{1}{2}\sum_{k=1}^{d} |\langle |P_k|^2 x, x\rangle + i\langle |S_k^*|^2 x, x\rangle|^2 \\
&= \frac{1}{2}\sum_{k=1}^{d} |\langle \left(|P_k|^2 + i|S_k^*|^2\right) x, x\rangle|^2 \\
&\leq \frac{1}{2} w_e^2(|\mathbf{P}|^2 + i|\mathbf{S}^*|^2).
\end{aligned}$$

Therefore, taking supremum over $\|x\| = 1$, we get desired result. □

Using Theorem 7.4, we obtain the following bounds.

Corollary 7.6. *Let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathbb{B}^d(\mathcal{H})$, then*

$$\begin{aligned}
w_e(\mathbf{T}) &\leq \frac{1}{\sqrt{2}} w_e(|\mathbf{T}|^{2t} + i|\mathbf{T}^*|^{2(1-t)}) \\
&\leq \frac{1}{\sqrt{2}} \left\| \sum_{k=1}^{d} \left(|T_k^*|^{4(1-t)} + |T_k|^{4t} \right) \right\|^{1/2},
\end{aligned}$$

for all t, $0 \leq t \leq 1$. In particular,

$$\begin{aligned}
w_e(\mathbf{T}) &\leq \frac{1}{\sqrt{2}} w_e(|\mathbf{T}| + i|\mathbf{T}^*|) \\
&\leq \frac{1}{\sqrt{2}} \left\| \sum_{k=1}^{d} \left(|T_k^*|^2 + |T_k|^2 \right) \right\|^{1/2}.
\end{aligned}$$

Proof. Let $T_k = U_k|T_k|$ be the polar decomposition of T_k for each $k = 1, 2, \ldots, d$. Considering $S_k = U_k|T_k|^{1-t}$ and $P_k = |T_k|^t$ in Theorem 7.4, we get the desired bounds. □

7.3 Euclidean operator radius of operator matrices

In this section, first we develop an upper bound for the Euclidean operator radius of $n \times n$ operator matrix whose entries are d-tuple operators.

Theorem 7.5. *Let* $\mathbb{T} = \left[\mathbf{T_{ij}}\right]_{n\times n}$ *be an* $n \times n$ *operator matrix, where* $\mathbf{T_{ij}} \in \mathbb{B}^d(\mathcal{H})$, $1 \leq i, j \leq n$. *If* $f, g : [0, \infty) \to [0, \infty)$ *are continuous functions, satisfy* $f(\lambda)g(\lambda) = \lambda$, *for all* $\lambda \in [0, \infty)$, *then*

$$w_e(\mathbb{T}) \leq w\left(\left[a_{ij}\right]_{n\times n}\right),$$

where $a_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) \quad \text{when } i{=}j \\ \sqrt{w_e\left(f^2(|\mathbf{T_{ji}}|) + g^2(|\mathbf{T^*_{ij}}|)\right) w_e\left(f^2(|\mathbf{T_{ij}}|) + g^2(|\mathbf{T^*_{ji}}|)\right)} \quad \text{when } i < j \\ 0 \quad \text{when } i > j. \end{cases}$

Proof. Let $\mathbf{T_{ij}} = (T^1_{ij}, T^2_{ij}, \ldots, T^d_{ij}) \in \mathbb{B}^d(\mathcal{H})$, $1 \leq i, j \leq n$, and $u = (x_1, x_2, \ldots, x_n) \in \oplus_{i=1}^n \mathcal{H}$

