Ricci solitons and CPE conjecture on some differentiable manifolds

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that thesis entitled "**Ricci solitons and CPE conjecture on some differentiable manifolds**" submitted by Sri **Sumanjit Sarkar** who got his name registered on 16th February, 2018 (Index no.: 59/18/Maths./25) for the award of Doctor of Philosophy (Science) degree of Jadavpur University, is absolutely based upon his own work under the supervision of **Prof. Arindam Bhattacharyya** and that neither this thesis nor any part of it has been submitted for either any degree/diploma or any other academic award anywhere before.

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Dedicated to

my parents

Manindra Kumar Sarkar and Sumita Sarkar

and

my beloved wife

Susmita

for their patience, support and love.

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Preface

The aim of this doctoral thesis is to study Ricci solitons and CPE conjecture within the framework of various differentiable manifolds. The thesis consists of **five chapters**. An introduction of the Differential Geometry, Ricci soliton and critical point equation (shortly CPE) conjecture are presented in **Chapter 1**.

In the **second chapter**, we consider 3-dimensional trans-Sasakian manifold of type (α, β) to admit a Ricci soliton and characterize the covariant derivative of potential vector field along the Reeb vector field as well as the nature of the soliton. Later, we initiate the study of $*$-η-Ricci soliton and gradient almost $*$-η-Ricci soliton within the framework of Kenmotsu manifold and obtain some characteristics of the manifold and the potential vector field. Finally we deliberate $*$-η-Ricci soliton admitting $(\kappa, \mu)'$-almost Kenmotsu manifold and proved that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

In the **third chapter**, we establish some results regarding conformal η-Ricci soliton and conformal Ricci soliton on $(\mathrm{LCS})_n$ manifold satisfying some curvature conditions such as ξ-conharmonically semi-symmetric, ξ-concircularly semi-symmetric and ξ-quasi-conformally semi-symmetric and obtain some results regarding the nature of the manifold as well as the nature of the structural vector field ξ. Later, we initiate the study of conformal η-Ricci soliton and almost conformal η-Ricci soliton within the framework of para-Sasakian manifold and we are able to find some attributes of the manifold, the scalar curvature of the manifold and the potential vector field of the soliton. Further, we evolve the characterizations of the Kenmotsu manifold and the nature of the potential vector field when the manifold satisfies $*$-conformal η-Ricci soliton and gradient almost $*$-conformal η-Ricci soliton. Eventually, we have contrived $*$-conformal η-Ricci soliton admitting $(\kappa, \mu)'$-almost Kenmotsu manifold and proved that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

In **fourth chapter**, we demonstrate the nature of the soliton if Kenmotsu manifold admits almost conformal Ricci soliton. We also able to observe some properties of the scalar curvature of the manifold and the soliton function, and potential vector field of the soliton. Then we prove that if an η-Einstein para-Kenmotsu manifold admits conformal Ricci soliton and $*$-conformal Ricci soliton, then it is Einstein. Further, we acquire that 3-dimensional para-cosymplectic manifold is Ricci flat if the manifold satisfies conformal Ricci soliton where the soliton vector field is conformal. Next, we evolve the nature of scalar curvature when the 3-dimensional trans-Sasakian manifold of type (α, β), provided $\alpha \neq 0$ satisfies $*$-conformal Ricci soliton.

In the **fifth chapter**, we study the critical point equation (shortly CPE) conjecture and $*$-critical point equation (shortly $*$-CPE) conjecture within the framework of various contact metric manifolds. First, it is proved that if a compact Sasakian manifold admits CPE, then either the manifold is Einstein or the potential function is harmonic in an open subset. Later, It is shown that if the manifold satisfies $*$-CPE then the manifold is η-Einstein. Later, we establish that Kenmotsu manifold satisfying the CPE either becomes an Einstein manifold or the derivative of potential function along characteristic vector field satisfy a certain relation on the distribution of η. Next, we study CPE on $(\kappa, \mu)'$-almost Kenmotsu manifold and obtained that the manifold is Einstein. In case of 3-dimensional trans-Sasakian manifold, we get that either the manifold becomes α-Sasakian or it becomes Einstein.

The thesis contains the subject matter of the following papers whose titles, journal information and chapterwise distribution are given below:

Authors	Title of paper and journal information	Chapter
Santu Dey, Sumanjit Sarkar and Arindam Bhattacharyya	"$*$-η-Ricci soliton and contact geometry", **Ricerche di Matematica,**, Published online, https://doi.org/10.1007/S11587-021-00667-0, (2021).	2
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Sumanjit Sarkar, Sampa Pahan and Arindam Bhattacharyya	"Conformal η-Ricci soliton on a type of $(LCS)_n$ Manifold", **Acta Universitatis Apulensis**, No. 63/2020, PP. 21-33, (2020)	3
Sumanjit Sarkar and Santu Dey	"$*$-Conformal η-Ricci soliton within the framework of Kenmotsu manifolds", Communicated.	3
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Sumanjit Sarkar, Santu Dey and Xiaomin Chen	"Certain results of conformal and $*$-conformal Ricci soliton on para-cosymplectic and para-Kenmotsu manifolds",**Filomat**, Vol 35, No. 15, PP 5001-5015, (2021).	4
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Sumanjit Sarkar, Sampa Pahan and Arindam Bhattacharyya	"Critical point equation within the framework of various contact metric manifolds", Communicated.	5
Sumanjit Sarkar	"A study of critical point equation and $*$-critical point equation on Sasakian manifold", Communicated.	5

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Introduction

1.1 Introduction to differentiable manifolds

The concept of a manifold is central to many parts of geometry and modern mathematical physics because it allows more complicated structures to be described and understood in terms of the relatively well-understood properties of Euclidean spaces. A manifold is defined as below,

Definition 1.1.1 (Manifold)**.** *[62] A topological space M is said to be a n-dimensional manifold if it is Hausdorff, second countable and each point of M has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^n$ i.e., locally Euclidean of dimension n.*

1-dimensional manifolds include lines and circles. 2-dimensional manifolds are also called surfaces. The unit n-sphere, n-dimensional real projective space are some examples of n-dimensional manifolds.

A *chart* on a n-dimensional manifold M is a pair (U, φ), where the domain U is an open subset of M and φ is a homeomorphism from U to an open subset of $\mathbb{R}^n$. An *atlas* is a collection of charts whose domains cover M. Moreover, an atlas $\mathcal{A}$ is called *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible to each other. A maximal smooth atlas defines a *differentiable structure* or *smooth structure* on a manifold.

In mathematics, a differentiable manifold is a type of manifold that is locally similar enough to a linear space to allow one to do calculus. One may apply ideas from calculus while working within the individual charts, since image of each chart lies within a linear space to which the usual rules of calculus apply. If the charts are smoothly compatible, then computations done in one chart are valid in any other differentiable chart.

Let M be a smooth manifold with or without boundary. A *Riemannian metric*, usually denoted by g, is a smooth, symmetric, covariant 2-tensor field on M that is positive definite at each point. The pair (M, g) is called a *Riemannian manifold*. A Riemannian metric is not the same thing as a metric in the sense of metric spaces, although the two concepts are related.

If g is a Riemannian metric on M, then for each point $p \in M$, the tensor g_p is an inner product on the tangent space of M at the point p, denoted by T_pM. We often define the real number $g_p(X, Y)$ by $\langle X, Y \rangle_g$ for $X, Y \in T_pM$. In any smooth local coordinates (x^i), the Riemannian metric g can be expressed as

$$g = g_{ij} dx^i dx^j,$$

where (g_{ij}) is a symmetric, positive definite matrix of smooth functions. It is well known that every smooth manifold with or without boundary admits a Riemannian metric. There may have enormous number of Riemannian metrics which can be defined in a manifold. For any point $p \in M$, we can define length or norm of a tangent vector and angle between two nonzero tangent vectors on T_pM using the Riemannian metric g.

In differential geometry, a *pseudo-Riemannian manifold* or a *semi-Riemannian manifold*, is a differentiable manifold M with a covariant 2-metric tensor g that is smooth, symmetric and everywhere nondegenerate. This is a generalization of a Riemannian manifold in which the requirement of positive-definiteness is relaxed. A well known result from linear algebra permits us to make a change of basis such that in the new base, g is represented by a diagonal matrix with -1 or 1 elements in the diagonal. If there are i, -1 elements and j, 1 elements in the diagonal, the tensor is said to have signature (i, j). The signature will be invariant in every connected component of M, but usually the restriction that it be a global invariant is added to the definition of a pseudo-Riemannian manifold. Unlike a Riemannian metric, some manifolds do not admit a pseudo-Riemannian metric.

Pseudo-Riemannian manifolds are crucial in Physics and in particular in General Theory of Relativity where space-time is modeled as a 4-pseudo Riemannian manifold with signature $(1, 3)$. Intuitively pseudo-Riemannian manifolds are generalizations of Minkowski's space just as a Riemannian manifold is a generalization of a vector space with a positive definite metric. The fundamental theorem of Riemannian geometry is true for pseudo-Riemannian manifolds as well. This allows one to speak of the Levi-Civita connection on a pseudo-Riemannian manifold along with the associated curvature tensor.

Definition 1.1.2 (Einstein manifold)**.** *[108] An* Einstein manifold *which is named after Albert Einstein, is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor is proportional to the metric. In other language, a Riemannian or pseudo-Riemannian manifold M is called an Einstein manifold if there exists some constant k such that $S = kg$, where the Ricci tensor of the manifold is denoted by S. Furthermore, if $k = 0$ then the manifold is called Ricci flat manifold.*

Definition 1.1.3 (Killing vector field)**.** *[108] A vector field X on a Riemannian or pseudo-Riemannian manifold is said to be Killing vector field if the Lie derivative with respect to X of the metric g vanishes, i.e., $\mathcal{L}_X g = 0$. Killing vector field is named after the German mathematician Wilhelm Karl Joseph Killing.*

Definition 1.1.4 (Koszul's formula)**.** *[29] If (M, g) is a Riemannian manifold (or pseudo-Riemannian manifold) with a Levi-Civita connection ∇, then for arbitrary vector fields X, Y and Z on M the Koszul's formula is defined by,*

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \tag{1.1.1}$$

We want to revisit some well known formulas from Yano[107] which are used extensively through out the entire thesis. On a Riemannian manifold (or semi-Riemannian manifold) (M, g) the following properties hold,

$$(\mathcal{L}_V \nabla)(X, Y) = \mathcal{L}_V \nabla_X Y - \nabla_X \mathcal{L}_V Y - \nabla_{[V,X]} Y, \tag{1.1.2}$$

$$(\mathcal{L}_V \nabla_Z g - \nabla_Z \mathcal{L}_V g - \nabla_{[V,Z]} g)(X, Y) = -g((\mathcal{L}_V \nabla)(X, Z), Y) - g((\mathcal{L}_V \nabla)(Y, Z), X), \tag{1.1.3}$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1.4}$$

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z), \tag{1.1.5}$$

where $X, Y, Z \in \chi(M)$ and R is the Riemannian curvature tensor.

Now we revisit some of the important contact and para-contact manifolds which form the basis of this thesis. These manifolds have been used broadly throughout the thesis.

1.1.1 Contact manifold

A differentiable manifold M of dimension $(2n + 1)$ is said to have an almost contact structure or (ϕ, ξ, η) structure if M admits a $(1, 1)$ tensor field ϕ, a vector field ξ, an 1-form η satisfying,

$$\phi^2 = -I + \eta \otimes \xi, \tag{1.1.6}$$

$$\eta(\xi) = 1, \tag{1.1.7}$$

where I is the identity mapping. Generally, ξ and η are called *characteristic vector field* or *Reeb vector field* and *almost contact 1-form* respectively (for more details we refer to [17], [85]).

A Riemannian metric g is said to be *compatible metric* if it satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.1.8}$$

for arbitrary vector fields X and Y on M. A manifold having almost contact structure along with compatible Riemannian metric is called *almost contact metric manifold*.

An almost contact metric manifold (M, ϕ, ξ, η, g) has the following properties,

$$\phi\xi = 0, \tag{1.1.9}$$

$$\eta \circ \phi = 0, \tag{1.1.10}$$

$$g(X, \xi) = \eta(X), \tag{1.1.11}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{1.1.12}$$

for arbitrary $X, Y \in \chi(M)$.

A $(2n + 1)$-dimensional manifold M is said to have a contact structure and it is called a contact manifold [86] if it carries a global 1-form η such that the volume form $\eta \wedge (d\eta)^n$ is non-zero everywhere on M. This 1-form η is called a contact form on M. For a contact form η there exists a global vector field ξ satisfying $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. This vector field ξ is called associated vector field to η. Every contact manifold M with contact form η admits an almost contact structure (ϕ, ξ, η, g) such that

$$g(X, \phi Y) = d\eta(X, Y), \tag{1.1.13}$$

hold for arbitrary vector fields X and Y on M.

The fundamental 2-form Φ is defined on an almost contact metric structure (ϕ, ξ, η, g) by $\Phi(X, Y) = g(X, \phi Y)$. An almost contact structure constructed from a contact form η is called contact metric structure associated to η and a manifold with such a structure is called a contact metric manifold. An almost contact metric structure with $\Phi = d\eta$ is a contact metric structure (for details see [47]).

An almost complex structure J is defined on $M \times \mathbb{R}$ where M is a $(2n+1)$-dimensional almost contact metric structure and $\mathbb{R}$ is the real line, by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where f is a smooth function on $M \times \mathbb{R}$. It is easy to verify that $J^2 = -I$. If J is integrable, we say that the almost contact structure is normal. The normality of an almost contact structure is equivalent with the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ (for more details we refer to [17]).

Sasakian manifold

Sasakian manifold is named after the great Japanese geometer Shigeo Sasaki and is defined as below.

Definition 1.1.5 (Sasakian manifold)**.** *If the contact structure of a differentiable manifold is normal, then the manifold is called Sasakian manifold or normal contact manifold.*

It is equivalent to say that, an almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is a Sasakian manifold if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.1.14}$$

holds for any vector fields X and Y of $\chi(M)$.

A contact manifold is called a *K-contact manifold* if the characteristic vector field ξ is Killing vector field. A Sasakian manifold is a K-contact manifold. The converse is also true but only for 3-dimensional manifold.

On a $(2n+1)$-dimensional Sasakian manifold the following relations hold ([10], [81], [40])

$$\nabla_X \xi = -\phi X, \tag{1.1.15}$$

$$(\nabla_X \eta)Y = g(X, \phi Y), \tag{1.1.16}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{1.1.17}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{1.1.18}$$
$$Q\phi = \phi Q, \tag{1.1.19}$$
$$S(X,\xi) = 2n\eta(X) \quad \Leftrightarrow \quad Q\xi = 2n\xi, \tag{1.1.20}$$
$$S(\phi X,\phi Y) = S(X,Y) - 2n\eta(X)\eta(Y), \tag{1.1.21}$$

for all $X, Y \in \chi(M)$, where ∇, R, Q, S are Levi-Civita connection with respect to the metric g, Riemannian curvature tensor, Ricci operator and Ricci tensor respectively.

Kenmotsu manifold

In 1969, S. Tanno[96] classified the connected almost contact metric manifolds whose automorphism groups have maximal dimensions as follows,

(i) Homogeneous normal contact Riemannian manifolds with constant ϕ-holomorphic sectional curvature if $k(\xi, X) > 0$;

(ii) Global Riemannian product of a line or a circle and a Kählerian manifold with constant holomorphic sectional curvature if $k(\xi, X) = 0$;

(iii) A warped product space $\mathbb{R} \times_f N$, where $\mathbb{R}$ is the real line and N is a Kählerian manifold, if $k(\xi, X) < 0$;

where $k(\xi, X)$ denotes the sectional curvature of the plane section containing the characteristic vector field ξ and an arbitrary vector field X.

In 1972, K. Kenmotsu in [58] obtained some tensor equations to characterize the manifolds of the third class using the wraping function $f(t) = ce^t$ on the interval $J = (-\epsilon, \epsilon)$. Since then the manifolds of the third class were called Kenmotsu manifolds. Conversely, every point on a Kenmotsu manifold has a neighbourhood which is locally a warped product $J \times_f N$, where f is given by the above mentioned relation.

Definition 1.1.6 (Almost Kenmotsu manifold). *An almost Kenmotsu manifold is an almost contact metric manifold where η is closed, i.e., $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$.*

Definition 1.1.7 (Kenmotsu manifold). *A normal almost Kenmotsu manifold is called Kenmotsu manifold.*

By [17], if in an almost contact metric manifold M the 1-form η and the (1,1)-tensor field ϕ satisfy the following condition for arbitrary $X, Y \in \chi(M)$

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{1.1.22}$$

then the manifold M is called a Kenmotsu manifold. It is easy to verify that the above mentioned relation is equivalent with the normality condition of the manifold.

In Kenmotsu manifold of dimension $(2n+1)$ the following relations hold,

$$\nabla_X \xi = X - \eta(X)\xi, \tag{1.1.23}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{1.1.24}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{1.1.25}$$

$$S(X, \xi) = -2n\eta(X) \Leftrightarrow Q\xi = -2n\xi, \tag{1.1.26}$$

$$\mathcal{L}_\xi g(X, Y) = 2g(X, Y) - 2\eta(X)\eta(Y), \tag{1.1.27}$$

for arbitrary $X, Y \in \chi(M)$, R is Riemannian curvature tensor, S is the Ricci tensor and $\mathcal{L}$ is the lie derivative operator.

$(\kappa, \mu)'$ almost Kenmotsu manifold

On an almost contact manifold we consider two $(1,1)$-type tensor fields $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ and an operator $\ell = R(., \xi)\xi$, where $\mathcal{L}_\xi \phi$ is the Lie derivative of ϕ along the direction ξ.

Some renowned mathematicians defined many nullity distributions on contact manifolds. In 1995, Blair et al. in [18] have defined (κ, μ)-nullity distribution on a contact metric manifold $M^{2n+1}(\phi, \xi\eta, g)$, for two real numbers κ and μ, by

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{Z \in T_pM | R(X, Y)Z = \kappa(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)hX - g(X, Z)hY)\},$$

for arbitrary vector fields X and Y on M. So, if the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, then

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

This nullity distribution is a generalization of κ-nullity distribution. If we consider $\mu = 0$, then (κ, μ)-nullity distribution reduces to κ-nullity distribution. A contact metric manifold whose characteristic vector field ξ belongs to κ-nullity distribution, i.e., the relation $R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y)$ holds, is called a $N(\kappa)$-contact metric manifold. A $N(\kappa)$-contact metric manifold is Sasakian if and only if $\kappa = 1$.

In 2009, Dileo and Pastore [35] first considered some nullity distributions like (κ,μ)-nullity distribution, $(\kappa,\mu)'$-nullity distribution on almost Kenmotsu manifold.

The tensor fields h and h' play vital roles in almost Kenmotsu manifold. Both of them are symmetric and satisfy the following relations,

$$\nabla_X \xi = X - \eta(X)\xi + h'X, \tag{1.1.28}$$

$$h\xi = h'\xi = 0, \tag{1.1.29}$$

$$h\phi = -\phi h, \tag{1.1.30}$$

$$tr(h) = tr(h') = 0,$$

for any $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection of the manifold M. In addition the following curvature property is also satisfied,

$$R(X,Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X, \tag{1.1.31}$$

where R is the Riemannian curvature tensor of (M, g).

Definition 1.1.8 ($(\kappa,\mu)'$-almost Kenmotsu manifold)**.** *An almost Kenmotsu manifold whose the characteristic vector field ξ satisfies the $(\kappa,\mu)'$-nullity distribution,*

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y), \tag{1.1.32}$$

for any $X, Y \in \chi(M)$, where κ and μ are real constants, is called $(\kappa,\mu)'$-almost Kenmotsu manifold.

On a $(\kappa,\mu)'$-almost Kenmotsu manifold M we have (see [35]),

$$h'^2(X) = -(\kappa + 1)[X - \eta(X)\xi], \tag{1.1.33}$$

$$h^2(X) = -(\kappa + 1)[X - \eta(X)\xi], \tag{1.1.34}$$

for $X \in \chi(M)$. From previous relation it follows that $h' = 0$ if and only if $\kappa = -1$ and $h' \neq 0$ otherwise. Let $X \in Ker(\eta)$ be an eigenvector field of h' orthogonal to ξ with respect to the eigenvalue α. Then, from (1.1.33) we get $\alpha^2 = -(\kappa + 1)$ which implies $\kappa \leq -1$. Dileo and Pastore proved that on a $(\kappa,\mu)'$-almost Kenmotsu manifold with

$\kappa < -1$, we have $\mu = -2$ (Proposition 4.1 of [35]). We use $(\kappa, -2)'$-almost Kenmotsu manifold throughout.

We recall some useful results on a $(2n+1)$ dimensional $(\kappa, -2)'$-almost Kenmotsu manifold M with $\kappa < -1$ as follows,

$$\begin{aligned} R(\xi, X)Y &= \kappa(g(X,Y)\xi - \eta(Y)X) - 2(g(h'X,Y)\xi - \eta(Y)h'X), \quad (1.1.35) \\ QX &= -2nX + 2n(\kappa+1)\eta(X)\xi - 2nh'(X), \quad (1.1.36) \\ r &= 2n(\kappa - 2n), \quad (1.1.37) \\ (\nabla_X \eta)Y &= g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y), \quad (1.1.38) \\ (\nabla_X h')Y &= -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X), \quad (1.1.39) \end{aligned}$$

where $X, Y \in \chi(M)$, Q, r are the Ricci operator and scalar curvature of M respectively.

Trans-Sasakian manifold

An almost contact metric manifold M is called a trans-Sasakian manifold if $(M \times \mathbb{R}, J, G)$, where G is the product metric on $M \times \mathbb{R}$, belongs to the class W_4 (see [46]).

Definition 1.1.9 (Trans-Sasakian manifold)**.** *An almost contact manifold $M(\phi, \xi, \eta, g)$ is called trans-Sasakian manifold of type (α, β) if there are smooth functions α, β satisfying,*

$$(\nabla_X \phi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \quad (1.1.40)$$

where $X, Y \in \chi(M)$ are arbitrary.

α, β are called structure functions of the manifold. Trans-Sasakian manifolds of type $(0,0), (\alpha,0), (0,\beta)$ are called cosymplectic, α-Sasakian, β-Kenmotsu manifolds respectively. Form (1.1.40) we can deduce that,

$$\nabla_X \xi = -\alpha(\phi X) + \beta(X - \eta(X)\xi). \quad (1.1.41)$$

Marrero [65] showed that a trans-Sasakian manifold of dimension ≥ 5 is either cosymplectic or α-Sasakian or β-Kenmotsu. So proper trans-Sasakian manifold exists for dimension

3. In a 3-dimensional trans-Sasakian manifold the following relations hold,

$$\begin{aligned}R(X,Y)Z =&(\frac{r}{2}+2\xi\beta-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y)-g(Y,Z)[(\frac{r}{2}+\xi\beta-\\&3(\alpha^2-\beta^2))\eta(X)\xi-\eta(X)(\phi D\alpha-D\beta)+(X\beta+(\phi X)\alpha)\xi]+g(X,Z)\\&[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\xi-\eta(Y)(\phi D\alpha-D\beta)+(Y\beta+(\phi Y)\alpha)\xi)]\\&-[(Z\beta+(\phi Z)\alpha)\eta(Y)+(Y\beta+(\phi Y)\alpha)\eta(Z)+(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\\&\eta(Y)\eta(Z)]X+[(Z\beta+(\phi Z)\alpha)\eta(X)+(X\beta+(\phi X)\alpha)\eta(Z)+(\frac{r}{2}+\xi\beta\\&-3(\alpha^2-\beta^2))\eta(X)\eta(Z)]Y, \qquad (1.1.42)\end{aligned}$$

$$\begin{aligned}S(X,Y) =&(\frac{r}{2}+\xi\beta-(\alpha^2-\beta^2))g(X,Y)-(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Y))-\\&(Y\beta+(\phi Y)\alpha)\eta(X)-(X\beta+(\phi X)\alpha)\eta(Y), \qquad (1.1.43)\end{aligned}$$

where Df denotes the gradient of the smooth function f and α, β are smooth functions on the manifold (for details see [72]).

If we restrict the smooth functions α, β to be constant functions ([37]), then we got some special relations compatible to our restrictions,

$$R(X,Y)\xi = (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y), \qquad (1.1.44)$$

$$S(X,Y) = (\frac{r}{2}-(\alpha^2-\beta^2))g(X,Y)-(\frac{r}{2}-3(\alpha^2-\beta^2))\eta(X)\eta(Y)). \qquad (1.1.45)$$

1.1.2 Para-contact manifold

The notion of almost para-contact manifold was first introduced by Sato [87]. Later Kaneyuki and Williams [57] associated pseudo-Riemannian metric with an almost para-contact manifold after Takahashi [95] intoduced pseudo-Riemannian metric in contact manifold, in particular, in Sasakian manifold.

A $(2n+1)$-dimensional smooth manifold M is said to have an almost para-contact structure if it admits a vector field ξ, (1,1)-tensor field ϕ and an 1-form η satisfying the following conditions

i) $\phi^2 = I - \eta\otimes\xi$, (1.1.46)

ii) $\eta(\xi) = 1$, (1.1.47)

iii) ϕ induces on the 2n-dimensional distribution $\mathcal{D} \equiv ker(\eta)$, an almost paracomplex structure $\mathcal{P}$ i.e., $\mathcal{P}^2 \equiv I_{\chi(M)}$ and the eigensubbundles $\mathcal{D}^+$ and $\mathcal{D}^-$, corresponding to the eigenvalues 1, -1 of $\mathcal{P}$ respectively, have equal dimensions n; hence $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$.

The vector field ξ is called characteristic vector field or Reeb vector field. An immediate consequence of those relations are

$$\phi\xi = 0, \tag{1.1.48}$$

$$\eta \circ \phi = 0. \tag{1.1.49}$$

The tensor field ϕ induces an almost paracomplex structure on each fibre of $Ker(\eta)$ i.e., the eigendistributions corresponding to eigenvalues 1 and -1 have same dimension n.

Zamkovoy in [109] proved that any almost para-contact structure admits a pseudo-Riemannian metric. If a manifold with an almost para-contact structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{1.1.50}$$

holds for any $X, Y \in \chi(M)$, then g is called compatible metric and the manifold (M, ϕ, ξ, η, g) is called almost para-contact metric manifold.

The fundamental 2-form Φ is defined on an almost para-contact metric manifold (M, ϕ, ξ, η, g) by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M. Clearly the skew-symmetricness of the 2-form Φ inherits from ϕ. An almost para-contact metric manifold for which

$$\Phi(X, Y) = d\eta(X, Y) = g(X, \phi Y), \tag{1.1.51}$$

is said to be para-contact metric manifold. In this case, η becomes a contact form i.e., $\eta \wedge (d\eta)^n \neq 0$. On a para-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ we consider a self-adjoint operator $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}_\xi$ denotes the Lie derivative along ξ. This operator h is symmetric and satisfies

$$h\phi = -\phi h, \tag{1.1.52}$$

$$h\xi = 0, \tag{1.1.53}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{1.1.54}$$

where ∇ is the operator of covariant differentiation w.r.t. the metric g. The normality of a para-contact metric manifold (M, ϕ, ξ, η, g) is equivalent to vanishing of the $(1, 2)$-

torsion tensor defined by $N_\phi(X,Y) = [\phi,\phi](X,Y) - 2d\eta(X,Y)\xi$, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[X,\phi Y] - \phi[\phi X,Y]$ for any $X, Y \in \chi(M)$.

Para-Kenmotsu manifold

On the analogy of Kenmotsu manifold, Welyczko [104] introduced the notion of para-Kenmotsu manifold (in short p-Kenmotsu manifold).

Definition 1.1.10 (almost para-Kenmotsu manifold). *If an almost para-contact metric manifold satisfies*

$$(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X, \tag{1.1.55}$$

for arbitrary vector fields X and Y, then the manifold is called almost para-Kenmotsu manifold.

Definition 1.1.11 (para-Kenmotsu manifold). *A normal almost para-Kenmotsu manifold is called para-Kenmotsu manifold.*

The following properties hold on a $(2n+1)$-dimensional para-Kenmotsu manifold

$$\nabla_X\xi = X - \eta(X)\xi, \tag{1.1.56}$$

$$(\nabla_X\eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{1.1.57}$$

$$Q\xi = -2n\xi, \tag{1.1.58}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{1.1.59}$$

$$(\mathcal{L}_\xi g)(X,Y) = 2[g(X,Y) - \eta(X)\eta(Y)], \tag{1.1.60}$$

for any $X, Y \in \chi(M)$ where, $\mathcal{L}$ and ∇ are the operators of Lie differentiation and covariant differentiation of g respectively. Q denotes the Ricci operator associated with the Ricci tensor S defined by $S(X,Y) = g(QX,Y)$ and R denotes the Riemannian curvature tensor.

Para-Sasakian manifold

Sato and Matsumoto [88] defined and studied a para-Sasakian manifold (in short p-Sasakian manifold) as special case of an almost paracontact manifold. Adati et al. [1] deduced some fundamental properties of para-Sasakian manifold.

Definition 1.1.12 (para-Sasakian manifold). *A normal para-contact metric manifold is called a para-Sasakian metric manifold.*

It is equivalent to say, an almost para-contact metric manifold is called a para-Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{1.1.61}$$

for arbitrary $X, Y \in \chi(M)$. In a para-Sasakian manifold the operator h vanishes and the manifold satisfies,

$$\nabla_X \xi = -\phi X, \tag{1.1.62}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{1.1.63}$$

$$Q\xi = -2n\xi, \tag{1.1.64}$$

for all vector fields X and Y on M and R, Q denote Riemannian curvature tensor and Ricci operator associated with the Ricci tensor S defined by $S(X,Y) = g(QX,Y)$.

Para-cosymplectic manifold

In 2004, Dacko [26] introduced the notion of para-cosymplectic manifold.

Definition 1.1.13 (para-cosymplectic manifold). *An almost para-contact metric manifold is said to be almost para-cosymplectic if the forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$ respectively. In addition if the normality of almost para-cosymplectic manifold is fulfilled then the it is called para-cosymplectic manifold.*

Equivalently we can say an almost para-contact metric manifold is para-cosymplectic if the forms η and Φ are parallel with respect to the corresponding Levi-Civita connection ∇ of the metric g i.e., $\nabla\eta = 0$ and $\nabla\Phi = 0$ respectively. We recall some useful relations which are satisfied for any para-cosymplectic manifold.

$$R(X,Y)\xi = 0, \tag{1.1.65}$$

$$(\nabla_X \phi) = 0, \tag{1.1.66}$$

$$\nabla_X \xi = 0, \tag{1.1.67}$$

$$Q\xi = 0, \tag{1.1.68}$$

where X is an arbitrary vector field and R, ∇, S and Q are the usual notations mentioned earlier.

Lorentzian concircular structure manifold

The notion of Lorentzian concircular structure manifold, briefly $(\text{LCS})_n$ manifold, is first introduced in 2003 by Shaikh (for details see [89]). A n-dimensional smooth connected para-contact Hausdorff manifold is called a *Lorentzian manifold* if it admits a Lorentzian metric. Lorentzian metric is named after great Dutch Physicist Hendrik Lorentz. A *Lorentzian metric tensor* g is a smooth symmetric tensor field of type (0,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a non degenerate inner product of signature (−,+,......,+) , where T_pM is the tangent space of M at p and $\mathbb{R}$ is the real number space. A non-zero tangent vector $v \in T_pM$ is said to be *timelike, non-spacelike, null* or *spacelike* if it satisfies $g_p(v, v) < 0, \leq 0, = 0$ or > 0 respectively.

In a Lorentzian manifold (M, g) a vector field ρ is defined by $g(X, \rho) = \eta(X)$, is said to be *concircular vector field* if,

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \omega(X)\eta(Y)\},$$

is satisfied where α is a non-zero scalar field, ω is a closed 1-form and ∇ is the covariant derivative operator w.r.t. Lorentzian metric g.

Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called *charecteristic vector field* or the *generator* of the manifold, then we have,

$$g(\xi, \xi) = -1.$$

Since ξ is a concircular vector field there must exists a non-zero 1-form η, such that,

$$g(X, \xi) = \eta(X), \tag{1.1.69}$$

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \tag{1.1.70}$$

hold for arbitrary vector fields $X, Y \in \chi(M)$ and α is a non-zero scalar field which satisfies,

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X).$$

ρ being certain scalar function which is given by $\rho = -(\xi\alpha)$. If we define $\phi X = \frac{1}{\alpha}\nabla_X \xi$, then from (1.1.70), we can deduce that

$$\phi^2 X = X + \eta(X)\xi. \tag{1.1.71}$$

Clearly ϕ is a symmetric $(1, 1)$ tensor which is called *structure tensor* of the manifold.

Definition 1.1.14 (Lorentzian concircular structure manifold). *A n-dimensional Lorentzian manifold M together with the unit timelike concircular vector field ξ, 1-form η and $(1,1)$ tensor ϕ is said to be Lorentzian concircular structure (briefly $(LCS)_n$) manifold.*

If we take $\alpha = 1$, then the manifold reduces to LP-Sasakian manifold of Matsumoto [67].

A $(\text{LCS})_n$ manifold satisfies the following properties,

$$\eta(\xi) = -1, \tag{1.1.72}$$

$$\eta \circ \phi = 0, \tag{1.1.73}$$

$$\phi\xi = 0, \tag{1.1.74}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{1.1.75}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)(\eta(Y)X - \eta(X)Y), \tag{1.1.76}$$

$$(\mathcal{L}_\xi g)(X, Y) = 2\alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \tag{1.1.77}$$

for arbitrary vector fields X and Y on M, where R is the Riemannian curvature tensor.

Here, we want to evoke some useful important definitions,

Definition 1.1.15 (η-Einstein manifold). *[108] An almost contact (or almost para-contact) metric manifold M is said to be η-Einstein manifold if there exists two constants a and b which satisfies the following relation,*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{1.1.78}$$

for all $X, Y \in \chi(M)$. Clearly if $b = 0$ then η-Einstein manifold reduces to Einstein manifold.

Definition 1.1.16 (Contact vector field). *[44] A vector field X on a contact manifold is said to be a contact vector field if it preserve the contact form η i.e., if there exist a smooth function f such that $\mathcal{L}_X\eta = f\eta$. When $f = 0$ on M, the vector field X is called a strict contact vector field.*

Definition 1.1.17 (Infinitesimal contact transformation). *[108] On an almost contact (or almost para-contact) metric manifold M, a vector field X is said to be infinitesimal contact transformation if $\mathcal{L}_X\xi = f\xi$, for some smooth function f. In particular, we call X as a strict infinitesimal contact transformation if $\mathcal{L}_X\xi = 0$.*

Definition 1.1.18 (Torse forming vector field). *A vector field ξ is called torse forming if it satisfies*

$$\nabla_X \xi = fX + \gamma(X)\xi, \tag{1.1.79}$$

for a smooth function $f \in C^\infty(M)$, 1-form γ and for all vector field X on M. A torse forming vector field is called recurrent if $f = 0$.

Definition 1.1.19 (Conformal vector field). *[108] On an almost contact (or almost para-contact) metric manifold M, a vector field V is said to be conformal Killing vector field or simply conformal vector field if there is a smooth function ρ such that*

$$\mathcal{L}_V g = \rho g.$$

ρ is called the conformal coefficient. If we consider the conformal coefficient ρ to be zero then the conformal vector field reduces to Killing vector field.

1.2 Introduction to Ricci solitons

A Riemannian manifold (or pseudo-Riemannian manifold) (M, g) is said to admit a Ricci soliton, which is a generalization of Einstein metric (i.e, $S = ag$ for some constant a), if there exists a smooth non-zero vector field V and a constant λ such that,

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0, \tag{1.2.80}$$

where $\mathcal{L}_V$ denotes Lie derivative along the direction V and S denotes the Ricci curvature tensor of the manifold. The vector field V is called potential vector field and λ is called soliton constant.

The Ricci soliton is a self-similar solution of the Hamilton's Ricci flow [49] which is defined by the equation $\frac{\partial g(t)}{\partial t} = -2S(g(t))$ with initial condition $g(0) = g$, where $g(t)$ is a one-parameter family of metrices on M. The potential vector field V and soliton constant λ play vital roles while determining the nature of the soliton. A soliton is said to be shrinking, steady or expanding according as $\lambda < 0$,$\lambda = 0$ or $\lambda > 0$. Now if V is zero or Killing then the Ricci soliton reduces to Einstein manifold and the soliton is called trivial soliton.

If the potential vector field V is the gradient of a smooth function f, denoted by Df then the soliton equation reduces to,

$$Hess_f + S + \lambda g = 0, \tag{1.2.81}$$

where $Hess_f$ is Hessian of the smooth function f. Perelman [76] proved that a Ricci soliton on a compact manifold is a gradient Ricci soliton.

In 2014, Kaimakamis and Panagiotidou [56] modified the definition of Ricci soliton where they have used $*$-Ricci tensor S^* which was introduced by Tachibana [94] and Hamada [48] respectively, in place of Ricci tensor S. The $*$-Ricci tensor S^* is defined by

$$S^*(X,Y) = \frac{1}{2}(trace\{\phi.R(X,\phi Y)\})$$

for all vector fields X and Y on M. They have used the concept of $*$-Ricci soliton within the framework of real hypersurfaces of a complex space form. A pseudo-Riemannian metric g is called a $*$-Ricci soliton if there exists a constant λ and a vector field V such that

$$\mathcal{L}_V g + 2S^* + 2\lambda g = 0. \tag{1.2.82}$$

Note that, $*$-Ricci soliton is trivial if the vector field V is Killing, and in this case, the manifold becomes $*$-Einstein. By $*$-Einstein manifold we mean a manifold whose $*$-Ricci tensor is proportional to the metric. Thus, $*$-Ricci soliton is considered as a natural generalization of $*$-Einstein metric. A $*$-Ricci soliton is said to be almost $*$-Ricci soliton if λ is a smooth function on M. Moreover, an almost $*$-Ricci soliton is called shrinking, steady and expanding according to as λ is positive, zero and negative, respectively.

In [56], it was studied that a real hypersurfaces of a non-flat complex space form admitting a $*$-Ricci soliton whose potential vector field is the structure vector field and was proved that a real hypersurface in a complex projective space does not admit a $*$-Ricci soliton. They have also shown that a real hypersurface of complex hyperbolic space admitting a $*$-Ricci soltion is locally congruent to a geodesic hypersphere.

In 2005, Fischer [38] has introduced conformal Ricci flow which is a mere generalisation of the classical Ricci flow equation that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation was given by,

$$\begin{aligned} \frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) &= -pg, \\ r(g) &= -1, \end{aligned}$$

where $r(g)$ is the scalar curvature of the manifold, p is scalar non-dynamical field and n is the dimension of the manifold. Corresponding to the conformal Ricci flow equation in 2015, Basu and Bhattacharyya [9] introduced the notion of conformal Ricci soliton equation as a generalization of Ricci soliton equation given by,

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{1.2.83}$$

If we consider the potential vector field V to be gradient of a smooth function f defined on M, then the soliton becomes gradient conformal Ricci soliton and is defined by,

$$Hess_f + S + [\lambda - (\frac{p}{2} + \frac{1}{n})]g = 0. \tag{1.2.84}$$

Replacing the Ricci tensor S by $*$-Ricci tensor S^* in the previous two equations, we get the equations for $*$-conformal Ricci soliton and gradient $*$-conformal Ricci soliton, respectively and these equations are given by,

$$\mathcal{L}_V g + 2S^* + [2\lambda - (p + \frac{2}{n})]g = 0, \tag{1.2.85}$$

$$Hess_f + S^* + [\lambda - (\frac{p}{2} + \frac{1}{n})]g = 0. \tag{1.2.86}$$

Furthermore if λ is considered to be a smooth function then the above mentioned solitions are called almost solitons.

In 2009, Cho and Kimura [24] introduced the concept of η-Ricci soliton which is another generalization of classical Ricci soliton and is given by,

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.2.87}$$

where μ is a real constant, η is an 1-form defined as $\eta(X) = g(X, \xi)$ for any $X \in \chi(M)$. Clearly it can be noted that if $\mu = 0$ then the η-Ricci soliton reduces to Ricci soliton.

In 2020, S. Dey et al. [31] defined $*$-η-Ricci soliton as

$$\mathcal{L}_\xi g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0.$$

The results concerning $*$-η-Ricci soliton were studied when the potential vector field V is the characteristic vector field ξ. Motivated from this we generalize the definition by considering the potential vector field as arbitrary vector field V and define as,

$$\mathcal{L}_V g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0. \tag{1.2.88}$$

Now if we consider the potential vector field V as the gradient of a smooth function f, then the $*$-η-Ricci soliton equation can be rewritten as

$$Hessf + S^* + \lambda g + \mu\eta \otimes \eta = 0. \tag{1.2.89}$$

Recently Siddiqi [92] established the notion of conformal η-Ricci soliton which generalizes both conformal Ricci soliton and η-Ricci soliton. The equation for conformal η-Ricci soliton is given by,

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0. \tag{1.2.90}$$

In the foregoing equation if we consider the soliton vector field as a gradient of a smooth function f, then the soliton equation changes to

$$Hess_f + S + [\lambda - (\frac{p}{2} + \frac{1}{n})]g + \mu\eta \otimes \eta = 0, \tag{1.2.91}$$

and the soliton is called gradient conformal η-Ricci soliton.

Again by replacing Ricci tensor S by $*$-Ricci tensor S^* we can define $*$-conformal η-Ricci soliton and gradient $*$-conformal η-Ricci soliton by,

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0, \tag{1.2.92}$$

$$Hess_f + S + [\lambda - (\frac{p}{2} + \frac{1}{n})]g + \mu\eta \otimes \eta = 0. \tag{1.2.93}$$

All the solitons related to η-Ricci soliton are called almost solitons if we consider λ and μ to be smooth functions.

1.3 Introduction to critical point equation

Let M be a n-dimensional compact orientable manifold of unit volume and $\mathcal{M}$ be the set of all Riemannian metrics defined on M. The Einstein-Hilbert functional is the total scalar curvature functional restricted on $\mathcal{M}$, $\mathcal{S} : \mathcal{M} \to \mathbb{R}$, defined by,

$$\mathcal{S}(g) = \int_{\mathcal{M}} r_g dv_g,$$

where r_g is the scalar curvature and dv_g is the volume form determined by the metric and orientation. It is well known that the critical points of $\mathcal{S}$ are Einstein metrics (for details see chapter 2 of [13]).

The classical Yamabe problem ensures the existence of many Riemannian metrics with constant scalar curvature. Let us assume a subset $\mathcal{C}$ of $\mathcal{M}$ with constant scalar curvature. Restricting the Einstein-Hilbert functional to $\mathcal{C}$, we can restate the above equation as,

$$Hess_g\lambda - (\Delta_g\lambda)g - \lambda S_g = S_g - \frac{r_g}{n}g, \tag{1.3.94}$$

where λ is a smooth function and $Hess_g$, Δ_g, S_g stand for Hessian, Laplacian and Ricci tensor corresponding to the Riemannian metric g, respectively. The smooth function λ is called the potential function. Tracing the equation (1.3.94), we obtain $\Delta_g\lambda = -\frac{r_g\lambda}{n-1}g$. Using this relation we can rewrite (1.3.94) as,

$$Hess_g\lambda + (\frac{r_g}{n-1}g - S_g)\lambda = S_g - \frac{r_g}{n}g. \tag{1.3.95}$$

If we consider the potential function λ to be constant in (1.3.95), then g becomes Einstein. So, to obtain non-trivial solution of (1.3.95), we consider the potential function to be non-constant. The critical point equation (shortly CPE) is defined by,

Definition 1.3.1 (Critical point equation)**.** *A compact oriented Riemannian metric manifold (M^n, g) of unit volume and dimension $n \geq 3$ with constant scalar curvature together with non-constant smooth potential function λ satisfying the equation* (1.3.95) *is called a critical point equation.*

In 2019, Dey and Majhi [32] modified the definition of critical point equation where they have used $*$-Ricci tensor S^* in place of Ricci tensor S. The $*$-critical point equation on a $(2n+1)$-dimensional manifold is given by [32],

$$Hess_g\lambda - (S^* - \frac{r^*}{2n}g)\lambda = S^* - \frac{r^*}{2n+1}g, \tag{1.3.96}$$

where r^* is the $*$-scalar curvature, obtained by tracing the $*$-Ricci tensor.

Besse [13] conjectured that a critical point equation metric is always Einstein. Since then, proving the conjecture has become the motivation for many mathematicians. Till now, no one can prove it but some partial results are developed under some particular curvature assumptions. Lafontaine, in [61], proved that the conjecture is true for a locally conformally flat manifold. Then some improvements of the result were made under half conformally flat assumption by Barros and Ribeiro [7]. In [53], Hwang concluded that the CPE conjecture is true under the certain assumption on the bounds of the potential function. Neto [71], provided a necessary and sufficient condition on the norm of the gradient of potential function for a CPE metric to be Einstein.

Many authors considered CPE in the context of many odd dimensional Riemannian manifolds, e.g. on K-contact manifold and (κ, μ)-contact manifold [73]; on almost Kenmotsu manifold [75]; on 3-dimensional trans-Sasakian manifold [33]; on f-cosymplectic manifold [59] etc. Like CPE, $*$-critical point equation was also studied on almost Kenmotsu manifold [34], on $N(k)$-contact manifold [32] etc.

Apart from the introductory chapter, this thesis consists of four chapters. A brief summary is given of these chapters as follows,

In the **second chapter**, we consider 3-dimensional trans-Sasakian manifold of type (α, β) to admit a Ricci soliton and characterize the covariant derivative of potential vector field along the Reeb vector field as well as the nature of the soliton. Later, we initiate the study of $*$-η-Ricci soliton and gradient almost $*$-η-Ricci soliton within the framework of Kenmotsu manifold and obtain some characteristics of the manifold and the potential vector field. Finally we deliberate $*$-η-Ricci soliton admitting $(\kappa, \mu)'$-almost Kenmotsu manifold and proved that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

In the **third chapter**, we establish some results regarding conformal η-Ricci soliton and conformal Ricci soliton on $(\mathrm{LCS})_n$ manifold satisfying some curvature conditions such as ξ-conharmonically semi-symmetric, ξ-concircularly semi-symmetric and ξ-quasi-conformally semi-symmetric and obtain some results regarding the nature of the manifold as well as the nature of the structural vector field ξ. Later, we initiate the study of conformal η-Ricci soliton and almost conformal η-Ricci soliton within the framework of para-Sasakian manifold and we are able to find some attributes of the manifold, the scalar curvature of the manifold and the potential vector field of the soliton. Further, we evolve the characterizations of the Kenmotsu manifold and the nature of the potential vector field when the manifold satisfies $*$-conformal η-Ricci soliton and gradient almost $*$-conformal η-Ricci soliton. Eventually, we have contrived $*$-conformal η-Ricci soliton admitting $(\kappa, \mu)'$-almost Kenmotsu manifold and proved that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

In **fourth chapter**, we demonstrate the nature of the soliton if Kenmotsu manifold admits almost conformal Ricci soliton. We also able to observe some properties of the scalar curvature of the manifold and the soliton function, and potential vector field of the soliton. Then we prove that if an η-Einstein para-Kenmotsu manifold admits conformal Ricci soliton and $*$-conformal Ricci soliton, then it is Einstein. Further, we acquire that 3-dimensional para-cosymplectic manifold is Ricci flat if the manifold satisfies conformal Ricci soliton where the soliton vector field is conformal. Next, we evolve the nature of scalar curvature when the 3-dimensional trans-Sasakian manifold of type (α, β), provided $\alpha \neq 0$ satisfies $*$-conformal Ricci soliton.

In the **fifth chapter**, we study the critical point equation (shortly CPE) conjecture and $*$-critical point equation (shortly $*$-CPE) conjecture within the framework of various contact metric manifolds. First, it is proved that if a compact Sasakian manifold admits CPE, then either the manifold is Einstein or the potential function is harmonic in an open subset. Later, It is shown that if the manifold satisfies $*$-CPE then the manifold is η-Einstein. Later, we establish that Kenmotsu manifold satisfying the CPE either becomes an Einstein manifold or the derivative of potential function along characteristic vector field satisfy a certain relation on the distribution of η. Next, we study CPE on $(\kappa, \mu)'$-almost Kenmotsu manifold and obtained that the manifold is Einstein. In case of 3-dimensional trans-Sasakian manifold, we get that either the manifold becomes α-Sasakian or it becomes Einstein.

2

On Ricci soliton and $* - \eta$ Ricci soliton

2.1 Introduction

This chapter is divided into five sections. Fist two sections consist of introduction and preliminaries.

In the third section, we show that a 3-dimensional trans-Sasakian manifold of type (α, β) admits a Ricci soliton where the covariant derivative of potential vector field in the direction of unit vector field ξ is orthogonal to ξ. It is also shown that if the structure functions satisfy $\alpha^2 = \beta^2$ then the covariant derivative of the potential vector field in the direction of ξ is a constant multiple of ξ. Finally, we present an example to verify our findings.

In next section, we initiate the study of $*$-η-Ricci soliton within the framework of Kenmotsu manifold as a characterization of Einstein metric. Here we display that a Kenmotsu metric as a $*$-η-Ricci soliton is Einstein metric if the soliton vector field is contact. Further, we have developed the characterization of the Kenmotsu manifold or the nature of the potential vector field when the manifold satisfies gradient almost $*$-η-Ricci soliton. We also furnish two examples to support our findings.

In the last section, we deliberate $*$-η-Ricci soliton admitting $(\kappa, -2)'$-almost Kenmotsu manifold with $\kappa < -1$ and proved that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

2.2 Preliminaries

In the introductory chapter, definitions and some fundamental properties of trans-Sasakian manifold, Kenmotsu manifold and $(\kappa, -2)'$-almost Kenmotsu manifold are given. Here we look back on some pertinent results and use these in our work.

Lemma 2.2.1. *[30] On a 3-dimensional trans-Sasakian manifold, the $*$-Ricci tensor S^* satisfies,*

$$S^*(X,Y) = \frac{1}{2}(r - 4(\alpha^2 - \beta^2))[g(X,Y) - \eta(X)\eta(Y)]. \tag{2.2.1}$$

for arbitrary vector fields X and Y of $\chi(M)$.

