EX/ET/MATH/T/113/2019(Old)

B. ETCE 1st year 1st sem. 2019 (Old)
Mathematics-I G

Time : Three hours Full Marks : 100

Group-A
(Answer any five questions)

1. (i) If $y = a\cos(\log x) + b\sin(\log x)$ then prove that $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$. 5

(ii) If a function f is such that its derivative f' is continuous in $[a, b]$ and derivable in (a, b) then prove that there exists a number $c(a < c < b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c)$$

5

2. (i) Show that $\frac{x}{\sin x}$ increases monotonically from $x = 0$ to $x = \frac{\pi}{2}$. 5

(ii) Show that $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$. 5

3. (i)Find the maximum value of $\sin x(1+\cos x)$. 5

(ii) Show that of all rectangles of given area, the square has the smallest perimeter. 5

4. (i) Determine $\lim\limits_{x\to 0} \frac{x-\log(1+x)}{1-\cos x}$. 5

(ii) Find the asymptotes of the curve $x^3 + y^3 - 3axy = 0$. 5

5. Prove that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \ldots.$ Justify the convergence of the series. 10

6. (i) Prove that $\lim\limits_{n\to\infty} \frac{\{(n+1)(n+2)\ldots(2n)\}^{\frac{1}{n}}}{n} = \frac{4}{e}$. 5

(ii) Examine the convergence of the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots, x > 0$. 5

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7. (i) Let $f_n(x) = x^2 e^{-nx}, x \in [0, \infty)$. Show that $f_n(x)$ is uniformly convergent on $[0, \infty)$. 5

(ii) Prove that the series $\sum \frac{1}{n^3+n^4x^2}$ is uniformly convergent for all real x. 5

Group-B
(Answer any five questions)

8. Evaluate $\lim_{x\to 0} \lim_{y\to 0} f(x,y)$, $\lim_{y\to 0} \lim_{x\to 0} f(x,y)$ and $\lim_{(x,y)\to(0,0)} f(x,y)$, if they exist, for the following functions:

$$\begin{aligned} (i)\ f(x,y) &= \frac{x-y}{x+y}, \quad x+y \neq 0 \\ &= 0, \quad\quad x+y=0 \\ (ii)\ f(x,y) &= \frac{xy}{x^2+y^2} + y\sin\frac{1}{x}, \quad xy \neq 0 \\ &= 0, \quad\quad x=0,\ y=0 \end{aligned}$$

5+5

9. (i) Verify whether f is continuous at $(0,0)$, where

$$\begin{aligned} f(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0) \\ &= 0, \quad\quad (x,y) = (0,0) \end{aligned}$$

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(ii) If $u = f(x,y)$ and $x = r\cos\theta,\ y = r\sin\theta$ then prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

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10. (i) Verify whether f is continuous at $(0,0)$, where

$$\begin{aligned} f(x,y) &= \frac{x^2y^3}{x^2+y^2}, \quad (x,y) \neq (0,0) \\ &= 0, \quad\quad (x,y) = (0,0) \end{aligned}$$

(ii). Find the directional derivative of the function $f(x,y,z) = \log(x^2 + 2y^2 + z^2)$ at the point $(2,1,1)$ in the direction of $(-1,2,3)$. 5

11. (i) If $\sin u = \frac{x^2+y^2}{x+y}$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

5

(ii) Find the equation of the tangent plane to the surface defined by $z = \sqrt{x^2+y^2}$ at the point P(1, 2). 5

12. (i)If $u = \cos x, \quad v = \sin x \cos y, \quad w = \sin x \sin y \cos z$ then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -\sin^3 x \sin^2 y \sin z.$$

5

(ii) If $z = \frac{\sin u}{\cos v}, \quad u = \frac{\cos y}{\sin x}, \quad v = \frac{\cos x}{\sin y}$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. 5

13. Find and classify the extreme values (if any) of the function defined as : 10

$$f(x,y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$$

14. Find the maxima amd minima of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. 10