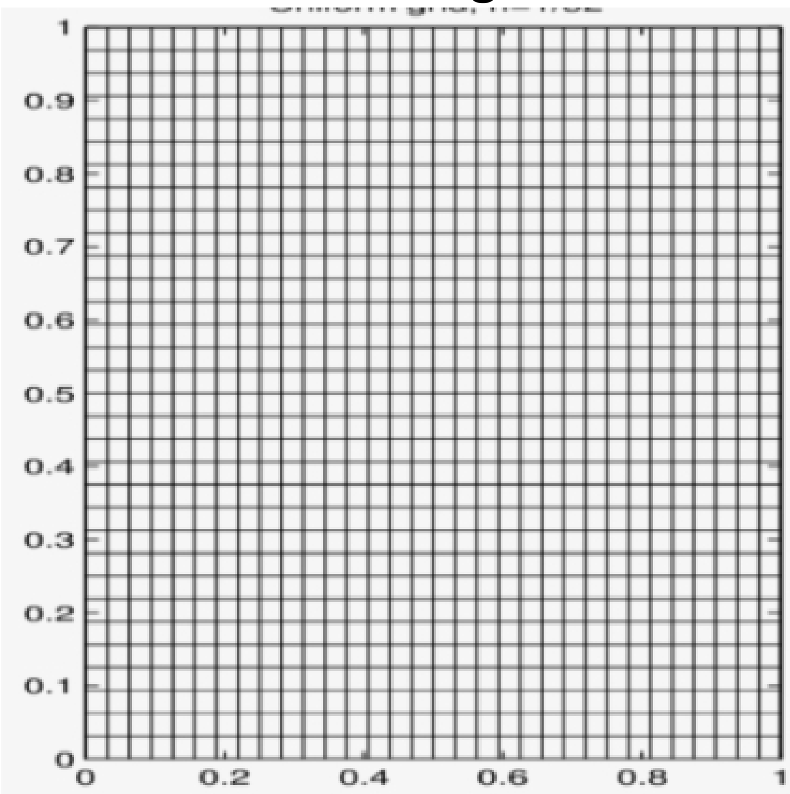
ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