Lemma 2.2.2. *[99] The Ricci operator Q on a $(2n+1)$-dimensional Kenmotsu manifold satisfies*

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{2.2.2}$$

$$(\nabla_\xi Q)X = -2QX - 4nX, \tag{2.2.3}$$

for an arbitrary vector field X on the manifold.

Lemma 2.2.3. *[99] The $*$-Ricci tensor S^* on a $(2n+1)$-dimensional Kenmotsu manifold is given by*

$$S^*(X,Y) = S(X,Y) + (2n-1)g(X,Y) + \eta(X)\eta(Y), \tag{2.2.4}$$

for arbitrary vector fields X and Y on the manifold.

Let a (2n+1)-dimensional Kenmotsu metric manifold be η-Einstein manifold. Now considering $X = \xi$ in (1.1.78) and using (1.1.26) we have, $a + b = -2n$. Contracting (1.1.78) over X and Y we get, $r = (2n+1)a + b$ where r denotes the scalar curvature of the manifold. Solving these two we have, $a = (1 + \frac{r}{2n})$ and $b = -(2n + 1 + \frac{r}{2n})$. Using these values we can rewrite (1.1.78) as,

$$S(X,Y) = (1 + \frac{r}{2n})g(X,Y) - (2n + 1 + \frac{r}{2n})\eta(X)\eta(Y). \tag{2.2.5}$$

Lemma 2.2.4. *[27] On a $(\kappa, -2)'$-almost Kenmotsu manifold with $\kappa < -1$ the $*$-Ricci tensor is given by*

$$S^*(X, Y) = -(\kappa + 2)(g(X, Y) - \eta(X)\eta(Y)), \quad (2.2.6)$$

for all vector fields X and Y.

2.3 A 3-dimesional trans-Sasakian manifold admitting a Ricci soliton

In this section we consider the metric of 3-dimensional trans-Sasakian manifold as a Ricci soliton and proved the following result. We also produce an example at the end of this section.

Theorem 2.3.1. *Let M be a 3-dimensional trans-Sasakian manifold of type (α, β) admitting a Ricci soliton where the structure functions α and β are constant. Then the following relations are satisfied,*

(i) If $\nabla_\xi V$ is orthogonal to ξ, then the soliton is shrinking for $\alpha^2 < \beta^2$, steady for $\alpha^2 = \beta^2$ and expanding for $\alpha^2 > \beta^2$.

(ii) If $\alpha^2 = \beta^2$, then the covariant derivative of the potential vector field V in the direction of ξ is a constant multiple of ξ.

Proof. In a 3-dimensional trans-Sasakian manifold where the structure functions α and β are constants, we know from (1.1.45) that the Ricci operator can be written as,

$$QX = (\frac{r}{2} - (\alpha^2 - \beta^2))X - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(X)\xi, \quad (2.3.1)$$

where r denotes the scalar curvature of the manifold and $X \in \chi(M)$ is any vector field. The aforementioned equation implies that it is an η-Einstein manifold. Now, considering covariant derivative of (2.3.1) along the direction of an arbitrary vector field Y, we acquire

$$\begin{aligned}(\nabla_Y Q)X &= \frac{1}{2}(Yr)X - \frac{1}{2}(Yr)\eta(X)\xi - (\frac{r}{2} - 3(\alpha^2 - \beta^2))[-\alpha g(\phi Y, X)\xi \\ &\quad + \beta g(X, Y)\xi - \alpha\eta(X)(\phi Y) + \beta\eta(X)Y - 2\beta\eta(X)\eta(Y)\xi]. \quad (2.3.2)\end{aligned}$$

Contracting X and using the well-known formula $trace\{X \to (\nabla_X Q)Y\} = \frac{1}{2}(Yr)$ in (2.3.2), we have

$$\xi r = -2r\beta + 12(\alpha^2 - \beta^2)\beta. \quad (2.3.3)$$

Using (1.1.45) in the definition of Ricci soliton (1.2.80), we have

$$(\mathcal{L}_V g)(Y,Z) = (2\lambda - r + 2(\alpha^2 - \beta^2))g(Y,Z) + (r - 6(\alpha^2 - \beta^2))\eta(Y)\eta(Z), \tag{2.3.4}$$

for all vector fields $Y, Z \in \chi(M)$. Now taking covariant derivative of (2.3.4) along an arbitrary vector field $X \in \chi(M)$, we get

$$\begin{aligned}(\nabla_X \mathcal{L}_V g)(Y,Z) = &- (Xr)g(Y,Z) + (Xr)\eta(Y)\eta(Z) + (r - 6(\alpha^2 - \beta^2))[-\alpha g(\phi X, Y)\eta(Z) \\ &+ \beta g(\phi X, \phi Y)\eta(Z) - \alpha g(\phi X, Z)\eta(Y) + \beta g(\phi X, \phi Z)\eta(Y)].\end{aligned} \tag{2.3.5}$$

Since ∇ is Riemannian metric connection, $\nabla g = 0$. Using symmetry of $\mathcal{L}_V \nabla$ in (1.1.3) and combining with (2.3.5), we have

$$\begin{aligned}(\mathcal{L}_V \nabla)(X,Y) = &- \frac{1}{2}(Xr)Y - \frac{1}{2}(Yr)X + \frac{1}{2}g(\phi X, \phi Y)Dr + \frac{1}{2}(Xr)\eta(Y)\xi + \frac{1}{2}(Yr)\eta(X)\xi \\ &+ (r - 6(\alpha^2 - \beta^2))[-\alpha\eta(Y)\phi X - \alpha\eta(X)\phi Y + \beta g(\phi X, \phi Y)\xi],\end{aligned} \tag{2.3.6}$$

for all vector fields X and Y on M. Taking covariant derivative with respect to arbitrary vector field X, we have

$$\begin{aligned}(\nabla_X \mathcal{L}_V \nabla)(Y,Z) = &- \frac{1}{2}g(Z, \nabla_X Dr)Y - \frac{1}{2}g(Y, \nabla_X Dr)Z + \frac{1}{2}g(\phi Y, \phi Z)(\nabla_X Dr) - \\ &\alpha\eta(Z)(Xr)\phi Y - \alpha\eta(Y)(Xr)\phi Z + \frac{1}{2}[(Zr)\eta(Y) + (Yr)\eta(Z)](\nabla_X \xi) \\ &+ \frac{1}{2}[g(Y, \nabla_X Dr)\eta(Z) - \alpha(Yr)g(\phi X, Z) + \beta g(\phi X, \phi Z)(Yr) + \\ &g(Z, \nabla_X Dr)\eta(Y) - \alpha(Zr)g(\phi X, Y) + \beta g(\phi X, \phi Y)(Zr) + \\ &2\beta g(\phi Y, \phi Z)(Xr)]\xi + \frac{1}{2}[\alpha g(\phi X, Y)\eta(Z) - \beta g(\phi X, \phi Y)\eta(Z) \\ &+ \alpha g(\phi X, Z)\eta(Y) - \beta g(\phi X, \phi Z)\eta(Y)]Dr + (r - 6(\alpha^2 - \beta^2)) \\ &[\{\alpha^2 g(\phi X, Z) - \alpha\beta g(\phi X, \phi Z)\}\phi Y + \{\alpha^2 g(\phi X, Y) - \\ &\alpha\beta g(\phi X, \phi Y)\}\phi Z - \alpha\eta(Z)((\nabla_X \phi)Y) - \alpha\eta(Y)((\nabla_X \phi)Z) \\ &+ \beta g(\phi Y, \phi Z)(\nabla_X \xi) + \{\alpha\beta g(\phi X, Y)\eta(Z) - \beta^2 g(\phi X, \phi Y)\eta(Z) \\ &+ \alpha\beta g(\phi X, Z)\eta(Y) - \beta^2 g(\phi X, \phi Z)\eta(Y)\}\xi].\end{aligned}$$

Using (1.1.5) in the last equation, we get

$$
\begin{aligned}
(\mathcal{L}_V R)(X,Y)Z =& \frac{1}{2}g(Z,\nabla_Y Dr)X - \frac{1}{2}g(Z,\nabla_X Dr)Y + \frac{1}{2}\{-\alpha(Yr)g(\phi X,Z) - \beta(Yr)\\
& g(\phi X,\phi Z) + g(Z,\nabla_X Dr)\eta(Y) - \alpha(Zr)g(\phi X,Y) + \alpha(Xr)g(\phi Y,Z)\\
& + \beta g(\phi Y,\phi Z)(Xr) - g(Z,\nabla_Y Dr)\eta(X) + \alpha(Zr)g(\phi Y,X)\}\xi +\\
& \alpha\eta(Z)(Yr)\phi X - \alpha\eta(Z)(Xr)\phi Y + \alpha\{\eta(X)(Yr) - \eta(Y)(Xr)\}\phi Z +\\
& \frac{1}{2}\{\alpha g(\phi X,Y)\eta(Z) + \alpha g(X,\phi Y)\eta(Z) - \alpha g(\phi X,Z)\eta(Y) - g(\phi X,\phi Z)\\
& \beta\eta(Y) - \alpha g(\phi Y,Z)\eta(X) + \beta g(\phi Y,\phi Z)\eta(X)\}Dr + \frac{1}{2}\{(Yr)\eta(Z) +\\
& (Zr)\eta(Y)\}(\nabla_X\xi) - \frac{1}{2}\{(Xr)\eta(Z) + (Zr)\eta(X)\}(\nabla_Y\xi) + \frac{1}{2}g(\phi Y,\phi Z)\\
& (\nabla_X Dr) - \frac{1}{2}g(\phi X,\phi Z)(\nabla_Y Dr) + (r - 6(\alpha^2-\beta^2))[\{\alpha\beta g(\phi Y,\phi Z) -\\
& \alpha^2 g(\phi Y,Z)\}\phi X - \{\alpha\beta g(\phi X,\phi Z) - \alpha^2 g(\phi X,Z)\}\phi Y + 2\alpha^2 g(\phi X,Y)\phi Z\\
& + \{2\alpha\beta g(\phi X,Y)\eta(Z) + \alpha\beta g(\phi X,Z)\eta(Y) - \beta^2 g(\phi X,\phi Z)\eta(Y) -\\
& \alpha\beta g(\phi Y,Z)\eta(X) + \beta^2 g(\phi Y,\phi Z)\eta(X)\}\xi + \beta g(\phi Y,\phi Z)(\nabla_X\xi) -\\
& \beta g(\phi X,\phi Z)(\nabla_Y\xi) - \alpha\eta(Z)((\nabla_X\phi)Y) - \alpha\eta(Y)((\nabla_X\phi)Z) + \alpha\eta(Z)\\
& ((\nabla_Y\phi)X) + \alpha\eta(X)((\nabla_Y\phi)Z). \qquad (2.3.7)
\end{aligned}
$$

This equation holds for any $X, Y, Z \in \chi(M)$. Contracting X in (2.3.7), we get

$$
(\mathcal{L}_V S)(Y,Z) = (\frac{\Delta r}{2} - 6\alpha^4 + 12\alpha^2\beta^2 - 6\beta^4 + r\alpha^2 - r\beta^2)g(\phi Y,\phi Z), \qquad (2.3.8)
$$

for any $Y, Z \in \chi(M)$. Again from (1.1.45) we have,

$$
\begin{aligned}
(\mathcal{L}_V S)(Y,Z) =& \frac{1}{2}g(\phi Y,\phi Z)(Vr) + (\frac{r}{2} - (\alpha^2-\beta^2))\{g(\nabla_Y V,Z) + g(Y,\nabla_Z V)\} -\\
& (\frac{r}{2} - 3(\alpha^2-\beta^2))\{\eta(Z)((\nabla_V\eta)Y) + \eta(Y)((\nabla_V\eta)Z) + \eta(Z)\eta(\nabla_Y V)\\
& + \eta(Y)\eta(\nabla_Z V)\}. \qquad (2.3.9)
\end{aligned}
$$

Comparing (2.3.8) with (2.3.9), yields

$$
\begin{aligned}
& (\frac{\Delta r}{2} - 6\alpha^4 + 12\alpha^2\beta^2 - 6\beta^4 + r\alpha^2 - r\beta^2)g(\phi Y,\phi Z) = \frac{1}{2}\{g(\phi Y,\phi Z)(Vr)\\
& +(\frac{r}{2} - (\alpha^2-\beta^2))\{g(\nabla_Y V,Z) + g(Y,\nabla_Z V)\} - (\frac{r}{2} - 3(\alpha^2-\beta^2))\{\eta(Z)\\
& ((\nabla_V\eta)Y) + \eta(Y)((\nabla_V\eta)Z) + \eta(Z)\eta(\nabla_Y V) + \eta(Y)\eta(\nabla_Z V)\}. \qquad (2.3.10)
\end{aligned}
$$

Now, letting $Y = Z = \xi$ gives rise to $(\alpha^2 - \beta^2)\eta(\nabla_\xi V) = 0$. Now, there will arise two cases either $\eta(\nabla_\xi V) = 0$ or $(\alpha^2 - \beta^2) = 0$. From the definition of Ricci soliton (1.2.80) we have,

$$\frac{1}{2}(g(\nabla_X V, Y) + g(\nabla_Y V, X)) + S(X, Y) = \lambda g(X, Y), \tag{2.3.11}$$

for any vector fields X and Y. For first case $\eta(\nabla_\xi V) = 0$ which implies $\nabla_\xi V$ is orthogonal to ξ, putting $X = Y = \xi$ in (2.3.11) gives $2(\alpha^2 - \beta^2) = \lambda$. It directly implies that the soliton is shrinking if $\alpha^2 < \beta^2$, steady if $\alpha^2 = \beta^2$ and expanding if $\alpha^2 > \beta^2$.

For the second case where $\alpha^2 = \beta^2$, then it follows from (2.3.11) that $\nabla_\xi V = \lambda\xi$ i.e., the covariant derivative of the potential vector field V in the direction of ξ is a λ multiple of ξ. □

Example 2.3.1. *We consider the manifold as $M = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields as defined below,*

$$e_1 = e^{2z}\frac{\partial}{\partial x}, \qquad e_2 = e^{2z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M. The Riemannian metric g is defined by,

$$g_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 3\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\xi = e_3$. Then the 1-form η is defined by $\eta(Z) = g(Z, e_3)$, for arbitrary $Z \in \chi(M)$, then we have the following relations,

$$\eta(e_1) = 0, \qquad \eta(e_2) = 0, \qquad \eta(e_3) = 1.$$

Let us define the (1,1)-tensor field ϕ as

$$\phi e_1 = e_2, \qquad \phi e_2 = -e_1, \qquad \phi e_3 = 0,$$

then it satisfies,

$$\begin{aligned} \phi^2(Z) &= -Z + \eta(Z)e_3, \\ g(\phi Z, \phi W) &= g(Z, W) - \eta(Z)\eta(W), \end{aligned}$$

for arbitrary $Z, W \in \chi(M)$. Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M. We can now easily conclude,

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = -2e_2, \qquad [e_1, e_3] = -2e_1.$$

Let ∇ be the Levi-Civita connection of M. Then from (1.1.1)*, we can have,*

$$\begin{array}{lll} \nabla_{e_1}e_1 = 2e_3, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = -2e_1, \\ \nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = 2e_3, & \nabla_{e_2}e_3 = -2e_2, \\ \nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = 0. \end{array}$$

From here we can easily verify that the relations (1.1.40) *and* (1.1.41) *are satisfied. Hence the considered manifold is trans-Sasakian manifold of type* $(0, -2)$*. The components of Riemannian curvature tensor are given by,*

$$\begin{array}{lll} R(e_1, e_2)e_1 = -4e_3, & R(e_1, e_2)e_2 = -4e_1, & R(e_1, e_2)e_3 = 0, \\ R(e_1, e_3)e_1 = 4e_2, & R(e_1, e_3)e_2 = 0, & R(e_1, e_3)e_3 = -4e_1, \\ R(e_2, e_3)e_1 = 0, & R(e_2, e_3)e_2 = -4e_2, & R(e_2, e_3)e_3 = -4e_2. \end{array}$$

And the components of Ricci tensor are given by,

$$S(e_1, e_1) = 0, \qquad S(e_2, e_2) = 0, \qquad S(e_3, e_3) = -8.$$

From here we can easily deduce that the scalar curvature of the manifold $r = \sum_{i=1}^{3} S(e_i, e_i) = -8$*. Let us define a vector field by,* $V = \xi$*. Then we can obtain,*

$$(\mathcal{L}_V g)(e_1, e_1) = -4, \qquad (\mathcal{L}_V g)(e_2, e_2) = -4, \qquad (\mathcal{L}_V g)(e_3, e_3) = 0.$$

Contracting (1.2.80) *and using the result* $r = -8$ *we deduce* $\lambda = 4$*. So* g *defines a Ricci soliton on this trans-Sasakian manifold for* $\lambda = 4$*.*

2.4 $* - \eta$-Ricci soliton on Kenmotsu manifold

In this section we consider that the metric g of a $(2n+1)$-dimensional Kenmotsu manifold represents a $*$-η-Ricci soliton and a gradient almost $*$-η-Ricci soliton.

Theorem 2.4.1. *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. If the metric g represents a $*$-η-Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein.*

Proof. Since the metric g of the Kenmotsu manifold represents a $*$-η-Ricci soliton so both of the equations (1.2.88) and (2.2.4) are satisfied. Combining these two, we have

$$(\mathcal{L}_V g)(X, Y) = -2S(X, Y) - (2\lambda + 4n - 2)g(X, Y) - 2(\mu + 1)\eta(X)\eta(Y). \tag{2.4.1}$$

Taking covariant derivative along arbitrary vector field Z and using (1.1.24), we obtain

$$\begin{aligned}(\nabla_Z \mathcal{L}_V g)(X,Y) &= -2(\nabla_Z S)(X,Y) - 2(\mu+1)\{g(X,Z)\eta(Y) \\ &\quad + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}, \qquad (2.4.2)\end{aligned}$$

for all $X, Y, Z \in \chi(M)$. Since ∇ is the metric connection i.e., $\nabla g = 0$. So (1.1.3) reduces to,

$$(\nabla_Z \mathcal{L}_V g)(X,Y) = g((\mathcal{L}_V \nabla)(X,Z),Y) + g((\mathcal{L}_V \nabla)(Y,Z),X), \qquad (2.4.3)$$

for all vector fields X, Y, Z on M. Combining (2.4.2) and (2.4.3) and by a combinatorial computation and applying the symmetry of $(\mathcal{L}_V \nabla)$, the foregoing equation yields

$$\begin{aligned}g((\mathcal{L}_V \nabla)(X,Y),Z) &= (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(Z,X) \\ &\quad -2(\mu+1)\{g(X,Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\}, \qquad (2.4.4)\end{aligned}$$

for arbitrary vector fields X, Y and Z on M. Using (2.2.2) and (2.2.3) in (2.4.4), we get

$$(\mathcal{L}_V \nabla)(X,\xi) = 2QX + 4nX, \qquad (2.4.5)$$

for all $X \in \chi(M)$. Now differentiating covariantly to (2.4.5) with respect to arbitrary vector field Y, we achieve

$$(\nabla_Y \mathcal{L}_\nabla)(X,\xi) = 2(\nabla_Y Q)X - (\mathcal{L}_V)(X,Y) + \eta(Y)(2QX + 4nX). \qquad (2.4.6)$$

In view of (2.4.6), the relation (1.1.5) can be rewritten as,

$$(\mathcal{L}_V R)(X,Y)\xi = 2\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + 2\eta(X)(QY + 2nY) - 2\eta(Y)(QX + 2nX), \qquad (2.4.7)$$

for arbitrary vector fields X and Y on M. Setting $Y = \xi$ in the aforementioned equation and using (1.1.26), (2.2.2) and (2.2.3) we get

$$(\mathcal{L}_V R)(X,\xi)\xi = 0. \qquad (2.4.8)$$

Now, taking (2.4.1) in account, the Lie derivative of $g(\xi,\xi) = 1$ along the potential vector field V yields

$$\eta(\mathcal{L}_V \xi) = \lambda + \mu. \qquad (2.4.9)$$

Plugging $Y = \xi$ and noting that (1.1.7) and (1.1.11), the equation (2.4.1) provides

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = -(2\lambda + 2\mu)\eta(X), \qquad (2.4.10)$$

which holds for arbitrary vector field X on M. From (1.1.25) we compute, $R(X,\xi)\xi = \eta(X)\xi - X$. Taking Lie derivative along the potential vector field V and inserting (2.4.9) and (2.4.10) in account, this reduces to

$$(\mathcal{L}_V R)(X,\xi)\xi = 2(\lambda+\mu)(X-\eta(X)\xi), \tag{2.4.11}$$

for all $X \in \chi(M)$. Finally comparing (2.4.8) and (2.4.11) we have, $2(\lambda+\mu)(X-\eta(X)\xi) = 0$. Since this holds for arbitrary $X \in \chi(M)$ so, we infer

$$\lambda = -\mu. \tag{2.4.12}$$

Invoking the relation (2.4.12) in (2.4.9), we easily obtain $\eta(\mathcal{L}_V\xi) = 0$. Since we have considered the potential vector field V as contact vector field so there must exists a smooth function f such that $\mathcal{L}_V\xi = f\xi$. Making use of this in (2.4.9), we get $f = \lambda + \mu$. Therefore by using the relation (2.4.12), we get $f = 0$ and thus $\mathcal{L}_V\xi = 0$. Finally the equation (2.4.10) reduces to

$$\mathcal{L}_V\eta = 0. \tag{2.4.13}$$

So, V is strictly infinitesimal contact transformation.

Inserting $Y = \xi$ and using (1.1.23), $\mathcal{L}_V\xi = 0$ and (2.4.13) in the relation (1.1.2) yields, $(\mathcal{L}_V\nabla)(X,\xi) = 0$. Substituting this in (2.4.5), we deduce $QX = -2nX \ \forall X \in \chi(M)$, which settles our claim. □

$*$-η-Ricci soliton is a generalisation of $*$-Ricci soliton, where we consider $\mu = 0$ in (1.2.88) to get $*$-Ricci soliton equation. We can rewrite the above theorem as:

Corollary 2.4.1. *Let $M^{(2n+1)}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. If the metric g represents a $*$-Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein.*

Example 2.4.1. *Let us consider the set $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$ as our manifold where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below:*

$$e_1 = e^{-v}\frac{\partial}{\partial x}, \qquad e_2 = e^{-v}\frac{\partial}{\partial y}, \qquad e_3 = e^{-v}\frac{\partial}{\partial z}, \qquad e_4 = e^{-v}\frac{\partial}{\partial u}, \qquad e_5 = \frac{\partial}{\partial v},$$

are linearly independent at each point of M. We define the metric g as

$$g(e_i,e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i,j \in \{1,2,3,4,5\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let η *be an 1-form defined by* $\eta(X) = g(X, e_5)$, *for arbitrary* $X \in \chi(M)$. *Let us define (1,1)-tensor field* ϕ *as,*

$$\phi(e_1) = e_3, \qquad \phi(e_2) = e_4, \qquad \phi(e_3) = -e_1, \qquad \phi(e_4) = -e_2, \qquad \phi(e_5) = 0.$$

Then the relations (1.1.6), (1.1.7) *and* (1.1.8) *are satisfied, where* $\xi = e_5$. *So,* (M, ϕ, ξ, η, g) *defines an almost contact structure on* M.

We can now deduce that,

$$\begin{array}{llll}
[e_1, e_2] = 0, & [e_1, e_3] = 0, & [e_1, e_4] = 0, & [e_1, e_5] = e_1, \\
[e_2, e_1] = 0, & [e_2, e_3] = 0, & [e_2, e_4] = 0, & [e_2, e_5] = e_2, \\
[e_3, e_1] = 0, & [e_3, e_2] = 0, & [e_3, e_4] = 0, & [e_3, e_5] = e_3, \\
[e_4, e_1] = 0, & [e_4, e_2] = 0, & [e_4, e_3] = 0, & [e_4, e_5] = e_4, \\
[e_5, e_1] = -e_1, & [e_5, e_2] = -e_2, & [e_5, e_3] = -e_3, & [e_5, e_4] = -e_4.
\end{array}$$

Let ∇ *be the Levi-Civita connection of* M. *Then from* (1.1.1), *we can have*

$$\begin{array}{lllll}
\nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\
\nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\
\nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\
\nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -e_5, & \nabla_{e_4} e_5 = e_4, \\
\nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0.
\end{array}$$

Therefore (1.1.22) *is satisfied for arbitrary* $X, Y \in \chi(M)$. *So* (M, ϕ, ξ, η, g) *becomes a Kenmotsu manifold.*
The non-vanishing components of curvature tensor are,

$$\begin{array}{lll}
R(e_1, e_2)e_2 = -e_1, & R(e_1, e_3)e_3 = -e_1, & R(e_1, e_4)e_4 = -e_1, \\
R(e_1, e_5)e_5 = -e_1, & R(e_1, e_2)e_1 = e_2, & R(e_1, e_3)e_1 = e_3, \\
R(e_1, e_4)e_1 = e_4, & R(e_1, e_5)e_1 = e_5, & R(e_2, e_3)e_2 = e_3, \\
R(e_2, e_4)e_2 = e_4, & R(e_2, e_5)e_2 = e_5, & R(e_2, e_3)e_3 = -e_2, \\
R(e_2, e_4)e_4 = -e_2, & R(e_2, e_5)e_5 = -e_2, & R(e_3, e_4)e_3 = e_4, \\
R(e_3, e_5)e_3 = e_5, & R(e_3, e_4)e_4 = -e_3, & R(e_4, e_5)e_4 = e_5, \\
R(e_5, e_3)e_5 = e_3, & R(e_5, e_4)e_5 = e_4. &
\end{array}$$

Now from the above results we have, $S(e_i, e_i) = -4$ *for* $i = 1, 2, 3, 4, 5$ *and*

$$S(X,Y) = -4g(X,Y) \quad \forall X, Y \in \chi(M). \tag{2.4.14}$$

Contracting this we have $r = \sum_{i=1}^{5} S(e_i, e_i) = -20 = -2n(2n+1)$ *where dimension of the manifold* $2n + 1 = 5$*. Also, we have*

$$S^*(e_i, e_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4, \\ 0, & \text{if } i = 5. \end{cases}$$

and $r^* = r + 4n^2 = -20 + 16 = -4$*. So*

$$S^*(X,Y) = -g(X,Y) + \eta(X)\eta(Y) \quad \forall X, Y \in \chi(M). \tag{2.4.15}$$

Now we consider a vector field V *as*

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{2.4.16}$$

Then from the above results we can justify that

$$(\mathcal{L}_V g)(X,Y) = 4\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{2.4.17}$$

which holds for all $X, Y \in \chi(M)$*. From* (2.4.15) *and* (2.4.17)*, we can conclude that* g *represents a* $*$*-*η*-Ricci soliton i.e., it satisfies* (1.2.88) *for potential vector field* V *defined by* (2.4.16)*,* $\lambda = -1$ *and* $\mu = 1$*.*

Theorem 2.4.2. *Let* $M^{(2n+1)}(\phi, \xi, \eta, g)$ *be a Kenmotsu manifold. If the metric* g *represents a gradient almost* $*$*-*η*-Ricci soliton then either* M *is Einstein or there exists an open set where the potential vector field* V *is pointwise collinear with the characteristic vector field* ξ*.*

Proof. In view of (2.2.4), from the gradient almost $*$-η-Ricci soliton equaion (1.2.89), we acquire

$$\nabla_X Df = -QX - (\lambda + 2n - 1)X - (\mu + 1)\eta(X)\xi, \tag{2.4.18}$$

for any vector field X on M. Taking covariant derivative along arbitrary vector Y and using (1.1.23), (1.1.24), yields

$$\begin{aligned} \nabla_Y \nabla_X Df = &- (\nabla_Y Q)X - Q(\nabla_Y X) - Y(\lambda)X - (\lambda + 2n - 1)(\nabla_Y X) \\ &- (\mu + 1)\{g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(\nabla_Y X)\xi + \eta(X)Y\}. \end{aligned} \tag{2.4.19}$$

Applying this in the expression of Riemannian curvature tensor (1.1.4), we obtain

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + Y(\lambda)X - X(\lambda)Y - (\mu+1)\{\eta(Y)X - \eta(X)Y\}. \tag{2.4.20}$$

Moreover an inner product with ξ and use of (2.2.2) and (2.2.3) yields

$$g(R(X,Y)Df,\xi) = Y(\lambda)\eta(X) - X(\lambda)\eta(Y), \tag{2.4.21}$$

for $X, Y \in \chi(M)$. Furthermore the inner product of (1.1.25) with the potential vector field Df provides

$$g(R(X,Y)Df,\xi) = \eta(Y)X(f) - \eta(X)Y(f), \tag{2.4.22}$$

for arbitrary X and Y on M. Comparing (2.4.21) and (2.4.22) and plugging $Y = \xi$, we have $X(f+\lambda) = \xi(f+\lambda)\eta(X)$. From this we achieve

$$d(f+\lambda) = \xi(f+\lambda)\eta. \tag{2.4.23}$$

So, $(f+\lambda)$ is invariant along the distribution $Ker(\eta)$ i.e., if $X \in Ker(\eta)$, then $X(f+\lambda) = d(f+\lambda)X = 0$.

Now, if we take inner product along arbitrary vector field Z after plugging $X = \xi$ in (2.4.20), we get

$$g(R(\xi,Y)Df,Z) = S(Y,Z) + (2n - \xi(\lambda) + \mu + 1)g(Y,Z) + Y(\lambda)\eta(Z) - (\mu+1)\eta(Y)\eta(Z). \tag{2.4.24}$$

Again noting that from (1.1.25), we can easily deduce for arbitrary vector fields Y and Z on M,

$$g(R(\xi,Y)Df,Z) = \xi(f)g(Y,Z) - Y(f)\eta(Z). \tag{2.4.25}$$

Comparing the equations (2.4.24) and (2.4.25) and applying (2.4.23), we obtain

$$S(Y,Z) = \{\xi(f+\lambda) - \mu - 2n - 1\}g(Y,Z) - \{\xi(f+\lambda) - \mu - 1\}\eta(Y)\eta(Z). \tag{2.4.26}$$

Since the above equation holds for arbitrary Y and Z, so the manifold is η-Einstein. Now tracing (2.4.26), we infer

$$\xi(f+\lambda) = \frac{r}{2n} + \mu + 2n + 2. \tag{2.4.27}$$

Plugging this in (2.4.26), we acquire

$$S(Y,Z) = (\frac{r}{2n} + 1)g(Y,Z) - (\frac{r}{2n} + 2n + 1)\eta(Y)\eta(Z),$$

for arbitrary vector fields Y and Z on M which is exactly same as (2.2.5). Contracting X in (2.4.20), the equation reduces to

$$S(Y, Df) = \frac{1}{2}Y(r) + 2nY(\lambda) - 2n(\mu+1)\eta(Y), \tag{2.4.28}$$

which holds for any $Y \in \chi(M)$. Now, taking into with (2.2.5), we compute

$$(r+2n)Y(f)-(r+2n(2n+1))\eta(Y)\xi(f)-nY(r)-4n^2Y(\lambda)+4n^2(\mu+1)\eta(y) = 0, \tag{2.4.29}$$

for all $Y \in \chi(M)$. Setting $Y = \xi$ and then in view of (2.4.27), we easily derive the relation

$$\xi(r) = -2(r + 2n(2n+1)). \tag{2.4.30}$$

Since $d^2 = 0$ and $d\eta = 0$, from (2.4.23) it follows $dr \wedge \eta = 0$ i.e., $dr(X)\eta(Y) - dr(Y)\eta(X) = 0$ for arbitrary $X, Y \in \chi(M)$. After inserting $Y = \xi$ and applying (2.4.30) it reduces to $X(r) = -2(r + 2n(2n+1))\xi$. Since X is an arbitrary vector field, we conclude that

$$Dr = -2(r + 2n(2n+1))\xi. \tag{2.4.31}$$

Let X be a vector field of the distribution $Ker(\eta)$. Then, (2.4.29) provides

$$(r + 2n)X(f) - 4n^2X(\lambda) = 0.$$

Invoking (2.4.23) and (2.4.27) we obtain, $(r+2n(2n+1))X(f) = 0$. From here we conclude

$$(r + 2n(2n+1))(Df - \xi(f)\xi) = 0.$$

If $r = -2n(2n+1)$, then from (2.2.5) we acquire that the manifold is Einstein with Einstein constant $-2n$.

If $r \neq -2n(2n+1)$ on some open set O of M, then $Df = \xi(f)\xi$ on that open set that is, the potential vector field is pointwise collinear with the characteristic vector field ξ, which completes the proof. □

Example 2.4.2. *Let us consider the set* $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ *as our manifold where* (x, y, z, u, v) *are the standard coordinates in* $\mathbb{R}^5$*. The vector fields defined below,*

$$e_1 = v\frac{\partial}{\partial x}, \qquad e_2 = v\frac{\partial}{\partial y}, \qquad e_3 = v\frac{\partial}{\partial z}, \qquad e_4 = v\frac{\partial}{\partial u}, \qquad e_5 = -v\frac{\partial}{\partial v},$$

forms a linearly independent set of vector fields on M. We define the metric g as

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We consider the reeb vector field $\xi = e_5$ then the 1-form η is defined by $\eta(X) = g(X, e_5)$, for arbitrary $X \in \chi(M)$ then, $\eta = dv$. Let us define (1,1)-tensor field ϕ as,

$$\phi(e_1) = e_2, \qquad \phi(e_2) = -e_1, \qquad \phi(e_3) = e_4, \qquad \phi(e_4) = -e_3, \qquad \phi(e_5) = 0.$$

Then the relations (1.1.6), (1.1.7) *and* (1.1.8) *are satisfied. So, (M, ϕ, ξ, η, g) defines an almost contact structure on M.*

Let ∇ be the Levi-Civita connection of M. Then from (1.1.1)*, we can have,*

$$\begin{array}{lllll}
\nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\
\nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\
\nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\
\nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -e_5, & \nabla_{e_4} e_5 = e_4, \\
\nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0.
\end{array}$$

Therefore $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ is satisfied for arbitrary $X, Y \in \chi(M)$. So (M, ϕ, ξ, η, g) becomes a Kenmotsu manifold.
The non-vanishing components of curvature tensor are,

$$\begin{array}{lll}
R(e_1, e_2)e_2 = -e_1, & R(e_1, e_3)e_3 = -e_1, & R(e_1, e_4)e_4 = -e_1, \\
R(e_1, e_5)e_5 = -e_1, & R(e_1, e_2)e_1 = e_2, & R(e_1, e_3)e_1 = e_3, \\
R(e_1, e_4)e_1 = e_4, & R(e_1, e_5)e_1 = e_5, & R(e_2, e_3)e_2 = e_3, \\
R(e_2, e_4)e_2 = e_4, & R(e_2, e_5)e_2 = e_5, & R(e_2, e_3)e_3 = -e_2, \\
R(e_2, e_4)e_4 = -e_2, & R(e_2, e_5)e_5 = -e_2, & R(e_3, e_4)e_3 = e_4, \\
R(e_3, e_5)e_3 = e_5, & R(e_3, e_4)e_4 = -e_3, & R(e_4, e_5)e_4 = e_5, \\
R(e_5, e_3)e_5 = e_3, & R(e_5, e_4)e_5 = e_4. &
\end{array}$$

Now from the above results we have, $S(e_i, e_i) = -4$ *for* $i = 1, 2, 3, 4, 5$ *and*

$$S(X, Y) = -4g(X, Y) \quad \forall X, Y \in \chi(M). \tag{2.4.32}$$

So, the manifold is Einstein. Also, we have

$$S^*(e_i, e_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4, \\ 0, & \text{if } i = 5. \end{cases}$$

and

$$S^*(X, Y) = -g(X, Y) + \eta(X)\eta(Y) \quad \forall X, Y \in \chi(M). \tag{2.4.33}$$

Let $f : M \to \mathbb{R}$ *be a smooth function defined by*

$$f(x, y, z, u, v) = x^2 + y^2 + z^2 + u^2 + \frac{v^2}{2}. \tag{2.4.34}$$

Then the gradient of f*,* Df *is given by*

$$Df = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{2.4.35}$$

Then from the above results we can verify that

$$(\mathcal{L}_{Df}g)(X, Y) = 2\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{2.4.36}$$

which holds for all $X, Y \in \chi(M)$*. From* (2.4.33) *and* (2.4.36) *we obtain that* g *represents a gradient almost* $*$ η*-Ricci soliton i.e., it satisfies* (1.2.89) *for* $V = Df$*, where* f *is defined by* (2.4.34)*,* $\lambda = 0$ *and* $\mu = 0$*.*

2.5 $*$-η-Ricci soliton on $(\kappa, -2)'$-almost Kenmotsu manifold with $\kappa < -1$

In this section we consider the manifold as a $(2n + 1)$-dimensional almost Kenmotsu manifold where the characteristic vector field ξ satisfies $(\kappa, -2)'$-nullity distribution. Then we let the metric g to represent a $*$-η-Ricci soliton.

Theorem 2.5.1. *Let* $M^{(2n+1)}(\phi, \xi, \eta, g)$ *be an almost Kenmotsu manifold such that* ξ *belongs to* $(\kappa, -2)'$*-nullity distribution where* $\kappa < -1$*. If the metric* g *represents a* $*$*-*η*-Ricci soliton satisfying* $\lambda + \mu \neq 0$ *then,* M *is Ricci-flat and is locally isometric to* $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$*.*

Proof. Combining (1.2.88) with (2.2.6), we derive

$$(\mathcal{L}_V g)(X,Y) = (2\kappa - 2\lambda + 4)g(X,Y) - 2(\kappa + \mu + 2)\eta(X)\eta(Y), \tag{2.5.1}$$

for all vector fields X and Y on M. Now taking covariant derivative of the foregoing equation along arbitrary vector field Z and using (1.1.38), we get

$$\begin{aligned}(\nabla_Z \mathcal{L}_V g)(X,Y) &= -2(\kappa+\mu+2)[\eta(Y)g(X,Z) + \eta(X)g(Y,Z) + \eta(Y)g(h'Z,X) \\ &\quad +\eta(X)g(h'Z,Y) - 2\eta(X)\eta(Y)\eta(Z)].\end{aligned}$$

By a straightforward combinatorial computation, using (2.4.3) and the symmetry of $(\mathcal{L}_V\nabla)$ in the aforementioned equation, we acquire

$$(\mathcal{L}_V\nabla)(X,Y) = -2(\kappa+\mu+2)[g(X,Y) + g(h'X,Y) - \eta(X)\eta(Y)]\xi, \tag{2.5.2}$$

for all $X, Y \in \chi(M)$. Replacing $Y = \xi$ and using (1.1.7), (1.1.11) and (1.1.29), we have

$$(\mathcal{L}_V\nabla)(X,\xi) = 0, \tag{2.5.3}$$

for arbitrary vector field X on M. Now taking (1.1.28) and (2.5.2) into account and differentiating (2.5.3) covariantly along arbitrary vector field Y, one can obtain

$$(\nabla_Y\mathcal{L}_V\nabla)(X,\xi) = 2(\kappa+\mu+2)[g(X,Y) - \eta(X)\eta(Y) + 2g(h'X,Y) + g(h'^2X,Y)]\xi, \tag{2.5.4}$$

for any vector fields X and Y on M. Setting $Z = \xi$ and using (2.5.4) repeatedly in (1.1.5), we achieve

$$(\mathcal{L}_V R)(X,Y)\xi = 0, \tag{2.5.5}$$

for arbitrary $X, Y \in \chi(M)$. Now taking Lie derivative of (1.1.32) along the potential vector field V and considering (1.1.7) and (1.1.29) into account, we get

$$\begin{aligned}(\mathcal{L}_V R)(X,\xi)\xi &= \kappa[g(X,\mathcal{L}_V\xi)\xi - 2\eta(\mathcal{L}_V\xi)X - ((\mathcal{L}_V\eta)X)\xi] + 2[2\eta(\mathcal{L}_V\xi)h'X \\ &\quad -\eta(X)(h'(\mathcal{L}_V\xi)) - g(h'X,\mathcal{L}_V\xi)\xi - ((\mathcal{L}_V h')X)],\end{aligned} \tag{2.5.6}$$

for any vector field X on M. Plugging $Y = \xi$ in (2.5.1), we infer

$$(\mathcal{L}_V\eta)X - g(X,\mathcal{L}_V\xi) = -2(\lambda+\mu)\eta(X), \tag{2.5.7}$$

for all $X \in \chi(M)$. Setting $X = \xi$ in the foregoing equation, we acquire

$$\eta(\mathcal{L}_V\xi) = \lambda + \mu. \tag{2.5.8}$$

With the help of (2.5.5),(2.5.7) and (2.5.8), one can rewrite the equation (2.5.6) as

$$\kappa(\lambda+\mu)(X-\eta(X)\xi)-2(\lambda+\mu)h'X+\eta(X)h'(\mathcal{L}_V\xi)+g(h'X,\mathcal{L}_V\xi)\xi+(\mathcal{L}_Vh')X=0. \quad (2.5.9)$$

Taking inner product of the foregoing equation with arbitrary vector field Y, we obtain

$$\begin{aligned}&(\lambda+\mu)[\kappa(g(X,Y)-\eta(X)\eta(Y))+g(h'X,Y)]+g((\mathcal{L}_Vh')X,Y)\\&+\eta(X)g(h'(\mathcal{L}_V\xi),Y)+g(h'X,\mathcal{L}_V\xi)\eta(Y)=0.\end{aligned}$$

Since the above equation holds for any vector fields X and Y on M, by replacing X by ϕX and Y by ϕY and taking (1.1.10) into account, we arrive at

$$(\lambda+\mu)[\kappa g(\phi X,\phi Y)-2g(h'\phi X,\phi Y)]+g((\mathcal{L}_Vh')\phi X,\phi Y)=0, \quad (2.5.10)$$

for all $X,Y\in\chi(M)$. Since $spec(h')=\{0,\alpha,-\alpha\}$, let X and V belong to the eigenspaces of $-\alpha$ and α denoted by $[-\alpha]'$ and $[\alpha]'$ respectively. Then $\phi X\in[\alpha]'$ (for more details we refer to [35]. Then (2.5.10) can be rewritten as

$$(\lambda+\mu)(\kappa-2\alpha)g(\phi X,\phi Y)+g((\mathcal{L}_Vh')\phi X,\phi Y)=0, \quad (2.5.11)$$

for all $X,Y\in\chi(M)$. It is remained to find the value of $g((\mathcal{L}_Vh')\phi X,\phi Y)$. To get this we prove a more generalized result: In a $(\kappa,-2)'$-almost Kenmotsu manifold $(\mathcal{L}_Xh')Y=0$, where X and Y belong to same eigenspaces.

Without loss of generality we assume that $X,Y\in[\alpha]'$, where $spec(h')=\{0,\alpha,-\alpha\}$. If we consider a local orthonormal ϕ-basis as $\{\xi,e_i,\phi e_i\}$, $i=1,2,...,n$ then

$$\nabla_XY=\sum_{i=1}^{n}g(\nabla_XY,e_i)e_i-(\alpha+1)g(X,Y)\xi,$$

and

$$\begin{aligned}(\mathcal{L}_Xh')Y &= \mathcal{L}_X(h'Y)-h'(\mathcal{L}_XY)\\&= \alpha(\mathcal{L}_XY)-h'(\mathcal{L}_XY)\\&= \alpha(\nabla_XY-\nabla_YX)-h'(\nabla_XY-\nabla_YX)\\&= \alpha(\alpha+1)g(X,Y)\xi-\alpha(\alpha+1)g(X,Y)\xi\\&= 0.\end{aligned}$$

Similarly we can prove that the above results hold if $X, Y \in [-\alpha]'$ (For more details we refer to [35]). Now (2.5.11) reduces to

$$(\lambda + \mu)(\kappa - 2\alpha)g(\phi X, \phi Y) = 0 \tag{2.5.12}$$

for any vector fields X and Y on M. Since by hypothesis $\lambda + \mu \neq 0$, from the foregoing equation we infer that $\kappa = 2\alpha$. Again form $\alpha^2 = -(\kappa + 1)$ we get $\alpha = -1$ and $\kappa = -2$. Plugging the value of κ in (2.2.6) we have $S^* = 0$, i.e., the manifold is Ricci-flat.

Again we get $spec(h') = \{0, 1, -1\}$. From corollary 4.2 of [35] we get M is locally symmetric. From proposition 4.1 of [35] we finally conclude that M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, where $\mathbb{H}^{n+1}(-4)$ is the hyperbolic space of constant curvature -4. So, the proof is completed. □

As we know, setting $\mu = 0$ in (1.2.88) gives rise to the equation of $*$-Ricci soliton, we can revisit the last theorem and can note the statement as:

Corollary 2.5.1. *Let $M(\phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost Kenmotsu manifold such that ξ belongs to $(\kappa, -2)'$-nullity distribution where $\kappa < -1$. If the metric g represents a $*$-Ricci soliton satisfying $\lambda \neq 0$ then, M is Ricci-flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

3

On conformal η-Ricci soliton

3.1 Introduction

This chapter consists of six sections. Fist two sections contain introduction and preliminaries, respectively.

In the third section, we establish some results regarding conformal η-Ricci soliton and conformal Ricci soliton on $(\text{LCS})_n$ manifold satisfying some curvature conditions such as ξ-conharmonically semi-symmetric, ξ-concircularly semi-symmetric and ξ-quasi-conformally semi-symmetric and obtained the nature of the soliton as well as the nature of the structural vector field ξ.

In the later section we initiate the study of conformal η-Ricci soliton and almost conformal η-Ricci soliton within the framework of para-Sasakian manifold. We prove that if para-Sasakian metric admits conformal η-Ricci soliton, then the manifold is η-Einstein and either the soliton vector field V is Killing or it leaves ϕ invariant. Here, we have shown the characteristics of the soliton vector field V and scalar curvature when the manifold admitting conformal η-Ricci soliton and vector field is pointwise collinear with the characteristic vector field ξ. Next, we show that a para-Sasakian metric endowed an almost conformal η-Ricci soliton is η-Einstein metric if the soliton vector field V is an infnitesimal contact transformation. We have also displayed that the manifold is Einstein if it represents a gradient almost conformal η-Ricci soliton. We have developed an example to display the alive of conformal η-Ricci soliton on 3-dimensional para-Sasakian manifold.

In the fifth section, we deliberate $*$-conformal η-Ricci soliton within the framework of Kenmotsu manifolds. Here we have shown that a Kenmotsu metric as a $*$-conformal

η-Ricci soliton is Einstein metric if the soliton vector field is contact. Further, we have evolved the characterization of the Kenmotsu manifold or the nature of the potential vector field when the manifold satisfies gradient almost $*$-conformal η-Ricci soliton. Then, we have constructed some examples to illustrate the existence of $*$-conformal η-Ricci soliton, gradient almost $*$-conformal η-Ricci soliton on Kenmotsu manifold.

In the last section, we contrive $*$-conformal η-Ricci soliton admitting $(\kappa, \mu)'$-almost Kenmotsu manifold with $\kappa < -1$ and proved that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. We also provide an example to establish our findings.

3.2 Preliminaries

Definitions and some properties of Lorentzian concircular structure manifold, para-Sasakian manifold, Kenmotsu manifold and $(\kappa, \mu)'$-almost Kenmotsu manifold are already stated in the first chapter. Here we want to recall some results on these manifolds proved by some eminent mathematicians which are useful to get the results stated in this chapter.

If (g, V, λ, p, μ) is a conformal η-Ricci soliton on an n-dimensional Lorentzian concircular structure manifold, then we can deduce the following,

$$S(X, Y) = (\frac{p}{2} + \frac{1}{n} - \lambda - \alpha)g(X, Y) - (\mu + \alpha)\eta(X)\eta(Y), \tag{3.2.1}$$

where S is the Ricci tensor of the manifold.

The conharmonic curvature tensor, denoted by H, is defined on an n-dimensional manifold M by [3],

$$H(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], \tag{3.2.2}$$

where X, Y and Z are arbitrary vector fields on M.

A manifold M is said to admit the ξ-conharmonically semi-symmetric curvature property if for the characteristic vector field ξ, the following property holds

$$R(\xi, X).H = 0,$$

for any vector field X in $\chi(M)$.

The concircular curvature tensor on an n-dimensional manifold M, denoted by C, is defined by [3],

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{3.2.3}$$

where X, Y and Z are arbitrary vector fields in $\chi(M)$.

A manifold M is said to admit the ξ-concircularly semi-symmetric curvature property if for the characteristic vector field ξ, the following property holds

$$R(\xi, X).C = 0,$$

for any vector field X.

The quasi-conformal curvature tensor on an n-dimensional manifold M, denoted by $\tilde{C}$, is defined by [3],

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{n}(\frac{a}{(n-1)} + 2b)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{3.2.4}$$

where a and b are non-zero constants and X, Y and Z are any vector fields in $\chi(M)$.

A manifold M is said to admit the ξ-quasi-conformally semi-symmetric curvature property if for the characteristic vector field ξ, the following property holds

$$R(\xi, X).\tilde{C} = 0,$$

for an arbitrary vector field X on M.

Lemma 3.2.1. *[70] In a para-Sasakian manifold, the Ricci operator Q commutes with the tensor field ϕ, i.e.,*

$$Q\phi = \phi Q. \tag{3.2.5}$$

Prakasha and Veeresha in established another beautiful result on para-Sasakian manifold (see lemma-1 of [80]) which using (3.2.5) can be restated as

$$(\nabla_\xi Q)X = 0, \tag{3.2.6}$$

$$(\nabla_X Q)\xi = Q\phi X + 2n\phi X. \tag{3.2.7}$$

3.3 Conformal η-Ricci soliton on Lorentzian concircular structure [or (LCS)$_n$] manifold

In this section we consider the metric g of a n-dimensional Lorentzian concircular structure manifold to represent a conformal η-Ricci soliton where the manifold satisfies some curvature properties like ξ-conharmonically semi-symmetric, ξ-concircullarly semi-symmetric and ξ-quasi-conformally semi-symmetric curvature properties.

