

SPECIAL TOPICS

Chapter - 14

Primitive-variables approach

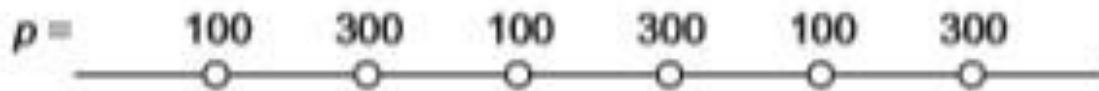
- The stream function–vorticity or biharmonic equation approach is suitable for solving two-dimensional flow problems. If the pressure distribution is also of interest, the problem becomes more involved.
- Stream function–vorticity formulation is difficult to implement in the problem of fluid flow with variable properties and in complex geometries.
- Also, ψ – ξ method is not applicable to three-dimensional problems.
- Therefore, if the pressure distribution is of interest and/or three-dimensional simulation is required, it is generally more efficient to solve the basic Navier–Stokes equations in terms of the primitive variables u , v , w , and p .

BASIC ISSUE WITH THE FDM GRID DURING PRIMITIVE VARIABLE APPROACH

Consider a 1D steady state momentum equation:

$$\rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2}$$

The FDM representation for the grid as shown in the figure below is:



Wavy pressure field for 1-D flow situation

$$\rho_i u_i \frac{u_{i+1} - u_{i-1}}{2\Delta x} = -\frac{p_{i+1} - p_{i-1}}{2\Delta x} + \mu_i \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

The momentum equation will contain the pressure difference between two alternate grid points and not between the adjacent ones.

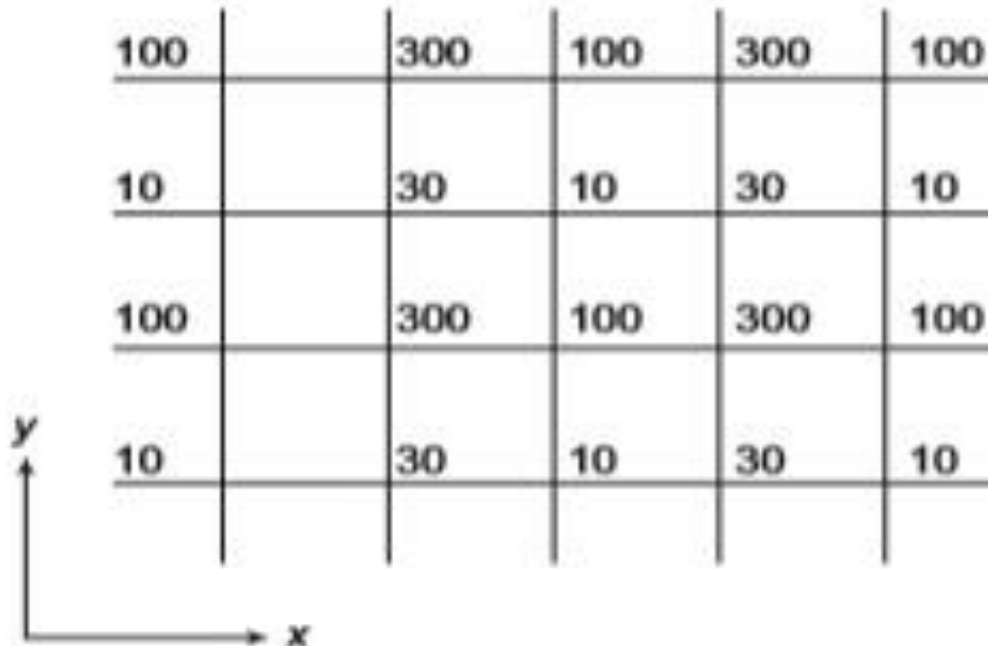
Implications

1. The pressure is taken from a coarser grid than the one actually employed. This would reduce the accuracy of the solution.
2. Even a wavy pressure (which can be realistic) field will be treated like a uniform pressure field by the momentum equation.

$$-\frac{p_{i+1} - p_{i-1}}{2\Delta x} = -\frac{100 - 100}{2\Delta x} = 0 \quad \text{and} \quad -\frac{p_{i+1} - p_{i-1}}{2\Delta x} = -\frac{300 - 300}{2\Delta x} = 0$$

No pressure field at all!

