ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

Chapter-3

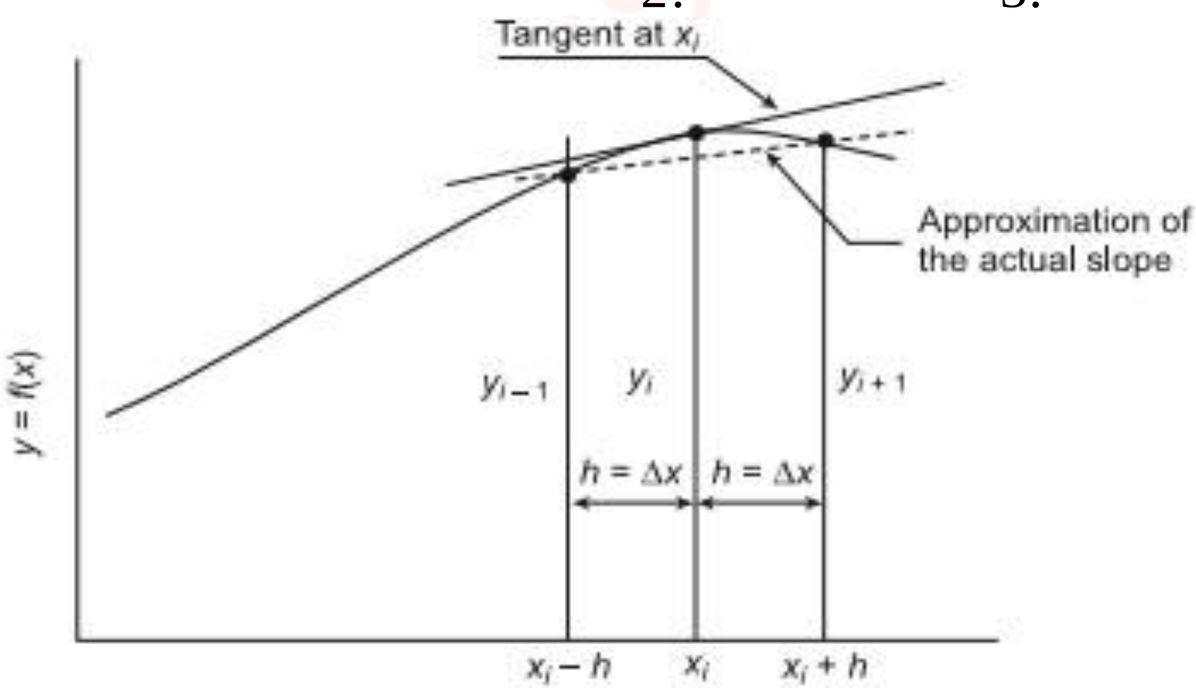
Introduction to the Finite Difference Method (FDM)

The finite difference method enables one to integrate a differential equation numerically by evaluating the values of the function at a discrete (finite) number of points. The origin of this method is Taylor series expansion, which assumes that the function is smooth, that is, continuous and differentiable.

Taylor series expansion

The Taylor series of a real or complex-valued function *f* (*x*) that is infinitely differentiable at a real or complex number *a* is the power series

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + ...$$



Central difference

The Taylor series expansion of a function *f* (*x*) at x_i+h expanded about x_i is

$$f\left(x_i+h\right)=f\left(x_i\right)+\left(x_i+h-x_i\right)f'\left(x_i\right)+\frac{\left(x_i+h-x_i\right)^2}{2!}f''\left(x_i\right)+\frac{\left(x_i+h-x_i\right)^3}{3!}f'''\left(x_i\right)+\ldots.$$

Let, $y_i=f\left(x_i\right) \quad y_{i+1}=f\left(x_i+h\right) \quad y_{i-1}=f\left(x_i-h\right)$

$$y_{i+1}=y_i+h\,y_i'+\frac{h^2}{2!}y_i''+\frac{h^3}{3!}y_i'''+\ldots. \qquad (1)$$

Then, the Taylor series expansion of a function *f* (*x*) at x_i-h expanded about x_i is

$$y_{i-1}=y_i-h\,y_i'+\frac{h^2}{2!}y_i''-\frac{h^3}{3!}y_i'''+\ldots. \qquad (2)$$

Subtracting (1)-(2), we get,

$$y_{i+1}-y_{i-1}=2h\,y_i'+\frac{2h^3}{3!}y_i'''+\ldots.$$

$$\therefore\ y_i'=\frac{y_{i+1}-y_{i-1}}{2h}-\frac{1}{6}\left(y_i'''h^2\right)+higher\ order\ terms$$