with $\|u\| = 1$, i.e., $\|x_1\|^2 + \|x_2\|^2 + \ldots + \|x_n\|^2 = 1$. Now,

$$\begin{aligned}
\|\langle \mathbb{T}u, u\rangle\| &= \left\| \sum_{i,j=1}^{n} \langle \mathbf{T_{ij}} x_j, x_i\rangle \right\| \\
&\leq \left\| \sum_{i=1}^{n} \langle \mathbf{T_{ii}} x_i, x_i\rangle \right\| + \left\| \sum_{\substack{i,j=1\\ i\neq j}}^{n} \langle \mathbf{T_{ij}} x_j, x_i\rangle \right\| \\
&\leq \sum_{i=1}^{n} \|\langle \mathbf{T_{ii}} x_i, x_i\rangle\| + \sum_{\substack{i,j=1\\ i<j}}^{n} \|\langle \mathbf{T_{ij}} x_j, x_i\rangle + \langle \mathbf{T_{ji}} x_i, x_j\rangle\| \\
&\leq \sum_{i=1}^{n} \|\langle \mathbf{T_{ii}} x_i, x_i\rangle\| + \sum_{\substack{i,j=1\\ i<j}}^{n} \left(\sum_{k=1}^{d} \left| \left\langle T_{ij}^k x_j, x_i \right\rangle + \left\langle T_{ji}^k x_i, x_j \right\rangle \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{i=1}^{n} \|\langle \mathbf{T_{ii}} x_i, x_i\rangle\| + \sum_{\substack{i,j=1\\ i<j}}^{n} \left(\sum_{k=1}^{d} \left(\left| \left\langle T_{ij}^k x_j, x_i \right\rangle \right| + \left| \left\langle T_{ji}^k x_i, x_j \right\rangle \right| \right)^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{i=1}^{n} w_e(\mathbf{T_{ii}}) \|x_i\|^2 \\
&+ \sum_{\substack{i,j=1\\ i<j}}^{n} \left(\sum_{k=1}^{d} \left(\left\| f(|T_{ij}^k|) x_j \right\| \left\| g(|T_{ij}^{k^*}|) x_i \right\| + \left\| f(|T_{ji}^k|) x_i \right\| \left\| g(|T_{ji}^{k^*}|) x_j \right\| \right)^2 \right)^{\frac{1}{2}} \qquad (7.2)
\end{aligned}$$

where the last inequality follows by using Lemma 7.5. Now, by Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\left(\sum_{k=1}^{d} \left(\left\| f(|T_{ij}^k|) x_j \right\| \left\| g(|T_{ij}^{k^*}|) x_i \right\| + \left\| f(|T_{ji}^k|) x_i \right\| \left\| g(|T_{ji}^{k^*}|) x_j \right\| \right)^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{d} \left\langle \left(f^2(|T_{ij}^k|) + g^2(|T_{ji}^{k^*}|) \right) x_j, x_j \right\rangle \left\langle \left(f^2(|T_{ji}^k|) + g^2(|T_{ij}^{k^*}|) \right) x_i, x_i \right\rangle \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{d} \left| \left\langle \left(f^2(|T_{ij}^k|) + g^2(|T_{ji}^{k^*}|) \right) x_j, x_j \right\rangle \right|^2 \right)^{\frac{1}{4}} \left(\sum_{k=1}^{d} \left| \left\langle \left(f^2(|T_{ji}^k|) + g^2(|T_{ij}^{k^*}|) \right) x_i, x_i \right\rangle \right|^2 \right)^{\frac{1}{4}} \\
&\qquad (\textit{using Cauchy-Schwarz inequality}) \\
&\leq w_e^{\frac{1}{2}} \left(f^2(|\mathbf{T_{ij}}|) + g^2(|\mathbf{T_{ji}^*}|) \right) w_e^{\frac{1}{2}} \left(f^2(|\mathbf{T_{ji}}|) + g^2(|\mathbf{T_{ij}^*}|) \right) \|x_j\| \|x_i\|.
\end{aligned}$$

Hence, from (7.2), we obtain that

$$\begin{aligned}
\|\langle \mathbb{T}u, u\rangle\| &\leq \sum_{i=1}^{n} w_e(\mathbf{T_{ii}})\|x_i\|^2 \\
&+ \sum_{\substack{i,j=1 \\ i<j}}^{n} w_e^{\frac{1}{2}}\left(f^2(|\mathbf{T_{ij}}|)+g^2(|\mathbf{T_{ji}^*}|)\right) w_e^{\frac{1}{2}}\left(f^2(|\mathbf{T_{ji}}|)+g^2(|\mathbf{T_{ij}^*}|)\right)\|x_j\|\|x_i\| \\
&= \langle T|u|, |u|\rangle\,,
\end{aligned}$$

where $|u| = \begin{bmatrix} \|x_1\| \\ \|x_2\| \\ \vdots \\ \|x_n\| \end{bmatrix} \in \mathbb{C}^n$ is an unit vector and $T = \left[a_{ij}\right]_{n\times n}$ is an $n \times n$ complex matrix with

$$a_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) \quad \textit{when } i{=}j \\ w_e^{\frac{1}{2}}\left(f^2(|\mathbf{T_{ji}}|)+g^2(|\mathbf{T_{ij}^*}|)\right) w_e^{\frac{1}{2}}\left(f^2(|\mathbf{T_{ij}}|)+g^2(|\mathbf{T_{ji}^*}|)\right) \quad \textit{when } i<j \\ 0 \quad \textit{when } i>j. \end{cases}$$

