Ref. No.: Ex/CSE/MATH/T/211A/2018

B.CSE, 2ND YR. 1ST SEMS EXAM, 2018

MATHEMATICS - IV

Full Marks: 100 Time: Three Hours

Answer question number 1 and any six questions from the rest.

1. Find the radius of convergence of the following series. (4)

   (a) $\sum_{n=1}^{n=\infty} \frac{(ax+b)^n}{c^n}$, where a, b and c are real numbers and $c \neq 0$.

   (b) $\sum_{n=1}^{\infty} \frac{2^{2n}x^n}{n^2}$

2. Solve Hermite differential equation

$$y'' - 2xy' + 2\alpha y = 0,$$

where α is a constant.

   (a) Find two linearly independent solutions near $x = 0$. Write first three terms in each series. (8)

   (b) Find radius of convergence for both the series. (2)

   (c) Show that there is a polynomial solution of degree n, in case $\alpha = n$, a non-negative integer. (2)

   (d) Find those particular polynomials of degree n denoted by $H_n(x)$ for $n = 0, 1, 2, 3$, such that coefficient of x^n in $H_n(x)$ is equal to 2^n. (4)

3. (a) Classify the singularity of the differential equation $xy'' - y' + 4x^3y = 0$. Find Frobenius series solution about the singular point of the equation. Write first three non-zero terms in each series. Also express the solution in terms of elementary functions. (10)

   (b) Find general solution of the Cauchy-Euler equation $9x^2y'' + 3xy' + y = 0$ (6)

4. (a) Find general solution of the differential equation (8)

$$y'' + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11.$$

   (b) Use the method of variation of parameters to find a particular integral of the differential equation (8)

$$y'' + 6y' + 9y = \frac{e^{-3x}}{x^3}.$$

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5. (a) Prove that (10)

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & ,m \neq n \\ \frac{2}{2n+1} & ,m = n \end{cases}$$

where $P_n(x)$ is the Legendra polynomial of degree n.

(b) Write generating function of Legendre ploynomials. Use that function to prove (6)

i. $P_n(-x) = (-1)^n P_n(x)$

ii. $P_{2n}(0) = (-1)^n \frac{1.3.5.........(2n-1)}{2^n n!}$

6. (a) State the orthogonality property of Chebyshev ploynomials of first kind. Plot first five Tchebyshev polynomials of first kind. Find the Tchebyshev series expansion of $sin(cos^{-1}x)$. Write first five terms of the series. (10)

(b) Prove that $T_n(x) = \cos(ncos^{-1}x)$ is a polynomial of degree n. Derive a recursion relation on $T_n(x)$. (6)

7. (a) Show that $f(z) = \bar{z}$ is nowhere differentiable. (4)

(b) Show that the function $f(z) = (x^2 + y) + i(y^2 - x)$ is not analytic at any point. (4)

(c) Calculate $\int_\gamma |z|^2 dz$, where γ denotes the contour that goes (6)
(i) vertically from 0 to i, then horizontally from i to $1 + i$,
(ii) horizontally from 0 to 1, then vertically from 1 to $1 + i$.

(d) Write all values of i^i in the form $a + ib$. (2)

8. (a) Use Cauchy integral formula to show that (4)

$$\int_\gamma |z+1|^2 dz = 2\pi i, \quad \text{where} \quad \gamma(t) = e^{it}, 0 \leq t \leq 2\pi$$

(b) Find Laurent series expansions of the following functions around $z = 0$ (8)
(i) $\frac{1}{z(1-z)}$ valid for $0 < |z| < 1$,
(ii) $z^3 e^{1/z}$ valid for $|z| > 0$.

(c) Consider a circle of radius 1, and let $Q_1, Q_2, ..., Q_n$ be the vertices of a regular n-gon inscribed in a circle. Join Q_1 to $Q_2, Q_3, ...Q_n$ by segments of lengths $\lambda_2, \lambda_3, ...\lambda_n$. Show that (4)

$$\prod_{i=2}^{n} \lambda_i = n.$$

9. (a) Find the poles and their orders of the functions (4)
(i) $\frac{1}{z^4+16}$ (ii) $\frac{1}{z^2+z-1}$

(b) Describe the type of singularity at $z = 0$ of each of the following functions (4)
(i) $z^3 \sin^2 z$ (ii) $\frac{\cos z - 1}{z^2}$

(c) By considering the function $f(z) = \frac{e^{iz}}{z^2+4z+5}$ integrated around suitable contour , find (8)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5} dx.$$

10. (a) Expand $f(x) = x^2, 0 < x < 2\pi$ in a Fourier series if the period of $f(x)$ is 2π. Hence find the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (10)

(b) Find Fourier series expansion of a function of period 10, given by (6)

$$f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$$