ELEMENTS OF COMPUTATIONAL FLUID DYNAMICS

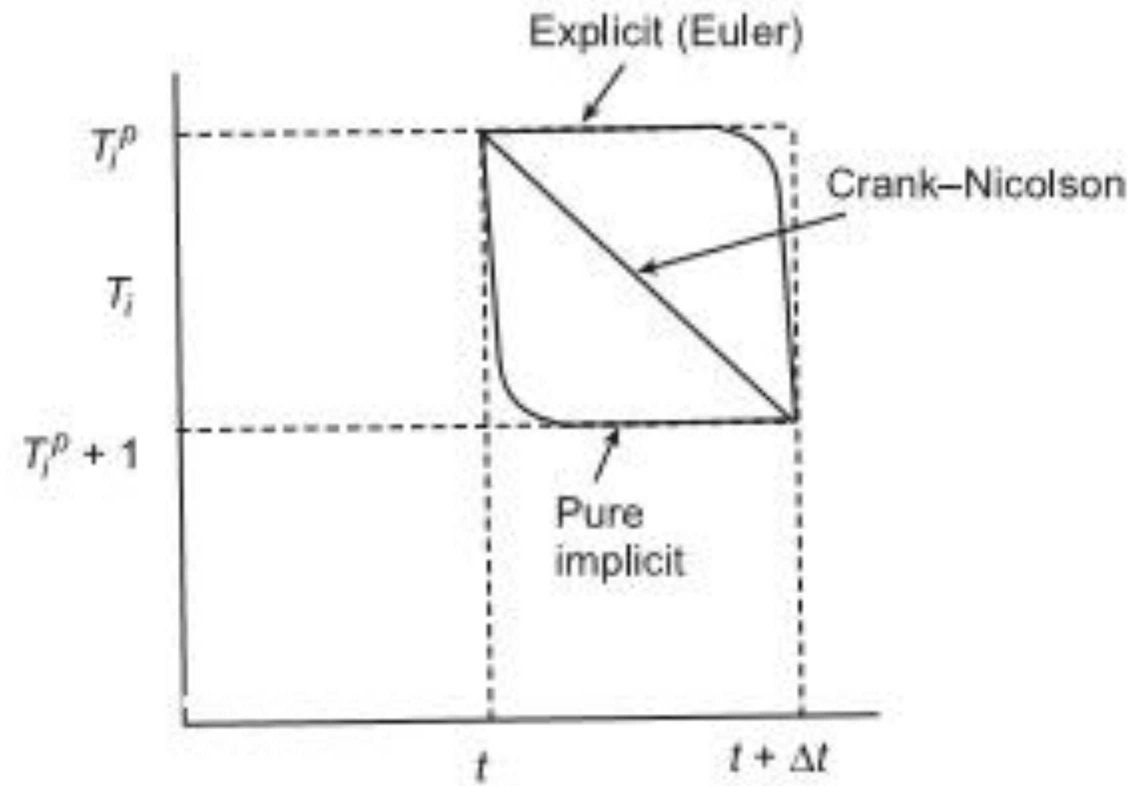
Chapter - 9

Mathematical representation of all three methods by a single discretization equation

$$\frac{\theta_i^{p+1}-\theta_i^{p}}{\Delta\tau}=\eta\frac{\theta_{i+1}^{p+1}-2\theta_i^{p+1}+\theta_{i-1}^{p+1}}{(\Delta X)^2}+(1-\eta)\frac{\theta_{i+1}^{p}-2\theta_i^{p}+\theta_{i-1}^{p}}{(\Delta X)^2}$$

- ✓ For ***η*** = 0, the equation represents the Euler method
- ✓ For ***η*** = 1, the equation represents the pure implicit method
- ✓ For ***η*** = 1/2, the equation represents the Crank−Nicolson method

Physical representation of all three methods



Stability: numerically induced oscillations

- ❖All the three schemes will give better results if time-steps are made smaller.
- ❖ In practice, however, one would usually like to take as large a time-step as one can to reduce the computational effort and time.
- ❖ In addition to decreasing the accuracy of the solution, large time steps can introduce some unwanted, numerically induced oscillations into the solution making it physically unrealistic.
- ❖ Such solutions are not acceptable and the method that produces such solution is called **unstable method**. This brings us to the formal definition of a stable numerical scheme, which is one for which errors from any source are not permitted to grow in the sequence of numerical procedures as the calculation proceeds from one marching step to the next.