Checker board pressure field for 2D flow situation



Implications

1. If a certain smooth pressure field is obtained as a solution, an infinite number of solutions can be constructed by adding a checker board pressure field to that source (Patankar, 1980).
2. The momentum equation would remain unaffected by this addition, since the checker board pressure field implies zero pressure force. A numerical method that allows such absurdities is certainly not desirable.

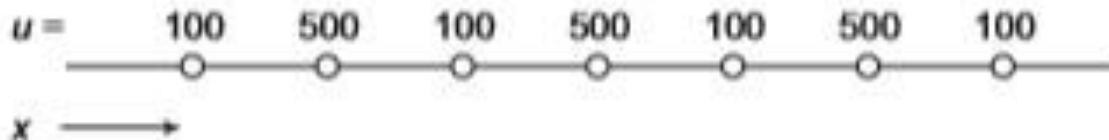
Representation of continuity equation

For 1D, steady, incompressible flow situation, the continuity equation is

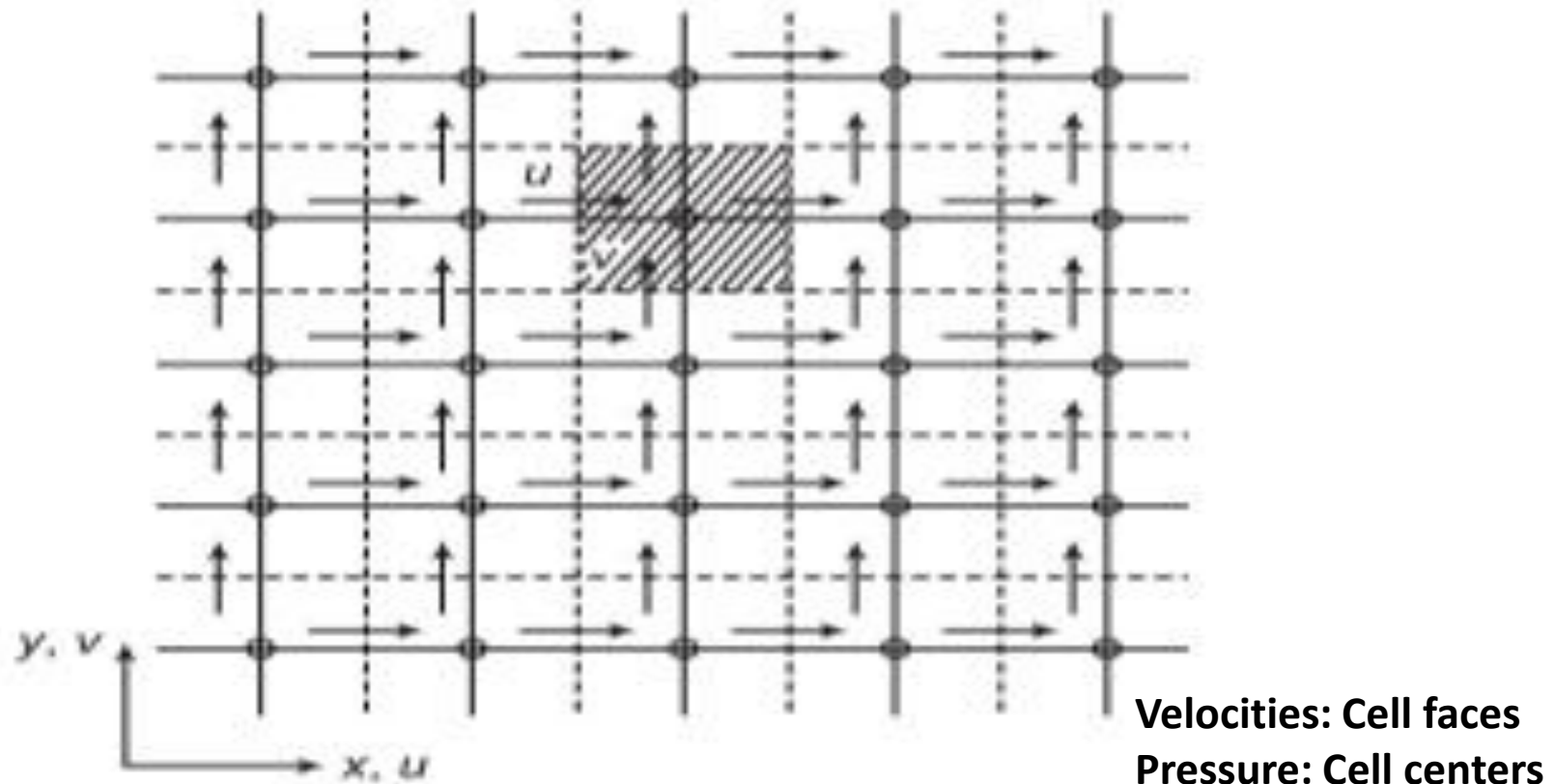
$$\begin{aligned}\frac{\partial u}{\partial x} &= 0 \\ \Rightarrow \frac{u_{i+1} - u_{i-1}}{2\Delta x} &= 0 \\ \Rightarrow u_{i+1} &= u_{i-1}\end{aligned}$$

