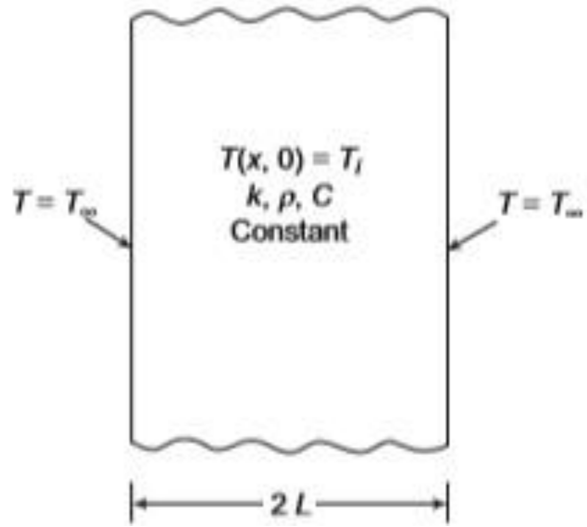
ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

Chapter - 8

Transient one-dimensional problem



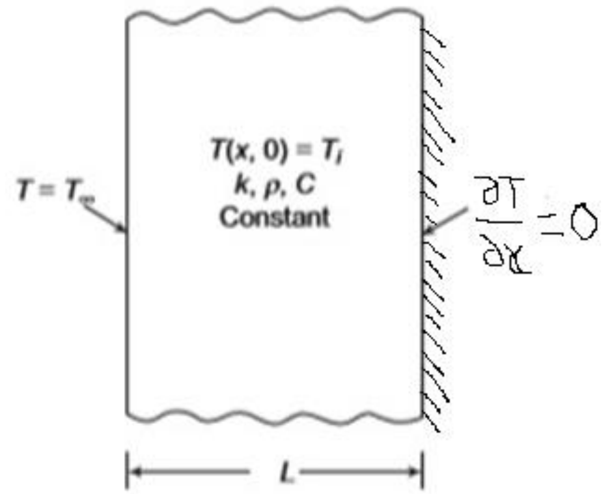
✓The case of a hot infinite plate of finite thickness 2L, is suddenly exposed to an atmosphere of temperature $T = T_\infty$.

✓The initial temperature is $T = T_i$.

✓ Heat transfer coefficient is large.

We may choose the consideration of symmetry.

Consideration of symmetry



Dimensionless form of governing equation

The energy equation in dimensionless form for constant thermophysical properties,

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2}$$

Where, $$\theta = \frac{T - T_\infty}{T_i - T_\infty}, \quad X = \frac{x}{L}, \tau = \frac{\alpha t}{L^2}$$

Initial and boundary conditions

$$IC:\ at\ \tau = 0,\quad \theta = 1,\ for\ all\ X$$

$$for\ \tau > 0,$$

$$BC1:\ at\ X = 0,\ \theta = 0$$

$$BC2:\ at\ X = 1,\ \frac{\partial \theta}{\partial X} = 0$$

Discretization

For any interior grid point, the FDM formulation gives

$$\frac{\partial \theta}{\partial \tau} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2} \qquad for\ \ i = 1, 2, \ldots\ldots, M$$

At the boundary point, *i*=*M*, we employ image-point technique to obtain,

$$\theta_{M+1} = \theta_{M-1}$$

$$\frac{\partial \theta_M}{\partial \tau} = \frac{2\theta_{M-1} - 2\theta_M}{(\Delta X)^2}$$

Methods of solution

There are three ways by which the initial-value problem can be solved. These are

(i) Euler method (or explicit method)

(ii) Crank–Nicolson method

(iii) Pure implicit method

Euler method (or explicit method)

Solution of temperature at the present time τ^p *is* θ^p

We seek the solution for the temperature at the future time, τ^{p+1} *is* θ^{p+1} with an increment in time as

$$\tau^{p+1} = \tau^p + \Delta\tau$$

The solution at the future time is obtained as

$$\theta^{p+1} = \theta^p + \left(\frac{d\theta}{d\tau}\right)^p \Delta\tau$$

Therefore, the discretized form,

$$\frac{\partial\theta}{\partial\tau} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2}$$