3.3.1 (LCS)$_n$ manifold admitting ξ-conharmonically semi symmetric curvature property

Theorem 3.3.1. *A conformal η-Ricci soliton in (LCS)$_n$ manifold (M, g, ξ, η, ϕ), admitting ξ-conharmonically semi-symmetric curvature property, satisfies the following properties,*

(i) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

(ii) ξ *is a geodesic vector field,*

(iii) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Proof. Taking inner product of (3.2.2) along ξ and using (1.1.76) and (3.2.1), we have

$$\eta(H(X,Y)Z) = (\alpha^2 - \rho - \frac{p}{(n-2)} - \frac{2}{n(n-2)} + \frac{2\lambda}{(n-2)} + \frac{\alpha}{(n-2)} - \frac{\mu}{(n-2)}) (g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \quad (3.3.1)$$

Here we have considered the ξ-conharmonically semi-symmetric curvature property, i.e., $R(\xi, X).H = 0$, which yields

$$R(\xi,X)H(Y,Z)W - H(R(\xi,X)Y,Z)W - H(Y,R(\xi,X)Z)W - H(Y,Z)R(\xi,X)W = 0.$$

Applying (1.1.76) in the above equation, we have

$$g(X,H(Y,Z)W)\xi - \eta(H(Y,Z)W)X - g(X,Y)H(\xi,Z)W + \eta(Y)H(X,Z)W - g(X,Z)H(Y,\xi)W + \eta(Z)H(Y,X)W - g(X,W)H(Y,Z)\xi + \eta(W)H(Y,Z)W = 0.$$

By taking inner product of the previous equation with ξ, we get

$$g(X, H(Y, Z)W) + \eta(H(Y, Z)W)\eta(X) + g(X, Y)\eta(H(\xi, Z)W)-$$
$$\eta(Y)\eta(H(X, Z)W) + g(X, Z)\eta(H(Y, \xi)W) - \eta(Z)\eta(H(Y, X)W)+$$
$$g(X, W)\eta(H(Y, Z)\xi) - \eta(W)\eta(H(Y, Z)W) = 0.$$

After using (3.3.1) the equation reduces to,

$$g(X, H(Y, Z)W) + [\alpha^2 - \rho - \frac{2}{n(n-2)} - \frac{p - 2\lambda - \alpha + \mu}{(n-2)}]$$
$$(g(X, Z)g(Y, W) - g(Z, W)g(X, Y)) = 0.$$

Let us consider the set $\{e_i\}_{i=1}^n$ as an orthonormal basis of the manifold. Then replacing $X = Y = e_i$ in the above equation, yields

$$\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}.$$

Hence (i) is proved.

Now considering $X = \xi$ we can rewrite (1.2.90) as,

$$g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) + 2S(Y, Z) + [2\lambda - (p + \frac{2}{n})]g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0,$$

for all $Y, Z \in \chi(M)$. Simplifying using (3.2.1), the above equation reduces to,

$$g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) - 2\alpha[g(Y, Z) + \eta(Y)\eta(Z)] = 0. \tag{3.3.2}$$

Considering $Z = \xi$ in the above equation, we get

$$g(\nabla_\xi \xi, Y) = 0.$$

Since the aforementioned relation holds for any $Y \in \chi(M)$, so $\nabla_\xi \xi = 0$. This concludes that ξ is a geodesic vector field. Thus (ii) is proved.

Taking covariant derivative of (3.2.1) we can find the general expressions of ∇S and ∇Q as,

$$(\nabla_X S)(Y, Z) = -(\mu + \alpha)[g(Y, \nabla_X \xi)\eta(Z) + g(Z, \nabla_X \xi)\eta(Y)], \tag{3.3.3}$$
$$(\nabla_X Q)Y = -(\mu + \alpha)[g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi], \tag{3.3.4}$$

for any $X, Y, Z \in \chi(M)$. Letting $X = \xi$ in (3.3.3) and (3.3.4) we get, $\nabla_\xi S = 0$ and $\nabla_\xi Q = 0$. It completes our results. □

Theorem 3.3.2. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold (M, g, ξ, η, ϕ) with dimension > 1, satisfying ξ-conharmonically semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. Let ξ be a torse forming vector field. Then we have from (1.1.79) that, $\nabla_X \xi = fX + \gamma(X)\xi$, for a smooth function $f \in C^\infty(M)$, 1-form γ and for all vector field X on M. Taking inner product with ξ, yields

$$g(\nabla_X \xi, \xi) = f\eta(X) - \gamma(X).$$

Hence we get $f\eta = \gamma$. After applying this result, (1.1.79) becomes

$$\nabla_X \xi = f[X + \eta(X)\xi]. \tag{3.3.5}$$

Applying (3.3.5) in (3.3.2), we get

$$2(f - \alpha)[g(Y, Z) - \eta(Y)\eta(Z)] = 0,$$

for all vector fields Y and Z. Since $n > 1$, contracting the foregoing equation, we get $f = \alpha$. Thus (3.3.5) reduces to,

$$\nabla_X \xi = \alpha[X + \eta(X)\xi] = \alpha\phi^2(X), \tag{3.3.6}$$

i.e., $\nabla_X \xi$ is collinear to $\phi^2(X)$ for all X. Hence we get $d\eta = 0$, which means that η is colsed.

Now let us consider ξ to be recurrent vector field. So, $f = \alpha = 0$. Thus (3.3.5) yields that ξ is a concurrent vector field i.e., $\nabla_X \xi = 0$ for all vector field X on M. Also we have,

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0,$$

for all X and Y on M. Thus we can conclude ξ is Killing vector field. □

We know conformal η-Ricci soliton reduces to conformal Ricci soliton if we consider μ to be zero in (1.2.90). Accordingly the results of theorem 3.3.1 change while the results of theorem 3.3.2 remain the same for conformal Ricci soliton. We can state the modified results of theorem 3.3.1 as,

Theorem 3.3.3. *A conformal Ricci soliton in (LCS)$_n$ manifold (M, g, ξ, η, ϕ), admitting the ξ-conharmonically semi-symmetric curvature property, satisfies the following properties,*

(i) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,

(ii) ξ *is a geodesic vector field,*

(iii) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

3.3.2 (LCS)$_n$ manifold admitting ξ-concircularly semi symmetric curvature property

Theorem 3.3.4. *A conformal η-Ricci soliton in (LCS)$_n$ manifold (M, g, ξ, η, ϕ), admitting ξ-concircularly semi-symmetric curvature property satisfies the following properties,*

(i) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

(ii) ξ *is a geodesic vector field,*

(iii) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Proof. Considering inner product of (3.2.3) w.r.t. ξ and using (1.1.76), we have

$$\eta(C(X,Y)Z) = (\alpha^2 - \rho - \frac{r}{n(n-1)})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \tag{3.3.7}$$

Here we have considered ξ-concircularly semi-symmetric curvature property i.e., $R(\xi, X).C = 0$, which yields

$$R(\xi,X)C(Y,Z)W - C(R(\xi,X)Y,Z)W - C(Y,R(\xi,X)Z)W - C(Y,Z)R(\xi,X)W = 0.$$

Applying (1.1.76) in the above equation, we have

$$g(X,C(Y,Z)W)\xi - \eta(C(Y,Z)W)X - g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W -$$
$$g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W - g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)W = 0.$$

By taking inner product in the previous equation with ξ, we get

$$g(X,C(Y,Z)W) + \eta(C(Y,Z)W)\eta(X) + g(X,Y)\eta(C(\xi,Z)W) -$$
$$\eta(Y)\eta(C(X,Z)W) + g(X,Z)\eta(C(Y,\xi)W) - \eta(Z)\eta(C(Y,X)W)$$
$$+g(X,W)\eta(C(Y,Z)\xi) - \eta(W)\eta(C(Y,Z)W) = 0.$$

After using (3.3.7) the equation reduces to,

$$g(X, C(Y, Z)W) + (\alpha^2 - \rho - \frac{r}{n(n-1)})(g(X, Z)g(Y, W) - g(Z, W)g(X, Y)) = 0.$$

Then replacing $X = Y = e_i$ in the above equation, where the set $\{e_i\}_{i=1}^n$ is an orthonormal basis of the manifold, yields

$$[\frac{p}{2} + \frac{1}{n} - \lambda - \alpha - (n-1)(\alpha^2 - \rho)]g(Z, W) - (\mu + \alpha)\eta(Z)\eta(W) = 0.$$

Since this holds for arbitrary $Z, W \in \chi(M)$, setting $Z = W = \xi$ we have,

$$\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}.$$

This proves (i), the first assertion of this theorem and the expression is identical with relation (i) of theorem 3.3.1. Other two outcomes (ii) and (iii) are immediate consequences and can be proved similarly like theorem 3.3.1. □

Theorem 3.3.5. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold (M, g, ξ, η, ϕ) with dimension $n > 1$, satisfying ξ-concircularly semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. Since the results of theorem 3.3.4 for concircular curvature tensor are same as of theorem 3.3.1 for conharmonic curvature tensor, the proof of this theorem is identical with the proof of theorem 3.3.2. □

We know conformal η-Ricci soliton is a mere generalisation conformal Ricci soliton. If we let μ to be zero in (1.2.90) then it reduces to conformal Ricci soliton. The results of theorem 3.3.4 change while the results of theorem 3.3.5 remain the same for conformal Ricci soliton. We can state the results of theorem 3.3.4 as,

Theorem 3.3.6. *A conformal Ricci soliton in $(LCS)_n$ manifold (M, g, ξ, η, ϕ), satisfying the ξ-concircularly semi-symmetric curvature property, satisfies the following properties,*

(i) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,

(ii) ξ *is a geodesic vector field,*

(iii) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

3.3.3 (LCS)$_n$ manifold admitting ξ-quasi-conformally semi symmetric curvature property

Theorem 3.3.7. *A conformal η-Ricci soliton in (LCS)$_n$ manifold (M, g, ξ, η, ϕ), satisfying ξ-quasi-conformally semi-symmetric curvature property, admits the following properties,*

(i) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

(ii) ξ *is a geodesic vector field,*

(iii) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Proof. Taking inner product of (3.2.4) w.r.t. ξ and using (1.1.76), we have

$$\begin{aligned}\eta(\tilde{C}(X,Y)Z) &= [a(\alpha^2-\rho)+b(p+\frac{2}{n}-2\lambda-\alpha+\mu)-\frac{r}{n}(\frac{a}{n-1}+2b)] \\ &\quad (g(Y,Z)\eta(X)-g(X,Z)\eta(Y)). \end{aligned} \tag{3.3.8}$$

Here we have considered ξ-quasi-conformally semi-symmetric curvature property i.e., $R(\xi, X).\tilde{C} = 0$, which yields

$$R(\xi,X)\tilde{C}(Y,Z)W - \tilde{C}(R(\xi,X)Y,Z)W - \tilde{C}(Y,R(\xi,X)Z)W - \tilde{C}(Y,Z)R(\xi,X)W = 0.$$

Applying (1.1.76) in the above equation, we have

$$\begin{aligned}&g(X,\tilde{C}(Y,Z)W)\xi - \eta(\tilde{C}(Y,Z)W)X - g(X,Y)\tilde{C}(\xi,Z)W + \eta(Y)\tilde{C}(X,Z)W - \\ &g(X,Z)\tilde{C}(Y,\xi)W + \eta(Z)\tilde{C}(Y,X)W - g(X,W)\tilde{C}(Y,Z)\xi + \eta(W)\tilde{C}(Y,Z)W = 0.\end{aligned}$$

By taking inner product in the previous equation w.r.t. ξ, we get

$$\begin{aligned}&g(X,\tilde{C}(Y,Z)W) + \eta(\tilde{C}(Y,Z)W)\eta(X) + g(X,Y)\eta(\tilde{C}(\xi,Z)W) - \\ &\eta(Y)\eta(\tilde{C}(X,Z)W) + g(X,Z)\eta(\tilde{C}(Y,\xi)W) - \eta(Z)\eta(\tilde{C}(Y,X)W) \\ &+g(X,W)\eta(\tilde{C}(Y,Z)\xi) - \eta(W)\eta(\tilde{C}(Y,Z)W) = 0.\end{aligned}$$

After using (3.3.8) the above equation reduces to,

$$\begin{aligned}&g(X,\tilde{C}(Y,Z)W) + [a(\alpha^2-\rho)+b(p+\frac{2}{n}-2\lambda-\alpha+\mu)-\frac{r}{n}(\frac{a}{n-1}+2b)] \\ &(g(X,Z)g(Y,W)-g(Z,W)g(X,Y)) = 0.\end{aligned}$$

Then replacing $X = Y = e_i$ in the above equation, where the set $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of the manifold, yields

$$[a+(n-2)b]S(Z,W)+[br-\frac{r}{n}(a+2(n-1)b)]g(Z,W)+[a(\alpha^2-\rho)+$$
$$b(p+\frac{2}{n}-2\lambda-\alpha+\mu)-\frac{r}{n}(\frac{a}{n-1}+2b)]g(Z,W)=0.$$

Since this holds for arbitrary $Z, W \in \chi(M)$, setting $Z = W = \xi$ we have,

$$\lambda+(n-1)\alpha^2=\frac{p}{2}+\mu+(n-1)\rho+\frac{1}{n}.$$

Hence first assertion of this theorem is proved.

(ii) and (iii) can be proved in similar manner like the proof of theorem 3.3.1. □

Theorem 3.3.8. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold (M, g, ξ, η, ϕ) with dimension $n > 1$, admitting ξ-quasi-conformally semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. The proof can be done in similar fashion like theorem 3.3.2. □

To get conformal Ricci soliton from conformal η-Ricci soliton we assume $\mu = 0$ in (1.2.90). Consequently the results of theorem 3.3.7 change while the results of theorem 3.3.8 remain unchanged for conformal Ricci soliton. We can rewrite the results of theorem 3.3.7 as,

Theorem 3.3.9. *A conformal Ricci soliton in $(LCS)_n$ manifold (M, g, ξ, η, ϕ), admitting ξ-quasi-conformally semi-symmetric curvature property satisfies the following properties,*

(i) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,

(ii) ξ *is a geodesic vector field,*

(iii) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

3.4 Para-Sasakian manifold admitting conformal η-Ricci soliton

In this section, we have studied conformal η-Ricci soliton and almost conformal η-Ricci soliton on $(2n+1)$-dimensional para-Sasakian manifold. It follows from (1.2.90) that almost conformal η-Ricci soliton is the generalization of almost η-Ricci soliton because it involve two smooth functions λ and μ. First we prove the following lemma which has been used to prove the next theorems.

Lemma 3.4.1. *If the metric g of a para-Sasakian manifold represents a conformal η-Ricci soliton, then*

$$\eta(\mathcal{L}_V\xi) = -(\mathcal{L}_V\eta)\xi = \lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu. \tag{3.4.1}$$

Proof. As the metric g satisfies conformal η-Ricci soliton equation (1.2.90), using (1.1.64), we can easily obtain

$$(\mathcal{L}_V g)(X,\xi) + 2(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)\eta(X) = 0,$$

for arbitrary vector field X. Lie differentiation of the relation $\eta(X) = g(X,\xi)$ along the soliton vector field V yields $(\mathcal{L}_V g)(X,\xi) = (\mathcal{L}_V\eta)X - g(X,\mathcal{L}_V\xi)$. Using this in the foregoing equation, we have

$$(\mathcal{L}_V\eta)X - g(X,\mathcal{L}_V\xi) = -2(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)\eta(X). \tag{3.4.2}$$

Finally taking Lie derivative of (1.1.47) along V into account, we can easily obtain our desired result (3.4.1). □

Theorem 3.4.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a para-Sasakian manifold. If the metric g represents a conformal η-Ricci soliton then the manifold is η-Einstein and either the soliton vector field V is Killing or it leaves ϕ invariant.*

Proof. Taking covariant derivative of (1.1.47) along arbitrary vector field Y and using (1.1.62), we can easily have $(\nabla_Y\eta)X = g(\phi X, Y)$.

Since the metric g of the manifold represents a conformal η-Ricci soliton, taking covariant derivative of (1.2.90) along arbitrary vector field Z, we obtain

$$(\nabla_Z\mathcal{L}_V g)(X,Y) = -2(\nabla_Z S)(X,Y) - 2\mu[g(\phi X,Z)\eta(Y) + g(\phi Y,Z)\eta(X)], \tag{3.4.3}$$

for all vector fields X, Y and Z on M. Now, ∇ is the metric connection on M i.e., $\nabla g = 0$. So the equation (1.1.3) reduces to,

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V \nabla)(X,Y),Z) + g((\mathcal{L}_V \nabla)(X,Z),Y), \tag{3.4.4}$$

for all vector fields X, Y, Z on M. Combining (3.4.3) and (3.4.4), we have

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X,Z),Y) + g((\mathcal{L}_V \nabla)(Y,Z),X) &= -2(\nabla_Z S)(X,Y) \\ &\quad - 2\mu[g(\phi X,Z)\eta(Y) + g(\phi Y,Z)\eta(X)]. \end{aligned}$$

By a straightforward combinatorial computation, the foregoing equation yields

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X,Y),Z) &= (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(Z,X) \\ &\quad + 2\mu[g(\phi X,Z)\eta(Y) + g(\phi Y,Z)\eta(X)], \end{aligned} \tag{3.4.5}$$

for all $X, Y, Z \in \chi(M)$. Setting $Y = \xi$ and making use of $(\nabla_Z S)(X,Y) = g((\nabla_Z Q)X, Y)$, (3.2.5), (3.2.6) and (3.2.7), we obtain

$$(\mathcal{L}_V \nabla)(X,\xi) = 2(\mu - 2n)(\phi X) - 2Q\phi X. \tag{3.4.6}$$

Differentiating the last equation covariantly with respect to arbitrary vector field Y and using (1.1.61) and (1.1.62), we acquire

$$\begin{aligned} (\nabla_Y \mathcal{L}_V \nabla)(X,\xi) &= (\mathcal{L}_V \nabla)(X,\phi Y) - 2(\nabla_Y Q)(\phi X) - 2\eta(X)(QY) \\ &\quad - 2\mu g(X,Y)\xi + 2(\mu - 2n)\eta(X)Y. \end{aligned} \tag{3.4.7}$$

Using (3.2.6), (3.4.6) and (3.4.7) in (1.1.5), we get

$$(\mathcal{L}_V R)(X,\xi)\xi = 4(\mu - 2n)X - 4QX - 4\mu\eta(X)\xi, \tag{3.4.8}$$

which holds for an arbitrary vector field X. Again, from (1.1.63) we get $R(X,\xi)\xi = \eta(X)\xi - X$. By virtue of (3.4.1) and (3.4.2), Lie differentiation of the last relation along V yields

$$(\mathcal{L}_V R)(X,\xi)\xi = 2(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)[X - \eta(X)\xi]. \tag{3.4.9}$$

Substituting the value of $(\mathcal{L}_V R)(X,\xi)\xi$ from (3.4.8) in the foregoing equation and taking inner product with arbitrary vector field Y, we obtain

$$S(X,Y) = \frac{1}{2}(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n - \mu)\eta(X)\eta(Y) - \frac{1}{2}(\lambda - \frac{p}{2} - \frac{1}{2n+1} + 2n - \mu)g(X,Y), \tag{3.4.10}$$

for all $X, Y \in \chi(M)$. This transforms the soliton equation (1.2.90) to

$$(\mathcal{L}_V g)(X,Y) = (\frac{p}{2} + \frac{1}{2n+1} + 2n - \lambda - \mu)[g(X,Y) + \eta(X)\eta(Y)]. \tag{3.4.11}$$

Differentiating (3.4.10) covariantly along arbitrary vector field Z, we get $(\nabla_Z S)(X,Y) = \frac{1}{2}(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n - \mu)[g(\phi X, Z)\eta(Y) + g(\phi Y, Z)\eta(X)]$. Repeated use of this in (3.4.3), gives rise to

$$(\mathcal{L}_V \nabla)(X,Y) = (\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)[\eta(Y)(\phi X) + \eta(X)(\phi Y)], \tag{3.4.12}$$

for arbitrary vector fields X and Y on M. Covariant differentiation of the aforementioned equation along arbitrary vector field Z and use of (1.1.61), yields

$$\begin{aligned}(\nabla_Z \mathcal{L}_V \nabla)(X,Y) =& (\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)[g(\phi Y, Z)(\phi X) + g(\phi X, Z) \\ & (\phi Y) - g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + 2\eta(X)\eta(Y)Z].\end{aligned}$$

Using this relation in (1.1.5) and contracting X, we get

$$(\mathcal{L}_V S)(Y,Z) = 2(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)[(2n+1)\eta(Y)\eta(Z) - g(Y,Z)]. \tag{3.4.13}$$

Taking Lie derivative of (3.4.10) along V and using (3.4.11), we achieve

$$\begin{aligned}(\mathcal{L}_V S)(Y,Z) =& \frac{1}{2}(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n - \mu)[((\mathcal{L}_V \eta)Y)\eta(Z) + \eta(Y)((\mathcal{L}_V \eta)Z)] + \frac{1}{2}(\lambda - \frac{p}{2} \\ & - \frac{1}{2n+1} + 2n - \mu)(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)[g(Y,Z) + \eta(Y)\eta(Z)].\end{aligned} \tag{3.4.14}$$

Comparisons of (3.4.13) and (3.4.14) gives

$$\begin{aligned}& 2(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)[(2n+1)\eta(Y)\eta(Z) - g(Y,Z)] = \\ & \frac{1}{2}(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n - \mu)[((\mathcal{L}_V \eta)Y)\eta(Z) + \eta(Y)((\mathcal{L}_V \eta)Z)] + \\ & \frac{1}{2}(\lambda - \frac{p}{2} - \frac{1}{2n+1} + 2n - \mu)(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)[g(Y,Z) + \eta(Y)\eta(Z)].\end{aligned} \tag{3.4.15}$$

Substituting Y and Z by $\phi^2 Y$ and ϕZ respectively and using (1.1.46), (1.1.49) and (1.1.51), we obtain

$$(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)(\lambda - \frac{p}{2} - \frac{1}{2n+1} + 2n - \mu + 4)d\eta(Y,Z) = 0, \tag{3.4.16}$$

$\forall Y, Z \in \chi(M)$. As we know, in para-Sasakian manifold $d\eta \neq 0$, we have $(\lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu)(\lambda - \frac{p}{2} - \frac{1}{2n+1} + 2n - \mu + 4) = 0$. This gives either $\lambda = \frac{p}{2} + \frac{1}{2n+1} + 2n - \mu$ or $\lambda = \frac{p}{2} + \frac{1}{2n+1} - 2n + \mu - 4$.

Case-I: If $\lambda = \frac{p}{2}+\frac{1}{2n+1}+2n-\mu$, then (3.4.10) reduces to $S(X,Y) = (\mu-2n)g(X,Y) - \mu\eta(X)\eta(Y)$ i.e., the manifold is η-Einstein. Also (3.4.11) gives $\mathcal{L}_V g = 0$. So, V is Killing vector field.

Case-II: Using $\lambda = \frac{p}{2} + \frac{1}{2n+1} - 2n + \mu - 4$ in (3.4.10), we get $S(X,Y) = 2g(X,Y) - 2(n+1)\eta(X)\eta(Y)$. So, the manifold is η-Einstein. Substituting Y by ϕY and setting $Z = \xi$ in (3.4.15), we obtain $(\mathcal{L}_V \eta)(\phi Y) = 0$. Further, replacing Y by ϕY and using (1.1.46), (3.4.1) and $\lambda = \frac{p}{2} + \frac{1}{2n+1} - 2n + \mu - 4$, we get

$$\mathcal{L}_V \eta = 2(2n - \mu + 2)\eta.$$

Exterior differentiation of the foregoing equation and use of well-known relation $d(\mathcal{L}_V \eta) = \mathcal{L}_V d\eta$ and (1.1.51), yields

$$(\mathcal{L}_V d\eta)(X,Y) = 2(2n - \mu + 2)g(X, \phi Y), \tag{3.4.17}$$

for arbitrary vector fields X and Y on M. Lie differentiation of (1.1.51) along V, infers

$$(\mathcal{L}_V d\eta)(X,Y) = 2(2n - \mu + 2)g(X, \phi Y) + g(X, (\mathcal{L}_V \phi)Y), \tag{3.4.18}$$

$\forall X, Y \in \chi(M)$. Comparing this with (3.4.17) gives $\mathcal{L}_V \phi = 0$, as X and Y are arbitrary vector fields. So, V leaves ϕ invariant. □

Theorem 3.4.2. *If the metric g of a para-Sasakian manifold M represents a conformal η-Ricci soliton and if the soliton vector field V is pointwise collinear with the characteristic vector field ξ then V is a constant multiple of ξ and the scalar curvature of the manifold is constant.*

Proof. Since the soliton vector field V is pointwise collinear with the characteristic vector field ξ, so, $V = f\xi$ where, f is a smooth function on M^{2n+1}. Substituting $V = f\xi$ in $(\mathcal{L}_V g)(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ and using (1.1.62), we get

$$(\mathcal{L}_V g)(X,Y) = (Xf)\eta(Y) + (Yf)\eta(X), \tag{3.4.19}$$

for arbitrary vector field X and Y on M. Using (3.4.19) in the soliton equation (1.2.90), we obtain

$$(Xf)\eta(Y)+(Yf)\eta(X)+2S(X,Y)+2\mu\eta(X)\eta(Y)+2(\lambda-\frac{p}{2}-\frac{1}{2n+1})g(X,Y)=0. \quad (3.4.20)$$

Setting $Y=\xi$ and using (1.1.47) and (1.1.64), the above equation becomes

$$Df = 2[2n+\frac{p}{2}+\frac{1}{2n+1}-\frac{1}{2}(\xi f)-\lambda-\mu]\xi. \quad (3.4.21)$$

Taking inner product with respect to Reeb vector field ξ, we acquire

$$\xi f = 2n+\frac{p}{2}+\frac{1}{2n+1}-\lambda-\mu. \quad (3.4.22)$$

From previous theorem we obtained either $\lambda=\frac{p}{2}+\frac{1}{2n+1}+2n-\mu$ or $\lambda=\frac{p}{2}+\frac{1}{2n+1}-2n+\mu-4$. If we consider $\lambda=\frac{p}{2}+\frac{1}{2n+1}+2n-\mu$, then from (3.4.22) we get $\xi f=0$. Substituting these values in (3.4.21), yields

$$Df = 0. \quad (3.4.23)$$

Now, if we consider $\lambda=\frac{p}{2}+\frac{1}{2n+1}-2n+\mu-4$, then from (3.4.22) we obtain $\xi f=2(2n-\mu+2)$ and from (3.4.21), we get

$$Df = (\xi f)\xi. \quad (3.4.24)$$

Taking (1.1.62) into account, differentiating the foregoing equation along X and then taking scalar product with arbitrary vector field Y, leads to

$$g(\nabla_X Df, Y) = (X(\xi f))\eta(Y)-(\xi f)g(\phi X, Y). \quad (3.4.25)$$

Anti-symmetrizing the last equation and using $g(\nabla_X Df, Y)=g(X,\nabla_Y Df)$, we have

$$(X(\xi f))\eta(Y)-(Y(\xi f))\eta(X)-2(\xi f)g(\phi X, Y)=0, \quad (3.4.26)$$

for arbitrary vector fields X and Y. If we let X to be unit vector (i.e., $g(X,X)=1$) in $Ker(\eta)$, then ϕX also becomes a unit vector with $g(\phi X,\phi Y)=-1$. Now replacing Y by ϕX in (3.4.26), we get $\xi f=0$. Substituting this value in (3.4.24) leads to

$$Df = 0. \quad (3.4.27)$$

Combining (3.4.23) and (3.4.27) we can conclude that $Df=0$ in entire manifold. Therefore f is constant. So, V is a constant multiple of ξ.
The equation (3.4.20) reduces to $QX+2(\lambda-\frac{p}{2}-\frac{1}{2n+1})X+2\mu\eta(X)\xi=0$. Tracing of this equation leads to $r=2-2\mu+(2n+1)(p-2\lambda)$, where r denotes the scalar curvature of the manifold. This completes the proof. □

Theorem 3.4.3. *Let M^{2n+1} be a para-Sasakian manifold with $n > 1$. If g represents an almost conformal η-Ricci soliton with the soliton vector field V as infinitesimal contact transformation, then the manifold is η-Einstein and either the soliton vector field V is Killing or it leaves ϕ invariant.*

Proof. Since V is of infinitesimal contact transformation, there exists a certain $a \in C^{\infty}(M)$ such that

$$\mathcal{L}_V \eta = a\eta. \tag{3.4.28}$$

Taking $d(\mathcal{L}_V \eta) = \mathcal{L}_V d\eta$ into account, exterior derivative of (3.4.28) gives

$$\mathcal{L}_V d\eta = (da) \wedge \eta + ad\eta.$$

Using (1.1.51), the foregoing equation can be rewritten as

$$(\mathcal{L}_V d\eta)(X, Y) = \frac{1}{2}[(Xa)\eta(Y) - \eta(X)(Ya)] + ag(X, \phi Y), \tag{3.4.29}$$

for arbitrary vector fields X and Y on M. Lie differentiation of (1.1.51) along the soliton vector field V, yields

$$(\mathcal{L}_V d\eta)(X, Y) = g(X, (\mathcal{L}_V \phi)Y) - 2g((\lambda - \frac{p}{2} - \frac{1}{2n+1})X + QX, \phi Y), \tag{3.4.30}$$

$\forall X, Y \in \chi(M)$. Comparing the aforementioned equation with (3.4.29), we obtain

$$2(\mathcal{L}_V \phi)Y = \eta(Y)(Da) - (Ya)\xi + (2a + 4\lambda - 2p - \frac{4}{2n+1})(\phi Y) + 4Q\phi Y, \tag{3.4.31}$$

for any vector Y in $\chi(M)$. Setting $Y = \xi$ and using (1.1.47) and (1.1.48), we get

$$2(\mathcal{L}_V \phi)\xi = Da - (\xi a)\xi. \tag{3.4.32}$$

Taking (1.2.90) and (3.4.28) into consideration, Lie differentiating $g(\xi, \xi) = 1$ and $g(X, \xi) = \eta(X)$ along V, we achieve

$$\eta(\mathcal{L}_V \xi) = \lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n + \mu, \tag{3.4.33}$$

$$\mathcal{L}_V \xi = (a + 2\lambda - p - \frac{2}{2n+1} - 4n + 2\mu)\xi. \tag{3.4.34}$$

Combining above two relations, we get

$$a = 2n - \lambda - \mu + \frac{p}{2} + \frac{1}{2n+1}. \tag{3.4.35}$$

Lie differentiating (1.1.48) along the soliton vector field V and using (3.4.35), we obtain $(\mathcal{L}_V\phi)\xi = 0$. Combining this with (3.4.32) we have $Da = (\xi a)\xi$, which further implies

$$da = (\xi a)\eta. \tag{3.4.36}$$

Operating the foregoing equation by exterior derivative operator d and using $d^2 = 0$, yields

$$d(\xi a) \wedge \eta + (\xi a)d\eta = 0.$$

Taking $\eta \wedge \eta = 0$ and $\eta \wedge d\eta \neq 0$ into account, wedge product of the above equation with respect to the 1-form η gives $\xi a = 0$. Substituting this in (3.4.36) we get $da = 0$ and so a is constant. Then the equation (3.4.31) reduces to

$$(\mathcal{L}_V\phi)Y = (a + 2\lambda - p - \frac{2}{2n+1})(\phi Y) + 2Q\phi Y, \tag{3.4.37}$$

for any vector field Y on M. Operating (1.1.46) by $\mathcal{L}_V$ and using (3.4.28) and (3.4.34) we get $\mathcal{L}_V\phi^2 = 0$. Consequently, we get $(\mathcal{L}_V\phi)(\phi X) + \phi(\mathcal{L}_V\phi)X = 0$ for an arbitrary vector field X. After repeated application of (3.4.37) and use of (1.1.46), (1.1.64) and (3.2.5), the aforementioned relation leads to

$$QX = -(\frac{a}{2} + \lambda - \frac{p}{2} - \frac{1}{2n+1})X + (\frac{a}{2} + \lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n)\eta(X)\xi, \tag{3.4.38}$$

for any vector field X of M. Covariant derivative of (3.4.38) along an arbitrary vector field Y yields

$$(\nabla_Y Q)X = (\frac{a}{2}+\lambda-\frac{p}{2}-\frac{1}{2n+1}-2n)[g(Y,\phi X)\xi-\eta(X)(\phi Y)]-(Y\lambda)[X-\eta(X)\xi]. \tag{3.4.39}$$

It is well-known that $Xr = g((\nabla_X Q)e_i, e_i)$ and $\frac{1}{2}Xr = g((\nabla_{e_i}Q)X, e_i)$, where $\{e_i\}_{i=1}^{2n+1}$ is an orthonormal basis of the manifold. Successive application of (3.4.39) in these two relations infers

$$Xr = -2n(X\lambda), \tag{3.4.40}$$

$$Xr = -2(X\lambda) + 2\eta(X)(\xi\lambda), \tag{3.4.41}$$

for an arbitrary vector field X. Since the characteristic vector field ξ is a Killing vector field in a para-Sasakian manifold, it follows that $\xi r = 0$. Plugging this in (3.4.40), we obtain $\xi\lambda = 0$ as we assume $n \neq 0$. Consequently, (3.4.41) reduces to $Xr = -2(X\lambda), \forall X \in \chi(M)$. Setting this in (3.4.40), we get

$$(n-1)(X\lambda) = 0.$$

Since $n > 1$ and X is an arbitrary vector field, we conclude that λ is a constant. Thus it follows from (3.4.35) that μ is also constant. Then the soliton reduces to conformal η-Ricci soliton and the result follows from theorem-3.4.1. □

Theorem 3.4.4. *Let M^{2n+1} be a para-Sasakian manifold of dimension > 3. If g represents a gradient almost conformal η-Ricci soliton, then the soliton reduces to gradient almost conformal Ricci soliton and the manifold is Einstein.*

Proof. From the gradient almost conformal η-Ricci soliton equation (1.2.91), we easily obtain

$$\nabla_X Df = -QX + (\frac{p}{2} + \frac{1}{2n+1} - \lambda)X - \mu\eta(X)\xi, \tag{3.4.42}$$

for an arbitrary vector field X of M. Taking covariant derivative along an arbitrary vector field Y, we acquire

$$\begin{aligned}\nabla_Y \nabla_X Df =& -(\nabla_Y Q)X - Q(\nabla_Y X) + (\frac{p}{2} + \frac{1}{2n+1} - \lambda)(\nabla_Y X) - (Y\lambda)X \\ &- (Y\mu)\eta(X)\xi - \mu[g(\phi X, Y)\xi + \eta(\nabla_Y X)\xi - \eta(X)(\phi Y)].\end{aligned}$$

Plugging the above equation along with (3.4.42) in (1.1.4), we infer

$$\begin{aligned}R(X,Y)Df =& (\nabla_Y Q)X - (\nabla_X Q)Y + (Y\lambda)X - (X\lambda)Y + (Y\mu)\eta(X)\xi \\ &- (X\mu)\eta(Y)\xi + \mu[2g(\phi X, Y)\xi - \eta(X)(\phi Y) + \eta(Y)(\phi X)],\end{aligned} \tag{3.4.43}$$

for all $X, Y \in \chi(M)$. Setting $Y = \xi$ in the foregoing equation and using (3.2.6) and (3.2.7), we get

$$R(X,\xi)Df = -Q\phi X + (\mu - 2n)(\phi X) + (\xi\lambda)X - (X\lambda)\xi + (\xi\mu)\eta(X)\xi - (X\mu)\xi.$$

Combining the last equation with (1.1.61) and (1.1.63), we obtain

$$\begin{aligned}g((\nabla_X\phi)Y, Df) =& -g(Q\phi X, Y) + (\mu - 2n)g(\phi X, Y) + (\xi\lambda)g(X,Y) \\ &- (X\lambda)\eta(Y) + (\xi\mu)\eta(X)\eta(Y) - (X\mu)\eta(Y),\end{aligned} \tag{3.4.44}$$

for arbitrary vector fields X and Y on M. Replacing X and Y by ϕX and ϕY in the foregoing equation, we achieve

$$g((\nabla_{\phi X}\phi)\phi Y, Df) = g(Q\phi X, Y) - (\mu - 2n)g(\phi X, Y) - (\xi\lambda)[g(X,Y) - \eta(X)\eta(Y)], \tag{3.4.45}$$

where we have used (1.1.46), (1.1.47), (1.1.50), (1.1.64) and (3.2.5). Subtraction of (3.4.44) from (3.4.45) yields

$$g((\nabla_{\phi X}\phi)\phi Y - (\nabla_X\phi)Y, Df) = 2g(Q\phi X, Y) - 2(\mu - 2n)g(\phi X, Y) - 2(\xi\lambda)g(X, Y) + (\xi\lambda)\eta(X)\eta(Y) + (X\lambda)\eta(Y) - (\xi\mu)\eta(X)\eta(Y) + (X\mu)\eta(Y). \quad (3.4.46)$$

From Zamkovoy [109], we know

$$(\nabla_{\phi X}\phi)\phi Y - (\nabla_X\phi)Y = 2g(X, Y)\xi - \eta(Y)[X + \eta(X)\xi], \quad (3.4.47)$$

holds for arbitrary vector fields X and Y in a para-Sasakian manifold (for proof see lemma-2.7 of [109] and here we have used $h = 0$ which holds in para-Sasakian manifold). Taking (3.4.47) into account, (3.4.46) can be rewritten as

$$2g(X, Y)(\xi f) - \eta(Y)(Xf) - \eta(X)\eta(Y)(\xi f) = 2g(Q\phi X, Y) - 2(\mu - 2n)g(\phi X, Y) - 2(\xi\lambda)g(X, Y) + (\xi\lambda)\eta(X)\eta(Y) + (X\lambda)\eta(Y) - (\xi\mu)\eta(X)\eta(Y) + (X\mu)\eta(Y). \quad (3.4.48)$$

Anti-symmetrizing the last equation and then replacing X and Y by ϕX and ϕY, respectively we get $Q\phi X = (\mu - 2n)(\phi X)$. Further, substitution of X by ϕX in the last relation yields

$$QX = (\mu - 2n)X - \mu\eta(X)\xi. \quad (3.4.49)$$

Covariant differentiation of (3.4.49) along an arbitrary vector field Y and using that expression of $(\nabla_Y Q)X$ in (3.4.43), we obtain

$$R(X, Y)Df = (Y\mu)X - (X\mu)Y + (Y\lambda)X - (X\lambda)Y, \quad (3.4.50)$$

$\forall X, Y \in \chi(M)$. Contraction of (3.4.43) and (3.4.50) over X, yields

$$Q(Df) = \frac{1}{2}(Dr) + 2n(D\lambda) + (D\mu) - (\xi\mu)\xi, \quad (3.4.51)$$

$$Q(Df) = 2n(D\mu + D\lambda). \quad (3.4.52)$$

Comparing the last two relations we get

$$(2n - 1)D\mu = \frac{1}{2}(Dr) - (\xi\mu)\xi. \quad (3.4.53)$$

Tracing (3.4.49), we find $r = 2n\mu - 2n(2n + 1)$. So, $Dr = 2nD\mu$. Plugging this relation in (3.4.53), we obtain

$$(n - 1)D\mu + (\xi\mu)\xi = 0. \quad (3.4.54)$$

As we know $g((\nabla_{e_i}Q)\xi, e_i) = \frac{1}{2}(\xi r)$, we can easily obtain $\xi r = 0$. Combining this result with $Dr = 2nD\mu$, we get $\xi\mu = 0$. Using this in (3.4.54), we have $D\mu = 0$ (since $n \neq 1$), so μ is constant. Scalar product of (3.4.52) with respect to the characteristic vector field ξ and use of (1.1.64), yields

$$\xi(f+\lambda) = 0, \tag{3.4.55}$$

as $n > 1$. Now, setting $Y = \xi$ in (3.4.50) and using (1.1.63), we obtain

$$(X(f+\lambda))\xi = (\xi(f+\lambda))X, \tag{3.4.56}$$

for an arbitrary vector field X. Plugging equation (3.4.55) in the above equation, we get that $f+\lambda$ is constant. As we have μ and $f+\lambda$ are constants, replacing X by Df in (3.4.49), we get

$$\mu[(Df) - (\xi f)\xi] = 0. \tag{3.4.57}$$

We suppose, μ is a non-zero constant. Then, from last equation, we get $Df = (\xi f)\xi$. Substitution of Df by $(\xi f)\xi$ in (3.4.42) and use of (1.1.62), yields

$$(X(\xi f))\xi - (\xi f)(\phi X) = (\frac{p}{2} + \frac{1}{2n+1} - \lambda - \mu + 2n)X, \tag{3.4.58}$$

for any vector field X of $\chi(M)$. Taking inner product of the foregoing equation with respect to ξ and substituting the resultant relation in (3.4.58), we get

$$(\xi f)(\phi X) + (\frac{p}{2} + \frac{1}{2n+1} - \lambda - \mu + 2n)[X - \eta(X)\xi] = 0.$$

Contracting the last equation over X and using $n \neq 0$, we obtain $\frac{p}{2} + \frac{1}{2n+1} - \lambda - \mu + 2n = 0$. Since μ is a constant, we have λ is also constant. As we know $f+\lambda$ is constant, this yields f is constant. This contradicts the fact that the soliton vector field V is non-zero as we get $V = Df = 0$.

So, μ must be identically equal to zero and the soliton reduces to gradient almost conformal Ricci soliton. Finally, the equation (3.4.49) reduces to $QX = -2nX \ \forall X \in \chi(M)$. So, the manifold becomes Einstein with Einstein constant $-2n$. Furthermore, the scalar curvature of the manifold can be expressed as $r = -2n(2n+1)$. □

Example 3.4.1. *We consider the example of the paper [70]. In this paper, authors considered the Euclidean space $M = \mathbb{R}^3$ with Cartesian coordinates (x, y, z) and defined the normal almost paracontact metric structure (φ, ξ, η, g) on M as follows,*

$$\varphi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}, \qquad \varphi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \qquad \varphi(\frac{\partial}{\partial z}) = 0,$$

$$\xi = 2\frac{\partial}{\partial z}, \qquad \eta = \frac{1}{2}(dz + ydx),$$

$$(g_{ij}) = \begin{pmatrix} \frac{y^2-1}{2} & 0 & \frac{y}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{y}{4} & 0 & \frac{1}{4} \end{pmatrix},$$

and authors have shown that the manifold M is para-Sasakian. Authors have taken pseudo-orthonormal φ-basis $e_1 = 2\partial y$, $e_2 = 2\partial x - 2y\partial z$ and $e_3 = \xi = 2\partial z$ and also obtained the expressions of the curvature tensor and the Ricci tensor respectively as follows,

$$R(e_1, e_2)e_1 = -3e_2, \qquad R(e_1, e_2)e_2 = -3e_1, \qquad R(e_1, e_2)e_3 = 0,$$
$$R(e_1, e_3)e_1 = \xi, \qquad R(e_1, e_3)e_2 = 0, \qquad R(e_1, e_3)e_3 = -e_1,$$
$$R(e_2, e_3)e_1 = 0, \qquad R(e_2, e_3)e_2 = -\xi, \qquad R(e_2, e_3)e_3 = -e_2,$$

and

$$S(e_1, e_1) = 2, \qquad S(e_2, e_2) = -2, \qquad S(e_3, e_3) = -2.$$

Also the scalar curvature r=2. Thus

$$S(X, Y) = 2g(X, Y) - 4\eta(X)\eta(Y), \qquad \forall X, Y \in \chi(M). \tag{3.4.59}$$

Let $f : M \to \mathbb{R}$ be a smooth function defined by,

$$f(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + z^2. \tag{3.4.60}$$

Then the gradient of f, Df is given by,

$$Df = (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}).$$

Now $(\mathcal{L}_{Df}g)(e_1, e_1) = -2g(e_1, e_1) = 2$, $(\mathcal{L}_{Df}g)(e_2, e_2) = 2g(e_2, e_2) = -2$ and $(\mathcal{L}_{Df}g)(e_3, e_3) = 4$. Then from the above results we can verify that,

$$(\mathcal{L}_{Df}g)(X, Y) = 2\{g(X, Y) + \eta(X)\eta(Y)\}, \tag{3.4.61}$$

for all $X, Y \in \chi(M)$. From (3.4.59) and (3.4.61) we obtain that g represents a gradient almost conformal η-Ricci soliton i.e., it satisfies (1.2.91) for $V = Df$, where f is defined by (3.4.60), $\lambda = \frac{p}{2} - \frac{8}{3}$ and $\mu = 3$.

3.5 $*$-conformal η-Ricci soliton on Kenmotsu manifold

In this section we consider that the metric g of a $(2n+1)$-dimensional Kenmotsu manifold represents a $*$-conformal η-Ricci soliton and a gradient almost $*$-conformal η-Ricci soliton.

Theorem 3.5.1. *Let $M^{(2n+1)}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. If the metric g represents a $*$-conformal η-Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein.*

Proof. Since the metric g of the Kenmotsu manifold represents a $*$-conformal η-Ricci soliton so both of the equations (1.2.92) and (2.2.4) are satisfied. Combining these two, we have

$$(\mathcal{L}_V g)(X,Y) = -2S(X,Y) - (2\lambda - p - \frac{2}{(2n+1)} + 4n - 2)g(X,Y) - 2(\mu+1)\eta(X)\eta(Y). \tag{3.5.1}$$

Taking covariant derivative w.r.t. arbitrary vector field Z and using (1.1.24), we have

$$\begin{aligned}(\nabla_Z\mathcal{L}_V g)(X,Y) &= -2(\nabla_Z S)(X,Y) - 2(\mu+1)\{g(X,Z)\eta(Y) \\ &\quad + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\},\end{aligned} \tag{3.5.2}$$

for all $X,Y,Z\in\chi(M)$. Since ∇ is the metric connection i.e., $\nabla g = 0$, the equation (1.1.3) reduces to,

$$(\nabla_Z\mathcal{L}_V g)(X,Y) = g((\mathcal{L}_V\nabla)(X,Z),Y) + g((\mathcal{L}_V\nabla)(Y,Z),X), \tag{3.5.3}$$

for all vector fields X, Y, Z on M. Combining (3.5.2) and (3.5.3) and by a straightforward combinatorial computation and using the symmetry of $(\mathcal{L}_V\nabla)$, the foregoing equation yields

$$\begin{aligned}g((\mathcal{L}_V\nabla)(X,Y),Z) &= (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(Z,X) \\ &\quad -2(\mu+1)\{g(X,Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\},\end{aligned} \tag{3.5.4}$$

for arbitrary vector fields X,Y and Z on M. Using (2.2.2) and (2.2.3), the foregoing equation yields

$$(\mathcal{L}_V\nabla)(X,\xi) = 2QX + 4nX, \tag{3.5.5}$$

for all $X \in \chi(M)$. Now differentiating covariantly this with respect to arbitrary vector field Y, we get

$$(\nabla_Y \mathcal{L}_\nabla)(X, \xi) = 2(\nabla_Y Q)X - (\mathcal{L}_V)(X, Y) + \eta(Y)(2QX + 4nX). \tag{3.5.6}$$

Using (3.5.6) in (1.1.5), we get

$$(\mathcal{L}_V R)(X, Y)\xi = 2\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + 2\eta(X)(QY + 2nY) - 2\eta(Y)(QX + 2nX), \tag{3.5.7}$$

for arbitrary vector fields X and Y on M. Setting $Y = \xi$ in the aforementioned equation and using (1.1.26),(2.2.2) and (2.2.3), we get

$$(\mathcal{L}_V R)(X, \xi)\xi = 0. \tag{3.5.8}$$

Now, taking (3.5.1) in account, the Lie derivative of $g(\xi, \xi) = 1$ along the potential vector field V gives rise to,

$$\eta(\mathcal{L}_V \xi) = \lambda - \frac{p}{2} - \frac{1}{(2n+1)} + \mu. \tag{3.5.9}$$

Setting $Y = \xi$ and using (1.1.7) and (1.1.11) in the equation (3.5.1), we get

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = (p + \frac{2}{(2n+1)} - 2\lambda - 2\mu)\eta(X), \tag{3.5.10}$$

which holds for arbitrary vector field X on M. From (1.1.25) we have, $R(X, \xi)\xi = \eta(X)\xi - X$. Taking Lie derivative along the potential vector field V and taking (3.5.9) and (3.5.10) in account, we acquire

$$(\mathcal{L}_V R)(X, \xi)\xi = (2\lambda + 2\mu - p - \frac{2}{(2n+1)})(X - \eta(X)\xi), \tag{3.5.11}$$

for all $X \in \chi(M)$. Finally comparing (3.5.8) and (3.5.11), we have $(2\lambda + 2\mu - p - \frac{2}{(2n+1)})(X - \eta(X)\xi) = 0$. Since this holds for arbitrary $X \in \chi(M)$, so it reduces to

$$\lambda = \frac{p}{2} + \frac{1}{(2n+1)} - \mu. \tag{3.5.12}$$

Using the relation (3.5.12) in (3.5.9) we easily obtain, $\eta(\mathcal{L}_V \xi) = 0$. Since we have considered the potential vector field V as contact vector field so there must exists a smooth function f such that $\mathcal{L}_V \xi = f\xi$. Making use of this in (3.5.9) we get $f = \lambda - \frac{p}{2} - \frac{1}{(2n+1)} + \mu$. Using (3.5.12), we get $f = 0$ and thus $\mathcal{L}_V \xi = 0$. Finally the equation (3.5.10) reduces to,

$$\mathcal{L}_V \eta = 0. \tag{3.5.13}$$

So, V is strictly infinitesimal contact transformation.

