B. ETCE 1st year 1st sem 2018
Mathematics-I G

Time : Three hours Full Marks : 100

(50 marks for each group)

Use seperate answer sheet for each group
Group-A
(Answer any five questions)

Q.1.(i) If $y = \tan^{-1} x$ then prove that

$$(x^2+1)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0.$$

5

Q.1.(ii) A twice differentiable function $f(x)$ on a closed interval $[a,b]$ is such that $f(a) = f(b) = 0$ and $f(x_0) > 0$ where $a < x_0 < b$. Prove that there exists at least one value of $x = c$(say) between a and b for which $f''(c) < 0$. **5**

Q.2.(i) Show that $2x - \cos x - \sin x = 0$ has exactly one solution. **5**
Q.2.(ii) Show that $2/\pi < \sin x/x < 1$ when $0 < x < \pi/2$. **5**

Q.3.(i) Examine the extreme value, if $f(x) = x^5 - 5x^4 + 5x^3 + 12$. **5**
Q.3.(ii) Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base. **5**

Q.4.(i) Find the values of p and q such that **5**

$$lim_{x \longrightarrow 0} \frac{x(1 - p\cos x) + q \sin x}{x^3} = \frac{1}{3}$$

Q.4.(ii) Expand $\log_e(1+x)$ in a finite series in powers of x, with remainder in Lagrange's form. **5**

Q.5. Find all the asymptotes of the curve $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$ **10**

[Turn over

Q.6.(i) Prove that the sequence $\{x_n\}$ defined by

$$x_1 = \sqrt{2},\ x_{n+1} = \sqrt{2x_n} \quad for \quad n \geq 1$$

converges to 2. 5

Q.6.(ii) Examine the convergence of the series 5

$$\frac{5}{1.2.4} + \frac{7}{2.3.5} + \frac{9}{3.4.6} + \frac{11}{4.5.7} + \dots.$$

Q.7.(i) Prove that the sequence $f_n(x)$ of functions where

$$f_n(x) = x^{n-1} - x^n, x \in [0,1]$$

is uniformly convergent on $[0,1]$. 5

Q.7.(ii) Prove that the series $\sum \frac{x}{n+n^2x^2}$ is uniformly convergent for all real x. 5

Group-B
(Answer any five questions)
Use seperate answer sheet for each group

Q.8. Evaluate $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$, $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ and $\lim_{(x,y)\to(0,0)} f(x,y)$, if they exist, for the following functions: 5+5

$$\begin{aligned}
(i)\ \ f(x,y) &= x\sin\frac{1}{y} + y\sin\frac{1}{x}\ , \quad xy \neq 0 \\
&= x\sin\frac{1}{y}\ , \qquad\qquad y \neq 0 \\
&= y\sin\frac{1}{x}\ , \qquad\qquad x \neq 0 \\
&= 0\ , \qquad\qquad\qquad x = 0,\ y = 0 \\
(ii)\ \ f(x,y) &= x^2\{\sin\frac{1}{y} + \frac{1}{x^2+y^2}\}\ , \quad y \neq 0 \\
&= 0\ , \qquad\qquad\qquad y = 0
\end{aligned}$$

Q.9.(i). Show that the function $f : R^2 \longrightarrow R$ defined by

$$\begin{aligned} f(x,y) &= \frac{x^2+3y^2}{3x^2+y^2}, (x,y) \neq (0,0) \\ &= 0, (x,y) = (0,0) \end{aligned}$$

is not continuous at the origin(0,0). 5

Q.9.(ii). If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

5

Q.10.(i). Let $f : R^2 \longrightarrow R$ defined as

$$\begin{aligned} f(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0) \\ &= 0, \qquad (x,y) = (0,0) \end{aligned}$$

Verify whether the function is differentiable at the origin. 5

Q.10.(ii). Find the directional derivative $D_{(\alpha)}f(x_0, y_0)$ of the function $f(x,y)$ at a given point (x_0, y_0) along the given direction α where 5

$$f(x,y) = x^2 - y^2, \quad x_0 = 1, y_0 = 2, \quad \alpha = 60^0.$$

Q.11.(i) If $z = f(x,y)$ and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ then prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

5

Q.11.(ii) Find the equation of the tangent plane to the surface defined by $f(x,y) = 2\cos(x-y) + 3\sin x$ at the point $(\pi, \pi/2)$. 5

Q.12. (i) If $u = x+y-z$, $v = x-y+z$, $w = x^2+y^2+z^2-2yz$ then show that 5

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

Q.12.(ii) State Weierstrass's M-test. Use it to test the uniform convergence of the series $\sum f_n(x)$, where $f_n(x) = \frac{\sin nx}{x^2+n^2}, x \geq 1$. 5

Q.13. Find and classify the extreme values (if any) of the function $f(x,y) = 2(x-y)^2 - x^4 - y^4$. 10

Q.14. A right cylindrical can is to have a volume of 0.25 cubic feet. Find the height h and radius r that will minimize the surface area of the can. 10