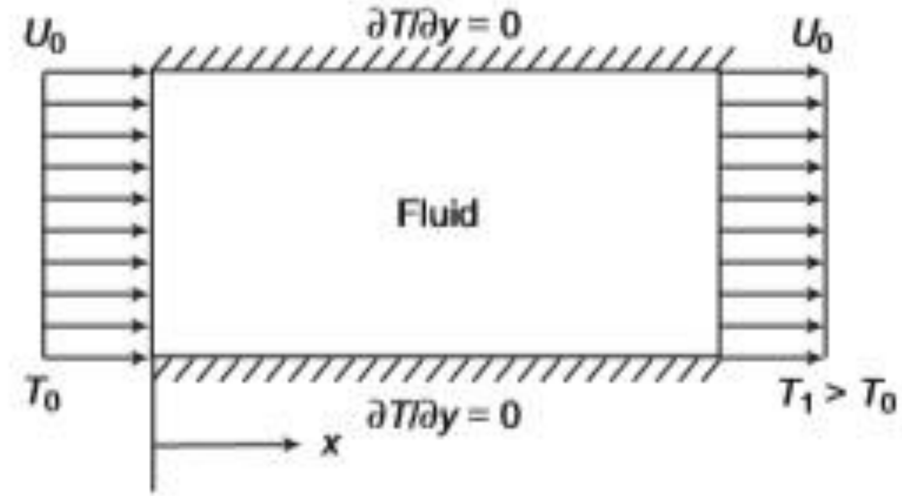
CONVECTION DIFFUSION EQUATIONS AND UPWIND SCHEME

Chapter - 12

Convection–diffusion (Steady, One Dimensional)



Convection diffusion problem for slug flow where the combined influences of steady axial convection and diffusion is examined. The fluid is flowing with a velocity U_0 and is subjected to specified temperatures as follows:

$$T_0 \quad \text{for} \quad x \le 0$$

$$T_1 \quad \text{for} \quad x \ge L$$

To determine, $T(x)$

Governing equation,

$$U_0 \frac{dT}{dx} = \alpha \frac{d^2T}{dx^2}$$

Dimensionless form

Non-dimensionalizing using the dimensionless variables:

$$\theta = \frac{T - T_0}{T_1 - T_0}, \quad X = \frac{x}{L}, \quad Pe = \frac{U_0 L}{\alpha}$$

Hence,

$$Pe \frac{d\theta}{dX} = \frac{d^2\theta}{dX^2}$$

$$BC1: \ at \ X = 0, \quad \theta = 0$$

$$BC2: \ at \ X = 1, \ \theta = 1$$

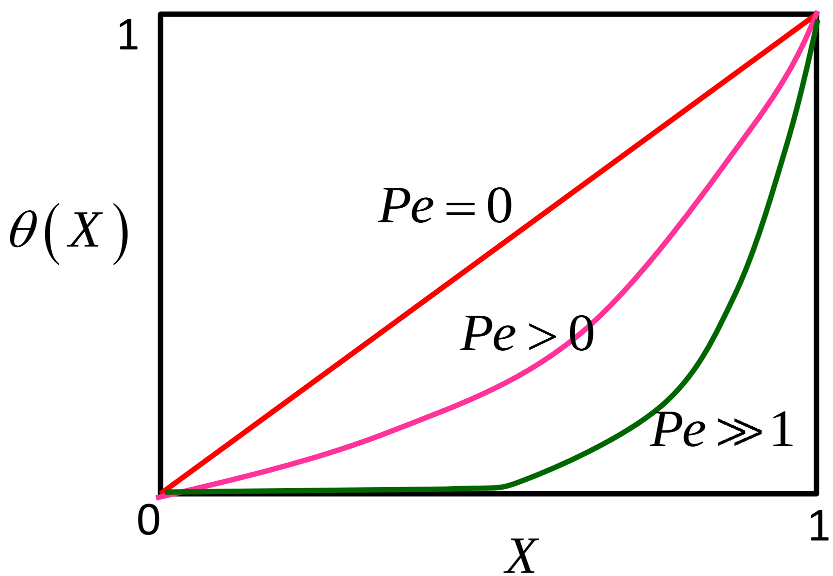
Analytical solution,

$$\theta(X) = \frac{\exp(Pe \cdot X) - 1}{\exp(Pe) - 1}$$

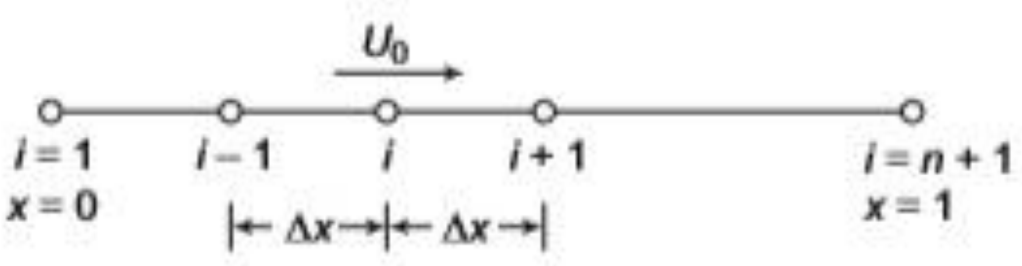
When, $Pe \to 0$ $\quad \theta(X) = X$ (Pure condition) And

When, $Pe \gg 1$ Convection dominates and the temperature changes from Θ=0 to Θ=1 is confined to a thin region X=1. This implies steep temperature gradient.