Setting $Y = \xi$ in (1.1.2) and using (1.1.23), $\mathcal{L}_V \xi = 0$ and (3.5.13), we get $(\mathcal{L}_V \nabla)(X, \xi) = 0$. Substituting this in (3.5.5), we finally obtain $QX = -2nX$ $\forall X \in \chi(M)$. This proves our result. □

$*$-conformal η-Ricci soliton is a mere generalisation of conformal $*$-Ricci soliton where we consider $\mu = 0$ in (1.2.92) to get conformal $*$-Ricci soliton equation. We can rewrite the above theorem as,

Corollary 3.5.1. *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. If the metric g represents a conformal $*$-Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein.*

Example 3.5.1. *Let us consider the set $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ as our manifold where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below,*

$$e_1 = e^{-v}\frac{\partial}{\partial x}, \qquad e_2 = e^{-v}\frac{\partial}{\partial y}, \qquad e_3 = e^{-v}\frac{\partial}{\partial z}, \qquad e_4 = e^{-v}\frac{\partial}{\partial u}, \qquad e_5 = \frac{\partial}{\partial v},$$

are linearly independent at each point of M. We define the metric g as,

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 5\}, \\ -1, & \text{if } i = j \text{ and } i, j \in \{3, 4\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let η be an 1-form defined by $\eta(X) = g(X, e_5)$, for arbitrary $X \in \chi(M)$. Let us define (1,1)-tensor field ϕ as,

$$\phi(e_1) = e_3, \qquad \phi(e_2) = e_4, \qquad \phi(e_3) = -e_1, \qquad \phi(e_4) = -e_2, \qquad \phi(e_5) = 0.$$

Then the relations (1.1.6), (1.1.7) *and* (1.1.8) *are satisfied, where $\xi = e_5$. So, (M, ϕ, ξ, η, g) defines an almost contact structure on M.*

We can now deduce that,

$$\begin{array}{llll} [e_1, e_2] = 0, & [e_1, e_3] = 0, & [e_1, e_4] = 0, & [e_1, e_5] = e_1, \\ [e_2, e_1] = 0, & [e_2, e_3] = 0, & [e_2, e_4] = 0, & [e_2, e_5] = e_2, \\ [e_3, e_1] = 0, & [e_3, e_2] = 0, & [e_3, e_4] = 0, & [e_3, e_5] = e_3, \\ [e_4, e_1] = 0, & [e_4, e_2] = 0, & [e_4, e_3] = 0, & [e_4, e_5] = e_4, \\ [e_5, e_1] = -e_1, & [e_5, e_2] = -e_2, & [e_5, e_3] = -e_3, & [e_5, e_4] = -e_4. \end{array}$$

Let ∇ be the Levi-Civita connection of M. Then from Koszul's formula (1.1.1)*, we can have,*

$$\begin{array}{lllll}
\nabla_{e_1}e_1 = -e_5, & \nabla_{e_1}e_2 = 0, & \nabla_{e_1}e_3 = 0, & \nabla_{e_1}e_4 = 0, & \nabla_{e_1}e_5 = e_1, \\
\nabla_{e_2}e_1 = 0, & \nabla_{e_2}e_2 = -e_5, & \nabla_{e_2}e_3 = 0, & \nabla_{e_2}e_4 = 0, & \nabla_{e_2}e_5 = e_2, \\
\nabla_{e_3}e_1 = 0, & \nabla_{e_3}e_2 = 0, & \nabla_{e_3}e_3 = -e_5, & \nabla_{e_3}e_4 = 0, & \nabla_{e_3}e_5 = e_3, \\
\nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = -e_5, & \nabla_{e_4}e_5 = e_4, \\
\nabla_{e_5}e_1 = 0, & \nabla_{e_5}e_2 = 0, & \nabla_{e_5}e_3 = 0, & \nabla_{e_5}e_4 = 0, & \nabla_{e_5}e_5 = 0.
\end{array}$$

Therefore (1.1.22) *is satisfied. So, (M, ϕ, ξ, η, g) becomes a Kenmotsu manifold.*
The non-vanishing components of curvature tensor are,

$$\begin{array}{lll}
R(e_1, e_2)e_2 = -e_1, & R(e_1, e_3)e_3 = -e_1, & R(e_1, e_4)e_4 = -e_1, \\
R(e_1, e_5)e_5 = -e_1, & R(e_1, e_2)e_1 = e_2, & R(e_1, e_3)e_1 = e_3, \\
R(e_1, e_4)e_1 = e_4, & R(e_1, e_5)e_1 = e_5, & R(e_2, e_3)e_2 = e_3, \\
R(e_2, e_4)e_2 = e_4, & R(e_2, e_5)e_2 = e_5, & R(e_2, e_3)e_3 = -e_2, \\
R(e_2, e_4)e_4 = -e_2, & R(e_2, e_5)e_5 = -e_2, & R(e_3, e_4)e_3 = e_4, \\
R(e_3, e_5)e_3 = e_5, & R(e_3, e_4)e_4 = -e_3, & R(e_4, e_5)e_4 = e_5, \\
R(e_5, e_3)e_5 = e_3, & R(e_5, e_4)e_5 = e_4. &
\end{array}$$

Now from the above results we have, $S(e_i, e_i) = -4$ for $i = 1, 2, 3, 4, 5$ and,

$$S(X, Y) = -4g(X, Y), \quad \forall X, Y \in \chi(M). \tag{3.5.14}$$

Contracting this we have $r = \sum_{i=1}^{5} S(e_i, e_i) = -20 = -2n(2n + 1)$ where dimension of the manifold $2n + 1 = 5$. Also, we have,

$$S^*(e_i, e_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4, \\ 0, & \text{if } i = 5. \end{cases}$$

and, $r^* = r + 4n^2 = -20 + 16 = -4$. So,

$$S^*(X, Y) = -g(X, Y) + \eta(X)\eta(Y) \quad \forall X, Y \in \chi(M). \tag{3.5.15}$$

Now we consider a vector field V as,

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{3.5.16}$$

Then from the above results we can justify that,

$$(\mathcal{L}_V g)(X,Y) = 4\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{3.5.17}$$

which holds for all $X, Y \in \chi(M)$. *From* (3.5.15) *and* (3.5.17) *we can conclude that* g *represents a* $*$*-conformal* η*-Ricci soliton i.e., it satisfies* (1.2.92) *for potential vector field* V *defined by* (3.5.16), $\lambda = \frac{p}{2} - \frac{4}{5}$ *and* $\mu = 1$.

Theorem 3.5.2. *Let* $M^{(2n+1)}(\phi, \xi, \eta, g)$ *be a Kenmotsu manifold. If the metric* g *represents a gradient almost* $*$*-conformal* η*-Ricci soliton then either* M *is Einstein or there exists an open set where the potential vector field* V *is pointwise collinear with the characteristic vector field* ξ.

Proof. Using (2.2.4) in the definition of gradient almost $*$-conformal η-Ricci soliton, given by the equation (1.2.93), we get

$$\nabla_X Df = -QX - (\lambda - \frac{p}{2} - \frac{1}{(2n+1)} + 2n - 1)X - (\mu + 1)\eta(X)\xi, \tag{3.5.18}$$

for any vector field X on M. Taking covariant derivative along arbitrary vector Y and using (1.1.23), (1.1.24), we get

$$\begin{aligned} \nabla_Y \nabla_X Df &= -(\nabla_Y Q)X - Q(\nabla_Y X) - Y(\lambda)X - (\lambda - \frac{p}{2} - \frac{1}{(2n+1)} + 2n - 1)(\nabla_Y X) \\ &\quad -(\mu+1)\{g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(\nabla_Y X)\xi + \eta(X)Y\}. \end{aligned} \tag{3.5.19}$$

Applying this in the expression of Riemannian curvature tensor, we get

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + Y(\lambda)X - X(\lambda)Y - (\mu+1)\{\eta(Y)X - \eta(X)Y\}. \tag{3.5.20}$$

Moreover an inner product with ξ and use of (2.2.2) and (2.2.3), yields

$$g(R(X,Y)Df, \xi) = Y(\lambda)\eta(X) - X(\lambda)\eta(Y), \tag{3.5.21}$$

for $X, Y \in \chi(M)$. Furthermore the inner product of (1.1.25) with the potential vector field Df gives,

$$g(R(X,Y)Df, \xi) = \eta(Y)X(f) - \eta(X)Y(f), \tag{3.5.22}$$

for arbitrary X and Y on M. Comparing (3.5.21) and (3.5.22) and setting $Y = \xi$ we have $X(f + \lambda) = \xi(f + \lambda)\eta(X)$. From this we obtain,

$$d(f + \lambda) = \xi(f + \lambda)\eta. \tag{3.5.23}$$

So, $(f+\lambda)$ is invariant along the distribution $Ker(\eta)$ i.e., if $X \in Ker(\eta)$ then $X(f+\lambda) = d(f+\lambda)X = 0$. Now, if we take inner product w.r.t. arbitrary vector field Z after plugging $X = \xi$ in (3.5.20), we get

$$g(R(\xi, Y)Df, Z) = S(Y, Z) + (2n - \xi(\lambda) + \mu + 1)g(Y, Z) + Y(\lambda)\eta(Z) - (\mu + 1)\eta(Y)\eta(Z). \tag{3.5.24}$$

Again from (1.1.25) we can easily obtain for arbitrary vector fields Y and Z on M,

$$g(R(\xi, Y)Df, Z) = \xi(f)g(Y, Z) - Y(f)\eta(Z). \tag{3.5.25}$$

Comparing the equations (3.5.24) and (3.5.25) and using (3.5.23), we obtain

$$S(Y, Z) = \{\xi(f + \lambda) - \mu - 2n - 1\}g(Y, Z) - \{\xi(f + \lambda) - \mu - 1\}\eta(Y)\eta(Z). \tag{3.5.26}$$

Since the above equation holds for arbitrary Y and Z, so the manifold is η-Einstein. Now contracting (3.5.26), we acquire

$$\xi(f + \lambda) = \frac{r}{2n} + \mu + 2n + 2. \tag{3.5.27}$$

Plugging this in (3.5.26), we achieve

$$S(Y, Z) = (\frac{r}{2n} + 1)g(Y, Z) - (\frac{r}{2n} + 2n + 1)\eta(Y)\eta(Z),$$

for arbitrary vector fields Y and Z on M which is exactly same as (2.2.5). Now contracting (3.5.20) with X gives rise to,

$$S(Y, Df) = \frac{1}{2}Y(r) + 2nY(\lambda) - 2n(\mu + 1)\eta(Y), \tag{3.5.28}$$

which holds for any $Y \in \chi(M)$. Now, comparing this with (2.2.5), we obtain

$$(r+2n)Y(f) - (r+2n(2n+1))\eta(Y)\xi(f) - nY(r) - 4n^2Y(\lambda) + 4n^2(\mu+1)\eta(Y) = 0, \tag{3.5.29}$$

for all $Y \in \chi(M)$. Now, setting $Y = \xi$ and using (3.5.27), we get the relation

$$\xi(r) = -2(r + 2n(2n + 1)). \tag{3.5.30}$$

Since $d^2 = 0$ and $d\eta = 0$, from (3.5.23) we obtain $dr \wedge \eta = 0$ i.e., $dr(X)\eta(Y) - dr(Y)\eta(X) = 0$ for arbitrary $X, Y \in \chi(M)$. After considering $Y = \xi$ and using (3.5.30) it reduces to $X(r) = -2(r + 2n(2n + 1))\xi$. Since X is an arbitrary vector field, so we conclude that

$$Dr = -2(r + 2n(2n + 1))\xi. \tag{3.5.31}$$

Let X be a vector field of the distribution $Ker(\eta)$. Then, (3.5.29) reduces to

$$(r+2n)X(f) - 4n^2 X(\lambda) = 0.$$

Using (3.5.23) and (3.5.27) we obtain, $(r+2n(2n+1))X(f) = 0$. From here we conclude,

$$(r+2n(2n+1))(Df - \xi(f)\xi) = 0.$$

If $r = -2n(2n+1)$, then from (2.2.5) we get that the manifold is Einstein with Einstein constant $-2n$.

If $r \neq -2n(2n+1)$ on some open set O of M, then $Df = \xi(f)\xi$ on that open set that is, the potential vector field is pointwise collinear with the characteristic vector field ξ. □

If we let the coefficient of $\eta \otimes \eta$ in (1.2.93) to be zero then the soliton reduces to gradient almost conformal $*$-Ricci soliton. The aforementioned result in the framework of gradient almost conformal $*$-Ricci soliton can be stated as,

Corollary 3.5.2. *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. If the metric g represents a gradient almost conformal $*$-Ricci soliton then either M is Einstein or the potential vector field V is pointwise collinear with the characteristic vector filed ξ on an open set on M.*

Example 3.5.2. *Let us consider the set $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ as our manifold where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below,*

$$e_1 = v\frac{\partial}{\partial x}, \qquad e_2 = v\frac{\partial}{\partial y}, \qquad e_3 = v\frac{\partial}{\partial z}, \qquad e_4 = v\frac{\partial}{\partial u}, \qquad e_5 = -v\frac{\partial}{\partial v},$$

forms a linearly independent set of vector fields on M. We define the metric g as,

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We consider the reeb vector field $\xi = e_5$ then the 1-form η is defined by $\eta(X) = g(X, e_5)$, for arbitrary $X \in \chi(M)$ then, $\eta = dv$. Let us define (1,1)-tensor field ϕ as,

$$\phi(e_1) = e_2, \qquad \phi(e_2) = -e_1, \qquad \phi(e_3) = e_4, \qquad \phi(e_4) = -e_3, \qquad \phi(e_5) = 0.$$

Then the relations (1.1.6)*,* (1.1.7) *and* (1.1.8) *are satisfied. So,* (M, ϕ, ξ, η, g) *defines an almost contact structure on* M*.*

Let ∇ *be the Levi-Civita connection of* M*. Then from Koszul's formula* (1.1.1)*, we can have,*

$$\begin{array}{lllll}
\nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\
\nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\
\nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\
\nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -e_5, & \nabla_{e_4} e_5 = e_4, \\
\nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0.
\end{array}$$

Therefore (1.1.22) *is satisfied for arbitrary* $X, Y \in \chi(M)$*. So* (M, ϕ, ξ, η, g) *becomes a Kenmotsu manifold.*
The non-vanishing components of curvature tensor are,

$$\begin{array}{lll}
R(e_1, e_2)e_2 = -e_1, & R(e_1, e_3)e_3 = -e_1, & R(e_1, e_4)e_4 = -e_1, \\
R(e_1, e_5)e_5 = -e_1, & R(e_1, e_2)e_1 = e_2, & R(e_1, e_3)e_1 = e_3, \\
R(e_1, e_4)e_1 = e_4, & R(e_1, e_5)e_1 = e_5, & R(e_2, e_3)e_2 = e_3, \\
R(e_2, e_4)e_2 = e_4, & R(e_2, e_5)e_2 = e_5, & R(e_2, e_3)e_3 = -e_2, \\
R(e_2, e_4)e_4 = -e_2, & R(e_2, e_5)e_5 = -e_2, & R(e_3, e_4)e_3 = e_4, \\
R(e_3, e_5)e_3 = e_5, & R(e_3, e_4)e_4 = -e_3, & R(e_4, e_5)e_4 = e_5, \\
R(e_5, e_3)e_5 = e_3, & R(e_5, e_4)e_5 = e_4. &
\end{array}$$

Now from the above results we have, $S(e_i, e_i) = -4$ *for* $i = 1, 2, 3, 4, 5$ *and,*

$$S(X, Y) = -4g(X, Y), \quad \forall X, Y \in \chi(M). \tag{3.5.32}$$

So, the manifold is Einstein. Also, we have,

$$S^*(e_i, e_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4, \\ 0, & \text{if } i = 5. \end{cases}$$

and,

$$S^*(X, Y) = -g(X, Y) + \eta(X)\eta(Y), \quad \forall X, Y \in \chi(M). \tag{3.5.33}$$

Let $f : M \to \mathbb{R}$ be a smooth function defined by,

$$f(x, y, z, u, v) = x^2 + y^2 + z^2 + u^2 + \frac{v^2}{2}. \tag{3.5.34}$$

Then the gradient of f, Df is given by,

$$Df = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{3.5.35}$$

Then from the above results we can verify that,

$$(\mathcal{L}_{Df}g)(X, Y) = 2\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{3.5.36}$$

which holds for all $X, Y \in \chi(M)$. From (3.5.33) *and* (3.5.36) *we obtain that g represents a gradient almost $*$-conformal η-Ricci soliton i.e., it satisfies* (1.2.93) *for $V = Df$, where f is defined by* (3.5.34), *$\lambda = \frac{p}{2} + \frac{1}{5}$ and $\mu = 0$.*

3.6 $*$-conformal η-Ricci soliton on $(\kappa, -2)'$-almost Kenmotsu manifold with $\kappa < -1$

In this section we consider the manifold as a $(2n + 1)$-dimensional almost Kenmotsu manifold where the characteristic vector field ξ satisfies $(\kappa, -2)'$-nullity distribution. Then we let the metric g to represent a $*$-conformal η-Ricci soliton.

Theorem 3.6.1. *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to $(\kappa, -2)'$-nullity distribution where $\kappa < -1$. If the metric g represents a $*$-conformal η-Ricci soliton satisfying $p \neq 2\lambda + 2\mu - \frac{2}{(2n+1)}$ then, M is Ricci-flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Proof. Combining (1.2.92) with (2.2.6), we get

$$(\mathcal{L}_V g)(X, Y) = (p + 2\kappa - 2\lambda + 4 + \frac{2}{(2n+1)})g(X, Y) - 2(\kappa + \mu + 2)\eta(X)\eta(Y), \tag{3.6.1}$$

for all vector fields X and Y on M. Now taking covariant derivative of the foregoing equation along arbitrary vector field Z and using (1.1.38), we get

$$\begin{aligned}(\nabla_Z \mathcal{L}_V g)(X, Y) &= -2(\kappa + \mu + 2)[\eta(Y)g(X, Z) + \eta(X)g(Y, Z) + \eta(Y) \\ &\quad g(h'Z, X) + \eta(X)g(h'Z, Y) - 2\eta(X)\eta(Y)\eta(Z)].\end{aligned}$$

By a straightforward combinatorial computation, use of (3.5.3), the symmetry of $(\mathcal{L}_V\nabla)$ in the aforementioned equation, we get

$$(\mathcal{L}_V\nabla)(X,Y) = -2(\kappa+\mu+2)[g(X,Y)+g(h'X,Y)-\eta(X)\eta(Y)]\xi, \tag{3.6.2}$$

for all $X, Y \in \chi(M)$. Replacing $Y = \xi$ and using (1.1.7), (1.1.11) and (1.1.29), we have

$$(\mathcal{L}_V\nabla)(X,\xi) = 0, \tag{3.6.3}$$

for arbitrary vector field X on M. Now taking (1.1.28) and (3.6.2) into account and differentiating (3.6.3) covariantly along arbitrary vector field Y, we get

$$(\nabla_Y\mathcal{L}_V\nabla)(X,\xi) = 2(\kappa+\mu+2)[g(X,Y)-\eta(X)\eta(Y)+2g(h'X,Y)+g(h'^2X,Y)]\xi \tag{3.6.4}$$

for arbitrary vector fields X and Y on M. Setting $Z = \xi$ in (1.1.5) and using (3.6.4) repeatedly, we obtain

$$(\mathcal{L}_V R)(X,Y)\xi = 0, \tag{3.6.5}$$

for arbitrary $X, Y \in \chi(M)$. Now taking Lie derivative of (1.1.32) along the potential vector field V, taking (1.1.7) and (1.1.29) into account, we get

$$\begin{aligned}(\mathcal{L}_V R)(X,\xi)\xi &= \kappa[g(X,\mathcal{L}_V\xi)\xi - 2\eta(\mathcal{L}_V\xi)X - ((\mathcal{L}_V\eta)X)\xi] + 2[2\eta(\mathcal{L}_V\xi)\\ &\quad h'X - \eta(X)(h'(\mathcal{L}_V\xi)) - g(h'X,\mathcal{L}_V\xi)\xi - ((\mathcal{L}_V h')X)],\end{aligned} \tag{3.6.6}$$

for any vector field X on M. Plugging $Y = \xi$ in (3.6.1), we obtain

$$(\mathcal{L}_V\eta)X - g(X,\mathcal{L}_V\xi) = (p - 2\lambda - 2\mu + \frac{2}{(2n+1)})\eta(X), \tag{3.6.7}$$

for all $X \in \chi(M)$. Setting $X = \xi$ in the foregoing equation, we get

$$\eta(\mathcal{L}_V\xi) = -(\frac{p}{2} - \lambda - \mu + \frac{1}{(2n+1)}). \tag{3.6.8}$$

With the help of (3.6.5),(3.6.7) and (3.6.8), we can rewrite the equation (3.6.6) as

$$\begin{aligned}&\kappa(p-2\lambda-2\mu+\frac{2}{(2n+1)})(X-\eta(X)\xi) - 2(p-2\lambda-2\mu+\frac{2}{(2n+1)})h'X -\\ &2\eta(X)h'(\mathcal{L}_V\xi) - 2g(h'X,\mathcal{L}_V\xi)\xi - 2(\mathcal{L}_V h')X = 0.\end{aligned} \tag{3.6.9}$$

Taking inner product of the foregoing equation with arbitrary vector field Y on M, we obtain

$$\begin{aligned}&(p-2\lambda-2\mu+\frac{2}{(2n+1)})[\kappa(g(X,Y)-\eta(X)\eta(Y)) - 2g(h'X,Y)]\\ &-2\eta(X)g(h'(\mathcal{L}_V\xi),Y) - 2g(h'X,\mathcal{L}_V\xi)\eta(Y) - 2g((\mathcal{L}_V h')X,Y) = 0.\end{aligned} \tag{3.6.10}$$

Since the above equation holds for any vector fields X and Y on M, by replacing X by $\phi(X)$ and Y by $\phi(Y)$ and taking (1.1.10) into account, we get

$$(p-2\lambda-2\mu+\frac{2}{(2n+1)})[\kappa g(\phi X,\phi Y)-2g(h'\phi X,\phi Y)]-2g((\mathcal{L}_V h')\phi X,\phi Y)=0, \quad (3.6.11)$$

for all $X, Y \in \chi(M)$. Since $spec(h') = \{0, \alpha, -\alpha\}$, let X and V belong to the eigenspaces of $-\alpha$ and α denoted by $[-\alpha]'$ and $[\alpha]'$ respectively. Then $\phi X \in [\alpha]'$ (for more details we refer to [35]). Then (3.6.12) can be rewritten as,

$$(p-2\lambda-2\mu+\frac{2}{(2n+1)})(\kappa-2)g(\phi X,\phi Y)-2g((\mathcal{L}_V h')\phi X,\phi Y)=0, \quad (3.6.12)$$

for all $X, Y \in \chi(M)$. It is remained to find the value of $g((\mathcal{L}_V h')\phi X, \phi Y)$. To get this we prove a more generalized result: In a $(\kappa, -2)'$-almost Kenmotsu manifold $(\mathcal{L}_X h')Y = 0$, where X and Y belong to same eigenspaces.

Without loss of generality we assume that $X, Y \in [\alpha]'$ where $spec(h') = \{0, \alpha, -\alpha\}$. If we consider a local orthonormal ϕ-basis as $\{\xi, e_i, \phi e_i\}, i = 1, 2, ..., n$ then,

$$\nabla_X Y = \sum_{i=1}^{n} g(\nabla_X Y, e_i)e_i - (\alpha+1)g(X,Y)\xi.$$

and,

$$\begin{aligned}
(\mathcal{L}_X h')Y &= \mathcal{L}_X(h'Y) - h'(\mathcal{L}_X Y) \\
&= \alpha(\mathcal{L}_X Y) - h'(\mathcal{L}_X Y) \\
&= \alpha(\nabla_X Y - \nabla_Y X) - h'(\nabla_X Y - \nabla_Y X) \\
&= \alpha(\alpha+1)g(X,Y)\xi - \alpha(\alpha+1)g(X,Y)\xi \\
&= 0.
\end{aligned}$$

Similarly we can prove that the above results hold if $X, Y \in [-\alpha]'$ (for more details we refer to [35]). Now (3.6.12) reduces to,

$$(p-2\lambda-2\mu+\frac{2}{(2n+1)})(\kappa-2)g(\phi X,\phi Y)=0, \quad (3.6.13)$$

for any vector fields X and Y on M. Since by hypothesis $p \neq 2\lambda + 2\mu - \frac{2}{(2n+1)}$, from the foregoing equation we infer that $\kappa = 2\alpha$. Again form $\alpha^2 = -(\kappa+1)$ we get $\alpha = -1$ and $\kappa = -2$. Plugging the value of κ in (2.2.6) we have $S^* = 0$, i.e., the manifold is Ricci-flat.

Again we get $spec(h') = \{0, 1, -1\}$. From corollary 4.2 of [35] we get M is locally symmetric. From proposition 4.1 of [35] we finally conclude that M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, where $\mathbb{H}^{n+1}(-4)$ is the hyperbolic space of constant curvature -4. □

As we know, setting $\mu = 0$ in (1.2.92) gives rise to the equation of conformal $*$-Ricci soliton, we can revisit the last theorem and can note the statement as:

Corollary 3.6.1. *Let $M(\phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost Kenmotsu manifold such that ξ blongs to $(\kappa, -2)'$-nullity distribution where $\kappa < -1$. If the metric g represents a conformal $*$-Ricci soliton satisfying $p \neq 2\lambda - \frac{2}{(2n+1)}$ then, M is Ricci-flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Example 3.6.1. *We consider the manifold as $M = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0\}$. We define three vector fields e_1, e_2 and e_3 as,*

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = 2x\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Then the set $\{e_1, e_2, e_3\}$ forms a linearly independent set of vector fields on M. We define the metric g as

$$g_{ij} = \delta_{ij}, \quad \forall i, j \in \{1, 2, 3\}.$$

Then it is easy to verify that $\{e_1, e_2, e_3\}$ forms an orthonormal basis on M. Let the 1-form η be defined by $\eta(X) = g(X, e_3)$, for arbitrary $X \in \chi(M)$. Let us define (1,1)-tensor field ϕ as,

$$\phi(e_1) = -e_2, \qquad \phi(e_2) = e_1, \qquad \phi(e_3) = 0.$$

Then we can verify that the relations (1.1.6), (1.1.7) *and* (1.1.8) *are satisfied where $\xi = e_3$. So, (M, ϕ, ξ, η, g) defines an almost contact structure on M.*

We also can compute that,

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = 0, \qquad [e_1, e_3] = 2e_1.$$

Let ∇ be the Levi-Civita connection of M. Then from Koszul's formula (1.1.1)*, we can have,*

$$\nabla_{e_1} e_1 = -2e_3, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_3 = 2e_1,$$
$$\nabla_{e_2} e_1 = 0, \qquad \nabla_{e_2} e_2 = 0, \qquad \nabla_{e_2} e_3 = 0,$$
$$\nabla_{e_3} e_1 = 0, \qquad \nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_3 = 0.$$

Therefore it is easy to verify that the structure (M, ϕ, ξ, η, g) is not Kenmotsu manifold, but almost Kenmotsu manifold. Now let us define the operator h' as,

$$h'(e_1) = e_1, \qquad h'(e_2) = -e_2, \qquad h'(e_3) = 0.$$

By straightforward computation we have the components of curvature tensor as,

$$R(e_1,e_2)e_1 = 0, \qquad R(e_1,e_2)e_2 = 0, \qquad R(e_1,e_2)e_3 = 0,$$
$$R(e_2,e_3)e_1 = 0, \qquad R(e_2,e_3)e_2 = 0, \qquad R(e_2,e_3)e_3 = 0,$$
$$R(e_1,e_3)e_1 = 4e_3, \qquad R(e_1,e_3)e_2 = 0, \qquad R(e_1,e_3)e_3 = -4e_1.$$

Now from the above results and taking (1.1.32) *in account we conclude that the Reeb vector field* ξ *belongs to the* $(\kappa, -2)'$*-nullity distribution with* $\kappa = -2$. *So, the manifold is* $(-2,-2)'$*-almost Kenmotsu manifold.*

Now from (2.2.6) *we get* $S^*(X,Y) = 0 \ \ \forall X, Y \in \chi(M)$. *Let us consider a vector field* V *as,*

$$V = e^{2z}\frac{\partial}{\partial x} + 4(y+z)\frac{\partial}{\partial y}. \tag{3.6.14}$$

Then from the above results one can get,

$$(\mathcal{L}_V g)(e_1,e_1) = 0, \qquad (\mathcal{L}_V g)(e_2,e_2) = 8, \qquad (\mathcal{L}_V g)(e_3,e_3) = 0,$$
$$(\mathcal{L}_V g)(e_1,e_2) = 0, \qquad (\mathcal{L}_V g)(e_2,e_3) = 0, \qquad (\mathcal{L}_V g)(e_1,e_3) = 0.$$

From here we can conclude that g *represents a* $*$*-conformal* η*-Ricci soliton i.e., it satisfies* (1.2.92) *for potential vector field* V *defined by* (3.6.14), $\lambda = \frac{p}{2} - \frac{11}{3}$ *and* $\mu = 4$. *From Dileo and Pastore[35] we can further conclude that the manifold is locally isometric to* $\mathbb{H}^2(-4) \times \mathbb{R}$.

With this example we conclude the chapter.

4

Some types of conformal Ricci solitons

4.1 Introduction

This chapter contains seven sections of which the first two sections are introduction and preliminaries.

In the third section, we study conformal almost Ricci soliton within the framework of Kenmotsu manifolds. First, we demonstrate that if the potential vector field is Jacobi along Reeb vector field, then the soliton reduces to conformal Ricci soliton. Then, we also show that either the manifold is of constant scalar curvature or the potential vector field is contact, if the manifold is η-Einstein Kenmotsu manifold. After that when we consider the soliton vector field as of infinitesimal contact transformation then either the gradient of λ is pointwise collinear with the Reeb vector field or the manifold becomes η-Einstein. Lastly, we develop an example of conformal almost Ricci soliton on Kenmotsu manifold.

In the later section, we deliberate conformal Ricci soliton within the framework of para-Kenmotsu manifold. Here we prove that if the manifold admits a conformal Ricci soliton then either the Lie derivative of Reeb vector field along potential vector field is orthogonal to ξ or the manifold becomes Einstein. We also show that if an η-Einstein para-Kenmotsu manifold admits conformal Ricci soliton then it is Einstein.

In the next section, we consider $*$-conformal Ricci soliton on para-Kenmotsu manifold. Here, we show that if an η-Einstein para-Kenmotsu manifold admits a $*$-conformal Ricci soliton then the manifold becomes Einstein with constant scalar curvature. We also provide an example to support our result.

In the sixth section, we show that 3-dimensional para-cosymplectic manifold is Ricci flat and the scalar curvature of the manifold is harmonic if the manifold satisfies conformal Ricci soliton where the soliton vector field is conformal.

In the final section, we evolve the nature of scalar curvature when the 3-dimensional trans-Sasakian manifold of type (α, β), provided $\alpha \neq 0$, satisfies $*$-conformal Ricci soliton.

4.2 Preliminaries

Discussions regarding Kenmotsu manifold, para-Kenmotsu manifold, para-cosymplectic manifold and trans-Sasakian manifold are made in the introductory chapter. Here, we want to take the opportunity to recall some useful results which are used to prove the results.

Lemma 4.2.1. *[58] Let M be a Kenmotsu manifold. The Riemannian curvature tensor R satisfies the following two conditions for arbitrary vector fields X, Y, Z on M,*

$$\begin{aligned} R(X,Y)\phi Z - \phi R(X,Y)Z &= g(Y,Z)\phi X - g(X,Z)\phi Y + \\ &\quad g(X,\phi Z)Y - g(Y,\phi Z)X, \qquad (4.2.1) \\ R(\phi X,\phi Y)Z - R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + \\ &\quad g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y. \qquad (4.2.2) \end{aligned}$$

Lemma 4.2.2. *[43] On a Kenmotsu manifold the Ricci operator and ϕ commutes, i.e. $Q\phi = \phi Q$.*

Lemma 4.2.3. *[43] Let $\{e_i : i = 1, \ldots\ldots, 2n+1\}$ be a local orthonormal frame on the Kenmotsu manifold. Then,*

$$\sum_{i=1}^{2n+1} g((\nabla_X Q)\phi e_i, e_i) = 0, \qquad (4.2.3)$$

$$\sum_{i=1}^{2n+1} g((\nabla_{\phi e_i} Q)\phi Y, e_i) = -\frac{1}{2}(Yr) - (r + 4n^2 + 2n)\eta(Y). \qquad (4.2.4)$$

Let a $(2n+1)$-dimensional almost para-Kenmotsu metric manifold be η-Einstein manifold. Considering $X = Y = \xi$ in (1.1.78) and using (1.1.58), we have $a + b = -2n$. Contracting (1.1.78) over X and Y we get $r = (2n+1)a + b$, where r denotes the scalar curvature of the manifold. Solving the last two equations, we get $a = (1 + \frac{r}{2n})$ and $b = -(2n + 1 + \frac{r}{2n})$.

Using these values we can rewrite (1.1.78) as

$$S(X,Y) = (1 + \frac{r}{2n})g(X,Y) - (2n + 1 + \frac{r}{2n})\eta(X)\eta(Y). \tag{4.2.5}$$

Lemma 4.2.4. *[98] The $*$-Ricci tensor on a $(2n+1)$-dimensional para-Kenmotsu manifold is given by,*

$$S^*(X,Y) = -S(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y), \tag{4.2.6}$$

for all vector fields X and Y on M.

If we consider the para-cosymplectic manifold is of dimension 3, then the Riemannian curvature tensor satisfies the following relation, for any vector fields X, Y, Z in $\chi(M)$

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.2.7}$$

Using this result we deduce that 3-dimensional para-cosymplectic manifold satisfies,

$$S(X,Y) = \frac{r}{2}[g(X,Y) - \eta(X)\eta(Y)], \tag{4.2.8}$$

$$QX = \frac{r}{2}[X - \eta(X)\xi]. \tag{4.2.9}$$

Lemma 4.2.5. *[60] For a 3-dimensional para-cosymplectic manifold, we have*

$$\xi(r) = 0. \tag{4.2.10}$$

4.3 Conformal almost Ricci soliton on Kenmotsu manifold

Theorem 4.3.1. *Let $M(\phi,\xi,\eta,g)$ be a $(2n+1)$-dimensional Kenmotsu manifold where $n > 1$. If the metric g represents a conformal almost Ricci soliton (g,V,λ) whose potential vector field V is Jacobi along characteristic vector field ξ, then the soliton reduces to conformal Ricci soliton.*

Proof. We differentiate covariantly the soliton equation (1.2.84) along an arbitrary vector field Z to find,

$$(\nabla_Z \mathcal{L}_V g)(X,Y) + 2(\nabla_Z S)(X,Y) + 2(Z\lambda)g(X,Y) = 0, \tag{4.3.1}$$

for any vector fields X and Y on M. We know ∇ is the metric connection, i.e., $\nabla g = 0$. So, the equation (1.1.3) reduces to

$$(\nabla_Z \mathcal{L}_V g)(X,Y) = g((\mathcal{L}_V \nabla)(X,Z),Y) + g((\mathcal{L}_V \nabla)(Y,Z),X), \tag{4.3.2}$$

for all vector fields X, Y, Z on M. We combine the identities (4.3.1) and (4.3.2) and by a combinatorial computation and utilizing the symmetry of $(\mathcal{L}_V \nabla)$, the foregoing equation yields

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X,Y),Z) \quad = \quad & (Z\lambda)g(X,Y) - (X\lambda)g(Y,Z) - (Y\lambda)g(X,Z) + \\ & (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z), \end{aligned}$$

for arbitrary vector fields X, Y and Z on M. Setting $Y = \xi$ in the foregoing equation and using (1.1.11), (2.2.2) and (2.2.3), we acquire

$$(\mathcal{L}_V \nabla)(X,\xi) = 2QX + 4nX + \eta(X)(D\lambda) - (X\lambda)\xi - (\xi\lambda)X. \tag{4.3.3}$$

The above equation holds for any vector field X. Now, covariant derivative of this equation along arbitrary vector field Y and use of (1.1.23), provides

$$\begin{aligned} &(\nabla_Y \mathcal{L}_V \nabla)(X,\xi) + (\mathcal{L}_V \nabla)(X,Y) = g(X,Y)D\lambda + 2\eta(Y)QX - g(X,\nabla_Y D\lambda)\xi \\ &+ 2(\nabla_Y Q)X + \eta(X)(\nabla_Y D\lambda) + 4n\eta(Y)X - (X\lambda)Y - (Y\lambda)X - \eta(\nabla_Y D\lambda)X. \end{aligned} \tag{4.3.4}$$

Repeatedly using (4.3.4) in (1.1.5) and taking the symmetry of Hessian of a smooth function into account, we obtain

$$\begin{aligned} (\mathcal{L}_V R)(X,Y)\xi =& 2[(\nabla_X Q)Y - (\nabla_Y Q)X] + 2[\eta(X)QY - 2\eta(Y)QX] + 4n[\eta(X)Y - \\ & 2\eta(Y)X] + \eta(Y)(\nabla_X D\lambda) - \eta(X)(\nabla_Y D\lambda) - \eta(\nabla_X D\lambda)Y + \eta(\nabla_Y D\lambda)X, \end{aligned} \tag{4.3.5}$$

$\forall X, Y \in \chi(M)$. Making use of (1.1.26) in the soliton equation (1.2.84), we achieve

$$(\mathcal{L}_V \eta)X = g(X, \mathcal{L}_V \xi) + (p + \frac{2}{2n+1} + 4n - 2\lambda)\eta(X), \tag{4.3.6}$$

for any $X \in \chi(M)$. Lie differentiating (1.1.25) along the soliton vector field V and manipulating using (1.1.25) and (4.3.6), we acquire

$$(\mathcal{L}_V R)(X,\xi)\xi = (p + \frac{2}{2n+1} + 4n - 2\lambda)[\eta(X)\xi - X], \tag{4.3.7}$$

for arbitrary vector field X on M. Now, plugging $Y = \xi$ in (4.3.5) and bringing (1.1.7), (1.1.26), (2.2.2) and (2.2.3) into play, we have

$$(\mathcal{L}_V R)(X,\xi)\xi = \nabla_X D\lambda - \eta(X)(\nabla_\xi D\lambda) - \eta(\nabla_X D\lambda)\xi + \eta(\nabla_\xi D\lambda)X, \tag{4.3.8}$$

which holds for any vector field X. We compare the above two equations to infer,

$$\nabla_X D\lambda - \eta(X)(\nabla_\xi D\lambda) - \eta(\nabla_X D\lambda)\xi + \eta(\nabla_\xi D\lambda)X = (p + \frac{2}{2n+1} + 4n - 2\lambda)[\eta(X)\xi - X]. \tag{4.3.9}$$

We contract (4.3.9) over X to obtain,

$$div(D\lambda) = -(2n-1)\eta(\nabla_\xi D\lambda) - 2n(p + \frac{2}{2n+1} + 4n - 2\lambda). \tag{4.3.10}$$

Taking (4.3.6) into account, we can rewrite (1.1.25) as

$$\begin{aligned}(\mathcal{L}_V R)(X,Y)\xi =& g(X,\mathcal{L}_V\xi)Y - g(Y,\mathcal{L}_V\xi)X - R(X,Y)\mathcal{L}_V\xi + \\ & (p + \frac{2}{2n+1} + 4n - 2\lambda)[\eta(X)Y - \eta(Y)X].\end{aligned}$$

Comparing the previous equation with (4.3.5), we achieve

$$\begin{aligned}&2[(\nabla_X Q)Y - (\nabla_Y Q)X] + 2[\eta(X)QY - 2\eta(Y)QX] + 4n[\eta(X)Y - 2\eta(Y)X] + \\ &\eta(Y)(\nabla_X D\lambda) - \eta(X)(\nabla_Y D\lambda) - \eta(\nabla_X D\lambda)Y + \eta(\nabla_Y D\lambda)X = g(X,\mathcal{L}_V\xi)Y - \\ &g(Y,\mathcal{L}_V\xi)X - R(X,Y)\mathcal{L}_V\xi + (p + \frac{2}{2n+1} + 4n - 2\lambda)[\eta(X)Y - \eta(Y)X],\end{aligned} \tag{4.3.11}$$

for arbitrary vector fields X and Y on M. Tracing the foregoing equation over Y and using (1.1.26), (4.3.10), we find

$$\begin{aligned}S(Y,\mathcal{L}_V\xi) =& Yr + 2(r + 2n(2n+1))\eta(Y) + (2n-1)\eta(\nabla_\xi D\lambda)\eta(Y) \\ & - (2n-1)\eta(\nabla_Y D\lambda) - 2ng(Y,\mathcal{L}_V\xi),\end{aligned} \tag{4.3.12}$$

$\forall Y \in \chi(M)$. Replacing X and Y of (4.3.11) by ϕX and ϕY and using (1.1.10) and (1.1.12) and finally tracing over X, we have

$$S(Y,\mathcal{L}_V\xi) = Yr + 2(r + 2n(2n+1))\eta(Y) - \eta(\nabla_Y D\lambda) + \eta(Y)\eta(\nabla_\xi D\lambda) - 2ng(Y,\mathcal{L}_V\xi), \tag{4.3.13}$$

for arbitrary vector field Y, where we have used (1.1.6), (4.2.3) and (4.2.4). Comparing (4.3.12) and (4.3.13) and using the symmetry of Hessian of a smooth function, we can conclude that

$$\nabla_\xi D\lambda = (\xi(\xi\lambda))\xi, \tag{4.3.14}$$

as by hypothesis $n > 1$. As Hessian of a smooth function is symmetric, operating (4.3.9) by (4.3.14), we secure

$$\nabla_X D\lambda = (p + \frac{2}{2n+1} + 4n - 2\lambda + \xi(\xi\lambda))[\eta(X)\xi - X] + (\xi(\xi\lambda))\eta(X)\xi, \tag{4.3.15}$$

for any vector field X on M. Using (4.3.15) and (1.1.23) in (1.1.4), yields

$$\begin{aligned} R(X,Y)D\lambda =& (p + \frac{2}{2n+1} + 4n - 2\lambda + 2\xi(\xi\lambda))[\eta(Y)X - \eta(X)Y] + \\ & (2(X\lambda) - X(\xi(\xi\lambda)))[Y - \eta(Y)\xi] - (2(Y\lambda) - Y(\xi(\xi\lambda))) \\ & [X - \eta(X)\xi] + (X(\xi(\xi\lambda)))\eta(Y)\xi - (Y(\xi(\xi\lambda)))\eta(X)\xi, \end{aligned} \tag{4.3.16}$$

$\forall X, Y \in \chi(M)$. Taking (1.1.7) and (1.1.25) into account, setting $Y = \xi$ in the previous relation, we acquire

$$\begin{aligned} &(p + \frac{2}{2n+1} + 4n - 2\lambda - \xi\lambda + 2\xi(\xi\lambda) + \xi(\xi(\xi\lambda)))X - (X\lambda)\xi + (X(\xi(\xi\lambda)))\xi \\ &- (p + \frac{2}{2n+1} + 4n - 2\lambda - 2\xi\lambda + 2\xi(\xi\lambda) + 2\xi(\xi(\xi\lambda)))\eta(X)\xi = 0, \end{aligned} \tag{4.3.17}$$

for any vector field X. Again, considering inner product with respect to ξ, we have $d(\lambda - \xi(\xi\lambda)) = \xi(\lambda - \xi(\xi\lambda))\eta$, where d is the exterior derivative operator. Keeping this relation in mind, (4.3.16) can be revised as

$$\begin{aligned} R(X,Y)D\lambda =& (p + \frac{2}{2n+1} + 4n - 2\lambda - \xi\lambda + 2\xi(\xi\lambda) + \xi(\xi(\xi\lambda))) \\ & [\eta(Y)X - \eta(X)Y] + (X\lambda)Y - (Y\lambda)X. \end{aligned} \tag{4.3.18}$$

We replace X and Y by ϕX and ϕY, respectively, in the previous equation and use (4.2.2) to derive

$$R(X,Y)D\lambda = (X\lambda)Y - (Y\lambda)X, \tag{4.3.19}$$

$\forall X, Y \in \chi(M)$. Since X and Y are arbitrary vector fields on M, plugging (4.3.19) in (4.3.18), we achieve

$$p + \frac{2}{2n+1} + 4n - 2\lambda - \xi\lambda + 2\xi(\xi\lambda) + \xi(\xi(\xi\lambda)) = 0. \tag{4.3.20}$$

Since by hypothesis V is Jacobi along ξ, so, $\nabla_\xi \nabla_\xi V - R(\xi, V)\xi = 0$. Therefore, using this relation in (1.1.4), we obtain $(\mathcal{L}_V \nabla)(\xi, \xi) = 0$. Plugging (4.3.3), (1.1.7) and (1.1.26) in the last relation, we get $D\lambda = 2(\xi\lambda)\xi$. From here we can easily obtain

$$\xi\lambda = 0. \tag{4.3.21}$$

Taking (4.3.21) into account, (4.3.20) can be restated as $\lambda = \frac{p}{2} + \frac{1}{2n+1} + 2n$, i.e., λ is constant. So, the soliton reduces to conformal Ricci soliton. □

Theorem 4.3.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an η-Einstein Kenmotsu manifold of dimension greater than 1. If the metric g represents a conformal almost Ricci soliton (g, V, λ), then either the manifold is of constant scalar curvature or the potential vector field is a contact vector field.*

Proof. In light of (2.2.5), we infer

$$QX = (1 + \frac{r}{2n})X - (2n + 1 + \frac{r}{2n})\eta(X)\xi. \tag{4.3.22}$$

We differentiate covariantly the previous equation along arbitrary vector field Y and utilize the equation (1.1.23) to deduce

$$(\nabla_Y Q)X = \frac{Yr}{2n}[X - \eta(X)\xi] - (1 + 2n + \frac{r}{2n})[g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(X)Y] \tag{4.3.23}$$

$\forall X \in \chi(M)$. Setting $Y = \xi$ in (4.3.5) and using (1.1.26), (4.3.22) and (4.3.23), we achieve

$$\begin{aligned}(\mathcal{L}_V R)(X,\xi)\xi =& \frac{\xi r}{n}(\phi^2 X) + 4(1 + 2n + \frac{r}{2n})(\phi^2 X) + (\nabla_X D\lambda) - \\ & \eta(X)(\nabla_\xi D\lambda) - \eta(\nabla_X D\lambda)\xi + \eta(\nabla_\xi D\lambda)X,\end{aligned}$$

for any vector field X on M. Plugging (4.3.5), (1.1.26), (2.2.2) and (2.2.3) in the last relation, yields

$$\xi r + 2(r + 2n(2n+1)) = 0. \tag{4.3.24}$$

Let us consider an orthonormal basis $\{e_i : i = 1, 2....., 2n+1\}$ of the manifold. As it is known that $\sum_{i=1}^{2n+1}(\nabla_X S)(e_i, e_i) = Xr$ and $\sum_{i=1}^{2n+1}(\nabla_{e_i} S)(X, e_i) = \frac{1}{2}(Xr)$ for an arbitrary vector field X, using these results in (4.3.2), we get

$$(\mathcal{L}_V \nabla)(e_i, e_i) = (2n - 1)D\lambda. \tag{4.3.25}$$

Again manipulating (4.3.2) using (4.3.23) and (4.3.24), we acquire

$$(\mathcal{L}_V \nabla)(e_i, e_i) = (2n-1)D\lambda + (1 - \frac{1}{n})Dr - (1 - \frac{1}{n})(\xi r)\xi. \tag{4.3.26}$$

As by hypothesis $n > 1$, comparing these last two equations, we have

$$Dr = (\xi r)\xi. \tag{4.3.27}$$

We plug (4.3.23) in (4.3.2) and then replace Y by ξ to obtain

$$(\mathcal{L}_V\nabla)(X,\xi) = (D\lambda)\eta(X) - (X\lambda)\xi - (\xi\lambda)X - \frac{\xi r}{2n}[X - \eta(X)\xi], \tag{4.3.28}$$

for an arbitrary vector field X on M. Using (1.1.23),(4.3.27), (4.3.28) and keeping the symmetries of Hessian of λ and $\mathcal{L}_V\nabla$ in mind, we can rewrite (1.1.5) as

$$(\mathcal{L}_V R)(X,Y)\xi = (\nabla_X D\lambda)\eta(Y) - \eta(\nabla_X D\lambda)Y - (\nabla_Y D\lambda)\eta(X) + \eta(\nabla_Y D\lambda)X, \tag{4.3.29}$$

for an arbitrary vector fields X and Y. As Hessian of a smooth function is symmetric, tracing the foregoing equation over X and inserting (4.3.10), (4.3.14), we get

$$(\mathcal{L}_V S)(Y,\xi) = 2n(2\lambda - p - 4n - \frac{2}{2n+1})\eta(Y), \tag{4.3.30}$$

for any vector field Y of $\chi(M)$. Lie differentiating (1.1.26) along arbitrary vector field Y, from (4.3.6), we obtain $(\mathcal{L}_V S)(Y,\xi) = 2n(2\lambda - p - 4n - \frac{2}{2n+1})\eta(Y) - S(Y,\mathcal{L}_V\xi) - 2ng(Y,\mathcal{L}_V\xi)$. Plugging this relation along with (2.2.5), (1.2.84) and (1.1.26) in (4.3.30), we acquire

$$(1 + 2n + \frac{r}{2n})[g(Y,\mathcal{L}_V\xi) - (\lambda - 2n - \frac{p}{2} - \frac{1}{2n+1})\eta(Y)] = 0, \tag{4.3.31}$$

$\forall Y \in \chi(M)$. As $n \neq 0$, either $r = -2n(2n+1)$ or $\mathcal{L}_V\xi = (\lambda - 2n - \frac{p}{2} - \frac{1}{2n+1})\xi$. This completes the proof. □

Theorem 4.3.3. *Let the metric g of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$ represents a conformal almost Ricci soliton (g,V,λ). If the soliton vector field V is of infinitesimal contact transformation then,*

(i) gradient of the soliton function is pointwise collinear with the characteristic vector field, and

(ii) the manifold becomes η-Einstein manifold.

