ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

Chapter - 2

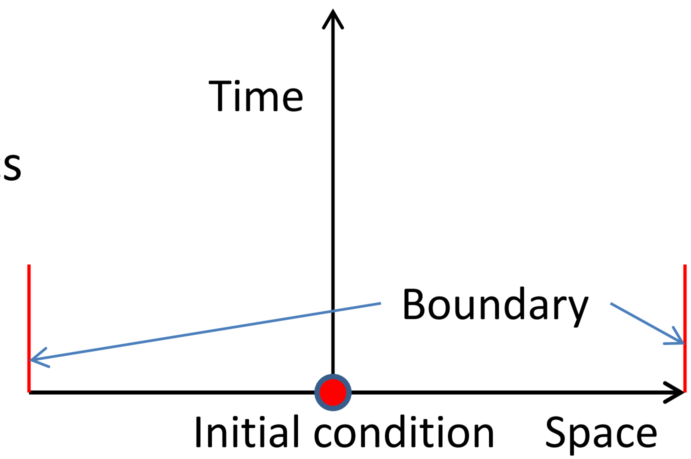
Examples of **elliptic, parabolic, and hyperbolic** equations

Hyperbolic PDE

$$\frac{\partial^2 \phi}{\partial t^2} = \kappa^2 \frac{\partial^2 \phi}{\partial x^2}$$ Wave equation

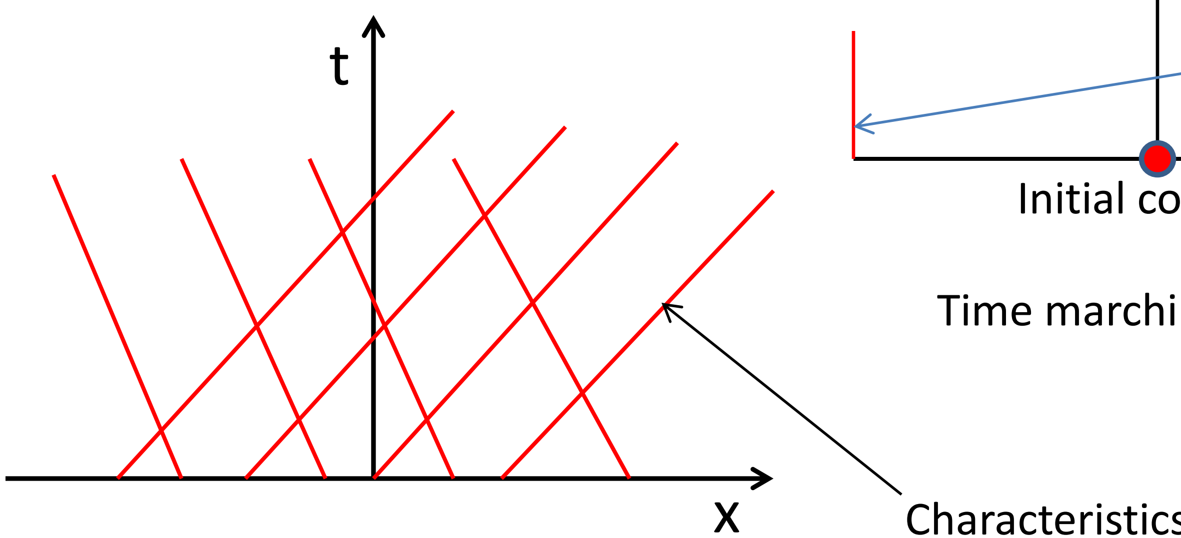
This equation requires both two initial and boundary conditions for the solution.

κ is real and positive constant.



The solution always follows some characteristics

Time marching problem.