$$\left[\frac{d\theta(X)}{dX}\right]_{X=1}=\left[\frac{Pe.\exp(Pe\cdot X)}{\exp(Pe)-1}\right]_{X=1}=\frac{Pe.\exp(Pe)}{\exp(Pe)-1}\approx Pe\ (\text{for } Pe\gg 1)$$



Numerical solution



Using central differencing FDM, we have

$$Pe\frac{d\theta}{dX}=\frac{d^2\theta}{dX^2}$$

$$Pe\left(\frac{\theta_{i+1}-\theta_{i-1}}{2\Delta X}\right)=\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{\left(\Delta X\right)^2}$$

Let us define cell or grid Peclet number as,

$$Pe_c=Pe\cdot\Delta X$$

Hence,

$$\frac{Pe_c}{2}\left(\theta_{i+1}-\theta_{i-1}\right)=\theta_{i+1}-2\theta_i+\theta_{i-1}$$

$$\Rightarrow\theta_i=\frac{1}{4}\left\{\left(2+Pe_c\right)\theta_{i-1}+\left(2-Pe_c\right)\theta_{i+1}\right\} \quad (1)$$

When, $Pe_c\to 0 \quad \Rightarrow\theta_i=\frac{1}{2}\left\{\theta_{i-1}+\theta_{i+1}\right\}$

When, $Pe_c\gg 1 \quad \Rightarrow\theta_i=\frac{1}{4}Pe_c\left\{\theta_{i-1}-\theta_{i+1}\right\}$

This shows that for ϴ increasing with X yields absurd and even negative values of Θ_i. Hence, Eq.(1) does not satisfy computational stability unless $Pe_c<2$.

If $Pe_c \geq 2$, the solution shows *wiggles*, a computational instability.

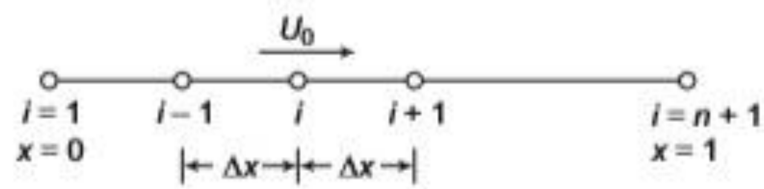
This problem in the numerical solution is circumvented by the **Upwind scheme** in the **Convection Term** .

Upwind scheme

Discretized form of equation,

$$Pe\frac{d\theta}{dX} = \frac{d^2\theta}{dX^2}$$

$$Pe\left(\frac{\theta_{i+1} - \theta_{i-1}}{2\Delta X}\right) = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta X)^2} \qquad (2)$$



Upwind scheme tells that,

if U_0 >0, then $\theta_{i+1} = \theta_i$

if U_0 <0, then $\theta_{i+1} = \theta_{i+2}$

(First order Upwinding scheme)

Therefore, Eq. (2) can be written as, for $U_0>0$

$$Pe\left(\frac{\theta_i-\theta_{i-1}}{2\Delta X}\right)=\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{\left(\Delta X\right)^2}$$

$$\Rightarrow \theta_i=\frac{\left(1+Pe_c\right)\theta_{i-1}+\theta_{i+1}}{2+Pe_c} \qquad (3)$$

When $Pe_c \rightarrow 0 \ \Rightarrow \theta_i=\frac{\theta_{i-1}+\theta_{i+1}}{2}$, (exact solution)

When $Pe_c \gg 1 \qquad \Rightarrow \theta_i=\theta_{i-1}$, (exact solution)

Hence, Eq.(3) is computationally stable.

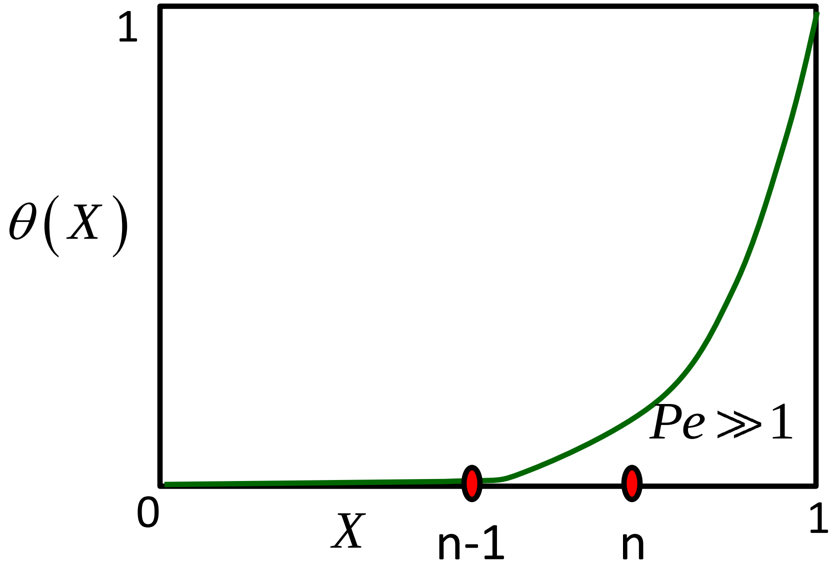
There are other higher order Upwind schemes,

a) Second Order Upwind scheme.

b) Third Order Upwind scheme or Quick.

Limitation of Upwind scheme: False Diffusion

In spite of the fact that the upwind scheme is computationally stable and gives physically realistic solution, it does not produce very accurate solution at high Peclet number (Pe >> 1) because of false diffusion.



When Pe is large, $d\theta/dX$ is nearly zero at X=0.5. Thus, the diffusion is almost absent. The upwind scheme always calculates the diffusion term and thus overestimates diffusion at high Peclet numbers. This is called False Diffusion.

To prevent such False Diffusion, local refinement in grid is necessary.