Proof. Since the soliton vector field V is of infinitesimal contact transformation, there exists a smooth function f on M such that

$$\mathcal{L}_V\eta = f\eta. \tag{4.3.32}$$

From (1.1.23), we can easily obtain for arbitrary vector fields X and Y of $\chi(M)$,

$$(\mathcal{L}_\xi g)(X,Y) = 2[g(X,Y) - \eta(X)\eta(Y)], \tag{4.3.33}$$

holds. Taking (1.2.84) and (1.1.26) in account, Lie derivative of (4.3.33) along soliton vector field V, yields

$$(\mathcal{L}_V\mathcal{L}_\xi g)(X,Y) = -4S(X,Y) - 2(2\lambda - p - \frac{2}{2n+1})g(X,Y) - 2g(X,\mathcal{L}_V\xi)\eta(Y) - 2g(Y,\mathcal{L}_V\xi)\eta(X) - 4(4n - 2\lambda + p + \frac{2}{2n+1})\eta(X)\eta(Y), \quad (4.3.34)$$

for any vector fields X and Y on M. Again, taking Lie derivative of the soliton equation (1.2.84) w.r.t. characteristic vector field ξ, we acquire

$$(\mathcal{L}_\xi\mathcal{L}_V g)(X,Y) = 2(p + \frac{2}{2n+1} + 4n - 2\lambda - \xi\lambda)g(X,Y) + 2(2\lambda - 4n - p - \frac{2}{2n+1})\eta(X)\eta(Y), \quad (4.3.35)$$

$\forall X,Y \in \chi(M)$. Since it is well known that $\mathcal{L}_X\mathcal{L}_Y g - \mathcal{L}_Y\mathcal{L}_X g = \mathcal{L}_{[X,Y]}g$, subtracting (4.3.35) from (4.3.34), we deduce

$$(\mathcal{L}_W g)(X,Y) = -2g(X,W)\eta(Y) - 2g(Y,W)\eta(X) - 2(4n - \xi\lambda)g(X,Y) - 4S(X,Y) - 2(4n - 2\lambda + p + \frac{2}{2n+1})\eta(X)\eta(Y), \quad (4.3.36)$$

where we have considered $W = [V,\xi] = \mathcal{L}_V\xi$. Manipulating (4.3.32) using (1.2.84) and (1.1.26), we have

$$\mathcal{L}_V\xi = (f + 2\lambda - 4n - p - \frac{2}{2n+1})\xi. \quad (4.3.37)$$

Now, we insert $X = \xi$ in (4.3.6) and plug (4.3.37) to yield

$$f = 2n + \frac{p}{2} + \frac{1}{2n+1} - \lambda. \quad (4.3.38)$$

Using the value of f in (4.3.37), we get $W = (\lambda - 2n - \frac{p}{2} - \frac{1}{2n+1})\xi$. Covariant differentiation of the foregoing equation along an arbitrary vector field X, yields

$$\nabla_X W = (X\lambda)\xi + (\lambda - 2n - \frac{p}{2} - \frac{1}{2n+1})[X - \eta(X)\xi]. \quad (4.3.39)$$

Taking (1.1.8) and (4.3.39) into account, we can easily get

$$(\mathcal{L}_W g)(X,Y) = (X\lambda)\eta(Y) + (Y\lambda)\eta(X) + (2\lambda - 4n - p - \frac{2}{2n+1})g(\phi X,\phi Y). \quad (4.3.40)$$

We combine the equation (4.3.40) with (4.3.36) to deduce

$$(X\lambda)\eta(Y) + (Y\lambda)\eta(X) + 4S(X,Y) + (p + 4n + \frac{2}{2n+1} - 2\lambda)\eta(X)\eta(Y)$$
$$[2\lambda - p + 4n - \frac{2}{2n+1} - 2(\xi\lambda)]g(X,Y) = 0, \quad (4.3.41)$$

for arbitrary vector fields X and Y on M. Setting $Y = \xi$ in the last equation and using (1.1.26), we have $D\lambda = (\xi\lambda)\xi$. Hence first part of our theorem is proved.

Replacing X and Y by ϕX and ϕY, respectively in (4.3.41) and using the result of lemma 2.5, (1.1.8), (1.1.10) and (1.1.26) we have $S(X,Y) = \frac{1}{4}[p + \frac{2}{2n+1} + 2(\xi\lambda) - 2\lambda - 4n]g(X,Y) + \frac{1}{4}[2\lambda - p - 4n - \frac{2}{2n+1} - 2(\xi\lambda)]\eta(X)\eta(Y)$. This completes the proof. $\square$

Example 4.3.1. *Let $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$ be a 5-dimensional manifold, where (x,y,z,u,v) be the standard coordinates in $\mathbb{R}^5$. Now, let us consider an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of vector fields on M, where,*

$$e_1 = v\frac{\partial}{\partial x}, \qquad e_2 = v\frac{\partial}{\partial y}, \qquad e_3 = v\frac{\partial}{\partial z}, \qquad e_4 = v\frac{\partial}{\partial u}, \qquad e_5 = -v\frac{\partial}{\partial v}.$$

Define $(1,1)$ tensor field ϕ as follows,

$$\phi(e_1) = e_2, \qquad \phi(e_2) = -e_1, \qquad \phi(e_3) = e_4, \qquad \phi(e_4) = -e_3, \qquad \phi(e_5) = 0.$$

The Riemannian metric is given by,

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

and $\eta(X) = g(X, e_5)$ for any $X \in \chi(M^5)$. Then for $\xi = e_5$, the relations (1.1.6), (1.1.7) and (1.1.8) are satisfied. Thus, (ϕ, ξ, η, g) is an almost contact structure.

Using (1.1.1) non-zero components of the Levi-Civita connection ∇ can be found as

$$\begin{gathered} \nabla_{e_1}e_1 = \nabla_{e_2}e_2 = \nabla_{e_3}e_3 = \nabla_{e_4}e_4 = -e_5, \\ \nabla_{e_1}e_5 = e_1, \ \nabla_{e_2}e_5 = e_2, \ \nabla_{e_3}e_5 = e_3, \ \nabla_{e_4}e_5 = e_4. \end{gathered} \tag{4.3.42}$$

By virtue of this we can verify (1.1.22) and therefore $M^5(\varphi, \xi, \eta, g)$ is a Kenmotsu manifold.

Using the well known expression of curvatute tensor (1.1.4), we now compute the following non-zero components

$$\begin{array}{llll} R(e_1,e_2)e_2 = -e_1, & R(e_1,e_3)e_3 = -e_1, & R(e_1,e_4)e_4 = -e_1, & R(e_1,e_5)e_5 = -e_1, \\ R(e_1,e_2)e_1 = e_2, & R(e_1,e_3)e_1 = e_3, & R(e_1,e_4)e_1 = e_4, & R(e_1,e_5)e_1 = e_5, \\ R(e_2,e_3)e_2 = e_3, & R(e_2,e_4)e_2 = e_4, & R(e_2,e_5)e_2 = e_5, & R(e_2,e_3)e_3 = -e_2, \\ R(e_2,e_4)e_4 = -e_2, & R(e_2,e_5)e_5 = -e_2, & R(e_3,e_4)e_3 = e_4, & R(e_3,e_5)e_3 = e_5, \\ R(e_3,e_4)e_4 = -e_3, & R(e_4,e_5)e_4 = e_5, & R(e_5,e_3)e_5 = e_3, & R(e_5,e_4)e_5 = e_4. \end{array}$$

Using this, we compute the components of the Ricci tensor as follows: $S(e_i, e_i) = -4,$ *for* $i = 1, 2, 3, 4, 5$ *and therefore*

$$S(X, Y) = -4g(X, Y), \quad for\ all\ X, Y \in \chi(M^5). \tag{4.3.43}$$

Let $f : M \to \mathbb{R}$ *be a smooth function defined by*

$$f(x, y, z, u, v) = x^2 + y^2 + z^2 + u^2 + \frac{v^2}{2}. \tag{4.3.44}$$

The potential vector field is given by,

$$V = Df = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{4.3.45}$$

Then with the help of (4.3.42) *we can show that*

$$(\mathcal{L}_V g)(X, Y) = 2\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{4.3.46}$$

for all $X,\ Y \in \chi(M^5)$. *So, combining* (4.3.43) *and* (4.3.46), *we observe that soliton Eq.* (1.2.84) *holds for* $\lambda = \frac{17}{5} + \frac{p}{2}$ *i.e., the metric g is a conformal almost Ricci soliton with this potential vector field* $V = Df$ *and* $\lambda = \frac{17}{5} + \frac{p}{2}$.

4.4 A para-Kenmotsu metric as conformal Ricci soliton

In this section we consider the metric of para-Kenmotsu manifold as a conformal Ricci soliton. The following lemma will be used to prove one of the our main results.

Lemma 4.4.1. *Let* (M, ϕ, ξ, η, g) *be a* $(2n + 1)$*-dimensional para-Kenmotsu manifold. Then the Ricci operator satisfies*

$$(\mathcal{L}_\xi Q)X = -2QX - 4nX = (\nabla_\xi Q)X \tag{4.4.1}$$

for any vector field X *on* M.

Proof. From (1.1.60), we have $(\mathcal{L}_\xi g)(Y, Z) = 2[g(Y, Z) - \eta(Y)\eta(Z)]$ for all $Y, Z \in \chi(M)$. Covariant derivative of that along an arbitrary vector field X on M and use of the equation (1.1.57), leads to

$$(\nabla_X \mathcal{L}_\xi g)(Y, Z) = 2[2\eta(X)\eta(Y)\eta(Z) - g(X, Y)\eta(Z) - g(X, Z)\eta(Y)] \tag{4.4.2}$$

for all $Y, Z \in \chi(M)$. As g is the metric connection i.e., $\nabla g = 0$, the relation (1.1.3) reduces to

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V \nabla)(X,Y),Z) + g((\mathcal{L}_V \nabla)(X,Z),Y). \tag{4.4.3}$$

for all vector fields X, Y, Z on M. Combining (4.4.2) and (4.4.3), we have

$$g((\mathcal{L}_\xi \nabla)(X,Y),Z)+g((\mathcal{L}_\xi \nabla)(X,Z),Y) = 2[2\eta(X)\eta(Y)\eta(Z)-g(X,Y)\eta(Z)-g(X,Z)\eta(Y)].$$

By a combinatorial computation, the foregoing equation yields

$$(\mathcal{L}_\xi \nabla)(Y,Z) = 2[\eta(Y)\eta(Z)\xi - g(Y,Z)\xi] \tag{4.4.4}$$

for all $Y, Z \in \chi(M)$. Taking covariant derivative of the above equation with respect to an arbitrary vector field X on M and using (1.1.56) and (1.1.57), we have

$$\begin{aligned}(\nabla_X \mathcal{L}_\xi \nabla)(Y,Z) =& 2[g(X,Y)\eta(Z)\xi + g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)X + \eta(Y)\eta(Z)X - 3\eta(X)\eta(Y)\eta(Z)].\end{aligned}$$

Using the foregoing relation in (1.1.5), we can compute

$$(\mathcal{L}_\xi R)(X,Y)Z = 2[g(X,Z)Y - g(Y,Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \tag{4.4.5}$$

for all vector fields X, Y, Z on M. Contracting (4.4.5) over X, we get

$$(\mathcal{L}_\xi S)(Y,Z) = 4n[\eta(Y)\eta(Z) - g(Y,Z)]. \tag{4.4.6}$$

The Lie derivative of $S(Y,Z) = g(QY,Z)$ along the direction of ξ, yields

$$(\mathcal{L}_\xi S)(Y,Z) = (\mathcal{L}_\xi g)(QY,Z) + g((\mathcal{L}_\xi Q)Y,Z). \tag{4.4.7}$$

On the other hand, replacing X and Y by QY and Z respectively in (1.1.60) and using (1.1.58), we have

$$(\mathcal{L}_\xi g)(QY,Z) = 2[g(QY,Z) + 2n\eta(Y)\eta(Z)]. \tag{4.4.8}$$

Combining (4.4.6), (4.4.7) and (4.4.8) all together, we infer

$$(\mathcal{L}_\xi Q)Y = -2QY - 4nY \tag{4.4.9}$$

for any $Y \in \chi(M)$. Again we know that

$$\begin{aligned}(\mathcal{L}_\xi Q)Y &= \mathcal{L}_\xi(QY) - Q(\mathcal{L}_\xi Y)\\ &= \nabla_\xi(QY) - \nabla_{QY}\xi - Q(\nabla_\xi Y) + Q(\nabla_Y \xi)\\ &= (\nabla_\xi Q)Y - \nabla_{QY}\xi + Q(\nabla_Y\xi).\end{aligned}$$

By virtue of (1.1.56) and (1.1.58) we see that $(\mathcal{L}_\xi Q)Y = (\nabla_\xi Q)Y$ for arbitrary vector field Y. Hence the result is proved. □

Theorem 4.4.1. *If the metric g of a para-Kenmotsu manifold (M, ϕ, ξ, η, g) of dimension > 3 represents a conformal Ricci soliton then either of the following properties holds:*

(i) The Lie derivative of ξ in the direction of the potential vector field V of the soliton i.e., $\mathcal{L}_V\xi$ is orthogonal to ξ.

(ii) The manifold is Einstein with Einstein constant $-2n$.

Proof. Let M be a $(2n+1)$ dimensional para-Kenmotsu manifold where $n > 1$. From (1.1.59), we have $R(X, \xi)\xi = \eta(X)\xi - X$. Now Lie derivative of the Riemannian curvature tensor along the vector field V, yields

$$(\mathcal{L}_V R)(X, \xi)\xi = ((\mathcal{L}_V\eta)X)\xi - g(X, \mathcal{L}_V\xi)\xi + 2\eta(\mathcal{L}_V\xi)X \tag{4.4.10}$$

for all vector fields X on M. Now the covariant derivative of (1.2.84) along an arbitrary vector field $Z \in \chi(M)$ provides

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = -2(\nabla_Z S)(X, Y) \tag{4.4.11}$$

for any $X, Y \in \chi(M)$. Using (4.4.3), we can rewrite (4.4.11) as

$$g((\mathcal{L}_V\nabla)(X, Y), Z) + g((\mathcal{L}_V\nabla)(X, Z), Y) = -2(\nabla_Z S)(X, Y).$$

By some combinatorial computation and using the symmetry of the (1,2)-tensor $\mathcal{L}_V\nabla$, the aforementioned yields

$$g((\mathcal{L}_V\nabla)(X, Y), Z) = (\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(Z, X). \tag{4.4.12}$$

Again differentiating the above equation covariantly with respect to an arbitrary vector field X of M and using (1.1.56), we can find from (1.1.58) that

$$(\nabla_X Q)\xi = -QX - 2nX \tag{4.4.13}$$

for all $X \in \chi(M)$. Making use of (4.4.1) and (4.4.13) and considering $Y = \xi$ in (4.4.12), we achieve

$$(\mathcal{L}_V \nabla)(X, \xi) = 2QX + 4nX \tag{4.4.14}$$

for any vector field X on M. Now considering covariant derivative of the last equation with respect to an arbitrary vector field Y of M and using (1.1.56), we acquire

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) = 2(\nabla_Y Q)X - (\mathcal{L}_V \nabla)(X, Y) + 2\eta(Y)QX + 4n\eta(Y)X. \tag{4.4.15}$$

Now letting $Z = \xi$ in (1.1.5) and using (4.4.15) in the foregoing equation, we have

$$(\mathcal{L}_V R)(X, Y)\xi = 4n[\eta(X)Y - \eta(Y)X] + 2[(\nabla_X Q)Y - (\nabla_Y Q)X] + 2[\eta(X)QY - \eta(Y)QX] \tag{4.4.16}$$

for all $X, Y \in \chi(M)$. Considering $Y = \xi$ in the aforementioned equation and using (1.1.58) and (4.4.1) in it, we obtain

$$(\mathcal{L}_V R)(X, \xi)\xi = 0. \tag{4.4.17}$$

Now, taking into account (1.2.84), the Lie derivative of $g(\xi, \xi) = 1$ along the direction of V, leads to

$$\eta(\mathcal{L}_V \xi) = \lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n. \tag{4.4.18}$$

Again, using (1.1.58) and letting $Y = \xi$, (1.2.84) implies

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = (4n - 2\lambda + p + \frac{2}{2n+1})\eta(X). \tag{4.4.19}$$

After using (4.4.17), (4.4.18) and (4.4.19), the equation (4.4.10) reduces to

$$(2\lambda - p - 4n - \frac{2}{2n+1})\phi^2 X = 0. \tag{4.4.20}$$

Since the last equation holds for any $X \in \chi(M)$, we can conclude that $\lambda = \frac{p}{2} + 2n + \frac{1}{2n+1}$. Using this result in (4.4.18) we have, $\eta(\mathcal{L}_V \xi) = 0$. From here the following two cases have arisen,

Case-I: $\mathcal{L}_V \xi$ is orthogonal to ξ.

Case-II: $\mathcal{L}_V \xi = 0$ for any vector field X of M. Then additionally using the value of λ, (4.4.19) reduces to $(\mathcal{L}_V \eta)X = 0$. Which further can be reduced to $\mathcal{L}_V \eta = 0$, since X

is an arbitrary vector field on M. Replacing Y by ξ and using (1.1.56) and the relations $\mathcal{L}_V\xi = 0$ and $\mathcal{L}_V\eta = 0$ in (1.1.2), we obtain

$$(\mathcal{L}_V\nabla)(X,\xi) = 0.$$

Finally substituting this in (4.4.14), we get $S(X,Y) = -2ng(X,Y)$ for any arbitrary vector fields X and Y on M. From this we can conclude that the manifold is Einstein with Einstein constant $-2n$. □

Theorem 4.4.2. *Let M be a (2n+1)-dimensional η-Einstein para-Kenmotsu manifold where $n > 1$. If the metric of the manifold represents a conformal Ricci soliton, then the manifold is Einstein.*

Proof. Let the metric g of an η-Einstein para-Kenmotsu manifold M whose dimension is greater than 3 represents a conformal Ricci soliton. Then clearly it satisfies (1.2.84) as well as (4.2.5). Combining these two relations, we have

$$(\mathcal{L}_Vg)(Y,Z) = (p - 2\lambda - \frac{r}{n} - \frac{4n}{2n+1})g(Y,Z) + (4n + 2 + \frac{r}{n})\eta(Y)\eta(Z), \qquad (4.4.21)$$

for all $Y, Z \in \chi(M)$. Covariant derivative of (4.4.21) with respect to an arbitrary vector field X on M and use of (4.4.3), leads to

$$\begin{aligned} g((\mathcal{L}_V\nabla)(X,Y),Z) + g((\mathcal{L}_V\nabla)(X,Z),Y) =&(4n+2+\frac{r}{n})[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - \\ &2\eta(X)\eta(Y)\eta(Z)] - \frac{Xr}{n}[g(Y,Z) + \eta(Y)\eta(Z)], \end{aligned} \qquad (4.4.22)$$

for any vector fields X, Y and Z on M. By combinatorial computation of the last equation, keeping the symmetry of $(\mathcal{L}_V\nabla)$ in mind, provides

$$\begin{aligned} 2n(\mathcal{L}_V\nabla)(X,Y) \;=\;& (Xr)\eta(Y)\xi - (Xr)Y + (Yr)\eta(X)\xi - (Yr)X + (Dr)g(X,Y) - \\ & (Dr)\eta(X)\eta(Y) + 2(4n^2 + 2n + r)[g(X,Y)\xi - \eta(X)\eta(Y)\xi] \end{aligned} \qquad (4.4.23)$$

where Dr is the gradient of r. Let us consider a local orthonormal basis of the manifold as $\{e_i\}_{i=1}^{2n+1}$. Next, setting $X = Y = e_i$ and summing over $1 \leq i \leq 2n+1$ in the last equation, we infer

$$n(\mathcal{L}_V\nabla)(e_i, e_i) = (\xi r)\xi + (n-1)Dr + 2n(4n^2 + 2n + r)\xi. \qquad (4.4.24)$$

After considering $X = Y = e_i$ and summing over i, (4.4.12) reduces to $g((\mathcal{L}_V\nabla)(e_i, e_i), Z) = Zr - \frac{1}{2}Zr - \frac{1}{2}Zr{=}0$. Since this holds for an arbitrary vector field Z, this can be rewritten as

$$(\mathcal{L}_V\nabla)(e_i, e_i) = 0. \tag{4.4.25}$$

Comparing (4.4.24) and (4.4.25), we get $(\xi r)\xi + (n-1)Dr + 2n(4n^2+2n+r) = 0$. Taking inner product with ξ, we have

$$\xi r = -2(4n^2 + 2n + r). \tag{4.4.26}$$

Again it further implies that $Dr = (\xi r)\xi$. Next substituting Y by ξ in (4.4.23), we get

$$2n(\mathcal{L}_V\nabla)(X, \xi) = (\xi r)(-X + \eta(X)\xi). \tag{4.4.27}$$

Covariant derivative of the foregoing equation with respect to an arbitrary vector field Y and using (1.1.56), (1.1.57) and (4.4.27), leads to

$$\begin{aligned} 2n(\nabla_Y\mathcal{L}_V\nabla)(X, \xi) =& (Y(\xi r))(-X + \eta(X)\xi) - 2n(\mathcal{L}_V\nabla)(X, Y) + (\xi r)[g(X, Y)\xi \\ &+ \eta(X)Y - \eta(Y)X - \eta(X)\eta(Y)\xi]. \end{aligned} \tag{4.4.28}$$

Using the relation (4.4.28) in (1.1.5), we achieve

$$2n(\mathcal{L}_V R)(X, Y)\xi = (X(\xi r))(-Y+\eta(Y)\xi)-(Y(\xi r))(-X+\eta(X)\xi)+2(\xi r)(\eta(Y)X-\eta(X)Y). \tag{4.4.29}$$

Contracting this over X, we have $(\mathcal{L}_V S)(Y, \xi) = 0$, where we have used $Dr = (\xi r)\xi$. Finally using $(\mathcal{L}_V S)(Y, \xi) = 0$, (4.2.5) and (4.4.21) in the Lie derivative of $S(Y, \xi) = -2n\eta(Y)$, we obtain

$$2n\Big(p-2\lambda-\frac{4n}{2n+1}+4n+2\Big)\eta(Y)+\Big(1+2n+\frac{r}{2n}\Big)g(Y, \mathcal{L}_V\xi) = \Big(2n+1+\frac{r}{2n}\Big)\eta(Y)\eta(\mathcal{L}_V\xi) \tag{4.4.30}$$

for any vector field Y on M. Taking $Y = \xi$ in the last equation, we get $\lambda = \frac{p}{2} + 2n + \frac{1}{2n+1}$. Setting $Y = Z = \xi$ in (4.4.21) and using the value of λ, we obtain $\eta(\mathcal{L}_V\xi) = 0$. Using these two relations, the equation (4.4.30) can be written as

$$(2n(2n + 1) + r)\mathcal{L}_V\xi = 0. \tag{4.4.31}$$

We suppose $r \neq -2n(2n + 1)$ on some open set O of M. Then (4.4.31) implies that $\mathcal{L}_V\xi = 0$, which further implies with help of (1.1.56) that $\nabla_\xi V = V - \eta(V)\xi$. Using these relations along with (1.1.56), (4.4.21) and (4.4.27) in (1.1.2) we obtain $\xi r = 0$. As $Dr = (\xi r)\xi$, so, $Dr = 0$ i.e., the scalar curvature is constant. So, from (4.4.26), we can find that $r = -2n(2n+1)$ on O, which is a contradiction to our assumption that $r \neq -2n(2n+1)$ on O. Thus from (4.4.31), we can infer $r \neq -2n(2n+1)$ on the entire manifold. Finally from (4.2.5), we have $S(X,Y) = -2ng(X,Y)$ for all $X, Y \in \chi(M)$. So, the manifold is Einstein with Einstein constant $-2n$. □

4.5 A para-Kenmotsu metric as $*$-conformal Ricci soliton

In this section we assume that the metric of para-Kenmotsu manifold represents a $*$-conformal Ricci soliton.

Theorem 4.5.1. *Let $M^{2n+1}(\phi, \xi, \eta, g), n > 1$ be an η-Einstein para-Kenmotsu manifold. If g represents a $*$-conformal Ricci soliton, then the manifold is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof. Let M be a $(2n+1)$-dimensional η-Einstein para-Kenmotsu manifold of dimension > 3 whose metric g represents a $*$-conformal Ricci soliton. So, the relations (1.2.85), (4.2.5) and (4.2.6) are satisfied. Rewriting (1.2.85) with the help of the rest two relations, we have

$$(\mathcal{L}_V g)(Y,Z) = (p - 2\lambda + \frac{r}{n} + 4n + \frac{2}{2n+1})g(Y,Z) - (4n + \frac{r}{n})\eta(Y)\eta(Z) \tag{4.5.1}$$

for all $Y, Z \in \chi(M)$. Differentiating the above equation with respect to an arbitrary vector field X of M and using (1.1.57), we achieve

$$\begin{aligned}(\nabla_X \mathcal{L}_V g)(Y,Z) = &\frac{Xr}{n} g(Y,Z) - \frac{Xr}{n}\eta(Y)\eta(Z) - (4n + \frac{r}{n})[g(X,Y)\eta(Z) \\ &+ g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)]\end{aligned} \tag{4.5.2}$$

for any vector fields X, Y and Z of M. Again from (4.4.3), we know $(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V \nabla)(X,Y), Z) + g((\mathcal{L}_V \nabla)(X,Z), Y)$. Using this and by a combinatorial computation, keeping in mind that $\mathcal{L}_V \nabla$ is a symmetric operator, the foregoing equation gives

$$\begin{aligned}2n(\mathcal{L}_V \nabla)(X,Y) = &(Xr)[Y - \eta(Y)\xi] + (Yr)[X - \eta(X)\xi] - (Dr)[g(X,Y) - \eta(X)\eta(Y)] \\ &-2(4n^2 + r)[g(X,Y) - \eta(X)\eta(Y)]\xi.\end{aligned} \tag{4.5.3}$$

The covariant derivative of (1.2.85) with respect to an arbitrary vector field X, yields

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = -2(\nabla_X S^*)(Y,Z). \tag{4.5.4}$$

By computation and use of the relation (4.4.3) in the equation (4.5.4), leads to

$$g((\mathcal{L}_V \nabla)(X,Y),Z) = (\nabla_Z S^*)(X,Y) - (\nabla_X S^*)(Y,Z) - (\nabla_Y S^*)(Z,X). \tag{4.5.5}$$

Again, taking covariant derivative of (4.2.6) with respect to an arbitrary vector field Z of M and then using (1.1.57), we get

$$(\nabla_Z S^*)(X,Y) = -(\nabla_Z S)(X,Y) - g(X,Z)\eta(Y) - g(Y,Z)\eta(X) + 2\eta(X)\eta(Y)\eta(Z). \tag{4.5.6}$$

Combining (4.5.6) with (4.5.5), yields

$$\begin{aligned} g((\mathcal{L}_V \nabla)(X,Y),Z) =& (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) - (\nabla_Z S)(X,Y) \\ & + 2g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z). \end{aligned} \tag{4.5.7}$$

Now, let us consider a local orthonormal basis $\{e_i\}_{i=1}^{2n+1}$ of the manifold. Replacing $X = Y = e_i$ in (4.5.3), we have

$$2n(\mathcal{L}_V \nabla)(e_i,e_i) = -2(\xi r)\xi - 2(n-1)(Dr) - 4n(4n^2 + r)\xi. \tag{4.5.8}$$

Again, substituting X and Y by e_i in equation (4.5.7) and summing over i, we get

$$(\mathcal{L}_V \nabla)(e_i,e_i) = 4n\xi. \tag{4.5.9}$$

Combining the above two relations we directly have

$$(\xi r)\xi + (n-1)(Dr) + 2n(4n^2 + 2n + r)\xi = 0. \tag{4.5.10}$$

The inner product with respect to ξ, reduces the aforementioned equation to $\xi r = -2(2n(2n+1) + r)$. As $n > 1$, using this relation in the equation (4.5.10) we easily obtain $Dr = (\xi r)\xi$. After substituting Y by ξ in (4.5.3) and using (1.1.46), we infer

$$2n(\mathcal{L}_V \nabla)(X,\xi) = (\xi r)\phi^2(X), \tag{4.5.11}$$

for all $X \in \chi(M)$. Differentiating (4.5.11) with respect to an arbitrary vector field Y and using (1.1.56), (1.1.57) and (4.5.11), we get

$$\begin{aligned} 2n(\nabla_Y \mathcal{L}_V \nabla)(X,\xi) + 2n(\mathcal{L}_V \nabla)(X,Y) =& (Y(\xi r))\phi^2 X - (\xi r)[g(X,Y)\xi + \\ & \eta(X)Y - \eta(Y)X - \eta(X)\eta(Y)\xi]. \end{aligned} \tag{4.5.12}$$

Using this in the well known formula (1.1.5), we have

$$2n(\mathcal{L}_V R)(X,Y)\xi = (X(\xi r))\phi^2 Y - (Y(\xi r))\phi^2 X - 2(\xi r)[\eta(Y)X - \eta(X)Y], \qquad (4.5.13)$$

for all $X, Y \in \chi(M)$. Contracting the above equation over X and using the relation $Dr = (\xi r)\xi$, we have $(\mathcal{L}_V S)(Y,\xi) = 0$. Using (4.2.5), (4.5.1) and $(\mathcal{L}_V S)(Y,\xi) = 0$ in the Lie derivative of $S(Y,\xi) = -2n\eta(Y)$, we get

$$2n\big(p - 2\lambda + \frac{2}{2n+1}\big)\eta(Y) + \big(2n+1+\frac{r}{2n}\big)[g(Y,\mathcal{L}_V\xi) - \eta(Y)\eta(\mathcal{L}_V\xi)] = 0. \qquad (4.5.14)$$

In the last equation considering $Y = \xi$, we obtain $\lambda = \frac{p}{2} + \frac{1}{2n+1}$ as $n > 1$. Again setting $Y = Z = \xi$ in (4.5.1), we have $\eta(\mathcal{L}_V\xi) = 0$. Applying these relations, we can rewrite (4.5.14) as

$$(2n(2n+1) + r)\mathcal{L}_V\xi = 0. \qquad (4.5.15)$$

We suppose $r \neq -2n(2n+1)$ on some open set O of M. Then from (4.5.15), directly we obtain $\mathcal{L}_V\xi = 0$. From (1.1.56), we deduce that $\nabla_\xi V = V - \eta(V)\xi$. Again taking $Z = \xi$ in (4.5.1) and using $\lambda = \frac{p}{2} + \frac{1}{2n+1}$, we have $\mathcal{L}_V\eta = 0$. Using these relations along with (1.1.56) and (4.5.11) in the identity (1.1.2), we obtain $\xi r = 0$. As $Dr = (\xi r)\xi$, so, $Dr = 0$ i.e., the scalar curvature r is constant. So, from the relation $\xi r = -2(2n(2n+1) + r)$, we can find that $r = -2n(2n+1)$ on O, which is a contradiction to our assumption that $r \neq -2n(2n+1)$ on O. Thus from (4.5.15), we can conclude that $r = -2n(2n+1)$ on the entire manifold M. Moreover from (4.2.5), we have $S(X,Y) = -2ng(X,Y)$ for all $X, Y \in \chi(M)$. So, the manifold is Einstein with Einstein constant $-2n$. □

Example 4.5.1. *We consider the manifold as $M = \{(x,y,x) \in \mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields are defined by*

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

are linearly independent at each point on M. The metric g is defined by

$$g(e_1,e_1) = g(e_3,e_3) = 1, \quad g(e_2,e_2) = -1, \quad g(e_1,e_2) = g(e_2,e_3) = g(e_3,e_1) = 0.$$

Let $\xi = e_3$. Then the 1-form η is defined by $\eta(X) = g(X,e_3)$, for arbitrary $X \in \chi(M)$, then we have the following relations

$$\eta(e_1) = 0, \qquad \eta(e_2) = 0, \qquad \eta(e_3) = 1.$$

Let us define the $(1,1)$*-tensor field* ϕ *as*

$$\phi e_2 = e_1, \qquad \phi e_1 = e_2, \qquad \phi e_3 = 0,$$

then the relations (1.1.46)*,* (1.1.47) *and* (1.1.50) *are satisfied. Thus* (ϕ, ξ, η, g) *defines an almost paracontact metric structure on* M*. We can now easily conclude*

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = e_2, \qquad [e_1, e_3] = e_1.$$

Let ∇ *be the Levi-Civita connection of* M*. Then by Koszul's formula* (1.1.1)*, we obtain*

$$\nabla_{e_1} e_1 = -e_3, \qquad \nabla_{e_1} e_2 = 0, \qquad \nabla_{e_1} e_3 = e_1,$$
$$\nabla_{e_2} e_1 = 0, \qquad \nabla_{e_2} e_2 = e_3, \qquad \nabla_{e_2} e_3 = e_2,$$
$$\nabla_{e_3} e_1 = 0, \qquad \nabla_{e_3} e_2 = 0, \qquad \nabla_{e_3} e_3 = 0.$$

From here we can easily verify that the relation (1.1.55) is satisfied. Hence the considered manifold is para-Kenmotsu manifold. The components of the Riemannian curvature tensor are given by

$$R(e_1, e_2)e_1 = e_2, \qquad R(e_1, e_2)e_2 = e_1, \qquad R(e_1, e_2)e_3 = 0,$$
$$R(e_1, e_3)e_1 = e_3, \qquad R(e_1, e_3)e_2 = 0, \qquad R(e_1, e_3)e_3 = -e_1,$$
$$R(e_2, e_3)e_1 = 0, \qquad R(e_2, e_3)e_2 = -e_3, \qquad R(e_2, e_3)e_3 = -e_2.$$

And the components of Ricci tensor and $*$*-Ricci tensor are given by*

$$S(e_1, e_1) = -2, \qquad S(e_2, e_2) = 2, \qquad S(e_3, e_3) = -2,$$
$$S^*(e_1, e_1) = 1, \qquad S^*(e_2, e_2) = -1, \qquad S^*(e_3, e_3) = 0.$$

From here we can easily deduce that the scalar curvature of the manifold $r = -6$ *and* $S(X, Y) = -2g(X, Y)\ \forall X, Y \in \chi(M)$*. Let us define a vector field by*

$$V = (x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Then we can obtain

$$(\mathcal{L}_V g)(e_1, e_1) = 2, \qquad (\mathcal{L}_V g)(e_2, e_2) = -2, \qquad (\mathcal{L}_V g)(e_3, e_3) = 0.$$

Contracting (1.2.84) and using the result $r = -6$ *we deduce* $\lambda = \frac{p}{2} + \frac{19}{3}$*. So* g *defines a conformal Ricci soliton on this para-Kenmotsu manifold for* $\lambda = \frac{p}{2} + \frac{19}{3}$*.*
Again contracting (4.2.6) we get, $r^* = -r - 4 = 2$ *(as* $r = -6$*). Now contracting (1.2.85) and using the previous result we obtain* $\lambda = \frac{p}{2} - \frac{5}{3}$*. So,* g *defines a* $*$*-conformal Ricci soliton on this para-Kenmotsu manifold for* $\lambda = \frac{p}{2} - \frac{5}{3}$*.*

4.6 A 3-dimensional para-cosymplectic metric as conformal Ricci soliton

In this section, we first prove some lemmas whose results are used to deduce our main result.

Lemma 4.6.1 ([107]). *If an n-dimensional Riemannian manifold admits a conformal vector field V then we have*

$$(\mathcal{L}_V S)(X,Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X,Y), \tag{4.6.1}$$
$$\mathcal{L}_V r = 2(n-1)\Delta\rho - 2\rho r, \tag{4.6.2}$$

for any vector fields X and Y, where D and Δ denote the gradient and Laplacian operator with respect to g respectively and r represents the scalar curvature of the manifold.

Lemma 4.6.2. *If the metric g of a 3-dimensional para-cosymplectic manifold represents a conformal Ricci soliton then the following properties hold*

$$\eta(\mathcal{L}_V\xi) = \lambda - \frac{p}{2} - \frac{1}{3}, \tag{4.6.3}$$
$$(\mathcal{L}_V\eta)\xi = -\lambda + \frac{p}{2} + \frac{1}{3}. \tag{4.6.4}$$

Proof. As the vector field ξ is a unit vector field, we have $g(\xi,\xi) = 1$. Taking Lie derivative of the previous relation with respect to vector field V, we have $(\mathcal{L}_V g)(\xi,\xi) + 2\eta(\mathcal{L}_V\xi) = 0$. Using (1.2.84), (1.1.47) and (4.2.8), we acquire

$$\eta(\mathcal{L}_V\xi) = \lambda - \frac{p}{2} - \frac{1}{3}.$$

Taking Lie derivative of (1.1.47) along the direction of the vector field V and using (4.6.3), we achieve

$$(\mathcal{L}_V\eta)\xi = -\lambda + \frac{p}{2} + \frac{1}{3}.$$

□

Theorem 4.6.1. *If the metric g of a 3-dimensional para-cosymplectic manifold (M^3,ϕ,ξ,η,g) which admits a conformal vector field V, represents a conformal Ricci soliton then the scalar curvature of the manifold is harmonic and the manifold is Ricci flat.*

Proof. Since, V is a conformal vector field, there exists a smooth function ρ such that

$$\mathcal{L}_V g = 2\rho g. \tag{4.6.5}$$

Combining (1.2.84) and (4.6.5) for 3-dimensional para-cosymplectic manifold, we have

$$\big(2\rho + 2\lambda - p - \frac{2}{3}\big)g(X,Y) + 2S(X,Y) = 0$$

for any $X, Y \in \chi(M)$. Contracting the above equation, we get

$$\rho = \frac{1}{6}(3p - 6\lambda - 2r + 2). \tag{4.6.6}$$

Using (4.6.6) in (4.6.1) and (4.6.2), we get

$$\begin{aligned} (\mathcal{L}_V S)(X,Y) &= \frac{1}{3}g(\nabla_X Dr, Y) - \frac{1}{3}(\Delta r)g(X,Y), & (4.6.7)\\ \mathcal{L}_V r &= -\frac{1}{3}(3p - 6\lambda - 2r + 2)r - \frac{4}{3}(\Delta r). & (4.6.8) \end{aligned}$$

Taking Lie derivative of (4.2.8) in the direction of the vector field V and using (1.2.84), (4.2.8), (4.6.7) and (4.6.8), we have

$$\begin{aligned} g(\nabla_X Dr, Y) = &-\big(\Delta r + \frac{r^2}{2}\big)g(X,Y) + \big[\frac{r}{2}(3p - 6\lambda + r + 2) + 2(\Delta r)\big]\eta(X)\eta(Y)\\ &- \frac{3r}{2}\big[((\mathcal{L}_V\eta)X)\eta(Y) + \eta(X)((\mathcal{L}_V\eta)Y)\big]. \end{aligned} \tag{4.6.9}$$

Covariant derivative of (4.2.10) along an arbitrary vector field X, yields $g(\nabla_X Dr, \xi) = 0$. Now setting $X = Y = \xi$ in the equation (4.6.9) and using the aforementioned relation along with the equation (4.6.4), we get

$$\Delta r = 0. \tag{4.6.10}$$

Hence the scalar curvature r of the manifold is harmonic.

Now considering $Y = \xi$ in (4.6.9) and using the relation $g(\nabla_X Dr, \xi) = 0$, (4.6.10), (4.6.4), we obtain the following relation

$$r((\mathcal{L}_V\eta)X) = r\big(\frac{p}{2} + \frac{1}{3} - \lambda\big)\eta(X) \tag{4.6.11}$$

for an arbitrary vector field X on M. Making use of the last equation, (4.2.9) and (4.6.10) in (4.6.9), we achieve

$$\nabla_X Dr = -rQX \tag{4.6.12}$$

for any arbitrary $X \in \chi(M)$. Now contracting it with respect to X, we get $\Delta r = -r^2$ and combining with (4.6.10), we infer $r = 0$ i.e., the manifold is Ricci flat. □

4.7 $*$-Conformal Ricci soliton on 3-dimensional trans-Sasakian manifold

Theorem 4.7.1. *Let M be a 3-dimensional trans-Sasakian manifold of type (α, β) where the structure functions α and β are constant with $\alpha \neq 0$. If the metric g represents a $*$-conformal Ricci soliton then the scalar curvature of the manifold is given by $r = (1 - \frac{\beta^2}{\alpha^2})(\frac{p}{2} + \frac{1}{3} - \lambda + 4\alpha^2)$.*

Proof. Since the metric g represents a $*$-conformal Ricci solition, using (2.2.1) in the definition of $*$-conformal Ricci solition (1.2.85), we get

$$(\mathcal{L}_V g)(X,Y) = (p + \frac{2}{3} + 4(\alpha^2 - \beta^2) - r - 2\lambda)g(X,Y) + (r - 4(\alpha^2 - \beta^2))\eta(X)\eta(Y) \quad (4.7.1)$$

for all vector fields X and Y on M. If we consider covariant derivative along an arbitrary vector field Z, then (4.7.1) reduces to

$$\begin{aligned}(\nabla_Z \mathcal{L}_V g)(X,Y) =& (Zr)[\eta(X)\eta(Y) - g(X,Y)] - (r - 4(\alpha^2 - \beta^2))[\alpha g(\phi Z, X)\eta(Y) - \\ & \beta g(\phi X, \phi Z)\eta(Y) + \alpha g(\phi Z, Y)\eta(X) - \beta g(\phi Y, \phi Z)\eta(X)], \quad (4.7.2)\end{aligned}$$

for all $X, Y, Z \in \chi(M)$. As ∇ is the metric connection i.e., $\nabla g = 0$, so (1.1.3) reduces to

$$(\nabla_Z \mathcal{L}_V g)(X,Y) = g((\mathcal{L}_V \nabla)(X,Z), Y) + g((\mathcal{L}_V \nabla)(Y,Z), X), \quad (4.7.3)$$

for all vector fields X, Y, Z on M. Combining (4.7.2) and (4.7.3) and by a straightforward combinatorial computation and using the symmetry of $(\mathcal{L}_V \nabla)$ along with (1.1.12), the foregoing equation yields

$$\begin{aligned}(\mathcal{L}_V \nabla)(X,Y) =& \frac{1}{2}(Dr)[g(X,Y) - \eta(X)\eta(Y)] - \frac{1}{2}(Xr)[Y - \eta(Y)\xi] - \frac{1}{2}(Yr)[X - \eta(X)\xi] \\ & + (r - 4(\alpha^2 - \beta^2))[\beta g(\phi X, \phi Y)\xi - \alpha\eta(Y)(\phi X) - \alpha\eta(X)(\phi Y)], \quad (4.7.4)\end{aligned}$$

for arbitrary vector fields X and Y on M. Setting $Y = \xi$ in (4.7.4), we have

$$(\mathcal{L}_V \nabla)(X,\xi) = -\frac{1}{2}(\xi r)[X - \eta(X)\xi] - \alpha(r - 4(\alpha^2 - \beta^2))(\phi X). \quad (4.7.5)$$

Applying covariant derivative w.r.t. arbitrary vector field Y and making use of (1.1.40) and (1.1.41), we obtain

$$
\begin{aligned}
(\nabla_Y \mathcal{L}_V \nabla)(X,\xi) =& \alpha(\mathcal{L}_V \nabla)(X,\phi Y) - \beta(\mathcal{L}_V \nabla)(X,Y) - \frac{1}{2}(Y(\xi r))[X - \eta(X)\xi] + \\
& \frac{1}{2}(\xi r)[\alpha g(\phi X, Y)\xi + \beta g(\phi X, \phi Y)\xi - \alpha\eta(X)(\phi Y) + \beta\eta(X)Y - \\
& \beta\eta(Y)X] - \alpha(Yr)(\phi X) - \alpha(r - 4(\alpha^2 - \beta^2))[\alpha g(X,Y)\xi - \\
& \alpha\eta(X)Y + \beta g(\phi Y, X)\xi - \beta\eta(X)(\phi Y) + \beta\eta(Y)(\phi X)]. \qquad (4.7.6)
\end{aligned}
$$

Using (4.7.6) in (1.1.5), we get

$$
\begin{aligned}
(\mathcal{L}_V R)(X,Y)\xi =& \alpha(\mathcal{L}_V \nabla)(\phi X, Y) - \alpha(\mathcal{L}_V \nabla)(X, \phi Y) - \frac{1}{2}(X(\xi r))[Y - \eta(Y)\xi] + \frac{1}{2}(Y(\xi r)) \\
& [X - \eta(X)\xi] + \frac{1}{2}(\xi r)[2\alpha g(X, \phi Y)\xi - \alpha\eta(Y)(\phi X) + \alpha\eta(X)(\phi Y) + \\
& 2\beta\eta(Y)X - 2\beta\eta(X)Y] - \alpha(Xr)(\phi Y) + \alpha(Yr)(\phi X) - \alpha(r - 4(\alpha^2 - \beta^2)) \\
& [\alpha\eta(X)Y - \alpha\eta(Y)X + 2\beta g(\phi X, Y)\xi + 2\beta\eta(X)(\phi Y) - 2\beta\eta(Y)(\phi X)].
\end{aligned}
$$

Setting $Y = \xi$ in the foregoing equation, we get

$$
\begin{aligned}
(\mathcal{L}_V R)(X,\xi)\xi =& \frac{1}{2}(\xi(\xi r))[X - \eta(X)\xi] + \beta(\xi r)[X - \eta(X)\xi] - \\
& 2\alpha(r - 4(\alpha^2 - \beta^2))[-\alpha X + \alpha\eta(X)\xi - \beta(\phi X). \qquad (4.7.7)
\end{aligned}
$$

Again, Lie differentiation of the equation (1.1.44) along soliton vector field V and use of (1.1.42) and (1.1.44), leads to

$$
(\mathcal{L}_V R)(X,\xi)\xi = (\alpha^2 - \beta^2)[g(X, \mathcal{L}_V \xi)\xi - ((\mathcal{L}_V \eta)X)\xi - 2\eta(\mathcal{L}_V \xi)X] \qquad (4.7.8)
$$

which holds good for arbitrary vector field X on M. Setting $Y = \xi$ in (4.7.1), implies

$$
(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = (p + \frac{2}{3} - 2\lambda)\eta(X). \qquad (4.7.9)
$$

Taking (4.7.9) into account, Lie derivative of $\eta(\xi) = 1$ along the direction of V, leads to

$$
2\eta(\mathcal{L}_V \xi) = -(p + \frac{2}{3} - 2\lambda). \qquad (4.7.10)
$$

After using (4.7.9) and (4.7.10), the equation (4.7.8) reduces to

$$
(\mathcal{L}_V R)(X,\xi)\xi = (\alpha^2 - \beta^2)(p + \frac{2}{3} - 2\lambda)[X - \eta(X)\xi], \qquad (4.7.11)
$$

for all $X \in \chi(M)$. Comparing (4.7.7) with (4.7.11), we achieve

$$(\alpha^2-\beta^2)(p+\frac{2}{3}-2\lambda)[X-\eta(X)\xi]=\frac{1}{2}(\xi(\xi r))[X-\eta(X)\xi]+\beta(\xi r)$$
$$[X-\eta(X)\xi]-2\alpha(r-4(\alpha^2-\beta^2))[-\alpha X+\alpha\eta(X)\xi-\beta(\phi X), \tag{4.7.12}$$

for $X \in \chi(M)$. Inner product with arbitrary vector field Y, gives

$$[\frac{1}{2}(\xi(\xi r))+\beta(\xi r)+2\alpha^2(r-4(\alpha^2-\beta^2))-(\alpha^2-\beta^2)(p+\frac{2}{3}-2\lambda)]$$
$$[g(X,Y)-\eta(X)\eta(Y)]+2\alpha\beta(r-4(\alpha^2-\beta^2))g(\phi X,Y)=0. \tag{4.7.13}$$

Anti-symmetrizing the foregoing equation yields,

$$[\frac{1}{2}(\xi(\xi r))+\beta(\xi r)+2\alpha^2(r-4(\alpha^2-\beta^2))-(\alpha^2-\beta^2)(p+\frac{2}{3}-2\lambda)]g(\phi X,\phi Y)=0. \tag{4.7.14}$$

Since this equation holds for arbitrary vector fields ϕX and ϕY and as we know from (2.3.3) that $\xi r=-2r\beta+12\beta(\alpha^2-\beta^2)$ holds in 3-dimensional trans-Sasakian manifold, we can easily conclude that the scalar curvature of the manifold satisfies $r=(1-\frac{\beta^2}{\alpha^2})(\frac{p}{2}+\frac{1}{3}-\lambda+4\alpha^2)$. □

5

CPE conjecture on some differentiable manifolds

5.1 Introduction

In this chapter we study critical point equation (shortly CPE) and $*$-critical point equation (shortly $*$-CPE) conjectures within the framework of various contact metric manifolds. This chapter is divided into six sections. First two sections being introduction and preliminaries, the rest four sections are arranged as follows,

In the third section, it is proved that if a compact Sasakian manifold admits CPE, then either the manifold is Einstein or the potential function is harmonic in an open subset. Later, It is shown that if the manifold satisfies $*$-CPE then the manifold is η-Einstein and obtain an expression for $*$-Ricci tensor. Finally we have constructed an examples to illustrate the existence of CPE and $*$-CPE on Sasakian manifold.

In later section, we establish that Kenmotsu manifold satisfying the CPE either becomes an Einstein manifold or the derivative of potential function along characteristic vector field satisfy a certain relation on the distribution of η.

In the next section, we study CPE on $(\kappa, \mu)'$-almost Kenmotsu manifold and obtained that the manifold is Einstein. We also construct an example to verify our outcome.

In the final section, in case of 3-dimensional trans-Sasakian manifold satisfying CPE, we get that either the manifold becomes α-Sasakian or it becomes Einstein. Finally we give an example of 3-dimensional trans-Sasakian manifold to support our result.

5.2 Preliminaries

In the previous chapters, we have considered the definitions and properties of Sasakian manifold, Kenmotsu manifold, $(\kappa, \mu)'$-almost Kenmotsu manifold and trans-Sasakian manifold. Here we want to bring some lights on some special relations.

The $*$-Ricci tensor in a $(2n+1)$-dimensional Sasakian manifold is given by [Lemma 3.1 of [40]],

$$S^*(X,Y) = S(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y), \tag{5.2.1}$$

for arbitrary vector fields X and Y. Tracing the foregoing equation we obtain

$$r^* = r - 4n^2. \tag{5.2.2}$$

On a $(2n+1)$-dimensional Kenmotsu manifold, Ricci tensor S satisfies,

$$S(\phi X, \phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{5.2.3}$$

for arbitrary vector fields X and Y on M.

5.3 CPE and $*$-CPE on Sasakian manifold

This section deals with the Sasakian manifold which satisfy the critical point equation (1.3.95) and $*$-critical point equation (1.3.96). First we prove a lemma which is used to deduce the later results,

Lemma 5.3.1. *Let (g, λ) be a non-constant solution of the critical point equation on a $(2n+1)$-dimensional Riemannian manifold M. Then the Riemannian curvature tensor R can be expressed as,*

$$R(X,Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + (\lambda+1)(\nabla_X Q)Y - (\lambda+1)(\nabla_Y Q)X + (Xf)Y - (Yf)X, \tag{5.3.1}$$

for any vector fields $X, Y \in \chi(M)$ and $f = -r(\frac{\lambda}{2n} + \frac{1}{2n+1})$.