Therefore,

$$\|\langle \mathbb{T}u, u\rangle\| \;\leq\; \langle T|u|, |u|\rangle \leq w(T),$$

holds for all $u \in \oplus_{i=1}^{n}\mathcal{H}$ with $\|u\| = 1$. This completes the proof. □

By Considering $f(\lambda) = \lambda^{\alpha}$ and $g(\lambda) = \lambda^{(1-\alpha)}$, $0 \leq \alpha \leq 1$ in Theorem 7.5, we obtain the following corollary.

Corollary 7.7. *Let $\mathbb{T} = \left[\mathbf{T_{ij}}\right]_{n\times n}$ be an $n\times n$ operator matrix, where $\mathbf{T_{ij}} \in \mathbb{B}^d(\mathcal{H})$, $1 \leq i, j \leq n$. Then*

$$w_e(\mathbb{T}) \leq w\left(\left[b_{ij}\right]_{n\times n}\right),$$

where $b_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) \quad \textit{when } i{=}j \\ w_e^{\frac{1}{2}}\left(|\mathbf{T_{ji}}|^{\mathbf{2}\alpha}+|\mathbf{T_{ij}^*}|^{2(1-\alpha)}\right) w_e^{\frac{1}{2}}\left(|\mathbf{T_{ij}}|^{\mathbf{2}\alpha}+|\mathbf{T_{ji}^*}|^{2(1-\alpha)}\right) \quad \textit{when } i<j \\ 0 \quad \textit{when } i>j, \end{cases}$

for all α, $0 \leq \alpha \leq 1$. In particular,

$$w_e(\mathbb{T}) \leq w\left(\left[b'_{ij}\right]_{n\times n}\right),$$

$$\text{where } b'_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) & \text{when } i{=}j \\ \sqrt{w_e\left(|\mathbf{T_{ji}}| + |\mathbf{T^*_{ij}}|\right) w_e\left(|\mathbf{T_{ij}}| + |\mathbf{T^*_{ji}}|\right)} & \text{when } i < j \\ 0 & \text{when } i > j. \end{cases}$$

Since $w_e(\mathbf{T}) \leq \|\mathbf{T}\|$ for every $\mathbf{T} \in \mathbb{B}^d(\mathcal{H})$ (see (7.1)) and $w\left([a_{ij}]_{n\times n}\right) \leq w\left([b_{ij}]_{n\times n}\right)$, when $0 \leq a_{ij} \leq b_{ij}$ for all i, j, the following corollaries are immediate from Theorem 7.5 and Corollary 7.7, respectively.

Corollary 7.8. *Let $\mathbb{T} = \left[\mathbf{T_{ij}}\right]_{n\times n}$ be an $n\times n$ operator matrix, where $\mathbf{T_{ij}} \in \mathbb{B}^d(\mathcal{H})$, $1 \leq i, j \leq n$. If $f, g : [0,\infty) \to [0,\infty)$ are continuous functions, satisfy $f(\lambda)g(\lambda) = \lambda$, for all $\lambda \in [0,\infty)$, then*

$$w_e\left(\mathbb{T}\right) \leq w\left(\left[c_{ij}\right]_{n\times n}\right),$$

$$\text{where } c_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) & \text{when } i{=}j \\ \left\|f^2(|\mathbf{T_{ji}}|) + g^2(|\mathbf{T^*_{ij}}|)\right\|^{\frac{1}{2}} \left\|f^2(|\mathbf{T_{ij}}|) + g^2(|\mathbf{T^*_{ji}}|)\right\|^{\frac{1}{2}} & \text{when } i < j \\ 0 & \text{when } i > j. \end{cases}$$

