ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

Chapter - 5

Numerical solution methods

Numerical methods for the diffusion problem: Conduction heat transfer

Steady and unsteady conduction

All conduction processes are divided broadly into two categories: **steady and unsteady**.

Steady state means that temperature, density, etc., at all points of the conduction region is independent of time.

Unsteady state means a change with time, usually only of the temperature. Unsteady state problems can be further split into two categories: **periodic and transient**.

Daily variation of earth's temperature due to solar effects exemplifies a typical **periodic** heat conduction problem. The immersion of hot steel plate in a cold quenching tank is an example of **transient** conduction. Transient periodic heat conduction is also not uncommon.

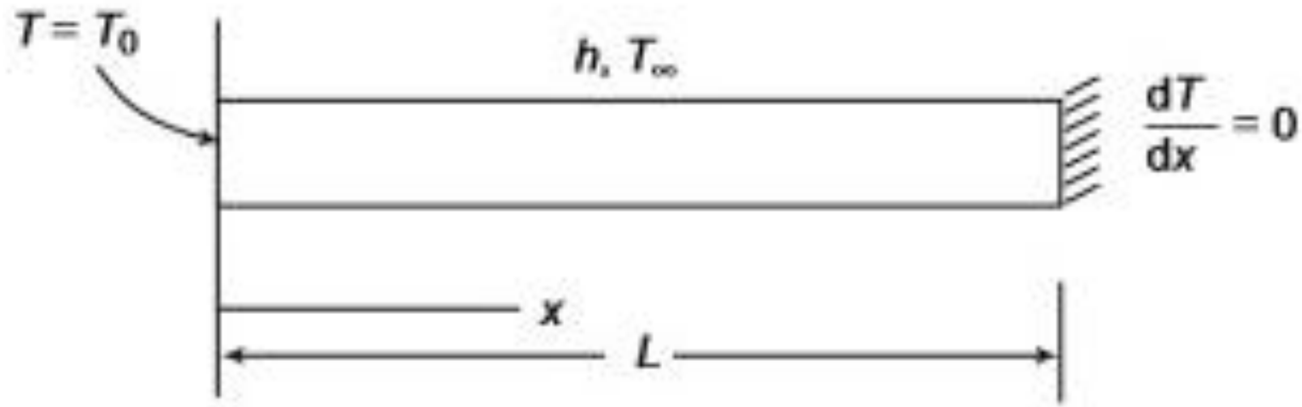
Dimensionality in conduction

Depending on the physical processes involved, it may be one, two, or three dimensional.

Basic approach in numerical heat conduction

Many difficult problems arise in conduction, for example, variable thermal conductivity, distributed energy sources, etc. for which analytical solutions are not available. Approximate solution is then obtained by **numerical method**. The basic approach is to arrive at the relevant governing differential equation based on the physics of the particular problem.

Step by step numerical solution of one dimensional steady conduction problem



Problem definition:

✓ Consider the one-dimensional steady-state heat conduction in an isolated rectangular horizontal fin. The base temperature is maintained at *T = T0* and the tip of the fin is insulated.

✓ The fin is exposed to a convective environment (neglecting radiation heat transfer from the fin), which is at *T∞ (T∞ < T0).* The average heat transfer coefficient of the fin to the ambient is h. The length of the fin is L, and the coordinate axis begins at the base of the fin.

✓ The one dimensionality arises from the fact that thickness of the fin is much small as compared to its length, and width can be considered either too long or the sides of the fin to be insulated.

Governing differential equation

The energy equation for the fin at the steady state (assuming constant k)

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}\left(T - T_\infty\right) = 0$$

where P= perimeter and A = cross-sectional area of the fin

Boundary conditions

Since the above equation is a linear, second-order ordinary differential equation, two boundary conditions are needed to completely describe this problem (which is a boundary value problem or elliptic problem). Boundary conditions are

$$BC1: \ at \ x = 0, \quad T = T_0$$

$$BC2: \ at \ x = L, \ \frac{dT}{dx} = 0$$

Dimensionless form

Non-dimensionalizing using the dimensionless variables: $\theta = \dfrac{T - T_\infty}{T_0 - T_\infty}, \ X = \dfrac{x}{L}$

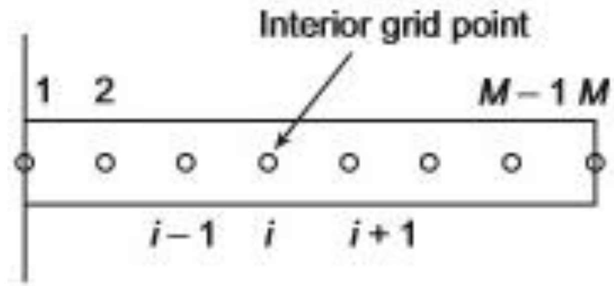
Hence, $$\frac{d^2\theta}{dX^2} - (mL)^2 \theta = 0$$ Where, $m^2 = hP/kA$

