ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

Chapter - 11

General algorithm for solution by Stream function-Vorticity method

Time marching sequential procedure to obtain velocity field:

Step 1: Specify the initial values for ξ and Ψ at t=0.

Step 2: Solve vorticity transport equation for ξ at each grid point at time t+Δt.

Step 3: Solve for new ψ values at all points by solving the Poisson equation using new ξ at each grid point.

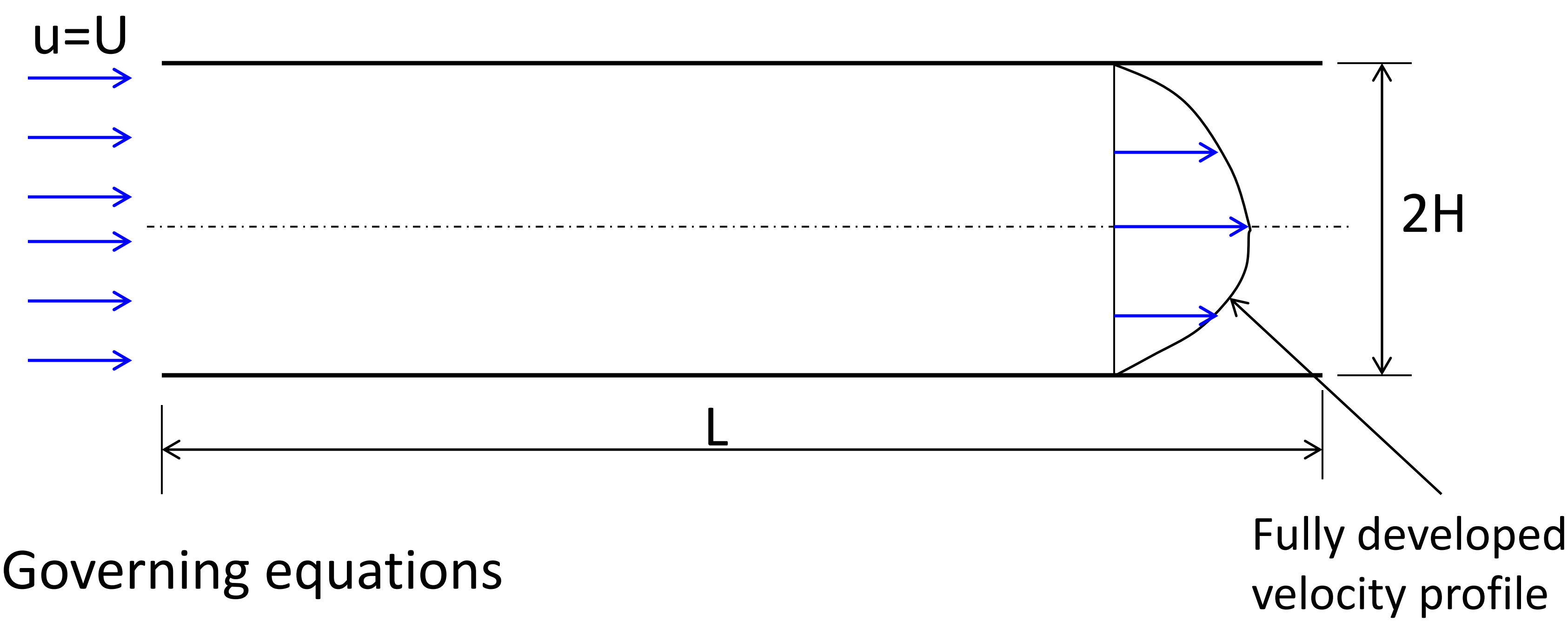
Step 4: Find the velocity components using $u = \partial\psi/\partial x,\ v = -\partial\psi/\partial y$

Step 5: Update the wall vorticity using vorticity equation at the wall.

Step 6: Return to **Step 2** if the steady state is not reached.

