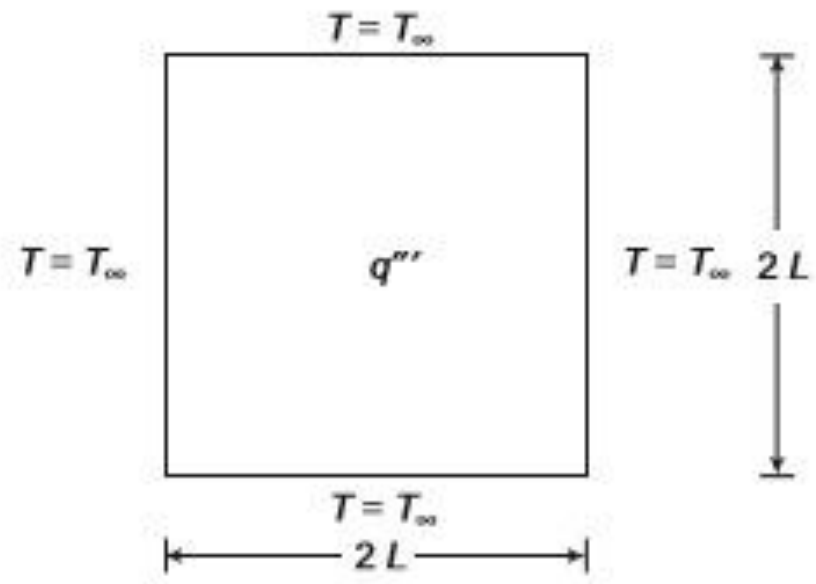
ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

Chapter - 7

Two-dimensional steady-state problem

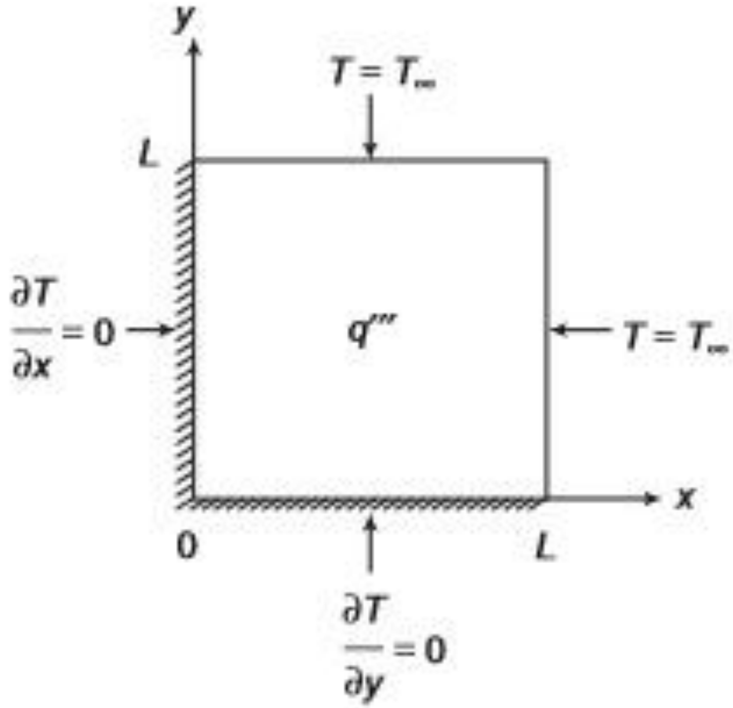


✓The case of steady heat conduction in a long square slab *(2L × 2L*) in which heat is generated at a uniform rate of q''' W/m3.

✓The problem can be assumed to be a two dimensional as the dimension of the slab is much longer in the direction normal to the cross-sectional plane; therefore, end effects can be neglected.

✓ All four sides are maintained at $T = T_{\infty}$, temperature of the surrounding fluid, assuming a large heat transfer coefficient.

Consideration of symmetry



➢A close look at the physics of the problem reveals that the problem is geometrically and thermally symmetric.

➢ Therefore, from the temperature distribution in any quarter of the physical domain by mirror-imaging, one can get the solution for the entire region.