von Neumann Stability Analysis

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2}$$

$$IC: \ at \ \tau = 0, \quad \theta = 1, \ for \ all \ X$$

$$for \ \tau > 0,$$

$$BC1: \ at \ X = 0, \ \theta = 0$$

$$BC2: \ at \ X = 1, \ \frac{\partial \theta}{\partial X} = 0$$

Assume a general Solution:

$$\theta(X, \tau) = \sum_{n=1}^{\infty} C_n \psi_n(\tau) e^{i\beta X}$$

Here, $\beta = \text{positive constant}, \ i = \sqrt{-1}$

Let us check stability conditions for Explicit, Implicit, and Cranck-Nicolson methods

Explicit method

$$\frac{\theta_i^{p+1}-\theta_i^p}{\Delta\tau}=\frac{\theta_{i+1}^p-2\theta_i^p+\theta_{i-1}^p}{(\Delta X)^2}$$

Substituting, $\theta_i^p(X,\tau)=\psi(\tau)e^{i\beta X}$

$$\frac{\psi(\tau+\Delta\tau)e^{i\beta X}-\psi(\tau)e^{i\beta X}}{\Delta\tau}=\frac{\psi(\tau)}{(\Delta X)^2}\left[e^{i\beta(X+\Delta X)}-2e^{i\beta X}+e^{i\beta(X-\Delta X)}\right]$$

$$\Rightarrow\frac{\psi(\tau+\Delta\tau)}{\psi(\tau)}=1+\frac{r}{e^{i\beta X}}\left[e^{i\beta(X+\Delta X)}-2e^{i\beta X}+e^{i\beta(X-\Delta X)}\right]$$

Here, $r=\dfrac{\Delta\tau}{(\Delta X)^2}$

Therefore,

$$\Rightarrow\frac{\psi(\tau+\Delta\tau)}{\psi(\tau)}=1-2r+r\left[e^{-i\beta\Delta X}+e^{i\beta\Delta X}\right]=1-2r+r(2\cos\beta\Delta X)$$

$$=1+2r(\cos\beta\Delta X-1)=1-4r\sin^2\left(\frac{\beta\Delta X}{2}\right)$$

Define amplification factor

$$\xi = \frac{\psi(\tau + \Delta\tau)}{\psi(\tau)}$$

Therefore,

$$\xi = 1 - 4r\sin^2\left(\frac{\beta\Delta X}{2}\right)$$

For stability, we require that

$$|\xi| \leq 1$$

$$\Rightarrow \left|1 - 4r\sin^2\left(\frac{\beta\Delta X}{2}\right)\right| \leq 1$$

Now, $\sin^2\left(\frac{\beta\Delta X}{2}\right) \approx 1$ for particular choice of β ΔX.

Therefore , to satisfy the above stability equation

$$r \leq 1/2$$

$$\Rightarrow \frac{\Delta\tau}{(\Delta X)^2} \leq 1/2$$

Implicit method

$$\frac{\theta_i^{p+1}-\theta_i^p}{\Delta\tau}=\frac{\theta_{i+1}^{p+1}-2\theta_i^{p+1}+\theta_{i-1}^{p+1}}{(\Delta X)^2}$$

Substituting, $\theta_i^p\left(X,\tau\right)=\psi\left(\tau\right)e^{i\beta X}$

We obtain following similar procedures, $\xi=\dfrac{1}{1+4r\sin^2\left(\dfrac{\beta\Delta X}{2}\right)}$

Which shows, $|\xi|\le 1$ for all r, unconditionally stable.

Crank-Nicolson

$$\frac{\theta_i^{p+1}-\theta_i^p}{\Delta\tau}=\frac{1}{2}\left[\frac{\theta_{i+1}^{p}-2\theta_i^{p}+\theta_{i-1}^{p}}{(\Delta X)^2}+\frac{\theta_{i+1}^{p+1}-2\theta_i^{p+1}+\theta_{i-1}^{p+1}}{(\Delta X)^2}\right]$$

Substituting, $\theta_i^p\left(X,\tau\right)=\psi\left(\tau\right)e^{i\beta X}$

We obtain following similar procedures, $\xi=\dfrac{1-2r\sin^2\left(\dfrac{\beta\Delta X}{2}\right)}{1+2r\sin^2\left(\dfrac{\beta\Delta X}{2}\right)}$

Which shows, $|\xi|\le 1$ for all r, unconditionally stable.