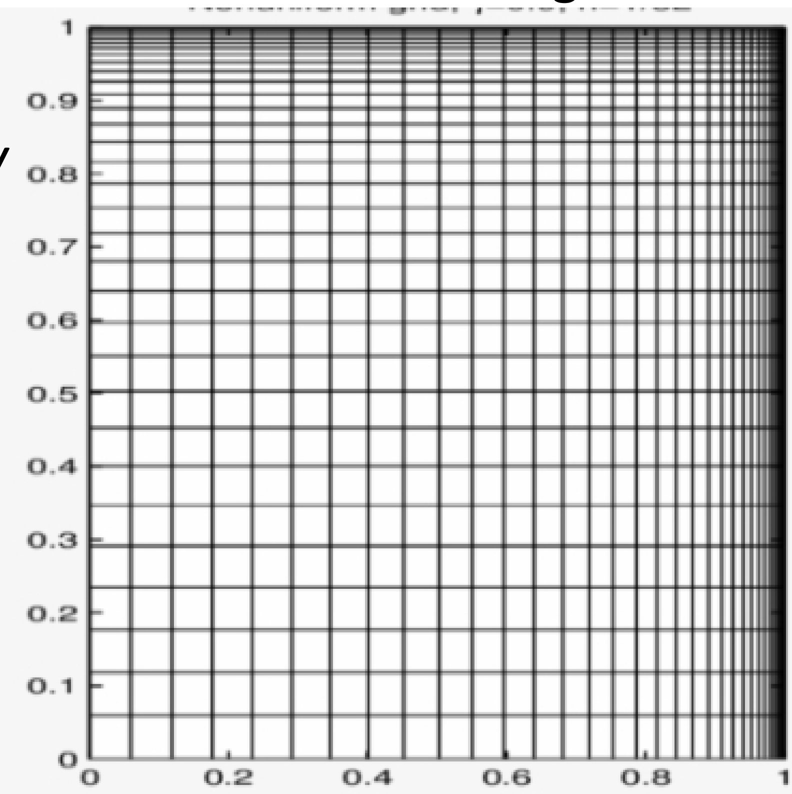
Chapter - 4

Concept of uniform and non-uniform grids

Uniform grid

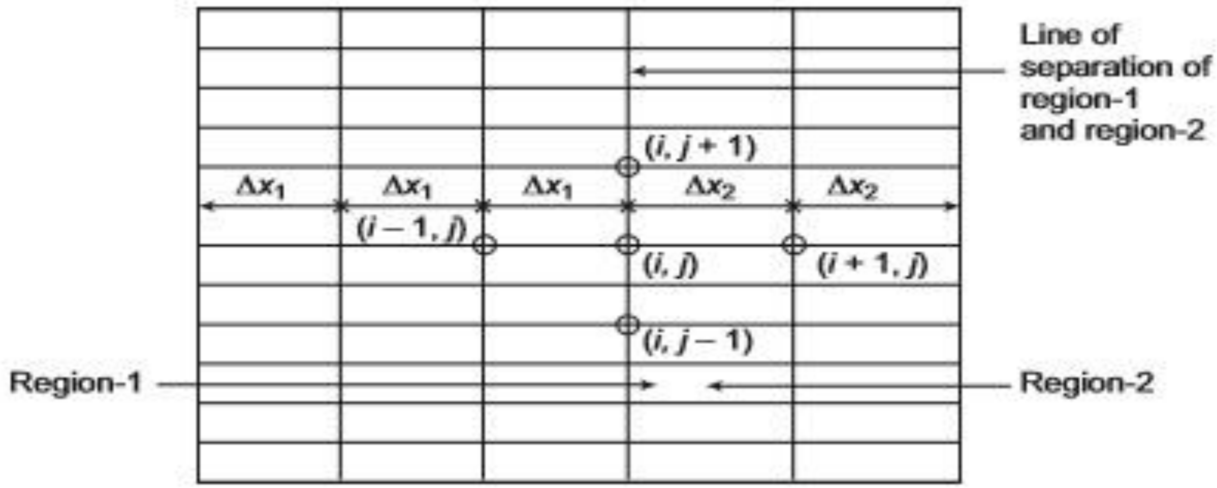


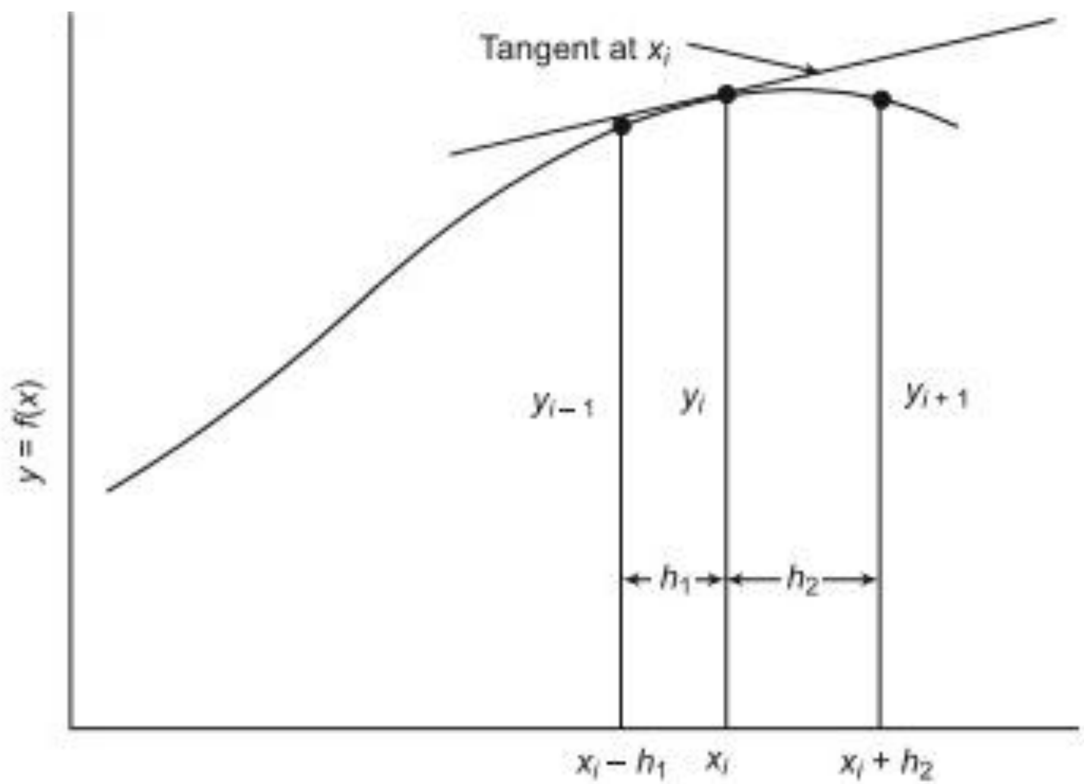
Non-uniform grid



Non-uniform grid is used in the region where gradients of the are expected to be high. One can save on computational memory and execution time with the help of fine grid in the region.

Central difference expressions for a non-uniform grid





The Taylor series for the function y = f(x) at $x_i + h_2$ and $x_i - h_1$ expanded about x_i are:

$$y_{i+1} = y_i + h_2\, y_i' + \frac{h_2^{\,2}}{2!} y_i'' + \frac{h_2^{\,3}}{3!} y_i''' + \ldots.$$

$$y_{i-1} = y_i - h_1\, y_i' + \frac{h_1^{\,2}}{2!} y_i'' - \frac{h_1^{\,3}}{3!} y_i''' + \ldots.$$

In terms of $h_2/h_1 = R$, the equation can be written as,

$$y_{i+1} = y_i + \left(Rh_1\right) y_i' + \frac{\left(Rh_1\right)^2}{2!} y_i'' + \frac{\left(Rh_1\right)^3}{3!} y_i''' + \ldots. \qquad (1)$$

$$y_{i-1} = y_i - h_1\, y_i' + \frac{h_1^{\,2}}{2!} y_i'' - \frac{h_1^{\,3}}{3!} y_i''' + \ldots. \qquad (2)$$

Now, (1) – R^2X(2) gives

$$y_{i+1} + R^2 y_{i-1} = y_i - R^2 y_i + \left(Rh_1\right) y_i' + \left(R^2 h_1\right) y_i' + O(h_1^{\,3})$$

$$\therefore\ y_i' = \frac{y_{i+1} - y_i\left(1 - R^2\right) - R^2 y_{i-1}}{R\left(1+R\right)h_1} + O(h_1^{\,2})$$

Checking for accuracy: For uniform grid, R = 1; hence, $h_1 = h_2 = h$. Substituting R = 1, we get

$$\therefore y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O\left(h^2\right)$$