➢The use of symmetry enables the numerical analyst to obtain the solution much faster as the number of grid points is greatly reduced.

Governing differential equation

The energy equation at the steady state (assuming constant k)

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+q^{'''}=0$$

Boundary conditions

Boundary conditions are

$$BC1:\ at\ \ x=0,\ \ \frac{\partial T}{\partial x}=0$$

$$BC2:\ at\ \ x=L,\ \ T=T_\infty$$

$$BC3:\ at\ \ y=0,\ \ \frac{\partial T}{\partial y}=0$$

$$BC4:\ at\ \ y=L,\ \ T=T_\infty$$

Dimensionless form

$$\theta=\frac{T-T_\infty}{\left(q^{'''}L^2/k\right)},\ \ X=\frac{x}{L},Y=\frac{y}{L}$$

Non-dimensionalizing using the dimensionless variables:

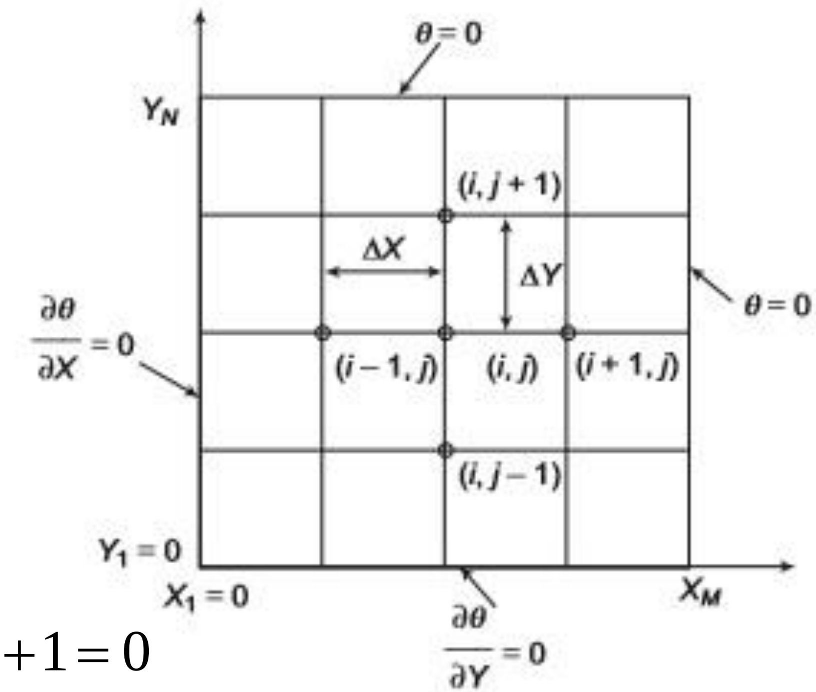
$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+1=0$$

$$BC1:\ at\ \ X=0,\ \ \frac{\partial\theta}{\partial X}=0$$

$$BC2:\ at\ \ X=1,\ \ \theta=0$$

$$BC3:\ at\ Y=0,\ \ \frac{\partial\theta}{\partial Y}=0$$

$$BC4:\ at\ Y=1,\ \ \theta=0$$



Discretization

The equation is discretized at any interior grid point (*i,j)* using central difference as follows:

$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+1=0$$

$$\Rightarrow \frac{\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{\left(\Delta X\right)^2}+\frac{\theta_{i,j+1}-2\theta_{i,j}+\theta_{i,j-1}}{\left(\Delta Y\right)^2}+1=0$$

For a uniform grid, ΔX=ΔY

$$\Rightarrow -\theta_{i-1,j}-\theta_{i,j-1}+4\theta_{i,j}-\theta_{i,j+1}-\theta_{i+1,j}=\left(\Delta X\right)^2 \qquad \text{(a)}$$

❖**Boundary condition along X = 0:**

Using image point technique $\theta_{i-1,\,j} = \theta_{i+1,\,j}$

Setting i=1 in Eq. (a), we have

$$-2\theta_{2,\,j} - \theta_{1,\,j-1} + 4\theta_{1,\,j} - \theta_{1,\,j+1} = \left(\Delta X\right)^2$$