Therefore,

$$\therefore y_i' = \frac{y_{i+1} - y_{i-1}}{2h} + O\left(h^2\right)$$

Neglecting the O(h^2) terms, which is called the **truncation error (error due to truncated Taylor series)**, we obtain the central difference FDM expression

$$\therefore y_i' = \left(\frac{dy}{dx}\right)_{x=x_i} = \frac{y_{i+1} - y_{i-1}}{2h}$$

Now, adding Eqs. (1) and (2)

$$y_{i+1} + y_{i-1} = 2y_i + h^2 y_i'' + \frac{h^4}{12} y_i'''' + \ldots.$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{1}{12}\left(y_i'''' h^2\right) + \text{higher order terms}$$

$$\therefore y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O\left(h^2\right)$$

Neglecting the O(h^2) terms

$$y_i'' = \left(\frac{d^2 y}{dx^2}\right)_{x=x_i} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

- It can also be expressed as

$$y_i'' = \frac{\dfrac{y_{i+1} - y_i}{h} - \dfrac{y_i - y_{i-1}}{h}}{h} \qquad y_i'' = \frac{\dfrac{y_{i+1} - y_i}{h} - \dfrac{y_i - y_{i-1}}{h}}{h}$$

- where $y'_{i+1/2}$ and $y'_{i-1/2}$ represent the slope of the tangent to the curve at $x_i + (h/2)$ and $x_i - (h/2)$, respectively.

- The afore-derived central difference expressions reveal that the first and second derivatives of the function involve values of function on both sides of the x-value at which the derivative of the function is to be evaluated.

Forward difference

From Taylor series expansions, it is also easy to obtain expressions for the derivatives that are entirely in terms of values of function at x_i and points to the right of x_i. These are called forward difference expressions.

$$y_{i+1} = y_i + h\,y_i' + \frac{h^2}{2!}\,y_i'' + \frac{h^3}{3!}\,y_i''' + \ldots.$$

$$\Rightarrow h\,y_i' = y_{i+1} - y_i - \frac{h^2}{2!}\,y_i'' - \frac{h^3}{3!}\,y_i''' - \ldots.$$

$$\Rightarrow y_i' = \frac{y_{i+1} - y_i}{h} - \frac{h}{2!}\,y_i'' - \frac{h^2}{3!}\,y_i''' - \ldots.$$

$$\therefore y_i' = \frac{y_{i+1} - y_i}{h} + O(h)$$

Neglecting the O(h) terms

$$\therefore y_i' = \left(\frac{dy}{dx}\right)_{x=x_i} = \frac{y_{i+1} - y_i}{h}$$

Similarly,

$$y_{i+2} = y_i + (2h)\, y_i' + \frac{(2h)^2}{2!}\, y_i'' + \frac{(2h)^3}{3!}\, y_i''' + \ldots. \qquad (1)$$

Also,

$$y_{i+1} = y_i + (h)\, y_i' + \frac{(h)^2}{2!}\, y_i'' + \frac{(h)^3}{3!}\, y_i''' + \ldots. \qquad (2)$$

Now, (1) – 2X(2) gives

$$\therefore y_i'' = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} + O(h)$$

Neglecting the O(h) terms

$$y_i'' = \left(\frac{d^2 y}{dx^2}\right)_{x=x_i} = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2}$$

Backward difference

With this approach, one can easily obtain derivative expressions that are entirely in terms of values of the function at x_i and points to the left of x_i. These are known as backward difference expressions, which are given below for y'_i and y''_i.

$$y_{i-1} = y_i - h\,y_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \ldots.$$

$$\Rightarrow h\,y_i' = y_i - y_{i-1} + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \ldots.$$

$$\Rightarrow y_i' = \frac{y_i - y_{i-1}}{h} + \frac{h}{2!}y_i'' - \frac{h^2}{3!}y_i''' + \ldots.$$

$$\therefore y_i' = \frac{y_i - y_{i-1}}{h} + O(h)$$

Neglecting the O(h) terms

$$\therefore y_i' = \left(\frac{dy}{dx}\right)_{x=x_i} = \frac{y_{i+1} - y_i}{h}$$

Similarly, one can obtain

$$y_i'' = \left(\frac{d^2y}{dx^2}\right)_{x=x_i} = \frac{y_i - 2y_{i-1} + y_{i-2}}{h^2} \qquad \text{(Home work)}$$

Employing forward, backward, and central difference expressions

I. Forward difference expressions are used when data to the left of a point at which a derivative is desired are not available.

II. Backward difference expressions are used when data to the right of the desired point are not available.

III. Central difference expressions are used when data on both sides of the desired point are available and are more accurate than either forward or backward difference expressions.