Central difference for y''_i: Eliminating y'_i between the first two equations, we have

$$\therefore y_i'' = \frac{y_{i+1} - \left(1+R\right) y_i + R y_{i-1}}{\left(R/2\right)\left(1+R\right) h_1^2} + O(h_1)$$

The order of accuracy for the central difference expression of is reduced by 1 in the case of non-uniform grid.

Checking for accuracy: For uniform grid, R = 1; hence, $h_1 = h_2 = h$. Substituting R = 1, we get

$$\therefore y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O\left(h^2\right)$$

Home assignment

Derive the FDM expressions at any point (i,j):

(1)
$$\left.\frac{\partial^4 y}{\partial x^4}\right|_{i,j} = \frac{y_{i-2,j} - 4y_{i-1,j} + 6y_{i,j} - 4y_{i+1,j} + y_{i+2,j}}{(\Delta x)^4} + O(\Delta x)^2$$

(2)
$$\left.\frac{\partial^2 \phi}{\partial x \partial y}\right|_{i,j} = \frac{\phi_{i+1,j+1} - \phi_{i-1,j+1} - \phi_{i+1,j-1} + \phi_{i-1,j-1}}{4\Delta x \Delta y} + O[(\Delta x)^2, (\Delta y)^2]$$

Types of errors in numerical solutions

Round-off error: It is introduced due to computer's inability to handle a large number of significant digits. The number of significant figures retained ranges from 7 to 16, although it may vary in different computer systems. The round-off error arises because a finite number of significant digits or decimal places are retained and all real numbers are rounded off by the computer. The last retained digit is rounded off if the first discarded digit is equal to or greater than 5. Otherwise, it is unchanged.