**Pressure-velocity
decoupling**

A consequence is that a wavy velocity field, which is not at all realistic, does satisfy the discretized continuity equation.



A Remedy: The Staggered Grid



1. The aforementioned difficulties can be circumvented by recognizing that all the variables need not be computed for the same grid points.
2. It is possible to employ a different grid for each dependent variable. In the staggered grid, the velocity components are calculated for the points that are located on the faces of the control volume (shaded), whereas the pressure is calculated for the regular grid points.
3. For uniformly spaced grid points, the control volume faces are situated exactly at the midway between the grid points.

Advantages

1. For a typical control volume shown (shaded with hatched lines), the discretized continuity equation would contain the differences of the adjacent velocity components; hence, a wavy velocity field would not be satisfied.
2. The pressure difference between the successive grid points now becomes the natural driving force for the velocity component located between the grid points. Hence, a non-uniform or a wavy pressure field will not be treated as a uniform pressure field and cannot arise as possible solutions.

Disadvantages

1. It is now obvious that a computer program based on the staggered grid must carry all indexing and geometric information about the location of the velocity components and must perform rather tedious interpolation.

SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) Algorithm of Patankar and Spalding (1972)

1. The procedure is based on a cyclic series of guess-and-correct operations to solve the governing equations.
2. The velocity components are first calculated from the momentum equations using a guessed pressure field.
3. The pressure and velocities are then corrected so as to satisfy continuity.
4. A staggered grid as shown is used.

**Pressure-velocity coupling
is achieved**

The SIMPLE Algorithm of Patankar and Spalding (1972)

Predictor step

X-momentum equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$
$$\Rightarrow \frac{u^* - u^p}{\Delta t} + \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)^{p+1} = -\frac{1}{\rho} \left(\frac{\partial p^*}{\partial x} \right)^{p+1} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^{p+1} \quad (i)$$

Y-momentum equation

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$
$$\Rightarrow \frac{v^* - v^p}{\Delta t} + \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right)^{p+1} = -\frac{1}{\rho} \left(\frac{\partial p^*}{\partial y} \right)^{p+1} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)^{p+1} \quad (ii)$$

u^* , v^* , p^* = predicted velocities and *guessed* pressure

Corrector step

$$\Rightarrow \frac{u^{p+1} - u^p}{\Delta t} + \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)^{p+1} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)^{p+1} + v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^{p+1} \quad (\text{iii})$$

$$\Rightarrow \frac{v^{p+1} - v^p}{\Delta t} + \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right)^{p+1} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial y} \right)^{p+1} + v \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)^{p+1} \quad (\text{iv})$$