Proof. Since (g,λ) is a non-constant solution of the critical point equation, so, $S - \frac{r}{2n+1}g = (\frac{r}{2n}g - S)\lambda - Hess(\lambda)$. Tracing this equation we get, $\Delta_g \lambda = -r_g \frac{\lambda}{2n}$. Thus the above mentioned equation can be exhibited as,

$$\nabla_X D\lambda = (\lambda+1)QX + fX,$$

for an arbitrary vector field X, where $f = -r(\frac{\lambda}{2n} + \frac{1}{2n+1})$. Now taking covariant derivative of the above equation with respect to an arbitrary vector field Y, we obtain

$$\nabla_Y(\nabla_X D\lambda) = (Y\lambda)QX + (\lambda+1)[(\nabla_Y Q)X + Q(\nabla_Y X))] + (Yf)X + f\nabla_Y X.$$

Then we apply the expression in (1.1.4) to get our required result. □

Theorem 5.3.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a compact Sasakian manifold. If (g, λ) is a non-constant solution of the critical point equation then either the manifold is Einstein or λ is harmonic in a scalar flat open subset.*

Proof. First, differentiating (1.1.20) covariantly along arbitrary vector field X and using (1.1.15), we get

$$(\nabla_X Q)\xi = Q\phi X - 2n\phi X. \tag{5.3.2}$$

Since, ξ is Killing in a Sasakian manifold, we have $\mathcal{L}_\xi S = 0$, i.e., $(\mathcal{L}_\xi Q)X = 0$ for arbitrary vector field X on M. Which further follows that,

$$\begin{aligned}0 =&\mathcal{L}_\xi(QX) - Q(\mathcal{L}_\xi X)\\ =&(\nabla_\xi Q)X + \nabla_{QX}\xi + Q(\nabla_X \xi).\end{aligned}$$

Plugging (1.1.15), (1.1.19) and (5.3.2) in the above relation, we acquire

$$\nabla_\xi Q = 0. \tag{5.3.3}$$

Keeping (1.1.17), (1.1.19), (1.1.20) and (5.3.2) into account, scalar product of the result of lemma-5.3.1 with ξ, yields

$$\begin{aligned}&(2n+1)[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] + (Xf)\eta(Y) - (Yf)\eta(X)\\ &+ 2(\lambda+1)[S(\phi X, Y) - 2ng(\phi X, Y)] = 0,\end{aligned} \tag{5.3.4}$$

$\forall X, Y \in \chi(M)$. Substituting ϕX and ξ in place of X and Y respectively in the foregoing equation and using (1.1.9), (1.1.10), (1.1.21), we obtain

$$(2n+1)\phi(D\lambda) + \phi(Df) = 0. \tag{5.3.5}$$

Since the scalar curvature r is constant, as (g, λ) is a non-constant solution of CPE, the above equation reduces to

$$[r - 2n(2n+1)]\phi(D\lambda) = 0, \tag{5.3.6}$$

where $n > 1$. So, from here two cases arise, either $r = 2n(2n+1)$ or $\phi(D\lambda) = 0$.

Considering $r = 2n(2n+1)$, we can rewrite $f = -r(\frac{\lambda}{2n} + \frac{1}{2n+1})$ as $f = -(2n+1)\lambda - 2n$. It easily follows that

$$Df = -(2n+1)D\lambda. \tag{5.3.7}$$

Keeping (1.1.18), (1.1.20), (5.3.2), (5.3.3) and (5.3.7) into account, plugging $Y = \xi$, we can manipulate (5.3.1) as

$$(\lambda+1)[Q(\phi X) - 2n(\phi X)] - (\xi\lambda)[QX - 2nX] = 0,$$

$\forall X \in \chi(M)$. Further, considering scalar product with an arbitrary vector field Y, we acquire

$$(\lambda+1)[S(\phi X, Y) - 2ng(\phi X, Y)] - (\xi\lambda)[S(X,Y) - 2ng(X,Y)] = 0. \tag{5.3.8}$$

Interchanging X and Y in the foregoing equation and then adding with that equation, we get

$$(\xi\lambda)[S(X,Y) - 2ng(X,Y)] = 0, \tag{5.3.9}$$

for any vector fields X and Y on M, where we have used (1.1.19). If we consider $\xi\lambda = 0$ in (5.3.8), as λ is a non-constant solution, we obtain $S(\phi X, Y) = 2ng(\phi X, Y)$ for any vector fields X and Y of $\chi(M)$. Replacing X by ϕX, the last relation yields $S(X,Y) = 2ng(X,Y)$, where we take (1.1.6) and (1.1.20) into consideration. So, the manifold is Einstein and this proves our first assertion.

Now, if possible suppose $r \neq 2n(2n+1)$ on some open subset O of M. Then (5.3.6) implies that $\phi(D\lambda) = 0$ in that open subset O. Using (1.1.6), we can easily achieve $D\lambda = (\xi\lambda)\xi$. Differentiation of this relation along an arbitrary vector field X, yields

$$\nabla_X D\lambda = (X(\xi\lambda))\xi - (\xi\lambda)(\phi X). \tag{5.3.10}$$

Plugging (5.3.10) in (1.3.94), we have

$$(\lambda+1)QX + [\Delta\lambda - \frac{r}{2n+1}]X = (X(\xi\lambda))\xi - (\xi\lambda)(\phi X),$$

for an arbitrary vector field X of $\chi(M)$. Setting $X = \xi$ in the above equation, we get the following relation,

$$\xi(\xi\lambda) = 2n(\lambda+1) + \Delta\lambda - \frac{r}{2n+1}. \tag{5.3.11}$$

Contracting X in (5.3.10), we achieve

$$\Delta\lambda = \xi(\xi\lambda). \tag{5.3.12}$$

Comparing the last two relations, we can conclude $r = 2n(2n+1)(\lambda+1)$. Differentiating this along the Reeb vector field ξ we get $\xi\lambda = 0$, as the scalar curvature r is constant and $n > 1$. Putting this in (5.3.12), we have $\Delta\lambda = 0$. Again tracing (1.3.94) and using $\Delta\lambda = 0$, we acquire $r = 0$ as λ is a non-constant function. So, we can conclude that λ is harmonic in a scalar flat open subset. □

Theorem 5.3.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a compact Sasakian manifold. If (g, λ) is a non-constant solution of the $*$-critical point equation then,*

(i) the manifold is η-Einstein,

(ii) the $*$-Ricci tensor is given by $S^* = (\frac{r}{2n} - 2n - 1)[g - \eta \otimes \eta]$.

Proof. Using (5.2.1) and (5.2.2) in the $*$-critical point equation (1.3.96), we obtain

$$Hess_g\lambda(X,Y) = (1+\lambda)[S(X,Y) - \eta(X)\eta(Y)] - (\frac{r-1}{2n+1} + \frac{\lambda r}{2n} - \lambda)g(X,Y),$$

for any vector fields X and Y of $\chi(M)$, which further yields

$$\nabla_X D\lambda = (1+\lambda)[QX - \eta(X)\xi] - (\frac{r-1}{2n+1} + \frac{\lambda r}{2n} - \lambda)X, \tag{5.3.13}$$

for an arbitrary vector field X on M. Differentiating along an arbitrary vector field Y, keeping (1.1.15) and (1.1.16) into account, we achieve

$$\begin{aligned}\nabla_Y\nabla_X D\lambda =& (Y\lambda)[QX - (\frac{r}{2n} - 1)X - \eta(X)\xi] + (1+\lambda)\nabla_Y(QX) - (1+\lambda)\\ & [g(\phi X, Y)\xi + \eta(\nabla_Y X)\xi - \eta(X)\phi Y] - (\frac{r-1}{2n+1} + \frac{\lambda r}{2n} - \lambda)(\nabla_Y X).\end{aligned} \tag{5.3.14}$$

Using (5.3.13) and (5.3.14) in (1.1.4), we have

$$\begin{aligned}R(X,Y)D\lambda =& (X\lambda)[QY - \eta(Y)\xi] - (Y\lambda)[QX - \eta(X)\xi] + (1+\lambda)[(\nabla_X Q)Y - (\nabla_Y Q)X]\\ & - (1+\lambda)[2g(\phi Y, X)\xi - \eta(Y)\phi X + \eta(X)\phi Y] - (\frac{r}{2n} - 1)[(X\lambda)Y - (Y\lambda)X],\end{aligned} \tag{5.3.15}$$

for arbitrary vector fields X and Y on M. Plugging $Y=\xi$ in the foregoing equation and making use of (1.1.7), (1.1.9), (1.1.20), (5.3.2) and (5.3.3), we obtain

$$\begin{aligned} R(X,\xi)D\lambda =&(2n-1)(X\lambda)\xi-(\xi\lambda)[QX-\eta(X)\xi]+(1+\lambda)[Q(\phi X)-(2n-1)\phi X] \\ &-(\frac{r}{2n}-1)[(X\lambda)\xi-(\xi\lambda)X]. \end{aligned} \tag{5.3.16}$$

Use of the above equation in (1.1.18), yields

$$(2n+1-\frac{r}{2n})(X\lambda)\xi-(\xi\lambda)[QX-\eta(X)\xi]+(\frac{r}{2n}-2)(\xi\lambda)X+(1+\lambda)[Q(\phi X)-(2n-1)\phi X]=0, \tag{5.3.17}$$

for any vector field X of $\chi(M)$. Taking scalar product with an arbitrary vector field Y, we get

$$\begin{aligned} &(2n+1-\frac{r}{2n})(X\lambda)\eta(Y)-(\xi\lambda)[S(X,Y)-\eta(X)\eta(Y)]+(1+\lambda) \\ &[S(\phi X,Y)-(2n-1)g(\phi X,Y)]+(\frac{r}{2n}-2)(\xi\lambda)g(X,Y)=0. \end{aligned} \tag{5.3.18}$$

Anti-symmetrizing this equation, we achieve

$$(2n+1-\frac{r}{2n})[(X\lambda)\eta(Y)-(Y\lambda)\eta(X)]=2(1+\lambda)[(2n-1)g(\phi X,Y)-S(\phi X,Y)]. \tag{5.3.19}$$

As $n\neq 0$, keeping (1.1.7) and (1.1.9) into account, setting $X=\xi$ in the foregoing equation, we have

$$(r-2n(2n+1))[(\xi\lambda)\eta(Y)-(Y\lambda)]=0, \tag{5.3.20}$$

for an arbitrary vector field Y on M.

If possible suppose $r=2n(2n+1)$ on an open set O of M. Then (5.3.19) reduces to $S(\phi X,Y)=(2n-1)g(\phi X,Y)$, $\forall X,Y\in\chi(M)$, as λ is a non-constant function. Replacing X by ϕX, in the above equation and using (1.1.6), (1.1.11) and (1.1.20), we get $S(X,Y)=(2n-1)g(X,Y)+\eta(X)\eta(Y)$. Tracing the foregoing equation we obtain $r=4n^2$, which is a contradiction to our assumption.

So, $r\neq 2n(2n+1)$ in the entire manifold. Therefore, (5.3.20) equation reduces to

$$D\lambda=(\xi\lambda)\xi. \tag{5.3.21}$$

Using (5.3.21), we can rewrite (5.3.18) as

$$\begin{aligned} &(2(n+1)-\frac{r}{2n})(\xi\lambda)\eta(X)\eta(Y)-(\xi\lambda)S(X,Y)+(\frac{r}{2n}-2)(\xi\lambda)g(X,Y) \\ &+(1+\lambda)[S(\phi X,Y)-(2n-1)g(\phi X,Y)]=0, \end{aligned} \tag{5.3.22}$$

for any vector fields X and Y on M. Interchanging X and Y in the previous equation and taking sum with (5.3.22) and keeping (1.1.12), (1.1.19) into account, we achieve

$$(\xi\lambda)[S(X,Y) - (\frac{r}{2n} - 2)g(X,Y) - (2(n+1) - \frac{r}{2n})\eta(X)\eta(Y)] = 0. \qquad (5.3.23)$$

If $\xi\lambda = 0$, from (5.3.21) we get, $D\lambda = 0$. This is a contradiction as λ is not a constant function. So, from (5.3.23), we acquire

$$S(X,Y) = (\frac{r}{2n} - 2)g(X,Y) + (2(n+1) - \frac{r}{2n})\eta(X)\eta(Y),$$

for any vector fields X and Y of $\chi(M)$. So, the manifold is η-Einstein, which proves the first assertion of our theorem. Using the above relation in (5.2.1), we obtain

$$S^*(X,Y) = (\frac{r}{2n} - 2n - 1)[g(X,Y) - \eta(X)\eta(Y)],$$

which completes this theorem. □

Example 5.3.1. *Let us consider the set $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$ as our manifold where (a,b,c,d,e) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below,*

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = 2(x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}), \qquad e_3 = \frac{\partial}{\partial z}, \qquad e_4 = \frac{\partial}{\partial u}, \qquad e_5 = 2(u\frac{\partial}{\partial z} - \frac{\partial}{\partial v}),$$

are linearly independent at each point of M. We define the metric g as

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

Let η be a 1-form defined by $\eta(X) = g(X, e_3)$, for arbitrary $X \in \chi(M)$. Let us define (1,1)-tensor field ϕ as,

$$\phi(e_1) = e_2, \qquad \phi(e_2) = -e_1, \qquad \phi(e_3) = 0, \qquad \phi(e_4) = e_5, \qquad \phi(e_5) = -e_4.$$

Then it is easy to verify that the relations (1.1.6)-(1.1.8) *are satisfied. So, for $\xi = e_3$, (M, ϕ, ξ, η, g) defines an almost contact structure on M.*

We can now deduce that,

$$\begin{array}{lllll}
[e_1,e_1] = 0, & [e_1,e_2] = 2e_3, & [e_1,e_3] = 0, & [e_1,e_4] = 0, & [e_1,e_5] = 0, \\
[e_2,e_1] = -2e_3, & [e_2,e_2] = 0, & [e_2,e_3] = 0, & [e_2,e_4] = 0, & [e_2,e_5] = 0, \\
[e_3,e_1] = 0, & [e_3,e_2] = 0, & [e_3,e_3] = 0, & [e_3,e_4] = 0, & [e_3,e_5] = 0, \\
[e_4,e_1] = 0, & [e_4,e_2] = 0, & [e_4,e_3] = 0, & [e_4,e_4] = 0, & [e_4,e_5] = 2e_3, \\
[e_5,e_1] = 0, & [e_5,e_2] = 0, & [e_5,e_3] = 0, & [e_5,e_4] = -2e_3, & [e_5,e_5] = 0.
\end{array}$$

Let ∇ *be the Levi-Civita connection of* M*. Then from* (1.1.1)*, we can compute the following relations,*

$$
\begin{array}{lllll}
\nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = e_3, & \nabla_{e_1}e_3 = -e_2 & \nabla_{e_1}e_4 = 0, & \nabla_{e_1}e_5 = 0, \\
\nabla_{e_2}e_1 = -e_3, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = e_1, & \nabla_{e_2}e_4 = 0, & \nabla_{e_2}e_5 = 0, \\
\nabla_{e_3}e_1 = -e_2, & \nabla_{e_3}e_2 = e_1, & \nabla_{e_3}e_3 = 0, & \nabla_{e_3}e_4 = -e_5, & \nabla_{e_3}e_5 = e_4, \\
\nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = -e_5, & \nabla_{e_4}e_4 = 0, & \nabla_{e_4}e_5 = e_3, \\
\nabla_{e_5}e_1 = 0, & \nabla_{e_5}e_2 = 0, & \nabla_{e_5}e_3 = e_4, & \nabla_{e_5}e_4 = -e_3, & \nabla_{e_5}e_5 = 0.
\end{array}
$$

Therefore (1.1.14) *is satisfied and* (M, ϕ, ξ, η, g) *becomes a Sasakian manifold.*

Using (1.1.4)*, we can compute the non-vanishing components of curvature tensor as*

$$
\begin{array}{lll}
R(e_1, e_2)e_1 = 3e_2, & R(e_1, e_2)e_2 = -3e_1, & R(e_1, e_2)e_4 = 2e_5, \\
R(e_1, e_2)e_5 = -2e_4, & R(e_1, e_3)e_1 = -e_3, & R(e_1, e_3)e_3 = e_1, \\
R(e_1, e_4)e_2 = e_5, & R(e_1, e_4)e_5 = -e_2, & R(e_1, e_5)e_2 = -e_4, \\
R(e_1, e_5)e_4 = e_2, & R(e_2, e_3)e_2 = -e_3, & R(e_2, e_3)e_3 = -e_2, \\
R(e_2, e_4)e_1 = -e_5, & R(e_2, e_4)e_5 = e_1, & R(e_2, e_5)e_1 = e_4, \\
R(e_2, e_5)e_4 = -e_1, & R(e_3, e_4)e_3 = -e_4, & R(e_3, e_4)e_4 = e_3, \\
R(e_3, e_5)e_3 = -e_5, & R(e_3, e_5)e_5 = e_3, & R(e_4, e_5)e_1 = 2e_2, \\
R(e_4, e_5)e_2 = -2e_1, & R(e_4, e_5)e_4 = 3e_5, & R(e_4, e_5)e_5 = -3e_4.
\end{array}
$$

Now from the above results we have,

$$
S(e_i, e_j) = \begin{cases} -2, & \text{if } i = j, \text{ where } i, j \in \{1, 2, 4, 5\} \\ 2, & \text{if } i = j, \text{ where } i, j \in \{3\} \\ 0, & \text{otherwise.} \end{cases}
$$

Contracting this we have $r = \sum_{i=1}^{5} S(e_i, e_i) = -6$*. Also, we have,*

$$
S^*(e_i, e_j) = \begin{cases} -5, & \text{if } i = j, \text{ where } i, j \in \{1, 2, 4, 5\} \\ -2, & \text{if } i = j, \text{ where } i, j \in \{3\} \\ 0, & \text{otherwise.} \end{cases}
$$

and, $r^* = \sum_{i=1}^{5} S^*(e_i, e_i) = -22 = r - 4n^2 = -6 - 16$*, where* $n = 2$*. So,* (5.2.1) *and* (5.2.2) *are satisfied.*

Now we consider a non-constant smooth function λ as, $\lambda = e^u$. Then we get $D\lambda = \lambda e_4$ and $\Delta\lambda = \lambda$. From here we can easily verify that, the critical point equation (1.3.95) *and $*$-critical point equation* (1.3.96) *are satisfied. This example verifies our results.*

5.4 CPE on Kenmotsu manifold

Theorem 5.4.1. *Let $M(\phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Kenmotsu manifold. If (g, λ) is a non-constant solution of the CPE, then*

(i) the scalar curvature of the manifold is $-2n(2n+1)$,

(ii) either the manifold is Einstein or $\xi\lambda = 1 + \lambda$ in kernel of η.

Proof. Plugging $Y = \xi$ in equation (5.3.1) and using (1.1.26), we acquire

$$R(X,\xi)D\lambda = [(Xf)-2n(X\lambda)]\xi+(\lambda+1)[(\nabla_X Q)\xi-(\nabla_\xi Q)X]-(\xi\lambda)QX-(\xi f)X. \quad (5.4.1)$$

for an arbitrary vector field X on M. Using (1.1.25), (2.2.2) and (2.2.3) in (5.4.1), we achieve

$$[(Xf) - (2n+1)(X\lambda)]\xi + (\xi\lambda)[X - QX] - (\xi f)X + (\lambda+1)[QX + 2nX] = 0, \quad (5.4.2)$$

for any vector field X in $\chi(M)$. Considering inner product of (5.4.2) along ξ and taking (1.1.26) into account, we get

$$Xf - (2n+1)(X\lambda) + (2n+1)(\xi\lambda)\eta(X) - (\xi f)\eta(X) = 0, \quad (5.4.3)$$

$\forall X \in \chi(M)$. Since (g, λ) is a non-constant solution of CPE, the scalar curvature r is constant and therefore $df = -\frac{r}{2n}d\lambda$. As $n \neq 0$, using this in (5.4.3), we obtain

$$[r + 2n(2n+1)](X\lambda - (\xi\lambda)\eta(X)) = 0. \quad (5.4.4)$$

As, X is an arbitrary vector field in the above relation, from here two cases arise, either $r = -2n(2n+1)$ or $D\lambda = (\xi\lambda)\xi$.

Let us consider the case $D\lambda = (\xi\lambda)\xi$. Covariant derivative along an arbitrary vector field X, yields

$$\nabla_X D\lambda = (X(\xi\lambda))\xi + (\xi\lambda)X - (\xi\lambda)\eta(X)\xi, \quad (5.4.5)$$

where we have used (1.1.23). From critical point equation (1.3.95), using (5.4.5), we acquire

$$(1+\lambda)QX = \frac{\lambda r}{2n}X + \frac{r}{2n+1}X + (X(\xi\lambda))\xi + (\xi\lambda)X - (\xi\lambda)\eta(X)\xi.$$

Considering $X = \xi$ in the last equation and using (1.1.7) and (1.1.26), we have

$$\xi(\xi\lambda) + \frac{r}{2n+1} + \frac{\lambda r}{2n} + 2n(1+\lambda) = 0. \tag{5.4.6}$$

Contracting X in (5.4.5), we get $\Delta\lambda = \xi(\xi\lambda) + 2n(\xi\lambda)$. Again, in the proof of lemma 5.3.1, we get $\Delta\lambda = -r\frac{\lambda}{2n}$. Combining these last two relations we get

$$\xi(\xi\lambda) + \frac{r\lambda}{2n} + 2n(\xi\lambda) = 0. \tag{5.4.7}$$

Since $n \neq 0$, manipulating (5.4.6) using (5.4.7), yields

$$\xi\lambda = \frac{r}{2n(2n+1)} + 1 + \lambda. \tag{5.4.8}$$

Differentiating (5.4.8) along ξ, as r is constant, we have $\xi(\xi\lambda) = \xi\lambda$. Using this relation in (5.4.7), we obtain $(1+\lambda)[r + 2n(2n+1)] = 0$. Since λ is a non-constant function, we can conclude that $r = -2n(2n+1)$.

Considering (5.4.2) along an arbitrary vector field Y and the replacing X and Y by ϕX and ϕY, respectively, we have

$$(1+\lambda-\xi\lambda)[S(X,Y) + 2ng(X,Y)] = 0.$$

$\forall X, Y \in \chi(M)$, where we have used (1.1.8), (1.1.10), (5.2.3) and $r = -2n(2n+1)$. So, either the manifold is Einstein with Einstein constant $-2n$, or $\xi\lambda = 1+\lambda$.

Let us consider the case where $\xi\lambda = 1+\lambda$. Differentiating this relation along an arbitrary vector field X and using (1.1.23), we get $Hess_\lambda(X,\xi) = (\xi\lambda)\eta(X)$. Replacing Y by ξ in (1.3.95) and using (1.1.23) and the expression of Hessian we get $\eta(X) = 0$. From here we can easily obtain our desired result. □

Note: Now if we consider a ϕ-basis $\{e_i, \phi e_i, \xi\}, i = 1, 2, 3, ..., n$ of M such that $Qe_i = \rho_i e_i$. Then we have $\phi Qe_i = \rho_i \phi e_i$. Using the ϕ-basis and (1.1.26) we can conclude that the scalar curvature r of the manifold is

$$r = g(Q\xi,\xi) + \sum_{i=1}^{n}[g(Qe_i, e_i) + g(Q\phi e_i, \phi e_i)] = -2n + 2\sum_{i=1}^{n}\rho_i.$$

Let us suppose $\sum_{i=1}^{n} \rho_i = \kappa$. Then from the critical point equation, we have

$$\begin{aligned} f =& -r(\frac{\lambda}{2n} + \frac{1}{2n+1}) \\ =& \frac{n-\kappa}{n}\lambda + \frac{2n-2\kappa}{2n+1}. \end{aligned}$$

5.5 CPE on $(\kappa, -2)'$-Kenmotsu manifold

Theorem 5.5.1. *Let $M(\phi, \xi, \eta, g)$ be an $(2n+1)$-dimensional $(\kappa, -2)'$-almost Kenmotsu manifold with $\kappa \leq -1$. If (g, λ) is a non-constant solution of the critical point equation, then M is an Einstein manifold.*

Proof. From (1.1.32), we have

$$R(\xi, Y)D\lambda = \kappa[(Y\lambda)\xi - (\xi\lambda)Y] - 2[((h'Y)\lambda)\xi - (\xi\lambda)h'Y], \tag{5.5.1}$$

for an arbitrary vector field Y of $\chi(M)$. Differentiating (1.1.36) and using (1.1.29), (1.1.28), (1.1.33) and (1.1.39), we obtain the following two relations

$$(\nabla_X Q)\xi = 2n(\kappa+2)h'X, \tag{5.5.2}$$

$$(\nabla_\xi Q)X = 0, \tag{5.5.3}$$

for any vector field X on M. Using these above two relations along with (1.1.36) in (5.3.1), we acquire

$$R(\xi, Y)D\lambda = (\xi\lambda)QY - 2n\kappa(Y\lambda)\xi - 2n(\lambda+1)(\kappa+2)h'Y + (\xi f)Y - (Yf)\xi, \tag{5.5.4}$$

$\forall Y \in \chi(M)$. As the scalar curvature r is constant in critical point equation, comparing (5.5.1) with (5.5.4) and using $f = -r(\frac{\lambda}{2n} + \frac{1}{2n+1})$, (1.1.36) and (1.1.37), we achieve

$$n(\kappa+1)[(\xi\lambda)\eta(Y)\xi - (Y\lambda)\xi] + ((h'Y)\lambda)\xi = [(n+1)(\xi\lambda) + n(\lambda+1)(\kappa+2)](h'Y), \tag{5.5.5}$$

for any $Y \in \chi(M)$. Considering scalar product with the Reeb vector field ξ, the foregoing equation reduces to

$$(h'Y)\lambda = n(\kappa+1)[(Y\lambda) - (\xi\lambda)\eta(Y)], \tag{5.5.6}$$

where we have used (1.1.29). Taking (5.5.6) into account, from (5.5.5), we have

$$\xi\lambda = -\frac{n}{(n+1)}(\lambda+1)(\kappa+2). \tag{5.5.7}$$

Differentiating the above equation along the direction of the characteristic vector field ξ, we get

$$Hess_{\lambda}(\xi,\xi) = -\frac{n}{(n+1)}(\kappa+2)(\xi\lambda).$$

Using this relation in the critical point equation (1.3.95) and considering (1.1.36), (1.1.37) and (5.5.7), we obtain

$$4n^2(n+1)(\kappa+1) + (2n+1)(2n^2\kappa - \kappa + 2n^2)\lambda = 0.$$

Since κ is a constant, we get λ is a constant. So, we can conclude that the manifold is Einstein. $\square$

Example 5.5.1. *We consider the manifold as* $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$, *where* (x,y,z,u,v) *are the standard coordinates in* $\mathbb{R}^5$. *Let* e_1, e_2, e_3, e_4, e_5 *are 5 vector fields which satisfy,*

$$[e_1,e_2]=0, \qquad [e_1,e_3]=-2e_3, \qquad [e_1,e_4]=-2e_4, \qquad [e_1,e_5]=0,$$

$$[e_i,e_j]=0 \quad \textit{where} \quad i,j=2,3,4,5.$$

Now we define a metric g *on* M *as,*

$$g(e_i,e_j) = \begin{cases} 1, & \textit{if } i=j \\ 0, & \textit{otherwise.} \end{cases}$$

Let η *be an 1-form defined by* $\eta(X) = g(X,e_1)$ *for arbitrary* $X \in \chi(M)$, *then we have the following relations,*

$$\eta(e_1)=1, \qquad \eta(e_i)=0; \textit{where} \quad i=2,3,4,5.$$

Let us define a $(1,1)$*-tensor field* ϕ *as,*

$$\phi(e_1)=0, \qquad \phi(e_2)=e_3, \qquad \phi(e_3)=-e_2, \qquad \phi(e_4)=e_5, \qquad \phi(e_5)=-e_4.$$

Then the relations (1.1.6), (1.1.7) *and* (1.1.8) *are satisfied. Thus for* $e_1=\xi$, (ϕ,ξ,η,g) *defines an almost contact metric structure on* M. *Moreover,*

$$h'\xi=0, \qquad h'e_2=e_2, \qquad h'e_3=e_3, \qquad h'e_4=0, \qquad h'e_5=0.$$

Let ∇ be the Levi-Civita connection of g. Then from (1.1.1) we can have,

$$
\begin{array}{lllll}
\nabla_\xi \xi = 0, & \nabla_\xi e_2 = 0, & \nabla_\xi e_3 = 0, & \nabla_\xi e_4 = 0, & \nabla_\xi e_5 = 0, \\
\nabla_{e_2} \xi = 0, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = 0, \\
\nabla_{e_3} \xi = 2e_3, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -2\xi, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = 0, \\
\nabla_{e_4} \xi = 2e_4, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -2\xi, & \nabla_{e_4} e_5 = 0, \\
\nabla_{e_5} \xi = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0.
\end{array}
$$

From the above relations we can easily conclude that the relation $\nabla_X \xi = -\phi^2 X + h'X$ holds for arbitrary $X \in \chi(M)$. So M is an almost Kenmotsu manifold.

By the above results, we can easily calculate the components of the curvature tensor R as follows

$$
\begin{array}{lll}
R(\xi, e_3)\xi = 4e_3, & R(\xi, e_4)\xi = 4e_4, & R(\xi, e_3)e_3 = -4\xi, \\
R(\xi, e_4)e_4 = -4\xi, & R(e_3, e_4)e_3 = 4e_4, & R(e_3, e_4)e_4 = -4e_3.
\end{array}
$$

From here we can conclude that the characteristic vector field ξ belongs to the $(\kappa, -2)'$-nullity distribution with $\kappa = -2$. So the manifold is a $(\kappa, -2)'$-almost Kenmotsu manifold.

5.6 CPE on trans-Sasakian manifold

Theorem 5.6.1. *Let $M(\phi, \xi, \eta, g)$ be a 3-dimensional trans-Sasakian manifold where the structure functions α, β are constant. If (g, λ) is a non-constant solution of the critical point equation then either the manifold becomes α-Sasakian manifold or the manifold becomes Einstein.*

Proof. From (1.1.44), we deduce

$$R(\xi, Y)D\lambda = (\alpha^2 - \beta^2)[(Y\lambda)\xi - (\xi\lambda)Y], \tag{5.6.1}$$

for an arbitrary vector field Y on M. Again, setting $X = \xi$ in (5.3.1), we obtain

$$R(\xi, Y)D\lambda = (\xi\lambda)QY - (Y\lambda)Q\xi + (\lambda + 1)[(\nabla_\xi Q)Y - (\nabla_Y Q)\xi] + (\xi f)Y - (Yf)\xi, \tag{5.6.2}$$

From (1.1.45) we have these following two relations,

$$QY = (\frac{r}{2} - (\alpha^2 - \beta^2))Y - (\frac{r}{2} - 3(\alpha^2 - \beta^2))\eta(Y)\xi, \tag{5.6.3}$$

$$Q\xi = 2(\alpha^2 - \beta^2)\xi, \tag{5.6.4}$$

for any vector field Y of $\chi(M)$. As the scalar curvature r is constant in a critical point equation, taking covariant derivative of (1.1.45) along an arbitrary vector field X, we get

$$(\nabla_X Q)Y = -(\frac{r}{2} - 3(\alpha^2 - \beta^2))[((\nabla_X \eta)Y)\xi + \eta(Y)(\nabla_X \xi)].$$

Making use of (1.1.41) in the above equation, yields

$$(\nabla_\xi Q)Y = 0, \tag{5.6.5}$$

$$(\nabla_Y Q)\xi = (\frac{r}{2} - 3(\alpha^2 - \beta^2))[\alpha\phi Y - \beta Y + \beta\eta(Y)\xi], \tag{5.6.6}$$

$\forall Y \in \chi(M)$. Using (5.6.3), (5.6.4), (5.6.5) and (5.6.6) in (5.6.2), we acquire

$$\begin{aligned} R(\xi, Y)D\lambda =& [(\xi\lambda)(\frac{r}{2} - (\alpha^2 - \beta^2)) + \beta(\lambda + 1)(\frac{r}{2} - 3(\alpha^2 - \beta^2)) + (\xi f)]Y - \\ & [(\frac{r}{2} - 3(\alpha^2 - \beta^2))((\xi\lambda) + \beta(\lambda + 1))\eta(Y) + 2(\alpha^2 - \beta^2)(Y\lambda) - \\ & (Yf)]\xi - \alpha(\lambda + 1)(\frac{r}{2} - 3(\alpha^2 - \beta^2))\phi(Y), \end{aligned} \tag{5.6.7}$$

for any vector field Y on M. As we know $f = -r(\frac{\lambda}{2} + \frac{1}{3})$ and the scalar curvature r is constant in a critical point equation, comparing (5.6.1) with (5.6.7), we get

$$(\frac{r}{2} - 3(\alpha^2 - \beta^2))\big[\beta(\lambda + 1)(Y - \eta(Y)\xi) - (\xi\lambda)\eta(Y)\xi + (Y\lambda)\xi - \alpha(\lambda + 1)\phi Y\big] = 0.$$

Taking scalar product of the aforementioned equation with arbitraty vector field X, we get

$$\begin{aligned} &(\frac{r}{2} - 3(\alpha^2 - \beta^2))\big[\beta(\lambda + 1)\{g(X, Y) - \eta(X)\eta(Y)\} - (\xi\lambda)\eta(X)\eta(Y) \\ &+ (Y\lambda)\eta(X) - \alpha(\lambda + 1)g(X, \phi Y)\big] = 0. \end{aligned}$$

Interchange of X and Y in the last equation, yields

$$\begin{aligned} &(\frac{r}{2} - 3(\alpha^2 - \beta^2))\big[\beta(\lambda + 1)\{g(X, Y) - \eta(X)\eta(Y)\} - (\xi\lambda)\eta(X)\eta(Y) \\ &+ (X\lambda)\eta(Y) - \alpha(\lambda + 1)g(\phi X, Y)\big] = 0. \end{aligned}$$

Adding the above two equations and using (1.1.12), we obtain

$$\begin{aligned} &(\frac{r}{2} - 3(\alpha^2 - \beta^2))\big[2\beta(\lambda + 1)\{g(X, Y) - \eta(X)\eta(Y)\} - 2(\xi\lambda)\eta(X)\eta(Y) \\ &+ (Y\lambda)\eta(X) + (X\lambda)\eta(Y)\big] = 0. \end{aligned} \tag{5.6.8}$$

Tracing (5.6.8), we have

$$\beta(\lambda+1)[r-6(\alpha^2-\beta^2)]=0.$$

Since λ is a non-constant function, from here two cases arise. If $\beta=0$, the manifold becomes α-Sasakian. If $r=6(\alpha^2-\beta^2)$, from (1.1.45) we can conclude that the manifold is Einstein with Einstein constant $2(\alpha^2-\beta^2)$. □

Note: If we further assume $\alpha=\beta$ where α,β are constant functions. If (g,λ) is a non-constant solution of the critical point equation then either the manifold becomes α-Sasakian manifold or the manifold is scalar flat.

Example 5.6.1. *Let $M=\mathbb{R}^3$. The vector fields defined as*

$$e_1=2\frac{\partial}{\partial z}, \qquad e_2=2\frac{\partial}{\partial y}, \qquad e_3=2\frac{\partial}{\partial x}+2y\frac{\partial}{\partial z},$$

are linearly independent at each point on M. Now we define a metric g on M as,

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1,$$
$$g(e_1,e_2)=g(e_2,e_3)=g(e_3,e_1)=0.$$

Let η be an 1-form defined by $\eta(X)=g(X,e_1)$, for arbitrary $X\in\chi(M)$, then we have $\eta=\frac{1}{2}(dz-ydx)$ which satisfies the following relations,

$$\eta(e_1)=1, \qquad \eta(e_2)=0, \qquad \eta(e_3)=0.$$

Let us define a (1,1)-tensor field ϕ as

$$\phi(e_1)=0, \qquad \phi(e_2)=e_3, \qquad \phi(e_3)=-e_2.$$

Then the relations (1.1.6)*,* (1.1.7) *and* (1.1.8) *are satisfied. Thus for $\xi=e_1$, (ϕ,ξ,η,g) defines an almost contact metric structure on M. We can now easily conclude*

$$[e_1,e_2]=0, \qquad [e_2,e_3]=2e_1, \qquad [e_1,e_3]=0.$$

Let ∇ be the Levi-Civita connection of M. Then from Koszul's formula (1.1.1)*, we can have,*

$$\nabla_{e_1}e_1=0, \qquad \nabla_{e_1}e_2=-e_3, \qquad \nabla_{e_1}e_3=e_2,$$
$$\nabla_{e_2}e_1=-e_3, \qquad \nabla_{e_2}e_2=0, \qquad \nabla_{e_2}e_3=e_1,$$
$$\nabla_{e_3}e_1=e_2, \qquad \nabla_{e_3}e_2=-e_1, \qquad \nabla_{e_3}e_3=0.$$

Finally from (1.1.40) *we can conclude that* $\alpha = 1$ *and* $\beta = 0$ *and* M *is a 3-dimensional trans-Sasakian Manifold.*

Now, from (1.1.4) *we obtain the following components of Riemannian curvature tensor,*

$$R(e_1, e_2)e_1 = -e_2, \qquad R(e_1, e_2)e_2 = e_1, \qquad R(e_1, e_2)e_3 = 0,$$
$$R(e_1, e_3)e_1 = -e_3, \qquad R(e_1, e_3)e_2 = 0, \qquad R(e_1, e_3)e_3 = e_1,$$
$$R(e_2, e_3)e_1 = 0, \qquad R(e_2, e_3)e_2 = 3e_3, \qquad R(e_2, e_3)e_3 = -3e_2.$$

The non-zero components of Ricci tensor are given by,

$$S(e_1, e_1) = 2, \qquad S(e_2, e_2) = -2, \qquad S(e_3, e_3) = -2.$$

So, the scalar curvature is obtained as $r = -2$.*If we consider* $\lambda = e^y$, *then* $D\lambda = \frac{1}{2}\lambda e_2$ *and we can easily verify that it satisfies the trace of* (1.3.95) *and the manifold is* α*-Sasakian manifold where* $\alpha = 1$. *It verifies the last theorem.*

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$*$-η-Ricci soliton and contact geometry

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Abstract
In the present paper, we initiate the study of $*$-η-Ricci soliton within the framework of Kenmotsu manifolds as a characterization of Einstein metrics. Here we display that a Kenmotsu metric as a $*$-η-Ricci soliton is Einstein metric if the soliton vector field is contact. Further, we have developed the characterization of the Kenmotsu manifold or the nature of the potential vector field when the manifold satisfies gradient almost $*$-η-Ricci soliton. Next, we deliberate $*$-η-Ricci soliton admitting $(\kappa, \mu)'$-almost Kenmotsu manifold and proved that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Finally we present some examples to decorate the existence of $*$-η-Ricci soliton, gradient almost $*$-η-Ricci soliton on Kenmotsu manifold.

Keywords Ricci flow · η-Ricci soliton · $*$-η-Ricci soliton · Gradient almost $*$-η-Ricci soliton · Kenmotsu manifold · $(\kappa, \mu)'$-almost Kenmotsu manifold

Mathematics Subject Classification 53C15 · 53C21 · 53C25 · 53C44

1 Motivations and background

In recent years, geometric flows, in particular, the Ricci flow have been an interesting research topic in differential geometry and geometric analysis as it naturally extends Einstein metric. Many mathematicians and physicists are very interested in the Ricci

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soliton as it has significant applications in mathematical fluid dynamics, string theory, general relativity etc. So, it enhances a motivational and predominant concern to explore Ricci soliton and also to allocate them topologically and geometrically in the field of Riemannian manifolds. Further, in modern mathematics, the methods of contact geometry plays an important role. Contact geometry has evolved from the mathematical formalism of classical mechanics. In 1969, Tanno [26] classified the connected almost contact metric manifolds whose automorphism groups have maximal dimensions as follows.

(a) Homogeneous normal contact Riemannian manifolds with constant ϕ-holomorphic sectional curvature if $k(\xi, X) > 0$;
(b) Global Riemannian product of a line or a circle and a Kählerian manifold with constant holomorphic sectional curvature if $k(\xi, X) = 0$;
(c) A warped product space $\mathbb{R} \times_f N$, where $\mathbb{R}$ is the real line and N is a Kählerian manifold, if $k(\xi, X) < 0$; where $k(\xi, X)$ denotes the sectional curvature of the plane section containing the characteristic vector field ξ and an arbitrary vector field X.

In [16], K. Kenmotsu first introduced and studied a particular class of almost contact metric manifolds, obtained some tensor equations to characterize the manifolds of the third class using the wraping function $f(t) = ce^t$ on the interval $J = (-\epsilon, \epsilon)$. Since then the manifolds of the third class were called Kenmotsu manifolds. Conversely, every point on a Kenmotsu manifold has a neighbourhood which is locally a warped product $J \times_f N$, where f is given by the above mentioned relation.

A pseudo-Riemannian manifold (M, g) admits a Ricci soliton which is a generalization of Einstein metric (i.e, $S = ag$ for some constant a) if there exists a smooth vector field V and a constant λ such that

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0,$$

where $\mathcal{L}_V$ denotes Lie derivative along the direction V and S denotes the Ricci curvature tensor of the manifold. The vector field V is called potential vector field and λ is called soliton constant.

The Ricci soliton is a self-similar solution of the Hamilton's Ricci flow [13] which is defined by the equation $\frac{\partial g(t)}{\partial t} = -2S(g(t))$ with initial condition $g(0) = g$, where $g(t)$ is a one-parameter family of metrics on M. The potential vector field V and soliton constant λ plays an indispensable role while determining the nature of the soliton. A soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Now if V is zero or Killing then the Ricci soliton reduces to Einstein manifold and the soliton is called trivial soliton.

If the potential vector field V is the gradient of a smooth function f, denoted by Df then the soliton equation reduces to

$$Hessf + S + \lambda g = 0,$$

where $Hessf$ is Hessian of f. Perelman [17] proved that a Ricci soliton on a compact manifold is a gradient Ricci soliton.

In 2009, Cho and Kimura [4] introduced the concept of η-Ricci soliton which is another generalization of classical Ricci soliton and is given by

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where μ is a real constant, η is a 1-form defined as $\eta(X) = g(X, \xi)$ for any $X \in \chi(M)$. Clearly it can be noted that if $\mu = 0$ then the η-Ricci soliton reduces to Ricci soliton.

In 2014, Kaimakamis and Panagiotidou [15] modified the definition of Ricci soliton where they have used $*$-Ricci tensor S^* which was introduced by Tachibana [25], in place of Ricci tensor S. The $*$-Ricci tensor S^* is defined by

$$S^*(X, Y) = \frac{1}{2}(trace\{\phi \cdot R(X, \phi Y)\})$$

for all vector fields X and Y on M, where ϕ is a $(1, 1)-$tensor field. They have used the concept of $*$-Ricci soliton within the framework of real hypersurfaces of a complex space form.

In 2020, Dey et al. [7] defined $*$-η-Ricci soliton as

$$\mathcal{L}_\xi g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0.$$

As per the authors knowledge, the results concerning $*$-η-Ricci soliton were studied when the potential vector field V is the characteristic vector field ξ. Motivated from this we generalize the definition by considering the potential vector field as arbitrary vector field V and define as:

$$\mathcal{L}_V g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where we considered the manifold as $(2n + 1)$-dimensional. Now if we consider the potential vector field V as the gradient of a smooth function f, then the $*$-η-Ricci soliton equation can be rewritten as

$$Hessf + S^* + \lambda g + \mu\eta \otimes \eta = 0. \tag{1.2}$$

By gradient almost $*$-η-Ricci soliton we mean, gradient $*$-η-Ricci soliton, where we consider λ as a smooth function.

Recent years, many authors have been studied Ricci soliton and η-Ricci soliton on contact and paracontact geometries. First, Sharma [24] initiated the study of Ricci solitons in contact geometry. Further, Ghosh [11] designed Ricci soliton on 3-dimensional Kenmotsu manifold. Later Călin and Crasmareanu [2], Ghosh [12], Wang [29] etc. measured Kenmotsu metric in term of Ricci soliton on contact manifolds. In [27], authors have strained $*$-Ricci solitons and gradient almost $*$-Ricci solitons on Kenmotsu manifolds and obtained some needful results. They have developed if a 3-Kenmotsu manifold admits a $*$-Ricci soliton, then the manifold is of constant negative curvature -1. They also get that if an η-Einstein Kenmotsu manifold of $dim > 3$ admits a $*$-Ricci soliton then the manifold becomes Einstein.

Next, Chen [3] considered a real hypersurface of a non-flat complex space form which admits a $*$-Ricci soliton whose potential vector field belongs to the principal curvature space and the holomorphic distribution. Many authors have been studied $*$-Ricci soliton and thier generalizations on contact and paracontact metric manifolds (e.g., see [5,6,9,10,18–23,28,30]). Recently, Wang [28] proved that if the metric of a Kenmotsu 3-manifold represents a $*$-Ricci soliton, then the manifold is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$. Based on the above facts and discussions in the research of contact geometry, a natuaral **question** arises.

Are there contact metric almost manifolds, whose metrics are $*$-η-Ricci soliton?

In later sections, we show that indeed the answer to this question is affirmative. The paper is organized as follows: in Sect. 2, the basic definitions and facts about contact metric manifolds, Kenmotsu manifolds and $(\kappa, \mu)'$-almost Kenmotsu manifolds are given. In the later section, we consider Kenmotsu metric as $*$-η-Ricci soliton and gradient almost $*$-η-Ricci soliton and obtain some useful results. We also provide some examples to support our findings in that section. In Sect 4, we consider the metric of $(\kappa, \mu)'$-almost Kenmotsu manifold to represent $*$-η-Ricci soliton along with a special condition and obtained that the manifold is Ricci flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

2 Notes on contact metric manifolds

By [1], a differentiable manifold M of dimension $(2n+1)$ is called an almost contact structure or (ϕ, ξ, η) structure if M admits a (1, 1) tensor field ϕ, a vector field ξ, a 1-form η satisfying:

$$\phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

where I is the identity mapping. Generally, ξ and η are called *characteristic vector field* or *Reeb vector field* and *almost contact 1-form* respectively.

A Riemannian metric g is said to be *compatible metric* if it satisfies:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for arbitrary vector fields X and Y on M. An almost contact manifold endowed with a compatible Riemannian metric is said to be an *almost contact metric manifold* and is denoted by (M, ϕ, ξ, η, g).

In a almost contact metric manifold (M, ϕ, ξ, η, g) the following conditions are satisfied:

$$\phi\xi = 0, \tag{2.4}$$

$$\eta \circ \phi = 0, \tag{2.5}$$

$$g(X, \xi) = \eta(X), \tag{2.6}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.7}$$

for arbitrary $X, Y \in \chi(M)$. The normality of an almost contact structure is equivalent with the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ (for more details we refer to [1]).

Definition 2.1 On an almost contact metric manifold M, a vector field X is said to be contact vector field if there exist a smooth function f such that $\mathcal{L}_X \xi = f\xi$.

Definition 2.2 On an almost contact metric manifold M, a vector field X is said to be infinitesimal contact transformation if $\mathcal{L}_X \eta = f\eta$ for some function f. In particular, we call X as a strict infinitesimal contact transformation if $\mathcal{L}_X \eta = 0$.

2.1 Kenmotsu manifold

We define the fundamental 2-form Φ on an almost contact metric manifold M by $\Phi(X, Y) = g(X, \phi Y)$ for arbitrary $X, Y \in \chi(M)$. We recall from [14], an *almost Kenmotsu manifold* is an almost contact metric manifold, where η is closed, i.e., $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. A normal almost Kenmotsu manifold is called *Kenmotsu manifold*.

By [1] if in a almost contact metric manifold M the 1-form η and the (1, 1)-tensor field ϕ satisfy the following condition for arbitrary $X, Y \in \chi(M)$:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.8}$$

where ∇ denotes the Riemannian connection of g, then the manifold M is called a Kenmotsu manifold. It is easy to verify that the above mentioned relation is equivalent with the normality condition of the manifold.

In Kenmotsu manifold of dimension $(2n + 1)$ the following relations hold:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.9}$$
$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.10}$$
$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$
$$S(X, \xi) = -2n\eta(X), \tag{2.12}$$
$$(\mathcal{L}_\xi)g(X, Y) = 2g(X, Y) - 2\eta(X)\eta(Y) \tag{2.13}$$

for arbitrary $X, Y, Z, W \in \chi(M)$, where $\mathcal{L}$ is the Lie derivative operator, R is the Riemannian curvature tensor and S is the Ricci tensor.

A (2n+1)-dimensional Kenmotsu metric manifold is said to be a η-Einstein Kenmotsu manifold if there exists two smooth functions a and b which satisfies the following relation

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{2.14}$$

for all $X, Y \in \chi(M)$. Clearly if $b = 0$ then η-Einstein manifold reduces to Einstein manifold. Now considering $X = \xi$ in the last equation and using (2.12) we have, $a + b = -2n$. Contracting (2.14) over X and Y we get, $r = (2n + 1)a + b$, where r

denotes the scalar curvature of the manifold. Solving these two we have, $a = (1 + \frac{r}{2n})$ and $b = -(2n + 1 + \frac{r}{2n})$. Using these values we can rewrite (2.14) as

$$S(X, Y) = \left(1 + \frac{r}{2n}\right) g(X, Y) - \left(2n + 1 + \frac{r}{2n}\right) \eta(X)\eta(Y). \tag{2.15}$$

2.2 $(\kappa, \mu)'$ almost Kenmotsu manifold

On an almost Kenmotsu manifold we consider two $(1, 1)$-type tensor fields $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ and an operator $\ell = R(., \xi)\xi$, where $\mathcal{L}_\xi \phi$ is the Lie derivative of ϕ along the direction ξ. The tensor fields h and h' plays an important role in an almost Kenmotsu manifold. Both of them are symmetric and satisfy the following relations:

$$\nabla_X \xi = X - \eta(X)\xi + h'X, \tag{2.16}$$

$$h\xi = h'\xi = 0, \tag{2.17}$$

$$h\phi = -\phi h, tr(h) = tr(h') = 0 \tag{2.18}$$

for any $X, Y \in \chi(M)$, where ∇ is the Levi-Civita connection of the metric g. In addition the following curvature property is aslo satisfied:

$$R(X, Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X, \tag{2.19}$$

where R is the Riemannian curvature tensor of (M, g).