And $$BC1: \ at \ X = 0, \quad \theta = 1$$

$$BC2: \ at \ x = 1, \ \frac{d\theta}{dX} = 0$$

Discretization

The equation given below is discretized at any interior grid point *i* using central difference for $d^2\Theta/dX^2$ as follows:



$$\frac{d^2\theta}{dX^2} - (mL)^2\theta = 0$$

$$\Rightarrow \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta x)^2} - (mL)^2\theta_i = 0$$

$$\therefore \theta_{i-1} - D\theta_i + \theta_{i+1} = 0 \qquad \text{for } i = 1, 2, \ldots\ldots, M$$

$$\text{where, } D = 2 + (mL)^2(\Delta x)^2$$

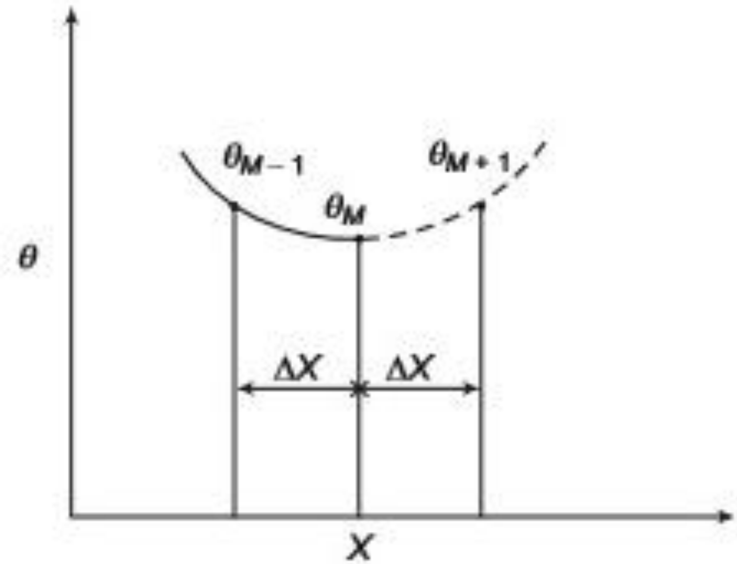
Handling of the boundary condition: At *x = L, i.e., at i = M,* reduces to:

$$\theta_{M-1} - D\theta_M + \theta_{M+1} = 0$$

By observing above equation it is revealed that Θ_{M+1} represents a fictitious temperature Θ at point M + 1, which lies outside the computational domain.

Method 1: image point technique

It is assumed that *Θ vs. X* curve extends beyond *X = 1* so that at *X = 1,* the condition *dΘ/dX = 0* is satisfied.



The dotted line represents the mirror-image extension of the solid line, indicating that a minima exists at X = 1. Mirror-image extension of Θ vs. X curve near the fin tip.

At point M

$$\left(\frac{d\theta}{dX}\right)_M = 0$$

$$\Rightarrow \frac{\theta_{M+1} - \theta_{M-1}}{2\Delta X} = 0$$

$$\therefore \theta_{M+1} = \theta_{M-1}$$

Hence,

$$\theta_{M-1} - D\theta_M + \theta_{M+1} = 0$$
$$\Rightarrow \theta_{M-1} - D\theta_M + \theta_{M-1} = 0$$
$$\therefore 2\theta_{M-1} - D\theta_M = 0$$

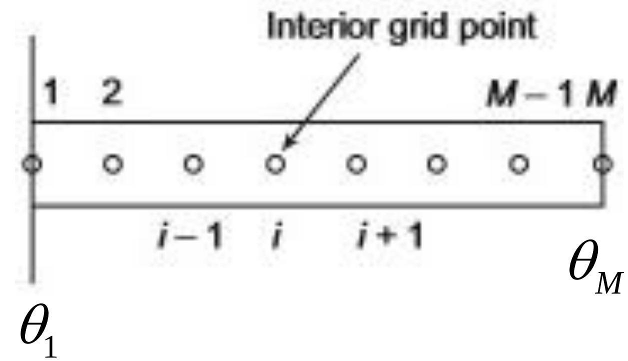
Method 2: use of higher order backward difference expression

An alternative to image point technique is to use a second-order backward difference for

$$\left(\frac{d\theta}{dX}\right)_M = 0$$
$$\Rightarrow \frac{3\theta_M - 4\theta_{M-1} + \theta_{M-2}}{2(\Delta X)} + O(\Delta X^2) = 0$$
$$\Rightarrow \frac{3\theta_M - 4\theta_{M-1} + \theta_{M-2}}{2(\Delta X)} = 0$$
$$\therefore 3\theta_M - 4\theta_{M-1} + \theta_{M-2} = 0$$

The above equation is valid at the grid point on the right boundary. The second-order scheme is used to match the order of accuracy of the central difference scheme used for interior points.