By (iii)-(i) and (iv)-(ii), we obtain

$$\Rightarrow \frac{u^{p+1} - u^*}{\Delta t} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)^{p+1} + \frac{1}{\rho} \left(\frac{\partial p^*}{\partial x} \right)^{p+1} = -\frac{1}{\rho} \frac{\partial}{\partial x} \left(p - p^* \right)^{p+1} = -\frac{1}{\rho} \left(\frac{\partial p'}{\partial x} \right)^{p+1}$$

$$\therefore u^{p+1} = u^* - \frac{\Delta t}{\rho} \left(\frac{\partial p'}{\partial x} \right)^{p+1} \quad (\text{v})$$

Hence,

$$\left(\frac{\partial u}{\partial x} \right)^{p+1} = \frac{\partial u^*}{\partial x} - \frac{\Delta t}{\rho} \left(\frac{\partial^2 p'}{\partial x^2} \right)^{p+1} \quad (\text{vi})$$

Also,

$$\Rightarrow \frac{v^{p+1} - v^*}{\Delta t} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial y} \right)^{p+1} + \frac{1}{\rho} \left(\frac{\partial p^*}{\partial y} \right)^{p+1} = -\frac{1}{\rho} \frac{\partial}{\partial y} (p - p^*)^{p+1} = -\frac{1}{\rho} \left(\frac{\partial p'}{\partial y} \right)^{p+1}$$

$$\therefore v^{p+1} = v^* - \frac{\Delta t}{\rho} \left(\frac{\partial p'}{\partial y} \right)^{p+1} \quad (\text{vii})$$

Hence,

$$\left(\frac{\partial v}{\partial y} \right)^{p+1} = \frac{\partial v^*}{\partial y} - \frac{\Delta t}{\rho} \left(\frac{\partial^2 p'}{\partial y^2} \right)^{p+1} \quad (\text{viii})$$

$$p' = p - p^* = \text{Pressure correction}$$

From continuity equation we have,

$$\left(\frac{\partial u}{\partial x}\right)^{p+1} + \left(\frac{\partial v}{\partial y}\right)^{p+1} = 0 \quad (\text{ix})$$

Substituting Eq. (vi) and Eq. (viii) in Eq. (ix), we have

$$\left(\frac{\partial u}{\partial x}\right)^{p+1} + \left(\frac{\partial v}{\partial y}\right)^{p+1} = 0$$

$$\Rightarrow \frac{\partial u^*}{\partial x} - \frac{\Delta t}{\rho} \left(\frac{\partial^2 p'}{\partial x^2}\right)^{p+1} + \frac{\partial v^*}{\partial y} - \frac{\Delta t}{\rho} \left(\frac{\partial^2 p'}{\partial y^2}\right)^{p+1} = 0$$

$$\Rightarrow \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} = \frac{\Delta t}{\rho} \left(\frac{\partial^2 p'}{\partial y^2}\right)^{p+1} + \frac{\Delta t}{\rho} \left(\frac{\partial^2 p'}{\partial x^2}\right)^{p+1}$$

$$\Rightarrow \left(\frac{\partial^2 p'}{\partial x^2}\right)^{p+1} + \left(\frac{\partial^2 p'}{\partial y^2}\right)^{p+1} = \frac{\rho}{\Delta t} \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right)$$

$$\therefore \nabla^2 (p')^{p+1} = \frac{\rho}{\Delta t} \nabla \cdot \vec{u}^* \quad (\text{x})$$