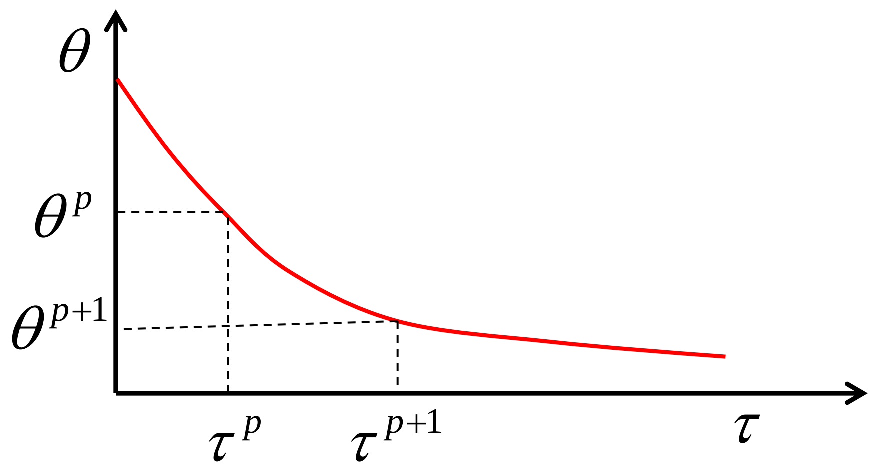
$$\Rightarrow \frac{\theta_i^{p+1} - \theta_i^p}{\Delta\tau} = \frac{\theta_{i+1}^p - 2\theta_i^p + \theta_{i-1}^p}{(\Delta X)^2}$$

$$\therefore \theta_i^{p+1} = \left\{\frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_{i+1}^p + \left(1 - \frac{2\Delta\tau}{(\Delta X)^2}\right)\theta_i^p + \left\{\frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_{i-1}^p$$

This method has some disadvantage as the solution becomes unstable when,

$$\left(1 - \frac{2\Delta\tau}{(\Delta X)^2}\right) \le 0$$

Stability criteria, restriction on time-step



Crank–Nicolson method

Solution of temperature at the present time τ^p is θ^p

We seek the solution for the temperature at the future time, τ^{p+1} is θ^{p+1} with an increment in time is assumed as an arithmetic mean value of the derivates between the beginning and end of the time interval as

$$\theta^{p+1} = \theta^p + \frac{1}{2}\left[\left(\frac{d\theta}{d\tau}\right)^p + \left(\frac{d\theta}{d\tau}\right)^{p+1}\right]\Delta\tau$$

Therefore, the discretized form,

$$\frac{\partial\theta}{\partial\tau} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2}$$

$$\Rightarrow \frac{\theta_i^{p+1} - \theta_i^p}{\Delta\tau} = \frac{1}{2}\left[\frac{\theta_{i+1}^p - 2\theta_i^p + \theta_{i-1}^p}{(\Delta X)^2} + \frac{\theta_{i+1}^{p+1} - 2\theta_i^{p+1} + \theta_{i-1}^{p+1}}{(\Delta X)^2}\right]$$

$$\therefore \left\{1 + \frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_i^{p+1} = \left\{\frac{\Delta\tau}{2(\Delta X)^2}\right\}\theta_{i+1}^p + \left(1 + \frac{\Delta\tau}{(\Delta X)^2}\right)\theta_i^p$$

$$+ \left\{\frac{\Delta\tau}{2(\Delta X)^2}\right\}\theta_{i-1}^p + \left\{\frac{\Delta\tau}{2(\Delta X)^2}\right\}\left(\frac{\theta_{i+1}^{p+1} + \theta_{i-1}^{p+1}}{(\Delta X)^2}\right)$$

The Crank–Nicolson method is **unconditionally stable** and therefore no restriction in the time step.

Pure Implicit method:

Solution of temperature at the present time τ^{p} *is* θ^{p}
We seek the solution for the temperature at the future time, τ^{p+1} *is* θ^{p+1} is assumed that the time derivates at the new time is used to move ahead with the time. It can be expressed as

$$\theta^{p+1} = \theta^{p} + \left(\frac{d\theta}{d\tau}\right)^{p+1} \Delta\tau$$

Therefore, the discretized form,

$$\frac{\partial\theta}{\partial\tau} = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2}$$

$$\Rightarrow \frac{\theta_i^{p+1} - \theta_i^{p}}{\Delta\tau} = \frac{\theta_{i+1}^{p+1} - 2\theta_i^{p+1} + \theta_{i-1}^{p+1}}{(\Delta X)^2}$$

$$\therefore \left[1 + \frac{2\Delta\tau}{(\Delta X)^2}\right]\theta_i^{p+1} = \theta_i^{p} + \left\{\frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_{i+1}^{p+1} + \left\{\frac{\Delta\tau}{(\Delta X)^2}\right\}\theta_{i-1}^{p+1}$$

The implicit method is **unconditionally stable** and therefore no restriction in the time step.

Accuracy of the methods

Euler method (or explicit method): FTCS

Space accuracy: $(\Delta X)^2$

Time accuracy: $(\Delta\tau)$

Crank–Nicolson method: CTCS

Space accuracy: $(\Delta X)^2$

Time accuracy: $(\Delta\tau)^2$

Pure Implicit method: BTCS

Space accuracy: $(\Delta X)^2$

Time accuracy: $(\Delta\tau)$