ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

Chapter - 10

Numerical methods for incompressible fluid flow

Types of flow problems

❖Three classes of flow problems are:

(i) Creeping flow (the limiting case of very large viscosity, that is, very small Reynolds number)

(ii) Boundary layer flow (the limiting case of very small viscous forces, that is, very large Reynolds number)

(iii) Inviscid flow or frictionless flow [ideal fluid, ($\mu = 0$)]. In all three cases, the flow geometry is taken as rectangular.

❖ Flow is assumed as laminar and isothermal, and viscosity is not a function of temperature. It may also be noted that gases may be treated as incompressible fluids when Mach Number < 0.3.

Governing Equations

Governing equations are the Navier-Stokes equations and can be written for a two-dimensional case as

Continuity:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

x-Momentum:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

y-Momentum:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Difficulties in solving Navier–Stokes equations

Nonlinearity: The convection part of the momentum equations involves nonlinear terms.

❖Starting with a guessed velocity field, one could iteratively solve the momentum equation to arrive at the converged solution for the velocity components.

❖ Therefore, nonlinearity poses no problems as such. It only makes the computations more involved.

Pressure gradient: The main hurdle to overcome in the calculation of velocity field is the unknown pressure field.

❖The pressure gradient behaves like a source term for a momentum equation. But, there is no equation for obtaining pressure. The challenging task is to determine the correct pressure distribution.

❖The pressure field is indirectly linked with the continuity equation. When the correct pressure field is plugged into the momentum equations, the resulting velocity field satisfies the continuity equation.

Stream function–Vorticity method

In this method, the difficulty associated with the computation of pressure is circumvented by eliminating the pressure gradient terms from the momentum equations by cross-differentiation, which leads to a vorticity−transport equation. This, when coupled with the definition of stream function for steady two-dimensional situations, is the basis of the well-known stream function−vorticity method.

Advantages

1. The pressure makes no appearance.
2. Instead of having to deal with the continuity and two momentum equations, we need to solve only two equations to obtain stream function and vorticity.

Disadvantages

1. Calculation of pressure.
2. Difficulty in specification of vorticity at a wall.
3. The method is **valid for two-dimensional** problems as the definition of stream function applies to two-dimensional flow field.

Stream function–Vorticity method

In this method, the vorticity is eliminated by differentiating the N-S equation consecutively with respect to x and y and thereby subtracting it.

x-Momentum: $\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$

Differentiating x-momentum equation with respect to y, we obtain,

$$\rho\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial^2 p}{\partial x\partial y}+\mu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (1)$$

y-Momentum: $\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$

Differentiating y-momentum equation with respect to x, we obtain,

$$\rho\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial^2 p}{\partial x\partial y}+\mu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \qquad (2)$$

Now, subtracting (1)-(2), we have

$$\rho\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)-\rho\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)$$

$$=-\frac{\partial^2 p}{\partial x\partial y}+\mu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial^2 p}{\partial x\partial y}-\mu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right]$$

$$=\mu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right] \qquad (3)$$

Let us define vorticity field,

$$-\xi=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x} \qquad (4)$$

Substituting Eq.(4) in Eq. (3), we obtain,

$$\rho\left(\frac{\partial \xi}{\partial t}+u\frac{\partial \xi}{\partial x}+v\frac{\partial \xi}{\partial y}\right)=\mu\left(\frac{\partial^2 \xi}{\partial x^2}+\frac{\partial^2 \xi}{\partial y^2}\right) \qquad (5)$$

$$\rho\frac{D\xi}{Dt}=\mu\nabla^2\xi$$

Vorticity-transport equation

Let us stream function,

$$v = -\frac{\partial \psi}{\partial x}$$

$$u = \frac{\partial \psi}{\partial y}$$

The stream function automatically satisfies continuity equation

Now, the vorticity field, $$-\xi = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

On substituting the definition of stream function in the vorticity field we obtain,

$$-\xi = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\xi$$

$$\nabla^2 \psi = -\xi$$

Therefore, the stream function and vorticity equations:

$$\nabla^2\psi = -\xi$$

$$\rho\frac{D\xi}{Dt} = \mu\nabla^2\xi$$

For inviscid flow, $\mu = 0, \Rightarrow \frac{D\xi}{Dt} = 0$

Therefore, for a steady inviscid flow, ξ is constant along a streamline.

For irrotational flow, $\nabla^2\psi = 0$

For rotational flow, $\nabla^2\psi = -\xi$

The vorticity transport equation becomes,

$$\rho\left(\frac{\partial\xi}{\partial t} + u\frac{\partial\xi}{\partial x} + v\frac{\partial\xi}{\partial y}\right) = \mu\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right) = \mu\nabla^2\xi$$

$$\Rightarrow \rho\left(\frac{\partial\xi}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial\xi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\xi}{\partial y}\right) = \mu\nabla^2\left(-\nabla^2\psi\right)$$

$$\therefore \frac{\partial}{\partial t}\left(\nabla^2\psi\right) - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} = \nu\nabla^4\psi$$

Determination of pressure for viscous flow

For obtaining pressure field, differentiating x-momentum equation with respect to x, we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)+\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2}+\nu\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{6}$$

Differentiating y-momentum equation with respect to y, we obtain,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right)+\left(\frac{\partial v}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x\partial y}=-\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2}+\nu\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{7}$$

Adding (6)+(7) we have,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+v\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right)\\ =-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)+\nu\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\nu\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\end{aligned} \tag{8}$$