Example problem Stream function-Vorticity method

A Newtonian incompressible viscous fluid enters a channel of height 2H and length L with uniform velocity U. Far downstream the velocity profile is fully developed parabolic). Formulate the problem by stream function-vorticity method. Assume the flow is steady and laminar. Show the solution algorithm in the form of flow chart.



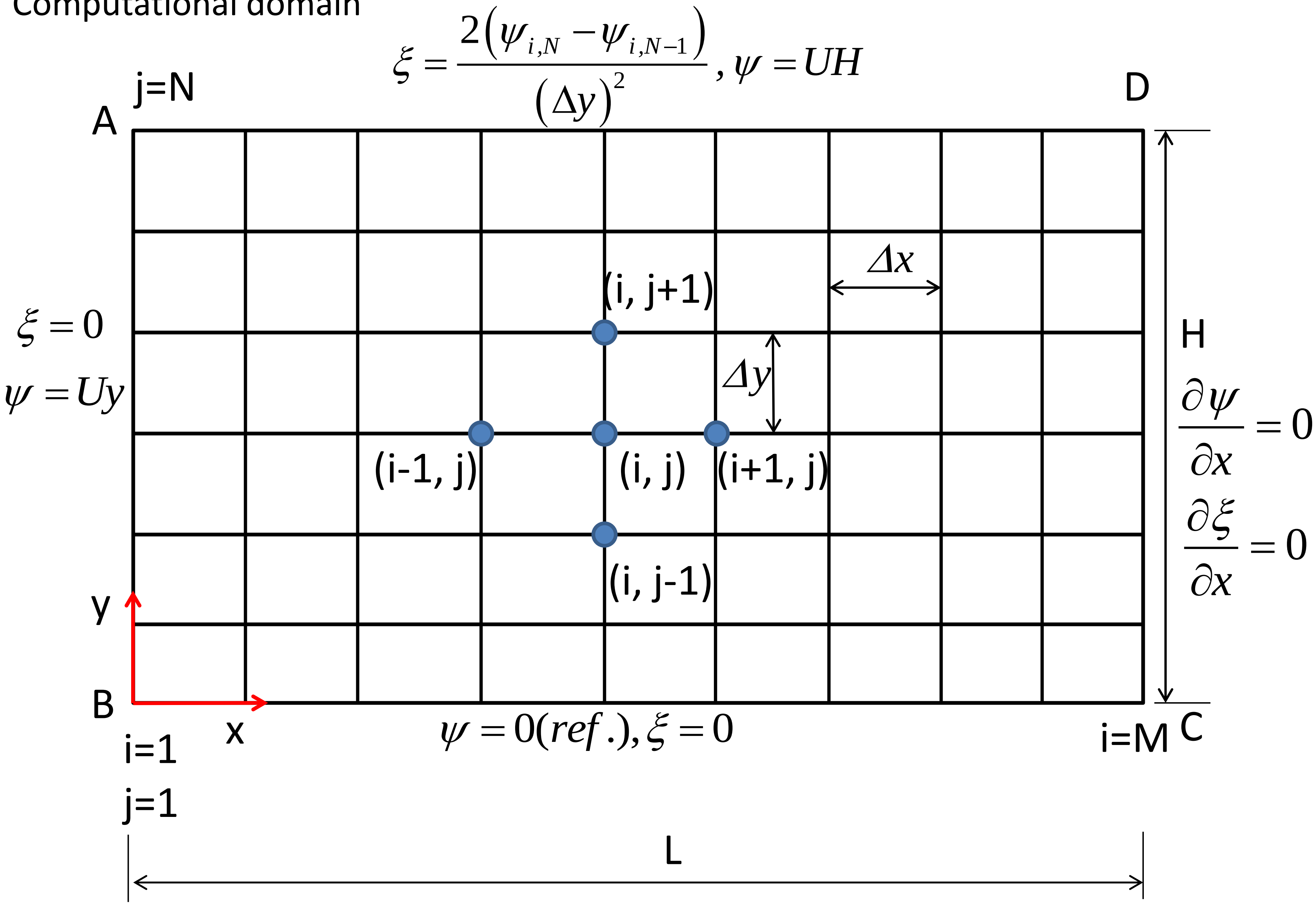
Governing equations

Stream function $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\xi$$

Vorticity $$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} = \nu\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right)$$