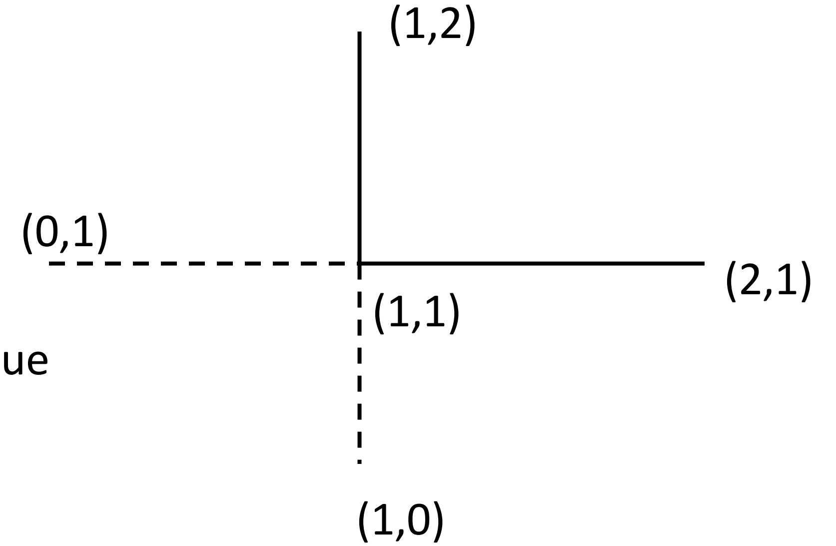
❖**Boundary condition along Y= 0:**

Using image point technique $\theta_{i,\,j-1} = \theta_{i,\,j+1}$

Setting j=1 in Eq. (a), we have

$$-\theta_{i+1,1} - 2\theta_{i,2} + 4\theta_{i,1} - \theta_{i-1,1} = \left(\Delta X\right)^2$$

Handling of corner points



Using image point technique

$$\theta_{0,1} = \theta_{2,1}$$

$$\theta_{1,0} = \theta_{1,2}$$

$$\left(\frac{\partial^2\theta}{\partial X^2}\right)_{1,1} + \left(\frac{\partial^2\theta}{\partial Y^2}\right)_{1,1} + 1 = 0$$

$$\Rightarrow \frac{\theta_{2,1} - 2\theta_{1,1} + \theta_{0,1}}{(\Delta X)^2} + \frac{\theta_{1,2} - 2\theta_{1,1} + \theta_{1,0}}{(\Delta Y)^2} + 1 = 0$$

$$\Rightarrow \frac{\theta_{2,1} - 2\theta_{1,1} + \theta_{2,1}}{(\Delta X)^2} + \frac{\theta_{1,2} - 2\theta_{1,1} + \theta_{1,2}}{(\Delta Y)^2} + 1 = 0$$

For a uniform grid, $\Delta X = \Delta Y$

$$2\theta_{2,1} - 4\theta_{1,1} + 2\theta_{1,2} + (\Delta X)^2 = 0$$

Solution method   Gauss-Seidel method

Equations:

$$-\theta_{i-1,j}-\theta_{i,j-1}+4\theta_{i,j}-\theta_{i,j+1}-\theta_{i+1,j}=(\Delta X)^2$$

$$-2\theta_{2,j}-\theta_{1,j-1}+4\theta_{1,j}-\theta_{1,j+1}=(\Delta X)^2$$

$$-\theta_{i+1,1}-2\theta_{i,2}+4\theta_{i,1}-\theta_{i-1,1}=(\Delta X)^2$$

$$2\theta_{2,1}-4\theta_{1,1}+2\theta_{1,2}+(\Delta X)^2=0$$

Initial guess,

$for\ i=1,M,\ J=1,N$

$\theta_{i,j}=0$

Pseudo code:

$for\ i=1,\ j=1$

$$\theta_{1,1}=\frac{1}{4}\left(2\theta_{2,1}+2\theta_{1,2}+(\Delta X)^2\right)$$