Corollary 7.9. *Let $\mathbb{T} = \left[\mathbf{T_{ij}}\right]_{n\times n}$ be an $n\times n$ operator matrix, where $\mathbf{T_{ij}} \in \mathbb{B}^d(\mathcal{H})$, $1 \leq i, j \leq n$. Then*

$$w_e\left(\mathbb{T}\right) \leq w\left(\left[d_{ij}\right]_{n\times n}\right),$$

$$\text{where } d_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) & \text{when } i{=}j \\ \left\||\mathbf{T_{ji}}|^{\mathbf{2}\alpha} + |\mathbf{T^*_{ij}}|^{2(1-\alpha)}\right\|^{\frac{1}{2}} \left\||\mathbf{T_{ij}}|^{\mathbf{2}\alpha} + |\mathbf{T^*_{ji}}|^{2(1-\alpha)}\right\|^{\frac{1}{2}} & \text{when } i < j \\ 0 & \text{when } i > j, \end{cases}$$

for all α, $0 \leq \alpha \leq 1$. In particular,

$$w_e\left(\mathbb{T}\right) \leq w\left(\left[d'_{ij}\right]_{n\times n}\right),$$

$$\text{where } d'_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) & \text{when } i{=}j \\ \left\||\mathbf{T_{ji}}| + |\mathbf{T^*_{ij}}|\right\|^{\frac{1}{2}} \left\||\mathbf{T_{ij}}| + |\mathbf{T^*_{ji}}|\right\|^{\frac{1}{2}} & \text{when } i < j \\ 0 & \text{when } i > j. \end{cases}$$

Now, it is well known that if $B = [b_{ij}]_{n\times n}$ is an $n \times n$ complex matrix with $b_{ij} \geq 0$ for all $i, j = 1, 2, ..., n$, then

$$w(B) = w\left(\left[\tfrac{b_{ij}+b_{ji}}{2}\right]_{n\times n}\right),$$

see in [52, p. 44]. By employing this argument, the bounds in Corollary 7.7 and 7.9 can be written as in the following remarks, respectively.

Remark 7.6. *Let* $\mathbb{T} = \left[\mathbf{T_{ij}}\right]_{n\times n}$ *be an* $n \times n$ *operator matrix, where* $\mathbf{T_{ij}} \in \mathbb{B}^d(\mathcal{H})$, $1 \le i, j \le n$. *Then*

$$w_e(\mathbb{T}) \le w\left(\left[e_{ij}\right]_{n\times n}\right),$$

$$\text{where } e_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) & \text{when } i{=}j \\ \frac{1}{2} w_e^{\frac{1}{2}}\left(|\mathbf{T_{ji}}|^{2\alpha} + |\mathbf{T_{ij}^*}|^{2(1-\alpha)}\right) w_e^{\frac{1}{2}}\left(|\mathbf{T_{ij}}|^{2\alpha} + |\mathbf{T_{ji}^*}|^{2(1-\alpha)}\right) & \text{when } i \neq j. \end{cases}$$

for all α, $0 \le \alpha \le 1$. *In particular,*

$$w_e(\mathbb{T}) \le w\left(\left[e'_{ij}\right]_{n\times n}\right),$$

$$\text{where } e'_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) & \text{when } i{=}j \\ \frac{1}{2} w_e^{\frac{1}{2}}\left(|\mathbf{T_{ji}}| + |\mathbf{T_{ij}^*}|\right) w_e^{\frac{1}{2}}\left(|\mathbf{T_{ij}}| + |\mathbf{T_{ji}^*}|\right) & \text{when } i \neq j. \end{cases}$$

Remark 7.7. *Let* $\mathbb{T} = \left[\mathbf{T_{ij}}\right]_{n\times n}$ *be an* $n \times n$ *operator matrix, where* $\mathbf{T_{ij}} \in \mathbb{B}^d(\mathcal{H})$, $1 \le i, j \le n$. *Then*

$$w_e(\mathbb{T}) \le w\left(\left[f_{ij}\right]_{n\times n}\right),$$

$$\text{where } f_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) & \text{when } i{=}j \\ \frac{1}{2}\left\||\mathbf{T_{ji}}|^{2\alpha} + |\mathbf{T_{ij}^*}|^{2(1-\alpha)}\right\|^{\frac{1}{2}}\left\||\mathbf{T_{ij}}|^{2\alpha} + |\mathbf{T_{ji}^*}|^{2(1-\alpha)}\right\|^{\frac{1}{2}} & \text{when } i \neq j. \end{cases}$$

for all α, $0 \le \alpha \le 1$. *In particular,*

$$w_e(\mathbb{T}) \le w\left(\left[f'_{ij}\right]_{n\times n}\right),$$

$$\text{where } f'_{ij} = \begin{cases} w_e(\mathbf{T_{ij}}) & \text{when } i{=}j \\ \frac{1}{2}\left\||\mathbf{T_{ji}}| + |\mathbf{T_{ij}^*}|\right\|^{\frac{1}{2}}\left\||\mathbf{T_{ij}}| + |\mathbf{T_{ji}^*}|\right\|^{\frac{1}{2}} & \text{when } i \neq j. \end{cases}$$

Considering $n = 2$ in Remark 7.6 and Remark 7.7, we develop the following bounds for the Euclidean operator radius of 2×2 operator matrices whose entries are d-tuple operators.