Consistency of Numerical Scheme

$$Original\ PDE = \text{Discretized PDE} + O\left(\Delta X, \Delta Y, \Delta\tau\right)^n$$

Now, a numerical scheme is consistent, if for $\left(\Delta X, \Delta Y, \Delta\tau\right) \to 0$

$$\text{Discretized PDE} = Original\ PDE$$

Now, a numerical scheme is inconsistent, if for $\left(\Delta X, \Delta Y, \Delta\tau\right) \to 0$

$$\text{Discretized PDE} \neq Original\ PDE$$

Two-dimensional transient problem

$$\frac{\partial T}{\partial t}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

Euler or explicit method of solution leads to

$$\frac{T_{i,j}^{p+1}-T_{i,j}^{p}}{\Delta t}=\alpha\left[\frac{T_{i+1,j}^{p}-2T_{i,j}^{p}+T_{i-1,j}^{p}}{\left(\Delta x\right)^2}+\frac{T_{i,j+1}^{p}-2T_{i,j}^{p}+T_{i,j-1}^{p}}{\left(\Delta y\right)^2}\right]$$

The solution to the above equation face no difficulties if the stability condition is satisfied

$$\Delta t\le\frac{1}{2\alpha\left[\left(\Delta x\right)^{-2}+\left(\Delta y\right)^{-2}\right]}$$

Similarly pure implicit method of solution leads to

$$\frac{T_{i,j}^{p+1}-T_{i,j}^{p}}{\Delta t}=\alpha\left[\frac{T_{i+1,j}^{p+1}-2T_{i,j}^{p+1}+T_{i-1,j}^{p+1}}{\left(\Delta x\right)^2}+\frac{T_{i,j+1}^{p+1}-2T_{i,j}^{p+1}+T_{i,j-1}^{p+1}}{\left(\Delta y\right)^2}\right]$$

For $\Delta x=\Delta y$

$$-rT_{i-1,j}^{p+1}-rT_{i,j-1}^{p+1}+(1+4r)T_{i,j}^{p+1}-rT_{i,j+1}^{p+1}-rT_{i+1,j}^{p+1}=T_{i,j}^{p},\quad \text{where, } r=\alpha\Delta t/(\Delta x)^2$$

Alternating Direction Implicit (ADI) method

This method employs two difference equations, which are used in turn over successive time-steps of duration Δt/2. The first equation is implicit in x-direction, whereas, the second one is implicit y-direction.

Let, $T^*_{i,j}$ is an intermediate value at the end of first Δt/2 time step, then,

$$\frac{T^*_{i,j} - T^p_{i,j}}{(\Delta t/2)} = \alpha\left[\frac{T^*_{i+1,j} - 2T^*_{i,j} + T^*_{i-1,j}}{(\Delta x)^2} + \frac{T^p_{i,j+1} - 2T^p_{i,j} + T^p_{i,j-1}}{(\Delta y)^2}\right]$$

Followed by,

$$\frac{T^{p+1}_{i,j} - T^*_{i,j}}{(\Delta t/2)} = \alpha\left[\frac{T^*_{i+1,j} - 2T^*_{i,j} + T^*_{i-1,j}}{(\Delta x)^2} + \frac{T^{p+1}_{i,j+1} - 2T^{p+1}_{i,j} + T^{p+1}_{i,j-1}}{(\Delta y)^2}\right]$$

For $\Delta x = \Delta y$

$$-T^*_{i-1,j} + 2\left(\frac{1}{r}+1\right)T^*_{i,j} - T^*_{i+1,j} = T^p_{i,j-1} + 2\left(\frac{1}{r}-1\right)T^p_{i,j} + T^p_{i,j+1}, \quad \text{where, } r = \alpha\Delta t/(\Delta x)^2$$

$$-T^{p+1}_{i,j-1} + 2\left(\frac{1}{r}+1\right)T^{p+1}_{i,j} - T^{p+1}_{i,j+1} = T^*_{i-,j} + 2\left(\frac{1}{r}-1\right)T^*_{i,j} + T^*_{i+1,j}$$

Accuracy $O\left[(\Delta x)^2, (\Delta y)^2, (\Delta t)^2\right]$

False transient approach

In this method, the solution of an elliptic problem can be obtained by using the methods for the solution of parabolic problem.

Example: For a steady elliptic problem,

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=0 \quad \text{in a region } R$$

Can be solved by assuming,

$$\frac{\partial T}{\partial t}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

Some initial condition is first applied throughout *R.*

Here, all the schemes of transient problem can be applied and can be time-stepped till the steady state is reached.

False transient: Although the problem is not time-dependent, it is solved in a manner as if it is time-dependent.