$for\ i=M,\ j=2,N-1$

$$\theta_{1,j}=\frac{1}{4}\left[(\Delta X)^2+2\theta_{2,j}+\theta_{1,j-1}+\theta_{1,j+1}\right]$$

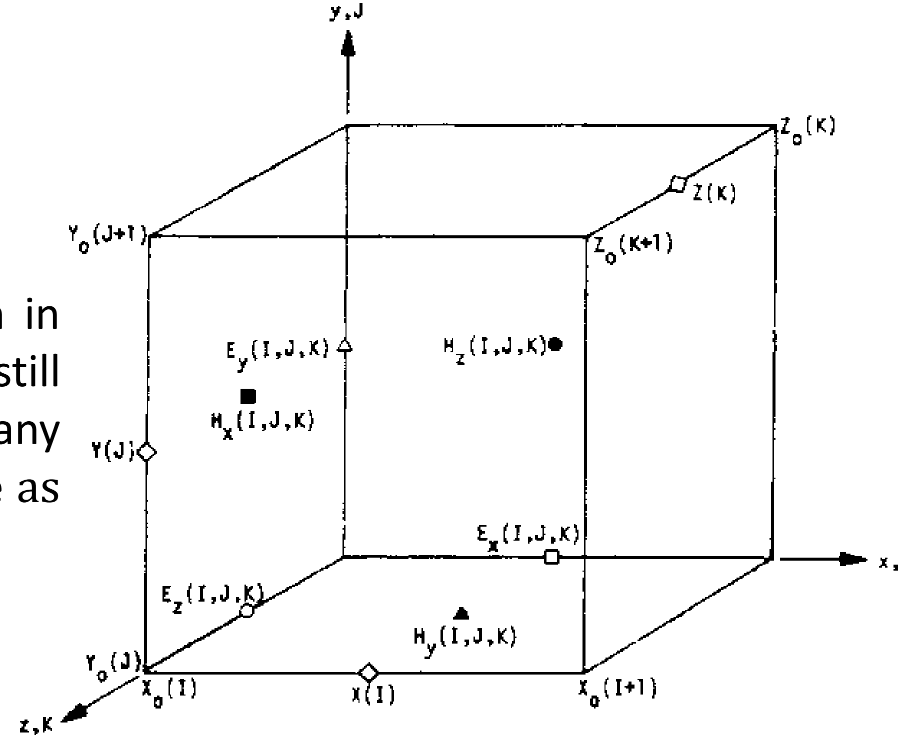
$for\ i=2,M-1,\ j=1$

$$\theta_{i,1}=\frac{1}{4}\left[(\Delta X)^2+\theta_{i+1,1}+2\theta_{i,2}+\theta_{i-1,1}\right]$$

$for\ i=2,M-1,\ j=2,N-1$

$$\theta_{i,j}=\frac{1}{4}\left[(\Delta X)^2+\theta_{i-1,j}+\theta_{i,j-1}+\theta_{i,j+1}+\theta_{i+1,j}\right]$$

Three-dimensional problems



For three-dimensional steady heat conduction in Cartesian coordinates, the basic approach is still the same. The equation is discretized at any interior grid point (*i,j,k)* using central difference as follows:

$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+\frac{\partial^2\theta}{\partial Z^2}1=0$$

$$\Rightarrow \frac{\theta_{i+1,j,k}-2\theta_{i,j,k}+\theta_{i-1,j,k}}{\left(\Delta X\right)^2}+\frac{\theta_{i,j+1,k}-2\theta_{i,j,k}+\theta_{i,j-1,k}}{\left(\Delta Y\right)^2}+\frac{\theta_{i,j,k+1}-2\theta_{i,j,k}+\theta_{i,j,k-1}}{\left(\Delta Z\right)^2}+1=0$$

For a uniform grid, ΔX=ΔY= ΔZ

$$\Rightarrow \theta_{i+1,j,k}+\theta_{i-1,j,k}+\theta_{i,j+1,k}+\theta_{i,j-1,k}+\theta_{i,j,k+1}+\theta_{i,j,k-1}-6\theta_{i,j,k}+\left(\Delta X\right)^2=0$$