Classification of Navier–Stokes equations

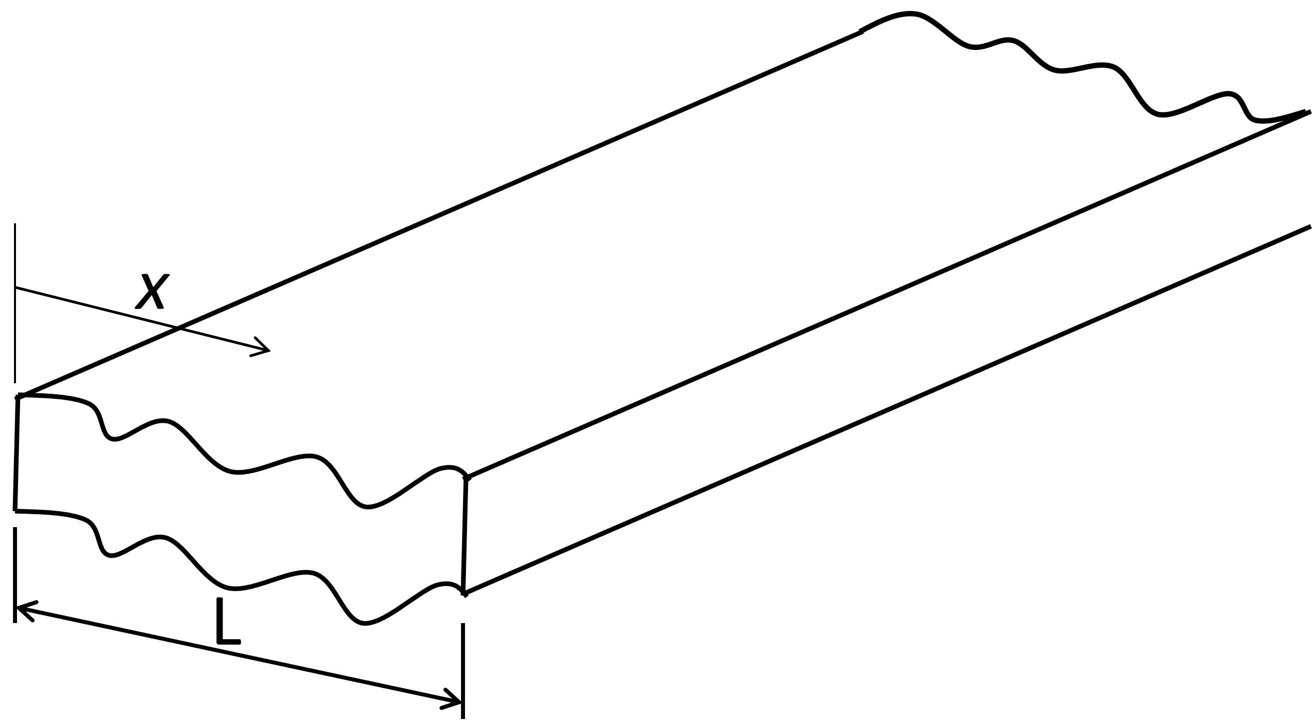
The complete Navier–Stokes equations in three space coordinates (x, y, z) and time (t) are a system of three nonlinear second-order equations in four independent variables. So, the normal classification rules do not apply directly to them. Nevertheless, they do possess properties such as hyperbolic, parabolic, and elliptic:

Hyperbolic Flows	Parabolic Flows	Elliptic Flows	Mixed Flows
• unsteady, inviscid compressible flow. A compressible flow can sustain sound and shock waves, and the Navier–Stokes equations are essentially hyperbolic in nature. •For steady inviscid compressible flows, the equations are hyperbolic if the speed is supersonic, and elliptic for subsonic speed.	•The boundary layer flows have essentially parabolic character. The solution marches in the downstream direction, and the numerical methods used for solving parabolic equations are appropriate.	• The subsonic inviscid flow falls under this category. •If a flow has a region of recirculation, information may travel upstream as well as downstream. Therefore, specification of boundary conditions only at the upstream end of the flow is not sufficient. The problem then becomes elliptic in nature.	There is a possibility that a flow may not be characterized purely by one type. For example, in a steady transonic flow, both supersonic and subsonic regions exist. The supersonic regions are hyperbolic, whereas subsonic regions are elliptic.

Initial and Boundary conditions

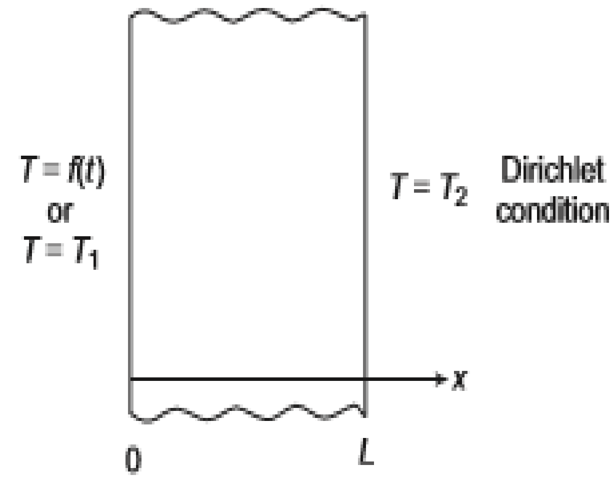
The initial and boundary conditions must be specified to obtain unique numerical solution to PDEs: Following Eq. depicts a problem in which the temperature within a large solid slab having finite thickness changes in the x-direction, as a function of time till steady state (corresponding to t → ∞) is reached:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$