Axisymmetric, half of the domain is considered (only upper half)

Computational domain



Specification of initial and boundary condition

Initial condition

$$\text{At } t = 0,\ \psi = 0,\ \xi = 0 \,(\text{assumed})$$

Vorticity boundary condition

Inlet (x=0): u=U, v=0

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Outlet (x=L):

$$\frac{\partial \xi}{\partial x} = 0$$

Centerline (y=0):

$$v = 0, \text{ and } \frac{\partial u}{\partial y} = 0 \Rightarrow \quad \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Wall (y=H): $$\xi_{i,N} = \frac{2\left(\psi_{i,N} - \psi_{i,N-1}\right)}{\left(\Delta y\right)^2}$$

Stream function boundary condition

Inlet (x=0): u=U, v=0

$$v = 0 \Rightarrow \; -\frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = \text{constant}$$

$$u = U \Rightarrow \; \frac{\partial \psi}{\partial y} = U \Rightarrow \psi = Uy + \text{constant}$$

$$\text{At, y = 0}, \; \psi = 0 \Rightarrow \text{constant} = 0$$

$$\text{Hence, } \psi = Uy$$

Outlet (x=L): $\dfrac{\partial \psi}{\partial x} = 0,\;$ since $v = 0$ at the exit

Centerline (y=0):

$$\psi = 0 \text{ (arbitrarily chosen)}$$

Wall (y=H):

$$\psi = UH \;(\text{Since at the entry plane } \psi = Uy\,)$$

At the interior grid point (i, j)

Employing pure implicit scheme FDM

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\xi$$

$$\Rightarrow \frac{\psi_{i+1,j}^{p+1} - 2\psi_{i,j}^{p+1} + \psi_{i-1,j}^{p+1}}{(\Delta x)^2} + \frac{\psi_{i,j+1}^{p+1} - 2\psi_{i,j}^{p+1} + \psi_{i,j-1}^{p+1}}{(\Delta y)^2} = -\xi_{i,j}^{p+1}$$

And

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} = \nu\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right)$$

$$\Rightarrow \frac{\xi_{i,j}^{p+1} - \xi_{i,j}^{p}}{\Delta t} + \left(\frac{\psi_{i,j+1}^{p+1} - \psi_{i,j-1}^{p+1}}{2\Delta y}\right)\left(\frac{\xi_{i+1,j}^{p+1} - \xi_{i-1,j}^{p+1}}{2\Delta x}\right) - \left(\frac{\psi_{i+1,j}^{p+1} - \psi_{i-1,j}^{p+1}}{2\Delta x}\right)\left(\frac{\xi_{i,j+1}^{p+1} - \xi_{i,j-1}^{p+1}}{2\Delta y}\right)$$

$$= \nu\left\{\left(\frac{\xi_{i+1,j}^{p+1} - 2\xi_{i,j}^{p+1} + \xi_{i-1,j}^{p+1}}{(\Delta x)^2}\right) + \left(\frac{\xi_{i,j+1}^{p+1} - 2\xi_{i,j}^{p+1} + \xi_{i,j-1}^{p+1}}{(\Delta y)^2}\right)\right\}$$

Calculations of velocities at grid point point (i, j)

$$u = \frac{\partial \psi}{\partial y}$$

$$\Rightarrow u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{(2\Delta y)}$$

And

$$v = -\frac{\partial \psi}{\partial x}$$

$$\Rightarrow v_{i,j} = -\frac{\psi_{i+1,j} - \psi_{i-1,j}}{(2\Delta x)}$$

Care should be taken at the exit and at the centerline.

Flowchart of the solution algorithm