By $(\kappa, \mu)'$-almost Kenmotsu manifold we mean almost Kenmotsu manifold where the characteristic vector field ξ satisfies the $(\kappa, \mu)'$-nullity distribution (for details see [8]), i.e.,

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \tag{2.20}$$

for any $X, Y \in \chi(M)$, where κ and μ are real constants. On a $(\kappa, \mu)'$-almost Kenmotsu manifold M, we have (see [8])

$$h'^2(X) = -(\kappa + 1)[X - \eta(X)\xi], \tag{2.21}$$

$$h^2(X) = -(\kappa + 1)[X - \eta(X)\xi] \tag{2.22}$$

for $X \in \chi(M)$. From previous relation it follows that $h' = 0$ if and only if $\kappa = -1$ and $h' \neq 0$ otherwise. Let $X \in Ker(\eta)$ be an eigenvector field of h' orthogonal to ξ w.r.t. the eigenvalue α. Then, from (2.21) we get $\alpha^2 = -(\kappa + 1)$ which implies $\kappa \leq -1$. Dileo and Pastore proved that on a $(\kappa, \mu)'$-almost Kenmotsu manifold with $\kappa < -1$, we have $\mu = -2$ (Proposition 4.1 of [8]). Since the same symbol μ is used in the coefficient of $\eta \otimes \eta$ in the definition of $*$-η-Ricci soliton and in $(\kappa, \mu)'$-almost Kenmotsu manifold, so to reduce the complications in notations we use $(\kappa, -2)'$-almost Kenmotsu manifold throughout this paper.

We recall some useful results on a $(2n+1)$ dimensional $(\kappa, -2)'$-almost Kenmotsu manifold M with $\kappa < -1$ as follows:

$$R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X) - 2(g(h'X, Y)\xi - \eta(Y)h'X) \tag{2.23}$$
$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - 2nh'(X), \tag{2.24}$$
$$r = 2n(\kappa - 2n), \tag{2.25}$$
$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) + g(h'X, Y), \tag{2.26}$$

where $X, Y \in \chi(M)$, Q, r are the Ricci operator and scalar curvature of M respectively.

3 $*$-η-Ricci soliton on Kenmotsu manifold

In this section we consider that the metric g of a $(2n + 1)$-dimensional Kenmotsu manifold represents a $*$-η-Ricci soliton and a gradient almost $*$-η-Ricci soliton. We recall some important lemmas relevant to our results.

Lemma 3.1 [27] *The Ricci operator Q on a $(2n+1)$-dimensional Kenmotsu manifold satisfies*

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{3.1}$$
$$(\nabla_\xi Q)X = -2QX - 4nX \tag{3.2}$$

for arbitrary vector field X on the manifold.

Lemma 3.2 [27] *The $*$-Ricci tensor S^* on a $(2n+1)$-dimensional Kenmotsu manifold is given by*

$$S^*(X, Y) = S(X, Y) + (2n - 1)g(X, Y) + \eta(X)\eta(Y) \tag{3.3}$$

for arbitrary vector fields X and Y on the manifold.

Theorem 3.3 *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. If the metric g represents a $*$-η-Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein.*

Proof Since the metric g of the Kenmotsu manifold represents a $*$-η-Ricci soliton so both of the Eqs. (1.1) and (3.3) are satisfied. Combining these two we have

$$(\mathcal{L}_V g)(X, Y) = -2S(X, Y) - (2\lambda + 4n - 2)g(X, Y) \\ -2(\mu + 1)\eta(X)\eta(Y). \tag{3.4}$$

Taking covariant derivative w.r.t. arbitrary vector field Z and using (2.10), we obtain

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = -2(\nabla_Z S)(X, Y) - 2(\mu + 1)\{g(X, Z)\eta(Y) \\ +g(Y, Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\} \tag{3.5}$$

for all $X, Y, Z \in \chi(M)$. Again from Yano [31] we have the following commutation formula

$$(\mathcal{L}_V \nabla_Z g - \nabla_Z \mathcal{L}_V g - \nabla_{[V,Z]} g)(X, Y) = -g((\mathcal{L}_V \nabla)(X, Z), Y) \\ -g((\mathcal{L}_V \nabla)(Y, Z), X),$$

where g is the metric connection i.e., $\nabla g = 0$. So the above equation reduces to

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = g((\mathcal{L}_V \nabla)(X, Z), Y) + g((\mathcal{L}_V \nabla)(Y, Z), X) \tag{3.6}$$

for all vector fields X, Y, Z on M. Combining (3.5) and (3.6) and by a straightforward combinatorial computation and applying the symmetry of $(\mathcal{L}_V \nabla)$ the foregoing equation yields

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(Z, X) \\ -2(\mu + 1)\{g(X, Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\} \tag{3.7}$$

for arbitrary vector fields X, Y and Z on M. Using (3.1) and (3.2), the foregoing equation yields

$$(\mathcal{L}_V \nabla)(X, \xi) = 2QX + 4nX \tag{3.8}$$

for all $X \in \chi(M)$. Now differentiating covariantly this with respect to arbitrary vector field Y, we achieve

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) = 2(\nabla_Y Q)X - (\mathcal{L}_V \nabla)(X, Y) + \eta(Y)(2QX + 4nX). \tag{3.9}$$

We know that, $(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z)$. In view of (3.9) in the previous relation we acquire

$$(\mathcal{L}_V R)(X, Y)\xi = 2\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + 2\eta(X)(QY + 2nY) \\ -2\eta(Y)(QX + 2nX) \tag{3.10}$$

for arbitrary vector fields X and Y on M. Setting $Y = \xi$ in the aforementioned equation and using (2.12), (3.1) and (3.2) we get

$$(\mathcal{L}_V R)(X, \xi)\xi = 0. \tag{3.11}$$

Now, taking (3.4) in account, the Lie derivative of $g(\xi, \xi) = 1$ along the potential vector field V yields

$$\eta(\mathcal{L}_V \xi) = \lambda + \mu. \tag{3.12}$$

Plugging $Y = \xi$ and noting that (2.2) and (2.6), the Eq. (3.4) provides

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = -(2\lambda + 2\mu)\eta(X), \tag{3.13}$$

which holds for arbitrary vector field X on M. From (2.11) we compute, $R(X,\xi)\xi = \eta(X)\xi - X$. Taking Lie derivative along the potential vector field V and inserting (3.12) and (3.13) in account, this reduces to

$$(\mathcal{L}_V R)(X,\xi)\xi = 2(\lambda+\mu)(X - \eta(X)\xi) \tag{3.14}$$

for all $X \in \chi(M)$. Finally comparing (3.11) and (3.14) we have, $2(\lambda+\mu)(X - \eta(X)\xi) = 0$. Since this holds for arbitrary $X \in \chi(M)$ so, we infer

$$\lambda = -\mu. \tag{3.15}$$

Invoking the relation (3.15) in (3.12), we easily obtain $\eta(\mathcal{L}_V\xi) = 0$. Since we have considered the potential vector field V as contact vector field so there must exists a smooth function f such that $\mathcal{L}_V\xi = f\xi$. Making use of this in (3.12) we get $f = \lambda+\mu$. Therefore by using the relation (3.15), we get $f = 0$ and thus $\mathcal{L}_V\xi = 0$. Finally the Eq. (3.13) reduces to

$$\mathcal{L}_V\eta = 0. \tag{3.16}$$

So, V is strictly infinitesimal contact transformation.

We know the well-known formula from Yano [31] that $(\mathcal{L}_V\nabla)(X,Y) = \mathcal{L}_V\nabla_X Y - \nabla_X\mathcal{L}_V Y - \nabla_{[V,X]}Y$. Inserting $Y = \xi$ and using (2.9), $\mathcal{L}_V\xi = 0$ and (3.16) yields, $(\mathcal{L}_V\nabla)(X,\xi) = 0$. Substituting this in (3.8), we deduce $QX = -2nX\ \forall X \in \chi(M)$, which settles our claim. □

$*$-η-Ricci soliton is a generalisation of $*$-Ricci soliton, where we consider $\mu = 0$ in (1.1) to get $*$-Ricci soliton equation. We can rewrite the above theorem as:

Corollary 3.4 *Let $M^{(2n+1)}(\phi,\xi,\eta,g)$ be a Kenmotsu manifold. If the metric g represents a $*$-Ricci soliton and if the soliton vector field V is contact, then V is strictly infinitesimal contact transformation and the manifold is Einstein.*

Example 3.5 Let us consider the set $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$ as our manifold where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below:

$$e_1 = e^{-v}\frac{\partial}{\partial x},\quad e_2 = e^{-v}\frac{\partial}{\partial y},\quad e_3 = e^{-v}\frac{\partial}{\partial z},\quad e_4 = e^{-v}\frac{\partial}{\partial u},\quad e_5 = \frac{\partial}{\partial v}$$

are linearly independent at each point of M. We define the metric g as

$$g(e_i,e_j) = \begin{cases} 1, & \text{if } i = j \textit{ and } i,j \in \{1,2,3,4,5\} \\ 0, & \text{otherwise.} \end{cases}$$

Let η be a 1-form defined by $\eta(X) = g(X,e_5)$, for arbitrary $X \in \chi(M)$. Let us define (1,1)-tensor field ϕ as:

$$\phi(e_1) = e_3,\quad \phi(e_2) = e_4,\quad \phi(e_3) = -e_1,\quad \phi(e_4) = -e_2,\quad \phi(e_5) = 0.$$

Then it satisfy the relations $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, where $\xi = e_5$ and X, Y is arbitrary vector field on M. So, (M, ϕ, ξ, η, g) defines an almost contact structure on M.

We can now deduce that,

$$\begin{array}{llll}
[e_1, e_2] = 0 & [e_1, e_3] = 0 & [e_1, e_4] = 0 & [e_1, e_5] = e_1 \\
[e_2, e_1] = 0 & [e_2, e_3] = 0 & [e_2, e_4] = 0 & [e_2, e_5] = e_2 \\
[e_3, e_1] = 0 & [e_3, e_2] = 0 & [e_3, e_4] = 0 & [e_3, e_5] = e_3 \\
[e_4, e_1] = 0 & [e_4, e_2] = 0 & [e_4, e_3] = 0 & [e_4, e_5] = e_4 \\
[e_5, e_1] = -e_1 & [e_5, e_2] = -e_2 & [e_5, e_3] = -e_3 & [e_5, e_4] = -e_4.
\end{array}$$

Let ∇ be the Levi–Civita connection of g. Then from $Koszul's formula$ for arbitrary $X, Y, Z \in \chi(M)$ given by:

$$\begin{aligned}
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\
-g(Y, [X, Z]) + g(Z, [X, Y]),
\end{aligned}$$

we can have:

$$\begin{array}{lllll}
\nabla_{e_1} e_1 = -e_5 & \nabla_{e_1} e_2 = 0 & \nabla_{e_1} e_3 = 0 & \nabla_{e_1} e_4 = 0 & \nabla_{e_1} e_5 = e_1 \\
\nabla_{e_2} e_1 = 0 & \nabla_{e_2} e_2 = -e_5 & \nabla_{e_2} e_3 = 0 & \nabla_{e_2} e_4 = 0 & \nabla_{e_2} e_5 = e_2 \\
\nabla_{e_3} e_1 = 0 & \nabla_{e_3} e_2 = 0 & \nabla_{e_3} e_3 = -e_5 & \nabla_{e_3} e_4 = 0 & \nabla_{e_3} e_5 = e_3 \\
\nabla_{e_4} e_1 = 0 & \nabla_{e_4} e_2 = 0 & \nabla_{e_4} e_3 = 0 & \nabla_{e_4} e_4 = -e_5 & \nabla_{e_4} e_5 = e_4 \\
\nabla_{e_5} e_1 = 0 & \nabla_{e_5} e_2 = 0 & \nabla_{e_5} e_3 = 0 & \nabla_{e_5} e_4 = 0 & \nabla_{e_5} e_5 = 0.
\end{array}$$

Therefore $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ is satisfied for arbitrary $X, Y \in \chi(M)$. So (M, ϕ, ξ, η, g) becomes a Kenmotsu manifold.

The non-vanishing components of curvature tensor are:

$$\begin{array}{lll}
R(e_1, e_2)e_2 = -e_1 & R(e_1, e_3)e_3 = -e_1 & R(e_1, e_4)e_4 = -e_1 \\
R(e_1, e_5)e_5 = -e_1 & R(e_1, e_2)e_1 = e_2 & R(e_1, e_3)e_1 = e_3 \\
R(e_1, e_4)e_1 = e_4 & R(e_1, e_5)e_1 = e_5 & R(e_2, e_3)e_2 = e_3 \\
R(e_2, e_4)e_2 = e_4 & R(e_2, e_5)e_2 = e_5 & R(e_2, e_3)e_3 = -e_2 \\
R(e_2, e_4)e_4 = -e_2 & R(e_2, e_5)e_5 = -e_2 & R(e_3, e_4)e_3 = e_4 \\
R(e_3, e_5)e_3 = e_5 & R(e_3, e_4)e_4 = -e_3 & R(e_4, e_5)e_4 = e_5 \\
R(e_5, e_3)e_5 = e_3 & R(e_5, e_4)e_5 = e_4. &
\end{array}$$

Now from the above results we have, $S(e_i, e_i) = -4$ for $i = 1, 2, 3, 4, 5$ and

$$S(X, Y) = -4g(X, Y) \ \forall X, Y \in \chi(M). \tag{3.17}$$

Contracting this we have $r = \sum_{i=1}^{5} S(e_i, e_i) = -20 = -2n(2n+1)$ where dimension of the manifold $2n + 1 = 5$. Also, we have

$$S^*(e_i, e_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4 \\ 0, & \text{if } i = 5. \end{cases}$$

and $r^* = r + 4n^2 = -20 + 16 = -4$. So

$$S^*(X, Y) = -g(X, Y) + \eta(X)\eta(Y) \;\; \forall X, Y \in \chi(M). \tag{3.18}$$

Now we consider a vector field V as

$$V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{3.19}$$

Then from the above results we can justify that

$$(\mathcal{L}_V g)(X, Y) = 4\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{3.20}$$

which holds for all $X, Y \in \chi(M)$. From (3.18) and (3.20), we can conclude that g represents a $*$-η-Ricci soliton i.e., it satisfies (1.1) for potential vector field V defined by (3.19), $\lambda = -1$ and $\mu = 1$.

Theorem 3.6 *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a Kenmotsu manifold. If the metric g represents a gradient almost $*$-η-Ricci soliton then either M is Einstein or there exists an open set where the potential vector field V is pointwise collinear with the characteristic vector field ξ.*

Proof In view of (3.3) in the definition of gradient almost $*$-η-Ricci soliton give by Eq. (1.2), we acquire

$$\nabla_X Df = -QX - (\lambda + 2n - 1)X - (\mu + 1)\eta(X)\xi \tag{3.21}$$

for any vector field X on M. Taking covariant derivative along arbitrary vector Y and using (2.9), (2.10) yields

$$\begin{aligned} \nabla_Y \nabla_X Df = &-(\nabla_Y Q)X - Q(\nabla_Y X) - Y(\lambda)X - (\lambda + 2n - 1)(\nabla_Y X) \\ &-(\mu + 1)\{g(X, Y)\xi - 2\eta(X)\eta(Y)\xi \\ &+\eta(\nabla_Y X)\xi + \eta(X)Y\}. \end{aligned} \tag{3.22}$$

Applying this in the expression of Riemannian curvature tensor we obtain

$$\begin{aligned} R(X, Y)Df = &(\nabla_Y Q)X - (\nabla_X Q)Y + Y(\lambda)X - X(\lambda)Y \\ &-(\mu + 1)\{\eta(Y)X - \eta(X)Y\}. \end{aligned} \tag{3.23}$$

Moreover an inner product w.r.t. ξ and use of (3.1) and (3.2) yields

$$g(R(X,Y)Df,\xi) = Y(\lambda)\eta(X) - X(\lambda)\eta(Y) \tag{3.24}$$

for $X, Y \in \chi(M)$. Furthermore the inner product of (2.11) with the potential vector field Df provides

$$g(R(X,Y)Df,\xi) = \eta(Y)X(f) - \eta(X)Y(f) \tag{3.25}$$

for arbitrary X and Y on M. Comparing (3.24) and (3.25) and plugging $Y = \xi$, we have $X(f+\lambda) = \xi(f+\lambda)\eta(X)$. From this we achieve

$$d(f+\lambda) = \xi(f+\lambda)\eta. \tag{3.26}$$

So, $(f+\lambda)$ is invariant along the distribution $Ker(\eta)$ i.e., if $X \in Ker(\eta)$ then $X(f+\lambda) = d(f+\lambda)X = 0$.

Now, if we takhe inner product w.r.t. arbitrary vector field Z after plugging $X = \xi$ in (3.23) we get

$$\begin{aligned} g(R(\xi,Y)Df,Z) = {} & S(Y,Z) + (2n - \xi(\lambda) + \mu + 1)g(Y,Z) + Y(\lambda)\eta(Z) \\ & -(\mu+1)\eta(Y)\eta(Z). \end{aligned} \tag{3.27}$$

Again noting that from (2.11), we can easily deduce for arbitrary vector fields Y and Z on M

$$g(R(\xi,Y)Df,Z) = \xi(f)g(Y,Z) - Y(f)\eta(Z). \tag{3.28}$$

Comparing the Eqs. (3.27) and (3.28) and applying (3.26), we obtain

$$S(Y,Z) = \{\xi(f+\lambda) - \mu - 2n - 1\}g(Y,Z) - \{\xi(f+\lambda) - \mu - 1\}\eta(Y)\eta(Z). \tag{3.29}$$

Since the above equation holds good for arbitrary Y and Z, so the manifold is η-Einstein. Now contracting (3.29), we infer

$$\xi(f+\lambda) = \frac{r}{2n} + \mu + 2n + 2. \tag{3.30}$$

Plugging this in (3.29), we acquire

$$S(Y,Z) = \left(\frac{r}{2n} + 1\right) g(Y,Z) - \left(\frac{r}{2n} + 2n + 1\right) \eta(Y)\eta(Z)$$

for arbitrary vector fields Y and Z on M which is exactly same as (2.15). Now contracting (3.23) w.r.t. X reduces to

$$S(Y,Df) = \frac{1}{2}Y(r) + 2nY(\lambda) - 2n(\mu+1)\eta(Y), \tag{3.31}$$

which holds for any $Y \in \chi(M)$. Now, taking into with (2.15), we compute

$$(r + 2n)Y(f) - (r + 2n(2n + 1))\eta(Y)\xi(f) - nY(r)$$
$$-4n^2Y(\lambda) + 4n^2(\mu + 1)\eta(y) = 0 \tag{3.32}$$

for all $Y \in \chi(M)$. Now, setting $Y = \xi$ and then in view of (3.30), we easily derive the relation

$$\xi(r) = -2(r + 2n(2n + 1)). \tag{3.33}$$

Since $d^2 = 0$ and $d\eta = 0$, from (3.26) it follows $dr \wedge \eta = 0$ i.e., $dr(X)\eta(Y) - dr(Y)\eta(X) = 0$ for arbitrary $X, Y \in \chi(M)$. After inserting $Y = \xi$ and applying (3.33) it reduces to $X(r) = -2(r + 2n(2n + 1))\xi$. Since X is an arbitrary vector field so we conclude that

$$Dr = -2(r + 2n(2n + 1))\xi. \tag{3.34}$$

Let X be a vector field of the distribution $Ker(\eta)$. Then, (3.32) provides

$$(r + 2n)X(f) - 4n^2X(\lambda) = 0.$$

Invoking (3.26) and (3.30) we obtain, $(r + 2n(2n + 1))X(f) = 0$. From here we conclude

$$(r + 2n(2n + 1))(Df - \xi(f)\xi) = 0.$$

If $r = -2n(2n + 1)$, then from (2.15) we acquire that the manifold is Einstein with Einstein constant $-2n$.

If $r \neq -2n(2n + 1)$ on some open set O of M, then $Df = \xi(f)\xi$ on that open set that is, the potential vector field is pointwise collinear with the characteristic vector field ξ, which finishes the proof. □

Example 3.7 Let us consider the set $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ as our manifold where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below:

$$e_1 = v\frac{\partial}{\partial x}, \qquad e_2 = v\frac{\partial}{\partial y}, \qquad e_3 = v\frac{\partial}{\partial z}, \qquad e_4 = v\frac{\partial}{\partial u}, \qquad e_5 = -v\frac{\partial}{\partial v}$$

forms a linearly independent set of vector fields on M. We define the metric g as

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We consider the reeb vector field $\xi = e_5$ then the 1-form η is defined by $\eta(X) = g(X, e_5)$, for arbitrary $X \in \chi(M)$ then, $\eta = dv$. Let us define (1,1)-tensor field ϕ as:

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = e_4, \quad \phi(e_4) = -e_3, \quad \phi(e_5) = 0.$$

Then it satisfy the relations $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ where X, Y is arbitrary vector field on M. So, (M, ϕ, ξ, η, g) defines an almost contact structure on M.

Let ∇ be the Levi–Civita connection of g. Then from $Koszul'sformula$ for arbitrary $X, Y, Z \in \chi(M)$ given by:

$$\begin{aligned} 2g(\nabla_X Y, Z) = & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ & -g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

we can have:

$$\begin{array}{lllll} \nabla_{e_1} e_1 = -e_5 & \nabla_{e_1} e_2 = 0 & \nabla_{e_1} e_3 = 0 & \nabla_{e_1} e_4 = 0 & \nabla_{e_1} e_5 = e_1 \\ \nabla_{e_2} e_1 = 0 & \nabla_{e_2} e_2 = -e_5 & \nabla_{e_2} e_3 = 0 & \nabla_{e_2} e_4 = 0 & \nabla_{e_2} e_5 = e_2 \\ \nabla_{e_3} e_1 = 0 & \nabla_{e_3} e_2 = 0 & \nabla_{e_3} e_3 = -e_5 & \nabla_{e_3} e_4 = 0 & \nabla_{e_3} e_5 = e_3 \\ \nabla_{e_4} e_1 = 0 & \nabla_{e_4} e_2 = 0 & \nabla_{e_4} e_3 = 0 & \nabla_{e_4} e_4 = -e_5 & \nabla_{e_4} e_5 = e_4 \\ \nabla_{e_5} e_1 = 0 & \nabla_{e_5} e_2 = 0 & \nabla_{e_5} e_3 = 0 & \nabla_{e_5} e_4 = 0 & \nabla_{e_5} e_5 = 0. \end{array}$$

Therefore $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ is satisfied for arbitrary $X, Y \in \chi(M)$. So (M, ϕ, ξ, η, g) becomes a Kenmotsu manifold.

The non-vanishing components of curvature tensor are:

$$\begin{array}{lll} R(e_1, e_2)e_2 = -e_1 & R(e_1, e_3)e_3 = -e_1 & R(e_1, e_4)e_4 = -e_1 \\ R(e_1, e_5)e_5 = -e_1 & R(e_1, e_2)e_1 = e_2 & R(e_1, e_3)e_1 = e_3 \\ R(e_1, e_4)e_1 = e_4 & R(e_1, e_5)e_1 = e_5 & R(e_2, e_3)e_2 = e_3 \\ R(e_2, e_4)e_2 = e_4 & R(e_2, e_5)e_2 = e_5 & R(e_2, e_3)e_3 = -e_2 \\ R(e_2, e_4)e_4 = -e_2 & R(e_2, e_5)e_5 = -e_2 & R(e_3, e_4)e_3 = e_4 \\ R(e_3, e_5)e_3 = e_5 & R(e_3, e_4)e_4 = -e_3 & R(e_4, e_5)e_4 = e_5 \\ R(e_5, e_3)e_5 = e_3 & R(e_5, e_4)e_5 = e_4. & \end{array}$$

Now from the above results we have, $S(e_i, e_i) = -4$ for $i = 1, 2, 3, 4, 5$ and

$$S(X, Y) = -4g(X, Y) \ \forall X, Y \in \chi(M). \tag{3.35}$$

So, the manifold is Einstein. Also, we have

$$S^*(e_i, e_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4 \\ 0, & \text{if } i = 5. \end{cases}$$

and

$$S^*(X,Y) = -g(X,Y) + \eta(X)\eta(Y) \ \ \forall X, Y \in \chi(M). \tag{3.36}$$

Let $f : M \to \mathbb{R}$ be a smooth function defined by

$$f(x,y,z,u,v) = x^2 + y^2 + z^2 + u^2 + \frac{v^2}{2}. \tag{3.37}$$

Then the gradient of f, Df is given by

$$Df = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{3.38}$$

Then from the above results we can verify that

$$(\mathcal{L}_{Df}g)(X,Y) = 2\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{3.39}$$

which holds for all $X, Y \in \chi(M)$. From (3.36) and (3.39) we obtain that g represents a gradient almost $*\,\eta$-Ricci soliton i.e., it satisfies (1.2) for $V = Df$, where f is defined by (3.37), $\lambda = 0$ and $\mu = 0$.

4 $*$-η-Ricci soliton on $(\kappa, \mu)'$-almost Kenmotsu manifold with $\kappa < -1$

In this section we consider the manifold as a $(2n+1)$-dimensional almost Kenmotsu manifold where the characteristic vector field ξ satisfies $(\kappa, -2)'$-nullity distribution. Then we let the metric g to represent a $*$-η-Ricci soliton. Here we look back on some pertinent results and used these in our work.

Lemma 4.1 [6] *On a $(\kappa, -2)'$-almost Kenmotsu manifold with $\kappa < -1$ the $*$-Ricci tensor is given by*

$$S^*(X,Y) = -(\kappa+2)(g(X,Y) - \eta(X)\eta(Y)) \tag{4.1}$$

for any vector fields X and Y.

Theorem 4.2 *Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to $(\kappa, -2)'$-nullity distribution where $\kappa < -1$. If the metric g represents a $*$-η-Ricci soliton satisfying $\lambda + \mu \neq 0$ then, M is Ricci-flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Proof Combining (1.1) with (4.1), we derive

$$(\mathcal{L}_V g)(X,Y) = (2\kappa - 2\lambda + 4)g(X,Y) - 2(\kappa + \mu + 2)\eta(X)\eta(Y) \tag{4.2}$$

for all vector fields X and Y on M. Now taking covariant derivative of the foregoing equation along arbitrary vector field Z and using (2.26) we get

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = -2(\kappa + \mu + 2)[\eta(Y)g(X, Z) + \eta(X)g(Y, Z) + \eta(Y) \\ g(h'Z, X) + \eta(X)g(h'Z, Y) - 2\eta(X)\eta(Y)\eta(Z)]. \quad (4.3)$$

By a straightforward combinatorial computation, use of (3.6), the symmetry of $(\mathcal{L}_V \nabla)$ in the aforementioned equation we acquire

$$(\mathcal{L}_V \nabla)(X, Y) = -2(\kappa + \mu + 2)[g(X, Y) + g(h'X, Y) - \eta(X)\eta(Y)]\xi \quad (4.4)$$

for all $X, Y \in \chi(M)$. Replacing $Y = \xi$ and using (2.2), (2.6) and (2.17), we have

$$(\mathcal{L}_V \nabla)(X, \xi) = 0 \quad (4.5)$$

for arbitrary vector field X on M. Now taking (2.16) and (4.4) into account and differentiating (4.5) covariantly along arbitrary vector field Y one can obtain

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) = 2(\kappa + \mu + 2)[g(X, Y) - \eta(X)\eta(Y) + 2g(h'X, Y) + g(h'^2 X, Y)]\xi \quad (4.6)$$

for any vector fields X and Y on M. Again from Yano we have the well-known curvature property, $(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z)$. Setting $Z = \xi$ and using (4.6) repeatedly we achieve

$$(\mathcal{L}_V R)(X, Y)\xi = 0 \quad (4.7)$$

for arbitrary $X, Y \in \chi(M)$. Now taking Lie derivative of (2.20) along the potential vector field V, taking (2.2) and (2.17) into account we get

$$(\mathcal{L}_V R)(X, \xi)\xi = \kappa[g(X, \mathcal{L}_V \xi)\xi - 2\eta(\mathcal{L}_V \xi)X - ((\mathcal{L}_V \eta)X)\xi] + 2[2\eta(\mathcal{L}_V \xi) \\ h'X - \eta(X)(h'(\mathcal{L}_V \xi)) - g(h'X, \mathcal{L}_V \xi)\xi - ((\mathcal{L}_V h')X)] \quad (4.8)$$

for any vector field X on M. Plugging $Y = \xi$ in (4.2), we infer

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = (-2\lambda - 2\mu)\eta(X), \quad (4.9)$$

for all $X \in \chi(M)$. Setting $X = \xi$ in the foregoing equation, we acquire

$$\eta(\mathcal{L}_V \xi) = -(-\lambda - \mu). \quad (4.10)$$

With the help of (4.7), (4.9) and (4.10), one can rewrite the Eq. (4.8) as

$$\kappa(-2\lambda - 2\mu)(X - \eta(X)\xi) - 2(-2\lambda - 2\mu)h'X \\ -2\eta(X)h'(\mathcal{L}_V \xi) - 2g(h'X, \mathcal{L}_V \xi)\xi - 2(\mathcal{L}_V h')X = 0. \quad (4.11)$$

Taking inner product of the foregoing equation with arbitrary vector field Y on M, we obtain

$$\begin{aligned}(-2\lambda-2\mu)[\kappa(g(X,Y)-\eta(X)\eta(Y))-2g(h'X,Y)]\\ -2\eta(X)g(h'(\mathcal{L}_V\xi),Y)-2g(h'X,\mathcal{L}_V\xi)\eta(Y)-2g((\mathcal{L}_Vh')X,Y)=0.\end{aligned} \tag{4.12}$$

Since the above equation holds for any vector fields X and Y on M, by replacing X by $\phi(X)$ and Y by $\phi(Y)$ and taking (2.5) into account we arrive at

$$(-2\lambda-2\mu)[\kappa g(\phi X,\phi Y)-2g(h'\phi X,\phi Y)]-2g((\mathcal{L}_Vh')\phi X,\phi Y)=0 \tag{4.13}$$

for all $X, Y \in \chi(M)$. Since $spec(h')=\{0,\alpha,-\alpha\}$, let X and V belong to the eigenspaces of $-\alpha$ and α denoted by $[-\alpha]'$ and $[\alpha]'$ respectively. Then $\phi X\in[\alpha]'$ (for more details we refer to [8]). Then (4.13) can be rewritten as

$$(-2\lambda-2\mu)(\kappa-2)g(\phi X,\phi Y)-2g((\mathcal{L}_Vh')\phi X,\phi Y)=0 \tag{4.14}$$

for all $X, Y \in \chi(M)$. It is remained to find the value of $g((\mathcal{L}_Vh')\phi X,\phi Y)$. To get this we prove a more generalized result: In a $(\kappa,\mu)'$-almost Kenmotsu manifold $(\mathcal{L}_Xh')Y=0$, where X and Y belong to same eigenspaces.

Without loss of generality we assume that $X, Y \in [\alpha]'$, where $spec(h')=\{0,\alpha,-\alpha\}$. If we consider a local orthonormal ϕ-basis as $\{\xi, e_i, \phi e_i\}$, $i=1,2,...,n$ then

$$\nabla_XY=\sum_{i=1}^{n}g(\nabla_XY,e_i)e_i-(\alpha+1)g(X,Y)\xi.$$

and

$$\begin{aligned}(\mathcal{L}_Xh')Y&=\mathcal{L}_X(h'Y)-h'(\mathcal{L}_XY)\\ &=\alpha(\mathcal{L}_XY)-h'(\mathcal{L}_XY)\\ &=\alpha(\nabla_XY-\nabla_YX)-h'(\nabla_XY-\nabla_YX)\\ &=\alpha(\alpha+1)g(X,Y)\xi-\alpha(\alpha+1)g(X,Y)\xi\\ &=0.\end{aligned}$$

Similarly we can prove that the above results hold good if $X, Y \in [-\alpha]'$. For more details we refer to [8]. Now (4.14) reduces to

$$(-2\lambda-2\mu)(\kappa-2)g(\phi X,\phi Y)=0 \tag{4.15}$$

for any vector fields X and Y on M. Since by hypothesis $\lambda+\mu\neq 0$, from the foregoing equation we infer that $\kappa=2\alpha$. Again form $\alpha^2=-(\kappa+1)$ we get $\alpha=-1$ and $\kappa=-2$. Plugging the value of κ in (4.1) we have $S^*=0$, i.e., the manifold is Ricci-flat.

Again we get $spec(h') = \{0, 1, -1\}$. From corollary 4.2 of [8] we get M is locally symmetric. From proposition 4.1 of [8] we finally conclude that M is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$, where $\mathbb{H}^{n+1}(-4)$ is the hyperbolic space of constant curvature -4. So, the proof is completed. □

As we know, setting $\mu = 0$ in (1.1) gives rise to the equation of $*$-Ricci soliton, we can revisit the theorem-4.2 and can note the statement as:

Corollary 4.3 *Let $M(\phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost Kenmotsu manifold such that ξ blongs to $(\kappa, -2)'$-nullity distribution where $\kappa < -1$. If the metric g represents a $*$-Ricci soliton satisfying $\lambda \neq 0$ then, M is Ricci-flat and is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

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Certain Results of Conformal and ∗-Conformal Ricci Soliton on Para-Cosymplectic and Para-Kenmotsu Manifolds

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Abstract. The goal of the paper is to deliberate conformal Ricci soliton and ∗-conformal Ricci soliton within the framework of paracontact geometry. Here we prove that if an η-Einstein para-Kenmotsu manifold admits conformal Ricci soliton and ∗-conformal Ricci soliton, then it is Einstein. Further we have shown that 3-dimensional para-cosymplectic manifold is Ricci flat if the manifold satisfies conformal Ricci soliton where the soliton vector field is conformal. We have also constructed some examples of para-Kenmotsu manifold that admits conformal and ∗-conformal Ricci soliton and verify our results.

1. Introduction

The notion of almost paracontact manifold was first introduced by Sato [23]. Later Kaneyuki and Williams [15] associated pseudo-Riemannian metric with an almost paracontact manifold after Takahashi [26] intoduced pseudo- Riemannian metric in contact manifold, in particular, in Sasakian manifold. Zamkovoy in [30] proved that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n+1, n)$. In recent years paracontact geometry has become area of interest for many authors ([5], [18], [16]). On the analogy of Kenmotsu manifold, Welyczko [28] introduced the notion of para-Kenmotsu manifold. Para-Kenmotsu manifold (in short p-Kenmotsu manifold) and special para-Kenmotsu manifold (briefly sp-Kenmotsu manifold) was studied by many authors, namely: Blaga [4], Adigond and Bagewadi [1], Prakasha and Vikas [20], Sinha and Prasad [24] and many others.

A pseudo-Riemannian manifold (M, g) admits a Ricci soliton which is a generalization of Einstein metric if there exists a smooth vector field V and a constant λ such that

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0,$$

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where $\mathcal{L}_V$ denotes Lie derivative along the direction V and S denotes the Ricci curvature tensor of the manifold. The vector field V is called potential vector field and λ is called soliton constant.

The Ricci soliton is a self-similar solution of the Hamilton's Ricci flow [12] which is defined by the geometric evolution equation $\frac{\partial g(t)}{\partial t} = -2S(g(t))$ with initial condition $g(0) = g$ where $g(t)$ is a one-parameter family of metrices on M. The potential vector field V and soliton constant λ play vital roles while determining the nature of the soliton. A soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Now if V is zero or Killing then the Ricci soliton reduces to Einstein manifold and the soliton is called trivial soliton.

In 2005, Fischer [10] has introduced conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation was given by

$$\begin{aligned} \frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) &= -pg, \\ r(g) &= -1, \end{aligned}$$

where $r(g)$ is the scalar curvature of the manifold, p is scalar non-dynamical field and n is the dimension of the manifold. Corresponding to the aforementioned conformal Ricci flow equation, Basu and Bhattacharyya [2] introduced the notion of conformal Ricci soliton equation as a generalization of Ricci soliton equation is given by

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{1}$$

In 2014, Kaimakamis and Panagiotidou [14] modified the definition of Ricci soliton where they have used $*$-Ricci tensor S^* which was introduced by Tachibana [25], in place of Ricci tensor S. The $*$-Ricci tensor S^* is defined by

$$S^*(X, Y) = \frac{1}{2}(trace\{\phi.R(X, \phi Y)\})$$

for all vector fields X and Y on M. They have used the concept of $*$-Ricci soliton within the framework of real hypersurfaces of a complex space form. A pseudo-Riemannian metric g is called a $*$-Ricci soliton if there exists a constant λ and a vector field V such that

$$\mathcal{L}_V g + 2S^* + 2\lambda g = 0.$$

Further Majhi and Dey [17] in 2020 revised the aforementioned definition of $*$-Ricci soliton with the help of (1) and defined $*$-conformal Ricci soliton as

$$\mathcal{L}_V g + 2S^* + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{2}$$

As follows in the literature, Ricci soliton on paracontact geometry studied by many authors ([3], [6], [21]). In particular, Calvaruso and Perrone [6] explicitly studied Ricci soliton on 3-dimensional almost paracontact manifolds. Conformal Ricci solitons have been studied in many contexts: on Kenmotsu manifold [2], on 3- dimensional trans-Sasakian manifold [8], on f-Kenmotsu manifold ([13], [19]) etc. by many authors. In 2018, Ghosh and Patra [11] first studied $*$-Ricci soliton on almost contact metric manifolds. The case of $*$-Ricci soliton in para-Sasakian manifold was treated by Prakasha and Veeresha in [22]. Recently in 2019, Venkatesha, Kumara and Naik [27] considered the metric of η-Einstein para-Kenmotsu manifold as $*$-Ricci soliton and proved that the manifold is Einstein. Erken [9] in 2019 considered Yamabe solitons on 3-dimensional para-cosymplectic manifold and proved some vital results like the manifold is either η-Einstein or Ricci flat.

Motivated by above mentioned works, in this paper, we consider conformal Ricci soliton and $*$-conformal Ricci soliton in the framework of para-Kenmotsu manifold and conformal Ricci soliton in the framework of 3- dimensional para-cosymplectic manifold. We have organized this paper as follows: in first section we look back on some elementary properties of para-Kenmotsu manifolds; in later section first we prove that if a para-Kenmotsu manifold satisfies conformal Ricci soliton then $\mathcal{L}_V \xi$ is orthogonal to ξ or the manifold is Einstein, secondly we prove that an η-Einstein para-kenmotsu manifold is Einstein if it admits a conformal Ricci soliton and then we prove the same for $*$-Conformal Ricci soliton. In the next section, we consider 3-dimensional para-coysmplectic manifold with a conformal Ricci soliton and deduce some relations on the scalar curvature of the manifold and finally, we provide some examples to verify our results.

2. Some preliminaries on para-Kenmotsu manifold

A $(2n+1)$-dimensional smooth manifold M is said to have an almost paracontact structure if it admits a vector field ξ, (1,1)-tensor field ϕ and a 1-form η satisfying the following conditions

$$i)\phi^2 = I - \eta \otimes \xi, \tag{3}$$

$$ii)\eta(\xi) = 1. \tag{4}$$

iii) ϕ induces on the 2n-dimensional distribution $\mathcal{D} \equiv ker(\eta)$, an almost paracomplex structure $\mathcal{P}$ i.e., $\mathcal{P}^2 \equiv I_{\chi(M)}$ and the eigensubbundles $\mathcal{D}^+$ and $\mathcal{D}^-$, corresponding to the eigenvalues $1, -1$ of $\mathcal{P}$ respectively, have equal dimension n; hence $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$.

If a manifold with an almost paracontact structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{5}$$

holds for any $X, Y \in \chi(M)$, then g is called compatible metric and the manifold (M, ϕ, ξ, η, g) is called almost paracontact metric manifold. If an almost paracontact metric manifold satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{6}$$

for arbitrary vector fields X and Y, then the manifold is called almost para-Kenmotsu manifold. The normality of an almost paracontact structure (M, ϕ, ξ, η) is equivalent to vanishing of the (1,2)-torsion tensor defined by $N_\phi(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi$, where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ and is defined by $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ for any $X, Y \in \chi(M)$. A normal almost para-Kenmotsu manifold is called para-Kenmotsu manifold.

The following properties hold on a $(2n+1)$-dimensional para-Kenmotsu manifold

$$\phi(\xi) = 0, \tag{7}$$

$$\eta \circ \phi = 0, \tag{8}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{9}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{10}$$

$$Q\xi = -2n\xi, \tag{11}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{12}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{13}$$

$$(\mathcal{L}_\xi g)(X, Y) = 2[g(X, Y) - \eta(X)\eta(Y)], \tag{14}$$

for any $X, Y \in \chi(M)$ where, $\mathcal{L}$ and ∇ are the operators of Lie differentiation and covariant differentiation of g respectively. Q denotes the Ricci operator associated with the Ricci tensor S defined by $S(X, Y) = g(QX, Y)$ and R denotes the Riemannian curvature tensor.

3. A para-Kenmotsu metric as conformal Ricci soliton

In this section we consider the metric of para-Kenmotsu manifold as a conformal Ricci soliton. The following lemma will be used to prove one of the our main results.

Lemma 3.1. *Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional para-Kenmotsu manifold. Then the Ricci operator satisfies*

$$(\mathcal{L}_\xi Q)X = -2QX - 4nX = (\nabla_\xi Q)X \tag{15}$$

for any vector field X on M.

Proof. From (14), we have $(\mathcal{L}_\xi g)(Y,Z) = 2[g(Y,Z) - \eta(Y)\eta(Z)]$ for all $Y, Z \in \chi(M)$. Covariant derivative of that along an arbitrary vector field X on M and use of the equation (10), leads to

$$(\nabla_X \mathcal{L}_\xi g)(Y,Z) = 2[2\eta(X)\eta(Y)\eta(Z) - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)] \tag{16}$$

for all $Y, Z \in \chi(M)$. Again from Yano [29], we have the following commutation formula

$$(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\mathcal{L}_V \nabla)(X,Y), Z) - g((\mathcal{L}_V \nabla)(X,Z), Y), \tag{17}$$

where g is the metric connection i.e., $\nabla g = 0$. So, the above equation reduces to

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V \nabla)(X,Y), Z) + g((\mathcal{L}_V \nabla)(X,Z), Y). \tag{18}$$

for all vector fields X, Y, Z on M. Combining (16) and (18), we have

$$g((\mathcal{L}_\xi \nabla)(X,Y), Z) + g((\mathcal{L}_\xi \nabla)(X,Z), Y) = 2[2\eta(X)\eta(Y)\eta(Z) - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)].$$

By a straightforward combinatorial computation, the foregoing equation yields

$$(\mathcal{L}_\xi \nabla)(Y,Z) = 2[\eta(Y)\eta(Z)\xi - g(Y,Z)\xi] \tag{19}$$

for all $Y, Z \in \chi(M)$. Taking covariant derivative of the above equation with respect to an arbitrary vector field X on M and using (9) and (10), we have

$$(\nabla_X \mathcal{L}_\xi \nabla)(Y,Z) = 2[g(X,Y)\eta(Z)\xi + g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi - g(Y,Z)X + \eta(Y)\eta(Z)X - 3\eta(X)\eta(Y)\eta(Z)].$$

From Yano [29], we have the well known commutation formula

$$(\mathcal{L}_V R)(X,Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y,Z) - (\nabla_Y \mathcal{L}_V \nabla)(X,Z). \tag{20}$$

From here we can compute

$$(\mathcal{L}_\xi R)(X,Y)Z = 2[g(X,Z)Y - g(Y,Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \tag{21}$$

for all vector fields X, Y, Z on M. Contracting (21) over X we get

$$(\mathcal{L}_\xi S)(Y,Z) = 4n[\eta(Y)\eta(Z) - g(Y,Z)]. \tag{22}$$

The Lie derivative of $S(Y,Z) = g(QY,Z)$ along the direction of ξ, yields

$$(\mathcal{L}_\xi S)(Y,Z) = (\mathcal{L}_\xi g)(QY,Z) + g((\mathcal{L}_\xi Q)Y, Z). \tag{23}$$

On the other hand, replacing X and Y by QY and Z respectively in (14) and using (11), we have

$$(\mathcal{L}_\xi g)(QY,Z) = 2[g(QY,Z) + 2n\eta(Y)\eta(Z)]. \tag{24}$$

Combining (22), (23) and (24) all together, we infer

$$(\mathcal{L}_\xi Q)Y = -2QY - 4nY \tag{25}$$

for any $Y \in \chi(M)$. Again we know that

$$\begin{aligned}(\mathcal{L}_\xi Q)Y &= \mathcal{L}_\xi(QY) - Q(\mathcal{L}_\xi Y)\\ &= \nabla_\xi(QY) - \nabla_{QY}\xi - Q(\nabla_\xi Y) + Q(\nabla_Y \xi)\\ &= (\nabla_\xi Q)Y - \nabla_{QY}\xi + Q(\nabla_Y\xi).\end{aligned}$$

By virtue of (9) and (11) we see that $(\mathcal{L}_\xi Q)Y = (\nabla_\xi Q)Y$ for arbitrary vector field Y. Hence the result is proved. □

Theorem 3.2. *If the metric g of a para-Kenmotsu manifold (M, ϕ, ξ, η, g) of dimension > 3 represents a conformal Ricci soliton then either of the following properties holds:*

i) The Lie derivative of ξ in the direction of the potential vector field V of the soliton i.e., $\mathcal{L}_V\xi$ is orthogonal to ξ.
ii) The manifold is Einstein with Einstein constant $-2n$.

Proof. Let M be a $(2n+1)$ dimensional para-Kenmotsu manifold where $n > 1$. From (12), we have $R(X, \xi)\xi = \eta(X)\xi - X$. Now Lie derivative of the Riemannian curvature along the vector field V, yields

$$(\mathcal{L}_V R)(X, \xi)\xi = ((\mathcal{L}_V\eta)X)\xi - g(X, \mathcal{L}_V\xi)\xi + 2\eta(\mathcal{L}_V\xi)X \tag{26}$$

for all vector fields X on M. Now the covariant derivative of (1) along an arbitrary vector field $Z \in \chi(M)$ provides

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = -2(\nabla_Z S)(X, Y) \tag{27}$$

for any $X, Y \in \chi(M)$. Using (18), we can rewrite (27) as

$$g((\mathcal{L}_V\nabla)(X, Y), Z) + g((\mathcal{L}_V\nabla)(X, Z), Y) = -2(\nabla_Z S)(X, Y).$$

By a straightforward combinatorial computation and using the symmetry of the (1,2)-tensor $\mathcal{L}_V\nabla$, the aforementioned yields

$$g((\mathcal{L}_V\nabla)(X, Y), Z) = (\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(Z, X). \tag{28}$$

Again differentiating the above equation covariantly with respect to an arbitrary vector field X of M and using (9), we can find from (11) that

$$(\nabla_X Q)\xi = -QX - 2nX \tag{29}$$

for all $X \in \chi(M)$. Making use of (15) and (29) and considering $Y = \xi$ in (28), we achieve

$$(\mathcal{L}_V\nabla)(X, \xi) = 2QX + 4nX \tag{30}$$

for any vector field X on M. Now considering covariant derivative of the last equation with respect to an arbitrary vector field Y of M and using (9), we acquire

$$(\nabla_Y \mathcal{L}_V\nabla)(X, \xi) = 2(\nabla_Y Q)X - (\mathcal{L}_V\nabla)(X, Y) + 2\eta(Y)QX + 4n\eta(Y)X. \tag{31}$$

Now letting $Z = \xi$ in (20) and using (31) in the foregoing equation, we have

$$(\mathcal{L}_V R)(X, Y)\xi = 4n[\eta(X)Y - \eta(Y)X] + 2[(\nabla_X Q)Y - (\nabla_Y Q)X] + 2[\eta(X)QY - \eta(Y)QX] \tag{32}$$

for all $X, Y \in \chi(M)$. Considering $Y = \xi$ in the aforementioned equation and using (11) and (15) in it, we obtain

$$(\mathcal{L}_V R)(X, \xi)\xi = 0. \tag{33}$$

Now, taking into account (1), the Lie derivative of $g(\xi, \xi) = 1$ along the direction of V leads to

$$\eta(\mathcal{L}_V \xi) = \lambda - \frac{p}{2} - \frac{1}{2n+1} - 2n. \tag{34}$$

Again, using (11) and letting $Y = \xi$, (1) implies

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = (4n - 2\lambda + p + \frac{2}{2n+1})\eta(X). \tag{35}$$

After using (33), (34) and (35), the equation (26) reduces to

$$(2\lambda - p - 4n - \frac{2}{2n+1})\phi^2 X = 0. \tag{36}$$

Since the last equation holds for any $X \in \chi(M)$, we can conclude that $\lambda = \frac{p}{2} + 2n + \frac{1}{2n+1}$. Using this result in (34) we have, $\eta(\mathcal{L}_V \xi) = 0$. From here the following two cases have arisen

Case-I: $\mathcal{L}_V \xi$ is orthogonal to ξ.