1. Dirichlet Conditions (First Kind):

The values of the dependent variables are specified at the boundaries in the figure:



- Boundary Conditions of first kind can be expressed as

  B.C. 1 *T=f (t) or T1 at x=0*

  B.C.2 T= T2 at x=L

  $\left.\right\}$ *t > 0*

- Initial Condition

  T= f(x) at t= 0 0<= x =<L

  or T= T0

2. Neumann Conditions (Second Kind)

The derivative of the dependent variable is given as a constant or as a function of the independent variable on one boundary:

$$\frac{\partial T}{\partial x} = 0 \ldots at \ldots x = L \ldots and \ldots t \geq 0$$

This condition specifies that the temperature gradient at the right boundary is zero (insulation condition).

Cauchy conditions: A problem that combines both Dirichlet and Neumann conditions is considered to have Cauchy conditions:

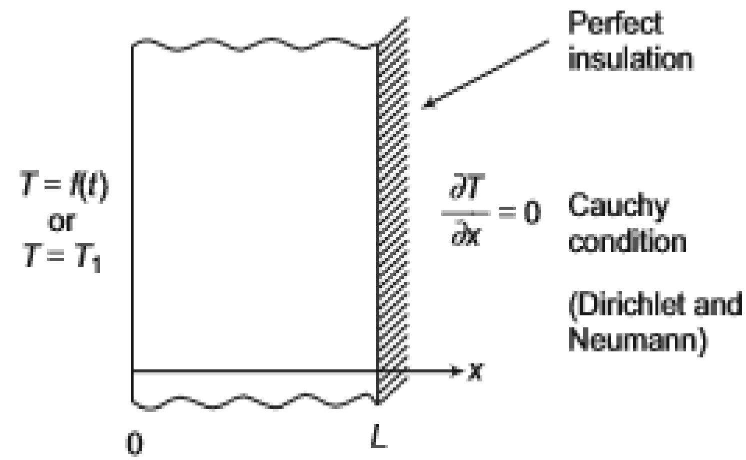
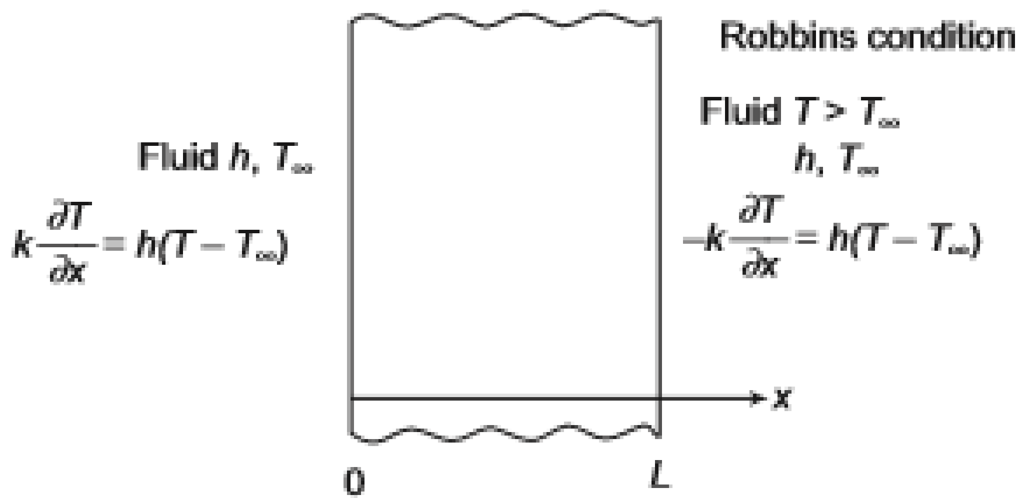


Fig: Cauchy conditions

3. Robbins Conditions (Third Kind)

The derivative of the dependent variable is given as a function of the dependent variable on the boundary. For the heat conduction problem, this may correspond to the case of cooling of a large steel slab of finite thickness *"L"* by water or oil, the heat transfer coefficient *h* being finite:



Initial and Boundary Value Problems

On the basis of their initial and boundary conditions, PDEs may be further classified into initial value or boundary value problems.

Initial Value Problems:

In this case, at least one of the independent variables has an open region. In the unsteady state heat conduction problem, the time variable has the range $0 \le t \le \infty$, where no condition has been specified at $t = \infty$; therefore, this is an initial value problem.

Boundary Value Problems:

When the region is closed for all independent variables and conditions are specified at all boundaries, then the problem is of the boundary value type. An example of this is the three-dimensional steady-state heat conduction (with no heat generation) problem, which is mathematically represented by the equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$