Corollary 7.10. *Let* $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S} \in \mathbb{B}^d(\mathcal{H})$. *Then*

$$w_e\left(\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}\right) \le \frac{1}{2}\left(w_e(\mathbf{P}) + w_e(\mathbf{S}) + \sqrt{\left(w_e(\mathbf{P}) - w_e(\mathbf{S})\right)^2 + \beta^2}\right),$$

where $\beta = \sqrt{w_e\left((|\mathbf{Q}| + |\mathbf{R}^*|\right) w_e\left((|\mathbf{R}| + |\mathbf{Q}^*|\right)}$.

Corollary 7.11. *Let* $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S} \in \mathbb{B}^d(\mathcal{H})$. *Then*

$$w_e\left(\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}\right) \leq \frac{1}{2}\left(w_e(\mathbf{P}) + w_e(\mathbf{S}) + \sqrt{(w_e(\mathbf{P}) - w_e(\mathbf{S}))^2 + \gamma^2}\right),$$

where $\gamma = \||\mathbf{Q}| + |\mathbf{R}^*|\|^{\frac{1}{2}} \||\mathbf{R}| + |\mathbf{Q}^*|\|^{\frac{1}{2}}$.

In [59], we proved that $w_e(\mathbf{T}^n) \leq \sqrt{d}\ w_e^n(\mathbf{T})$ for every $\mathbf{T} \in \mathbb{B}^d(\mathcal{H})$ and for every positive integer n. Using this power inequality, we develop an upper bound for the joint numerical radius of the product of two d-tuple operators.

Theorem 7.8. *If* $\mathbf{P}$, $\mathbf{Q} \in \mathbb{B}^d(\mathcal{H})$, *then*

$$w_e(\mathbf{PQ}) \leq \frac{\sqrt{d}}{4} w_e\left((|\mathbf{P}| + |\mathbf{Q}^*|\right) w_e\left((|\mathbf{Q}| + |\mathbf{P}^*|\right).$$

Proof. Following [59, Lemma 3.1], we have

$$\begin{aligned} w_e(\mathbf{PQ}) \leq \max\{w_e(\mathbf{PQ}), w_e(\mathbf{QP})\} &= w_e\left(\begin{bmatrix} \mathbf{PQ} & 0 \\ 0 & \mathbf{QP} \end{bmatrix}\right) \\ &= w_e\left(\begin{bmatrix} 0 & \mathbf{P} \\ \mathbf{Q} & 0 \end{bmatrix}^2\right) \\ &\leq \sqrt{d}\ w_e^2\left(\begin{bmatrix} 0 & \mathbf{P} \\ \mathbf{Q} & 0 \end{bmatrix}\right). \end{aligned}$$

Using Corollary 7.10, we have

$$w_e(\mathbf{PQ}) \leq \frac{\sqrt{d}}{4} w_e((|\mathbf{P}| + |\mathbf{Q}^*|)) w_e((|\mathbf{Q}| + |\mathbf{P}^*|)).$$

□

Finally, using the above theorem we develop an upper bound for the Euclidean operator radius of d-tuple operators.

Corollary 7.12. *If* $\mathbf{T} \in \mathbb{B}^d(\mathcal{H})$, *then*

$$w_e(\mathbf{T}) \leq \frac{\sqrt{d}}{4} w_e\left(|\mathbf{T}|^{1-t} + |\mathbf{T}|^t\right) w_e\left(|\mathbf{T}|^t + |\mathbf{T}^*|^{1-t}\right),$$

for all t, $0 \leq t \leq 1$.

Proof. Suppose $\mathbf{T} = (T_1, T_2, \ldots, T_d)$, $\mathbf{P} = (P_1, P_2, \ldots, P_d)$, $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_d) \in \mathbb{B}^d(\mathcal{H})$. Let $T_k = U_k|T_k|$ be the polar decomposition of T_k for all $k = 1, 2, \ldots, d$. Considering $P_k = U_k|P_k|^{1-t}$ and $Q_k = |T_k|^t$ in Theorem 7.8, we get the desired result. □

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