Therefore, from (8) we obtain,

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+v\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$=-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)+\nu\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\nu\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\Rightarrow\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2-2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+u\left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+v\left\{\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}$$

$$=-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)+\nu\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{9}$$

Now, from continuity equation, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

Hence, $$-2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}=-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)$$

$$\Rightarrow\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=2\rho\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)=2\rho\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right]$$

$$\therefore\nabla^2 p=2\rho\left[\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2\right]$$

Creeping flow (very small Reynolds number)

In very slow motions or in motions with very large viscosity, the viscous forces are much greater than the inertia forces. Therefore, it is reasonable to neglect the inertia terms with respect to the viscous terms. We obtain for steady flow

$$\therefore \underbrace{\cancel{\frac{\partial}{\partial t}\left(\nabla^2\psi\right)}}_{=0} - \underbrace{\cancel{\frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}}}_{=0} + \underbrace{\cancel{\frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}}}_{=0} = \nu\nabla^4\psi$$

$$\Rightarrow \nabla^4\psi = 0 \qquad (10)$$

Where, $$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^2}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}$$

Eq. (10) is called *biharmonic equation* for stream function and is also a linear PDE.

Applications of creeping flow

(i) Hydrodynamic theory of lubrication

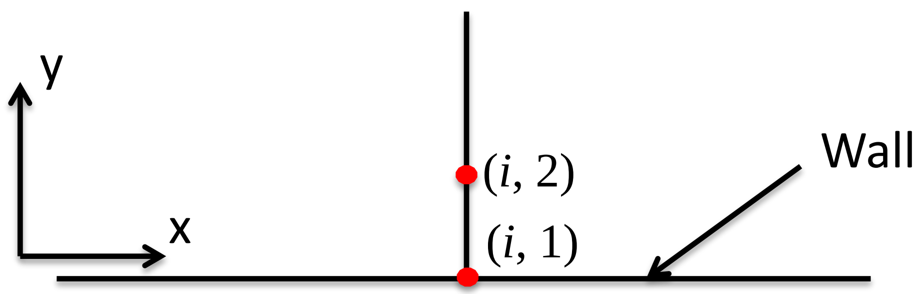
(ii) Polymer and food extrusion

(iii) Flow in porous media

Boundary conditions for stream function and vorticity

- ✓ Stream function boundary conditions are obtained from the velocity distribution.
- ✓ Vorticity boundary conditions are also obtained from velocity distributions except at the walls where a special treatment is required.

a) Vorticity boundary condition at a stationary non-sloping wall



To determine ξ at the wall, a Taylor series expansion of ψ about the wall point (*i*, 1) reads,

$$\psi_{i,2} = \psi_{i,1} + \left.\frac{\partial \psi}{\partial y}\right|_{i,1} \Delta y + \frac{(\Delta y)^2}{2} \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,1} + O(\Delta y)^3$$

Since, no-slip wall satisfies,

$$u_{i,1} = \left.\frac{\partial \psi}{\partial y}\right|_{i,1} = 0$$

Therefore, $$\left.\frac{\partial^2\psi}{\partial y^2}\right|_{i,1} = \left.\frac{\partial u}{\partial y}\right|_{i,1}$$ and, $$\xi_{i,1} = \left.\frac{\partial v}{\partial x}\right|_{i,1} - \left.\frac{\partial u}{\partial y}\right|_{i,1}$$

Since, for non-porous wall, $$v = 0 \Rightarrow \left.\frac{\partial v}{\partial x}\right|_{i,1} = 0$$

Hence, $$\xi_{i,1} = -\left.\frac{\partial u}{\partial y}\right|_{i,1} = -\left.\frac{\partial^2\psi}{\partial y^2}\right|_{i,1}$$

Now, $$\psi_{i,2} = \psi_{i,1} + \left.\frac{\partial \psi}{\partial y}\right|_{i,1}\Delta y + \frac{(\Delta y)^2}{2}\left.\frac{\partial^2\psi}{\partial y^2}\right|_{i,1} + O(\Delta y)^3$$

$$\Rightarrow \psi_{i,2} = \psi_{i,1} + 0\times\Delta y + \frac{(\Delta y)^2}{2}\left(-\xi_{i,1}\right) + O(\Delta y)^3$$

$$\therefore \xi_{i,1} = \frac{2\left(\psi_{i,1} - \psi_{i,2}\right)}{(\Delta y)^2} + O(\Delta y)$$

First-order accurate expression.

Therefore, in general,

$$\therefore \xi_s = \frac{2\left(\psi_s - \psi_{s+1}\right)}{\left(\Delta n\right)^2} + O\left(\Delta n\right)$$

Here, Δn is the normal distance between the grid points s at the wall and one grid point (s+1) away from the wall.

b) Vorticity boundary condition for moving wall

If the wall is moving with a velocity u=U and non-porous, v=0, one can adopt similar procedure of Taylor series expansion to obtain vorticity boundary condition for a moving wall as,

$$\therefore -\xi_{i,j} = \frac{2\left(\psi_{i,j+1} - \psi_{i,j} - U\Delta y\right)}{\left(\Delta y\right)^2} + O\left(\Delta y\right)$$

Here, y is positive away from the wall, and (*i, j*) represents the point at the wall.

c) Vorticity boundary condition at corners of a block

At corners, the velocity derivatives are not continuous at corners and vorticity becomes singular. The remedy is to exclude corners during computations and local refinement of the mesh at corners.