Matrix equations



$$\theta_1 = 1 \qquad \text{for } i = 1$$

$$\theta_{i-1} - D\theta_i + \theta_{i+1} = 0 \qquad \text{for } i = 2, \ldots\ldots, M-1$$

$$2\theta_{M-1} - D\theta_M = 0 \qquad \text{for } i = M$$

The above sets of algebraic equations can be written as,

$$\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 1 & -D & 1 & 0 & . & . & 0 \\ 0 & 1 & -D & 1 & . & . & 0 \\ 0 & 0 & 1 & -D & 1 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 2 & -D \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ . \\ . \\ \theta_{M-1} \\ \theta_M \end{Bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A\theta = b$$

Matrix equations

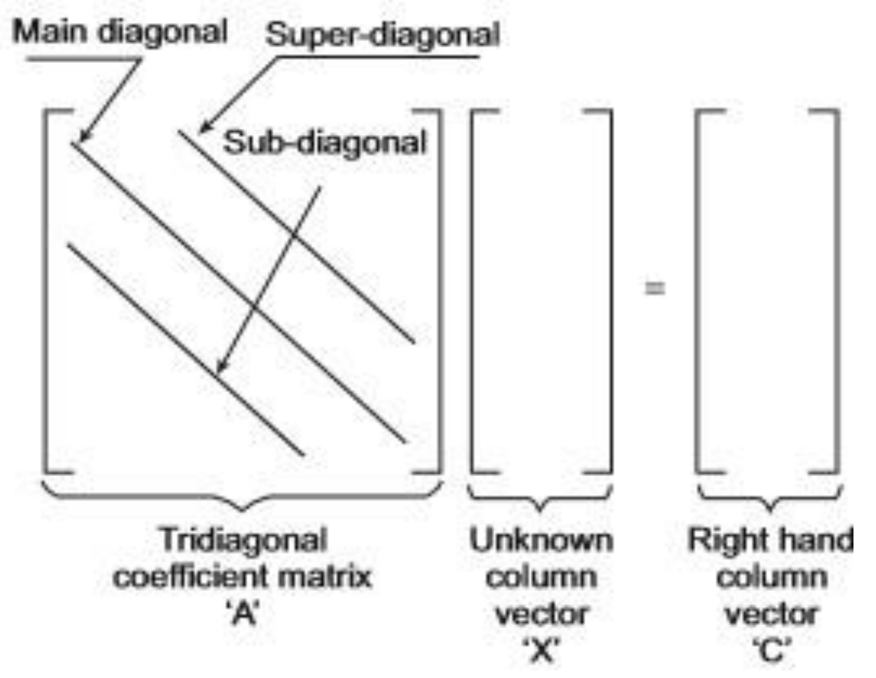
$$A=\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 1 & -D & 1 & 0 & . & . & 0 \\ 0 & 1 & -D & 1 & . & . & 0 \\ 0 & 0 & 1 & -D & 1 & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 2 & -D \end{bmatrix}, \quad \theta=\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ . \\ . \\ \theta_{M-1} \\ \theta_M \end{Bmatrix}, \quad b=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Tridiagonal matrix

Tridiagonal system of equations

Method of Solution

The coefficient matrix has three diagonals: main diagonal, sub-diagonal, and super-diagonal; hence, the name tridiagonal matrix.



The set of equations can be solved by any of the following three methods:

1. Gaussian elimination.
2. Thomas algorithm (or tridiagonal matrix algorithm or simply TDMA).
3. Gauss−Seidel iterative method.

1. Gaussian elimination

- ✓ This method reduces a given set of N equations to an equivalent triangular set, so that one of the equations has only one unknown.
- ✓ This unknown is determined and the remaining unknowns are obtained by the process of back substitution.

The basic approach is shown in a step-by-step form as given.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

a_{11} is called the *pivot* below which the terms are to be made zero.

Step I

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22}^{(1)} & a_{23}^{(1)} \\ & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \end{bmatrix}$$

The superscript represents the step number, $a_{22}^{(1)}$ is now the pivot for the next operation.

Step II

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22}^{(1)} & a_{23}^{(1)} \\ & & a_{33}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \end{bmatrix}$$

Solution accuracy

- ✓ Round-off error may significantly affect the accuracy if a large number of equations is involved.
- ✓ The round-off error is cumulative because the errors are carried on from one step to the other during the elimination process.
- ✓ G-E is generally used if the number of equations is typically less than 20 when the coefficient matrix is dense. For sparse coefficient matrix, however, a large number of equations can be solved.

This method needs only $O(n^3)$ arithmetic operation.