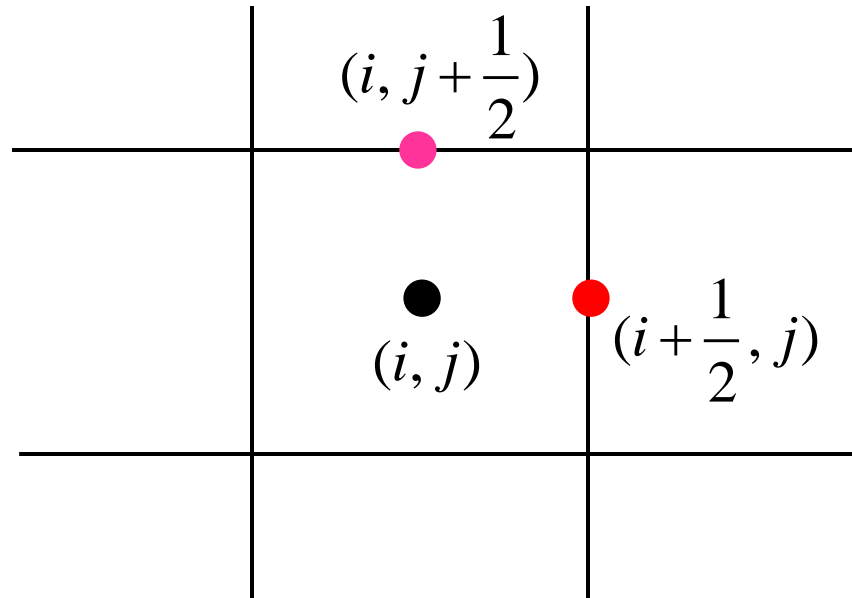
Pressure correction equation

SPECIAL TOPICS CONTD..

Lecture-16

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Discretized equations using the SIMPLE Algorithm



Predictor equation

X-momentum equation

$$\frac{u_{i+\frac{1}{2},j}^* - u_{i+\frac{1}{2},j}^p}{\Delta t} + \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)_{i+\frac{1}{2},j}^{p+1} = -\frac{1}{\rho} \left(\frac{\partial p^*}{\partial x} \right)_{i,j}^{p+1} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i+\frac{1}{2},j}^{p+1}$$

Now,

$$\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right)_{i+\frac{1}{2},j}^{p+1} = \left(\frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \right)_{i+\frac{1}{2},j}^{p+1} = \left(\frac{\partial u^2}{\partial x} \right)_{i+\frac{1}{2},j}^{p+1} + \left(\frac{\partial uv}{\partial y} \right)_{i+\frac{1}{2},j}^{p+1}$$

$$\left(\frac{\partial u^2}{\partial x} \right)_{i+\frac{1}{2},j}^{p+1} = \frac{\left(u^2 \right)_{i+\frac{1}{2}+1,j}^{p+1} - \left(u^2 \right)_{i+\frac{1}{2}-1,j}^{p+1}}{2\Delta x} = \frac{\left(u^2 \right)_{i+\frac{3}{2},j}^{p+1} - \left(u^2 \right)_{i-\frac{1}{2},j}^{p+1}}{2\Delta x}$$

$$\left(\frac{\partial uv}{\partial y} \right)_{i+\frac{1}{2},j}^{p+1} = \frac{\left(uv \right)_{i+\frac{1}{2},j+1}^{p+1} - \left(uv \right)_{i+\frac{1}{2},j-1}^{p+1}}{2\Delta y} = \frac{u_{i+\frac{1}{2},j+1}^{p+1} v_{i+\frac{1}{2},j+1}^{p+1} - u_{i+\frac{1}{2},j-1}^{p+1} v_{i+\frac{1}{2},j-1}^{p+1}}{2\Delta y}$$

Were, (by linear interpolation)

$$v_{i+\frac{1}{2},j+1}^{p+1} = \frac{v_{i+\frac{1}{2},j+\frac{1}{2}}^{p+1} + v_{i+\frac{1}{2},j+\frac{3}{2}}^{p+1}}{2}$$

$$v_{i+\frac{1}{2},j+\frac{1}{2}}^{p+1} = \frac{v_{i,j+\frac{1}{2}}^{p+1} + v_{i+1,j+\frac{1}{2}}^{p+1}}{2}$$