Truncation error: It takes place due to the replacement of an exact mathematical expression by a numerical approximation. It is the difference between an exact expression and the corresponding truncated form used in the numerical solution.

Discretization error : It is the error in the overall solution that results from the truncation error assuming the round-off error to be negligible.

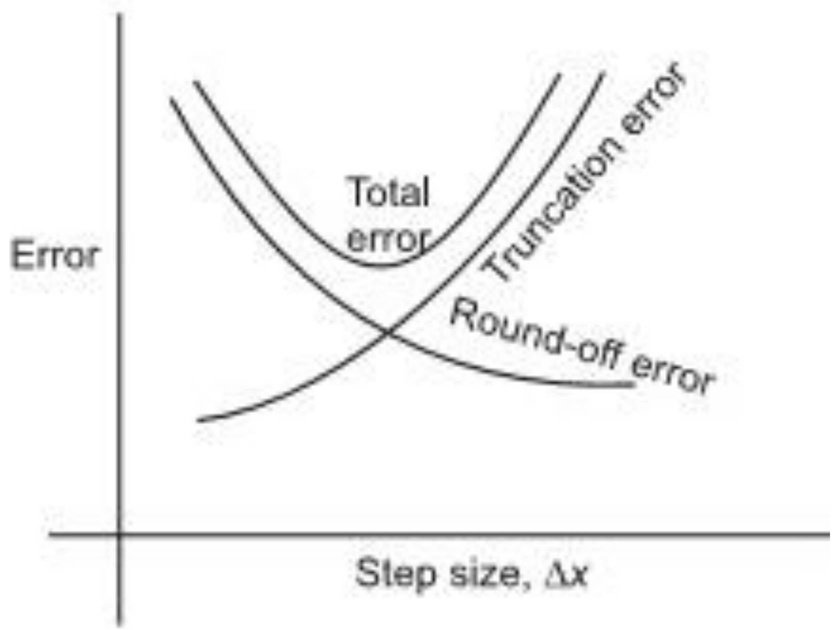
Discretization error = Exact solution –Numerical solution with no round-off error

Accuracy of solution: Optimum step size

- ✓ The accuracy of a numerical solution is determined by its total error, which is the sum of the round-off error and truncation error