Case-II: $\mathcal{L}_V \xi = 0$ for any vector field X of M. Then additionally using the value of λ, (35) reduces to $(\mathcal{L}_V \eta)X = 0$. Which further can be reduced to $\mathcal{L}_V \eta = 0$, since X is an arbitrary vector field on M.
On other hand, we have a renowned relation (see [29]):

$$(\mathcal{L}_V \nabla)(X, Y) = \mathcal{L}_X \nabla_X Y - \nabla_X \mathcal{L}_V Y - \nabla_{[V,X]} Y, \tag{37}$$

which holds for arbitrary vector fields X and Y of M. Now replacing Y by ξ and using (9) and the relations $\mathcal{L}_V \xi = 0$ and $\mathcal{L}_V \eta = 0$ in the foregoing equation we obtain

$$(\mathcal{L}_V \nabla)(X, \xi) = 0.$$

Finally substituting this in (30), we get $S(X, Y) = -2ng(X, Y)$ for any arbitrary vector fields X and Y on M. From this we can conclude that the manifold is Einstein with Einstein constant $-2n$. □

A (2n+1)-dimensional almost para-Kenmotsu metric manifold is said to be η-Einstein para-Kenmotsu manifold if there exists two smooth functions a and b which satisfies the following relation

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{38}$$

for all $X, Y \in \chi(M)$. Clearly, if $b = 0$ then η-Einstein manifold reduces to Einstein manifold. Now considering $X = Y = \xi$ in the last equation and using (11), we have $a + b = -2n$. Contracting (38) over X and Y we get $r = (2n + 1)a + b$, where r denotes the scalar curvature of the manifold. Solving the last two equations, we get $a = (1 + \frac{r}{2n})$ and $b = -(2n + 1 + \frac{r}{2n})$. Using these values we can rewrite (38) as

$$S(X, Y) = (1 + \frac{r}{2n})g(X, Y) - (2n + 1 + \frac{r}{2n})\eta(X)\eta(Y). \tag{39}$$

Theorem 3.3. *Let M be a (2n+1)-dimensional η-Einstein para-Kenmotsu manifold where $n > 1$. If the metric of the manifold represents a conformal Ricci soliton, then the manifold is Einstein.*

Proof. Let the metric g of an η-Einstein para-Kenmotsu manifold M whose dimension is greater than 3 represents a conformal Ricci soliton. Then clearly it satisfies (1) as well as (39). Combining these two relations, we have

$$(\mathcal{L}_V g)(Y,Z) = (p - 2\lambda - \frac{r}{n} - \frac{4n}{2n+1})g(Y,Z) + (4n + 2 + \frac{r}{n})\eta(Y)\eta(Z) \tag{40}$$

for all $Y, Z \in \chi(M)$. Covariant derivative of (40) with respect to an arbitrary vector field X on M and use of (18), leads to

$$\begin{aligned} g((\mathcal{L}_V\nabla)(X,Y),Z) + g((\mathcal{L}_V\nabla)(X,Z),Y) =& (4n + 2 + \frac{r}{n})[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] \\ &- \frac{Xr}{n}[g(Y,Z) + \eta(Y)\eta(Z)] \end{aligned} \tag{41}$$

for any vector fields X, Y and Z on M. By straightforward computation of the last equation, keeping the symmetry of $(\mathcal{L}_V\nabla)$ in mind, provides

$$\begin{aligned} 2n(\mathcal{L}_V\nabla)(X,Y) \;=\;& (Xr)\eta(Y)\xi - (Xr)Y + (Yr)\eta(X)\xi - (Yr)X + (Dr)g(X,Y) - (Dr)\eta(X)\eta(Y) \\ &+2(4n^2 + 2n + r)[g(X,Y)\xi - \eta(X)\eta(Y)\xi], \end{aligned} \tag{42}$$

where Dr is the gradient of r. Let us consider a local orthonormal basis of the manifold as $\{e_i\}_{i=1}^{2n+1}$. Next, setting $X = Y = e_i$ and summing over $1 \le i \le 2n + 1$ in the last equation, we infer

$$n(\mathcal{L}_V\nabla)(e_i, e_i) = (\xi r)\xi + (n-1)Dr + 2n(4n^2 + 2n + r)\xi. \tag{43}$$

After considering $X = Y = e_i$ and summing over i, (28) reduces to $g((\mathcal{L}_V\nabla)(e_i, e_i), Z) = Zr - \frac{1}{2}Zr - \frac{1}{2}Zr=0$. Since this holds for an arbitrary vector field Z, this can be rewritten as

$$(\mathcal{L}_V\nabla)(e_i, e_i) = 0. \tag{44}$$

Comparing (43) and (44), we get $(\xi r)\xi + (n-1)Dr + 2n(4n^2 + 2n + r) = 0$. Taking inner product with ξ this implies that

$$\xi r = -2(4n^2 + 2n + r). \tag{45}$$

Again it further implies that $Dr = (\xi r)\xi$. Next substituting Y by ξ in (42), we get

$$2n(\mathcal{L}_V\nabla)(X,\xi) = (\xi r)(-X + \eta(X)\xi). \tag{46}$$

Covariant derivative of the foregoing equation with respect to an arbitrary vector field Y and using (9), (10) and (46), leads to

$$2n(\nabla_Y\mathcal{L}_V\nabla)(X,\xi) = (Y(\xi r))(-X+\eta(X)\xi) - 2n(\mathcal{L}_V\nabla)(X,Y) + (\xi r)[g(X,Y)\xi + \eta(X)Y - \eta(Y)X - \eta(X)\eta(Y)\xi]. \tag{47}$$

Using the relation (47) in (20), we achieve

$$2n(\mathcal{L}_V R)(X,Y)\xi = (X(\xi r))(-Y + \eta(Y)\xi) - (Y(\xi r))(-X + \eta(X)\xi) + 2(\xi r)(\eta(Y)X - \eta(X)Y). \tag{48}$$

Contracting this over X, we have $(\mathcal{L}_V S)(Y,\xi) = 0$, where we have used $Dr = (\xi r)\xi$. Finally using $(\mathcal{L}_V S)(Y,\xi) = 0$, (39) and (40) in the Lie derivative of $S(Y,\xi) = -2n\eta(Y)$, we obtain

$$2n\Big(p - 2\lambda - \frac{4n}{2n+1} + 4n + 2\Big)\eta(Y) + \Big(1 + 2n + \frac{r}{2n}\Big)g(Y, \mathcal{L}_V\xi) = \Big(2n + 1 + \frac{r}{2n}\Big)\eta(Y)\eta(\mathcal{L}_V\xi) \tag{49}$$

for any vector field Y on M. Taking $Y = \xi$ in the last equation, we get $\lambda = \frac{p}{2} + 2n + \frac{1}{2n+1}$. Setting $Y = Z = \xi$ in (40) and using the value of λ, we obtain $\eta(\mathcal{L}_V\xi) = 0$. Using these two relations, the equation (49) can be written as

$$(2n(2n+1) + r)\mathcal{L}_V\xi = 0. \tag{50}$$

We suppose $r \neq -2n(2n+1)$ on some open set O of M. Then (50) implies that $\mathcal{L}_V \xi = 0$, which further implies with help of (9) that $\nabla_\xi V = V - \eta(V)\xi$. Using these relations along with (9), (40) and (46) in (37) we obtain $\xi r = 0$. As $Dr = (\xi r)\xi$, so, $Dr = 0$ i.e., the scalar curvature is constant. So, from (45), we can find that $r = -2n(2n+1)$ on O, which is a contradiction to our assumption that $r \neq -2n(2n+1)$ on O. Thus from (50), we can infer $r \neq -2n(2n+1)$ on the entire manifold. Finally from (39), we have $S(X,Y) = -2ng(X,Y)$ for all $X, Y \in \chi(M)$. So, the manifold is Einstein with Einstein constant $-2n$. □

4. A para-Kenmotsu metric as $*$-conformal Ricci soliton

In this section we assume that the metric of para-Kenmotsu manifold represents a $*$-conformal Ricci soliton. Venkatesha, Kumara and Naik[27] have deduced the expression of $*$-Ricci tensor for para-Kenmotsu manifold as

$$S^*(X,Y) = -S(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y) \tag{51}$$

for all vector fields X and Y on M.

Theorem 4.1. *Let $M^{2n+1}(\phi, \xi, \eta, g), n > 1$ be a η-Einstein para-Kenmotsu manifold. If g represents a $*$-conformal Ricci soliton, then the manifold is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof. Let M be a $(2n+1)$-dimensional η-Einstein para-Kenmotsu manifold of dimension > 3 whose metric g represents a $*$-conformal Ricci soliton. So, the relations (2), (39) and (51) are satisfied. Rewriting (2) with the help of the rest two relations, we have

$$(\mathcal{L}_V g)(Y,Z) = (p - 2\lambda + \frac{r}{n} + 4n + \frac{2}{2n+1})g(Y,Z) - (4n + \frac{r}{n})\eta(Y)\eta(Z) \tag{52}$$

for all $Y, Z \in \chi(M)$. Differentiating the above equation with respect to an arbitrary vector field X of M and using (10), we achieve

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = \frac{Xr}{n} g(Y,Z) - \frac{Xr}{n}\eta(Y)\eta(Z) - (4n + \frac{r}{n})[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] \tag{53}$$

for any vector fields X, Y and Z of M. Again from (18), we know $(\nabla_X \mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V \nabla)(X,Y), Z) + g((\mathcal{L}_V \nabla)(X,Z), Y)$. Using this and by a combinatorial computation, keeping in mind that $\mathcal{L}_V \nabla$ is a symmetric operator, the foregoing equation gives

$$\begin{aligned} 2n(\mathcal{L}_V \nabla)(X,Y) &= (Xr)[Y - \eta(Y)\xi] + (Yr)[X - \eta(X)\xi] - (Dr)[g(X,Y) - \eta(X)\eta(Y)] \\ &\quad -2(4n^2 + r)[g(X,Y) - \eta(X)\eta(Y)]\xi. \end{aligned} \tag{54}$$

The covariant derivative of (2) with respect to an arbitrary vector field X, yields

$$(\nabla_X \mathcal{L}_V g)(Y,Z) = -2(\nabla_X S^*)(Y,Z). \tag{55}$$

The straightforward computation and use of the relation (18) in the equation (55), leads to

$$g((\mathcal{L}_V \nabla)(X,Y), Z) = (\nabla_Z S^*)(X,Y) - (\nabla_X S^*)(Y,Z) - (\nabla_Y S^*)(Z,X). \tag{56}$$

Again, taking covariant derivative of (51) with respect to an arbitrary vector field Z of M and then using (10), we get

$$(\nabla_Z S^*)(X,Y) = -(\nabla_Z S)(X,Y) - g(X,Z)\eta(Y) - g(Y,Z)\eta(X) + 2\eta(X)\eta(Y)\eta(Z). \tag{57}$$

Combining (57) with (56), yields

$$g((\mathcal{L}_V \nabla)(X,Y), Z) = (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) - (\nabla_Z S)(X,Y) + 2g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z). \tag{58}$$

Now, let us consider a local orthonormal basis $\{e_i\}_{i=1}^{2n+1}$ of the manifold. Replacing $X = Y = e_i$ in (54), we have

$$2n(\mathcal{L}_V\nabla)(e_i, e_i) = -2(\xi r)\xi - 2(n-1)(Dr) - 4n(4n^2 + r)\xi. \tag{59}$$

Again, substituting X and Y by e_i in equation (58) and summing over i, we get

$$(\mathcal{L}_V\nabla)(e_i, e_i) = 4n\xi. \tag{60}$$

Combining the above two relations we directly have

$$(\xi r)\xi + (n-1)(Dr) + 2n(4n^2 + 2n + r)\xi = 0. \tag{61}$$

The inner product with respect to ξ, reduces the aforementioned equation to $\xi r = -2(2n(2n+1) + r)$. As $n > 1$, using this relation in the equation (61) we easily obtain $Dr = (\xi r)\xi$. After substituting Y by ξ in (54) and using (3), we infer

$$2n(\mathcal{L}_V\nabla)(X, \xi) = (\xi r)\phi^2(X) \tag{62}$$

for all $X \in \chi(M)$. Differentiating (62) with respect to an arbitrary vector field Y and using (9), (10) and (62), we get

$$2n(\nabla_Y\mathcal{L}_V\nabla)(X, \xi) + 2n(\mathcal{L}_V\nabla)(X, Y) = (Y(\xi r))\phi^2X - (\xi r)[g(X, Y)\xi + \eta(X)Y - \eta(Y)X - \eta(X)\eta(Y)\xi]. \tag{63}$$

Using this in the well known formula (20), we have

$$2n(\mathcal{L}_VR)(X, Y)\xi = (X(\xi r))\phi^2Y - (Y(\xi r))\phi^2X - 2(\xi r)[\eta(Y)X - \eta(X)Y] \tag{64}$$

for all $X, Y \in \chi(M)$. Contracting the above equation over X and using the relation $Dr = (\xi r)\xi$, we have $(\mathcal{L}_VS)(Y, \xi) = 0$. Using (39), (52) and $(\mathcal{L}_VS)(Y, \xi) = 0$ in the Lie derivative of $S(Y, \xi) = -2n\eta(Y)$, we get

$$2n\Big(p - 2\lambda + \frac{2}{2n+1}\Big)\eta(Y) + \Big(2n + 1 + \frac{r}{2n}\Big)[g(Y, \mathcal{L}_V\xi) - \eta(Y)\eta(\mathcal{L}_V\xi)] = 0. \tag{65}$$

In the last equation considering $Y = \xi$, we obtain $\lambda = \frac{p}{2} + \frac{1}{2n+1}$ as $n > 1$. Again setting $Y = Z = \xi$ in (52), we have $\eta(\mathcal{L}_V\xi) = 0$. Applying these relations, we can rewrite (65) as

$$(2n(2n+1) + r)\mathcal{L}_V\xi = 0. \tag{66}$$

We suppose $r \neq -2n(2n+1)$ on some open set O of M. Then from (66), directly we obtain $\mathcal{L}_V\xi = 0$. From (9), we deduce that $\nabla_\xi V = V - \eta(V)\xi$. Again taking $Z = \xi$ in (52) and using $\lambda = \frac{p}{2} + \frac{1}{2n+1}$, we have $\mathcal{L}_V\eta = 0$. Using these relations along with (9) and (62) in the identity (37), we obtain $\xi r = 0$. As $Dr = (\xi r)\xi$, so, $Dr = 0$ i.e., the scalar curvature r is constant. So, from the relation $\xi r = -2(2n(2n+1) + r)$, we can find that $r = -2n(2n+1)$ on O, which is a contradiction to our assumption that $r \neq -2n(2n+1)$ on O. Thus from (66), we can conclude that $r = -2n(2n+1)$ on the entire manifold M. Moreover from (39), we have $S(X, Y) = -2ng(X, Y)$ for all $X, Y \in \chi(M)$. So, the manifold is Einstein with Einstein constant $-2n$. □

5. A 3-dimensional para-cosymplectic metric as conformal Ricci soliton

In 2004, Dacko [7] introduced the notion of para-cosymplectic manifold. The fundamental 2-form Φ is defined on an almost paracontact metric manifold (M, ϕ, ξ, η, g) by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M. Clearly the skew-symmetricness of the 2-form Φ inherits from ϕ.
An almost paracontact metric manifold is said to be almost para- coymplectic if the forms η and Φ are closed, i.e., $d\eta = 0$ and $d\Phi = 0$ respectively. In addition if the normality of almost para-cosymplectic manifold is fulfilled then the it is called para-cosymplectic manifold. Equivalently we can say an almost paracontact

metric manifold is para-cosymplectic if the forms η and Φ are parallel with respect to the corresponding Levi-Civita connection ∇ of the metric g i.e., $\nabla\eta = 0$ and $\nabla\Phi = 0$ respectively. We recall some useful relations which are satisfied for any para-cosymplectic manifold.

$$R(X,Y)\xi = 0, \tag{67}$$
$$(\nabla_X\phi) = 0, \tag{68}$$
$$\nabla_X\xi = 0, \tag{69}$$
$$S(X,\xi) = 0, \tag{70}$$
$$Q\xi = 0, \tag{71}$$

where X is an arbitrary vector field and R, ∇, S and Q are the usual notations. For the 3-dimensional case, we have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{72}$$

Using this result we deduce that 3-dimensional para-cosymplectic manifold satisfies

$$S(X,Y) = \frac{r}{2}[g(X,Y) - \eta(X)\eta(Y)], \tag{73}$$
$$QX = \frac{r}{2}[X - \eta(X)\xi] \tag{74}$$

for any $X, Y \in \chi(M)$.

A vector field V is said to be conformal Killing vector field or simply conformal vector field if there is a smooth function ρ such that

$$\mathcal{L}_V g = 2\rho g. \tag{75}$$

ρ is called the conformal coefficient. If we consider the conformal coefficient ρ to be zero then the conformal vector field reduces to Killing vector field. Now we first prove some lemmas whose results are used to deduce our main result.

Lemma 5.1 ([29]). *If a n-dimensional Riemannian manifold admits a conformal vector field V then we have*

$$(\mathcal{L}_V S)(X,Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X,Y), \tag{76}$$
$$\mathcal{L}_V r = 2(n-1)\Delta\rho - 2\rho r \tag{77}$$

for any vector fields X and Y, where D and Δ denote the gradient and Laplacian operator of g respectively and r represents the scalar curvature of the manifold.

Lemma 5.2. *If the metric g of a 3-dimensional para-cosymplectic manifold represents a conformal Ricci soliton then the following properties hold*

$$\eta(\mathcal{L}_V\xi) = \lambda - \frac{p}{2} - \frac{1}{3}, \tag{78}$$
$$(\mathcal{L}_V\eta)\xi = -\lambda + \frac{p}{2} + \frac{1}{3}. \tag{79}$$

Proof. As the vector field ξ is a unit vector field, we have $g(\xi,\xi) = 1$. Taking Lie derivative of the previous relation with respect to vector field V, we have $(\mathcal{L}_V g)(\xi,\xi) + 2\eta(\mathcal{L}_V\xi) = 0$. Using (1), (4) and (73), we acquire

$$\eta(\mathcal{L}_V\xi) = \lambda - \frac{p}{2} - \frac{1}{3}.$$

Taking Lie derivative of (4) along the direction of the vector field V and using (78), we achieve

$$(\mathcal{L}_V\eta)\xi = -\lambda + \frac{p}{2} + \frac{1}{3}.$$

□

Lemma 5.3. *For a 3-dimensional para-cosymplectic manifold, we have*

$$\xi(r) = 0. \tag{80}$$

Proof. For proof we refer to [9]. □

Theorem 5.4. *If the metric g of a 3-dimensional para-cosymplectic manifold $(M^3, \phi, \xi, \eta, g)$ which admits a conformal vector field V, represents a conformal Ricci soliton then the scalar curvature of the manifold is Harmonic and the manifold is Ricci flat.*

Proof. Combining (1) and (75) for 3-dimensional para-cosymplectic manifold, we have

$$\Big(2\rho + 2\lambda - p - \frac{2}{3}\Big)g(X,Y) + 2S(X,Y) = 0$$

for any $X, Y \in \chi(M)$. Contracting the above equation, we get

$$\rho = \frac{1}{6}(3p - 6\lambda - 2r + 2). \tag{81}$$

Using (81) in (76) and (77), we get

$$(\mathcal{L}_V S)(X,Y) = \frac{1}{3}g(\nabla_X Dr, Y) - \frac{1}{3}(\Delta r)g(X,Y), \tag{82}$$

$$\mathcal{L}_V r = -\frac{1}{3}(3p - 6\lambda - 2r + 2)r - \frac{4}{3}(\Delta r). \tag{83}$$

Taking Lie derivative of (73) in the direction of the vector field V and using (1), (73), (82) and (83), we have

$$g(\nabla_X Dr, Y) = -\Big(\Delta r + \frac{r^2}{2}\Big)g(X,Y) + \Big[\frac{r}{2}(3p - 6\lambda + r + 2) + 2(\Delta r)\Big]\eta(X)\eta(Y) - \frac{3r}{2}\Big[((\mathcal{L}_V\eta)X)\eta(Y) + \eta(X)((\mathcal{L}_V\eta)Y)\Big]. \tag{84}$$

Covariant derivative of (80) along an arbitrary vector field X, yields $g(\nabla_X Dr, \xi) = 0$. Now setting $X = Y = \xi$ in the equation (84) and using the aforementioned relation along with the equation (79), we get

$$\Delta r = 0. \tag{85}$$

Hence the scalar curvature r of the manifold is Harmonic.
Now considering $Y = \xi$ in (84) and using the relation $g(\nabla_X Dr, \xi) = 0$, (85), (79), we obtain the following relation

$$r((\mathcal{L}_V\eta)X) = r\Big(\frac{p}{2} + \frac{1}{3} - \lambda\Big)\eta(X) \tag{86}$$

for an arbitrary vector field X on M. Making use of the last equation, (74) and (85) in (84), we achieve

$$\nabla_X Dr = -rQX \tag{87}$$

for any arbitrary $X \in \chi(M)$. Now contracting it with respect to X, we get $\Delta r = -r^2$ and combining with (85), we infer $r = 0$ i.e., the manifold is Ricci flat. □

6. Examples

In this section we provide some examples to verify our outcomes.

Example 6.1. *We consider the manifold as* $M = \{(x, y, x) \in \mathbb{R}^3\}$, *where* (x, y, z) *are the standard coordinates in* $\mathbb{R}^3$. *The vector fields are defined by*

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point on M. *The metric* g *is defined by*

$$g(e_1, e_1) = g(e_3, e_3) = 1, \qquad g(e_2, e_2) = -1, \qquad g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let $\xi = e_3$. *Then the 1-form* η *is defined by* $\eta(X) = g(X, e_3)$, *for arbitrary* $X \in \chi(M)$, *then we have the following relations*

$$\eta(e_1) = 0, \qquad \eta(e_2) = 0, \qquad \eta(e_3) = 1.$$

Let us define the (1,1)-tensor field ϕ *as*

$$\phi e_2 = e_1, \qquad \phi e_1 = e_2, \qquad \phi e_3 = 0,$$

then it satisfies

$$\begin{aligned} \phi^2(X) &= X - \eta(X)e_3, \\ g(\phi X, \phi Y) &= -g(X, Y) + \eta(X)\eta(Y) \end{aligned}$$

for arbitrary $X, Y \in \chi(M)$. *Thus* (ϕ, ξ, η, g) *defines an almost paracontact metric structure on* M. *We can now easily conclude*

$$[e_1, e_2] = 0, \qquad [e_2, e_3] = e_2, \qquad [e_1, e_3] = e_1.$$

Let ∇ *be the Levi-Civita connection of* g. *Then the Koszul's formula for arbitrary* $X, Y, Z \in \chi(M)$ *is given by*

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using this we obtain

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = e_1, \\ &\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = e_3, && \nabla_{e_2} e_3 = e_2, \\ &\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0. \end{aligned}$$

From here we can easily verify that the relation (6) is satisfied. Hence the considered manifold is para-Kenmotsu manifold. The components of the Riemannian curvature tensor are given by

$$\begin{aligned} &R(e_1, e_2)e_1 = e_2, && R(e_1, e_2)e_2 = e_1, && R(e_1, e_2)e_3 = 0, \\ &R(e_1, e_3)e_1 = e_3, && R(e_1, e_3)e_2 = 0, && R(e_1, e_3)e_3 = -e_1, \\ &R(e_2, e_3)e_1 = 0, && R(e_2, e_3)e_2 = -e_3, && R(e_2, e_3)e_3 = -e_2. \end{aligned}$$

And the components of Ricci tensor and $*$*-Ricci tensor are given by*

$$\begin{aligned} &S(e_1, e_1) = -2, && S(e_2, e_2) = 2, && S(e_3, e_3) = -2, \\ &S^*(e_1, e_1) = 1, && S^*(e_2, e_2) = -1, && S^*(e_3, e_3) = 0. \end{aligned}$$

From here we can easily deduce that the scalar curvature of the manifold $r = -6$ *and* $S(X, Y) = -2g(X, Y)\ \forall X, Y \in \chi(M)$. *Let us define a vector field by*

$$V = (x - 1)\frac{\partial}{\partial x} + (y - 1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Then we can obtain

$$(\mathcal{L}_V g)(e_1, e_1) = 2, \qquad (\mathcal{L}_V g)(e_2, e_2) = -2, \qquad (\mathcal{L}_V g)(e_3, e_3) = 0.$$

Contracting (1) and using the result $r = -6$ *we deduce* $\lambda = \frac{p}{2} + \frac{19}{3}$. *So* g *defines a conformal Ricci soliton on this para-Kenmotsu manifold for* $\lambda = \frac{p}{2} + \frac{19}{3}$.
Again Contracting (51) we get, $r^* = -r - 4 = 2$ *(as* $r = -6$*). Now contracting (2) and using the previous result we obtain* $\lambda = \frac{p}{2} - \frac{5}{3}$. *So,* g *defines a* $*$*-conformal Ricci soliton on this para-Kenmotsu manifold for* $\lambda = \frac{p}{2} - \frac{5}{3}$.

Example 6.2. *Let us consider the set* $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ *as our manifold where* (x, y, z, u, v) *are the standard coordinates in* $\mathbb{R}^5$. *The vector fields defined below*

$$e_1 = e^{-v}\frac{\partial}{\partial x}, \qquad e_2 = e^{-v}\frac{\partial}{\partial y}, \qquad e_3 = e^{-v}\frac{\partial}{\partial z}, \qquad e_4 = e^{-v}\frac{\partial}{\partial u}, \qquad e_5 = \frac{\partial}{\partial v},$$

are linearly independent at each point of M. We define the metric g *as*

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i, j \in \{1, 2, 5\} \\ -1, & \text{if } i = j \text{ and } i, j \in \{3, 4\} \\ 0, & \text{otherwise.} \end{cases}$$

Let η *be a 1-form defined by* $\eta(X) = g(X, e_5)$, *for arbitrary* $X \in \chi(M)$. *Let us define (1,1)-tensor field* ϕ *as*

$$\phi(e_1) = e_3, \qquad \phi(e_2) = e_4, \qquad \phi(e_3) = e_1, \qquad \phi(e_4) = e_2, \qquad \phi(e_5) = 0.$$

Then it satisfies the relations $\phi^2(X) = X - \eta(X)\xi$ *and* $\eta(\xi) = 1$, *where* $\xi = e_5$ *and X is an arbitrary vector field on M. So,* (M, ϕ, ξ, η, g) *defines an almost paracontact structure on M.*
We can now deduce that

$$\begin{array}{llll} [e_1, e_2] = 0, & [e_1, e_3] = 0, & [e_1, e_4] = 0, & [e_1, e_5] = e_1, \\ [e_2, e_1] = 0, & [e_2, e_3] = 0, & [e_2, e_4] = 0, & [e_2, e_5] = e_2, \\ [e_3, e_1] = 0, & [e_3, e_2] = 0, & [e_3, e_4] = 0, & [e_3, e_5] = e_3, \\ [e_4, e_1] = 0, & [e_4, e_2] = 0, & [e_4, e_3] = 0, & [e_4, e_5] = e_4, \\ [e_5, e_1] = -e_1, & [e_5, e_2] = -e_2, & [e_5, e_3], = -e_3, & [e_5, e_4] = -e_4. \end{array}$$

Let ∇ *be the Levi-Civita connection of* g. *Then Koszul's formula is given by*

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

for arbitrary $X, Y, Z \in \chi(M)$. *Using this we get*

$$\begin{array}{lllll} \nabla_{e_1} e_1 = -e_5, & \nabla_{e_1} e_2, = 0 & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = e_5, & \nabla_{e_4} e_5 = e_4, \\ \nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0. \end{array}$$

Therefore $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ *is satisfied for arbitrary* $X, Y \in \chi(M)$. *So* (M, ϕ, ξ, η, g) *is an almost para-Kenmotsu manifold. The previous outcomes can easily be verified using this example.*

7. Conclusion

In this article, we have used the methods of local Riemannian or semi-Riemannian geometry to interpretation solutions of (1) and (2) and impregnate Einstein metrics in a large class of metrics of conformal Ricci solitons and $*$-conformal Ricci solitons on paracontact geometry, specially on para-Kenmotsu and para-cosymplectic manifold. Our results will not only play an indispensable and incitement role in paracontact geometry but also it has significant and motivational contribution in the area of further research of complex geometry, specially on Kähler and para-Kähler manifold etc. and we can think about the physical interpretation of conformal Ricci solitons and $*$-conformal Ricci solitons also in differential geometry. There are some questions which arise from our article to study in further research:

(i) Are the results of theorem 3.2 and theorem 3.3 true if we assume the dimension of the manifold as 3?
(ii) Does theorem 4.1 hold without assuming η-Einstein condition?
(iii) If we consider the dimension more than 3, then is theorem 5.4 true?
(iv) What can we say about theorem 5.4 if we assume vector field V is not conformal?
(v) Which results of the our paper are also true in nearly Kenmotsu manifolds or f-Kenmotsu manifolds or f-cosymplectic manifolds?

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CONFORMAL η-RICCI SOLITON ON A TYPE OF $(\text{LCS})_N$ MANIFOLD

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Abstract. An n-dimensional Lorentzian concircular structural manifold (in short $(\text{LCS})_n$ manifold) has enormous applications in Mathematical Physics as it has Lorentzian metric g as well as a contact form η. In this note we have established some results regarding conformal η-Ricci soliton and conformal Ricci soliton on $(\text{LCS})_n$ manifold satisfying some curvature conditions like ξ-conharmonically semi-symmetric, ξ-concircularly semi-symmetric and ξ-Quasi-conformally semi-symmetric and obtained the nature of the soliton as well as the nature of the structural vector field ξ.

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Keywords: Ricci flow, conformal η-Ricci soliton, conformal Ricci soliton, $(\text{LCS})_n$ manifold.

1. Introduction

Richard S. Hamilton introduced the concept of Ricci flow (for details see [17]) which was named after great Italian mathematician Gregorio Ricci-Curbastro. If we take a smooth closed (compact without boundary) Riemannian manifold M equipped with a smooth Riemannian metric g then the Ricci flow is defined by the geometric evolution equation,

$$\frac{\partial g(t)}{\partial t} = -2S(g(t)) \tag{1}$$

where S is the Ricci curvature tensor of the manifold and $g(t)$ is a one-parameter family of metrices on M.

A Riemannian manifold (M, g) is called a Ricci soliton if there exists a vector field V and a constant λ such that the following equation holds,

$$\frac{1}{2}\mathcal{L}_V g + S + \lambda g = 0 \tag{2}$$

where $\mathcal{L}_V$ denotes Lie derivative along the direction V and λ is a non-zero constant. The vector field V is called potential vector field and λ is called soliton constant. Ricci soliton which is a natural extension of Einstein manifold is a self-similar solution of Ricci flow. The potential vector field V and soliton constant λ play vital roles while determining the nature of the soliton. A soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Now if V is Killing then the Ricci soliton reduces to Einstein manifold. Compact Ricci solitons are the fixed points of the Ricci flow (1.1) projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds.

In 2005, A. E. Fischer [2] has introduced conformal Ricci flow which is a variation of the classical Ricci flow equation (1.1) that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation was given by,

$$\begin{aligned} \frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) &= -pg \\ r(g) &= -1 \end{aligned} \tag{3}$$

where $r(g)$ is the scalar curvature of the manifold, p is scalar non-dynamical field and n is the dimension of the manifold. Corresponding to the aforementioned conformal Ricci flow equation N. Basu and A. Bhattacharyya [15] introduced the notion of conformal Ricci soliton equation as a generalization of Ricci soliton equation given by,

$$\mathcal{L}_V g + 2S + [2\lambda - (p + \frac{2}{n})]g = 0. \tag{4}$$

In 2009, J. T. Cho and M. Kimura [11] introduced the concept of η-Ricci soliton which is another generalization of classical Ricci soliton and is given by,

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0 \tag{5}$$

where μ is a real constant, η is a 1-form defined as $\eta(X) = g(X, V)$ for any $X \in \chi(M)$. Clearly it can be noted that if $\mu = 0$ then the η-Ricci soliton (g, V, λ, μ) reduces to Ricci soliton.

Recently Md. D. Siddiqi [14] established the notion of conformal η-Ricci soliton which generalizes both conformal Ricci soliton and η-Ricci soliton. The equation for conformal η-Ricci soliton is given by,

$$\mathcal{L}_X g + 2S + [2\lambda - (p + \frac{2}{n})]g + 2\mu\eta \otimes \eta = 0. \tag{6}$$

If we take $\mu = 0$ in (1.6) then it reduces to conformal Ricci soliton (1.4).

Ricci solitons have been studied in many contexts: on Kähler manifolds[16], on contact and Lorentzian manifolds [5],[6], on K-contact manifolds [18] etc. by many authors. Nagaraja and Premalatha [10] studied the nature of Ricci soliton on Kenmotsu manifold; Călin and Crasmareanu [7] on f-Kenmotsu manifold; He and Zhu [8] on Sasakian manifold; Ingalahalli and Bagewadi [9] on α-Sasakian manifold; Y. Wang [24] on 3-dimensional cosymplectic manifold and S. Pahan and A. Bhattacharyya on 3-dimensional trans-Sasakian manifold [21].In 2016, T. Dutta, N. Basu and A. Bhattacharyya studied conformal Ricci soliton on 3-dimensional trans-Sasakian manifold[23].

S. R. Ashoka, C. S. Bagewadi and G. Ingalahalli [22] gave some insight on Ricci soliton in $(\mathrm{LCS})_n$ manifold. Many authors have developed several results on many context of $(\mathrm{LCS})_n$ manifolds like: Yadav, Chaubey, Suthar[20]; Hui and Chakraborty [19]; Baishya [12]; Blaga [3] etc. on η-Ricci Soliton. Chaubey and Siddiqi have studied almost conformal η-Ricci solitons in 3-dimensional $(\mathrm{LCS})_3$ manifolds.

Motivated from above mentioned well praised works we have studied behaviour of conformal η-Ricci soliton on n-dimensional Lorentzian concircular structure manifold (briefly $(\mathrm{LCS})_n$ manifold) satisfying certain curvature properties such as ξ-conharmonically semi-symmetric, ξ-concircularly semi-symmetric and ξ-Quasi-conformally semi-symmetric which are represented by,

$$R(\xi, X).H = 0, \qquad R(\xi, X).C = 0, \qquad R(\xi, X).\tilde{C} = 0$$

respectively. In the later section we have revisited some definitions and important properties of $(\mathrm{LCS})_n$ manifold and there after the main results of this paper have been described.

2. Some preliminaries on $(\mathrm{LCS})_n$ manifold

The notion of Lorentzian concircular structure manifold, briefly $(\mathrm{LCS})_n$ manifold is first introduced in 2003 by Shaikh (for details see [1]). An n-dimensional smooth connected paracontact Hausdorff manifold is called a *Lorentzian manifold* if it admits a Lorentzian metric. Lorentzian metric is named after great Dutch Physicist Hendrik Lorentz. A *Lorentzian metric tensor* g is a smooth symmetric tensor field of type (0,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a non degenerate inner product of signature $(-,+,......,+)$, where T_pM is the tangent space of M at p and $\mathbb{R}$ is the real number space. A non-zero tangent vector $v \in T_pM$ is said to be *timelike*, *non-spacelike*, *null* or *spacelike* if it satisfies $g_p(v, v) < 0$, ≤ 0, $= 0$ or > 0 respectively.

In a Lorentzian manifold (M, g) a vector field ρ is defined by $g(X, \rho) = \eta(X)$, is said to be *concircular vector field* if,

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \omega(X)\eta(Y)\} \tag{7}$$

is satisfied where α is a non-zero scalar field, ω is a closed 1-form and ∇ is the covariant derivative operator w.r.t. Lorentzian metric g.

Let M be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called *charecteristic vector field* or the *generator* of the manifold, then we have,

$$g(\xi,\xi) = -1. \tag{8}$$

Since ξ is a concircular vector field there must exists a non-zero 1-form η, such that,

$$g(X,\xi) = \eta(X) \tag{9}$$

$$(\nabla_X \eta)(Y) = \alpha g(X,Y) + \eta(X)\eta(Y) \tag{10}$$

hold for arbitrary vector fields $X, Y \in \chi(M)$ and α is a non-zero scalar field which satisfies,

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X) \tag{11}$$

ρ being certain scalar function which is given by $\rho = -(\xi\alpha)$. If we define $\phi X = \frac{1}{\alpha}\nabla_X \xi$, then from (10) we can deduce that,

$$\phi X = X + \eta(X)\xi. \tag{12}$$

Clearly ϕ is a symmetric (1,1) tensor which is called *structure tensor* of the manifold. Thus the n-dimensional Lorentzian manifold M together with the unit timelike concircular vector field ξ, 1-form η and (1,1) tensor ϕ is said to be Lorentzian concircular structure (briefly $(\mathrm{LCS})_n$) manifold. If we take $\alpha = 1$, then the manifold reduces to LP-Sasakian manifold of Matsumoto [13].

A $(\mathrm{LCS})_n$ manifold satisfies the following properties,

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \eta\circ\phi = 0, \quad \phi\xi = 0 \tag{13}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{14}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) \tag{15}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)(\eta(Y)X - \eta(X)Y) \tag{16}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)(g(X,Y)\xi - \eta(Y)X) \tag{17}$$

$$(\mathcal{L}_\xi g)(X,Y) = 2\alpha(g(X,Y) + \eta(X)\eta(Y)) \tag{18}$$

where R is the Riemannian curvature tensor. Furthermore if (g, V, λ, p, μ) is a conformal η-Ricci soliton then we can deduce the following,

$$S(X,Y) = (\frac{p}{2} + \frac{1}{n} - \lambda - \alpha)g(X,Y) - (\mu + \alpha)\eta(X)\eta(Y) \quad (19)$$

$$QX = (\frac{p}{2} + \frac{1}{n} - \lambda - \alpha)X - (\mu + \alpha)\eta(X)\xi \quad (20)$$

$$r = (\frac{p}{2} + \frac{1}{n} - \lambda - \alpha)n + (\mu + \alpha) \quad (21)$$

where S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature of the manifold. We now want to recall some useful definitions [4],

Definition 1. *A vector field ξ is called $torseforming$ if it satisfies*

$$\nabla_X \xi = fX + \gamma(X)\xi \quad (22)$$

for a smooth function $f \in C^{\infty}(M)$, 1-form γ and for all vector field X on M. A torse forming vector field is called $recurrent$ if $f = 0$.

3. Main results

Theorem 1. *A conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-conharmonically semi-symmetric curvature property, satisfies the following properties,*

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

b) ξ is a geodesic vector field,

c) $\nabla_\xi S = 0$ and $\nabla_\xi Q = 0$.

Proof. The conharmonic curvature tensor H is defined by,

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \quad (23)$$

Now taking inner product w.r.t. ξ and using (15), (19) and (20) we have,

$$\eta(H(X,Y)Z) = (\alpha^2 - \rho - \frac{p}{(n-2)} - \frac{2}{n(n-2)} + \frac{2\lambda}{(n-2)} + \frac{\alpha}{(n-2)} - \frac{\mu}{(n-2)})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \quad (24)$$

Here we have considered ξ-conharmonically semi-symmetric curvature property, i.e., $R(\xi, X).H = 0$, which yields,

$$R(\xi,X)H(Y,Z)W - H(R(\xi,X)Y,Z)W - H(Y,R(\xi,X)Z)W - H(Y,Z)R(\xi,X)W = 0. \quad (25)$$

Applying (17) in the above equation we have,

$$g(X,H(Y,Z)W)\xi - \eta(H(Y,Z)W)X - g(X,Y)H(\xi,Z)W + \eta(Y)H(X,Z)W - g(X,Z)H(Y,\xi)W + \eta(Z)H(Y,X)W - g(X,W)H(Y,Z)\xi + \eta(W)H(Y,Z)W = 0. \tag{26}$$

By taking inner product of the previous equation with ξ we get,

$$g(X,H(Y,Z)W) + \eta(H(Y,Z)W)\eta(X) + g(X,Y)\eta(H(\xi,Z)W) - \eta(Y)\eta(H(X,Z)W) + g(X,Z)\eta(H(Y,\xi)W) - \eta(Z)\eta(H(Y,X)W) + g(X,W)\eta(H(Y,Z)\xi) - \eta(W)\eta(H(Y,Z)W) = 0. \tag{27}$$

After using (24) the equation reduces to,

$$g(X,H(Y,Z)W) + (\alpha^2 - \rho - \frac{p}{(n-2)} - \frac{2}{n(n-2)} + \frac{2\lambda}{(n-2)} + \frac{\alpha}{(n-2)} - \frac{\mu}{(n-2)})(g(X,Z)g(Y,W) - g(Z,W)g(X,Y)) = 0. \tag{28}$$

Let us consider the set $\{e_i\}_{i=1}^{n}$ as a basis of the manifold. Then replacing $X = Y = e_i$ in the above equation yields,

$$\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}. \tag{29}$$

Hence (a) is proved.

Now considering $X = \xi$ we can rewrite (6) as,

$$g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) + 2S(Y,Z) + [2\lambda - (p + \frac{2}{n})]g(Y,Z) + 2\mu\eta(Y)\eta(Z) = 0 \tag{30}$$

for all $Y, Z \in \chi(M)$. Simplifying using (19), the above equation reduces to,

$$g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) - 2\alpha[g(Y,Z) + \eta(Y)\eta(Z)] = 0. \tag{31}$$

Considering $Z = \xi$ in the above equation, we get,

$$g(\nabla_\xi \xi, Y) = 0. \tag{32}$$

Since the aforementioned relation holds for any $Y \in \chi(M)$, so $\nabla_\xi \xi = 0$. This concludes that ξ is a geodesic vector field. Thus (b) is proved.

Taking covariant derivative of (19) and (20) we can find the general expressions of ∇S and ∇Q as,

$$(\nabla_X S)(Y,Z) = -(\mu + \alpha)[g(Y, \nabla_X \xi)\eta(Z) + g(Z, \nabla_X \xi)\eta(Y)] \tag{33}$$

$$(\nabla_X Q)Y = -(\mu + \alpha)[g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi] \tag{34}$$

for any $Y, Z \in \chi(M)$. Letting $X = \xi$ in (33) and (34) we get, $\nabla_\xi S = 0$ and $\nabla_\xi Q = 0$. It completes our results.

Theorem 2. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), satisfying ξ-conharmonically semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. Let ξ be a torse forming vector field. Then we have from (22) that $\nabla_X \xi = fX + \gamma(X)\xi$, for a smooth function $f \in C^\infty(M)$, 1-form γ and for all vector field X on M. Taking inner product w.r.t. ξ it yields,

$$g(\nabla_X \xi, \xi) = f\eta(X) - \gamma(X).$$

Hence we get $f\eta = \gamma$. After applying this result, (22) becomes,

$$\nabla_X \xi = f[X + \eta(X)\xi]. \tag{35}$$

Applying (35) in (31) we get,

$$2(f - \alpha)[g(Y, Z) - \eta(Y)\eta(Z)] = 0,$$

for all vector fields Y and Z and hence we get $f = \alpha$. Thus (35) reduces to,

$$\nabla_X \xi = \alpha[X + \eta(X)\xi] = \alpha\phi^2(X), \tag{36}$$

i.e., $\nabla_X \xi$ is collinear to $\phi^2(X)$ for all X. Hence we get $d\eta = 0$, which means that η is colsed.

Now let us consider ξ to be recurrent vector field. So, $f = \alpha = 0$. Thus (35) yields that ξ is a concurrent vector field i.e., $\nabla_X \xi = 0$ for all vector field X on M. Also we have,

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0 \tag{37}$$

for all X and Y on M. Thus we can conclude ξ is Killing vector field.

Remark 1. *We know conformal η-Ricci soliton reduces to conformal Ricci soliton if we consider μ to be zero in (6). Accordingly the results of theorem 1 change while the results of theorem 2 remain the same for conformal Ricci soliton. We can state the modified results of theorem 1 as:*

A conformal Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-conharmonically semi-symmetric curvature property, satisfies the following properties,

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,
b) ξ is a geodesic vector field,
c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Theorem 3. *A conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-concircularly semi-symmetric curvature property satisfies the following properties,*

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

b) ξ *is a geodesic vector field,*

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Proof. The concircular curvature tensor C is defined by,

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{38}$$

Now taking inner product w.r.t. ξ and using (15) we have,

$$\eta(C(X,Y)Z) = (\alpha^2 - \rho - \frac{r}{n(n-1)})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \tag{39}$$

Here we have considered ξ-concircularly semi-symmetric curvature property i.e., $R(\xi, X).C = 0$, which yields,

$$R(\xi,X)C(Y,Z)W - C(R(\xi,X)Y,Z)W - C(Y,R(\xi,X)Z)W - C(Y,Z)R(\xi,X)W = 0. \tag{40}$$

Applying (17) in the above equation we have,

$$\begin{aligned}&g(X,C(Y,Z)W)\xi - \eta(C(Y,Z)W)X - g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W - \\ &g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W - g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)W = 0.\end{aligned} \tag{41}$$

By taking inner product in the previous equation with ξ we get,

$$\begin{aligned}&g(X,C(Y,Z)W) + \eta(C(Y,Z)W)\eta(X) + g(X,Y)\eta(C(\xi,Z)W) - \\ &\eta(Y)\eta(C(X,Z)W) + g(X,Z)\eta(C(Y,\xi)W) - \eta(Z)\eta(C(Y,X)W) \\ &+g(X,W)\eta(C(Y,Z)\xi) - \eta(W)\eta(C(Y,Z)W) = 0.\end{aligned} \tag{42}$$

After using (33) the equation reduces to,

$$g(X,C(Y,Z)W) + (\alpha^2 - \rho - \frac{r}{n(n-1)})(g(X,Z)g(Y,W) - g(Z,W)g(X,Y)) = 0. \tag{43}$$

Then replacing $X = Y = e_i$ in the above equation, where the set $\{e_i\}_{i=1}^n$ is a basis of the manifold, yields,

$$[\frac{p}{2} + \frac{1}{n} - \lambda - \alpha - (n-1)(\alpha^2 - \rho)]g(Z,W) - (\mu + \alpha)\eta(Z)\eta(W) = 0. \tag{44}$$

Since this holds for arbitrary $Z, W \in \chi(M)$, setting $Z = W = \xi$ we have,

$$\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}. \tag{45}$$

This proves (a) and the expression is identical with (29). Other two outcomes (b) and (c) are immediate consequences and can be proved similarly like theorem 1.

Theorem 4. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), satisfying ξ-concircularly semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. Since the results of theorem 3 for concircular curvature tensor are same as of theorem 1 for conharmonic curvature tensor, the proof of this theorem is identical with the proof of theorem 2.

Remark 2. *We know conformal η-Ricci soliton is a mere generalisation conformal Ricci soliton. If we let μ to be zero in (6) then it reduces to conformal Ricci soliton. The results of theorem 3 change while the results of theorem 4 remain the same for conformal Ricci soliton. We can modify the results of theorem 3 as:*

A conformal Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), satisfying ξ-concircularly semi-symmetric curvature property, satisfies the following properties,

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,

b) ξ *is a geodesic vector field,*

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Theorem 5. *A conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), satisfying ξ-Quasi-conformally semi-symmetric curvature property, admits the following properties,*

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}$,

b) ξ *is a geodesic vector field,*

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Proof. The Quasi-conformal curvature tensor $\tilde{C}$ is defined by,

$$\begin{aligned} \tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n}\left(\frac{a}{(n-1)} + 2b\right)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{46}$$

where a and b are non-zero constants. Now taking inner product w.r.t. ξ and using (15) we have,

$$\begin{aligned} \eta(\tilde{C}(X,Y)Z) &= [a(\alpha^2 - \rho) + b(p + \frac{2}{n} - 2\lambda - \alpha + \mu) - \frac{r}{n}(\frac{a}{n-1} + 2b)] \\ &\quad (g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \end{aligned} \tag{47}$$

Here we have considered ξ-Quasi-conformally semi-symmetric curvature property i.e., $R(\xi, X).\tilde{C} = 0$, which yields,

$$R(\xi, X)\tilde{C}(Y, Z)W - \tilde{C}(R(\xi, X)Y, Z)W - \tilde{C}(Y, R(\xi, X)Z)W - \tilde{C}(Y, Z)R(\xi, X)W = 0. \tag{48}$$

Applying (17) in the above equation we have,

$$\begin{aligned} &g(X, \tilde{C}(Y, Z)W)\xi - \eta(\tilde{C}(Y, Z)W)X - g(X, Y)\tilde{C}(\xi, Z)W + \eta(Y)\tilde{C}(X, Z)W - \\ &g(X, Z)\tilde{C}(Y, \xi)W + \eta(Z)\tilde{C}(Y, X)W - g(X, W)\tilde{C}(Y, Z)\xi + \eta(W)\tilde{C}(Y, Z)W = 0. \end{aligned} \tag{49}$$

By taking inner product in the previous equation w.r.t. ξ we get,

$$\begin{aligned} &g(X, \tilde{C}(Y, Z)W) + \eta(\tilde{C}(Y, Z)W)\eta(X) + g(X, Y)\eta(\tilde{C}(\xi, Z)W) - \\ &\eta(Y)\eta(\tilde{C}(X, Z)W) + g(X, Z)\eta(\tilde{C}(Y, \xi)W) - \eta(Z)\eta(\tilde{C}(Y, X)W) \\ &+g(X, W)\eta(\tilde{C}(Y, Z)\xi) - \eta(W)\eta(\tilde{C}(Y, Z)W) = 0. \end{aligned} \tag{50}$$

After using (43) the above equation reduces to,

$$\begin{aligned} &g(X, \tilde{C}(Y, Z)W) + [a(\alpha^2 - \rho) + b(p + \frac{2}{n} - 2\lambda - \alpha + \mu) - \frac{r}{n}(\frac{a}{n-1} + 2b)] \\ &(g(X, Z)g(Y, W) - g(Z, W)g(X, Y)) = 0. \end{aligned} \tag{51}$$

Then replacing $X = Y = e_i$ in the above equation, where the set $\{e_i\}_{i=1}^n$ is a basis of the manifold, yields,

$$\begin{aligned} &[a + (n-2)b]S(Z, W) + [br - \frac{r}{n}(a + 2(n-1)b)]g(Z, W) + [a(\alpha^2 - \rho) + \\ &b(p + \frac{2}{n} - 2\lambda - \alpha + \mu) - \frac{r}{n}(\frac{a}{n-1} + 2b)]g(Z, W) = 0. \end{aligned} \tag{52}$$

Since this holds for arbitrary $Z, W \in \chi(M)$, setting $Z = W = \xi$ we have,

$$\lambda + (n-1)\alpha^2 = \frac{p}{2} + \mu + (n-1)\rho + \frac{1}{n}. \tag{53}$$

Hence (a) is proved.

(b) and (c) can be proved in similar manner like the proof of theorem 1.

Theorem 6. *If ξ is a torse forming conformal η-Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-Quasi-conformally semi-symmetric curvature property, then η is colsed. Furthermore if ξ is a recurrent torse forming vector field then it is Killing vector field.*

Proof. The proof can be done in similar fashion like theorem 2.

Remark 3. *To get conformal Ricci soliton from conformal η-Ricci soliton we assume $\mu = 0$ in (6). Consequently the results of theorem 5 change while the results of theorem 6 remain unchanged for conformal Ricci soliton. We can revise the results of theorem 5 as:*

A conformal Ricci soliton in $(LCS)_n$ manifold, say (M, g, ξ, η, ϕ), admitting ξ-Quasi-conformally semi-symmetric curvature property satisfies the following properties,

a) $\lambda + (n-1)\alpha^2 = \frac{p}{2} + (n-1)\rho + \frac{1}{n}$,

b) ξ *is a geodesic vector field,*

c) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

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