Again,

$$-\frac{1}{\rho} \left(\frac{\partial p^*}{\partial x} \right)_{i,j}^{p+1} = -\frac{1}{\rho} \left(\frac{p_{i+1,j}^* - p_{i,j}^*}{\Delta x} \right)^{p+1}$$

and,

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{i+\frac{1}{2},j}^{p+1} = \left(\frac{\partial^2 u}{\partial x^2} \right)_{i+\frac{1}{2},j}^{p+1} + \left(\frac{\partial^2 u}{\partial y^2} \right)_{i+\frac{1}{2},j}^{p+1}$$

$$\left(\frac{\partial^2 u}{\partial x^2} \right)_{i+\frac{1}{2},j}^{p+1} = \frac{u_{i+\frac{1}{2}+1,j}^{p+1} - 2u_{i+\frac{1}{2},j}^{p+1} + u_{i+\frac{1}{2}-1,j}^{p+1}}{(\Delta x)^2} = \frac{u_{i+\frac{3}{2},j}^{p+1} - 2u_{i+\frac{1}{2},j}^{p+1} + u_{i-\frac{1}{2},j}^{p+1}}{(\Delta x)^2}$$

$$\left(\frac{\partial^2 u}{\partial y^2} \right)_{i+\frac{1}{2},j}^{p+1} = \frac{u_{i+\frac{1}{2},j+1}^{p+1} - 2u_{i+\frac{1}{2},j}^{p+1} + u_{i+\frac{1}{2},j-1}^{p+1}}{(\Delta y)^2}$$

Similarly for v momentum equation

Pressure correction equation

$$\begin{aligned} \left(\frac{\partial^2 p'}{\partial x^2} \right)^{p+1} + \left(\frac{\partial^2 p'}{\partial y^2} \right)^{p+1} &= \frac{\rho}{\Delta t} \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) \\ \Rightarrow \left(\frac{p'_{i+1,j} - 2p'_{i,j} + p'_{i-1,j}}{(\Delta x)^2} \right)^{p+1} + \left(\frac{p'_{i,j+1} - 2p'_{i,j} + p'_{i,j-1}}{(\Delta y)^2} \right)^{p+1} \\ &= \frac{\rho}{\Delta t} \left(\frac{u^*_{i+\frac{3}{2},j} - u^*_{i-\frac{1}{2},j}}{2\Delta x} \right) + \frac{\rho}{\Delta t} \left(\frac{v^*_{i,j+\frac{3}{2}} - v^*_{i,j-\frac{1}{2}}}{2\Delta y} \right) \end{aligned}$$

Velocity correction equation

$$\begin{aligned} u^{p+1} &= u^* - \frac{\Delta t}{\rho} \left(\frac{\partial p'}{\partial x} \right)^{p+1} \\ \Rightarrow u^{p+1}_{i+\frac{1}{2},j} &= u^*_{i+\frac{1}{2},j} - \frac{\Delta t}{\rho} \left(\frac{p'_{i+1,j} - p'_{i,j}}{\Delta x} \right)^{p+1} \end{aligned}$$

Boundary conditions:

At walls

$$u = 0, u^* = 0$$

$$v = 0, v^* = 0$$

No-slip

$$\frac{\partial p}{\partial n} = 0, \frac{\partial p'}{\partial n} = 0$$

At outlet

$$\frac{\partial u}{\partial n} = 0$$

Fully-developed

$$\frac{\partial v}{\partial n} = 0$$

$$p = \text{specified} \Rightarrow p' = 0$$

(n is the surface normal)

Flowchart of the SIMPLE algorithm

Step 1: Initialize velocities at the staggered grid points. Guess p^* at the grid points.

Step 2:

First inner loop: Solve velocity prediction equations with boundary conditions for u^* and v^* till convergence.

Second inner loop: Solve pressure correction equation with boundary conditions for p' with the predicted velocities till convergence.

Step 3: Correct velocities and pressure:

$$u^{p+1} = u^* - \frac{\Delta t}{\rho} \left(\frac{\partial p'}{\partial x} \right)^{p+1}$$

$$v^{p+1} = v^* - \frac{\Delta t}{\rho} \left(\frac{\partial p'}{\partial y} \right)^{p+1}$$

$$p^{p+1} = p' + p^*$$

Step 4: Go to step 2 and iterate till the convergence is achieved, that is all corrections becomes negligibly small and the mass balance,

$$\nabla \cdot \vec{u}^* < \varepsilon \text{ is satisfied.}$$

Step 5: Go to the next time step, $t=t+\Delta t$ and repeat the cycle till steady state is achieved.