Total error = Round-off error + Truncation error

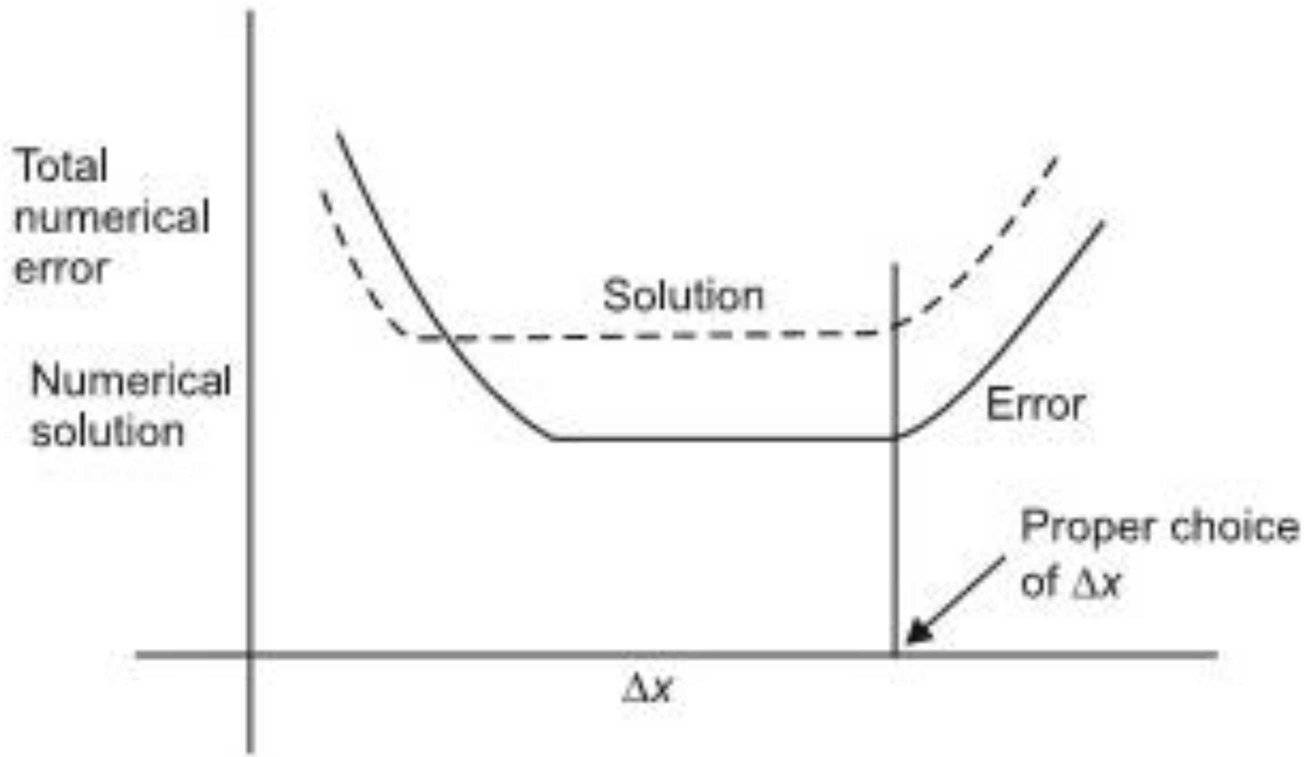
- ✓ Round-off error is directly proportional to the total number of arithmetic operations
- ✓ The total number of arithmetic operations is inversely proportional to the step size
- ✓ Round-off error is inversely proportional to the step size
- ✓ Truncation error is directly proportional to the step size



Variation of round-off error, truncation error, and total error with step size

Optimum grid size-The grid/mesh independence/sensitivity test

The test is carried out by experimenting with various grid sizes and watching how the solution changes with respect to the changes in grid sizes. A stage will come when changing the grid spacing will not affect the solution. In other words, the solution has now become independent of grid spacing. Grid sizes are changed discretely, one does not get an exact minimum, but a range of grid sizes for which total error remains more or less unchanged. This is called **grid independence test.**



Example: Calculate the numerical value of $d(7x^3)/dx$ at x=1, using central difference scheme and check its accuracy. Use Δx=0.5 and 0.1. Which answer is more accurate?

SOLN: **Exact solution at x=1,**

$$\left(\frac{dy}{dx}\right)_{x=1} = \left[21x^2\right]_{x=1} = 21$$

Central difference with Δx=0.5

$$\therefore \left(y_i'\right)_{x=1} = \left(\frac{dy}{dx}\right)_{x=1} = \frac{y_{i+1} - y_{i-1}}{2h} = \frac{y_{x+\Delta x} - y_{x-\Delta x}}{2\times \Delta x}$$

$$\left(y_i'\right)_{x=1} = \frac{y_{1+0.5} - y_{1-0.5}}{2\times 0.5} = \frac{y_{1.5} - y_{0.5}}{2\times 0.5} = \frac{7\left(1.5\right)^3 - 7\left(0.5\right)^3}{1} = 22.75$$

$$\boldsymbol{Error} = \left|\frac{22.75 - 21}{21}\right| \times 100 = 8.33\%$$

Central difference with Δx=0.1

$$\left(y_i'\right)_{x=1} = \frac{y_{1+0.1} - y_{1-0.1}}{2\times 0.1} = \frac{y_{1.1} - y_{0.9}}{2\times 0.1} = \frac{7\left(1.1\right)^3 - 7\left(0.9\right)^3}{0.2} = 21.07$$

$$\boldsymbol{Error} = \left|\frac{21.07 - 21}{21}\right| \times 100 = 0.33\%$$